Combinatorial aspects of bow varieties

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Summary

First introduced by Cherkis in theoretical physics, bow varieties form a rich family of symplectic varieties generalizing Nakajima quiver varieties. An algebro-geometric definition was later given by Nakajima and Takayama via moduli spaces of quiver representations. The main goal of this thesis is to study the torus equivariant cohomology of bow varieties. Our study is motivated by classical Schubert calculus and lays the foundation for a Schubert calculus for bow varieties, where the underlying quiver is of finite type A.

The crucial main mathematical tool we use is the theory of *stable envelopes* of Maulik and Okounkov. We show that this theory applies to bow varieties and study it with the main focus on explicit calculations. As a main result of this thesis we generalize a fundamental ingredient of classical Schubert calculus to the world of bow varieties: *The Chevalley–Monk formula*. Our generalization of this formula characterizes the multiplication of tautological divisors with respect to the stable envelope basis.

In the first part of this thesis, we give a self-contained introduction to the construction of bow varieties and their geometric properties following the work of Nakajima and Takayama. In particular, we recall the classification of torus fixed points of bow varieties and use a similar method to prove that this classification also holds for generic one-parameter torus actions.

In the second part, we redevelop the theory of stable envelopes in the framework of bow varieties. The main result is a self-contained reproof of the existence of stable envelopes for bow varieties using the deformation to the normal cone construction due to Fulton. This proof provides in particular an algorithm which computes the stable basis elements as linear combinations of the fundamental classes of attracting cells.

Motivated by the localization principle in torus equivariant cohomology, we prove in the next part a formula which determines the equivariant multiplicities of stable basis elements at torus fixed points via a diagrammatic calculus of permutations and symmetric groups. The main ingredient for this formula is the Resolution Theorem due to Botta and Rimányi.

In the final part of this thesis, we state and prove the Chevalley–Monk formula theorem for bow varieties. In the proof we use orthogonality properties of stable basis elements which are similar to the orthogonality properties of (equivariant) Schubert classes. A further important ingredient in the proof is a certain divisibility theorem for equivariant multiplicities of stable basis elements which we prove using our diagrammatic calculus of symmetric groups.

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Chapter 1

Introduction

This thesis describes and studies algebraic invariants of bow varieties. Bow varieties are certain generalizations of Nakajima quiver varieties which form a rich family of holomorphic symplectic moduli spaces of quiver representations. Their origin goes back to theoretical physics, where they were introduced by Cherkis, [Che09], [Che10], [Che11]. In the context of quiver representations, they were defined by Nakajima and Takayama in [NT17]. The name bow variety indicates the important part of the quiver which looks like a bow, see [Che09]. The easiest examples of bow varieties are cotangent bundles of Grassmannians and (more generally) partial flag varieties. Other famous varieties from geometric representation theory which can be realized as bow varieties are Nakajima quiver varieties, see [NT17], and equivariant Slodowy slices, see [Los06], [RR23]. In general, bow varieties can become very involved.

The main goal of this thesis is to give a detailed and explicit description of the torus equivariant cohomology of bow varieties corresponding to finite type A quivers. Motivated by the classical Schubert calculus, which is in particular a powerful tool to describe and understand cohomology rings of Grassmannians, we develop the foundations of a *Schubert calculus for bow varieties*. The generalizations of the Schubert classes are in this picture the stable envelope classes introduced by Maulik and Okounkov in [MO19].

More precisely, the purpose of this thesis is twofold:

- In the framework of bow varieties, we bring together the general theory of stable envelopes of Maulik and Okounkov with concrete combinatorial models. In this way, we provide a treatment of bow varieties from a new Schubert calculus point of view.
- As a main result (Theorem 10.26), we give then a general Chevalley–Monk formula. It describes the multiplication in equivariant cohomology algebras of bow varieties in a combinatorial way.

1.1 Classical Chevalley–Monk formula

The classical *Chevalley–Monk formula*, see [Mon59], [Che94], characterizes the multiplication of tautological divisors in the singular cohomology of partial flag varieties. Thereby, this formula uniquely determines the ring structure of these singular cohomology rings. The formula is best known in the special case of Grassmannians, where the cohomology can be described in combinatorial terms using symmetric functions and partitions. For Grassmannians, the Chevalley–Monk formula coincides with a special case of the much older Pieri's formula, see e.g. [EH16, Proposition 4.9]. We will now describe this formula as well as some applications and generalizations in the special case of Grassmannians. In this framework, the formula can be conveniently described using the combinatorics of partitions.

Enumerative problems reformulated via divisors

We go back to the origins of Schubert calculus and consider the following classical question from enumerative geometry, see [Sch79, Chapter 19]:

(Q1) Given 4 lines $L_1, \dots, L_4$ in the complex projective space $\mathbb{P}^3$ in general position, how many lines in $\mathbb{P}^3$ meet all the four lines $L_1, \dots, L_4$?

It is a well-known result that the answer to this question is 2. We now explain how the answer can be obtained via the intersection theory on Grassmannians: Let $\mathrm{Gr} = \mathrm{Gr}(2,4)$ be the Grassmannian parameterizing 2-dimensional subspaces of $\mathbb{C}^4$. To relate to (Q1), we identify Gr with the moduli space of lines in $\mathbb{P}^3$. The question (Q1) can then be reformulated as:

(Q1') What is the order of the set $\bigcap_{i=1}^4 \{V \in G \mid V \cap L_i \neq \emptyset\}$?

The variety Gr admits the tautological bundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$, where

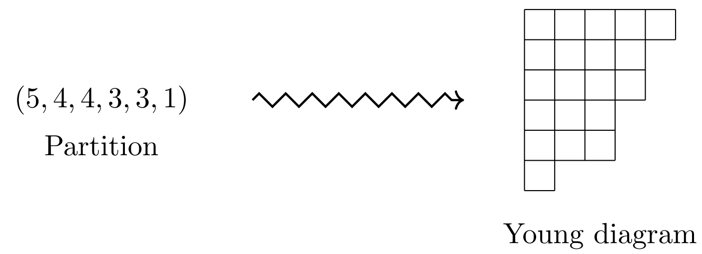
$$\mathcal{S} = \{(L, v) \mid v \in L\} \subset \mathrm{Gr} \times \mathbb{C}^4, \quad \mathcal{Q} = (\mathrm{Gr} \times \mathbb{C}^n)/\mathcal{S}.$$

The global sections of $\mathcal{Q}$ are parameterized by $\mathbb{C}^4$. For two linear independent vectors $v, w \in \mathbb{C}^4$, the global section $v \wedge w$ of the exterior power $\bigwedge^2 \mathcal{Q}$ vanishes at a point $V \in \mathrm{Gr}$ if and only if V intersects the plane spanned by v and w. There is a characteristic class of Gr in the singular cohomology $H^*(\mathrm{Gr})$ which is exactly the cohomology class corresponding to this vanishing locus: The first Chern class of the quotient bundle $c_1(\mathcal{Q}) \in H^2(\mathrm{Gr})$. Thus, by Kleiman's Transversality Theorem [Kle74], we can reformulate (Q1') as follows:

(Q1") What is the degree of $c_1(\mathcal{Q})^4$ in $H^*(\mathrm{Gr})$?

This degree can be determined in terms of *combinatorial intersection theory*, the main ingredient of Schubert calculus. Going back to Hermann Schubert [Sch79], it describes the enumerative geometry of subspaces of a vector space which allows to describe in particular the cohomology ring of Grassmannians. For a general introduction to Schubert calculus, see e.g. [KL72], [Ful97], [EH16].

The Gauss decomposition algorithm implies that the singular cohomology $H^*(\mathrm{Gr}(k,n))$ of any Grassmannian $\mathrm{Gr}(k,n)$ is a free $\mathbb{Z}$-module and is equipped with a homogeneous basis $(\mathfrak{S}_\lambda)_{\lambda \in \mathcal{P}(k,n)}$ which is called the *Schubert basis*, see e.g. [Ful97] for more details on the construction of the Schubert basis. This basis is naturally labeled by the set $\mathcal{P}(k,n)$ of all partitions, where the corresponding Young diagram has at most k rows and $n-k$ columns.



The cohomological degree of the corresponding Schubert class $\mathfrak{S}_\lambda$ equals $2 \cdot |\lambda|$, where $|\lambda|$ is the number of boxes in λ. In the special case $(k, n) = (2, 4)$ this leads to the following basis:

$H^0(\mathrm{Gr})$	$H^2(\mathrm{Gr})$	$H^4(\mathrm{Gr})$	$H^6(\mathrm{Gr})$	$H^8(\mathrm{Gr})$
$\mathfrak{S}_\emptyset$	$\mathfrak{S}_{\ydiagram{1}}$	$\mathfrak{S}_{\ydiagram{1,1}}$, $\mathfrak{S}_{\ydiagram{2}}$	$\mathfrak{S}_{\ydiagram{2,1}}$	$\mathfrak{S}_{\ydiagram{2,2}}$

The construction of the Schubert basis implies that in general $\mathfrak{S}_{\ydiagram{1}} = c_1(\mathcal{Q}_{k,n})$, where $\mathcal{Q}_{k,n}$ is the quotient bundle on $\mathrm{Gr}(k, n)$. By Pieri's formula, the multitplication of $c_1(\mathcal{Q}_{k,n})$ with respect to the Schubert basis is given as follows:

$$c_1(\mathcal{Q}_{k,n}) \cdot \mathfrak{S}_\lambda = \sum_{\mu \in M_\lambda} \mathfrak{S}_\mu. \tag{1.1}$$

Here, M_λ is the set of $\mu \in \mathcal{P}(k, n)$ with the property that λ can be obtained from μ by removing a single box.

Using (1.1), we can now easily determine the product $\mathfrak{S}_{\ydiagram{1}}^4$ in $H^*(\mathrm{Gr})$:

$$\mathfrak{S}_{\ydiagram{1}} \xmapsto{\mathfrak{S}_{\ydiagram{1}}\cdot} \mathfrak{S}_{\ydiagram{1,1}} + \mathfrak{S}_{\ydiagram{2}} \xmapsto{\mathfrak{S}_{\ydiagram{1}}\cdot} 2\mathfrak{S}_{\ydiagram{2,1}} \xmapsto{\mathfrak{S}_{\ydiagram{1}}\cdot} 2\mathfrak{S}_{\ydiagram{2,2}}.$$

By construction, $\mathfrak{S}_{\ydiagram{2,2}} \in H^8(\mathrm{Gr})$ is the cohomology class corresponding to a point. Thus, the degree of $\mathfrak{S}_{\ydiagram{1}}^4$ indeed equals 2 which gives the desired answer to (Q1).

Overall, due to this well established understanding of the multiplication of the tautological divisor with respect to the Schubert basis a variety of enumerative questions in the flavor of (Q1) can be solved, see e.g. the exposition given in [EH16].

Generalization to torus equivariant cohomology

We may more generally study the torus equivariant cohomology ring $H_T^*(\mathrm{Gr}(k, n))$, where the action of the torus $T = (\mathbb{C}^*)^n$ on $\mathrm{Gr}(k, n)$ is induced by the standard T-action on $\mathbb{C}^n$. The localization principle in torus equivariant cohomology, see e.g. [AF23, Chapter 7], provides a strong interplay between local and global data of equivariant cohomology classes which makes torus equivariant cohomology a powerful cohomology theory which is well-suited for explicit computations. By construction, $H_T^*(\mathrm{Gr}(k, n))$ is a module over $H_T^*(\mathrm{pt})$ which is in fact free. The $\mathbb{Z}$-algebra $H_T^*(\mathrm{pt})$ is isomorphic to the polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$, where the variable t_i corresponds to the i-th factor of T. There is also a T-equivariant version of Schubert classes $(\mathfrak{S}_\lambda^T)_{\lambda \in \mathcal{P}(k,n)}$ which form a basis of $H_T^*(\mathrm{Gr}(k, n))$ over $H_T^*(\mathrm{pt})$, see e.g. [AF23, Chapter 9].

If we set all equivariant parameters equal to 0, the T-equivariant cohomology $H_T^*(\mathrm{Gr}(k,n))$ specializes to the usual singular cohomology $H^*(\mathrm{Gr}(k,n))$ and each torus equivariant Schubert class $\mathfrak{S}_\lambda^T$ specializes to the classical Schubert class $\mathfrak{S}_\lambda$.

The formula (1.1) extends to the torus equivariant setting by just adding a diagonal term, see e.g. [AF23, Theorem 9.6.2]:

$$c_1^T(\mathcal{Q}_{k,n}) \cdot \mathfrak{S}_\lambda^T = \Big(\sum_{i \in E_\lambda} t_i\Big) \cdot \mathfrak{S}_\lambda^T + \sum_{\mu \in M_\lambda} \mathfrak{S}_\mu^T. \tag{1.2}$$

Here, $c_1^T(\mathcal{Q}_{k,n})$ is the first T-equivariant Chern class of $\mathcal{Q}_{k,n}$ and

$$E_\lambda = \{1, \dots, n\} \setminus \{\lambda_1 + k, \lambda_2 + (k-1), \dots, \lambda_k + 1\}.$$

For instance, if $(k,n) = (2,4)$ and $\lambda = \square$ then

$$c_1^T(\mathcal{Q}_{2,4}) \cdot \mathfrak{S}_\square^T = (t_2 + t_4)\mathfrak{S}_\square^T + \mathfrak{S}_{\substack{\square\\\square}}^T + \mathfrak{S}_{\square\square}^T.$$

Setting $t_1 = \ldots = t_n = 0$ in (1.2) gives back the classical formula (1.1). In general, the multiplication rules in the torus equivariant cohomology of Grassmannians are connected, and in fact geometric incarnations, of deep combinatorial results in the theory of symmetric functions, see e.g. [MS99], [KT03], [MS99], [AF23].

Generalization to the cotangent bundle

Now, we leave classical Schubert calculus and enter the world of holomorphic symplectic varieties by passing from $\mathrm{Gr}(k,n)$ to its cotangent bundle $T^*\mathrm{Gr}(k,n)$. As $T^*\mathrm{Gr}(k,n)$ is a vector bundle over $\mathrm{Gr}(k,n)$, this variety admits a further $\mathbb{C}^*$-action given by scaling the fibers and we like to study its $(T \times \mathbb{C}^*)$-equivariant cohomology. The fact that we deal with a holomorphic symplectic variety provides us a very deep and powerful tool: *The stable envelope bases* of Maulik and Okounkov from [MO19]. These bases should be viewed as analogues of the equivariant Schubert bases for $\mathrm{Gr}(k,n)$. More precisely, the family of stable envelope bases should be viewed as analogue of the family of equivariant Schubert bases indexed by different choices of Borels or cocharacters respectively, see [GKS20].

Stable envelopes exist for a large family of symplectic varieties with torus action. They provide families of bases of the torus equivariant cohomology depending on the choice of a generic cocharacter. These bases are uniquely characterized by three stability conditions: a *normalization*, a *support* and a *smallness* condition. These stability conditions are similar to the uniqueness conditions appearing in equivariant Schubert calculus, see e.g. [KT03], [GKS20]. Just as in equivariant Schubert calculus, the base change matrix of different stable envelope bases produces a solution of the Yang–Baxter equation and thus provide interesting braid group actions on cohomology. The usual FRT-construction [FRT88], [Kas95] then can be applied and provides a Hopf algebra acting on the torus equivariant cohomology of the respective symplectic variety.

In the special case of cotangent bundles of Grassmannians, this Hopf algebra is a one-parameter deformation of the Yangian for $\mathfrak{gl}_2$, see [MO19, Chapter 11] and also [RTV15]. The multiplication of $c_1^{T\times\mathbb{C}^*}(\mathcal{Q}_{k,n})$ with respect to the stable envelope basis was determined in [MO19, Theorem 10.1.1], see also [Su16, Theorem 3.1]. As explained in [RTV15], these matrices coincide with certain limits of the dynamical Hamiltonians for the XXX model for $\mathfrak{gl}_2$ as defined earlier by Tarasov and Varchenko, see [TV00], [TV05]. Amazingly, this formula can be expressed in terms of partitions as follows, see the explanation in Section 10.8:

$$c_1^{T\times\mathbb{C}^*}(\mathcal{Q}_{k,n})\cdot \mathrm{Stab}_\sigma(\lambda) = \Big(\sum_{i\in E_\lambda} t_i\Big)\cdot \mathrm{Stab}_\sigma(\lambda) + \sum_{\mu\in \mathrm{RH}_\lambda} (-1)^{|\mu|-|\lambda|-1}\cdot h\cdot \mathrm{Stab}_\sigma(\mu). \qquad (1.3)$$

Here, $\sigma\colon \mathbb{C}^*\to T$, $t\mapsto (t,t^2,\dots,t^n)$ is the chosen cocharacter, h is the equivariant parameter corresponding to the additional $\mathbb{C}^*$-factor in $T\times\mathbb{C}^*$ and RH_λ is the set of all partitions $\mu\in\mathcal{P}(k,n)$ such that λ can be obtained from μ by deleting a single rim hook. Recall that a rim hook of a Young diagram μ is a collection of contiguous boxes running along the border of μ:

For instance, if again $k=2$, $n=4$ and $\lambda=\square$ then

$$c_1^{T\times\mathbb{C}^*}(\mathcal{Q}_{2,4})\cdot \mathrm{Stab}_\sigma(\square) = (t_2+t_4)\mathrm{Stab}_\sigma(\square) + h\mathrm{Stab}_\sigma(\ydiagram{1,1}) + h\mathrm{Stab}_\sigma(\ydiagram{2}) + h\mathrm{Stab}_\sigma(\ydiagram{2,2}).$$

The stable envelope bases not only share characterizing properties of equivariant Schubert bases, but can moreover be viewed as one-parameter deformations of Schubert bases: As explained in e.g. [AMSS23], the stable enevlope basis element $\mathrm{Stab}_\sigma(\lambda)$ is a one parameter deformation of $\mathfrak{S}_\lambda^T$ and $\mathrm{Stab}_\sigma(\lambda)$ degenerates to $\mathfrak{S}_\lambda^T$ via the limit

$$\lim_{h\to\infty}\Big(h^{|\lambda|-k(n-k)}\mathrm{Stab}_\sigma(\lambda)\Big) = \mathfrak{S}_\lambda^T.$$

This limit implies that (1.3) degenerates to (1.2).

1.2 New picture: bow varieties

Nakajima quiver varieties are a fundamental part of geometric representation theory as they for instance geometrize universal enveloping algebras of Kac–Moody algebras and their corresponding integrable models. For an exposition to the topic see e.g. [Gin12] or [Kir16]. As explained in [NT17, Theorem 2.15], the family of bow varieties extends the family of Nakajima quiver varieties. Thus, we can in particular realize cotangent bundles of partial flag varieties as bow varieties, see Theorem 2.67 for a precise statement.

In this thesis, we will only consider bow varieties corresponding to finite type A quivers, see [Gai24] for other types.

Passing from Nakajima quiver varieties to bow varieties adds new features to the theory. In particular, the family of bow varieties is equipped with a class of isomorphisms between bow varieties corresponding to different input data. These isomorphisms correspond to certain transition moves on the input datum of bow varieties, see [NT17] and the exposition in Section 2.4. These transition moves are ultimately motivated by the Hanany–Witten transition in string theory from [HW97], see also the explanation in [NT17]. Thus, the corresponding isomorphisms of bow varieties are called *Hanany–Witten isomorphisms*.

By their construction, bow varieties are endowed with an action of a torus $\mathbb{T}$ which scales the symplectic form. In the case of cotangent bundles of partial flag varieties, this torus action matches with the $(T \times \mathbb{C}^*)$-action from the previous section. It was proved in [Nak18, Theorem A.5] that bow varieties admit finitely many $\mathbb{T}$-fixed points which can be classified via matrices with entries in $\{0, 1\}$ with fixed row and column vectors, see Section 3.2 for precise statements. This explicit classification result makes the family of bow varieties well-suited for explicit computations in torus equivariant cohomology. In particular, the torus fixed point combinatorics of bow varieties naturally generalizes the torus fixed point combinatorics of partial flag varieties.

In fact, the classification torus fixed points of bow varieties of finite type A appears as a shadow of a more general Fock space combinatorics in the affine type A case, see [Nak18].

Chevalley–Monk formula for bow varieties

The aforementioned classification of $\mathbb{T}$-fixed points is the starting point of our treatment. We now give an exposition of the main result of this thesis: A Chevalley–Monk formula for bow varieties (Theorem 10.26).

As mentioned already, bow varieties fit well in the framework of Maulik and Okounkov's theory which provides in particular the existence of stable envelope bases for bow varieties. The elements in a stable envelope basis are labeled by the torus fixed points of the respective bow variety.

The construction of bow varieties provides a family of tautological bundles $(\xi_X)_X$. As explained in Section 10.1, the mutliplication with its first Chern classes uniquely determines the ring structure on the localized equivariant cohomology rings of bow varieties. In the special case of cotangent bundles of Grassmannians, the tautological bundles ξ_X are all isomorphic to the quotient bundle, see Theorem 2.69.

Theorem 10.26 gives a formula for the multiplication of the $\mathbb{T}$-equivariant first Chern classes of tautological bundles on bow varieties with respect to the stable envelope basis:

Theorem A (Chevalley–Monk formula for bow varieties)**.** *Let M be a matrix labeling a $\mathbb{T}$-fixed point of a bow variety and let ξ_X be a tautological bundle. Then, we have*

$$c_1^{\mathbb{T}}(\xi_X) \cdot \mathrm{Stab}_\sigma(M) = \iota_M^*(c_1^{\mathbb{T}}(\xi_X)) \cdot \mathrm{Stab}_\sigma(M) + \sum_{M' \in \mathrm{SM}_{M,X}} \mathrm{sgn}(M, M') \cdot h \cdot \mathrm{Stab}_\sigma(M'). \quad (1.4)$$

Here, $\iota_M^*(c_1^{\mathbb{T}}(\xi_X))$ is the equivariant multiplicity of $c_1^{\mathbb{T}}(\xi_X)$ at the $\mathbb{T}$-fixed point of $\mathcal{C}(\mathcal{D})$ labeled by the matrix M and h is an equivariant parameter corresponding to the scaling of the symplectic form. The set $\mathrm{SM}_{M,X}$ is a certain set of matrices that differ from M by replacing a $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$-minor of M with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We refer to the replacement move defining the elements in $\mathrm{SM}_{M,X}$ as a *simple move*, see Definition 10.3 for a precise treatment.

Application to classical setting

Consider again the cotangent bundle $T^*\mathrm{Gr}(k,n)$. As we explain in Section 3.2, the bow variety realization of $T^*\mathrm{Gr}(k,n)$ comes with a labeling of the $(T \times \mathbb{C}^*)$-fixed points of $T^*\mathrm{Gr}(k,n)$ by the set $\mathrm{B}_{k,n}$ of $(2 \times n)$-matrices M with entries in $\{0,1\}$ satisfying the following row and column sum conditions:

$$\sum_{i=1}^{n} M_{1,i} = k, \quad \sum_{i=1}^{n} M_{2,i} = n-k, \quad \sum_{i=1}^{2} M_{i,j} = 1, \quad \text{for } j = 1, \dots, n.$$

We then discuss in Section 10.8 that we have a bijection

$$\mathrm{B}_{k,n} \xleftrightarrow{1:1} \mathcal{P}(k,n)$$

and under this bijection simple moves correspond to rim hook removals. Hence, the formula (1.4) naturally generalizes (1.3).

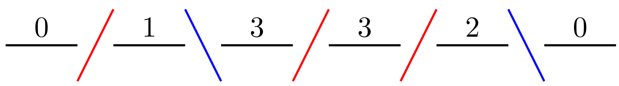
1.3 Structure of the thesis

This thesis consists of four parts:

Part 1: geometry of bow varieties

In the first part of the thesis, we give a self-contained introduction to the geometry of bow varieties. In Chapter 2, we consider important ingredients of the theory of GIT quotients. Thereby, we place special focus on the characterization of geometric points of GIT quotients via stability conditions due to King [Kin94]. We also consider GIT quotients in the framework of symplectic algebraic geometry and hamiltonian reduction. In particular, we discuss an algebro-geometric version of the Marsden–Weinstein Theorem, see Theorem 2.17.

Hereafter, we recall the construction of bow varieties from [NT17] as GIT quotients of specific moduli spaces of quiver representation. For this, we use the language of brane combinatorics which was developed in [RS20], [Sho21]. The input data for the construction of a bow variety is a *brane diagram* which is an object like this:

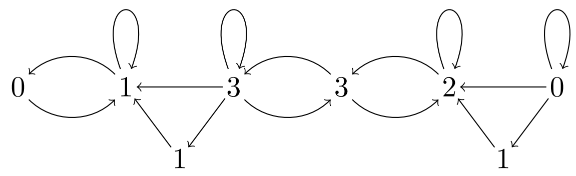


That is, a brane diagram is a certain configuration of horizontal black lines with integral labels and between each adjacent pair of horizontal lines there is a red or a blue line. As explained in detail in [RR23, Section 2.4], these brane diagrams are motivated from theoretical physics and arise from brane projections from $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ to the plane, but in this thesis we just consider them as combinatorial data.

The construction of bow varieties goes in three steps:

$$\mathcal{D} \rightsquigarrow Q_{\mathcal{D}} \rightsquigarrow \widetilde{\mathcal{M}}(\mathcal{D}) \rightsquigarrow \mathcal{C}(\mathcal{D})$$

In the first step one assigns to a brane diagram $\mathcal{D}$ a specific quiver $Q_{\mathcal{D}}$ with dimension vector. If for instance $\mathcal{D}$ is as above, $Q_{\mathcal{D}}$ is as follows:



To $Q_{\mathcal{D}}$, one assigns an affine moduli space $\widetilde{\mathcal{M}}(\mathcal{D})$ of quiver representations. It is endowed with a holomorphic symplectic structure. The bow variety $\mathcal{C}(\mathcal{D})$ is finally obtained as a GIT quotient from $\widetilde{\mathcal{M}}(\mathcal{D})$ via hamiltonian reduction, see Definition 2.36.

The explicit construction of $\mathcal{C}(\mathcal{D})$ gives that this variety satisfies many convenient properties, some of which we explain in Section 2.3. In particular, $\mathcal{C}(\mathcal{D})$ is smooth and inherits a holomorphic symplectic structure from $\widetilde{\mathcal{M}}(\mathcal{D})$. A further important tool will be explained in Proposition 2.47, namely the action of a torus $\mathbb{T} = \mathbb{A} \times \mathbb{C}_{h}^{*}$ on $\mathcal{C}(\mathcal{D})$ which was introduced in [NT17, Section 6]. The construction of this $\mathbb{T}$-action is such that the $\mathbb{A}$-action leaves the symplectic form on $\mathcal{C}(\mathcal{D})$ invariant whereas the $\mathbb{C}_{h}^{*}$-action scales the symplectic form.

Chapter 3 is then devoted to the study of the $\mathbb{T}$-fixed points of bow varieties. First, we recall the classification theorem of $\mathbb{T}$-fixed points from [Nak18, Theorem A.5] via the language of tie diagrams from [RS20], [Sho21]. These tie diagrams are extensions of brane diagrams by adding ties between the colored lines in a specific way. The incidence matrices of these tie diagrams are the matrices with entries in $\{0,1\}$ with fixed row and column sums depending on the brane diagram $\mathcal{D}$ which were mentioned already above. From a tie diagram, one can read off an explicit representation of the quiver $Q_{\mathcal{D}}$ which yields a $\mathbb{T}$-fixed point of $\mathcal{C}(\mathcal{D})$. The Classification Theorem (Theorem 3.7) then states that this construction gives a bijection

$$\{\text{Tie diagrams of } \mathcal{D}\} \xleftrightarrow{1:1} \mathcal{C}(\mathcal{D})^{\mathbb{T}}.$$

As an outcome of Section 3.2, this explicit $\mathbb{T}$-fixed point combinatorics makes explicit computations in the $\mathbb{T}$-equivariant cohomology of $\mathcal{C}(\mathcal{D})$ possible.

Hereafter, we follow [Nak18, Theorem A.5] to prove the Generic Cocharacter Theorem (Theorem 3.23) which improves the Classification Theorem of $\mathbb{T}$-fixed points. This theorem states that that the $\mathbb{T}$-fixed locus of $\mathcal{C}(\mathcal{D})$ coincides with the $\mathbb{C}^{*}$-fixed locus which is induced by a generic cocharacter of $\mathbb{A}$.

Part 2: stable envelopes for bow varieties

In the second part of the thesis, we redevelop the theory of stable envelopes from [MO19] in the setup of bow varieties with the focus on a more direct treatment. The theory of stable envelopes builds on the theory of attracting cells. In Chapter 4, we give a self-contained introduction to attracting cells for bow varieties following [MO19, Chapter 3]. The classical Białynicki-Birula decomposition, see [BB73], gives that the attracting cells of $\mathbb{T}$-fixed points with respect to generic cocharacters of $\mathbb{A}$ are affine and locally closed subvarieties of $\mathcal{C}(\mathcal{D})$. Their closures are then (possibly singular) lagrangian subvarieties of $\mathcal{C}(\mathcal{D})$. As in classical Schubert calculus, the closure relation defines a partial order $\preceq$ on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$, see Section 4.1 and Section 4.4 for precise statements. We lay out the theory and illustrate it with examples.

In Section 4.5, we compare attracting cells of a generic cocharacter σ of $\mathbb{A}$ with the attracting cells of the opposite cocharacter σ^{-1}. A crucial result is Theorem 4.24 which states that the intersection on attracting cells corresponding to opposite cocharacters is always proper. A further useful result is Theorem 4.23 which states that the attracting cells of σ and σ^{-1} induce opposite partial orders on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$. We prove Theorem 4.23 using a smooth compactification of $\mathcal{C}(\mathcal{D})$.

In Chapter 5, we come to some of the main actors in this thesis: The stable envelopes. In Theorem 5.10, we give a self-contained reproof of the existence and uniqueness of stable envelopes from [MO19, Chapter 3] in the case of bow varieties. The uniqueness proof is a direct application of the defining conditions of stable envelopes and torus equivariant intersection theory. The existence proof is based on a result which gives that the $\mathbb{A}$-equivariant multiplicities of lagrangian subvarieties at a $\mathbb{T}$-fixed point $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ are uniquely determined (up to a factor in $\mathbb{Z}$) by the tangent weights at p, see Theorem 5.15. We refer to this theorem as *Langrangian Multiplicity Theorem.* Its proof is based on the deformation to the tangent cone construction due to Fulton [Ful84] and on further deformation results for conical lagrangian subvarieties of symplectic vector spaces which we state in Proposition 5.29.

In particular, the proof of the existence of stable envelopes yields an algorithm to compute stable basis elements as $\mathbb{Z}$-linear combinations of the $\mathbb{T}$-equivariant cohomology classes of attracting cell closures, see Corollary 5.19. In Section 5.6 and in Chapter 6, we then compute several stable basis elements using this algorithm.

A convenient consequence of the uniqueness property of stable envelopes is that they are compatible with Hanany–Witten isomorphisms, i.e. if $\Phi\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})$ is a Hanany–Witten isomorphism then the induced isomorphism $H^*_{\mathbb{T}}(\mathcal{C}(\tilde{\mathcal{D}})) \xrightarrow{\sim} H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ maps the stable basis elements of $\mathcal{C}(\tilde{\mathcal{D}})$ to the stable basis elements of $\mathcal{C}(\mathcal{D})$, see Proposition 5.13.

In Chapter 7, we study properties of stable basis elements with respect to the virtual intersection pairing on $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$. As bow varieties are quasi-projective varieties, their torus equivariant cohomologoy is endowed with a virtual intersection pairing which mimics the Atiya–Bott–Berline–Vergne integration formula for projective varieties:

$$(.,.)_{\mathrm{vir}}\colon H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D})) \times H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D})) \xrightarrow{\ \sim\ } S^{-1}H^*_{\mathbb{T}}(\mathrm{pt}), \quad (\alpha,\beta)_{\mathrm{vir}} = \sum_{p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}} \frac{\iota_p^*(\alpha\cdot\beta)}{e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D}))}.$$

Here, $e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D}))$ is the $\mathbb{T}$-equivariant Euler class of the tangent space $T_p\mathcal{C}(\mathcal{D})$ and S is the multiplicative set generated by all tangent weights of torus fixed points of $\mathcal{C}(\mathcal{D})$.

An important result is the following Polynomiality Theorem (Theorem 7.6):

Theorem B (Polynomiality)**.** *The virtual intersection pairings of the form*

$$(\alpha \cdot \mathrm{Stab}_\sigma(p), \mathrm{Stab}_{\sigma^{-1}}(q))_{\mathrm{vir}}, \quad p,\ q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}, \quad \alpha \in H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$$

are all contained in the non-localized equivariant cohomology ring $H^*_{\mathbb{T}}(\mathrm{pt})$.

In other words, this theorem states that $(\alpha\cdot\mathrm{Stab}_\sigma(p), \mathrm{Stab}_{\sigma^{-1}}(q))_{\mathrm{vir}}$ is always a polynomial in their equivariant parameters which motivates the name *Polynomiality Theorem.* Its proof involves the properness statement from Theorem 4.24.

We finish Chapter 7 by giving a self-contained reproof of the Orthogonality Theorem (Theorem 7.8) from [MO19, Theorem 4.4.1]. This theorem states that stable basis elements corresponding to inverse cocharacters are orthogonal with respect to the virtual intersection pairing:

Theorem C (Orthogonality)**.** *For all p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have*

$$(\mathrm{Stab}_\sigma(p), \mathrm{Stab}_{\sigma^{-1}}(q))_{\mathrm{vir}} = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$$

This orthogonality property is again analogous to the orthogonality of opposite Schubert classes in (equivariant) Schubert calculus, see e.g. [Ful97].

Part 3: equivariant multiplicities of stable basis elements

By the Localization Theorem, the equivariant multiplicities of $\mathbb{T}$-equivariant cohomology classes in $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ at the $\mathbb{T}$-fixed points carry important information. We therefore deal in the third part of this thesis with the central question:

How can we compute equivariant multiplicities of stable basis elements of bow varieties?

The compatibility of stable envelopes with Hanany–Witten isomorphisms yields that we can restrict our attention to *separated* brane diagrams, these are brane diagrams of the shape $/\!/ \dots /\!/ \ \backslash\backslash \dots \backslash\backslash$. As explained in Section 2.4, the quiver representations defining points of bow varieties corresponding to separated brane diagrams satisfy useful nilpotency and surjectivity properties. These properties simplify the description of the points on these bow varieties and also calculations in the equivariant cohomology rings.

In Chapter 8, we study local moves on separated brane diagrams of the form

We refer to the left move above as *red extension move*, whereas the right move is called *blue extension move*. The goal of Chapter 8 is to compare the stable basis elements of the bow varieties corresponding to red and blue extension moves respectively. This should be compared with [BR23, Section 5.10], where similar questions were considered in the framework of elliptic cohomology.

If $\mathcal{D}'$ is obtained from a brane diagram $\mathcal{D}$ by a red extension move, we show in Proposition 8.3 that the bow varieties $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ are torus equivariantly isomorphic and hence the stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ coincide.

On the other hand, in the case where $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a blue extension move, the bow varieties $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ are in general *not* isomorphic, see Lemma 8.13. However, we prove in Theorem 8.15 that there is still a closed embedding $\mathcal{C}(\mathcal{D}) \hookrightarrow \mathcal{C}(\mathcal{D}')$. Using this embedding we then prove in Theorem 8.38 the following crucial result:

Theorem D. *The equivariant multiplicities of stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ coincide up to multiplication by a uniform constant factor.*

For the proof of this theorem, we introduce a certain comparison cocharacter of the torus $\mathbb{T}'$ acting on $\mathcal{C}(\mathcal{D}')$. The corresponding $\mathbb{C}^*$-fixed locus X_0 is contained in the subvariety $\mathcal{C}(\mathcal{D})$. The proof is then based on a comparison of the attracting cells of X_0, $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$.

Chapter 9 connects back to the classical setting and deals with relations between stable basis elements of bow varieties and stable basis elements of cotangent bundles of partial flag varieties. We begin with recalling the localization formula from [Su17] which expresses equivariant multiplicities of stable basis elements of cotangent bundles of partial flag varieties via combinatorics of symmetric groups. In Proposition 9.13, we give an equivalent reformulation of this formula in terms of string diagrams which gives an illustrative approach to and a diagrammatic version of this formula.

Hereafter, we study a bijection between the $\mathbb{T}$-fixed points of bow varieties and certain double cosets of symmetric groups with respect to Young subgroups which we call *fully separated*. As we will show in Section 9.5, fully separated double cosets satisfy strong uniqueness properties which distinguish them from usual double cosets. One important result that will turn out to be crucial in the proof of the Chevalley–Monk formula for bow varieties in the fourth part is Theorem 9.35 which states the following:

Theorem E. *Let λ, μ be partitions of n and S_λ, $S_\mu \subset S_n$ be the corresponding Young subgroups. Let $w \in S_n$ such that $S_\lambda w S_\mu$ is fully separated. Given u, $u' \in S_\lambda$ and v, $v' \in S_\mu$ with $uwv = u'wv'$, then we have $u = u'$ and $v = v'$.*

In particular, as we will discuss in Section 10.5, this theorem simplifies computations in the context of the diagrammatic localization formula from Proposition 9.13.

We finish Chapter 9, by combining this symmetric group calculus for bow varieties with the D5 Resolution Theorem of Botta and Rimányi [BR23, Theorem 6.13]. In this way, we derive in Theorem 9.44 a formula which determines the equivariant multiplicities $\iota_q^*(\mathrm{Stab}_\sigma(p))$ of stable basis elements of bow varieties in terms of equivariant multiplicities $\iota_{\tilde{q}}^*(\mathrm{Stab}_{\tilde{\sigma}}(\tilde{p}))$ of stable basis elements of cotangent bundles of partial flag varieties:

Theorem F. *Given a bow variety $\mathcal{C}(\mathcal{D})$, then there exists a partial flag variety F together with an inclusion of torus fixed points*

$$\mathcal{C}(\mathcal{D})^{\mathbb{T}} \longrightarrow (T^*F)^{T\times\mathbb{C}^*}, \quad p \mapsto \tilde{p}$$

such that we have for all p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$:

$$P_{\mathcal{D}} \cdot \iota_q^*(\mathrm{Stab}_\sigma(p)) = \Psi(\iota_{\tilde{q}}^*(\mathrm{Stab}_{\tilde{\sigma}}(\tilde{p}))),$$

for some constant factor $P_{\mathcal{D}} \in H_{\mathbb{T}}^*(\mathrm{pt})$ depending only on $\mathcal{D}$. Here, $\tilde{\sigma}$ is a cocharacter of T depending on σ and $\Psi\colon H_{T\times\mathbb{C}^*}^*(\mathrm{pt}) \to H_{\mathbb{T}}^*(\mathrm{pt})$ is a substitution homomorphism.

As an application of this result and Proposition 9.13, we prove diagrammatic approximation formulas of stable basis elements of bow varieties in Proposition 9.50. These formulas allow to diagrammatically compute equivariant multiplicities modulo powers of the parameter h.

Part 4: Chevalley–Monk formula for bow varieties

The last part of this thesis is devoted to the Chevalley–Monk formula for bow varieties (Theorem A). The proof of this formula combines several different results that appeared in the previous chapters. First, we use the Orthogonality Theorem C to deduce that the coefficients appearing in the stable envelope basis expansion of the products $c_1^{\mathbb{T}}(\xi_X) \cdot \mathrm{Stab}_\sigma(p)$ can be determined via virtual intersection pairings of the form

$$(c_1^{\mathbb{T}}(\xi_X) \cdot \mathrm{Stab}_\sigma(p), \mathrm{Stab}_{\sigma^{-1}}(q))_{\mathrm{vir}}, \quad p,\ q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}. \tag{1.5}$$

Then, using a degree argument, we deduce from the Polynomiality Theorem B that it suffices to compute these virtual intersection pairings modulo h^2, where h is the equivariant parameter corresponding to the torus action which scales the symplectic form on $\mathcal{C}(\mathcal{D})$. Via Theorem D and Theorem F, we then prove in Theorem 10.12 the following divisibility result:

Theorem G. *Let $p,\ q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with corresponding matrices M_p and M_q. Suppose $p \neq q$ and that M_q is not obtained from M_p via a simple move. Then, $\iota_q^*(\mathrm{Stab}_\sigma(p))$ is divisible by h^2.*

This result gives that (1.5) vanishes if $p \neq q$ and M_q is not obtained from M_p via a simple move. The remaining cases, i.e. $p = q$ or M_q is obtained from M_p via a simple move, are then covered in Theorem 10.15 which states the following:

Theorem H. *For all $p,\ q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have*

$$(c_1^{\mathbb{T}}(\xi_X) \cdot \mathrm{Stab}_\sigma(p), \mathrm{Stab}_{\sigma^{-1}}(q))_{\mathrm{vir}} = \begin{cases} \iota^*(c_1^{\mathbb{T}}(\xi_X)) & \text{if } p = q, \\ \mathrm{sgn}(M_p, M_q)h & \text{if } M_q \in \mathrm{SM}_{M_p,X}, \\ 0 & \text{otherwise.} \end{cases}$$

The main tools in the proof of Theorem H are the uniqueness properties of fully separated double cosets from Theorem E and the approximation formulas resulting from Theorem F.

Parts of this thesis are already submitted and available as preprint versions:

1. [Weh23], ArXiv number: 2310.11235,
2. [SW23], ArXiv number: 2312.03144.

Conventions

If not stated otherwise, all varieties and vector spaces in this thesis are over $\mathbb{C}$. Varieties are not necessarily irreducible. If $y \in Y$ is a smooth point of a variety Y, we denote by T_yY the tangent space of Y at y. If V is a finite dimensional $\mathbb{C}^*$-representation, we denote the corresponding weight space decomposition as

$$V = \bigoplus_{a \in \mathbb{Z}} V_a, \quad V_a = \{v \in V \mid t.v = t^a v \text{ for all } t \in \mathbb{C}^*\}.$$

We denote the weight spaces of positive, negative, non-negative and non-positive weights as

$$V^+ = \bigoplus_{a>0} V_a, \quad V^- = \bigoplus_{a<0} V_a, \quad V^{\geq 0} = \bigoplus_{a\geq 0} V_a, \quad V^{\leq 0} = \bigoplus_{a \leq 0} V_a.$$

We also denote by V^0 the subspace of T-fixed vectors of V. If $T = (\mathbb{C}^*)^r$ is a torus and $\chi_1,\ \chi_2 \colon T \to \mathbb{C}^*$ are characters of T, we denote their product

$$T \longrightarrow \mathbb{C}^*, \quad t \mapsto \chi_1(t)\chi_2(t)$$

with the sum notation $\chi_1 + \chi_2$.

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Chapter 2

Geometry of bow varieties

In this chapter, we give a self-contained introduction to bow varieties and their geometric properties based in particularly on [NT17]. We begin with recalling fundamental ingredients of the theory of geometric invariant theory (short GIT) with special emphasis on the characterizations of stability and semistability conditions. Then, we construct bow varieties as GIT quotients of certain moduli spaces of quiver representations using hamiltonian reduction following [NT17, Section 2]. To describe the underlying combinatorics, we use the language of brane diagrams and brane combinatorics from [RS20, Section 2].

The construction of bow varieties implies that these varieties are equipped with many desirable features: They are smooth, quasi-projective and admit a holomorphic symplectic form. They come with a torus action which scales this symplectic form and moreover they also come family of torus equivariant tautological vector bundles.

The combinatorics of brane diagrams is in particular used to describe Hanany–Witten transitions between different brane diagrams. These transition moves provide isomorphisms between related bow varieties of these brane diagrams.

We close this chapter with a detailed exposition of the realization of cotangent bundles of partial flag varieties as bow varieties which form an important family of examples of bow varieties.

2.1 Reminders on GIT quotients

In this section, we recall the definition of GIT quotients and some of their most important geometric properties. For more details on this subject, see [MFK94] as well as the expository works [Muk03] and [New09]. For the convenience of the reader, we give proofs of many of the presented results.

We begin with fixing some notation: Let X be an affine variety with coordinate ring $\mathcal{O}(X)$. Let G be a reductive group acting on X with action map $G \times X \to X, (g, x) \mapsto g.x$. We denote the algebra of G-invariants of $\mathcal{O}(X)$ by $\mathcal{O}(X)^G$ and let $X/\!\!/G = \mathrm{Spec}(\mathcal{O}(X)^G)$ be the corresponding categorical quotient. Recall that due to a theorem of Hilbert (see e.g. [Muk03, Theorem 4.51]), $X/\!\!/G$ is an affine variety.

Definition and basic properties

GIT quotients are defined as projective spectra of algebras of semi-invariants. These are defined as follows: For a rational character $\chi\colon G \to \mathbb{C}^*$, the $\mathbb{N}_0$*-graded algebra of semi-invariants of* $\mathcal{O}(X)$ is defined as

$$\mathcal{O}(X)_\chi \coloneqq \bigoplus_{n\geq 0} \mathcal{O}(X)_{\chi^n},$$

where $\mathcal{O}(X)_{\chi^n} = \{f \in \mathcal{O}(X) \mid f(g.x) = \chi(g)^n f(x), \text{ for all } x \in X,\, g \in G\}$. Note that the degree 0 piece of $\mathcal{O}(X)_\chi$ is given as $\mathcal{O}(X)_{\chi^0} = \mathcal{O}(X)^G$.

The algebra $\mathcal{O}(X)_\chi$ can also be interpreted as algebra of invariants on the variety $\hat{X} \coloneqq X \times \mathbb{C}$. For this, we extend the G-action to $\hat{X}$ via

$$g.(x, z) = (g.x, \chi^{-1}(g)z). \tag{2.1}$$

Then, there is an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{O}(X)_\chi \xrightarrow{\sim} \mathcal{O}(\hat{X})^G, \quad f \mapsto \hat{f}, \tag{2.2}$$

where $\hat{f}(x, z) = f(x)z^n$, for all $(x, z) \in \hat{X}$. As $\mathcal{O}(\hat{X})^G$ is a finitely generated $\mathbb{C}$-algebra, so is $\mathcal{O}(X)_\chi$.

Definition 2.1. The *GIT quotient* $X/\!\!/_\chi G$ is defined as the scheme

$$X/\!\!/_\chi G \coloneqq \operatorname{Proj}(\mathcal{O}(X)_\chi).$$

Since $\mathcal{O}(X)_\chi$ is a reduced algebra, we conclude that $X/\!\!/_\chi G$ is also a reduced scheme. Moreover, the construction of $X/\!\!/_\chi G$ as projective spectrum yields the following:

Proposition 2.2. *The following holds:*

(i) The scheme $X/\!\!/_\chi G$ is a quasi-projective variety.

(ii) The morphism $\pi : X/\!\!/_\chi G \to X/\!\!/G$ induced by the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)_\chi$ is projective.

Proof. Let $a_1, \dots, a_r$ be homogeneous generators of $\mathcal{O}(X)_\chi$ over $\mathcal{O}(X)^G$, where we denote the degree of a_i by $|a_i|$. Let $B \coloneqq \mathbb{C}[x_1, \dots, x_r]$ be the graded polynomial algebra, where x_i is homogeneous of degree $|a_i|$. The algebra homomorphism $B \to \mathcal{O}(X)_\chi$ given by $x_i \mapsto a_i$ induces a closed immersion

$$X/\!\!/_\chi G \hookrightarrow X/\!\!/G \times \operatorname{Proj}(B).$$

By e.g. [BR86, Theorem 4.B7], the weighted projective space $\operatorname{Proj}(B)$ admits a closed immersion $\operatorname{Proj}(B) \hookrightarrow \mathbb{P}^N$, for some $N \gg 0$. Thus, we obtain a closed immersion

$$\iota\colon X/\!\!/_\chi G \hookrightarrow X/\!\!/G \times \mathbb{P}^N.$$

Thus, $X/\!\!/_\chi G$ is a quasi-projective variety. Let $\operatorname{pr}\colon X/\!\!/G \times \mathbb{P}^N \to X/\!\!/G$ be the projection to the first factor. Then, $\operatorname{pr} \circ \iota = \pi$ which proves that the morphism π is projective. □

Stable and semistable points

Next, we recall the notion of χ-(semi)stable points of X and how they characterize the geometric points of the GIT quotient $X/\!\!/_\chi G$ from [MFK94] and in particular in the version of [Kin94].

Definition 2.3. Let $x \in X$.

(i) We call x *χ-semistable* if there exist $n \geq 1$ and $f \in \mathcal{O}(X)_{\chi^n}$ such that $x \in D(f)$, where $D(f) = \{x \in X \mid f(x) \neq 0\}$.

(ii) We call x *χ-stable* if there exist $n \geq 1$ and $f \in \mathcal{O}(X)_{\chi^n}$ such that

  (a) $x \in D(f)$,

  (b) the action $G \times D(f) \to D(f)$ is a closed morphism and

  (c) the isotropy group G_x is finite.

We write X^{ss} resp. X^{s} for the subset of χ-semistable resp. χ-stable points of X. By construction, X^{ss} is an open subset of X. By the upper-semicontinuity of fiber dimensions, see e.g. [Mum88, Corollary I.8.3], the sets

$$\{x \in X \mid \dim(G_x) \leq n\}, \quad \text{for } n \geq 0$$

are open in X. In particular, the subset of all points in X with 0-dimensional isotropy group is open in X. Thus, we conclude that X^{s} is also an open subset of X.

There are several equivalent definitions for χ-semistability resp. χ-stability. For instance, by [MFK94, Amplification 1.11 and Lemma 0.3], the χ-stability condition can be reformulated as follows:

Lemma 2.4. *For a χ-semistable point $x \in X^{\mathrm{ss}}$ the following are equivalent:*

(i) x is χ-stable,

(ii) the isotropy group G_x is finite and the orbit $G.x$ is closed in X^{ss},

(iii) the orbit morphism $a_x \colon G \to X^{\mathrm{ss}}$, $g \mapsto g.x$ is proper.

In addition, King introduced the following topological criteria for χ-(semi)stability [Kin94, Lemma 2.2]:

Proposition 2.5 (King's stability)**.** *As above, let $\hat{X} = X \times \mathbb{C}$ with the G-action from* (2.1)*. Let $x \in X$ and $z \in \mathbb{C} \setminus \{0\}$.*

(i) The point x is χ-semistable if and only if the orbit closure $\overline{G.(x,z)}$ does not intersect $X \times \{0\}$.

(ii) The point $x \in X$ is χ-stable if and only if the orbit $G.(x,z)$ is closed in $\hat{X}$ and the isotropy group $G_{(x,z)}$ is finite for all $z \neq 0$.

Proof. For (i), suppose that x is χ-semistable. Thus, there exist $n \geq 1$ and $f \in \mathcal{O}(X)_{\chi^n}$ such that $f(x) \neq 0$. Let $\hat{f} \in \mathcal{O}(\hat{X})^G$ be the G-invariant regular function on $\hat{X}$ associated to f from (2.2). Then, $\hat{f}$ vanishes on $X \times \{0\}$ and equals $z^n f(x)$ on $\overline{G.(x,z)}$. Thus, $G.(x,z)$ does not intersect $X \times \{0\}$. Conversely, if $\overline{G.(x,z)}$ and $X \times \{0\}$ are disjoint then, as G is reductive, there exists $h \in \mathcal{O}(\hat{X})^G$ such that h vanishes on $X \times \{0\}$ and h equals 1 on $\overline{G.(x,z)}$. By (2.2), $h(y,w) = \sum_{n \geq 0} h_n(y) w^n$, for all $(y,w) \in \hat{X}$, where $h_n \in \mathcal{O}(X)_{\chi^n}$, for all $n \geq 0$. As h vanishes on $X \times \{0\}$, we have $h_0 = 0$. Thus there exists $m > 0$ such that $h_m(x) \neq 0$ which proves that x is χ-semistable. For (ii), we begin with the following:

Observation 2.6. Let $x \in X^{\mathrm{ss}}$, $n \geq 1$ and $f \in \mathcal{O}(X)_{\chi^n}$ with $x \in D(f)$. We set

$$Z_f \coloneqq \{(y,w) \in \hat{X} \mid \hat{f}(y,w) = f(x) z^n\}.$$

By definition, Z_f is a G-invariant closed subvariety of $\hat{X}$ containing (x,z). Let $\mathrm{pr}\colon Z_f \to X$ be the projection to the first factor. Then, the image of pr is contained in $D(f)$ and $\mathrm{pr}\colon Z_f \to D(f)$ is a finite G-equivariant morphism. Thus, as properness is a property which is local on target, we deduce that the orbit morphism $a_x\colon G \to X^{\mathrm{ss}}$, $g \mapsto g.x$ is proper if and only if $a_{(x,z)}\colon G \to \hat{X}$, $g \mapsto g.(x,z)$ is proper.

Now, assume that x is χ-stable. Then, by Lemma 2.4.(iii), a_x is proper. Hence, by Observation 2.6, also $a_{(x,z)}$ is proper. Therefore, the orbit $G.(x,z)$ is closed in $\hat{X}$. By definition, the stabilizer $G_{(x,z)}$ equals the preimage $a_{(x,z)}^{-1}(x,z)$. As $a_{(x,z)}$ is a proper morphism, we conclude that $G_{(x,z)}$ is proper over $\mathbb{C}$. Since G is affine and $G_{(x,z)}$ is a closed subvariety of G, we conclude that also $G_{(x,z)}$ is affine. Hence, $G_{(x,z)}$ must be finite. Conversely, assume that $G.(x,z)$ is closed in $\hat{X}$ and the isotropy group $G_{(x,z)}$ is finite. Since $X \times \{0\}$ does not intersect $G.(x,z)$, we know by (i) that x is χ-semistable. Let f, Z_f and pr be as in Observation 2.6. Since $\mathrm{pr}(G.(x,z)) = G.x$ and pr is finite, we know that $\dim(G.(x,z)) = \dim(G.x)$ which gives $\dim(G_{(x,z)}) = \dim(G_x)$. Thus, G_x is finite. Moreover, the finiteness of pr implies that $G.x$ is closed in $D(f)$. As this is true for all $n \geq 1$ and $f \in \mathcal{O}(X)_{\chi^n}$ with $x \in D(f)$, we conclude that $G.x$ is closed in X^{ss}. Hence, by Lemma 2.4.(ii), this proves that x is χ-stable. $\square$

Via Observation 2.6 and Lemma 2.4.(iii) we immediately obtain a further equivalent definition of χ-stability:

Corollary 2.7. *With the assumptions of Proposition 2.5, a χ-semistable point $x \in X^{\mathrm{ss}}$ is χ-stable if and only if the orbit morphism $a_{(x,z)}\colon G \to \hat{X}$ is proper.*

After this discussion about equivalent definitions of χ-(semi)stability, we now come to the geometric points of the GIT quotient $X/\!\!/_\chi G$. They are characterized by the χ-(semi)stable points of X as follows: For a semi-invariant function $f \in \mathcal{O}(X)_{\chi^n}$, let $D_+(f) \subset X/\!\!/_\chi G$ be the corresponding principal open subset. That is,

$$D_+(f) = \mathrm{Spec}(\mathcal{O}_{\chi,(f)}), \quad \text{where } \mathcal{O}_{\chi,(f)} = \Big\{ \frac{a}{f^r} \mid r \geq 0,\ a \in \mathcal{O}_{\chi^{rn}} \Big\}.$$

By construction, we have an obvious identification $\mathcal{O}_{\chi,(f)} \cong \mathcal{O}(D(f))^G$ and hence an isomorphism of schemes $D(f)/\!\!/G \cong D_+(f)$. The canonical quotient morphisms $D(f) \to D_+(f)$ glue to a morphism $F\colon X^{\mathrm{ss}} \to X/\!\!/_\chi G$. Then, we have the following theorem, see [MFK94, Theorem 1.10]:

Theorem 2.8 (GIT-Theorem)**.** *The following holds:*

(i) *The morphism* $F\colon X^{\mathrm{ss}} \to X/\!\!/_{\chi}G$ *is a categorical quotient and surjective.*

(ii) *Given* $x,\, y \in X^{\mathrm{ss}}$*, we have* $F(x) = F(y)$ *if and only if the orbit closures* $\overline{G.x}$ *and* $\overline{G.y}$ *in* X *intersect non-trivially in* X^{ss}*, i.e.* $\overline{G.x} \cap \overline{G.y} \cap X^{\mathrm{ss}} \neq \emptyset$.

(iii) *Let* $U = F(X^{\mathrm{s}})$*. Then,* $U \subset X/\!\!/_{\chi}G$ *is an open subvariety and the morphism of varieties* $F_{|X^{\mathrm{s}}}\colon X^{\mathrm{s}} \to U$ *is a geometric quotient. In particular,* F *induces a bijection*

$$\{G\text{-orbits in } X^{\mathrm{s}}\} \xleftrightarrow{1:1} \{\text{Points of } U\}.$$

Mumford's Numerical Criterion

Mumford introduced a numerical criterion for χ-(semi)stability [MFK94, Chapter 2] which proved to be very practical in explicit computations. Our formulation of this criterion is following [Kin94, Proposition 2.6].

Recall that a *one-parameter subgroup of* G is an algebraic cocharacter $\lambda\colon \mathbb{C}^* \to G$. Let $\langle \lambda, \chi \rangle$ be the unique integer such that $\chi(\lambda(t)) = t^{\langle \lambda, \chi \rangle}$, for all $t \in \mathbb{C}^*$. For a given point $x \in X$, we say that *the limit* $\lim_{t\to 0} \lambda(t).x$ *exists in* X if and only if the morphism $\mathbb{C}^* \to X$, $t \mapsto \lambda(t).x$ extends to a morphism $\mathbb{C} \to X$.

Theorem 2.9 (Mumford's Numerical Criterion)**.** *A point* $x \in X$ *is* χ*-semistable (resp.* χ*-stable) if and only if for all non-trivial one-parameter subgroups* λ *such that* $\lim_{t\to 0} \lambda(t).x$ *exists in* X*, we have* $\langle \lambda, \chi \rangle \geq 0$ *(resp.* $\langle \lambda, \chi \rangle > 0$*).*

Mumford's Numerical Criterion can be proved using the following three auxiliary statements. The first one states that one-parameter subgroups detect points on the boundary of orbits, see [Kem78, Theorem 1.4]:

Lemma 2.10. *Let* $x \in X$ *and* $Y \subset X$ *be a closed and* G*-invariant subvariety such that* $Y \cap \overline{G.x} \neq \emptyset$*. Then, there exist* $y \in Y$ *and a one-parameter subgroup* λ *of* G *such that* $\lim_{t\to 0} \lambda(t).x = y$.

The second auxiliary statement states that properness orbit morphisms can be detected via the non-trivial one-parameter subgroups of G, see [MFK94, Step (i) in proof of Theorem 2.1]:

Lemma 2.11. *Suppose* G *acts linearly on* $\mathbb{C}^n$ *and let* $x \in \mathbb{C}^n \setminus \{0\}$*. Then, the orbit morphism* $G \to \mathbb{C}^n$*,* $g \mapsto g.x$ *is not proper if and only if for some non-trivial one-parameter subgroup* λ *of* G*, the limit* $\lim_{t\to 0} \lambda(t).x$ *exists in* $\mathbb{C}^n$.

Proof of Theorem 2.9. Suppose x is χ-semistable and λ is a one-parameter subgroup of G such that $\lim_{t\to 0} \lambda(t).x$ exists in X. Let $x_0 \in X$ be this limit and fix some $z \in \mathbb{C}$ with $z \neq 0$. If $\langle \lambda, \chi \rangle < 0$ then $\lim_{t\to 0} \lambda(t).(x,z) = (x_0, 0)$ and hence $\overline{G.(x,z)}$ intersects $X \times \{0\}$ non-trivially. By Proposition 2.5.(i), this contradicts the χ-semistability of x. Thus, we have $\langle \lambda, \chi \rangle \geq 0$. Conversely, if x is not χ-semistable then $\overline{G.(x,z)}$ intersects $X \times \{0\}$. Hence, by Lemma 2.10, there exists a one-parameter subgroup λ of G such that $\lim_{t\to 0} \lambda(t).(x,z)$ is contained in $X \times \{0\}$. This is equivalent to the conditions that $\lim_{t\to 0} \lambda(t).x$ exists in X and $\langle \lambda, \chi \rangle < 0$. Now, we prove the statement about stability. To apply Lemma 2.11 recall from e.g. [Bri18, Proposition 2.2.5] that there exists a G-equivariant closed immersion $X \hookrightarrow W$ into a finite-dimensional G-representation W. By Corollary 2.7, a χ-semistable point x is not χ-stable if and only if the orbit morphism $a_{x,z} \colon G \to \hat{X}$, $g \mapsto g.(x,z)$ is not proper. By Lemma 2.11, this is equivalent to the condition that there exists a non-trivial one-parameter subgroup λ of G such that $\lim_{t\to 0} \lambda(t).(x,z)$ exists in $\hat{X}$. This is equivalent to $\lim_{t\to 0} \lambda(t).x$ exists in X and $\langle \lambda, \chi \rangle \geq 0$ which completes the proof. □

In the following example, we apply Mumford's Numerical Criterion to explicitly determine χ-(semi)stable points.

Example 2.12. Let $X = \mathbb{C}^3$ where we denote the canonical basis vectors by e_1, e_2, e_3. We equip X with the $G = (\mathbb{C}^* \times \mathbb{C}^*)$-action

$$(t_1, t_2).(a_1 e_1 + a_2 e_2 + a_3 e_3) = t_1^2 a_1 e_1 + t_1 t_2 a_2 e_2 + t_2^2 a_3 e_3.$$

Let $\chi \colon G \to \mathbb{C}^*$, $(t_1, t_2) \mapsto t_1 t_2$. The non-trivial one-parameter subgroups of G are

$$\lambda_{b_1,b_2} : \mathbb{C}^* \longrightarrow G, \quad t \mapsto t^{b_1} t^{b_2}, \quad (b_1, b_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}.$$

The following table records for which $x \in X$ and parameters b_1, b_2 the limit $\lim_{t\to 0} \lambda_{b_1,b_2}(t).x$ exists in X:

Coordinates of $x = a_1e_1 + a_2e_2 + a_3e_3$	Parameters b_1, b_2 such that the limit $\lim_{t\to 0} \lambda_{b_1,b_2}(t).x$ exists in X
$a_1, a_3 \neq 0$	$b_1,\ b_2 \geq 0$
$a_1 \neq 0,\ a_3 = 0$	$b_1 \geq 0,\ b_1 + b_2 \geq 0$
$a_3 \neq 0,\ a_1 = 0$	$b_2 \geq 0,\ b_1 + b_2 \geq 0$
$a_1, a_3 = 0,\ a_2 \neq 0$	$b_1 + b_2 \geq 0$
$a_1 = a_2 = a_3 = 0$	$b_1,\ b_2$ arbitrary

From this, we deduce that if $\lim_{t\to 0} \lambda_{b_1,b_2}(t).x$ exists in X then $\langle \lambda_{b_1,b_2}, \chi \rangle \geq 0$ if and only if $x \neq 0$. Thus, Mumford's Numerical Criterion implies that the χ-semistable locus of X equals $X \setminus \{0\}$. Likewise, the above table gives that if $\lim_{t\to 0} \lambda_{b_1,b_2}(t).x$ exists in X then $\langle \lambda_{b_1,b_2}, \chi \rangle > 0$ if and only if x is of the form $x = a_1e_1 + a_2e_2 + a_3e_3$ with a_1, $a_3 \neq 0$. Thus, by Mumford's Numerical Criterion, the χ-stable locus of X equals $\{x \in X \mid x = a_1e_1 + a_2e_2 + a_3e_3 \text{ with } a_1,\ a_3 \neq 0\}$.

Compatibility with algebraic group actions

Suppose X is endowed with a further action of an affine algebraic group H that commutes with the G-action. As the pullback $\mathcal{O}(X) \to \mathcal{O}(H) \otimes \mathcal{O}(X)$ restricts to a morphism on the G-invariants $\mathcal{O}(X)^G \to \mathcal{O}(H) \otimes \mathcal{O}(X)^G$, we obtain an H-action on the categorical quotient $X/\!\!/G$. This H-action on $X/\!\!/G$ is the unique H-action such that the quotient morphism $X \to X/\!\!/G$ is H-equivariant.

We now use the characterization of the GIT quotient $X/\!\!/_\chi G$ from Theorem 2.8 to show that also $X/\!\!/_\chi G$ inherits an H-action:

Proposition 2.13. *There exists a unique H-action on $X/\!\!/_\chi G$ such that the quotient morphism $F\colon X^{\mathrm{ss}} \to X/\!\!/_\chi G$ from Theorem 2.8 is H-equivariant. Moreover, the projection morphism $\pi\colon X/\!\!/_\chi G \to X/\!\!/G$ is H-equivariant.*

Proof. Let $a\colon H \times X \to X$ be the action morphism. As the G- and H-action commute, we have $f \circ h \in \mathcal{O}(X)_{\chi^n}$, for all semi-invariants $f \in \mathcal{O}(X)_{\chi^n}$ and $h \in H$. Thus, X^{ss} is invariant under the H-action. For $f \in \mathcal{O}(X)_{\chi^n}$, we denote by $a_f\colon H \times D(f) \to X^{\mathrm{ss}}$ the restriction of a to $H \times D(f)$. Since the composition $F \circ a_f\colon H \times D(f) \to X^{\mathrm{ss}} \to X/\!\!/_\chi G$ is G-invariant, there exists a unique a morphism $a'_f\colon H \times D(f)/\!\!/G \to X/\!\!/_\chi G$ such that the following diagram commutes:

$$\begin{array}{ccc} H \times D(f) & \longrightarrow & H \times D(f)/\!\!/G \\ {\scriptstyle a_f}\downarrow & & \downarrow{\scriptstyle a'_f} \\ X^{\mathrm{ss}} & \xrightarrow{\;F\;} & X/\!\!/_\chi G \end{array} \tag{2.3}$$

Given a further semi-invariant function $f' \in \mathcal{O}(X)_{\chi^{n'}}$, we have commutative diagrams:

$$\begin{array}{ccccc} H \times D(f) & \xleftarrow{\quad} & H \times D(ff') & \xrightarrow{\quad} & H \times D(f') \\ & {\scriptstyle a_f}\searrow & \downarrow{\scriptstyle a_{ff'}} & \swarrow{\scriptstyle a_{f'}} & \\ & & X^{\mathrm{ss}} & & \end{array} \qquad \begin{array}{ccccc} H \times D(f)/\!\!/G & \xleftarrow{\quad} & H \times D(ff')/\!\!/G & \xrightarrow{\quad} & H \times D(f')/\!\!/G \\ & {\scriptstyle a'_f}\searrow & \downarrow{\scriptstyle a'_{ff'}} & \swarrow{\scriptstyle a'_{f'}} & \\ & & X/\!\!/_\chi G & & \end{array}$$

Note that the commutativity of the left diagram implies the commutativity of the right diagram. We conclude that the morphisms a'_f glue to a morphism $a'\colon H \times X/\!\!/_\chi G \to X/\!\!/_\chi G$. The morphism a' defines an H-action on $X/\!\!/_\chi G$ since identity and associativity conditions for a' follow from the respective conditions for a. By (2.3), the quotient morphism F is H-equivariant. Since F is surjective, a' is the unique H-action on $X/\!\!/_\chi G$ such that F is H-equivariant. To see that π is H-equivariant, note that for all $h \in H$ and $f \in \mathcal{O}(X)_{\chi^n}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(X)^G & \xrightarrow{\;\pi^*\;} & \mathcal{O}(D(f))^G \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ \mathcal{O}(X)^G & \xrightarrow{\;\pi^*\;} & \mathcal{O}(D(f \circ h))^G \end{array}$$

Therefore, π is compatible with the action of h which implies that π is H-equivariant. □

Compatibility with Poisson structures

We now restrict our attention to GIT quotients associated to symplectic varieties and consider the hamiltonian reduction mechanism from symplectic geometry in the framework of algebraic geometry. In particular, we show that GIT quotients of vanishing loci of moment maps always inherit a Poisson bracket. For more details on symplectic forms in algebraic geometry, see e.g. [CG97, Chapter 1] and [Kir16, Chapter 9].

Let Y be a smooth and affine variety with algebraic G-action. We further assume that Y admits an algebraic and G-invariant symplectic form ω. Let f be a regular function on U. We denote by X_f the corresponding hamiltonian vector field of f, i.e. X_f is the unique vector field on Y such that $\omega(\cdot, X_f) = df$. The G-invariant symplectic form ω induces a G-invariant Poisson bracket on $\{.,.\}$ on Y via

$$\{f,g\} = \omega(X_g, X_f), \quad f,\ g \in \mathcal{O}(Y).$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual Lie algebra of $\mathfrak{g}$. Then, G-acts on $\mathfrak{g}$ via the adjoint action and on $\mathfrak{g}^*$ via the coadjoint action. Let

$$\langle .,. \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \longrightarrow \mathbb{C}, \quad (f,g) \mapsto f(g)$$

be the evaluation pairing. A G-equivariant morphism of varieties $m\colon Y \to \mathfrak{g}^*$ is called a *moment map for the G-action on Y* if

$$dH_g = \omega(\vec{g}, \cdot), \quad \text{for all } g \in \mathfrak{g}. \tag{2.4}$$

Here, $H_g := \langle m(\cdot), g\rangle$ and $\vec{g}$ is the vector field generated by g. The moment map condition (2.4) is equivalent to

$$\{H_g, f\} = -df(\vec{g}), \quad \text{for all } g \in \mathfrak{g},\ f \in \mathcal{O}(Y). \tag{2.5}$$

In the following, we show that GIT quotients of $X := m^{-1}(0)$ always inherit a Poisson bracket from Y. First, we consider the categorical quotient

$$X/\!\!/G = \operatorname{Spec}((\mathcal{O}(Y)/I)^G),$$

where I is the ideal generated by all H_g, for $g \in \mathfrak{g}$. For $f \in \mathcal{O}(Y)$, we denote by $[f]$ its residue class in $\mathcal{O}(Y)/I$.

Proposition 2.14. *There exists a unique Poisson bracket $\{.,.\}'$ on $X/\!\!/G$ such that*

$$\{[f],[f']\}' = [\{f,f'\}], \quad \text{for all } [f],\ [f'] \in (\mathcal{O}(Y)/I)^G. \tag{2.6}$$

We begin with the following auxiliary statement:

Lemma 2.15. *Let $f \in \mathcal{O}(Y)$ such that $[f] \in (\mathcal{O}(Y)/I)^G$. Then, we have $\{f, I\} \subset I$.*

Proof. It suffices to show $df(\vec{g}) = \{f, H_g\} \in I$, for all $g \in \mathfrak{g}$. Recall from e.g. [Bri18, Proposition 2.2.5] that we can choose a G-equivariant embedding $Y \hookrightarrow W$ into a finite dimensional G-representation W. Hence, there exists a finite dimensional G-subrepresentation $V \subset \mathcal{O}(Y)$ containing f and $df(\vec{g})$. We equip V with the usual euclidean topology. For $t \in \mathbb{C}^*$, set $F_t := t^{-1}(f - f \circ \exp(tg))$. Then, $F_t \in V$ and F_t converges pointwise to $df(\vec{g})$ for $t \to 0$. Hence, we also have $\lim_{t\to 0} F_t = df(\vec{g})$ in the equclidean topology on V. Note that $I \cap V$ is a subvector space of V and hence $I \cap V$ is closed in V. Since $[f] \in (\mathcal{O}(Y)/I)^G$, we have $F_t \in V \cap I$, for all t. Thus, also $df(\vec{g}) \in V \cap I$ which completes the proof. $\square$

Proof of Proposition 2.14. By Lemma 2.15, $\{.,.\}'$ is a well-defined regular function on X. For all $g \in G$ and $[f], [f'] \in (\mathcal{O}(Y)/I)^G$, we have

$$g.\{[f],[f']\}' = g.\{f, f'\} = \{g.f, g.f'\} = \{f, f'\} + \{f, h'\} + \{h, f'\} + \{h, h'\},$$

for some h, $h' \in I$. Thus, Lemma 2.15 implies that $\{[f],[f']\}'$ is G-invariant and hence $\{.,.\}'$ indeed takes values in $(\mathcal{O}(Y)/I)^G$. The Poisson bracket conditions for $\{.,.\}'$ follow immediately from the Poisson bracket conditions for $\{.,.\}$. □

Next, we generalize Proposition 2.14 to GIT quotients. For this, note that if χ is a rational character of G, $f \in \mathcal{O}(X)_{\chi^n}$ a semi-invariant function and $\tilde{f} \in \mathcal{O}(Y)$ a lift of f then Proposition 2.14 gives that there is a unique Poisson bracket $\{.,.\}'$ on $D(f)/\!\!/G$ such that

$$\{[h],[h']\}' = [\{h, h'\}], \quad \text{for all } [h],\, [h'] \in (\mathcal{O}(D(\tilde{f})/I)^G. \tag{2.7}$$

Proposition 2.16. *The GIT quotient $X/\!\!/_\chi G$ admits a unique Poisson bracket $\{.,.\}'$ such that for all semi-invariant functions $f \in \mathcal{O}(X)_{\chi^n}$, the restriction of $\{.,.\}'$ to $D(f)/\!\!/G$ coincides with* (2.7)*.*

Proof. Given semi-invariants $f \in \mathcal{O}(X)_{\chi^n}$ and $f' \in \mathcal{O}(X)_{\chi^{n'}}$ then, by (2.7), the restrictions of the Poisson structure from $D(f)/\!\!/G$ and $D(f')/\!\!/G$ to $D(ff')/\!\!/G$ coincide. Hence, the locally defined Poisson brackets from (2.7) glue to a global Poisson bracket on $X/\!\!/_\chi G$. □

We close this section with the following algebro-geometric version of the Marsden–Weinstein Theorem which gives sufficient conditions under which the Poisson bracket on $X/\!\!/_\chi G$ is non-degenerated. For more details on the Marsden–Weinstein in the context of symplectic geometry, see e.g. [AM78, Chapter 4].

Theorem 2.17 (Marsden–Weinstein Theorem)**.** *Suppose that X^{ss} is a smooth variety of dimension* $\dim(Y) - \dim(G)$ *and that the quotient morphism $\pi\colon X^{\mathrm{ss}} \to X/\!\!/_\chi G$ is a principal G-bundle (in the Zariski topology). Then, $X/\!\!/_\chi G$ admits a unique algebraic symplectic form ω' such that*

$$\pi^*\omega' = \iota^*\omega, \tag{2.8}$$

where $\iota\colon X^{\mathrm{ss}} \hookrightarrow Y$ is the inclusion. The Poisson bracket corresponding to ω' coincides with the Poisson bracket $\{.,.\}'$ from Proposition 2.16.

Proof. Since $\pi\colon X^{\mathrm{ss}} \to X/\!\!/_\chi G$ is a principal G-bundle, we have a short exact sequence of G-equivariant vector bundles

$$0 \to X^{\mathrm{ss}} \times \mathfrak{g} \xrightarrow{\alpha} TX^{\mathrm{ss}} \xrightarrow{d\pi} \pi^*T(X/\!\!/_\chi G) \to 0.$$

Here, α maps (x, g) to $(x, \vec{g}_x)$. By construction, $\mathrm{im}(\alpha_x) = T_xG.x$, for all $x \in X^{\mathrm{ss}}$. Let ${}^\omega T_xG.x$ denote the orthogonal complement of $T_xG.x$ in T_xY with respect to ω_x. By (2.4), $T_xX^{\mathrm{ss}} = \ker(d_xm) \subset {}^\omega T_xG.x$. Thus, $\iota^*\omega$ induces an algebraic bilinear form ξ on $\pi^*T(X/\!\!/_\chi G)$. Since ω is G-invariant, there exists an algebraic bilinear form ω' on $T(X/\!\!/_\chi G)$ such that

$\pi^*\omega' = \xi$. Thus, ω' satisfies (2.8). It is left to show that ω' is indeed a symplectic form. By construction, for $x \in X^{\mathrm{ss}}$, the bilinear form $\omega'_{\pi(x)}$ on $T_{\pi(x)}(X/\!\!/_\chi G) \cong T_xX^{\mathrm{ss}}/T_xG.x$ is given as

$$\omega'_{\pi(x)}([v],[w]) = \omega(v,w), \quad v,w \in T_xX^{\mathrm{ss}}. \tag{2.9}$$

Since for all $x \in X^{\mathrm{ss}}$, we have $T_xX^{\mathrm{ss}} \subset {}^{\omega}T_xG.x$ and both vector spaces are of dimension $\dim(Y) - \dim(G)$, we have $T_xX^{\mathrm{ss}} = {}^{\omega}T_xG.x$. This implies ${}^{\omega}T_xX^{\mathrm{ss}} = T_xG.x$. Therefore, (2.9) gives that ω' is non-degenerated. So it is left to show that ω' is closed. By naturality of the exterior derivative, we have the following equality in $\Omega^3(X^{\mathrm{ss}})$:

$$\pi^*d\omega' = d\pi^*\omega' = d\iota^*\omega = \iota^*\omega = 0. \tag{2.10}$$

Here, $\Omega^i(X^{\mathrm{ss}})$ denotes the sheaf of i-forms on X^{ss}. Since π is a principal G-bundle, we conclude that $\pi^*\colon \Omega^3(X/\!\!/_\chi G) \to \Omega^3(X^{\mathrm{ss}})$ is injective. Thus, (2.10) gives $d\omega' = 0$. Therefore, ω' is an algebraic symplectic form on $X/\!\!/_\chi G$. Finally, (2.8) implies that the Poisson bracket of ω' coincides with the Poisson bracket $\{.,.\}'$ from Proposition 2.16. □

Remark. If for all $x \in X^{\mathrm{ss}}$ the differential $d_xm\colon T_xX \to T_{m(x)}\mathfrak{g}^*$ is surjective then the Regular Value Theorem implies that X^{ss} is a smooth variety of dimension $\dim(Y) - \dim(G)$. In particular, with this assumption the dimension condition from Theorem 2.17 is always satisfied.

2.2 Triangle parts

Triangle parts are symplectic varieties that are essential building blocks in the construction of bow varieties. They emerged from the representation theory of chainsaw quivers, see [NT17] and the references therein. In this section we recall the definition and fundamental properties of triangle parts following [Tak16, Section 2] and [NT17, Sections 3 and 5].

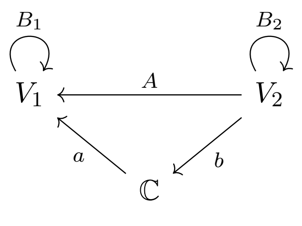
Geometry of triangle parts

Fix finite dimensional vector spaces V_1 and V_2 and set $m_1 := \dim(V_1)$ and $m_2 := \dim(V_2)$. Let $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ and $\mathfrak{g} = \mathrm{End}(V_1) \oplus \mathrm{End}(V_2)$ be the corresponding Lie algebra, where the Lie bracket is given by the commutator bracket $[A,B] = AB - BA$ on $\mathrm{End}(V_1)$, $\mathrm{End}(V_2)$. As usual, G acts on $\mathfrak{g}$ via base change: $(g_1,g_2)(B_1,B_2) = (g_1B_1g_1^{-1}, g_2B_2g_2^{-1})$.

We define the vector space

$$\mathbb{N}_{V_1,V_2} := \mathrm{Hom}(V_2,V_1) \oplus \mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \oplus \mathrm{Hom}(\mathbb{C},V_1) \oplus \mathrm{Hom}(V_2,\mathbb{C}). \tag{2.11}$$

The elements of $\mathbb{N}_{V_1,V_2}$ are tuples (A,B_1,B_2,a,b) of linear maps as illustrated in the diagram:



Then, G acts on $\mathbb{N}_{V_1,V_2}$ via

$$(g_1, g_2).(A, B_1, B_2, a, b) = (g_1 A g_2^{-1}, g_1 B_1 g_1^{-1}, g_2 B_2 g_2^{-1}, g_1 a, b g_2^{-1}).$$

This G-action induces also an action of the Lie algebra $\mathfrak{g}$ on $\mathbb{N}_{V_1,V_2}$:

$$(\gamma_1, \gamma_2).(A, B_1, B_2, a, b) = (\gamma_1 A - A\gamma_2, \gamma_1 B_1 - B_1\gamma_1, \gamma_2 B_2 - B_2\gamma_2, \gamma_1 a, -b\gamma_2).$$

Definition 2.18. The *triangle part* $\mathrm{tri}(V_1, V_2)$ is defined as the θ-semistable locus of $\mu^{-1}(0)$, that is $\{x \in \mu^{-1}(0) \mid x \text{ is } \theta\text{-semistable}\}$, where

$$\mu\colon \mathbb{N}_{V_1,V_2} \longrightarrow \mathrm{Hom}(V_2, V_1), \quad (A, B_1, B_2, a, b) \mapsto B_1 A - A B_2 + ab \tag{2.12}$$

and

$$\theta\colon G \longrightarrow \mathbb{C}^*, \quad (g_1, g_2) \mapsto \frac{\det(g_1)}{\det(g_2)}.$$

Note that, by definition, $\mathrm{tri}(V_1, V_2)$ is a locally closed G-invariant subvariety of $\mathbb{N}_{V_1,V_2}$. We like to employ Mumford's Numerical Criterion to characterize the θ-semistable points $x = (A, B_1, B_2, a, b)$ of $\mu^{-1}(0)$. For this, we introduce the following *subspace conditions*, see [Tak16, Section 2]:

(S1) If $S \subset V_2$ is a subspace with $B_2(S) \subset S$, $A(S) = 0$, $b(S) = 0$ then $S = 0$.

(S2) If $T \subset V_1$ is a subspace with $B_1(T) \subset T$, $\mathrm{im}(A) + \mathrm{im}(a) \subset T$ then $T = V_1$.

Property (S1) is a useful criterion to check vanishing of subspaces of V_2 whereas (S2) is useful for proving that subspaces of V_1 actually coincide with V_1. For instance, a direct application of these conditions is the following non-degeneracy result:

Proposition 2.19. *If $x = (A, B_1, B_2, a, b) \in \mu^{-1}(0)$ satisfies (S1) and (S2) then A has full rank.*

Proof. Choose bases $(v_{1,i})_i$ resp. $(v_{2,j})_j$ of V_1 resp. V_2 and view A, B_1, B_2, a, b as matrices with respect to these bases. Let $A^T\colon V_1^* \to V_2^*$ and $a^T\colon V_1^* \to \mathbb{C}$ be the transpose of A and a. Note that if $f \in V_1^*$ and $v \in V_2$ such that $f \in \ker(A^T)$ and $v \in \ker(A)$ then, by (2.12), we have

$$a^T(f) \cdot b(v) = f((B_1 A - A B_2 + ab)(v)) = 0. \tag{2.13}$$

Suppose that A has not full rank. Then, either $\ker(A) \subset \ker(b)$ or $\ker(A^T) \subset \ker(a^T)$ by (2.13). If $\ker(A) \subset \ker(b)$ then $\ker(A)$ satisfies (S1) and hence $\ker(A) = 0$. If $\ker(A^T) \subset \ker(a^T)$ then $T = \{w \in V_1 \mid \ker(A^T)(w) = 0\}$ satisfies (S2) and thus $T = V_1$ and $\ker(A^T) = 0$. Therefore, A has full rank. □

We further introduce the following *triangle part conditions*:

(T1) If $S_1 \subset V_1, S_2 \subset V_2$ are subspaces with $B_1(S_1) \subset S_1$, $B_2(S_2) \subset S_2$, $A(S_2) \subset S_1$ and $b(S_2) = 0$ then $\dim(S_1) \geq \dim(S_2)$.

(T2) If $T_1 \subset V_1, T_2 \subset V_2$ are subspaces with $B_1(T_1) \subset T_1$, $B_2(T_2) \subset T_2$, $A(T_2) \subset T_1$ and $\mathrm{im}(a) \subset T_1$ then $\mathrm{codim}(T_1) \leq \mathrm{codim}(T_2)$.

Clearly, the condition (T1) implies (S1) by setting $S_1 = 0$, $S_2 = S$ and the condition (T2) implies (S2) by setting $T_1 = T$, $T_2 = V_2$. The next proposition gives that these conditions are actually equivalent to θ-semistability:

Proposition 2.20. *For $x = (A, B_1, B_2, a, b) \in \mu^{-1}(0)$, the following are equivalent:*

(i) x is θ-semistable,

(ii) x satisfies (T1) and (T2),

(iii) x satisfies (S1) and (S2).

Proof. We begin with (ii) ⇒ (i). To apply Mumford's Numerical Criterion, let $\lambda\colon \mathbb{C}^* \to G$ be a one-parameter subgroup such that $\lim_{t\to 0} \lambda(t).x$ exists in $\mu^{-1}(0)$. The vector spaces V_1 and V_2 decompose into weight spaces

$$V_i = \bigoplus_{n\in\mathbb{Z}} V_i^n, \quad \text{where } V_i^n = \{v \in V_i \mid \lambda(t).v = t^n v \text{ for all } t \in \mathbb{C}^*\}, \quad i = 1, 2,$$

with corresponding vector space filtrations

$$F_m V_i = \bigoplus_{n\geq m} V_i^n, \quad m \in \mathbb{Z},\ i = 1, 2.$$

Let $n_0 < 0$, $n_1 > 0$ such that $F_{n_0} V_i = V_i$ and $F_{n_1} V_i = 0$ for $i = 1, 2$. By Mumford's Numerical Criterion, x is θ-semistable if and only if $\langle \lambda, \theta \rangle \geq 0$ which is equivalent to

$$\sum_{j=n_0}^{n_1} j \dim(V_1^j) \geq \sum_{j=n_0}^{n_1} j \dim(V_2^j). \tag{2.14}$$

View $\mathbb{C}$ as filtered vector space with filtration $F_m\mathbb{C} = 0$ if $m > 0$ and $F_m\mathbb{C} = \mathbb{C}$ if $m \leq 0$. Then the existence of the limit $\lim_{t\to 0} \lambda(t).x$ is equivalent to the condition that all the operators A, B_1, B_2, a, b are morphisms of filtered vector spaces. Thus, we can apply (T1) to the pairs $(F_m V_1, F_m V_2)$ with $m > 0$ which yields $\sum_{j=m}^{n_1} \dim(V_1^j) \geq \sum_{j=m}^{n_1} \dim(V_2^j)$. This directly implies

$$\sum_{j=1}^{n_1} j \dim(V_1^j) \geq \sum_{j=1}^{n_1} j \dim(V_2^j). \tag{2.15}$$

Similarly, applying (T2) to the pairs $(F_m V_1, F_m V_2)$ with $m \leq 0$ gives $\sum_{j=n_0}^{m-1} \dim(V_1^j) \leq \sum_{j=n_0}^{m-1} \dim(V_2^j)$. Hence, we obtain

$$\sum_{j=n_0}^{-1} j \dim(V_1^j) \geq \sum_{j=n_0}^{-1} j \dim(V_2^j). \tag{2.16}$$

Combining (2.15) and (2.16) then gives (2.14). Thus, x is θ-semistable. To show (i) ⇒ (iii), suppose that x is θ-semistable and we are given $S \subset V_2$ satisfying the conditions of (S1). Let $W \subset V_2$ be a vector space complement of S and define a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$ via

$$\lambda(t)_{|V_1} = \mathrm{id}_{V_1}, \quad \lambda(t)_{|S} = t\,\mathrm{id}_S, \quad \lambda(t)_{|W} = \mathrm{id}_W\,.$$

Then, the limit $\lim_{t\to 0} \lambda(t).x$ exists in $\mu^{-1}(0)$ and hence Mumford's Numerical Criterion yields

$$0 \le \langle \lambda, \theta \rangle = -\dim(S).$$

Thus, $S = 0$. Analogously, if $T \subset V_1$ satisfies the conditions of (S2) then pick a vector space complement $V \subset V_1$ of T and define a one-parameter subgroup $\lambda' \colon \mathbb{C}^* \to G$ via

$$\lambda'(t)_{|T} = \mathrm{id}_T, \quad \lambda'(t)_{|V} = t^{-1}\,\mathrm{id}_V, \quad \lambda'(t)_{|V_2} = \mathrm{id}_{V_2}.$$

Again, $\lim_{t\to 0} \lambda'(t).x$ exists in $\mu^{-1}(0)$ and Mumford's Numerical Criterion gives

$$0 \le \langle \lambda', \theta \rangle = -\dim(V).$$

Hence, $V = 0$ and $T = V_1$. To prove (iii) $\Rightarrow$ (ii), we first consider the case $\dim(V_1) \le \dim(V_2)$. Suppose $S_1 \subset V_1, S_2 \subset V_2$ satisfy the conditions of (T1). From $\mu(x) = 0$ follows that B_2 maps $\ker(A) \cap \ker(b)$ to $\ker(A)$. As S_2 is contained in $\ker(b)$, this implies that $S_2 \cap \ker(A)$ is B_2-invariant. Hence, $S_2 \cap \ker(A)$ satisfies the conditions of (S1) which gives $S_2 \cap \ker(A) = 0$. Thus, $A_{|S_2}$ is injective and we conclude $\dim(S_2) \ge \dim(S_1)$ as $A(S_2) \subset S_1$ which gives (T1). The property (T2) follows immediately from the surjectivity of A which follows from Proposition 2.19. It remains to consider the case $\dim(V_1) > \dim(V_2)$. By Proposition 2.19, A is injective which directly implies (T1). Assume $T_1 \subset V_1, T_2 \subset V_2$ satisfy the conditions of (T2). Since $\mu(x) = 0$, the operator B_1 maps $\mathrm{im}(A)$ to $\mathrm{im}(A) + \mathrm{im}(a)$. As $\mathrm{im}(a) \subset T_1$ and T_1 is B_1-invariant, we conclude that $T_1 + \mathrm{im}(A)$ is B_1-invariant. Hence, $T_1 + \mathrm{im}(A)$ satisfies the conditions of (S2) which yields $T_1 + \mathrm{im}(A) = V_1$. As $A(T_2) \subset T_1$ and A is injective, we can choose a vector space decomposition $T_2 \oplus W' \oplus W'' = V_2$ such that $T_2 \oplus W' = A^{-1}(T_1 \cap \mathrm{im}(A))$. Since $A(W'') \cap T_1 = 0$, we deduce that

$$\mathrm{codim}(T_1) = \dim(W'') \le \dim(W') + \dim(W'') = \mathrm{codim}(T_2)$$

which completes the proof. □

A further direct application of the conditions (S1), (S2) is that $\mathrm{tri}(V_1, V_2)$ is smooth:

Proposition 2.21. *The variety* $\mathrm{tri}(V_1, V_2)$ *is smooth and each irreducible component of* $\mathrm{tri}(V_1, V_2)$ *is of dimension*

$$\dim(\mathbb{N}_{V_1,V_2}) - \dim(\mathrm{Hom}(V_2, V_1)) = m_1^2 + m_2^2 + m_1 + m_2.$$

For the proof, recall that for finite dimensional vector spaces V and W we have the perfect trace pairing

$$\langle .,. \rangle \colon \mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V), \quad \langle A, B \rangle = \mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Proof of Proposition 2.21. For $x = (A, B_1, B_2, a, b) \in \mathrm{tri}(V_1, V_2) \subset \mathbb{N}_{V_1,V_2}$, the differential of μ at x is given as $d_x\mu \colon \mathbb{N}_{V_1,V_2} \to \mathrm{Hom}(V_2, V_1)$,

$$(A', B_1', B_2', a', b') \mapsto B_2A' + B_2'A - A'B_1 - AB_1' + a'b + ab'. \tag{2.17}$$

Suppose $f \in \mathrm{Hom}(V_1, V_2)$ is orthogonal to the image of $d_x\mu$ with respect to the trace pairing, i.e. $\langle d_x\mu(x'), f\rangle = 0$, for all $x' \in \mathbb{N}_{V_1,V_2}$. Thanks to (2.17), we have

$$Af = 0, \quad fA = 0, \quad B_2 f = B_1 f, \quad bf = 0, \quad fa = 0.$$

This gives that $\mathrm{im}(f)$ satisfies (S1) and hence $f = 0$. Thus, $d_x\mu$ is surjective. By the Regular Value Theorem, x is a smooth point of $\mu^{-1}(0)$ and the dimension of the tangent space $T_x\mu^{-1}(0)$ equals $\dim(\mathbb{N}_{V_1,V_2}) - \dim(\mathrm{Hom}(V_2, V_1))$. Hence, $\mathrm{tri}(V_1, V_2)$ is smooth and each irreducible component is of dimension $\dim(\mathbb{N}_{V_1,V_2}) - \dim(\mathrm{Hom}(V_2, V_1))$. □

Affine structure

Using Proposition 2.19 and the stability conditions (S1) and (S2), Takayama constructed in [Tak16, Proposition 2.20] certain normal forms for the points in $\mathrm{tri}(V_1, V_2)$ which we recall in this subsection. In particular, these normal forms reveal that $\mathrm{tri}(V_1, V_2)$ is an affine variety.

In the case $m_1 = m_2$, these normal forms can be directly obtained from Proposition 2.19:

Proposition 2.22. *For each $m \in \mathbb{N}_0$, there is an isomorphism of varieties*

$$H\colon \mathrm{GL}(m) \times \mathrm{Mat}_{m,m}(\mathbb{C}) \times \mathrm{Mat}_{1,m}(\mathbb{C}) \times \mathrm{Mat}_{m,1}(\mathbb{C}) \xrightarrow{\sim} \mathrm{tri}(\mathbb{C}^m, \mathbb{C}^m)$$

given by

$$H(u, h, I, J) = (u, u^{-1}hu, h - IJ, I, Ju).$$

In particular, $\mathrm{tri}(\mathbb{C}^m, \mathbb{C}^m)$ *is an affine variety.*

Proof. By Proposition 2.19,

$$\mathrm{tri}(\mathbb{C}^m, \mathbb{C}^m) = \{x = (A, B_1, B_2, a, b) \in \mu^{-1}(0) \mid A \in \mathrm{GL}(m)\}.$$

Thus, we deduce that H is a well-defined bijective morphism. Since $\mathrm{GL}(m) \times \mathrm{Mat}_{m,m}(\mathbb{C})$ is connected and $\mathrm{tri}(\mathbb{C}^m, \mathbb{C}^m)$ is smooth, Proposition 2.25 from the next subsection gives that H is an isomorphism of varieties. □

The crucial ingredient for the normal forms in the case $m_1 \neq m_2$ is a result from [Tak16]. To formulate it let $M_{m,n} \subset \mathrm{Mat}_{n,n}(\mathbb{C})$ with $m < n$ be the set of matrices of the form

$$\eta(h, g, f, e_0, e) := \begin{pmatrix} h & 0 & g \\ f & 0 & e_0 \\ 0 & \mathrm{id} & e \end{pmatrix}, \tag{2.18}$$

where

$$h \in \mathrm{Mat}_{m,m}(\mathbb{C}), \quad g \in \mathrm{Mat}_{m,1}(\mathbb{C}), \quad f \in \mathrm{Mat}_{1,m}(\mathbb{C})$$
$$e_0 \in \mathrm{Mat}_{1,1}(\mathbb{C}), \quad e \in \mathrm{Mat}_{n-m-1,1}(\mathbb{C}).$$

Note that $M_{m,n}$ is an affine closed subvariety of $\mathrm{Mat}_{n,n}(\mathbb{C})$. The points of $\mathrm{tri}(V_1, V_2)$ can be described via matrices $\eta(h, g, f, e_0, e)$ as follows, see [Tak16, Proposition 2.20]:

Lemma 2.23. *Let $x = (A, B_1, B_2, a, b) \in \mathrm{tri}(V_1, V_2)$.*

(i) If $m_1 < m_2$ then there exist bases

$$(v_{1,1}, \dots, v_{1,m_1}), \quad (v_{2,1}, \dots, v_{2,m_1}, v'_{2,1}, \dots, v'_{2,m_2-m_1-1}, v''_2)$$

of V_1, V_2 and $\eta = \eta(h, g, f, e_0, e) \in M_{m_1,m_2}$ such that with respect to these bases we have

$$(A, B_1, B_2, a, b) = ((\mathrm{id}\ 0\ 0), h, \eta, g, (0\ 0\ 1)).$$

(ii) If $m_1 > m_2$ then there exist bases

$$(v_{1,1}, \dots, v_{1,m_2}, v'_1, v''_{1,1}, \dots, v''_{1,m_1-m_2-1}), \quad (v_{2,1}, \dots, v_{2,m_2})$$

of V_1, V_2 and $\eta = \eta(h, g, f, e_0, e) \in M_{m_2,m_1}$ such that with respect to these bases we have

$$(A, B_1, B_2, a, b) = \left(\begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix}, \eta, h, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, -f \right).$$

Using Lemma 2.23, we obtain the following normal forms for points of triangle parts:

Proposition 2.24 (Normal forms of triangle parts)**.** *The following holds:*

(i) If $m_1 < m_2$ then there exists an isomorphism of varieties

$$H \colon \mathrm{GL}(m_2) \times M_{m_1,m_2} \xrightarrow{\sim} \mathrm{tri}(\mathbb{C}^{m_1}, \mathbb{C}^{m_2})$$

given by $H(u, \eta(h, g, f, e_0, e)) = ((\mathrm{id}\ 0\ 0)u^{-1}, h, u\eta(h, g, f, e_0, e)u^{-1}, g, (0\ 0\ 1)u^{-1})$.

(ii) If $m_1 > m_2$ then there exists an isomorphism of varieties

$$H \colon \mathrm{GL}(m_1) \times M_{m_2,m_1} \xrightarrow{\sim} \mathrm{tri}(\mathbb{C}^{m_1}, \mathbb{C}^{m_2})$$

given by $H(u, \eta(h, g, f, e_0, e)) = \left(u \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix}, u\eta(h, g, f, e_0, e)u^{-1}, h, u \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, -f \right)$.

In particular, $\mathrm{tri}(\mathbb{C}^{m_1}, \mathbb{C}^{m_2})$ *is always an affine variety.*

Proof. We only proof (i) as (ii) follows along similar lines. Given $(u, \eta) \in \mathrm{GL}(m_1) \times M_{m_2,m_1}$, an easy calculation shows that $(A, B_1, B_2, a, b) = H(u, \eta) \in \mu^{-1}(0)$. As A is surjective, $H(u, \eta)$ satisfies (S2). By construction of H, we have

(1) $(u_{m_1+1}, B_2(u_{m_1+1}), \dots, B_2^{m_2-m_1-1}(u_{m_1+1}))$ is a basis of $\ker(A)$,

(2) $(u_{m_1+1}, B_2(u_{m_1+1}), \dots, B_2^{m_2-m_1-2}(u_{m_1+1}))$ is a basis of $\ker(A) \cap \ker(b)$.

Here, u_i is the i-th column vector of u. Suppose $S \subset V_2$ satisfies the conditions of (S1) and assume $v \in S \setminus \{0\}$. Then, by (1) and (2), we have $B_2^j(v) \notin \ker(b)$, for some $j \geq 1$. This constradicts the assumption that S is a B_2-invariant subspace of $\ker(b)$. Hence, $S = 0$ and therefore, $H(u, \eta)$ satisfies (S1). Thus, H is a well-defined morphism of varieties. As $\mathrm{GL}(m_2) \times M_{m_1,m_2}$ is connected and $\mathrm{tri}(\mathbb{C}^{m_1}, \mathbb{C}^{m_2})$ is smooth, Proposition 2.25 implies that H is an isomorphism of varieties if and only if H is bijective. For surjectivity, recall from Lemma 2.23.(i) that for all $(A, B_1, B_2, a, b) \in \mathrm{tri}(\mathbb{C}^{m_1}, \mathbb{C}^{m_2})$, there exists $\eta(h, g, f, e_0, e) \in M_{m_1,m_2}$ and $(g_1, g_2) \in \mathrm{GL}(m_1) \times \mathrm{GL}(m_2)$ such that

$$(A, B_1, B_2, a, b) = (g_1(\mathrm{id}\ 0\ 0)g_2^{-1}, g_1 h g_1^{-1}, g_2\eta(h, g, f, e_0, e)g_2^{-1}, g_1 g, (0\ 0\ 1)g_2^{-1}).$$

Hence, if we set $u = g_2\left(\begin{smallmatrix} g_1^{-1} & 0 \\ 0 & \mathrm{id}\end{smallmatrix}\right)$, we get

$$H(u, \eta(g_1 h g_1^{-1}, g_1 g, f g_1^{-1}, e_0, e)) = (A, B_1, B_2, a, b).$$

Thus, H is surjective. For injectivity, suppose $H(u, \eta) = (A, B_1, B_2, a, b) = H(u', \eta')$. First, we show that the column vectors $u_1, \ldots, u_{m_2}$ and $u'_1, \ldots, u'_{m_2}$ of u and u' coincide. From $(\mathrm{id}\ \ 0\ \ 0)u = A = (\mathrm{id}\ \ 0\ \ 0)u'$ follows $u_i = u'_i$, for $i = 1, \ldots, m_1$. Next, note that the vector space $\ker(A) \cap \ker(b) \cap \ker(bB_2) \cap \ldots \cap \ker(bB_2^{m_2-m_1-2})$ is of dimension 1. In addition, u_{m_1+1} and u'_{m_1+1} are both generators of this vector space. Since $bB_2^{m_2-m_1-1}(u_{m_1+1}) = 1 = B_2^{m_2-m_1-1}b(u'_{m_1+1})$, we therefore conclude $u_{m+1} = u'_{m+1}$. From this, we deduce

$$u_{m_1+1+i} = B_2^i u_{m_1+1} = B_2^i u'_{m_1+1} = u'_{m_1+1+i}, \quad \text{for } i = 1, \ldots, m_2 - m_1 - 1.$$

Hence, $u = u'$. As $u\eta u^{-1} = u'\eta'(u')^{-1}$, we also have $\eta = \eta'$. This proves that H is injective. □

Remark. The normal forms from Proposition 2.22 and Proposition 2.24 are also called *Hurtubise normal forms.* This is due to the fact that Takayama matched in [Tak16, Section 2] the normal forms for triangle parts with Hurtubise's normal forms of solutions of Nahm's equation over intervals from [Hur89]. In this way, Takayama gave an interpretation of these moduli spaces of solutions of differential equations in terms of moduli of representations of handsaw quivers.

Isomorphism criterion

In the proof of Proposition 2.22 and Proposition 2.24, we used the following general result in algebraic geometry:

Proposition 2.25. *Let $f\colon X \to Y$ be a bijective morphism of varieties. If the connected components of X are all of the same dimension and Y is irreducible and normal then f is an isomorphism of varieties.*

We first prove the following auxiliary statement:

Lemma 2.26. *Let $f\colon X \to Y$ be a bijective morphism of varieties. Then,* $\dim(X) = \dim(Y)$.

Proof. By Grothendieck's version of Zariski's Main Theorem, see [EGA, Chapter IV, Corollary 18.12.13], there exists a factorization

$$\begin{array}{ccc} X & \xrightarrow{\quad \iota \quad} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Y & \end{array} \tag{2.19}$$

where ι is an open dense immersion and f' is a finite morphism. As ι is an open dense immersion, we have $\dim(X) = \dim(Y')$. Since f' is finite and surjective, we also have $\dim(Y') = \dim(Y)$. □

Proof of Proposition 2.25. Let Y', ι, f' be as in (2.19). Since f' is surjective, there exists an irreducible component Y_0 of Y' such that the restriction $f'_{|Y_0}\colon Y_0 \to Y$ is a dominant morphism. As f' is finite and $f'_{|Y_0}$ is dominant, we conclude that also $f'_{|Y_0}$ is finite. Since $f'_{|Y_0}$ is injective on the open dense subvariety $X \cap Y_0$, Lemma 2.27 below gives that $f'_{|Y_0}$ is birational. Thus, by e.g. [Liu06, Corollary 4.4.6], $f'_{|Y_0}$ is an isomorphism of varieties. Next, we show that $X \cap Y_0 = X$. Assume $X \cap Y_0 \neq X$ and let $Z' \subset Y'$ be the union of all irreducible components of Y' which are different to Y_0. Set $Z \coloneqq f'(Z')$ and $U \coloneqq X \cap (Z' \setminus Y_0)$. Then, U is a dense open subvariety of Z' and since f' is finite, Z is a closed subvariety of Y. As f is injective, $f(U)$ is disjoint from the open subvariety $f(Y_0 \cap X) \subset Y$. Hence, $f(Y_0 \cap X)$ is also disjoint from the Zariski closure $\overline{f(U)}$ in Y. As f' is closed, we have $Z = \overline{f(U)}$. Thus, $f(X \cap Y_0 \cap Z') = \emptyset$. Since f is bijective, $X \cap Y_0$ and $X \cap Z'$ are disjoint closed subvarieties of X. This implies $\dim(X \cap Z') = \dim(X)$ and that f restricts to a bijective morphism

$$f_{|X\cap Z'}\colon X \cap Z' \longrightarrow Y \setminus V,$$

where $V \coloneqq f'(X \cap Y_0)$. As, Y is irreducible and $V \subset Y$ is a non-empty open subvariety, we have $\dim(Y \setminus V) < \dim(Y)$. However, Lemma 2.26 gives

$$\dim(Y \setminus V) = \dim(X \cap Z') = \dim(X) = \dim(Y).$$

This contradicts $\dim(Y \setminus V) < \dim(Y)$. Hence, we must have $Y_0 \cap X = X$. Since X is an open subvariety of Y_0 and the isomorphism $f'_{|Y_0}$ restricts to the bijection $f\colon X \to Y$, we must have $X = Y_0$. Therefore, f is an isomorphism of varieties. □

Lemma 2.27. *Let X, Y be irreducible varieties and $f\colon X \to Y$ be a injective and dominant morphism. Then, f is birational.*

Proof. By definition, the induced morphism of schemes $\Delta_f\colon X \to X \times_Y X$ is a locally closed immersion. Since f is an injective morphism of varieties, we conclude that $\mathrm{im}(\Delta_f)$ contains all the closed points of $X \to X \times_Y X$. Thus, Δ_f is a surjective morphism of schemes. This implies that f is universally injective and hence, by e.g. [Stacks, Lemma 01S4], the extension of function field $f^*\colon \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ is purely inseparable. As we are in characteristic 0, this field extension has to be of degree 1 which proves that f is birational. □

Symplectic structure

It was shown in [FR14] that the vanishing locus $\mu^{-1}(0)$ admits an algebraic Poisson structure. Then, Nakajima and Takayama proved in [NT17, Proposition 5.7] that the restriction of this Poisson bracket to $\mathrm{tri}(V_1, V_2)$ is non-degenerate and therefore corresponds to a symplectic form on $\mathrm{tri}(V_1, V_2)$. In this subsection, we recall the definition and important properties of this Poisson bstructure.

Set

$$\mathfrak{n} \coloneqq \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_1, \mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C}, V_2).$$

Let $[.,.]$ be the unique Lie bracket on $\mathfrak{n}$ which satisfies

(a) $\mathrm{Hom}(V_1, V_2)$ is central,

(b) we have $[\mathrm{Hom}(V_1,\mathbb{C}),\mathrm{Hom}(V_1,\mathbb{C})]=0=[\mathrm{Hom}(\mathbb{C},V_2),\mathrm{Hom}(\mathbb{C},V_2)]$,

(c) for all $f\in\mathrm{Hom}(V_1,\mathbb{C})$ and $a\in\mathrm{Hom}(\mathbb{C},V_2)$, we have

$$[a,f]\in\mathrm{Hom}(V_1,V_2),\quad [a,f](v)=f(v)a(1),\quad \text{for all } v\in V_1.$$

The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{n}$ via

$$(g_1,g_2).(A,a,b)=(g_1A-Ag_2,g_1a,-bg_2)$$

Thus, we can take the semi-direct product $\mathfrak{a}:=\mathfrak{g}\ltimes\mathfrak{n}$ where we denote the Lie bracket on $\mathfrak{a}$ by $[.,.]'$. The group G acts on $\mathfrak{a}$ via the usual base change action. A direct computation shows that $[.,.]'$ is G-invariant.

Next, we use $[.,.]'$ to induce a Poisson structure on $\mathbb{N}_{V_1,V_2}$. The trace pairing induces a perfect pairing $\langle .,.\rangle\colon\mathbb{N}_{V_1,V_2}\times\mathfrak{a}\to\mathbb{C}$ satisfying

$$\langle g.x,a\rangle=-\langle x,g.a\rangle,\quad \text{for all } x\in\mathbb{N}_{V_1,V_2},\ a\in\mathfrak{a},\ g\in\mathfrak{g}. \tag{2.20}$$

Via $\langle .,.\rangle$, we obtain identifications of vector spaces $\mathbb{N}_{V_1,V_2}\cong\mathfrak{a}^*$ and $\mathbb{N}^*_{V_1,V_2}\cong(\mathfrak{a}^*)^*\cong\mathfrak{a}$. We define the Poisson bracket $\{.,.\}'$ on $\mathbb{N}_{V_1,V_2}$ to be the unique Poisson bracket such that $\{f,g\}'=[f,g]'$, for all $f,\ g\in\mathfrak{a}$. By (2.20), this Poisson bracket admits the following moment map for the G-action:

Proposition 2.28. *The projection $m'\colon\mathbb{N}_{V_1,V_2}\to\mathfrak{g}$, $(A,B_1,B_2,a,b)\mapsto(B_1,B_2)$ is a moment map for the G-action on $\mathbb{N}_{V_1,V_2}$.*

Proof. By the Leibnitz rule, it suffices to prove (2.5) for $a\in\mathfrak{a}\cong\mathbb{N}^*_{V_1,V_2}$. If $g\in\mathfrak{g}$ and $x\in\mathbb{N}_{V_1,V_2}$ then $\{\langle m'(\cdot),g\rangle,a\}(x)=\langle x,g.a\rangle$. Let $\vec{g}_x$ be the fiber of $\vec{g}$ over x. Then, $da(\vec{g}_x)=\langle g.x,a\rangle$. Thus, the proof follows from (2.20). □

In general, the Poisson bracket $\{.,.\}'$ is not compatible with the defining equation $\mu=0$ of triangle parts from (2.12). Thus, in general, $\{.,.\}'$ does *not* induce a Poisson bracket on $\mathrm{tri}(V_1,V_2)$. However, this can be fixed, by twisting $\{.,.\}'$ as follows: The vector space automorphism of $\mathfrak{a}$ given by

$$(B_1,B_2,A,a,b)\to(B_1,-B_2,A,a,b)$$

induces an algebra automorphism $\nu\colon\mathcal{O}(\mathbb{N}_{V_1,V_2})\to\mathcal{O}(\mathbb{N}_{V_1,V_2})$ via the identification $\mathbb{N}^*_{V_1,V_2}\cong\mathfrak{a}$. Then, we define the Poisson bracket $\{.,.\}$ on $\mathbb{N}_{V_1,V_2}$ as

$$\{f,g\}:=\{\nu(f),\nu(g)\}',\quad f,\ g\in\mathcal{O}(\mathbb{N}_{V_1,V_2}). \tag{2.21}$$

By [FR14, Proposition 3.15], we have the following result:

Proposition 2.29. *The Poisson bracket $\{.,.\}$ on $\mathbb{N}_{V_1,V_2}$ from (2.21) induces a G-invariant Poisson bracket on $\mu^{-1}(0)$ and hence also on $\mathrm{tri}(V_1,V_2)$.*

The Poisson bracket $\{.,.\}$ admits an explicit description via coordinate functions: Choose bases $(v_{1,i})_i$ resp. $(v_{2,j})_j$ of V_1 resp. V_2 and let $A_{k,l} \in \mathcal{O}(\mathbb{N}_{V_1,V_2})$ be the regular function which assigns to a point $x = (A, B_1, B_2, a, b)$ the (k,l)-entry of the matrix of A with respect to the bases $(v_{1,i})_i$ and $(v_{2,j})_j$. Define $(B_1)_{k,l}$, $(B_2)_{k,l}$, a_k and $b_k \in \mathcal{O}(\mathbb{N}_{V_1,V_2})$ in the same way. Then, inserting these coordinate functions in (2.21) yields the following formulas for $\{.,.\}$:

$$\begin{gathered} \{A_{i,j}, A_{k,l}\} = 0, \quad \{a_i, a_j\} = 0 = \{b_i, b_j\}, \\ \{(B_2)_{i,j}, (B_2)_{k,l}\} = \delta_{i,l}(B_2)_{k,j} - \delta_{j,k}(B_2)_{i,l}, \\ \{(B_1)_{ij}, (B_1)_{kl}\} = \delta_{i,l}(B_1)_{k,j} - \delta_{j,k}(B_1)_{i,l}, \\ \{(B_2)_{i,j}, a_k\} = 0 = \{(B_1)_{i,j}, b_k\}, \\ \{(B_2)_{i,j}, b_k\} = \delta_{i,k} b_j, \quad \{(B_1)_{i,j}, a_k\} = -\delta_{j,k} a_i, \\ \{(B_2)_{i,j}, A_{k,l}\} = \delta_{i,l} A_{k,j}, \quad \{(B_1)_{i,j}, A_{k,l}\} = -\delta_{j,k} A_{i,l}, \\ \{b_i, a_j\} = A_{j,i}, \quad \{A_{i,j}, b_k\} = 0 = \{A_{i,j}, a_k\}. \end{gathered} \tag{2.22}$$

Via the explicit formulas from (2.22), we deduce the following crucial result:

Proposition 2.30. *The restriction of* $\{.,.\}$ *to* $\mathrm{tri}(V_1, V_2)$ *is non-degenerate.*

Proof. We only prove the case $m_1 = m_2 =: m$, as the case $m_1 \neq m_2$ is similar. Set $R := \mathcal{O}(\mathrm{tri}(V_1, V_2))$ and let $\mathrm{Der}(R, R)$ be the R-module of $\mathbb{C}$-derivations. Set

$$\Theta\colon R \longrightarrow \mathrm{Der}(R, R), \quad f \mapsto \{f, .\}$$

and let $E \subset \mathrm{Der}(R, R)$ be the R-module generated by the image of Θ. By definition, $\{.,.\}$ is non-degenerated if and only if $E = \mathrm{Der}(R, R)$. By Proposition 2.22, it suffices to show that E contains all the derivations

$$\frac{\partial}{\partial A_{i,j}}, \quad \frac{\partial}{\partial (B_1)_{i,j}}, \quad \frac{\partial}{\partial a_i}, \quad \frac{\partial}{\partial b_i}, \quad \text{for } 1 \leq i, j \leq m. \tag{2.23}$$

By (2.22), we have

$$\Theta(A_{i,j}) = \sum_{k=1}^{m} A_{k,j} \frac{\partial}{\partial (B_1)_{k,i}}, \quad \text{for } 1 \leq i, j \leq m. \tag{2.24}$$

As the matrix $(A_{i,j})_{i,j} \in \mathrm{Mat}_{m,m}(R)$ is invertible over R, (2.24) implies that E contains all $\frac{\partial}{\partial (B_1)_{i,j}}$. From (2.22) follows

$$\Theta(b_i) \equiv \sum_{k=1}^{m} A_{k,i} \frac{\partial}{\partial a_k} \mod E, \quad -\Theta(a_i) \equiv \sum_{k=1}^{m} A_{i,k} \frac{\partial}{\partial b_k} \mod E, \quad \text{for } i = 1, \ldots, m.$$

Thus, E also contains all $\frac{\partial}{\partial a_i}$ and $\frac{\partial}{\partial b_i}$. Finally, (2.22) gives

$$-\Theta((B_1)_{i,j}) \equiv \sum_{k=1}^{m} A_{i,k} \frac{\partial}{\partial A_{j,k}} \mod E, \quad \text{for } 1 \leq i, j \leq m$$

which implies that E contains also all $\frac{\partial}{\partial A_{i,j}}$. Hence, all derivations from (2.23) are contained in E. This proves that $\{.,.\}$ is non-degenerated. ☐

In particular, Proposition 2.30 implies that there exists a unique symplectic form ω on $\mathrm{tri}(V_1, V_2)$ such that

$$\omega(X_g, X_f) = \{f, g\}, \quad \text{for all } f, g \in \mathcal{O}(\mathrm{tri}(V_1, V_2)). \tag{2.25}$$

Here, X_f, X_g denote the hamiltonian vector fields of f, g with respect to ω.

By Proposition 2.28, we deduce that ω admits the following moment map for the G-action on $\mathrm{tri}(V_1, V_2)$:

Corollary 2.31. *The G-equivariant morphism*

$$m\colon \mathrm{tri}(V_1, V_2) \longrightarrow \mathfrak{g}, \quad (A, B_1, B_2, a, b) \mapsto (B_1, -B_2) \tag{2.26}$$

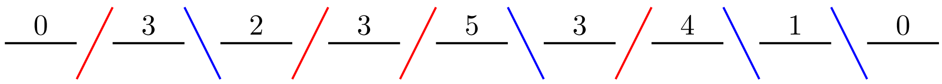
is a moment map for the G-action on $\mathrm{tri}(V_1, V_2)$.

2.3 Bow varieties

In this section, we recall the construction and geometric properties of bow varieties from [NT17, Section 2]. Bow varieties are defined as hamiltonian reductions of certain moduli spaces of quiver representations that we call *affine brane varieties*. To construct these affine brane varieties, we use the language of brane diagrams from [RS20].

Brane diagrams

A *brane diagram* is an object like this:

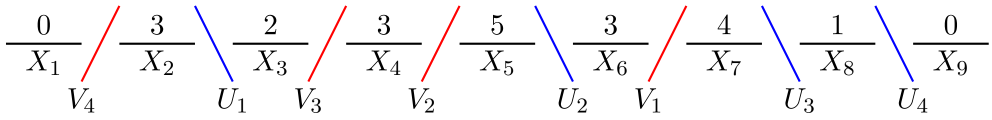


That is, a *brane diagram* is a finite sequence of black horizontal lines drawn from left to right. Between each consecutive pair of horizontal lines there is either a blue SE-NW line \ or a red SW-NE line /. Each horizontal line X is labeled by a non-negative integer d_X. We further demand that the first and the last horizontal line is labeled by 0.

Remark. Our terminology is slightly different to the terminology in [NT17] [RS20] and [RR23] which is motivated from string theory. There the horizontal lines are called D3 branes, the blue lines D5 branes and the red lines NS5 branes, see in particular the explanation in [RR23, Section 2.4]. However, for our purposes, it suffices to view brane diagrams as purely combinatorial objects. Hence, we will refer to the lines in brane diagrams just by their colors.

Notation 2.32. Let $\mathrm{h}(\mathcal{D})$, $\mathrm{b}(\mathcal{D})$ and $\mathrm{r}(\mathcal{D})$ denote the set of black, blue and red lines in a given brane diagram $\mathcal{D}$. We denote the number of red lines in $\mathcal{D}$ by M and the individual red lines by $V_1, \dots, V_M$ numbered from *right to left*. Likewise, let N be the number of blue lines in $\mathcal{D}$ and we denote by $U_1, \dots, U_N$ these lines, numbered from *left to right*. The black lines in $\mathcal{D}$ are denoted by $X_1, \dots, X_{M+N+1}$ also numbered from *left to right*.

Thus, the lines in the above brane diagram are labeled as follows



Remark. The convention to number the red lines from right to left differs from the one in [RS20].

Given two lines Y_1, Y_2 in $\mathcal{D}$, we write

$$Y_1 \triangleleft Y_2 \tag{2.27}$$

if Y_1 is to the left of Y_2. If Y is a colored line, we denote the black line directly to the left resp. to the right of Y by Y^- resp. Y^+. Similarly, the colored lines directly left and right to a black line X in $\mathcal{D}$ are denoted by X^- and X^+.

Affine brane varieties

We continue with the definition of affine brane varieties. For this, we assign to each horizontal line X in $\mathcal{D}$ the vector space $W_X := \mathbb{C}^{d_X}$. Further, we set $W_{\mathcal{D}} := \bigoplus_{X\in \mathrm{h}(\mathcal{D})} W_X$. We also denote W_{X_i} just by W_i. For any red line $V \in \mathrm{r}(\mathcal{D})$, define the variety

$$\mathbb{M}_V := \mathrm{Hom}(W_{V^+}, W_{V^-}) \oplus \mathrm{Hom}(W_{V^-}, W_{V^+}).$$

We denote the elements of $\mathbb{M}_V$ as tuples $y_V = (C_V, D_V)$ and equip $\mathbb{M}_V$ with the usual $(\mathrm{GL}(W_{V^-}) \times \mathrm{GL}(W_{V^+}))$-action

$$(g_-, g_+).(C_V, D_V) = (g_- C_V g_+^{-1}, g_+ D_V g_-^{-1}), \quad g_- \in \mathrm{GL}(W_{V^-}),\ g_+ \in \mathrm{GL}(W_{V^+}).$$

It is well-known, see e.g.[Gin12, Section 4], that $\mathbb{M}_V$ admits a non-degenerated $(\mathrm{GL}(W_{V^-}) \times \mathrm{GL}(W_{V^+}))$-invariant Poisson bracket $\{.,.\}$ that is uniquely determined by

$$\{C_{V,i,j}, D_{V,k,l}\} = -\delta_{i,l}\delta_{j,k}. \tag{2.28}$$

The corresponding symplectic form on $\mathbb{M}_V$ is $\sum_{i=1}^{d_{V^+}} \sum_{j=1}^{d_{V^-}} dC_{V,i,j} \wedge dD_{V,j,i}$ and admits the following moment map for the $(\mathrm{GL}(W_{V^-}) \times \mathrm{GL}(W_{V^+}))$-action:

$$m_V \colon \mathbb{M}_V \longrightarrow \mathrm{End}(W_{V^-}) \oplus \mathrm{End}(W_{V^+}), \quad (C, D) \mapsto (-CD, DC). \tag{2.29}$$

To any blue line U, we attach the triangle part $\mathbb{M}_U := \mathrm{tri}(W_{U^-}, W_{U^+})$. We write the elements of $\mathbb{M}_U$ as tuples $x_U = (A_U, B_U^-, B_U^+, a_U, b_U)$ and denote by

$$m_U \colon \mathbb{M}_U \longrightarrow \mathrm{End}(W_{U^-}) \oplus \mathrm{End}(W_{U^+}), \quad (A_U, B_U^-, B_U^+, a_U, b_U) \mapsto (B_U^-, -B_U^+) \tag{2.30}$$

the moment map from Corollary 2.31.

Definition 2.33. The *affine brane variety associated to* $\mathcal{D}$ is defined as the affine variety

$$\widetilde{\mathcal{M}}(\mathcal{D}) := \Big(\prod_{U \in \mathrm{b}(\mathcal{D})} \mathbb{M}_U \Big) \times \Big(\prod_{V \in \mathrm{r}(\mathcal{D})} \mathbb{M}_V \Big).$$

Additionally, we define the vector space

$$\mathbb{V}_{\mathcal{D}} := \Big(\bigoplus_{U \in \mathrm{b}(\mathcal{D})} \mathbb{N}_{W_{U^-}, W_{U^+}} \Big) \oplus \Big(\bigoplus_{V \in \mathrm{r}(\mathcal{D})} \mathbb{M}_V \Big). \tag{2.31}$$

Here, $\mathbb{N}_{W_{U^-}, W_{U^+}}$ is defined as in (2.11). Note that $\widetilde{\mathcal{M}}(\mathcal{D})$ is a locally closed subvariety of $\mathbb{V}_{\mathcal{D}}$. We denote points of $\widetilde{\mathcal{M}}(\mathcal{D})$ and $\mathbb{V}_{\mathcal{D}}$ as tuples $((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V)$. By Proposition 2.24, all factors of $\widetilde{\mathcal{M}}(\mathcal{D})$ are smooth and affine varieties which yields that also $\widetilde{\mathcal{M}}(\mathcal{D})$ is smooth and affine. We endow $\widetilde{\mathcal{M}}(\mathcal{D})$ with an algebraic (base change) action of the group

$$\mathcal{G} := \prod_{X \in h(\mathcal{D})} \mathrm{GL}(W_X) \tag{2.32}$$

given as

$$\begin{aligned}&(g_X)_X.((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V)\\&\quad= ((g_{U^-} A_U g_{U^+}^{-1}, g_{U^-} B_U^- g_{U^-}^{-1}, g_{U^+} B_U^+ g_{U^+}^{-1}, g_{U^-} a_U, b_U g_{U^+}^{-1})_U, (g_{V^-} C_V g_{V^+}^{-1}, g_{V^+} D_V g_{V^-}^{-1})_V).\end{aligned}$$

The Poisson brackets on the factors $\mathbb{M}_U$ and $\mathbb{M}_V$ induce a non-degenerated $\mathcal{G}$-invariant Poisson bracket $\{.,.\}$ on $\widetilde{\mathcal{M}}(\mathcal{D})$. We denote the corresponding algebraic symplectic form on $\widetilde{\mathcal{M}}(\mathcal{D})$. The moment maps m_U and m_V for $\mathbb{M}_U$ and $\mathbb{M}_V$ from (2.30) and (2.29) induce the following moment map for the $\mathcal{G}$-action on $\widetilde{\mathcal{M}}(\mathcal{D})$:

$$\tilde{m} \colon \widetilde{\mathcal{M}}(\mathcal{D}) \longrightarrow \bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X), \quad ((x_U)_U, (y_V)_V) \mapsto \sum_{U \in \mathrm{b}(\mathcal{D})} m_U(x_U) + \sum_{V \in \mathrm{r}(\mathcal{D})} m_V(y_V).$$

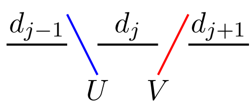
More explicitly, for a black line $X \in \mathrm{h}(\mathcal{D})$, the corresponding component $\tilde{m}((x_U)_U, (y_V)_V)_X$ is given by

$$\tilde{m}((x_U)_U, (y_V)_V)_X = \begin{cases} B_{X^+}^- - B_{X^-}^+ & \text{if } X^+,\ X^- \text{ are both blue,} \\ D_{X^-} C_{X^-} - C_{X^+} D_{X^+} & \text{if } X^+,\ X^- \text{ are both red,} \\ D_{X^-} C_{X^-} + B_{X^+}^- & \text{if } X^+ \text{ is blue and } X^- \text{ is red,} \\ -C_{X^+} D_{X^+} - B_{X^-}^+ & \text{if } X^+ \text{ is red and } X^- \text{ is blue.} \end{cases} \tag{2.33}$$

The conditions (S1) and (S2) for triangle parts yield that the points of $\tilde{m}^{-1}(0)$ satisfy the injectivity and surjectivity conditions:

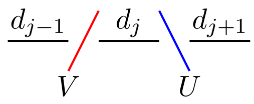
Proposition 2.34. *Let $y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \tilde{m}^{-1}(0)$.*

(i) Given a local configuration in $\mathcal{D}$ of the form:



Then, the map $F \colon W_j \to W_{j-1} \oplus W_{j+1} \oplus \mathbb{C}$, $v \mapsto (A_U(v), D_V(v), b_U(v))$ is injective.

(ii) Given a local configuration in $\mathcal{D}$ of the form:



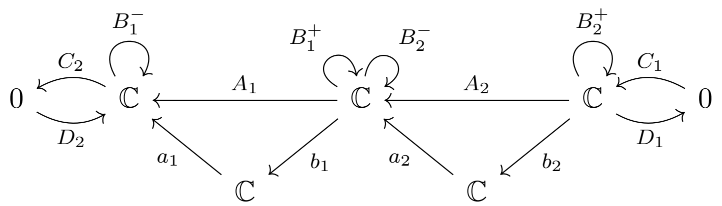
Then, the map $F' \colon W_{j-1} \oplus W_{j+1} \oplus \mathbb{C} \to W_j$, $(v_1, v_2, v_3) \mapsto (D_V(v_1), A_U(v_2), a_U(v_3))$ is surjective.

Proof. We begin with (i). By the moment map equation (2.33), we have $C_V D_V = -B_U^+$ which implies $\ker(D_V) \subset \ker(B_U^+)$. Thus, $\ker(F)$ is B_U^+-invariant and therefore satisfies the conditions of (S1). Hence, $\ker(F) = 0$. For (ii), note that as $D_V C_V = -B_U^-$, we have $\mathrm{im}(B_U^-) \subset \mathrm{im}(D_V)$ and hence $\mathrm{im}(F')$ is B_U^--invariant. This implies that $\mathrm{im}(F')$ satisfies conditions of (S2) and thus $\mathrm{im}(F') = W_j$. □

Example 2.35. Let $\mathcal{D} = 0/1\backslash 1\backslash 1/0$. We write elements of $\widetilde{\mathcal{M}}(\mathcal{D})$ as tuples

$$((A_i, B_i^-, B_i^+, a_i, b_i)_{i=1,2}, (C_i, D_i)_{i=1,2})$$

according to the diagram:



(2.34)

By Proposition 2.19, the conditions (S1) and (S2) are equivalent to A_1, $A_2 \neq 0$. Thus, we have an isomorphism of varieties $\widetilde{\mathcal{M}}(\mathcal{D}) \xrightarrow{\sim} (\mathbb{C}^* \times \mathbb{C}^3)^2$ given by

$$((A_i, B_i^-, B_i^+, a_i, b_i)_{i=1,2}, (C_i, D_i)_{i=1,2}) \mapsto (A_1, B_1^+, a_1, b_1, A_2, B_2^+, a_2, b_2).$$

Construction of bow varieties

Let $\mathcal{D}$ be a brane diagram and $\widetilde{\mathcal{M}}(\mathcal{D})$ be the corresponding affine brane variety. Let $\mathcal{G}$ and $\tilde{m}$ be as in the previous subsection.

Definition 2.36. The *bow variety associated to $\mathcal{D}$* is defined as

$$\mathcal{C}(\mathcal{D}) := \tilde{m}^{-1}(0) /\!\!/_{\chi} \mathcal{G},$$

where

$$\chi \colon \mathcal{G} \longrightarrow \mathbb{C}^*, \quad (g_X)_X \mapsto \prod_{X \in \mathrm{h}'(\mathcal{D})} \det(g_X)$$

and $\mathrm{h}'(\mathcal{D})$ is the set of black lines X in $\mathcal{D}$ such that X^- is red.

By Proposition 2.2.(i), $\mathcal{C}(\mathcal{D})$ is a quasi-projective variety. Moreover, Proposition 2.16 implies that the Poisson bracket $\{.,.\}$ on $\widetilde{\mathcal{M}}(\mathcal{D})$ induces a Poisson bracket $\{.,.\}'$ on $\mathcal{C}(\mathcal{D})$.

Geometric properties

Recall from Theorem 2.8 that the points of $\mathcal{C}(\mathcal{D})$ are characterized by χ-(semi)stability conditions. By applying Mumford's Numerical Criterion, Nakajima and Takayama proved the following useful χ-(semi)stability conditions for points on $\tilde{m}^{-1}(0)$, see [NT17, Proposition 2.8]:

Proposition 2.37 ((Semi-)Stability for bow varieties)**.** *Let*

$$x = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V)) \in \tilde{m}^{-1}(0).$$

Then the following holds:

(i) The point x is χ-semistable if and only if x satisfies the following condition: For all graded subspaces $T = \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X \subset W_{\mathcal{D}}$ such that $\mathrm{im}(a_U) + \mathrm{A_U} \subset T_{U^-}$ and A_U induces an isomorphism $W_{U^+}/T_{U^+} \to W_{U^-}/T_{U^-}$, for all $U \in \mathrm{b}(\mathcal{D})$, we have

$$\sum_{X \in \mathrm{h}'_{\mathcal{D}}} \mathrm{codim}(T_X) \leq 0. \tag{2.35}$$

(ii) The point x is χ-stable if and only if we have an strict inequality in (2.35) *unless $T = W_{\mathcal{D}}$.*

Proposition 2.37 has many useful consequences. As a direct consequence we get that the χ-semistable and the χ-stable locus of $\tilde{m}^{-1}(0)$ coincide:

Corollary 2.38 (Semistable=stable)**.** *We have $\tilde{m}^{-1}(0)^{\mathrm{ss}} = \tilde{m}^{-1}(0)^{\mathrm{s}}$.*

Applying Theorem 2.8 then directly gives:

Corollary 2.39. *The quotient morphism $\pi\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to \mathcal{C}(\mathcal{D})$ is a geometric quotient.*

Next, we employ Proposition 2.37 to deduce that the χ-stable locus $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is actually smooth and we also have a convenient dimension formula:

Proposition 2.40. *The variety $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is smooth and each irreducible component is of dimension*

$$\dim(\widetilde{M}(\mathcal{D})) - \dim(\mathcal{G}) = \Big(\sum_{U \in \mathrm{b}(\mathcal{D})} (d_{U^-}^2 + d_{U^+}^2 + d_{U^-} + d_{U^+})\Big) + \Big(\sum_{V \in \mathrm{r}(\mathcal{D})} 2d_{V^-}d_{V^+}\Big) - \Big(\sum_{X \in \mathrm{h}(\mathcal{D})} d_X^2\Big).$$

For the proof, recall the definition of $\mathbb{V}_{\mathcal{D}}$ from (2.31) and let $m'\colon \mathbb{V}_{\mathcal{D}} \to \bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X)$ be given by the formula (2.33). We also define

$$\mu'\colon \mathbb{V}_{\mathcal{D}} \longrightarrow \bigoplus_{U \in \mathrm{b}(\mathcal{D})} \mathrm{Hom}(W_{U^+}, W_{U^-}) \tag{2.36}$$

as $\mu'((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) = (B_U^- A_U - A_U B_U^+ + a_U b_U)_U$. Combining these two morphisms, we set

$$\mathbb{N}_{\mathcal{D}} \coloneqq \Big(\bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X)\Big) \oplus \Big(\bigoplus_{U \in \mathrm{b}(\mathcal{D})} \mathrm{Hom}(W_{U^+}, W_{U^-})\Big) \tag{2.37}$$

and

$$\beta\colon \mathbb{V}_{\mathcal{D}} \xrightarrow{\binom{m'}{\mu'}} \mathbb{N}_{\mathcal{D}}. \tag{2.38}$$

Note that, by construction, $\tilde{m}^{-1}(0) = \beta^{-1}(0)$ and hence $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is an open subvariety of $\beta^{-1}(0)$. To employ the Regular Value Theorem, we use the following lemma:

Lemma 2.41. *For $y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \tilde{m}^{-1}(0)^{\mathrm{s}}$, the differential $d_y\beta$ is surjective.*

Proof. The differentials

$$d_y m'\colon \mathbb{V}_{\mathcal{D}} \longrightarrow \bigoplus_{X\in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X), \quad d_y\mu'\colon \mathbb{V}_{\mathcal{D}} \longrightarrow \bigoplus_{U\in \mathrm{b}(\mathcal{D})} \mathrm{Hom}(W_{U^+}, W_{U^-})$$

map a point $y' = ((A'_U, (B'_U)^-, (B'_U)^+, a'_U, b'_U)_U, (C'_V, D'_V)_V) \in \mathbb{V}_{\mathcal{D}}$ to

$$d_y m'(y')_X = \begin{cases} (B'_{X^+})^- - (B'_{X^-})^+ & \text{if } X^+,\ X^- \in \mathrm{b}(\mathcal{D}),\\ D'_{X^-}C_{X^-} + D_{X^-}C'_{X^-} - C'_{X^+}D_{X^+} - C_{X^+}D'_{X^+} & \text{if } X^+,\ X^- \in \mathrm{r}(\mathcal{D}),\\ D'_{X^-}C_{X^-} + D'_{X^-}C_{X^-} + (B'_{X^+})^- & \text{if } X^+ \in \mathrm{b}(\mathcal{D}),\ X^- \in \mathrm{r}(\mathcal{D}),\\ -C'_{X^+}D_{X^+} - C_{X^+}D'_{X^+} - (B'_{X^-})^+ & \text{if } X^+ \in \mathrm{r}(\mathcal{D}),\ X^- \in \mathrm{b}(\mathcal{D}). \end{cases}$$

and

$$d_y\mu'(y')_U = B_U^- A'_U + (B'_U)^- A_U - A'_U B_U^+ - A_U (B'_U)^+ + a'_U b_U + a_U b'_U.$$

Suppose

$$(f, h) = ((f_X)_X, (h_U)_U) \in \Big(\bigoplus_{X\in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X)\Big) \oplus \Big(\bigoplus_{U\in \mathrm{b}(\mathcal{D})} \mathrm{Hom}(W_{U^-}, W_{U^+})\Big)$$

is orthogonal to $\mathrm{im}(d_y\beta)$ with respect to the trace pairing. By the above description of $d_y m'$ and $d_y\mu'$, we conclude that for $U \in \mathrm{b}(\mathcal{D})$, $V \in \mathrm{r}(\mathcal{D})$ holds

$$\begin{gathered} B_U^- h_U = h_U B_U^+, \quad h_U a_U = 0, \quad b_U h_U = 0,\\ A_U h_U = f_{U^-}, \quad h_U A_U = f_{U^+},\\ C_V f_{V^+} = f_{V^-} C_V, \quad D_V f_{V^-} = f_{V^+} D_V. \end{gathered} \tag{2.39}$$

Combining (2.36) and (2.39), we get

$$B_U^+ f_{U^+} = f_U^+ B_{U^+}, \quad B_U^- f_{U^-} = f_{U^-} B_U^-, \quad \text{for } U \in \mathrm{b}(\mathcal{D}). \tag{2.40}$$

Set $T_X \coloneqq \ker(f_X)$, $S_X \coloneqq \mathrm{im}(f_X)$ and fix $U \in \mathrm{b}(\mathcal{D})$. By (2.40), T_{U^-}, S_{U^-} are B_U^--invariant and T_{U^+}, S_{U^+} are B_U^+-invariant. By (2.39), we have $A_U(T_{U^+}) \subset T_{U^-}$ and $A_U(S_{U^+}) \subset S_{U^-}$. In addition, (2.39) also gives $\mathrm{im}(a_U) \subset T_{U^-}$ and $S_{U^+} \subset \ker(b_U)$. Thus, we can apply (T2) to the pair (T_{U^-}, T_{U^+}) and (T1) to the pair (S_{U^-}, S_{U^+}). Hence, we have

$$\mathrm{codim}(T_{U^+}) \leq \mathrm{codim}(T_{U^-}), \quad \dim(S_{U^-}) \leq \dim(S_{U^+}).$$

By definition, $\mathrm{codim}(T_X) = \dim(S_X)$, for all X. Therefore,

$$\mathrm{codim}(T_{U^+}) = \mathrm{codim}(T_{U^-}), \quad \dim(S_{U^-}) = \dim(S_{U^+}). \tag{2.41}$$

By (2.36) and (2.40), we can infer that $T_{U^-} + \operatorname{im}(A_U)$ is B^-_U-invariant and therefore $T_{U^-} + \operatorname{im}(A_U) = W_{U^-}$ by (S2). By (2.41), we deduce that A_U induces an isomorphism of vector spaces $W_{U^+}/T_{U^+} \xrightarrow{\sim} W_{U^-}/T_{U^-}$. Hence, $T \coloneqq \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X \subset W_{\mathcal{D}}$ satisfies the conditions of Proposition 2.37 which gives $T = W_{\mathcal{D}}$. Thus, $f_X = 0$, for all X. Finally, (2.39) gives that $\operatorname{im}(h_U)$ satisfies the conditions of (S1). Therefore, $h_U = 0$, for all U and hence $(f, h) = 0$. This proves that $d_y\beta$ is surjective. □

Proof of Proposition 2.40. Let $y \in \tilde{m}^{-1}(0)^{\mathrm{s}} \subset \beta^{-1}(0)$. By Lemma 2.41 the differential $d_y\beta$ is surjective. Therefore, the Regular Value Theorem gives that y is a smooth point of $\tilde{m}^{-1}(0) = \beta^{-1}(0)$ and the tangent space $T_y\tilde{m}^{-1}(0)$ is of dimension $\dim(\mathbb{V}_{\mathcal{D}}) - \dim(\mathbb{N}_{\mathcal{D}})$. By Proposition 2.21, we have

$$\dim(\mathbb{V}_{\mathcal{D}}) - \dim(\mathbb{N}_{\mathcal{D}}) = \dim(\widetilde{\mathcal{M}}(\mathcal{D})) - \dim(\mathcal{G}).$$

Thus, we deduce that $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is smooth and all irreducible components are of dimension $\dim(\widetilde{\mathcal{M}}(\mathcal{D})) - \dim(\mathcal{G})$. □

A similar argument as in the proof of Lemma 2.41 gives that the $\mathcal{G}$-action on $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is free:

Proposition 2.42. *The $\mathcal{G}$-action on $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is free.*

Proof. Let $y = ((A_U, B^-_U, B^+_U, a_U, b_U)_U, (C_V, D_V)_V) \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ and $g = (g_X)_X \in \mathcal{G}$ with $g.y = y$. Set $T_X \coloneqq \ker(g_X - \mathrm{id}_{W_X}) \subset W_X$ and $T_X \coloneqq \operatorname{im}(g_X - \mathrm{id}_{W_X}) \subset W_X$. Note that $\operatorname{codim}(T_X) = \dim(T'_X)$, for all $X \in \mathrm{h}(\mathcal{D})$. By construction, $T \coloneqq \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X$ and $T' \coloneqq \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T'_X$ are A_U, B^-_U, B^+_U, C_V, D_V-invariant subspaces of $W_{\mathcal{D}}$. In addition, we have $\operatorname{im}(a_U) \subset T$ and $T'_{U^+} \subset \ker(b_U)$, for all $U \in \mathrm{b}(\mathcal{D})$. Hence, if $U \in \mathrm{b}(\mathcal{D})$ then $T_{U^-} + \operatorname{im}(A_U)$ satisfies (S1) which gives $T_{U^-} + \operatorname{im}(A_U) = W_{U^-}$. Likewise, $T'_{U^+} \cap \ker(A_U)$ satisfies (S1) which yields that the restriction $A_{|T'_{U^+}}$ is injective. Since $A(T'_{U^+}) \subset T'_{U^-}$, we get $\dim(T'_{U^+}) \leq \dim(T'_{U^-})$. Equivalently, $\operatorname{codim}(T_{U^+}) \leq \dim(T_{U^-})$. As $T_{U^-} + \operatorname{im}(A_U) = W_{U^-}$, we conclude $\operatorname{codim}(T_{U^+}) = \dim(T_{U^-})$ and that A_U induces a vector space isomorphism $W_{U^+}/T_{U^+} \xrightarrow{\sim} W_{U^-}/T_{U^-}$. Thus, T satisfies the conditions of Proposition 2.37 and therefore $T = W_{\mathcal{D}}$. Thus, $g = \mathrm{id}_{W_{\mathcal{D}}}$. □

We now deduce some geometric properties for bow varieties:

Proposition 2.43 (Geometric properties of bow varieties)**.** *The following holds:*

(i) The bow variety $\mathcal{C}(\mathcal{D})$ is smooth.

(ii) The quotient morphism $\pi\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to \mathcal{C}(\mathcal{D})$ is a principal $\mathcal{G}$-bundle (in the Zariski topology).

(iii) The Poisson bracket $\{.,.\}'$ on $\mathcal{C}(\mathcal{D})$ is non-degenerated and the corresponding symplecitc form ω' on $\mathcal{C}(\mathcal{D})$ satisfies

$$\pi^*\omega' = \iota^*\omega, \tag{2.42}$$

where $\iota\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \hookrightarrow \widetilde{\mathcal{M}}(\mathcal{D})$ is the inclusion and ω the symplectic form on $\widetilde{\mathcal{M}}(\mathcal{D})$.

Proof. By Proposition 2.42 and Proposition 2.40, $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is smooth and the $\mathcal{G}$-action on $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is free. By Corollary 2.38, $\tilde{m}^{-1}(0)^{\mathrm{s}} = \tilde{m}^{-1}(0)^{\mathrm{ss}}$ and hence $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is covered by $\mathcal{G}$-invariant open affine subvarieties. Thus, Luna's Slice Theorem gives that $\mathcal{C}(\mathcal{D})$ is smooth and $\pi\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to \mathcal{C}(\mathcal{D})$ is a principal $\mathcal{G}$-bundle in the étale topology. Since $\mathcal{G}$ is a special group, see e.g. [Mil13, Theorem 11.4], we deduce that π is actually a principal $\mathcal{G}$-bundle in the Zariski topology. Thus, we proved (i) and (ii). Finally, (iii) is an immediate consequence of Theorem 2.17. □

Proposition 2.43.(ii) and Proposition 2.40 yield the following dimension formula for $\mathcal{C}(\mathcal{D})$:

Corollary 2.44. *Each irreducible component of $\mathcal{C}(\mathcal{D})$ is of dimension*

$$\dim(\mathcal{C}(\mathcal{D})) = \Big(\sum_{U \in \mathrm{b}(\mathcal{D})} (d_{U^-}^2 + d_{U^+}^2 + d_{U^-} + d_{U^+}) \Big) + \Big(\sum_{V \in \mathrm{r}(\mathcal{D})} 2d_{V^-} d_{V^+} \Big) - \Big(\sum_{X \in \mathrm{h}(\mathcal{D})} 2d_X^2 \Big). \quad (2.43)$$

From Proposition 2.43.(ii), it follows that bow varieties admit a family of tautological bundles:

Corollary 2.45. *Let $X \in \mathrm{h}(\mathcal{D})$. Then, the diagonal action of $\mathcal{G}$ on $\tilde{m}^{-1}(0)^{\mathrm{s}} \times W_X$ is free and the geometric quotient*

$$\xi_X \coloneqq (\tilde{m}^{-1}(0)^{\mathrm{s}} \times W_X)/\mathcal{G} \quad (2.44)$$

is a vector bundle over $\mathcal{C}(\mathcal{D})$ (in the Zariski toplology).

Definition 2.46. The vector bundle ξ_X from (2.44) is called the *tautological bundle corresponding to X*. We call $\xi_{\mathcal{D}} \coloneqq \bigoplus_{X \in \mathrm{h}(\mathcal{D})} \xi_X$ the *full tautological bundle of $\mathcal{C}(\mathcal{D})$*.

Remark. In [NT17], Nakajima and Takayama give a more general definition of bow varieties depending on more stability parameters $\nu_\sigma^{\mathbb{C}}$ and $\nu_\sigma^{\mathbb{R}}$. For simplicity, we only consider bow varieties corresponding to the specializations $\nu_\sigma^{\mathbb{C}} = 0$ and $\nu_\sigma^{\mathbb{R}} = -1$. One convenient feature of this family of bow varieties is that they are smooth, which is not true in general.

Explicit example $T^*\mathbb{P}^1$

We show now that the bow variety $\mathcal{C}(\mathcal{D})$, where $\mathcal{D}$ is as in Example 2.35 is isomorphic to a very familiar quasi-projective variety: The cotangent bundle of the projective line $T^*\mathbb{P}^1$.

Recall from e.g. [CG97, Lemma 1.4.9] that $T^*\mathbb{P}^1$ is isomorphic to the total space of the vector bundle $\mathrm{Hom}(\mathcal{Q}, \mathcal{S})$, where $\mathcal{S}$ denotes the tautological bundle on $\mathbb{P}^1$ and $\mathcal{Q} = (\mathbb{P}^1 \times \mathbb{C})/\mathcal{S}$ the universal quotient bundle of $\mathbb{P}^1$. Thus, the points of $T^*\mathbb{P}^1$ are given by

$$T^*\mathbb{P}^1 = \{(V, f) \mid V \in \mathbb{P}^1,\ f \in \mathrm{End}(\mathbb{C}^2),\ \mathrm{im}(f) \subset V,\ V \subset \ker(f)\}.$$

Recall the data to specify elements of $\mathcal{C}(\mathcal{D})$ from (2.34) and that for a tuple

$$y = ((A_i, B_i^+, B_i^-, a_i, b_i)_{i=1,2}, (C_i, D_i)_{i=1,2}) \in \widetilde{\mathcal{M}}(\mathcal{D}).$$

The conditions (S1) and (S2) are equivalent to $A_1,\ A_2 \neq 0$. By (2.12) and (2.33), y is contained in $\tilde{m}^{-1}(0)$ if and only the following equations are satisfied:

$$B_1^- A_1 - A_1 B_1^+ + a_1 b_1 = 0, \quad B_2^- A_2 - A_2 B_2^+ + a_2 b_2 = 0, \quad (2.45)$$

$$B_1^- = 0, \quad B_1^+ = B_2^-, \quad B_2^+ = 0. \quad (2.46)$$

These equations imply $a_1b_1 + A_1a_2b_2A_2^{-1} = 0$. By Proposition 2.37, the χ-stability condition is equivalent to $(a_1, a_2) \neq (0, 0)$. Hence, $\ker(a_1\ a_2)$ is of dimension 1. From this, we deduce that we have a surjective morphism of varieties $H\colon \mathcal{C}(\mathcal{D}) \to T^*\mathbb{P}^1$ given as

$$[(A_i, B_i^-, B_i^+, a_i, b_i)_{i=1,2}, (C_i, D_i)_{i=1,2}] \mapsto \left(\ker\left(a_1A_1^{-1}a_2\right), \begin{pmatrix} b_1 \\ b_2A_2^{-1} \end{pmatrix} \begin{pmatrix} a_1A_1^{-1} & a_2 \end{pmatrix}\right).$$

To conclude that H is an isomorphism, it suffices by Proposition 2.25 to show that H is injective. Let y, $y' \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ with $H([y]) = H([y'])$. Write

$$y = ((A_i, B_i^-, B_i^+, a_i, b_i)_{i=1,2}, (C_j, D_j)_{i=1,2})$$

and

$$y' = ((A_i', (B_i')^-, (B_i')^+, a_i', b_i')_{i=1,2}, (C_j', D_j')_{j=1,2}).$$

We may assume $A_1 = A_2 = A_1' = A_2' = 1$. Suppose $a_2 \neq 0$. In this case, we can additionally assume $a_2 = a_2'$. Since $\ker(a_1\ a_2) = \ker(a_1'\ a_2')$, we conclude that also $a_1 = a_1'$. As

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} b_1' \\ b_2' \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix}$$

and $(a_1\ a_2)$ is surjective, we deduce $b_1 = b_1'$ and $b_2 = b_2'$. Then, (2.45) yields $[y] = [y']$. The case $a_1 \neq 0$ follows along similar lines. Thus, we proved that H is injective and hence an isomorphism of varieties.

Remark. This example shows a very special instance of the general fact that each Nakajima quiver variety of type A is isomorphic to a bow variety, see [NT17, Theorem 2.15]. In particular, cotangent bundles of of partial flag varieties can be realized as bow varieties. We will explicitly discuss this realization in Section 2.5.

Torus actions

As we discuss in this subsection, there are two kinds of torus actions on bow varieties. The first one follows easily from the construction of bow varieties. Thus, we refer to this action as the *obvious action*. The second one was introduced in [NT17, Section 6.9.3] and scales the symplectic form. We therefore refer to this action as the *scaling action*. In our exposition, we follow the conventions from [RS20, Section 3.1], for the precise connection to the definition of Nakajima and Takayama see [RS20, Section 3.4].

Recall from Notation 2.32 that $N = |\mathrm{b}(\mathcal{D})|$ is the number of blue lines in $\mathcal{D}$. The following two tori will be used:

- $\mathbb{A} = (\mathbb{C}^*)^N$ and its elements are denoted by $(t_1, \dots, t_N)$ or $(t_U)_{U\in b(\mathcal{D})}$ or just by $(t_U)_U$.
- $\mathbb{C}_h^* = \mathbb{C}^*$ and its elements are usually denoted by h.

We set

$$\mathbb{T} := \mathbb{A} \times \mathbb{C}_h^*. \tag{2.47}$$

The obvious action

Recall the definition of $\mathbb{V}_{\mathcal{D}}$ from (2.31). The torus $\mathbb{A}$ acts algebraically on $\mathbb{V}_{\mathcal{D}}$ via

$$\begin{aligned}(t_U)_U.((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \\ = ((A_U, B_U^-, B_U^+, a_U t_U^{-1}, t_U b_U)_U, (C_V, D_V)_V).\end{aligned} \tag{2.48}$$

By definition, the $\mathbb{A}$-action and $\mathcal{G}$-action on $\mathbb{V}_{\mathcal{D}}$ commute. Let

$$\mu' \colon \mathbb{V}_{\mathcal{D}} \longrightarrow \bigoplus_{U \in \mathrm{b}(\mathcal{D})} \mathrm{Hom}(W_{U^+}, W_{U^-})$$

be as in (2.36). A direct computation gives that μ' is $\mathbb{A}$-invariant and that the $\mathbb{A}$-action is also compatible with (S1) and (S2). Hence, the $\mathbb{A}$-action restricts to the affine brane variety $\widetilde{\mathcal{M}}(\mathcal{D})$. By construction, the moment map $\tilde{m}$ from (2.33) is $\mathbb{A}$-invariant. Since the $\mathbb{A}$-action is also compatible with the χ-stability criterion from Proposition 2.37, we get an induced $\mathbb{A}$-action on $\mathcal{C}(\mathcal{D})$ which is explicitly given by

$$\begin{aligned}(t_U)_U.[(A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V] \\ = [(A_U, B_U^-, B_U^+, a_U t_U^{-1}, t_U b_U)_U, (C_V, D_V)_V].\end{aligned} \tag{2.49}$$

Via the explicit description of the Poisson bracket on $\widetilde{\mathcal{M}}(\mathcal{D})$ from (2.22) and (2.28), we conclude that the Poisson bracket on $\widetilde{\mathcal{M}}(\mathcal{D})$ (and equivalently the symplectic form ω on $\widetilde{\mathcal{M}}(\mathcal{D})$) is $\mathbb{A}$-invariant. Hence, by (2.42), we conclude that also the Poisson bracket on $\mathcal{C}(\mathcal{D})$ (and equivalently the symplectic form ω' on $\mathcal{C}(\mathcal{D})$) is $\mathbb{A}$-invariant.

The scaling action

We have an algebraic $\mathbb{C}_h^*$-action on $\mathbb{V}_{\mathcal{D}}$ via

$$\begin{aligned}h.((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \\ = ((A_U, hB_U^-, hB_U^+, a_U, hb_U)_U, (hC_V, D_V)_V).\end{aligned} \tag{2.50}$$

Again, the $\mathbb{C}_h^*$-action and the $\mathcal{G}$-action commute. One can easily check that μ' is $\mathbb{C}_h^*$-equivariant, where $\mathbb{C}_h^*$ acts on $\bigoplus_X \mathrm{Hom}(W_{U^+}, W_{U^-})$ via $h.(f_X)_X = (hf_X)_X$. As the $\mathbb{C}_h^*$-action is also compatible with (S1) and (S2), we get an induced $\mathbb{C}_h^*$-action on $\widetilde{\mathcal{M}}(\mathcal{D})$. The moment map $\tilde{m}$ is also $\mathbb{C}_h^*$-equivariant where again $\mathbb{C}_h^*$ acts on $\bigoplus_X \mathrm{End}(W_X)$ via $h.(f_X)_X = (hf_X)_X$. Since the $\mathbb{C}_h^*$-action is further compatible with the χ-stability condition from Proposition 2.37, we get an induced $\mathbb{C}_h^*$-action on the bow variety $\mathcal{C}(\mathcal{D})$:

$$\begin{aligned}h.[(A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \\ = [(A_U, hB_U^-, hB_U^+, a_U, hb_U)_U, (hC_V, D_V)_V].\end{aligned} \tag{2.51}$$

Again employing (2.22) and (2.28) yields $h^*\omega = h\omega$, for all $h \in \mathbb{C}_h^*$. By (2.42), this implies that the $\mathbb{C}_h^*$ also scales the symplectic form on $\mathcal{C}(\mathcal{D})$, i.e. $h^*\omega' = h\omega'$.

Finally, note that the $\mathbb{A}$-action and the $\mathbb{C}_h^*$-action on $\mathbb{V}_{\mathcal{D}}$ commute. Thus, we have the following result:

Proposition 2.47. *The $\mathbb{A}$- and $\mathbb{C}_h^*$-action on $\mathbb{V}_\mathcal{D}$ induce a $\mathbb{T} = (\mathbb{A} \times \mathbb{C}_h^*)$-action on $\widetilde{\mathcal{M}}(\mathcal{D})$, $\tilde{m}^{-1}(0)^{\mathrm{s}}$ and $\mathcal{C}(\mathcal{D})$.*

In addition, each tautological bundle ξ_X of $\mathcal{C}(\mathcal{D})$ carries the structure of a $\mathbb{T}$-equivariant vector bundle via

$$(t,h).[y,v] = [(t,h).y, v], \quad (t,h) \in \mathbb{T},\ v \in W_X,\ y \in \tilde{m}^{-1}(0)^{\mathrm{s}}. \tag{2.52}$$

Tangent bundle via tautological bundles

Next, we employ the characterization of tangent spaces of the χ-stable locus of $\tilde{m}^{-1}(0)$ from Lemma 2.41 to describe tangent bundles of bow varieties via tautological bundles. In particular, we deduce a formula of the $\mathbb{T}$-equivariant K-theory classes of tangent bundles in terms of tautological bundles. This formula was given in [RS20, Section 3.2] and [Sho21, Theorem 3.1.15] as a consequence of [NT17, Proposition 2.20]. In this subsection, we give a self-contained reproof of this formula. We lay our focus on the involved morphisms of vector bundles.

For $i \in \mathbb{Z}$, we denote by $\mathbb{C}_{h^i}$ the $\mathbb{T}$-representation corresponding to the character $\mathbb{T} \to \mathbb{C}^*$, $(t_1, \dots, t_N, h) \mapsto h^i$. If W is a $\mathbb{T}$-representation, we denote the tensor product $W \otimes \mathbb{C}_{h^i}$ also just by $h^i W$.

Recall from Proposition 2.43.(ii) that the projection $\pi\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to \mathcal{C}(\mathcal{D})$ is a principal $\mathcal{G}$-bundle. Thus, we have a short exact sequence of $\mathcal{G}$-equivariant vector bundles over $\tilde{m}^{-1}(0)^{\mathrm{s}}$:

$$0 \to \tilde{m}^{-1}(0)^{\mathrm{s}} \times \Big(\bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X) \Big) \xrightarrow{\alpha} T\tilde{m}^{-1}(0)^{\mathrm{s}} \xrightarrow{d\pi} \pi^* T\mathcal{C}(\mathcal{D}) \to 0. \tag{2.53}$$

Here,

$$\alpha\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \times \Big(\bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X) \Big) \longrightarrow T\tilde{m}^{-1}(0)^{\mathrm{s}}, \quad g = (g_X)_X \mapsto \vec{g},$$

where $\vec{g}$ is the vector field assigned to g. If we view $T\tilde{m}^{-1}(0)^{\mathrm{s}}$ as locally closed subvariety of $T\mathbb{V}_\mathcal{D} \cong \mathbb{V}_\mathcal{D} \times \mathbb{V}_\mathcal{D}$ then for

$$y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \mathbb{V}_\mathcal{D},$$

the induced morphism on the fibers $\alpha_y\colon \bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X) \to \mathbb{V}_\mathcal{D}$ is given as

$$\begin{aligned}\alpha_y(g) =& ((g_{U^-} A_U - A_U g_{U^+}, g_{U^-} B_U^- - B_U^- g_{U^-}, g_{U^+} B_U^+ - B_U^+ g_{U^+}, g_{U^-} a_U, -b_U g_{U^+})_U, \\ & (g_{V^-} C_V - C_V g_{V^+}, g_{V^+} D_V - D_V g_{V^-})_V).\end{aligned}$$

As the $\mathcal{G}$-action commutes with the $\mathbb{T}$-action on $\tilde{m}^{-1}(0)^{\mathrm{s}}$, we conclude that (2.53) is a short exact sequence of $\mathbb{T}$-equivariant vector bundles. Thus, (2.53) induces a short exact sequence of $\mathbb{T}$-equivariant vector bundles over $\mathcal{C}(\mathcal{D})$:

$$0 \to \Big(\bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(\xi_X) \Big) \to (T\tilde{m}^{-1}(0)^{\mathrm{s}})/\mathcal{G} \to T\mathcal{C}(\mathcal{D}) \to 0. \tag{2.54}$$

Thus, we have the following identity in $K_\mathbb{T}(\mathcal{C}(\mathcal{D}))$:

$$[T\mathcal{C}(\mathcal{D})] = [(T\tilde{m}^{-1}(0)^{\mathrm{s}})/\mathcal{G}] - \Big(\sum_{X \in \mathrm{h}(\mathcal{D})} [\mathrm{End}(\xi_X)] \Big). \tag{2.55}$$

Next, we characterize the $\mathbb{T}$-equivariant K-theory class $[(T\tilde{m}^{-1}(0)^{\mathrm{s}})/\mathcal{G}]$. Recall the morphism of varieties $\beta\colon \mathbb{V}_{\mathcal{D}} \to \mathbb{N}_{\mathcal{D}}$ from (2.38). By Lemma 2.41, we have an isomorphism of $\mathbb{T}$-equivariant vector bundles $T\tilde{m}^{-1}(0)^{\mathrm{s}} \cong \iota^*\ker(d\beta)$, where $\iota\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \hookrightarrow \mathbb{V}_{\mathcal{D}}$ is the inclusion. Lemma 2.41 further implies that we have a short exact sequence of $\mathcal{G}$- and $\mathbb{T}$-equivariant vector bundles over $\tilde{m}^{-1}(0)^{\mathrm{s}}$:

$$0 \to \iota^*\ker(d\beta) \to \tilde{m}^{-1}(0)^{\mathrm{s}} \times \mathbb{V}_{\mathcal{D}} \xrightarrow{d\beta} \tilde{m}^{-1}(0)^{\mathrm{s}} \times h\mathbb{N}_{\mathcal{D}} \to 0.$$

Thus, we get an induced short exact sequence of $\mathbb{T}$-equivariant vector bundles over $\mathcal{C}(\mathcal{D})$:

$$0 \to \iota^*\ker(d\beta)/\mathcal{G} \to (\tilde{m}^{-1}(0)^{\mathrm{s}} \times \mathbb{V}_{\mathcal{D}})/\mathcal{G} \to (\tilde{m}^{-1}(0)^{\mathrm{s}} \times h\mathbb{N}_{\mathcal{D}})/\mathcal{G} \to 0.$$

The above quotient bundles can be $\mathbb{T}$-equivariantly expressed via tautological bundles over $\mathcal{C}(\mathcal{D})$ as follows:

$$\begin{aligned}(\tilde{m}^{-1}(0)^{\mathrm{s}} \times \mathbb{V}_{\mathcal{D}})/\mathcal{G} =& \Big(\bigoplus_{U \in \mathrm{b}(\mathcal{D})} \mathrm{Hom}(\xi_{U^+}, \xi_{U^-}) \oplus h\,\mathrm{End}(\xi_{U^-}) \oplus h\,\mathrm{End}(\xi_{U^+}) \\ &\oplus \mathrm{Hom}(\mathbb{C}_U, \xi_{U^-}) \oplus h\,\mathrm{Hom}(\xi_{U^+}, \mathbb{C}_U)\Big) \\ &\oplus \Big(\bigoplus_{V \in \mathrm{r}(\mathcal{D})} h\,\mathrm{Hom}(\xi_{V^+}, \xi_{V^-}) \oplus \mathrm{Hom}(\xi_{V^-}, \xi_{V^+})\Big), \\ (\tilde{m}^{-1}(0)^{\mathrm{s}} \times h\mathbb{N}_{\mathcal{D}})/\mathcal{G} =& \bigoplus_{X \in \mathrm{h}(\mathcal{D})} h\,\mathrm{End}(\xi_X) \;\oplus \bigoplus_{U \in \mathrm{b}(\mathcal{D})} h\,\mathrm{Hom}(\xi_{U^+}, \xi_{U^-}).\end{aligned}$$

Thus, we have the following identity in $K_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$:

$$[(T\tilde{m}^{-1}(0)^{\mathrm{s}})/\mathcal{G}] = \sum_{U \in \mathrm{b}(\mathcal{D})} T_U \;+ \sum_{V \in \mathrm{r}(\mathcal{D})} T_V \;- \sum_{X \in \mathrm{h}(\mathcal{D})} T_X, \tag{2.56}$$

where

$$\begin{aligned} T_U &= (1-h)[\mathrm{Hom}(\xi_{U^+}, \xi_{U^-})] + h[\mathrm{End}(\xi_{U^-})] + h[\mathrm{End}(\xi_{U^+})] \\ &\qquad + [\mathrm{Hom}(\mathbb{C}_U, \xi_{U^-})] + h[\mathrm{Hom}(\xi_{U^+}, \mathbb{C}_U)], \\ T_V &= h[\mathrm{Hom}(\xi_{V^+}, \xi_{V^-})] + [\mathrm{Hom}(\xi_{V^-}, \xi_{V^+})], \\ T_X &= [\mathrm{End}(\xi_X)]. \end{aligned} \tag{2.57}$$

Inserting (2.56) into (2.55) then immediately gives the following formula for $[T\mathcal{C}(\mathcal{D})]$ in terms of tautological bundles:

Corollary 2.48. *We have the following identity in* $K_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$*:*

$$[T\mathcal{C}(\mathcal{D})] = \sum_{U \in \mathrm{b}(\mathcal{D})} T_U \;+ \sum_{V \in \mathrm{r}(\mathcal{D})} T_V \;- \sum_{X \in \mathrm{h}(\mathcal{D})} (1+h) T_X,$$

where T_U*,* T_V *and* T_X *are defined as in* (2.57)*.*

Emptyness conditions

The injectivity and surjectivity constrains for points on $\tilde{m}^{-1}(0)$ from Proposition 2.34 yield that $\mathcal{C}(\mathcal{D})$ is empty unless $\mathcal{D}$ satisfies the following combinatorial properties:

Corollary 2.49. *If $\mathcal{C}(\mathcal{D}) \neq \emptyset$ then we have $d_j \leq d_{j-1} + d_{j+1} + 1$, for all local configurations $d_{j-1}/d_j\backslash d_{j+1}$ and $d_{j-1}\backslash d_j/d_{j+1}$ in $\mathcal{D}$.*

Proof. If the condition is violated, Proposition 2.34 gives $\tilde{m}^{-1}(0) = \emptyset$. Thus, $\mathcal{C}(\mathcal{D}) = \emptyset$. □

Definition 2.50. A brane diagram $\mathcal{D}$ is called *admissible* if $d_j \leq d_{j-1} + d_{j+1} + 1$, for all local configurations $d_{j-1}/d_j\backslash d_{j+1}$ and $d_{j-1}\backslash d_j/d_{j+1}$ in $\mathcal{D}$.

Assumption. From now on we assume that each brane diagram $\mathcal{D}$ is admissible.

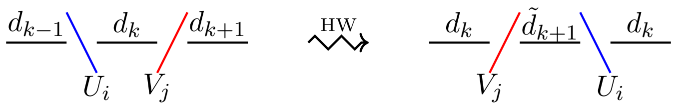
2.4 Hanany–Witten transition

It was shown in [NT17, Proposition 7.1] that the family of bow varieties comes with an interesting collection of isomorphisms between bow varieties, called Hanany–Witten isomorphisms. These isomorphisms correspond to certain moves, called Hanany–Witten transitions, on brane diagrams and are well-behaved with respect to the torus action. In this section, we recall some important properties of Hanany–Witten isomorphisms. For their explicit construction see [NT17, Section 7] and also the exposition in [RS20, Section 3.3].

Hanany–Witten isomorphisms

We begin with describing the underlying combinatorics of Hanany–Witten isomorphisms.

Definition 2.51. Let $\mathcal{D}$ and $\tilde{\mathcal{D}}$ be brane diagrams. We say that $\tilde{\mathcal{D}}$ *is obtained from $\mathcal{D}$ via a Hanany–Witten transition* if $\tilde{\mathcal{D}}$ differs from $\mathcal{D}$ by performing a local move of the form



where $d_{k-1}+d_{k+1}+1 = d_k+\tilde{d}_k$. If $\tilde{\mathcal{D}}$ is obtained from $\mathcal{D}$ via a finite number of Hanany–Witten transitions, we write $\mathcal{D} \overset{\text{HW}}{\rightsquigarrow} \tilde{\mathcal{D}}$ and call $\mathcal{D}$ and $\tilde{\mathcal{D}}$ *Hanany–Witten equivalent.*

The following proposition (see [NT17, Proposition 7.1] and [RS20, Theorem 3.9]) characterizes the isomorphism corresponding to a Hanany–Witten transition as well as the interplay of tautological bundles under this isomorphism:

Proposition 2.52. *Suppose $\tilde{\mathcal{D}}$ is obtained from $\tilde{\mathcal{D}}$ via Hanany–Witten transition, where the blue line U_i is exchanged with the red line V_j. Let X_k be the black line in $\mathcal{D}$ with $X_k^- = U_i$ and $X_k^+ = V_j$. Then, there exists a ρ_i-equivariant isomorphism of varieties*

$$\Phi\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}}), \tag{2.58}$$

where ρ_i is the algebraic group automorphism

$$\rho_i\colon \mathbb{T} \xrightarrow{\sim} \mathbb{T}, \quad (t_1, \dots, t_N, h) \mapsto (t_1, \dots, t_{i-1}, ht_i, t_{i+1}, \dots, t_N, h).$$

Furthermore, the following holds:

(i) *We have $\mathbb{T}$-equivariant isomorphisms of vector bundles $\xi_{\mathcal{D},X_l} \cong \Phi^*\xi_{\tilde{\mathcal{D}},X_l}$, for $l \neq k$.*

(ii) *There is a short exact sequence of $\mathbb{T}$-equivariant vector bundles*

$$0 \to \xi_{\mathcal{D},X_k} \to \xi_{\mathcal{D},X_{k-1}} \oplus \xi_{\mathcal{D},X_{k+1}} \oplus h\mathbb{C}_{U_i} \to \Phi^*\xi_{\tilde{\mathcal{D}},X_l} \to 0. \tag{2.59}$$

Here, $\Phi^*\xi_{\tilde{\mathcal{D}},X_l}$ is the $\mathbb{T}$-equivariant pullback of $\xi_{\tilde{\mathcal{D}},X_l}$ via Φ and $\mathbb{C}_{U_i}$ denotes the trivial bundle on $\mathcal{C}(\mathcal{D})$ corresponding to the character $(t_1,\dots,t_N,h) \mapsto t_i$.

Example 2.53. The brane diagram $\tilde{\mathcal{D}} = 0/1\backslash 0$ is obtained from $\mathcal{D} = 0\backslash 0/0$ by the Hanany–Witten transition which switches the blue with the red line in $\mathcal{D}$. By construction, $\mathcal{C}(\mathcal{D})$ is isomorphic to a single point $\mathcal{C}(\mathcal{D}) \cong \{\mathrm{pt}\}$. Thus, by Proposition 2.52, $\mathcal{C}(\tilde{\mathcal{D}})$ is also isomorphic to $\{\mathrm{pt}\}$.

Hanany–Witten transition allows to move all red lines in a brane diagram to the left of all blue lines not changing the isomorphism type of the respective bow variety. As we will discuss in the following subsection, the realization of bow varieties corresponding to this particular type of brane diagrams admits some useful properties.

Separated brane diagrams

Definition 2.54. For a given brane diagram $\mathcal{D}$ the *separation degree of* $\mathcal{D}$ is defined as

$$\mathrm{sdeg}(\mathcal{D}) := |\{(U,V) \in \mathrm{b}(\mathcal{D}) \times \mathrm{r}(\mathcal{D}) | U \triangleleft V\}|.$$

We call $\mathcal{D}$ *separated* if $\mathrm{sdeg}(\mathcal{D}) = 0$, i.e. all red lines are in $\mathcal{D}$ to the left of all blue lines.

Via Hanany–Witten transition, we deduce that any brane diagram is Hanany–Witten equivalent to a separated brane diagram:

Proposition 2.55 (Reduction argument)**.** *There exists a separated brane diagram $\tilde{\mathcal{D}}$ such that $\mathcal{D} \overset{\mathrm{HW}}{\rightsquigarrow} \tilde{\mathcal{D}}$.*

Proof. Suppose $\mathrm{sdeg}(\mathcal{D}) > 0$. Then, there exist $U \in \mathrm{b}(\mathcal{D}), V \in \mathrm{r}(\mathcal{D})$ such that U is directly to the left of V. Since $\mathcal{D}$ is admissible, we can apply a Hanany–Witten transition reducing the separatedness degree by 1. Now just repeat this argument. □

For a separated brane diagram the operators defining points of $\mathcal{C}(\mathcal{D})$ satisfy the following nilpotency conditions:

Proposition 2.56 (Nilpotency)**.** *Suppose $\mathcal{D}$ is separated and let*

$$((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \tilde{m}^{-1}(0).$$

Then, the following holds:

(i) *We have $(C_{V_j}D_{V_j})^{M-j} = 0$ and $(D_{V_j}C_{V_j})^{M-j+1} = 0$, for $j = 1,\dots,M-1$.*

(ii) *We have $(B_{U_1}^-)^M = 0$.*

Proof. By the moment map equations (2.33), we have

$$\begin{aligned} (C_{V_j}D_{V_j})^{M-j} &= C_{V_j}(C_{V_{j+1}}D_{V_{j+1}})^{M-j-1}D_{V_j}, \\ (D_{V_j}C_{V_j})^{M-j+1} &= D_{V_j}(D_{V_{j+1}}C_{V_{j+1}})^{M-j}D_{V_j}, \end{aligned} \quad \text{for all } j = 1, \dots, M-1.$$

Thus, (i) follows from $C_{V_M} = 0$, $D_{V_M} = 0$ via induction on j. The assertion (ii) follows from (i) since $B^-_{U_1} = -C_{V_1}D_{V_1}$. □

We also have the following surjectivity property of the C-operators:

Proposition 2.57 (Surjectivity of C-operators)**.** *Suppose $\mathcal{D}$ is separated and let*

$$((A_U, B^-_U, B^+_U, a_U, b_U)_U, (C_V, D_V)_V) \in \tilde{m}^{-1}(0)^{\mathrm{s}}.$$

Then, all C_V are surjective.

Proof. Define the graded subspace $T = \bigoplus_{X\in \mathrm{h}(\mathcal{D})} T_X \subset W_{\mathcal{D}}$ as $T_{U^\pm} = W_{U^\pm}$, for all $U \in \mathrm{b}(\mathcal{D})$ and $T_{V^-} = \mathrm{im}(C_V)$, for all $V \in \mathrm{r}(\mathcal{D})$. By the moment map equation (2.33), T satisfies the conditions of Proposition 2.37 and therefore $T = W_{\mathcal{D}}$. Hence, all operators C_V are surjective. □

Margin vectors

We recall some invariants of brane diagrams from [RS20, Section 2] and [Sho21, Section 2] that are stable under Hanany–Witten transition. Recall the conventions from Notation 2.32.

Definition 2.58. Given a brane diagram $\mathcal{D}$, we assign the following invariants to $\mathcal{D}$:

$$r_i(\mathcal{D}) \coloneqq d_{V_i^+} - d_{V_i^-} + |\{U \in \mathrm{b}(\mathcal{D}) \mid U \triangleleft V_i\}|, \quad c_j(\mathcal{D}) \coloneqq d_{U_j^-} - d_{U_j^+} + |\{V \in \mathrm{r}(\mathcal{D}) \mid V \triangleright U_j\}|,$$

where $i \in \{1, \dots, M\}$, $j \in \{1, \dots, N\}$. In addition, we set

$$R_l(\mathcal{D}) \coloneqq \sum_{i=1}^{l} r_i(\mathcal{D}), \quad C_l(\mathcal{D}) \coloneqq \sum_{j=1}^{l} c_j(\mathcal{D}), \quad \bar{R}_l(\mathcal{D}) \coloneqq \sum_{i=l}^{M} r_i(\mathcal{D}), \quad \bar{C}_l(\mathcal{D}) \coloneqq \sum_{j=l}^{N} c_j(\mathcal{D}).$$

As $\mathcal{D}$ is usually a fixed brane diagram, we just denote $r_i(\mathcal{D})$, $c_j(\mathcal{D})$, $R_i(\mathcal{D})$, $\bar{R}_i(\mathcal{D})$, $C_j(\mathcal{D})$ and $\bar{C}_j(\mathcal{D})$ by r_i, c_j, R_i, $\bar{R}_i$, C_j and $\bar{C}_j$. The vectors $\mathbf{r} = \mathbf{r}(\mathcal{D}) = (r_1, \dots, r_M)$ and $\mathbf{c} = \mathbf{c}(\mathcal{D}) = (c_1, \dots, c_N)$ are called *margin vectors of $\mathcal{D}$*.

The next proposition gives that margin vectors are invariant under Hanany–Witten transition:

Proposition 2.59. *If $\mathcal{D} \overset{\mathrm{HW}}{\rightsquigarrow} \tilde{\mathcal{D}}$ then $\mathbf{r}(\mathcal{D}) = \mathbf{r}(\tilde{\mathcal{D}})$ and $\mathbf{c}(\mathcal{D}) = \mathbf{c}(\tilde{\mathcal{D}})$.*

Proof. It suffices to consider the case where $\tilde{\mathcal{D}}$ is obtained from $\mathcal{D}$ by a single Hanany–Witten transition switching a blue line U_j with a red line V_i. We denote the red resp. blue lines of $\tilde{\mathcal{D}}$ by $\tilde{V}$ resp. $\tilde{U}$ to distinguish them from the colored lines of $\mathcal{D}$. By definition, we have

$$|\{\tilde{U} \in \mathrm{b}(\tilde{\mathcal{D}}) \mid U \triangleleft \tilde{V}_l\}| = \begin{cases} |\{U \in \mathrm{b}(\mathcal{D}) \mid U \triangleleft V_l\}| - 1 & \text{if } l = i, \\ |\{U \in \mathrm{b}(\mathcal{D}) \mid U \triangleleft V_l\}| & \text{if } l \neq i. \end{cases}$$

Since $d^{\pm}_{\tilde{V}_l} = d^{\pm}_{V_l}$, for $l \neq i$, we conclude

$$r_l(\tilde{\mathcal{D}}) = d_{V_l^+} - d_{V_l^-} + |\{U \in \mathrm{b}(\tilde{\mathcal{D}}) \mid U \triangleleft V_l\}| = r_l(\mathcal{D}).$$

By the definition of Hanany–Witten transition, we have

$$d_{\tilde{V}_i^+} - d_{\tilde{V}_i^-} = d_{V_l^+} - d_{V_l^-} + 1.$$

This implies

$$r_i(\tilde{\mathcal{D}}) = d_{V_i^+} - d_{V_i^-} + |\{U \in \mathrm{b}(\tilde{\mathcal{D}}) \mid U \triangleleft V_i\}| = r_i(\mathcal{D}).$$

Thus, we proved $\mathbf{r}(\mathcal{D}) = \mathbf{r}(\tilde{\mathcal{D}})$. The proof for $\mathbf{c}(\mathcal{D}) = \mathbf{c}(\tilde{\mathcal{D}})$ is analogous. □

If $\mathcal{D}$ is separated, we have $r_i = d_{V_i^+} - d_{V_i^-}$ and $c_j = d_{U_j^-} - d_{U_j^+}$, for all $V_i \in \mathrm{r}(\mathcal{D})$, $U_j \in \mathrm{b}(\mathcal{D})$. Thus, we can easily read off the labels of the black lines can be easily read off from the margin vectors:

$$d_{V_i^+} = \bar{R}_i = \sum_{l=i}^{M} r_l, \quad d_{U_j^-} = \bar{C}_j = \sum_{l=j}^{N} c_l, \quad i = 1, \dots, M,\ j = 1, \dots, N. \tag{2.60}$$

In particular, we have $\bar{R}_1(\mathcal{D}) = \bar{C}_1(\mathcal{D})$.

From (2.60), we deduce the following improvement of Proposition 2.55:

Corollary 2.60. *A brane diagram $\mathcal{D}$ is Hanany–Witten equivalent to the separated brane diagram*

$$\tilde{\mathcal{D}} = 0 \,/\, \bar{R}_M \,/\, \bar{R}_{M-1} \,/ \cdots /\, \bar{R}_2 \,/\, \bar{R}_1 = \bar{C}_1 \,\backslash\, \bar{C}_2 \,\backslash \cdots \backslash\, \bar{C}_N \,\backslash\, 0$$

Here, $\bar{R}_i = \bar{R}_i(\mathcal{D})$ and $\bar{C}_j = \bar{C}_j(\mathcal{D})$ for all i, j. In addition, $\tilde{\mathcal{D}}$ is the unique separated brane diagram such that $\mathcal{D} \overset{\mathrm{HW}}{\rightsquigarrow} \tilde{\mathcal{D}}$.

Proof. By Proposition 2.55, there exists a separated brane diagram $\mathcal{D}'$ such that $\mathcal{D} \overset{\mathrm{HW}}{\rightsquigarrow} \mathcal{D}'$. By (2.60) and the invariance of margin vectors under Hanany–Witten transformation, we conclude $\mathcal{D}' = \tilde{\mathcal{D}}$. □

Example 2.61. As in Example 2.35, let $\mathcal{D} = 0/1\backslash 1\backslash 1/0$. As one red line is to the right of the both blue lines, we have $\mathbf{c}(\mathcal{D}) = (1, 1)$. Likewise, since there are two blue lines to the left of V_1, we have $r_1(\mathcal{D}) = d_{V_1^+} - d_{V_1^-} + 2 = 0 - 1 + 2 = 1$. There is no blue line to the left of V_2. Thus, we deduce $r_2(\mathcal{D}) = d_{V_2^+} - d_{V_2^-} = 1 - 0 = 1$. Consequently, $\bar{R}_1 = \bar{C}_1 = 2$ and $\bar{R}_2 = \bar{C}_2 = 1$. Therefore, by Corollary 2.60, $\tilde{\mathcal{D}} = 0/1/2\backslash 1\backslash 0$ is the unique separated brane diagram such that $\mathcal{D} \overset{\mathrm{HW}}{\rightsquigarrow} \tilde{\mathcal{D}}$.

Note that margin vectors can have negative entries. For example for $\mathcal{D} = 0\backslash 1\backslash 0$ we have $\mathbf{c}(\mathcal{D}) = (-1, 1)$. However, Corollary 2.60 ensures that the entries of the numbers $\bar{R}_i$ and $\bar{C}_j$ are always non-negative.

2.5 Cotangent bundles of flag varieties as bow varieties

We now consider a key player in geometric representation theory: The cotangent bundles of partial flag varieties. These varieties form a rich family of symplectic varieties. For their relevance in geometric representation theory see in particular the exposition in [CG97] and the references therein.

Given natural numbers $0 < d_1 < d_2 < \ldots < d_m < n$, we denote by $F(d_1, \ldots, d_m; n)$ the partial flag variety parameterizing inclusions of vector subspaces

$$\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset \mathbb{C}^n$$

with $\dim(E_i) = d_i$, for $i = 1, \ldots, m$. We denote the cotangent bundle of $F(d_1, \ldots, d_m; n)$ by $T^*F(d_1, \ldots, d_m; n)$.

There are several ways in which the variety $T^*F(d_1, \ldots, d_m; n)$ can be constructed. In this section, we consider its realization as homogeneous space and as bow variety. As explained in [NT17, Theorem 2.15], the bow variety realization is equivalent to the well-known realization as Nakajima quiver variety from [Nak94, Theorem 7.3].

Realization of cotangent bundles via parabolic subgroups

We like to characterize the points of $T^*F(d_1, \ldots, d_m; n)$ in terms of linear operators. For this, we recall the following well-known realization of $T^*F(d_1, \ldots, d_m; n)$ via parabolic subgroups of general linear groups, see e.g. [CG97, Section 1.4].

Set $d_0 = 0$, $d_{m+1} = n$, $E_0 = 0$, $E_{m+1} = \mathbb{C}^n$ and $\delta_i = d_i - d_{i-1}$, for $i = 1, \ldots, m+1$. Let $G = \mathrm{GL}_n$ and $P \subset G$ be the parabolic subgroup of block matrices of the shape

$$\begin{pmatrix} P_{1,1} & P_{1,2} & \ldots & P_{1,m+1} \\ & P_{2,2} & \ldots & P_{2,m+1} \\ & & \ddots & \vdots \\ & & & P_{m+1,m+1} \end{pmatrix}, \quad P_{i,j} \in \mathrm{Mat}_{\delta_i,\delta_j}(\mathbb{C}).$$

Then, P acts on G via $p.g = gp^{-1}$. It is a well-known result, see e.g. [Spr98], that the geometric quotient G/P exists and there is an isomorphism of varieties $G/P \xrightarrow{\sim} F(d_1, \ldots, d_m; n)$ given as

$$[g] \mapsto \mathcal{F}_g \coloneqq (\{0\} \subset \langle g_1, \ldots, g_{d_1} \rangle \subset \cdots \subset \langle g_1, \ldots, g_{d_m} \rangle \subset \mathbb{C}^n). \tag{2.61}$$

Here, g_i denotes the i-th column vector of g.

Let $\mathfrak{g} = \mathrm{End}(\mathbb{C}^n)$ be the Lie algebra of G and $\mathfrak{p} \subset \mathfrak{g}$ be the Lie-subalgebra corresponding to P. We denote by $\mathfrak{p}^\perp$ the annihilator of $\mathfrak{p}$ with respect to the trace pairing on $\mathfrak{g}$. That is, $\mathfrak{p}^\perp$ is the Lie subalgebra of $\mathfrak{g}$ consisting of block matrices of the form

$$\begin{pmatrix} 0 & P_{1,2} & P_{1,3} & \ldots & P_{1,m+1} \\ & 0 & P_{2,3} & \ldots & P_{2,m+1} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & P_{m,m+1} \\ & & & & 0 \end{pmatrix}, \quad P_{i,j} \in \mathrm{Mat}_{\delta_i,\delta_j}(\mathbb{C}).$$

Note that the parabolic subgroup P acts algebraically on $\mathfrak{p}^\perp$ via conjugation, i.e. $p.g = pgp^{-1}$, where $p \in P$ and $g \in \mathfrak{p}^\perp$. Moreover, we equip $G \times \mathfrak{p}^\perp$ with the diagonal P-action $p.(g_1, g_2) = (p.g_1, p.g_2)$.

We then have the following realization of $T^*F(d_1, \dots, d_m; n)$ as geometric quotient, see [CG97, Lemma 1.4.9]:

Proposition 2.62. *The geometric quotient $(G \times \mathfrak{p}^\perp)/P$ exists and $(G \times \mathfrak{p}^\perp)/P$ is a vector bundle over $F(d_1, \dots, d_m; n)$ via the projection $(G \times \mathfrak{p}^\perp)/P \to G/P \cong F(d_1, \dots, d_m; n)$. In addition, we have an isomorphism of vector bundles over $(G \times \mathfrak{p}^\perp)/P \xrightarrow{\sim} T^*F(d_1, \dots, d_m; n)$.*

This result characterizes the points of $T^*F(d_1, \dots, d_m; n)$ in terms of linear operators:

Corollary 2.63. *Let $T_{d_1,\dots,d_m;n} \subset F(d_1, \dots, d_m; n) \times \mathrm{End}(\mathbb{C}^n)$ be the closed subbundle over $F(d_1, \dots, d_m; n)$ given by all pairs $(\mathcal{F}, f)$ such that $f(E_i) \subset E_{i-1}$, for all $i = 1, \dots, m$, where $\mathcal{F} = (0 \subset E_1 \subset \ldots \subset E_m \subset \mathbb{C}^n)$. Then, we have an isomorphism of vector bundles*

$$(G \times \mathfrak{p}^\perp)/P \xrightarrow{\sim} T_{d_1,\dots,d_m;n}, \quad [g,p] \mapsto (\mathcal{F}_g, gpg^{-1})$$

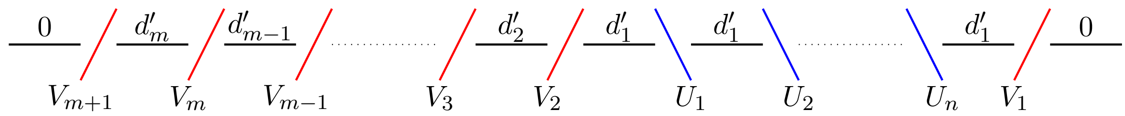
over $F(d_1, \dots, d_m; n)$. Here, $\mathcal{F}_g$ is defined as in (2.61)*.*

In the following, we will always implicitly identify $T^*F(d_1, \dots, d_m; n)$ with the variety $T_{d_1,\dots,d_m;n}$ via Corollary 2.63.

Realization of cotangent bundles via bow varieties

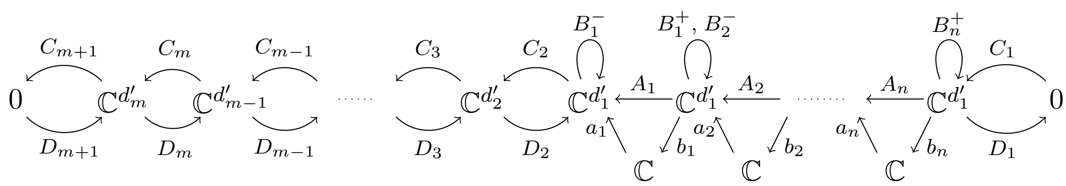
Next, we realize $T^*F(d_1, \dots, d_m; n)$ as bow variety. F

Definition 2.64. Let $\tilde{\mathcal{D}}(d_1, \dots, d_m; n)$ be the brane diagram:



where $d'_i = n - d_i$ for $i = 1, \dots, m$.

We denote elements of the affine brane variety $\widetilde{\mathcal{M}}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$ and the bow variety $\mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$ according to the diagram



Recall the moment map $\tilde{m}$ on $\widetilde{\mathcal{M}}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$ from (2.33). Let

$$y = ((A_i, B_i^-, B_i^+, a_i, b_i)_i, (C_j, D_j)_j) \in \tilde{m}^{-1}(0).$$

Proposition 2.19 gives that $A_1, \ldots, A_n$ are vector space isomorphisms. By (2.12), an easy induction argument yields

$$B_i^- = -\left(\sum_{j=i}^{n}(A_iA_{i+1}\cdots A_{j-1})a_jb_j(A_j^{-1}A_{j-1}^{-1}\cdots A_i^{-1})\right), \quad i = 1, \ldots, n. \tag{2.62}$$

We define operators

$$a_y\colon \mathbb{C}^n \longrightarrow \mathbb{C}^{d'_m}, \quad b_y\colon \mathbb{C}^{d'_m} \longrightarrow \mathbb{C}^n \tag{2.63}$$

via the matrices

$$a_y = \begin{pmatrix} a_1 & A_1a_2 & \ldots & A_1\cdots A_{n-1}a_n \end{pmatrix}, \quad b_y = \begin{pmatrix} b_1A_1^{-1} \\ b_2A_2^{-1}A_1^{-1} \\ \vdots \\ b_{n-1}A_{n-1}^{-1}\ldots A_1^{-1} \\ b_nA_n^{-1}\ldots A_1^{-1} \end{pmatrix}.$$

Note that (2.62) in case $i = 1$ is equivalent to $B_1^- = -a_yb_y$.

The χ-stability criterion from Proposition 2.37 can be reformulated in terms of the following surjectivity conditions:

Lemma 2.65. *The point y is χ-stable if and only if the operators a_y, $C_2, \ldots, C_m$ are all surjective.*

Proof. We write $W = W_{\tilde{\mathcal{D}}(d_1,\ldots,d_m;n)}$. If y is χ-stable then, by Proposition 2.37, we have

$$\operatorname{im}(C_m\cdots C_2a_y) \oplus \ldots \oplus \operatorname{im}(C_2a_y) \oplus \operatorname{im}(a_y) \oplus W_{X_{m+2}} \oplus \ldots \oplus W_{X_{m+n+1}} = W.$$

This is equivalent to the surjectivity of a_y, $C_2, \ldots, C_m$. Conversely, suppose a_y, $C_2, \ldots, C_m$ are all surjective. Let $W' \subset W$ be a graded subspace satisfying the conditions of Proposition 2.37. Then, as W' is invariant under all A_i and contains $a_1(1), \ldots, a_n(1)$, we deduce $W'_{m+1} = \operatorname{im}(a_y) = W_{m+1}$. Since $C_2, \ldots, C_m$ are surjective, we get $W'_j = W_j$, for $j < m$. As all A_i induce vector space isomorphisms $W_{m+1+i}/W'_{m+1+i} \xrightarrow{\sim} W_{m+i}/W'_{m+i}$, we also get $W'_i = W_i$, for $i > m$ □

In particular, Lemma 2.65 implies that we have a morphism of varieties

$$\tilde{m}^{-1}(0)^{\mathrm{s}} \longrightarrow F(d_1, \ldots, d_m; n)$$

given as

$$y = ((A_i, B_i^-, B_i^+, a_i, b_i)_i, (C_j, D_j)_j) \mapsto \mathcal{F}_y = (0 \subset E_{y,1} \subset \ldots \subset E_{y,m} \subset \mathbb{C}^n), \tag{2.64}$$

where $E_{y,i} \coloneqq \ker(C_i\cdots C_2a_y)$. As before, let $E_{y,0} = 0$ and $E_{y,m+1} = \mathbb{C}^n$. Form (2.62) follows that the operator b_ya_y is compatible with respect to the flag $\mathcal{F}_y$ in the following sense:

Lemma 2.66. *We have $b_ya_y(E_{y,i}) \subset E_{y,i-1}$ for $i = 1, \ldots, m+1$.*

Proof. If $v \in \ker(C_i\cdots C_2a_y)$ then (2.62) implies $C_{i-1}\cdots C_2a_yb_ya_yv = -C_{i-1}\cdots C_2B_1^-a_yv$. By (2.33), we have $-C_{i-1}\cdots C_2B_1^-a_yv = C_{i-1}\cdots C_2D_2C_2a_yv = D_iC_i\cdots C_2a_yv = 0$ which completes the proof. □

By combining (2.64) and Lemma 2.66, we can realize $T^*F(d_1,\dots,d_m;n)$ as bow variety:

Theorem 2.67. *There is an isomorphism of varieties*

$$H\colon \mathcal{C}(\tilde{\mathcal{D}}(d_1,\dots,d_m;n)) \xrightarrow{\sim} T^*F(d_1,\dots,d_m;n),$$

given by

$$[y] = [(A_i, B_i^-, B_i^+, a_i, b_i)_i, (C_j, D_j)_j] \mapsto (\mathcal{F}_y, b_y a_y).$$

For the proof, we use the following basic lemma from linear algebra:

Lemma 2.68. *Let* $0 \le d \le n$ *and* $f,\ f' \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^d)$ *be surjective with* $\ker(f) = \ker(f')$. *Then, there exists* $g \in \mathrm{GL}(d)$ *such that* $gf = f'$.

Proof. Choose standard basis vectors $e_{i_1},\dots,e_{i_d}$ of $\mathbb{C}^n$ such that $(f(e_{i_1}),\dots,f(e_{i_d}))$ is a basis of $\mathbb{C}^d$. Thus,

$$\langle e_{i_1},\dots,e_{i_d}\rangle \oplus \ker(f) = \mathbb{C}^n. \tag{2.65}$$

As $\ker(f) = \ker(f')$, (2.65) gives that also $(f'(e_{i_1}),\dots,f'(e_{i_d}))$ is a basis of $\mathbb{C}^d$. Define $g \in \mathrm{GL}(d)$ via $g(f(e_{i_j})) = f'(e_{i_j})$, for $j = 1,\dots,d$. To conclude $gf = f'$, it remains to show $g(f(e_k)) = f'(e_k)$, for $k \neq i_1,\dots,i_d$. By (2.65), we can write

$$e_k = \Big(\sum_{j=1}^{d} \lambda_{i_j} e_{i_j}\Big) + v, \quad \lambda_{i_j} \in \mathbb{C}, v \in \ker(f).$$

Hence, as $\ker(f) = \ker(f')$, we conclude

$$g(f(e_k)) = \sum_{j=1}^{d} \lambda_{i_j} g(f(e_{i_j})) = \sum_{j=1}^{d} \lambda_{i_j} f'(e_{i_j}) = f'(e_k)$$

which proves $gf = f'$. □

Proof of Theorem 2.67. By (2.64) and Lemma 2.66, H is a well-defined morphism of varieties. By Proposition 2.25, H is an isomorphism if and only if H is bijective. For injectivity, suppose that $H([y]) = H([y'])$, where

$$[y] = [(A_i, B_i^-, B_i^+, a_i, b_i)_i, (C_j, D_j)_j]$$

and

$$[y'] = [(A_i', (B')_i^-, (B')_i^+, a_i', b_i')_i, (C_j', D_j')_j].$$

Let a_y, $a'_{y'}$, b_y and $b'_{y'}$ be as in (2.63). By Lemma 2.65, the operators $a_y, C_2,\dots,C_m$ and $a'_{y'}$, $C_2',\dots,C_m'$ are all surjective. Thus, as $\mathcal{F}_y = \mathcal{F}_{y'}$, Lemma 2.68 gives that after applying the action of a suitable element in $\mathcal{G}$, we have $A_1 = \ldots = A_n = A_1' = \ldots = A_n' = \mathrm{id}$, $a_y = a_{y'}$ and $C_j = C_j'$, for all j. Since $b_y a_y = b'_{y'} a'_{y'} = b'_{y'} a_y$, we conclude $b_y = b'_{y'}$. Thus, by (2.62), $B_i^- = (B')_i^-$ as well as $B_i^+ = (B')_i^+$, for all i. Finally, we prove via induction on j that $D_j = D_j'$, for $j = 2,\dots,m$. The base case $j = 2$ follows from (2.33):

$$D_2 C_2 = -B_1^- = -(B')_1^- = D_2' C_2' = D_2 C_2.$$

As C_2 is surjective, we conclude $D_2 = D'_2$. If $j > 2$, (2.33) and the induction hypothesis yield

$$D_j C_j = C_{j-1} D_{j-1} = C'_{j-1} D'_{j-1} = D'_j C'_j = D'_j C_j.$$

Again, the surjectivity of C_j gives $D_j = D'_j$. Hence, we proved $[y] = [y']$ which gives that H is injective. For surjectivity, let $(\mathcal{F}, f) \in T^*F(d_1, \dots, d_m; n)$. Write $\mathcal{F} = (0 \subset E_1 \subset \dots \subset E_m \subset \mathbb{C}^n)$. Choose $a\colon \mathbb{C}^n \to \mathbb{C}^{d'_1}$ and $C_i\colon \mathbb{C}^{d'_{i-1}} \to \mathbb{C}^{d'_i}$, for $i = 2, \dots, m$ such that $E_1 = \ker(a)$ and $E_i = \ker(C_i \dots C_2 a)$, for $i = 2, \dots, m$. This implies that a, $C_2, \dots, C_m$ are all surjective. Since a is surjective and $E_1 \subset \ker(f)$, there exists $b\colon \mathbb{C}^{d'_1} \to \mathbb{C}^n$ such that $ba = f$. Let $a_1, \dots, a_n$ be the column vectors of a and $b_1, \dots, b_n$ be the row vectors of b. Then, we set

$$A_i \coloneqq \mathrm{id}_{\mathbb{C}^{d'_1}}, \quad B_i^- \coloneqq -\Big(\sum_{j=i}^{n} a_i b_j\Big), \quad B_i^+ \coloneqq B_{i-1}^-, \quad \text{for } i = 1, \dots, n. \tag{2.66}$$

Finally, we inductively construct operators $D_i\colon \mathbb{C}^{d'_i} \to \mathbb{C}^{d'_{i-1}}$, for $i = 2, \dots, m$ such that

$$C_2 D_2 = -B_1^-, \quad \text{and} \quad C_i D_i = D_{i-1} C_{i-1}, \quad i = 2, \dots, m. \tag{2.67}$$

For the base case $i = 2$ note that as $f(E_1) \subset f(E_2)$, we have $\ker(C_2 a) \subset \ker(B_1^- a)$. Thus, $\ker(C_2) \subset \ker(B_1^-)$ and hence there exists a unique $D_2\colon \mathbb{C}^{d'_2} \to \mathbb{C}^{d'_1}$ such that $D_2 C_2 = -B_1^-$. Now, for $i > 0$ the induction hypothesis gives

$$C_{i-1} D_{i-1} C_{i-1} C_{i-2} \dots C_2 a = C_{i-1} C_{i-2} \dots C_2 a f.$$

As $f(E_i) \subset E_{i-1}$, we conclude that the kernel of the above operator contains E_i. Since $C_{i-1}, \dots, C_2, a$ are all surjective, this implies $\ker(C_i) \subset \ker(C_{i-1} D_{i-1})$. Thus, there exists a unique $D_i\colon \mathbb{C}^{d'_i} \to \mathbb{C}^{d'_{i-1}}$ such that $D_i C_i = C_{i-1} D_{i-1}$. Now, by (2.66), the point

$$y = ((A_i, B_i^-, B_i^+, a_i, b_i)_i, (C_j, D_j)_j) \in \mathbb{V}_{\tilde{\mathcal{D}}(d_1, \dots, d_m; n)}$$

satisfies (2.12). As all A_i are isomorphisms, y also satisfies (S1) and (S2). Thus, y is contained in $\widetilde{\mathcal{M}}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$. By (2.67), we further have $y \in \tilde{m}^{-1}(0)$ and Lemma 2.65 gives that y is χ-stable. Hence, $[y]$ is indeed a point on $\mathcal{C}(\mathcal{D})$ which satisfies $H([y]) = (\mathcal{F}, f)$. Thus, H is also surjective which proves that H is an isomorphism of varieties. □

The $\mathbb{T} = (\mathbb{A} \times \mathbb{C}_h^*)$-action from (2.49) and (2.51) on $\mathcal{C}(\mathcal{D})$ induces the following $\mathbb{T}$-action on $T^*F(d_1, \dots, d_m; n)$:

$$t.(\mathcal{F}, f) = (d(t)(\mathcal{F}), d(t) f d(t)^{-1}), \quad h.(\mathcal{F}, f) = (\mathcal{F}, hf),$$

where $t = (t_1, \dots, t_n) \in \mathbb{A}$, $(\mathcal{F}, f) \in T^*F(d_1, \dots, d_m; n)$ and $d(t)$ is the diagonal operator such that $d(t)(e_i) = t_i e_i$ for $i = 1, \dots n$, where $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbb{C}^n$.

Matching of tautological bundles

For $i = 1, \dots, m$, let

$$\mathcal{S}_i = \{(\mathcal{F} = (0 \subset E_1 \subset \dots \subset E_m \subset \mathbb{C}^n), v) \mid v \in E_i\} \subset F(d_1, \dots, d_m; n) \times \mathbb{C}^n$$

be the i-th tautological bundle and $\mathcal{Q}_i = (F(d_1, \dots, d_m; n) \times \mathbb{C}^n)/\mathcal{S}_i$ the corresponding quotient bundle. By abuse of language, we also denote the pullbacks of $\mathcal{S}_i$ and $\mathcal{Q}_i$ to $T^*F(d_1, \dots, d_m; n)$ by $\mathcal{S}_i$ and $\mathcal{Q}_i$. The $\mathbb{T}$-action on $T^*F(d_1, \dots, d_m; n)$ induces a $\mathbb{T}$-action on $\mathcal{S}_i$:

$$(t,h).(\mathcal{F}, f, v) = (d(t)(\mathcal{F}), hd(t)fd(t)^{-1}, d(t)v), \quad (t,h) \in \mathbb{T}.$$

Likewise, we get an induced $\mathbb{T}$-action on $\mathcal{Q}_i$:

$$(t,h).(\mathcal{F}, f, [w]) = (d(t)(\mathcal{F}), hd(t)fd(t)^{-1}, [d(t)w]), \quad (t,h) \in \mathbb{T}.$$

In this way, $\mathcal{S}_i$ and $\mathcal{Q}_i$ become $\mathbb{T}$-equivariant vector bundles.

The next theorem states that (up to an equivariant twist) the quotient bundles $\mathcal{Q}_i$ correspond to tautological bundles of $\mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$:

Theorem 2.69. *There is an isomorphism*

$$\tilde{H} \colon \xi_{m+2-i} \xrightarrow{\sim} \mathcal{Q}_i \otimes \mathbb{C}_{h^{1-i}},$$

of $\mathbb{T}$-equivariant vector bundles over $T^*F(d_1, \dots, d_m; n)$ given as

$$\tilde{H}[(y, w)] = (\mathcal{F}_y, b_y a_y, I_{y,i}^{-1}(w)),$$

where $y = ((A_i, B_i^-, B_i^+, a_i, b_i)_i, (C_j, D_j)_j) \in \tilde{m}^{-1}(0)^{\mathrm{s}}$, $w \in W_{m+2-i} = \mathbb{C}^{d_i'}$ and the vector space isomorphism $I_{y,i} \colon \mathbb{C}^n / E_{y,i} \xrightarrow{\sim} \mathbb{C}^{d_i'}$ is induced by $C_i \dots C_2 a$.

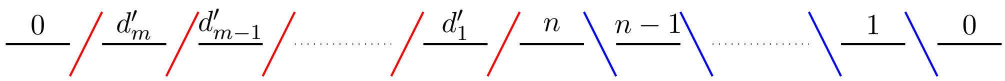
Proof. Theorem 2.67 gives that $\tilde{H}$ is bijective and hence an isomorphism by Proposition 2.25. The identity $I_{(t,h).y} = h^{i-1} I_y d(t)^{-1}$ for all $(t,h) \in \mathbb{T}$ implies that $\tilde{H}$ is indeed $\mathbb{T}$-equivariant. □

Realization via separated brane diagrams

In some situations, it is convenient to work with the following bow variety realization of $T^*F(d_1, \dots, d_m; n)$ corresponding to a separated brane diagram: By construction, the brane diagram $\tilde{\mathcal{D}}(d_1, \dots, d_m; n)$ admits the margin vectors

$$\mathbf{r} = (n - d_1', d_1' - d_2', d_2' - d_3', \dots, d_{m-1}' - d_m', d_m'), \quad \mathbf{c} = (1, 1, \dots, 1).$$

Thus, by Corollary 2.60, $\tilde{\mathcal{D}}(d_1, \dots, d_m; n)$ is Hanany–Witten equivalent to the separated brane diagram $\mathcal{D}(d_1, \dots, d_m; n)$ which is defined as



(2.68)

Note that $\mathcal{D}(d_1, \dots, d_m; n)$ is obtained from $\tilde{\mathcal{D}}(d_1, \dots, d_m; n)$ via Hanany–Witten transitions by moving V_1 to the left of $U_1, \dots, U_n$. Let

$$\Phi \colon \mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n)) \xrightarrow{\sim} \mathcal{C}(\mathcal{D}(d_1, \dots, d_m; n)). \tag{2.69}$$

be the corresponding Hanany–Witten isomorphism from Proposition 2.52. Then,

$$H' \coloneqq H \circ \Phi^{-1} \colon \mathcal{C}(\mathcal{D}(d_1, \dots, d_m; n)) \xrightarrow{\sim} T^*F(d_1, \dots, d_m; n) \tag{2.70}$$

is a ρ-equivariant isomorphism of varieties, where ρ is the automorphism of $\mathbb{T}$ given by $(t_1, \dots, t_n, h) \mapsto (t_1 h^{-1}, \dots, t_n h^{-1}, h)$.

Chapter 3

Torus fixed points of bow varieties

In this chapter, we study the action and in particular the torus fixed points if the torus action from Section 2.3. Recall that a bow varietiy $\mathcal{C}(\mathcal{D})$ is equipped with an action of a torus $\mathbb{T} = \mathbb{A} \times \mathbb{C}^*_{\hbar}$, where $\mathbb{A}$ leaves the symplectic form ω' on $\mathcal{C}(\mathcal{D})$ invariant whereas $\mathbb{C}^*_{\hbar}$ scales ω'. Nakajima proved in [Nak18, Theorem A.5] that this torus action admits only finitely many fixed points by first giving a classification of the $\mathbb{A}$-fixed points in terms of certain versions of Maya diagrams or equivalently partitions. The fact that $\mathcal{C}(\mathcal{D})^{\mathbb{A}}$ is finite directly implies $\mathcal{C}(\mathcal{D})^{\mathbb{A}} = \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.

In this chapter, we recall this classification of $\mathbb{T}$-fixed points of bow varieties using the language of tie diagrams from [RS20] and [Sho21]. We begin by describing the underlying combinatorics and the resulting explicit construction of $\mathbb{T}$-fixed points. We will in particular see in Section 3.3 that this classification of $\mathbb{T}$-fixed points is well-behaved with respect to Hanany–Witten transition. Hereafter, we consider in Section 3.4 the classification of $\mathbb{T}$-fixed points in the special case of cotangent bundles of partial flag varieties. We match the classification of torus fixed points in terms of tie diagrams with the classical parameterization in terms of symmetric (i.e. Weyl) group elements. In the last part of this chapter we follow [Nak18, Theorem A.5] to prove the Generic Cocharacter Theorem (Theorem 3.23) which states that the fixed point locus corresponding to any generic one-parameter subgroup of $\mathbb{A}$ coincides with $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$. This result will be crucial in the study of attracting cells in Chapter 4 and of stable envelopes of bow varieties in Chapter 5.

3.1 Tie data and tie diagrams

Following [RS20, Section 4], we associate combinatorial objects to brane diagrams.

Definitions

Let $\mathcal{D}$ be a brane diagram and recall the total order $\triangleleft$ on the set of lines in $\mathcal{D}$ from (2.27). Given a pair (Y_1, Y_2) of colored lines and a black line X in $\mathcal{D}$ with $Y_1 \triangleleft Y_2$, we say that the pair (Y_1, Y_2) *covers* X if $Y_1 \triangleleft X \triangleleft Y_2$.

Definition 3.1. A *Tie data with underlying brane diagram* $\mathcal{D}$ is the data of $\mathcal{D}$ together with a set D of pairs of colored lines of $\mathcal{D}$ such that the following holds:

- If $(Y_1, Y_2) \in D$ then $Y_1 \triangleleft Y_2$.
- If $(Y_1, Y_2) \in D$ then either Y_1 is blue and Y_2 is red, or Y_1 is red and Y_2 is blue.
- For all black lines X of $\mathcal{D}$, the number of pairs in D covering X is equal to d_X.

We denote by $\mathrm{Tie}(\mathcal{D})$ the set of all tie data associated to $\mathcal{D}$.

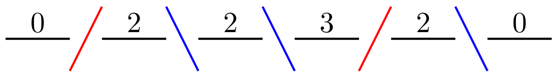
As $\mathcal{D}$ is usually a fixed underlying brane diagram, we just refer to a tie data associated to $\mathcal{D}$ just by the set D.

We visualize such a tie data D by attaching to the brane diagram $\mathcal{D}$ dotted curves connecting a red line with a blue line according to the following algorithm: We consider all pairs $(Y_1, Y_2) \in D$ of one red and one blue line.

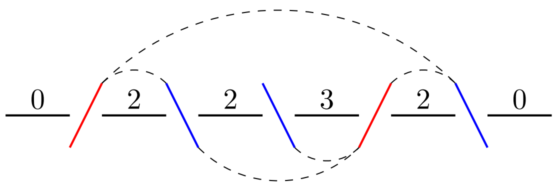
- If Y_1 is blue and Y_2 is red, we draw a dotted connection *below* the diagram $\mathcal{D}$.
- If Y_1 is red and Y_2 is blue, we draw a dotted connection *above* the diagram $\mathcal{D}$.

The resulting diagram is called the *tie diagram of* D and the dotted curves are called *ties*. Conversely, a diagram with connections between red and blue lines, drawn from red to blue at the top and from blue to red at the bottom, is a visualization of a tie data, if and only if each black line X is covered from the top and bottom by a total number of d_X arcs.

Example 3.2. Let $\mathcal{D}$ be the brane diagram



Then, the pairs $D = \{(V_2, U_1), (V_2, U_3), (U_1, V_1), (U_2, V_1), (V_1, U_3)\}$ give the data of a tie diagram which is visualized by



Note that for instance the first label 2 corresponds to the two curves covering it from above, whereas the second label 2 corresponds to one curve running above and one curve running below it.

As tie data and their corresponding tie diagrams are in obvious one-to-one correspondence, we do not distinguish between them.

Non-negativity of margin vectors

One can easily verify that not every brane diagram can be extended to a tie diagram. However, if we can extend a given brane diagram $\mathcal{D}$ to a tie diagram, we have the following non-negativity result for margin vectors: For this, recall that $M = |\mathrm{r}(\mathcal{D})|$ and $N = |\mathrm{b}(\mathcal{D})|$ from Notation 2.32 and the definition of the margin vectors $\mathbf{r} = (r_1, \ldots, r_M)$ and $\mathbf{c} = (c_1, \ldots, c_N)$ associated to $\mathcal{D}$ from Definition 2.58.

Lemma 3.3. *If $\mathcal{D}$ can be extended to a tie diagram then $r_i \geq 0$ and $c_j \geq 0$, for all $i = 1, \ldots, M$, $j = 1, \ldots, N$.*

Proof. For $D \in \mathrm{Tie}(\mathcal{D})$ and $V_i \in \mathrm{r}(\mathcal{D})$, we have

$$d_{V_i^+} - d_{V_i^-} = |\{U \in \mathrm{b}(U) \mid (V_i, U) \in D\}| - |\{U \in \mathrm{b}(U) \mid (U, V_i) \in D\}|.$$

Since $r_i = d_{V_i^+} - d_{V_i^-} + |\{U \in \mathrm{b}(U) \mid U \triangleleft V_i\}|$, we thus deduce

$$r_i = |\{U \in \mathrm{b}(U) \mid (V_i, U) \in D\}| + |\{U \in \mathrm{b}(U) \mid (U, V_i) \notin D\}|.$$

This implies $r_i \geq 0$. The proof for $c_j \geq 0$ is analogous. □

Binary contingency tables

Let $\mathcal{D}$ still be a fixed brane diagram with margin vectors $\mathbf{r}$ and $\mathbf{c}$.

Next, we give an equivalent definition of tie diagrams in terms of matrices with entries in $\{0, 1\}$. As we will see in Proposition 3.18, these matrices are well-behaved with respect to Hanany–Witten transition.

Definition 3.4. Let $\mathrm{bct}(\mathcal{D})$ denote the set of all $M \times N$ matrices B with entries in $\{0, 1\}$ satisfying following row and column sum conditions:

- $\sum_{j=1}^{N} B_{i,j} = r_i$, for all $i \in \{1, \ldots, M\}$,
- $\sum_{i=1}^{M} B_{i,j} = c_j$, for all $j \in \{1, \ldots, N\}$.

The elements of $\mathrm{bct}(\mathcal{D})$ are called *binary contingency tables of* $\mathcal{D}$.

The binary contingency tables of $\mathcal{D}$ encode the tie diagrams of $\mathcal{D}$:

Proposition 3.5. *There is a bijection*

$$M \colon \mathrm{Tie}(\mathcal{D}) \xleftrightarrow{1:1} \mathrm{bct}(\mathcal{D}), \quad D \mapsto M(D),$$

where

$$M(D)_{i,j} = \begin{cases} 1 & \text{if } (V_i, U_j) \in D,\ V_i \triangleleft U_j, \\ 1 & \text{if } (U_j, V_i) \notin D,\ U_j \triangleleft V_i, \\ 0 & \text{if } (V_i, U_j) \notin D,\ V_i \triangleleft U_j, \\ 0 & \text{if } (U_j, V_i) \in D,\ U_j \triangleleft V_i. \end{cases} \tag{3.1}$$

The inverse of M is given by $\mathrm{bct}(\mathcal{D}) \to \mathrm{Tie}(\mathcal{D})$, $B \mapsto D_B$, *where* $D_B := D'_B \cup D''_B$ *and* $D'_B = \{(V_i, U_j) \mid V_i \triangleleft U_j,\ B_{i,j} = 1\}$, $D''_B = \{(U_j, V_i) \mid U_j \triangleleft V_i,\ B_{i,j} = 0\}$.

Proof. We first show that for all $D \in \mathrm{Tie}(\mathcal{D})$, the matrix $M(D)$ is indeed contained in $\mathrm{bct}(\mathcal{D})$, i.e. $M(D)$ satisfies the required row and column sum conditions. Let $V_i \in \mathrm{r}(\mathcal{D})$. Then, we have

$$\sum_{j=1}^{N} M(D)_{i,j} = |\{U \in \mathrm{b}(\mathcal{D}) \mid (V_i, U) \in D\}| + |\{U \in \mathrm{b}(\mathcal{D}) \mid (U, V_i) \notin D\}|. \tag{3.2}$$

Since D is a tie diagram, we have

$$d_{V_i^+} - d_{V_i^-} = |\{U \in \mathrm{b}(\mathcal{D}) \mid (V_i, U) \in D\}| - |\{U \in \mathrm{b}(\mathcal{D}) \mid (U, V_i) \in D\}|.$$

This implies (3.2) $= d_{V_i^+} - d_{V_i^-} + |\{U \in \mathrm{b}(U) \mid U \triangleleft V_i\}| = r_i$ and thus proves the row sum condition for $M(D)$. The column sum condition follows along similar lines and hence $M(D) \in \mathrm{bct}(\mathcal{D})$. Next, we prove that $D_B \in \mathrm{Tie}(\mathcal{D})$, for all $B \in \mathrm{bct}(\mathcal{D})$. For this, we show via induction on l that

$$d_{X_l} = |\{(Y_1, Y_2) \in D_B \mid Y_1 \triangleleft X_l \triangleleft Y_2\}|, \quad \text{for all } X_l. \tag{3.3}$$

The case $l = 1$ is clear. For the induction step, suppose that $V = X_l^-$ is red. The row sum condition for B gives

$$|\{U \in \mathrm{b}(\mathcal{D}) \mid (V, U) \in D_B\}| + |\{U \in \mathrm{b}(\mathcal{D}) \mid (U, V) \notin D_B\}| = d_{V^+} - d_{V^-} + |\{U \in \mathrm{b}(\mathcal{D}) \mid U \triangleleft V\}|.$$

This is equivalent to $|\{U \in \mathrm{b}(\mathcal{D}) \mid (V, U) \in D_B\}| + |\{U \in \mathrm{b}(\mathcal{D}) \mid (U, V) \in D_B\}| = d_{V^+} - d_{V^-}$. With the induction hypothesis,

$$d_{V^-} = |\{(Y_1, Y_2) \in D_B \mid Y_1 \triangleleft X_{l-1},\ X_l \triangleleft Y_2\}| + |\{U \in \mathrm{b}(\mathcal{D}) \mid (U, V) \in D_B\}|.$$

Thus, we obtain

$$\begin{aligned} d_{V^+} &= |\{(Y_1, Y_2) \in D_B \mid Y_1 \triangleleft X_{l-1},\ X_l \triangleleft Y_2\}| + |\{U \in \mathrm{b}(\mathcal{D}) \mid (V, U) \in D_B\}| \\ &= |\{(Y_1, Y_2) \in D_B \mid Y_1 \triangleleft X_l \triangleleft Y_2\}|. \end{aligned}$$

Thus, we proved (3.3). The case where X_l^- is blue is analogous. Hence, we conclude that D_B is indeed a tie diagram over $\mathcal{D}$. Finally, (3.1) yields $M(D_B) = B$, for all $B \in \mathrm{bct}(\mathcal{D})$ and $D_{M(D)} = D$, for all $D \in \mathrm{Tie}(\mathcal{D})$ which proves the proposition. □

Separating line

The separating line of a binary contingency table B is a useful tool to illustrate the corresponding tie diagram D_B from Proposition 3.5. It is constructed as follows: Draw the matrix B into a coordinate system, where the entry $B_{i,j}$ is put into the square box with side length 1 and south-west corner at $(M - i, j - 1)$. Then, we define points $p_0, \dots, p_{M+N}$ in this coordinate system via $p_0 = (0, 0)$ and

$$p_i = \begin{cases} p_{i-1} + (1, 0) & \text{if } X_i^- \text{ is blue,} \\ p_{i-1} + (0, 1) & \text{if } X_i^- \text{ is red,} \end{cases}$$

for $i = 1, \dots, M + N$. The *separating line S_B of B* is then obtained by connecting each p_i with p_{i+1} by a straight line.

We can illustrate D_B using the following algorithm:

- For each (i, j) such that $B_{i,j} = 1$ and the entry $B_{i,j}$ lies below S_D, draw a dotted curve connecting V_i and U_j.

- For each (i, j) such that $B_{i,j} = 0$ and the entry $B_{i,j}$ lies above S_D, draw a dotted curve connecting V_i and U_j.

Example 3.6. Let $\mathcal{D}$ and D be as in Example 3.2. The corresponding binary contingency table $M(D)$ with separating line is given as

	U_1	U_2	U_3	U_4
V_1	1	0	0	1
V_2	0	1	1	0
V_3	0	0	1	1

3.2 Classification of torus fixed points

Next, we follow [RS20, Section 4] to associate to each tie diagram D over a given brane diagram $\mathcal{D}$ a $\mathbb{T}$-fixed point $x_D \in \mathcal{C}(\mathcal{D})$. This assignment then gives the desired classification result from [Nak18, Theorem A.5]:

Theorem 3.7 (Classification of $\mathbb{T}$-fixed points)**.** *There is a bijection*

$$\mathrm{Tie}(\mathcal{D}) \xleftrightarrow{1:1} \mathcal{C}(\mathcal{D})^{\mathbb{T}}.$$

The explicit assignment $D \mapsto x_D$ is given in Definition 3.12 below.

We prove Theorem 3.7 in Section 3.5 as a consequence of the Generic Cocharacter Theorem.

We now come to the explicit construction of the $\mathbb{T}$-fixed point x_D, for $D \in \mathrm{Tie}(\mathcal{D})$. For the convenience of the reader, we give self-contained reproofs of statements used in the construction of x_D with special emphasis on the stability properties of the involved quiver representations.

Butterfly diagrams

Given a tie diagram $D \in \mathrm{Tie}(\mathcal{D})$, we first assign to D a family of colored graphs which are called *butterfly diagrams*. Based on the structure of these butterfly diagrams, we then define in the subsequent subsection the $\mathbb{T}$-fixed point x_D in terms of matrices.

We first define the vertex set of the butterfly diagrams. For this, recall notation from Notation 2.32.

Definition 3.8. Let D be a tie diagram and U be a blue line in $\mathcal{D}$. Let $J \in \{1, \ldots, M+N\}$ such that $U^- = X_J$. The set $V(D,U)$ of *butterfly vertices corresponding to $\mathcal{D}$ and U* is a finite subset of $\mathbb{Z}^2$, where a point $(j_1, j_2) \in \mathbb{Z}^2$ is contained in $V(D,U)$ if and only if the following conditions (i)–(iii) are satisfied:

(i) $2 - J \leq j_1 \leq M + N - J$,

(ii) $c_{D,U,X_{j_1+J}} \leq j_2 < c_{D,U,X_{j_1+J}} + d_{D,U,X_{j_1+J}}$,

(iii) $d_{D,U,X_{j_1+J}} \neq 0$.

Here, $c_{D,U,X}$ and $d_{D,U,X}$ are integers, depending on a black line $X = X_j$, defined as follows:

$$d_{D,U,X} \coloneqq \begin{cases} |\{V \in \mathrm{r}(\mathcal{D}) \mid (V,U) \in D, V \triangleleft X\}| & \text{if } X \triangleleft U, \\ |\{V \in \mathrm{r}(\mathcal{D}) \mid (U,V) \in D, V \triangleright X\}| & \text{if } X \triangleright U. \end{cases}$$

For $X_j \triangleleft U$, we define c_{D,U,X_j} recursively via $c_{D,U,X_J} = 0$ and for $2 \leq j < J$ as

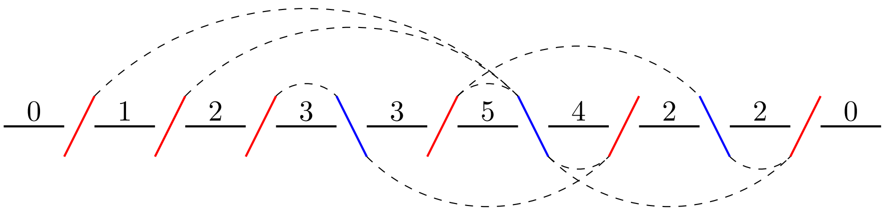
$$c_{D,U,X_j} \coloneqq \begin{cases} c_{D,U,X_{j+1}} & \text{if } X_j^+ \text{ is blue,} \\ c_{D,U,X_{j+1}} & \text{if } X_j^+ \text{ is red and } d_{D,U,X_j} + 1 = d_{D,U,X_{j+1}}, \\ c_{D,U,X_{j+1}} - 1 & \text{if } X_j^+ \text{ is red and } d_{D,U,X_j} = d_{D,U,X_{j+1}}. \end{cases}$$

In case $X_j \triangleright U$, we set

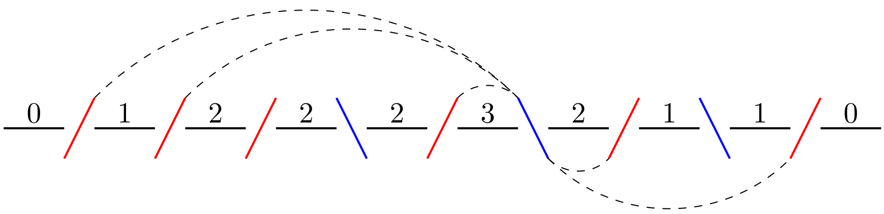
$$c_{D,U,X_j} \coloneqq d_{D,U,X_{J+1}} - d_{D,U,X_j}.$$

We call the elements of $V(D,U)$ the *butterfly vertices*, the integers c_{D,U,X_j} the *column bottom indices* and the d_{D,U,X_j} the *column heights of D and U*.

Example 3.9. Let D be the following tie diagram:



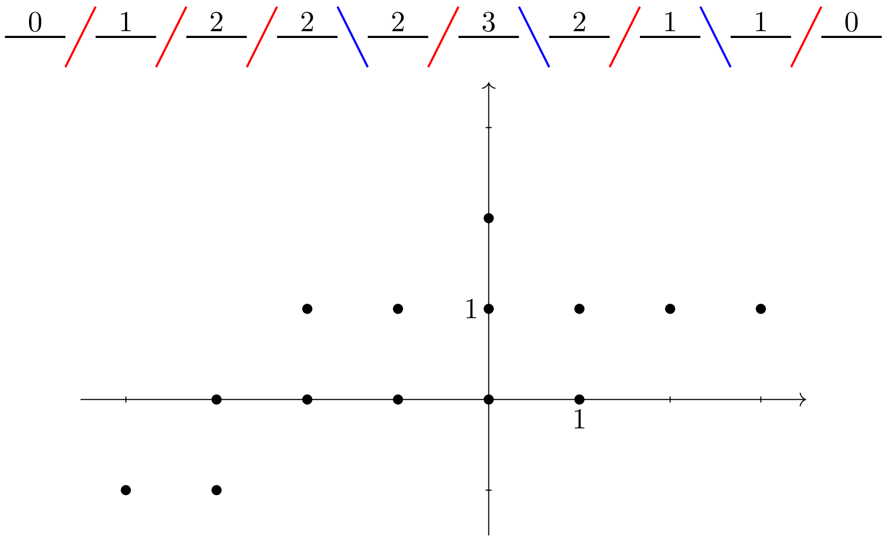
We pick $U = U_2$. In order to determine the integers $d_{D,U,X}$, we remove all ties *not* connected to U_2 and count for each black line X the number of ties which cover X:



The resulting numbers d_{D,U,X_j} are the new labels. We denote the underlying brane diagram by $\mathcal{D}_{D,U}$. The column bottom indices c_{D,U,X_j} are

j	2	3	4	5	6	7	8	9
c_{D,U,X_j}	-1	-1	0	0	0	0	1	1

Following Definition 3.8, we draw the elements of $V(D,U)$ as dots into the coordinate plane. For better illustration, we draw the coordinate plane below the brane diagram $\mathcal{D}_{D,U}$.



Let still D be a tie diagram and U a fixed blue line in $\mathcal{D}$.

Definition 3.10. A *butterfly diagram for* (D,U) is a finite, directed, colored graph with colors black, blue, red, violet and green with vertex set $V(D,U)$.

We assign to each pair (D,U) a butterfly diagram $\mathrm{b}(D,U)$. To encode the vertices in the diagram, we first define subsets of $V(D,U)$:

$$V_{\mathrm{b}}^{+} = \{(i,j) \in V(D,U) \mid X_{i+J}^{+} \in \mathrm{b}(\mathcal{D})\}, \qquad V_{\mathrm{b}}^{-} = \{(i,j) \in V(D,U) \mid X_{i+J}^{-} \in \mathrm{b}(\mathcal{D})\},$$
$$V_{\mathrm{r}}^{+} = \{(i,j) \in V(D,U) \mid X_{i+J}^{+} \in \mathrm{r}(\mathcal{D})\}, \qquad V_{\mathrm{r}}^{-} = \{(i,j) \in V(D,U) \mid X_{i+J}^{-} \in \mathrm{r}(\mathcal{D})\}.$$

In addition, we set $V_{\mathrm{b}} = V_{\mathrm{b}}^{+} \cup V_{\mathrm{b}}^{-}$ and $V_{\mathrm{r}} = V_{\mathrm{r}}^{+} \cup V_{\mathrm{r}}^{-}$. The colored arrows of $\mathrm{b}(D,U)$ are recorded in Table 3.1.

Color	Arrows of $\mathrm{b}(D,U)$	
black	$(i,j-1) \leftarrow (i,j)$	$(i,j),(i,j-1) \in V_b$
blue	$(i-1,j) \leftarrow (i,j)$	$(i,j) \in V_b^-$, $(i-1,j) \in V_b^+$
red	$(i+1,j) \leftarrow (i,j)$	$(i,j) \in V_r^+$, $(i+1,j) \in V_r^+$
violet	$(i-1,j-1) \leftarrow (i,j)$	$(i,j) \in V_r^-$, $(i-1,j-1) \in V_r^+$
green	$(0,d_{D,U,U^-}) \leftarrow *$	if $d_{D,U,U^-} \neq 0$
	$* \leftarrow (1,d_{D,U,U^-}+1)$	if $d_{D,U,U^-} < d_{D,U,U^+}$

Table 3.1: Arrows of the butterfly diagram $\mathrm{b}(D,U)$.

If for instance (D,U) are as in Example 3.9 then the corresponding butterfly diagram $\mathrm{b}(D,U)$ is illustrated in Figure 3.1. For further examples of butterfly diagrams see [RS20, Section 4.6].

From butterfly diagrams to torus fixed points

Next, we associate to a given tie diagram D and its butterfly diagrams $\mathrm{b}(\mathcal{D},U)$ the associated $\mathbb{T}$-fixed point x_D by interpreting the dots as basis elements and the arrows as vector space

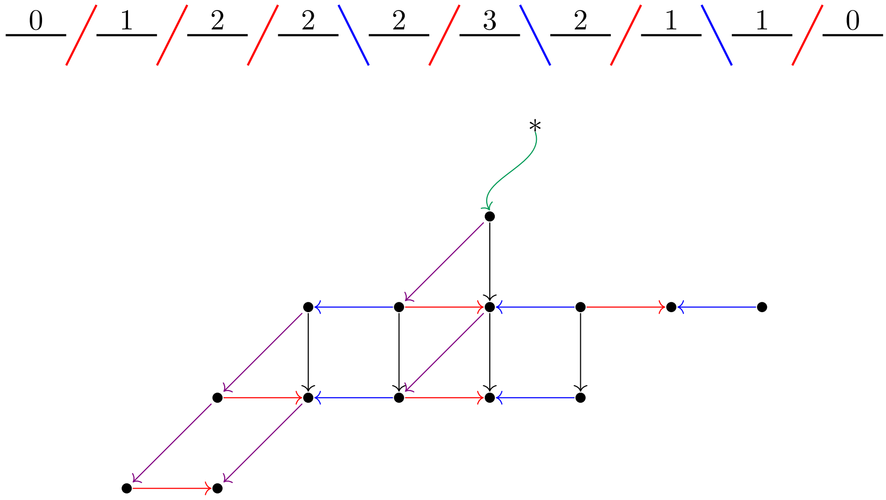


Figure 3.1: Butterfly diagram $\mathrm{b}(D, U)$, for (D, U) as in Example 3.9

.

homomorphisms: Let $F_{D,U} = \bigoplus_{i,j\in\mathbb{Z}} \mathbb{C}e_{U,i,j}$ and let $\mathbb{C}_{D,U} = \mathbb{C}$. Assume a is an arrow in $\mathrm{b}(D, U)$ which is not green. Denote by (i_1, j_1) the source of a and by (i_2, j_2) the target of a. Then, we assign to a the vector space endomorphism

$$\varphi_a\colon F_{D,U} \longrightarrow F_{D,U}, \quad \varphi_a(e_{U,i,j}) = \begin{cases} e_{U,i_2,j_2} & \text{if } i = i_1,\ j = j_1, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\mathrm{b}(D, U)$ admits at most one green arrow starting in $*$ and at most one green arrow ending in $*$. If $\mathrm{b}(D, U)$ admits a green arrow a starting in $*$ and ending in (i, j) we assign to a the vector space homomorphism

$$\psi_a\colon \mathbb{C}_{D,U} \longrightarrow F_{D,U}, \quad \psi_a(1) = e_{U,i,j}.$$

If $\mathrm{b}(D, U)$ admits a green arrow b starting in (i_1, j_1) and ending in $*$, we assign to b the vector space homomorphism

$$\psi_b'\colon F_{D,U} \longrightarrow \mathbb{C}_{D,U}, \quad \psi_b'(e_{U,i,j}) = \begin{cases} 1 & \text{if } i = i_1,\ j = j_1, \\ 0 & \text{otherwise.} \end{cases}$$

The column indices of the butterfly vertices define finite dimensional subspaces of $F_{D,U}$:

$$F_{D,U,X_i} \coloneqq \langle e_{U,i-J,j} \mid (i - J, j) \in V(D, U)\rangle, \quad \text{for all } X_i \in h(\mathcal{D}).$$

In addition, we set

$$F_X \coloneqq \bigoplus_{U\in b(\mathcal{D})} F_{D,U,X}, \quad \text{for } X \in \mathrm{h}(\mathcal{D}).$$

Let U' be a blue line of $\mathcal{D}$ and $J' \in \{1, \ldots, M + N\}$ such that $X_{J'} = (U')^-$. Using the colored arrows of $\mathrm{b}(D, U)$, we define linear operators

$$A_{D,U,U'} \in \mathrm{Hom}(F_{D,U,X_{J'+1}}, F_{D,U,X_{J'}}), \;\; B^+_{D,U,U'} \in \mathrm{End}(F_{D,U,X_{J'+1}}), \;\; B^-_{D,U,U'} \in \mathrm{End}(F_{D,U,X_{J'}})$$

as

$$A_{D,U,U'}(e_{U,J'-J+1,j}) = \sum_{a \in \mathrm{blue}(D,U,J')} \varphi_a(e_{U,J'-J+1,j}),$$
$$B^+_{D,U,U'}(e_{U,J'+1-J,j}) = \sum_{a \in \mathrm{black}(D,U,J'+1)} (-1)\varphi_a(e_{U,J'+1-J,j}),$$
$$B^-_{D,U,U'}(e_{U,J'-J,j}) = \sum_{a \in \mathrm{black}(D,U,J')} (-1)\varphi_a(e_{U,J'-J,j}).$$

Here and in the following, for any color c, we denote by $\mathrm{c}(D, U, j)$ the set of arrows colored c in $\mathrm{b}(D, U)$ with first coordinate of the target equal to j.

Next, we analogously construct linear operators for each red line. Given a red line V in $\mathcal{D}$ and $I \in \{1, \dots, M + N\}$ such that $X_I = V^-$, we define linear operators:

$$C_{D,U,V} \in \mathrm{Hom}(F_{D,U,X_{I+1}}, F_{D,U,X_I}) \quad \text{and} \quad D_{D,U,V} \in \mathrm{Hom}(F_{D,U,X_I}, F_{D,U,X_{I+1}})$$

via the formulas

$$C_{D,U,V}(e_{U,I-J+1,j}) = \sum_{a \in \mathrm{violet}(D,U,I-J)} \varphi_a(e_{U,I-J+1,j}), \quad D_{D,U,V}(e_{U,I-J,j}) = \sum_{a \in \mathrm{red}(D,U,I-J+1)} \varphi_a(e_{U,I-J,j}).$$

Finally, we also define homomorphisms

$$a_{D,U} \in \mathrm{Hom}(\mathbb{C}_U, F_{D,U,U^-}) \quad \text{and} \quad b_{D,U} \in \mathrm{Hom}(F_{D,U,U^+}, \mathbb{C}_U), \tag{3.4}$$

$$a_{D,U}(1) = \begin{cases} \psi_a(1) & \text{if } \mathrm{green}^{\mathrm{out}}(D,U) = \{a\}, \\ 0 & \text{if } \mathrm{green}^{\mathrm{out}}(D,U) = \emptyset, \end{cases}$$
$$b_{D,U}(e_{U,i,j}) = \begin{cases} \psi'_b(e_{U,i,j}) & \text{if } \mathrm{green}^{\mathrm{in}}(D,U) = \{b\}, \\ 0 & \text{if } \mathrm{green}^{\mathrm{in}}(D,U) = \emptyset, \end{cases}$$

where $\mathrm{green}^{\mathrm{in}}(D, U)$ and $\mathrm{green}^{\mathrm{out}}(D, U)$ are the sets of green arrows starting respectively ending in the additional vertex $*$.

Combining the above pieces, we now define the point x_D. For this, recall the notation from Section 2.3.

Proposition 3.11. *For $D \in \mathrm{Tie}(\mathcal{D})$, we set*

$$y_D := ((A_{D,U}, B^-_{D,U}, B^+_{D,U}, a_{D,U}, b_{D,U})_U, (C_{D,V}, D_{D,V})_V) \in \mathbb{V}_{\mathcal{D}},$$

where

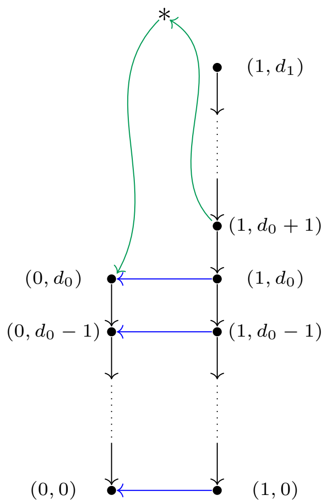
$$A_{D,U} = \bigoplus\nolimits_{U' \in \mathrm{b}(\mathcal{D})} A_{D,U',U}, \quad B^+_{D,U} = \bigoplus\nolimits_{U' \in \mathrm{b}(\mathcal{D})} B^+_{D,U',U}, \quad B^-_{D,U} = \bigoplus\nolimits_{U' \in \mathrm{b}(\mathcal{D})} B^-_{D,U',U},$$
$$C_{D,V} = \bigoplus\nolimits_{U' \in \mathrm{b}(\mathcal{D})} C_{D,U',V}, \quad D_{D,V} = \bigoplus\nolimits_{U' \in \mathrm{b}(\mathcal{D})} D_{D,U',V}$$

and a_U, b_U are defined as in (3.4)*. Then, $y_D \in \tilde{m}^{-1}(0)^{\mathrm{s}}$.*

Proof. We have to show that y_D satisfies (2.33) and (2.36) as well as the stability conditions (S1), (S2) and the χ-stability criterion from Proposition 2.37. The equations (2.33) and (2.36) can be directly shown using the definition of y_D. So we proceed with proving (S1) and (S2). Let $U \in \mathrm{b}(\mathcal{D})$. By definition, $(A_{D,U}, B^-_{D,U}, B^+_{D,U}, a_{D,U}, b_{D,U})$ satisfies (S1) if $A_{D,U}$ is injective. So suppose $A_{D,U}$ is not injective. Since the $A_{D,U',U}$ are vector space isomorphisms for $U' \neq U$, we deduce that the operators

$$(A_{D,U,U}, B^-_{D,U,U}, B^+_{D,U,U}, a_{D,U}, b_{D,U})$$

correspond to the diagram



We conclude

$$\ker(A_{D,U}) = \ker(A_{D,U,U}) = \langle e_{U,1,d_0+1}, \dots, e_{U,1,d_1} \rangle.$$

Let $w \in \ker(A_{D,U}) \setminus \{0\}$ and write

$$w = \lambda_1 e_{U,1,d_0+1} + \ldots + \lambda_{d_1-d_0} e_{U,1,d_1}, \quad \lambda_i \in \mathbb{C}.$$

Choose l such that $\lambda_l \neq 0$ and $\lambda_i = 0$, for $i < l$. Then, we have

$$b_{D,U}((B^+_{D,U})^{l-1}(w)) = (-1)^{l-1}\lambda_l \neq 0.$$

Thus, $\ker(A_{D,U}) \cap \ker(b_{D,U})$ admits no non-trivial $B^+_{D,U}$-invariant subspaces which is equivalent to (S1). For (S2) note that $\langle (B^-_{D,U})^i(a_{D,U}(1)) \mid i \geq 0 \rangle = F_{D,U,U^-}$. Since all $A_{D,U,U'}$ with $U \neq U'$ are vector space isomorphisms, this implies that the only $B^-_{D,U}$-invariant subspace of W_{U^-} containing $\mathrm{im}(A_{D,U})$ and $\mathrm{im}(a_{D,U})$ is W_{U^-} which gives (S2). Finally, we prove that y_D is χ-stable. Suppose $T = \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X$ satisfies the conditions of Proposition 2.37. To prove that $T = W_{\mathcal{D}}$, we show that T contains all $F_{D,U,X}$, for $X \in \mathrm{h}(\mathcal{D})$ and $U \in \mathrm{b}(\mathcal{D})$. Let $U \in \mathrm{b}(\mathcal{D})$ and write $U^- = X_J$. Since $a_{D,U}(1) \in T_{U^-}$, we have $F_{D,U,U^-} \subset T_{U^-}$. As all $A_{D,U,U'}$ and $C_{D,U,V}$ are surjective, for all U', $V \lhd U$, we deduce $F_{D,U,X} \subset T_X$, for all $X \in \mathrm{h}(\mathcal{D})$ with $X \lhd U$. By construction, all $A_{D,\tilde{U},U} \colon F_{D,\tilde{U},U^+} \xrightarrow{\sim} F_{D,\tilde{U},U^-}$ are vector space isomorphisms, for $\tilde{U} \neq U$. Thus, as $A_{D,U}$ induces an vector space isomorphism $W_{U^+}/T_{U^+} \xrightarrow{\sim} W_{U^-}/T_{U^-}$, we conclude that $F_{D,U,U^+} \subset T_{U^+}$. Finally, we prove via induction on i that $F_{D,U,X_{J+i}} \subset T_{X_{J+i}}$ for $i \geq 1$. The case $i = 1$ is clear as $X_{J+1} = U^+$.

So suppose $i \geq 2$. If $V = X^-_{J+i}$ is red then we have $F_{D,U,X_{J+i}} \subset T_{X_{J+i}}$, as $D_{D,U,V}$ is surjective. If $U' = X^+_{J+i}$ is blue then $A_{D,U,U'} \colon F_{D,U,X_{J+i}} \xrightarrow{\sim} F_{D,U,X_{J+i-1}}$ is an isomorphism of vector spaces. By the induction hypothesis, we have $F_{D,U,X_{J+i-1}} \subset T_{X_{J+i-1}}$. Hence, we conclude $F_{D,U,X_{J+i}} \subset A^{-1}_{D,U'}(T_{X_{J+i-1}})$. Since $A_{D,U'}$ induces an isomorphism $W_{X_{J+i}}/T_{X_{J+i}} \xrightarrow{\sim} W_{X_{J+i-1}}/T_{X_{J+i-1}}$, we thus have $F_{D,U,X_{J+i}} \subset T_{X_{J+i}}$. Therefore, we proved that T contains all $F_{D,U,X}$. Hence, $T = W_{\mathcal{D}}$ and y_D is χ-stable by Proposition 2.37. □

Definition 3.12. We set

$$x_D \coloneqq [y_D] = [(A_{D,U}, B^+_{D,U}, B^-_{D,U}, a_{D,U}, b_{D,U})_U, (C_{D,V}, D_{D,V})_V] \in \mathcal{C}(\mathcal{D}),$$

where y_D is defined as in Proposition 3.11. We call x_D the $\mathbb{T}$-*fixed point corresponding to* D.

The next proposition gives that x_D is indeed a $\mathbb{T}$-fixed point of $\mathcal{C}(\mathcal{D})$.

Proposition 3.13. *Let* $t = (t_1, \ldots, t_N) \in \mathbb{A}$ *and* $h \in \mathbb{C}^*_h$.

(i) We have $t.y_D = g_t.y_D$, *where* $g_t = \bigoplus_{i=1}^N g_{U_i,t} \in \mathcal{G}$ *and* $g_{U_i,t} \colon F_{D,U_i} \to F_{D,U_i}$, $v \mapsto t_i v$.

(ii) We have $h.y_D = g_h.y_D$, *where* $g_h = \bigoplus_{i=1}^N g_{U_i,h} \in \mathcal{G}$ *and*

$$g_{U_i,h} \colon F_{D,U_i} \to F_{D,U_i}, \quad e_{U,i,j} \mapsto h^{j-d_{D,U,U^-}} e_{U,i,j}.$$

Proof. The assertion (i) follows from the fact that the action of t only affects the operators $a_{D,U}$ and $b_{D,U}$. The assertion (ii) is a consequence of the fact that the operators $B^\pm_{D,U}$ and $C_{D,U}$ correspond to arrows that lower the second coordinate of the respective vertices by 1, whereas the operators $A_{D,U}$ and $D_{D,U}$ correspond to arrows whose source and target have the same second coordinate. □

3.3 Associated weight spaces

Next, we consider the fibers of tautological bundles of the points x_D. Since x_D is a $\mathbb{T}$-fixed point, these fibers are $\mathbb{T}$-representations. Using the structure of the corresponding weight spaces, we prove in Proposition 3.16 that two $\mathbb{T}$-fixed points x_D and $x_{D'}$ coincide if and only if $D = D'$. Then, we apply this result to characterize the images of the $\mathbb{T}$-fixed points x_D under Hanany–Witten isomorphism.

Recall the definition of the full tautological bundle $\xi_{\mathcal{D}}$ from Definition 2.46 and its $\mathbb{T}$-equivariant structure from (2.52). Let

$$p = [(A_U, B^-_U, B^+_U, a_U, b_U)_U, (C_V, D_V)_V] \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}.$$

We identify the fiber $\xi_{\mathcal{D},p}$ with $W_{\mathcal{D}} = \bigoplus_{X \in \mathrm{h}(\mathcal{D})} W_X$. In this way, we obtain an induced graded $\mathbb{T}$-action on $W_{\mathcal{D}}$

$$\rho \colon \mathbb{T} \longrightarrow \mathcal{G}, \quad (t,h) \mapsto (\rho_X(t,h))_X \tag{3.5}$$

satisfying the following *action identity* in $\mathbb{V}_{\mathcal{D}}$, for all $(t,h) \in \mathbb{T}$:

$$\begin{aligned} &\rho(t).((A_U, B^-_U, B^+_U, a_U, b_U)_U, (C_V, D_V)_V) \\ &\qquad = ((A_U, hB^-_U, hB^+_U, a_U t_U^{-1}, h t_U b_U)_U, (hC_V, D_V)_V). \end{aligned} \tag{3.6}$$

We denote the representations $(W_{\mathcal{D}}, \rho)$ resp. (W_X, ρ_X) also just by W_p resp. $W_{p,X}$ to stress the dependence on the choice of $\mathbb{T}$-fixed point p.

For a character $\tau\colon \mathbb{T} \to \mathbb{C}^*$ and a black line X in $\mathcal{D}$, let $W_{p,\tau}$ and $W_{p,\tau,X}$ be the corresponding weight space of W_p and $W_{p,X}$ respectively. The finite-dimensionality of $W_{\mathcal{D}}$ implies

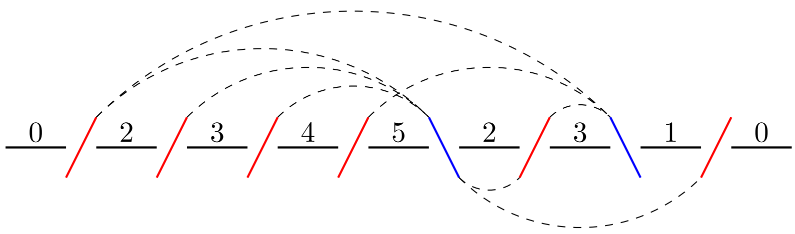
$$W_p = \bigoplus_{\tau} W_{p,\tau} = \bigoplus_{\tau} \bigoplus_{X \in \mathrm{h}(\mathcal{D})} W_{p,\tau,X}. \tag{3.7}$$

Suppose now that $p = x_D$ for some $D \in \mathrm{Tie}(\mathcal{D})$. Then, by Proposition 3.13, we have the following weight space decomposition

$$W_{x_D,X} = \bigoplus_{U \in \mathrm{b}(\mathcal{D})} \bigoplus_{l = c_{D,U,X}}^{d_{D,U,X}} h^{l+1-d_{D,U,U^-}} \mathbb{C}_U, \quad \text{for } X \in \mathrm{h}(\mathcal{D}), \tag{3.8}$$

where, as in Section 2.4, $\mathbb{C}_U$ denotes the $\mathbb{T}$-representation corresponding to the character $((t_{U'})_U, h) \mapsto t_U$, for $U \in \mathrm{b}(\mathcal{D})$.

Example 3.14. Consider the brane diagram $0/2/3/4/5\backslash 2/3\backslash 1/0$ with tie diagram D:



We like to employ (3.8) to determine the $\mathbb{T}$-weight space decomposition of all $W_{x_D,X}$. By Definition 3.8, one can easily read off the column heights d_{D,U,X_j} from this illustration:

j	2	3	4	5	6	7	8
d_{D,U_1,X_j}	1	2	3	3	2	1	1
d_{D,U_2,X_j}	1	1	1	2	2	3	0

The resulting indices $c_{D,U,j}$ are then given as follows:

j	2	3	4	5	6	7	8
c_{D,U_1,X_j}	-1	-1	-1	0	0	1	1
c_{D,U_2,X_j}	-2	-1	0	0	0	0	0

To determine for instance the $\mathbb{T}$-weight decomposition of W_{x_D,X_3}, note that since $c_{D,U_1,X_3} = -1$ and $d_{D,U_1,X_3} = 2$, the contribution of U_1 in (3.8) is given by $h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1}$. Likewise, as $c_{D,U_2,X_3} = -1$ and $d_{D,U_2,X_3} = 1$, the contribution of U_1 in (3.8) is given by $h^{-2}\mathbb{C}_{U_2}$. Consequently, $W_{x_D,X_3} = h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2}$. The other $\mathbb{T}$-weight space decompositions of the W_{x_D,X_j} can be computed in exactly the same way and are recorded in the following table:

j	W_{x_D,X_j}
2	$h^{-3}\mathbb{C}_{U_1} \oplus h^{-4}\mathbb{C}_{U_2}$
3	$h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1} \oplus h^{-3}\mathbb{C}_{U_2}$
4	$h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2}$
5	$h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1} \oplus \mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2} \oplus h^{-1}\mathbb{C}_{U_2}$
6	$h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2} \oplus h^{-1}\mathbb{C}_{U_2}$
7	$h^{-1}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2} \oplus h^{-1}\mathbb{C}_{U_2} \oplus \mathbb{C}_{U_2}$
8	$h^{-1}\mathbb{C}_{U_1}$

Reconstructing tie diagrams from weight spaces

We now restrict our attention to weight spaces of W_p corresponding to characters of the subtorus $\mathbb{A} \subset \mathbb{T}$. We begin with the following invariance property:

Lemma 3.15 (Invariance property)**.** *Let $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ and τ be a character of $\mathbb{A}$. Then, the weight space $W_{p,\tau}$ is invariant under all operators A_U, B_U^-, B_U^+, C_V, D_V.*

Proof. We only show that $W_{p,\tau}$ is A_U-invariant, since the proof for the other operators is analogous. From (3.6) we deduce $\rho_{U^-}(t)A_U(w) = A_U\rho_{U^+}(t)(w) = \tau(t)A_U(w)$, for $t \in \mathbb{C}^*$, $w \in W_{p,\tau,U^+}$. Hence, $W_{p,\tau}$ is A_U-invariant. □

Now, let $p = x_D$, for some tie diagram D of $\mathcal{D}$. Then, Proposition 3.13 gives

$$W_{p,t_U} = F_{D,U}, \quad \text{for all } U \in \mathrm{b}(\mathcal{D}). \tag{3.9}$$

Thus, the weight space decomposition (3.7) can be used to distinguish $\mathbb{T}$-fixed points for different tie diagrams:

Proposition 3.16 (Reconstruction of tie diagrams)**.** *We have*

$$x_D = x_{D'} \quad \textit{if and only if} \quad D = D'.$$

In the proof we use that a tie diagram is uniquely determined by its column heights:

Lemma 3.17. *Let D, $D' \in \mathrm{Tie}(\mathcal{D})$. Then, $D = D'$ if and only if $d_{D,U,X} = d_{D',U,X}$, for all $U \in \mathrm{b}(\mathcal{D})$, $X \in \mathrm{h}(\mathcal{D})$.*

Proof. The lemma follows form the fact that $(V,U) \in D$ if and only if $d_{D,U,V^+} = d_{D,U,V^-} + 1$ and that $(U,V) \in D$ if and only if $d_{D,U,V^+} = d_{D,U,V^-} - 1$. □

Proof of Proposition 3.16. If $x_D = x_{D'}$ then (3.9) gives

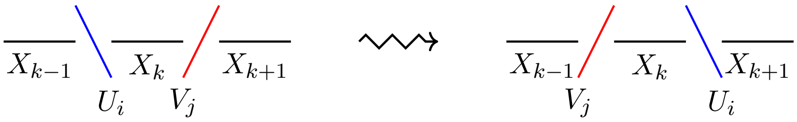
$$d_{D,U,X} = \dim(F_{D,U,X}) = \dim(F_{D',U,X}) = d_{D',U,X}, \quad \text{for } U \in \mathrm{b}(\mathcal{D}),\ X \in \mathrm{h}(\mathcal{D}).$$

Thus, Lemma 3.17 yields $D = D'$. □

Compatibility with Hanany–Witten transition

Next, we apply (3.9) to prove in Proposition 3.18 certain matching identities for the $\mathbb{T}$-fixed points x_D under Hanany–Witten isomorphisms. These matching identities were shown in [Sho21, Theorem 3.2.10] by analyzing the explicit construction of Hanany–Witten isomorphisms from [NT17]. Our proof of these matching identities avoids the explicit construction of Hanany–Witten isomorphisms and just uses the reconstruction result from Proposition 3.16.

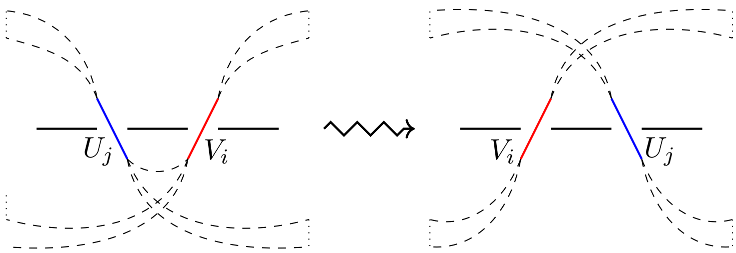
Suppose $\mathcal{D} \overset{\mathrm{HW}}{\rightsquigarrow} \tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}$ is obtained from $\mathcal{D}$ via a local move of the form



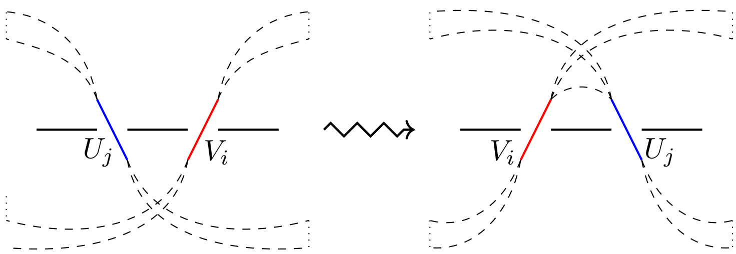
We denote the corresponding Hanany–Witten isomorphism by $\Phi\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})$. Corresponding to this Hanany-Witten transition, we define a map

$$\phi\colon \mathrm{Tie}(\mathcal{D}) \longrightarrow \mathrm{Tie}(\tilde{\mathcal{D}}), \quad \phi(D) = \begin{cases} D \setminus \{(U_j, V_i)\} & \text{if } (U_i, V_j) \in D, \\ D \cup \{(V_i, U_j)\} & \text{if } (U_i, V_j) \notin D. \end{cases} \tag{3.10}$$

We refer to ϕ as *Hanany–Witten transition for tie diagrams*. In pictures, $\phi(D)$ is obtained from D by performing, in case $(U_j, V_i) \in D$, the local move



and, in case $(U_j, V_i) \notin D$, the local move



Note that in the first case we removed the tie between U_j and V_i and in the second case we created a tie between U_j and V_i. This pictorial description of $\phi(D)$ gives that D and $\phi(D)$ admit the same binary contingency tables, i.e. we have

$$M(D) = M(\phi(D)), \quad \text{for all } D \in \mathrm{Tie}(\mathcal{D}), \tag{3.11}$$

where $M(D)$ and $M(\phi(D))$ are defined as in Proposition 3.5. In particular, we conclude that the map ϕ in (3.10) is a bijection.

From the illustration of ϕ we further deduce that the column heights of D and $\phi(D)$ are connected as follows:

$$d_{\phi(D),U,X} = \begin{cases} d_{D,U,X} & \text{for } X \neq X_k, \\ d_{D,U,X_{k-1}} + d_{D,U,X_{k+1}} - d_{D,U,X_k} & \text{for } X = X_k,\, U \neq U_j, \\ d_{D,U_i,X_{k-1}} + d_{D,U_i,X_{k+1}} + 1 - d_{D,U_i,X_k} & \text{for } X = X_k,\, U = U_j. \end{cases} \tag{3.12}$$

Since tie diagrams are uniquely determined by their column heights, we deduce the following compatibility result between the Hanany–Witten isomorphism Φ and the Hanany–Witten transition ϕ for tie diagrams:

Proposition 3.18. *For all $D \in \mathrm{Tie}(\mathcal{D})$, we have $\Phi(x_D) = x_{\phi(D)}$.*

Proof. By Theorem 3.7, there exists $\tilde{D} \in \mathrm{Tie}(\tilde{\mathcal{D}})$ such that $\Phi(x_D) = x_{\tilde{D}}$. By Proposition 2.52, we have $\mathbb{A}$-equivariant isomorphisms of vector bundles $\Phi^*(\xi_{\tilde{\mathcal{D}},X}) \cong \xi_{\mathcal{D},X}$, for $X \neq X_k$ and a sort exact sequence of $\mathbb{A}$-equivariant vector bundles

$$0 \to \xi_{\mathcal{D},X_k} \to \xi_{\mathcal{D},X_{k-1}} \oplus \xi_{\mathcal{D},X_{k+1}} \oplus \mathbb{C}_{U_j} \to \Phi^*(\xi_{\tilde{\mathcal{D}},X_k}) \to 0. \tag{3.13}$$

Hence, we obtain $\dim(W_{x_{\tilde{D}},t_U,X}) = \dim(W_{x_D,t_U,X_l})$, for $X \neq X_k$. From (3.13) we conclude

$$\dim(W_{x_{\tilde{D}},t_U,X_k}) = \dim(W_{x_D,t_U,X_{k-1}}) + \dim(W_{x_D,t_U,X_{k+1}}) - \dim(W_{x_D,t_U,X_k}), \quad \text{for } U \neq U_j$$

and also

$$\dim(W_{x_{\tilde{D}},t_{U_j},X_k}) = \dim(W_{x_D,t_{U_j},X_{k-1}}) + \dim(W_{x_D,t_{U_j},X_{k+1}}) + 1 - \dim(W_{x_D,t_{U_j},X_k}).$$

By (3.9) and (3.12), we conclude $d_{\tilde{D},U,X} = d_{\phi(D),U,X}$, for all U, X. Therefore, Lemma 3.17 gives $\tilde{D} = \phi(D)$. □

3.4 Torus fixed points of cotangent bundles of flag varieties

It is well-known that the torus fixed points of cotangent bundles of flag varieties are parameterized by left cosets of symmetric groups with respect to Young subgroups. In this section, we illustrate an equivalence between this classification and the classification in terms of tie diagrams via the realization as bow variety from Theorem 2.67. For this, recall the notation of Section 2.5.

Consider $T^*F(d_1, \ldots, d_m; n)$ and let S_n be the symmetric group on n letters. We usually denote permutations $w \in S_n$ in one line notation $w = w(1)w(2)\ldots w(n)$. To each $w \in S_n$, we assign the flag

$$\mathcal{F}_w := (\{0\} \subset \langle e_{w(1)}, \ldots, e_{w(d_1)} \rangle \subset \cdots \subset \langle e_{w(1)}, \ldots, e_{w(d_m)} \rangle \subset \mathbb{C}^n). \tag{3.14}$$

By construction, $(\mathcal{F}_w, 0)$ is a $\mathbb{T}$-fixed point of $T^*F(d_1, \ldots, d_m; n)$. Moreover, note that $\mathcal{F}_w = \mathcal{F}_{w'}$ if and only if $w' \in wS_{\boldsymbol{\delta}}$, where $S_{\boldsymbol{\delta}} = S_{\delta_1} \times \ldots \times S_{\delta_{m+1}}$ is the Young subgroup. Thus, we denote $\mathcal{F}_w$ also by $\mathcal{F}_{wS_{\boldsymbol{\delta}}}$. By e.g. [Ful97, Section 10.1], we have a bijection

$$S_n/S_{\boldsymbol{\delta}} \xrightarrow{\sim} (T^*F(d_1, \ldots, d_m; n))^{\mathbb{T}}, \quad wS_{\boldsymbol{\delta}} \mapsto (\mathcal{F}_{wS_{\boldsymbol{\delta}}}, 0).$$

In the following, we illustrate an equivalence between this classification of torus fixed points of $T^*F(d_1,\dots,d_m;n)$ and the classification of $T^*F(d_1,\dots,d_m;n)$ in terms of tie diagrams via its realization as bow variety from Theorem 2.67. Let $\tilde{\mathcal{D}}(d_1,\dots,d_m;n)$ be the brane diagram from Definition 2.64 and

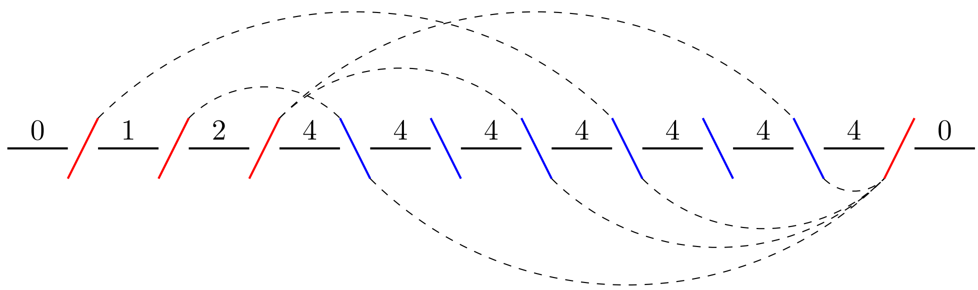
$$H\colon \mathcal{C}(\tilde{\mathcal{D}}(d_1,\dots,d_m;n)) \xrightarrow{\sim} T^*F(d_1,\dots,d_m;n)$$

be the $\mathbb{T}$-equivariant isomorphism from Theorem 2.67. Given $w\in S_n$, we define a tie diagram $\tilde{D}_w \in \mathrm{Tie}(\mathcal{D})$ as $\tilde{D}_w = \tilde{D}'_w \cup \tilde{D}''_w$, where

$$\begin{aligned}\tilde{D}'_w &= \{(V_i,U_j)\mid i\in\{2,3,\dots,m+1\},\ j\in\{w(d_{i-1}+1),\dots,w(d_i)\}\},\\ \tilde{D}''_w &= \{(U_j,V_1)\mid j\in\{w(d_1+1),w(d_1+2)\dots,w(n)\}\}.\end{aligned}$$

Note that $\tilde{D}_w = \tilde{D}_{w'}$, for all $w'\in wS_{\boldsymbol{\delta}}$. Thus, we also denote $\tilde{D}_w$ by $\tilde{D}_{wS_{\boldsymbol{\delta}}}$.

Example 3.19. Let $m=3$, $d_1=2$, $d_2=4$, $d_3=5$ and $n=6$. Then, the brane diagram $\tilde{\mathcal{D}}(2,4,5;6)$ equals $0/1/2/4\backslash 4\backslash 4\backslash 4\backslash 4\backslash 4\backslash 4/0$. Let $w\in S_6$ be the permutation $w=253614$. To construct $\tilde{D}_w$, note that since $d_1=2$, there are no ties in $\tilde{D}_w$ which are connected to U_2 and U_5. As $d_2=4$, the blue lines U_3, U_6 are both connected to V_1 and V_2. Likewise, since $d_3=5$, the blue line U_1 is connected to V_1 and V_3. Finally, $n=6$ implies that there are ties between U_4 and V_1, V_4. Hence, $\tilde{D}_w$ is illustrated as follows:



Lemma 3.20. *We have a bijection*

$$S_n/S_{\boldsymbol{\delta}} \xrightarrow{\sim} \mathrm{Tie}(\tilde{\mathcal{D}}(d_1,\dots,d_m;n)),\quad wS_{\boldsymbol{\delta}} \mapsto D_{wS_{\boldsymbol{\delta}}}.$$

Proof. Let $D\in\mathrm{Tie}(\tilde{\mathcal{D}}(d_1,\dots,d_m;n))$. We set

$$L_1 := \{j \mid (U_j,V_1)\in D\},\quad L_i := \{j\mid (V_i,U_j)\in D\},\quad \text{for } i=2,\dots,m+1.$$

In addition, $L_0 := \{1,\dots,n\}$. By construction of $\tilde{\mathcal{D}}(d_1,\dots,d_m;n)$, we have $|L_1| = n-d_1$ and that for each $j\in L_1$, there exists exactly one $i\in\{2,\dots,m+1\}$ with $(V_i,U_j)\in D$. As $\mathbf{r}(\mathcal{D})=\boldsymbol{\delta}$, we deduce that $|L_i\setminus L_{i+1}| = \delta_{i+1}$, for $i=0,\dots,m$. Thus, there exists a unique left coset $c_D\in S_n/S_{\boldsymbol{\delta}}$ such that for all $w\in c_D$ holds

$$L_i\setminus L_{i+1} = \{w(d_i+1),\dots,w(d_{i+1})\},\quad \text{for } i=0,\dots,m.$$

This yields $D_{c_D}=D$ as well as $c_{D_c}=c$, for all $c\in S_n/S_{\boldsymbol{\delta}}$ which proves the lemma. ☐

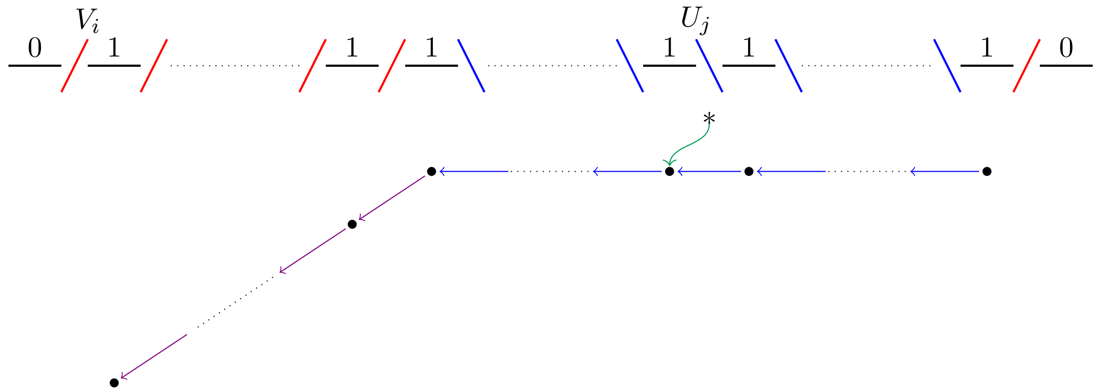
The isomorphism H is compatible with the bijection from Lemma 3.20:

Proposition 3.21. *For all $w \in S_n$, we have $H(D_{wS_\delta}) = (\mathcal{F}_{wS_\delta}, 0)$.*

Proof. As H is $\mathbb{T}$-equivariant, $H(D_{wS_\delta}) = (\mathcal{F}_{zS_\delta}, 0)$, for some $z \in S_n$. We write

$$\mathcal{F}_{zS_\delta} = (0 \subset E_1 \subset \ldots \subset E_m \subset \mathbb{C}^n).$$

Given $j \in \{w(d_i+1), \ldots w(d_{i+1})\}$, the butterfly diagram corresponding to D_{wS_δ} and U_j is given as



Hence, $e_j \in E_i$ and $e_j \notin E_1, \ldots, E_{i-1}$ which implies $\mathcal{F}_{wS_\delta} = \mathcal{F}_{z_\delta}$. Thus, we have $wS_\delta = zS_\delta$. ☐

Fixed point matching in the separated case

Let $\mathcal{D}(d_1, \ldots, d_m; n)$ be the brane diagram from (2.68). Let

$$\Phi \colon \mathcal{C}(\tilde{\mathcal{D}}(d_1, \ldots, d_m; n)) \xrightarrow{\sim} \mathcal{C}(\mathcal{D}(d_1, \ldots, d_m; n)),$$
$$\phi \colon \mathrm{Tie}(\tilde{\mathcal{D}}(d_1, \ldots, d_m; n)) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}(d_1, \ldots, d_m; n))$$

be the associated Hanany–Witten isomorphism and the Hanany–Witten transition on tie diagrams. For $w \in S_n$, we define a tie diagram $D_w \in \mathrm{Tie}(\mathcal{D})$ via the rule

$$(V_i, U_j) \in D_w \quad \Leftrightarrow \quad j \in \{w(d_{i-1}+1), \ldots, w(d_i)\}. \tag{3.15}$$

By construction, $D_w = D_{w'}$ if and only if $w' \in wS_\delta$. Thus, we also denote D_w by D_{wS_δ}. From Proposition 3.18 follows that ϕ is given by

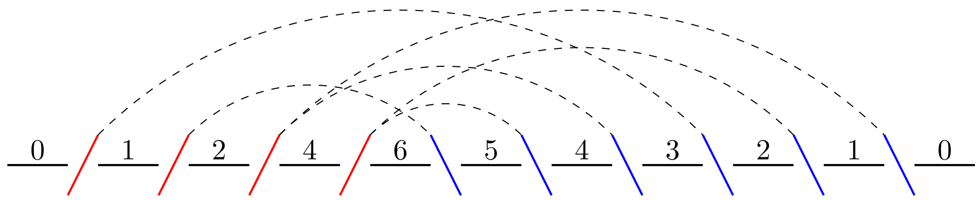
$$\phi(\tilde{D}_{wS_\delta}) = D_{wS_\delta}, \quad w \in S_n.$$

Consequently, the isomorphism $H' = H \circ \Phi^{-1} \colon \mathcal{C}(\mathcal{D}(d_1, \ldots, d_m; n)) \xrightarrow{\sim} T^*F(d_1, \ldots, d_m; n)$ satisfies

$$H'(x_{D_{wS_\delta}}) = (\mathcal{F}_{wS_\delta}, 0), \quad \text{for all } w \in S_n. \tag{3.16}$$

Example 3.22. As in Example 3.19, we choose $m = 3$, $d_1 = 2$, $d_2 = 4$, $d_3 = 5$ and $n = 6$. Thus, $\mathcal{D}(2, 4, 5; 6)$ is given by 0/1/2/4/6\5\4\3\2\1\0. Choose w as in Example 3.19. Since $d_1 = 2$, the blue lines U_2 and U_5 are connected in D_w to V_1. As $d_2 = 4$, the blue lines U_3 and

U_6 are connected to V_2. Likewise, $d_3 = 5$ gives that there is a tie between U_1 and V_3. Finally, $n = 6$ implies that U_4 is connected to V_4. Therefore, we can illustrate D_w as follows:



3.5 The Generic Cocharacter Theorem

Let $\mathcal{D}$ be a fixed brane diagram and

$$\sigma \colon \mathbb{C}^* \longrightarrow \mathbb{A}, \quad t \mapsto (\sigma_U(t))_U$$

be a cocharacter. We call σ *generic* if $\sigma_U \neq \sigma_{U'}$, for $U \neq U'$. In addition, we set

$$\mathcal{C}(\mathcal{D})^\sigma := \{x \in \mathcal{C}(\mathcal{D}) \mid \sigma(t).x = x, \text{ for all } t \in \mathbb{C}^*\}.$$

Theorem 3.23 (Generic Cocharacter Theorem)**.** *Let $\sigma \colon \mathbb{C}^* \to \mathbb{A}$ be generic. Then,*

$$\mathcal{C}(\mathcal{D})^\sigma = \{x_D \mid D \in \operatorname{Tie}(\mathcal{D})\}.$$

We prove the Generic Cocharacter Theorem in the five subsequent subsections. Before this, we prove some applications. As a direct consequence, we obtain a proof of Theorem 3.7:

Proof of Theorem 3.7. Let σ be a generic cocharacter of $\mathbb{A}$. As $\mathbb{A} \subset \mathbb{T}$, we have $\mathcal{C}(\mathcal{D})^\mathbb{T} \subset \mathcal{C}(\mathcal{D})^\sigma$. By Proposition 3.13, every x_D is also a $\mathbb{T}$-fixed point. Thus, $\mathcal{C}(\mathcal{D})^\sigma \subset \mathcal{C}(\mathcal{D})^\mathbb{T}$. □

We also obtain a following statement about tangent weights of torus fixed points:

Corollary 3.24 (Tangent weights)**.** *Let $p \in \mathcal{C}(\mathcal{D})^\mathbb{T}$ and τ be a $\mathbb{T}$-weight of $T_p\mathcal{C}(\mathcal{D})$. Then, there exist $i, j \in \{1, \dots, N\}$ with $i \neq j$ and $m \in \mathbb{Z}$ such that $\tau = t_i - t_j + mh$.*

Proof. By Corollary 2.48, all tangent weights of $T_p\mathcal{C}(\mathcal{D})$ are of the form $\tau' - \tau''$, where τ', τ'' are $\mathbb{T}$-weights of W_p. Thus, (3.8) implies that all $\mathbb{T}$-weights are of the form $t_i - t_j + mh$, with $i, j \in \{1, \dots, N\}$ and $m \in \mathbb{Z}$. By the Generic Cocharacter Theorem, p is an isolated $\mathbb{A}$-fixed point. Thus, the equivariant slice theorem, see e.g. [AF23, Theorem 5.1.4], yields that no $\mathbb{A}$-weight of $T_p\mathcal{C}(\mathcal{D})$ is trivial. Thus, no $\mathbb{T}$-weight of $T_p\mathcal{C}(\mathcal{D})$ is of the form mh, for $m \in \mathbb{Z}$ which proves the corollary. □

Remark. In [FS23, Theorem 3.2], Foster and Shou give an explicit formula for the tangent weights at $\mathbb{T}$-fixed points for bow varieties corresponding to separated brane diagrams. Its proof relies on a detailed study of the expression of the $\mathbb{T}$-equivariant K-theory class the tangent bundle in terms of classes tautological bundles from Corollary 2.48.

Reduction to separated case

Let σ be a generic cocharacter of $\mathbb{A}$. The next lemma gives that it suffices to prove the Generic Cocharacter Theorem for bow varieties corresponding to separated brane diagrams.

Lemma 3.25. *If the Generic Cocharacter Theorem holds for all $\mathcal{C}(\mathcal{D})$ with $\mathcal{D}$ separated then it holds for all $\mathcal{C}(\mathcal{D})$, where $\mathcal{D}$ is not necessarily separated.*

Proof. If $\mathcal{C}(\mathcal{D})$ is not empty then, by Proposition 2.55, there exists a separated brane diagram $\tilde{\mathcal{D}}$ with $\mathcal{D} \overset{\mathrm{HW}}{\rightsquigarrow} \tilde{\mathcal{D}}$. Let $\Phi\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})$ be the corresponding Hanany–Witten isomorphism. As Φ is $\mathbb{A}$-equivariant, we deduce

$$|\mathrm{Tie}(\tilde{\mathcal{D}})| = |\mathcal{C}(\tilde{\mathcal{D}})^\sigma| = |\mathcal{C}(\mathcal{D})^\sigma|.$$

In addition, $|\mathrm{Tie}(\tilde{\mathcal{D}})| = |\mathrm{Tie}(\mathcal{D})|$ by Proposition 3.5. Hence, $|\mathcal{C}(\mathcal{D})^\sigma| = |\mathrm{Tie}(\mathcal{D})|$. Recall from Lemma 3.16, that if $D \neq D'$ then also $x_D \neq x_{D'}$. Thus, $\{x_D \mid D \in \mathrm{Tie}(\mathcal{D})\}$ and $\mathcal{C}(\mathcal{D})^\sigma$ have the same cardinality. Therefore, the inclusion $\{x_D \mid D \in \mathrm{Tie}(\mathcal{D})\} \subset \mathcal{C}(\mathcal{D})^\sigma$ is an equality. □

Assumption. From now on, we assume that $\mathcal{D}$ is a separated brane diagram.

Weight spaces for generic cocharacters

Let $p = [(A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V] \in \mathcal{C}(\mathcal{D})^\sigma$. Via σ we view the $\mathbb{T}$-representations W_p and $W_{p,X}$ from (3.5) as $\mathbb{C}^*$-representations. For simplicity, we denote weight spaces of W_p and $W_{p,X}$ corresponding to a character $\tau\colon \mathbb{C}^* \to \mathbb{C}^*$ just by W_τ and $W_{\tau,X}$. By finite-dimensionality, we have weight space decompositions

$$W_{\mathcal{D}} = \bigoplus_\tau W_\tau, \quad W_X = \bigoplus_\tau W_{\tau,X}, \quad \text{for } X \in \mathrm{h}(\mathcal{D}). \tag{3.17}$$

By Lemma 3.15, the weight spaces W_τ are all invariant under the operators A_U, $B_U^\pm$, C_V and D_V for $U \in \mathrm{b}(\mathcal{D})$, $V \in \mathrm{r}(\mathcal{D})$. In the following, we study the weight spaces W_τ and provide a diagrammatic description of the actions of the operators A_U, $B_U^\pm$, C_V, D_V on them.

Proposition 3.26. *Let $U \in \mathrm{b}(\mathcal{D})$. Then, the following holds:*

(i) $\mathrm{im}(a_U) \subset W_{\sigma_U,U^-}$,

(ii) $\bigoplus_{\tau\neq\sigma_U} W_{\tau,U^+} \subset \ker(b_U)$,

(iii) the operator A_U induces a $\mathbb{C}$-linear isomorphism $W_{\tau,U^+} \xrightarrow{\sim} W_{\tau,U^-}$, for all $\tau \neq \sigma_U$.

Proof. By (3.6), we have $\rho(t)_{U^+} a_U(\sigma_U(t)^{-1}1) = a_U(1)$ and

$$\sigma_U(t) b_U(\rho(t)_{U^+}^{-1} w) = b_U(w), \quad \text{for all } w \in W_{U^+},\ t \in \mathbb{C}^*$$

which implies (i), (ii). By (i) (or (ii)), we now know that $a_U b_U$ vanishes on W_{τ,U^+}. Hence, (2.12) gives

$$B_U^- A_U(w) = A_U B_U^+(w), \quad \text{for all } w \in W_{\tau,U^+}. \tag{3.18}$$

In particular, $\ker(A_{U|W_{\tau,U^+}})$ is B_U^+-invariant and thus, $\ker(A_{U|W_{\tau,U^+}}) = 0$ by (S1). Next, we show that $A_{U|W_{\tau,U^+}}$ surjects onto W_{τ,U^-}. By (3.18), $\mathrm{im}(A_{U|W_{\tau,U^+}})$ is stable under the B_U^--action. By Lemma 3.15 and (i), the subspace

$$\mathrm{im}(A_{U|W_{\tau,U^+}}) \oplus \bigoplus_{\nu\neq\tau} W_{\nu,U^-} \subset W_{U^-}$$

satisfies (S2) and thus equals W_{U^-} which proves (iii). □

Proposition 3.26 gives the following improvement of (3.17):

Corollary 3.27. *We have* $W_{\mathcal{D}} = \bigoplus_{U\in \mathrm{b}(\mathcal{D})} W_{\sigma_U}$.

Proof. We have to show $W_\tau = 0$, for each τ with $\tau \neq \sigma_U$, for all $U \in \mathrm{b}(\mathcal{D})$. But by Proposition 3.26 and Lemma 3.15, the direct sum $\bigoplus_{\nu\neq\tau} W_\nu \subset W_{\mathcal{D}}$ satisfies the conditions of Proposition 2.37 and hence equals $W_{\mathcal{D}}$. Thus, its complement is zero which gives $W_\tau = 0$. □

Bases and diagrammatics for the blue part

Let U_i be a blue line in $\mathcal{D}$. Next, we employ Proposition 3.26 and the stability condition (S2) to determine bases of the spaces $W_{\sigma_{U_i},X_j}$, for $j = M+1,\dots,M+N+1$. We further describe the restrictions of the operators A_U, B_U^-, B_U^+ with respect to these bases, for all $U \in \mathrm{b}(\mathcal{D})$.

Corollary 3.28. *The following holds:*

(i) We have $W_{\sigma_{U_i},X_{M+i+1+j}} = 0$, *for* $j \geq 1$.

(ii) The $\mathbb{C}$*-vector space* $W_{\sigma_{U_i},U_i^-}$ *is generated by* $\{(B_{U_i}^-)^i a_{U_i}(1) | i \geq 0\}$.

(iii) The operator A_{U_j} *induces an isomorphism of vector spaces* $W_{\sigma_{U_i},X_j+1} \xrightarrow{\sim} W_{\sigma_{U_i},X_j}$ *for* $M+1 \leq j \leq M+i-1$.

Proof. By Proposition 3.26.(iii), the subspaces $W_{\sigma_{U_i},X_{M+i+1+j}}$ are mutually isomorphic, for $j \geq 1$. Hence, (i) follows from $W_{X_{M+N+1}} = 0$. For (ii), let $E := \langle (B_{U_i}^-)^l a_{U_i}(1) | l \geq 0\rangle$. Since $W_{\sigma_{U_i},U_i^+} = 0$, the subspace $E \oplus \bigoplus_{\tau\neq\sigma_{U_i}} W_\tau \subset W_{\mathcal{D}}$ equals $W_{\mathcal{D}}$ by (S2). This implies $E = W_{\sigma_{U_i},U_i^-}$. Statement (iii) is immediate from Proposition 3.26.(iii). □

Consider $W_{\sigma_{U_i},X_j}$, where $M+1 \leq j \leq M+i$. Using Corollary 3.28 we define a basis for $W_{\sigma_{U_i},X_j}$ as follows: Let $r = \dim(W_{\sigma_{U_i},X_{M+i}})$ and we set $y_{M+i} := a_{U_i}(1) \in W_{\sigma_{U_i},X_{M+i}}$. In addition, we define recursively $y_{M+i-k} := A_{U_{i-k}} \dots A_{U_{i-1}} y_{M+i} \in W_{\sigma_{U_i},X_{M+i-k}}$, for $1 \leq k < i$ and we set $y_{M+l,k} := (-B_{U_l}^-)^k y_{M+l}$, for $l = 1,\dots,i$, $k \geq 0$.

Corollary 3.29. *Let* $l \in \{1,\dots,i\}$. *Then,* $(y_{M+l,0},\dots,y_{M+l,r-1})$ *is a basis of* $W_{\sigma_{U_i},X_{M+l}}$.

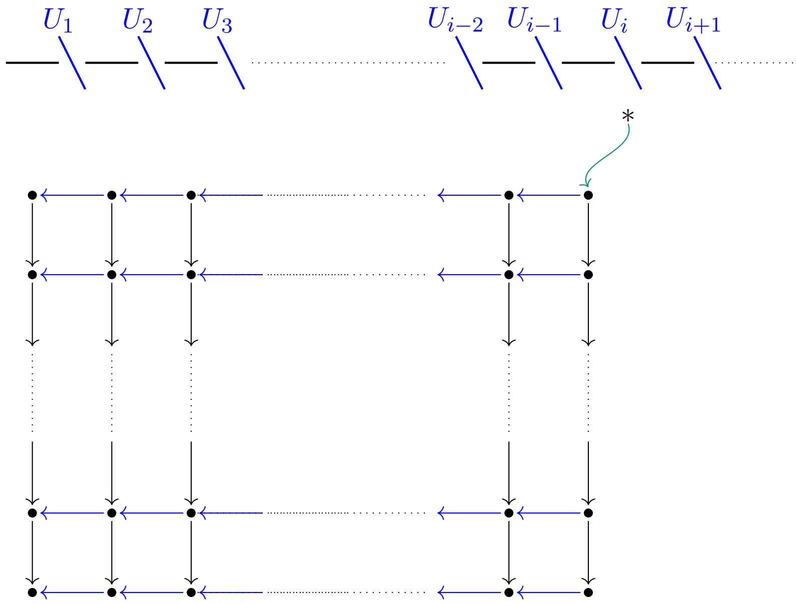
Proof. By (2.12) and Proposition 3.26.(ii), we have

$$A_{U_{j-1}} B_{U_j}^- w = B_{U_{j-1}}^- A_{U_{j-1}} w, \quad \text{for } j = 2,\dots,i \text{ and } w \in W_{\sigma_{U_i},U_j^-}. \tag{3.19}$$

Thus, the corollary follows from the statements (ii) and (iii) from Corollary 3.28. □

We denote the basis $(y_{M+l,0}, \dots, y_{M+l,r-1})$ of $W_{\sigma_{U_i}, X_{M+l}}$ by $\mathfrak{B}_{M+l}$, for $l = 1, \dots, i$. Our previous considerations lead to the following diagrammatic description of the operators A_U, B_U^-, B_U^+ with respect to this choice of bases:

Corollary 3.30 (Blue operators). *The restrictions of the operators $(A_U, -B_U^-, -B_U^+)_U$ and a_{U_i}, b_{U_i} to $W_{\sigma_{U_i}}$ with respect to the bases $\mathfrak{B}_{M+1}, \dots, \mathfrak{B}_{M+i}$ are illustrated by the following diagram, where each column contains r dots:*



Proof. By Corollary 3.28.(iii), the dimension of the vector spaces $W_{\sigma_{U_i}, X_j}$ match with the diagram. Proposition 2.56 gives that the operators B_U^- are nilpotent. Thus, by Corollary 3.29, the operators $-B_U^-$ (and equivalently $-B_U^+$) act on the chosen basis as in the diagram. It follows from (3.19) that also the operators A_U act as illustrated in the diagram. By definition, we have $a_{U_i}(1) = y_{M+i,0}$ and since $W_{\sigma_{U_i}, X_{M+i+1}} = 0$, we also have $b_{U_i} = 0$ by Proposition 3.26.(ii). $\square$

Bases and diagrammatics for the red part

Let still U_i be a fixed blue line in $\mathcal{D}$ and $r = \dim(W_{\sigma_U, U^-})$. Similar to the previous subsection, we now characterize bases for the weight spaces $W_{\sigma_{U_i}, X_j}$, for $1 \leq j \leq M$. Then, we give diagrammatic descriptions of the restriction of the operators C_V, D_V with respect to these particular bases.

At first, we set up some notation. Set

$$z_{M+1} := y_{M+1} \in W_{\sigma_{U_i}, X_{M+1}}$$

and define $z_{M+1-j} \in W_{\sigma_{U_i}, X_{M+1-j}}$ recursively as $z_{M+1-j} = C_{V_j} z_{M+2-j}$, for $j = 1, \dots, M$. Let $z_{l,k} := (D_{V_{M+2-l}} C_{V_{M+2-l}})^k z_l$, for $l = 2, \dots, M+1$, $k \geq 0$. We set $E_l := \langle z_{l,k} | k \geq 0 \rangle$ and

$$E := \Big(\bigoplus_{l=2}^{M} E_l \Big) \oplus \Big(\bigoplus_{l=1}^{N} W_{\sigma_{U_i}, X_{M+1+l}} \Big) \subset W_{\sigma_{U_i}}.$$

Note that by the moment map equation, we have $z_{M+1,k} = y_{M+1,k}$, for all $k \geq 0$.

Proposition 3.31. *We have $E = W_{\sigma_{U_i}}$.*

At first, we investigate how the operators C_V, D_V act on the elements $z_{k,l}$:

Lemma 3.32. *The following holds:*

(i) We have $C_{V_{M+1-l}} z_{l+1,k} = z_{l,k}$, for $k \geq 0$, $l = 1, \ldots, M$.

(ii) We have $D_{V_{M+1-l}} z_{l,k} = z_{l+1,k+1}$, for $k \geq 0$, $l = 1, \ldots, M$.

Moreover, E is invariant under all A_U, $B_U^{\pm}$, C_V, D_V.

Proof. The assertions (i) and (ii) are immediate from the moment map equations. The invariance of E under all A_U, B_U^-, B_U^+ follows directly from Lemma 3.15. Furthermore, (i) and (ii) imply that E is invariant under all C_V, D_V. □

The proof of Proposition 3.31 follows now from the stability criterion for bow varieties.

Proof of Proposition 3.31. By Lemma 3.32, the subspace $E' = E \oplus \bigoplus_{\tau \neq \sigma_{U_i}} W_\tau \subset W$ satisfies the conditions of Proposition 2.37 and hence equals W. Thus, we have $E = W_{\sigma_{U_i}}$. □

Proposition 3.31 and its proof lead to the following useful observation:

Corollary 3.33. *For $V \in \mathrm{r}(\mathcal{D})$, the following holds:*

(i) The operator C_V induces a surjection $W_{\sigma_{U_i}, V^+} \to W_{\sigma_{U_i}, V^-}$.

(ii) We have either $\dim(W_{\sigma_{U_i}, V^+}) = \dim(W_{\sigma_{U_i}, V^-})$ or $\dim(W_{\sigma_{U_i}, V^+}) = \dim(W_{\sigma_{U_i}, V^-}) + 1$.

Proof. According to Lemma 3.32.(ii) and Proposition 3.31, the image of C_V contains a generating system of $W_{\sigma_{U_i}, V^-}$ which gives (i). For (ii), write $V = V_l$, where $l = 1, \ldots, M$. By Lemma 3.32.(i), D_{V_l} surjects onto $\langle z_{M+2-l,k} | k \geq 1 \rangle$. By Proposition 2.56, this is a subspace of $W_{\sigma_{U_i}, V^+}$ of codimension 1. Combining this with (i), we obtain $\dim(W_{\sigma_{U_i}, V^+}) - 1 \leq \dim(W_{\sigma_{U_i}, V^-}) \leq \dim(W_{\sigma_{U_i}, X_{V^+}})$. Thus, we conclude (ii). □

Now, by Corollary 3.33, there exist $k_0 := 1 \leq k_1 < \ldots < k_r \leq k_{r+1} := M$ such that $\dim(W_{\sigma_{U_i}, V_{k_j}^+}) = \dim(W_{\sigma_{U_i}, V_{k_j}^-}) + 1$ for $j = 1, \ldots, r$ and $\dim(W_{\sigma_{U_i}, V_l^+}) = \dim(W_{\sigma_{U_i}, V_l^-})$ in case $k_j < l < k_{j+1}$ with $j = 0, \ldots, r$. The following corollary is immediate from Proposition 3.31:

Corollary 3.34 (Combinatorial bases)**.** *Let $X_l \in \mathrm{h}(\mathcal{D})$ with $V_{k_{j+1}} \triangleleft X_l \triangleleft V_{k_j}$. Then, the vector space $W_{\sigma_{U_i}, X_l}$ has basis $(z_{l,0}, \ldots, z_{l,r-j-1})$. We denote this basis by $\mathfrak{B}_{U_i,l}$.*

We now give a diagrammatic description of the operators C_V, D_V:

Corollary 3.35 (Red operators)**.** *The operators C_{V_l}, D_{V_l}, where $l = k_j \ldots, k_{j+1} - 1$, with respect to the bases $\mathfrak{B}_{k_j}, \ldots, \mathfrak{B}_{k_{j+1}}$ are illustrated by the diagram in Figure 3.2.*

Proof. By Proposition 2.56, we have $z_{l,k} = 0$, for $k \geq \dim(W_{\sigma_{U_i}, X_l})$. Hence, Lemma 3.32 gives that the stated operators act on the given bases exactly as illustrated in the diagram. □

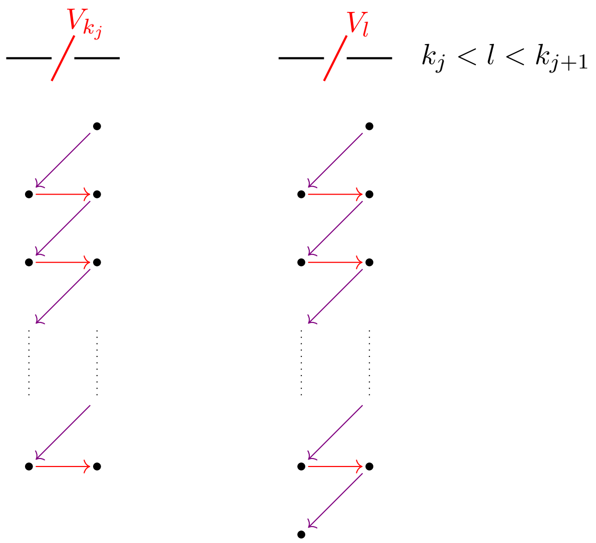


Figure 3.2: Diagrammatic description of C_{V_l}, D_{V_l} in case $l = k_j, \dots, k_{j+1} - 1$.

Proof of the Generic Cocharacter Theorem

The proof of the Generic Cocharacter Theorem is essentially a consequence of the diagrammatic description of the operators A_U, $B_U^{\pm}$, C_V, D_V from Corollary 3.30 and Corollary 3.35.

Proof of Theorem 3.23. For a given $p \in \mathcal{C}(\mathcal{D})^\sigma$, we define a tie diagram D via

$$(V, U) \in D \quad \Leftrightarrow \quad \dim(W_{\sigma_U, V^+}) = \dim(W_{\sigma_U, V^-}) + 1.$$

For $U_i \in \mathrm{b}(\mathcal{D})$, let r, $k_0, \dots, k_{r+1}$ be defined as in the previous subsection. By Corollary 3.28, we have $d_{D,U_i,X_l} = 0$, for $l > M + i$ and $d_{D,U_i,X_l} = r$, for $M + 1 \le l \le M + i$. Moreover, Corollary 3.34 gives $d_{D,U_i,X_l} = r - j$, for $j = 0, \dots, r$ and $V_{k_j+1} \lhd X_l \lhd V_{k_j}$. Thus, $d_{D,U_i,X_l} = \dim(W_{\sigma_{U_i}, X_l})$, for all $X_l \in \mathrm{h}(\mathcal{D})$ which implies that D is indeed a tie diagram of $\mathcal{D}$. The corresponding column bottom indices are given by $c_{D,U_i,X_l} = 0$, for $M + 1 \le l \le M + i$ and $c_{D,U_i,X_l} = l - M - 1 + j$, for $V_{k_{j+1}} \lhd X_l \lhd V_{k_j}$, $j = 0, \dots, r$. Therefore, the vector spaces F_{D,U_i,X_l} have bases $(e_{U_i,l-M-i,0}, \dots, e_{U_i,l-M-i,r-1})$, for $M+1 \le l \le M+i$. In case $V_{k_{j+1}} \lhd X_l \lhd V_{k_j}$, for some $j = 0, \dots, r$, the vector space F_{D,U_i,X_l} has the basis $(\tilde{e}_{U_i,X_l,0}, \dots, \tilde{e}_{U_i,X_l,r-1-j})$, where we set $\tilde{e}_{U_i,X_l,k} \coloneqq e_{U_i,l-M-i,l-M-2+r-k}$. Consequently, we can define isomorphisms of vector spaces $\phi_{U_i,X_l} \colon W_{\sigma_{U_i},X_l} \xrightarrow{\sim} F_{D,U,X_l}$ via

$$\phi_{U_i,X_l}(y_{l,k}) = e_{U_i,l-M-i,r-k-1}, \quad \text{for } M + 1 \le l \le M + i,\ k = 0, \dots, r - 1$$

and

$$\phi_{U_i,X_l}(z_{l,k}) = \tilde{e}_{U_i,X_l,k}, \quad \text{for } V_{k_{j+1}} \lhd X_l \lhd V_{k_j},\ k = 0, \dots, j - 1.$$

For the other X_l, we have $W_{\sigma_{U_i},X_l} = 0$, so we set $\phi_{U_i,X_l} \coloneqq 0$, for $X_l \lhd V_{k_M}$ and $X_l \rhd U_i$. By Corollary 3.30,

$$\phi_{U_i,U^-} A_U(w_+) = A_{D,U_i,U} \phi_{U_i,U^+}(w_+), \quad \phi_{U_i,U^-} B_U^-(w_-) = B_{D,U_i,U}^- \phi_{U_i,U^-}(w_-),$$
$$\phi_{U_i,U^+} B_U^+(w_+) = B_{D,U_i,U}^+ \phi_{U_i,U^+}(w_+),$$

for all $U \in \mathrm{b}(\mathcal{D})$, $w_- \in W_{\sigma_{U_i},U^-}$, $w_+ \in W_{\sigma_{U_i},U^+}$. In addition, Corollary 3.30 gives $\phi_{U_i,U_i^-} a_{U_i} = a_{D,U_i}$ and $b_{U_i} = b_{D,U_i} = 0$. Likewise, Corollary 3.35 implies

$$\phi_{U_i,V^-} C_V(v_+) = C_{D,U_i,V} \phi_{U_i,V^+}(v_+), \quad \phi_{U_i,V^+} D_V(v_-) = D_{D,U_i,V} \phi_{U_i,V^-}(v_-),$$

for all $V \in r(\mathcal{D})$, $v_- \in W_{\sigma_{U_i},V^-}$, $v_+ \in W_{\sigma_{U_i},V^+}$. Thus, we proved that p equals the $\mathbb{T}$-fixed point x_D and hence $\mathcal{C}(\mathcal{D})^\sigma = \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. $\square$

Chapter 4

Attracting cells for bow varieties

In this chapter, we recall several aspects of the theory of attracting cells from [MO19] in the framework of bow varieties. This theory is an important ingredient for the theory of stable envelopes which we will discuss in the subsequent chapter.

First, we study the affine structure of attracting cells of torus fixed points of bow varieties $\mathcal{C}(\mathcal{D})$ and show that they are always isomorphic to an affine space of dimension $\frac{1}{2}\dim(\mathcal{C}(\mathcal{D}))$, see Proposition 4.4. Then, we discuss and study in Sections 4.4-4.6 the partial ordering on $\mathbb{T}$-fixed points induced from the closure relations of attracting cells. Finally, we show in Theorem 4.24 that certain intersections of closures of attracting cells are proper closed subvarieties of $\mathcal{C}(\mathcal{D})$, despite the closures of attracting cells being in general not proper.

In [MO19], Maulik and Okounkov consider attracting cells of smooth and symplectic varieties X with a torus action which leaves the symplectic structure invariant, and further assume that X is quasi-projective and that X admits a torus equivariant proper morphism to an affine variety. By construction, as GIT quotients, bow varieties satisfy all these properties and hence all the results from [MO19] apply. As we have seen in Section 3.2, we are additionally in the preferable situation that bow varieties have finitely many torus fixed points and these fixed points can be described combinatorially. These facts simplify some aspects of the theory of attracting cells and make the computation of attracting cells in some examples possible.

To have a well-behaved theory, we require, as in the classical situation of flag varieties, that the fixed locus with respect to generic one-parameter subgroups of the torus $\mathbb{A}$ is non-empty. By the Generic Cocharacter Theorem (Theorem 3.23), this is equivalent to the following assumption:

Assumption. From now on, we assume that $\mathcal{C}(\mathcal{D})$ is a bow variety with $\mathcal{C}(\mathcal{D})^{\mathbb{T}} \neq \emptyset$.

Recall from Theorem 3.7 that $\mathcal{C}(\mathcal{D})^{\mathbb{T}} \neq \emptyset$ if and only if the brane diagram $\mathcal{D}$ can be extended to a tie diagram.

4.1 Attracting cells

Before we go into details, we prove two general propositions about bow varieties. The first one is about the existence of important $\mathbb{T}$-equivariant morphisms.

Proposition 4.1. *There exists a smooth and projective variety X with $\mathbb{T}$-action and a finite dimensional $\mathbb{T}$-representation V such that there exists*

(i) a $\mathbb{T}$-equivariant open dense immersion $\mathcal{C}(\mathcal{D}) \hookrightarrow X$ and

(ii) a $\mathbb{T}$-equivariant closed immersion $X \hookrightarrow \mathbb{P}(V)$.

Proof. By Proposition 2.2.(i) and Proposition 2.43.(i), $\mathcal{C}(\mathcal{D})$ is a smooth and quasi-projective variety. Hence, by [Sum74, Theorem 2], there exists a $\mathbb{T}$-equivariant locally closed immersion $\mathcal{C}(\mathcal{D}) \hookrightarrow \mathbb{P}(V')$, where V' is a finite dimensional $\mathbb{T}$-representation. Let X' be the Zariski closure of $\mathcal{C}(\mathcal{D})$ in $\mathbb{P}(V')$. Then, X' is a $\mathbb{T}$-invariant closed subvariety of $\mathbb{P}(V')$ containing $\mathcal{C}(\mathcal{D})$ as open dense subvariety. Let $\mathcal{I}$ be the ideal sheaf on X' corresponding to the closed subvariety $X' \setminus \mathcal{C}(\mathcal{D})$ of X'. Applying the Equivariant Hironaka Theorem, see e.g. [Wło05, Theorem 1.0.2], to the pair $(X, \mathcal{I})$ yields that there exists a smooth and projective variety X with $\mathbb{T}$-action and a birational $\mathbb{T}$-equivariant morphism $f\colon X \to X'$ such that the restriction $f^{-1}(\mathcal{C}(\mathcal{D})) \to \mathcal{C}(\mathcal{D})$ is a $\mathbb{T}$-equivariant isomorphism. This gives (i). The assertion (ii) follows from applying [Sum74, Theorem 2] to the variety X. □

The next proposition is a useful statement about the existence of $\mathbb{T}$-equivariant proper morphisms to affine spaces:

Proposition 4.2. *There exists a proper and $\mathbb{T}$-equivariant morphism $\mathcal{C}(\mathcal{D}) \to V$, where V is a finite dimensional $\mathbb{T}$-representation.*

Proof. Let $\tilde{m}\colon \widetilde{\mathcal{M}}(\mathcal{D}) \to \bigoplus_{X \in \mathrm{h}(\mathcal{D})} \mathrm{End}(W_X)$ be the moment map from (2.33). By Proposition 2.43.(ii) and Proposition 2.13 the projection $\pi\colon \mathcal{C}(\mathcal{D}) \to \tilde{m}^{-1}(0)/\!\!/G$ is a projective and $\mathbb{T}$-equivariant morphism. Since $\tilde{m}^{-1}(0)/\!\!/G$ is affine, there exists a $\mathbb{T}$-equivariant closed immersion $\iota\colon \tilde{m}^{-1}(0)/\!\!/G \hookrightarrow V$, where V is a finite dimensional $\mathbb{T}$-representation. Thus, $\iota \circ \pi\colon \mathcal{C}(\mathcal{D}) \to V$ is a proper and $\mathbb{T}$-equivariant morphism. □

Definition and affine structure

Let $\sigma\colon \mathbb{C}^* \to \mathbb{A}$ be a generic cocharacter and $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. By Corollary 3.24, we have a splitting $T_p\mathcal{C}(\mathcal{D}) = T_p\mathcal{C}(\mathcal{D})^+_\sigma \oplus T_p\mathcal{C}(\mathcal{D})^-_\sigma$ in the subspace of strictly positive respectively strictly negative weights corresponding to σ. As the symplectic form on $\mathcal{C}(\mathcal{D})$ is $\mathbb{A}$-invariant, the vector spaces $T_p\mathcal{C}(\mathcal{D})^+_\sigma$ and $T_p\mathcal{C}(\mathcal{D})^-_\sigma$ are both of dimension $\frac{1}{2}\dim(\mathcal{C}(\mathcal{D}))$.

Definition 4.3. The *attracting cell of p with respect to σ* is defined as

$$\mathrm{Attr}_\sigma(p) := \{z \in \mathcal{C}(\mathcal{D}) \mid \lim_{t\to 0} \sigma(t).z = p\}.$$

By definition, $\mathrm{Attr}_\sigma(p)$ is just a $\mathbb{T}$-invariant subset of $\mathcal{C}(\mathcal{D})$. However, the following proposition shows that it actually carries the structure of a locally closed affine subvariety of $\mathcal{C}(\mathcal{D})$.

Proposition 4.4. *The attracting cell* $\mathrm{Attr}_\sigma(p)$ *is a locally closed $\mathbb{T}$-invariant subvariety of $\mathcal{C}(\mathcal{D})$ which is $\mathbb{T}$-equivariantly isomorphic to the affine space $T_p\mathcal{C}(\mathcal{D})^+_\sigma$.*

For the proof, we use the classical Białynicki-Birula Theorem from [BB73, Theorem 4.3] and some general applications.

The classical Białynicki-Birula Theorem

Let X be a smooth and projective variety with algebraic $T = \mathbb{C}^*$-action. Recall from [Ive72, Theorem 1] that X^T is a smooth closed subvariety of X. We denote by $F_1, \dots, F_r$ the irreducible components of X^T and set

$$X_{F_i}^+ := \{x \in X \mid \lim_{t\to 0} t.x \text{ is contained in } F_i\}.$$

Theorem 4.5 (Bialynicki-Birula). *The following holds:*

(i) Each $X_{F_i}^+$ is a locally closed subvariety of X.

(ii) The limit map $\pi_i \colon X_{F_i}^+ \to F_i$, $\pi(x) = \lim_{t\to 0} t.x$ is an affine fiber bundle of F_i, where all fibers are affine spaces.

(iii) For each $p \in F_i$, we have an isomorphism of T-equivariant varieties $\pi_i^{-1}(p) \cong T_pX^+$.

Theorem 4.5 generalizes to the quasi-projective setting as follows: Let Y be a smooth and quasi-projective variety with algebraic $T = \mathbb{C}^*$-action. Then, Y^T is again a smooth closed subvariety of Y. Let $F_1', \dots, F_s'$ be the irreducible components of Y^T and we set

$$Y_{F_i'}^+ := \{y \in Y \mid \lim_{t\to 0} t.y \text{ exists in } Y \text{ and is contained in } F_i'\}.$$

Corollary 4.6. *The following holds:*

(i) Each $Y_{F_i'}^+$ is a locally closed subvariety of Y.

(ii) The limit map $\pi_i' \colon Y_{F_i'}^+ \to F_i'$, $\pi(y) = \lim_{t\to 0} t.y$ is an affine fiber bundle of F_i'.

(iii) For each $p \in F_i'$, we have an isomorphism of T-equivariant varieties $(\pi_i')^{-1}(p) \cong T_pY^+$.

Proof. As in Proposition 4.1, we can choose an open dense and T-equivariant embedding $Y \hookrightarrow X$ into a smooth and projective variety X with algebraic T-action. As before, let $F_1, \dots, F_r$ be the irreducible components of X^T and $\pi_i \colon X_{F_i}^+ \to F_i$ the limit morphism. As Y is an open dense and T-invariant subvariety of X, we can assume that $F_i' = F_i \cap Y$ for $i = 1, \dots, s$. To conclude (i), (ii) and (iii), it suffices by Theorem 4.5 to show

$$\pi_i^{-1}(F_i') = Y_{F_i'}^+, \quad \text{for } i = 1, \dots, s.$$

For each $p \in F_i'$, we have that $Y \cap \pi_i^{-1}(p)$ is a T-invariant and open subvariety of $\pi_i^{-1}(p)$. Thus, as $\pi_i^{-1}(p) \cong T_pX^+$, Lemma 4.7 below yields $Y \cap \pi_i^{-1}(p) = \pi_i^{-1}(p)$. Hence, we conclude $\pi_i^{-1}(F_i') = Y_{F_i'}^+$. □

Lemma 4.7. *Let W be a finite dimensional T-representation such that all T-weights of W are strictly positive. Let $U \subset W$ be an open T-invariant subvariety containing the origin of W. Then, $U = W$.*

Proof. Let $w \in W$ and $\overline{T.w}$ be the Zariski closure of the T-orbit of w in W. As all T-weights of W are strictly positive, $\overline{T.w}$ contains the origin. So $U \cap \overline{T.w}$ is a non-empty open T-invariant subvariety of $\overline{T.w}$ which implies $w \in U$. □

Suppose $T' = (\mathbb{C}^*)^m$ is a further torus acting algebraically on Y such that the T- and T'-actions commute. This assumtion gives that $Y^+_{F'_1}, \dots, Y^+_{F'_s}$ are T'-invariant locally closed subvarieties of Y. Also, each limit morphism π'_i is T'-equivariant and hence $Y^+_{F'_i}$ is a T'-equivariant affine bundle over F'_i. The next proposition gives that if $p \in F'_i \cap Y^{T'}$ then the identification of fibers $(\pi'_i)^{-1}(p) \cong T_pY^+$ from Corollary 4.6.(iii) can be chosen to be T'-equivariant:

Proposition 4.8. *For each $p \in F'_i \cap Y^{T'}$, we have a T'-equivariant isomorphism of varieties $(\pi'_i)^{-1}(p) \xrightarrow{\sim} T_pY^+$.*

For the proof, we use the following result from [Kon96]:

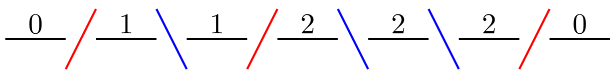
Theorem 4.9. *Let V be a finite dimensional T-representation such that all T-weights appearing in the weight space decomposition of V are strictly positive. Let $Z \subset V$ be a smooth and T-invariant closed subvariety containing the origin p of V. Consider $T_pZ \subset V$ as T-subrepresentations and let $\mathrm{pr}\colon V \to T_pZ$ be any T-equivariant projection. Then,* pr *restricts to a T-equivariant isomorphism $Z \xrightarrow{\sim} T_pZ$.*

Proof of Proposition 4.8. By Corollary 4.6.(iii), $Z := (\pi'_i)^{-1}(p)$ is a smooth and affine variety with algebraic $(T' \times T)$-action. Thus, there exists a $(T' \times T)$-equivariant closed immersion $Z \hookrightarrow V$ into a finite dimensional $(T' \times T)$-representation. Since all points of $(\pi'_i)^{-1}(p)$ are attracted to p under the T-action, we can assume that p is mapped to the origin in V and that all T-weights of V are strictly positive. We view the tangent space T_pZ as $(T' \times T)$-subrepresentation of V. Choose a $(T' \times T)$-equivariant projection $\mathrm{pr}\colon V \to T_pZ$. Then, by Theorem 4.9, pr restricts to a $(T' \times T)$-equivariant isomorphism of varieties $Z \xrightarrow{\sim} T_pZ$. $\square$

Proof of Proposition 4.4. By Corollary 4.6, $\mathrm{Attr}_\sigma(p)$ is a locally closed $\mathbb{T}$-invariant subvariety of $\mathcal{C}(\mathcal{D})$. Then, by Proposition 4.8, $\mathrm{Attr}_\sigma(p)$ is $\mathbb{T}$-equivariantly isomorphic to $T_p\mathcal{C}(\mathcal{D})^+_\sigma$. $\square$

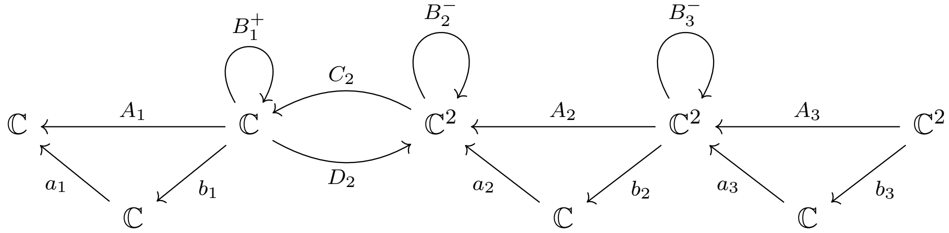
4.2 Attracting cells in a concrete example

Let $\mathcal{D}$ be the brane diagram



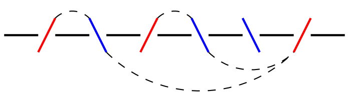
(4.1)

We encode the elements of $\widetilde{\mathcal{M}}(\mathcal{D})$, $\tilde{m}^{-1}(0)$ and the bow variety $\mathcal{C}(\mathcal{D})$ as tuples of endomorphisms with the notation given by the following diagram:

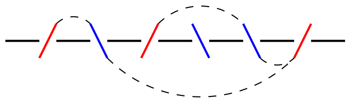


Here, we dropped the operators C_1, C_3, D_1, D_3, B_1^- and B_3^+ from the picture as they always vanish. We also identified B_2^+ and B_3^- according to the moment map equation. Moreover, note that A_1, A_2, A_3 are isomorphisms by Proposition 2.19.

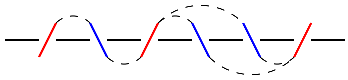
One can easily check that $\mathcal{D}$ can be extended to exactly five different tie diagrams:



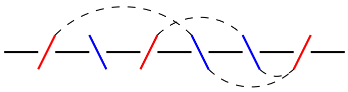
D_1



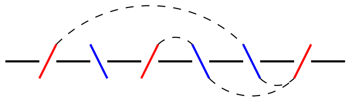
D_2



D_3



D_4



D_5

By Theorem 3.7, the fixed point locus is $\mathcal{C}(\mathcal{D})^{\mathbb{T}} = \{x_{D_1}, x_{D_2}, x_{D_3}, x_{D_4}, x_{D_5}\}$, see Section 3.2 for the explicit construction of the x_{D_i}. In order to determine the attracting cells of these $\mathbb{T}$-fixed points, we first describe a covering of $\mathcal{C}(\mathcal{D})$ by open affine $\mathbb{T}$-invariant subvarieties.

We start with the following observation:

Claim 4.10. *A point*

$$x = (A_1, A_2, A_3, B_1^+, B_2^-, B_3^-, C_2, D_2, a_1, a_2, a_3, b_1, b_2, b_3) \in \tilde{m}^{-1}(0)$$

is χ-stable if and only if the following equalities hold

$$\mathrm{im}(a_1)+\mathrm{Im}(A_1C_2a_2)+\mathrm{im}(A_1C_2A_2a_3) = \mathbb{C}, \quad \mathrm{im}(a_2)+\mathrm{im}(A_2a_3)+\mathrm{im}(D_2A_1^{-1}a_1) = \mathbb{C}^2. \quad (4.2)$$

Proof. Define vector spaces

$$T_1' \coloneqq \mathrm{im}(a_1), \quad T_2' \coloneqq \mathrm{im}(A_1^{-1}a_1), \quad T_3' \coloneqq \mathrm{im}(a_2) + \mathrm{Im}(A_2a_3),$$
$$T_4' \coloneqq \mathrm{im}(A_2^{-1}a_2) + \mathrm{Im}(a_3), \quad T_5' \coloneqq \mathrm{im}(A_3^{-1}A_2^{-1}a_2) + \mathrm{Im}(A_3^{-1}a_3),$$

as well as

$$T_1'' = \mathrm{im}(A_1C_2a_2) + \mathrm{Im}(A_1C_2A_2a_3), \quad T_2'' = \mathrm{im}(C_2a_2) + \mathrm{Im}(C_2A_2a_3),$$
$$T_3'' = \mathrm{im}(D_2A_1^{-1}a_1), \quad T_4'' = \mathrm{im}(A_2^{-1}D_2A_1^{-1}a_1), \quad T_5'' = \mathrm{im}(A_3^{-1}A_2^{-1}D_2A_1^{-1}a_1).$$

Set $T' \coloneqq \bigoplus_{i=1}^5 T_i'$, $T'' \coloneqq \bigoplus_{i=1}^5 T_i''$ and consider T' and T'' as graded subspaces of $W_{\mathcal{D}}$. In addition, let $T_i \coloneqq T_i' + T_i''$ and $T \coloneqq \bigoplus_{i=1}^5 T_i \subset W_{\mathcal{D}}$. Note that by construction, x satisfies (4.2) if and only if $T = W_{\mathcal{D}}$. Next, we show that T satisfies the conditions of Proposition 2.37. Since T contains the images of all a_U-operators, it is left to show that T is invariant under all A_U, $B_U^{\pm}$, C_V, D_V and that each A_U induces $\mathbb{C}$-linear isomorphisms $W_{U^+}/T_{U^+} \xrightarrow{\sim} W_{U^-}/T_{U^-}$. By definition, we have

$$A_1(T_2) = T_1, \quad A_2(T_4) = T_3, \quad A_3(T_5) = T_4. \quad (4.3)$$

Therefore, T is invariant under all A_U. As all A_U are vector space isomorphisms, (4.3) also yields that each A_U induces a vector space isomorphism $W_{U^+}/T_{U^+} \xrightarrow{\sim} W_{U^-}/T_{U^-}$. By (2.12), $B_3^-A_3 = -a_3b_3$ and $B_1^+ = A_1^{-1}a_1b_1$. Therefore, we conclude

$$\mathrm{im}(B_3^-) = \mathrm{im}(B_3^-A_3) \subset \mathrm{im}(a_3) \subset T_4', \quad \mathrm{im}(B_1^+) \subset \mathrm{Im}(A_1^{-1}a_1) = T_2'.$$

Applying again (2.12) gives $B_2^- A_2 = A_2 B_3^- - a_2 b_2$ which implies

$$\mathrm{im}(B_3^-) = \mathrm{im}(B_3^- A_3) \subset \mathrm{im}(a_2) + \mathrm{im}(A_3 B_3^-) \subset T_3'.$$

Thus, we proved that T is invariant under the operators B_1^+, B_2^-, B_3^-. By definition, $C_2(T_3') \subset T_2''$ and $D_2(T_2') \subset T_3''$. From (2.33) follows that $B_1^+ = -C_2 D_2$ and $B_2^- = -D_2 C_2$. Hence, we have $C_2(T_3'') \subset \mathrm{im}(B_1^+) \subset T_2'$ and $D_2(T_2'') \subset \mathrm{im}(B_2^-) \subset T_3'$. This implies that T is invariant under C_2 and D_2 and consequently T satisfies the conditions of Proposition 2.37. Now, if x does not satisfy (4.2) then $T \neq W_{\mathcal{D}}$ and x is not χ-stable by Proposition 2.37. Conversely, if x is not χ-stable, there exists a graded subspace $S = \bigoplus_{i=1}^5 S_i \subset W_{\mathcal{D}}$ satisfying the conditions of Proposition 2.37 with $S \neq W_{\mathcal{D}}$. Since S contains the images of all a_U, the $A_U^{\pm 1}$, $B_U^{\pm}$, C_V, D_V-invariance of S implies $T \subset S$. Consequently $T \neq W_{\mathcal{D}}$ and S does not satisfy (4.2). □

From Claim 4.10, we deduce the following explicit conditions for χ-stability:

Claim 4.11. *A point*

$$x = (A_1, A_2, A_3, B_1^+, B_2^-, B_3^-, C_2, D_2, a_1, a_2, a_3, b_1, b_2, b_3) \in \tilde{m}^{-1}(0)$$

is χ-stable if and only if one of the following five conditions is satisfied:

(cov-1) $a_1 \neq 0$ *and* $\det(a_2\ D_2) \neq 0$,

(cov-2) $a_1 \neq 0$ *and* $\det(A_2 a_3\ D_2) \neq 0$,

(cov-3) $a_1 \neq 0$ *and* $\det(a_2\ A_2 a_3) \neq 0$,

(cov-4) $C_2 a_2 \neq 0$ *and* $\det(a_2\ A_2 a_3) \neq 0$,

(cov-5) $C_2 A_2 a_3 \neq 0$ *and* $\det(a_2\ A_2 a_3) \neq 0$.

Proof. Suppose $a_1 \neq 0$. By Claim 4.10, x is χ-stable if and if and only if one of the following pairs is a basis of $\mathbb{C}^2$:

$$(a_2(1), D_2 A_1^{-1} a_1(1)), \quad (A_2 a_3(1), D_2 A_1^{-1} a_1(1)), \quad (a_2(1), A_2 a_3(1)).$$

Thus, we conclude that x is χ-stable if one of (cov-1), (cov-2) and (cov-3) holds. If $a_1 = 0$ then Claim 4.10 yields that x is χ-stable if and only if $(a_2(1), A_2 a_3(1))$ is a basis of $\mathbb{C}^2$ and $(C_2 a_2(1), C_2 A_2 a_3(1))$ is a generating system for $\mathbb{C}$. Therefore, x is χ-stable if and only if (cov-4) or (cov-5) is satisfied. □

We use Claim 4.11, to cover $\mathcal{C}(\mathcal{D})$ by $\mathbb{T}$-invariant affine open subvarieties

$$\mathcal{C}(\mathcal{D}) = \bigcup_{i=1}^{5} W_i. \tag{4.4}$$

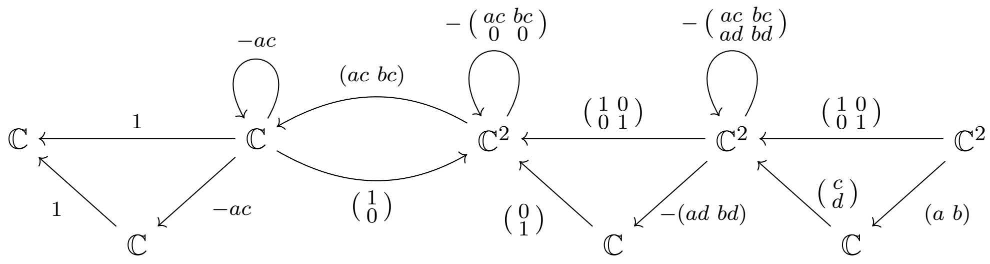
By Claim 4.11, $\tilde{m}^{-1}(0)^{\mathrm{s}}$ is covered by the following open affine $\mathbb{T}$- and $\mathcal{G}$-invariant subvarieties

$$\tilde{m}^{-1}(0)^{\mathrm{s}} = \widetilde{W}_1 \cup \widetilde{W}_2 \cup \widetilde{W}_3 \cup \widetilde{W}_4 \cup \widetilde{W}_5,$$

where $\widetilde{W}_i = \{x \in m^{-1}(0) \mid x \text{ satisfies (cov-i)}\}$, for $i = 1, \dots, 5$. Setting $W_i := \widetilde{W}_i/\mathcal{G} \subset \mathcal{C}(\mathcal{D})$, provides a cover (4.4) of $\mathcal{C}(\mathcal{D})$ by open affine $\mathbb{T}$-invariant subvarieties. Note that $x_{D_i} \in W_i$, for each i.The next claim contains explicit normal forms of the elements in the W_i. Via these normal forms, we deduce that each W_i is isomorphic to $\mathbb{C}^4$.

Claim 4.12. *The parameterization from Figure 4.1 on the next page gives, for any* $i = 1, \dots, 5$, *an isomorphism of varieties* $\eta_i \colon \mathbb{C}^4 \xrightarrow{\sim} W_i$ *with* $\eta_i(0) = x_{D_i}$. *In particular,* (4.4) *is a covering by affine spaces.*

Proof. We only prove the case $i = 1$ since the other cases can be proved analogously. Let $\tilde{\eta}_1 \colon \mathbb{C}^4 \to \tilde{m}^{-1}(0)^{\mathrm{s}}$ be the morphism of varieties which maps a point $(a, b, c, d) \in \mathbb{C}^4$ to the tuple displayed by the diagram:



Let

$$\eta_1 \colon \mathbb{C}^4 \longrightarrow \mathcal{C}(\mathcal{D}) \tag{4.5}$$

be the induced morphism. Clearly, $\mathrm{im}(\eta_1) \subset W_1$. Conversely, given

$$x = [A_1, A_2, A_3, B_1^+, B_2^-, B_3^-, C_2, D_2, a_1, a_2, a_3, b_1, b_2, b_3] \in W_1,$$

we may assume by the defining conditions of W_1 that

$$a_1 = 1, \quad A_1 = 1, \quad A_2 = A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $b_3 = (a\ b)$ and $a_3 = \left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)$. Then, (2.12) implies $B_3^- = -\left(\begin{smallmatrix} ac & bc \\ ad & db \end{smallmatrix}\right)$. Moreover, let $b_2 = (x\ y)$ and $D_2 = \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)$. By (2.12) and the moment map equation (2.33), we deduce

$$-\begin{pmatrix} ac & bc \\ ad + x & db + y \end{pmatrix} = B_2^- = \begin{pmatrix} -z & -w \\ 0 & 0 \end{pmatrix}.$$

Hence, $x = -ad$, $y = -bd$, $z = ac$ and $w = bc$. Finally, (2.12) and (2.33) also give $b_1 = B_1^+ = -z = -ac$. This implies $x \in \mathrm{im}(\eta_1)$ and thus, $\mathrm{im}(\eta_1) = W_1$. To show that η_1 is an isomorphism, it now suffices by Proposition 2.25 to show that η_1 is injective. Assume $\eta_1(a, b, c, d) = \eta_1(a', b', c', d')$, so there exists $g = (g_1, g_2, g_3, g_4, g_5) \in \mathcal{G}$ such that $g.\tilde{\eta}_1(a, b, c, d) = \tilde{\eta}_1(a', b', c', d')$. This directly implies $g_1 = g_2 = 1$ and $g_3 = g_4 = g_5$. In addition, the conditions $g_3\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)g_2^{-1} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $g_3\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ imply that g_3 is the identity matrix. Hence, $(a, b, c, d) = (a', b', c', d')$. Thus, η_1 is injective and therefore an isomorphism. □

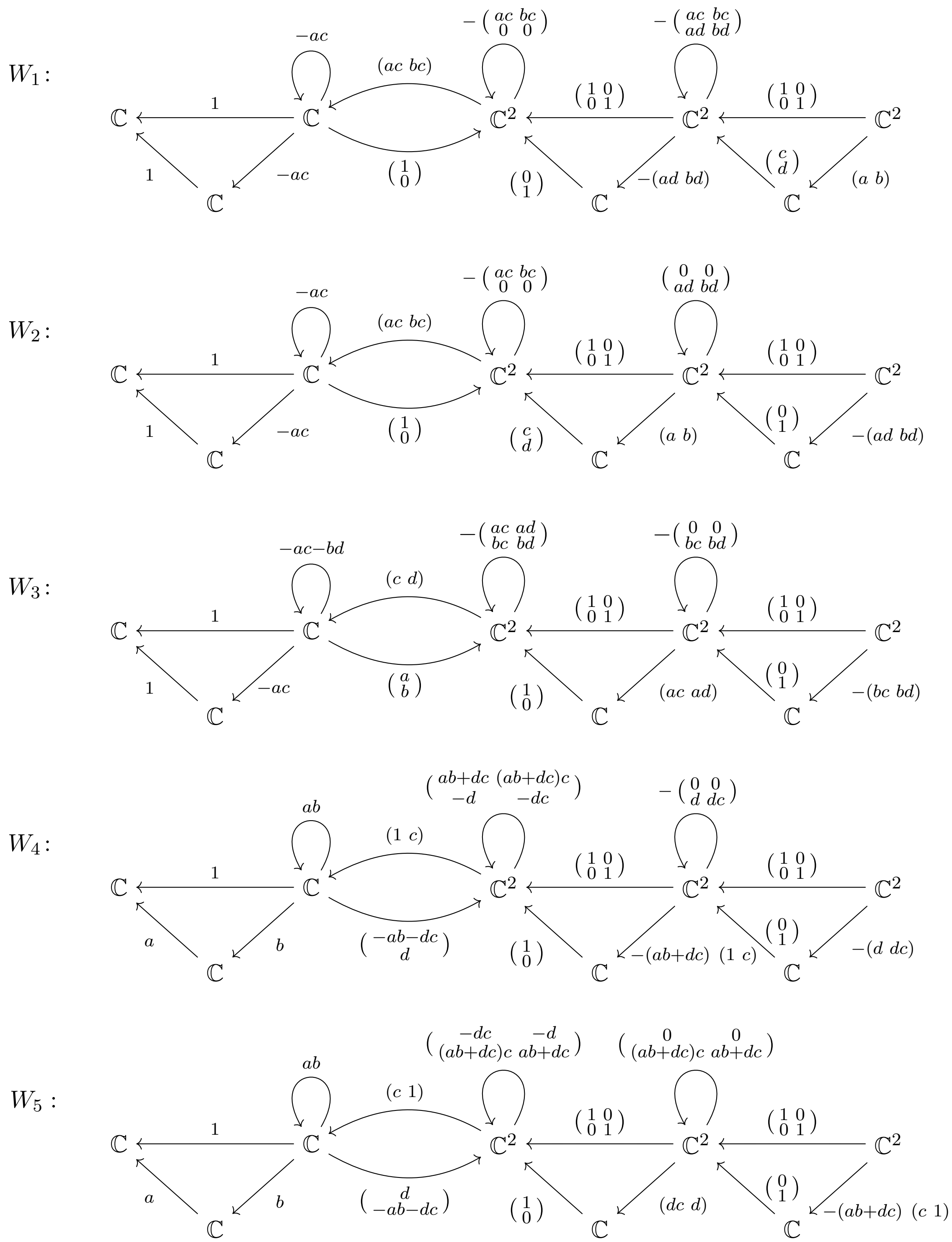
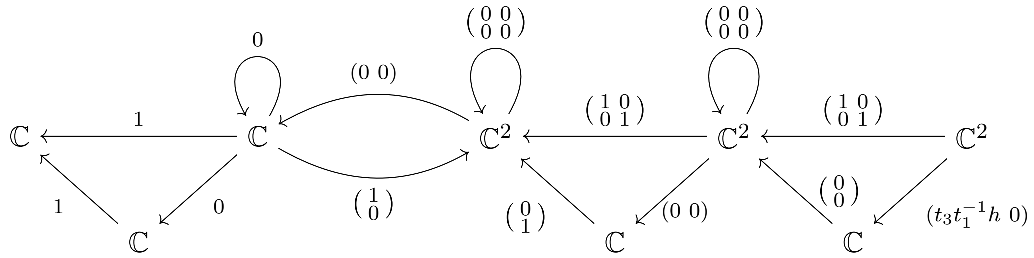


Figure 4.1: Parametrizations of the $\mathbb{T}$-invariant affine open neighborhoods $W_1, \ldots, W_5$ of the $\mathbb{T}$-fixed points $x_{D_1}, \ldots, x_{D_5}$. Here, $(a, b, c, d) \in \mathbb{C}^4$.

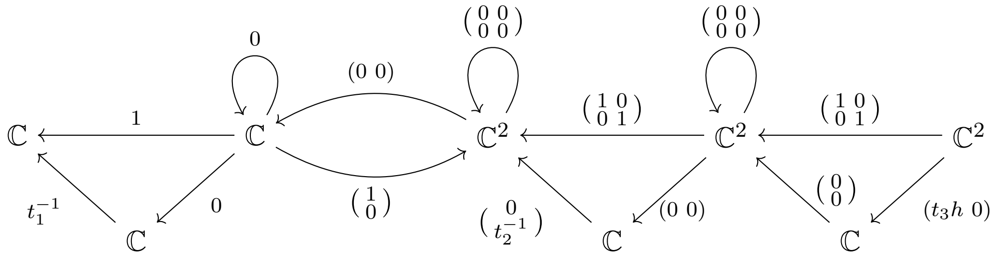
Via the explicit parameterizations of the coordinate charts $W_i \cong \mathbb{C}^4$ from Claim 4.12, we can now determine the induced $\mathbb{T}$-action on $\mathbb{C}^4$. It turns out that the $\mathbb{T}$-action on $\mathbb{C}^4$ is always linear and has the following weight space decomposition:

Claim 4.13. *The $\mathbb{T}$-action on $\mathbb{C}^4$ induced by the isomorphism $\eta_i \colon \mathbb{C}^4 \xrightarrow{\sim} W_i$ is linear and the standrad basis vectors $e_1, \dots, e_4$ are weight vectors. The corresponding weights are recorded in Table 4.1 below.*

Proof. We only show that e_1 is a weight vector of the induced $\mathbb{T}$-action from $\eta_1 \colon \mathbb{C}^4 \xrightarrow{\sim} W_1$. The other statements of the claim can be shown in a similar way. We have to show that for all $(t_1, t_2, t_3, h) \in \mathbb{T}$, we have $(t_1, t_2, t_3, h).\eta_1(e_1) = \eta_1(t_3 t_1^{-1} h \cdot e_1)$. Let $\tilde{\eta}_1 \colon \mathbb{C}^4 \to \tilde{m}^{-1}(0)$ be as in the proof of Claim 4.12. Then, $\tilde{\eta}_1(t_3 t_1^{-1} h \cdot e_1)$ is the tuple associated to the diagram:



Likewise, $(t_1, t_2, t_3, h).\tilde{\eta}_1(e_1)$ corresponds to



Thus, we have $g.((t_1, t_2, t_3, h).\tilde{\eta}_1(e_1)) = \tilde{\eta}_1(t_3 t_1^{-1} h \cdot e_1)$, where

$$g = (g_1, g_2, g_3, g_4, g_5), \quad g_1 = g_2 = t_1, \quad g_3 = g_4 = g_5 = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}.$$

Hence, we indeed have $(t_1, t_2, t_3, h).\eta_1(e_1) = \eta_1(t_3 t_1^{-1} h \cdot e_1)$. □

Open affine \ Standard basis vector	e_1	e_2	e_3	e_4
W_1	$t_3 - t_1 + h$	$t_3 - t_2 + h$	$t_1 - t_3$	$t_2 - t_3$
W_2	$t_2 - t_1 + h$	$t_2 - t_3 + h$	$t_1 - t_2$	$t_3 - t_2$
W_3	$t_2 - t_1$	$t_3 - t_1$	$t_1 - t_2 + h$	$t_2 - t_3 + h$
W_4	$t_2 - t_1 - h$	$t_1 - t_2 + 2h$	$t_2 - t_3$	$t_3 - t_2 + h$
W_5	$t_3 - t_1 - h$	$t_1 - t_3 + 2h$	$t_3 - t_2$	$t_2 - t_3 + h$

Table 4.1: $\mathbb{T}$-weight space decomposition of $W_1, \dots, W_5$.

The affine covering (4.4) with its $\mathbb{T}$-weight space decomposition from Claim 4.13 lets us now get a hand on the attracting cells:

Claim 4.14. *For a given generic cocharacter $\sigma\colon \mathbb{C}^* \to \mathbb{A}$, we have*

$$\operatorname{Attr}_\sigma(x_{D_i}) = W_{i,\sigma}^+, \quad \text{for i=1,\dots,5.}$$

Here, $W_{i,\sigma}^+$ is the subvector space of W_i generated by all positive weight spaces of W_i with respect to σ.

Proof. By construction, $W_{i,\sigma}^+ = W_i \cap \operatorname{Attr}_\sigma(x_{D_i})$ is an open and $\mathbb{T}$-invariant subvariety of $\operatorname{Attr}_\sigma(x_{D_i})$. Since $W_{i,\sigma}^+$ contains x_{D_i}, Proposition 4.4 and Lemma 4.7 imply $W_{i,\sigma}^+ = \operatorname{Attr}_\sigma(x_{D_i})$. ☐

Using Claim 4.14, we can now easily determine the attracting cells via Table 4.1. Take for instance the cocharacter $\sigma_0(t) = (t, t^2, t^3)$. Then, by Claim 4.14, the attracting cell $\operatorname{Attr}_{\sigma_0}(x_{D_1})$ is equal to the subspace of W_1 generated by all weight space which are positive with respect to σ_0. By Table 4.1, we have

$$W_1 \cong \mathbb{C}_{t_3-t_1+h} \oplus \mathbb{C}_{t_3-t_2+h} \oplus \mathbb{C}_{t_1-t_3} \oplus \mathbb{C}_{t_2-t_3}.$$

Pairing these characters with σ_0 gives

$$\langle \sigma_0, t_3 - t_1 + h\rangle = 2, \quad \langle \sigma_0, t_3 - t_2 + h\rangle = 1, \quad \langle \sigma_0, t_1 - t_3\rangle = -2, \quad \langle \sigma_0, t_1 - t_2\rangle = -1.$$

Thus, $\operatorname{Attr}_{\sigma_0}(x_{D_1}) = \mathbb{C}_{t_3-t_1+h} \oplus \mathbb{C}_{t_3-t_2+h}$. The remaining attracting cells $\operatorname{Attr}_{\sigma_0}(x_{D_i})$ can be determined in the same way:

$$\begin{aligned} \operatorname{Attr}_{\sigma_0}(x_{D_2}) &= \mathbb{C}_{t_2-t_1+h} \oplus \mathbb{C}_{t_3-t_2}, & \operatorname{Attr}_{\sigma_0}(x_{D_3}) &= \mathbb{C}_{t_2-t_1} \oplus \mathbb{C}_{t_3-t_1}, \\ \operatorname{Attr}_{\sigma_0}(x_{D_4}) &= \mathbb{C}_{t_2-t_1-h} \oplus \mathbb{C}_{t_3-t_2+h}, & \operatorname{Attr}_{\sigma_0}(x_{D_5}) &= \mathbb{C}_{t_3-t_1-h} \oplus \mathbb{C}_{t_3-t_2}. \end{aligned}$$

We leave it as an exercise to the reader to consider other choices of cocharacters and to determine the respective attracting cells.

We return now to the general framework and will see that the attracting cells are in fact constant along certain chambers inside the space of cocharacters.

4.3 Independence of choice of chamber

Let Λ be the cocharacter lattice of $\mathbb{A}$ and consider the vector space $\Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$. For $1 \leq i, j \leq N$ with $i \neq j$, we define the following hyperplanes:

$$H_{i,j} := \{(t_1, \dots, t_N) \mid t_i = t_j\} \subset \Lambda_\mathbb{R}.$$

The connected components of

$$\Lambda_\mathbb{R} \setminus \Big(\bigcup_{\substack{1\leq i,j\leq N \\ i\neq j}} H_{i,j} \Big)$$

are called *chambers*. There is a (well-known from Lie theory) one-to-one correspondence

$$\{\text{Chambers}\} \xleftrightarrow{1:1} S_N,$$

where we assign to a permutation $\pi \in S_N$ the connected component

$$\mathfrak{C}_\pi \coloneqq \{(t_1, \ldots, t_N) \mid t_{\pi(1)} < t_{\pi(2)} < \ldots < t_{\pi(N)}\}.$$

The chamber $\mathfrak{C}_{\mathrm{id}}$ is called the *antidomonant chamber* and denoted by $\mathfrak{C}_-$. The *dominant chamber* is defined as $\mathfrak{C}_+ \coloneqq -\mathfrak{C}_-$.

Remark. This correspondence allows to connect the chambers with the combinatorics of the symmetric group. Moreover note the parallel to the more Lie theoretic description of attracting cells, a.k.a. Schubert cells, of Grassmannians, see [GKS20]. For readers new to the subject it might be helpful to keep for the following this analogous framework in mind.

We have the following independence result for attracting cells:

Proposition 4.15 (Invariance of chambers)**.** *Let $\mathfrak{C}$ be a chamber and σ, $\tau \in \Lambda \cap \mathfrak{C}$. Then, we have*

$$T_p\mathcal{C}(\mathcal{D})^+_\sigma = T_p\mathcal{C}(\mathcal{D})^+_\tau, \quad T_p\mathcal{C}(\mathcal{D})^-_\sigma = T_p\mathcal{C}(\mathcal{D})^-_\tau. \tag{4.6}$$

Moreover, attracting cells are constant along chamber, i.e. for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have

$$\mathrm{Attr}_\sigma(p) = \mathrm{Attr}_\tau(p). \tag{4.7}$$

Proof. Let $\pi \in S_N$ such that $\mathfrak{C}_\pi = \mathfrak{C}$ and fix a $\mathbb{T}$-fixed point p. Recall from Corollary 3.24 that the $\mathbb{A}$-weights of $T_p\mathcal{C}(\mathcal{D})$ are of the form $t_i - t_j$, where $i \neq j$. It follows

$$T_p\mathcal{C}(\mathcal{D})^+_\sigma = \bigoplus_{\substack{1\leq i,j\leq n \\ \pi^{-1}(i)>\pi^{-1}(j)}} T_p\mathcal{C}(\mathcal{D})_{t_i-t_j} = T_p\mathcal{C}(\mathcal{D})^+_\tau$$

and

$$T_p\mathcal{C}(\mathcal{D})^-_\sigma = \bigoplus_{\substack{1\leq i,j\leq n \\ \pi^{-1}(i)<\pi^{-1}(j)}} T_p\mathcal{C}(\mathcal{D})_{t_i-t_j} = T_p\mathcal{C}(\mathcal{D})^-_\tau.$$

Thus, we proved (4.6). The equality (4.7) follows directly from (4.6) and Proposition 4.4. □

By Proposition 4.15, the $T_p\mathcal{C}(\mathcal{D})^+_\sigma$, $T_p\mathcal{C}(\mathcal{D})^-_\sigma$ and $\mathrm{Attr}_\sigma(p)$ only depend on the chamber $\mathfrak{C}$ containing σ. Thus, we also denote them respectively by $T_p\mathcal{C}(\mathcal{D})^+_\mathfrak{C}$, $T_p\mathcal{C}(\mathcal{D})^-_\mathfrak{C}$ and $\mathrm{Attr}_\mathfrak{C}(p)$.

Remark. In [MO19], Maulik and Okounkov defined chambers in a slightly different way. They defined them as connected components of the complement of the union of all hyperplanes orthogonal to the $\mathbb{A}$-tangent weights of $\mathbb{A}$-fixed points. Corollary 3.24 implies that the chambers defined in this subsections refine the chambers in the sense of [MO19]. The inclusion may be strict as for instance the bow variety $\mathcal{C}(0/1/3\backslash 1\backslash 0)$ is just a single point. Hence, there exists only a single chamber in the sense of Maulik and Okounkov whereas the chambers defined in this subsection are in one-to-one correspondence with the elements of the symmetric group on two letters.

4.4 Partial order by attraction

Given a chamber $\mathfrak{C}$, we define a preorder $\preceq_{\mathfrak{C}}$ on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$ as the transitive closure of the relation

$$p \in \overline{\mathrm{Attr}_{\mathfrak{C}}(q)} \quad \Rightarrow \quad p \preceq_{\mathfrak{C}} q, \tag{4.8}$$

where $\overline{\mathrm{Attr}_{\mathfrak{C}}(q)}$ denotes the Zariski closure of $\mathrm{Attr}_{\mathfrak{C}}(q)$ in $\mathcal{C}(\mathcal{D})$.

As we usually work with a fixed choice of chamber, we denote $\preceq_{\mathfrak{C}}$ also just by $\preceq$.

Lemma 4.16 (Fixed point ordering)**.** *The preorder $\preceq$ is a partial order on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$.*

Proof. Evidently, the preorder $\preceq$ is reflexive and transitive. Hence, it is left to show that $\preceq$ is antisymmetric. Let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $p \preceq q$ and $q \preceq p$. For the sake of contradiction, assume $p \neq q$. Given $\sigma \in \mathfrak{C}$, we can choose by Proposition 4.1 a smooth σ-equivariant compactification $\mathcal{C}(\mathcal{D}) \hookrightarrow X$. Let $F_1, \dots, F_r$ be the $\mathbb{C}^*$-fixed components of X and $X_{F_i}^+, \dots, X_{F_i}^+$ the corresponding attracting cells. By Lemma 4.17 below, we can order the fixed points components in such a way such that the subsets

$$Y_i := \bigsqcup_{1 \leq j \leq i} X_{F_j}^+ \subset X$$

are closed subvarieties of X. By the Generic Cocharacter Theorem, we have $F_i = \{p\}$ and $F_j = \{q\}$ for some i, j. Without loss of generality, we may assume $i < j$. In particular, we have $q \notin Y_i$. Note that $Y_i \cap \mathcal{C}(\mathcal{D})$ is Zariski closed in $\mathcal{C}(\mathcal{D})$ and stable under attraction, i.e. if $x \in \mathcal{C}(\mathcal{D})$ and $\lim_{t\to 0} \sigma(t).x$ is contained in $Y_i \cap \mathcal{C}(\mathcal{D})$ then $x \in Y_i \cap \mathcal{C}(\mathcal{D})$. Therefore, if $p' \preceq p$ for some $p' \in \mathcal{C}(\mathcal{D})$, we conclude $p' \in Y_i$. Thus, we must have $q \in Y_i$ which contradicts $i < j$. Thus, $p = q$ and hence, $\preceq$ is antisymmetric. $\square$

Lemma 4.17. *Let V be a finite dimensional $T = \mathbb{C}^*$-representation with weight space decomposition $V = \bigoplus_{m \in \mathbb{Z}} V_m$. Let $X \hookrightarrow \mathbb{P}(V)$ be a smooth and T-invariant closed subvariety. We denote the irreducible components of X^T by $F_1, \dots, F_r$ and the corresponding attracting cells by $X_{F_1}^+, \dots, X_{F_r}^+$. Define a function*

$$\mathrm{wt}\colon \{F_1, \dots, F_r\} \longrightarrow \mathbb{Z},$$

where $\mathrm{wt}(F_i)$ is the unique integer such that $F_i \subset \mathbb{P}(V_{\mathrm{wt}(F_i)})$. Then,

$$Y_F := X_F^+ \sqcup \Big(\bigsqcup_{\mathrm{wt}(F') > \mathrm{wt}(F)} X_{F'}^+ \Big)$$

is a closed subvariety of X, for all $F = F_i$ and $i = 1, \dots, r$.

Proof. Suppose $\mathrm{wt}(F) = m$. Let

$$\mathrm{pr}\colon \mathbb{P}(V^{\geq m}) \setminus \mathbb{P}(V^{>m}) \to \mathbb{P}(V_m)$$

be the linear projection. Then, $X_F^+ = X \cap \mathrm{pr}^{-1}(F)$ and thus, X_F^+ is a closed subvariety of $X \cap (\mathbb{P}(V^{\geq m}) \setminus \mathbb{P}(V^{>m}))$. Hence, $X_F^+ \cup (X \cap \mathbb{P}(V^{>m}))$ is a closed subvariety of X. From

$$X \cap \mathbb{P}(V^{>m}) = \bigsqcup_{\mathrm{wt}(F') > m} X_{F'}^+$$

follows that $Y_F = X_F^+ \cup (X \cap \mathbb{P}(V^{>m}))$ and thus, Y_F is a closed subvariety of X. $\square$

Partial order by attraction in a concrete example

Let $\mathcal{D}$ be the brane diagram from (4.1) and $\mathcal{C}(\mathcal{D})$ the corresponding bow variety. Recall the cover of $\mathcal{C}(\mathcal{D})$ by $\mathbb{T}$-invariant affine opens from (4.4). Again, we choose the generic cocharacter $\sigma_0 = (t, t^2, t^3)$. To characterize the corresponding partial order, we first compute all intersections $\overline{\mathrm{Attr}_{\sigma_0}(x_{D_i})} \cap W_j$.

Claim 4.18. *For i, $j \in \{1, \dots, 5\}$, the intersection $\overline{\mathrm{Attr}_{\sigma_0}(x_{D_i})} \cap W_j$ is a $\mathbb{T}$-invariant linear subspace of W_j whose weight space decomposition is recorded in Table 4.2 below.*

Proof. We only prove the case $i = 2$ and $j = 1$ since all other cases are similar. From Claim 4.12 we know $\mathrm{Attr}_{\sigma_0}(x_{D_i}) = \{\eta_2(a, 0, 0, d) \mid a, d \in \mathbb{C}\}$. By Claim 4.11, we have that $\eta_2(a, 0, 0, d) \in W_1$ if and only if $d \neq 0$. A direct computation shows

$$\left(1, 1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & d^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & d^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & d^{-1} \end{smallmatrix}\right)\right).\tilde{\eta}_2(a, 0, 0, d) = \tilde{\eta}_1(-a, 0, 0, d^{-1}).$$

Therefore, we have $\overline{\mathrm{Attr}_{\sigma_0}(x_{D_2})} \cap W_1 = \{\eta_1(a, 0, 0, d) \mid a, d \in \mathbb{C}\}$. Finally, Claim 4.13 yields that $\{\eta_1(a, 0, 0, d) \mid a, d \in \mathbb{C}\} = \mathbb{C}_{t_3 - t_1 + h} \oplus \mathbb{C}_{t_2 - t_3}$. □

$i \backslash j$	1	2	3	4	5
1	$\mathbb{C}_{t_3-t_1+h} \oplus \mathbb{C}_{t_3-t_2+h}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\mathbb{C}_{t_3-t_1+h} \oplus \mathbb{C}_{t_2-t_3}$	$\mathbb{C}_{t_2-t_1+h} \oplus \mathbb{C}_{t_3-t_2}$	$\emptyset$	$\emptyset$	$\emptyset$
3	$\mathbb{C}_{t_1-t_3} \oplus \mathbb{C}_{t_2-t_3}$	$\mathbb{C}_{t_1-t_2} \oplus \mathbb{C}_{t_3-t_2}$	$\mathbb{C}_{t_2-t_1} \oplus \mathbb{C}_{t_3-t_1}$	$\emptyset$	$\emptyset$
4	$\mathbb{C}_{t_3-t_2+h} \oplus \mathbb{C}_{t_1-t_3}$	$\emptyset$	$\mathbb{C}_{t_3-t_1} \oplus \mathbb{C}_{t_1-t_2+h}$	$\mathbb{C}_{t_2-t_1-h} \oplus \mathbb{C}_{t_3-t_2+h}$	$\emptyset$
5	$\emptyset$	$\emptyset$	$\mathbb{C}_{t_1-t_2+h} \oplus \mathbb{C}_{t_1-t_3+h}$	$\mathbb{C}_{t_2-t_3} \oplus \mathbb{C}_{t_2-t_1-h}$	$\mathbb{C}_{t_3-t_1-h} \oplus \mathbb{C}_{t_3-t_2}$

Table 4.2: Intersections $\overline{\mathrm{Attr}_{\sigma_0}(x_{D_i})} \cap W_j$ as subspaces of W_j.

Our computations yield that the partial order corresponding to σ_0 is given as follows:

Claim 4.19. *We have an isomorphism of partially ordered sets*

$$(\{1, 2, 3, 4, 5\}, \leq) \xrightarrow{\sim} (\mathcal{C}(\mathcal{D})^{\mathbb{T}}, \preceq), \quad i \mapsto x_{D_i},$$

where $\leq$ is the usual ordering on $\{1, 2, 3, 4, 5\}$.

Proof. We only prove that $x_{D_2} \preceq x_{D_4}$ as all other cases can be shown in a similar way. By Claim 4.18, $\overline{\mathrm{Attr}_{\sigma_0}(x_{D_4})}$ contains x_{D_3}. Thus, $x_{D_3} \preceq x_{D_4}$. Likewise, Claim 4.18 gives that $\overline{\mathrm{Attr}_{\sigma_0}(x_{D_3})}$ contains x_{D_2} and therefore $x_{D_2} \preceq x_{D_3}$. Hence, we also have $x_{D_2} \preceq x_{D_4}$. □

Full attraction cells

Lemma 4.17 implies that for a smooth projective variety with a one-parameter torus action, we can order the attracting cells such that the successive unions of the attracting cells are all Zariski closed. Motivated by this general result, we prove an analogous statement for bow varieties.

Let $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. The *full attracting cell of* p *with respect to the chamber* $\mathfrak{C}$ is defined as

$$\operatorname{Attr}^f_{\mathfrak{C}}(p) := \bigsqcup_{q \preceq p} \operatorname{Attr}_{\mathfrak{C}}(q).$$

We then have the following result:

Proposition 4.20. *The full attracting cell* $\operatorname{Attr}^f_{\mathfrak{C}}(p)$ *is Zariski closed in* $\mathcal{C}(\mathcal{D})$.

For the proof, we use the following basic property of proper morphisms:

Lemma 4.21. *Let* X, X' *be algebraic varieties with* $T = \mathbb{C}^*$*-actions* ρ, ρ' *and let* $f\colon X \to X'$ *be a proper* T*-equivariant morphism. Then, we have*

$$\{x \in X \mid \lim_{t\to 0} \rho(t).x \textit{ exists in } X\} = f^{-1}(\{x' \in X' \mid \lim_{t\to 0} \rho'(t).x' \textit{ exists in } X'\}).$$

Proof. The inclusion $\subset$ is clear. For the converse inclusion, let $x \in X$ such that $\lim_{t\to 0} t.f(x)$ exists in X'. Let $\mathcal{O}_{\mathbb{C},0}$ be the stalk of the structure sheaf of $\mathbb{C}$ at the origin and K be the function field of $\mathbb{C}$. We denote by $a_x\colon T \to X$, $t \mapsto t.x$ the orbit morphism. By assumption, we can extend the composition $f \circ h$ to a morphism $a_x'\colon \mathbb{C} \to X'$. Thus, we have a commuting diagram:

$$\begin{array}{ccccc} \operatorname{Spec}(K) & \xrightarrow{\kappa} & T & \xrightarrow{a_x} & X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(\mathcal{O}_{\mathbb{C},0}) & \xrightarrow{\iota_0} & \mathbb{C} & \xrightarrow{a_x'} & X' \end{array}$$

Here, κ, ι, ι_0 are the obvious morphisms. By the valuative criterion for properness, see e.g. [Har77, Theorem II.4.7], there exists a morphism $g\colon \operatorname{Spec}(\mathcal{O}_{\mathbb{C},0}) \to X$ such that $f \circ g = a_x' \circ \iota_0$ and $g \circ j = a_x \circ \kappa$. Since X is of finite type over $\mathbb{C}$, there exists an open subvariety $U \subset \mathbb{C}$ and a morphism $g'\colon U \to X$ such that $g' \circ \iota_0 = g$. Since $g \circ j = a_x \circ \kappa$, we conclude that the restriction of g' to $U \cap T$ equals the orbit morphism a_x. Hence, we can extend a_x to $\mathbb{C}$ which gives that the limit $\lim_{t\to 0} t.x$ exists in X. □

Proof of Proposition 4.20. To prove that $\operatorname{Attr}^f_{\mathfrak{C}}$ is closed, we show

$$\operatorname{Attr}^f_{\mathfrak{C}}(p) = \bigcup_{q \preceq p} \overline{\operatorname{Attr}_{\mathfrak{C}}(q)}. \tag{4.9}$$

The inclusion $\subset$ is clear. For the converse, let $q \in \mathcal{C}(\mathcal{D})$ with $q \preceq p$ and let $x \in \overline{\operatorname{Attr}_{\mathfrak{C}}(q)}$. In addition, we fix $\sigma \in \mathfrak{C}$. By Proposition 4.2, there exists a proper σ-equivariant morphism $\pi\colon \mathcal{C}(\mathcal{D}) \to V$ to a finite dimensional $\mathbb{C}^*$-representation V. Since $\pi(\operatorname{Attr}_{\mathfrak{C}}(q)) \subset V^{\geq 0}$, we have $\overline{\operatorname{Attr}_{\mathfrak{C}}(q)} \subset \pi^{-1}(V^{\geq 0})$. Hence, by Lemma 4.21, the limit $\lim_{t\to 0} \sigma(t).x$ exists in $\mathcal{C}(\mathcal{D})$. Let $q' := \lim_{t\to 0} \sigma(t).x$. By the Generic Cocharacter Theorem, we have $q' \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. As $\operatorname{Attr}_{\mathfrak{C}}(q)$ is σ-invariant, $\overline{\operatorname{Attr}_{\mathfrak{C}}(q)}$ is also σ-invariant and hence $\sigma(\mathbb{C}^*).x \subset \overline{\operatorname{Attr}_{\mathfrak{C}}(q)}$. Therefore, also the limit point q' is contained in $\overline{\operatorname{Attr}_{\mathfrak{C}}(q)}$ which gives $q' \preceq q$. Hence, we deduce $x \in \operatorname{Attr}^f_{\mathfrak{C}}(p)$ which completes the proof. □

4.5 Opposite attracting cells

We define opposite chambers and attracting cells in analogy to the respective notions in Schubert calculus:

Definition 4.22. The *opposite chamber* of $\mathfrak{C}$ is defined as

$$\mathfrak{C}^{\mathrm{op}} := \{a \in \mathfrak{a}_{\mathbb{R}} \mid -a \in \mathfrak{C}\}.$$

For $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we call $\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(p)$ the *opposite attracting cell* of $\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(p)$.

Note that $\sigma \in \mathfrak{C}$ if and only if $\sigma^{-1} \in \mathfrak{C}^{\mathrm{op}}$. The next theorem states that the partial order $\preceq_{\mathfrak{C}^{\mathrm{op}}}$ is in fact opposite to the partial order of $\preceq_{\mathfrak{C}}$:

Theorem 4.23 (Opposite order)**.** *Let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. Then, $p \preceq_{\mathfrak{C}} q$ if and only if $q \preceq_{\mathfrak{C}^{\mathrm{op}}} p$.*

We prove Theorem 4.23 using an analytic limit argument and properness properties of intersections of opposite attracting cells which we consider in the following subsection.

Properness of intersections of opposite cells

In general, the closure of the attracting cells corresponding to $\mathfrak{C}$ or to $\mathfrak{C}^{\mathrm{op}}$ need not be proper. However, the next theorem gives that their intersection is always proper:

Theorem 4.24 (Properness)**.** *For p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, the intersection $\overline{\mathrm{Attr}_{\mathfrak{C}}(p)} \cap \overline{\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(q)}$ is a proper variety over $\mathbb{C}$.*

We immediately conclude, using (4.9), the analogous result for full attracting cells:

Corollary 4.25. *Let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. Then, $\mathrm{Attr}^f_{\mathfrak{C}}(p) \cap \mathrm{Attr}^f_{\mathfrak{C}^{\mathrm{op}}}(q)$ is proper over $\mathbb{C}$.*

For the proof of Theorem 4.24, we set up some notation: Pick a cocharacter $\sigma \in \mathfrak{C}$ and, as in the proof of Proposition 4.20, a proper σ-equivariant morphism $\pi\colon \mathcal{C}(\mathcal{D}) \to V$ to a finite dimensional $\mathbb{C}^*$-representation V. Let $\mathrm{pr}_0\colon V \to V^0$ be the linear projection corresponding to the direct sum decomposition $V = V^- \oplus V^0 \oplus V^+$ and set $\bar{\pi} := \mathrm{pr}_0 \circ \pi$. Note that we have $\bar{\pi}(p) = \pi(p)$, for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. We first establish a technical tool:

Lemma 4.26. *Let $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, $v := \pi(p) \in V^0$ and $\mathfrak{C}' \in \{\mathfrak{C}, \mathfrak{C}^{\mathrm{op}}\}$. If $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}} \cap \overline{\mathrm{Attr}_{\mathfrak{C}'}(p)}$ then $\overline{\mathrm{Attr}_{\mathfrak{C}'}(q)} \subset \bar{\pi}^{-1}(v)$.*

Proof. We only prove the case $\mathfrak{C}' = \mathfrak{C}$ as the proof for $\mathfrak{C}' = \mathfrak{C}^{\mathrm{op}}$ is analogous. Since $\bar{\pi}$ is σ-equivariant, we have $\mathrm{Attr}_{\mathfrak{C}}(p) \subset \bar{\pi}^{-1}(v)$ and hence $\overline{\mathrm{Attr}_{\mathfrak{C}}(p)} \subset \bar{\pi}^{-1}(v)$. Thus, $\pi(q) = v$. Using again that $\bar{\pi}$ is σ-equivriant, we conclude $\overline{\mathrm{Attr}_{\mathfrak{C}}(q)} \subset \bar{\pi}^{-1}(v)$. □

Proof of Theorem 4.24. As above, we set $v := \pi(p)$. If $\pi(q) \neq v$ then Lemma 4.26 implies

$$\overline{\mathrm{Attr}_{\mathfrak{C}}(p)} \cap \overline{\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(q)} \subset \bar{\pi}^{-1}(v) \cap \bar{\pi}^{-1}(\pi(q)) = \emptyset.$$

So let us assume $\pi(q) = v$. Since $\overline{\mathrm{Attr}_{\mathfrak{C}}(p)} \subset \pi^{-1}(V^{\geq 0})$ and $\overline{\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(q)} \subset \pi^{-1}(V^{\leq 0})$ we conclude $\overline{\mathrm{Attr}_{\mathfrak{C}}(p)} \cap \overline{\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(q)} \subset \pi^{-1}(V^0)$. Applying Lemma 4.26 gives

$$\overline{\mathrm{Attr}_{\mathfrak{C}}(p)} \cap \overline{\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(q)} \subset \pi^{-1}(v).$$

Since π is proper, we know that the scheme theoretic fiber $\pi^{-1}(v)$ is proper over $\mathbb{C}$. As $\overline{\mathrm{Attr}_{\mathfrak{C}}(p)} \cap \overline{\mathrm{Attr}_{\mathfrak{C}^{\mathrm{op}}}(q)}$ is a closed subvariety of $\pi^{-1}(v)$, it is also proper over $\mathbb{C}$. $\square$

Example 4.27. Consider again the bow variety $\mathcal{C}(\mathcal{D})$ and let σ_0 be the cocharacter (t, t^2, t^3). By Claim 4.18, the intersection $\overline{\mathrm{Attr}_{\sigma_0}(x_{D_2})} \cap \overline{\mathrm{Attr}_{\sigma_0^{-1}}(x_{D_1})}$ is isomorphic to the complex projective line $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$; an isomorphism is given by $\infty \mapsto x_{D_2}$, $x \mapsto \eta_1(0,0,0,x)$, for $x \in \mathbb{C}$. Here, $\eta_1 \colon \mathbb{C}^4 \xrightarrow{\sim} W_1$ is the isomorphism of varieties from Claim 4.12.

Proof of Theorem 4.23

We fix a similar setup as in the proof of Proposition 4.4: Let $\sigma \in \mathfrak{C}$ and we choose a locally closed $\mathbb{T}$-equivariant immersion $\iota \colon \mathcal{C}(\mathcal{D}) \hookrightarrow \mathbb{P}(V)$, where V is a finite dimensional $\mathbb{T}$-representation. For $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we denote by X_p the Zariski closure of $\iota(\mathrm{Attr}_\sigma(p))$ in $\mathbb{P}(V)$. Thus, X_p is a closed $\mathbb{T}$-invariant subvariety of $\mathbb{P}(V)$ that contains $\iota(\mathrm{Attr}_\sigma(p))$ as open dense $\mathbb{T}$-invariant subvariety.

In this and the subsequent subsection, we usually view the varieties $\mathcal{C}(\mathcal{D})$, X_p and $\mathbb{P}(V)$ as $\mathbb{C}^*$-varieties via the generic cocharacter σ. Also, we just view V as $\mathbb{C}^*$-representation. As V is finite dimensional, we have the usual weight space decomposition

$$V = \bigoplus_{a \in \mathbb{Z}} V_a, \quad \text{where } V_a = \{v \in V \mid t.v = t^a v, \text{ for all } t \in \mathbb{C}^*\}.$$

We denote the dimension of V_a by n_a. The $\mathbb{C}^*$-fixed point locus of $\mathbb{P}(V)$ is given as

$$\mathbb{P}(V)^{\mathbb{C}^*} = \{[v] \mid v \in V_a \setminus \{0\}, \text{ for some } a \in \mathbb{Z}\}.$$

Given $a \in \mathbb{Z}$ and $v \in V_a \setminus \{0\}$, the attracting cell of $[v]$ in $\mathbb{P}(V)$ equals

$$\{x \in \mathbb{P}(V) \mid \lim_{t \to 0} t.x = [v]\} = \Big\{[v + w] \mid w \in \bigoplus_{a' > a} V_{a'}\Big\}. \tag{4.10}$$

Its Zariski closure in $\mathbb{P}(V)$ is the projective subspace $\mathbb{P}(\langle v \rangle \oplus \bigoplus_{a'>a} V_{a'})$.

For each $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, there exists a weight vector v_p such that $\iota(p) = [v_p]$. Let $a_p \in \mathbb{Z}$ be the weight of v_p. Suppose $p \in \overline{\mathrm{Attr}_\sigma(q)}$, for some $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. Then, (4.10) implies $a_q \leq a_p$ and we have equality if and only if $p = q$.

The following lemma will be crucial in the proof of Theorem 4.23:

Lemma 4.28. *Let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $p \in \overline{\mathrm{Attr}_\sigma(q)}$ and $p \neq q$. Then, there exists $p' \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ and $z' \in \mathcal{C}(\mathcal{D})$ such that*

(i) p', $z' \in \overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)}$,

(ii) $a_q \leq a_{p'} < a_p$,

(iii) $\lim_{t \to 0} t.z' = p'$ *and* $\lim_{t \to \infty} t.z' = p$.

Proof of Theorem 4.23. Assuming Lemma 4.28, let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ be distinct with $p \preceq_{\mathfrak{C}} q$. Thus, by definition of $\preceq_{\mathfrak{C}}$, there exists pairwise distinct $q_1, \dots, q_r \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $q_1 = q$, $q_r = p$ and $q_{i+1} \in \overline{\mathrm{Attr}_\sigma(q_i)}$, for all i. In order to show $q \preceq_{\mathfrak{C}^{\mathrm{op}}} p$, we prove that $q_i \in \overline{\mathrm{Attr}_{\sigma^{-1}}(q_{i+1})}$, for all i. For given i, there exists, by Lemma 4.28, a sequence $p_{i,n}$ in $\overline{\mathrm{Attr}_\sigma(q)} \cap \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ such that

(a) $p_{i,n} \in \overline{\operatorname{Attr}_\sigma(q_i)} \cap \overline{\operatorname{Attr}_{\sigma^{-1}}(q_{i+1})}$, for all n and

(b) $a_{p_{i,n}} = a_{q_i}$, for almost all n.

Since $\{p' \in \overline{\operatorname{Attr}_\sigma(q_i)} \cap \mathcal{C}(\mathcal{D})^{\mathbb{T}} \mid a_{p'} = a_{q_i}\} = \{q_i\}$, we have $p_{i,n} = q_i$, for almost all n. This yields $q_i \in \overline{\operatorname{Attr}_{\sigma^{-1}}(q_{i+1})}$. □

To prove Lemma 4.28, we use an analytic limit argument. Let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $p \in \overline{\operatorname{Attr}_\sigma(q)}$ and $p \neq q$. We choose bases $(v_{a,1}, \ldots, v_{a,n_a})$ of the weight spaces V_a. Without loss of generality, $v_p = v_{a_p,1}$ and $v_q = v_{a_q,1}$. Moreover, let

$$W := \Big(\bigoplus_{a<a_p} V_a\Big) \oplus \langle v_{a_p,2}, \ldots, v_{a_p,n_{a_p}}\rangle \oplus \Big(\bigoplus_{a>a_p} V_a\Big). \tag{4.11}$$

Let $Z_p = \{[v_p + w] \mid w \in W\} \subset \mathbb{P}(V)$ be the coordinate chart with origin $[v_p]$. We have that Z_p is $\mathbb{C}^*$-invariant and $t.[v_p + v] = v_p + t^{a-a_p}v$, for all $v \in V_a$, $a \in \mathbb{Z}$ and $t \in \mathbb{C}^*$. We equip W with a hermitian product with unitary basis given by (4.11) and thus view W as metric space. Via the isomorphism of (analytic) varieties $W \xrightarrow{\sim} Z_p, w \mapsto [v_p + w]$, we also view Z_p as metric space and denote by $|\cdot|$ the induced absolute value and by $\operatorname{dist}(.,.)$ the induced distance function on Z_p.

We set

$$W' := \{[v_p + \lambda v_q + w] \mid \lambda \in \mathbb{C}, w \in \bigoplus_{a_q<a<a_p} V_a\} \subset Z_p.$$

Note that W' is a $\mathbb{C}^*$-invariant linear subspace of Z_p.

Proof of Lemma 4.28. We want to construct a sequence of elements in $\operatorname{Attr}_\sigma(q) \cap Z_p$ which approaches W' but is far away from $[v_p]$. First, we show that for all $\varepsilon > 0$, there exists $z \in \iota(\operatorname{Attr}_\sigma(q)) \cap Z_p$ such that

$$|z| \in [1,2] \quad \textit{and} \quad \operatorname{dist}(z, W') < \varepsilon. \tag{4.12}$$

By Lemma 4.29 below, there exists a path $\gamma\colon [0,1] \to X_q \cap Z_p$, continuous in the analytic topology, such that $\gamma([0,1)) \subset \iota(\operatorname{Attr}_\sigma(q))$ and $\gamma(1) = [v_p]$. According to our choice of basis, we can write

$$\gamma(s) = \Big[\gamma_{a_q}(s)v_q + \Big(\sum_{a_q<a'<a_p}\sum_{i=1}^{n_{a'}}\gamma_{a',i}(s)v_{a',i}\Big) + v_p \\ + \Big(\sum_{i=2}^{n_{a_p}}\gamma_{a_p,i}(s)v_{a_p,i}\Big) + \Big(\sum_{a'>a_p}\sum_{i=1}^{n_{a'}}\gamma_{a',i}(s)v_{a',i}\Big)\Big].$$

The property $\gamma([0,1)) \subset \iota(\operatorname{Attr}_\sigma(q))$ implies $\gamma_{a_q}(s) \neq 0$, for $s \in [0,1)$. Since $\gamma(1) = [v_p]$, we have $\gamma_{i,j}(s) \to 0$, for $s \to 1$ and all i, j. Hence, we may assume that all $\gamma_{a_{p+i},j}$ with $i \geq 0$ satisfy for all $s \in [0,1]$

$$|\gamma_{a_{p+i},j}(s)| < n^{-1}\varepsilon, \quad \text{where } n = \sum_{a'\geq a_p} n_{a'}. \tag{4.13}$$

Choose $t_0 \in \mathbb{C}^*$ with $|t_0| < 1$ such that $|t_0.\gamma(0)| > 2$. Thus, as $t_0.\gamma(1) = [v_p]$, the Intermediate Value Theorem implies that there exists $s_0 \in (0,1)$ such that $|t_0.\gamma(s_0)| \in [1,2]$. In addition, (4.13) yields

$$\operatorname{dist}(t_0.\gamma(s_0), W') = \Big|\Big(\sum_{i=2}^{n_{a_p}} \gamma_{a_p,i}(s) v_{a_p,i}\Big) + \Big(\sum_{a'>a_p}\sum_{i=1}^{n_{a'}} t_0^{a'-a_p}\gamma_{a',i}(s)v_{a',i}\Big)\Big| < \varepsilon.$$

Hence $z := t_0.\gamma(s_0)$ satisfies (4.12). Since $\iota(\mathrm{Attr}_\sigma(q)) \cap Z_p$ is $\mathbb{C}^*$-invariant, we conclude $z \in \iota(\mathrm{Attr}_\sigma(q))$. Thus, z satisfies all desired properties.

As a direct consequence of (4.12), we conclude that there exist a sequence $z_m \in \iota(\mathrm{Attr}_\sigma(q))$ such that $\operatorname{dist}(z_m, [v_p]) \in [1,2]$, for all m and $\operatorname{dist}(z_m, W') \to 0$, for $m \to \infty$. By the Heine–Borel Theorem, z_m has a convergent subsequence with limit $z' \in Z_p \cap X_q$. As $\operatorname{dist}(z_n, W') \to 0$, we also have $z' \in W'$. The condition $\operatorname{dist}(z_m, [v_p]) \in [1,2]$ yields $z' \neq [v_p]$. So by the definition of W', we can write

$$z' = [w_{a_q} + w_{a_q+1} + \ldots + w_{a_p-1} + v_p], \quad w_{a_q} \in \langle v_q\rangle,\ w_{a_q+i} \in V_{a_q+i}, \quad \text{for } i > 1. \tag{4.14}$$

As $z' \neq [v_p]$, we have $w_{a_q+r} \neq 0$, for some $r \in \{0, \ldots, a_p - a_q - 1\}$. Set

$$r_0 := \min(\{r \in \{0, \ldots, a_p - a_q - 1\} \mid w_{a_q+r} \neq 0\}). \tag{4.15}$$

By construction,

$$\lim_{t\to 0} t.z' = [w_{a_q+r_0}] \quad \text{and} \quad \lim_{t\to\infty} t.z' = [v_p]. \tag{4.16}$$

Recall that $\iota(\mathrm{Attr}_\sigma(q))$ is an open dense $\mathbb{C}^*$-subvariety of X_q. Since $[v_p]$ is contained in the orbit closure $\overline{\mathbb{C}^*.z'}$, the intersection $\iota(\mathcal{C}(\mathcal{D})) \cap \overline{\mathbb{C}^*.z'}$ is a non-empty open $\mathbb{C}^*$-invariant subvariety of $\overline{\mathbb{C}^*.z'}$. Hence, $z' \in \iota(\mathcal{C}(\mathcal{D}))$. As $z' \in X_q$, we have $z' \in \iota\big(\overline{\mathrm{Attr}_\sigma(q)}\big)$. Since $\lim_{t\to\infty} t.z' = [v_p]$, we also have $z' \in \iota\big(\overline{\mathrm{Attr}_{\sigma^{-1}}(p)}\big)$. By Theorem 4.24, $\overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)}$ is a closed proper $\mathbb{C}^*$-invariant subvariety of $\mathcal{C}(\mathcal{D})$. This implies

$$\lim_{t\to 0} t.z' = [w_{a_q+r_0}] \in \iota\big(\overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)}\big). \tag{4.17}$$

Set $p' := \iota^{-1}([w_{a_q+r_0}])$. Then, as $[w_{a_q+r_0}]$ is a $\mathbb{C}^*$-fixed point of $\mathbb{P}(V)$, we have $p' \in \mathcal{C}(\mathcal{D})^\sigma$. The Generic Cocharacter Theorem then gives $p' \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. By (4.17), we know that (i) is satisfied. Moreover, (4.15) yields $a_{p'} = a_q + r_0 < a_p$ which implies (ii). Finally, (4.16) yields that z' satisfies (iii). So p' and z' satisfy all desired properties. □

Approximation of boundary points via paths

In the proof of Lemma 4.28, we used the following statement:

Lemma 4.29. *Let Y be a smooth algebraic variety of dimension d which is embedded into a projective variety X as open dense subvariety. Then, for all $y \in Y \backslash X =: Z$, there exists a path $\gamma\colon [0,1] \to Y$ continuous with respect to the analytic topology on Y such that $\gamma([0,1)) \subset X$ and $\gamma(1) = y$.*

Proof. By the Monomalization Theorem, see e.g. [Kol09, Theorem 3.35], there exists a smooth projective variety Y' and a morphism of varieties $f\colon Y' \to Y$ such that

(a) f restricts to an isomorphism $f^{-1}(X) \xrightarrow{\sim} X$ and

(b) $f^{-1}(Z)$ is a simple normal crossing divisor.

Thus, we may assume that Z is a simple normal crossing divisor. Given $y \in Z$. Then, as Z is a simple normal crossing divisor, there exists an analytic neighborhood of y in Y which is analytically isomorphic to a neighborhood U of the origin in $\mathbb{C}^d$ such that under this isomorphism y is identified with the origin and Z equals the vanishing locus of functions $f_1 \cdots f_r$, where $f_1, \dots, f_r \colon U \to \mathbb{C}$ are holomorphic functions with $r \leq d$ and $l_1, \dots, l_r$ are linearly independent, where l_i denotes the first order approximation of f_i. After applying a linear transformation, we may assume that l_i is the projection the i-th coordinate in $\mathbb{C}^d$. By further shrinking U, we can thus assume that there exists a constant $C > 0$ such that

$$|f_i(z) - z_i| < C|z|^2, \quad \text{for } z = (z_1, \dots, z_d) \in U,\ i = 1, \dots, r.$$

Hence, we conclude $\{z \in U \mid |z_i| > C|z|^2\} \cap Z = \emptyset$. It follows that $\mu(1, \dots, 1) \notin Z$, for $0 < \mu < (C\sqrt{d})^{-1}$. By choosing C large enough, we may assume that the closed ball centered at the origin with radius $C' = \frac{1}{2}(C\sqrt{d})^{-1}$ is entirely contained in U. Thus, if we set

$$\gamma \colon [0,1] \longrightarrow U, \quad s \mapsto s \cdot C' \cdot (1, \dots, 1)$$

then γ yields a path with the desired properties. □

4.6 Partial order via invariant curves

In [BFR23, Section 4], an equivalent description of $\preceq_{\mathfrak{C}}$ via $\mathbb{T}$-invariant curves was given. This description of $\preceq_{\mathfrak{C}}$ is particularly useful as the computation of $\mathbb{T}$-invariant curves is usually easier than the computation of all closures of attracting cells. In this section, we give a self-contained reproof of this result using Lemma 4.28 and deformation techniques involving Hilbert schemes.

As before, let $\sigma \in \mathfrak{C}$. We have the following fundamental result:

Proposition 4.30. *For $p, q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have $p \preceq_{\mathfrak{C}} q$ if and only if there exists $x_1, \dots, x_k \in \mathcal{C}(\mathcal{D})$ and $p_1, \dots, p_k \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ such that $p_1 = p$ and $p_k = q$ and for all i, the following conditions are satisfied*

(i) $\lim_{t \to 0} \sigma(t).x_i = p_{i+1}$ *and* $\lim_{t \to \infty} \sigma(t).x_i = p_i$,

(ii) $\mathbb{T}.x_i = \sigma(\mathbb{C}^*).x_i$.

The implication $\Rightarrow$ of Proposition 4.30 is immediate from the definition of $\preceq_{\mathfrak{C}}$. We prove the converse implication in the subsequent subsections using deformation techniques.

Remark. In the recent work [FS23], Foster and Shou provide a classification of the $\mathbb{T}$-invariant curves of bow varieties. Then, in [BFR23], this classification is used to explicitly identify the partial order $\preceq_{\mathfrak{C}}$ with the secondary Bruhat order on $(0,1)$-matrices.

Smoothness of one-dimensional $\mathbb{T}$-orbits

We start with the following smoothness result for one-dimensional $\mathbb{T}$-orbits.

Proposition 4.31. *Let $x \in \mathcal{C}(\mathcal{D})$ and $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ such that the orbit $\mathbb{T}.x$ is one-dimensional and $p \in \overline{\mathbb{T}.x}$. Then, there exists a character $\tau\colon \mathbb{T} \to \mathbb{C}^*$ and a $\mathbb{T}$-equivariant open immersion $\iota\colon \mathbb{C} \hookrightarrow \overline{\mathbb{T}.x}$, where $\mathbb{T}$ acts on $\mathbb{C}$ via τ and $\iota(0) = p$.*

We use the following auxiliary statement:

Lemma 4.32. *Let V be a finite dimensional $\mathbb{A}$-representation with $\mathbb{A}$-weight decomposition of the form*

$$V = \bigoplus_{\substack{1 \leq i,j \leq N \\ i \neq j}} V_{t_i - t_j}.$$

Let $Y \subset V$ be an irreducible $\mathbb{A}$-invariant closed subvariety with $\dim(Y) = 1$. Suppose we have $0 \in Y$. Then, we have $Y = \langle v \rangle$, for some $\mathbb{A}$-weight vector $v \in V$.

Proof. Suppose Y contains a point $w = w_1 + \ldots + w_r$, where $r \geq 2$ and $w_1, \ldots, w_r \in V$ are $\mathbb{A}$-weight vectors corresponding to pairwise distinct weights. Let $\tau_l = t_{i_l} - t_{j_l}$ be the $\mathbb{A}$-weight of w_l. Since $0 \in Y$, we can assume that $t_{i_2} - t_{j_2} \neq t_{j_1} - t_{i_1}$. Thus, there exist cocharacters $\sigma_1, \sigma_2\colon \mathbb{C}^* \to \mathbb{A}$ such that

$$\begin{aligned} \langle \sigma_1, \tau_1 \rangle &= 1, & \langle \sigma_2, \tau_1 \rangle &= 0, \\ \langle \sigma_1, \tau_2 \rangle &= 0, & \langle \sigma_2, \tau_2 \rangle &= 1. \end{aligned}$$

This yields that $\overline{\sigma_1(\mathbb{C}^*).w}$ and $\overline{\sigma_2(\mathbb{C}^*).w}$ are two distinct connected components of Y. This contradicts the assumption that Y is irreducible. Hence, we conclude that $Y = \langle v \rangle$, for some $\mathbb{A}$-weight vector $v \in V$. □

Proof of Proposition 4.31. Set $Z \coloneqq \overline{\mathbb{T}.x}$. It is a general fact, see e.g. [AF23, Section 7.2], that there exists a character $\tau\colon \mathbb{T} \to \mathbb{C}^*$ and a $\mathbb{T}$-equivariant injective morphism $\iota\colon \mathbb{C} \to Z$ onto an open subvariety Y of Z such that $\iota(0) = p$ and ι restricts to an isomorphism of varieties $\mathbb{C}^* \xrightarrow{\sim} \mathbb{T}.x$. Therefore, to conclude the proposition, it suffices to prove that p is a smooth point of Z. By the Slice Theorem, see e.g. [AF23, Theorem 5.1.4], there exists a $\mathbb{T}$-invariant open subvariety U containing p and a $\mathbb{T}$-equivariant étale morphism $f\colon U \to V$ to a $\mathbb{T}$-invariant open subvariety of $T_p\mathcal{C}(\mathcal{D})$ with $f(p) = 0$. As the orbit $\mathbb{T}.x$ is one-dimensional, $\mathbb{T}.f(x)$ is also a one-dimensional $\mathbb{T}$-orbit of $T_p\mathcal{C}(\mathcal{D})$. Since $\overline{\mathbb{T}.x}$ contains p, we conclude that $\overline{\mathbb{T}.f(x)}$ contains 0. Thus, by Corollary 3.24 and Lemma 4.32, we have $\overline{\mathbb{T}.f(x)} = \langle w \rangle$, for some $\mathbb{T}$-weight vector $w \in T_p\mathcal{C}(\mathcal{D})$. Therefore, 0 is a smooth point of $\overline{\mathbb{T}.f(x)}$. As f is étale, we conclude that p is also a smooth point of Z. □

Proposition 4.31 directly implies the following isomorphism types for closures of one-dimensional $\mathbb{T}$-orbits:

Corollary 4.33. *Let $Z \subset \mathcal{C}(\mathcal{D})$ be the Zariski closure of a one-dimensional $\mathbb{T}$-orbit. Then, Z is a smooth subvariety of $\mathcal{C}(\mathcal{D})$. In particular, Z is isomorphic to $\mathbb{C}^*$, $\mathbb{C}$ or $\mathbb{P}^1$.*

Thanks to Corollary 4.33, we call the one-dimensional $\mathbb{T}$-orbits of $\mathcal{C}(\mathcal{D})$ also $\mathbb{T}$-*invariant curves.*

Reminders on Hilbert schemes

Before we prove Proposition 4.30, we briefly recall the definition and some properties of Hilbert schemes. For a general introduction to Hilbert schemes see e.g. the expository works [Str96] and [Nit05].

Let X be a projective variety with a fixed ample line bundle $\mathcal{L}$. Moreover, let $P \in \mathbb{Q}[x]$ be a polynomial. The *Hilbert functor*

$$\mathcal{H}ilb_X^P \colon \{\text{Schemes over } \mathbb{C}\} \longrightarrow \{\text{Sets}\}$$

is defined as

$$\mathcal{H}ilb_X^P(Y) = \{Z \subset X \times Y \mid Z \text{ satisfies (Hilb-1)–(Hilb-3)}\},$$

where

(Hilb-1) Z is a closed subscheme of $X \times Y$,

(Hilb-2) the projection $\mathrm{pr}_Y \colon Z \to Y$ is flat,

(Hilb-3) for all closed points $y \in Y$, the fiber $\mathrm{pr}_Y^{-1}(y)$ admits Hilbert polynomial P.

The following fundamental result is due to Grothendieck, see e.g. [Str96, Theorem 8.1]:

Theorem 4.34 (Existence of Hilbert schemes)**.** *The functor $\mathcal{H}ilb_X^P$ is representable by a projective scheme* Hilb_X^P.

This theorem implies that there exists a universal family $\mathcal{Z} \subset X \times \mathrm{Hilb}_X^P$ such that for every family $Z \subset X \times Y$ satisfying (Hilb-1)–(Hilb-3), there exists a unique morphism $f \colon Y \to \mathrm{Hilb}_X^P$ such that $f^*\mathcal{Z} = Z$. If Z' is a closed subscheme of X with Hilbert polynomial P, we denote by $[Z']$ the corresponding closed point on Hilb_X^P.

If G is an algebraic group acting on X then this action induces a G-action on Hilb_X^P. On the closed points of Hilb_X^P, this action is given as $g.[Z'] = [g.Z']$, where g and $[Z']$ are closed points of G and Hilb_X^P.

Deformation of torus invariant varieties

Consider the general situation, where X is a normal and quasi-projective variety with an algebraic action of a torus T. The representability of Hilbert functors allows us to deform closed subvarieties of X into T-invariant closed subschemes of X as follows:

Lemma 4.35 (Deformation Lemma)**.** *Let $T' \subset T$ be a subtorus such that $T/T' \cong \mathbb{C}^*$ and let $\tau \colon \mathbb{C}^* \to T$ be a chocharacter such that the induced map $\mathbb{C}^* \to T/T'$ is an isomorphism of algebraic groups. For a T'-invariant closed irreducible subvariety $Y \subset X$, set*

$$\Gamma' := \{(\tau(t).y, t) \mid y \in Y,\ t \in \mathbb{C}^*\} \subset X \times \mathbb{C}^*.$$

Let Γ be the Zariski closure of Γ' in $X \times \mathbb{C}$ and let Γ_0 be the scheme theoretic fiber of 0 with respect to the projection $X \times \mathbb{C} \to \mathbb{C}$. Then, the following holds

(i) the projection $\Gamma \to \mathbb{C}$ is flat,

(ii) Γ_0 *is a* T*-invariant closed subscheme of* X*,*

(iii) the irreducible components of Γ_0 *are* T*-invariant closed subvarieties of* X *all of dimension* $\dim(Y)$*.*

Proof. By [Sum74, Theorem 2], there exists a T-equivariant embedding $X \hookrightarrow X'$ into a projective variety X' with a T-action. Let $\overline{Y}$ be the Zariski closure of Y in X'. Then, $\overline{Y}$ is a T-invariant closed subvariety of X'. Set

$$\overline{\Gamma}' := \{(\tau(t).y, t) \mid y \in \overline{Y}, t \in \mathbb{C}^*\} \subset X' \times \mathbb{C}^*.$$

Then, $\overline{\Gamma}'$ contains Γ' as open dense subvariety. Likewise, let $\overline{\Gamma}$ be the Zariski closure of $\overline{\Gamma}'$ in $X' \times \mathbb{C}$ and $\overline{\Gamma}_0$ be the fiber of 0 with respect to the projection $X' \times \mathbb{C} \to \mathbb{C}$. By construction, $\overline{\Gamma}$ resp. $\overline{\Gamma}_0$ contains Γ resp. Γ_0 as open dense subscheme. Fix an ample line bundle on X' and let P be the corresponding Hilbert polynomial of $\overline{Y}$. By definition, the flat family $\overline{\Gamma}' \to \mathbb{C}^*$ corresponds to the τ-orbit of $[\overline{Y}]$ in $\mathrm{Hilb}^P_{X'}$. Since $\overline{\Gamma}'$ is a closed subvariety of $X' \times \mathbb{C}^*$ and $X' \times \mathbb{C}^*$ is an open subvariety of $X' \times \mathbb{C}$, we have that $\overline{\Gamma}'$ is a locally closed subvariety of $X' \times \mathbb{C}$. We conclude that $\overline{\Gamma}$ equals the scheme theoretic closure of $\overline{\Gamma}'$ in $X' \times \mathbb{C}$. Thus, by e.g. [Har77, Proposition III.9.7], the projection $\pi\colon \overline{\Gamma} \to \mathbb{C}$ is a flat morphism. Since Γ is an open subvariety of $\overline{\Gamma}$, we deduce that also the projection $\Gamma \to \mathbb{C}$ is flat which proves (i). Let $\bar{\tau}\colon \mathbb{C}^* \xrightarrow{\sim} T/T'$ be the algebraic group isomorphism induced by τ. We equip $\mathbb{C}$ with the T-action $t.x = \bar{\tau}^{-1}([t]) \cdot x$, where $[t]$ denotes the class of t in T/T'. Then, the flat family $\pi\colon \overline{\Gamma} \to \mathbb{C}$ corresponds to a T-equivariant morphism $\mathbb{C} \to \mathrm{Hilb}^P_{X'}$. This implies that $\overline{\Gamma}_0$ is a T-invariant closed subscheme of X'. As the T-action continuously permutes the finitely many irreducible components of $\overline{\Gamma}_0$, we conclude that all irreducible components of $\overline{\Gamma}_0$ are T-invariant. The flatness of π implies that the dimension of all irreducible components of $\overline{\Gamma}_0$ equals $\dim(Y)$, see e.g. [Har77, Corollary III.9.6]. Since $\Gamma_0 = \overline{\Gamma}_0 \cap X$, we conclude that Γ_0 is a T-invariant open dense subscheme of $\overline{\Gamma}_0$. Thus, (ii) and (iii) follow from the respective properties of $\overline{\Gamma}_0$. □

In the following subsection, we use the Deformation Lemma to deform orbits of generic cocharacters into one dimensional orbits with respect to the torus $\mathbb{T}$.

Proof of Proposition 4.30

As before, fix $\sigma \in \mathfrak{C}$. Again, choose a $\mathbb{T}$-equivariant locally closed immersion $\iota\colon \mathcal{C}(\mathcal{D}) \hookrightarrow \mathbb{P}(V)$ and let a_p be defined as in the proof of Theorem 4.23. In particular, note that if p, p', $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ such that $p' \neq p$, q and $p' \in \overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)}$ then we have $a_p < a_{p'} < a_p$.

Proof of Proposition 4.30. To prove that $\preceq_{\mathfrak{C}}$ is equivalently characterized by $\mathbb{T}$-invaraint curves as described in Proposition 4.30, it suffices to show that for p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $p \neq q$ and $p \in \overline{\mathrm{Attr}_\sigma(q)}$ there exist $x_1, \dots, x_k \in \mathcal{C}(\mathcal{D})$ and $p_1, \dots, p_k \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ such that $p_1 = p$ and $p_k = q$ and for all i, we have

(a) $\lim_{t\to 0} \sigma(t).x_i = p_{i+1}$ and $\lim_{t\to\infty} \sigma(t).x_i = p_i$,

(b) $\mathbb{T}.x_i = \sigma(\mathbb{C}^*).x_i$.

We prove this statement for fixed q and arbitrary p via induction on a_p. If $a_p = a_q$ then $p \in \overline{\mathrm{Attr}_\sigma(q)}$ gives $p = q$ and the statement is trivial. If $a_p > a_q$ then, by Lemma 4.28, there exists $x \in \overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)}$ with $\lim_{t\to\infty} \sigma(t).x = p$. Applying the Deformation Lemma to $\overline{\sigma(\mathbb{C}^*).x}$ gives that there exists a one-dimensional $\mathbb{T}$-invariant irreducible subvariety Y of $\overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)}$ containing p. Let $y \in Y \setminus \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. Note that we have

$$a_{p'} < a_p, \quad \text{for all } p' \in \overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)} \cap \mathcal{C}(\mathcal{D})^{\mathbb{T}}, \quad p' \neq p.$$

Thus, we conclude that $\lim_{t\to\infty} \sigma(t).y = p$. The properness of $\overline{\mathrm{Attr}_\sigma(q)} \cap \overline{\mathrm{Attr}_{\sigma^{-1}}(p)}$ implies that $p_0 := \lim_{t\to 0} \sigma(t).y$ exists in $\mathcal{C}(\mathcal{D})$. By the Generic Cocharacter Theorem, we have $p_0 \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. As $p_0 \in \overline{\mathrm{Attr}_\sigma(q)}$ and $a_{p_0} < a_p$, we can apply the induction hypothesis to p_0 which yields a chain of one dimensional $\mathbb{T}$-orbits with the desired properties (a) and (b). ☐

Example of invariant curves

Consider again the bow variety $\mathcal{C}(\mathcal{D})$ from (4.1). In this subsection, we characterize the $\mathbb{T}$-invariant curves of $\mathcal{C}(\mathcal{D})$. Via Proposition 4.30, this determines the partial orders $\preceq_{\mathfrak{C}}$ on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$.

The classification of $\mathbb{T}$-invariant curves of $\mathcal{C}(\mathcal{D})$ can be conveniently illustrated via the *GKM-graph* $\Gamma_{\mathcal{C}(\mathcal{D})}$ of $\mathcal{C}(\mathcal{D})$, see e.g. [Tym05] for an introduction to GKM-theory. This graph, named after Goresky, Kottwitz and MacPherson, is defined as follows:

- The vertex set of $\Gamma_{\mathcal{C}(\mathcal{D})}$ is $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$.
- For each $\mathbb{T}$-invariant curve which is isomorphic to $\mathbb{P}^1$, draw a an edge between the $\mathbb{T}$-fixed points corresponding to 0 and ∞.
- For each $\mathbb{T}$-invariant curve which is isomorphic to $\mathbb{C}$, draw en edge with one open end and one end adjacent to the $\mathbb{T}$-fixed point corresponding to 0.

Additionally, we decorate each pair (γ, p), where γ is an edge of $\Gamma_{\mathcal{C}(\mathcal{D})}$ and p is a vertex of $\Gamma_{\mathcal{C}(\mathcal{D})}$ that is adjacent to γ with the tangent weight $T_p\gamma$.

Remark. As $\mathcal{C}(\mathcal{D})$ is only quasi-projective there exist some $\mathbb{T}$-invariant curves that contain only one $\mathbb{T}$-fixed point. Hence, the GKM-graph of $\mathcal{C}(\mathcal{D})$ contains edges that are just adjacent to one vertex and admit an open end.

Recall the $\mathbb{T}$-invariant affine open subvarieties $\mathcal{C}(\mathcal{D}) = \bigcup_{i=1}^5 W_i$ from (4.4) and the parameterizations $\eta_i \colon \mathbb{C}^4 \xrightarrow{\sim} W_i$ from Claim 4.12.

Claim 4.36. *The $\mathbb{T}$-invariant curves of $\mathcal{C}(\mathcal{D})$ are exactly the Zariski closures of the orbits $\mathbb{T}.\eta_i(e_j)$, where $e_1, \dots, e_4$ are the standard basis vectors of $\mathbb{C}^4$.*

In the proof, we use the following statement:

Claim 4.37. *Let $x \in W_i$, for some $i = 1, \dots, 5$. Then, $\overline{\mathbb{T}.x}$ contains $\eta_i(0)$.*

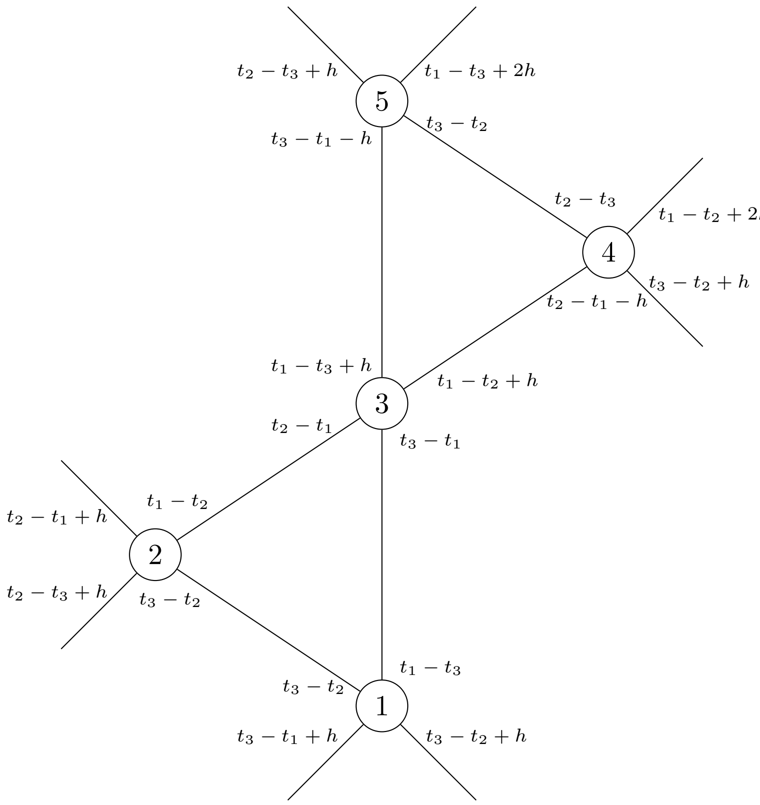
Proof. We only prove the case $i = 1$ as the other cases are similar. By Claim 4.13, W_1 admits the weight space decomposition

$$W_1 = \mathbb{C}_{t_3 - t_1 + h} \oplus \mathbb{C}_{t_3 - t_2 + h} \oplus \mathbb{C}_{t_1 - t_2} \oplus \mathbb{C}_{t_2 - t_3}.$$

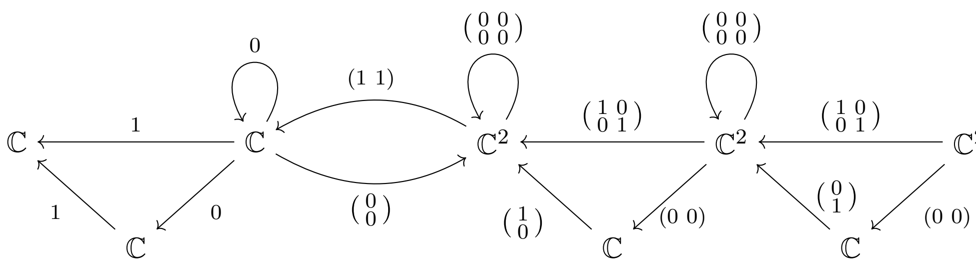
Denote the elements of W_1 as $x = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ according to this decomposition. By using the $\mathbb{C}_h^*$-action, we deduce that $\overline{\mathbb{T}.x}$ contains $x' := (0, 0, \lambda_3, \lambda_4)$. Let $\sigma\colon \mathbb{C}^* \to \mathbb{A}$, $t \mapsto (t^3, t^2, t)$. Then, $\lim_{t\to 0} \sigma(t).x' = 0$ and thus $0 \in \overline{\mathbb{T}.x}$. □

Proof of Claim 4.36. Let $x \in \mathcal{C}(\mathcal{D})$ such that $\gamma := \overline{\mathbb{T}.x}$ is one-dimensional. We have $x \in W_i$, for some $i = 1, \dots, 5$. By Claim 4.37, we have $\eta_i(0) \in \gamma$. Thus, Lemma 4.32 and Claim 4.13 imply that $\mathbb{T}.x = \eta_i(e_j)$, for some $j = 1, \dots, 4$. □

Claim 4.38. *The GKM-graph $\Gamma_{\mathcal{C}(\mathcal{D})}$ is given as follows:*



Proof. By Claim 4.36, the $\mathbb{T}$-invariant curves of $\mathcal{C}(\mathcal{D})$ are exactly the Zariski closures of the $\mathbb{T}.\eta_i(e_j)$, for $i = 1, \dots, 5$ and $j = 1, \dots, 4$. We just determine the Zariski closure of $\mathbb{T}.\eta_5(e_3)$ as all other closures can be determined in the same way. By Claim 4.12, $\eta_5(e_3)$ corresponds to the diagram



Hence, $\eta_5(e_3)$ also satisfies (cov-4) and therefore $\eta_5(e_3) \in W_4$. Thus, Claim 4.37 gives $\gamma = \mathbb{T}.\eta_5(e_3) \cup \{x_{D_5}, x_{D_4}\}$. Thus, the GKM-graph of $\mathcal{C}(\mathcal{D})$ indeed contains an edge e connecting the vertices labeled by 5 and 4. By Claim 4.13, the tangent weight of γ at x_{D_5} is $t_3 - t_2$. Hence, the pair $(5, e)$ is decorated with $t_3 - t_2$, whereas the pair $(4, e)$ is decorated with $t_2 - t_3$. □

By Proposition 4.30, the partial order $\preceq_{\mathfrak{C}}$ corresponding to an arbitrary choice of chamber can be read off from $\Gamma_{\mathcal{C}(\mathcal{D})}$ as follows: For p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have $p \preceq_{\mathfrak{C}} q$ if and only if there exists a finite sequence of vertices $p_1, \ldots, p_r$ in $\Gamma_{\mathcal{C}(\mathcal{D})}$ such that

(i) $p_1 = q$ and $p_r = p$,

(ii) for all i, p_i and p_{i+1} are connected by an edge γ such that $T_{p_i}\gamma$ is positive and $T_{p_{i+1}}\gamma$ is negative with respect to $\mathfrak{C}$.

The reader is invited to use this criterion to give another proof that for the cocharacter $\sigma_0 = (t, t^2, t^3)$ the corresponding partial order matches with the usual order $\leq$ on $\{1, 2, 3, 4, 5\}$.

Remark. The GKM graph of the bow variety $\mathcal{C}(\mathcal{D})$ can also be found in [RS20].

Remark. In general bow varieties may admit infinitely many $\mathbb{T}$-invariant curves, see e.g. the examples in [FS23].

Chapter 5

Stable envelopes

Stable envelopes were introduced by Maulik and Okounkov in [MO19, Chapter 3] as families of torus equivariant cohomology classes satisfying a certain set of stability conditions which are analogous to to the stability conditions from equivariant Schubert calculus, see e.g. [KT03, Lemma 1], [GKS20, Lemma 3.8]. The stability conditions provide different families of stable envelope bases in localized equivariant cohomology (similar to the equivariant Schubert bases corresponding to different choices of Borel subgroups). Just like in equivariant Schubert calculus, the base change between different stable envelope bases produces solutions of the Yang–Baxter equations which provides a connection between algebraic geometry and integrable systems, see [MO19, Chapter 5]. It in particular allows interesting braid group actions on equivariant cohomology algebras and constructions of geometric quantum groups.

Maulik and Okounkov consider in their setup smooth algebraic varieties X admitting a holomorphic symplectic form ω satisfying the following conditions on torus actions:

(Torus-1) There exists a pair of tori $A \subset T$ acting algebraically on X such that ω is fixed by A and scaled by T.

(Torus-2) There exists an affine variety X_0 with algebraic T-action and an T-equivariant proper morphism $X \to X_0$.

Bow varieties satisfy these conditions, namely (Torus-1) by discussion on the torus action in Section 2.3 and (Torus-2) by Proposition 4.2. As a consequence, stable envelopes exists for bow varieties and the results of Maulik and Okounkov's theory can be applied to them.

The current chapter is devoted to a study of these stable envelopes. We give a detailed reproof of the uniqueness and existence of stable envelopes in the framework of bow varieties following [MO19, Chapter 3]. As we are in the preferable situation of finitely many torus fixed points, some arguments simplify in our setup. The uniqueness property can be shown via a direct argument which combines the stability properties of stable envelopes and excess intersection theory.

The proof of the existence of stable envelopes follows the algorithmic procedure from [MO19, Section 3.5]. In the case of bow varieties, this procedure expresses the stable basis elements as $\mathbb{Z}$-linear combinations of the Poincaré duals of the fundamental classes of the attracting cell closures. The crucial input for this algorithmic procedure is a result which controls the equivariant multiplicities of torus invariant lagrangian subvarieties. We take this result from the proof of [MO19, Lemma 3.4.2] and refer to it as Lagrangian Multiplicity Theorem, see Theorem 5.15. The Sections 5.4 and 5.5 are devoted to a detailed reproof of Theorem 5.15 following the general outline from [MO19, Section 3.4]. Central ingredients of the proof are the deformation to the normal cone construction from [Ful84] and a further deformation result, see Proposition 5.29, where we deform lagrangian conic subvarieties of symplectic vector spaces into lagrangian hyperplanes.

We close this chapter with an example where we explicitly compute stable basis elements using the algorithm provided by the existence proof.

5.1 Torus equivariant cohomology

Before we discuss stable envelopes, we recall important ingredients of torus equivariant cohomology and torus equivariant intersection theory which are crucial for the theory of stable envelopes. For more details on equivariant cohomology see e.g. [Hsi75], [tD87] and [AF23]. For an introduction to equivariant intersection theory, see e.g. [Bri97] and [EG96]. For the convenience of the reader, prove some of the presented statements.

Let X be a variety with an algebraic action of a torus $T = (\mathbb{C}^*)^r$. We denote by $H^*_T(X)$ the T-equivariant cohomology and by $\overline{H}^T_*(X)$ the T-equivariant Borel–Moore homology of X with coefficients in $\mathbb{Q}$. Via the cup product, $H^*_T(X)$ is equipped with a ring structure denoted by $(\alpha, \beta) \mapsto \alpha \cdot \beta$. Furthermore, we denote the standard action of an element $\alpha \in H^*_T(X)$ on $a \in \overline{H}^T_*(X)$ by $\alpha.a$.

Given a T-equivariant morphism $f\colon X \to Y$ of varieties with algebraic T-actions, we denote the respective pullback and pushforward morphisms in T-equivariant cohomology and T-equivariant Borel–Moore homology (whenever they are defined) by f^* and f_*.

If X is additionally smooth and Y, Y' are closed T-invariant subvarieties then via the usual cup product construction, see e.g. [CG97, Section 2.6.15], we obtain the corresponding *T-equivariant intersection product*

$$\cap\colon \overline{H}^T_*(Y) \times \overline{H}^T_*(Y') \longrightarrow \overline{H}^T_*(Y \cap Y').$$

This pairing satisfies the action identity

$$(\alpha.a) \cap b = i^*(\alpha).(a \cap b), \tag{5.1}$$

where $\alpha \in H^*_T(Y)$, $a \in \overline{H}^T_*(Y)$, $b \in \overline{H}^T_*(Y')$ and $i\colon Y \cap Y' \hookrightarrow Y$ is the inclusion.

Approximation

The T-equivariant cohomology of a variety X with algebraic T-action can be approximated via singular cohomology as follows: Set $\mathbb{E}_n := (\mathbb{C}^n \setminus \{0\})^r$ and equip $\mathbb{E}_n$ with the $T = (\mathbb{C}^*)^r$-action

$$(t_1, \dots, t_r).(v_1, \dots, v_r) = (t_1 v_1, \dots, t_r v_r), \quad \text{for } (t_1, \dots, t_r) \in T,\ (v_1, \dots, v_r) \in \mathbb{E}_n.$$

Then, T-acts freely on $\mathbb{E}_n$. Set

$$U_i \coloneqq \{(x_1, \dots, x_n) \in \mathbb{C}^n \setminus \{0\} \mid x_i \neq 0\}$$

and note that we have an isomorphism of varieties

$$\mathbb{C}^{n-1} \times \mathbb{C}^* \xrightarrow{\sim} U_i, \quad (v_1, \dots, v_{n-1}, t) \mapsto (tv_1, \dots, tv_{i-1}, t, tv_i, \dots, tv_{n-1}). \tag{5.2}$$

Set $U_{\mathbf{i}} \coloneqq U_{i_1} \times \ldots \times U_{i_r}$, for $\mathbf{i} = (i_1, \dots, i_r)$. Then, we have a cover of $\mathbb{E}_n$ by T-invariant open subvarieties

$$\mathbb{E}_n = \bigcup_{\mathbf{i} \in I_n} U_{\mathbf{i}}, \quad \text{where } I_n = \{1, \dots, n\}^r. \tag{5.3}$$

Via (5.2), we obtain T-equivariant isomorphisms $\mathbb{C}^{r(n-1)} \times T \cong U_{\mathbf{i}}$, where T acts trivially on $\mathbb{C}^{r(n-1)}$ and via the regular action on T. Thus, the geometric quotient $\mathbb{E}_n/T$ exists and and we have an obvious isomorphism of varieties $\mathbb{E}_n/T \cong (\mathbb{P}^n)^r$. By (5.3), the projection $\mathrm{pr}\colon \mathbb{E}_n \to \mathbb{P}^n$ is a principal T-bundle in the Zariski topology.

Given a complex scheme $\tilde{X}$ with algebraic T-action, we conclude by (5.3) that $\mathbb{E}^n \times \tilde{X}$ (equipped with the diagonal T-action) admits a cover by T-invariant opens

$$\mathbb{E}_n \times \tilde{X} = \bigcup_{\mathbf{i} \in I_n} U_{\mathbf{i}} \times \tilde{X}$$

and we have T-equivariant isomorphisms $U_{\mathbf{i}} \times \tilde{X} \cong \mathbb{C}^{r(n-1)} \times \tilde{X} \times T$. Thus, T acts freely on $\mathbb{E}_n \times \tilde{X}$. This gives that the geometric quotient $(\mathbb{E}_n \times \tilde{X})/T$ exists and the projection $\mathbb{E}_n \times \tilde{X} \to (\mathbb{E}_n \times \tilde{X})/T$ is a principal T-bundle in the Zariski topology. Moreover, $(\mathbb{E}_n \times \tilde{X})/T$ is covered by open subschemes

$$(\mathbb{E}_n \times \tilde{X})/T = \bigcup_{\mathbf{i} \in I_n} (U_{\mathbf{i}} \times \tilde{X})/T \tag{5.4}$$

and we have isomorphisms of schemes $(U_{\mathbf{i}} \times \tilde{X})/T \cong \mathbb{C}^{r(n-1)} \times \tilde{X}$. Note that the singular cohomology groups (with rational coefficients) $H^i(\mathbb{E}_n)$ vanish for $1 < i < 2n-1$. Thus, by e.g. [AF23, Proposition 2.2.2], we conclude that there exist natural isomorphisms of $\mathbb{Q}$-vector spaces

$$f_i\colon H^i_T(X) \xrightarrow{\sim} H^i((\mathbb{E}_n \times X)/T), \quad \text{for } i < 2n-1. \tag{5.5}$$

These isomorphism are compatible with the ring structure on $H^*_T(X)$ and $H^*((\mathbb{E}_n \times X)/T)$ in this range, i.e. if $\alpha \in H^i_T(X)$ and $\beta \in H^j_T(X)$ with $i+j < 2n-1$ then $f_{i+j}(\alpha \cdot \beta) = f_i(\alpha) \cdot f_j(\beta)$.

The analogous result to (5.5) for T-equivariant Borel–Moore homology, see e.g. [AF23, Section 17.1], states that there are natural isomorphisms of $\mathbb{Q}$-vector spaces

$$\overline{H}^T_i(X) \xrightarrow{\sim} \overline{H}_{i+r(2n-1)}((\mathbb{E}_n \times X)/T), \quad \text{for } i > -2n+1, \tag{5.6}$$

where $\overline{H}_*$ denotes the usual Borel–Moore homology with coefficient in $\mathbb{Q}$.

Fundamental classes

Suppose now that X is additionally irreducible. Let d be the dimension of X (as complex variety). As explained in e.g. [AF23, Section 17.1], there exists a unique class $[X]^T \in \overline{H}^T_{2d}(X)$ such that for all $n \geq 1$, $[X]^T$ corresponds to the fundamental class $[(\mathbb{E}_n \times X)/T]$ under the identification $\overline{H}^T_{2d}(X) \cong \overline{H}_{2d+r(2n-1)}((\mathbb{E}_n \times X)/T)$ from (5.6). Consequently, $[X]^T$ is called the *T-equivariant fundamental class* of X.

If X is reducible then the *T-equivariant fundamental class* of X is defined as

$$[X^T] := \sum_{l=1}^{s} j_{l*}[X_l]^T,$$

where $X_1, \ldots, X_s$ are the irreducible components of X and $j_l \colon X_l \hookrightarrow X$ are the respective inclusions.

If X is additionally smooth, we have the *T-equivariant Poincaré duality isomorphism*, see e.g. [AF23, Section 3.4]:

$$H^*_T(X) \xrightarrow{\sim} \overline{H}^T_*(X), \quad \alpha \mapsto \alpha.[X].$$

If $Y \subset X$ is a closed T-invariant subvariety then, by definition, the map

$$\overline{H}^T_*(Y) \longrightarrow \overline{H}^T_*(Y), \quad a \mapsto [X]^T \cap a \tag{5.7}$$

equals the identity on $\overline{H}^T_*(Y)$.

Fundamental classes of schemes

Let $\tilde{X}$ be a separated scheme of finite type over $\mathbb{C}$ with algebraic T-action and let $X = \tilde{X}_{\text{red}}$. As before, denote by $X_1, \ldots, X_s$ the irreducible components of X. Recall from e.g. [Ful84, Section 1.5] that the fundamental class $[\tilde{X}] \in \overline{H}_*(X)$ of $\tilde{X}$ in the usual non-equivariant Borel–Moore homology is defined as

$$[\tilde{X}] := \sum_{i=1}^{s} m(X_i, \tilde{X}) j_{i*}[X_i],$$

where $j_i \colon X_i \hookrightarrow X$ is the inclusion. Here, $m(X_i, \tilde{X})$ are the respective *geometric multiplicities* which are defined as follows: Let $\mathcal{O}_{\tilde{X},X_i}$ be the stalk of the structure sheaf of $\tilde{X}$ at X_i. Since X_i and $\tilde{X}$ have equal dimension, the Krull dimension of $\mathcal{O}_{\tilde{X},X_i}$ is 0. As $\mathcal{O}_{\tilde{X},X_i}$ is also noetherian, $\mathcal{O}_{\tilde{X},X_i}$ is artinian. Thus, every finitely generated $\mathcal{O}_{\tilde{X},X_i}$-module M has finite length which we denote by $l_{\mathcal{O}_{\tilde{X},X_i}}(M)$.

Definition 5.1. The *geometric multiplicity of X_i in $\tilde{X}$* is defined as

$$m(X_i, \tilde{X}) := l_{\mathcal{O}_{\tilde{X},X_i}}(\mathcal{O}_{\tilde{X},X_i}).$$

We now consider how the definition of fundamental classes generalize to the T-equivariant framework. The next proposition gives that geometric multiplicities are well-behaved with respect to the quotient construction from (5.4).

Proposition 5.2. *Let $n \geq 1$. Then, the following holds* *(i)* $(\mathbb{E}_n \times X_1)/T, \ldots, (\mathbb{E}_n \times X_s)/T$ *are the irreducible components of* $(\mathbb{E}_n \times \tilde{X})/T$,

(ii) *we have* $m((\mathbb{E}_n \times X_i)/T, (\mathbb{E}_n \times \tilde{X})/T) = m(X_i, \tilde{X})$, *for* $i = 1, \ldots, s$.

Proof. Recall from (5.4) that we have a cover by open subschemes

$$(\mathbb{E}_n \times \tilde{X})/T = \bigcup_{\mathbf{i} \in I_n} (U_{\mathbf{i}} \times \tilde{X})/T$$

and the isomorphisms $(U_{\mathbf{i}} \times \tilde{X})/T \cong \mathbb{C}^{r(n-1)} \times \tilde{X}$. Under these isomorphisms $(U_{\mathbf{i}} \times X_j)/T$ corresponds to $\mathbb{C}^{r(n-1)} \times X_j$. Thus, we conclude (i). For (ii), we set

$$A \coloneqq \mathcal{O}_{\tilde{X}, X_j}, \quad B \coloneqq \mathcal{O}_{\mathbb{C}^{r(n-1)} \times \tilde{X}, \mathbb{C}^{r(n-1)} \times X_j}.$$

Note that B is isomorphic as $\mathbb{C}$-algebra to the function field $A(x_1, \ldots, x_{r(n-1)})$. Pick an arbitrary $U_{\mathbf{i}} \subset \mathbb{E}_n$. Then, we have

$$m((\mathbb{E}_n \times X_j)/T, (\mathbb{E}_n \times \tilde{X})/T) = m((U_{\mathbf{i}} \times X_j)/T, (U_{\mathbf{i}} \times \tilde{X})/T) = l_B(B).$$

Since the inclusion $A \hookrightarrow B$ is flat, we conclude from e.g. [Ful84, Lemma A.4.1] that

$$l_B(B) = l_A(A) + l_B(B/\mathfrak{m}B),$$

where $\mathfrak{m} \subset A$ is the unique maximal ideal. We have $B/\mathfrak{m}B \cong k(x_1, \ldots, x_{r(n-1)})$ as rings, where $k = A/\mathfrak{m}$ is the residue field. Thus, $l_B(B/\mathfrak{m}B) = 0$ and we conclude

$$m((\mathbb{E}_n \times X_j)/T, (\mathbb{E}_n \times \tilde{X})/T) = l_A(A) = m(X_j, \tilde{X})$$

which completes the proof. □

Thus, we derive the following definition of T-equivariant fundamental classes:

Definition 5.3. The *T-equivariant fundamental class of* $\tilde{X}$ is defined as

$$[\tilde{X}]^T \coloneqq \sum_{i=1}^{s} m(X_i, \tilde{X}) j_{i*}[X_i]^T,$$

where $j_i \colon X_i \hookrightarrow X$ is the inclusion. Note that $[\tilde{X}]^T$ is an element in $\overline{H}_*^T(X)$.

One useful aspect of scheme theoretic fundamental classes is the following general result:

Proposition 5.4. *Let $U \subset \mathbb{P}^1$ be an open subvariety equipped with the trivial T-action and let $Z \subset X \times U$ be an irreducible T-invariant closed subvariety such that the projection $\pi \colon Z \to U$ is flat. Let p, $q \in U$ and $\pi^{-1}(p)$, $\pi^{-1}(q)$ be the scheme theoretic fibers. Then, in $\overline{H}_*^T(X)$ holds*

$$i_{p*}[\pi^{-1}(p)]^T = i_{q*}[\pi^{-1}(q)]^T,$$

where $i_p \colon \pi^{-1}(p) \hookrightarrow X$ and $i_q \colon \pi^{-1}(q) \hookrightarrow X$ are the respective closed immersions.

Proof. Let $\dim(Z) = d + 1$ and choose $n > d$. By (5.4), $(\mathbb{E}_n \times Z)/T$ is an irreducible closed subvariety of $((\mathbb{E}_n \times X)/T) \times U$ and the projection $\tilde{\pi}\colon (\mathbb{E}_n \times Z)/T \to U$ is flat. Hence, we deduce from e.g. [Ful84, Proposition 19.1.1] that $i_{p*}[\tilde{\pi}^{-1}(p)] = i_{q*}[\tilde{\pi}^{-1}(q)]$ in $\overline{H}_{2d}(\mathbb{E}_n \times X)$. By definition, we have

$$\tilde{\pi}^{-1}(p) = (\mathbb{E}_n \times \pi^{-1}(p))/T, \quad \tilde{\pi}^{-1}(q) = (\mathbb{E}_n \times \pi^{-1}(q))/T.$$

Hence, by Proposition 5.2, the isomorphism $\overline{H}_{2d+r(2n-1)}((\mathbb{E}_n \times X)/T) \cong \overline{H}^T_{2d}(X)$ maps $i_{p*}[\tilde{\pi}^{-1}(p)]$ to $i_{p*}[\pi^{-1}(p)]^T$ and $i_{q*}[\tilde{\pi}^{-1}(q)]$ to $i_{q*}[\pi^{-1}(q)]^T$. Thus, we conclude $i_{p*}[\pi^{-1}(p)]^T = i_{q*}[\pi^{-1}(q)]^T$. □

Localization Theorem

The Localization Theorem is a central ingredient of torus equivariant cohomology which provides a crucial exchange of local and global data. For its formulation, we set up some notation: Let pt be the topological space consisting of one single point. We view pt as variety with trivial T-action. Recall from e.g. [AF23, Example 1.1.2] that we have an isomorphism of $\mathbb{Q}$-algebras

$$H^*_T(\mathrm{pt}) \cong H^*((\mathbb{P}^\infty)^r) \cong \mathbb{Q}[t_1, \dots, t_r],$$

where the variable t_i corresponds to the first Chern class of the tautological bundle on the i-th factor of $(\mathbb{P}^\infty)^r$. In particular, each t_i is homogeneous of degree 2. Note that for every variety X with algebraic T-action, $H^*_T(X)$ is an algebra over $H^*_T(\mathrm{pt})$. Let $\mathrm{Char}(T)$ be the character lattice of T. We embed $\mathrm{Char}(T)$ into $H_T(\mathrm{pt})$, where we map the character

$$\tau_{a_1,\dots,a_r}\colon T \longrightarrow \mathbb{C}^*, \quad (x_1, \dots, x_r) \mapsto x_1^{a_1} \cdots x_r^{a_r}, \quad \text{for } a_1, \dots, a_r \in \mathbb{Z}$$

to the linear polynomial $a_1t_1 + \ldots + a_rt_r$. Let $S \subset H^*_T(\mathrm{pt})$ be the multiplicative set generated by the set

$$(\mathrm{Char}(T) \setminus \{0\}) = \{a_1t_1 + \ldots + a_rt_r \mid (a_1, \dots, a_r) \in \mathbb{Z}^r \setminus \{0\}\}. \tag{5.8}$$

Let $H^*_T(X)_{\mathrm{loc}} \coloneqq S^{-1}H^*_T(X)$ be the *localized T-equivariant cohomology of X*.

The following version of the Localization Theorem can be found in [AF23, Theorem 7.1.1]:

Theorem 5.5 (Localization)**.** *The inclusion $\iota\colon X^T \hookrightarrow X$ restricts to an isomorphism of $H^*_T(\mathrm{pt})_{\mathrm{loc}}$-algebras*

$$S^{-1}\iota\colon H^*_T(X)_{\mathrm{loc}} \xrightarrow{\sim} H^*_T(X^T)_{\mathrm{loc}}.$$

In the important special case where X^T is the disjoint union of isolated points, we have

$$H^*_T(X^T)_{\mathrm{loc}} \cong \prod_{p \in X^T} H^*_T(\{p\})_{\mathrm{loc}}.$$

For $p \in X^T$, let $\iota_p\colon \{p\} \hookrightarrow X$ be the inclusion. For $\alpha \in H^*_T(X)$, the restriction $\iota_p^*(\alpha) \in H^*_T(\{p\})$ is called the *equivariant multiplicity of α at p*. According to the Localization Theorem, the image of α in $H^*_T(X)_{\mathrm{loc}}$ is uniquely determined by the equivariant multiplicities of α at all T-fixed points of X. In the next subsection, we discuss possibilities to explicitly compute equivariant multiplicities.

Equivariant multiplicities

Assume that X is a smooth variety with algebraic T-action and let $p \in X^T$ be a T-fixed point of X. In the following, we recall important facts about equivariant multiplicities of Poincaré duals of fundamental classes of T-invariant subvarieties of X. For this, note that if $j\colon Y \hookrightarrow X$ is the inclusion of a closed T-invariant subvariety and $\alpha \in H_T^*(X)$ is the Poincaré dual of $j_*[Y]^T$ then

$$\iota_p^*(\alpha).[p]^T = [Y]^T \cap [p]^T.$$

We begin with the following important special case: Let V be a finite dimensional T-representation of dimension d. We denote the origin of V by p. $j\colon W \hookrightarrow V$ be the inclusion of a T-subrepresentation. We like to explicitly determine the T-equivariant intersection product $[W]^T \cap [p]^T$.

For this, let $\pi\colon V \to \{p\}$ be the projection and $\iota_p\colon \{p\} hookrightarrow V$ be the inclusion. By e.g. [AF23, Proposition 17.4.1], we have that $\pi^*\colon H_T^*(\mathrm{pt}) \xrightarrow{\sim} H_T^*(V)$ and $\iota_p^*\colon H_T^*(V) \xrightarrow{\sim} H_T^*(\mathrm{pt})$ are inverse isomorphisms of $\mathbb{Q}$-algebras. Let $s^*\colon \overline{H}_*^T(V) \xrightarrow{\sim} \overline{H}_*^T(\{p\})$ be the isomorphism of $H_T^*(\mathrm{pt})$-modules corresponding to ι_p^* via Poincaré duality. The isomorphism s^* is called the *T-equivariant Gysin isomorphism*. By definition, s^* is homogeneous of degree $-2d$, i.e. s^* maps $\overline{H}_i^T(V)$ to $\overline{H}_{i-2d}^T(\{p\})$, for all i. Furthermore, s^* satisfies $s^*[V]^T = [p]^T$ and we have $s^*(j_*[W]^T) = [W]^T \cap [p]^T$. By e.g. [AF23, Proposition 17.4.1], we have the following explicit formula for $s^*(j_*[W]^T)$: Let $V = \bigoplus_{i=1}^s \mathbb{C}_{\tau_i}^{m_i}$ and $W = \bigoplus_{i=1}^s \mathbb{C}_{\tau_i}^{n_i}$ be the respective T-weight space decompositions. Then, we have

$$s^*(j_*[W]^T) = e_T(V/W).[p]^T = \Big(\prod_{i=1}^s \tau_i^{m_i-n_i}\Big).[p]^T. \tag{5.9}$$

Here, e_T denotes the T-equivariant Euler class.

We now come to the general setup where X is smooth and p is a T-fixed point of X. Next, we recall how equivariant multiplicities on X can be characterized via Gysin pullbacks of fundamental classes of tangent cones. For this, let $Y \subset X$ be a closed T-invariant subvariety. Suppose Y contains p and let $\mathcal{I}$ be the ideal sheaf over Y corresponding to p. Let

$$C_pY \coloneqq \mathrm{Spec}_p\Big(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\Big)$$

be the tangent cone. Here, Spec_p denotes the relative spectrum over $\{p\}$. Note that C_pY is a possibly reduced scheme which admits a closed immersion $i\colon C_pY \hookrightarrow T_pX$. By construction, each irreducible component Z of C_pY is a conical subvariety of T_pX, i.e. Z is invariant under the $\mathbb{C}^*$-action on T_pX given by the usual scalar multiplication. Via the T-action on Y, C_pY is equipped with an algebraic T-action and the closed immersion i is T-equivariant. If p is a smooth point of Y then C_pY is canonically isomorphic to the tangent space T_pY.

By e.g. [AF23, Proposition 17.4.1], we have the following result:

Proposition 5.6. *We have $[Y]^T \cap [p]^T = s^*(i_*[C_pY]^T)$.*

As a direct consequence, we obtain the following formula for equivariant multiplicities for smooth subvarieties:

Corollary 5.7. *Let $j\colon Y \hookrightarrow X$ be a closed T-invariant subvariety and suppose $p \in Y$ is a smooth point of Y. Then, we have*

$$[Y]^T \cap [p]^T = e_T(T_pX/T_pY).[p]^T.$$

Proof. As p is a smooth point of Y, we have $T_pY = C_pY$. Let $i\colon T_pY \hookrightarrow T_pX$ be the corresponding inclusion. By Proposition 5.6, we have $[Y]^T \cap [p]^T = s^*(i_*[T_pY]^T)$. From (5.9) then follows $s^*(i_*[T_pY]^T) = e_T(T_pX/T_pY).[p]^T$. □

From Corollary 5.7, we deduce the following technical result that will be applied later in the uniqueness proof of stable envelopes:

Corollary 5.8. *Let $Y \subset X$ be a T-invariant closed subvariety, $U \subset Y$ be a smooth T-invariant open subvariety, $p \in U$ be a T-fixed point and*

$$\{p\} \xhookrightarrow{i} U \xhookrightarrow{j} Y \xhookrightarrow{\kappa} X$$

be the respective inclusions. Let $a \in \overline{H}_*^T(Y)$ and $\alpha \in H_T^*(X)$ be the Poincaré dual of $\kappa_*(a)$. Denote by $\beta \in H_T^*(U)$ the Poincaré dual of $j^*(a)$. Then, we have

$$\iota_p^*(\alpha) = e_T(T_pX/T_pU) \cdot i^*(\beta).$$

Proof. Since there exists an open T-invariant subvariety $V \subset X$ with $V \cap Y = U$, we may assume by Proposition 5.6 that $U = Y$. By (5.7) and (5.1),

$$\iota_p^*(\alpha).[p]^T = \iota_p^*(\alpha).([X]^T \cap [p]^T) = (\alpha.[X]^T) \cap [p]^T = \kappa_*(a) \cap [p]^T. \tag{5.10}$$

By the definition of the T-equivariant intersection product, we have

$$(5.10) = a \cap [p]^T. \tag{5.11}$$

Applying first (5.1) and then Corollary 5.7 yields

$$(5.11) = (\beta.[Y]^T) \cap [p]^T = i^*(\beta).([Y]^T \cap [p]^T) = (i^*(\beta) \cdot e_T(T_pX/T_pY)).[p]^T.$$

Hence, $\iota_p^*(\alpha) = i^*(\beta) \cdot e_T(T_pX/T_pY)$ which proves the corollary. □

For better readability, we use the following convention: Given an inclusion of a T-invariant subvariety $j\colon Y \hookrightarrow X$, we also denote the pushforward of a T-equivariant fundamental class $j_*[Y]^T$ in $\overline{H}_*^T(X)$ just by $[Y]^T$.

5.2 Stable envelopes

We return to the setup where X is a bow variety $\mathcal{C}(\mathcal{D})$ and T is either $\mathbb{A}$ or $\mathbb{T}$. We denote the equivariant parameters by $H_{\mathbb{A}}^*(\mathrm{pt}) = \mathbb{Q}[t_1, \ldots, t_N]$ and $H_{\mathbb{T}}^*(\mathrm{pt}) = \mathbb{Q}[t_1, \ldots, t_N, h]$ respectively.

A crucial definition from [MO19] is the following:

Definition 5.9. Let d be the dimension of $\mathcal{C}(\mathcal{D})$ as complex variety. *Stable envelopes* are maps, depending on a choice of a chamber $\mathfrak{C}$ of $\mathbb{A}$:

$$\mathcal{C}(\mathcal{D})^{\mathbb{T}} \xrightarrow{\mathrm{Stab}_{\mathfrak{C}}} H^d_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$$

which are uniquely characterized by the properties (Stab-1)-(Stab-3) from Theorem 5.10, called the *normalization*, *support* and *smallness* condition, respectively.

Theorem 5.10 (Stable envelopes)**.** *Fix a chamber $\mathfrak{C}$ of $\mathbb{A}$. Then, there exist a unique family $(\mathrm{Stab}_{\mathfrak{C}}(p))_{p\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ of elements in $H^d_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ satisfying the following conditions:*

(Stab-1) We have $\iota_p^*(\mathrm{Stab}_{\mathfrak{C}}(p)) = e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}})$, for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.

(Stab-2) We have that $\mathrm{Stab}_{\mathfrak{C}}(p)$ is supported on $\mathrm{Attr}^f_{\mathfrak{C}}(p)$, for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.

(Stab-3) Let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $q \prec p$. Then, $\iota_q^*(\mathrm{Stab}_{\mathfrak{C}}(p))$ is divisible by h.

Recall that a $\mathbb{T}$-equivariant cohomology class $\gamma \in H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ is supported on a $\mathbb{T}$-invariant closed subvariety $A \subset \mathcal{C}(\mathcal{D})$ if and only if $j^*(\gamma) = 0$, where $j\colon \mathcal{C}(\mathcal{D}) \setminus A \hookrightarrow \mathcal{C}(\mathcal{D})$ is the inclusion.

The normalization and support condition directly imply that stable envelopes provide a basis of the localized equivariant cohomology ring:

Corollary 5.11 (Stable envelope basis)**.** *For a fixed chamber $\mathfrak{C}$ of $\mathbb{A}$, the $\mathbb{T}$-equivariant cohomology classes $(\mathrm{Stab}_{\mathfrak{C}}(p))_{p\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ form a $H^*_{\mathbb{T}}(\mathrm{pt})_{\mathrm{loc}}$-basis of $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$.*

Definition 5.12. We refer to $(\mathrm{Stab}_{\mathfrak{C}}(p))_{p\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ as *stable envelope basis corresponding to* $\mathfrak{C}$ and to the individual $\mathbb{T}$-equivariant cohomology classes $\mathrm{Stab}_{\mathfrak{C}}(p)$ as *stable basis elements.*

Remark. The stable envelope maps $\mathrm{Stab}_{\mathfrak{C}}$ provide a map

$$\{\text{Chambers}\} \longrightarrow \{\text{Bases of } H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}\}.$$

It is a central result of Maulik and Okounkov that the base change matrices with respect to adjacent chambers give solutions to Yang–Baxter equations, providing an interesting connection to the theory of integrable systems. In the special case of cotangent bundles of partial flag varieties the corresponding integrable system is the inhomogeneous XXX model for general linear Lie algebras; see [MO19] for more details.

Remark. In the case of Nakajima quiver varieties, the definition of stable envelopes in [MO19] also includes a choice of signs in the normalization axiom, that corresponds to a choice of polarization of the involved Nakajima quiver variety. Polarizations can be defined in the setting of bow varieties too, see [Sho21, Section 4.4.1] and one could work with the more general definition. For simplicity, we however choose here all signs to be 1.

Matching under Hanany–Witten transition

Before we come to the proof of the uniqueness and existence of stable envelopes, we show that stable envelopes are compatible with Hanany–Witten transitions in the following way: Suppose $\tilde{\mathcal{D}}$ is obtained from $\mathcal{D}$ by Hanany–Witten transition. Let $\Phi\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})$ be the corresponding Hanany–Witten isomorphism and $\Phi^*\colon H^*_{\mathbb{T}}(\mathcal{C}(\tilde{\mathcal{D}})) \xrightarrow{\sim} H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ be the induced isomorphism. Furthermore, let $\rho\colon \mathbb{T} \xrightarrow{\sim} \mathbb{T}$ be the automorphism of algebraic groups from Proposition 2.52 and $\rho^*\colon H^*_{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\sim} H^*_{\mathbb{T}}(\mathrm{pt})$ be the induced isomorphism.

Proposition 5.13. *For all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have*

$$\mathrm{Stab}_{\mathfrak{C}}(p) = \Phi^*(\mathrm{Stab}_{\mathfrak{C}}(\Phi(p))).$$

Proof. Since $\Phi^*(h) = h$, we conclude that $\Phi^*(\mathrm{Stab}_{\mathfrak{C}}(\Phi(p)))$ satisfies the smallness condition. As Φ is $\mathbb{A}$-equivariant, we have

$$\Phi^{-1}(\mathrm{Attr}_{\mathfrak{C}}(\Phi(q))) = \mathrm{Attr}_{\mathfrak{C}}(q), \quad \text{for all } q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}. \tag{5.12}$$

This implies the support condition for $\Phi^*(\mathrm{Stab}_{\mathfrak{C}}(\Phi(p)))$. Denote by $\Lambda_p \in H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ and $\tilde{\Lambda}_p \in H^*_{\mathbb{T}}(\mathcal{C}(\tilde{\mathcal{D}}))$ the Poincaré duals of the $\mathbb{T}$-equivariant fundamental classes of $\overline{\mathrm{Attr}_{\mathfrak{C}}(p)}$ and $\overline{\mathrm{Attr}_{\mathfrak{C}}(\Phi(p))}$ respectively. Then, (5.12) yields $\Phi^*(\tilde{\Lambda}_p) = \Lambda_p$. Thus, we conclude

$$e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}}) = \iota_p^*(\Lambda_p) = \iota_p^*(\Phi^*(\tilde{\Lambda}_p)). \tag{5.13}$$

The normalization condition for $\mathrm{Stab}_{\mathfrak{C}}(\Phi(p))$ yields

$$\iota^*_{\Phi(p)}(\tilde{\Lambda}_p) = e_{\mathbb{T}}(T_{\Phi(p)}\mathcal{C}(\tilde{\mathcal{D}})^-_{\mathfrak{C}}) = \iota^*_{\Phi(p)}(\mathrm{Stab}_{\mathfrak{C}}(\Phi(p))).$$

Thus, we conclude that

$$(5.13) = \rho^*(\iota^*_{\Phi(p)}(\mathrm{Stab}_{\mathfrak{C}}(\Phi(p)))) = \iota_p^*(\Phi^*(\mathrm{Stab}_{\mathfrak{C}}(\Phi(p)))).$$

This proves the normalization condition for $\Phi^*(\mathrm{Stab}_{\mathfrak{C}}(\Phi(p)))$. □

In the remainder of this chapter, we give a proof of Theorem 5.10 following [MO19, Chapter 3].

Uniqueness of stable envelopes

We now prove the uniqueness statement of Theorem 5.10.

Proof of Theorem 5.10 (Uniqueness). If $(\mathrm{Stab}_{\mathfrak{C}}(p))_{p\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ and $(\mathrm{Stab}'_{\mathfrak{C}}(p))_{p\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ satisfy the conditions of Theorem 5.10 then the family $(\mathrm{Stab}_{\mathfrak{C}}(p) - \mathrm{Stab}'_{\mathfrak{C}}(p))_{p\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ satisfies the conditions of Lemma 5.14 below. Hence, we have $\mathrm{Stab}_{\mathfrak{C}}(p) = \mathrm{Stab}'_{\mathfrak{C}}(p)$, for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. □

Lemma 5.14. *Assume $(\gamma_p)_{p\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ is a family of homogeneous equivariant cohomology classes in $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ of degree $d = \dim(\mathcal{C}(\mathcal{D}))$ satisfying the following two conditions:*

(a) For all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, the class γ_p is supported on $\mathrm{Attr}^f_{\mathfrak{C}}(p)$.

(b) Let $p, q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $q \preceq p$. Then, $\iota_q^*(\gamma_p)$ is divisible by h.

Then, $\gamma_p = 0$, for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.

Proof. Fix $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ and let $\leq$ be a total order on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$ which refines $\preceq$. Let $n = |\mathcal{C}(\mathcal{D})^{\mathbb{T}}|$ and denote the $\mathbb{T}$-fixed points of $\mathcal{C}(\mathcal{D})$ by $q_1, \dots, q_n$, where $q_i \leq q_j$ if and only if $i \leq j$. Let $i_0 \in \{1, \dots, n\}$ such that $p = q_{i_0}$. Furthermore, we set

$$A_i := \bigsqcup_{j=1}^{i} \mathrm{Attr}_{\mathfrak{C}}(q_j), \quad \text{for } i = 1, \dots, n$$

and $A_0 := \emptyset$. According to Proposition 4.20, each A_i is a closed $\mathbb{T}$-invariant subvariety of $\mathcal{C}(\mathcal{D})$ and contains $\mathrm{Attr}_{\mathfrak{C}}(q_i)$ as open subvariety. To prove $\gamma_p = 0$, we show that

if γ_p is supported on A_i for some $i \in \{1, \dots, n\}$, then, γ_p is also supported on A_{i-1}.

This implies $\gamma_p = 0$, since γ_p is supported on A_0 and thus has empty support.

So let us prove the above statement. Let $\kappa\colon A_i \hookrightarrow \mathcal{C}(\mathcal{D})$ be the inclusion. Since γ_p is supported on A_i, there exists $a \in \overline{H}_*^{\mathbb{T}}(A_i)$ such that $\kappa_*(a)$ is the Poincaé dual of γ_p. Let $f\colon \{q_i\} \hookrightarrow \mathrm{Attr}_{\mathfrak{C}}(q_i)$ and $j\colon \mathrm{Attr}_{\mathfrak{C}}(q_i) \hookrightarrow A_i$ denote the inclusions and let $\beta \in H_{\mathbb{T}}^*(\mathrm{Attr}_{\mathfrak{C}}(q_i))$ be the Poincaré dual of $j^*(a)$. By Corollary 5.8, we have

$$\iota_{q_i}^*(\gamma_p) = e_{\mathbb{T}}(T_{q_i}\mathcal{C}(\mathcal{D})_{\mathfrak{C}}^-) \cdot f^*(\beta). \tag{5.14}$$

By Corollary 3.24, $e_{\mathbb{T}}(T_{q_i}\mathcal{C}(\mathcal{D})_{\mathfrak{C}}^-)$ is homogeneous of degree d and not divisible by h. Thus, by condition (b) and (5.14), we conclude $f^*(\beta) = 0$. By Proposition 4.4, f^* is an isomorphism of rings. Therefore, $\beta = 0$ which is equivalent to a being supported on A_{i-1}. Hence, γ_p is supported on A_{i-1} as well. □

5.3 Existence of stable envelopes

The remainder of this chapter is devoted to the existence of stable envelopes. We will see that they can be constructed using an iterative procedure based on general properties of equivariant multiplicities of lagrangian subvarieties which are stated in Theorem 5.15. This theorem will be proved in Sections 5.4 and 5.5 using the deformation to the tangent cone construction from [Ful84, Section 5.1] and further deformation techniques which are similar to those from Section 4.6. In Section 5.6, we finally illustrate the explicit construction of stable basis elements in an example.

Equivariant multiplicities of lagrangian subvarieties

We now come to the main ingredient in the proof of the existence of stable envelopes. We call this the *Langrangian Multiplicity Theorem* since it characterizes the equivariant multiplicity of lagrangian subvarieties of symplectic varieties at points in therms of the tangent weights at p.

Let X be a smooth symplectic variety of dimension $2n$. We assume that X is endowed with an algebraic action of a torus $T = (\mathbb{C}^*)^r$ such that the symplectic form ω of X is invariant under the T-action. Recall that a closed subvariety $L \subset X$ is called *isotropic* if the restriction of ω to the smooth locus L_{sm} of L vanishes. We call L *lagrangian* if $\dim(L) = n$ and moreover, L is isotropic.

Given an isolated T-fixed point $p \in X$, we can choose a decomposition into T-invariant subspaces $T_pX = V_1 \oplus V_2$ such that ω_p induces an isomorphism $V_1 \cong V_2^*$.

Theorem 5.15 (Langrangian Multiplicity Theorem)**.** *Suppose V_1 admits the T-weight decomposition $V_1 = \bigoplus_{i=1}^n \mathbb{C}_{\chi_i}$, where $\chi_1, \dots, \chi_n$ are characters of T which we view as homogeneous elements in $H_T^*(\{p\})$. Given a T-invariant lagrangian subvariety $L \subset X$, we can find $a_{p,L} \in \mathbb{Z}$ such that the following equality holds in $\overline{H}_*^T(\{p\})$:*

$$[L]^T \cap [p]^T = \Big(a_{p,L}\Big(\prod_{i=1}^n \chi_i\Big)\Big).[p]^T.$$

We prove Theorem 5.15 in Section 5.5. First, we apply this result to the setting of bow varieties. For this, set $L_p \coloneqq \overline{\mathrm{Attr}_{\mathfrak{C}}(p)}$, for $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. The next proposition gives that L_p is indeed a lagrangian subvariety of $\mathcal{C}(\mathcal{D})$.

Proposition 5.16. *For all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, the variety L_p is a lagrangian subvariety of $\mathcal{C}(\mathcal{D})$.*

We first prove an auxiliary statement:

Lemma 5.17. *Let V be a finite dimensional $\mathbb{C}^*$-represenation with all $\mathbb{C}^*$-weight spaces strictly positive. Suppose $\varpi\colon TV \times TV \to \mathbb{C}$ is a $\mathbb{C}^*$-invariant bilinear form. Then, $\varpi = 0$.*

Proof. We use the identification $TV \cong V \times V$. Suppose $w \in V$ and v_1, $v_2 \in V$ are weight vectors of repsective weights a_1, a_2. Then, we have

$$\varpi_w(v_1, v_2) = \varpi_{t.w}(t^{a_1}v_1, t^{a_2}v_2), \quad \text{for all } t \in \mathbb{C}^*.$$

By continuity (in the analytic topology), we deduce

$$\varpi_w(v_1, v_2) = \lim_{t \to 0}\big(\varpi_{t.w}(t^{a_1}v_1, t^{a_2}v_2)\big) = 0.$$

Hence, $\varpi_w = 0$. □

Proof of Proposition 5.16. Recall from Proposition 4.4 that we have a $\mathbb{T}$-equivariant isomorphism of varieties $\mathrm{Attr}_{\mathfrak{C}}(p) \cong T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}}$. Let ϖ be the restriction of the symplectic form ω' of $\mathcal{C}(\mathcal{D})$ to $T\,\mathrm{Attr}_{\mathfrak{C}}(p) \times T\,\mathrm{Attr}_{\mathfrak{C}}(p)$. As ω' is $\mathbb{A}$-invariant, so is ϖ. Choose $\sigma \in \mathfrak{C}$ and view $\mathrm{Attr}_{\mathfrak{C}}(p)$ as $\mathbb{C}^*$-representation via σ. Since all $\mathbb{C}^*$-weights of $\mathrm{Attr}_{\mathfrak{C}}(p)$ are strictly positive, Lemma 5.17 implies $\varpi = 0$. As $\mathrm{Attr}_{\mathfrak{C}}(p)$ is an open dense subvariety of L_p, we conclude that L_p is lagrangian. □

Combining Theorem 5.15 and Proposition 5.16 gives the following consequence:

Corollary 5.18. *Let p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ and suppose $p \in L_q$. Then,*

$$[L_q]^{\mathbb{A}} \cap [p]^{\mathbb{A}} = a_{p,q}e_{\mathbb{A}}(T_p\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}}).[p]^{\mathbb{A}},$$

where $a_{p,q}$ is an integer depending on p and q.

Proof. By Proposition 5.16, $L_q \subset \mathcal{C}(\mathcal{D})$ is a lagrangian subvariety. Applying Theorem 5.15 according to the decomposition $T_p\mathcal{C}(\mathcal{D}) = T_p\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}} \oplus T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}}$ then finishes the proof. □

Proof of existence of stable envelopes

We now use Corollary 5.18 to give a direct construction of stable envelopes. For this, we use the following notation: For $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, let $\Lambda_p \in H^d_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ be the Poincaré dual of $[L_p]^{\mathbb{T}}$.

Proof of Theorem 5.10. Let $\preceq'$ be a total order on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$ refining $\preceq$ and let s be the cardinality of $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$. Denote the elements of $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$ by $p_1, \ldots, p_s$, where the labeling is compatible with our choice of total ordering, i.e. we have $p_i \preceq' p_j$ if and only if $i \leq j$. For each $i \in \{1, \ldots, s\}$, we construct a family of cohomology classes $\gamma_{i,1}, \ldots, \gamma_{i,i} \in H^d_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ such that each $\gamma_{i,j}$ satisfies the following three properties:

(a) $\iota^*_{p_i}(\gamma_{i,j}) = e_{\mathbb{T}}(T_{p_i}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}})$,

(b) there exist $a_{i,j,1}, \ldots, a_{i,j,i-j} \in \mathbb{Z}$ such that $\gamma_{i,j} = \Lambda_{p_i} + \sum_{l=1}^{i-j} a_{i,j,l}\Lambda_{p_{i-l}}$,

(c) we have that $\iota^*_{p_l}(\gamma_{i,j})$ is divisible by h, for $l = i-1, i-2, \ldots, j$.

We set $\gamma_{i,i} := \Lambda_{p_i}$. Then, $\gamma_{i,i}$ clearly satisfies the properties (a)-(c). Suppose $\gamma_{i,i}, \ldots, \gamma_{i,j}$ have been constructed. Then, we define $\gamma_{i,j-1}$ as follows: Since $\gamma_{i,j}$ satisfies (b), we know by Corollary 5.18 that there exists $a \in \mathbb{Z}$ such that

$$\iota^*_{p_{j-1}}(\gamma_{i,j}) \equiv a e_{\mathbb{T}}(T_{p_{j-1}}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}}) \mod h.$$

Set $\gamma_{i,j-1} := \gamma_{i,j} - a\Lambda_{p_{j-1}}$. By construction, $\gamma_{i,j-1}$ satisfies (b) and $\iota^*_{p_{j-1}}(\gamma_{i,j-1})$ is divisible by h. Hence, properties (a) and (c) follow from $p_i, p_{i-1}, \ldots, p_j \notin L_{p_{j-1}}$. Thus, $\gamma_{i,j-1}$ satisfies all the desired properties.

Now, set $\mathrm{Stab}_{\mathfrak{C}}(p_i) := \gamma_{i,1}$, for $i = 1, \ldots, s$. Then, the normalization condition follows immediately from (a), the support condition from (b) and the smallness condition from (c). This completes the proof of Theorem 5.10. □

The proof of Theorem 5.10 directly gives the following consequence:

Corollary 5.19. *For all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have*

$$\mathrm{Stab}_{\mathfrak{C}}(p) = \sum_{q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}} a_{p,q}\Lambda_q,$$

where $a_{p,q} \in \mathbb{Z}$ with $a_{p,p} = 1$ and $a_{p,q} = 0$ if $q \not\preceq p$.

Note that the coefficients $a_{p,q}$ from Corollary 5.19 are uniquely determined by Corollary 5.11.

5.4 Tangent cones of lagrangian subvarieties

In this section, we pass to the analytic setup. The definition of tangent cones in the analytic framework is analogous to the definition of normal cones in the algebraic framework: Let X be a complex manifold and $Y \subset X$ closed analytic subvariety. Let $p \in Y$ be a point and $\mathcal{I}$ the ideal sheaf over Y corresponding to p. The *tangent cone of p in Y* is defined as

$$\mathrm{Specan}\Big(\bigoplus_{n \geq 0} \mathcal{I}^n_p / \mathcal{I}^{n+1}_p\Big).$$

Here, $\mathcal{I}_p^n$ is the stalk of $\mathcal{I}^n$ at p and Specan denotes the analytic spectrum, see e.g. [GPR94, Section II.3] for a definition.

By e.g. [Ser56, Proposition 3], we have that if X is a smooth (algebraic) variety and $Y \subset X$ a closed subvariety with respective analytifications X^{an}, Y^{an} then the canonical isomorphism $T_pX^{\mathrm{an}} \cong (T_pX)^{\mathrm{an}}$ restricts to an isomorphism of analytic spaces $C_pY^{\mathrm{an}} \cong (C_pY)^{\mathrm{an}}$.

The main result of this section is the following proposition:

Proposition 5.20. *Let X be a complex symplectic manifold with symplectic form ω and let $L \subset X$ be a lagrangian subvariety. Then, for all $p \in L$, we have that all irreducible components of the tangent cone C_pL are lagrangian subvarieties of T_pX. Here, we view T_pX as complex symplectic manifold via the symplectic form ω_p.*

For the proof, recall the holomorphic Darboux Theorem:

Theorem 5.21 (Darboux)**.** *Let X be a symplectic complex manifold of dimension $2n$. We equip $\mathbb{C}^{2n}$ with the standard symplectic form*

$$\xi \coloneqq \sum_{i=1}^{n} dx_i \wedge dx_{i+n}, \tag{5.15}$$

where $(x_1, \dots, x_{2n})$ are the coordinates of $\mathbb{C}^{2n}$. Then, for all $x \in X$, there exists an analytic neighborhood $U \subset X$ such that U is isomorphic as symplectic complex manifold to an open subset of $\mathbb{C}^{2n}$.

From now on, let $X \subset \mathbb{C}^{2n}$ be an open subset containing the origin which we denote by p. Via (5.15), we view X as symplectic complex manifold. To prove Proposition 5.20, we use the deformation to the tangent cone construction from [Ful84, Section 5.1]. For this, we recall some basic properties of blow ups in the analytic framework. For more details on this subject, see e.g. [GPR94, Section VII.2] and [Fis76, Section 4.1].

Definition 5.22. Let $Y \subset X$ be a closed analytic subvariety containing p. The *blow up* Bl_pY *of Y at p* is defined as the analytic closure of

$$\{((y_1, \dots, y_{2n}), [y_1 \colon \dots \colon y_{2n}]) \mid (y_1, \dots, y_{2n}) \in Y \setminus \{p\}\} \subset Y \times \mathbb{P}^{2n-1}.$$

By construction, Bl_pY is a closed analytic subvariety of $Y \times \mathbb{P}^{2n-1}$. If $Z \subset X$ is a closed analytic subvariety containing Y then Bl_pY is a closed analytic subvariety of Bl_pZ. Let $\pi_Y \colon \mathrm{Bl}_pY \to Y$ be the projection. Then, π induces an isomorphism $\pi_Y^{-1}(Y \setminus \{p\}) \xrightarrow{\sim} Y \setminus \{p\}$.

Definition 5.23. Let $Y \subset X$ be a closed analytic subvariety containing p. The *deformation to the tangent cone of X at p* is defined as

$$M_p^0Y \coloneqq (\mathrm{Bl}_{\hat{p}}(Y \times \mathbb{C})) \setminus \mathrm{Bl}_pY,$$

where $\hat{p}$ is the origin in $Y \times \mathbb{C}$. We view Y as closed analytic subvariety of $Y \times \mathbb{C}$ via $y \mapsto (y, 0)$.

If $\mathbb{P}^{2n-1} \hookrightarrow \mathbb{P}^{2n}$ is the inclusion $[x_1 \colon \dots \colon x_{2n}] \mapsto [x_1 \colon \dots \colon x_{2n} \colon 0]$ then

$$M_p^0Y = (\mathrm{Bl}_{\hat{p}}(Y \times \mathbb{C})) \setminus (Y \times \mathbb{P}^{2n-1}).$$

The following proposition, see [Ful84, Section 5.1], gives that M_p^0Y indeed transforms Y into the tangent cone C_pY:

Proposition 5.24. *Let $\hat{\pi}_Y \colon M_p^0 Y \to Y \times \mathbb{C}$ be the projection.*

(i) We have that $\hat{\pi}_Y$ induces an isomorphism of complex analytic spaces

$$\hat{\pi}_Y^{-1}(Y \times \mathbb{C}^*) \xrightarrow{\sim} Y \times \mathbb{C}^*. \tag{5.16}$$

(ii) The fiber $\hat{\pi}_Y^{-1}(\hat{p})$ is a hypersurface of $M_p^0 Y$.

(iii) There exists an isomorphism $\hat{\pi}_Y^{-1}(\hat{p}) \cong C_p Y$ of complex analytic spaces such that the following diagram commutes

$$\begin{array}{ccc} \hat{\pi}_Y^{-1}(\hat{p}) & \xrightarrow{\sim} & C_p Y \\ \big\downarrow & & \big\downarrow \\ \hat{\pi}_X^{-1}(\hat{p}) & \xrightarrow{\sim} & T_p X \end{array}$$

Here, $C_p Y \hookrightarrow T_p X$ is the standard inclusion.

Remark. In [Ful84, Section 5.1], the deformation to the tangent cone is considered in the setup of algebraic varieties. The results transfer directly to the analytic setup.

The complex manifold structure on $M_p^0 X$ can be characterized as follows: Set

$$X' \coloneqq \{(t^{-1}x_1, \ldots, t^{-1}x_{2n}, t) \mid (x_1, \ldots, x_{2n}) \in X,\ t \in \mathbb{C}^*\} \cup (\mathbb{C}^{2n} \times \{0\}) \subset \mathbb{C}^{2n+1}. \tag{5.17}$$

Then, there is an isomorphism of complex manifolds

$$M_p^0(X) \xrightarrow{\sim} X'$$

given by

$$((x_1, \ldots, x_{2n}, t), [x_1 \colon \ldots \colon x_{2n} \colon t]) \mapsto (t^{-1}x_1, \ldots, t^{-1}x_{2n}, t), \quad \text{for } t \neq 0$$

and

$$((0, \ldots, 0), [x_1 \colon \ldots \colon x_{2n} \colon 1]) \mapsto (x_1, \ldots, x_{2n}, 0).$$

Under this identification, the inclusion $T_p X \hookrightarrow M_p^0 X$ corresponds to

$$\iota \colon T_p X \longhookrightarrow X', \quad \frac{\partial}{\partial x_i} \mapsto e_i, \tag{5.18}$$

where e_i denotes the i-th standard basis vector in $\mathbb{C}^{2n+1}$.

A further convenient property of the deformation to the tangent cone construction is that it deforms the symplectic form ξ from (5.15) into the symplectic form ξ_p on $T_p X$ in the following sense: Consider the holomorphic bilinear form

$$\hat{\xi} = \mathrm{pr}^* \xi \colon TX' \times TX' \longrightarrow \mathbb{C},$$

where $\mathrm{pr} \colon X' \to \mathbb{C}^{2n}$ is the projection. Then, by (5.18), we have

$$\iota^* \hat{\xi} = \xi_p. \tag{5.19}$$

Let $f\colon X \times \mathbb{C}^* \hookrightarrow X'$ be the open embedding from (5.16) using the identification $M_pX \cong X'$. Let $\iota_t\colon X \hookrightarrow X'$ be the inclusion

$$\iota_t\colon X \xrightarrow{\sim} X \times \{t\} \xrightarrow{f} X'.$$

Then, we have

$$\iota_t^* \hat{\xi} = t^{-2}\xi. \tag{5.20}$$

Combining these pieces, we deduce a proof of Proposition 5.20:

Proof of Proposition 5.20. By the holomorphic Darboux Theorem, we may assume that X is an analytic open neighborhood of the origin $p \in \mathbb{C}^2$ and X admits the symplectic structure ξ from (5.15). Let $L \subset X$ be a lagrangian subvariety containing p. With the above notation, (5.20) implies that $\hat{\xi}$ vanishes on the tangent bundle $T(L_{\mathrm{sm}} \times \mathbb{C}^*)$. Let Z be an irreducible component of C_pL. By Proposition 5.24, $L_{\mathrm{sm}} \times \mathbb{C}^*$ is an open dense analytic subvariety of M_p^0L and $\dim(Z) < \dim(L_{\mathrm{sm}} \times \mathbb{C}^*)$. Hence, Proposition 5.25 below gives that TZ_{sm} is contained in the analytic closure of $T(L_{\mathrm{sm}} \times \mathbb{C}^*)$ in TM_p^0L. Thus, $\hat{\xi}$ also vanishes on TZ_{sm} which implies that Z is a lagrangian subvariety of T_pX by (5.19). $\square$

Approximation tangent vectors of analytic varieties

In the proof of Proposition 5.20 we used the following result:

Proposition 5.25. *Let X be a complex manifold and Y, $Z \subset X$ be locally closed smooth irreducible analytic subvarieties such that Z is contained in the closure of Y in X. Then, the closure of TY in TX contains TZ.*

We prove Proposition 5.25 via a regularity result due to Whitney. For its formulation, we first recall some properties of Grassmannians. Let $\mathrm{Gr}(k,n)$ be the Grassmannian parameterizing k-dimensional subvector spaces of $\mathbb{C}^n$. Via the analytic topology, we consider consider $\mathrm{Gr}(k,n)$ as compact complex manifold. The analytic topology on $\mathrm{Gr}(k,n)$ is metrizable via the distance function

$$\mathrm{dist}(V,W) = \max_{v\in V,\ |v|=1} \Big(\min_{w\in W} |v-w| \Big), \quad \text{for } V,\ W \in \mathrm{Gr}(k,n),$$

where $|\cdot|$ is the usual euclidean absolute value on $\mathbb{C}^n$.

Lemma 5.26. *Suppose V_n is a sequence in $\mathrm{Gr}(k,n)$ converging to $W \in \mathrm{Gr}(k,n)$. Then, for all $w \in W$, there exists a sequence v_n in $\mathbb{C}^n$ with $v_n \in V_n$ and v_n converges to w in $\mathbb{C}^n$.*

Proof. We may assume $|w| = 1$. Choose $v_n \in V_n$ with $|v_n - w| \leq \mathrm{dist}(W, V_n)$. Then, the sequence v_n converges to w. $\square$

Suppose now that X is an open analytic subvariety of $\mathbb{C}^n$ and $Y \subset X$ is a locally closed irreducible smooth analytic subvariety of dimension k. For each point $y \in Y$ there exists an open neighborhood $U \subset X$ of y and holomorphic functions $f_1, \ldots, f_r\colon U \to \mathbb{C}$ such that $Y \cap U$ equals the vanishing locus of $f_1, \ldots, f_r$. Via the identification of vector spaces

$$T_yY \cong \{v \in \mathbb{C}^n \mid d_yf_i(v) = 0,\ \text{for i=1,\ldots,r}\},$$

we consider T_yY as element in $\mathrm{Gr}(k,n)$ and TY as locally closed smooth subvariety of $TX \cong X \times \mathbb{C}^n$.

The following definition is due to Whitney [Whi65, Section 19]:

Definition 5.27. In the above situation, let $Z \subset X$ be a further smooth locally closed irreducible smooth subvariety. We say that Y is *(a)-regular over Z* if for all sequences y_n in Y such that y_n converges to a point $z \in Z$ and the sequence $T_{y_n}Y$ converges to some $W \in \mathrm{Gr}(k,n)$, we have $T_zZ \subset W$.

We have the following fundamental lemma, see [Whi65, Lemma 19.3]:

Lemma 5.28. *Let Y, $Z \subset X$ be locally closed irreducible smooth analytic subvarieties with $\dim(Z) < \dim(Y)$. Then, there exists an open dense subvariety $U \subset Z$ such that Y is (a)-regular over U.*

Proof of Proposition 5.25. We may assume that X is an open analytic subvariety of $\mathbb{C}^n$. As before, let $k = \dim(Y)$. Since Z is contained in the closure of Y, we have $\dim(Z) < k$. Hence, by Lemma 5.28, there exists open dense subvariety $U \subset Z$ such that Y is (a)-regular over U. Let $u \in U$ and $v \in T_uU$. Let y_n be a sequence in Y such that y_n converges to u. Since $\mathrm{Gr}(k,n)$ is compact, we can assume that the sequence $T_{y_n}Y$ converges to some W in $\mathrm{Gr}(k,n)$. By (a)-regulariy, $T_uU \subset W$. Thus, by Lemma 5.26, there exist $v_n \in T_{y_n}Y$ such that the sequence (y_n, v_n) converges to (u,v) in TX. Hence, TU is contained in the closure of TY in TX. As U is dense in Z, the closure of TU in TX contains TZ. Therefore, the closure of TY in TX contains TZ. □

5.5 Flat deformations of conical lagrangian subvarieties

We now return to the algebraic setting of Theorem 5.15: Let X be a smooth symplectic variety with algebraic T-action. We assume that the symplectic form ω on X is T-invariant. Let $p \in X$ and L be a lagrangian subvariety containing p. In Proposition 5.20, we proved that the irreducible components of the tangent cone C_pL are lagrangian subvarieties of T_pX. The next proposition shows that it is in fact possible to deform C_pL into a possibly non-reduced union of lagrangian hyperplanes. This enables us to characterize the equivariant multiplicity of L at p.

Proposition 5.29. *We have $[C_pL]^T = \sum_{i=1}^s m_i[H_i]^T$ in $\overline{H}_*^T(T_pX)$, where $H_1, \dots, H_s \subset T_pX$ are T-invariant lagrangian hyperplanes and $m_1, \dots, m_s \in \mathbb{N}_0$.*

Assuming Proposition 5.29, we obtain directly a proof of Theorem 5.15.

Proof of Theorem 5.15. Recall, with the notation of Theorem 5.15, that T_pX admits the T-weight space decomposition $T_pX \cong \bigoplus_{i=1}^n (\mathbb{C}_{\chi_i} \oplus \mathbb{C}_{-\chi_i})$. Thus, if $H \subset T_pX$ is a T-invariant lagrangian hyperplane then H admits the weight space decomposition $H \cong \bigoplus_{i=1}^n \mathbb{C}_{\varepsilon_i\chi_i}$, where $\varepsilon_i \in \{\pm 1\}$. Hence, Corollary 5.7 implies

$$s^*([H]^T) = \Big(\prod_{i=1}^n \varepsilon_i\chi_i\Big).[p]^T, \tag{5.21}$$

where $s^*\colon \overline{H}^T_*(T_pX) \xrightarrow{\sim} \overline{H}^T_*(\{p\})$ is the Gysin isomorphism. Recall from Proposition 5.6 that $[L]^T \cap [p]^T = s^*([C_pL]^T)$. Finally, by Proposition 5.29 and (5.21), we conclude that $s^*([C_pL]^T)$ is an integer multiple of $(\prod_{i=1}^n \chi_i).[p]^T$ which completes the proof. □

The remainder of this section is devoted to the proof of Proposition 5.29. Consider the symplectic vector space $\mathbb{C}^{2n}$ with basis $e_1, \dots, e_{2n}$ and standard symplectic form

$$\varpi(e_i, e_{n+j}) = -\varpi(e_{n+j}, e_i) = \delta_{i,j}, \quad \varpi(e_i, e_j) = \varpi(e_{n+i}, e_{n+j}) = 0, \quad \text{for } i,\, j \in \{1, \dots, n\}.$$

As before, we view $\mathbb{C}^{2n}$ as symplectic variety. Note that the symplectic structure on $\mathbb{C}^{2n}$ induced by ϖ equals ξ from (5.15). Suppose the torus $T = (\mathbb{C}^*)^r$ acts diagonally on $\mathbb{C}^{2n}$ preserving the symplectic form. Moreover, let $T' = (\mathbb{C}^*)^{n+2}$ acting on $\mathbb{C}^{2n}$ via

$$(t_1, \dots, t_{n+2}) \mapsto \begin{pmatrix} t_1 t_{n+1} & & & & & \\ & \ddots & & & & \\ & & t_n t_{n+1} & & & \\ & & & t_1^{-1} t_{n+2} & & \\ & & & & \ddots & \\ & & & & & t_n^{-1} t_{n+2} \end{pmatrix}.$$

Proposition 5.29 is basically a consequence of the following lemmas:

Lemma 5.30. *Let $C \subset \mathbb{C}^{2n}$ be an T-invariant conical and lagrangian subvariety. Then, there exist irreducible, T'-invariant, conical and lagrangian subvarieties $Z_1, \dots, Z_s \subset \mathbb{C}^{2n}$ and natural numbers $m_1, \dots, m_s$ such that $[C]^T = \sum_{i=1}^s m_i [Z_i]^T$ in $\overline{H}^T_*(\mathbb{C}^{2n})$.*

Lemma 5.31. *Let $Z \subset \mathbb{C}^{2n}$ be an irreducible, T'-invariant, lagrangian subvariety. Then, there exist $v_1, \dots, v_n \in \mathbb{C}^{2n}$ with $v_i \in \{e_i, e_{i+n}\}$, for $i = 1, \dots, n$, such that $Z = \langle v_1, \dots, v_n \rangle$.*

Proof of Proposition 5.29. Choose a symplectic identification $T_pX \cong \mathbb{C}^{2n}$, where the symplectic form on T_pX gets identified with ϖ. By Lemma 5.30, there exist irreducible, T'-invariant, conical and lagrangian subvarieties $Z_1, \dots, Z_s \subset \mathbb{C}^{2n}$ and natural numbers $m_1, \dots, m_s$ such that $[C_pL]^T = \sum_{i=1}^s m_i [Z_i]^T$. By Lemma 5.31, $Z_1, \dots, Z_s$ are T'-invariant lagrangian hyperplanes. □

We finish this section with the proofs of Lemmas 5.30 and 5.31.

Proof of Lemma 5.30. Define subtori $T_0, \dots, T_{n+2} \subset T'$ as

$$T_i \coloneqq \{(t_1, \dots, t_i, 1, \dots, 1) \in T' \mid t_1, \dots, t_i \in \mathbb{C}^*\}, \quad i = 0, \dots, n+2.$$

Note that T_0 is the trivial subgroup and $T_{n+2} = T'$. For $j = 1, \dots, n+2$, define cocharacters $\sigma_j\colon \mathbb{C}^* \to T'$ as

$$\sigma_i(t)_j = \begin{cases} t & \text{if } j = i, \\ 1 & \text{otherwise.} \end{cases} \tag{5.22}$$

Claim 5.32. *Let $i \in \{1, \dots, n+2\}$ and $C \subset \mathbb{C}^{2n}$ be a T_{i-1}- and T-invariant conical and lagrangian subvariety. Then, there exist irreducible, T_i and T-invariant, conical and lagrangian subvarieties $Z_1, \dots, Z_s \subset \mathbb{C}^{2n}$ and natural numbers $m_1, \dots, m_s$ such that $[C]^T = \sum_{i=1}^s m_i [Z_i]^T$ in $\overline{H}^T_*(\mathbb{C}^{2n})$.*

Proof of Claim 5.32. As in Lemma 4.35, let Γ be the Zariski closure of

$$\Gamma' = \{(\sigma_i(t).x, t) \mid x \in C,\ t \in \mathbb{C}^*\} \subset \mathbb{C}^{2n} \times \mathbb{C}.$$

Recall from Lemma 4.35.(i) that the projection $\pi\colon \Gamma \to \mathbb{C}$ is flat. Let $\Gamma_0 = \pi^{-1}(0)$ be the scheme theoretic fiber. We denote the irreducible components of Γ_0 by $Z_1, \dots, Z_s$ and let $m_i = m(Z_i, \Gamma_0)$ be the respective geometric multiplicity. By Lemma 4.35.(ii), $Z_1, \dots, Z_s$ are T_i- and T-invariant conical subvarieties of $\mathbb{C}^{2n}$. As in the proof of Proposition 5.20, let $\mathrm{pr}\colon \mathbb{C}^{2n} \times \mathbb{C} \to \mathbb{C}$ be the projection and $\hat{\xi} := \mathrm{pr}^* \xi$ the induced bilinear form on the tangent bundle $T(\mathbb{C}^{2n} \times \mathbb{C})$. Since C is a lagrangian subvariety of $\mathbb{C}^{2n}$ and the T_i-action scales the symplectic structure on $\mathbb{C}^{2n}$, we conclude that the restriction of $\hat{\xi}$ to $T\Gamma'_{\mathrm{sm}}$ vanishes. Recall from e.g. [Mum76, Theorem 2.33] that, as Γ' is an open dense subset of Γ in the Zariski-topology, Γ' is also an open dense subset of Γ in the analytic topology. By Lemma 4.35.(iii), we have $\dim(\Gamma') = \dim(Z_i) + 1$, for all $i = 1, \dots, s$. Hence, by Proposition 5.25, $T(Z_i)_{\mathrm{sm}}$ is contained in the analytic closure of $T\Gamma'_{\mathrm{sm}}$ in $T(\mathbb{C}^{2n} \times \mathbb{C})$. Thus, ξ vanishes on $T(Z_i)_{\mathrm{sm}}$ and hence Z_i is also a lagrangian subvariety of $\mathbb{C}^{2n}$. Finally, as π is flat, Proposition 5.4 gives

$$[C]^T = [\Gamma_0]^T = \sum_{i=1}^{s} m_i [Z_i]^T$$

which completes the proof of the claim. □

Using Claim 5.32, we can now easily deduce Lemma 5.30 using a repetitive argument. By applying Claim 5.32 to T_1 and C, we conclude that there exist irreducible, T_1- and T-invariant, conical, lagrangian subvarieties $Z_{1,1}, \dots, Z_{1,s_1} \subset \mathbb{C}^{2n}$ as well as natural numbers $m_{1,1}, \dots, m_{1,s_1}$ such that $[C]^T = \sum_{i=1}^{r_1} m_{1,i}[Z_{1,i}]^T$. Now, repeat this procedure by applying the claim to T_2 and $Z_{1,1}, \dots, Z_{1,r_1}$ and continue repeating. After $n+2$ repetitions, we obtain subvarieties $Z_1, \dots, Z_s \subset \mathbb{C}^{2n}$ satisfying the desired conditions of Lemma 5.30. □

Proof of Lemma 5.31. Since $\dim(Z) = n$, there exists a smooth point $z = (z_1, \dots, z_{2n}) \in Z$ such that at least n coordinates of z are non-zero. We show that for each $i \in \{1, \dots, n\}$, exactly one of the coordinates z_i, z_{n+i} is zero and the other non-zero. Suppose that there exists $i \in \{1, \dots, n\}$ such that z_i and z_{n+i} are both non-zero. Let $\sigma_i\colon \mathbb{C}^* \to T'$ be the cocharacter from (5.22) and $\sigma\colon \mathbb{C}^* \to T'$ be the cocharacter given by

$$\sigma(t)_j = \begin{cases} 1 & \text{if } j \neq i,\ n+2, \\ t & \text{if } j = i,\ n+2. \end{cases}$$

Let $\gamma_1 = \sigma_i(\mathbb{C}^*).z$ and $\gamma_2 = \sigma(\mathbb{C}^*).z$ be the respective $\mathbb{C}^*$-orbits in Z. Then, $T_z\gamma_1 = \langle w_1 \rangle$ and $T_z\gamma_2 = \langle w_2 \rangle$, where

$$w_1 = z_i e_i - z_{n+i} e_{n+i}, \quad w_2 = z_i e_i + \Big(\sum_{\substack{1 \le j \le n \\ j \neq i}} z_{n+j} e_{n+j} \Big).$$

Here, we used the standard identification of symplectic vector spaces $T_z\mathbb{C}^{2n} \cong \mathbb{C}^{2n}$. By definition, $\varpi(w_1, w_2) = z_i z_{n+i} \neq 0$ which contradicts the assumption that Z is lagrangian. Thus, exactly one of z_i, z_{n+i} is zero and the other non-zero.

For $i \in \{1, \dots, n\}$, we define

$$v_i := \begin{cases} e_i & \text{if } z_i \neq 0, \\ e_{n+i} & \text{if } z_{n+i} \neq 0. \end{cases}$$

As Z is T'-invariant, we conclude $\langle v_1, \dots, v_n \rangle \subset Z$. Since Z is irreducible and of dimension n, the inclusion must be an equality. □

5.6 Example computation of stable envelopes

We illustrate the construction of stable envelopes from Section 5.3 for the bow variety $\mathcal{C}(0/1\backslash 1/2\backslash 2\backslash 2/0)$ from Section 4.2. Recall the labeling of tie diagrams of $\mathcal{D}$ from there. Let $\mathfrak{C}$ be the chamber containing the cocharacter $\sigma_0 = (t, t^2, t^3)$. We denote by $\Lambda_i \in H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ the Poincaré dual of $[L_{x_{D_i}}]^{\mathbb{T}}$, where $L_{x_{D_i}} = \overline{\mathrm{Attr}_{\mathfrak{C}}(x_{D_i})}$, for $i = 1, \dots, 5$.

To apply the algorithmic procedure from Section 5.3, we first compute all $\mathbb{T}$-equivariant equivariant multiplicites $\iota^*_{x_{D_j}}(\Lambda_i)$.

Claim 5.33. *The equivariant multiplicities $\iota^*_{x_{D_j}}(\Lambda_i)$, for i, $j = 1, \dots, 5$, are recorded in the following table:*

$i \backslash j$	1	2	3	4	5
1	$(t_1 - t_3) \cdot (t_2 - t_3)$	0	0	0	0
2	$(t_1 - t_3) \cdot (t_3 - t_2 + h)$	$(t_1 - t_2) \cdot (t_2 - t_3 + h)$	0	0	0
3	$(t_3 - t_1 + h) \cdot (t_3 - t_2 + h)$	$(t_2 - t_1 + h) \cdot (t_2 - t_3 + h)$	$(t_1 - t_2 + h) \cdot (t_1 - t_3 + h)$	0	0
4	$(t_2 - t_3) \cdot (t_3 - t_1 + h)$	0	$(t_2 - t_1) \cdot (t_1 - t_3 + h)$	$(t_2 - t_3) \cdot (t_1 - t_2 + 2h)$	0
5	0	$(t_3 - t_2) \cdot (t_2 - t_1 + h)$	$(t_2 - t_1) \cdot (t_3 - t_1)$	$(t_1 - t_2 + 2h) \cdot (t_3 - t_2 + h)$	$(t_1 - t_3 + 2h) \cdot (t_2 - t_3 + h)$

Table 5.1: Equivariant multiplicities $\iota^*_{x_{D_j}}(\Lambda_i)$

Proof of Claim 5.33. We only compute $\iota^*_{x_{D_1}}(\Lambda_3)$ as all the other equivariant multiplicities can be determined analogously. By Claim 4.13, the open subvariety W_1 containing x_{D_1} is $\mathbb{T}$-equivariantly isomorphic to

$$\mathbb{C}_{t_3-t_1+h} \oplus \mathbb{C}_{t_3-t_2+h} \oplus \mathbb{C}_{t_1-t_3} \oplus \mathbb{C}_{t_2-t_3},$$

where x_{D_1} gets identified with the origin. Then, Claim 4.18 yields that $\overline{\mathrm{Attr}_{\mathfrak{C}}(x_{D_3})} \cap W_1$ corresponds to the subspace $\mathbb{C}_{t_1-t_3} \oplus \mathbb{C}_{t_2-t_3}$. Hence, (5.9) implies

$$\iota^*_{x_{D_1}}(\Lambda_3) = e_{\mathbb{T}}(\mathbb{C}_{t_3-t_1+h} \oplus \mathbb{C}_{t_3-t_2+h}) = (t_3 - t_1 + h)(t_3 - t_2 + h)$$

which equals the corresponding entry in Table 5.1. □

Now, to compute for instance the stable basis element $\mathrm{Stab}_{\mathfrak{C}}(x_{D_3})$, we first set $\gamma_{3,3} \coloneqq \Lambda_3$. By Claim 5.33, we have

$$\iota^*_{x_{D_2}}(\gamma_{3,3}) = (t_2 - t_1 + h)(t_2 - t_3 + h), \quad \iota^*_{x_{D_2}}(\Lambda_2) = (t_1 - t_2)(t_2 - t_3 + h).$$

Thus, we set $\gamma_{3,2} \coloneqq \Lambda_3 + \Lambda_2$. By construction, $\iota^*_{x_{D_1}}(\gamma_{3,2}) = h(t_3 - t_2 + h)$. So $\iota^*_{x_{D_1}}(\gamma_{3,2})$ is already divisible by h and hence we have

$$\mathrm{Stab}_{\mathfrak{C}}(x_{D_3}) = \gamma_{3,2} = \Lambda_3 + \Lambda_2.$$

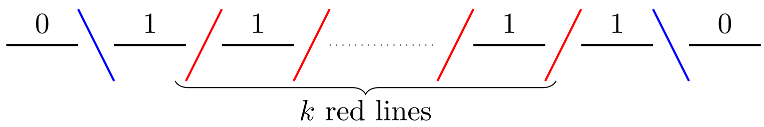
The other stable basis elements can be computed in exactly the same way using Claim 5.33. They are given by

$$\mathrm{Stab}_{\mathfrak{C}}(x_{D_1}) = \Lambda_1, \qquad \mathrm{Stab}_{\mathfrak{C}}(x_{D_4}) = \Lambda_4 + \Lambda_3 + \Lambda_2 + \Lambda_1,$$
$$\mathrm{Stab}_{\mathfrak{C}}(x_{D_2}) = \Lambda_2 + \Lambda_1, \qquad \mathrm{Stab}_{\mathfrak{C}}(x_{D_5}) = \Lambda_5 + \Lambda_4 + \Lambda_2.$$

Chapter 6

Explicit examples of stable basis elements

In this chapter, we explicitly apply the algorithmic procedure from Section 5.3 to the bow varieties $\mathcal{C}_k$ attached to the brane diagrams of the form

$\mathcal{D}_k :=$ 

(6.1)

We express the stable basis elements of $\mathcal{C}_k$ as $\mathbb{Z}$-linear combinations of the Poincaré duals of the fundamental classes of attracting cell closures, see Proposition 6.5 for the precise formula.

These explicit results are applied later in Section 9.6, where we compute equivariant multiplicities of stable basis elements of arbitrary bow varieties.

As we show in Proposition 6.2, the bow variety $\mathcal{C}_k$ can be covered with affine and torus invariant coordinate charts. These charts enable us to explicitly determine the attracting cells of $\mathcal{C}_k$ as well as their Zariski closures and their respective equivariant multiplicities, see Proposition 6.3 and (6.5). These results then allow us to employ the iterative construction procedure from Section 5.3 to compute the stable basis elements of $\mathcal{C}_k$.

Note that the multiplicities of stable basis elements for these specific examples of bow varieties were determined in the framework of elliptic cohomology in [RSVZ22, Section 4]. The technical tools, in particular the elliptic abelianization procedure from [AO21], used there are however different from ours.

As described in [NT17], the variety $\mathcal{C}_k$ also appears in theoretical physics where it can be interpreted as a Coulomb branch which is connected to the cotangent bundle of the Grassmannian $\mathrm{Gr}(k-1, k)$ via 3d mirror symmetry. This is a theory from string theory which connects $N = 4$ supersymmetric gauge theories.

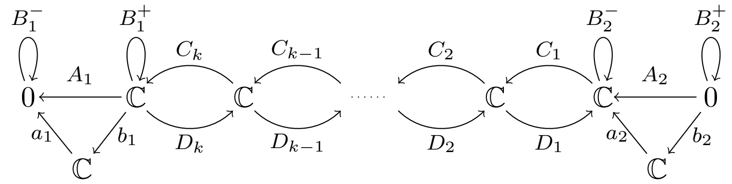
6.1 Open coordinate charts

In this section, we determine open affine coordinate charts of $\mathcal{C}_k$ which are also torus invariant. For this, we use similar methods as in Section 4.2.

Recall the definition of the affine brane variety $\widetilde{\mathcal{M}}(\mathcal{D}_k)$ from Definition 2.33, the corresponding gauge group $\mathcal{G}$ from (2.32) and the moment map $\tilde{m}$ from (2.33). We denote elements of $\widetilde{\mathcal{M}}(\mathcal{D}_k)$ as tuples

$$x = ((A_i, B_i^-, B_i^+, a_i, b_i)_{i=1,2}, (C_j, D_j)_{j=1,\dots,k})$$

according to the diagram



We view all operators A_i, B_i^-, B_i^+, a_i, b_i, C_j and D_j as elements in $\mathbb{C}$. Note that the condition (S1) for $\widetilde{\mathcal{M}}(\mathcal{D}_k)$ is equivalent to $b_1 \neq 0$ and (S2) is equivalent to $a_2 \neq 0$. By definition, $\tilde{m}(x) = 0$ if and only if

$$B_1^+ = -C_k D_k, \quad B_2^- = -D_1 C_1, \quad C_i D_i = D_{i+1} C_{i+1}, \quad \text{for } i = 1, \dots, k-1. \tag{6.2}$$

Therefore, by dropping vanishing and tautological operators, we denote elements of $\tilde{m}^{-1}(0)$ just as tuples

$$(b_1, a_2, C_1, D_1, \dots, C_k, D_k).$$

The rank 3 torus $\mathbb{T}$ acts on $\tilde{m}^{-1}(0)^{\mathrm{s}}$ via

$$(t_1, t_2, h).(b_1, a_2, C_1, D_1, \dots, C_k, D_k) = (ht_1 b_1, a_2 t_2^{-1}, hC_1, D_1, \dots, hC_k, D_k).$$

Similar to Section 4.2, we now construct covers of the χ-stable locus $\tilde{m}^{-1}(0)^{\mathrm{s}}$. Set

$$\widetilde{\Omega}_i := \{(b_1, a_2, C_1, D_1, \dots, C_k, D_k) \in \tilde{m}^{-1}(0) \mid C_1, \dots, C_{i-1} \neq 0,\ D_{i+1}, \dots, D_k \neq 0\}$$

and $\widetilde{\Omega} := \bigcup_{i=1}^k \widetilde{\Omega}_i$.

Lemma 6.1. *We have* $\tilde{m}^{-1}(0)^{\mathrm{s}} = \widetilde{\Omega}$.

Proof. Let $x = (b_1, a_2, C_1, D_1, \dots C_k, D_k) \in \tilde{m}^{-1}(0)$. Note that a graded subspace $T = \bigoplus_{X \in \mathrm{h}(\mathcal{D}_k)} T_X \subset W_{\mathcal{D}} = \bigoplus_{X \in \mathrm{h}(\mathcal{D}_k)} W_X$ satisfies the conditions of Proposition 2.37 if and only if

(a) T is invariant under the operators C_1, $D_1, \dots, C_k$, D_k and

(b) $T_{V_1^+} = \mathbb{C}$ and $T_{V_k^-} = \mathbb{C}$.

Suppose $x \in \widetilde{\Omega}_i$, for some i and that T satisfies (b) and (a). Then, as $C_1, \dots, C_{i-1} \neq 0$, we have $T_{V_j^-} = \mathbb{C}$, for $j = 1, \dots, i-1$. Likewise, $D_{i+1}, \dots, D_k \neq 0$ implies $T_{V_j^+} = \mathbb{C}$, for $j = i+1, \dots, k$. Hence, $T = W_{\mathcal{D}}$ and $x \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ by Proposition 2.37. Conversely, if $x \in \tilde{m}^{-1}(0)^{\mathrm{s}}$, we define $T' = \bigoplus_{X \in \mathrm{h}(\mathcal{D}_k)} T'_X \subset W_{\mathcal{D}}$ via

$$T'_{V_1^+} = \mathbb{C}, \quad T'_{V_k^-} = \mathbb{C}, \quad T'_{V_j^-} = \mathrm{Im}(C_j \dots C_1) + \mathrm{Im}(D_{j+1} \dots D_k), \quad \text{for } j = 1, \dots, k-1.$$

By (6.2), T' satisfies (b) and (a). If $x \notin \widetilde{\Omega}$, we have $C_{i-1} \cdots C_1 = 0$ as well as $D_{i+1} \cdots D_k = 0$, for some i. Consequently, $T'_{V_i^-} = 0$ and x is not χ-stable by Proposition 2.37. This gives $\tilde{m}^{-1}(0)^{\mathrm{s}} \subset \widetilde{\Omega}$. $\square$

By definition, each $\widetilde{\Omega}_i$ is a $\mathcal{G}$- and $\mathbb{T}$-invariant open subvariety of $\tilde{m}^{-1}(0)^{\mathrm{s}}$. Thus, we have a cover by $\mathbb{T}$-invariant open subvarieties

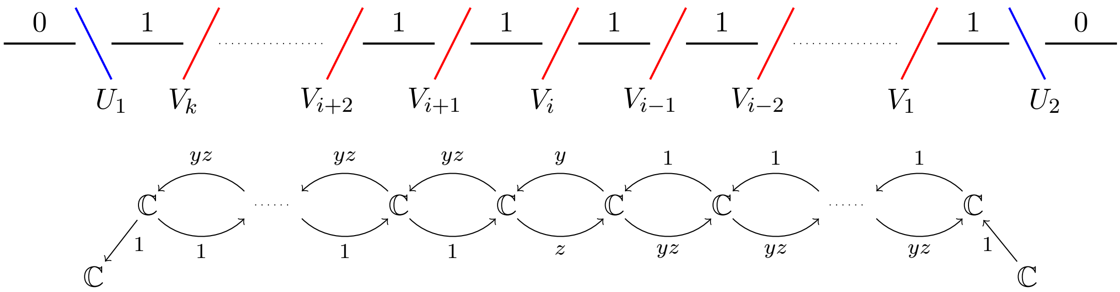
$$\mathcal{C}_k = \bigcup_{i=1}^{k} \Omega_i, \quad \text{where } \Omega_i := \widetilde{\Omega}_i/\mathcal{G}. \tag{6.3}$$

Next, we show that each Ω_i is $\mathbb{T}$-equivariantly isomorphic to a two dimensional affine space with linear $\mathbb{T}$-action.

Proposition 6.2. *For each $i = 1, \ldots, k$, we have a $\mathbb{T}$-equivariant isomorphism of varieties*

$$\eta_i \colon \mathbb{C}_{t_1 - t_2 + (i+1)h} \oplus \mathbb{C}_{t_2 - t_1 - ih} \xrightarrow{\sim} \Omega_i,$$

where $\eta_i(y,z) = [\tilde{\eta}_i(y,z)]$ and $\tilde{\eta}_i(y,z) \in \widetilde{\Omega}_i$ is represented by the following diagram:



Proof. By Proposition 2.25, η_i is an isomorphism of varieties if and only if η_i is bijective. For injectivity, suppose $\eta_i(y,z) = \eta_i(y',z')$. Hence, there exists $g = (g_X)_X \in \mathcal{G}$ such that $g.\eta_i(y,z) = \eta_i(y',z')$. We write

$$\tilde{\eta}_i(y,z) = (b_1, a_2, C_1, D_1, \ldots C_k, D_k), \quad \tilde{\eta}_i(y',z') = (b_1', a_2', C_1', D_1', \ldots C_k', D_k').$$

Since $b_1 = b_1' = 1$ and $a_2 = a_2' = 1$, we have $g_{V_1^+} = 1$ and $g_{V_k^-} = 1$. Then,

$$C_1 = \ldots = C_{i-1} = C_1' = \ldots = C_{i-1}' = 1, \quad D_{i+1} = \ldots = D_k = D_{i+1}' = \ldots = D_k' = 1$$

implies $g_{V_j^+} = g_{V_j^-} = 1$, for all $j \neq i$. Thus, g equals the identity which yields $y = C_i = C_i' = y'$ and $z = D_i = D_i' = z'$. Hence, η_i is injective. For surjectivity, let $x = (b_1, a_2, C_1, D_1, \ldots C_k, D_k) \in \widetilde{\Omega}_i$. Then, we have

$$g.x = \tilde{\eta}_i(b_1 D_k^{-1} \cdots D_{i+1}^{-1} C_i \cdots C_1 a_2, a_2^{-1} C_1^{-1} \cdots C_{i-1}^{-1} D_i \cdots D_k b_1^{-1}),$$

where $g = (g_X)_X \in \mathcal{G}$ is defined as $g_{V_k^-} = b_1$, $g_{V_1^+} = a_2^{-1}$ and

$$g_{V_j^-} = a_2^{-1} C_1^{-1} \cdots C_j^{-1}, \quad g_{V_l^+} = b_1 D_k^{-1} \cdots D_l^{-1}, \quad \text{for } j = 1, \ldots, i-1,\ l = i+1, \ldots, k.$$

Hence, $[x] = \eta_i(b_1 D_k^{-1} \cdots D_{i+1}^{-1} C_i \cdots C_1 a_2, a_2^{-1} C_1^{-1} \cdots C_{i-1}^{-1} D_i \cdots D_k b_1^{-1})$ which proves the surjectivity of η_i. Hence, η_i is an isomorphism of varieties. To see that η_i is $\mathbb{T}$-equivariant, note that for $t = (t_1, t_2, h)$ and $(y,z) \in \mathbb{C}^2$, we have

$$g_t.(t.\tilde{\eta}_i(y,z)) = \tilde{\eta}_i(t_1 t_2^{-1} h^{i+1} y, t_2 t_1^{-1} h^{-i} z),$$

where $g_t = (g_{t,X})_X \in \mathcal{G}$ is defined as

$$g_{t,V_j^+} = t_2 h^{1-j}, \quad g_{t,V_l^-} = t_1 h, \quad \text{for } j = 1, \dots, i,\ l = i, \dots, k.$$

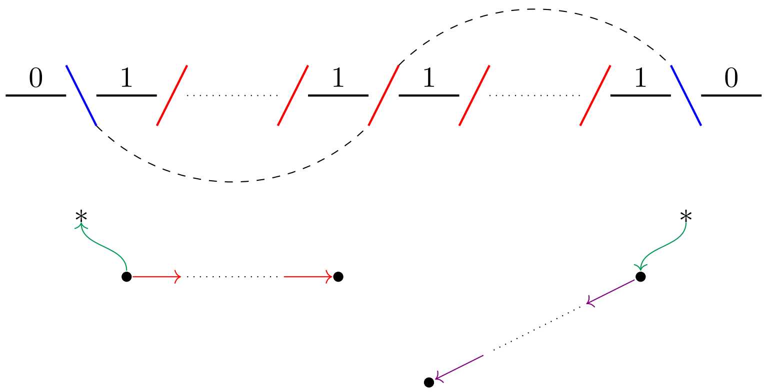
Thus, η_i is $\mathbb{T}$-equivariant. □

6.2 Torus fixed points

By construction, the brane diagram $\mathcal{D}_k$ admits exactly k tie diagrams:

$$\mathrm{Tie}(\mathcal{D}_k) = \{D_1, \dots, D_k\}, \quad \text{where } D_i = \{(U_1, V_i), (V_i, U_2)\}. \tag{6.4}$$

The corresponding visualization and butterfly diagrams of D_i are given as follows:



Hence, the corresponding $\mathbb{T}$-fixed point x_{D_i} from Definition 3.12 equals $\eta_i(0,0)$.

6.3 Attracting cells

As the torus $\mathbb{A}$ corresponding to $\mathcal{D}_k$ is of rank 2, there are only two chambers assigned to $\mathbb{A}$: The dominant chamber $\mathfrak{C}_+$ and the antidominant chamber $\mathfrak{C}_-$. The attracting cells of the dominant chamber and their Zariski closures are characterized as follows:

Proposition 6.3. *For $i = 1, \dots, k$, let $L_i^+ := \overline{\mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i})}$ be the Zariski closure. Then,*

(i) $\mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i}) = \eta_i(\mathbb{C}_{t_1 - t_2 + (i+1)h})$, *where η_i is defined as in Proposition 6.2,*

(ii) for $j \neq i$, we have

$$L_i^+ \cap \Omega_j = \begin{cases} \eta_{i+1}(\mathbb{C}_{t_2 - t_1 - (i+1)h}), & \textit{for } j = i+1, \\ \emptyset, & \textit{otherwise}, \end{cases}$$

(iii) $L_i^+ = \mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i}) \cup \{x_{D_{i-1}}\}$.

In particular, L_i^+ is smooth and hence isomorphic to $\mathbb{P}^1$.

Proof. As η_i is $\mathbb{T}$-equivariant, we have $\eta_i(\mathbb{C}_{t_1-t_2+(i+1)h}) = \mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i}) \cap \Omega_i$. By Proposition 4.4, $\mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i}) \cong \mathbb{C}$. Hence, the inclusion $\eta_i(\mathbb{C}_{t_1-t_2+(i+1)h}) \subset \mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i})$ has to be an equality which proves (i). For (ii), note that for $y \in \mathbb{C} \setminus \{0\}$, we have

$$g.\tilde{\eta}_i(y,0) = \tilde{\eta}_{i+1}(0,y^{-1}),$$

where $g = (g_X)_X \in \mathcal{G}$ is defined as $g_{V_i^-} = y^{-1}$ and $g_X = 1$, for $X \neq V_i^-$. Thus, we have

$$\mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i}) \cap \Omega_{i+1} = \eta_{i+1}(\mathbb{C}_{t_2-t_1-(i+1)h} \setminus \{0\}), \quad L_i^+ \cap \Omega_{i+1}\eta_{i+1}(\mathbb{C}_{t_2-t_1-(i+1)h}).$$

Now, by (i), for each $x = [b_1, a_2, C_1, D_1, \dots C_k, D_k] \in \mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i})$, the operators $C_{i+1}, \dots, C_k$ and $D_1, \dots, D_{i-1}$ vanish. Thus, $\mathrm{Attr}_{\mathfrak{C}_+}(x_{D_i}) \cap \Omega_j = \emptyset$, for $j \neq i, i+1$. Hence, we deduce (ii). The assertion (iii) follows directly from (ii). □

The same proof gives the corresponding statements for the antidominant chamber:

Proposition 6.4. *For $i = 1, \dots, k$, let $L_i^- := \overline{\mathrm{Attr}_{\mathfrak{C}_-}(x_{D_i})}$ be the Zariski closure. Then,*

(i) $\mathrm{Attr}_{\mathfrak{C}_-}(x_{D_i}) = \eta_i(\mathbb{C}_{t_2-t_1-ih})$,

(ii) for $j \neq i$, we have

$$L_i^- \cap \Omega_j = \begin{cases} \eta_{i-1}(\mathbb{C}_{t_1-t_2+ih}), & \text{for } j = i-1, \\ \emptyset, & \text{otherwise,} \end{cases}$$

(iii) $L_i^- = \mathrm{Attr}_{\mathfrak{C}_-}(x_{D_i}) \cup \{x_{D_{i-1}}\}$.

In particular, L_i^- is isomorphic to $\mathbb{P}^1$.

Let $\Lambda_i^\pm \in H^*_{\mathbb{T}}(\mathcal{C}_k)$ be the Poincaré dual of the fundamental class $[L_i^\pm]^{\mathbb{T}}$. By Corollary 5.7, we can directly read off the equivariant multiplicities of Λ_i^+ from Proposition 6.3:

$$\iota^*_{x_{D_j}}(\Lambda_i^+) = \begin{cases} t_2 - t_1 - ih & \text{if } j = i, \\ t_1 - t_2 + (i+2)h & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Likewise, the equivariant multiplicities of Λ_i^- are

$$\iota^*_{x_{D_j}}(\Lambda_i^-) = \begin{cases} t_1 - t_2 + (i+1)h & \text{if } j = i, \\ t_2 - t_1 - (i-1)h & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

6.4 Stable basis elements

Using the explicit description of the attracting cells from the previous subsection, we now determine the stable basis elements of $\mathcal{C}_k$ as well as their equivariant multiplicities:

Proposition 6.5. *For $i = 1, \dots, k$, we have*

$$\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_i}) = \Lambda_i^+ + \Lambda_{i+1}^+ + \ldots + \Lambda_k^+$$

and

$$\iota_{x_{D_j}}^*(\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_i})) = \begin{cases} 0 & \text{if } j < i, \\ t_2 - t_1 - ih & \text{if } j = i, \\ h & \text{if } j > i. \end{cases} \tag{6.7}$$

Similarly, we have

$$\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_i}) = \Lambda_i^- + \Lambda_{i-1}^- + \ldots + \Lambda_1^-$$

and

$$\iota_{x_{D_j}}^*(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_i})) = \begin{cases} h & \text{if } j < i, \\ t_1 - t_2 + (i+1)h & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases} \tag{6.8}$$

Proof. We only prove the proposition for the dominant chamber $\mathfrak{C}_+$. Since each Λ_j^+ is supported on L_j^+, we conclude that $\Lambda_i^+ + \Lambda_{i+1}^+ + \ldots + \Lambda_k^+$ is supported on $\bigcup_{j=i}^k L_j^+$. By Proposition 4.20, we have $\bigcup_{j=i}^k L_j^+ = \mathrm{Attr}_{\mathfrak{C}_+}^f(x_{D_i})$ and thus $\Lambda_i^+ + \Lambda_{i+1}^+ + \ldots + \Lambda_k^+$ satisfies the support condition. From (6.5), we immediately obtain (6.7) which implies the normalization and smallness condition. Thus, $\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_i}) = \Lambda_i^+ + \Lambda_{i+1}^+ + \ldots + \Lambda_k^+$. □

Chapter 7

Polynomiality and Orthogonality Theorems

Even though bow varieties are in general not projective, they still admit a *virtual* intersection pairing $(.,.)_{\mathrm{vir}}$, see Definition 7.4. This pairing is modeled on the Atiyah–Bott–Berline–Vergne integration formula and takes values in the localized torus equivariant cohomology of a point.

In this chapter, we study properties of the stable basis elements of bow varieties with respect to the virtual intersection pairing. We first prove in Theorem 7.6 that virtual intersection pairings of the form

$$(\alpha \cdot \mathrm{Stab}_{\mathfrak{C}}(p), \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))_{\mathrm{virt}}, \quad \alpha \in H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D})), \quad p,\ q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$$

have in fact values in the non-localized cohomology $H^*_{\mathbb{T}}(\mathrm{pt})$. In other words, those intersection pairings are polynomials in the equivariant parameters of $H^*_{\mathbb{T}}(\mathrm{pt})$. We therefore refer to Theorem 7.6 as *the Polynomiality Theorem.* The main ingredient of the proof of Theorem 7.6 is the properness result from Theorem 4.24.

The second main purpose of this chapter is a self-contained reproof of the Orthogonality Theorem from [MO19, Theorem 4.4.1] which states that the stable basis elements of opposite chambers are orthogonal with respect to the virtual intersection pairing. This theorem provides a parallel between stable basis elements and (equivariant) Schubert classes which also have an analogous orthogonality property, namely Schubert classes and opposite Schubert classes are orthogonal with respect to the Poincaré pairing, see e.g. [Ful97, Section 10.2].

7.1 The (virtual) intersection pairing

In this section, we recall the definition of intersection pairings in torus equivariant cohomology and their virtual versions for quasi-projective varieties. Given a projective variety Y with an algebraic action of a torus $T = (\mathbb{C}^*)^r$, we denote by $\varepsilon^Y : Y \to \mathrm{pt}$ the projection. Let $\varepsilon^Y_* : \overline{H}^T_*(Y) \to \overline{H}^T_*(\mathrm{pt})$ be the corresponding pushforward in T-equivariant Borel–Moore homology. Assume that Y is additionally smooth. In this case, we also obtain a pushforward $H^*_T(Y) \to H^*_T(\mathrm{pt})$ via Poincaré duality which we also denote by ε^Y_*. Then, the T*-equivariant intersection pairing* is defined as

$$(.,.)\colon H^*_T(Y) \times H^*_T(Y) \longrightarrow H^*_T(\mathrm{pt}), \quad (\alpha, \beta) = \varepsilon^Y_*(\alpha \cdot \beta).$$

An effective tool to compute T-equivariant intersection pairings is the Atiyah–Bott–Berline–Vergne integration formula, see [AB84, Eq(3.8)], is based on the localization principle in T-equivariant cohomology.

The Atiyah–Bott–Berline–Vergne integration formula

Let X be a smooth projective variety with an algebraic T-action. Recall from (5.8) the definition of $H_T^*(X)_{\mathrm{loc}}$.

Before we state the integration formula, we recall two general results about the T-fixed components of X. The first one is the following theorem from [Ive72, Theorem 1]:

Theorem 7.1. *We have that X^T is a smooth closed subvariety of X. If $F \subset X^T$ is an irreducible component then we have $T_xF = (T_xX)^T$, for all $x \in F$.*

The second result is the following lemma about the invertibility of Euler classes:

Lemma 7.2. *Let $F \subset X^T$ be an irreducible component and E be a T-equivariant vector bundle over F. Then, $e_T(E)$ is invertible in $H_T^*(F)_{\mathrm{loc}}$.*

Proof. As T acts trivially on F, we have a canonical isomorphism of graded $\mathbb{Q}$-algebras $H_T^*(F) \cong H^*(F) \otimes H_T^*(\mathrm{pt})$, where $H^*(F)$ is the usual singular cohomology and the tensor product is over $\mathbb{Q}$. In particular, the graded pieces of $H_T^*(F)$ are given as

$$H_T^j(F) \cong \bigoplus_{i=0}^{j} H^i(F) \otimes H_T^{j-i}(\mathrm{pt}). \tag{7.1}$$

Since F is fixed under T, we have a T-equivariant splitting $E \cong \bigoplus_{i=1}^k \mathbb{C}_{\tau_i}^{m_i}$, where $\tau_1, \ldots, \tau_k$ are characters of T and $\mathbb{C}_\tau$ denotes the T-equivariant vector bundle $F \times \mathbb{C}$ with T-action $t.(x, v) = (x, \tau(t)v)$. Let m be the rank of E. Then, by e.g. [EG98, Lemma 3], the $H_T^{2m}(\mathrm{pt})$-component of $e_T(E)$ under the identification (7.1) is given by $\tau := \prod_{i=1}^k \tau_i^{m_i}$. Since all elements in $H^j(F)$, for $j > 0$ are nilpotent, we conclude that $e_T(E)$ becomes invertible in $H_T^*(F)_{\mathrm{loc}} \cong H^*(F) \otimes H_T^*(\mathrm{pt})_{\mathrm{loc}}$. □

Theorem 7.3 (Atiyah–Bott–Berline–Vergne integration formula)**.** *Let $F_1, \ldots, F_s \subset X^T$ be the irreducible components of X^T. Then, we have the following equality in $H_T^*(\mathrm{pt})_{\mathrm{loc}}$:*

$$\varepsilon_*^X(\alpha) = \sum_{i=1}^{s} \varepsilon_*^{F_i}\Big(\frac{\iota_{F_i}^*(\alpha)}{e_T(N_{F_i})}\Big), \quad \text{for } \alpha \in H_T^*(X),$$

where $\iota_{F_i} \colon F_i \hookrightarrow X$ are the inclusions and $N_{F_i} = (\iota_{F_i}^* TX)/TF_i$ are the respective normal bundles.

We now pass to the quasi-projective setup and discuss the notion of the virtual intersection pairing which is modeled on the Atiyah–Bott–Berline–Vergne integration formula.

The virtual intersection pairing

Suppose now that X is a smooth quasi-projective variety with algebraic T-action. We additionally assume that X^T is a proper variety over $\mathbb{C}$.

Definition 7.4. The *virtual pushforward* $\varepsilon^X_{*,\mathrm{virt}}\colon H^*_T(X) \to H^*_T(\mathrm{pt})_{\mathrm{loc}}$ is defined as

$$\varepsilon^X_{*,\mathrm{virt}}(\alpha) = \sum_{i=1}^{s} \varepsilon^{F_i}_* \Big(\frac{\iota^*_{F_i}(\alpha)}{e_T(N_{F_i})}\Big), \quad \textit{for } \alpha \in H^*_T(X),$$

where $F_1, \dots, F_s \subset X$ are the irreducible components of X^T and $\iota_{F_i}\colon F_i \hookrightarrow X$ the respective inclusions. The *virtual intersection pairing on X* is defined as

$$(.,.)_{\mathrm{virt}}\colon H^*_T(X) \times H^*_T(X) \longrightarrow H^*_T(\mathrm{pt})_{\mathrm{loc}}, \quad (\alpha,\beta)_{\mathrm{virt}} = \varepsilon^X_{*,\mathrm{virt}}(\alpha\cdot\beta).$$

By definition, the virtual pushforward and the virtual intersection pairing take values in the localized T-equivariant cohomology ring $H^*_T(\mathrm{pt})_{\mathrm{loc}}$. However, for T-equivariant cohomology classes in $H^*_T(X)$ with proper support, the virtual pushforward is actually contained in $H^*_T(\mathrm{pt})$.

Lemma 7.5. *Let $i\colon Y \hookrightarrow X$ be a T-invariant closed subvariety which is proper over $\mathbb{C}$. Let $a \in \overline{H}^T_*(Y)$ and $\alpha \in H^*_T(X)$ be the Poincaré dual of $i_*(a)$. Then, under the identification of $H^*_T(\mathrm{pt})$-modules $H^*_T(\mathrm{pt}) \cong \overline{H}^*_T(\mathrm{pt})$, we have*

$$\varepsilon^X_{*,\mathrm{virt}}(\alpha) = \varepsilon^Y_*(a).$$

In particular, $\varepsilon^X_{*,\mathrm{virt}}(\alpha)$ is contained in $H^*_T(\mathrm{pt})$.

Proof. As in Proposition 4.1, there exists a T-equivariant open immersion $X \subset \tilde{X}$, where $\tilde{X}$ is a smooth projective variety with T-action. Denote by $j\colon Y \hookrightarrow \tilde{X}$ the inclusion and let $\beta \in H^*_T(\tilde{X})$ be the Poincaré dual of $j_*(a)$. As before, let $F_1, \dots, F_s \subset X^T$ be the irreducible components of X^T and $\iota_{F_i} : F_i \hookrightarrow X$ the respective inclusions. Since X^T is proper, $F_1, \dots, F_s$ are also irreducible components of $\tilde{X}^T$. We denote the remeining irreducible components of $\tilde{X}^T$ by $\tilde{F}_1, \dots, \tilde{F}_{\tilde{s}}$. If $F \subset \tilde{X}^T$ is an irreducible component, let $j_F : F \hookrightarrow \tilde{X}$ be the respective inclusion. As Y does not intersect $\tilde{F}_i$, we have $j^*_{\tilde{F}_i}(\beta) = 0$, for $i = 1, \dots, \tilde{s}$. In addition, since $F_i \subset X$, we conclude $j^*_{F_i}(\beta) = \iota^*_{F_i}(\alpha)$. Thus, the Atiyah–Bott–Berline–Vergne integration formula yields

$$\varepsilon^Y_*(a) = \varepsilon^{\tilde{X}}_*(j_*(a)) = \varepsilon^{\tilde{X}}_*(\beta) = \sum_{i=1}^{s} \varepsilon^{F_i}_*\Big(\frac{j^*_{F_i}(\beta)}{e_T(N_{F_i})}\Big) = \sum_{i=1}^{s} \varepsilon^{F_i}_*\Big(\frac{\iota^*_{F_i}(\alpha)}{e_T(N_{F_i})}\Big) = \varepsilon^X_{*,\mathrm{virt}}(\alpha)$$

which completes the proof. □

Let $X = \mathcal{C}(\mathcal{D})$ be a bow variety and $T = \mathbb{T} = \mathbb{A} \times \mathbb{C}^*_h$ be the torus of rank $N+1$ from (2.47). Then, Corollary 3.24 implies that the virtual intersection form $\mathcal{C}(\mathcal{D})$ takes values in $S_0^{-1}H^*_T(\mathrm{pt}) \subset H^*_T(\mathrm{pt})_{\mathrm{loc}}$, where S_0 is the multiplicative set generated by

$$\{t_i - t_j + mh \mid 1 \le i,j \le N,\ i \ne j,\ m \in \mathbb{Z}\}. \tag{7.2}$$

A crucial difference between $S_0^{-1}H^*_T(\mathrm{pt})$ and $H^*_T(\mathrm{pt})_{\mathrm{loc}}$ is that the equivariant parameter h is a prime element in $S_0^{-1}H^*_T(\mathrm{pt})$ which is an important ingredient of the proof of the Ortogonality Theorem.

7.2 Polynomiality Theorem

Fix a chamber $\mathfrak{C}$ of $\mathbb{A}$.

Theorem 7.6 (Polynomiality)**.** *We have*

$$(\alpha \cdot \mathrm{Stab}_{\mathfrak{C}}(p), \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))_{\mathrm{virt}} \in H^*_{\mathbb{T}}(\mathrm{pt}),$$

for all $\alpha \in H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ *and* $p,\ q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.

Proof. By the support condition, the product $\mathrm{Stab}_{\mathfrak{C}}(p) \cdot \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q)$ is supported on the intersection $A_{p,q} := \mathrm{Attr}^f_{\mathfrak{C}}(p) \cap \mathrm{Attr}^f_{\mathfrak{C}^{\mathrm{op}}}(q)$. Hence, also $\alpha \cdot \mathrm{Stab}_{\mathfrak{C}}(p) \cdot \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q)$ is supported on $A_{p,q}$. By Corollary 4.25, $A_{p,q}$ is proper over $\mathbb{C}$. Therefore, we have

$$(\alpha \cdot \mathrm{Stab}_{\mathfrak{C}}(p), \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))_{\mathrm{virt}} = \varepsilon^{\mathcal{C}(\mathcal{D})}_{*,\mathrm{virt}}(\alpha \cdot \mathrm{Stab}_{\mathfrak{C}}(p) \cdot \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q)) \in H^*_{\mathbb{T}}(\mathrm{pt}),$$

by Lemma 7.5. □

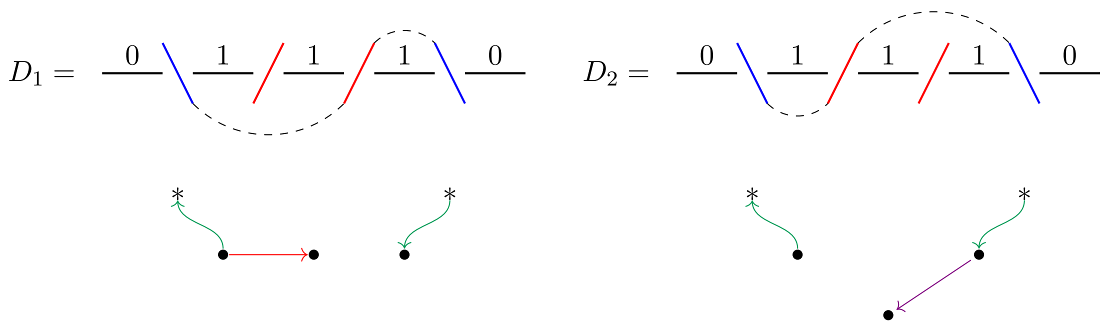
Example 7.7. As in Chapter 6, let $\mathcal{C}_2$ be the bow variety corresponding to the brane diagram $\mathcal{D}_2 = 0\backslash 1/1/1\backslash 0$. We have

$$\mathrm{Tie}(\mathcal{D}_2) = \{D_1, D_2\}, \quad \text{where } D_1 = \{(U_1, V_1), (V_1, U_2)\},\ D_2 = \{(U_1, V_2), (V_2, U_2)\}.$$

Let $\alpha = c_1(\xi_{X_3})$ be the first $\mathbb{T}$-equivariant Chern class of the tautological bundle ξ_{X_3}. We now show by a direct computation that

$$(\alpha \cdot \mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2}), \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1}))_{\mathrm{vir}} \tag{7.3}$$

is indeed a polynomial in the equivariant parameters. For this, we determine the necessary equivariant multiplicities and tangent weights. The butterfly diagrams of D_1, D_2 are as follows:



Thus, the restriction formula for tautological bundles (3.8) gives $\iota^*_{x_{D_1}}(\xi_{X_3}) = \mathbb{C}_{t_1+h}$ and $\iota^*_{x_{D_2}}(\xi_{X_3}) = \mathbb{C}_{t_2-h}$. Hence, $\iota^*_{x_{D_1}}(\alpha) = t_1 + h$ and $\iota^*_{x_{D_2}}(\alpha) = t_2 - h$. By Proposition 6.2, the tangent weights at x_{D_1} and x_{D_2} are

$$T_{x_{D_1}}\mathcal{C}_2 = \mathbb{C}_{t_1-t_2+2h} \oplus \mathbb{C}_{t_2-t_1-h}, \quad T_{x_{D_2}}\mathcal{C}_2 = \mathbb{C}_{t_1-t_2+3h} \oplus \mathbb{C}_{t_2-t_1-2h}. \tag{7.4}$$

Proposition 6.5 gives the equivariant multiplicities of $\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1})$ and $\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2})$:

$$\begin{aligned} \iota^*_{x_{D_1}}(\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1})) &= t_2 - t_1 - h, & \iota^*_{x_{D_1}}(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2})) &= h, \\ \iota^*_{x_{D_2}}(\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1})) &= h, & \iota^*_{x_{D_2}}(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2})) &= t_1 - t_2 + 3h. \end{aligned} \tag{7.5}$$

Inserting this into the definition of virtual intersection products yields

$$\begin{aligned}(7.3) &= \frac{\iota^*_{x_{D_1}}(\alpha \cdot \mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2}) \cdot \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1}))}{e_{\mathbb{T}}(T_{x_{D_1}}\mathcal{C}_2)} + \frac{\iota^*_{x_{D_2}}(\alpha \cdot \mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2}) \cdot \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1}))}{e_{\mathbb{T}}(T_{x_{D_2}}\mathcal{C}_2)} \\ &= \frac{(t_1+h)h}{t_1-t_2+2h} + \frac{(t_2-h)h}{t_2-t_1-2h} \\ &= h.\end{aligned}$$

Therefore, we proved that (7.3) is indeed a polynomial in the equivariant parameters.

7.3 Orthogonality Theorem

The Orthogonality Theorem states that stable basis elements corresponding to opposite chambers are orthogonal with respect to the virtual intersection pairing on $\mathcal{C}(\mathcal{D})$:

Theorem 7.8 (Orthogonality Theorem)**.** *We have*

$$(\mathrm{Stab}_{\mathfrak{C}}(p), \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))_{\mathrm{virt}} = \delta_{p,q},$$

for all p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.

Proof. Recall that the virtual intersection pairing on $\mathcal{C}(\mathcal{D})$ takes values in $S_0^{-1}H^*_{\mathbb{T}}(\mathrm{pt})$, where S_0 is defined as in (7.2) and that the equivariant parameter h is a prime element in $S_0^{-1}H^*_{\mathbb{T}}(\mathrm{pt})$. By definition, we have

$$(\mathrm{Stab}_{\mathfrak{C}}(p), \mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))_{\mathrm{virt}} = \sum_{z\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}} \frac{\iota^*_z(\mathrm{Stab}_{\mathfrak{C}}(p)) \cdot \iota^*_z(\mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))}{e_{\mathbb{T}}(T_z\mathcal{C}(\mathcal{D}))}. \tag{7.6}$$

Theorem 7.6 implies that (7.6) is actually contained in $H^*_{\mathbb{T}}(\mathrm{pt})$. If $p \neq q$, we know by the smallness condition that h divides $\iota^*_z(\mathrm{Stab}_{\mathfrak{C}}(p)) \cdot \iota^*_z(\mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))$, for all $z \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. However, Corollary 3.24 gives $h \nmid e_{\mathbb{T}}(T_z\mathcal{C}(\mathcal{D}))$, for all $z \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$. It follows that (7.6) is divisible by h in $S_0^{-1}H^*_{\mathbb{T}}(\mathrm{pt})$ and hence also in $H^*_{\mathbb{T}}(\mathrm{pt})$. As $\iota^*_z(\mathrm{Stab}_{\mathfrak{C}}(p)) \cdot \iota^*_z(\mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(q))$ and $e_{\mathbb{T}}(T_z\mathcal{C}(\mathcal{D}))$ are homogeneous of the same degree, we conclude that (7.6) is a degree 0 polynomial in the equivariant parameters. Hence, (7.6) has to vanish. Now, let us consider the case $p = q$. By the normalization condition, we can infer

$$\frac{i^*_p(\mathrm{Stab}_{\mathfrak{C}}(p)) \cdot \iota^*_p(\mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(p))}{e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D}))} = \frac{e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}}) \cdot e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}})}{e_{\mathbb{T}}(T_p\mathcal{C}(\mathcal{D}))} = 1.$$

In addition, the same argument as in the case $p \neq q$ gives

$$\sum_{\substack{z\in\mathcal{C}(\mathcal{D})^{\mathbb{T}} \\ z\neq p}} = \frac{\iota^*_z(\mathrm{Stab}_{\mathfrak{C}}(p)) \cdot \iota^*_z(\mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(p))}{e_{\mathbb{T}}(T_z\mathcal{C}(\mathcal{D}))} = 0.$$

Thus, we deduce

$$\sum_{z\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}} \frac{\iota^*_z(\mathrm{Stab}_{\mathfrak{C}}(p)) \cdot \iota^*_z(\mathrm{Stab}_{\mathfrak{C}^{\mathrm{op}}}(p))}{e_{\mathbb{T}}(T_z\mathcal{C}(\mathcal{D}))} = 1.$$

This finishes the proof of the Orthogonality Theorem. □

Example 7.9. Let $\mathcal{C}_2$, D_1, D_2 be as in Example 7.7. We show by direct computations that the stable bases $(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_i}))_{i=1,2}$ and $(\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_i}))_{i=1,2}$ are indeed orthogonal with respect to $(.,.)_{\mathrm{vir}}$. Recall from Proposition 6.5 that the equivariant mutiplicities $\iota^*_{x_{D_1}}(\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_2}))$ and $\iota^*_{x_{D_2}}(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_1}))$ both vanish. Thus, all summands contributing to the virtual intersection product $(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_1}), \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_2}))_{\mathrm{vir}}$ also vanish. Consequently

$$(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_1}), \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_2}))_{\mathrm{vir}} = 0.$$

Likewise, we deduce that

$$(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_i}), \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_i}))_{\mathrm{vir}} = \frac{\iota^*_{x_{D_i}}(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_i}) \cdot \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_i}))}{e_{\mathbb{T}}(T_{x_{D_i}}\mathcal{C}_2)} = 1,$$

where the second equality follows from the normalization condition. Finally, using the formulas for tangent weight from (7.4) and the equivariant multiplicities of $\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1})$ and $\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2})$ from (7.5), we conclude

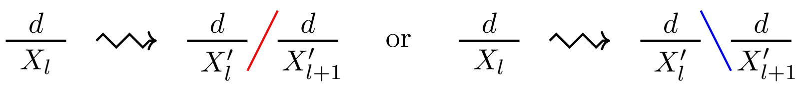
$$\begin{aligned}(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_2}), \mathrm{Stab}_{\mathfrak{C}_+}(x_{D_1}))_{\mathrm{vir}} &= \frac{h(t_2 - t_1 - h)}{(t_1 - t_2 + 2h)(t_2 - t_1 - h)} + \frac{(t_1 - t_2 + 3h)h}{(t_1 - t_2 + 3h)(t_2 - t_1 - 2h)} \\ &= \frac{h}{t_1 - t_2 + 2h} + \frac{h}{t_2 - t_1 - 2h} \\ &= 0.\end{aligned}$$

Hence, we showed that $(\mathrm{Stab}_{\mathfrak{C}_-}(x_{D_i}))_{i=1,2}$ and $(\mathrm{Stab}_{\mathfrak{C}_+}(x_{D_i}))_{i=1,2}$ are orthogonal with respect to $(.,.)_{\mathrm{vir}}$.

Chapter 8

Extension moves for bow varieties

In this chapter, we relate the stable basis elements of different bow varieties whose associated brane diagrams differ by a small local change. More precisely, assume that we are given two separated brane diagrams $\mathcal{D}$ and $\mathcal{D}'$ such that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by performing one of the following local moves involving a red or blue line:



We refer to these moves as *extension moves.* The central question of this chapter is:

How are the stable basis elements of $\mathcal{C}(\mathcal{D})$ *and* $\mathcal{C}(\mathcal{D}')$ *connected?*

It turns out that the answer to this question strongly depends on the color of the line which is added in the extension move.

If the extension move adds a red line, we prove in Proposition 8.3 that there is a torus equivariant isomorphism of varieties between $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$. The induced isomorphism in torus equivariant cohomology respects stable basis elements as explained in Corollary 8.9.

In case the extension move adds a blue line, the dimension of $\mathcal{C}(\mathcal{D}')$ is in general strictly greater than the dimension of $\mathcal{C}(\mathcal{D})$, see Lemma 8.13. Thus, $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ are in general *not* isomorphic. We prove however in Theorem 8.15 that there is a torus equivariant embedding $\iota\colon \mathcal{C}(\mathcal{D}) \hookrightarrow \mathcal{C}(\mathcal{D}')$ which induces a bijection on torus fixed points. This embedding allows a comparison between the attracting cells of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$. In particular, we prove in Proposition 8.27 that the equivariant multiplicities of closures of attracting cells just differ by multiplication with a uniform constant factor. Using this result, we prove in Theorem 8.38 that the stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ are connected as follows. By Corollary 5.19, the stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ can be realized as $\mathbb{Z}$-linear combinations

$$\mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p) = \sum_{q\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}} a_{p,q}\cdot\Lambda_q, \qquad \mathrm{Stab}_{\mathfrak{C}'}^{\mathcal{C}(\mathcal{D}')}(p') = \sum_{q'\in\mathcal{C}(\mathcal{D}')^{\mathbb{T}'}} a'_{p',q'}\cdot\Lambda'_{q'},$$

where Λ_q and $\Lambda'_{q'}$ are the Poincaré duals of the fundamental class of the Zariski closures of $\mathrm{Attr}_{\mathfrak{C}}(q)$ and $\mathrm{Attr}_{\mathfrak{C}'}(q')$ in $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ respectively. Theorem 8.38 now states that if $\mathfrak{C}'$ restricts to $\mathfrak{C}$ (see Definition 8.25 for a precise definition) then we have

$$a_{p,q} = a'_{\iota(p),\iota(q)}, \quad \text{for all } p,\, q\in\mathcal{C}(\mathcal{D})^{\mathbb{T}}.$$

Since ι induces a bijection $\mathcal{C}(\mathcal{D})^{\mathbb{T}} \xrightarrow{\sim} \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$ on torus fixed points, this equality uniquely determines all the coefficients $a'_{p',q'}$. We therefore call Theorem 8.38 the *Coefficient Theorem.* As an application of Theorem 8.38, we then deduce that the stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ also just differ by a uniform constant factor, see Corollary 8.39.

These results might be compared with similar observations in the context of elliptic cohomology from [BR23, Section 5.10].

Assumption. In this chapter, all brane diagrams are assumed to be separated.

We use the following terminology: Given a brane diagram $\mathcal{D}$, we call a colored line Y in $\mathcal{D}$ *chargeless* if $d_{Y^-} = d_{Y^+}$. If Y is not chargeless, we call Y *essential.* We call the brane diagram $\mathcal{D}$ *essential* if all colored lines of $\mathcal{D}$ are essential.

Note that in an extension move either a red or a blue chargeless line is added. Also note that if D is a tie diagram of $\mathcal{D}$ and Y is a chargeless line of $\mathcal{D}$ then, as we assumed that $\mathcal{D}$ is separated, no tie in D is attached to Y.

8.1 Red extension moves

Let $\mathcal{D}$ and $\mathcal{D}'$ be brane diagrams. To distinguish the colored lines of $\mathcal{D}$ and $\mathcal{D}'$, we denote the red, blue and black lines of $\mathcal{D}$ by V, U and X whereas the red, blue and black lines of $\mathcal{D}'$ are denoted by V', U' and X' respectively.

Definition 8.1. Given a black line X_l in $\mathcal{D}$, we say that $\mathcal{D}'$ *is obtained from* $\mathcal{D}$ *via a red extension move at* X_l if we obtain $\mathcal{D}'$ from $\mathcal{D}$ by replacing the black line X_l with label $d = d_{X_l}$ with the following local configuration:

$$\frac{d}{X_l} \quad \rightsquigarrow \quad \frac{d}{X'_l} \,/\, \frac{d}{X'_{l+1}}$$

For instance, 0/1/3/3/5\3\2\0 is obtained from 0/1/3/5\3\2\0 via a red extension move at X_3 as the black line X_3 of $\mathcal{D}$ is replaced by the local configuration 3/3.

Assumption. Throughout this section, we assume that $\mathcal{D}'$ is obtained from $\mathcal{D}$ via a red extension move at X_l.

As we assumed that $\mathcal{D}$, $\mathcal{D}'$ are separated, we have a bijection between the respective sets of tie diagrams:

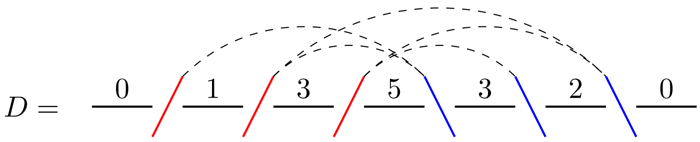
$$\mathrm{f}\colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}'), \tag{8.1}$$

where for $D \in \mathrm{Tie}(\mathcal{D})$ the corresponding tie diagram $\mathrm{f}(D)$ is given as

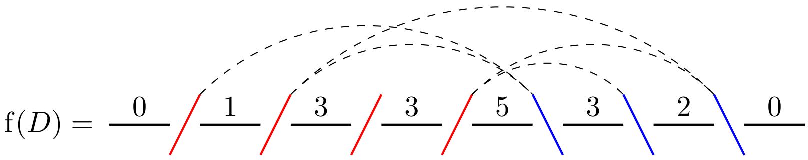
$$\mathrm{f}(D) = \{(V'_i, U'_j) \mid V_i \triangleright X_l,\ (V_i, U_j) \in D\} \cup \{(V'_i, U'_j) \mid V_{i-1} \triangleleft X_l,\ (V_{i-1}, U_j) \in D\}.$$

Pictorially, $\mathrm{f}(D)$ is obtained from D by just replacing the black line X_l with the local configuration d/d leaving all ties unchanged.

For example, consider $\mathcal{D} =$ 0/1/3/5\3\2\0, $\mathcal{D}' =$ 0/1/3/3/5\3\2\0 as above and choose $D \in \mathrm{Tie}(\mathcal{D})$ as follows:



Then, $\mathrm{f}(D)$ is obtained from D by replacing the black line X_3 with the local configuration $3/3$ where we do not change any of the ties:



In the next subsection, we compare the corresponding bow varieties of $\mathcal{D}$ and $\mathcal{D}'$.

Isomorphism of bow varieties

Recall from Notation 2.32 that M is the number of red and N is the number of blues lines in $\mathcal{D}$. Let $k \in \{0, \dots, M\}$ such that $(X_l')^+ = V_{k+1}'$, i.e. V_{k+1}' is the red line which is added to $\mathcal{D}'$ via the red extension move. Since $\mathcal{D}$ and $\mathcal{D}'$ are both separated, we have $1 \leq l \leq M+1$ and $k = M+1-l$.

As $\mathcal{D}$ and $\mathcal{D}'$ have the same number of blue lines, their respective bow varieties $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ both admit an action of the torus $\mathbb{T} = \mathbb{A} \times \mathbb{C}_h^*$, where $\mathbb{A} = (\mathbb{C}^*)^N$.

In the following, we show that there exists a $\mathbb{T}$-equivariant isomorphism between $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$. For this, recall the definition of the affine brane varieties $\widetilde{\mathcal{M}}(\mathcal{D})$, $\widetilde{\mathcal{M}}(\mathcal{D}')$, their ambient spaces $\mathbb{V}_{\mathcal{D}}$, $\mathbb{V}_{\mathcal{D}'}$ and the associated gauge groups $\mathcal{G}$, $\mathcal{G}'$ from (2.31), Definition 2.33 and (2.32). Recall also the moment maps $\tilde{m}$ resp. $\tilde{m}'$ of $\widetilde{\mathcal{M}}(\mathcal{D})$ resp. $\widetilde{\mathcal{M}}(\mathcal{D}')$ from (2.33) and that the vanishing loci $\tilde{m}^{-1}(0)$ resp. $(\tilde{m}')^{-1}(0)$ are locally closed subvariety of $\mathbb{V}_{\mathcal{D}}$ resp. $\mathbb{V}_{\mathcal{D}'}$. Let χ resp. χ' be the character of $\mathcal{G}$ resp. $\mathcal{G}'$ from Definition 2.36 and $\tilde{m}^{-1}(0)^{\mathrm{s}}$ resp. $(\tilde{m}')^{-1}(0)^{\mathrm{s}}$ the corresponding χ- resp. χ'-stable locus.

We have the following crucial results:

Lemma 8.2. *Let $\tilde{\Theta}\colon \mathbb{V}_{\mathcal{D}} \to \mathbb{V}_{\mathcal{D}'}$ be the morphism of varieties which maps a point*

$$y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \mathbb{V}_{\mathcal{D}}$$

to

$$\tilde{\Theta}(y) = ((A_{U'}, (B_{U'})^-, (B_{U'})^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}),$$

where $A_{U_i'} = A_{U_i}$, $B_{U_i'}^- = B_{U_i}^-$, $B_{U_i'}^+ = B_{U_i}^+$, $a_{U_i'} = a_{U_i}$, $b_{U_i'} = b_{U_i}$, for all $i = 1, \dots, N$ and

$$C_{V_j'} = \begin{cases} C_{V_j} & \text{if } j \leq k, \\ \mathrm{id} & \text{if } j = k+1, \\ C_{V_{j-1}} & \text{if } j > k+1, \end{cases} \qquad D_{V_j'} = \begin{cases} D_{V_j} & \text{if } j \leq k, \\ C_{V_k} D_{V_k} & \text{if } j = k+1, \\ D_{V_{j-1}} & \text{if } j > k+1. \end{cases} \tag{8.2}$$

if $l < M+1$ and

$$C_{V'_j} = \begin{cases} \mathrm{id} & \text{if } j = 1, \\ C_{V_{j-1}} & \text{if } j > 1, \end{cases} \qquad D_{V'_j} = \begin{cases} -B^-_{U_1} & \text{if } j = 1, \\ D_{V_{j-1}} & \text{if } j > 1, \end{cases} \tag{8.3}$$

if $l = M+1$. Then, $\tilde{\Theta}$ restricts to a morphism of varieties $\tilde{\Theta}\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to (\tilde{m}')^{-1}(0)^{\mathrm{s}}$.

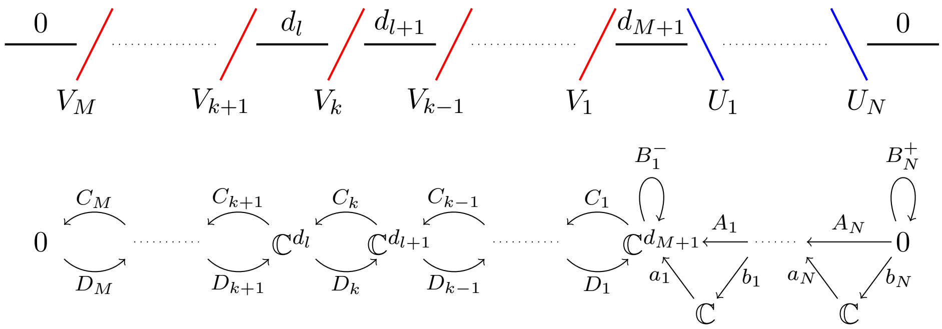
Proposition 8.3. *The morphism of varieties $\tilde{\Theta}\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ from Lemma 8.2 induces a $\mathbb{T}$-equivariant isomorphism of varieties*

$$\Theta\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\mathcal{D}').$$

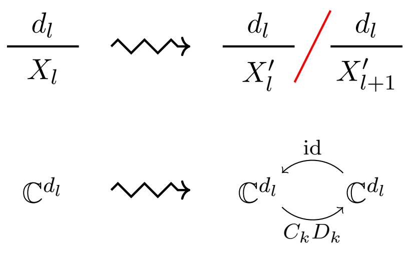
Remark. The morphism $\tilde{\Theta}$ from Lemma 8.2 can be illustrated as follows: Let

$$y = ((A_U, B^-_U, B^+_U, a_U, b_U)_U, (C_V, D_V)_V) \in \mathbb{V}_{\mathcal{D}}.$$

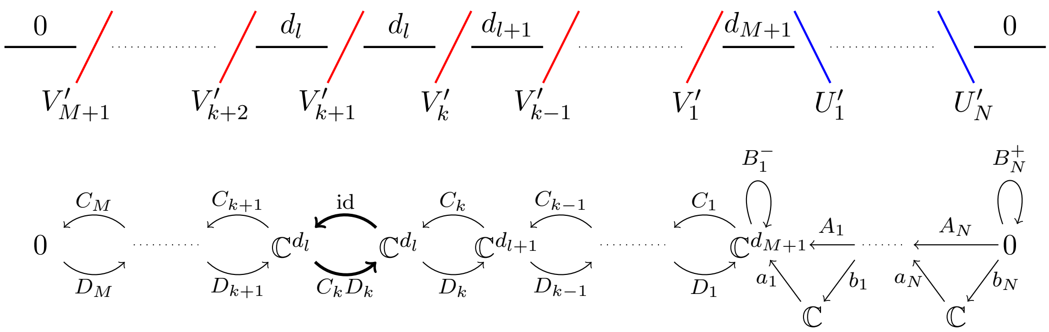
Then, y is presented by the following diagram:



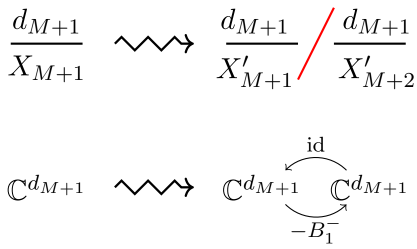
Here, as in Proposition 8.3, $k = M - l + 1$ and hence $V_k^- = X_l$. We also denote A_{U_i} by A_i, $B^-_{U_i}$ by B^-_i and similarly for the other operators. In case $l < M+1$, we obtain a diagram for $\tilde{\Theta}(y)$ by performing the following replacements:



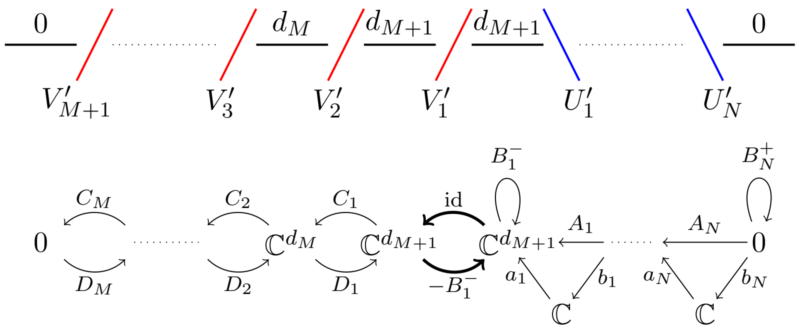
Hence, $\tilde{\Theta}(y)$ corresponds to the following diagram:



Here, we highlighted the newly added arrows. In the special case $l = M+1$, we perform the following replacement:



Thus, in this case, $\tilde{\Theta}(y)$ corresponds to the following diagram:



The next two subsections are devoted to the proof of Lemma 8.2 and Proposition 8.3.

Basis theorem for the blue part

We begin with the following general basis theorem for bow varieties corresponding to separated brane diagrams. The statement is similar to Corollary 3.29.

Recall the margin vectors $\mathbf{c} = \mathbf{c}(\mathcal{D}) = (c_1, \dots, c_N)$ from Definition 2.58 and that $d_{M+i} = \sum_{j=i}^{N} c_j$, for $i = 1, \dots, N+1$.

Proposition 8.4. *Let* $y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in (\tilde{m})^{-1}(0)$ *and set*

$$s_{y,j,r}^{(i)} \coloneqq A_{U_i} A_{U_{i+1}} \cdots A_{U_{j-2}} A_{U_{j-1}} (B_{U_j}^-)^r a_{U_j}(1) \in \mathbb{C}^{d_{M+i}} \tag{8.4}$$

for $i = 1, \dots, N$, $j \geq i$ *and* $r \geq 0$. *Then,*

$$(s_{y,j,r}^{(i)} \mid j = i, i+1, \dots, N, \; r = 0, \dots, c_j - 1)$$

is a basis of $\mathbb{C}^{d_{M+i}}$, *for all* $i = 1, \dots, N+1$.

Proof. We prove the statement via induction on i. The case $i = N+1$ is clear as in this case $d_{M+i} = 0$. If $i < N+1$ then by Lemma 2.23.(ii), we have a decomposition

$$\mathbb{C}^{d_{M+i}} = \operatorname{im}(A_{U_i}) \oplus \operatorname{im}(a_{U_i}) \oplus \operatorname{im}(B_{U_i}^- a_{U_i}) \oplus \ldots \oplus \operatorname{im}((B_{U_i}^-)^{c_i - 1} a_{U_i}). \tag{8.5}$$

By Proposition 2.19, A_{U_i} is injective. Hence, by the induction hypothesis, $\mathbb{C}^{d_{M+i+1}}$ admits the basis $(s_{y,j,r}^{(i+1)} \mid j = i+1, \dots, N, \; r = 0, \dots, c_j - 1)$. Therefore, $\operatorname{im}(A_{U_i})$ admits the basis $(s_{y,j,l}^{(i)} \mid j = i+1, \dots, N, \; r = 0, \dots, c_j - 1)$ which completes the proof. $\square$

From Proposition 8.4, we immediately deduce the following consequence about invariant subspaces of $W_{\mathcal{D}}$:

Corollary 8.5. *Let $T = \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X \subset W_{\mathcal{D}}$ be an A_U- and $B_U^{\pm}$-invariant graded subspace such that $a_U(1) \in T$, for all $U \in \mathrm{b}(\mathcal{D})$. Then, $T_{U^\pm} = W_{U^\pm}$, for all $U \in \mathrm{b}(\mathcal{D})$.*

In particular, Corollary 8.5 simplifies applications of the χ-stability criterion from Proposition 2.37.

Proofs of Lemma 8.2 and Proposition 8.3

As before, suppose that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a red extension move at X_l and let V'_{k+1} be the added red line.

Proof of Lemma 8.2. Since $\tilde{\Theta}$ leaves the operators corresponding to blue lines unchanged, $\tilde{\Theta}$ respects (2.12) as well as the conditions (S1) and (S2). Thus, we conclude $\tilde{\Theta}(\widetilde{\mathcal{M}}(\mathcal{D})) \subset \widetilde{\mathcal{M}}(\mathcal{D}')$. By (8.2) and (8.3), $\tilde{\Theta}$ also respects the moment map equation (2.33) and therefore $\tilde{\Theta}(\tilde{m}^{-1}(0)) \subset (\tilde{m}')^{-1}(0)$. Let

$$y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \tilde{m}^{-1}(0)$$

be χ-stable. To see that $\tilde{\Theta}(y) = ((A_{U'}, B_{U'}^-, B_{U'}^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'})$ is also χ'-stable, let $T = \bigoplus_{X' \in \mathrm{h}(\mathcal{D}')} T_{X'} \subset W_{\mathcal{D}'}$ be a graded subspace satisfying the conditions of Proposition 2.37. Note that

$$(A_{U_i}, B_{U_i}^-, B_{U_i}^+, a_{U_i}, b_{U_i}) = (A_{U'_i}, B_{U'_i}^-, B_{U'_i}^+, a_{U'_i}, b_{U'_i}), \quad \text{for } i = 1, \ldots, N.$$

Thus, as T is $A_{U'}$- and $B_{U'}^{\pm}$-invariant and contains all $a_{U'}(1)$, Corollary 8.5 yields $T_{(U')^\pm} = W_{(U')^\pm}$, for all $U' \in \mathrm{b}(\mathcal{D})$. By Proposition 2.57, all $C_{V'}$ are surjective which implies $T_{(V')^-} = W_{(V')^-}$, for all $V' \in \mathrm{r}(\mathcal{D})$. Hence $T = W_{\mathcal{D}'}$ and $\tilde{\Theta}(y)$ is χ'-stable by Proposition 2.37. Consequently, $\tilde{\Theta}$ restricts to a morphism of varieties $\tilde{\Theta}\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to (\tilde{m}')^{-1}(0)^{\mathrm{s}}$. □

For the proof of Proposition 8.3, we define a surjection

$$\pi_{\mathrm{h}}\colon \mathrm{h}(\mathcal{D}') \longrightarrow \mathrm{h}(\mathcal{D}), \quad \pi_{\mathrm{h}}(X'_i) = \begin{cases} X_i & \text{if } i \leq l, \\ X_{i-1} & \text{if } i > l. \end{cases} \tag{8.6}$$

We get an induced map $\iota_{\mathrm{h}}\colon \mathcal{G} \to \mathcal{G}'$, $(g_X)_X \mapsto (g'_{X'})_{X'}$, where $g'_{X'} = g_{\pi_{\mathrm{h}}(X')}$, for all $X' \in \mathrm{h}(\mathcal{D}')$.

The next lemma gives that $\tilde{\Theta}$ indeed induces a surjective morphism on the associated bow varieties:

Lemma 8.6. *The morphism of varieties $\tilde{\Theta}\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ induces a surjective morphism of varieties $\Theta\colon \mathcal{C}(\mathcal{D}) \to \mathcal{C}(\mathcal{D}')$.*

Proof. If we are given y_1, $y_2 \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ such that $g.y_1 = y_2$ for some $g = (g_X)_X \in \mathcal{G}$ then, by (8.2) and (8.2), $\iota_{\mathrm{h}}(g).\tilde{\Theta}(y_1) = \tilde{\Theta}(y_2)$. Hence, $\tilde{\Theta}$ induces a morphism $\Theta\colon \mathcal{C}(\mathcal{D}) \to \mathcal{C}(\mathcal{D}')$. For surjectivity, let $y' = ((A_{U'}, B_{U'}^-, B_{U'}^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$. As $d_{X'_l} = d_{X'_{l+1}}$, we deduce from Proposition 2.57 that $C_{V'_{k+1}}$ is an isomorphism of vector spaces. Define $\tilde{g} = (\tilde{g}_{X'})_{X'} \in \mathcal{G}'$ as $\tilde{g}_{(V'_{k+1})^+} := C_{V'_{k+1}}$ and $\tilde{g}_{X'} := \mathrm{id}$, for $X' \neq (V'_{k+1})^+$. Write

$$\tilde{g}.y' = ((\tilde{A}_{U'}, \tilde{B}^-_{U'}, \tilde{B}^+_{U'}, \tilde{a}_{U'}, \tilde{b}_{U'})_{U'}, (\tilde{C}_{V'}, \tilde{D}_{V'})_{V'}).$$

Since $\tilde{C}_{V'_{k+1}} = \mathrm{id}$, there exists $y \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ with $\tilde{\Theta}(y) = \tilde{g}.y'$ by Lemma 8.7 below. Thus, Θ is surjective. □

Lemma 8.7. *Let* $y' = ((A_{U'}, B^-_{U'}, B^+_{U'}, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ *with* $C_{V'_{k+1}} = \mathrm{id}$. *Define* $F(y') := ((A_U, B^-_U, B^+_U, a_U, b_U)_U, (C_V, D_V)_V) \in \mathbb{V}_{\mathcal{D}}$ *via* $A_{U_i} = A_{U'_i}$, $B^-_{U_i} = B^-_{U'_i}$, $B^+_{U_i} = B^+_{U'_i}$, $a_{U_i} = a_{U'_i}$, $b_{U_i} = b_{U'_i}$, *for all* $i = 1, \ldots, N$ *and*

$$C_{V_j} = \begin{cases} C_{V'_j} & \text{for } j \leq k, \\ C_{V'_{j+1}} & \text{for } j > k, \end{cases} \qquad D_{V_j} = \begin{cases} D_{V'_j} & \text{for } j \leq k, \\ D_{V'_{j+1}} & \text{for } j > k. \end{cases} \tag{8.7}$$

Then, $\tilde{\Theta}(F(y')) = y'$ *and* $F(y')$ *is contained in* $\tilde{m}^{-1}(0)^{\mathrm{s}}$.

Proof. By (2.33), (8.2) and (8.3), we have $\tilde{\Theta}(F(y')) = y'$. Thus, it is left to show that $F(y')$ is contained in $\tilde{m}^{-1}(0)^{\mathrm{s}}$. As F leaves the operators corresponding to blue lines unchanged, we conclude that $F(y')$ satisfies (2.12), (S1) and (S2). Thus, $F(y') \in \widetilde{\mathcal{M}}(\mathcal{D})$. From (8.7) follows that $F(y')$ also satisfies the moment map equation (2.33) and therefore $F(y') \in \tilde{m}^{-1}(0)$. To see that $F(y')$ is χ-stable let $T = \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X \subset W_{\mathcal{D}}$ be a graded subspace satisfying the conditions of Proposition 2.37. Since y' is χ'-stable, Corollary 8.5 gives $T_{U^\pm} = W_{U^\pm}$, for all $U \in \mathrm{b}(\mathcal{D})$. As also all C_V are surjective by Proposition 2.57, we deduce $T = W_{\mathcal{D}}$ and hence $F(y')$ is χ-stable by Proposition 2.37. □

The next lemma states that the morphism Θ is indeed $\mathbb{T}$-equivariant.

Lemma 8.8. *The morphism of varieties* $\Theta\colon \mathcal{C}(\mathcal{D}) \to \mathcal{C}(\mathcal{D}')$ *is* $\mathbb{T}$*-equivariant.*

Proof. Note that $\tilde{\Theta}$ leaves the operators attached to blue lines unchanged. Hence (2.48) implies that $\tilde{\Theta}$ (and hence also Θ) is $\mathbb{A}$-equivariant. So it is left to show that Θ is $\mathbb{C}^*_h$-equivariant. Let $h \in \mathbb{C}^*_h$ and $y \in \tilde{m}^{-1}(0)^{\mathrm{s}}$. By (8.2) and (8.3), we have $\tilde{\Theta}(h.y) = g_h.(h.\tilde{\Theta}(y))$, where

$$(g_h)_{X'_i} = \begin{cases} h^{-1} \cdot \mathrm{id} & \text{for } i \leq l, \\ \mathrm{id} & \text{for } i > l. \end{cases}$$

Thus, Θ is $\mathbb{C}^*_h$- and hence also $\mathbb{T}$-equivariant. □

Proof of Proposition 8.3. By Lemma 8.6 and Lemma 8.8, we know that Θ is a surjective and $\mathbb{T}$-equivariant morphism of varieties. By Proposition 2.25, Θ is an isomorphism of varieties if and only if Θ is bijective. Thus, it is left to show that Θ is injective. Suppose there are y_1, $y_2 \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ with $g'.\tilde{\Theta}(y_1) = \tilde{\Theta}(y_2)$, for some $g' = (g'_{X'})_{X'} \in \mathcal{G}'$. Then, (8.2) and (8.3) yield $g.y_1 = y_2$, where $g = (g_X)_X \in \mathcal{G}$ is given by

$$g_{X_i} = \begin{cases} g'_{X'_i} & \text{for } i \leq l, \\ g'_{X'_{i+1}} & \text{for } i > l. \end{cases}$$

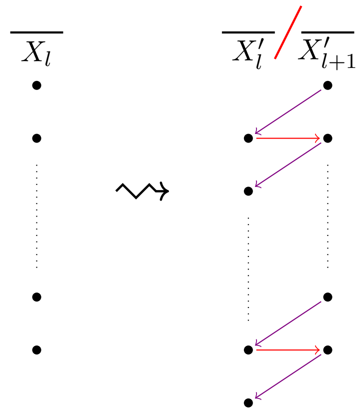
Hence, Θ is indeed injective. □

Invariance of stable basis elements

Since the isomorphism $\Theta\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\mathcal{D}')$ is $\mathbb{T}$-equivariant, Θ induces a bijection $\mathcal{C}(\mathcal{D})^{\mathbb{T}} \xrightarrow{\sim} \mathcal{C}(\mathcal{D}')^{\mathbb{T}}$. The next corollary states that this bijection corresponds to the bijection $\mathrm{f}\colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}')$ from (8.1).

Corollary 8.9. *We have* $\Theta(x_D) = x_{\mathrm{f}(D)}$*, for all* $D \in \mathrm{Tie}(\mathcal{D})$*.*

Proof. For all $i = 1, \dots, N$, the butterfly diagram $\mathrm{b}(U_i', \mathrm{f}(D))$ is obtained from the butterfly diagram $\mathrm{b}(U_i, D)$ by first replacing the column corresponding to X_l by the following diagram:



and then shifting all dots and arrows corresponding to lines which are to the left of X_l' down by one. Thus, by Lemma 8.2, we have $\tilde{\Theta}(y_D) = y_{\mathrm{f}(D)}$, where $y_D \in \tilde{m}^{-1}(0)^{\mathrm{s}}$, $y_{\mathrm{f}(D)} \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ are defined as in Proposition 3.11. Since $x_D = [y_D]$ and $x_{\mathrm{f}(D)} = [y_{\mathrm{f}(D)}]$, we conclude $\Theta(x_D) = x_{\mathrm{f}(D)}$. □

From Proposition 8.3 and Corollary 8.9, we now deduce that the induced isomorphism $\Theta^*\colon H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}')) \xrightarrow{\sim} H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ yields the following bijection on the respective sets of stable basis elements:

Corollary 8.10. *For all* $D \in \mathrm{Tie}(\mathcal{D})$ *and any choice of chamber* $\mathfrak{C}$ *of* $\mathbb{A}$*, we have*

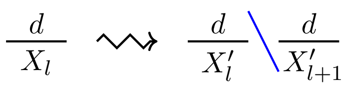
$$\Theta^*(\mathrm{Stab}_{\mathfrak{C}}(x_{\mathrm{f}(D)}) = \mathrm{Stab}_{\mathfrak{C}}(x_D).$$

Proof. Since the stability conditions for stable basis elements from Theorem 5.10 are invariant with respect to $\mathbb{T}$-equivariant isomorphisms, $\Theta^*(\mathrm{Stab}_{\mathfrak{C}}(x_{\mathrm{f}(D)})$ is a stable basis element. As $\Theta(x_D) = x_{\mathrm{f}(D)}$, we must have $\Theta^*(\mathrm{Stab}_{\mathfrak{C}}(x_{\mathrm{f}(D)}) = \mathrm{Stab}_{\mathfrak{C}}(x_D)$. □

8.2 Blue extension moves

Again, let $\mathcal{D}$ and $\mathcal{D}'$ be brane diagrams.

Definition 8.11. Given a black line X_l in $\mathcal{D}$, we say that $\mathcal{D}'$ *is obtained from* $\mathcal{D}$ *via a blue extension move at* X_l if we obtain $\mathcal{D}'$ from $\mathcal{D}$ by replacing the black line X_l with label $d = d_{X_l}$ with the following local configuration:



For example, the brane diagram $0/1/3/5\backslash 3\backslash 2\backslash 2\backslash 0$ is obtained from $0/1/3/5\backslash 3\backslash 2\backslash 0$ via a blue extension move at X_6 since X_6 is replaced with the configuration $2\backslash 2$.

Assumption. From now until Section 8.7, we assume that $\mathcal{D}'$ is obtained from $\mathcal{D}$ via a blue extension move at X_l.

Just as in (8.1), we have a bijection

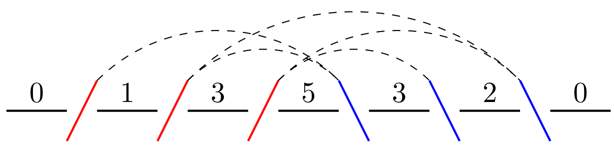
$$\mathrm{f}'\colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}'), \tag{8.8}$$

where for $D \in \mathrm{Tie}(\mathcal{D})$ the tie diagram $\mathrm{f}'(D)$ is defined as

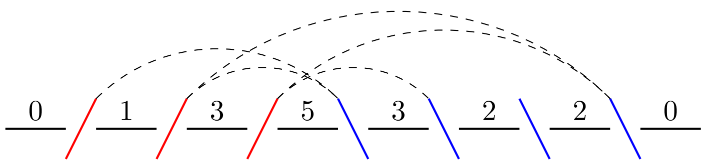
$$\mathrm{f}'(D) = \{(V_i', U_j') \mid U_j \triangleleft X_l,\ (V_i, U_j) \in D\} \cup \{(V_i', U_j') \mid U_{j-1} \triangleright X_l,\ (V_i, U_{j-1}) \in D\}.$$

Pictorially, $\mathrm{f}'(D)$ is obtained from D by replacing the black line X_l with the local configuration $d\backslash d$ and we leave all ties unchanged.

Example 8.12. Choose as above $\mathcal{D} = 0/1/3/5\backslash 3\backslash 2\backslash 0$ and $\mathcal{D}' = 0/1/3/5\backslash 3\backslash 2\backslash 2\backslash 0$. Let $D \in \mathrm{Tie}(\mathcal{D})$ be the tie diagram

$D =$ 

Then, we obtain $\mathrm{f}'(D)$ from D by just replacing the black line X_6 with the local configuration $2\backslash 2$ leaving all ties unchanged:

$\mathrm{f}'(D) =$ 

In contrast to the previous section, the next lemma gives that the bow varieties $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ are in general of different dimension. Hence, $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ are in general not isomorphic as varieties.

Lemma 8.13. *We have* $\dim(\mathcal{C}(\mathcal{D}')) = \dim(\mathcal{C}(\mathcal{D})) + 2d$.

For example, if $\mathcal{D}$ and $\mathcal{D}'$ are as in Example 8.12, we have $d = 2$ and thus $\dim(\mathcal{C}(\mathcal{D}')) = \dim(\mathcal{C}(\mathcal{D})) + 4$.

Proof of Lemma 8.13. By (2.43), we have

$$\dim(\mathcal{C}(\mathcal{D}')) - \dim(\mathcal{C}(\mathcal{D})) = 2d(d+1) - 2d^2 = 2d$$

which proves Lemma 8.13. □

However, as we will discuss in the next subsection, there exists a closed embedding of $\mathcal{C}(\mathcal{D})$ into $\mathcal{C}(\mathcal{D}')$. The construction of this embedding is similar to the construction of the isomorphism Θ from Proposition 8.3.

8.3 Embedding Theorem

Let $\mathbb{V}_{\mathcal{D}}$, $\mathbb{V}_{\mathcal{D}'}$, $\mathcal{G}, \mathcal{G}'$, $\widetilde{\mathcal{M}}(\mathcal{D})$, $\widetilde{\mathcal{M}}(\mathcal{D}')$, $\tilde{m}$, $\tilde{m}'$, χ and χ' be as in Section 8.1. Let $k \in \{1, \dots, N+1\}$ such that $(X_l')^+ = U_k'$. That is, U_k' is the blue line that is added in the blue extension move.

The bow variety $\mathcal{C}(\mathcal{D})$ resp. $\mathcal{C}(\mathcal{D}')$ is endowed with the action of the torus $\mathbb{T} = \mathbb{A} \times \mathbb{C}_h^*$ resp. $\mathbb{T}' = \mathbb{A}' \times \mathbb{C}_h^*$. Note that $\mathbb{A}$ has rank N and $\mathbb{A}'$ has rank $N+1$. We view $\mathbb{T}$ as subtorus of $\mathbb{T}'$ via

$$\mathbb{T} \hookrightarrow \mathbb{T}', \quad (t_1, \dots, t_N, h) \mapsto (t_1, \dots, t_{k-1}, 1, t_k, \dots, t_N, h).$$

On the other hand, via the quotient map

$$\mathbb{T}' \longrightarrow \mathbb{T}, \quad (t_1, \dots, t_{N+1}, h) \mapsto (t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_{N+1}, h),$$

we view $\mathcal{C}(\mathcal{D})$ as $\mathbb{T}'$-variety.

We set

$$Z_0 \coloneqq \{[(A_{U'}, B_{U'}^-, B_{U'}^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}] \mid a_{U_k'} = 0,\ b_{U_k'} = 0\} \subset \mathcal{C}(\mathcal{D}').$$

Then, Z_0 is a $\mathbb{T}'$-equivariant closed subvariety of $\mathcal{C}(\mathcal{D}')$. The next two results are the main results of this section. In particular, they provide an isomorphism of varieties $\mathcal{C}(\mathcal{D}) \xrightarrow{\sim} Z_0$.

Lemma 8.14. *Let $\iota' \colon \mathbb{V}_{\mathcal{D}} \to \mathbb{V}_{\mathcal{D}'}$ be the morphism which maps a point*

$$y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \mathbb{V}_{\mathcal{D}}$$

to

$$\iota'(y) = ((A_{U'}, B_{U'}^-, B_{U'}^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}),$$

where $(C_{V_j'}, D_{V_j'}) = (C_{V_j}, D_{V_j})$, for all j and

$$(A_{U_i'}, B_{U_i'}^-, B_{U_i'}^+, a_{U_i'}, b_{U_i'}) = \begin{cases} (A_{U_i}, B_{U_i}^-, B_{U_i}^+, a_{U_i}, b_{U_i}) & \text{if } i < k, \\ (\mathrm{id}, B_{U_{k-1}}^+, B_{U_{k-1}}^+, 0, 0) & \text{if } i = k, \\ (A_{U_{i-1}}, B_{U_{i-1}}^-, B_{U_{i-1}}^+, a_{U_{i-1}}, b_{U_{i-1}}) & \text{if } i > k, \end{cases} \tag{8.9}$$

if $k > 1$ and

$$(A_{U_i'}, B_{U_i'}^-, B_{U_i'}^+, a_{U_i'}, b_{U_i'}) = \begin{cases} (\mathrm{id}, -D_{V_1}C_{V_1}, -D_{V_1}C_{V_1}, 0, 0) & \text{if } i = 1, \\ (A_{U_{i-1}}, B_{U_{i-1}}^-, B_{U_{i-1}}^+, a_{U_{i-1}}, b_{U_{i-1}}) & \text{if } i > 1, \end{cases} \tag{8.10}$$

if $k = 1$. Then, ι' restricts to a morphism $\iota' \colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to (\tilde{m}')^{-1}(0)^{\mathrm{s}}$.

Theorem 8.15 (Embedding Theorem)**.** *The morphism $\iota' \colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ from Lemma 8.14 induces a $\mathbb{T}'$-equivariant closed immersion*

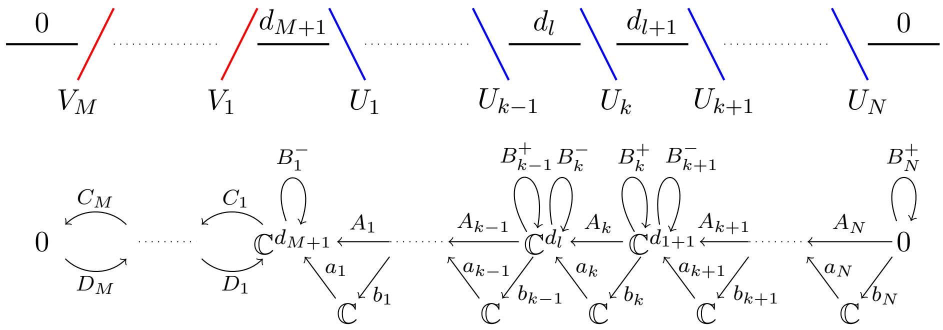
$$\iota \colon \mathcal{C}(\mathcal{D}) \hookrightarrow \mathcal{C}(\mathcal{D}')$$

which restricts to a $\mathbb{T}'$-equivariant isomorphism $\mathcal{C}(\mathcal{D}) \xrightarrow{\sim} Z_0$.

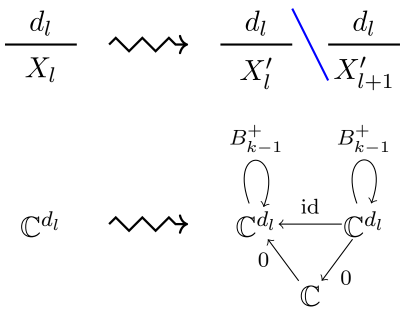
Remark. The morphism $\iota' \colon \mathbb{V}_{\mathcal{D}} \to \mathbb{V}_{\mathcal{D}'}$ from Lemma 8.14 can be illustrated as follows: Let

$$y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \mathbb{V}_{\mathcal{D}}.$$

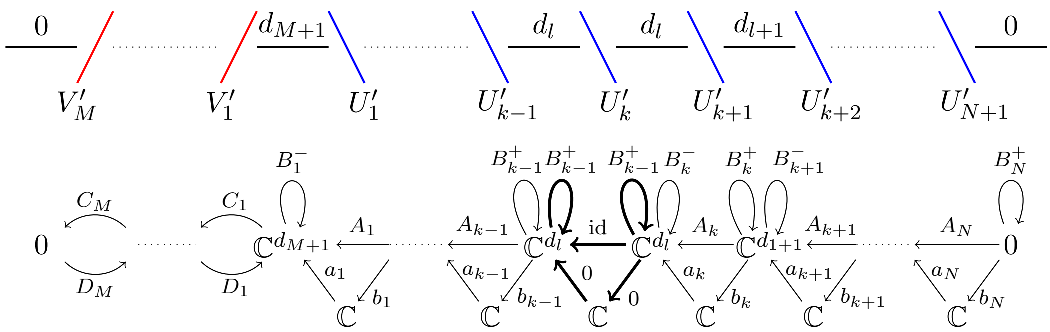
Then, y is presented by the diagram:



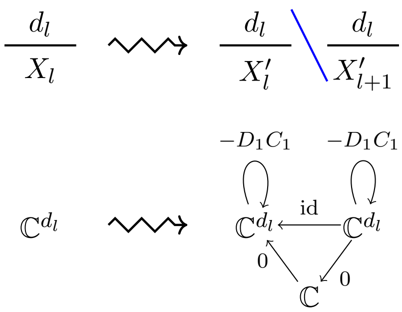
Again, we write A_i for A_{U_i}, B_i^- for $B_{U_i}^-$ etc. In case $k > 1$, we obtain the diagram for $\iota'(y)$ by performing the following replacement:



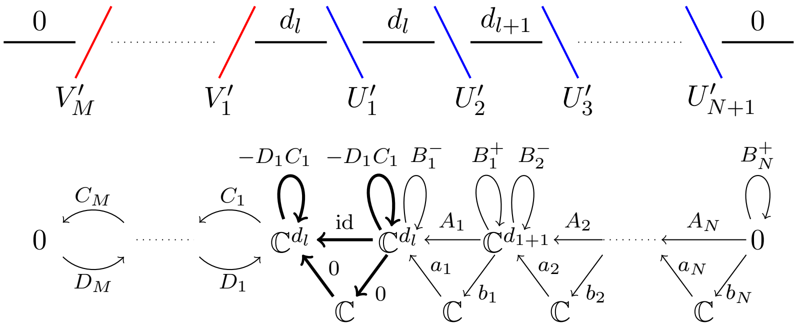
That is, $\iota'(y)$ is given by the diagram:



Here, we highlighted the newly added arrows. In case $k = 1$, we replace $\mathbb{C}^{d_l}$ as follows:



Then, $\iota'(y)$ corresponds to the diagram



The next two subsections are devoted to the proofs of Lemma 8.14 and Theorem 8.15.

Z_0 as torus fixed locus

In this subsection, we apply Proposition 8.4 to show that the subvariety $Z_0 \subset \mathcal{C}(\mathcal{D}')$ from Theorem 8.15 is the fixed locus corresponding to the cocharacter

$$\sigma_0 \colon \mathbb{C}^* \longrightarrow \mathbb{A}', \quad t \mapsto (\sigma_{0,U'}(t))_{U'}, \quad \text{where } \sigma_{0,U'}(t) = \begin{cases} t & \text{if } U' = U'_k, \\ 1 & \text{if } U' \neq U'_k. \end{cases} \tag{8.11}$$

Proposition 8.16. *We have $Z_0 = \mathcal{C}(\mathcal{D}')^{\sigma_0}$.*

As a direct consequence of Proposition 8.16, we get the following:

Corollary 8.17. *We have that Z_0 is a smooth subvariety of $\mathcal{C}(\mathcal{D}')$.*

Proof. Recall from Proposition 2.2.(i) that $\mathcal{C}(\mathcal{D}')$ is quasi-projective. Thus, by Theorem 7.1, the fixed locus $\mathcal{C}(\mathcal{D}')^{\sigma_0}$ is a smooth subvariety of $\mathcal{C}(\mathcal{D}')$. Hence, Z_0 is smooth by Proposition 8.16. □

The following auxiliary statement will be used in the proof of Proposition 8.16:

Lemma 8.18. *Let $y = ((A_{U'}, B^-_{U'}, B^+_{U'}, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$. Suppose there exist $g = (g_{X'})_{X'} \in \mathcal{G}'$ and $t \in \mathbb{C}^*$ such that $g.y = \sigma_0(t).y$. Then, we have $g = \mathrm{id}$.*

Proof. Denote the margin vector of $\mathcal{D}'$ by $\mathbf{c}(\mathcal{D}') = (c'_1, \ldots, c'_{N+1})$. By construction,

$$c'_i = \begin{cases} c_i & \text{if } i < k, \\ 0 & \text{if } i = k, \\ c_{i-1} & \text{if } i > k. \end{cases}$$

As in Proposition 8.4, let

$$s^{(i)}_{y,j,r} \coloneqq A_{U'_i} A_{U'_{i+1}} \cdots A_{U'_{j-2}} A_{U'_{j-1}} (B^-_{U'_j})^r a_{U'_j}(1) \in \mathbb{C}^{d'_{M+i}},$$

for $i = 1, \ldots, N+1$, $j \geq i$ and $r \geq 0$. From $g.y = \sigma_0(t).y$, we deduce

$$g_{(U')^-} A_{U'} = A_{U'} g_{(U')^+}, \quad g_{(U')^\pm} B^\pm_{U'} = B^\pm_{U'} g_{(U')^\pm}, \quad g_{(U')^-} a_{U'}(1) = \sigma_{0,U'}(t) \cdot a_{U'}(1),$$

for all $U' \in \mathrm{b}(\mathcal{D}')$. Thus, if $j \neq k$, we have

$$\begin{aligned} g_{(U_i')^-} s_{y,j,r}^{(i)} &= g_{(U_i')^-} A_{U_i'} \cdots A_{U_{j-1}'} (B_{U_j}^-)^r a_{U_j'}(1) \\ &= A_{U_i'} \cdots A_{U_{j-1}'} (B_{U_j'}^-)^r (\sigma_{0,U_j'}(t) \cdot a_{U_j'}(1)) \\ &= s_{y,j,r}^{(i)}. \end{aligned} \tag{8.12}$$

Since $c_k' = 0$, Proposition 8.4 says that

$$(s_{y,j,r}^{(i)} \mid j = i, i+1, \ldots, N+1,\ j \neq k,\ r = 0, \ldots, c_j' - 1)$$

is a basis for $\mathbb{C}^{d'_{M+i}}$ for $i = 1, \ldots, N+1$. Hence, (8.12) yields $g_{(U')^\pm} = \mathrm{id}$, for all $U' \in \mathrm{b}(\mathcal{D}')$. Next, we prove via induction on j that also all $g_{(V_j')^+}$ are equal to the identity. The case $j = 1$ is clear since $(V_j')^+ = (U_j')^-$. Let $j > 1$ and recall from Proposition 2.57 that $C_{V_{j-1}'}$ is surjective. From $g.y = \sigma_0(t).y$, we deduce $g_{(V_j')^+} C_{V_{j-1}'} = C_{V_{j-1}'} g_{(V_{j-1}')^+}$. By the induction hypothesis $g_{(V_{j-1}')^+} = \mathrm{id}$ and hence, as $C_{V_j'}$ is surjective, we also have $g_{(V_j')^+} = \mathrm{id}$. Thus, we proved $g = \mathrm{id}$. □

Proof of Proposition 8.16. Let $y = ((A_{U'}, B_{U'}^-, B_{U'}^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ with $a_{U_k'} = 0$ and $b_{U_k'} = 0$. By (8.11), $\sigma_0(t).y = y$, for all $t \in \mathbb{C}^*$. This proves $Z_0 \subset \mathcal{C}(\mathcal{D})^{\sigma_0}$. Conversely, let $t \in \mathbb{C}^* \setminus \{1\}$ and $y = ((A_{U'}, B_{U'}^-, B_{U'}^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ with $\sigma_0(t).y = g.y$, for some $g = (g_{X'})_{X'} \in \mathcal{G}'$. By Lemma 8.18, we conclude $g = \mathrm{id}$. Therefore, we have $t^{-1} a_{U_k'} = a_{U_k'}$ and $t b_{U_k'} = b_{U_k'}$. Consequently, $a_{U_k'} = 0$ and $b_{U_k'} = 0$ which yields $\mathcal{C}(\mathcal{D})^{\sigma_0} \subset Z_0$. □

Proofs of Lemma 8.14 and Theorem 8.15

Define the surjection

$$\pi_{\mathrm{h}} \colon \mathrm{h}(\mathcal{D}') \longrightarrow \mathrm{h}(\mathcal{D}), \quad \pi_{\mathrm{h}}(X_i') = \begin{cases} X_i & \text{if } i \leq l, \\ X_{i-1} & \text{if } i > l \end{cases}$$

and the inclusion $\iota_{\mathrm{h}} \colon \mathcal{G} \to \mathcal{G}'$, $(g_X)_X \mapsto (g'_{X'})_{X'}$, where $g'_{X'} = g_{\pi_{\mathrm{h}}(X')}$, for all $X' \in \mathrm{h}(\mathcal{D}')$.

Proof of Lemma 8.14. For $y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in \tilde{m}^{-1}(0)^{\mathrm{s}}$, we write

$$\iota'(y) = ((A_{U'}, B_{U'}^-, B_{U'}^+, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}).$$

By (8.9) resp. (8.10), $\iota'(y)$ satisfies (2.12), (S1) and (S2). Thus, $\iota'(y) \in \widetilde{\mathcal{M}}(\mathcal{D}')$. Since also $(C_{V_j}, D_{V_j}) = (C_{V_j'}, D_{V_j'})$, for all j, we conclude $\iota'(y) \in (\tilde{m}')^{-1}(0)$. To see that $\iota'(y)$ is χ'-satble, suppose that $T' = \bigoplus_{X' \in \mathrm{h}(\mathcal{D}')} T'_{X'} \subset W_{\mathcal{D}'}$ is a graded subspace satisfying the conditions of Proposition 2.37. Define the graded subspace $T = \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X \subset W_{\mathcal{D}}$ as $T_{V^\pm} = T'_{(V')^\pm}$ and

$$T_{U_i^+} = \begin{cases} T'_{(U_i')^+} & \text{if } i < k, \\ T'_{(U_{i+1}')^+} & \text{if } i \geq k. \end{cases}$$

As T contains all $a_U(1)$ and is invariant under all A_U and $B_U^\pm$, Corollary 8.5 gives $T_{U^\pm} = W_{U^\pm}$, for all $U \in \mathrm{b}(\mathcal{D})$. As T is also invariant under all C_V and all C_V are surjective by

Proposition 2.57, we conclude $T = W_{\mathcal{D}}$ which implies $T'_{X'} = W_{X'}$, for $X' \neq (U'_k)^-$. Since $A'_{U'_k} = \mathrm{id}$ and T' is invariant under $A'_{U'_k}$, we also deduce $T'_{(U'_k)^-} = W_{(U'_k)^-}$. Thus, $T' = W_{\mathcal{D}'}$ and $\iota'(y)$ is χ'-stable by Proposition 2.37. $\square$

The following lemma will be used in the proof of Theorem 8.15.

Lemma 8.19. *The morphism of varieties $\iota' \colon \tilde{m}^{-1}(0)^{\mathrm{s}} \to (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ from Lemma 8.14 induces a morphism of varieties $\iota \colon \mathcal{C}(\mathcal{D}) \to \mathcal{C}(\mathcal{D}')$ and the image of ι equals Z_0.*

Proof. If $y \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ and $g = (g_X)_X \in \mathcal{G}$ then (8.9) and (8.10) imply $\iota'(g.y) = \iota_{\mathrm{h}}(g).\iota'(y)$. Thus, ι' induces a morphism of varieties $\iota \colon \mathcal{C}(\mathcal{D}) \to \mathcal{C}(\mathcal{D}')$. Again, by (8.9) and (8.10), the image of ι is contained in Z_0. Conversely, let $y' = ((A_{U'}, B^-_{U'}, B^+_{U'}, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ with $a_{U'_k} = 0$ and $b_{U'_k} = 0$. Since $d_{(U'_k)^-} = d_{(U'_k)^+}$, Proposition 2.19 gives that $A_{U'_k}$ is an isomorphism of vector spaces. Define $\tilde{g} = (\tilde{g}_{X'})_{X'} \in \mathcal{G}'$ as

$$\tilde{g}_{X'} = \begin{cases} A_{U'_k} & \text{if } X' = (U'_k)^+, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Set $\tilde{y} \coloneqq \tilde{g}.y$ and write

$$\tilde{y} = ((\tilde{A}_{U'}, \tilde{B}^-_{U'}, \tilde{B}^+_{U'}, \tilde{a}_{U'}, \tilde{b}_{U'})_{U'}, (\tilde{C}_{V'}, \tilde{D}_{V'})_{V'}).$$

By construction, $\tilde{A}_{U'_k} = \mathrm{id}$. Hence, Lemma 8.20 below gives that there exists $y \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ with $\iota'(y) = \tilde{y}$. Thus, $\iota([y]) = [y']$ which proves $\mathrm{im}(\iota) = Z_0$. $\square$

Lemma 8.20. *Let $y' = ((A_{U'}, B^-_{U'}, B^+_{U'}, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ with $a_{U'_k} = 0$, $b_{U'_k} = 0$ and $A'_{U'_k} = \mathrm{id}$. Define*

$$F(y') \coloneqq ((A_U, B^-_U, B^+_U, a_U, b_U)_U, (C_V, D_V)_V) \in \mathbb{V}_{\mathcal{D}}$$

via

$$(A_{U_i}, B^-_{U_i}, B^+_{U_i}, a_{U_i}, b_{U_i}) = \begin{cases} (A_{U'_i}, B^-_{U'_i}, B^+_{U'_i}, a_{U'_i}, b_{U'_i}) & \text{if } i < k, \\ (A_{U'_{i+1}}, B^-_{U'_{i+1}}, B^+_{U'_{i+1}}, a_{U'_{i+1}}, b_{U'_{i+1}}) & \text{if } i \geq k \end{cases} \tag{8.13}$$

and $(C_{V_j}, D_{V_j}) = (C_{V'_j}, D_{V'_j})$, for all j. Then, $F(y') \in \tilde{m}^{-1}(0)^{\mathrm{s}}$ and $\iota' F(y') = y'$.

Proof. By (8.9) and (8.10), we conclude $\iota' F(y') = y'$. From (8.13), we deduce that $F(y')$ satisfies (2.12), (S1), (S2) and hence $F(y') \in \widetilde{\mathcal{M}}(\mathcal{D})$. Since F further leaves all operators corresponding to red lines invariant, we deduce that $F(y') \in \tilde{m}^{-1}(0)$. To conclude that $F(y')$ is χ-stable, let $T = \bigoplus_{X \in \mathrm{h}(\mathcal{D})} T_X \subset W_{\mathcal{D}}$ be a graded subspace satisfying the conditions of Proposition 2.37. We define the graded subspace $T' = \bigoplus_{X' \in \mathrm{h}(\mathcal{D}')} T'_{X'} \subset W_{\mathcal{D}'}$ as

$$T'_{X'} \coloneqq T_{\pi_{\mathrm{h}}(X')}, \quad \text{for } X' \in \mathrm{h}(\mathcal{D}').$$

As $a_{U'_k} = 0$, we deduce that T' contains all $a_{U'}(1)$. Since T' is invariant under all $A_{U'}$ and $B^{\pm}_{U'}$, Corollary 8.5 yields $T'_{(U')^{\pm}} = W_{(U')^{\pm}}$, for all $U' \in \mathrm{b}(\mathcal{D}')$. As T' is also invariant under all $C_{V'}$, Proposition 2.57 implies that $T'_{(V')^{\pm}} = W_{(V')^{\pm}}$, for all $V' \in \mathrm{r}(\mathcal{D}')$. Thus, $T' = W_{\mathcal{D}'}$ which also implies $T = W_{\mathcal{D}}$. Thus, $F(y')$ is χ-stable by Proposition 2.37. $\square$

Proof of Theorem 8.15. By Lemma 8.19, we have $\mathrm{im}(\iota) = Z_0$. As Z_0 is smooth, Proposition 2.25 implies that ι restricts to an isomorphism $\iota\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} Z_0$ if and only if ι is injective. Suppose $g'.\iota'(y_1) = \iota'(y_2)$, for some $y_1, y_2 \in \tilde{m}^{-1}(0)^{\mathrm{s}}$, $g' = (g'_{X'})_{X'} \in \mathcal{G}'$. By (8.9) and (8.10), we have $g.y_1 = y_2$, where $g = (g_X)_X \in \mathcal{G}$ is defined as

$$g_{X_j} = \begin{cases} g'_{X'_j} & \text{if } j \leq l, \\ g'_{X'_{j+1}} & \text{if } j > l. \end{cases}$$

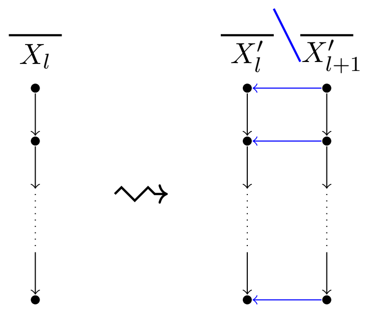
Thus, ι is injective and hence induces an isomorphism $\iota\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} Z_0$. Finally, (8.9) and (8.10) give that ι' is $\mathbb{T}'$-equivariant. Thus, also ι is $\mathbb{T}'$-equivariant which completes the proof. □

Matching of torus fixed points

Recall the bijection $\mathrm{f}'\colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}')$ from (8.8).

Corollary 8.21. *We have $\iota(x_D) = x_{\mathrm{f}'(D)}$, for all $D \in \mathrm{Tie}(\mathcal{D})$.*

Proof. Given $i = 1, \ldots, N$, the butterfly diagram $\mathrm{b}(U'_i, f'(D))$ is obtained from the butterfly diagram $\mathrm{b}(U_i, D)$ by replacing the column and black arrows corresponding to X_l by the diagram:



Therefore, (8.9) and (8.10) imply $\iota'(y_D) = y_{\mathrm{f}'(D)}$, where $y_D \in \tilde{m}^{-1}(0)^{\mathrm{s}}$, $y_{\mathrm{f}'(D)} \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ are defined as in Proposition 3.11. This gives $\iota(x_D) = x_{\mathrm{f}'(D)}$. □

8.4 Application of basis theorem

As we saw in the proofs of Proposition 8.3 and Theorem 8.15, Proposition 8.4 proved to be useful to prove stability conditions for points on bow varieties. In this section, we prove a further useful application of Proposition 8.4 about the triviality of tautological bundles of $\mathcal{C}(\mathcal{D})$.

We say that a black line $X \in \mathrm{h}(\mathcal{D})$ *belongs to the blue part of* $\mathcal{D}$ if $X = U^{\pm}$, for some $U \in \mathrm{b}(\mathcal{D})$. Likewise, we say that the tautological bundle $\xi_X = \xi_{\mathcal{D},X}$ *belongs to the blue part of* $\mathcal{D}$ if X belongs to the blue part of $\mathcal{D}$.

Proposition 8.22. *Suppose X belongs to the blue part of $\mathcal{D}$. Then, we have an isomorphism of $\mathbb{T}$-equivariant vector bundles*

$$\xi_X \cong \bigoplus_{U_j \triangleright X} \bigoplus_{j=0}^{c_i - 1} h^{-j} \mathbb{C}_{U_i}.$$

Proof. Suppose $X = U_i^-$, for some $U_i \in \mathrm{b}(\mathcal{D})$. For

$$y = ((A_U, B_U^-, B_U^+, a_U, b_U)_U, (C_V, D_V)_V) \in (\tilde{m})^{-1}(0)^{\mathrm{s}},$$

let $s^{(i)}_{y,j,r} \in W_X$ be defined as in (8.4). Given $g = (g_X)_X \in \mathcal{G}$, we have

$$\begin{aligned} s^{(i)}_{g.y,j,r} &= g_{U_i^-} A_{U_i} g_{U_i^+}^{-1} \cdots g_{U_{j-1}^-} A_{U_{j-1}} g_{U_{j-1}^+}^{-1} (g_{U_j^-} B_{U_j}^- g_{U_j^-}^{-1})^r g_{U_j^-} a_{U_j}(1) \\ &= g_{U_i^-} A_{U_i} \cdots A_{U_{j-1}} (B_{U_j}^-)^r a_{U_j}(1) \\ &= g_{U_i^-} s^{(i)}_{y,j,r}. \end{aligned}$$

Thus, the morphism of varieties

$$\tilde{m}^{-1}(0)^{\mathrm{s}} \times \mathbb{C} \longrightarrow \tilde{m}^{-1}(0)^{\mathrm{s}} \times W_X, \quad (y, \lambda) \mapsto (y, \lambda \cdot s^{(i)}_{y,j,r})$$

induces a section

$$s^{(i)}_{j,r} \colon \mathcal{C}(\mathcal{D}) \times \mathbb{C} \longrightarrow \xi_X, \quad ([y], \lambda) \mapsto [y, \lambda \cdot s^{(i)}_{y,j,r}].$$

Let $t = (t_1, \ldots, t_N, h) \in \mathbb{T}$. Then, we have

$$\begin{aligned} s^{(i)}_{t.y,j,r} &= A_{U_i} \cdots A_{U_{j-1}} (h^r B_{U_j}^-)^r a_{U_j}(t_j^{-1}) \\ &= h^r t_j^{-1} A_{U_i} \cdots A_{U_{j-1}} (B_{U_j}^-)^r a_{U_j}(1) \\ &= h^r t_j^{-1} s^{(i)}_{y,j,r}. \end{aligned}$$

Thus, we conclude that $s^{(i)}_{j,r}$ is actually a $\mathbb{T}$-equivariant section $s^{(i)}_{j,r} \colon \mathcal{C}(\mathcal{D}) \times (h^{-r}\mathbb{C}_{U_j}) \to \xi_X$. Therefore, we have a morphism of $\mathbb{T}$-equivariant vector bundles

$$\bigoplus_{U_j \triangleright X} \bigoplus_{r=0}^{c_j - 1} h^{-r}\mathbb{C}_{U_j} \xrightarrow{\sum_{j,r} s^{(i)}_{j,r}} \xi_X$$

which is surjective by Proposition 8.4. As both vector bundles are of the same rank, this morphism is an isomorphism of $\mathbb{T}$-equivariant vector bundles. □

Remark. The triviality of the tautological bundles which belong to the blue part of $\mathcal{D}$ was also observed in [BR23].

8.5 Restrictions of tautological bundles

Next, we consider restrictions of tautological bundles from $\mathcal{C}(\mathcal{D}')$ to $\mathcal{C}(\mathcal{D})$ via the embedding $\iota \colon \mathcal{C}(\mathcal{D}) \hookrightarrow \mathcal{C}(\mathcal{D}')$ from Theorem 8.15. In particular, we show in Proposition 8.23 that the tautological bundles on $\mathcal{C}(\mathcal{D}')$ restrict to tautological bundles on $\mathcal{C}(\mathcal{D})$. As an application, we show in Proposition 8.24 that the torus equivariant K-theory class of the normal bundle of this embedding can be expressed as a sum of trivial bundles. This result will be useful in the study of the attracting cells on $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$.

Proposition 8.23. *For each $X' \in \mathrm{h}(\mathcal{D}')$, there is an isomorphism of $\mathbb{T}'$-equivariant vector bundles $\iota^*\xi_{\mathcal{D}',X'} \cong \xi_{\mathcal{D},\pi_{\mathrm{h}}(X')}$.*

Proof. Note that $W_{X'} = W_{\pi_{\mathrm{h}}(X')}$. Let $\iota'\colon \tilde{m}^{-1}(0)^{\mathrm{s}} \hookrightarrow (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ be as in Lemma 8.14. Given $g = (g_X)_X \in \mathcal{G}$, we have

$$(g.y, g_{\pi_{\mathrm{h}}(X')}.v) = (\iota_{\mathrm{h}}(g).\iota'(y), \iota_{\mathrm{h}}(g)_{X'}.v), \quad \text{for all } y \in \tilde{m}^{-1}(0)^{\mathrm{s}},\ v \in W_{X'}.$$

Thus, the morphism of varieties

$$\tilde{m}^{-1}(0)^{\mathrm{s}} \times W_{\pi_{\mathrm{h}}(X')} \longrightarrow (\tilde{m}')^{-1}(0)^{\mathrm{s}} \times W_{X'}, \quad (y, v) \mapsto (\iota'(y), v)$$

induces a surjective morphism of vector bundles

$$f\colon \xi_{\mathcal{D},\pi_{\mathrm{h}}(X')} \longrightarrow \iota^* \xi_{\mathcal{D}',X'}.$$

As $\xi_{\mathcal{D},\pi_{\mathrm{h}}(X')}$ and $\iota^*\xi_{\mathcal{D}',X'}$ have the same rank, f is an isomorphism of vector bundles. Since ι' is $\mathbb{T}'$-equivariant, so is f. □

K-theory class of the normal bundle

Let N_ι be the normal bundle of the embedding $\iota\colon \mathcal{C}(\mathcal{D}) \hookrightarrow \mathcal{C}(\mathcal{D}')$.

Proposition 8.24. *In $K_{\mathbb{T}'}(\mathcal{C}(\mathcal{D}))$ holds*

$$[N_\iota] = \bigoplus_{U'_i \triangleright X_l} \bigoplus_{j=0}^{c'_i - 1} \Big(h^{-j} [\mathbb{C}_{U'_i} \otimes \mathbb{C}^\vee_{U'_k}] + h^{j+1} [\mathbb{C}_{U'_k} \otimes \mathbb{C}^\vee_{U'_i}] \Big).$$

Here, $\mathbb{C}^\vee_{U'_i}$ denotes the dual vector bundle of $\mathbb{C}_{U'_i}$. For the proof of Proposition 8.24, recall from Corollary 2.48 that in $K_{\mathbb{T}'}(\mathcal{C}(\mathcal{D}))$ holds

$$[T\mathcal{C}(\mathcal{D})] = \sum_{U \in \mathrm{b}(\mathcal{D})} T_U + \sum_{V \in \mathrm{r}(\mathcal{D})} T_V - \sum_{X \in \mathrm{h}(\mathcal{D})} T_X, \tag{8.14}$$

where

$$\begin{aligned} T_U &= (1-h)[\mathrm{Hom}(\xi_{\mathcal{D},U^+}, \xi_{\mathcal{D},U^-})] + h[\mathrm{End}(\xi_{\mathcal{D},U^-})] + h[\mathrm{End}(\xi_{\mathcal{D},U^+})] \\ &\qquad + [\mathrm{Hom}(\mathbb{C}_U, \xi_{\mathcal{D},U^-})] + h[\mathrm{Hom}(\xi_{\mathcal{D},U^+}, \mathbb{C}_U)], \\ T_V &= h[\mathrm{Hom}(\xi_{\mathcal{D},V^+}, \xi_{\mathcal{D},V^-})] + [\mathrm{Hom}(\xi_{\mathcal{D},V^-}, \xi_{\mathcal{D},V^+})], \\ T_X &= (1+h)[\mathrm{End}(\xi_{\mathcal{D},X})]. \end{aligned} \tag{8.15}$$

Likewise, in $K_{\mathbb{T}'}(\mathcal{C}(\mathcal{D}'))$ holds

$$[T\mathcal{C}(\mathcal{D}')] = \sum_{U' \in \mathrm{b}(\mathcal{D}')} T'_{U'} + \sum_{V' \in \mathrm{r}(\mathcal{D}')} T'_{V'} - \sum_{X' \in \mathrm{h}(\mathcal{D}')} T'_{X'}, \tag{8.16}$$

where

$$\begin{aligned} T'_{U'} &= (1-h)[\mathrm{Hom}(\xi_{\mathcal{D}',(U')^+}, \xi_{\mathcal{D}',(U')^-})] + h[\mathrm{End}(\xi_{\mathcal{D}',(U')^-})] + h[\mathrm{End}(\xi_{\mathcal{D}',(U')^+})] \\ &\qquad + [\mathrm{Hom}(\mathbb{C}_{U'}, \xi_{\mathcal{D}',(U')^-})] + h[\mathrm{Hom}(\xi_{\mathcal{D}',(U')^+}, \mathbb{C}_{U'})], \\ T'_{V'} &= h[\mathrm{Hom}(\xi_{\mathcal{D}',(V')^+}, \xi_{\mathcal{D}',(V')^-})] + [\mathrm{Hom}(\xi_{\mathcal{D}',(V')^-}, \xi_{(V')^+})], \\ T'_{X'} &= (1+h)[\mathrm{End}(\xi_{\mathcal{D}',X'})]. \end{aligned}$$

Proof of Proposition 8.24. By Proposition 8.23, we have

$$\iota^* T_{X'} = T_{\pi_{\mathrm{h}}(X)}, \quad \iota^* T_{V_i'} = T_{V_i},$$

for all $X' \in \mathrm{h}(\mathcal{D}')$ and $V_i' \in \mathrm{r}(\mathcal{D}')$. Likewise, for $j \neq k$, we have

$$\iota^* T_{U_j'} = \begin{cases} T_{U_j} & \text{if } j < k, \\ T_{U_{j-1}} & \text{if } j > k. \end{cases}$$

Hence, we deduce from (8.14) and (8.16) that

$$[N_\iota] = \iota^*[T\mathcal{C}(\mathcal{D}')] - [T\mathcal{C}(\mathcal{D})] = \iota^* T_{U_k'} - \iota^* T_{X_l'}.$$

Proposition 8.23 gives $\iota^*\xi_{\mathcal{D}',(U_k')^\pm} \cong \xi_{\mathcal{D},X_l} \cong \iota^*\xi_{\mathcal{D}',X_l'}$. Therefore,

$$\iota^* T_{U_k'} - \iota^* T_{X_l'} = [\mathrm{Hom}(\mathbb{C}_{U_k'}, \xi_{\mathcal{D}',X_l'})] + h[\mathrm{Hom}(\xi_{\mathcal{D}',X_l'}, \mathbb{C}_{U_k'}).$$

By Proposition 8.22, we have a $\mathbb{T}'$-equivariant isomorphism of vector bundles

$$\xi_{\mathcal{D}',X_l'} \cong \bigoplus_{U_i' \triangleright X_l} \bigoplus_{j=0}^{c_i'-1} h^{-j}\mathbb{C}_{U_i'}.$$

Hence, we conclude

$$[N_\iota] = \bigoplus_{U_i' \triangleright X_l} \bigoplus_{j=0}^{c_i'-1} \Big(h^{-j}[\mathbb{C}_{U_i'} \otimes \mathbb{C}_{U_k'}^\vee] + h^{j+1}[\mathbb{C}_{U_k'} \otimes \mathbb{C}_{U_i'}^\vee] \Big)$$

which completes the proof. □

8.6 Comparison of attracting cells

From now on, we view $\mathcal{C}(\mathcal{D})$ as closed subvariety of $\mathcal{C}(\mathcal{D}')$ via the closed immersion ι from Theorem 8.15. In this section, we compare the attracting cells of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$. The main result of this section is Proposition 8.27 which states that the equivariant multiplicities of the attracting cell closures of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ just differ by the multiplication with a uniform constant factor.

First, we consider restrictions of chambers from the torus $\mathbb{A}'$ to the subtorus $\mathbb{A}$.

Restrictions of chambers

Define the inclusion

$$\mathrm{inc}_{\mathrm{b}} \colon \mathrm{b}(\mathcal{D}) \longrightarrow \mathrm{b}(\mathcal{D}'), \quad U_i \mapsto \begin{cases} U_i' & \text{if } i < k, \\ U_{i+1}' & \text{if } i \geq k. \end{cases}$$

For instance, if $\mathcal{D} = 0/1/3/5\backslash 3\backslash 2\backslash 0$ and $\mathcal{D}' = 0/1/3/5\backslash 3\backslash 2\backslash 2\backslash 0$ then, as $k = 3$, the injection $\mathrm{inc}_{\mathrm{b}}$ is given as

$$U_1 \mapsto U_1', \quad U_2 \mapsto U_2', \quad U_3 \mapsto U_4'.$$

Definition 8.25. For a chamber $\mathfrak{C}'$ of $\mathbb{A}'$, the set

$$\iota^*\mathfrak{C}' := \{\sigma\colon \mathbb{C}^* \to \mathbb{A} \mid \text{there exists } \sigma' \in \mathfrak{C}' \text{ with } \sigma_U = \sigma'_{\mathrm{inc}_{\mathrm{b}}(U)}, \text{ for all } U \in \mathrm{b}(\mathcal{D})\} \tag{8.17}$$

is a chamber of $\mathbb{A}$ which we call the *restriction of* $\mathfrak{C}'$ *to* $\mathbb{A}$.

Example 8.26. As above, choose $\mathcal{D} = 0/1/3/5\backslash 3\backslash 2\backslash 0$ and $\mathcal{D}' = 0/1/3/5\backslash 3\backslash 2\backslash 2\backslash 0$. Let $\mathfrak{C}'$ is the chamber $\{t_1 < t_3 < t_2 < t_4\}$. Then, $\iota^*\mathfrak{C}'$ is obtained from $\mathfrak{C}'$ by first forgetting the coordinate t_3 which belongs to the chargeless blue line U_3 and then relabel the indices of the t_i via

$$t_1 \mapsto t_1, \quad t_2 \mapsto t_2, \quad t_4 \mapsto t_3.$$

That is, we obtain $\iota^*\mathfrak{C}'$ from $\mathfrak{C}'$ as follows:

$$\{t_1 < t_3 < t_2 < t_4\} \quad \rightsquigarrow \quad \{t_1 < t_2 < t_4\} \quad \rightsquigarrow \quad \{t_1 < t_2 < t_3\}$$

Thus, $\iota^*\mathfrak{C}' = \{t_1 < t_2 < t_3\}$.

Equivariant multiplicities of closures of attracting cells

Let $\mathfrak{C}'$ be a chamber of $\mathbb{A}'$ and $\mathfrak{C} = \iota^*\mathfrak{C}'$ be the restriction to $\mathbb{A}$. Denote the respective attracting cells of $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}} = \mathcal{C}(\mathcal{D})^{\mathbb{T}'}$ by $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)$ and $\mathrm{Attr}^{\mathcal{C}(\mathcal{D}')}_{\mathfrak{C}'}(p)$ The respective Zariski closures in $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ are denoted by L_p and L'_p. Likewise, let $\Lambda_p \in H^*_{\mathbb{T}'}(\mathcal{C}(\mathcal{D}))$ and $\Lambda'_p \in H^*_{\mathbb{T}'}(\mathcal{C}(\mathcal{D}'))$ be the Poincaré dual of $[L_p]^{\mathbb{T}'}$ and $[L'_p]^{\mathbb{T}'}$ respectively. We further view $H^*_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Q}[t_1, \dots, t_N, h]$ as $\mathbb{Q}$-subalgebra of $H^*_{\mathbb{T}'}(\mathrm{pt}) \cong \mathbb{Q}[t_1, \dots, t_{N+1}, h]$ via the embedding

$$t_i \mapsto \begin{cases} t_i & \text{if } i < k, \\ t_{i+1} & \text{if } i \geq k, \end{cases} \qquad h \mapsto h. \tag{8.18}$$

Proposition 8.27. *For all p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have*

$$\iota_q^*(\Lambda'_p) = e' \cdot \iota_q^*(\Lambda_p)$$

in $H^*_{\mathbb{T}'}(\mathrm{pt})$. Here,

$$e' = \Big(\prod_{\substack{i>k \\ U_i \in \mathrm{b}^+_{U'_k,\mathfrak{C}'}}} \prod_{j=0}^{c'_i - 1} (t_k - t_i + (j+1)h) \Big) \cdot \Big(\prod_{\substack{i>k \\ U_i \in \mathrm{b}^-_{U'_k,\mathfrak{C}'}}} \prod_{j=0}^{c'_i - 1} (t_i - t_k - jh) \Big),$$

where

$$\begin{aligned} \mathrm{b}^+_{U'_k,\mathfrak{C}'} &= \{U'_i \in \mathrm{b}(\mathcal{D}') \mid \langle \sigma', t_i - t_k \rangle > 0, \text{ for all } \sigma' \in \mathfrak{C}'\}, \\ \mathrm{b}^-_{U'_k,\mathfrak{C}'} &= \{U'_i \in \mathrm{b}(\mathcal{D}') \mid \langle \sigma', t_i - t_k \rangle < 0, \text{ for all } \sigma' \in \mathfrak{C}'\}. \end{aligned} \tag{8.19}$$

Remark. By Proposition 8.24, we have

$$e' = e_{\mathbb{T}'}(N^-_{\iota,p,\mathfrak{C}'}), \quad \text{for all } p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}. \tag{8.20}$$

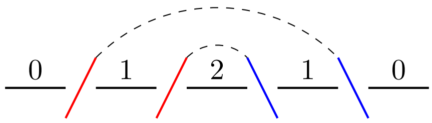
Here, $N_{\iota,p}$ is the fiber of the normal bundle N_ι over p and $N^-_{\iota,p,\mathfrak{C}'}$ is the negative part of $N_{\iota,p}$ with respect to the chamber $\mathfrak{C}'$.

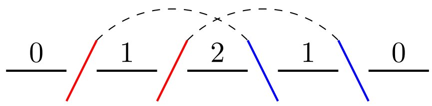
Equivariant multiplicities via Proposition 8.27 in a concrete example

Let $\mathcal{D}' = 0/1/2\backslash 1\backslash 1\backslash 0$. Consider the Zariski closures of the attracting cells $L'_p = \overline{\mathrm{Attr}_{\mathfrak{C}'}(p)}$, for $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}'}$, where we choose $\mathfrak{C}' = \{t_1 < t_2 < t_3\}$. Our goal is to determine all equivariant multiplicities of all the L'_p.

For this, let $\mathcal{D} = 0/1/2\backslash 1\backslash 0$. So $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a blue extension move at the black line X_4. We denote by $\iota\colon \mathcal{C}(\mathcal{D}) \hookrightarrow \mathcal{C}(\mathcal{D})$ the inclusion from Theorem 8.15. Note that the restriction $\mathfrak{C} := \iota^*\mathfrak{C}'$ equals the chamber $\{t_1 < t_2\}$.

By Proposition 8.27, the equivariant multiplicities of the closures of attracting cells of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ just differ by multiplication with a uniform constant factor. Hence, we first determine the equivariant multiplicities of closures of attracting cells of $\mathcal{C}(\mathcal{D})$. The brane diagram $\mathcal{D}$ admits two tie diagrams

$D_1 =$ 

$D_2 =$ 

Recall from Theorem 2.67 and (2.70) that there exists a ρ-equivariant isomorphism f varieties $H'\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} T^*\mathbb{P}^1$, where $\rho\colon \mathbb{T} \xrightarrow{\sim} \mathbb{T}$ is the automorphism of algebraic groups given as

$$(t_1, t_2, h) \mapsto (t_1 h^{-1}, t_2 h^{-1}, h).$$

Let $\pi\colon T^*\mathbb{P}^1 \to \mathbb{P}^1$ be the projection. We view $\mathbb{P}^1$ as subvariety of $T^*\mathbb{P}^1$ via the zero-section and denote the elements of $\mathbb{P}^1$ via homogeneous coordinates $[x : y]$, where x, $y \in \mathbb{C}$ with $(x, y) \neq (0, 0)$. By (3.16), we have $H'(x_{D_1}) = [1 : 0]$ and $H'(x_{D_2}) = [0 : 1]$. Their respective attracting cells are given as

$$\mathrm{Attr}_{\mathfrak{C}}([1:0]) = \{[1:x] \mid x \in \mathbb{C}\}, \quad \mathrm{Attr}_{\mathfrak{C}}([0:1]) = \pi^{-1}([0:1]).$$

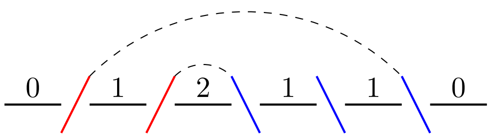
The respective Zariski closures are

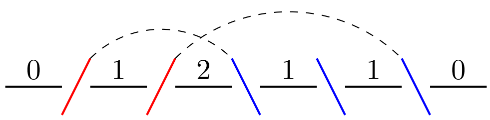
$$L_{[1:0]} = \mathbb{P}^1, \quad L_{[0:1]} = \pi^{-1}([0\colon 1]). \tag{8.21}$$

Let $\Lambda_{[1:0]}$ and $\Lambda_{[0:1]}$ be the Poincaré dual of $[L_{[1:0]}]^{\mathbb{T}}$ and $[L_{[0:1]}]^{\mathbb{T}}$ respectively. By (8.21), the equivariant multiplicities $\Lambda_{[1:0]}$ and $\Lambda_{[0:1]}$ are

$$\begin{aligned} \iota_{[1:0]}^*(\Lambda_{[1:0]}) &= t_1 - t_2 + h, & \iota_{[1:0]}^*(\Lambda_{[0:1]}) &= 0, \\ \iota_{[0:1]}^*(\Lambda_{[1:0]}) &= t_2 - t_1 + h, & \iota_{[0:1]}^*(\Lambda_{[0:1]}) &= t_1 - t_2. \end{aligned} \tag{8.22}$$

Now, we come to the attraction cells of $\mathcal{C}(\mathcal{D}')$. The brane diagram $\mathcal{D}'$ admits the tie diagrams

$D'_1 =$ 

$D'_2 =$ 

By (8.8), we have $\mathrm{f}'(x_{D_1}) = x_{D'_1}$ and $\mathrm{f}'(x_{D_2}) = x_{D'_2}$. Let Λ'_p be the Poincaré dual of $[L'_p]^{\mathbb{T}'}$, for $p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$. By (8.18), we view $H^*_{\mathbb{T}}(\mathrm{pt})$ as subalgebra of $H^*_{\mathbb{T}'}(\mathrm{pt})$ via the embedding

$$H^*_{\mathbb{T}}(\mathrm{pt}) \longhookrightarrow H^*_{\mathbb{T}'}(\mathrm{pt}), \quad \text{where } t_1 \mapsto t_1,\ t_2 \mapsto t_3,\ h \mapsto h.$$

By Proposition 8.27, we have

$$\iota^*_{x_{D'_i}}(\Lambda'_{x_{D'_i}}) = (t_2 - t_3 + h) \cdot \iota^*_{x_{D_i}}(\Lambda_{x_{D_i}}), \quad \text{for } i,\ j = 1, 2. \tag{8.23}$$

Hence, inserting (8.22) into (8.23) yields the following explicit equivariant multiplicities:

$$\begin{aligned}
\iota^*_{x_{D'_1}}(\Lambda_{x_{D'_1}}) &= (t_2 - t_3 + h)(t_1 - t_3 + h), & \iota^*_{x_{D'_1}}(\Lambda_{x_{D'_2}}) &= 0,\\
\iota^*_{x_{D'_2}}(\Lambda_{x_{D'_1}}) &= (t_2 - t_3 + h)(t_3 - t_1 + h), & \iota^*_{x_{D'_2}}(\Lambda_{x_{D'_2}}) &= (t_2 - t_3 + h)(t_1 - t_3).
\end{aligned}$$

Thus, we computed all equivariant multiplicities of the Poincaré duals of attracting cell closures of $\mathcal{C}(\mathcal{D}')$.

The next four subsections are devoted to the proof of Proposition 8.27. In particular, we define an auxiliary cocharacter to which we refer as *comparison cocharacter* and study geometric properties of the corresponding fixed point locus.

Comparison cocharacter

Recall that we assumed that $\mathcal{D}'$ is obtained from $\mathcal{D}$ via a blue extension move at the black line X_l and U'_k is the chargeless line added to $\mathcal{D}$.

Fix a chamber $\mathfrak{C}'$ of the torus $\mathbb{A}'$.

Definition 8.28. We define the cocharacter $\tau = \tau_{k,\mathfrak{C}'}$ of $\mathbb{A}'$ as

$$t \mapsto (\tau_{U'}(t))_{U'}, \quad \tau_{U'}(t) = \begin{cases} t & \text{if } U' \in \mathrm{b}^+_{U'_k,\mathfrak{C}'},\\ 1 & \text{if } U' = U'_k,\\ t^{-1} & \text{if } U' \in \mathrm{b}^-_{U'_k,\mathfrak{C}'}.\end{cases} \tag{8.24}$$

We call τ *the comparison cocharacter of* $\mathfrak{C}'$ *with respect to* U'_k.

Example 8.29. As in Example 8.12, let $\mathcal{D} = 0/1/3/5\backslash 3\backslash 2\backslash 0$ and $\mathcal{D}' = 0/1/3/5\backslash 3\backslash 2\backslash 2\backslash 0$. Let $\mathfrak{C}'$ be the chamber $\{t_1 < t_3 < t_2 < t_4\}$. Then, as $t_1 < t_3$ and $t_3 < t_2$, $t_3 < t_4$, we have

$$\mathrm{b}^+_{U'_3,\mathfrak{C}'} = \{U'_2, U'_4\}, \quad \mathrm{b}^-_{U'_3,\mathfrak{C}'} = \{U'_1\}.$$

Consequently, the comparison cocharacter $\tau = \tau_{3,\mathfrak{C}'}$ is given as $(t^{-1}, t, 1, t)$.

The next proposition contains useful positivity (resp. negativity) properties the comparison cocharacter:

Proposition 8.30. *Let* $p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$. *Then, the following holds*

(i) We have $N^+_{\iota,p,\mathfrak{C}'} = N^+_{\iota,p,\tau}$ *and* $N^-_{\iota,p,\mathfrak{C}'} = N^-_{\iota,p,\tau}$.

(ii) All $\mathbb{A}'$*-weights of* $T_p\mathcal{C}(\mathcal{D})^+_\tau$ *and* $T_p\mathcal{C}(\mathcal{D}')^+_\tau$ *are strictly positive with respect to* $\mathfrak{C}'$.

(iii) All $\mathbb{A}'$-weights of $T_p\mathcal{C}(\mathcal{D})^-_\tau$ and $T_p\mathcal{C}(\mathcal{D}')^-_\tau$ are strictly negative with respect to $\mathfrak{C}'$.

Proof. We just prove (i) as the other statements are similar. By Proposition 8.24, all $\mathbb{A}'$-weights of $N_{\iota,p}$ are of the form $\pm(t_k - t_i)$. Such a weight is positive (resp. negative) with respect to $\mathfrak{C}'$ if and only if it is positive (resp. negative) with respect to τ. Thus, we have $N^+_{\iota,p,\mathfrak{C}'} = N^+_{\iota,p,\tau}$ and $N^-_{\iota,p,\mathfrak{C}'} = N^-_{\iota,p,\tau}$. □

Fixed locus of comparison charcacter

We set

$$X_0 := \mathcal{C}(\mathcal{D}')^\tau = \{x \in \mathcal{C}(\mathcal{D}') \mid \tau(t).x = x, \text{ for all } t \in \mathbb{C}^*\}.$$

By Theorem 7.1, X_0 is a smooth and $\mathbb{T}'$-invariant closed subvariety of $\mathcal{C}(\mathcal{D}')$. The next proposition gives that X_0 is actually also contained in $\mathcal{C}(\mathcal{D})$.

Proposition 8.31. *We have $X_0 \subset \mathcal{C}(\mathcal{D})$.*

As a preparation, we use the following auxiliary lemma:

Lemma 8.32. *Let $y = ((A_{U'}, B^-_{U'}, B^+_{U'}, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)$ and $g = (g)_{X'} \in \mathcal{G}'$ and $t \in \mathbb{C}^*$ such that $g.y = \tau(t).y$. As in (8.4), set*

$$s^{(i)}_{y,j,r} := A_{U'_i} A_{U'_{i+1}} \cdots A_{U'_{j-2}} A_{U'_{j-1}} (B^-_{U'_j})^r a_{U'_j}(1) \in W_{(U'_i)^-},$$

for $i = 1, \ldots, N+1$, $j \geq i$, $r \geq 0$. Then, we have

$$g_{(U'_i)^-} s^{(i)}_{y,j,r} = \tau_{U'_j}(t) s^{(i)}_{y,j,r}.$$

Proof. From $g.y = \tau(t).y$ follows

$$g_{(U')^-} A_{U'} = A_{U'} g_{(U')^+}, \quad g_{(U')^\pm} B^\pm_{U'} = B^\pm_{U'} g_{(U')^\pm}, \quad g_{(U')^-} a_{U'} = \tau_{U'}(t) a_{U'}$$

for all $U' \in \mathrm{b}(\mathcal{D}')$. Thus, (8.4) gives

$$\begin{aligned} g_{(U'_i)^-} s^{(i)}_{y,j,r} &= g_{(U'_i)^-} A_{U'_i} \cdots A_{U'_{j-1}} (B^-_{U_j})^r a_{U'_j}(1) \\ &= A_{U'_i} \cdots A_{U'_{j-1}} (B^-_{U'_j})^r \tau_{U'_j}(t) a_{U'_j}(1) \\ &= \tau_{U'_j}(t) s^{(i)}_{y,j,r}. \end{aligned}$$

This finishes the proof. □

Proof of Proposition 8.31. Let $y = ((A_{U'}, B^-_{U'}, B^+_{U'}, a_{U'}, b_{U'})_{U'}, (C_{V'}, D_{V'})_{V'}) \in (\tilde{m}')^{-1}(0)^{\mathrm{s}}$ such that $[y] \in X_0$. This means that for all $t \in \mathbb{C}^*$, there exists $g_t = (g_{t,X'})_{X'} \in \mathcal{G}'$ with $g_t.y = \tau(t).y$. In the following, we assume that $t \neq 1$. By Theorem 8.15, $[y]$ is contained in $\mathcal{C}(\mathcal{D})$ if and only if $a_{U'_k} = 0$ and $b_{U'_k} = 0$. With the notation from Lemma 8.32, we set

$$\begin{aligned} \mathfrak{B}^- &:= (s^{(k)}_{y,j,l} \mid j = k+1, \ldots, N+1,\ l = 0, \ldots, c'_j - 1), \\ \mathfrak{B}^+ &:= (s^{(k+1)}_{y,j,l} \mid j = k+1, \ldots, N+1,\ l = 0, \ldots, c'_j - 1). \end{aligned}$$

By Proposition 8.4, $\mathfrak{B}^\pm$ is a basis of $W_{(U'_k)^\pm}$. Lemma 8.32 implies that $\mathfrak{B}^\pm$ is an eigen basis of $g_{t,(U'_k)^\pm}$ and each eigenvalue is either t or t^{-1}. By Lemma 8.32, we know that $a_{U'_k}(1)$ is an eigenvector of $g_{t,(U'_k)^-}$ with eigenvalue 1. Hence, we conclude $a_{U'_k} = 0$. To see that also $b_{U'_k} = 0$, note that $g_t.y = \tau(t).y$. This implies $b_{U'_k} g^{-1}_{t,(U'_k)^+} = b_{U'_k}$ and thus we have

$$b_{U'_k}(s^{(k+1)}_{y,j,l}) = b_{U'_k} g^{-1}_{t,(U'_k)^+}(s^{(k+1)}_{y,j,l}) = t^{\pm 1} b_{U'_k}(s^{(k+1)}_{y,j,l}), \quad \text{for all } s^{(k+1)}_{y,j,l} \in \mathfrak{B}^+.$$

Hence, $b_{U'_k}$ vanishes on all elements in $\mathfrak{B}^+$ and therefore $b_{U'_k} = 0$. Thus, we deduce $[y] \in \mathcal{C}(\mathcal{D})$. □

Attracting fiber bundles over X_0

Consider the attraction sets

$$Z^+ := \{x \in \mathcal{C}(\mathcal{D}) \mid \lim_{t\to 0} \tau(t).x \in X_0\}, \quad \tilde{Z}^+ := \{x \in \mathcal{C}(\mathcal{D}') \mid \lim_{t\to 0} \tau(t).x \in X_0\}.$$

By Corollary 4.6, Z^+ and $\tilde{Z}^+$ are locally closed subvarieties of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ respectively. A similar argument as in the proof of Proposition 4.20 actually gives that Z^+ and $\tilde{Z}^+$ are both closed subvarieties:

Lemma 8.33. *We have that Z^+ is a $\mathbb{T}'$-invariant closed subvariety of $\mathcal{C}(\mathcal{D})$ and $\tilde{Z}^+$ is a $\mathbb{T}'$-invariant closed subvariety of $\mathcal{C}(\mathcal{D}')$.*

Proof. By Proposition 4.2, there exists a proper and $\mathbb{T}'$-equivariant morphism $f\colon \mathcal{C}(\mathcal{D}) \to V$, where V is a finite dimensional $\mathbb{T}'$-representation. Then, by Lemma 4.21, $Z^+ = f^{-1}(V^{\geq 0}_\tau)$ and hence, Z^+ is a $\mathbb{T}'$-invariant closed subvariety of $\mathcal{C}(\mathcal{D})$. The same proof works for $\tilde{Z}^+$. □

According to Corollary 4.6, the limit maps

$$\pi\colon Z^+ \longrightarrow X_0, \qquad z \mapsto \lim_{t\to 0} \tau(t).z,$$
$$\tilde{\pi}\colon \tilde{Z}^+ \longrightarrow X_0, \qquad z \mapsto \lim_{t\to 0} \tau(t).z$$

are morphisms of varieties and Z^+, $\tilde{Z}^+$ are both $\mathbb{T}'$-equivariant affine bundles over X_0. Moreover, for all $x \in X_0$, we have isomorphisms of varieties $\pi^{-1}(x) \cong T_x\mathcal{C}(\mathcal{D})^+_\tau$ and $\tilde{\pi}^{-1}(x) \cong T_x\mathcal{C}(\mathcal{D}')^+_\tau$. If additionally $x \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$ then these isomorphisms can be chosen to be $\mathbb{T}'$-equivariant by Proposition 4.8.

Next we like to relate the attracting cells of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ to the affine bundles Z^+ and $\tilde{Z}^+$. For this, note that for all $p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$ and each cocharacter $\sigma' \in \mathfrak{C}'$, we have

$$\mathrm{Attr}^{X_0}_{\sigma'}(p) = \mathrm{Attr}^{\mathcal{C}(\mathcal{D}')}_{\mathfrak{C}'}(p) \cap X_0.$$

Thus, $\mathrm{Attr}^{X_0}_{\sigma'}(p)$ is independent from the choice of σ' and we denote $\mathrm{Attr}^{X_0}_{\sigma'}(p)$ also by $\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p)$. We further denote the Zariski closure of $\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p)$ in X_0 by $L^{(0)}_p$.

Proposition 8.34. *Let $p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$. Then, the following holds:*

(i) We have $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p) = \pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$ and $L_p = \pi^{-1}(L^{(0)}_p)$.

(ii) We have $\mathrm{Attr}^{\mathcal{C}(\mathcal{D}')}_{\mathfrak{C}'}(p) = \tilde{\pi}^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$ and $L'_p = \tilde{\pi}^{-1}(L^{(0)}_p)$.

For the proof, we use the following auxiliary lemma:

Lemma 8.35. *For all $p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$, we have*

$$T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}'} = (T_p\mathcal{C}(\mathcal{D})^0_\tau)^+_{\mathfrak{C}'} \oplus (T_p\mathcal{C}(\mathcal{D})^+_\tau)^+_{\mathfrak{C}'}, \quad T_p\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}'} = (T_p\mathcal{C}(\mathcal{D})^0_\tau)^-_{\mathfrak{C}'} \oplus (T_p\mathcal{C}(\mathcal{D})^-_\tau)^-_{\mathfrak{C}'}$$

and

$$T_p\mathcal{C}(\mathcal{D}')^+_{\mathfrak{C}'} = (T_p\mathcal{C}(\mathcal{D}')^0_\tau)^+_{\mathfrak{C}'} \oplus (T_p\mathcal{C}(\mathcal{D}')^+_\tau)^+_{\mathfrak{C}'}, \quad T_p\mathcal{C}(\mathcal{D}')^-_{\mathfrak{C}'} = (T_p\mathcal{C}(\mathcal{D}')^0_\tau)^-_{\mathfrak{C}'} \oplus (T_p\mathcal{C}(\mathcal{D}')^-_\tau)^-_{\mathfrak{C}'}.$$

Proof. We just prove $T_p\mathcal{C}(\mathcal{D}')^+_{\mathfrak{C}'} = (T_p\mathcal{C}(\mathcal{D}')^0_\tau)^+_{\mathfrak{C}'} \oplus (T_p\mathcal{C}(\mathcal{D}')^+_\tau)^+_{\mathfrak{C}'}$ as the other assertions are similar. By definition, $(T_p\mathcal{C}(\mathcal{D})^0_\tau)^+_{\mathfrak{C}'}$ and $(T_p\mathcal{C}(\mathcal{D})^+_\tau)^+_{\mathfrak{C}'}$ are contained in $T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}'}$. Conversely, let $v \in T_p\mathcal{C}(\mathcal{D}')^+_{\mathfrak{C}'}$ be an $\mathbb{A}'$-weight vector of some weight λ. By Corollary 3.24, λ is of the form $t_i - t_j$ with $i \neq j$ and $i, j \neq k$. Suppose $\langle \tau, t_i - t_j \rangle < 0$. Then, by (8.24), we have $U_i' \in \mathrm{b}^-_{U_k',\mathfrak{C}'}$ and $U_j' \in \mathrm{b}^+_{U_k',\mathfrak{C}'}$, where $\mathrm{b}^\pm_{U_k',\mathfrak{C}'}$ are defined as in (8.19). Equivalently, if $\sigma' \in \mathfrak{C}'$, we have

$$\langle \sigma', t_i - t_k \rangle < 0, \quad \langle \sigma', t_j - t_k \rangle > 0.$$

Consequently,

$$\langle \sigma', t_i - t_j \rangle = \langle \sigma', t_i - t_k \rangle - \langle \sigma', t_j - t_k \rangle < 0.$$

This contradicts the assumption $v \in T_p\mathcal{C}(\mathcal{D}')^+_{\mathfrak{C}'}$. Hence, $\langle \tau, t_i - t_j \rangle \geq 0$ and therefore $T_p\mathcal{C}(\mathcal{D}')^+_{\mathfrak{C}'} = (T_p\mathcal{C}(\mathcal{D}')^0_\tau)^+_{\mathfrak{C}'} \oplus (T_p\mathcal{C}(\mathcal{D}')^+_\tau)^+_{\mathfrak{C}'}$. □

An important consequence of Lemma 8.35 is the following statement about dimensions:

Lemma 8.36. *For all $p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$ holds*

$$\dim(\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)) = \dim(\pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))), \quad \dim(\mathrm{Attr}^{\mathcal{C}(\mathcal{D}')}_{\mathfrak{C}'}(p)) = \dim(\tilde{\pi}^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))).$$

Proof. Note that $T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}'} = T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}}$. Hence, Corollary 4.6 gives $\dim(\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)) = \dim(T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}'})$. Likewise, Corollary 4.6 implies

$$\dim(\pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))) = \dim((T_pX_0)^+_{\mathfrak{C}'}) + \dim(T\mathcal{C}(\mathcal{D})^+_\tau). \tag{8.25}$$

By Proposition 8.30.(ii), $T\mathcal{C}(\mathcal{D})^+_\tau = (T\mathcal{C}(\mathcal{D})^+_\tau)^+_{\mathfrak{C}'}$. Since $T_pX_0 = T_p\mathcal{C}(\mathcal{D})^0_\tau$, we conclude $(T_pX_0)^+_{\mathfrak{C}'}) = (T\mathcal{C}(\mathcal{D})^0_\tau)^+_{\mathfrak{C}'}$. Thus, Lemma 8.35 yields

$$(8.25) = \dim((T_p\mathcal{C}(\mathcal{D})^0_\tau)^+_{\mathfrak{C}'}) + \dim((T_p\mathcal{C}(\mathcal{D})^+_\tau)^+_{\mathfrak{C}'}) = \dim(\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)).$$

The statement for $\dim(\mathrm{Attr}^{\mathcal{C}(\mathcal{D}')}_{\mathfrak{C}'}(p))$ can be shown in a similar way. □

Proof of Proposition 8.34. We just prove (i) as the proof of (ii) is analogous. We first show $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p) \subset \pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$. By Lemma 8.33, $T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}'} \subset T_p\mathcal{C}(\mathcal{D})^{\geq 0}_\tau$. Thus, since $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p) \cong T_p\mathcal{C}(\mathcal{D})^+_{\mathfrak{C}'}$, we know $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p) \subset Z^+$. Since the projection π is $\mathbb{T}'$-equivariant, we conclude that $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)$ is contained in $\pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$. By construction, $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)$ is a locally closed subvariety of $\pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$. By Lemma 8.36, we have

$$\dim(\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)) = \dim(\pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))).$$

Thus, $\mathrm{Attr}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)$ is an open dense subvariety of $\pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$. Now let $x \in \pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$ and let $Y \coloneqq \overline{\tau(\mathbb{C}^*).x}$ be the Zariski closure of the orbit $\tau(\mathbb{C}^*).x$ in $\pi^{-1}(\mathrm{Attr}^{X_0}_{\mathfrak{C}'}(p))$. Via τ, we view Y variety with algebraic $\mathbb{C}^*$-action. Since $\mathrm{Attr}_{\mathfrak{C}'}^{X_0}(p) \subset \mathrm{Attr}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p)$, we know that $\pi(x) \in \mathrm{Attr}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p)$ and thus $\mathrm{Attr}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p) \cap Y$ is an non-empty open and $\mathbb{C}^*$-invariant subvariety of Y. Hence, we conclude $\mathrm{Attr}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p) \cap Y = Y$ and in particular $x \in \mathrm{Attr}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p)$. Thus, we proved $\mathrm{Attr}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p) = \pi^{-1}(\mathrm{Attr}_{\mathfrak{C}'}^{X_0}(p))$. This equality also implies $L_p \subset \pi^{-1}(L_p^{(0)})$ and hence $L_p = \pi^{-1}(L_p^{(0)})$ as both varieties are irreducible and of the same dimension. This completes the proof of (i). □

Proof of Proposition 8.27

Let $\Lambda_p^{(0)} \in H_{\mathbb{T}'}(X_0)$ be the Poincaré dual of $[L_p^{(0)}]^{\mathbb{T}'}$. Using Proposition 8.34, we now show that the equivariant multiplicities of Λ_p and Λ'_p just differ from the equivariant multiplicities of $\Lambda_p^{(0)}$ by multiplying with a certain Euler class:

Lemma 8.37. *For all p, $q \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$, we have*

$$\iota_q^*(\Lambda_p) = e_{\mathbb{T}'}(T_q\mathcal{C}(\mathcal{D})_\tau^-) \cdot \iota_q^*(\Lambda_p^{(0)}) \tag{8.26}$$

and

$$\iota_q^*(\Lambda'_p) = e_{\mathbb{T}'}(T_q\mathcal{C}(\mathcal{D}')_\tau^-) \cdot \iota_q^*(\Lambda_p^{(0)}). \tag{8.27}$$

Proof. We just prove (8.26) as the proof for (8.27) follows along similar lines. Recall from Proposition 8.34.(i) that $L_p = \pi^{-1}(L_p^0)$. Let $\pi_1\colon T_qZ^+ \to T_q\mathcal{C}(\mathcal{D})_\tau^0$ and $\pi_2\colon T_qZ^+ \to T_q\mathcal{C}(\mathcal{D})_\tau^+$ be the projections corresponding to

$$T_qZ^+ = T_qX_0 \oplus T_q(\pi^{-1}(q)) = T_q\mathcal{C}(\mathcal{D})_\tau^0 \oplus T_q\mathcal{C}(\mathcal{D})_\tau^+.$$

Since Z^+ is a $\mathbb{T}'$-equivariant affine fiber bundle, these projections induce a $\mathbb{T}'$-equivariant isomorphism of schemes

$$\pi_1 \times \pi_2\colon C_q(\pi^{-1}(L_p^{(0)})) \xrightarrow{\sim} C_qL_p^{(0)} \times T_q\mathcal{C}(\mathcal{D})_\tau^+.$$

Thus, Proposition 5.6 gives

$$\iota_q^*(\Lambda_p) = e_{\mathbb{T}'}((T_q\mathcal{C}(\mathcal{D})_\tau^+ \oplus T_q\mathcal{C}(\mathcal{D})_\tau^-)/T_q\mathcal{C}(\mathcal{D})_\tau^+) \cdot \iota_q^*(\Lambda_p^{(0)}) = e_{\mathbb{T}'}(T_q\mathcal{C}(\mathcal{D})_\tau^-) \cdot \iota_q^*(\Lambda_p^{(0)})$$

which proves (8.26). □

Proof of Proposition 8.27. Let p, $q \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$. By Lemma 8.37, we have

$$\iota_q^*(\Lambda'_p) = e_{\mathbb{T}'}(T_q\mathcal{C}(\mathcal{D}')) \cdot \iota_q^*(L_p^{(0)}) = e_{\mathbb{T}'}(N_{\iota,q,\tau}^-) \cdot e_{\mathbb{T}'}(T_q\mathcal{C}(\mathcal{D}')) \cdot \iota_q^*(L_p^{(0)}) = e_{\mathbb{T}'}(N_{\iota,q,\tau}^-) \cdot \iota_q^*(\Lambda_p).$$

By Proposition 8.30.(i), we conclude $e_{\mathbb{T}'}(N_{\iota,q,\tau}^-) = e_{\mathbb{T}'}(N_{\iota,q,\mathfrak{C}^-}^-)$ and by (8.20), $e' = e_{\mathbb{T}'}(N_{\iota,q,\mathfrak{C}'}^-)$. Therefore, $\iota_q^*(\Lambda'_p) = e' \cdot \iota_q^*(\Lambda_p)$ which completes the proof. □

8.7 Comparison of stable basis elements

We now pass from the attracting cells of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ to their respective stable basis elements. Again, let $\mathfrak{C}'$ be a chamber of $\mathbb{A}'$ and $\mathfrak{C} = \iota^*\mathfrak{C}'$ be its restriction to $\mathbb{A}$. We denote the respective partial orders of $\mathfrak{C}$ and $\mathfrak{C}'$ on $\mathcal{C}(\mathcal{D})^{\mathbb{T}} = \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$ by $\preceq$ and $\preceq'$. Recall from Corollary 5.19 that for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have

$$\mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p) = \sum_{q \preceq p} a_{p,q}\Lambda_q \tag{8.28}$$

for uniquely determined $a_{p,q} \in \mathbb{Z}$ with $a_{p,p} = 1$.

We now come to the Coefficient Theorem which is the main result of this section:

Theorem 8.38 (Coefficient Theorem)**.** *For all $p \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$, we have*

$$\mathrm{Stab}_{\mathfrak{C}'}^{\mathcal{C}(\mathcal{D}')}(p) = \sum_{q \preceq p} a_{p,q}\Lambda'_q,$$

where $a_{p,q}$ are as in (8.28)*.*

Remark. A good name for Theorem 8.38 would be *Universal Coefficient Theorem*. Unfortunately, this name is already taken by algebraic topology.

As a direct consequence of Theorem 8.38, we deduce that the equivariant multiplicities of stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}')$ just differ by multiplication with a uniform constant factor.

Corollary 8.39. *For all p, $q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we have*

$$\iota_q^*(\mathrm{Stab}_{\mathfrak{C}'}^{\mathcal{C}(\mathcal{D}')}(p)) = e' \cdot \iota_q^*(\mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p))$$

in $H^*_{\mathbb{T}'}(\mathrm{pt})$, where e' is defined as in Proposition 8.27.

Proof. By Theorem 8.38 and Proposition 8.27,

$$\iota_q^*(\mathrm{Stab}_{\mathfrak{C}'}^{\mathcal{C}(\mathcal{D}')}(p)) = \iota_q^*\Big(\sum_{q \preceq p} a_{p,q}\Lambda'_q\Big) = \sum_{q \preceq p} e' a_{p,q}\iota_q^*(\Lambda_q) = e' \cdot \iota_q^*(\mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(p))$$

which proves the corollary. □

Proof of Theorem 8.38

First we show that the partial orders $\preceq$ and $\preceq'$ coincide.

Lemma 8.40. *For all p, $q \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$, we have $q \preceq p$ if and only if $q \preceq' p$.*

Proof. With the notation from Proposition 8.34, we have $q \in L_p$ if and only if $q \in L_p^{(0)}$, for all p, $q \in \mathcal{C}(\mathcal{D}')^{\mathbb{T}'}$. Likewise Proposition 8.34.(ii) gives $q \in L'_p$ if and only if $q \in L_p^{(0)}$. Hence, $q \in L_p$ if and only if $q \in L'_p$ and therefore the partial orders $\preceq$ and $\preceq'$ coincide. □

Proof of Theorem 8.38. We prove that $\tilde{\Lambda}_p := \sum_{q \preceq p} a_{p,q} \Lambda'_q$ satisfies the stability conditions from Theorem 5.10. As Λ'_q is supported on L'_q, we conclude that $\tilde{\Lambda}_p$ is supported on $\bigcup_{q \preceq p} L_q$ which equals the full attracting cell $\mathrm{Attr}^f_{\mathfrak{C}'}(p)$ by Prposition 4.20 and Lemma 8.40. Thus, the support condition is satisfied. Next, we prove the normalization condition. As $\preceq = \preceq'$, we have $\iota_p^*(\Lambda'_q) = 0$ for $q \prec p$. Thus, as $a_{p,p} = 1$,

$$\iota_p^*(\tilde{\Lambda}_p) = \iota_p^*(\Lambda_p) = e_{\mathbb{T}'}(T_p \mathcal{C}(\mathcal{D}')^-_{\mathfrak{C}'})$$

which proves the normalization condition. Finally, by Proposition 8.27, we have

$$\iota_q^*(\tilde{\Lambda}_p) = e' \cdot \iota_q^*(\mathrm{Stab}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p)).$$

If $q \neq p$ then h divides $\iota_q^*(\mathrm{Stab}^{\mathcal{C}(\mathcal{D})}_{\mathfrak{C}}(p))$ and hence also $\iota_q^*(\tilde{\Lambda}_p)$. Thus, $\tilde{\Lambda}_p$ satisfies the smallness condition which implies $\tilde{\Lambda}_p = \iota_q^*(\mathrm{Stab}^{\mathcal{C}(\mathcal{D}')}_{\mathfrak{C}'}(p))$. □

8.8 Reduction to essential brane diagrams

In this section, we assign to each brane diagram $\mathcal{D}$ an essential brane diagram $\mathcal{D}_{\mathrm{ess}}$. Then, by employing Corollary 8.39, we show in Proposition 8.44 that the equivariant multiplicities of the stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}_{\mathrm{ess}})$ coincide up to multiplication with a uniform constant.

Underlying essential brane diagrams

Given a general brane diagram $\mathcal{D}$, we denote by $\mathrm{b}_{\mathrm{ess}}(\mathcal{D})$ and $\mathrm{r}_{\mathrm{ess}}(\mathcal{D})$ the respective sets of essential blue and essential red lines in $\mathcal{D}$. We set $N_{\mathrm{ess}} := |\mathrm{b}_{\mathrm{ess}}(\mathcal{D})|$, $M_{\mathrm{ess}} := |\mathrm{r}_{\mathrm{ess}}(\mathcal{D})|$ and write

$$\mathrm{b}_{\mathrm{ess}}(\mathcal{D}) = \{U_{j_1}, \dots, U_{j_{N_{\mathrm{ess}}}}\}, \quad \mathrm{r}_{\mathrm{ess}}(\mathcal{D}) = \{V_{i_1}, \dots, U_{i_{M_{\mathrm{ess}}}}\}.$$

Definition 8.41. The *underlying essential brane diagram* $\mathcal{D}_{\mathrm{ess}}$ is defined as the unique (separated) brane diagrams with N_{ess} blue lines $U_1^{\mathrm{ess}}, \dots, U_{N_{\mathrm{ess}}}^{\mathrm{ess}}$ and M_{ess} red lines $V_1^{\mathrm{ess}}, \dots, V_{M_{\mathrm{ess}}}^{\mathrm{ess}}$ and the labels of the horizontal lines are given as

$$d_{(U_k^{\mathrm{ess}})^-} = d_{U_{j_k}^-}, \quad d_{(U_k^{\mathrm{ess}})^+} = d_{U_{j_k}^+}, \quad d_{(V_l^{\mathrm{ess}})^-} = d_{V_{i_l}^-}, \quad d_{(V_l^{\mathrm{ess}})^+} = d_{V_{i_l}^+},$$

for $k = 1, \dots, N_{\mathrm{ess}}$ and $l = 1, \dots, M_{\mathrm{ess}}$.

For instance, if $\mathcal{D} = 0/1/3/3/5\backslash 3\backslash 2\backslash 2\backslash 0$ then the chargeless colored lines of $\mathcal{D}$ are V_2 and U_3. The underlying essential brane diagram of $\mathcal{D}$ is then obtained from $\mathcal{D}$ by replacing the local configuration $3/3$ with 3 and $2\backslash 2$ with 2. That is, $\mathcal{D}_{\mathrm{ess}}$ equals $0/1/3/5\backslash 3\backslash 2\backslash 0$.

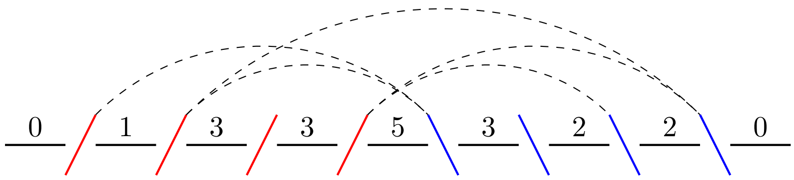
Just as in (8.1) and (8.8), we have a bijection

$$\mathrm{f}_{\mathrm{ess}} \colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}_{\mathrm{ess}}), \tag{8.29}$$

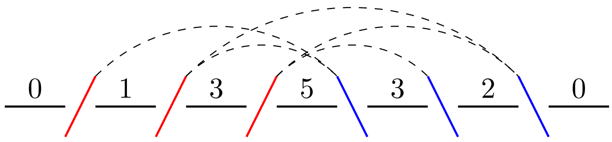
where for $D \in \mathrm{Tie}(\mathcal{D})$, the tie diagram $\mathrm{f}_{\mathrm{ess}}(D)$ is defined as

$$(V_l, U_k) \in \mathrm{f}_{\mathrm{ess}}(D) \quad \Leftrightarrow \quad (V_{i_l}, U_{j_k}) \in D.$$

Example 8.42. If we choose again $\mathcal{D} = 0{\color{red}/}1{\color{red}/}3{\color{red}/}3{\color{red}/}5{\color{blue}\backslash}3{\color{blue}\backslash}2{\color{blue}\backslash}2{\color{blue}\backslash}0$ and $D \in \mathrm{Tie}(\mathcal{D})$ as follows:

$D =$ 

Then, $\mathrm{f}_{\mathrm{ess}}(D)$ is obtained from D be replacing the local configurations $3{\color{red}/}3$ and $2{\color{blue}\backslash}2$ by black lines labeled by 3 and 2. We leave the ties of D unchanged. Thus, $\mathrm{f}_{\mathrm{ess}}(D)$ is given as follows:

$\mathrm{f}_{\mathrm{ess}}(D) =$ 

Comparison of equivariant multiplicities

We denote the tori acting on $\mathcal{C}(\mathcal{D})$ resp. $\mathcal{C}(\mathcal{D}_{\mathrm{ess}})$ by $\mathbb{A}$, $\mathbb{T}$ resp. $\mathbb{A}_{\mathrm{ess}}$, $\mathbb{T}_{\mathrm{ess}}$. We have an inclusion

$$\mathbb{A}_{\mathrm{ess}} \hookrightarrow \mathbb{A}, \quad t_k \mapsto t_{j_k}, \quad \text{for } k = 1, \dots, N_{\mathrm{ess}}.$$

Given a chamber $\mathfrak{C}$ of $\mathbb{A}$, we define $\mathfrak{C}_{\mathrm{ess}}$ as the unique chamber of $\mathbb{A}_{\mathrm{ess}}$ such that for all k, $l \in \{1, \dots, N_{\mathrm{ess}}\}$, we have

$$\langle \sigma, t_k - t_l \rangle > 0 \quad \text{for all } \sigma \in \mathfrak{C}_{\mathrm{ess}} \quad \Leftrightarrow \quad \langle \sigma, t_{j_k} - t_{j_l} \rangle > 0 \quad \text{for all } \sigma \in \mathfrak{C}.$$

Example 8.43. If $\mathcal{D} = 0{\color{red}/}2{\color{red}/}4{\color{blue}\backslash}4{\color{blue}\backslash}3{\color{blue}\backslash}3{\color{blue}\backslash}1{\color{blue}\backslash}0$ then the chargeless lines of $\mathcal{D}$ are U_1 and U_3. Thus, $\mathcal{D}_{\mathrm{ess}} = 0{\color{red}/}2{\color{red}/}4{\color{blue}\backslash}3{\color{blue}\backslash}1{\color{blue}\backslash}0$. As the essential blue lines of $\mathcal{D}$ are U_2, U_4, U_5, the inclusion $\mathbb{A}_{\mathrm{ess}} \hookrightarrow \mathbb{A}$ is given as $(t_1, t_2, t_3) \mapsto (t_2, t_4, t_5)$. Consider now the chamber $\mathfrak{C} = \{t_2 < t_1 < t_3 < t_5 < t_5\}$ of $\mathbb{A}$. Then, we obtain $\mathfrak{C}_{\mathrm{ess}}$ from $\mathfrak{C}$ by first forgetting the coordinates t_1 and t_3 as they correspond to chargeless lines. Then, we relabel the coordinates according to $(t_2, t_4, t_5) \mapsto (t_1, t_2, t_3)$. Thus, $\mathfrak{C}_{\mathrm{ess}}$ is given as $\{t_1 < t_3 < t_2\}$.

Next, we employ Corollary 8.10 and Corollary 8.39 to compare the equivariant multiplicities of stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{D}_{\mathrm{ess}})$. For this, we view $H^*_{\mathbb{T}_{\mathrm{ess}}}(\mathrm{pt})$ as subalgebra of $H^*_{\mathbb{T}}(\mathrm{pt})$ via the embedding, $H^*_{\mathbb{T}_{\mathrm{ess}}}(\mathrm{pt}) \hookrightarrow H^*_{\mathbb{T}}(\mathrm{pt})$ given by $h \mapsto h$ and $t_k \mapsto t_{j_k}$, for $k = 1, \dots, N_{\mathrm{ess}}$.

Proposition 8.44. *For all $D \in \mathrm{Tie}(\mathcal{D})$ and all choices of chambers $\mathfrak{C}$ of $\mathbb{A}$, we have*

$$\mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(x_D) = e_{\mathrm{ess}}(\mathcal{D}, \mathfrak{C}) \cdot \mathrm{Stab}_{\mathfrak{C}_{\mathrm{ess}}}^{\mathcal{C}(\mathcal{D}_{\mathrm{ess}})}(x_{\mathrm{f}_{\mathrm{ess}}(D)})$$

in $H^*_{\mathbb{T}}(\mathrm{pt})$. Here,

$$e_{\mathrm{ess}}(\mathcal{D}, \mathfrak{C}) = \prod_{U_k \in \mathrm{b}_{\mathrm{cl}}(\mathcal{D})} \Big(\Big(\prod_{\substack{i > k \\ U_i \in \mathrm{b}^+_{U_k, \mathfrak{C}}}} \prod_{j=0}^{c_i - 1} (t_k - t_i + (j+1)h) \Big) \cdot \Big(\prod_{\substack{i > k \\ U_i \in \mathrm{b}^-_{U_k, \mathfrak{C}}}} \prod_{j=0}^{c_i - 1} (t_i - t_k - jh) \Big) \Big),$$

where $\mathrm{b}_{\mathrm{cl}}(\mathcal{D})$ is the set of chargeless blue lines in $\mathcal{D}$ and $\mathrm{b}^{\pm}_{U_k, \mathfrak{C}}$ is defined as in (8.19).

Proof. We prove the statement by induction on the number of chargeless lines in $\mathcal{D}$. If $\mathcal{D}$ admits no chargeless lines then $\mathcal{D} = \mathcal{D}_{\text{ess}}$ and the proposition is trivial. Suppose now that $\mathcal{D}$ admits $n \geq 0$ chargeless lines and $\mathcal{D}'$ is obtained from $\mathcal{D}$ by adding a chargeless line Y. Let $\mathrm{f}'_{\text{ess}} \colon \mathrm{Tie}(\mathcal{D}') \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}_{\text{ess}})$ be the corresponding bijection from (8.29). Suppose Y is red. Then, by definition, $e_{\text{ess}}(\mathcal{D}, \mathfrak{C}) = e_{\text{ess}}(\mathcal{D}', \mathfrak{C})$. Let $\mathrm{f} \colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}')$ be as in (8.1). By construction, $\mathrm{f}'_{\text{ess}} \circ \mathrm{f} = \mathrm{f}_{\text{ess}}$. Thus, Corollary 8.10 gives

$$\begin{aligned} \mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D}')}(x_{\mathrm{f}(D)}) &= \mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(x_D) \\ &= e_{\text{ess}}(\mathcal{D}, \mathfrak{C}) \cdot \mathrm{Stab}_{\mathfrak{C}_{\text{ess}}}^{\mathcal{C}(\mathcal{D}_{\text{ess}})}(x_{\mathrm{f}_{\text{ess}}(D)}) \\ &= e_{\text{ess}}(\mathcal{D}', \mathfrak{C}) \cdot \mathrm{Stab}_{\mathfrak{C}_{\text{ess}}}^{\mathcal{C}(\mathcal{D}_{\text{ess}})}(x_{\mathrm{f}'_{\text{ess}}(\mathrm{f}(D))}), \end{aligned}$$

for all $D \in \mathrm{Tie}(\mathcal{D})$. This proves the proposition for $\mathcal{D}'$ in case Y is red. Suppose now that Y is blue. Let $\mathbb{A} \hookrightarrow \mathbb{A}'$, $\mathbb{T} \hookrightarrow \mathbb{T}'$ and $H^*_{\mathbb{T}}(\mathrm{pt}) \hookrightarrow H^*_{\mathbb{T}'}(\mathrm{pt})$ be as in Section 8.3 and Section 8.6. Fix a chamber $\mathfrak{C}'$ of $\mathbb{A}'$ and let $\mathfrak{C} := \iota^*\mathfrak{C}'$ be the restriction of $\mathfrak{C}'$ to $\mathbb{A}$ from (8.17). Let $\mathrm{f}' \colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\mathcal{D}')$ be as in (8.8). By definition, $\mathrm{f}'_{\text{ess}} \circ \mathrm{f}' = \mathrm{f}_{\text{ess}}$. Thus, Corollary 8.39 yields that in $H^*_{\mathbb{T}'}(\mathrm{pt})$ holds

$$\begin{aligned} \mathrm{Stab}_{\mathfrak{C}'}^{\mathcal{C}(\mathcal{D}')}(x_{\mathrm{f}'(D)}) &= e' \cdot \mathrm{Stab}_{\mathfrak{C}}^{\mathcal{C}(\mathcal{D})}(x_D) \\ &= e' \cdot e_{\text{ess}}(\mathcal{D}, \mathfrak{C}) \cdot \mathrm{Stab}_{\mathfrak{C}_{\text{ess}}}^{\mathcal{C}(\mathcal{D}_{\text{ess}})}(x_{\mathrm{f}_{\text{ess}}(D)}) \\ &= e' \cdot e_{\text{ess}}(\mathcal{D}, \mathfrak{C}) \cdot \mathrm{Stab}_{\mathfrak{C}_{\text{ess}}}^{\mathcal{C}(\mathcal{D}_{\text{ess}})}(x_{\mathrm{f}'_{\text{ess}}(\mathrm{f}'(D))}), \end{aligned}$$

for all $D \in \mathrm{Tie}(\mathcal{D})$. Here, e' is defined as in Proposition 8.27. By construction, $e' \cdot e_{\text{ess}}(\mathcal{D}, \mathfrak{C}) = e_{\text{ess}}(\mathcal{D}', \mathfrak{C}')$. Hence, we conclude the proposition for $\mathcal{D}'$. □

Chapter 9

Equivariant multiplicities via a symmetric group calculus

In this chapter, we consider the question:

How can we compute the equivariant multiplicities of stable basis elements of bow varieties?

First, observe by Proposition 5.13 and Proposition 8.44 that we can restrict our attention to separated and essential brane diagrams.

To give an answer to the above question, we first consider the special case of cotangent bundles of partial flag varieties. In this case, Su proved in [Su17, Theorem 1.1] a formula for the equivariant multiplicities of stable basis elements in terms simple roots and subwords of reduced expressions of permutations. In Proposition 9.13, we give an equivalent diagrammatic version of this formula in terms of string diagrams of permutations. This alternative illustrative approach turns out to be practical in explicit calculations.

We continue in Section 9.5 with studying a bijection between the set of tie diagrams of a given brane diagram $\mathcal{D}$ and certain double cosets of the symmetric group S_N to which we refer as *fully separated* double cosets, see Definition 9.26. As we will show in Theorem 9.35, the permutations which are contained in fully separated double cosets satisfy strong uniqueness properties.

In Theorem 9.44, we combine the correspondence between tie diagrams and fully separated double cosets with the D5 Resolution Theorem [BR23, Theorem 6.13]. As a consequence, we derive a formula for the equivariant multiplicities of the stable basis elements of bow varieties in terms of the equivariant multiplicities of the stable basis elements of cotangent bundles of partial flag varieties. Thus, Theorem 9.44 provides a way to compute equivariant multiplicities via the combinatorics of symmetric groups.

Assumption. All brane diagram in this chapter are assumed to be separated and essential.

9.1 Symmetric groups and their diagrammatics

In this section, we recall the illustration of permutations via string diagrams and important combinatorial properties of these diagrams that will be useful in the study of equivariant multiplicities of stable basis elements of bow varieties.

Preliminaries and notation

We briefly recall some basic facts on symmetric groups. For more details, see e.g. [Hum90] or [Sag01].

Denote the simple transpositions of the symmetric group S_n by $s_1, \dots, s_{n-1}$, where $s_i = (i, i+1)$. Every permutation can be written as $w = \sigma_1 \cdots \sigma_r$, where all σ_i are simple transpositions. If r is as small as possible, we call the expression $\sigma_1 \cdots \sigma_r$ for w *reduced* and call r the *length* of w and denote it by $l(w)$. It is well-known that $l(w)$ is equal to the number of inversions of w:

$$l(w) = |\mathrm{Inv}(w)|, \quad \mathrm{Inv}(w) = \{(i,j) \mid 1 \leq i < j \leq n,\ w(i) > w(j)\}. \tag{9.1}$$

By definition, a permutation w is larger than a permutation w' in the *Bruhat order* if some (not necessarily a consecutive) subword of a reduced expression for w is a reduced word for w'. It is a well-known fact that if w dominates w' in the Bruhat order then every reduced expression for w admits a subword which is a reduced expression for w', see e.g. [Hum90, Theorem 5.10].

Let $R^+ = \{t_i - t_j \mid 1 \leq i < j \leq n\} \subset \mathbb{Q}[t_1, \dots, t_n]$ be the set of positive roots and $R^- = \{t_i - t_j \mid 1 \leq j < i \leq n\}$ the set of negative roots. By (9.1), we have

$$l(w) = |\{\alpha \in R^+ \mid w.\alpha \in R^-\}|. \tag{9.2}$$

The set on the right hand side of (9.2) can also be characterized as follows: For $s = s_i$ we denote by $\alpha_s = t_i - t_{i+1} \in \mathbb{Q}[t_1, \dots, t_n]$ the corresponding simple root. Given a reduced expression $w = \sigma_1 \cdots \sigma_{l(w)}$, we set

$$\beta_i \coloneqq (\sigma_1 \cdots \sigma_{i-1}).\alpha_{\sigma_i}, \quad i = 1, \dots, l(w). \tag{9.3}$$

Then, by e.g. [Hum90, Section 5.6], the set of positive roots that gets mapped to negative roots by w^{-1} is given by

$$\{\beta_1, \dots, \beta_{l(w)}\} = \{\alpha \in R^+ \mid w^{-1}.\alpha \in R^-\}. \tag{9.4}$$

Example 9.1. Let $n = 5$ and $w = 35412$. Then, the set of inversions of w equals

$$\mathrm{Inv}(w) = \{(1,4), (2,4), (3,4), (1,5), (2,5), (3,5), (2,3)\}.$$

Thus, $l(w) = 7$ and one can directly check that $w = s_4 s_2 s_1 s_3 s_2 s_4 s_3 =: \sigma_1 \cdots \sigma_7$ is a reduced expression of w. To compute for instance β_4, note that $\sigma_4 = s_3$ and its corresponding simple root is $\alpha_{s_3} = t_3 - t_4$. Since $\sigma_1\sigma_2\sigma_3 = 31254$, (9.3) gives

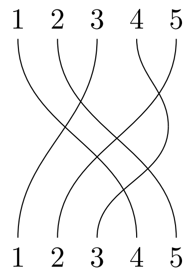
$$\beta_4 = (\sigma_1\sigma_2\sigma_3).\alpha_{s_3} = t_2 - t_5.$$

The remaining roots β_i are recorded in the following table:

i	1	2	3	4	5	6	7
β_i	$t_4 - t_5$	$t_2 - t_3$	$t_1 - t_3$	$t_2 - t_5$	$t_1 - t_5$	$t_2 - t_4$	$t_1 - t_4$

Diagrammatics of permutations

We illustrate permutations in the common way using string diagrams. For instance if $w = 35412 \in S_5$ then the following string diagram d_w is a permutation diagram for w since it consists of 5 strands and each number i on the bottom is connected to the number $w(i)$ on the top by a strand:



(9.5)

To give a formal definition of permutation diagrams, we define a *strand* as a smooth embedding $\lambda\colon [0,1] \to \mathbb{R}^2$.

Definition 9.2. Let $w \in S_n$ be a permutation. A collection $\lambda_1, \dots, \lambda_n$ of n strands is called a *diagram of* w if the following holds:

(i) $\lambda_i(0) = (i, 0)$ and $\lambda_i(1) = (w(i), 1)$ for all $i = 1, \dots, n$,

(ii) every two strands intersect only in finitely many points and all of these intersections are transversal and

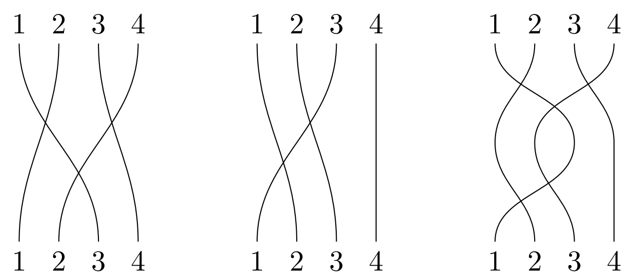
(iii) there are no triple or even higher intersections among the strands $\lambda_1, \dots, \lambda_n$.

A diagram is called *reduced* if the number of intersections among $\lambda_1, \dots, \lambda_n$ is equal to $l(w)$.

Example 9.3. Let $w = 35412 \in S_5$ be as in Example 9.1 and d_w as in (9.5). Since $l(w) = 7$ and d_w contains exactly 7 crossings, the diagram d_w is reduced.

For two diagrams d_w and d_z of permutations w, $z \in S_n$ the composition is $d_w \circ d_z$ is defined as follows: First draw d_w on top of d_z and then apply the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, \frac{1}{2}y)$. The resulting diagram $d_w \circ d_z$ is a permutation diagram for the product wz.

Example 9.4. Let $w = 2413$ and $z = 3124$. The following picture shows diagrams for d_w, d_z and the composition $d_w \circ d_z$:



d_w $\qquad$ d_z $\qquad$ $d_w \circ d_z$

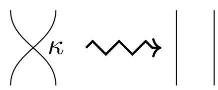
We now relate the diagrammatics of permutations to (reduced) expressions of permutations and the Bruhat order.

Definition 9.5. We call the intersection of two strands a *crossing*. Given a diagram d_w of a permutation w, we denote by $K(d_w)$ the set of crossings in d_w. If all crossings of d_w are of pairwise distinct height, we call d_w a *coding diagram for* w and denote the crossings of d_w by $\kappa_1, \ldots, \kappa_{l(w)}$, where κ_1 is the highest crossing, κ_2 is the second highest crossing etc.

Suppose that d_w is a coding diagram for $w \in S_n$ and let k be the number of crossings in d_w. After applying a homotopy which does not change the heights of the crossings in d_w, we can view d_w as a composition $d_{\sigma_1} \circ \ldots \circ d_{\sigma_k}$, where $\sigma_1, \ldots, \sigma_k$ are simple transpositions and d_{σ_i} is a reduced diagram for σ_i. Hence, we have $w = \sigma_1 \cdots \sigma_k$. In this way, d_w encodes an expression of w in terms of simple transpositions.

Example 9.6. Let w and d_w be as in Example 9.3. Then, d_w is a coding diagram for w and corresponds to the reduced expression $w = s_4 s_2 s_1 s_3 s_2 s_4 s_3$.

Definition 9.7. Given a crossing $\kappa \in K(d_w)$, we refer to the local move



as the *resolving of* κ.

Resolving crossings connects permutation diagrams to the Bruhat order as follows:

Lemma 9.8. *Let w, $w' \in S_n$ and d_w be a diagram for w. Then, w is larger than w' in the Bruhat order if and only if we can obtain a diagram for w' by resolving crossings from d_w.*

Proof. After applying a homotopy, we can assume that d_w is a coding diagram. Let $w = \sigma_1 \cdots \sigma_k$ be the corresponding expression for w in terms of simple transpositions. If w is larger than w' in the Bruhat order then there exist $1 \leq r \leq k$ and $1 \leq i_1 < \ldots < i_r \leq k$ such that $w' = \sigma_{i_1} \cdots \sigma_{i_r}$. Thus, resolving all crossings in $K(d_w) \setminus \{\kappa_{i_1}, \ldots, \kappa_{i_r}\}$ from d_w gives a diagram for w'. Conversely, suppose that resolving crossings $\kappa_{j_1}, \ldots, \kappa_{j_s}$ from d_w gives a diagram $d_{w'}$ for w'. Then, set $r := k - s$ and write

$$\{\kappa_{i_1} < \ldots < \kappa_{i_r}\} = K(d_w) \setminus \{\kappa_{j_1}, \ldots, \kappa_{j_s}\}.$$

After possibly applying a homotopy, $d_{w'}$ equals the composition $d_{\sigma_{i_1}} \circ \ldots \circ d_{\sigma_{i_r}}$. Thus, we have $w' = \sigma_{i_1} \cdots \sigma_{i_r}$ and hence w is larger than w' in the Bruhat order. $\square$

For $w \in S_n$ with reduced diagram d_w, we define a function

$$\mathrm{wt}\colon K(d_w) \longrightarrow \mathbb{Q}[t_1, \ldots, t_n]$$

as follows: Let κ be a crossing between the strands λ and λ'. Let j resp. j' be the endpoints of λ resp. λ'. Assuming $j < j'$, we set

$$\mathrm{wt}(\kappa) := t_j - t_{j'}.$$

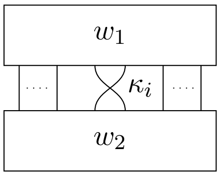
We call $\mathrm{wt}(\kappa)$ the *weight* of κ.

The next proposition gives that the weights of crossings coincide with the β_i from (9.3):

Proposition 9.9. *Let d_w be a reduced coding diagram and let $w = \sigma_1 \cdots \sigma_{l(w)}$ be the reduced expression corresponding to d_w. Then, we have*

$$\mathrm{wt}(\kappa_i) = \beta_i, \quad \text{for all } i = 1, \ldots, l(w).$$

Proof. For given $i \in \{1, \ldots, l(w)\}$, set $w_1 := \sigma_1 \cdots \sigma_{i-1}$, $w_2 := \sigma_{i+1} \cdots \sigma_{l(w)}$ and $\sigma_i = (j, j+1)$. After applying a homotopy, we can view d_w as a composition of a a reduced diagram of w_2, a reduced diagram of σ_i and a reduced diagram of w_1:



Thus, we have $\mathrm{wt}(\kappa_i) = t_{w_1(j)} - t_{w_1(j+1)} = w_1.(\alpha_{\sigma_i}) = \beta_i$ which completes the proof. □

Example 9.10. Let w and d_w be as in Example 9.3. The strands which are crossed in the crossing κ_4 end in 2 and 5. Thus, $\mathrm{wt}(\kappa_4) = t_2 - t_5$ which coincides with β_4 from Example 9.1.

9.2 Localization formula for full flag varieties

Let $F = F(1, 2, \ldots, n-1; n)$ be the full flag variety of $\mathbb{C}^n$ endowed with the $\mathbb{T}$-action from Section 2.5. For simplicity, we refer to a $\mathbb{T}$-fixed point $(\mathcal{F}_w, 0) \in (T^*F)^{\mathbb{T}}$ from (3.14) just by the permutation w. The localization formula from [Su17, Theorem 1.1] determines the $\mathbb{T}$-equivariant multiplicities of the stable basis elements of T^*F with respect to the antidominant chamber $\mathfrak{C}_-$.

Theorem 9.11 (Localization formula)**.** *Let $w \in S_n$ and $w = \sigma_1\sigma_2 \cdots \sigma_{l(w)}$ be a reduced expression. Then, for all $w' \in S_n$, we have*

$$\iota_w^*(\mathrm{Stab}_{\mathfrak{C}_-}(w')) = \Big(\prod_{\alpha \in L_w} (\alpha + h)\Big)\Big(\sum_{\substack{1 \leq i_1 < \cdots < i_k \leq l(w) \\ w' = \sigma_{i_1} \cdots \sigma_{i_k}}} h^{l(w)-k} \prod_{j=1}^{k} \beta_{i_j}\Big), \tag{9.6}$$

where the β_i are defined as in (9.3) *and*

$$L_w = R^+ \setminus \{\alpha \in R^+ \mid w^{-1}(\alpha) \in R^-\} = \{\alpha \in R^+ \mid \ \alpha \neq \beta_l, \text{ for all } l\}. \tag{9.7}$$

Remark. In [Su17], a different sign convention is used: There h is replaced by $-h$ and $\mathrm{Stab}_{\mathfrak{C}_-}(w)$ is replaced by $(-1)^{l(w)}\mathrm{Stab}_{\mathfrak{C}_-}(w)$.

Example 9.12. Let $n = 5$ and consider the permutations $w = 35412$ and $w' = 23415$. We now use Theorem 9.11 to compute the equivariant multiplicity $\iota_w^*(\mathrm{Stab}_{\mathfrak{C}_-}(w'))$. For this, we have to determine all the ingredients of the formula (9.6). As in Example 9.1, we pick $w = s_4s_2s_1s_3s_2s_4s_3$ as reduced expression for w. Checking all possible subwords of this expression for w gives that there are only two subwords that give w', namely $\sigma_1\sigma_3\sigma_5\sigma_6\sigma_7$ and

$\sigma_3\sigma_5\sigma_7$. We already computed the β_i in Example 9.1. The only positive roots which are not equal to one of the β_i are $(t_1 - t_2)$, $(t_3 - t_4)$ and $(t_3 - t_5)$. Thus, (9.7) gives

$$L_w = \{(t_1 - t_2), (t_3 - t_4), (t_3 - t_5)\}.$$

By Theorem 9.11, the subword $\sigma_1\sigma_3\sigma_5\sigma_6\sigma_7$ contributes the summand

$$(t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h)h^2\beta_1\beta_3\beta_5\beta_6\beta_7,$$

whereas the subword $\sigma_3\sigma_5\sigma_7$ contributes

$$(t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h)h^4\beta_3\beta_5\beta_7.$$

Hence, Theorem 9.11 yields

$$\iota_w^*(\mathrm{Stab}_{\mathfrak{C}_-}(w')) = (t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h) \cdot h^2(\beta_1\beta_6 + h^2)\beta_3\beta_5\beta_7. \tag{9.8}$$

Diagrammatic localization formula

Employing the diagrammatics from Section 9.1 leads to the following diagrammatic version of Theorem 9.11:

Proposition 9.13 (Diagrammatic localization formula)**.** *Let $w \in S_n$ and d_w be a reduced diagram of w. Then, for all $w' \in S_n$, we have*

$$\iota_w^*(\mathrm{Stab}_{\mathfrak{C}_-}(w')) = \Big(\prod_{\alpha \in L'_w} (\alpha + h)\Big)\Big(\sum_{K' \in K_{d_w,w'}} h^{|K'|} \prod_{\kappa \in K(d_w)\setminus K'} \mathrm{wt}(\kappa)\Big),$$

where $K_{d_w,w'}$ is the set of all subsets $K' \subset K(d_w)$ such that resolving all crossings of K' from d_w gives a diagram for w' and

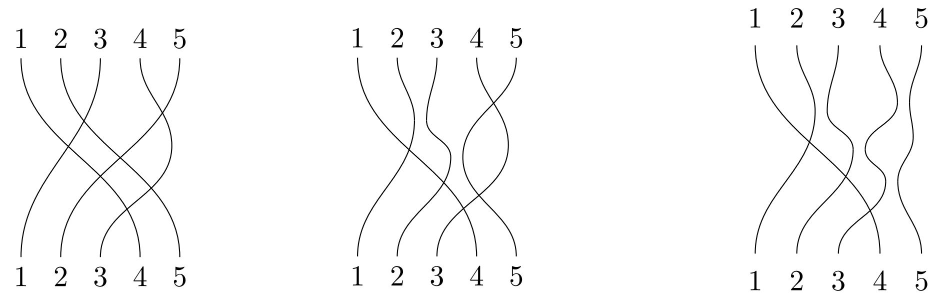
$$L'_w = \{\alpha \in R^+ \mid \alpha \neq \mathrm{wt}(\kappa), \text{ for all } \kappa \in K(d_w)\}.$$

Proof. We may assume without loss of generality that d_w is a coding diagram. Let $1 \leq i_1 < \cdots < i_k \leq l(w)$ and let d' be the diagram obtained from d_w by resolving all crossings κ_i with $i \neq i_1, \ldots, i_k$. By viewing d_w as composition of reduced diagrams corresponding to simple transpositions, we deduce that $w' = \sigma_{i_1} \cdots \sigma_{i_k}$ if and only if d' is a diagram for w'. Thus, Proposition 9.9 implies

$$\sum_{K' \in K_{d_w,w'}} h^{|K'|} \prod_{\kappa \in K(d_w)\setminus K'} \mathrm{wt}(\kappa) = \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq l(w) \\ w' = \sigma_{i_1} \cdots \sigma_{i_k}}} h^{l(w)-k} \prod_{j=1}^{k} \beta_{i_j}.$$

In addition, Proposition 9.9 also gives $L_w = L'_w$ which completes the proof. □

Example 9.14. Let w and w' be as in Example 9.12 and let d_w be as in Example 9.3. We now determine the equivariant multiplicity $\iota_w^*(\mathrm{Stab}_{\mathfrak{C}_-}(w'))$ using the diagrammatic localization formula from Proposition 9.13. One can easily check that there are just two possibilities to obtain a diagram for w' by resolving crossings from d_w. One is given by resolving the crossings κ_2 and κ_4, the other by resolving the crossings κ_1, κ_2, κ_4 and κ_6, in pictures:



d_w Resolving κ_2, κ_4 Resolving κ_1, κ_2, κ_4, κ_6

The diagram in the middle corresponds to the subword $\sigma_1\sigma_3\sigma_5\sigma_6\sigma_7$ and the diagram on the right hand side to $\sigma_3\sigma_5\sigma_7$. The diagram in the middle still contains the crossings κ_1, κ_3, κ_4, κ_5, κ_6, κ_7 and thus contributes the summand

$$(t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h)h^2\mathrm{wt}(\kappa_1)\mathrm{wt}(\kappa_3)\mathrm{wt}(\kappa_5)\mathrm{wt}(\kappa_6)\mathrm{wt}(\kappa_7),$$

whereas the diagram on the right only contains κ_3, κ_5, κ_7 and therefore contributes the summand

$$(t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h)h^4\mathrm{wt}(\kappa_3)\mathrm{wt}(\kappa_5)\mathrm{wt}(\kappa_7)$$

Hence, Proposition 9.13 yields

$$\begin{aligned}&\iota_w^*(\mathrm{Stab}_{\mathfrak{C}_-}(w')) = \\ &\quad (t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h) \cdot h^2(\mathrm{wt}(\kappa_1)\mathrm{wt}(\kappa_6) + h^2)\mathrm{wt}(\kappa_3)\mathrm{wt}(\kappa_5)\mathrm{wt}(\kappa_7).\end{aligned}$$

This agrees with the result (9.8) from the computation in Example 9.12.

9.3 Localization formula for partial flag varieties

Let $F = F(d_1, \dots, d_m; n)$ be a partial flag variety and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_{m+1})$ be as in Section 2.5. As before, for a given $w \in S_n$, we also denote the $\mathbb{T}$-fixed point $(\mathcal{F}_{wS_{\boldsymbol{\delta}}}, 0)$ by $wS_{\boldsymbol{\delta}}$.

It was proved in [Su17, Corollary 4.3] that the equivariant multiplicities of the stable basis elements of T^*F can be computed via equivariant multiplicities of stable basis elements of $T^*F(1, 2 \dots, n-1; n)$. For the formulation of the formula, recall from e.g. [Hum90, Section 5.12], that each left coset $wS_{\boldsymbol{\delta}}$ contains a unique element of minimal Bruhat length which is called the *shortest representative of* $wS_{\boldsymbol{\delta}}$.

Proposition 9.15. *For all* w, $w' \in S_n$ *we have*

$$\iota_{wS_{\boldsymbol{\delta}}}^*(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\boldsymbol{\delta}})) = \sum_{\substack{z \in S_n \\ zS_{\boldsymbol{\delta}} = wS_{\boldsymbol{\delta}}}} \frac{(-1)^{l(w'S_{\boldsymbol{\delta}}) + l(w')}\iota_z^*(\mathrm{Stab}_{\mathfrak{C}_-}(\mathrm{w}'))}{\prod_{\alpha \in R_{\boldsymbol{\delta}}} z.\alpha},$$

where $l(wS_{\boldsymbol{\delta}})$ *is the length of the shortest coset representative of* $wS_{\boldsymbol{\delta}}$ *and*

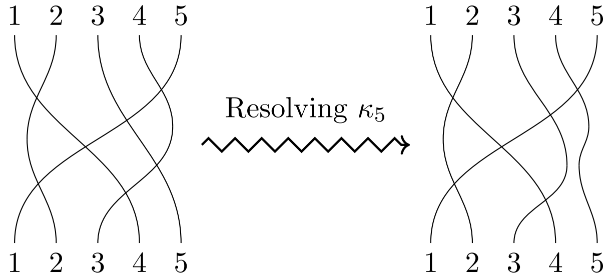
$$R_{\boldsymbol{\delta}} = \{t_i - t_j \mid \textit{there exist } l \in \{1, \dots, r\} \textit{ with } d_1 + \dots + d_{l-1} \le i < j \le d_1 + \dots + d_l\}.$$

Example 9.16. Let $\boldsymbol{\delta} = (2, 2, 1)$ and $w = 25143$, $w' = 52314$. We now apply Proposition 9.15, to determine the equivariant multiplicity $\iota_{wS_{\boldsymbol{\delta}}}^*(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\boldsymbol{\delta}}))$. For this, note that

w is the shortest coset representative of $wS_{\boldsymbol{\delta}}$. Since $l(w) = 5$ and $l(w') = 6$, we deduce that $z = w(s_1 \times s_1 \times \mathrm{id})$ is the only element in $wS_{\boldsymbol{\delta}}$ that dominates w' in the Bruhat order. Hence, Proposition 9.15 gives

$$\iota^*_{wS_{\boldsymbol{\delta}}}(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\boldsymbol{\delta}})) = \frac{\iota^*_z(\mathrm{Stab}_{\mathfrak{C}_-}(w'))}{(t_{z(1)} - t_{z(2)})(t_{z(3)} - t_{z(4)})} = \frac{\iota^*_z(\mathrm{Stab}_{\mathfrak{C}_-}(w'))}{(t_5 - t_2)(t_4 - t_1)}. \tag{9.9}$$

To compute $\iota^*_z(\mathrm{Stab}_{\mathfrak{C}_-}(w'))$ we use Proposition 9.13. The following figure shows a reduced diagram d_z for z. Since $l(z) = 7$ and $l(w') = 6$, there is only one possibility to obtain a diagram $d_{w'}$ for w' from d_z by resolving crossings:



(9.10)

We record the weights of the crossings of d_z in the following table:

i	1	2	3	4	5	6	7
$\mathrm{wt}(\kappa_i)$	$t_4 - t_5$	$t_1 - t_2$	$t_3 - t_5$	$t_1 - t_5$	$t_3 - t_4$	$t_2 - t_5$	$t_1 - t_4$

Thus, Proposition 9.13 yields

$$\iota^*_z(\mathrm{Stab}_{\mathfrak{C}_-}(w')) = (t_1 - t_3 + h)(t_2 - t_3 + h)(t_2 - t_4 + h)h \cdot \prod_{i \neq 5} \mathrm{wt}(\kappa_i).$$

Therefore, by Proposition 9.15, we have

$$(9.9) = (t_1 - t_3 + h)(t_2 - t_3 + h)(t_2 - t_4 + h)h(t_4 - t_5)(t_1 - t_2)(t_3 - t_5)(t_1 - t_5). \tag{9.11}$$

Symmetric group equivariance of stable basis elements

For $z \in S_n$, the vector space isomorphism $\psi_z\colon \mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^n$, $e_i \mapsto e_{z(i)}$ induces an isomorphism of varieties

$$\phi_z\colon T^*F \xrightarrow{\sim} T^*F, \quad (\mathcal{F}, f) \mapsto (\psi_z(\mathcal{F}), \psi_z f \psi_z^{-1}).$$

By construction, ϕ_z maps a $\mathbb{T}$-fixed point $(\mathcal{F}_{wS_{\boldsymbol{\delta}}}, 0)$ to $(\mathcal{F}_{zwS_{\boldsymbol{\delta}}}, 0)$. In addition, ϕ_z is equivariant with respect to the automorphism $\rho_z\colon \mathbb{T} \xrightarrow{\sim} \mathbb{T}$ given as $(t_1, \ldots, t_n, h) \mapsto (t_{z(1)}, \ldots, t_{z(n)}, h)$. We equip $H^*_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Q}[t_1, \ldots, t_n, h]$ with the S_n-action given by $w.h = h$ and $w.t_i = t_{w(i)}$, for $i = 1, \ldots, n$. Then, the induced automorphism ρ^*_z on $H^*_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Q}[t_1, \ldots, t_n, h]$ coincides with the action of z^{-1}.

The next proposition gives that the induced automorphism ϕ^*_z of $H^*_{\mathbb{T}}(T^*F)$ permutes stable basis elements as follows:

Proposition 9.17. *For all $w \in S_n$ and all choices of chambers $\mathfrak{C}$ of $\mathbb{A}$, it holds*

$$\mathrm{Stab}_{\mathfrak{C}}(wS_{\boldsymbol{\delta}}) = \phi^*_z(\mathrm{Stab}_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}})).$$

In particular, we have

$$\iota^*_{w'S_{\boldsymbol{\delta}}}(\mathrm{Stab}_{\mathfrak{C}}(wS_{\boldsymbol{\delta}})) = z^{-1}.\Big(\iota^*_{zw'S_{\boldsymbol{\delta}}}(\mathrm{Stab}_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}}))\Big), \quad \textit{for all } w' \in S_n.$$

Proof. We show that $\phi_z^*(\mathrm{Stab}_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}}))$ satisfies the stability conditions for $\mathrm{Stab}_{\mathfrak{C}}(wS_{\boldsymbol{\delta}})$ from Theorem 5.10. Since ϕ_z maps each $\mathrm{Attr}_{\mathfrak{C}}(yS_{\boldsymbol{\delta}})$ isomorphically onto $\mathrm{Attr}_{z.\mathfrak{C}}(zyS_{\boldsymbol{\delta}})$, we have $yS_{\boldsymbol{\delta}} \preceq_{\mathfrak{C}} y'S_{\boldsymbol{\delta}}$ if and only if $zyS_{\boldsymbol{\delta}} \preceq_{z.\mathfrak{C}} zy'S_{\boldsymbol{\delta}}$, for all y, $y' \in S_n$. Thus, ϕ_z maps $\mathrm{Attr}^f_{\mathfrak{C}}(wS_{\boldsymbol{\delta}})$ isomorphically onto $\mathrm{Attr}^f_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}})$. As $\mathrm{Stab}_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}})$ is supported on $\mathrm{Attr}^f_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}})$, we deduce that $\phi_z^*(\mathrm{Stab}_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}}))$ is supported on $\mathrm{Attr}^f_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}})$ and thus the support condition is satisfied. Let $\Lambda_{wS_{\boldsymbol{\delta}}} \in H^*_{\mathbb{T}}(T^*F)$ be the Poincaré dual of $[\mathrm{Attr}_{\mathfrak{C}}(wS_{\boldsymbol{\delta}})]^{\mathbb{T}}$. Then, we have

$$\iota^*_{wS_{\boldsymbol{\delta}}}(\phi_z^*(\mathrm{Stab}_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}}))) = \iota^*_{wS_{\boldsymbol{\delta}}}(\Lambda_{wS_{\boldsymbol{\delta}}}) = e_{\mathbb{T}}(T_{wS_{\boldsymbol{\delta}}}(T^*F)^-_{\mathfrak{C}}).$$

This proves the normalization condition. The smallness follows from $\rho_z^*(h) = h$. Hence, $\mathrm{Stab}_{\mathfrak{C}}(wS_{\boldsymbol{\delta}}) = \phi_z^*(\mathrm{Stab}_{z.\mathfrak{C}}(zwS_{\boldsymbol{\delta}}))$. □

Equivariant multiplicities of the bow variety realization

We like to view T^*F as bow variety via the realization $H' \colon \mathcal{C}(\mathcal{D}(d_1, \dots, d_m; n)) \xrightarrow{\sim} T^*F$ from (2.70). Recall from there that H' is equivariant with respect to the automorphism ρ of $\mathbb{T}$ given by $(t_1, \dots, t_n, h) \mapsto (t_1h^{-1}, \dots, t_nh^{-1}, h)$. The induced $\mathbb{Q}$-algebra automorphism $\rho^* : H^*_{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\sim} H^*_{\mathbb{T}}(\mathrm{pt})$ is given as $h \mapsto h$ and $t_i \mapsto t_i - h$ for $i = 1, \dots, n$.

Recall from (3.16) that for all $w \in S_n$, we have $H'(x_{D_{wS_{\boldsymbol{\delta}}}}) = (F_{wS_{\boldsymbol{\delta}}}, 0)$. Here, $D_{wS_{\boldsymbol{\delta}}}$ is defined as in (3.15). Thus, we conclude

$$\mathrm{Stab}_{\mathfrak{C}_-}(D_{wS_{\boldsymbol{\delta}}}) = (H')^*\mathrm{Stab}_{\mathfrak{C}_-}(wS_{\boldsymbol{\delta}}), \tag{9.12}$$

where we identified the tie diagram $D_{wS_{\boldsymbol{\delta}}}$ with its associated $\mathbb{T}$-fixed point $x_{D_{wS_{\boldsymbol{\delta}}}}$. From (9.12), we directly get

$$\iota^*_{D_{w'S_{\boldsymbol{\delta}}}}(\mathrm{Stab}_{\mathfrak{C}_-}(D_{wS_{\boldsymbol{\delta}}})) = \rho^*(\iota^*_{w'S_{\boldsymbol{\delta}}}(\mathrm{Stab}_{\mathfrak{C}_-}(wS_{\boldsymbol{\delta}}))), \quad \text{for all } w' \in S_n. \tag{9.13}$$

The localization formula implies that the equivariant multiplicities of stable basis elements of T^*F are actually ρ^*-invariant:

Proposition 9.18. *For all w, $w' \in S_n$, we have*

$$\iota^*_{D_{w'S_{\boldsymbol{\delta}}}}(\mathrm{Stab}_{\mathfrak{C}_-}(D_{wS_{\boldsymbol{\delta}}})) = \iota^*_{w'S_{\boldsymbol{\delta}}}(\mathrm{Stab}_{\mathfrak{C}_-}(wS_{\boldsymbol{\delta}})).$$

Proof. We have $\rho^*(t_i - t_j + mh) = t_i - t_j + mh$, for all $i, j \in \{1, \dots, n\}$ and $m \in \mathbb{Z}$. Thus, Theorem 9.11 and Proposition 9.15 imply

$$\rho^*(\iota^*_{wS_{\boldsymbol{\delta}}}(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\boldsymbol{\delta}}))) = \iota^*_{wS_{\boldsymbol{\delta}}}(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\boldsymbol{\delta}})).$$

Hence, the proposition follows from (9.13). □

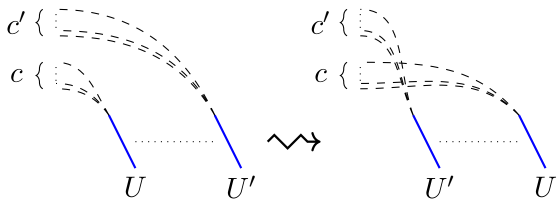
9.4 Symmetric group equivariance for bow varieties

We now return to the general setup of bow varieties. The main result of this section is the Equivairance Theorem (Theorem 9.20) which states that the symmetric group equivariance for stable basis elements of cotangent bundles of partial flag varieties from Proposition 9.17 extends to the framework of bow varieties.

Symmetric group action brane and tie diagrams

Recall the conventions from Notation 2.32 and the definition of margin vectors from Definition 2.58. Also recall that all brane diagrams in this chapter are assumed to be separated and essential.

Let $\mathcal{D}$ be a fixed brane diagram. If $D \in \mathrm{Tie}(\mathcal{D})$ is a tie diagram and $U, U' \in \mathrm{b}(\mathcal{D})$ are blue lines then swapping the blue lines U, U' with their connected ties gives a new tie diagram over a brane diagram that possibly differs from $\mathcal{D}$:



This gives S_N-actions on the sets

$$\mathrm{BD}_N := \{\text{Brane diagrams } \mathcal{D} \mid |\mathrm{b}(U)| = N\} \quad \text{and} \quad \bigsqcup_{\mathcal{D} \in \mathrm{BD}_N} \mathrm{Tie}(\mathcal{D}).$$

These S_N-actions can be characterized as follows: For a permutation $w \in S_N$, the brane diagram $w.\mathcal{D}$ is the brane diagram with M red lines, N blue lines and the numbers on the horizontal lines are given as

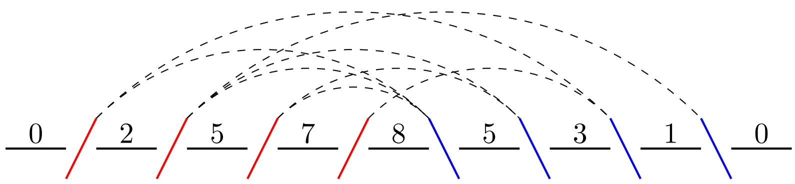
$$d_{X_i}(w.\mathcal{D}) = d_{X_i}(\mathcal{D}), \quad i = 1, \ldots, M+1, \quad d_{X_{M+j}} = \sum_{l=j}^{N} c_{w^{-1}(l)}, \quad j = 1, \ldots, N+1.$$

By construction, the $\mathbf{r}$-margin vectors of $\mathcal{D}$ and $w.\mathcal{D}$ coincide, i.e. $\mathbf{r}(w.\mathcal{D}) = \mathbf{r}(\mathcal{D})$. On the other side $\mathbf{c}(w.\mathcal{D})$ is obtained from $\mathbf{c}(\mathcal{D})$ via $\mathbf{c}(w.\mathcal{D}) = (c_{w^{-1}(1)}(\mathcal{D}), \ldots, c_{w^{-1}(N)}(\mathcal{D}))$. If $D \in \mathrm{Tie}(\mathcal{D})$, then the tie diagram $w.D \in \mathrm{Tie}(w.\mathcal{D})$ is given as

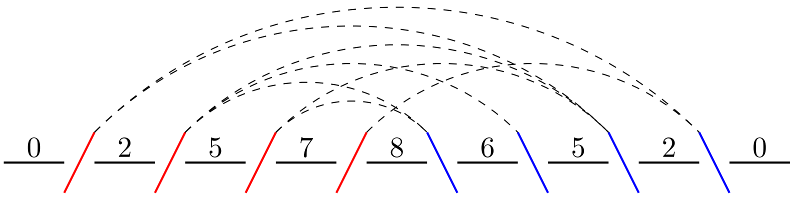
$$w.D = \bigcup_{(V_i, U_j) \in D} \{(V_i, U_{w(j)})\}.$$

Pictorially, the action of w on D is given by moving each blue line U_i with its attached ties to the position of $U_{w(i)}$.

Example 9.19. Consider the following tie diagram D with underlying brane diagram $\mathcal{D}$:



Let $w = 3142 \in S_4$. To obtain the tie diagram $w.D$, we permute the blue lines with the attached ties according to w, i.e. the blue line U_1 is moved with its three attached ties to the position of U_3 etc. The respective labels of the horizontal lines of $w.\mathcal{D}$ can then be easily determined by counting the number of ties above the horizontal lines:



Equivariance Theorem

We now come to the main theorem of this section:

Theorem 9.20 (Equivariance Theorem)**.** *For D, $D' \in \mathrm{Tie}(\mathcal{D})$ and $w \in S_N$ we have*

$$e_{\mathbb{T}}(N^-_{\mathcal{D},\mathfrak{C}}) \cdot \iota^*_{D'}(\mathrm{Stab}_{\mathfrak{C}}(D)) = w^{-1}.\Big(e_{\mathbb{T}}(N^-_{w.\mathcal{D},w.\mathfrak{C}}) \cdot \iota^*_{w.D'}(\mathrm{Stab}_{w.\mathfrak{C}}(w.D))\Big),$$

where $N^-_{\mathcal{D},\mathfrak{C}}$ (resp. $N^-_{w.\mathcal{D},w.\mathfrak{C}}$) is the negative part of the constant $\mathbb{T}$-equivariant bundle $N_{\mathcal{D}}$ (resp. $N_{w.\mathcal{D}}$) over $\mathcal{C}(\mathcal{D})$ (resp. $\mathcal{C}(w.\mathcal{D})$) from Definition 9.21 below.

We prove Theorem 9.20 in Section 9.8.

Remark. In [BR23, Proposition 6.18], a similar S_N-equivariance statement for stable basis elements with a different normalization is proved in the framework of elliptic cohomology.

Definition 9.21. The $\mathbb{T}$-equivariant vector bundle $N_{\mathcal{D}}$ over $\mathcal{C}(\mathcal{D})$ is defined as

$$N_{\mathcal{D}} := \Big(\bigoplus_{j=1}^{N}\bigoplus_{l=1}^{c_j-1} h^l(\xi_{U_j^+} \otimes \mathbb{C}^\vee_{U_j})\Big) \oplus \Big(\bigoplus_{j=1}^{N}\bigoplus_{l=1}^{c_j-1} h^{1-l}(\mathbb{C}_{U_j} \otimes \xi^\vee_{U_j^+})\Big). \tag{9.14}$$

Recall from Proposition 8.22 that for all $U_j \in \mathrm{b}(\mathcal{D})$, we have an isomorphism of $\mathbb{T}$-equivariant vector bundles

$$\xi_{U_j^+} \cong \bigoplus_{i=j+1}^{N}\bigoplus_{l=0}^{c_i-1} h^{-l}\mathbb{C}_{U_i}. \tag{9.15}$$

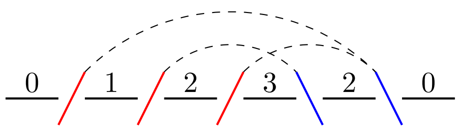
Thus, the positive and negative part of $N_{\mathcal{D}}$ with respect to a choice of chamber $\mathfrak{C}$ can be easily read off from the definition. For instance, if $\mathfrak{C}$ equals the antidominant chamber $\mathfrak{C}_-$ then the $N^\pm_{\mathcal{D},\mathfrak{C}_-}$ are given as follows:

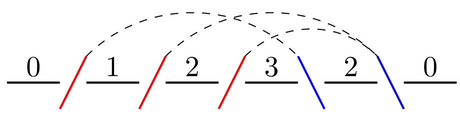
Proposition 9.22. *We have*

$$N^+_{\mathcal{D},\mathfrak{C}_-} = \bigoplus_{j=1}^{N}\bigoplus_{i=j+1}^{N}\bigoplus_{l=1}^{c_j-1}\bigoplus_{k=0}^{c_i-1} h^{l-k}(\mathbb{C}_{U_i} \otimes \mathbb{C}^\vee_{U_j}), \quad N^-_{\mathcal{D},\mathfrak{C}_-} = \bigoplus_{j=1}^{N}\bigoplus_{i=j+1}^{N}\bigoplus_{l=1}^{c_j-1}\bigoplus_{k=0}^{c_i-1} h^{k-l+1}(\mathbb{C}_{U_j} \otimes \mathbb{C}^\vee_{U_i})$$

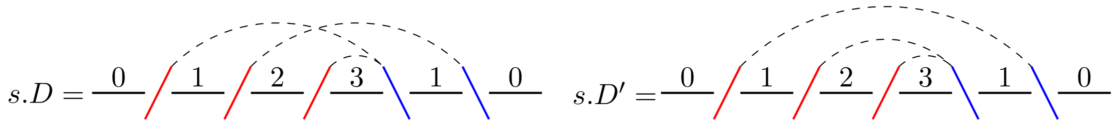
Proof. By (9.15), all $\mathbb{A}$-weights of $\xi_{U^+} \otimes \mathbb{C}^\vee_U$ are positive and all $\mathbb{A}$-weights of $\mathbb{C}_U \otimes \xi^\vee_{U^+}$ are negative with respect to $\mathfrak{C}_-$. Hence, the proposition follows by inserting (9.15) into (9.14). □

Example 9.23. Let $\mathcal{D} = 0/1/2/3\backslash 2\backslash 0$. We like to employ Theorem 9.20 to determine the equivariant multiplicity $\iota^*_{D'}(\mathrm{Stab}_{\mathfrak{C}_-}(D))$ in case D, $D' \in \mathrm{Tie}(\mathcal{D})$ are defined as

$D =$ 

$D' =$ 

Let $s = 21 \in S_2$. Then, $s.D$ (resp. $s.D'$) is obtained from D (resp. D') by switching the positions of the blue lines with its attached ties:



To apply Theorem 9.20, we first determine the vector bundles $N_{\mathcal{D}}$ and $N_{s.\mathcal{D}}$. From $c(\mathcal{D}) = (1,2)$, we deduce that $N_{\mathcal{D}} = 0$, whereas $c(s.\mathcal{D}) = (2,1)$ gives $N_{s.\mathcal{D}} = \mathbb{C}_{t_1-t_2} \oplus \mathbb{C}_{t_2-t_1+h}$. Since $s.\mathfrak{C}_- = \mathfrak{C}_+$, we have $N^-_{s.\mathcal{D},s.\mathfrak{C}_-} = \mathbb{C}_{t_2-t_1+h}$. Thus, Theorem 9.20 implies

$$\iota^*_{D'}(\mathrm{Stab}_{\mathfrak{C}_-}(D)) = s.\Big((t_2 - t_1 + h)\iota^*_{s.D'}(\mathrm{Stab}_{\mathfrak{C}_+}(s.D))\Big). \tag{9.16}$$

To determine the equivariant multiplicity $\iota^*_{s.D'}(\mathrm{Stab}_{\mathfrak{C}_+}(s.D))$ note that the brane diagram $\mathcal{D}_3 = 0\backslash 1/1/1/1\backslash 0$ from Section 6.1 is Hanany–Witten equivalent to $s.\mathcal{D}$. By Proposition 3.18, the corresponding Hanany–Witten isomorphism Φ satisfies $\Phi(x_{D_2}) = x_{s.D}$ and $\Phi(x_{D_3}) = x_{s.D'}$, where $D_2, D_3 \in \mathrm{Tie}(\mathcal{D})$ are defined as in (6.4). By Proposition 6.5, $\iota^*_{D_3}(\mathrm{Stab}_{\mathfrak{C}_+}(D_2)) = h$. Thus, Proposition 5.13 gives $\iota^*_{s.D'}(\mathrm{Stab}_{\mathfrak{C}_+}(s.D)) = h$. Inserting this into (9.16) finally yields

$$\iota^*_{D'}(\mathrm{Stab}_{\mathfrak{C}_-}(D)) = h(t_1 - t_2 + h).$$

Remark. As we will see in Proposition 9.59, the bundles $N_{\mathcal{D}}$ satisfies convenient compatibility relations which make them useful in practical computations.

Renormalized stable basis elements

Because of Theorem 9.20, it is sometimes more convenient to work this the following renormalized version of stable basis elements:

Definition 9.24. With the above notation, we set

$$\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D) := e_{\mathbb{T}}(N^-_{\mathcal{D},\mathfrak{C}}) \cdot \mathrm{Stab}_{\mathfrak{C}}(D), \quad \text{for all } D \in \mathrm{Tie}(\mathcal{D}). \tag{9.17}$$

We call the elements $\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D)$ the *renormalized stable basis elements of* $\mathcal{C}(\mathcal{D})$.

In the special case $\mathcal{D} = \mathcal{D}(d_1, \dots, d_m; n)$, where $\mathcal{D}(d_1, \dots, d_m; n)$ is defined as in (2.68), we have $c_1 = \ldots = c_n = 1$. Thus, by (9.14), $N_{\mathcal{D}} = 0$ which yields $\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D) = \mathrm{Stab}_{\mathfrak{C}}(D)$. Thus in this case the stable basis elements and the renormalized stable basis elements coincide.

9.5 Symmetric group calculus for bow varieties

Let $\mathcal{D}$ be a fixed brane diagram and let $n = \sum_{i=1}^M r_i = \sum_{j=1}^N c_j$. We denote by

$$S_{\mathbf{c}} := S_{c_1} \times \ldots \times S_{c_N} \subset S_n \quad \text{and} \quad S_{\mathbf{r}} := S_{r_1} \times \ldots \times S_{r_M} \subset S_n$$

the corresponding Young subgroups.

In this section, we describe a correspondence between the binary contingency tables (and equivalently the tie diagrams) of $\mathcal{D}$ and a special class of $(S_{\mathbf{c}}, S_{\mathbf{r}})$-double cosets which we call *fully separated*, see Definition 9.26. As we will see in Theorem 9.35 and Corollary 9.38, permutations that belong to fully separated double cosets satisfy strong uniqueness properties which distinguish fully separated double cosets from general double cosets.

Fully separated double cosets

The usual assignment of a $(S_{\mathbf{c}}, S_{\mathbf{r}})$-double coset to a matrix leads to the following well-known bijection, see e.g. [JK81, Theorem 1.3.10]:

Theorem 9.25. *Let $\Xi(\boldsymbol{r}, \boldsymbol{c})$ be the set of all $M \times N$-matrices A with entries in $\mathbb{Z}_{\geq 0}$ satisfying*

$$\sum_{l=1}^{N} A_{i,l} = r_i, \quad \sum_{l=1}^{M} A_{l,j} = c_j, \quad \textit{for all } i,\ j.$$

Then, the map $Z\colon S_n \to \Xi(\boldsymbol{r}, \boldsymbol{c})$ given by

$$Z(w)_{i,j} = |w(\{R_{i-1}+1, \dots, R_i\}) \cap \{C_{j-1}+1, \dots, C_j\}|$$

induces a bijection

$$\bar{Z}\colon S_{\boldsymbol{c}} \backslash S_n / S_{\boldsymbol{r}} \xrightarrow{\sim} \Xi(\boldsymbol{r}, \boldsymbol{c}), \quad S_{\boldsymbol{c}} w S_{\boldsymbol{r}} \mapsto Z(w). \tag{9.18}$$

By definition, the elements of $\mathrm{bct}(\mathcal{D})$ are exactly the matrices $\Xi(\mathbf{r}, \mathbf{c})$ with all entries contained in $\{0,1\}$. The following notion characterizes the double cosets that correspond to $\mathrm{bct}(\mathcal{D})$ under $\bar{Z}$:

Definition 9.26. A permutation $w \in S_n$ is called *fully separated (with respect to $(\boldsymbol{r}, \boldsymbol{c})$)* if

$$|w(\{R_{i-1}+1, \dots, R_i\}) \cap \{C_{j-1}+1, \dots, C_j\}| \leq 1, \quad \text{for all } i \in \{1, \dots, M\},\ j \in \{1, \dots, N\}.$$

If w is fully separated then so is every element in $S_{\mathbf{c}} w S_{\mathbf{r}}$. Hence, we call a double coset $S_{\mathbf{c}} w S_{\mathbf{r}}$ *fully separated* if all its elements are fully separated. Likewise, we call a left $S_{\mathbf{r}}$-coset (resp. right $\mathbf{c}$-coset) *fully separated* if all its elements are fully separated.

Clearly, a permutation w is fully separated if and only if $Z(w)$ is contained in $\mathrm{bct}(\mathcal{D})$. Thus, we have the following corollary:

Corollary 9.27. *The bijection $\bar{Z}$ from* (9.18) *restricts to a bijection*

$$\mathrm{fsep}_{\boldsymbol{c},\boldsymbol{r}} \xrightarrow{\sim} \mathrm{bct}(\mathcal{D}),$$

where $\mathrm{fsep}_{\boldsymbol{c},\boldsymbol{r}}$ denotes the set of fully separated $(S_{\boldsymbol{c}}, S_{\boldsymbol{r}})$-double cosets.

Example 9.28. Let $n = 5$ and $\mathbf{r} = (2,2,1)$, $\mathbf{c} = (1,2,2)$. To compute the matrix entries $Z(w_1)_{2,2}$ and $Z(w_2)_{2,2}$ for the permutations $w_1 = 14253$ and $w_2 = 14235$ note that $\{w_1(3), w_1(4)\} \cap \{2,3\} = \{2\}$. Hence, $Z(w_1)_{2,2} = 1$. Likewise, as $\{w_1(3), w_1(4)\} \cap \{2,3\} = \{2,3\}$, we have $Z(w_2)_{2,2} = 2$. The other entries of $Z(w_1)$ and $Z(w_2)$ can be computed in the same way:

$$Z(w_1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z(w_2) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, w_1 is fully separated, as all entries of $Z(w_1)$ are contained in $\{0,1\}$. On the other hand, $Z(w_2)_{2,2} = 2$ and hence w_2 is not fully separated.

In diagrammatic language the fully separatedness condition can be reformulated as follows:

Lemma 9.29. *Let $w \in S_N$ and d_w be a diagram for w. Then, w is fully separated if and only if for all $i \in \{1, \dots, M\}$, $j \in \{1, \dots, N\}$ there exists at most one strand in d_w with source in $\{R_{i-1}+1, \dots, R_i\}$ and target in $\{C_{j-1}+1, \dots, C_j\}$.*

Proof. By definition, w is fully separated if and only if $|w(\{R_{i-1}+1, \dots, R_i\}) \cap \{C_{j-1}+1, \dots, C_j\}| \leq 1$, for all i, j. This is equivalent to the condition that for all i, j, there exists at most one strand in d_w with source in $\{R_{i-1}+1, \dots, R_i\}$ and target in $\{C_{j-1}+1, \dots, C_j\}$. □

Remark. In [JK81] the fully separatedness condition is called *trivial intersection property.*

Shortest double coset representatives

Recall from e.g. [Hum90, Section 5.12] that each left coset $wS_{\mathbf{r}}$ (resp. right coset $S_{\mathbf{c}}w$) contains a unique representative of minimal Bruhat length w_l (resp. w_r). We have that w_l (resp. w_r) is uniquely determined by the condition $w_l(R_{i-1}+1) < \dots < w_l(R_i)$, for all i (resp. $w_r^{-1}(C_{j-1}+1) < \dots < w_r^{-1}(C_j)$, for all j). Likewise, each double coset $S_{\mathbf{c}}wS_{\mathbf{r}}$ contains a unique representative of shortest Bruhat length w_d which is uniquely characterized by the conditions

$$w_d(R_{i-1}+1) < \dots < w_d(R_i) \quad \text{and} \quad w_d^{-1}(C_{j-1}+1) < \dots < w_d^{-1}(C_j), \quad \text{for all } i, j.$$

In the following, we describe the shortest representative of $(S_{\mathbf{c}}, S_{\mathbf{r}})$-double cosets corresponding to binary contingency tables. We begin with a hopefully intuitive example:

Example 9.30. Let $n = 10$, $\mathbf{r} = (3, 2, 2, 3)$, $\mathbf{c} = (2, 3, 2, 1, 2)$ and

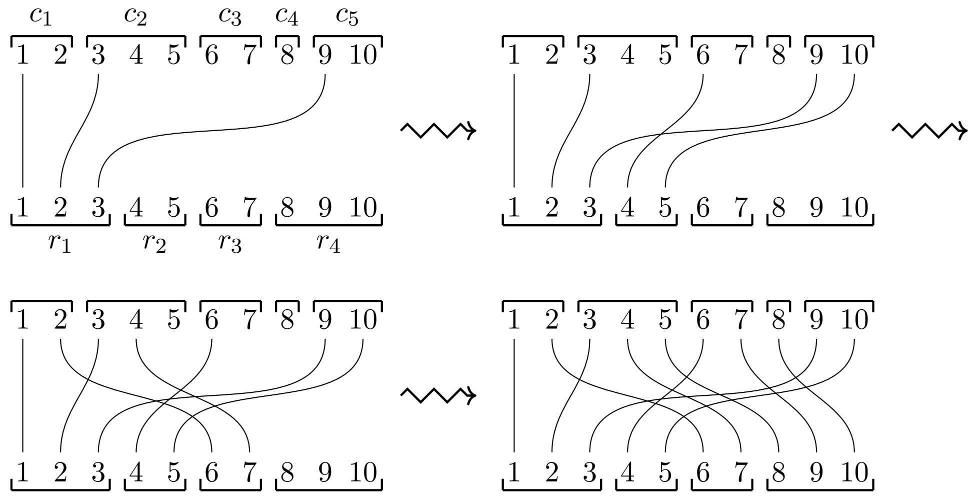
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

We draw a diagram for the shortest double coset representative of $\bar{Z}^{-1}(A)$ following the next steps: At first, we define functions

$$F_{A,i} : \{1, \dots, r_i\} \longrightarrow \{1, \dots, N\}, \quad i = 1, \dots, M, \tag{9.19}$$

where $F_{A,i}(l)$ is the column index of the l-th 1-entry in the i-th row of A. For instance, $F_{A,1} : \{1, 2, 3\} \to \{1, \dots, 5\}$ is given by $F_{A,1}(1) = 1$, $F_{A,1}(2) = 2$ and $F_{A,1}(3) = 5$.

We start drawing our diagram by drawing strands λ_l starting in $l = 1, \dots, r_1$ and ending in $C_{F_{A,1}(1)-1}+1, \dots, C_{F_{A,1}(r_1)-1}+1$. Then, we draw strands λ_{r_1+l} starting in $r_1+1, \dots, r_1+r_2$ and the endpoint of λ_{r_1+l} is the smallest element of $\{C_{F_{A,2}(l)-1}+1, \dots C_{F_{A,2}}(l)\}$ that is not already the endpoint of a strand. Continuing this procedure leads to the following permutation diagram:



We denote the resulting permutation by $\tilde{w}_A$, i.e. $\tilde{w}_A = 13961024578$. Our condition to pick always the smallest entry in $\{C_{j-1}+1, \dots, C_j\}$ that is not already the endpoint of a strand implies $\tilde{w}_A^{-1}(C_{j-1}+1) < \dots < \tilde{w}_A^{-1}(C_j)$, for all j. In addition, as the functions $F_{A,i}$ strictly increase, we also have $\tilde{w}_A(R_{i-1}+1) < \dots < \tilde{w}_A(R_i)$, for all i. Thus, $\tilde{w}_A$ is a shortest $(S_{\mathbf{c}}, S_{\mathbf{r}})$-double coset representative. As $\tilde{w}_A$ satisfies $\tilde{w}_A(R_{i-1}+l) \in \{C_{F_{A,i}(l)-1}+1, \dots C_{F_{A,i}(l)}\}$ for all i, l, we conclude

$$Z(\tilde{w}_A)_{i,j} = \begin{cases} 1 & \text{if } j = F_{A,i}(l), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $Z(\tilde{w}_A) = A$ which implies that $\tilde{w}_A$ is indeed the shortest representative of $\bar{Z}^{-1}(A)$.

We return to the general setup: Let $\mathcal{D}$ be a brane diagram and $A \in \operatorname{bct}(\mathcal{D})$. As in the previous example, let $F_{A,i} \colon \{1, \dots, r_i\} \to \{1, \dots, N\}$ be the function assigning to l the column index of the l-th 1-entry in the i-th row of A. Likewise, let $G_{A,j} \colon \{1, \dots, c_j\} \to \{1, \dots, M\}$ be the function assigning to l the row index of the l-th 1-entry in the j-th column of A. We also set $n_{A,i,j} \coloneqq \sum_{l=1}^{i} A_{l,j}$. That is, $n_{A,i,j}$ is the number of 1-entries that are in the j-th column of A and strictly above the entry $A_{i+1,j}$.

Definition 9.31. We define a map $\tilde{w}_A \colon \{1, \dots, n\} \to \{1, \dots, n\}$ as

$$\tilde{w}_A(R_{i-1}+l) = C_{F_{A,i}(l)-1} + n_{A,i,F_{A,i}(l)}, \quad \text{for } i = 1, \dots, M \text{ and } l = 1, \dots, r_i.$$

We now prove that $\tilde{w}_A$ is indeed contained in S_n:

Lemma 9.32. *The map $\tilde{w}_A$ is bijective.*

Proof. It suffices to show that $\tilde{w}_A$ is surjective. Let $j \in \{1, \dots, N\}$ and $l \in \{1, \dots, c_j\}$. Then, let $i = G_{A,j}(l)$ and $l' \in \{1, \dots, r_i\}$ such that $A_{i,j}$ corresponds to the l'-th 1-entry in the i-th row of A. By construction, we have $\tilde{w}_A(R_{i-1}+l') = C_{j-1}+l$ which proves the surjectivity of $\tilde{w}_A$. □

The next proposition lists important properties of $\tilde{w}_A$.

Proposition 9.33. *The following holds:*

(i) $Z(\tilde{w}_A) = A$,

(ii) $\tilde{w}_A$ *is the shortest representative of* $\bar{Z}^{-1}(A)$,

(iii) *we have* $l(\tilde{w}_A) = |\mathrm{Inv}(A)|$, *where*

$$\mathrm{Inv}(A) = \{((i_1, j_1), (i_2, j_2)) \mid A_{i_1,j_1} = A_{i_2,j_2} = 1,\ i_1 < i_2,\ j_2 < j_1\}.$$

Proof. By construction, $\tilde{w}_A(R_{i-1} + l) \in \{C_{F_{A,i}(l)-1} + 1, \ldots, C_{F_{A,i}(l)}\}$ which gives

$$Z(\tilde{w}_A)_{i,j} = \begin{cases} 1 & \text{if } j = F_{A,i}(l), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $Z(\tilde{w}_A) = A$. Moreover, we conclude $\tilde{w}_A(R_{i-1} + 1) < \ldots < \tilde{w}_A(R_i)$, for all i. By definition, we also have $\tilde{w}_A^{-1}(C_{j-1} + l) \in \{R_{G_{A,j}(l)-1} + 1, \ldots, R_{G_{A,j}(l)}\}$ which implies $\tilde{w}_A^{-1}(C_{j-1}+1) < \ldots < \tilde{w}_A^{-1}(C_j)$, for all j. Thus, $\tilde{w}_A$ is the shortest representative of $\bar{Z}^{-1}(A)$. Finally, note that since $\tilde{w}_A$ is a shortest left $S_{\mathbf{r}}$-coset representative, the inversions of $\tilde{w}_A$ are exactly the ordered pairs $(R_{i_1} + l_1, R_{i_2} + l_2)$ with

$$1 \leq i_1 < i_2 \leq M, \quad 1 \leq l_1 \leq r_{i_1}, \quad 1 \leq l_2 \leq r_{i_2}, \quad F_{A,i_1}(l_1) > F_{A,i_2}(l_2).$$

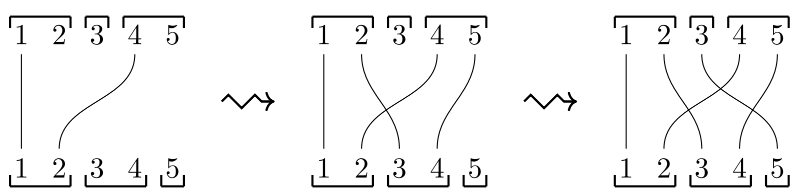
Therefore, we have a bijection $\mathrm{Inv}(\tilde{w}_D) \xrightarrow{\sim} \mathrm{Inv}(D)$, where an inversion $(R_{i_1} + l_1, R_{i_2} + l_2)$ of $\tilde{w}_A$ is mapped to $((i_1, F_{A,i_1}(l_1)), (i_2, F_{A,i_2}(l_2)))$. Hence, $l(\tilde{w}_D) = |\mathrm{Inv}(A)|$. □

Example 9.34. Let w, $w' \in S_5$ be as in Example 9.16 and choose $\mathbf{r} = (2, 2, 1)$, $\mathbf{c} = (2, 1, 2)$. Then, we have

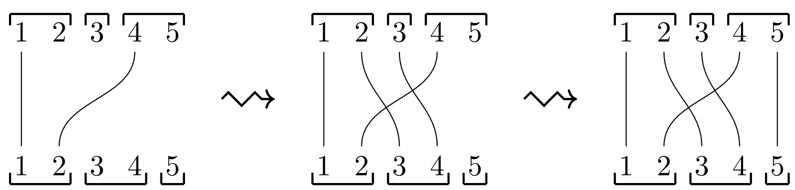
$$Z(w) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z(w') = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set $A \coloneqq Z(w)$ and $A' \coloneqq Z(w')$. To determine the permuations $\tilde{w}_A$ and $\tilde{w}_{A'}$, we first read off the functions $F_{A,i}$ and $F_{A',i}$ from A and A'. We have $F_{A,1} = F_{A,2} \colon \{1, 2\} \to \{1, 2, 3\}$, $1 \mapsto 1$, $2 \mapsto 3$ and $F_{A,3} \colon \{1\} \to \{1, 2, 3\}$, $1 \mapsto 2$. Likewise, we have $F_{A',1} = F_{A,1}$, $F_{A',2} \colon \{1, 2\} \to \{1, 2, 3\}$, $1 \mapsto 1$, $2 \mapsto 2$ and $F_{A,3} \colon \{1\} \to \{1, 2, 3\}$, $1 \mapsto 3$. Hence, the stepwise construction of $\tilde{w}_A$ and $\tilde{w}_{A'}$ can be illustrated as follows:

Construction of $\tilde{w}_A$:



Construction of $\tilde{w}_{A'}$:



Thus, we have $\tilde{w}_A = 14253$ and $\tilde{w}_{A'} = 14235$.

Uniqueness properties

In this subsection, we discuss strong uniqueness properties of fully separated permutations that distinguish them from general permutations. The central result is the following theorem:

Theorem 9.35. *Assume $w \in S_n$ is fully separated. Let v, $v' \in S_{\mathbf{r}}$ and u, $u' \in S_{\mathbf{c}}$ such that $uwv = u'wv'$. Then, $u = u'$ and $v = v'$.*

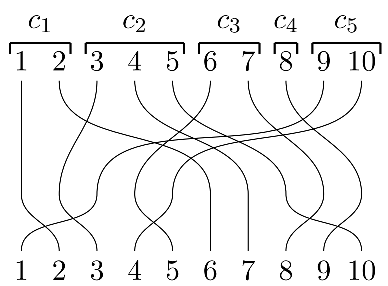
Before we prove Theorem 9.35, we illustrate the idea of the proof in the following example:

Example 9.36. Let n, $\mathbf{r}$, $\mathbf{c}$ and A be as in Example 9.30. For a permutation $w \in S_n$, we define the function

$$F_w : \{1, \dots, n\} \longrightarrow \{1, \dots, N\}, \quad i \mapsto F_w(i),$$

where $F_w(i)$ is the unique element on $\{1, \dots, N\}$ such that $C_{F_w(i)-1} + 1 \leq w(i) \leq C_{F_w(i)}$. In terms of diagrammatic calculus, the function F_w can be characterized as follows: Pick a diagram d_w for w. On the top of d_w, draw N square brackets around the intervals $\{1, \dots, C_1\}$, $\{C_1 + 1, \dots, C_2\}, \dots, \{C_{N-1} + 1, \dots, C_N\}$. Label these square brackets with $1, \dots, N$ from left to right. Then, $F_w(i)$ is the index of the square bracket containing the endpoint of the unique strand starting in i.

Let for instance $v = v_1 \times v_2 \times v_3 \times v_4 \in S_{\mathbf{r}}$, where $v_1 = 312$, $v_2 = 21$, $v_3 = 12$, $v_4 = 231$. A diagram for $\tilde{w}_A v$ is given by



The functions $F_{\tilde{w}_A}$ and $F_{\tilde{w}_A v}$ can be easily read of from their diagrams:

i	1	2	3	4	5	6	7	8	9	10
$F_{\tilde{w}_A}(i)$	1	2	5	3	5	1	2	2	3	4
$F_{\tilde{w}_A v}(i)$	5	1	2	5	3	1	2	3	4	5

Next, we show that if we know $F_{\tilde{w}_A v}$ then we can reconstruct the permutation v. We begin by reconstructing the factor $v_1 \in S_3$. The first three letters in the row of $F_{\tilde{w}_A v}$ give the word 512. Then, using the identification $1 \mapsto 1$, $2 \mapsto 2$, $5 \mapsto 3$, we see that 512 corresponds to $312 = v_1$. Next, the fourth and the fifth letters in the row of $F_{\tilde{w}_A v}$ give the word 53. Using the identification $3 \mapsto 1$, $5 \mapsto 2$, we get the word $21 = v_2$. In the same way one can construct v_3 and v_4 and thus the permutation v.

In our reasoning, the fully separatedness property was essential because this property ensures that the restriction of $F_{\tilde{w}_A v}$ to $\{1, 2, 3\}$, $\{4, 5\}$, $\{6, 7\}$, $\{8, 9, 10\}$ is injective.

We proceed with the general setup. As in Example 9.36, we define for given $w \in S_n$ the function

$$F_w : \{1, \dots, n\} \longrightarrow \{1, \dots, N\}, \quad i \mapsto F_w(i), \tag{9.20}$$

where $F_w(i)$ is the unique element in $\{1, \dots, N\}$ such that

$$C_{F_w(i)-1} + 1 \leq w(i) \leq C_{F_w(i)}.$$

Likewise, we define

$$G_w : \{1, \dots, n\} \longrightarrow \{1, \dots, M\}, \quad j \mapsto G_w(j),$$

where $G_w(j)$ is the unique element in $\{1, \dots, M\}$ such that

$$R_{G_w(j)-1} + 1 \leq w^{-1}(j) \leq R_{G_w(j)}.$$

Similarly as F_w, the function G_w admits the following diagrammatic interpretation: Pick a diagram for w and draw M square brackets on the bottom around the discrete intervals $\{1, \dots, R_1\}$, $\{R_1 + 1, \dots, R_2\}, \dots, \{R_{M-1} + 1, \dots, R_M\}$. Label the square brackets with $1, \dots, M$ from left to right. Then, $G_w(j)$ is the index of the square bracket containing the starting point of the unique strand with endpoint j.

If w is fully separated then the restrictions of F_w to the sets $\{R_{i-1}+1, \dots, R_i\}$ is injective, for $i = 1, \dots, M$. Likewise, the restriction of G_w to $\{C_{j-1} + 1, \dots, C_j\}$ is also injective, for $j = 1, \dots, N$.

The next lemma lists useful properties of the functions F_w and G_w.

Lemma 9.37. *Assume $w \in S_n$ is fully separated. For $u \in S_{\mathbf{c}}$ and $v \in S_{\mathbf{r}}$, we have*

(i) $F_{uw} = F_w$,

(ii) $G_{wv} = G_w$,

(iii) $F_{wv} = F_w$ *if and only if* $v = \mathrm{id}$,

(iv) $G_{uw} = G_w$ *if and only if* $u = \mathrm{id}$.

Proof. Since u leaves the sets $\{C_{j-1} + 1, \dots, C_j\}$ invariant, we get (i). Likewise, v leaves the sets $\{R_{i-1}+1, \dots, R_i\}$ invariant which gives (ii). For (iii), suppose that $v \neq \mathrm{id}$ and $F_{wv} = F_w$. Let $l \in \{1, \dots, n\}$ such that $v(l) \neq l$. Choose $i \in \{1, \dots, M\}$ such that $l \in \{R_{i-1}+1, \dots, R_i\}$. As $v \in S_{\mathbf{r}}$, we have that $v(l)$ is also contained in $\{R_{i-1} + 1, \dots, R_i\}$. By definition, $F_{wv}(l) = F_w(v(l))$ and hence $F_w(v(l)) = F_w(l)$. This contradicts the fact that the restriction of F_w to $\{R_{i-1} + 1, \dots, R_i\}$ is injective. The proof of (iv) is analogous. □

Proof of Theorem 9.35. We may assume $u' = \mathrm{id}$, $v' = \mathrm{id}$. By Lemma 9.37.(i), $F_w = F_{uwv} = F_{wv}$. Thus, Lemma 9.37.(iii) implies $v = \mathrm{id}$. Likewise, Lemma 9.37.(ii) gives $G_w = G_{uwv} = G_{uw}$ which implies $u = \mathrm{id}$ by Lemma 9.37.(iv). Hence, we proved $u = u'$ and $v = v'$. □

Let $A \in \operatorname{bct}(\mathcal{D})$. By construction of $F_{\tilde{w}_A}$ and $F_{A,i}$, we have

$$F_{\tilde{w}_A}(R_{i-1}+l) = F_{A,i}(l), \quad i = 1, \ldots, M,\ l = R_{i-1}+1, \ldots, R_i. \tag{9.21}$$

This observation combined with Lemma 9.37 leads to the following characterization of shortest representatives of fully separated left resp. right cosets:

Corollary 9.38. *For $A \in \operatorname{bct}(\mathcal{D})$, the following holds:*

(i) $u\tilde{w}_A$ is a shortest left $S_{\mathbf{r}}$-coset representative, for all $u \in S_{\mathbf{c}}$,

(ii) if $wS_{\mathbf{r}}$ is a fully separated left coset then there exist $A \in \operatorname{bct}(\mathcal{D})$, $u \in S_{\mathbf{c}}$ such that $u\tilde{w}_A$ is the shortest representative of $wS_{\mathbf{r}}$,

(iii) $\tilde{w}_A v$ is a shortest right $S_{\mathbf{c}}$-coset representative, for all $v \in S_{\mathbf{r}}$,

(iv) if $S_{\mathbf{c}}w$ is a fully separated right coset then there exist $A \in \operatorname{bct}(\mathcal{D})$, $v \in S_{\mathbf{r}}$ such that $\tilde{w}_A v$ is the shortest representative of $S_{\mathbf{c}}w$.

Proof. By Lemma 9.37.(i) and (9.21), we have

$$F_{u\tilde{w}_A}(R_{i-1}+1) < F_{u\tilde{w}_A}(R_{i-1}+2) < \ldots < F_{u\tilde{w}_A}(R_i), \quad \text{for } i = 1, \ldots, M.$$

This implies $u\tilde{w}_A(R_{i-1}+1) < u\tilde{w}_A(R_{i-1}+2) < \ldots < u\tilde{w}_A(R_i)$, for $i = 1, \ldots, M$. Hence, $u\tilde{w}_A$ is the shortest representative of $u\tilde{w}_A S_{\mathbf{r}}$ which gives (i). For (ii), we use that if $wS_{\mathbf{r}}$ is fully separated then $Z(w) \in \operatorname{bct}(\mathcal{D})$ and hence there exist $u \in S_{\mathbf{c}}$, $v \in S_{\mathbf{r}}$ such that $w = u\tilde{w}_{Z(w)}v$. Thus, (i) gives that $u\tilde{w}_{Z(w)}$ is the shortest representative of $wS_{\mathbf{r}}$. The proofs of (iii) and (iv) are analogous. □

9.6 Equivariant multiplicities via resolutions

The main results of this section are Theorem 9.42 and Theorem 9.44 which provide formulas that express the equivariant multiplicities of stable basis elements of bow varieties in terms of equivariant multiplicities of stable basis elements of cotangent bundles of partial flag varieties. In particular, these formulas allow to compute equivariant multiplicities for bow varieties via the diagrammatic calculus from Proposition 9.13.

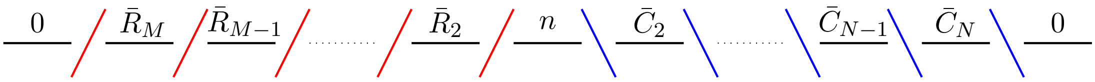
Theorem 9.42 deals with the stable basis elements corresponding to the dominant chamber, whereas Theorem 9.44 focuses on the antidominant chamber. As we discuss in Section 9.8, Theorem 9.42 and Theorem 9.44 are actually equivalent.

The formulation of Theorem 9.42 and Theorem 9.44 uses the language of symmetric group calculus for bow varieties which was developed in the previous section as well as the language of resolutions of tie diagrams from [BR23]. We therefore refer to these theorems as the *Equivariant Resolution Theorems.* More specifically, we call Theorem 9.42 the *Dominant Equivariant Resolution Theorem* and Theorem 9.44 the *Antidominant Equivariant Resolution Theorem.*

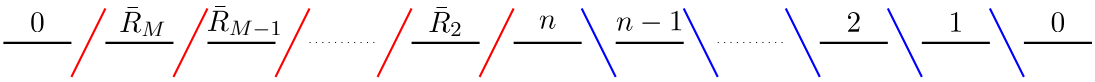
We begin with the underlying combinatorics on brane and tie diagrams.

Resolutions of brane and tie diagrams

As before, let $\mathcal{D}$ be a fixed brane diagram. Recall from Definition 2.58 that $\bar{R}_i = \sum_{l=i}^{M} r_l$ and $\bar{C}_j = \sum_{l=j}^{N} c_l$. Hence, the labels of the horizontal lines of $\mathcal{D}$ are given as follows:



Definition 9.39. The *resolution* $\mathrm{Res}(\mathcal{D})$ *of* $\mathcal{D}$ is the brane diagram defined as

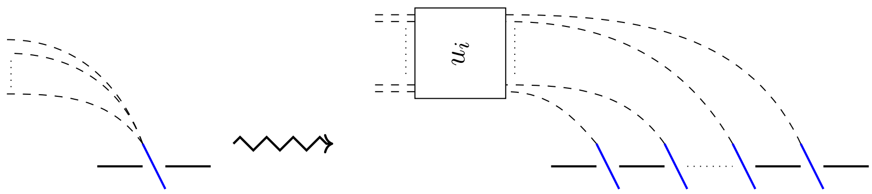


That is, the resolved brane diagram $\mathrm{Res}(\mathcal{D})$ is obtained from $\mathcal{D}$ by replacing the part $n\backslash\bar{C}_2\backslash\ldots\backslash\bar{C}_N\backslash 0$ with $n\backslash n-1\backslash\ldots\backslash 2\backslash 1\backslash 0$. Thus, $\mathrm{Res}(\mathcal{D})$ is equal to the brane diagram $\mathcal{D}(R_1,\ldots,R_{M-1};n)$ from (2.68). Hence, $\mathcal{C}(\mathrm{Res}(\mathcal{D}))$ is isomorphic to $T^*F(R_1,\ldots,R_{M-1};n)$.

Given $u = u_1 \times \cdots \times u_N \in S_{\mathbf{c}}$, we obtain an inclusion

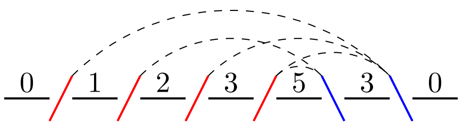
$$\mathrm{Res}_u \colon \mathrm{Tie}(\mathcal{D}) \longhookrightarrow \mathrm{Tie}(\mathrm{Res}(\mathcal{D})), \tag{9.22}$$

where for a tie diagram $D \in \mathrm{Tie}(\mathcal{D})$, the resolved tie diagram $\mathrm{Res}_u(D)$ is obtained via performing at each blue line U_i the local move:



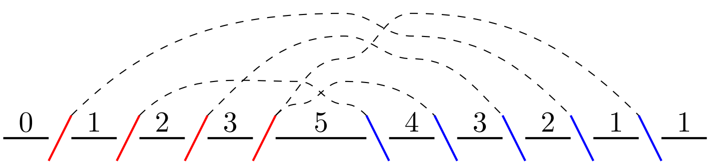
Here, the box around u_i represents an arbitrary diagram for u_i.

Example 9.40. Let D be the following tie diagram with corresponding binary contingency table:



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We choose $u_1 = 21$ and $u_2 = 231$. Then, $\mathrm{Res}_u(D)$ and its associated binary contingency table are given by



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Here, we see the rotated diagrams for u_1 and u_2, where the diagram for u_1 involves the ties of the first and the second blue line and the diagram for u_2 involves the ties of the third, fourth and fifth blue line.

As in Proposition 3.5, we denote by $M(D)$ the binary contingency table corresponding to a tie diagram $D \in \mathrm{Tie}(\mathcal{D})$. In addition, we set $\tilde{w}_D \coloneqq \tilde{w}_{M(D)}$, where $\tilde{w}_{M(D)}$ is the permutation from Definition 9.31.

In terms of left $S_{\mathbf{r}}$-cosets, we can characterize $\mathrm{Res}_u(D)$ as follows:

Proposition 9.41. *We have*

$$\mathrm{Res}_u(D) = D_{u\tilde{w}_D S_r},$$

where $D_{u\tilde{w}_D S_r}$ is defined as in (3.15).

Proof. Suppose the blue line U_j in D is connected to the red lines $V_{i_1}, \ldots V_{i_{c_j}}$, where $i_1 < \ldots < i_{c_j}$. Then, in $\mathrm{Res}_u(D)$, the blue line $U_{C_{j-1}+l}$ is connected to $V_{i_{u(l)}}$, for $l = 1, \ldots, c_j$. On the other hand, by the construction of $\tilde{w}_D$, we have $\tilde{w}_D^{-1}(C_{j-1}+l) \in \{R_{i_l-1}+1, \ldots, R_{i_1}\}$, for all l. Thus, $\tilde{w}_D^{-1}u^{-1}(C_{j-1}+l) \in \{R_{i_{u_j(l)}-1}+1, \ldots, R_{i_{u_j(l)}}\}$. Hence, the tie diagram $\mathrm{Res}_u(D)$ equals $D_{u\tilde{w}_D S_{\mathbf{r}}}$. □

Equivariant Resolution Theorems

We now come to the main results of this section. We begin with the Dominant Equivariant Resolution Theorem because its formulation is slightly easier. However, in practical computations, we will always use the Antidominant Equivariant Resolution Theorem in this thesis.

Theorem 9.42 (Dominant Equivariant Resolution Theorem)**.** *Let D, $D' \in \mathrm{Tie}(\mathcal{D})$ and $u \in S_{\boldsymbol{c}}$. Then, the equivariant multiplicities of the renormalized stable basis elements from Definition 9.24 can be computed via*

$$\Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i-1}(jh)^{c_i-j}\Big)\cdot \iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D')) = \Psi'_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\mathfrak{C}_+}(\mathrm{Res}_u(D')))),$$

where $\Psi'_{\mathcal{D}}\colon \mathbb{Q}[t_1,\ldots,t_n,h] \to \mathbb{Q}[t_1,\ldots,t_N,h]$ is the $\mathbb{Q}[h]$-algebra homomorphism given by $\Psi'_{\mathcal{D}}(t_{C_{i-1}+k}) = t_i - (c_i-1-k)h$, for $i = 1,\ldots,N$, $k = 1,\ldots,c_i$.

Theorem 9.42 is proved in Section 9.8. By employing the isomorphism $\mathcal{C}(\mathrm{Res}(\mathcal{D})) \cong T^*F(R_1,\ldots,R_{M-1};n)$, we obtain the following reformulation of Theorem 9.42:

Corollary 9.43. *With the notation of Theorem 9.42, we have*

$$\Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i-1}(jh)^{c_i-j}\Big)\cdot \iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D')) = \Psi'_{\mathcal{D}}(\iota^*_{\tilde{w}_D S_r}(\mathrm{Stab}_{\mathfrak{C}_+}(w'S_r))),$$

where the stable basis element on the right hand side is on $T^*F(R_1,\ldots,R_{M-1};n)$ and $w' \in S_{\boldsymbol{c}}\tilde{w}_{D'}S_{\boldsymbol{r}}$.

Proof. By Proposition 9.41, $D_{\tilde{w}_D S_{\mathbf{r}}} = \mathrm{Res}_{\mathrm{id}}(D)$ and $D_{w' S_{\mathbf{r}}} = \mathrm{Res}_u(D')$, for some $u \in S_{\mathbf{c}}$. Thus, Proposition 9.18 yields

$$\iota^*_{\tilde{w}_D S_{\mathbf{r}}}(\mathrm{Stab}_{\mathfrak{C}_+}(w' S_{\mathbf{r}})) = \iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\mathfrak{C}_+}(\mathrm{Res}_u(D'))).$$

Hence, the statement follows from Theorem 9.42. □

Antidominant Equivariant Resolution Theorem

For the formulation of the Antidominant Equivariant Resolution Theorem, we set

$$u_0 := (w_{0,c_1} \times \ldots \times w_{0,c_N}) \in S_{\mathbf{c}}, \tag{9.23}$$

where $w_{0,l}$ denotes the longest element in S_l, for all l.

Theorem 9.44 (Antidominant Equivariant Resolution Theorem)**.** *Let D and D' be tie diagrams of $\mathcal{D}$. Then, the equivariant multiplicities of the renormalized stable basis elements can be computed via*

$$\Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i-1}(jh)^{c_i-j}\Big)\cdot \iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')) = \Psi_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{u_0}(D)}(\mathrm{Stab}_{\mathfrak{C}_-}(\mathrm{Res}_u(D')))).$$

Here, $u \in S_{\boldsymbol{c}}$, u_0 is as in (9.23) *and the $\mathbb{Q}[h]$-algebra homomorphism $\Psi_{\mathcal{D}} \colon \mathbb{Q}[t_1,\ldots,t_n,h] \to \mathbb{Q}[t_1,\ldots,t_N,h]$ is given by $\Psi_{\mathcal{D}}(t_{C_{i-1}+k}) = t_i - (k-1)h$, for $i = 1,\ldots,N$, $k = 1,\ldots,c_i$.*

We also prove Theorem 9.44 in Section 9.8.

For $D \in \mathrm{Tie}(\mathcal{D})$, we set

$$w_D := u_0 \tilde{w}_D \in S_n. \tag{9.24}$$

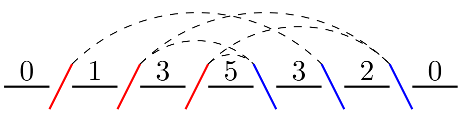
The next statement is a version of Corollary 9.43 for the antidominant chamber and follows along similar lines:

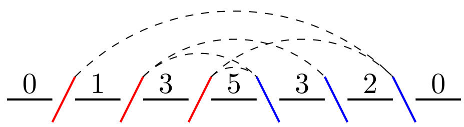
Corollary 9.45. *With the notation of Theorem 9.44, we have*

$$\Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i-1}(jh)^{c_i-j}\Big)\cdot \iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')) = \Psi_{\mathcal{D}}(\iota^*_{w_D S_{\boldsymbol{r}}}(\mathrm{Stab}_{\mathfrak{C}_-}(w' S_{\boldsymbol{r}}))),$$

where the stable basis element on the right hand side is on $T^*F(R_1,\ldots,R_{M-1};n)$, $w' \in S_{\boldsymbol{c}} \tilde{w}_{D'} S_{\boldsymbol{r}}$ and w_D is defined as in (9.24).

Example 9.46. Let D and D' be the following tie diagrams:

$D =$ 

$D' =$ 

In the following, we compute the equivariant multiplicity $\iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))$. Note that $n = 5$, $\mathbf{r} = (2,2,1)$ and $\mathbf{c} = (2,1,2)$. Let w, $w' \in S_5$ be as in Example 9.16. Then, we have $Z(w) = M(D)$ and $Z(w') = M(D')$. Hence, we know from Example 9.16 that $\tilde{w}_D = 14253$

and $\tilde{w}_{D'} = 14235$. Thus, $w = (s \times \mathrm{id} \times s)\tilde{w}_D$, where $s = 21 \in S_2$. This gives $w = w_D$. Therefore, by Corollary 9.45, we have

$$h^2 \cdot \iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')) = \Psi_{\mathcal{D}}(\iota_{wS_{\mathbf{r}}}^*(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\mathbf{r}}))), \tag{9.25}$$

where

$$\Psi_{\mathcal{D}} : \mathbb{Q}[t_1, t_2, t_3, t_4, t_5, h] \longrightarrow \mathbb{Q}[t_1, t_2, t_3, h]$$

is the $\mathbb{Q}[h]$-algebra homomorphism given by $t_1 \mapsto t_1$, $t_2 \mapsto t_1 - h$, $t_3 \mapsto t_2$, $t_4 \mapsto t_3$, $t_5 \mapsto t_3 - h$. From (9.11), we know

$$\iota_{wS_{\mathbf{r}}}^*(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\mathbf{r}})) = h(t_1 - t_3 + h)(t_2 - t_3 + h)(t_2 - t_4 + h)(t_4 - t_5)(t_1 - t_2)(t_3 - t_5)(t_1 - t_5).$$

Thus, we have

$$\Psi_{\mathcal{D}}(\iota_{wS_{\mathbf{r}}}^*(\mathrm{Stab}_{\mathfrak{C}_-}(w'S_{\mathbf{r}}))) = h^3(t_1 - t_2 + h)(t_1 - t_2)(t_1 - t_2 + h)(t_2 - t_3)(t_1 - t_3). \tag{9.26}$$

Finally, inserting (9.26) in (9.25) yields

$$\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')) = h(t_1 - t_2 + h)(t_1 - t_2)(t_1 - t_2 + h)(t_2 - t_3)(t_1 - t_3).$$

The following general divisibility statement will be useful in applications of Theorem 9.44:

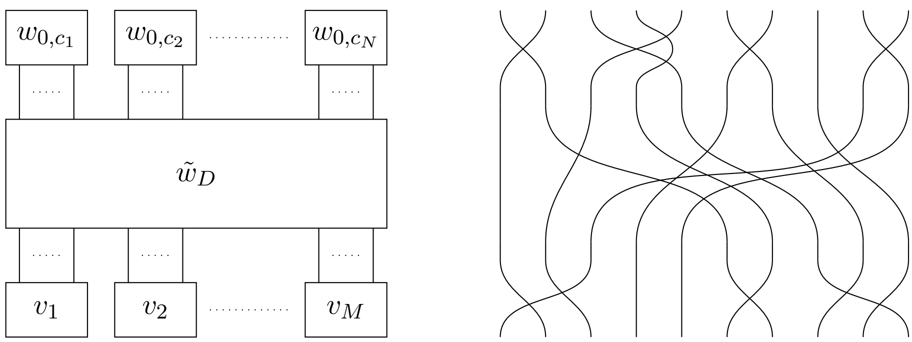
Lemma 9.47. *Let i, $j \in \{1, \dots, n\}$ with $i < j$. Then, h divides $\Psi_{\mathcal{D}}(t_i - t_j)$ if and only if $C_{l-1} + 1 \leq i < j \leq C_l$, for some $l = 1, \dots, N$.*

Proof. Suppose $C_{l-1} + 1 \leq i < j \leq C_l$, for some $l = 1, \dots, N$. Then, $\Psi_{\mathcal{D}}(t_i - t_j) = (j - i)h$. On the other hand, if $C_{l_0 - 1} + 1 \leq i \leq C_{l_0}$ and $C_{l_1 - 1} + 1 \leq j \leq C_{l_1}$, for some $l_0 < l_1$. Then, $\Psi_{\mathcal{D}}(t_i - t_j) \equiv t_{l_0} - t_{l_1} \mod h$. □

9.7 Approximations of equivariant multiplicities

Next, we combine the diagrammatic localization formula from Proposition 9.13 and Corollary 9.45 to approximate equivariant multiplicities of stable basis elements modulo powers of h.

For this, we like to choose the reduced diagrams for permutations of a particular form: Let $w = u_0 \tilde{w}_D (v_1 \times \cdots \times v_M)$, where, as in (9.23), $u_0 = w_{0,c_1} \times \ldots \times w_{0,c_N}$ and each $v_j \in S_{r_j}$ is an arbitrary element. By Corollary 9.38, we can choose a reduced diagram for w of the form:



(9.27)

Here, the boxes represent reduced diagrams of the respective permutations. The example on the right shows the permutation $u_0\tilde{w}_D v$, where $\tilde{w}_D$ is the shortest $(S_{\mathbf{c}}, S_{\mathbf{r}})$-double coset representative from Example 9.30 with $\mathbf{r} = (3, 2, 2, 3)$, $\mathbf{c} = (2, 3, 2, 1, 2)$ and $v \in S_{\mathbf{r}}$ is chosen as $v = v_1 \times v_2 \times v_3 \times v_4$ with $v_1 = 312$, $v_2 = 12$, $v_3 = 21$, $v_4 = 231$.

If d_w is a diagram of shape (9.27) then, according to their position in the diagram, we define the following subsets of crossings in d_w:

$$\begin{aligned} K_U(d_w) &= \{\kappa \in K(d_w) \mid \kappa \text{ belongs to some } w_{0,c_i}, \text{ for } i = 1, \dots, N\}, \\ K_W(d_w) &= \{\kappa \in K(d_w) \mid \kappa \text{ belongs to } \tilde{w}_D\}, \\ K_V(d_w) &= \{\kappa \in K(d_w) \mid \kappa \text{ belongs to some } v_i, \text{ for } i = 1, \dots, M\}. \end{aligned} \tag{9.28}$$

The next proposition shows that the weights of crossings in K_U precisely contribute the normalization factor which appears in Theorem 9.44.

Proposition 9.48. *The normalization factor from Theorem 9.44 can be expressed via weights of crossings as follows:*

$$\prod_{i=1}^{N} \prod_{j=1}^{c_i - 1} (jh)^{c_i - j} = \Psi_{\mathcal{D}}\Big(\prod_{\kappa \in K_U(d_w)} \mathrm{wt}(\kappa) \Big).$$

The proof is immediate from the following lemma:

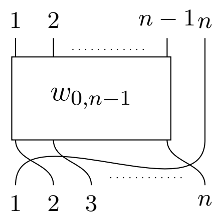
Lemma 9.49. *Let $w_{0,n} \in S_n$ be the longest element and $d_{w_{0,n}}$ a reduced diagram of $w_{0,n}$. Then, we have*

$$\Psi\Big(\prod_{\kappa \in K(d_{w_{0,n}})} \mathrm{wt}(\kappa) \Big) = \prod_{j=1}^{n-1} (jh)^{n-j},$$

where $\Psi : \mathbb{Q}[t_1, \dots, t_n, h] \to \mathbb{Q}[t, h]$ is the $\mathbb{Q}[h]$-algebra homomorphism given by

$$t_i \mapsto t - (i-1)h, \quad i = 1, \dots, n.$$

Proof. Note that by (9.4), the product $\prod_{\kappa \in K(d_{w_{0,n}})} \mathrm{wt}(\kappa)$ does not depend on the choice of reduced diagram. We prove the statement by induction on n. The case $n = 1$ is trivial. For $n > 1$, we choose $d_{w_{0,n}}$ to be of the following shape:



Here, the box represents a reduced diagram for $w_{0,n-1}$. Let K' be the set of crossings contained in the box of $w_{0,n-1}$ and K'' be the set of crossings outside of the box of $w_{0,n-1}$. From the diagram $d_{w_{0,n}}$, we can read off that the crossings in K'' have weights $t_1 - t_n, \dots, t_{n-1} - t_n$.

Thus, we have $\Psi(\prod_{\kappa\in K''}\mathrm{wt}(\kappa))=\prod_{i=1}^{n-1}(ih)$. Applying the induction hypothesis to K' yields

$$\begin{aligned}\Psi\Big(\prod_{\kappa\in K(d_{w_{0,n}})}\mathrm{wt}(\kappa)\Big)&=\Psi\Big(\prod_{\kappa\in K'}\mathrm{wt}(\kappa)\Big)\cdot\Psi\Big(\prod_{\kappa\in K''}\mathrm{wt}(\kappa)\Big)\\&=\Big(\prod_{j=1}^{n-2}(jh)^{n-1-j}\Big)\cdot\Psi\Big(\prod_{i=1}^{n-1}(t_i-t_n)\Big)\\&=\prod_{j=1}^{n-1}(jh)^{n-j}\end{aligned}$$

which finishes the proof. □

Proposition 9.50 (Approximation)**.** *Under the same assumptions as in Corollary 9.45, choose for all $z\in w_DS_{\mathbf{r}}$ a reduced diagram d_z of shape* (9.27)*. Then, we have*

$$\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))\equiv\sum_{z\in w_DS_{\mathbf{r}}}\frac{(-1)^{l(w')+l(w'S_r)}\Big(\prod_{\alpha\in L'_z}\Psi_{\mathcal{D}}(\alpha+h)\Big)\cdot P_{d_z,w',m}}{\prod_{\beta\in R_{\mathbf{r}}}\Psi_{\mathcal{D}}(z.\beta)}\mod h^m,\quad(9.29)$$

where L'_z is defined as in Proposition 9.13,

$$P_{d_z,w',m}=\sum_{K'\in K(d_z,w',m-1)}h^{|K'\setminus K_U(d_z)|}f_{K'}\cdot\Big(\prod_{\substack{\kappa\in K(d_z)\\ \kappa\notin K',K_U(d_z)}}\Psi_{\mathcal{D}}(\mathrm{wt}(\kappa))\Big)$$

and

$$\begin{aligned}K(d_z,w',m-1)&=\{K'\in K_{d_z,w'}\mid|K'\setminus K_U(d_z)|\le m-1\},\\ f_{K'}&=\frac{h^{|K'\cap K_U(d_z)|}\prod_{\kappa\in K_U(d_z)\setminus K'}\Psi_{\mathcal{D}}(\mathrm{wt}(\kappa))}{\prod_{i=1}^{N}\prod_{j=1}^{c_i-1}(jh)^{c_i-j}}.\end{aligned}$$

Remark. By Lemma 9.47 and Lemma 9.49, the factor $f_{K'}$ is always contained in $\mathbb{Q}$. Moreover, note that for all $t_i-t_j\in R_{\mathbf{r}}$ and $z\in S_{\mathbf{c}}\tilde{w}_DS_{\mathbf{r}}$, we have $F_z(i)\neq F_z(j)$. Here, F_z is defined as in (9.20). This implies that $\Psi_{\mathcal{D}}(\alpha)$ is of the form $t_{i_1}-t_{i_2}+mh$, where $1\le i_1<i_2\le N$ and $m\in\mathbb{Z}$.

Proof of Proposition 9.50. For $z\in w_DS_{\mathbf{r}}$ with reduced diagram d_z of shape (9.27), define $m_0(z):=|K(d_z)\setminus K_U(d_z)|$. By Corollary 9.45, we have

$$\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))=\sum_{z\in w_DS_{\mathbf{r}}}\frac{(-1)^{l(z)+l(w'S_{\mathbf{r}})}\Big(\prod_{\alpha\in L'_z}\Psi_{\mathcal{D}}(\alpha+h)\Big)\cdot P_{d_z,w',m_0(z)}}{\prod_{\beta\in R_{\mathbf{r}}}\Psi_{\mathcal{D}}(z.\beta)}.$$

If $K'\in K_{d_z,w'}\setminus K(d_z,w',m-1)$ then, by Proposition 9.48, the polynomial

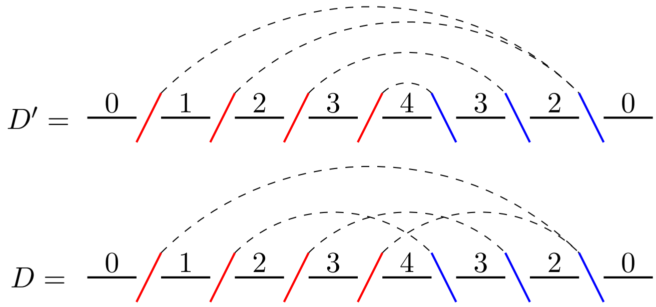
$$h^{|K'|}\prod_{\kappa\in K_U(d_z)\setminus K'}\Psi_{\mathcal{D}}(\mathrm{wt}(\kappa))$$

is divisible by $h^{\frac{1}{2}(c_1(c_1-1)+\cdots+c_N(c_N-1))+m}$. Thus, we have

$$P_{d_z,w',m_0(z)}\equiv P_{d_z,w',m}\mod h^m,\quad\text{for all }z\in w_DS_{\mathbf{r}}.$$

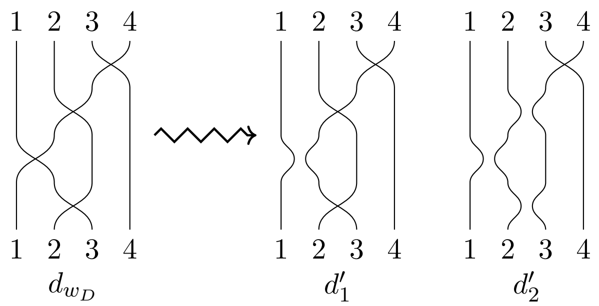
This proves the proposition. □

Example 9.51. Let D and $D' \in \mathrm{Tie}(\mathcal{D})$ be the following tie diagrams:



We now use Proposition 9.50 to determine the equivariant multiplicity $\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))$ modulo h^2. Following the stepwise construction from Definition 9.31, we deduce that $\tilde{w}_{D'} = \mathrm{id}$ and $\tilde{w}_D = 3214$. Since $\mathbf{c}(\mathcal{D}) = (1,1,2)$, we conclude $w_{D'} = 1243$ and $w_D = 4213$. The next picture contains our choice of reduced diagram d_{w_D} of shape (9.27) for w_D as well as the only two possibilities to obtain a diagram for $w_{D'}$ from d_{w_D} by resolving crossings:

Diagram for w_D: Diagrams for $w_{D'}$:



Note that the weights of the crossings of d_{w_D} are

$$\mathrm{wt}(\kappa_1) = t_3 - t_4, \quad \mathrm{wt}(\kappa_2) = t_2 - t_4, \quad \mathrm{wt}(\kappa_3) = t_1 - t_4, \quad \mathrm{wt}(\kappa_4) = t_1 - t_2.$$

Hence, $L'_{w_D} = \{(t_2 - t_3), (t_1 - t_3)\}$. The diagram d_1' is obtained from d_{w_D} by resolving the crossing κ_3 which does not belong to $K_U(d_{w_D}) = \{\kappa_1\}$. Likewise, the diagram d_2' is obtained from d_{w_D} by resolving three crossings which are all not contained in $K_U(d_{w_D})$. Hence, $K(d_{w_D}, w_{D'}, 1) = \{K'\}$, where $K' = \{\kappa_3\}$. Thus, as $\mathbf{r}(\mathcal{D}) = (1,1,1,1)$, Proposition 9.50 yields

$$\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')) \equiv \Psi_{\mathcal{D}}((t_2 - t_3 + h)(t_1 - t_3 + h)) \cdot h \cdot \Psi_{\mathcal{D}}(\mathrm{wt}(\kappa_2)\mathrm{wt}(\kappa_4)) \mod h^2.$$

Since the $\mathbb{Q}[h]$-algebra homomorphism $\Psi_{\mathcal{D}} : \mathbb{Q}[t_1, t_2, t_3, t_4, h] \to \mathbb{Q}[t_1, t_2, t_3, h]$ is given as

$$t_1 \mapsto t_1, \quad t_2 \mapsto t_2, \quad t_3 \mapsto t_3, \quad t_4 \mapsto t_3 - h,$$

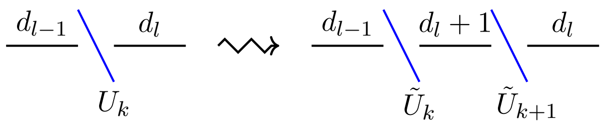
we conclude that $\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))$ is congruent to $h(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)^2$ modulo h^2.

9.8 Proofs of main theorems

We now prove Theorem 9.20, Theorem 9.44 and Theorem 9.42. For this, we recall two central results of [BR23].

Botta–Rimányi Embedding Theorem

Fix a brane diagram $\mathcal{D}$ and suppose there exists $k \in \{1, \dots, N\}$ such that $c_k > 1$. Let $\tilde{\mathcal{D}}$ be the brane diagram obtained from $\mathcal{D}$ by performing the local move



For instance, let $\mathcal{D} = 0/1/3/4/6/7\backslash5\backslash2\backslash0$. Note that we have $\mathbf{c}(\mathcal{D}) = (2, 3, 2)$. If we choose $k = 2$ then $\tilde{\mathcal{D}}$ is obtained from $\mathcal{D}$ by replacing the local configuration $5\backslash2$ with $5\backslash3\backslash2$, i.e. we have $\tilde{\mathcal{D}} = 0/1/3/4/6/7\backslash5\backslash3\backslash2\backslash0$.

As before, we denote the red, blue resp. black lines of $\mathcal{D}$ by V, U resp. X. Likewise, the red, blue resp. black lines of $\tilde{\mathcal{D}}$ are denoted by $\tilde{V}$, $\tilde{U}$ resp. $\tilde{X}$. We denote the respective margin coefficients of $\mathcal{D}$ and $\tilde{\mathcal{D}}$ by c_i, r_j and $\tilde{c}_i$, $\tilde{r}_j$. Let $\tilde{\mathbb{T}} = \tilde{\mathbb{A}} \times \mathbb{C}_h^*$ be the torus from (2.47) acting on $\mathcal{C}(\tilde{\mathcal{D}})$.

The following embedding theorem was proved in [BR23, Proposition 6.3]:

Theorem 9.52 (Botta–Rimányi Embedding Theorem)**.** *There exists a closed immersion $\iota : \mathcal{C}(\mathcal{D}) \hookrightarrow \mathcal{C}(\tilde{\mathcal{D}})$ which is equivariant with respect to*

$$\varphi : \mathbb{T} \longhookrightarrow \tilde{\mathbb{T}}, \quad (t_1, \dots, t_N, h) \mapsto (t_1, \dots, t_{k-1}, h^{-1}t_k, t_k, t_{k+1}, \dots, t_N, h).$$

Moreover, we have isomorphisms of $\mathbb{T}$-equivariant vector bundles $\iota^*\xi_{\tilde{X}} \cong \xi_{\tilde{\pi}_{\mathrm{h}}(\tilde{X})}$, for $\tilde{X} \neq \tilde{U}_k^+$. Here,

$$\tilde{\pi}_{\mathrm{h}} : \mathrm{h}(\tilde{\mathcal{D}}) \longrightarrow \mathrm{h}(\mathcal{D}), \quad \tilde{\pi}_{\mathrm{h}}(\tilde{X}_i) = \begin{cases} X_i & \text{if } \tilde{X}_i \triangleleft \tilde{U}_k, \\ X_{i-1} & \text{if } \tilde{X}_i \triangleright \tilde{U}_k. \end{cases}$$

By applying Corollary 2.48, we deduce a formula for the $\mathbb{T}$-equivariant K-theory class of the normal bundle N_ι of ι in terms of constant bundles:

Corollary 9.53. *In $K_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$, it holds*

$$[N_\iota] = (1 + h) + \sum_{j=k+1}^{N} \Big(\sum_{l=0}^{c_j-1} h^l [\mathbb{C}_{U_k} \otimes \mathbb{C}_{U_j}^\vee] + h^{1-l}([\mathbb{C}_{U_k}^\vee \otimes \mathbb{C}_{U_j}]) \Big).$$

Proof. Set $E_1 := \iota^*\xi_{\tilde{U}_k^-}$, $E_2 := \iota^*\xi_{\tilde{U}_k^+}$, $E_3 := \iota^*\xi_{\tilde{U}_{k+1}^+}$. By Corollary 2.48 and Theorem 9.52, we have

$$[N_\iota] = [\iota^* T\mathcal{C}(\tilde{\mathcal{D}})] - [T\mathcal{C}(\mathcal{D})] = T_1 + T_2 - T_3,$$

where

$$T_1 = (1 - h)[\mathrm{Hom}(E_2, E_1)] + [\mathrm{Hom}(h^{-1}\mathbb{C}_{U_k}, E_1)],$$

$$T_2 = (1 - h)[\mathrm{Hom}(E_3, E_2)] + (h - 1)[\mathrm{End}(E_2)] + h[\mathrm{Hom}(E_2, h^{-1}\mathbb{C}_{U_k})] + [\mathrm{Hom}(\mathbb{C}_{U_k}, E_2)],$$

$$T_3 = (1 - h)[\mathrm{Hom}(E_3, E_1)] + [\mathrm{Hom}(\mathbb{C}_{U_k}), E_1)].$$

Note that by Theorem 9.52, $E_1 \cong \xi_{U_k^-}$ and $E_3 \cong \xi_{U_k^+}$. By Proposition 8.22, we have

$$E_1 \cong \bigoplus_{j=k}^{N} \bigoplus_{l=0}^{c_j-1} h^l \mathbb{C}_{U_j}, \quad E_2 \cong E_3 \oplus \mathbb{C}_{U_k}, \quad E_3 \cong \bigoplus_{j=k+1}^{N} \bigoplus_{l=0}^{c_j-1} h^l \mathbb{C}_{U_j}. \tag{9.30}$$

From (9.30), we deduce $[\mathrm{Hom}(E_2, E_1)] = [\mathrm{Hom}(E_3, E_1)] + [\mathrm{Hom}(\mathbb{C}_{U_k}, E_1)]$ which gives $T_1 = T_3$. Hence, $[N_\iota] = T_2$. By (9.30), we have $[\mathrm{End}(E_2)] = [\mathrm{Hom}(E_3, E_2)] + [\mathrm{Hom}(\mathbb{C}_{U_k}, E_2)]$. Thus,

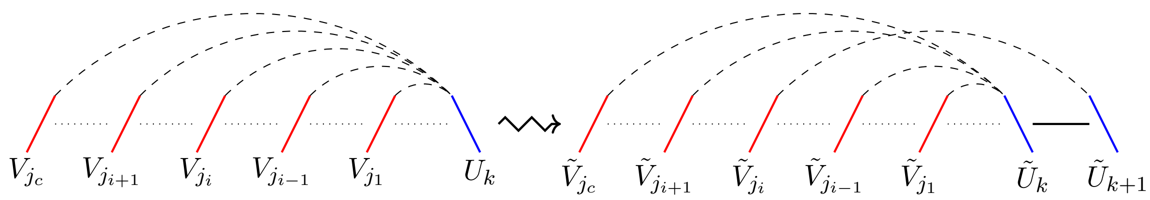
$$\begin{aligned} T_2 &= (h-1)[\mathrm{Hom}(\mathbb{C}_{U_k}, E_3)] + h[\mathrm{Hom}(E_2, h^{-1}\mathbb{C}_{U_k})] + [\mathrm{Hom}(\mathbb{C}_{U_k}, E_2)] \\ &= h[\mathrm{Hom}(\mathbb{C}_{U_k}, E_2)] + [\mathrm{Hom}(\mathbb{C}_{U_k}, E_2)] \\ &= h[\mathrm{Hom}(\mathbb{C}_{U_k}, E_3 \oplus \mathbb{C}_{U_k})] + [\mathrm{Hom}(\mathbb{C}_{U_k}, E_3 \oplus \mathbb{C}_{U_k})] \\ &= (1+h) + h[\mathrm{Hom}(\mathbb{C}_{U_k}, E_3)] + [\mathrm{Hom}(E_3, \mathbb{C}_{U_k})]. \end{aligned}$$

Inserting (9.30) then proves the corollary. □

The D5 Resolution Theorem

We now recall the D5 Resolution Theorem from [BR23, Theorem 6.13]. It states that it is in fact possible to determine the equivariant multiplicities of stable basis elements of $\mathcal{C}(\mathcal{D})$ via equivariant mutliplicities of the stable basis elements of $\mathcal{C}(\tilde{\mathcal{D}})$.

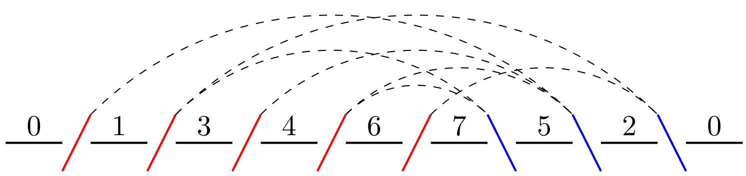
We first consider some combinatorial relations between the tie diagrams of $\mathcal{D}$ and $\tilde{\mathcal{D}}$. Let $c := c_k$. For $D \in \mathrm{Tie}(\mathcal{D})$, let $V_{j_1}, \ldots, V_{j_c}$ with $1 \leq j_1 < \ldots < j_c \leq M$ be the red lines in $\mathcal{D}$ that are connected to U_k with a tie. For $i = 1, \ldots, c$, we obtain a tie diagram $\mathrm{f}_i(D) \in \mathrm{Tie}(\tilde{\mathcal{D}})$ by performing the following local move in D:



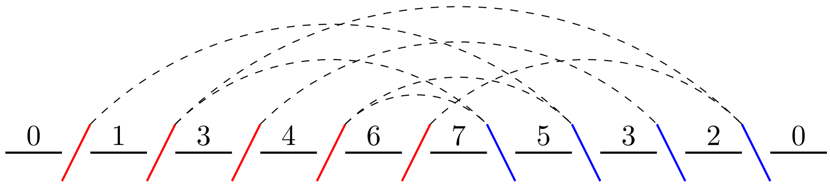
That is, $\tilde{U}_{k+1}$ is connected to $\tilde{V}_{j_i}$, whereas $\tilde{U}_k$ is connected to all $\tilde{V}_{j_l}$ except $\tilde{V}_{j_i}$. In this way, we obtain an inclusion

$$\mathrm{f}_i \colon \mathrm{Tie}(\mathcal{D}) \hookrightarrow \mathrm{Tie}(\tilde{\mathcal{D}}), \quad D \mapsto \mathrm{f}_i(D). \tag{9.31}$$

Example 9.54. Let $\mathcal{D} = 0/1/3/4/6/7\backslash 5\backslash 2\backslash 0$ and $\tilde{\mathcal{D}} = 0/1/3/4/6/7\backslash 5\backslash 3\backslash 2\backslash 0$. We choose $i = 2$ and the tie diagram $D \in \mathrm{Tie}(\mathcal{D})$ as follows:



Since U_2 is connected to V_2, V_3, V_5, we obtain $\mathrm{f}_2(D) \in \mathrm{Tie}(\tilde{\mathcal{D}})$ from D by first deleting the tie between V_3 and U_2. Then, we replace the black line in the blue part labeled by 3 with the local configuration $3\backslash 2$. Finally, we draw a tie between V_2 and the new blue line. Hence, $\mathrm{f}_2(D)$ equals



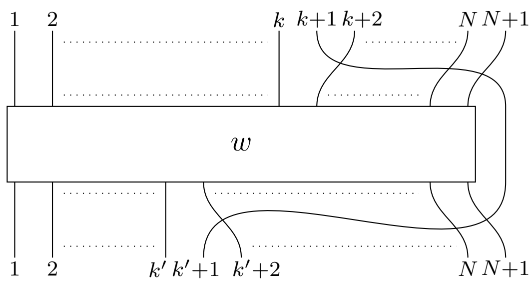
For simplicity, we denote the dominant chambers of $\mathbb{A}$ and $\tilde{\mathbb{A}}$ both by $\mathfrak{C}_+$. Likewise, we denote the antidominant chamber of $\mathbb{A}$ and $\tilde{\mathbb{A}}$ both by $\mathfrak{C}_-$. Next, we define a map

$$\{\text{Chambers of } \mathbb{A}\} \longrightarrow \{\text{Chambers of } \tilde{\mathbb{A}}\}, \quad \mathfrak{C} \mapsto \tilde{\mathfrak{C}} \tag{9.32}$$

as follows: Let $\mathfrak{C}$ be a chamber of $\mathbb{A}$ and $w \in S_N$ such that $\mathfrak{C} = w.\mathfrak{C}_+$. Then, the chamber $\tilde{\mathfrak{C}}$ of $\tilde{\mathbb{A}}$ is defined as $\tilde{\mathfrak{C}} := \tilde{w}.\mathfrak{C}_+$, where $\tilde{w} \in S_{N+1}$ is defined as $\tilde{w} := \tilde{w}_2(w \times \mathrm{id})\tilde{w}_1$, where

$$\tilde{w}_1(i) = \begin{cases} i & \text{if } i \leq k', \\ N+1 & \text{if } i = k'+1, \\ i-1 & \text{if } i > k'+1, \end{cases} \qquad \tilde{w}_2 = (i) = \begin{cases} i & \text{if } i \leq k, \\ i+1 & \text{if } i > k \text{ and } i \leq N, \\ k+1 & \text{if } i = N+1. \end{cases}$$

Here, $k' := w^{-1}(k)$. Diagrammatically, $\tilde{w}$ is obtained from w as follows:



Example 9.55. Let $w = 31425$ and $k = 2$. Then, $\tilde{w}$ is obtained from w by first increasing all entries which are larger than 2 by 1 and then replacing the entry 2 by the two entries 23. Hence, $\tilde{w} = 415236$.

Theorem 9.56 (D5 Resolution Theorem)**.** *For all D, $D' \in \mathrm{Tie}(\mathcal{D})$, $i \in \{1, \ldots, c\}$ and all choices of chambers $\mathfrak{C}$ of $\mathbb{A}$, we have*

$$\gamma_i \cdot e_{\mathbb{T}}(N^-_{\iota,\mathfrak{C}}) \cdot \iota_D^*(\mathrm{Stab}_{\mathfrak{C}}(D')) = \varphi^*(\iota^*_{\mathrm{f}_c(D)}(\mathrm{Stab}_{\tilde{\mathfrak{C}}}(\mathrm{f}_i(D')))), \tag{9.33}$$

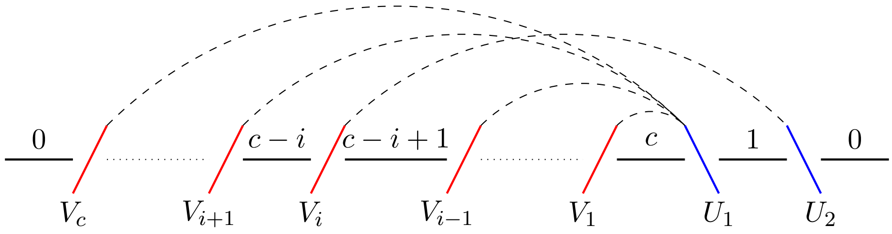
where $\tilde{\mathfrak{C}}$ is defined as in (9.32) *and the resolution coefficients $\gamma_i \in H^*_{\mathbb{T}}(\mathrm{pt})$ are defined in Definition 9.57 below.*

Remark. The D5 Resolution Theorem is proved in [BR23, Theorem 6.13] in the framework of elliptic cohomology. The same proof can also be used here in torus equivariant cohomology. The name D5 Resolution Theorem refers to the connection of the blue lines in brane diagrams to brane systems from theoretical physics.

We now come to the definition of the resolution coefficients γ_i: Let $\tilde{\mathcal{D}}_c$ be the brane diagram $0/1/2/\dots/c-1/c\backslash 1\backslash 0$ and let $\mathbb{T}' = (\mathbb{C}^*)^2 \times \mathbb{C}^*_h$ be the torus acting on $\mathcal{C}(\tilde{\mathcal{D}}_c)$. For $i = 1, \dots, c$, let $\tilde{D}_i \in \mathrm{Tie}(\tilde{\mathcal{D}}_c)$ be the tie diagram

$$\tilde{D}_i = \{(V_l, U_1) \mid l \neq i\} \cup \{(V_i, U_2)\}.$$

That is, $\tilde{D}_i$ can is illustrated as follows:



Definition 9.57. For $i \in \{1, \dots, c\}$, the resolution coefficient $\gamma_i \in H^*_{\mathbb{T}}(\mathrm{pt})$ is defined as the equivariant multiplicity

$$\gamma_i := \tilde{\varphi}^*(\iota^*_{\tilde{D}_c}(\mathrm{Stab}_{\mathfrak{C}_+}(\tilde{D}_i))),$$

where $\tilde{\varphi}\colon \mathbb{T} \to \mathbb{T}'$, $(t_1, \dots, t_N, h) \mapsto (h^{-1}t_k, t_k)$.

Equivariant multiplicities via the D5 Resolution Theorem

We now use Theorem 9.56 to connect the equivariant multiplicities of the renormalized stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\tilde{\mathcal{D}})$.

Proposition 9.58. *For all D, $D' \in \mathrm{Tie}(\mathcal{D})$ and $i \in \{1, \dots, c\}$, we have*

$$\Big(\prod_{i=1}^{c-1} ih\Big) \cdot \iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D')) = \varphi^*(\iota^*_{f_c(D)}(\widetilde{\mathrm{Stab}}_{\tilde{\mathfrak{C}}}(\mathrm{f}_i(D')))).$$

To prove Proposition 9.58, we use the following proposition which connects N_ι and the bundles $N_{\mathcal{D}}$, $N_{\tilde{\mathcal{D}}}$ from Definition 9.21.

Proposition 9.59. *We have*

$$\Big(\prod_{i=2}^{c-1} ih\Big) \cdot e_{\mathbb{T}}(N^-_{\mathcal{D},\mathfrak{C}}) = e_{\mathbb{T}}(N^-_{\iota,\mathfrak{C}}) \cdot e_{\mathbb{T}}(\iota^* N^-_{\tilde{\mathcal{D}},\tilde{\mathfrak{C}}}).$$

Proof. By Theorem 9.52, we have

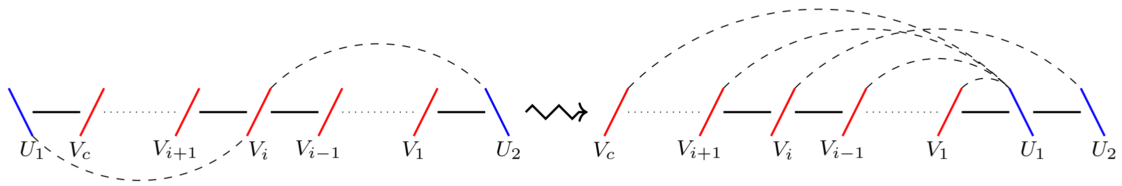
$$\begin{aligned} &\iota^*[N^-_{\tilde{\mathcal{D}},\tilde{\mathfrak{C}}}] - [N^-_{\mathcal{D},\mathfrak{C}}] \\ &\quad = \Big(\sum_{i=1}^{c-2} h^i \iota^*[\mathbb{C}^\vee_{\tilde{U}_k} \otimes \mathbb{C}_{\tilde{U}_{k+1}}]\Big) - \Big([\mathrm{Hom}(\xi_{U_k^+}, \mathbb{C}_{U_k})] + h[\mathrm{Hom}(\mathbb{C}_{U_k}, \xi_{U_k^+})]\Big)^-_{\mathfrak{C}}. \end{aligned} \tag{9.34}$$

Since ι is φ-equivariant, we have $\iota^*[\mathbb{C}^\vee_{\tilde{U}_k} \otimes \mathbb{C}_{\tilde{U}_{k+1}}] = h$ in $K_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$. Thus, we conclude

$$\sum_{i=1}^{c-2} h^i \iota^*[\mathbb{C}^\vee_{\tilde{U}_k} \otimes \mathbb{C}_{\tilde{U}_{k+1}}] = \sum_{i=2}^{c-1} h^i.$$

By Corollary 9.53, the other term in (9.34) equals $[N^-_{\iota,\mathfrak{C}}]$. Hence, taking Euler classes on both sides of (9.34) completes the proof. ☐

Next, we determine the resolution coefficients γ_i, for $i = 1, \ldots, c$. For this, note that the brane diagram $\mathcal{D}_c = 0\backslash 1/1/\ldots/1/1\backslash 0$ from (6.1) is Hanany–Witten equivalent to $\tilde{\mathcal{D}}_c$. Let $\Phi\colon \mathcal{C}(\mathcal{D}_c) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}}_c)$ be the the corresponding Hanany–Witten isomorphism. By Proposition 2.52, Φ is equivariant with respect to the automorphism $\varphi_c\colon \mathbb{T}' \xrightarrow{\sim} \mathbb{T}'$, $(t_1, t_2, h) \mapsto (t_1 h^c, t_2, h)$. Moreover, Proposition 3.18 yields $\Phi^{-1}(x_{\tilde{D}_i}) = x_{D_i}$, where $D_i \in \mathrm{Tie}(\mathcal{D})$ is as in (6.4). In pictures, D_i transforms into $\tilde{\mathcal{D}}_i$ as follows:



Proposition 9.60. *We have $\gamma_i = h$, for all $i = 1, \ldots, c$.*

Proof. Recall from Proposition 6.5 that

$$\iota_{D_c}^*(\mathrm{Stab}_{\mathfrak{C}_+}(D_i)) = \begin{cases} t_2 - t_1 + ch & \text{if } i = c, \\ h & \text{if } i \neq c. \end{cases}$$

Hence, Proposition 5.13 gives $\iota_{\tilde{\mathcal{D}}_c}^*(\mathrm{Stab}_{\mathfrak{C}_+}(\tilde{\mathcal{D}}_i)) = (\varphi_c^*)^{-1}(\iota_{D_c}^*(\mathrm{Stab}_{\mathfrak{C}_+}(D_i))) = t_2 - t_1$. Thus, we conclude $\gamma_i = \tilde{\varphi}^*(t_2 - t_1) = h$. □

Proof of Proposition 9.58. By Theorem 9.56, we have

$$\varphi^*(\iota_{\mathrm{f}_c(D)}^*\widetilde{\mathrm{Stab}}_{\tilde{\mathfrak{C}}}(\mathrm{f}_i(D'))) = \varphi^*(\gamma_i) \cdot e_{\mathbb{T}}(N_{\iota,\mathfrak{C}}^-) \cdot e_{\mathbb{T}}(\iota^* N_{\tilde{\mathcal{D}},\tilde{\mathfrak{C}}}^-) \cdot \iota_D^*(\mathrm{Stab}_{\mathfrak{C}}(D')). \tag{9.35}$$

By Proposition 6.5, $\varphi(\gamma_i) = h$ and by Proposition 9.59, we have

$$e_{\mathbb{T}}(N_{\iota,\mathfrak{C}}^-) \cdot e_{\mathbb{T}}(\iota^* N_{\tilde{\mathcal{D}},\tilde{\mathfrak{C}}}^-) = \left(\prod_{i=2}^{c-1} ih\right) \cdot e_{\mathbb{T}}(N_{\mathcal{D},\mathfrak{C}}^-).$$

Thus, we have

$$(9.35) = \left(\prod_{i=1}^{c-1} ih\right) \cdot e_{\mathbb{T}}(N_{\mathcal{D},\mathfrak{C}}^-) \cdot \iota_D^*(\mathrm{Stab}_{\mathfrak{C}}(D')) = \left(\prod_{i=1}^{c-1} ih\right) \cdot \iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D'))$$

which proves the proposition. □

Equivariant multiplicities via cotangent bundles of flag varieties

Next, we employ Proposition 9.58 to show that the equivariant multiplicities of stable basis elements of $\mathcal{C}(\mathcal{D})$ can be computed in terms of equivariant multiplicities of stable basis elements of cotangent bundles of partial flag varieties.

Define a map

$$\psi_{\mathcal{D}}\colon S_N \longrightarrow S_n, \tag{9.36}$$

where for $z \in S_N$, the permutation $\psi_{\mathcal{D}}(z)$ is defined as

$$(\psi_{\mathcal{D}}(z))\Big(\Big(\sum_{i=1}^{j-1} c_{z(i)}\Big) + l\Big) = C_{z(j)-1} + l, \quad \text{for } j = 1, \ldots, N,\ l = 1, \ldots, c_{z(j)}.$$

That is, $\psi_{\mathcal{D}}(z)$ maps $1,\dots,c_{z(1)}$ to $C_{z(1)-1}+1,\dots,C_{z(1)}$ and $c_{z(1)}+1,\dots,c_{z(1)}+c_{z(2)}$ to $C_{z(2)-1}+1,\dots,C_{z(2)}$ etc.

Example 9.61. Let $N=3$, $\mathbf{c}=(3,2,2)$ and $z=231$. Then, $\psi_{\mathcal{D}}(z)$ is obtained from z by replacing 2 with 45, 3 with 67 and 1 with 123. Hence, $\psi_{\mathcal{D}}(z)=4567123$.

In general, note that if $z=\mathrm{id}$ then also $\psi_{\mathcal{D}}(z)=\mathrm{id}$.

Proposition 9.62. *For $\mathfrak{C}=z.\mathfrak{C}_+$, D, $D'\in\mathrm{Tie}(\mathcal{D})$ and $u\in S_{\mathbf{c}}$, we have*

$$\Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i-1}(jh)^{c_i-j}\Big)\iota_D^*(\widetilde{\mathrm{Stab}_{\mathfrak{C}}}(D'))=\Psi'_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\psi_{\mathcal{D}}(z).\mathfrak{C}_+}(\mathrm{Res}_u(D')))),$$

where $\Psi'_{\mathcal{D}}$ is defined as in Theorem 9.42.

Proof. We prove the statement by induction on C_N-N. If $C_N-N=0$ then $\mathbf{c}=(1,\dots,1)$ and thus $\mathcal{D}=\mathcal{D}(R_1,\dots,R_{M-1};n)$. As $\mathcal{D}=\mathrm{Res}(\mathcal{D})$, the statement is trivial. Suppose now that $C_N-N>0$. As before, let $U_k\in\mathrm{b}(\mathcal{D})$ with $c_k(\mathcal{D})>1$. In addition, let $\tilde{\mathcal{D}}$, $\tilde{\mathfrak{C}}$, c, and φ be as in the previous subsection.

Claim 9.63. *Let $y\in S_{N+1}$ be the unique permutation such that $\tilde{\mathfrak{C}}=y.\mathfrak{C}_+$. Then, $\psi_{\mathcal{D}}(z)=\psi_{\tilde{\mathcal{D}}}(y)$.*

Proof of Claim 9.63. Let $k':=z^{-1}(k)$ and set $l_0:=\sum_{i=1}^{k'-1}c_{z(i)}$, $l_1:=\sum_{i=1}^{k'}c_{z(i)}$. Since $c_{z(j)}=\tilde{c}_{y(j)}$, for $j=1,\dots,k'-1$, we deduce

$$(\psi_{\mathcal{D}}(z))(i)=(\psi_{\tilde{\mathcal{D}}}(y))(i),\quad \text{for } i=1,\dots,l_0.$$

By construction, $c_{z(k')}=\tilde{c}_{y(k')}+1$ which yields

$$(\psi_{\mathcal{D}}(z))(i)=(\psi_{\tilde{\mathcal{D}}}(y))(i),\quad \text{for } i=l_0+1,\dots,l_1-1.$$

Moreover, $\tilde{c}_{y(k'+1)}=\tilde{c}_{k+1}=1$ gives $(\psi_{\mathcal{D}}(z))(l_1)=(\psi_{\tilde{\mathcal{D}}}(y))(l_1)$. Finally, $c_{z(j)}=\tilde{c}_{y(j+1)}$, for $j=k',\dots,N$ implies

$$(\psi_{\mathcal{D}}(z))(i)=(\psi_{\tilde{\mathcal{D}}}(y))(i),\quad \text{for } i=l_1+1,\dots,n.$$

Thus, we proved $\psi_{\mathcal{D}}(z)=\psi_{\tilde{\mathcal{D}}}(y)$. □

Write $u=u_1\times\ldots\times u_N$ and let $i_0:=u_k(c)$. Note that $\mathrm{Res}_{\mathrm{id}}(D)=\mathrm{Res}_{\mathrm{id}}(\mathrm{f}_c(D))$. Set

$$\tilde{u}:=u_1\times\ldots\times u_{k-1}\times\tilde{u}_k\times\mathrm{id}\times u_{k+1}\times\ldots\times u_N\in S_{\mathbf{c}(\tilde{\mathcal{D}})},$$

where $\tilde{u}_k\in S_{c-1}$ is defined as

$$\tilde{u}_k(j)=\begin{cases}u_k(j) & \text{if } u_k(j)<i_0,\\ u_k(j)-1 & \text{if } u_k(j)>i_0.\end{cases}$$

Then, $\mathrm{Res}_u(D)=\mathrm{Res}_{\tilde{u}}(\mathrm{f}_i(D))$. Hence, applying the induction hypothesis to $\mathcal{C}(\tilde{\mathcal{D}})$ gives

$$\Big(\prod_{l=1}^{N+1}\prod_{j=1}^{\tilde{c}_l-1}(jh)^{c_l-j}\Big)\iota^*_{\mathrm{f}_c(D)}(\widetilde{\mathrm{Stab}_{\tilde{\mathfrak{C}}}}(\mathrm{f}_i(D')))=\Psi'_{\tilde{\mathcal{D}}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\psi_{\tilde{\mathcal{D}}}(y).\mathfrak{C}_+}(\mathrm{Res}_u(D')))).\quad (9.37)$$

By Proposition 9.58,

$$\Big(\prod_{l=1}^{N}\prod_{j=1}^{c_l-1}(jh)^{c_l-j}\Big)\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D')) = \Big(\prod_{l=1}^{N+1}\prod_{j=1}^{\tilde{c}_l-1}(jh)^{\tilde{c}_l-j}\Big)\cdot\varphi^*(\iota^*_{f_c(D)}\widetilde{\mathrm{Stab}}_{\tilde{\mathfrak{C}}}(\mathrm{f}_i(D'))). \tag{9.38}$$

Since $\varphi^*\Psi'_{\tilde{\mathcal{D}}} = \Psi'_{\mathcal{D}}$, we obtain from (9.37) that

$$(9.38) = \Psi'_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\psi_{\tilde{\mathcal{D}}}(y).\mathfrak{C}_+}(\mathrm{Res}_u(D')))).$$

Finally, Claim 9.63 gives $\psi_{\tilde{\mathcal{D}}}(y) = \psi_{\mathcal{D}}(z)$ and hence completes the proof. □

Proofs of main theorems

We now come to the proofs of Theorem 9.42, Theorem 9.20 and Theorem 9.44.

Proof of Theorem 9.42. Let D, $D' \in \mathrm{Tie}(\mathcal{D})$ and $u \in S_{\mathbf{c}}$. Choosing $z = \mathrm{id}$ in Proposition 9.62 gives

$$\Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i-1}(jh)^{c_i-j}\Big)\cdot\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D')) = \Psi'_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\mathfrak{C}_+}(\mathrm{Res}_u(D'))))$$

which proves Theorem 9.42. □

Next, we prove the following special case of Theorem 9.20:

Proposition 9.64. *For all D, $D' \in \mathrm{Tie}(\mathcal{D})$ and $z \in S_N$, we have*

$$\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D')) = z^{-1}.\Big(\iota^*_{z.D}(\widetilde{\mathrm{Stab}}_{z.\mathfrak{C}_+}(z.D'))\Big).$$

We use the following property of the map $\psi_{\mathcal{D}}$ from (9.36):

Lemma 9.65. *Let $D \in \mathrm{Tie}(\mathcal{D})$ and $z \in S_N$. Then, we have*

$$\mathrm{Res}_{\mathrm{id}}(D) = \psi_{\mathcal{D}}(z).\mathrm{Res}_{\mathrm{id}}(z^{-1}.D).$$

Proof. Let $i \in \{1, \ldots, N\}$ and set $c := c_{z(i)}$. Suppose the blue line $U_{z(i)} \in \mathrm{b}(\mathcal{D})$ is connected to the red lines $V_{j_1}, \ldots, V_{j_c}$ with $j_1 < \ldots < j_c$. Let $l \in \{1, \ldots, c\}$. Then, in $\mathrm{Res}_{\mathrm{id}}(D)$, we have that $U_{C_{z(i)-1}+l}$ is just connected to V_{j_l}. Set $i_0 := \sum_{j=1}^{i-1} c_{z(j)}$. Note that in $\mathrm{Res}_{\mathrm{id}}(z^{-1}.D)$, the blue line U_{i_0+l} is just connected to V_{j_l}. As $\psi_{\mathcal{D}}(i_0 + l) = C_{z(i)-1} + l$, we deduce that also in $\psi_{\mathcal{D}}(z).\mathrm{Res}_{\mathrm{id}}(z^{-1}.D)$, the blue line $U_{C_{z(i)-1}+l}$ is just connected to V_{j_l}. Thus, we have $\mathrm{Res}_{\mathrm{id}}(D) = \psi_{\mathcal{D}}(z).\mathrm{Res}_{\mathrm{id}}(z^{-1}.D)$. □

Proof of Proposition 9.64. We set

$$P_h := \Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i(\mathcal{D})-1}(jh)^{c_i(\mathcal{D})-j}\Big) = \Big(\prod_{i=1}^{N}\prod_{j=1}^{c_i(z.\mathcal{D})-1}(jh)^{c_i(z.\mathcal{D})-j}\Big). \tag{9.39}$$

By Proposition 9.62, we have

$$\iota^*_{z.D}(\widetilde{\mathrm{Stab}}_{z.\mathfrak{C}_+}(z.D')) = P_h \cdot \Psi'_{z.\mathcal{D}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(z.D)}(\widetilde{\mathrm{Stab}}_{\psi_{z.\mathcal{D}}(z).\mathfrak{C}_+}(\mathrm{Res}_{\mathrm{id}}(z.D')))). \tag{9.40}$$

Lemma 9.65 implies

$$\mathrm{Res}_{\mathrm{id}}(z.D) = \psi_{z.\mathcal{D}}(z).\mathrm{Res}_{\mathrm{id}}(D), \quad \mathrm{Res}_{\mathrm{id}}(z.D') = \psi_{z.\mathcal{D}}(z).\mathrm{Res}_{\mathrm{id}}(D').$$

Thus, Proposition 9.17 gives

$$(9.40) = P_h \cdot \Psi'_{z.\mathcal{D}}\Big(\psi_{z.\mathcal{D}}(z).\Big(\iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\mathfrak{C}_+}(\mathrm{Res}_{\mathrm{id}}(D')))\Big)\Big). \tag{9.41}$$

Note that Lemma 9.65 yields $\Psi'_{z.\mathcal{D}}(\psi_{z.\mathcal{D}}(z).f) = z.(\Psi'_{\mathcal{D}}(f))$, for all $f \in \mathbb{Q}[t_1, \ldots, t_n, h]$. Therefore, we conclude

$$(9.41) = P_h \cdot z.\Big(\Psi'_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(D)}(\mathrm{Stab}_{\mathfrak{C}_+}(\mathrm{Res}_{\mathrm{id}}(D'))))\Big). \tag{9.42}$$

By Proposition 9.62, (9.42) equals $P_h \cdot z.(\iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D')))$ which completes the proof. □

Proof of Theorem 9.20. Let $z \in S_N$ and set $\mathfrak{C} := z.\mathfrak{C}_+$. Let D, $D' \in \mathrm{Tie}(\mathcal{D})$ and $w \in S_N$. By Proposition 9.64, we have

$$\iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D')) = z.\Big(\iota^*_{z^{-1}.D}(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(z^{-1}.D'))\Big)$$

and

$$\iota^*_{w.D}(\widetilde{\mathrm{Stab}}_{w.\mathfrak{C}}(w.D')) = wz.\Big(\iota^*_{z^{-1}.D}(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(z^{-1}.D'))\Big).$$

Thus, we deduce

$$\iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D')) = w^{-1}.\Big(\iota^*_{w.D}(\widetilde{\mathrm{Stab}}_{w.\mathfrak{C}}(w.D'))\Big)$$

which proves Theorem 9.20. □

Next, we combine Theorem 9.20 and Theorem 9.42 to prove Theorem 9.44.

Proof of Theorem 9.44. We have to show that for all D, $D' \in \mathrm{Tie}(\mathcal{D})$ and $u \in S_{\mathbf{c}}$, we have

$$P_h \cdot \iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')) = \Psi_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{u_0}(D)}(\mathrm{Stab}_{\mathfrak{C}_-}(\mathrm{Res}_u(D')))), \tag{9.43}$$

where $u_0 = (w_{0,c_1} \times \ldots \times w_{0,c_N}) \in S_{\mathbf{c}}$ and P_h is as in (9.39). By Theorem 9.20,

$$P_h \cdot \iota^*_D(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')) = P_h \cdot w_{0,N}.(\iota^*_{w_{0,N}.D}(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(w_{0,N}.D'))). \tag{9.44}$$

Theorem 9.42 then gives

$$(9.44) = w_{0,N}.\Big(\Psi'_{w_{0,N}.\mathcal{D}}(\iota^*_{\mathrm{Res}_{\mathrm{id}}(w_{0,N}.D)}(\mathrm{Stab}_{\mathfrak{C}_+}(\mathrm{Res}_{u'}(w_{0,N}.D'))))\Big), \tag{9.45}$$

where $u' := u_0 u$. Since $w_{0,n}.\mathrm{Res}_{\mathrm{id}}(w_{0,N}.D) = \mathrm{Res}_{u_0}(D)$ and $w_{0,n}.\mathrm{Res}_{u'}(w_{0,N}.D') = \mathrm{Res}_u(D')$, Proposition 9.17 yields

$$\iota^*_{w_{0,N}.\mathrm{Res}_{\mathrm{id}}(D)}(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(w_{0,N}.(\mathrm{Res}_{u'}(D'))) = w_{0,n}.\Big(\iota^*_{\mathrm{Res}_{u_0}(D)}(\mathrm{Stab}_{\mathfrak{C}_-}(\mathrm{Res}_u(D')))\Big).$$

Note that $w_{0,N}.(\Psi'_{w_{0,N}.\mathcal{D}}(w_{0,n}.f)) = \Psi_{\mathcal{D}}(f)$, for all $f \in \mathbb{Q}[t_1, \ldots, t_n, h]$. Therefore, we deduce

$$(9.45) = \Psi_{\mathcal{D}}(\iota^*_{\mathrm{Res}_{u_0}(D)}(\mathrm{Stab}_{\mathfrak{C}_-}(\mathrm{Res}_u(D')))).$$

Thus, we proved (9.43) and hence Theorem 9.20. □

Chapter 10

Chevalley–Monk fomulas for bow varieties

The classical Chevalley–Monk Formula [Mon59], [Che94] is a fundamental ingredient of Schubert calculus. This formula uniquely determines the ring structure of the singular cohomology of partial flag varieties by expressing products of first Chern classes of tautological bundles with Schubert classes as $\mathbb{Z}$-linear combination of Schubert classes. The coefficients appearing hereby admit a convenient description in terms of symmetric group calculus.

Passing from a partial flag variety $F = F(d_1, \ldots, d_M; n)$ to its cotangent bundle T^*F, the Chevalley–Monk Formula was generalized in [MO19, Theorem 10.1.1], see also [Su16, Theorem 3.1]. The formula here determines the stable basis expansion of products of torus equivariant first Chern classes of tautological bundles with stable basis elements. Since the stable basis elements of T^*F are one-parameter deformations of the (torus equivariant) Schubert classes of F, this formula degenerates to the classical Chevalley–Monk formula, see e.g. [AMSS23, Section 9.3] and therefore can be viewed as *Chevalley–Monk formula for cotangent bundles of partial flag varieties.*

In this chapter, we generalize this formula away from the classical context of flag varieties to the more general setup of bow varieties. The main result is Theorem 10.26, where we prove a new formula which determines the stable basis expansion of the products $c_1(\xi) \cdot \mathrm{Stab}_{\mathfrak{C}}(p)$, where ξ is a tautological bundle on a bow variety $\mathcal{C}(\mathcal{D})$. The appearing coefficients in this basis expansion are characterized by certain swap moves on tie diagrams to which we refer as *simple moves*, see Definition 10.3. In the special case, where the bow variety equals the cotangent bundle of a partial flag variety, Theorem 10.26 specializes to the formula from [MO19, Theorem 10.1.1]. Hence, we refer to the formula from Theorem 10.26 as *Chevalley–Monk formula for bow varieties.*

As we will show in Proposition 10.1, the localized $\mathbb{T}$-equivariant cohomology of any bow variety $\mathcal{C}(\mathcal{D})$ is generated by the $\mathbb{T}$-equivariant first Chern classes of the tautological bundles. As a consequence, Theorem 10.26 uniquely determines the ring structure of $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$.

To prove this Theorem 10.26, we employ two crucial ingredients. The first one is a divisibility result, see Theorem 10.12, which gives a sufficient criterion for equivariant multiplicities $\iota_q^*(\mathrm{Stab}_{\mathfrak{C}}(p))$ to be divisible by the parameter h^2. The second one is Theorem 10.15 which determines approximations of equivariant multiplicities $\iota_q^*(\mathrm{Stab}_{\mathfrak{C}}(p))$ in the case when they are not divisible by h^2. We prove Theorem 10.12 and Theorem 10.15 via the diagrammatic approximation formulas from Proposition 9.50.

10.1 Tautological divisors as generators

Let $\mathcal{D}$ be a fixed brane diagram. If $\mathcal{V}$ is a $\mathbb{T}$-equivariant vector bundle over $\mathcal{C}(\mathcal{D})$, we denote by $c_i(\mathcal{V})$ its i-th $\mathbb{T}$-equivaraint Chern class.

We refer to the first $\mathbb{T}$-equivariant Chern classes of the tautological bundles ξ_X on $\mathcal{C}(\mathcal{D})$ as *tautological divisors.*

Proposition 10.1. *The $H^*_{\mathbb{T}}(\mathrm{pt})_{\mathrm{loc}}$-algebra $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$ is generated by the elements $c_1(\xi_X)$, for $X \in \mathrm{h}(\mathcal{D})$.*

Proof. Set $R \coloneqq H^*_{\mathbb{T}}(\mathrm{pt})_{\mathrm{loc}}$, $A \coloneqq H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$. Also set $\lambda_{D,X} \coloneqq \iota_D^*(c_1(\xi_X)) \in R$, for $X \in \mathrm{h}(\mathcal{D})$ and $D \in \mathrm{Tie}(\mathcal{D})$. Let $R[u_X; X \in \mathrm{h}(\mathcal{D})]$ be the polynomial ring in formal variables u_X. We have to show that the R-algebra homomorphism $F\colon R[u_X; X \in \mathrm{h}(\mathcal{D})] \to A$, $u_X \mapsto c_1(\xi_X)$ is surjective. By Theorem 5.5, the inclusion $\iota\colon \mathcal{C}(\mathcal{D})^{\mathbb{T}} \hookrightarrow \mathcal{C}(\mathcal{D})$ induces an isomorphism of R-algebras $\iota^*\colon A \xrightarrow{\sim} H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D})^{\mathbb{T}})_{\mathrm{loc}} \cong \prod_{D\in\mathrm{Tie}(\mathcal{D})} H^*_{\mathbb{T}}(\{x_D\})_{\mathrm{loc}}$. Let $e_D \in A$ be the idempotent in A corresponding to the factor $H^*_{\mathbb{T}}(\{x_D\})_{\mathrm{loc}}$, i.e. e_D is the unique element in A such that

$$e_D \in \bigcap_{\substack{D'\in\mathrm{Tie}(\mathcal{D})\\ D'\neq D}} \ker(\iota_{D'}^*) \quad \text{and} \quad e_D \equiv 1 \bmod \ker(\iota_D^*).$$

Note that F is surjective if and only if all idempotents e_D are contained in the image of F. For $D \in \mathrm{Tie}(\mathcal{D})$, define ideals

$$\mathfrak{a}_D \coloneqq ((u_X - \lambda_{D,X}); X \in \mathrm{h}(\mathcal{D})) \subset R[u_X; X \in \mathrm{h}(\mathcal{D})].$$

Note that $F(\mathfrak{a}_D) \subset \ker(\iota_D^*)$. We claim that the ideals $\mathfrak{a}_D$ are pairwise coprime. Indeed, if $D \neq D'$ then there exists $U_i \in \mathrm{b}(\mathcal{D})$ and $X_j \in \mathrm{h}(\mathcal{D})$ with $d_{D,U_i,X_j} \neq d_{D',U_i,X_j}$. By (3.8),

$$\lambda_{D,X_j} = \sum_{U\in\mathrm{b}(\mathcal{D})} \sum_{l=c_{D,U,X}}^{d_{D,U,X}} (t_U + (l+1-d_{D,U,U^-})h),$$

$$\lambda_{D',X_j} = \sum_{U\in\mathrm{b}(\mathcal{D})} \sum_{l=c_{D',U,X}}^{d_{D',U,X}} (t_U + (l+1-d_{D',U,U^-})h).$$

Hence, $\lambda_{D,X_j} - \lambda_{D',X_j}$ is a unit in R. Since $\lambda_{D,X_j} - \lambda_{D',X_j} = (\lambda_{D,X_j} - u_{X_j}) + (u_{X_j} - \lambda_{D',X_j})$, we deduce $(\lambda_{D,X_j} - \lambda_{D',X_j}) \in \mathfrak{a}_D + \mathfrak{a}_{D'}$ and therefore $\mathfrak{a}_D + \mathfrak{a}_{D'} = R[u_X; X \in \mathrm{h}(\mathcal{D})]$. Thus, $\mathfrak{a}_D$ and $\mathfrak{a}_{D'}$ are coprime. Now, by the Chinese Remainder Theorem, we have that for all $D \in \mathrm{Tie}(\mathcal{D})$, there exists an element $f_D \in R[u_X; X \in \mathrm{h}(\mathcal{D})]$ such that

$$f_D \in \bigcap_{\substack{D'\in\mathrm{Tie}(\mathcal{D})\\ D'\neq D}} \mathfrak{a}_{D'} \quad \text{and} \quad f_D \equiv 1 \bmod \mathfrak{a}_D.$$

Consequently, $F(f_D) = e_D$ which proves the surjectivity of F. □

10.2 Chevalley–Monk formula in the separated case

In this section, we state and prove the Chevalley–Monk formulas for the tautological bundles associated to $\mathcal{D}$ in the case where $\mathcal{D}$ is separated. In the upcoming section, we then derive the general Chevalley–Monk formula for tautological bundles corresponding to general brane diagrams using Hanany–Witten transition.

Assumption. Until the end of Section 10.6, we assume that $\mathcal{D}$ is separated.

Recall from Proposition 8.22 that the tautological bundles $\xi_{M+1}, \dots, \xi_{M+N+1}$ are constant. Hence, we focus on characterizing the multiplication of $c_1(\xi_1), \dots, c_1(\xi_M)$.

Chevalley–Monk formula for the antidominant chamber

We first restrict our attention to the antidominant chamber $\mathfrak{C}_-$. In this case, the Chevalley–Monk formula is given as follows:

Theorem 10.2 (Chevalley–Monk formula for antidominant chamber)**.** *Let $D \in \mathrm{Tie}(\mathcal{D})$. Then, we have the following identity in $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$:*

$$c_1(\xi_i) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(D) = \iota_D^*(c_1(\xi_i)) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(D) + \sum_{D' \in \mathrm{SM}_{D,i}} \mathrm{sgn}(D, D')h \cdot \mathrm{Stab}_{\mathfrak{C}_-}(D'),$$

for $i = 1, \dots, M$. Here, the set of simple moves $\mathrm{SM}_{D,i}$ is defined in (10.1) *and the signs of simple moves $\mathrm{sgn}(D, D') \in \{\pm 1\}$ are defined in Definition 10.7.*

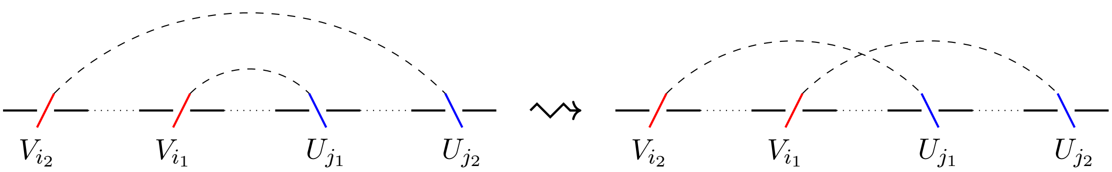
The proof of Theorem 10.2 is given in Section 10.4. We first give the definitions relevant for the theorem. We begin with the notion of simple moves and moving ties:

Definition 10.3. Let D, $D' \in \mathrm{Tie}(\mathcal{D})$. We say that D' is obtained from D via a *simple move* if there exist $1 \le i_1 < i_2 \le M$ and $1 \le j_1 < j_2 \le N$ such that $(V_{i_1}, U_{j_1}), (V_{i_2}, U_{j_2}) \in D$, $(V_{i_1}, U_{j_2}), (V_{i_2}, U_{j_1}) \in D'$ and

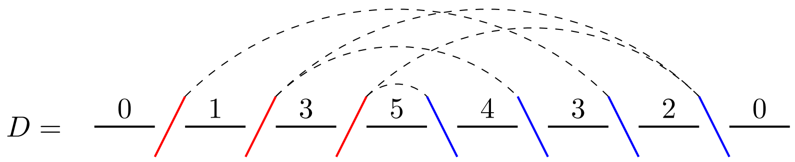
$$D \setminus \{(V_{i_1}, U_{j_2}), (V_{i_2}, U_{j_1})\} = D' \setminus \{(V_{i_1}, U_{j_1}), (V_{i_2}, U_{j_2})\}.$$

We call (V_{i_1}, U_{j_1}) the *right moving tie* and (V_{i_2}, U_{j_2}) the *left moving tie of* D. Let SM_D be the set of all tie diagrams that are obtained from D via a simple move.

Pictorially, simple moves can be described as switching two ties as illustrated:



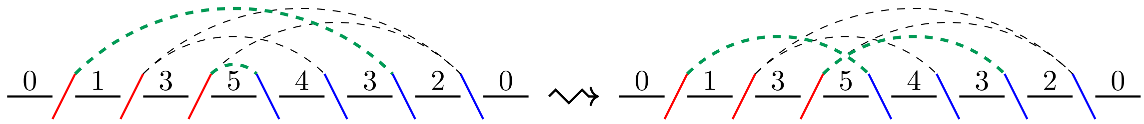
Example 10.4. Let $\mathcal{D} = 0/1/3/4/5\backslash 4\backslash 3\backslash 1\backslash 0$ and



Since (V_1, U_1), $(V_3, U_3) \in D$ and (V_3, U_1), $(V_1, U_3) \notin D$, the set

$$D' = (D \cup \{(V_3, U_1), (V_1, U_3)\}) \setminus \{(V_1, U_1), (V_3, U_3)\}$$

is a tie diagram that is obtained from D via a simple move which replaces the tie (V_2, U_1) with (V_2, U_3) and the tie (V_4, U_3) with (V_4, U_1). This simple move can be illustrated as follows:



Here, we highlighted the ties which are involved in the simple move in green.

The bijection between tie diagrams and binary contingency tables from Proposition 3.5 then gives the following equivalent characterization of simple moves:

Lemma 10.5. *Let D, $D' \in \mathrm{Tie}(\mathcal{D})$. Then, D' is obtained from D via a simple move if and only if there exist $1 \leq i_1 < i_2 \leq M$ and $1 \leq j_1 < j_2 \leq N$ such that*

(i) $M(D)_{i_1,j_1} = M(D)_{i_2,j_2} = 1$ *and* $M(D)_{i_1,j_2} = M(D)_{i_1,j_2} = 0$,

(ii) $M(D')_{i_1,j_1} = M(D')_{i_2,j_2} = 0$ *and* $M(D')_{i_1,j_2} = M(D')_{i_1,j_2} = 1$,

(iii) $M(D)_{l,k} = M(D')_{l,k}$, *for* $(l, k) \notin \{(i_1, j_1), (i_2, j_1), (i_1, j_2), (i_2, j_2)\}$.

Proof. By Proposition 3.5, the condition (i) is equivalent to

$$(V_{i_1}, U_{j_1}),\ (V_{i_2}, U_{j_2}) \in D \quad \text{and} \quad (V_{i_1}, U_{j_2}),\ (V_{i_2}, U_{j_1}) \notin D.$$

Likewise, (ii) is equivalent to

$$(V_{i_1}, U_{j_2}),\ (V_{i_2}, U_{j_1}) \in D' \quad \text{and} \quad (V_{i_1}, U_{j_1}),\ (V_{i_2}, U_{j_2}) \notin D'.$$

Finally, (iii) is equivalent to the condition

$$(V_l, U_k) \in D \quad \Leftrightarrow \quad (V_l, U_k) \in D', \quad \text{for } (l, k) \notin \{(i_1, j_1), (i_2, j_1), (i_1, j_2), (i_2, j_2)\}.$$

Thus, the conditions (i)-(iii) are satisfied if and only if D' is obtained from D via a simple move with right moving tie (V_{i_1}, U_{j_1}) and left moving tie (V_{i_2}, U_{j_2}). □

If $M(D)$, $M(D') \in \mathrm{bct}(\mathcal{D})$, we say that $M(D')$ is obtained from $M(D)$ via a *simple move* if and only if D' is obtained from D via a simple move. Equivalently, $M(D')$ is obtained from $M(D)$ via a simple move if and only if the conditions (i)-(iii) from Lemma 10.5 are satisfied.

Example 10.6. Let $\mathcal{D}$, D and D' be as in Example 10.4. The binary contingency tables of D and D' are given as

$$M(D) = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array} \qquad M(D') = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}$$

By Lemma 10.5, the simple move which turns D into D' corresponds to replacing the 0-entries $M(D)_{1,1}$, $M(D)_{3,3}$ with 1-entries and replacing the 0-entries $M(D)_{1,3}$, $M(D)_{3,1}$ with 1-entries:

$$\begin{array}{|c|c|c|c|}\hline \mathbf{1} & 0 & \mathbf{0} & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline \mathbf{0} & 0 & \mathbf{1} & 0 \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline \mathbf{0} & 0 & \mathbf{1} & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline \mathbf{1} & 0 & \mathbf{0} & 0 \\ \hline \end{array}$$

For $X_i \in \mathrm{h}(\mathcal{D})$ with $i \in \{1, \dots, M\}$, we define the *set of simple moves relative to* X_i as

$$\mathrm{SM}_{D,i} = \{D' \in \mathrm{Tie}(\mathcal{D}) \mid D' \text{ satisfies (a), (b) and (c)}\}, \tag{10.1}$$

where

(a) D' is obtained from D via a simple move,

(b) if (V_{i_1}, U_{j_1}) is the right moving tie of D then $X_i \lhd V_{i_1}$,

(c) if (V_{i_2}, U_{j_2}) is the left moving tie of D then $V_{i_2} \lhd X_i$.

For instance, if D and D' are in Example 10.4 then, as the moving ties of D are (V_1, U_1) and (V_3, U_3), D' is contained in $\mathrm{SM}_{D,i}$ for $i = 1, 2, 3$.

Next, we define the sign of a simple move:

Definition 10.7. Let $D' \in \mathrm{SM}_D$ with left moving tie (V_{i_1}, U_{j_1}) and right moving tie (V_{i_2}, U_{j_2}). Then, we define

$$\mathrm{sgn}(D, D') := \begin{cases} 1 & \text{if } n_1 + n_2 \text{ is even,} \\ -1 & \text{if } n_1 + n_2 \text{ is odd,} \end{cases}$$

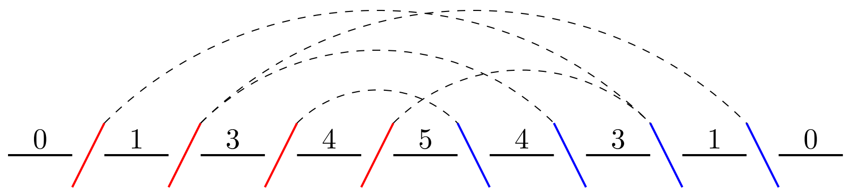
where

$$n_1 := |\{(V_{i_1}, U_j) \mid j_1 < j < j_2\}|, \quad n_2 := |\{(V_{i_2}, U_j) \mid j_1 < j < j_2\}|.$$

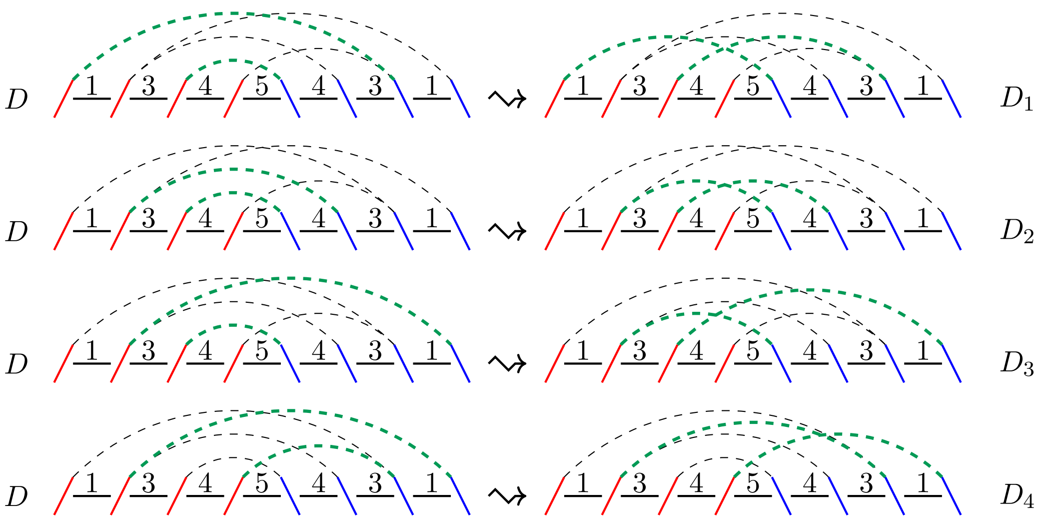
We call $\mathrm{sgn}(D, D')$ the *sign of the simple move between* D *and* D'.

Thus, all notions appearing in Theorem 10.2 are introduced.

Example 10.8. Let $\mathcal{D} = 0/1/3/4/5\backslash 4\backslash 3\backslash 1\backslash 0$ and

$D =$ 

We now use Theorem 10.2 to determine $c_1(\xi_i) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(D)$, for $i = 3$. The next picture shows all simple moves which are contained in $\mathrm{SM}_{D,i}$.



Here, we highlighted the moving ties of the respective simple moves and we omitted the horizontal black lines on the boundary of the tie diagrams. From the picture, we deduce that $\mathrm{SM}_{D,i} = \{D_1, D_2, D_3, D_4\}$, where

$$\begin{aligned} D_1 &= (D \cup \{(V_2, U_3), (V_4, U_1)\}) \setminus \{(V_2, U_1), (V_4, U_3)\}, \\ D_2 &= (D \cup \{(V_2, U_2), (V_3, U_1)\}) \setminus \{(V_2, U_1), (V_3, U_2)\}, \\ D_3 &= (D \cup \{(V_2, U_4), (V_3, U_1)\}) \setminus \{(V_2, U_1), (V_3, U_4)\}, \\ D_4 &= (D \cup \{(V_1, U_4), (V_3, U_3)\}) \setminus \{(V_1, U_3), (V_3, U_4)\}. \end{aligned} \tag{10.2}$$

From the diagram one can easily read off the respective signs:

$$\mathrm{sgn}(D, D_1) = \mathrm{sgn}(D, D_2) = \mathrm{sgn}(D, D_4) = 1, \quad \mathrm{sgn}(D, D_3) = -1.$$

By (3.8), we have an isomorphism of $\mathbb{T}$-representations $\iota_D^*(\xi_i) \cong h^{-2}\mathbb{C}_{U_2} \oplus h^{-2}\mathbb{C}_{U_3} \oplus h^{-2}\mathbb{C}_{U_4}$. Thus, $\iota_D^*(c_1(\xi_i)) = t_2 + t_3 + t_4 - 6h$ and hence, Theorem 10.2 gives

$$\begin{aligned} c_1(\xi_i) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(D) = (t_2 + t_3 + t_4 - 6h)\mathrm{Stab}_{\mathfrak{C}_-}(D) + h\mathrm{Stab}_{\mathfrak{C}_-}(D_1) + h\mathrm{Stab}_{\mathfrak{C}_-}(D_2) \\ - h\mathrm{Stab}_{\mathfrak{C}_-}(D_3) + h\mathrm{Stab}_{\mathfrak{C}_-}(D_4). \end{aligned}$$

The sign

In this subsection, we give an interpretation of $\mathrm{sgn}(D, D')$ in terms of permutations assigned to the double cosets of D and D'. From this, we deduce that after appropriate normalization of the stable basis all off-diagonal entries in the Chevalley–Monk formula become equal to $-h$.

Given a tie diagram D and $D' \in \mathrm{SM}_D$ with right moving tie (V_{i_1}, U_{j_1}) and left moving tie (V_{i_2}, U_{j_2}). Let $\tilde{w}_{D'} = \tilde{w}_{M(D')} \in S_n$ be the shortest $(S_{\mathbf{c}}, S_{\mathbf{c}})$-double coset representative from Definition 9.31. Since $(V_{i_1}, U_{j_2}) \in D'$, there exist a unique $f_1 \in \{R_{i_1-1} + 1, \dots, R_{i_1}\}$ such that $\tilde{w}_{D'}(f_1) \in \{C_{j_2-1} + 1, \dots, C_{j_2}\}$. Likewise, as $(V_{i_2}, U_{j_1}) \in D'$, there exists a unique $f_2 \in \{R_{i_2-1} + 1, \dots, R_{i_2}\}$ with $\tilde{w}_{D'}(f_2) \in \{C_{j_1-1} + 1, \dots, C_{j_1}\}$. We set

$$\tilde{y}_D \coloneqq \tilde{w}_{D'} \circ (f_1, f_2). \tag{10.3}$$

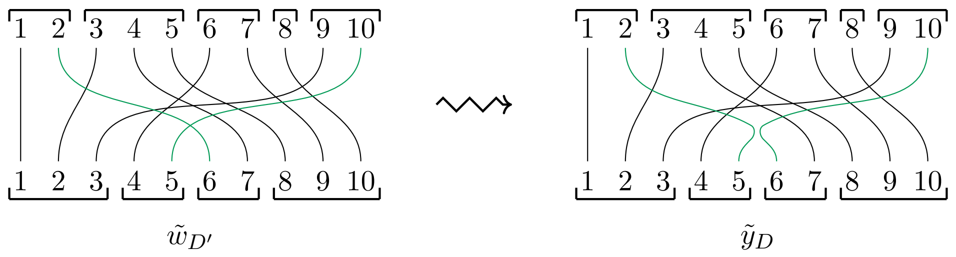
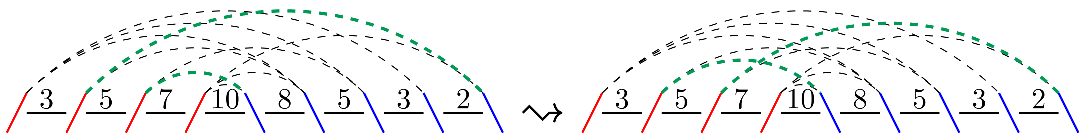


Figure 10.1: Construction of $\tilde{y}_D$ from Example 10.9. We highlighted the strands and the crossing whose resolution gives $\tilde{y}_D$.

Since D' is obtained from D via a simple move with moving ties (V_{i_1}, U_{j_1}) and (V_{i_2}, U_{j_2}), we conclude that $\tilde{y}_D \in \tilde{w}_D S_{\mathbf{r}}$.

The permutation $\tilde{y}_D$ has the following diagrammatic interpretation: Let $d_{\tilde{w}_{D'}}$ be a reduced diagram of $\tilde{w}_{D'}$. Since $(V_{i_1}, U_{j_2}) \in D'$ there exists a unique strand λ_1 in $d_{\tilde{w}_{D'}}$ starting in $\{R_{i_1-1}+1, \ldots, R_{i_1}\}$ and ending in $\{C_{j_2-1}+1, \ldots, C_{j_2}\}$. Likewise, as $(V_{i_2}, U_{j_1}) \in D'$ there is also a unique strand λ_2 in $d_{w_{D'}}$ which starts in $\{R_{i_2-1}+1, \ldots, R_{i_2}\}$ and ends in $\{C_{j_1-1}+1, \ldots, C_{j_1}\}$. As $i_1 < i_2$ and $j_1 < j_2$, the strands λ_1 and λ_2 intersect exactly once. Resolving the crossing of λ_1 and λ_2 then gives a diagram for the permutation $\tilde{y}_D$.

Example 10.9. Consider the simple move:



We denote the tie diagram on the left by D and the one on the right by D'. The moving ties are (V_2, U_1) and (V_3, U_5). Note that $n = 10$, $\mathbf{r} = (3,2,2,3)$ and $\mathbf{c} = (2,3,2,1,2)$. By construction, the binary contingency table $M(D')$ equals the matrix A from Example 9.30, where we also constructed the corresponding shortest $(S_{\mathbf{c}}, S_{\mathbf{r}})$-double coset representative $\tilde{w}_{D'} = 13961024578$. We have $\tilde{w}_{D'}(5) \in \{9,10\} = \{C_3+1, \ldots, C_4\}$ and $\tilde{w}_{D'}(6) \in \{1,2\} = \{C_0+1, \ldots, C_1\}$, so $f_1 = 5$ and $f_2 = 6$. This gives $\tilde{y}_D = 13962104578$. The diagrammatic construction of $\tilde{y}_D$ is illustrated in Figure 10.1.

Comparing the length of $\tilde{w}_D$ and $\tilde{y}_D$ gives the sign we attached to D and D':

Proposition 10.10. *We have* $(-1)^{l(\tilde{w}_D)+l(\tilde{y}_D)} = \operatorname{sgn}(D, D')$.

Proof. By construction, $\tilde{w}_{D'}(R_{i-1}+l) \in \{C_{F_{M(D'),i}(l)-1}+1, \ldots C_{F_{M(D'),i}(l)}\}$, for all i, l, where $F_{M(D'),i}$ is defined as in (9.19). This directly implies that the set

$$\{(i,j) \mid \text{there exists } l \text{ with } R_{l-1}+1 \le i < j \le R_l \text{ and } \tilde{y}_D(i) > \tilde{y}_D(j)\}$$

equals

$$\{(f_1, f_1+1), \ldots, (f_1, f_1+n_1)\} \cup \{(f_2-1, f_2), \ldots, (f_2-n_2, f_2)\}.$$

Here, n_1 and n_2 are as in Definition 10.7. Since $\tilde{w}_D$ is the shortest representative of $\tilde{y}_D S_{\mathbf{r}}$, we conclude $l(\tilde{y}_D) = l(\tilde{w}_D) + n_1 + n_2$ which proves the proposition. □

Theorem 10.2 implies now a Chevalley–Monk formula with simplified signs:

Corollary 10.11. *Let $D \in \mathrm{Tie}(\mathcal{D})$. Then, the following identity holds in $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$:*

$$c_1(\xi_i) \cdot \mathrm{Stab}'_{\mathfrak{C}_-}(D) = \iota_D^*(c_1(\xi_i)) \cdot \mathrm{Stab}'_{\mathfrak{C}_-}(D) - h \cdot \Big(\sum_{D' \in \mathrm{SM}_{D,i}} \mathrm{Stab}'_{\mathfrak{C}_-}(D') \Big),$$

for $i = 1, \dots, M$, where $\mathrm{Stab}'_{\mathfrak{C}_-}(T) = (-1)^{l(\tilde{w}_T)} \mathrm{Stab}_{\mathfrak{C}_-}(T)$ for $T \in \mathrm{Tie}(\mathcal{D})$.

Proof. By construction, $\tilde{y}_D$ is obtained from $\tilde{w}_{D'}$ by precomposition with a transposition. Hence, we have $(-1)^{l(\tilde{y}_D)} = (-1)^{l(\tilde{w}_{D'})+1}$. Thus, Proposition 10.10 yields

$$\mathrm{sgn}(D, D') = (-1)^{l(\tilde{w}_D)+l(\tilde{y}_D)} = (-1)^{l(\tilde{w}_D)+l(\tilde{w}_{D'})+1}$$

which proves the corollary. □

10.3 Divisibility and Approximation

We now consider divisibility and approximation theorems for equivariant multiplicities of stable basis elements. These results are essential ingredients of the proof of Theorem 10.2. We first formulate the theorems and deduce some consequences. The proofs are then given in Section 10.5.

Theorem 10.12 (h^2-Divisibility)**.** *The equivariant multiplicity $\iota_{D'}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))$ is divisible by h^2, for $D \in \mathrm{Tie}(\mathcal{D})$ and $D' \notin \mathrm{SM}_D \cup \{D\}$.*

By applying Theorem 9.20, we deduce the analogous result for the chamber $\mathfrak{C}_+$:

Corollary 10.13. *We have that $\iota_D^*(\mathrm{Stab}_{\mathfrak{C}_+}(D'))$ is divisible by h^2, for $D \in \mathrm{Tie}(\mathcal{D})$ and $D' \notin \mathrm{SM}_D \cup \{D\}$.*

Proof. As before, let $w_{0,N} \in S_N$ be the longest element. By Theorem 9.20, we have

$$\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D')) = w_{0,N}.\Big(\iota_{w_{0,N}.D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(w_{0,N}.D'))\Big). \tag{10.4}$$

Here, $\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D')$ and $\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(w_{0,N}.D')$ are the renormalized stable basis elements from Defintion 9.24. Since $D' \notin \mathrm{SM}_D$ if and only if $w_{0,N}.D \notin \mathrm{SM}_{w_{0,N}.D'}$, Theorem 10.12 implies that the right hand side of (10.4) is divisible by h^2. Thus, $\iota_D^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D'))$ is divisible by h^2 and hence also $\iota_D^*(\mathrm{Stab}_{\mathfrak{C}_+}(D'))$. □

Combining Theorem 10.12 and Corollary 10.13 gives the following divisibility result:

Corollary 10.14 (h^2-Divisibility of products)**.** *Let D, D', $T \in \mathrm{Tie}(\mathcal{D})$ such that $T \notin \{D, D'\}$ or $D' \notin \mathrm{SM}_D \cup \{D\}$. Then, we have*

$$\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_-}(D) \cdot \mathrm{Stab}_{\mathfrak{C}_+}(D')) \equiv 0 \mod h^2. \tag{10.5}$$

Proof. If $T \notin \{D, D'\}$ then the smallness condition implies that both $\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))$ and $\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_+}(D'))$ are divisible by h which gives that (10.5) is divisible by h^2. If $T = D$ and $D' \notin \mathrm{SM}_D \cup \{D\}$ then, by Corollary 10.13, $\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_+}(D'))$ is divisible by h^2 and so is (10.5). Likewise, if $T = D'$ and $D' \notin \mathrm{SM}_D \cup \{D\}$ then Theorem 10.12 implies that $\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))$ is divisible by h^2 and hence also (10.5). □

The next theorem determines h^2-approximations of equivariant multiplicities of stable basis elements labeled by tie diagrams which differ by a simple move:

Theorem 10.15 (h^2-Approximation)**.** *Let $D \in \mathrm{Tie}(\mathcal{D})$ and $D' \in \mathrm{SM}_D$ with (V_{i_1}, U_{j_1}) be the right moving tie and (V_{i_2}, U_{j_2}) be the left moving tie of D. Then, we have*

$$\frac{\iota_{D'}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))}{e_{\mathbb{T}}(T_{D'}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_-})} \equiv \mathrm{sgn}(D, D')\frac{h}{t_{j_1} - t_{j_2}} \mod h^2$$

in $S_0^{-1}H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$. Here, S_0 is defined as in (7.2)*.*

Using Theorem 9.20, we again deduce the analogous statement for $\mathfrak{C}_+$:

Corollary 10.16. *Let $D \in \mathrm{Tie}(\mathcal{D})$ and $D' \in \mathrm{SM}_D$ with (V_{i_1}, U_{j_1}) be the right moving tie and (V_{i_2}, U_{j_2}) be the left moving tie of D. Then, we have*

$$\frac{\iota_{D}^*(\mathrm{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_+})} \equiv \mathrm{sgn}(D, D')\frac{h}{t_{j_2} - t_{j_1}} \mod h^2$$

in $S_0^{-1}H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$.

Proof. By the definition of $\widetilde{\mathrm{Stab}}$, we have

$$\frac{\iota_{D}^*(\mathrm{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_+})} = \frac{\iota_{D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D'))}{\iota_{D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D))}, \quad \frac{\iota_{D}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D'))}{e_{\mathbb{T}}(T_{D}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_-})} = \frac{\iota_{D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))}{\iota_{D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D))}. \tag{10.6}$$

In addition, Theorem 9.20 yields

$$\frac{\iota_{D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D'))}{\iota_{D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_+}(D))} = w_{0,N}.\left(\frac{\iota_{w_{0,N}.D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(w_{0,N}.D'))}{\iota_{w_{0,N}.D}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(w_{0,N}.D))}\right). \tag{10.7}$$

The tie diagram $w_{0,N}.D$ is obtained from $w_{0,N}.D'$ via a simple move, where (V_{i_1}, U_{N-j_2+1}) is the right moving tie and (V_{i_2}, U_{N-j_1+1}) is the left moving tie of $w_{0,N}.D'$. Thus, we have

$$\begin{aligned}\frac{\iota_{D}^*(\mathrm{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_+})} &= w_{0,N}.\left(\frac{\iota_{w_{0,N}.D}^*(\mathrm{Stab}_{\mathfrak{C}_-}(w_{0,N}.D'))}{e_{\mathbb{T}}(T_{w_{0,N}.D}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_-})}\right) \\ &\equiv w_{0,N}.\left(\mathrm{sgn}(w_{0,N}.D', w_{0,N}.D)\frac{h}{t_{N-j_2+1} - t_{N-j_1+1}}\right) \mod h^2 \\ &\equiv \mathrm{sgn}(D, D')\frac{h}{t_{j_2} - t_{j_1}} \mod h^2,\end{aligned}$$

where the first equality follows from (10.6) and (10.7), the subsequent congruence from Theorem 10.15 and the final congruence from $\mathrm{sgn}(D, D') = \mathrm{sgn}(w_{0,N}.D', w_{0,N}.D)$. □

Example 10.17. Let D and D_1 be as in Example 10.8. To determine the (modulo h^2)-approximation of the fraction

$$\frac{\iota_{D_1}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))}{e_{\mathbb{T}}(T_{D_1}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_-})},$$

note that that D_1 is obtained from D via a simple move with right moving tie (V_1, U_1) and left moving tie (V_4, U_3). Moreover, we showed in Example 10.8 that $\mathrm{sgn}(D, D_1) = 1$. Thus, Theorem 10.15 yields

$$\frac{\iota_{D_1}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))}{e_{\mathbb{T}}(T_{D_1}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_-})} \equiv \frac{h}{t_1 - t_3} \mod h^2.$$

Remark. In the framework of partial flag varieties, the results of this section are contained in [Su17, Corollary 3.8].

10.4 Proof of Theorem 10.2

We begin with the following auxiliary statement:

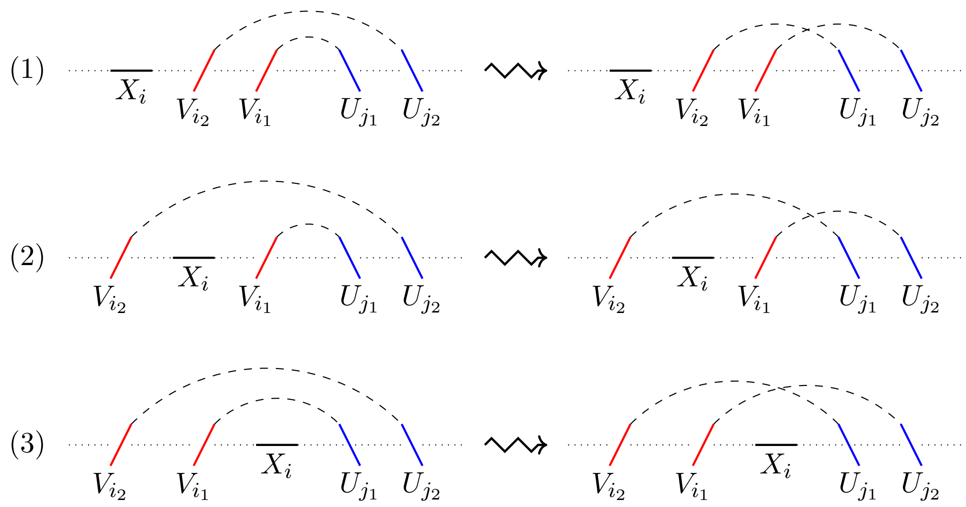
Lemma 10.18. *Let $D \in \mathrm{Tie}(\mathcal{D})$, $D' \in \mathrm{SM}_D$ with right moving tie (V_{i_1}, U_{j_1}) and left moving tie (V_{i_2}, U_{j_2}). Then, we have*

$$\iota_{D'}^*(c_1(\xi_i)) - \iota_D^*(c_1(\xi_i)) \equiv \begin{cases} t_{j_1} - t_{j_2} \bmod h & \text{if } D \in \mathrm{SM}_{D,i}, \\ 0 \bmod h & \text{if } D \notin \mathrm{SM}_{D,i}. \end{cases}$$

Proof. From (3.8), we obtain

$$\iota_T^*(c_1(\xi_i)) = \sum_{U \in \mathrm{b}(\mathcal{D})} d_{T,U,X_i} t_i \mod h, \quad \text{for all } T \in \mathrm{Tie}(\mathcal{D}), \tag{10.8}$$

where d_{T,U,X_i} is defined as in Definition 3.8. According to the relative position of X_i with respect to V_{i_1} and V_{i_2} we have the three cases illustrated as follows:



In the first and third case, we have $d_{D,U,X_i} = d_{D',U,X_i}$, for all $U \in \mathrm{b}(\mathcal{D})$. Hence, (10.8) yields $\iota_{D'}^*(c_1(\xi_i)) - \iota_D^*(c_1(\xi_i)) \equiv 0 \bmod h$, for $D' \notin \mathrm{SM}_{D,i}$. The second case is equivalent to $D' \in \mathrm{SM}_{D,i}$. In this case, we have

$$d_{D,U,X_i} = \begin{cases} d_{D',U,X_i} & \text{if } U \in \mathrm{b}(\mathcal{D}) \setminus \{U_{j_1}, U_{j_2}\}, \\ d_{D',U,X_i} - 1 & \text{if } U = U_{j_1}, \\ d_{D',U,X_i} + 1 & \text{if } U = U_{j_2}. \end{cases}$$

Thus, (10.8) gives $\iota_{D'}^*(c_1(\xi_i)) - \iota_D^*(c_1(\xi_i)) \equiv t_{j_1} - t_{j_2} \bmod h$, for $D' \in \mathrm{SM}_{D,i}$. □

Proof of Theorem 10.2. By Theorem 7.8, we have to show

$$(c_1(\xi_i) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(D), \mathrm{Stab}_{\mathfrak{C}_+}(D'))_{\mathrm{virt}} = \begin{cases} \iota_D^*(c_1(\xi_i)) & \text{if } D' = D, \\ \mathrm{sgn}(D,D')h & \text{if } D' \in \mathrm{SM}_{D,i}, \\ 0 & \text{otherwise.} \end{cases} \tag{10.9}$$

By Theorem 7.6, the virtual scalar product in (10.9) is a linear polynomial in the equivariant parameters. Hence, it suffices to prove the above equality in $H^*_{\mathbb{T}}(\mathrm{pt})/(h^2)$. Suppose first that $D = D'$. By Theorem 4.23, the partial orders $\preceq_+ = \preceq_{\mathfrak{C}_+}$ and $\preceq_- = \preceq_{\mathfrak{C}_-}$ as defined in (4.8) are opposite. Thus, the support condition for stable basis elements implies that $\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_-}(D)) = 0$, for $T \npreceq_- D$ and $\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_+}(D)) = 0$, for $T \prec_- D$. Therefore, we have

$$\begin{aligned}(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D), \mathrm{Stab}_{\mathfrak{C}_+}(D))_{\mathrm{virt}} &= \sum_{D''\in \mathrm{C}(\mathcal{D})^{\mathbb{T}}} \frac{\iota_{D''}^*(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D)\cdot \mathrm{Stab}_{\mathfrak{C}_+}(D))}{e_{\mathbb{T}}(T_{D''}\mathcal{C}(\mathcal{D}))} \\ &= \frac{\iota_{D}^*(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D)\cdot \mathrm{Stab}_{\mathfrak{C}_+}(D))}{e_{\mathbb{T}}(T_{D}\mathcal{C}(\mathcal{D}))}.\end{aligned} \tag{10.10}$$

Then, the normalization condition yields

$$(10.10) = \frac{\iota_D^*(c_1(\xi_i))\cdot e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_-})\cdot e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_+})}{e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D}))} = \iota_D^*(c_1(\xi_i)).$$

This proves (10.9) for $D = D'$. Next, we assume $D' \in \mathrm{SM}_D$ and let (V_{i_1}, U_{j_1}) be the right moving tie and (V_{i_2}, U_{j_2}) be the left moving tie of D. By Corollary 10.14, we have modulo h^2:

$$\begin{aligned}(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D), \mathrm{Stab}_{\mathfrak{C}_+}(D'))_{\mathrm{virt}} \equiv\ & \frac{\iota_{D}^*(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D)\cdot \mathrm{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D}\mathcal{C}(\mathcal{D}))} \\ &+ \frac{\iota_{D'}^*(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D)\cdot \mathrm{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D'}\mathcal{C}(\mathcal{D}))}.\end{aligned} \tag{10.11}$$

Then, Corollary 10.13 and the normalization condition imply

$$\frac{\iota_{D}^*(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D)\cdot \mathrm{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D}\mathcal{C}(\mathcal{D}))} \equiv \mathrm{sgn}(D,D')h\frac{\iota_D^*(c_1(\xi_i))}{t_{j_2}-t_{j_1}} \mod h^2, \tag{10.12}$$

whereas Theorem 10.15 combined with the normalization condition gives

$$\frac{\iota_{D'}^*(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D)\cdot \mathrm{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D'}\mathcal{C}(\mathcal{D}))} \equiv \mathrm{sgn}(D,D')h\frac{\iota_{D'}^*(c_1(\xi_i))}{t_{j_1}-t_{j_2}} \mod h^2. \tag{10.13}$$

Inserting (10.12) and (10.13) in (10.11) yields

$$(10.11) \equiv \mathrm{sgn}(D,D')h\frac{\iota_D^*(c_1(\xi_i)) - \iota_{D'}^*(c_1(\xi_i))}{t_{j_2}-t_{j_1}} \mod h^2. \tag{10.14}$$

Now, Lemma 10.18 implies

$$(10.14) \equiv \begin{cases} \mathrm{sgn}(D,D')h \mod h^2 & \text{if } D \in \mathrm{SM}_{D,i}, \\ 0 \mod h^2 & \text{if } D \notin \mathrm{SM}_{D,i}. \end{cases}$$

Thus, we proved (10.9) for $D' \in \mathrm{SM}_D$. Finally, it remains to prove (10.9) for $D' \notin \mathrm{SM}_D \cup \{D\}$. By Corollary 10.14, this assumption implies that h^2 divides all equivariant multiplicities $\iota_T^*(\mathrm{Stab}_{\mathfrak{C}_-}(D)\cdot \mathrm{Stab}_{\mathfrak{C}_+}(D'))$. Thus, we conclude

$$(c_1(\xi_i)\cdot \mathrm{Stab}_{\mathfrak{C}_-}(D), \mathrm{Stab}_{\mathfrak{C}_+}(D'))_{\mathrm{virt}} \equiv 0 \mod h^2$$

which proves (10.9) for $D' \notin \mathrm{SM}_D \cup \{D\}$. This completes the proof of Theorem 10.2. □

10.5 Proofs of Theorem 10.12 and Theorem 10.15

Recall the notation from Section 9.6 and Section 9.7.

Note that by Proposition 8.44, it suffices to show Theorem 10.12 for essential brane diagrams as defined in introduction of Chapter 8. Hence, we assume throughout this subsection that $\mathcal{D}$ is essential.

Our crucial tool is the following:

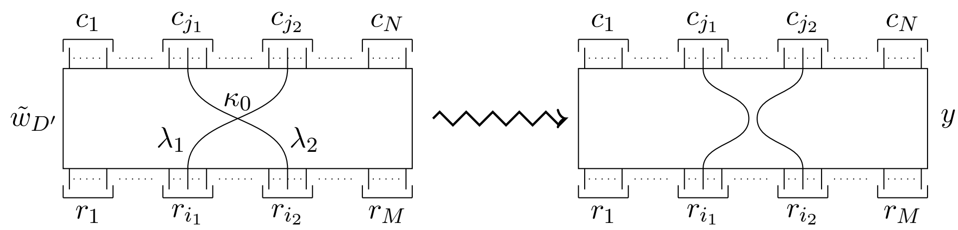
Lemma 10.19. *Let D, $D' \in \mathrm{Tie}(\mathcal{D})$, $z \in w_{D'}S_{\mathbf{r}}$ and d_z a reduced diagram of shape* (9.27)*. Suppose $K(d_z, w_D, 1) \neq \emptyset$, where $K(d_z, w_D, 1)$ is defined as in Proposition 9.50. Then, we have $D' \in \mathrm{SM}_D \cup \{D\}$.*

Proof. Given $K \in K(d_z, w_D, 1)$ then, as d_z is of shape (9.27), we distinguish between the following two cases:

(1) all crossings in K are contained in the boxes corresponding to $w_{0,c_1}, \dots, w_{0,c_N}$ and $v_1, \dots, v_M$,

(2) exactly one crossing $\kappa_0 \in K$ is contained in the box corresponding to $\tilde{w}_{D'}$ and the remaining crossings of K are contained in the boxes corresponding to $w_{0,c_1}, \dots, w_{0,c_N}$.

If (1) is satisfied then resolving all crossings contained in K from d_z still gives a permutation in $S_{\mathbf{c}} w_{D'} S_{\mathbf{r}}$. Thus, we have $S_{\mathbf{c}} w_D S_{\mathbf{r}} = S_{\mathbf{c}} w_{D'} S_{\mathbf{r}}$ which implies $D = D'$ by Corrolary 9.27. Assume now that (2) is satisfied. Let $d_{\tilde{w}_{D'}}$ be the reduced diagram for $\tilde{w}_{D'}$ contained in d_z. We denote by λ_1, λ_2 the strands in $d_{\tilde{w}_{D'}}$ that intersect in κ_0 and let $y \in S_n$ be the permutation that is obtained from $d_{\tilde{w}_{D'}}$ by resolving the crossing κ_0. In pictures, y is obtained as follows:



Let f_1, f_2 resp. g_1, g_2 the starting resp. endpoints of λ_1, λ_2 in $d_{\tilde{w}_{D'}}$. As in the above picture, we assume $f_1 < f_2$. As $\tilde{w}_{D'}$ is the shortest element in $S_{\mathbf{c}} \tilde{w}_{D'} S_{\mathbf{r}}$ there exist $i_1 < i_2$ and $j_1 < j_2$ such that

$$R_{i_1-1} < f_1 \leq R_{i_1}, \quad R_{i_2-1} < f_2 \leq R_{i_2}, \quad C_{j_1-1} < g_2 \leq C_{j_1}, \quad C_{j_2-1} < g_1 \leq C_{j_2}.$$

Thus, we conclude

$$F_y(f_1) = F_{\tilde{w}_{D'}}(f_2) = j_1, \quad F_y(f_2) = F_{\tilde{w}_{D'}}(f_1) = j_2, \quad F_y(i) = F_{\tilde{w}_{D'}}(i), \tag{10.15}$$

for $i \neq f_1, f_2$. Here, F_y, $F_{\tilde{w}_{D'}}$ are defined as in (9.20). By assumption, $y \in S_{\mathbf{c}} \tilde{w}_D$. Thus, we have $F_y = F_{\tilde{w}_D}$. Hence, by passing to the associated matrices of these double cosets, we deduce that (10.15) is equivalent to

$$M(D)_{i_1,j_1} = M(D)_{i_2,j_2} = 1, \qquad M(D')_{i_1,j_1} = M(D')_{i_2,j_2} = 0,$$
$$M(D)_{i_1,j_2} = M(D)_{i_1,j_2} = 0, \qquad M(D')_{i_1,j_2} = M(D')_{i_1,j_2} = 1,$$

as well as $M(D)_{l,k} = M(D')_{l,k}$ for $(l,k) \notin \{(i_1,j_1),(i_1,j_2),(i_2,j_1),(i_2,j_2)\}$. Therefore, by Lemma 10.5, D' is obtained from D via a simple move. □

The proof of Theorem 10.12 follows now from Proposition 9.50 and Lemma 10.19:

Proof of Theorem 10.12. We have to show that the equivariant multiplicity $\iota^*_{D'}(\mathrm{Stab}_{\mathfrak{C}_-}(D))$ is divisible by h^2, for all $D' \notin \mathrm{SM}_D \cup \{D\}$. Assume $\iota^*_{D'}(\mathrm{Stab}_{\mathfrak{C}_-}(D))$ is not divisible by h^2 for some $D' \notin \mathrm{SM}_D \cup \{D\}$. By definition, this implies that $\iota^*_{D'}(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D))$ is also not divisible by h^2. Thus, by Proposition 9.50, there exists $z \in w_{D'}S_{\mathbf{r}}$ with reduced diagram d_z of shape (9.27) such that $K(d_z, w_D, 1) \neq \emptyset$. Hence, Lemma 10.19 yields $D' \in \mathrm{SM}_D \cup \{D\}$ which contradicts our assumption $D' \notin \mathrm{SM}_D \cup \{D\}$. □

Proof of Theorem 10.15

Again, we can assume that $\mathcal{D}$ is essential.

We need some further notation: Let $D \in \mathrm{Tie}(\mathcal{D})$ and $D' \in \mathrm{SM}_D$ with right moving tie (V_{i_1}, U_{j_1}) and left moving tie (V_{i_2}, U_{j_2}). Let $z \in w_{D'}S_{\mathbf{r}}$ with a reduced diagram d_z of shape (9.27). Then, there exist unique strands λ_1, λ_2 in d_z with starting points f_1, f_2 and endpoints g_1, g_2 such that

$$R_{i_1-1} < f_1 \leq R_{i_1}, \quad R_{i_2-1} < f_2 \leq R_{i_2}, \quad C_{j_1-1} < g_2 \leq C_{j_1}, \quad C_{j_2-1} < g_1 \leq C_{j_2}.$$

Let κ_0 denote the crossing of λ_1 and λ_2.

To prove Theorem 10.15 we utilize the approximation formula of Proposition 9.50. To apply this formula appropriately, we use the following lemma:

Lemma 10.20. *Let $\tilde{y}_D \in S_n$ be as in* (10.3) *and $y_D = (w_{0,c_1} \times \ldots \times w_{0,c_N})\tilde{y}_D$. Then, we have*

$$K(d_z, y_D, 1) = \begin{cases} \{\kappa_0\} & \text{if } z = w_{D'}, \\ \emptyset & \text{if } z \neq w_{D'}, \end{cases}$$

where $K(d_z, y_D, 1)$ is defined as in Proposition 9.50.

Proof. Let $z = w_{D'}v$ where $v \in S_{\mathbf{r}}$ and suppose $K \in K(d_z, y_D, 1)$. As in (9.28), we denote by $K_U(d_z)$ the crossings in d_z corresponding to the boxes of $w_{0,c_1}, \ldots, w_{0,c_N}$. By assumption $|K \setminus K_U(d_z)| \leq 1$. Thus, as z is fully separated, we have $K \setminus K_U(d_z) = \{\kappa_0\}$. By construction, resolving the crossing κ_0 from d_z gives a diagram for $y_D v$. Hence, Theorem 9.35 implies that $v = \mathrm{id}$ and $K \cap K_U(d_z) = \emptyset$ which proves the lemma. □

By combining Proposition 9.50 and Lemma 10.20, we obtain the following consequence:

Corollary 10.21. *We have that $\iota^*_{D'}(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D))$ is congruent modulo h^2 to*

$$\frac{\mathrm{sgn}(D,D') \cdot h \cdot \left(\prod_{\alpha \in L'_{w_{D'}}} \Psi_{\mathcal{D}}(\alpha + h)\right) \cdot \left(\prod_{\kappa \in K_W(d_{w_{D'}}) \setminus \{\kappa_0\}} \Psi_{\mathcal{D}}(\mathrm{wt}(\kappa))\right)}{\prod_{\beta \in R_r} \Psi_{\mathcal{D}}(w_{D'}.\beta)}.$$

Here, we used the notation from Proposition 9.50 and $K_W(d_{w_{D'}})$ is defined as in (9.28).

Proof. If we choose $w' = y_D$ in (9.29) then, by Lemma 10.20, the only set of crossings that contributes to (9.29) is $K(d_{w_{D'}}, y_D, 1) = \{\kappa_0\}$. Thus, by Proposition 9.50, $\iota_{D'}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D))$ is congruent modulo h^2 to

$$\frac{(-1)^{l(y_D)+l(w_D)} \cdot h \cdot \left(\prod_{\alpha \in L'_{w_{D'}}} \Psi_{\mathcal{D}}(\alpha + h)\right) \cdot \left(\prod_{\kappa \in K_W(d_{w_{D'}})\setminus\{\kappa_0\}} \Psi_{\mathcal{D}}(\mathrm{wt}(\kappa))\right)}{\prod_{\beta \in R_{\mathbf{r}}} \Psi_{\mathcal{D}}(w_{D'}.\beta)}.$$

By Proposition 10.10, $(-1)^{l(w_D)+l(y_D)} = \mathrm{sgn}(D, D')$ which proves the corollary. □

Proof of Theorem 10.15. By definition, we have

$$\frac{\iota_{D'}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))}{e_{\mathbb{T}}(T_{D'}\mathcal{C}(\mathcal{D})^-_{\mathfrak{C}_-})} = \frac{\iota_{D'}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D))}{\iota_{D'}^*(\mathrm{Stab}_{\mathfrak{C}_-}(D'))} = \frac{\iota_{D'}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D))}{\iota_{D'}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))}. \tag{10.16}$$

Proposition 9.50 gives that $\iota_{D'}^*(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D'))$ is congruent modulo h to

$$\frac{\left(\prod_{\alpha \in L'_{w_{D'}}} \Psi_{\mathcal{D}}(\alpha)\right) \cdot \left(\prod_{\kappa \in K_W(d_{w_{D'}})} \Psi_{\mathcal{D}}(\mathrm{wt}(\kappa))\right)}{\prod_{\beta \in R_{\mathbf{r}}} \Psi_{\mathcal{D}}(w_{D'}.\beta)}. \tag{10.17}$$

Combining (10.17) and Corollary 10.21 then yields

$$(10.16) \equiv \frac{\mathrm{sgn}(D, D') \cdot h}{\Psi_{\mathcal{D}}(\mathrm{wt}(\kappa_0))} \equiv \frac{\mathrm{sgn}(D, D') \cdot h}{t_{j_1} - t_{j_2}} \mod h^2$$

which proves Theorem 10.15. □

10.6 Chevalley–Monk formula for arbitrary chamber

Employing Theorem 9.20, generalizes the Chevalley–Monk formula for the antidominant chamber from Theorem 10.2 to any choice of chamber:

Theorem 10.22. *Let $\mathfrak{C} = z^{-1}.\mathfrak{C}_-$ for $z \in S_N$, D be a tie diagram of $\mathcal{D}$ and $i \in \{1, \dots, M\}$. Then, the following identity holds in $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$:*

$$c_1(\xi_i) \cdot \mathrm{Stab}_{\mathfrak{C}}(D) = \iota_D^*(c_1(\xi_i)) \cdot \mathrm{Stab}_{\mathfrak{C}}(D) + \sum_{D' \in \mathrm{SM}_{D,z,i}} \mathrm{sgn}_z(D, D') \cdot h \cdot \mathrm{Stab}_{\mathfrak{C}}(D'),$$

where $\mathrm{SM}_{D,z,i} = \{D' \in \mathrm{Tie}(\mathcal{D}) \mid z.D' \in \mathrm{SM}_{z.D,i}\}$ *and* $\mathrm{sgn}_z(D, D') = \mathrm{sgn}(z.D, z.D')$.

Let $\mathrm{SM}_{D,z} = \bigcup_{i=1}^M \mathrm{SM}_{D,z,i}$. If $D' \in \mathrm{SM}_{D,z}$ then we say that D' *is obtained from D via a z-twisted simple move.*

Proof of Theorem 10.22. Note that (3.8) implies

$$\iota_T^*(\xi_{\mathcal{D},X_i}) = w^{-1}.\left(\iota_{w.T}^*(\xi_{w.\mathcal{D}}, X_i)\right), \quad \text{for all } w \in S_N \text{ and } T \in \mathrm{Tie}(\mathcal{D}). \tag{10.18}$$

Employing (10.18) and Theorem 9.20 for a given $T \in \mathrm{Tie}(\mathcal{D})$ yields

$$\iota_T^*(c_1(\xi(\mathcal{D})) \cdot \widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D)) = z^{-1}.\left(\iota_{z.T}^*(c_1(\xi_{z.\mathcal{D},X_i}) \cdot \widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(z.D))\right). \tag{10.19}$$

Then, Theorem 10.2 gives that (10.19) is equal to

$$z^{-1}.\Big(\iota^*_{z.T}\Big(\iota^*_{z.D}(c_1(\xi_{z.\mathcal{D},X_i}))\cdot(\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(z.D))+\sum_{D'\in\mathrm{SM}_{z.D,i}}\mathrm{sgn}(z.D,D')\cdot h\cdot\widetilde{\mathrm{Stab}}_{\mathfrak{C}_-}(D')\Big)\Big). \quad (10.20)$$

Applying again (10.18) and Theorem 9.20 then gives

$$(10.20)=\iota^*_T\Big(\iota^*_D(c_1(\xi_{\mathcal{D},X_i}))\cdot\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D)+\sum_{D'\in\mathrm{SM}_{D,z,i}}\mathrm{sgn}_z(D,D')\cdot h\cdot\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D')\Big).$$

Thus, Theorem 5.5 implies

$$c_1(\xi_i)\cdot\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D)=\iota^*_D(c_1(\xi_i))\cdot\mathrm{Stab}_{\mathfrak{C}}(D)+\sum_{D'\in\mathrm{SM}_{D,z,i}}\mathrm{sgn}_z(D,D')\cdot h\cdot\widetilde{\mathrm{Stab}}_{\mathfrak{C}}(D'). \quad (10.21)$$

As $\mathrm{Stab}_{\mathfrak{C}}$ and $\widetilde{\mathrm{Stab}}_{\mathfrak{C}}$ just differ by a uniform constant factor in $H^*_{\mathbb{T}}(\mathrm{pt})$, we conclude Theorem 10.22 from (10.21). □

10.7 Chevalley–Monk formulas in the general case

In the previous section, we proved the Chevalley–Monk formula for bow varieties corresponding to separated brane diagrams. Via Hanany–Witten transition, we finally deduce Chevalley–Monk formulas for bow varieties corresponding to arbitrary choices of brane diagram and chamber.

Simple moves for general brane diagrams

Fix a brane diagram $\mathcal{D}$. First, we generalize the notion of (twisted) simple moves:

Definition 10.23. For $D\in\mathrm{Tie}(\mathcal{D})$, we define the *set of simple moves* SM_D as the set of all $D'\in\mathrm{Tie}(\mathcal{D})$ such that there exist $1\le i_1<i_2\le M$ and $1\le j_1<j_2\le N$ satisfying

(i) $M(D)_{i_1,j_1}=M(D)_{i_2,j_2}=1$ and $M(D)_{i_1,j_2}=M(D)_{i_1,j_2}=0$,

(ii) $M(D')_{i_1,j_1}=M(D')_{i_2,j_2}=0$ and $M(D')_{i_1,j_2}=M(D')_{i_1,j_2}=1$,

(iii) $M(D)_{l,k}=M(D')_{l,k}$, for all $(l,k))\notin\{(i_1,j_1),(i_2,j_1),(i_1,j_2),(i_2,j_2)\}$.

If $D'\in\mathrm{SM}_D$ we say that D' *is obtained from D via a simple move.*

Given additionally $i\in\{1,\dots,M\}$, we define the *set of simple move relative to i* $\mathrm{SM}_{D,i}$ as the set of all tie diagrams D' of $\mathcal{D}$ such that there exists $1\le i_1\le M-i+1\le i_2\le M$ as well as $1\le j_1<j_2\le N$ satisfying (i)-(iii).

The graphical illustration of simple moves depends on the position of the separating line relative to the respective 2×2 submatrix where the simple move is performed. The six possible cases are recorded in Figure 10.2.

If $D'\in\mathrm{SM}_D$ then the *sign of the simple move between D and D'* is defined as

$$\mathrm{sgn}(D,D'):=(-1)^{n_1+n_2},\quad \text{where } n_1=\sum_{l=j_1+1}^{j_2-1}M(D)_{i_1,l},\ n_2=\sum_{l=j_1+1}^{j_2-1}M(D)_{i_2,l}.$$

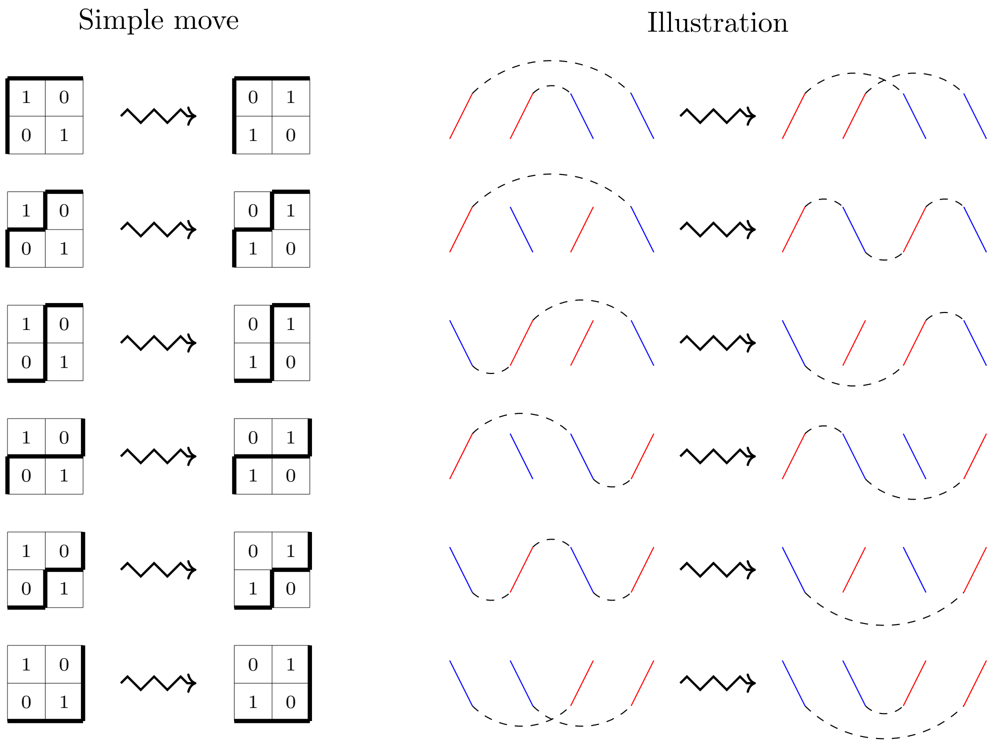


Figure 10.2: Illustration of simple moves for general brane diagrams.

The notion of twisted simple moves also generalizes as expected: for $z \in S_N$ we set

$$\mathrm{SM}_{D,z} := \{D' \in \mathrm{Tie}(\mathcal{D}) \mid z.D' \in \mathrm{SM}_{z.D}\}$$

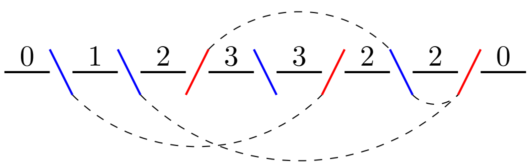
and

$$\mathrm{SM}_{D,z,i} := \{D' \in \mathrm{Tie}(\mathcal{D}) \mid z.D' \in \mathrm{SM}_{z.D,i}\}, \quad \text{for } i = 1, \dots, M.$$

If $D' \in \mathrm{SM}_{D,z}$, we say that *D' is obtained from D via a z-twisted simple move.* The corresponding *sign of the z-twisted simple move between D and D'* is defined as $\mathrm{sgn}_z(D, D') := \mathrm{sgn}(z.D, z.D')$.

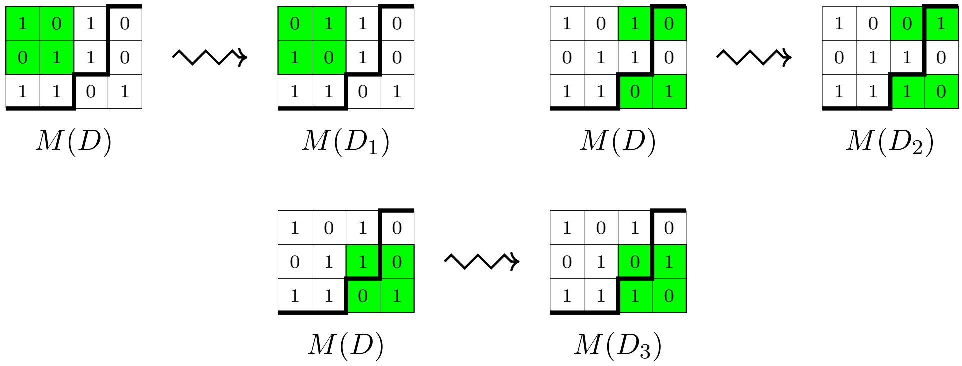
By Lemma 10.5, the definitions of (z-twisted) simple moves and the corresponding signs agree with the previous definitions for separated brane diagrams.

Example 10.24. Let $\mathcal{D} = 0\backslash 1\backslash 2/3\backslash 3/2\backslash 2/0$. As tie diagram D we choose

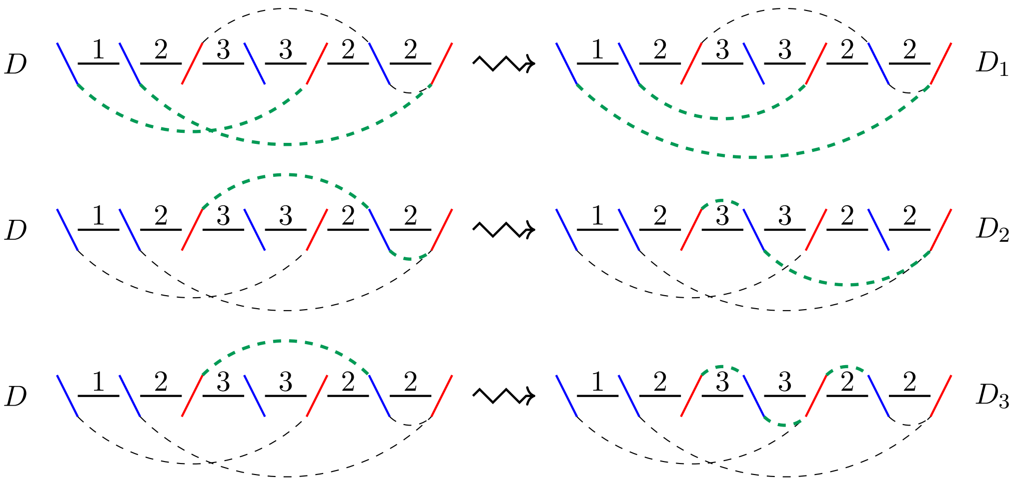


1	0	1	0
0	1	1	0
1	1	0	1

The tie diagrams that are obtained from D via a simple moves have the following binary contingency tables:



We highlighted the submatrices that are involved in the respective simple moves. These simple moves can be illustrated via tie diagrams as follows:



Simple moves are well-behaved with respect to Hanany–Witten transition: Let $\tilde{D}$ be the brane diagram obtained via Hanany–Witten transition from $\mathcal{D}$ by switching $U_{j_0} \in \mathrm{b}(\mathcal{D})$ and $V_{i_0} \in \mathrm{r}(\mathcal{D})$. Let $\Phi\colon \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})$ be the corresponding Hanany–Witten isomorphism (see Proposition 2.52) and let $\phi\colon \mathrm{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Tie}(\tilde{\mathcal{D}})$ be the induced bijection, see (3.10).

Lemma 10.25. *Let D, $D' \in \mathrm{Tie}(\mathcal{D})$, $z \in S_N$ and $i \in \{1, \dots, M\}$. Then, we have $D' \in \mathrm{SM}_{D,i,z}$ if and only if $\phi(D') \in \mathrm{SM}_{\phi(D),i,z}$.*

Proof. The proof is immediate from the fact that $M(D) = M(\phi(D))$, for all $D \in \mathrm{Tie}(\mathcal{D})$, see (3.11). □

Chevalley–Monk formula in the general case

We finally formulate and prove Chevalley–Monk formulas for bow varieties corresponding to not-necessarily separated brane diagrams.

We first set up some notation: given a brane diagram $\mathcal{D}$ and $i \in \{1, \dots, M-1\}$ then we set

$$I(\mathcal{D}, i) \coloneqq \{X \in \mathrm{h}(\mathcal{D}) \mid V_{i+1} \lhd X \lhd V_i\}.$$

In addition, we set

$$I(\mathcal{D}, 0) \coloneqq \{X \in \mathrm{h}(\mathcal{D}) \mid V_1 \lhd X\} \quad \text{and} \quad I(\mathcal{D}, M) \coloneqq \{X \in \mathrm{h}(\mathcal{D}) \mid X \lhd V_M\}.$$

For instance, let $\mathcal{D} = 0\backslash 1\backslash 2/3\backslash 3/2\backslash 2/0$ be as in Example 10.24. Then, one can easily check that $I(\mathcal{D},0) = \{X_8\}$, $I(\mathcal{D},1) = \{X_7, X_6\}$, $I(\mathcal{D},2) = \{X_5, X_4\}$ and $I(\mathcal{D},3) = \{X_3, X_2, X_1\}$.

Now, we finally state the general Chevalley–Monk formula:

Theorem 10.26 (Chevalley–Monk formula for bow varieties)**.** *Let $\mathfrak{C} = z^{-1}.\mathfrak{C}_-$ for $z \in S_N$ and $i \in \{0, \dots, M+1\}$. Then, we have the following identity in $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$:*

$$c_1(\xi_j) \cdot \mathrm{Stab}_{\mathfrak{C}}(D) = \iota_D^*(c_1(\xi_j)) \cdot \mathrm{Stab}_{\mathfrak{C}}(D) + \sum_{D' \in \mathrm{SM}_{D,z,i}} \mathrm{sgn}_z(D,D') \cdot h \cdot \mathrm{Stab}_{\mathfrak{C}}(D'),$$

for all $X_j \in I(\mathcal{D},i)$ and $D \in \mathrm{Tie}(\mathcal{D})$.

For the proof, we use the following notation: Given a $\mathbb{T}$-equivariant vector bundle E on $\mathcal{C}(\mathcal{D})$, we denote by $\mathrm{C}(\mathcal{D},E) = \mathrm{C}(\mathcal{D},E)_{D,D'}$ the matrix with entries in $H^*_{\mathbb{T}}(\mathrm{pt})_{\mathrm{loc}}$ corresponding to the operator of multiplication with $c_1(E)$ on $H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))_{\mathrm{loc}}$ with respect to the stable basis $(\mathrm{Stab}_{\mathfrak{C}}(D))_{D\in \mathrm{Tie}(\mathcal{D})}$.

We will use the following lemma:

Lemma 10.27. *Let $U_{j_0} \in \mathrm{b}(\mathcal{D})$, $V_{i_0} \in \mathrm{r}(\mathcal{D})$, $\tilde{\mathcal{D}}$, Φ and ϕ be as in Lemma 10.25. Let $X_l = U_{j_0}^+$ and D, $D' \in \mathrm{Tie}(\mathcal{D})$. Then, we have*

$$\varphi_{j_0}(\mathrm{C}(\tilde{\mathcal{D}}, \tilde{\xi}_j)_{\phi(D),\phi(D')}) = \mathrm{C}(\mathcal{D},\xi_j)_{D,D'}, \quad \text{for } j \neq l$$

and

$$\varphi_{j_0}(\mathrm{C}(\tilde{\mathcal{D}}, \tilde{\xi}_l)_{\phi(D),\phi(D')}) = \mathrm{C}(\mathcal{D},\xi_{l+1})_{D,D'} + \mathrm{C}(\mathcal{D},\xi_{l-1})_{D,D'} - \mathrm{C}(\mathcal{D},\xi_l)_{D,D'} + (t_{j_0}+h)\delta_{D,D'}.$$

Here, $\tilde{\xi}_i$ is the tautological bundle over $\mathcal{C}(\tilde{\mathcal{D}})$ corresponding to X_i and $\varphi_{j_0} : \mathbb{Q}[t_1,\dots,t_N,h] \xrightarrow{\sim} \mathbb{Q}[t_1,\dots,t_N,h]$ is the $\mathbb{Q}[h]$-algebra automorphism given by $t_{j_0} \mapsto t_{j_0} + h$ and $t_j \mapsto t_j$, for $j \neq j_0$.

Proof. Let $\Phi^* \colon H^*_{\mathbb{T}}(\mathcal{C}(\tilde{\mathcal{D}})) \xrightarrow{\sim} H^*_{\mathbb{T}}(\mathcal{C}(\mathcal{D}))$ be the induced ring isomorphism from Φ. By Proposition 5.13, we have $\Phi^*(\mathrm{Stab}_{\mathfrak{C}}(\phi(T))) = \mathrm{Stab}_{\mathfrak{C}}(T)$, for all $T \in \mathrm{Tie}(\mathcal{D})$. Thus,

$$\varphi_{j_0}(\mathrm{C}(\tilde{\mathcal{D}}, \tilde{\xi}_i)_{\phi(D),\phi(D')}) = \mathrm{C}(\mathcal{D}, \Phi^*(\tilde{\xi}_i))_{D,D'}.$$

Hence, the lemma follows from Proposition 2.52. □

Proof of Theorem 10.26. We prove the theorem via induction on the separation degree of $\mathcal{D}$, see Definition 2.54. The case $\mathrm{sdeg}(\mathcal{D}) = 0$ is exactly the statement of Theorem 10.22, so let us assume $\mathrm{sdeg}(\mathcal{D}) > 0$. As in the proof of Theorem 10.2, the support condition for stable basis elements directly implies

$$\mathrm{C}(\mathcal{D},\xi_j)_{D,D} = \iota_D^*(c_1(\xi_j)), \quad \text{for all } D \in \mathrm{Tie}(\mathcal{D}).$$

Hence, it is left to show that $\mathrm{C}(\mathcal{D},E)$ has the correct off-diagonal terms. As $\mathrm{sdeg}(\mathcal{D}) > 0$, there exist $U_{j_0} \in \mathrm{b}(\mathcal{D})$ and $V_{i_0} \in \mathrm{r}(\mathcal{D})$ as in Lemma 10.27. In the following, we use the notation from Lemma 10.27. Let D, $D' \in \mathrm{Tie}(\mathcal{D})$ with $D \neq D'$. Assume first that $X_j \neq U^+_{j_0}$. Then, we have $X_j \in I(\tilde{\mathcal{D}}, i)$ and hence, the induction hypothesis gives

$$\mathrm{C}(\tilde{\mathcal{D}}, \tilde{\xi}_j)_{\phi(D),\phi(D')} = \begin{cases} \mathrm{sgn}_z(\phi(D), \phi(D'))h & \text{if } \phi(D') \in \mathrm{SM}_{\phi(D),i,z}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, Lemma 10.25 and Lemma 10.27 imply

$$\mathrm{C}(\mathcal{D}, \xi_j)_{D,D'} = \begin{cases} \mathrm{sgn}_z(D, D')h & \text{if } D' \in \mathrm{SM}_{D,i,z}, \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathrm{C}(\mathcal{D}, \xi_j)$ has the correct off-diagonal terms. It remains to prove the case $X_j = U^+_{j_0}$. Note that in this case $i = i_0$. Since $X_{j+1} \in I(\mathcal{D}, i-1)$ and $X_j \in I(\tilde{\mathcal{D}}, i-1)$, the induction hypothesis and the previous case imply $\varphi_{j_0}(\mathrm{C}(\tilde{\mathcal{D}}, \tilde{\xi}_j)_{\phi(D),\phi(D')}) = \mathrm{C}(\mathcal{D}, \xi_{j+1})_{D,D'}$. Therefore, Lemma 10.25 and Lemma 10.27 again imply that $\mathrm{C}(\mathcal{D}, \xi_j)$ and $\mathrm{C}(\mathcal{D}, \xi_{j-1})$ have identical off-diagonal entries. By the first case, the latter are given by

$$\mathrm{C}(\mathcal{D}, \xi_{j-1})_{D,D'} = \begin{cases} \mathrm{sgn}_z(D, D')h & \text{if } D' \in \mathrm{SM}_{D,i,z}, \\ 0 & \text{otherwise} \end{cases}$$

which completes the proof. □

10.8 Simple moves and rim hook removals

Consider the Grassmannian $\mathrm{Gr}(k, n)$ with its cotangent bundle $T^*\mathrm{Gr}(k, n)$. As before, let $\mathcal{Q}$ be the pullback of the quotient bundle from $\mathrm{Gr}(k, n)$ to $T^*\mathrm{Gr}(k, n)$. Recall the bow variety realization $\mathcal{C}(\tilde{\mathcal{D}}(k; n)) \xrightarrow{\sim} T^*\mathrm{Gr}(k, n)$ from Theorem 2.67 and that $\mathcal{Q}$ is $\mathbb{T}$-equivariantly isomorphic to the tautological bundle ξ_2 on $\mathcal{C}(\tilde{\mathcal{D}}(k; n))$.

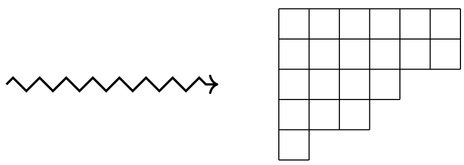
In the introduction, we stated the formula for the stable basis expansion of the products $c_1(\mathcal{Q}) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(p)$ from [MO19, Theorem 10.1.1] using the language of partitions, see (1.3). The main combinatorial tool in (1.3) were the rim hook removals on Young diagrams, see Definition 10.31.

In this section, we show that the formulas for the stable basis expansion of $c_1(\mathcal{Q}) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(p)$ from (1.3) and Theorem 10.26 are actually equivalent. For this, we use the well-known matching of partitions and binary contingency tables, see e.g. [Pos05], and show in Proposition 10.34 that under this correspondence rim hook removals correspond to simple moves.

Matching of partitions and binary contingency tables

We usually identify a partition with its corresponding Young diagram:

$(6, 6, 4, 3, 1)$
Partition λ



Young diagram of λ (10.22)

For a partition λ, we denote by $|\lambda|$ the number of boxes in the Young diagram of λ. Let $\mathcal{P}(k,n)$ be the set of all partitions whose Young diagram admits at most k rows and $n-k$ columns. To $\lambda \in \mathcal{P}(k,n)$, we assign its corresponding *row-vector* $(\lambda_1,\dots,\lambda_k)$, where λ_i is the number of boxes in the i-th row of λ. Likewise, we assign to λ its *column vector* $(\tilde{\lambda}_1,\dots,\tilde{\lambda}_{n-k})$, where $\tilde{\lambda}_i$ is the number of boxes in the i-th column of λ.

Example 10.28. Choose $(k,n)=(6,13)$ and let λ be as in (10.22). Then, λ is contained in $\mathcal{P}(k,n)$. The row vector and the column vector of λ are given as

$$(6,6,4,3,1,0) \quad \text{and} \quad (5,4,4,3,2,2,0).$$

We now match the elements of $\mathcal{P}(k,n)$ with the binary contingency tables of $\tilde{\mathcal{D}}(k;n)$. Since $\tilde{\mathcal{D}}(k;n)$ admits the margin vectors $\mathbf{r}=(k,n-k)$ and $\mathbf{c}=(1,\dots,1)$, the set $\mathrm{bct}(\tilde{\mathcal{D}}(k;n))$ is the set of $(2\times n)$-matrices A with entries in $\{0,1\}$ satisfying the following row and column sum conditions

$$\sum_{i=1}^{n} A_{1,i}=k, \quad \sum_{i=1}^{n} A_{2,i}=n-k, \quad \sum_{l=1}^{2} A_{l,j}=1, \quad \text{for } j=1,\dots,n. \tag{10.23}$$

Recall the functions

$$F_{A,1}\colon \{1,\dots,k\} \longrightarrow \{1,\dots,n\}, \quad F_{A,2}\colon \{1,\dots,n-k\} \longrightarrow \{1,\dots,n\}$$

from (9.19), where $F_{A,i}(j)$ is the column index of the j-th 1-entry in the i-th row of A. By e.g. [Pos05, Section 2], we have a bijection

$$\eta\colon \mathrm{bct}(\tilde{\mathcal{D}}(k;n)) \xrightarrow{\sim} \mathcal{P}(k,n), \quad A \mapsto \eta(A), \tag{10.24}$$

where $\eta(A)$ is the unique element in $\mathcal{P}(k,n)$ with row vector

$$(F_{A,1}(k)-k, F_{A,1}(k-1)-(k-1),\dots,F_{A,1}(1)-1). \tag{10.25}$$

Example 10.29. Let (k,n) and λ be as in Example 10.28. We want to determine $A \coloneqq \eta^{-1}(\lambda)$. As λ has the row vector $(6,6,4,3,1,0)$, we conclude that $F_{A,1}$ is given as

$$(F_{A,1}(1),\dots,F_{A,1}(6))=(1,3,6,8,11,12).$$

Hence, we deduce from (10.23) that $F_{A,2}$ is given as

$$(F_{A,2}(1),\dots,F_{A,2}(7))=(2,4,5,7,9,10,13).$$

Consequently, we have

$$A=\begin{pmatrix} 1&0&1&0&0&1&0&1&0&0&1&1&0\\ 0&1&0&1&1&0&1&0&1&1&0&0&1 \end{pmatrix}.$$

The bijection η from (10.24) can be equivalently characterized via column vectors:

Lemma 10.30. *For $A \in \mathrm{bct}(\tilde{\mathcal{D}}(k;n))$, we have that $\eta(A)$ is the unique element in $\mathcal{P}(k,n)$ with column vector*

$$(k+1-F_{A,2}(1), k+2-F_{A,2}(2),\dots,k+(n-k)-F_{A,2}(n-k)).$$

Proof. For $i \in \{1, \ldots, n-k\}$, the number of boxes in the i-th column of $\eta(A)$ equals $i_0 :=$ $\sum_{j=F_{A,2}(i)}^{n} A_{1,j}$. By (10.23), we have

$$i_0 = n - F_{A,2}(i) - \Big(\sum_{j=F_{A,2}(i)+1}^{n} A_{2,j} \Big). \tag{10.26}$$

As $F_{A,2}(i)$ is the column index of the i-th 1-entry in A, we have $\sum_{j=F_{A,2}(i)+1}^{n} A_{2,j} = n-k-i$. Thus, (10.26) implies that the i-th column of $\eta(A)$ contains $k+i-F_{A,2}(i)$ boxes. $\square$

Rim hook removals vs simple moves

Next, we recall the definition of rim hooks and show that simple moves correspond to rim hook removals under the bijection η from (10.24). For more details on rim hooks, see e.g. [BCFF99, Section 2].

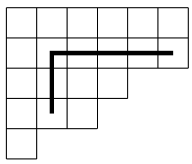
Definition 10.31. Let λ be a partition and (i,j) be a box in λ. The *hook* $\mathrm{H}_\lambda(i,j)$ *of* (i,j) *in* λ is defined as

$$\mathrm{H}_\lambda(i,j) := \mathrm{H}_\lambda^+(i,j) \cup \{(i,j)\},$$

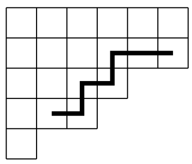
where $\mathrm{H}_\lambda^+(i,j)$ is the set of all boxes in λ to the right and below (i,j). The *rim hook* $\mathrm{RH}_\lambda(i,j)$ *of* (i,j) *in* λ is the collection of contiguous boxes running along the border of λ starting in the bottom-most box of $\mathrm{H}_\lambda(i,j)$ and ending in the right-most box of $\mathrm{H}_\lambda(i,j)$.

Given a hook $\mathrm{H}_\lambda(i,j)$, we define its *height* $\mathrm{ht}(\mathrm{H}_\lambda(i,j))$ as the number of boxed in the j-th column of $\mathrm{H}_\lambda(i,j)$. Likewise, if $\mathrm{RH}_\lambda(i,j)$ is a rim hook, we define the *height of* $\mathrm{RH}_\lambda(i,j)$ as $\mathrm{ht}(\mathrm{RH}_\lambda(i,j)) := \mathrm{ht}(\mathrm{H}_\lambda(i,j))$.

Example 10.32. Let λ be as in (10.22). The following picture shows the hook and the rim hook of the box $(2,2)$ in λ:



Hook of $(2,2)$



Rim hook of $(2,2)$

As the hook $\mathrm{H}_\lambda(2,2)$ contains 3 boxes in the second column, we have

$$\mathrm{ht}(\mathrm{RH}_\lambda(i,j)) = \mathrm{ht}(\mathrm{H}_\lambda(i,j)) = 3.$$

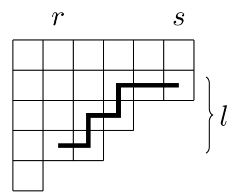
For $\lambda \in \mathcal{P}(k,n)$, we define RH_λ as the set of all $\mu \in \mathcal{P}(k,n)$ such that λ is obtained from μ by deleting a rim hook from μ. The elements of RH_λ can be uniquely characterized via their column vectors, see e.g. [BCFF99, Section 2]:

Lemma 10.33. *Let λ, $\mu \in \mathcal{P}(k,n)$. Then, we have $\mu \in \mathrm{RH}_\lambda$ if and only if there exist $1 \le r \le s \le n-k$ and $l \in \{1, \ldots, k\}$ such that*

$$(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-k}) = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{r-1}, \tilde{\mu}_{r+1}-1, \tilde{\mu}_{r+2}-1, \ldots, \tilde{\mu}_s - 1, \tilde{\mu}_r - l, \tilde{\mu}_{s+1}, \ldots, \tilde{\mu}_{n-k}).$$

If $\mu \in \mathcal{P}(k,n)$ and R is a rim hook in μ then the coefficients r, s, l from Lemma 10.33 are given as follows: r is the column index of the left-most box in R, s is the column index of the right-most box in R and l equals the height of R.

If for example, μ is the partition from Example 10.32 and R is the rim hook from there then as the picture below indicates, we have $r = 2$, $s = 6$ and $l = 3$.



The next proposition is the main result of this subsection.

Proposition 10.34. *For all $A \in \mathrm{bct}(\tilde{\mathcal{D}}(k;n))$, we have*

$$\eta(\mathrm{SM}_A) = \mathrm{RH}_{\eta(A)}.$$

Here, SM_A denotes the set of all elements in $\mathrm{bct}(\tilde{\mathcal{D}}(k;n))$ that are obtained from A via a simple move.

We begin with the following statement:

Lemma 10.35. *Let $A \in \mathrm{bct}(\tilde{\mathcal{D}}(k;n))$ and $A' \in \mathrm{SM}_A$. Then, $\eta(A') \in \mathrm{RH}_{\eta(A)}$.*

Proof. By assumption, there exist $1 \leq j_1 < j_2 \leq n$ such that

$$A_{1,j_1} = A_{2,j_2} = 1, \qquad A'_{1,j_1} = A'_{2,j_2} = 0,$$
$$A_{1,j_2} = A_{1,j_2} = 0, \qquad A'_{1,j_2} = A'_{1,j_2} = 1$$

and $A_{i,j} = A'_{i,j}$ for $(i,j) \neq (1,j_1)$, $(2,j_1)$, $(1,j_2)$, $(2,j_2)$. Set $\lambda := \eta(A)$. To show that $\mu := \eta(A')$ is contained in RH_λ, we show that the conditions of Lemma 10.33 are satisfied with

$$r = F_{A,1}^{-1}(j_1), \quad s = F_{A,2}^{-1}(j_2), \quad l = \sum_{i=j_1}^{j_2} A_{1,i}. \tag{10.27}$$

If $i < r$ or $i > s$, we have $F_{A,2}(i) = F_{A',2}(i)$ and hence $\tilde{\lambda}_i = \tilde{\mu}_i$. Likewise, for $r \leq i < s$, we have $F_{A,2}(i) = F_{A',2}(i+1)$ and hence $\tilde{\lambda}_i = \tilde{\mu}_{i+1} - 1$ by Lemma 10.30. Finally, since $F_{A',2}(r) = j_1$ and $F_{A,2}(s) = j_2$, Lemma 10.30 gives

$$\tilde{\mu}_r - \tilde{\lambda}_s = \sum_{i=j_1}^{j_2} A_{1,i}.$$

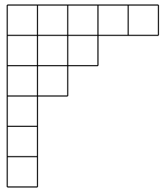
Hence, Lemma 10.33 yields $\mu \in \mathrm{RH}_\lambda$. □

Suppose $A' \in \mathrm{SM}_A$ where the simple move is performed in the j_1-th and j_2-th column of A with $j_1 < j_2$. Set $\lambda := \eta(A)$, $\mu := \eta(A')$. Then, (10.27) actually tells us the box (i_0, j_0) of μ such that λ is obtained from μ by removing $\mathrm{RH}_\mu(i_0,j_0)$. Namely, by (10.27), we have $j_0 = F_{A,1}^{-1}(j_1)$ and

$$i_0 = \tilde{\mu}_{j_0} - \Big(\sum_{i=j_1+1}^{j_2} A_{1,i} \Big). \tag{10.28}$$

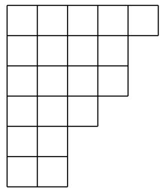
Example 10.36. Consider the following binary contingency table A with partition λ:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \lambda =$$

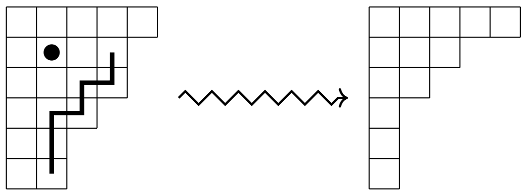


Let $A' \in \mathrm{SM}_A$ be obtained from A by performing a simple move in the second and ninth column of A and μ the partition corresponding to A'. That is, we have

$$A' = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mu =$$



Here, we highlighted the matrix entries that are involved in the simple move. Note that between the green columns in A', the first row contains four 1-entries. Hence, as $\tilde{\mu}_2 = 6$, we deduce from (10.27) that λ is obtained from μ by removing the rim hook $\mathrm{RH}_\mu(2,2)$.



Proof of Proposition 10.34. Let $A \in \mathrm{bct}(\tilde{\mathcal{D}}(k;n))$ and set $\lambda := \eta(A)$. By Lemma 10.35, we have $\eta(\mathrm{SM}_A) \subset \mathrm{RH}_\lambda$. Thus, it is left to show $\eta^{-1}(\mathrm{RH}_\lambda) \subset \mathrm{SM}_A$. Let $\mu \in \mathrm{RH}_\lambda$ and set $A' := \eta^{-1}(\mu)$. To show that A' is obtained from A via a simple move, we compare the functions $F_{A,2}$ and $F_{A',2}$. Let (i_0, j_0) be the box in μ such that we obtain λ by removing $\mathrm{RH}_\mu(i_0, j_0)$ from μ. Let s_0 be the column index of the right-most box in $\mathrm{RH}_\mu(i_0, j_0)$. By Lemma 10.33, we have

$$\tilde{\lambda}_i = \tilde{\mu}_i, \quad \text{for } i < i_0 \text{ or } i > s_0$$

and

$$\tilde{\lambda}_i = \tilde{\mu}_{i+1} - 1, \quad \text{for } i = i_0,\ i_0+1, \ldots, s_0 - 1.$$

Thus, Lemma 10.30 yields

$$F_{A,2}(i) = F_{A',2}(i), \quad \text{for } i < i_0 \text{ or } i > s_0$$

and

$$F_{A,2}(i) = F_{A',2}(i+1), \quad \text{for } i = i_0,\ i_0+1, \ldots, s_0 - 1.$$

Hence, if we write

$$(F_{A',2}(1), \ldots, F_{A',2}(n-k)) = (l_1, \ldots, l_{n-k})$$

then we have

$$(F_{A,2}(1), \ldots, F_{A,2}(n-k)) = (l_1, \ldots, l_{i_0-1}, l_{i_0+1}, l_{i_0+2}, \ldots, l_{s_0-1}, j, l_{s_0+1}, l_{s_0+2}, \ldots, l_{n-k}),$$

where $j = F_{A,2}(s_0)$. Lemma 10.33 gives $\tilde{\mu}_{i_0} - \tilde{\lambda}_{s_0} \geq 1$. By Lemma 10.30 this is equivalent to $(k + i_0 - l_{i_0}) - (k + s_0 - j) \geq 1$. Since $s_0 \geq i_0$, we deduce $j - l_{i_0} \geq 1$. Thus, we conclude that A' is obtained from A by a simple move which is performed in the l_{i_0}-th and j-th column of A. □

Reformulation of Theorem 10.26

We identify the bow variety $\mathcal{C}(\tilde{\mathcal{D}}(k;n))$ with $T^*\mathrm{Gr}(k,n)$ via the $\mathbb{T}$-equivariant isomorphism from Theorem 2.67. Via the bijection η from (10.24), we may label the stable basis elements of $T^*\mathrm{Gr}(k,n)$ by the elements of $\mathcal{P}(k,n)$, i.e. we set $\mathrm{Stab}_{\mathfrak{C}}(\lambda) \coloneqq \mathrm{Stab}_{\mathfrak{C}}(\eta^{-1}(\lambda))$, for $\lambda \in \mathcal{P}(k,n)$.

We now give an equivalent reformulation of Theorem 10.26 for $T^*\mathrm{Gr}(k,n)$ using the language of partitions. For simplicity, we restrict our attention to the antidominant chamber.

Corollary 10.37. *For all $\lambda \in \mathcal{P}(k,n)$, we have the following identity in $H_{\mathbb{T}}^*(T^*\mathrm{Gr}(k,n))$:*

$$c_1(\mathcal{Q}) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(\lambda) = \Big(\sum_{i \in E_\lambda} t_i\Big) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(\lambda) + \sum_{\mu \in \mathrm{RH}_\lambda} (-1)^{|\mu|-|\lambda|-1} \cdot h \cdot \mathrm{Stab}_{\mathfrak{C}_-}(\mu). \tag{10.29}$$

Here, $E_\lambda \coloneqq \{1,\dots,n\} \setminus \{\lambda_1 + k, \lambda_2 + (k-1), \dots, \lambda_k + 1\}$.

Proof. Let $A \in \mathrm{bct}(\tilde{\mathcal{D}}(k;n))$ with $\eta(A) = \lambda$ and write

$$(F_{A,2}(1), \dots, F_{A,2}(n-k)) = (l_1, \dots, l_{n-k}).$$

By (3.8), we have $\iota_A^*(\mathcal{Q}) = t_{l_1} + \ldots + t_{l_{n-k}}$. In addition, (10.25) gives $E_\lambda = \{l_1, \dots, l_{n-k}\}$. Thus, Theorem 10.26 yields that we have the following identity on $H_{\mathbb{T}}^*(T^*\mathrm{Gr}(k,n))_{\mathrm{loc}}$:

$$c_1(\mathcal{Q}) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(A) = \Big(\sum_{i \in E_\lambda} t_i\Big) \cdot \mathrm{Stab}_{\mathfrak{C}_-}(A) + \sum_{A' \in \mathrm{SM}_A} \mathrm{sgn}(A, A') \cdot h \cdot \mathrm{Stab}_{\mathfrak{C}_-}(A'). \tag{10.30}$$

Recall from e.g. [AF23, Corollary 3.3.3] that $T^*\mathrm{Gr}(k,n)$ is equivariantly formal. Hence, (10.30) also holds in the non-localized equivariant cohomology ring $H_{\mathbb{T}}^*(T^*\mathrm{Gr}(k,n))$. By Proposition 10.34, we have $\eta(\mathrm{SM}_A) = \mathrm{RH}_\lambda$. Thus, it is left to show that

$$\mathrm{sgn}(A, A') = (-1)^{|\eta(A')|-|\lambda|-1}, \quad \text{for } A' \in \mathrm{SM}_A. \tag{10.31}$$

Suppose A' is obtained from A via a simple move performed in the j_1-th and j_2-th column of A with $j_1 < j_2$. Then, by (10.23), we have $\mathrm{sgn}(A, A') = (-1)^{j_2 - j_1 - 1}$. On the other hand, let (i_0, j_0) be the box in $\mu \coloneqq \eta(A')$ such that λ is obtained from μ by removing $\mathrm{RH}_\mu(i_0, j_0)$. Let s_0 be the column index of the right-most box in $\mathrm{RH}_\mu(i_0, j_0)$. Then,

$$|\mu| - |\lambda| = s_0 - i_0 + \tilde{\mu}_{i_0} - \tilde{\lambda}_{s_0},$$

By Lemma 10.30, $\tilde{\mu}_{i_0} = k + i_0 - F_{A',2}(i_0)$ and $\tilde{\lambda}_{s_0} = k + s_0 - F_{A,2}(s_0)$. Hence, $|\mu| - |\lambda| = j_2 - j_1$ which proves (10.31). Thus, (10.30) is equivalent to (10.29). □

As desired, the formula (10.29) coincides with the formula (1.3) from the introduction.

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