Stability conditions on derived categories

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Zusammenfassung

Die vorliegende Arbeit gliedert sich in zwei Teile, denen jeweils ein Kapitel gewidmet ist. Im ersten Teil werden Stabilitätsbedingungen auf kompakten komplexen Mannigfaltigkeiten, die keine nichttrivialen komplex-analytischen Untervarietäten besitzen, konstruiert und klassifiziert.
Im zweiten Abschnitt des ersten Kapitels wird gezeigt, dass der Raum der Stabilitätsbedingungen $\mathrm{Stab}(X)$ für K3-Flächen mit $\mathrm{Pic}(X) = 0$ zusammenhängend und einfach zusammenhängend ist, und dass die Abbildung $\pi : \mathrm{Stab}(X) \longrightarrow \pi(\mathrm{Stab}(X))$, die jeder Stabilitätsbedingung ihre zentrale Ladung zuordnet, eine universelle Überlagerung ist. Mit Hilfe der Gruppenwirkungen von $\mathrm{Aut}(\mathrm{D^b}(X))$ und $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, der universellen Überlagerung von $\mathrm{GL}^+(2,\mathbb{R})$, auf $\mathrm{Stab}(X)$ gelingt es, eine vollständige (explizite) Beschreibung aller Stabilitätsbedingungen auf X anzugeben. Die Kenntnis der Struktur von $\mathrm{Stab}(X)$ gestattet es, die Gruppe $\mathrm{Aut}(\mathrm{D^b}(X))$ der Autoäquivalenzen vom Fourier–Mukai Typ zu bestimmen. Diese Gruppe zerfällt in die direkte Summe von $\mathrm{Aut}(X)$ und einer freien abelschen Gruppe vom Rank zwei, die durch den Shift-Funktor und den sphärischen Twist(-Funktor) zur Strukturgarbe $\mathcal{O}_X$ erzeugt wird. Der letzte Teil des zweiten Abschnittes beschäftigt sich mit einer Erweiterung der Definition einer Stabilitätsbedingung, die den Raum der Stabilitätsbedingungen zu dem komplexifizierten Kählerkegel in Bezug zu setzt.
Dem dritten Abschnitt des ersten Kapitels liegen generische komplexe Tori der Dimension d mit der Eigenschaft $\mathrm{ch}_k(F) = 0$ für alle coherenten Garben F und alle $0 < k < d$ zu Grunde. Wir werden eine einfach zusammenhängende Zusammenhangskomponente des Raumes $\mathrm{Stab}(X)$ konstruieren, die genau diejenigen Stabilitätsbedingungen enthält, für die alle Linienbündel und alle Wolkenkratzergarben $\mathbb{C}(x)$ für $x \in X$ stabil sind. All diese Stabilitätsbedingungen werden bis auf die Wirkung von $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ explizit konstruiert. Ein Repräsentantensystem wird durch eine diskrete Serie von $d = \dim(X)$ Stabilitätsbedingungen $\sigma_{(p)}, 0 \leq p < d$, und $d-1$ Einparameterfamilien $\sigma^{\gamma}_{(p)}, 1 \leq p < d, \gamma \in (0, 1/2)$ gegeben.
Im letzten Abschnitt des ersten Kapitels werden die Stabilitätsbedingungen $\sigma_{(p)}$ aus dem vorherigen Beispiel auf den Fall einfacher kompakter komplexer Mannigfaltigkeiten verallgemeinert. Die Herzen $\mathrm{Coh_{(p)}(X)}$ der Stabilitätsbedingungen $\sigma_{(p)}$ werden mittels lokaler Kohomologie definiert. Die Existenz der Stabilitätsbedingungen $\sigma_{(p)}, 0 \leq p < d$, wird im Wesentlichen aus der Tatsache folgen, dass die abelschen Kategorien $\mathrm{Coh_{(p)}(X)}$ für $0 < p < d$ von endlicher Länge, d.h. sowohl noethersch als auch artinsch sind.

Der zweite Teil dieser Arbeit entstand aus der Frage, wie groß die Menge der Komplexe aus $\mathrm{D^b}(X)$, die sich nicht längs einer Twistorgerade deformieren lassen, for projektive K3-Flächen X eigentlich ist. Da sich Wolkenkratzergarben $\mathbb{C}(x)$ und μ-stabile Vektorbündel vom Grad Null nach einem Resultat von

M. Verbitsky deformieren lassen, liegt es nahe, nach der von diesen Objekten erzeugten vollen triangulierten Unterkategorie von $\mathrm{D^b}(X)$ auszuteilen. Im zweiten Kapitel soll diese Quotientenkategorie $\mathcal{Q}$ sogar für beliebige glatte irreduzible projektive Varietäten der Dimension d beschreiben werden.
Der erste Schritt besteht darin, μ-Stabilität als Stabilitätsbedingung im Sinne des ersten Kapitels zu interpretieren. Dazu muss man erst nach allen Komplexen, deren Träger eine Kodimension größer gleich zwei hat, austeilen. Der erste Abschnitt des zweiten Kapitels beschäftigt sich mit den Eigenschaften der so entstehenden Quotientenkategorie $\mathrm{D}^b_{d,d-1}(X)$. Zunächst wird gezeigt, dass $\mathrm{D}^b_{d,d-1}(X)$ zur beschränkten derivierten Kategorie der abelschen Kategorie $\mathrm{Coh}_{d,d-1}(X)$ äquivalent ist und somit eine kanonische t-Struktur besitzt. Die Kategorie $\mathrm{Coh}_{d,d-1}(X)$ entsteht aus $\mathrm{Coh}(X)$ durch Austeilen aller Garben, deren Träger eine Kodimension größer gleich zwei hat. Es wird zumindest für $1 \leq d \leq 2$ gezeigt, dass $\mathrm{D}^b_{d,d-1}(X)$ bezüglich dieser t-Struktur die homologische Dimension eins hat. Dies impliziert z.B., dass jedes Objekt aus $\mathrm{D}^b_{d,d-1}(X)$ die direkte Summe seiner (verschobenen) Kohomologien ist. Ein weiteres Resultat besagt, dass Erweiterungen einer Torsionsgarbe durch eine torsionsfreie Garbe in $\mathrm{Coh}_{d,d-1}(X)$ trivial sind.
Mit Hilfe dieser beiden Aussagen wird im zweiten Abschnitt des zweiten Kapitels der Quotient $\mathcal{Q}$ von $\mathrm{D^b}(X)$ bzw. $\mathrm{D}^b_{d,d-1}(X)$ nach der vollen triangulierten Unterkategorie, die von allen μ-stabilen Objekten vom Grad Null erzeugt wird, bestimmt. Das Hauptresultat dieses Abschnittes besagt, dass diese Quotientenkategorie äquivalent zur beschränkten derivierten Kategorie der endlich dimensionalen Vektorräume über einer Divisionsalgebra ist, wenn die Untergruppe $\{\deg(E) \mid E \in \mathrm{Coh}(X)\} \subset \mathbb{Z}$ durch den Grad eines effektiven Divisors D erzeugt wird. Die Divisionsalgebra ist der Endomorphismenring von $\mathcal{O}_D$ in $\mathcal{Q}$.

Contents

Introduction

The notion of stability in the geometric context goes back to D. Mumford's classical work on geometric invariant theory [31]. He applied his general results to construct a moduli space of (semi)stable vector bundles of certain numerical invariants on a smooth projective curve. Doing this, he discovered that (semi)stable vector bundles can be characterized by the behaviour of the ratio $\deg / \operatorname{rk}$ on the set of its subbundles. This notion of stability was generalized by Takemoto [39],[40] for sheaves on higher dimensional varieties and is known as Mumford–Takemoto or μ-stability. Another approach to generalize Mumford's stability on curves to the case of higher dimensional projective varieties was presented by Gieseker. Gieseker-stability is related to the behaviour of the Hilbert polynomial

$$\mathbb{Z} \ni m \longmapsto \chi(X, E(mH)) \in \mathbb{Z},$$

where H is a fixed ample divisor, for large numbers m. The degree of this polynomial is less or equal the dimension d of X. Mumford's μ-stability is related to the behaviour of the degree d and degree $d-1$ term of this polynomial. Using this notion of stability, one can construct moduli spaces of (semi)stable sheaves. See [21] for more details. The notion of stability has a natural generalization for Kähler manifolds if one replaces the ample divisor by a Kähler class $[\omega]$ on X.

A couple of years ago physicists discovered the importance of stability in string theory. M. Douglas introduced the notion of Π-stability of D-branes [11],[12]. In order to understand Douglas' work, T. Bridgeland gave a precise mathematical definition of a stability condition [8],[4]. As suggested by physics, he proved that the space $\operatorname{Stab}(X)$ of all stability conditions with certain conditions has a natural structure of a complex manifold. Furthermore, the group of autoequivalences $\operatorname{Aut}(\mathrm{D^b}(X))$ of $\mathrm{D^b}(X)$ and the universal cover $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ of the group $\mathrm{GL}^+(2,\mathbb{R})$ act on $\operatorname{Stab}(X)$. From the view point of string theory, the quotient $(\operatorname{Aut}(\mathrm{D^b}(X))\backslash \operatorname{Stab}(X))/\mathbb{C}^*$ should be a good candidate for the complexified Kähler moduli space. Note that $\mathbb{C}^* \subset \mathrm{GL}^+(2,\mathbb{R})$ acts on the quotient $\operatorname{Aut}(\mathrm{D^b}(X))\backslash \operatorname{Stab}(X)$ because the fibre of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ over $id_{\mathbb{R}^2}$ acts like the subgroup $\mathbb{Z}[2] \subset \operatorname{Aut}(\mathrm{D^b}(X))$ on $\operatorname{Stab}(X)$. On the other hand, one can try to understand the group $\operatorname{Aut}(\mathrm{D^b}(X))$ by means of its action on $\operatorname{Stab}(X)$. Over the last years the space $\operatorname{Stab}(X)$ has been investigated for several manifolds X.

The aim of the first chapter is to construct stability conditions on Kähler manifolds without nontrivial (analytic) subvarieties. The definition of a stability condition and a list of examples including those of the next sections is contained in section 1.1.
The second section 1.2 is devoted to K3 surfaces with $\mathrm{Pic}(X) = 0$. It is easy to see, that the set of isomorphism classes of these K3 surfaces is the complement of countable infinite many hypersurfaces in the moduli space of K3 surfaces. Thus, every generic K3 surface is of this type. We have a full description of the space of stability conditions $\mathrm{Stab}(X)$ for these K3 surfaces.
In the third section 1.3 we construct stability conditions on the derived category of complex tori without nontrivial subvarieties. Similar to the case of K3 surfaces, every generic complex torus has this specific property. In contrast to the case of K3 surfaces, a complete understanding of $\mathrm{Stab}(X)$ is still missing. Nevertheless, we are able to describe a single connected component of $\mathrm{Stab}(X)$ characterized by a classifying property.
The aim of the last section 1.4 of the first chapter is the general construction of stability conditions on Kähler manifolds without nontrivial subvarieties. For this we need the techniques of local cohomology. Unfortunately, we do not know any component of $\mathrm{Stab}(X)$ in general. It is even unknown whether or not the constructed stability conditions belong to the same component. In the case of generic K3 surfaces and generic tori, they do.

The motivation for the second chapter was the following question. Can we use the results for generic K3 surfaces as presented in section 1.2 to complete our knowledge about $\mathrm{Stab}(X)$ for projective K3 surfaces X (see [7])? In order to do this, we have to connect the derived categories of both surfaces. Guided by M. Verbitsky's article [41] one might deform a complex of coherent sheaves on a generic K3 surface along a twistor line to a complex on a projective K3 surface X in the twistor space. Although there is no complete understanding of this procedure at the moment, we can true to figure out the expected subcategory of $\mathrm{D^b}(X)$ obtained in this way. First of all, every skyscraper sheaf $\mathbb{C}(x) \in \mathrm{D^b}(X)$ for $x \in X$ is a deformation of a skyscraper sheaf on the generic K3 surface due the existence of horizontal twistor lines in the twistor space. On the other hand, every stable vector bundle B on X of ω-degree zero admits a deformation along the twistor line corresponding to the Kähler form ω on X. This is a special case of Theorem 2.5 in [42]. Indeed, $\mathrm{c}_2(B)$ is always of Hodge type (2,2) on a surface, independent on the chosen complex structure of Kähler type. Furthermore, $\mathrm{c}_1(B)$ remains of type (1,1) along the twistor line if it is orthogonal with respect to the intersection pairing to every holomorphic symplectic 2-form σ_λ for λ in the twistor line $\mathbb{P}^1$. This is equivalent to the fact that $\mathrm{c}_1(B)$ is orthogonal to the 'usual' three Kähler forms $\omega_I = \omega, \omega_J = \mathrm{Re}\,\sigma$ and $\omega_K = \mathrm{Im}\,\sigma$, where σ is the holomorphic symplectic 2-form on X.
Every μ-stable torsionfree sheaf E of ω-degree zero fits into a short exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow T \longrightarrow 0$$

with $\dim \operatorname{supp}(T) = 0$ and $E^{\vee\vee}$ a stable vector bundle of degree zero. Thus, we expect that the image of the bounded derived category of a generic K3 surface under the deformation along the twistor line is a triangulated subcategory of $\mathrm{D^b}(X)$ containing all μ-stable sheaves of ω-degree zero. This leads to the following question.

Let X be an irreducible smooth projective variety of dimension $d \geq 1$. How 'big' is the full subcategory of $\mathrm{D^b}(X)$ consisting of complexes with μ-semistable cohomology sheaves of degree zero? To answer this question, we try to compute the quotient category. Since μ-stability is naturally defined on the quotient category of $\mathrm{D^b}(X)$ by the full triangulated subcategory of complexes whose support has codimension greater or equal two, we will investigate this quotient category in the first section 2.1 of chapter 2. Note that in the case of curves, this quotient category is just $\mathrm{D^b}(X)$. We will see, that the quotient category possesses some properties of the derived category of curves. The main result of this section is the fact that the quotient category has homological dimension one, at least for $d \leq 2$. Nevertheless, there are also differences. For example, it is unknown whether the quotient category has a Serre functor. Furthermore, the Hom-groups might be infinite-dimensional over the base field.

The main result of the last section 2.2 states that under some conditions on X the category $\mathrm{D^b}(X)$ modulo complexes with semistable cohomology sheaves of degree zero is equivalent to the bounded derived category of finite-dimensional vector spaces over a division algebra. We will give some examples to illustrate this result.

Notation. To make the notation more convenient for the reader, we will denote the skyscraper sheaf in $x \in X$ of length one on a complex manifold or on a variety X over the algebraically closed field k with $k(x)$, even if the field is $k = \mathbb{C}$. We will use the standard notation $\mathrm{Coh}(X)$ and $\mathrm{D^b}(X)$ for the abelian category of coherent sheaves and its bounded derived triangulated category.

Furthermore, $\mathrm{GL}^+(2, \mathbb{R})$ denotes the group of all orientation preserving automorphisms of $\mathbb{R}^2$, i.e. the group of all 2×2-matrices of positive determinant.

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Chapter 1

Stability conditions

1.1 Stability conditions and examples

Motivated by string theorists Tom Bridgeland introduced the notion of a stability condition on a triangulated category $\mathcal{D}$. See [14] for the backround in homological algebra. All triangles appearing in the text are assumed to be distinguished triangles.

Definition 1.1.1 ([8], Definition 1.1)**.** *A stability condition $(Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a linear map $Z : K(\mathcal{D}) \to \mathbb{C}$, called the central charge, and a full additive subcategory $\mathcal{P}(\phi) \subseteq \mathcal{D}$ for each real number $\phi \in \mathbb{R}$. Furthermore, the pair $(Z, \mathcal{P})$ satisfies the following axioms:*

(a) if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,

(b) for all $\phi \in \mathbb{R}, \mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

(d) for $0 \neq E \in \mathcal{D}$ there is a Harder–Narasimhan filtration, i.e. a finite sequence of real numbers $\phi_1 > \phi_2 > \ldots > \phi_n$ and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j.

A family $(\mathcal{P}(\phi))_{\phi \in \mathbb{R}}$ of full additive subcategories with the properties (b),(c) and (d) is called a slicing. For any interval $I \subseteq \mathbb{R}$, define $\mathcal{P}(I)$ to be the extension-closed full subcategory of $\mathcal{D}$ generated by the subcategories $\mathcal{P}(\phi)$ for $\phi \in I$. Bridgeland has shown that the categories $\mathcal{P}(I)$ are quasi-abelian for every interval $I \subset \mathbb{R}$ of length < 1 ([8], Lemma 4.3). A quasi-abelian category is a category with kernels and cokernels such that every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism. In contrast to an abelian category, the image of a morphism is not necessarily isomorphic to its coimage. Morphisms with this additional property are called strict. Subobjects with a strict embedding are called strict and similar for quotients. It can be shown that the additive subcategories $\mathcal{P}(\phi)$ and $\mathcal{P}((\phi,\phi+1])$ as well as $\mathcal{P}([\phi,\phi+1))$ are always abelian for every $\phi\in\mathbb{R}$. Moreover, the pair $(\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0}) := (\mathcal{P}((0,\infty)),\mathcal{P}((-\infty,1])$ is a bounded t-structure on $\mathcal{D}$ with heart $\mathcal{A} := \mathcal{P}((0,1])$. Furthermore, the linear map $Z : K(\mathcal{A}) = K(\mathcal{D}) \to \mathbb{C}$ satisfies

(i) *if* $0 \neq E \in \mathcal{A}$ *then* $Z(E) \in H = \{r\exp(i\pi\phi) \mid r > 0, 0 < \phi \leq 1\} \subseteq \mathbb{C}$*,*

(ii) *for* $0 \neq E \in \mathcal{A}$ *there is a Harder–Narasimhan filtration, i.e. a finite chain of subobjects* $0 = E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E$ *whose factors* $F_j = E_j/E_{j-1}$ *are semistable objects of* $\mathcal{A}$ *with* $\phi(F_1) > \phi(F_2) > \ldots > \phi(F_n)$*. An object* $F \in \mathcal{A}$ *is said to be semistable (with respect to* Z*) if* $\phi(G) \leq \phi(F)$ *for every subobject* $0 \neq G \subset F$*.*

Giving a stability condition on a triangulated category $\mathcal{D}$ is equivalent to giving a bounded t-structure $(\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0})$ on $\mathcal{D}$ with heart $\mathcal{A} := \mathcal{D}^{\leq 0}\cap\mathcal{D}^{\geq 0}$ and a linear map $Z : K(\mathcal{A}) \to \mathbb{C}$ satisfying the two properties (i) and (ii) ([8], Proposition 5.3). There is a very useful criterion to check the Harder–Narasimhan property (ii) (see [8], Proposition 2.4).

(ii.1) *There are no infinite sequences* $\ldots \subset E_{j+1} \subset E_j \subset \ldots \subset E_0$ *of subobjects in* $\mathcal{A}$ *with* $\phi(E_{j+1}) > \phi(E_j)$ *for all* j*, and*

(ii.2) *there are no infinite sequences* $E^0 \twoheadrightarrow \ldots \twoheadrightarrow E^j \twoheadrightarrow E^{j+1} \twoheadrightarrow \ldots$ *of quotients in* $\mathcal{A}$ *with* $\phi(E^j) > \phi(E^{j+1})$ *for all* j*.*

If it is clear from the context, we will write $(Z,\mathcal{A})$ for the stability condition $(Z,\mathcal{P})$, where $\mathcal{A}$ is the heart $\mathcal{P}((0,1])$ of the associated t-structure.
The following technical property is very important in order to control deformations of stability conditions.

Definition 1.1.2. *A stability condition* $(Z,\mathcal{P})$ *is called locally-finite if there exists a real number* $\eta > 0$ *such that for all* $t \in \mathbb{R}$ *the quasi-abelian category* $\mathcal{P}((t-\eta,t+\eta)) \subseteq \mathcal{D}$ *is of finite length.*

Note that a quasi-abelian category is called of finite length if it is artinian and noetherian, i.e. every decreasing sequence and every increasing sequence of strict subobjects becomes stationary. This is equivalent to the fact that every object of the category has a finite filtration by strict subobjects, the so-called Jordan–Hölder filtration, such that the successive quotients are simple, i.e. they have no strict subobjects.
If the central charge Z takes values in $\mathbb{R}$, the condition of locally-finiteness implies that the heart $\mathcal{P}((0,1])$ of the associated t-structure is of finite length. It is an easy exercise to check that for every stability condition with a central

charge Z whose image is a discrete subgroup of $\mathbb{C}$ the quasi-abelian subcategory $\mathcal{P}(I)$ is of finite length for every interval $I \subset \mathbb{R}$ of length < 1. In particular, such a stability condition is locally-finite and the abelian subcategories $\mathcal{P}(\phi)$ are of finite length. Stable objects of phase ϕ are by definition the simple objects of $\mathcal{P}(\phi)$. Thus, every semistable object has a Jordan–Hölder filtration by stable objects of the same phase if the image of the central charge is discrete.
There is a natural topology on the set of all locally-finite stability conditions on $\mathcal{D}$ ([8], section 6). In order to get a finite-dimensional space of stability conditions, one needs the notion of a numerical stability condition. For this we assume that $\mathcal{D}$ is a k-linear category and

$$\sum_{i\in\mathbb{Z}} \dim_k \mathrm{Hom}_{\mathcal{D}}(E, F[i]) < \infty$$

for all objects $E, F \in \mathcal{D}$. We denote the orthogonal complement of $K(\mathcal{D})$ with respect to the Euler form

$$\chi(E, F) := \sum_{i\in\mathbb{Z}} (-1)^i \dim_k \mathrm{Hom}_{\mathcal{D}}(E, F[i])$$

by $K(\mathcal{D})^\perp$ and the quotient $K(\mathcal{D})/K(\mathcal{D})^\perp$ by $\mathcal{N}(\mathcal{D})$. The category $\mathcal{D}$ is called numerically finite if the 'numerical Grothendieck group' $\mathcal{N}(\mathcal{D})$ has finite rank.

Definition 1.1.3. *Suppose that the triangulated category $\mathcal{D}$ is numerically finite. A stability condition $\sigma = (Z, \mathcal{P})$ is called numerical if the central charge Z factorizes over the quotient map $K(\mathcal{D}) \longrightarrow \mathcal{N}(\mathcal{D})$. Thus, $Z(E) = -\chi(\pi(\sigma), E)$ for a unique vector $\pi(\sigma)$ of $\mathcal{N}(\mathcal{D}) \otimes \mathbb{C}$.*

Using the Riemann–Roch theorem for a compact complex manifold X, we see that $\mathrm{D}^{\mathrm{b}}(X)$ is numerically finite and a stability condition $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}(X)$ is numerical if and only if $Z(E)$ depends only on the Mukai vector $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$ of $E \in \mathrm{D}^{\mathrm{b}}(X)$. Note that $v : K(X) \longrightarrow \mathrm{H}^*(X, \mathbb{Q})$ identifies the space $(\mathcal{N}(X) \otimes \mathbb{C}, -\chi)$ with a subspace of $(\mathrm{H}^*(X, \mathbb{C}), \langle\cdot,\cdot\rangle)$, where $\langle v, w\rangle = \int_X \exp(\mathrm{c}_1(X)) v^\vee w$ is the Mukai pairing and $v^\vee = \sum \sqrt{-1}^j v_j$ is the dual of $v = \sum v_j$ with $v_j \in \mathrm{H}^j(X, \mathbb{C})$. The set of all numerical and locally-finite stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ is denoted by $\mathrm{Stab}(X)$. In the following all stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ are assumed to be numerical and locally-finite.

Theorem 1.1.4 ([8], Corollary 1.3)**.** *For each connected component $\Sigma \subseteq \mathrm{Stab}(X)$ there is a complex linear subspace*

$$V(\Sigma) \subseteq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X), \mathbb{C}) = (\mathcal{N}(X) \otimes \mathbb{C})^\vee \overset{-\chi}{\cong} \mathcal{N}(X) \otimes \mathbb{C} \subseteq \mathrm{H}^*(X, \mathbb{C})$$

and a local homeomorphism $\pi : \Sigma \to V(\Sigma)$ which maps a stability condition to its central charge. In particular, $\mathrm{Stab}(X)$ *is a finite-dimensional complex manifold.*

A characterization of $V(\Sigma)$ can be found at the beginning of subsection 1.3.3. One would expect that the complex manifold $\mathrm{Stab}(X)$ contains some information about X.
A very important tool to analyse $\mathrm{Stab}(X)$ is the continuous action of two groups on $\mathrm{Stab}(X)$ ([8], Lemma 8.2). First of all, there is a natural action of the group of autoequivalences $\mathrm{Aut}(\mathrm{D^b}(X))$ of $\mathrm{D^b}(X)$ on $\mathrm{Stab}(X)$. An autoequivalence Ψ of $\mathrm{D^b}(X)$ acts on $\mathrm{Stab}(X)$ from the left by $\Psi\cdot(Z,\mathcal{P}) = (Z\circ\psi, \mathcal{P}')$, where ψ is the induced action on the K-group $K(\mathrm{D^b}(X))$ and $\mathcal{P}'(\phi) = \Psi(\mathcal{P}(\phi))$. Furthermore, there is a natural action of the universal covering group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ of $\mathrm{GL}^+(2,\mathbb{R})$ from the right. The group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ can be thought of as the set of pairs (g, f), where $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi+1) = f(\phi)+1$, and $g \in \mathrm{GL}^+(2,\mathbb{R})$ is an orientation-preserving linear isomorphism on $\mathbb{R}^2$ such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z} = \mathbb{R}^2\backslash\{0\}/\mathbb{R}_{>0}$ are the same. Then $(Z,\mathcal{P})\cdot(g,f) = (Z',\mathcal{P}')$, where $Z' = g^{-1}\circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$. The group $\mathrm{GL}^+(2,\mathbb{R})$ acts in a similar way on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X),\mathbb{C})$ and $\pi : \mathrm{Stab}(X) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X),\mathbb{C})$ intertwines both actions. The action of $\mathrm{Aut}(\mathrm{D^b}(X))$ on $\mathrm{Stab}(X)$ commutes with the one of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.
In string theory the quotient $\mathrm{Aut}(\mathrm{D^b}(X))\backslash\mathrm{Stab}(X)$ should be a $\mathbb{C}^*$-bundle over the stringy Kähler moduli space. The action of $\mathbb{C}^*$ on that quotient is induced by the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on $\mathrm{Stab}(X)$. Indeed, the group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ contains the universal cover $\mathbb{C}$ of $\mathbb{C}^*$ as a subgroup and $2\pi i \in \mathbb{C}$ acts like the double shift functor [2]. Thus, the $\mathbb{C}^*$-action is well defined on the quotient.
From a mathematical point of view one would like to understand $\mathrm{Aut}(\mathrm{D^b}(X))$ by means of its action on $\mathrm{Stab}(X)$. Due to this, the structure of $\mathrm{Stab}(X)$ has been studied by many authors for various types of Kähler manifolds X. In the following we will give a list of examples. Some of these will be studied in greater detail in subsequent sections.

Curves

Let X be a smooth compact complex curve of genus $g \geq 1$. We have got the standard t-structure on $\mathrm{D^b}(X)$ with heart $\mathrm{Coh}(X)$, the category of coherent sheaves on X. Putting $Z(E) = -\deg(E) + i\,\mathrm{rk}(E) = -\mathrm{ch}_1(E) + i\,\mathrm{rk}(E)$ one gets a stability condition $\sigma_{(0)}$ on $\mathrm{D^b}(X)$. Then, $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ acts freely on $\mathrm{Stab}(X)$ and

$$\mathrm{Stab}(X) = \sigma_{(0)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$$

is the orbit of $\sigma_{(0)}$. See [25] for a proof. The space of stability conditions $\mathrm{Stab}(\mathbb{P}^1)$ is isomorphic to $\mathbb{C}^2$. For more details in the case of $\mathbb{P}^1$ see [34] and [24].

μ-Stability

In contrast to the case of curves, the notion of μ-stability is not a stability condition on a Kähler manifold of dimension $d > 1$ because there are torsion sheaves whose support has codimension ≥ 2. For those sheaves E we get $Z(E) = -\deg(E) + i\,\mathrm{rk}(E) = 0$ which contradicts the axioms of a stability condition. The degree of the sheaf is defined with respect to a fixed Kähler class $[\omega]$ on X. In order to avoid this difficulty, one can take the derived category $\mathrm{D^b}(\mathrm{Coh}_{d,d-1}(X))$ of the abelian quotient category of coherent sheaves modulo those sheaves whose support has codimension ≥ 2. The standard t-structure together with the central charge $Z = -\deg + i\,\mathrm{rk}$ with respect to a rational Kähler class $[\omega]$ is a numerical and locally-finite stability condition on $\mathrm{D^b}(\mathrm{Coh}_{d,d-1}(X))$. The associated $\mathrm{GL}^+(2,\mathbb{R})$-orbit is a connected component of the complex manifold $\mathrm{Stab}(\mathrm{Coh}_{d,d-1}(X))$ of numerical and locally-finite stability conditions on $\mathrm{D^b}(\mathrm{Coh}_{d,d-1}(X))$ which depends only on the ray $\mathbb{R}_{>0}[\omega]$ through $[\omega]$. Thus, every rational ray in the Kähler cone of X defines a connected component of $\mathrm{Stab}(\mathrm{Coh}_{d,d-1}(X))$. See [35] for more details. We will go back to μ-stability in this context in section 2.2.

Generic K3 surfaces
Let X be a K3 surface with $\mathrm{Pic}(X) = 0$. Although the case of these generic K3 surfaces has already been studied by Huybrechts, Macrì and Stellari in [22], we will give a slightly different approach to $\mathrm{Stab}\,X$ in the next section. Here is a summary of the results.
In addition to the standard t-structure, there is another t-structure on $\mathrm{D^b}(X)$. Its heart $\mathrm{Coh}_{(1)}(X)$ consists of complexes E of length 2, where $H^0(E)$ is a torsion sheaf and $H^{-1}(E)$ is a torsionfree sheaf. In contrast to $\mathrm{Coh}(X) =: \mathrm{Coh}_{(0)}(X)$, this abelian category has finite length. The standard t-structure together with the function $Z_{(0)}(E) = -\,\mathrm{ch}_2(E) + i\,\mathrm{rk}(E)$ defines a stability condition $\sigma_{(0)}$ on $\mathrm{D^b}(X)$. For any real number $\gamma \in (0, 1/2)$ the pair consisting of our new t-structure together with the function $Z^{\gamma}_{(1)}(E) = -\,\mathrm{ch}_2(E) + \cot(\pi\gamma)\,\mathrm{rk}(E)$ gives another stability condition $\sigma^{\gamma}_{(1)}$ on $\mathrm{D^b}(E)$. In order to get all stability conditions on $\mathrm{D^b}(X)$, one needs the spherical twist $T_{\mathcal{O}_X}$ by $\mathcal{O}_X$. This is an autoequivalence of $\mathrm{D^b}(X)$ which maps $E \in \mathrm{D^b}(X)$ to the cone of the triangle

$$\mathrm{Hom}^{\bullet}(\mathcal{O}_X, E) \otimes \mathcal{O}_X \xrightarrow{ev} E \longrightarrow T_{\mathcal{O}_X}(E) \longrightarrow \mathrm{Hom}^{\bullet}(\mathcal{O}_X, E) \otimes \mathcal{O}_X\,[1]\,.$$

The space of all stability conditions is simply connected and given by

$$\mathrm{Stab}(X) = \bigcup_{k\in\mathbb{Z}} T^k_{\mathcal{O}_X}\Big(\sigma_{(0)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \cup \bigcup_{\gamma\in(0,1/2)} \sigma^{\gamma}_{(1)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})\Big).$$

The orbits in this expression are disjoint. The isotropy group of $\sigma_{(0)}$ is trivial whereas the isotropy groups of $\sigma^{\gamma}_{(1)}$ are of real dimension 2. One can imagine $\mathrm{Stab}(X)$ as an infinite helix similar to the picture in the case of generic tori on page 26. The autoequivalence $T_{\mathcal{O}_X}$ maps one 'semicircle' in that picture to its neighbour. Note that a point in the helix represents a simply connected 2-dimensional subspace in the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit of some stability condition.

Projective K3 surfaces
Note that the K3 surfaces of the previous example are highly non-projective. The case of projective K3 surfaces has been studied by Bridgeland in [7]. In order to state his results, we need some notation. In the case of K3 surfaces the

Mukai vector (map) v identifies the numerical Grothendieck lattice $(\mathcal{N}(X), -\chi)$ with the extended Néron–Severi lattice $\mathbb{Z} \oplus NS(X) \oplus \mathbb{Z} \subseteq \mathrm{H}^*(X, \mathbb{Z})$ together with the Mukai pairing

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

We introduce the root system of the lattice

$$\Delta(X) = \{\delta \in \mathcal{N}(X) \mid \langle \delta, \delta \rangle = -2\}$$

and we denote by $\delta^\perp$ the orthogonal complement of δ in $\mathcal{N}(X) \otimes \mathbb{C}$ with respect to the $\mathbb{C}$-linear extended Mukai pairing. Furthermore, we write $\mathcal{P}^{\pm}(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ for the two connected components of the set of those vectors in $\mathcal{N}(X) \otimes \mathbb{C}$ whose real and imagenary part span a positive definite two-plane in $\mathcal{N}(X) \otimes \mathbb{R}$. Using this notation, we can state the main theorem of Bridgeland's article [7].

Theorem 1.1.5 ([7], Theorem 1.1)**.** *There is a connected component* $\Sigma(X) \subset \mathrm{Stab}(X)$ *which is mapped by* π *onto the open subset*

$$\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp \subset \mathcal{N}(X) \otimes \mathbb{C}.$$

Moreover, the induced map $\pi : \Sigma(X) \longrightarrow \mathcal{P}_0^+(X)$ *is a regular covering map and the subgroup of* $\mathrm{Aut}(\mathrm{D^b}(X))$*, which acts trivial on the cohomology* $\mathrm{H}^*(X, \mathbb{Z})$ *and preserves the connected component* $\Sigma(X)$*, acts freely on* $\Sigma(X)$ *and is the group of deck transformations of* π*.*

The ideas of the proof apply also to the case of abelian surfaces. The results are even better than in the case of K3 surfaces.

Theorem 1.1.6 ([7], Theorem 15.2)**.** *Let* X *be an abelian surface over* $\mathbb{C}$*. There is a connected component* $\mathrm{Stab}^\dagger(X) \subset \mathrm{Stab}(X)$ *which is mapped by* π *onto the open subset* $\mathcal{P}^+(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$*. Moreover, the induced map*

$$\pi : \mathrm{Stab}^\dagger(X) \longrightarrow \mathcal{P}^+(X)$$

is the universal cover, and the group of deck transformations is generated by the double shift functor [2]. In particular, $\mathrm{Stab}^\dagger(X)$ *is simply connected.*

Generic complex tori

Let X be a generic complex torus such that $\mathrm{ch}_k(\mathcal{F}) = 0$ for all $0 < k < \dim(X) =: d$ and all coherent sheaves $\mathcal{F}$ on X. Due to our first example, we can assume $d \geq 2$. A detailed investigation of $\mathrm{Stab}(X)$ will be given in section 1.3. At this point we will give a short summary of the main results. See also [28].

As in the K3 case there are two t-structures on $\mathrm{D^b}(X)$ with hearts $\mathrm{Coh}(X) = \mathrm{Coh}_{(0)}(X)$ and $\mathrm{Coh}_{(1)}(X)$. For $\dim(X) = d > 2$ one has to consider further $d-2$ bounded t-structures. Their hearts $\mathrm{Coh}_{(p)}(X)$, $2 \leq p \leq d-1$, are the full subcategories of direct sums $\mathcal{F}[p] \oplus \mathcal{T}$, where $\mathcal{F}$ is a locally free sheaf and $\mathcal{T}$ is a torsion sheaf.

Theorem 1.1.7. *The abelian categories* $\mathrm{Coh}_{(p)}(X)$ *are of finite length for* $1 \leq p \leq d-1$.

To each of these t-structures we associate a function $Z_{(p)}(E) = -\operatorname{ch}_d(E) + (-1)^p i \operatorname{rk}(E)$ and obtain d stability conditions $\sigma_{(p)}$, $0 \leq p \leq d-1$. The $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-isotropy group of $\sigma_{(p)}$ is trivial and the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit is open in $\mathrm{Stab}(X)$. The orbit of $\sigma_{(p)}$ is connected with the orbit of $\sigma_{(p-1)}$ by a wall of real dimension 3 for $0 < p \leq d-1$. Similar to the case of K3 surfaces, this wall is a 1-parameter family of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbits of some stability conditions $\sigma^\gamma_{(p)}$, $\gamma \in (0,1/2)$, with isotropy groups of real dimension 2. The t-structure underlying $\sigma^\gamma_{(p)}$ is the one with heart $\mathrm{Coh}_{(p)}$ and the central charge is $Z^\gamma_{(p)}(E) = -\operatorname{ch}_d(E) - (-1)^p \cot(\pi\gamma)\operatorname{rk}(E)$.

Theorem 1.1.8. *The set*

$$U(X) := \left(\bigcup_{0 \leq p < d} \sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \;\cup \bigcup_{\substack{1 \leq p < d \\ \gamma \in (0,1/2)}} \sigma^\gamma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \right)$$

is a simply connected component of $\mathrm{Stab}(X)$ *and* $U(X) = \mathrm{Stab}(X)$ *for* $\dim(X) \leq 2$. *The orbits in this formula are disjoint. The map* $\pi : U(X) \to \pi(U(X)) \subset \mathrm{H}^*(X,\mathbb{C})$ *is a covering only in dimension* $d \leq 2$. *Furthermore,* $U(X)$ *is the set of all stability conditions on* $\mathrm{D^b}(X)$ *such that* $k(y) \in \mathcal{P}(\phi)$ $\forall\, y \in X$ *and* $L \in \mathcal{P}(\psi)$ $\forall\, L \in \mathrm{Pic}^0(X) = \hat{X}$ *for two real numbers* ϕ, ψ.

Note that the latter characterization of $U(X)$ is invariant under Fourier–Mukai transform with respect to the Poincaré bundle. Hence, $U(X)$ and $U(\hat{X})$ are canonically isomorphic. It is unknown whether there are more stability conditions on $\mathrm{D^b}(X)$ for $\dim(X) \geq 3$. The picture on page 26 illustrates $U(X)$ and $\pi(U(X))$. Note that a point in the helix represents a simply connected 2-dimensional subspace in the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit of some stability condition, whereas a point in the annulus below represents a 2-dimensional subspace in the $\mathrm{GL}^+(2,\mathbb{R})$-orbit with fundamental group $\mathbb{Z}$.

Kähler manifolds without nontrivial subvarieties

We will generalize the bounded t-structures with heart $\mathrm{Coh}_{(p)}$ to compact complex Kähler manifolds without any nontrivial (analytic) subvarieties in section 1.4. Theorem 1.1.7 is still valid. Together with the central charges $Z_{(p)}(E) = -\operatorname{ch}_d(E) + (-1)^p i \operatorname{rk}(E)$ one obtains d stability conditions $\sigma_{(p)}$, $0 \leq p \leq d-1$, on $\mathrm{D^b}(X)$ as in the case of generic tori. Note that curves, generic K3 surfaces and generic tori are Kähler manifolds of this type. More examples of Kähler manifolds having this property are given by general deformations of the Hilbert scheme of a K3 surface.

$\mathbb{P}^n$ and del Pezzo surfaces

Stability conditions on the projective spaces $\mathbb{P}^n$ and on del Pezzo surfaces have been studied by E. Macrì in [24].

Stability and group actions

Let X be a smooth projective variety and G a finite group acting on X. The group G acts also by autoequivalences on $\mathrm{D^b}(X)$ and we obtain a G-action on $\mathrm{Stab}(X)$. Using this, one can construct stability conditions on the equivariant derived category $\mathrm{D}^b_G(X)$ due to the following theorem proved by E. Macrì, S. Mehrotra and P. Stellari.

Theorem 1.1.9 ([26], Theorem 1.1)**.** *The subset of invariant stability conditions in* $\mathrm{Stab}(\mathrm{D^b}(X)) = \mathrm{Stab}(X)$ *is a closed submanifold with a closed embedding into* $\mathrm{Stab}(\mathrm{D}^b_G(X))$ *via the forgetful functor.*

The authors apply this theorem to the case of some weighted projective lines, Kummer surfaces and Enriques surfaces. See [26] for more details.

Non-compact examples

There are also examples of stability conditions for non-compact Kähler manifolds in [5] and [6].

1.2 Stability conditions on generic K3 surfaces

In this section we give a full description of the space of stability conditions on the derived category of a generic K3 surface X. A generic K3 surface is a K3 surface with a vanishing Picard group. We will prove that the space of all stability conditions is connected and simply connected and we will explain the formula

$$\mathrm{Stab}(X) = \bigcup_{k\in\mathbb{Z}} T^k_{\mathcal{O}_X}\Big(\sigma_{(0)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \cup \bigcup_{\gamma\in(0,1/2)} \sigma^{\gamma}_{(1)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})\Big).$$

Furthermore, the map π is the universal cover of its image.

1.2.1 General results about stability conditions on generic K3 surfaces

In this subsection we collect some general statements about stability conditions on generic K3 surfaces. Let X be a K3 surface with $\mathrm{Pic}(X) = 0$. A K3 surface of this kind is called generic because the set of K3 surfaces with non-vanishing Picard group is a countable union of hypersurfaces in the moduli space of K3 surfaces. The Mukai vector $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$ of $E \in \mathrm{D^b}(X)$ is given by the pair $v(E) = (\mathrm{rk}(E), s(E)) \in \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})$ with $s(E) = \mathrm{ch}_2(E) + \mathrm{rk}(E)$. As $\mathrm{Pic}(X) = \mathrm{H}^2(X,\mathbb{Z}) \cap \mathrm{H}^{1,1}(X) = 0$, we can drop $c_1(E) = 0$ in the notation. Let $\langle(r_1, s_1), (r_2, s_2)\rangle = -r_1s_2 - s_1r_2$ be the Mukai pairing on $\mathcal{N}(X) \cong \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})$. Using the shorthands $\mathrm{Hom}^i(E,F) := \mathrm{Hom}(E,F[i]) \cong \mathrm{Hom}(E[-i],F)$ and $\hom^i(E,F) := \dim_{\mathbb{C}} \mathrm{Hom}^i(E,F)$ for $E,F \in \mathrm{D^b}(X)$, the Grothendieck–Riemann–Roch theorem yields

$$-\chi(E,F) = -\sum_{i\in\mathbb{Z}}(-1)^i \hom^i(E,F) = \langle v(E), v(F)\rangle \tag{1.1}$$

for $E,F \in \mathrm{D^b}(X)$. If E and F are contained in the heart $\mathcal{A}$ of some bounded t-structure, e.g. $\mathcal{A} = \mathcal{P}((0,1])$ for some stability condition $\sigma = (Z,\mathcal{P})$ on X, we get $\mathrm{Hom}^i(E,F) = 0$ for $i < 0$ and $\mathrm{Hom}^i(E,F) = 0$ for $i > 2$ by Serre duality

$$\mathrm{Hom}^i(E,F) \cong \mathrm{Hom}^{2-i}(F,E)^\vee \qquad \forall i \in \mathbb{Z}.$$

In that case equation (1.1) simplifies to

$$-\chi(E,F) = \hom^1(E,F) - \hom(E,F) - \hom(F,E) = \langle v(E), v(F)\rangle. \tag{1.2}$$

The following lemma is a simple consequence of the previous equation.

Lemma 1.2.1 ([7], Lemma 4.1)**.** *If E is stable in some stability condition $\sigma = (Z,\mathcal{P})$ on X, then $\langle v(E), v(E)\rangle = v(E)^2 \geq 0$ or $v(E)^2 = -2$, where the last case occurs if and only if E is spherical, i.e.* $\hom(E,E) = \hom^2(E,E) = 1$ *and* $\hom^1(E,E) = 0$.

Proof. Since E is stable, it is simple in some of the abelian categories $\mathcal{P}(\phi)$. Hence, $\mathrm{Hom}(E,E) = \mathbb{C} \cdot id_E$. The square $v(E)^2$ is always an even number, and formula (1.2) gives the first statement of the lemma. The case $v(E)^2 = -2$ occurs if and only if $\hom^1(E,E) = 0$. Since $\hom(E,E) = \hom^2(E,E) = 1$ and $\hom^i(E,E) = \hom^{2-i}(E,E) = 0$ for $i < 0$, this is exactly the case if E is spherical. □

The next two lemmas will provide us with stable objects.

Lemma 1.2.2 ([22], Lemma 3.1 and Proposition 1.11)**.**
The sheaf $\mathcal{O}_X$ is stable in every stability condition $\sigma = (Z, \mathcal{P})$ on X.

Corollary 1.2.3 ([22], Lemma 2.3)**.** *If two skyscraper sheaves $k(x)$ and $k(y)$ are stable in some stability condition $\sigma = (Z, \mathcal{P})$ on X, they have the same phase.*

Proof. Let us denote by ϕ_x, ϕ_y and ψ the phases of the stable objects $k(x), k(y)$ and $\mathcal{O}_X$ in σ, where we have already used the previous Lemma 1.2.2. Since $0 \neq \hom(\mathcal{O}_X, k(x)) = \hom(k(x)[-2], \mathcal{O}_X)$, we obtain

$$\phi_x - 2 \leq \psi \leq \phi_x$$

due to condition (b) and (c) of Definition 1.1.1 and similar for ϕ_y. Equality cannot occur. For example, $\phi_x = \psi$ would imply that $k(x)$ and $\mathcal{O}_X$ are simple objects of the same abelian category $\mathcal{P}(\psi)$, and since $\hom(\mathcal{O}_X, k(x)) \neq 0$, we obtain the contradiction $\mathcal{O}_X \cong k(x)$. Hence,

$$\phi_x - 2 < \psi < \phi_x \quad \text{and} \quad \phi_y - 2 < \psi < \phi_x\,. \tag{1.3}$$

Since $Z(k(x)) = Z(k(y))$, we get $\phi_x = \phi_y + 2k_{x,y}$ with $k_{x,y} \in \mathbb{Z}$. Due to the upper inequalities, this is only possible for $k_{x,y} = 0$, and the assertion follows. □

One might ask whether the skyscraper sheaves $k(x)$ are stable with respect to all stability condition on X. This is not the case as we will see in subsection 1.2.3, but there is a weaker assertion. In order to state this, we need the spherical twist T on the derived category $\mathrm{D^b}(X)$ induced by the spherical object $\mathcal{O}_X$. The spherical twist is an autoequivalence (see [37] or Prop. 8.6 in [20]), and for every object $E \in \mathrm{D^b}(X)$ there is a distinguished triangle

$$\bigoplus_{i \in \mathbb{Z}} \mathcal{O}_X[i] \otimes \mathrm{Hom}(\mathcal{O}_X[i], E) \longrightarrow E \longrightarrow T(E) \longrightarrow \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_X[i+1] \otimes \mathrm{Hom}(\mathcal{O}_X[i], E),$$

where the first map is the 'evaluation map'.

Lemma 1.2.4 ([22], Corollary 3.7)**.** *For every stability condition on X there exists an integer $n \in \mathbb{Z}$ such that for every point $x \in X$ the complex $T^n(k(x))$ is stable of the same phase.*

1.2.2 Stability conditions with stable points

In this subsection we construct all stability conditions such that all skyscraper sheaves $k(x)$ are stable. We will identify the central charge $Z \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X), \mathbb{C})$ with a real 2×2 matrix

$$Z \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} r \\ s \end{pmatrix} \longrightarrow \begin{pmatrix} ar + bs \\ cr + ds \end{pmatrix} \simeq (ar + bs) + i(cr + ds)\,.$$

The stability conditions σ^+ and $\sigma_{(\mathbf{0})}$
We consider the abelian category $\mathrm{Coh}(X)$ together with its canonical t-structure on $\mathrm{D}^{\mathrm{b}}(X)$. Define $Z^+(r,s) = -s + ir$ on $\mathcal{N}(X) \cong \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})$. Using the fact that every torsion sheaf F has zero-dimensional support and $\mathrm{ch}_2(F) = \mathrm{H}^0(X,F)$, we see $Z^+(E) \in H = \{r \exp(i\pi\phi) \mid r > 0, 0 < \phi \leq 1\} \subseteq \mathbb{C}$ for all $0 \neq E \in \mathrm{Coh}(X)$. The pair $\sigma^+ = (Z^+, \mathrm{Coh}(X))$ defines a stability condition if Z^+ has the Harder–Narasimhan property. In order to check this, it is enough to show that for every sheaf E there are no infinite sequences

$$\cdots \subseteq E^{j+1} \subseteq E^j \subseteq \cdots \subseteq E^1 \subseteq E^0 = E$$

with $\phi(E^{j+1}) > \phi(E^j)$ for all $j \geq 0$ and

$$E = E_0 \twoheadrightarrow E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots$$

with $\phi(E_j) > \phi(E_{j+1})$ for all $j \geq 0$ due to [8] Proposition 2.4. Since $\mathrm{Coh}(X)$ is noetherian, we only need to prove the first assertion. Let us assume the existence of such a sequence. Since $\mathrm{rk}(E^{j+1}) \leq \mathrm{rk}(E^j)$, there must be an integer $n \geq 0$ such that $\mathrm{rk}(E^j) = \mathrm{rk}(E^n)$ for all $j \geq n$. Thus, $E^j/E^{j+1} = T^j$ is a torsion sheaf for all $j \geq n$ and $Z(E^j) - Z(E^{j+1}) = Z(T^j) < 0$ or $\phi(E^{j+1}) \leq \phi(E^j)$ which contradicts the assumption.
Let us denote by $\mathcal{P}^+$ the slicing of the stability condition σ^+. Obviously, $\mathcal{P}^+$ is locally-finite since Z^+ has a discrete image. By construction $\mathcal{P}^+(1) =: \mathcal{T}$ is the abelian category of torsion sheaves on X. Thus, every semistable sheaf E of phase $\phi \in (0,1)$ is torsionfree, and $\mathcal{P}^+((0,1)) =: \mathcal{F}$ is the quasi-abelian category of torsionfree sheaves. The abelian category $\mathrm{Coh}^\sharp(X) := \mathcal{P}^+([1,2))$ consists of those complexes E of length two with

$$H^0(E) \in \mathcal{P}^+(1) = \mathcal{T} \qquad \text{and} \qquad H^{-1}(E) = H^0(E[-1]) \in \mathcal{P}^+((0,1)) = \mathcal{F}\,.$$

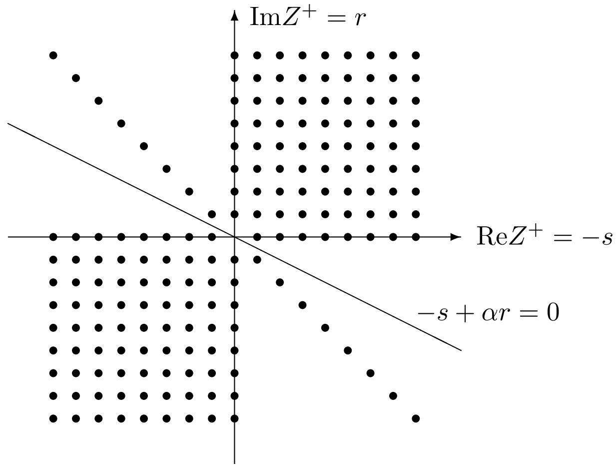
Furthermore, $(\mathcal{T}, \mathcal{F})$ is a torsion pair and $\mathrm{Coh}^\sharp(X) = \mathcal{P}^+([1,2))$ is the tilt of $\mathrm{Coh}(X)$ with respect to the pair $(\mathcal{T}, \mathcal{F})$. For the definition of a torsion pair and the corresponding tilt we refer to Definition 1.3.7 and Lemma 1.3.8 or to the article [16].
Moreover, every ideal sheaf is stable in σ^+ because every proper quotient sheaf is a torsion sheaf. In particular, we have the following stable objects

$$k(x) \in \mathcal{P}^+(1) \quad , \quad \mathcal{O}_X \in \mathcal{P}^+(3/4) \quad , \quad T(k(x)) = I_x[1] \in \mathcal{P}^+(3/2).$$

The last equation follows from the triangle $\mathcal{O}_X \to k(x) \to I_x[1] \to \mathcal{O}_X[1]$ and the triangle at the end of subsection 1.2.1. Using Lemma 1.2.1 we obtain

$$v(E)^2 = -2r(E)s(E) \geq 0 \text{ or } -2$$

for every stable object E in σ^+, and we see that for every semistable object E the value $Z^+(E) = -s(E) + i \cdot \mathrm{rk}(E)$ is one of the dots in the following picture.



Thus, $\mathcal{F} = \mathcal{P}^+((0,1)) = \mathcal{P}^+((0,3/4])$ and $\mathrm{Coh}^\sharp(X) = \mathcal{P}^+(([1,2)) = \mathcal{P}^+([1,7/4])$. It is easy to prove that the stability condition $\sigma_{(0)} = (Z_{(0)}, \mathrm{Coh}(X))$ with $Z_{(0)}(E) = -\mathrm{ch}_2(E) + i\,\mathrm{rk}(E) = -s(E) + (1+i)\,\mathrm{rk}(E)$ is contained in the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of σ^+. Moreover, $\det(Z^+) > 0$ and $\det(Z_{(0)}) > 0$.

The stability conditions σ^- and $\sigma_{(1)}$
The next example is the stability condition

$$\sigma^- = (Z^-, \mathcal{P}^-) := T^{-1} \cdot \sigma^+ \cdot \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, f(t) = t + 1/2 \right).$$

The spherical twist T acts like $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ on the (r,s)-plane. By the definitions of the actions, we get

$$Z^- \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{\simeq Z^+} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and, therefore, $Z^-(r,s) = -s - ir$. We obtain the following stable objects

$$\begin{aligned} \mathcal{O}_X[1] = T^{-1}(\mathcal{O}_X) &\quad \text{with phase} \quad \phi^+(\mathcal{O}_X) - 1/2 = 1/4, \\ k(x) = T^{-1}T(k(x)) = T^{-1}(I_x[1]) &\quad \text{with phase} \quad \phi^+(I_x[1]) - 1/2 = 1, \\ T^{-1}(k(x)) &\quad \text{with phase} \quad \phi^+(k(x)) - 1/2 = 1/2. \end{aligned}$$

We will show shortly that the heart $\mathcal{P}^-((0,1])$ of σ^- is the abelian category $\mathrm{Coh}^\sharp(X)$ and if we apply the transformation $s \mapsto s, r \mapsto -r$ to the picture above, we get the possible values of $Z^-(E)$ for semistable objects E in σ^-. In particular, $\mathrm{Coh}^\sharp(X) = \mathcal{P}^-([1/4,1])$. Furthermore, the stability condition $\sigma_{(1)} = (Z_{(1)}, \mathrm{Coh}^\sharp(X))$ with $Z_{(1)}(E) = -\mathrm{ch}_2(E) - i\,\mathrm{rk}(E) = -s(E) + (1-i)\,\mathrm{rk}(E)$ is contained in the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of σ^-. Moreover, $\det(Z^-) < 0$ and $\det(Z_{(1)}) < 0$.

The stability conditions $\sigma_\alpha = \sigma^\gamma_{(\mathbf{1})}$

Fix a real number $\alpha \in (1,\infty)$. Let us consider the t-structure of the previous example with heart $\mathrm{Coh}^\sharp(X)$ and the function $Z_\alpha(r,s) = -s + \alpha r$. Looking at the previous picture we see that $Z_\alpha(E) < 0$ for all $0 \neq E \in \mathrm{Coh}^\sharp(X) = \mathcal{P}^+([1,7/4])$. Hence, Z_α is a stability function on $\mathrm{Coh}^\sharp(X)$, and the pair $\sigma_\alpha = (Z_\alpha, \mathrm{Coh}^\sharp(X))$ defines a stability condition if Z_α has the Harder–Narasimhan property. But the latter is clear since every object $E \in \mathrm{Coh}^\sharp(X)$ is semistable of phase one with respect to Z_α. Furthermore, the abelian category $\mathrm{Coh}^\sharp(X)$ has finite length because the image $Z_\alpha(\mathrm{Coh}^\sharp(X)) \subseteq \mathbb{R}_{<0}$ is a discrete subset. This is because for $m > 0$ the inverse image $Z_\alpha^{-1}([-m,0])$ in the integral $(-s,r)$-plane of the picture is the finite set bounded by the three lines

$$\{r=0\} \quad , \quad \{r=s\} \quad \text{and} \quad \{-s+\alpha r = -m\}.$$

Therefore, σ_α is a locally-finite stability condition on X, and we will denote its slicing by $\mathcal{P}_\alpha$. Hence, $\mathcal{P}_\alpha(\phi) = \mathrm{Coh}^\sharp(X)[\phi-1]$ for $\phi \in \mathbb{Z}$ and $\mathcal{P}_\alpha(\phi) = 0$ else. Obviously, the slicing $\mathcal{P}_\alpha$ does not depend on α. The objects $k(x)$ and $\mathcal{O}_X[1]$ are stable in σ_α because their Mukai vectors are primitive and extremal in the domain $1 \leq \phi \leq 7/4$ of the picture. In contrast to this, we have the following exact sequence in $\mathrm{Coh}^\sharp(X) = \mathcal{P}_\alpha(1)$

$$0 \longrightarrow k(x) \longrightarrow T(k(x)) = I_x[1] \longrightarrow \mathcal{O}_X[1] \longrightarrow 0$$

which is the Jordan–Hölder filtration of the semistable but not stable object $T(k(x))$. For every $\alpha > 1$ there is a unique $\gamma \in (0,1/2)$ with $\alpha = 1 + \cot(\pi\gamma)$, i.e. $Z_\alpha(E) = -\mathrm{ch}_2(E) + \cot(\pi\gamma)\,\mathrm{rk}(E)$. If we write the central charge like this, we denote the stability condition by $\sigma^\gamma_{(1)}$. The reason for this will become clear in the next section. Moreover, $\det(Z\alpha) = \det(Z^\gamma_{(1)}) = 0$.

The rest of this subsection is dedicated to the proof of the following proposition.

Proposition 1.2.5. *The set $U(X) \subseteq \mathrm{Stab}(X)$ of all stability conditions on X such that for every point $x \in X$ the sheaf $k(x)$ is stable is given by the following disjoint union of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbits*

$$U(X) \;=\; \underbrace{\sigma^+\widetilde{\mathrm{GL}}^+(2,\mathbb{R})}_{=:\,U^+(X)} \;\cup\; \underbrace{\cup_{\alpha>1}\sigma_\alpha\widetilde{\mathrm{GL}}^+(2,\mathbb{R})}_{=:\,U^0(X)} \;\cup\; \underbrace{\sigma^-\widetilde{\mathrm{GL}}^+(2,\mathbb{R})}_{=:\,U^-(X)}\;.$$

Remark. The maximal real dimension of $\mathrm{Stab}(X)$ is less or equal $\dim_{\mathbb{R}}(\mathcal{N}(X)\otimes\mathbb{C}) = 4$. Since the $\mathrm{GL}^+(2,\mathbb{R})$-orbits of Z^+ and Z^- are of real dimension 4, the orbits $U^+(X)$ and $U^-(X)$ are open in $\mathrm{Stab}(X)$ and of real dimension 4. The $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit of σ_α is two-dimensional, and the family $(\sigma_\alpha)_{\alpha>1}$ varies continuously since the slicing $\mathcal{P}_\alpha$ is constant and the central charge Z_α depends continuously on α. Therefore, $U^0(X)$ is of real dimension 3. One should imagine the real hyperplane $U^0(X)$ as the boundary between the open sets $U^+(X)$ and $U^-(X)$ as we will see in the next subsection.

In order to prove this proposition, we need the following important lemma.

Lemma 1.2.6 ([7], Lemma 6.1)**.** *Suppose $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$ is a stability condition on X such that for each point $x \in X$ the sheaf $k(x)$ is stable in σ of phase 1. Let E be an object of $\mathrm{D^b}(X)$.*

(a) if $E \in \mathcal{P}((0,1])$ then the cohomology sheaves $H^i(E)$ vanish unless $i \in \{-1,0\}$, and, moreover, the sheaf $H^{-1}(E)$ is torsionfree,

(b) if $E \in \mathcal{P}(1)$ is stable then either $E = k(x)$ for some $x \in X$, or $E[-1]$ is a locally-free sheaf,

(c) if $E \in \mathrm{Coh}(X)$ is a sheaf on X then $E \in \mathcal{P}((-1,1])$.

Proof. The only crucial point in the proof of Lemma 6.1 in [7] is the requirement of a finite locally-free resolution for every coherent sheaf on X. Such a resolution exists in the projective case. Due to a result of [36], we can find a finite locally-free resolution of a coherent sheaf even on a non-projective compact complex surface. □

Corollary 1.2.7 ([7], Proposition 6.2)**.** *Suppose $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$ is a stability condition on X such that for each point $x \in X$ the sheaf $k(x)$ is stable in σ of phase one. Then, the categories*

$$\mathcal{T}' := \mathrm{Coh}(X) \cap \mathcal{P}((0,1]) \quad \textit{and} \quad \mathcal{F}' := \mathrm{Coh}(X) \cap \mathcal{P}((-1,0])$$

form a torsion pair in $\mathrm{Coh}(X)$, i.e.

1. $\mathrm{Hom}(\mathcal{T}', \mathcal{F}') = 0$ *and*

2. *for every $E \in \mathrm{Coh}(X)$ there is a short exact sequence* $0 \to T \to E \to F \to 0$ *with $T \in \mathcal{T}'$ and $F \in \mathcal{F}'$.*

Furthermore, $\mathcal{P}((0,1])$ is the corresponding tilt, i.e. $\mathcal{P}((0,1])$ is the full subcategory of $\mathrm{D^b}(X)$ with objects $\{E \in \mathrm{D^b}(X) |\, H^0(E) \in \mathcal{T}', H^{-1}(E) \in \mathcal{F}', H^i(E) = 0$ otherwise$\}$.

Proof. The first property of the pair $(\mathcal{T}', \mathcal{F}')$ follows from the definition of a slicing. Due to (c) of the previous lemma, there is a triangle $T \to E \to F \to T[1]$ with $T \in \mathcal{P}((0,1])$ and $F \in \mathcal{P}((-1,0])$. We apply the cohomology functor to this triangle. Using (a) of the lemma and $E \in \mathrm{Coh}(X)$, we obtain the exact cohomology sequence

$$0 \to H^{-1}(T) \to 0 \to 0 \to H^0(T) \to E \to H^0(F) \to 0 \to 0 \to H^1(F) \to 0\,.$$

Hence, $T \cong H^0(T) \in \mathcal{T}'$, $F \cong H^0(F) \in \mathcal{F}'$ and the triangle gives the required short exact sequence in $\mathrm{Coh}(X)$.
In order to prove the last statement, we denote the heart $\{E \in \mathrm{D^b}(X) | \, H^0(E) \in \mathcal{T}', H^{-1}(E) \in \mathcal{F}', H^i(E) = 0 \text{ otherwise}\}$ of the tilted t-structure by $\mathcal{P}'$ and mention that $\mathcal{T}' \subseteq \mathcal{P}((0,1])$ and $\mathcal{F}'[1] \subseteq \mathcal{P}((0,1])$. Since every object $E \in \mathcal{P}'$ is an extension of the object $H^0(E) \in \mathcal{T}'$ by $H^{-1}(E)[1] \in \mathcal{F}'[1]$, and since $\mathcal{P}((0,1])$ is closed under extension, we get $\mathcal{P}' \subseteq \mathcal{P}((0,1])$. It is a standard argument to conclude $\mathcal{P}' = \mathcal{P}((0,1])$ because both abelian subcategories are the hearts of bounded t-structures. □

Finally, we prove the Proposition 1.2.5.

Proof. Suppose $\sigma = (Z, \mathcal{P})$ is a stability condition such that all the skyscraper sheaves $k(x)$ are stable. By Corollary 1.2.3 the phase does not depend on the point $x \in X$. There are three cases.

1. $\det(Z) > 0$: After applying some element $(g, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ to σ we can assume $Z(r, s) = -s + ir$, and all sheaves $k(x)$ are stable of phase one. For $E \in \mathrm{Coh}(X) \cap \mathcal{P}((-1, 0])$ we get $\mathrm{rk}(E) = 0$. On the other hand, all torsion sheaves are contained in $\mathcal{P}(1)$. Hence, $E = 0$. With the notation of the last corollary we get $\mathcal{F}' = 0$ and, therefore, $\mathcal{T}' = \mathrm{Coh}(X)$. Thus, $\mathcal{P}((0,1]) = \mathrm{Coh}(X)$, i.e. $\sigma = \sigma^+$.

2. $\det(Z) < 0$: After applying some element $(g, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ to σ we can assume $Z(r, s) = -s - ir$, and all sheaves $k(x)$ are stable of phase one. For $E \in \mathrm{Coh}(X) \cap \mathcal{P}((0, 1])$ we get $\mathrm{rk}(E) = 0$ and E must be a torsion sheaf. The latter are contained in the intersection. Therefore, $\mathcal{T}' = \{\text{torsion sheaves}\}$ and we obtain $\mathcal{F}' = \{\text{torsionfree sheaves}\}$. Thus, $\mathcal{P}((0,1]) = \mathrm{Coh}^\sharp(X)$, i.e. $\sigma = \sigma^-$.

3. $\det(Z) = 0$: After applying some element $(g, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ to σ we can assume $Z(r, s) = -s + \alpha r$, and all sheaves $k(x)$ are stable of phase one. Since $k(x) \in \mathcal{P}(1)\ \forall x \in X$, the torsion sheaves are contained in the intersection $\mathcal{T}' = \mathrm{Coh}(X) \cap \mathcal{P}((0,1]) = \mathrm{Coh}(X) \cap \mathcal{P}(1)$. If $E \in \mathcal{T}'$ is not a torsion sheaf, there is an epimorphism $E \to k(x)$ in $\mathrm{Coh}(X)$ with nontrivial kernel E' for some point $x \in X$. This map is also an epimorphism in $\mathcal{P}(1)$ since $k(x)$ is stable. Therefore, the kernel E' is also in $\mathcal{T}'$ and, of course, not a torsion sheaf. But $Z(E') = Z(E) - Z(k(x)) = Z(E) + 1$. Repeating this argument with E' if necessary, we obtain some $\widetilde{E} \in \mathcal{T}'$ with $Z(\widetilde{E}) > 0$ which is a contradiction. Hence, $\mathcal{T}' = \{\text{torsion sheaves}\}$, and as in the second case we conclude $\mathcal{P}((0,1]) = \mathrm{Coh}^\sharp(X)$. Looking at the previous picture, we see $Z(E) < 0$ for all $E \in \mathrm{Coh}^\sharp(X)$ if and only if $\alpha > 1$. Thus, $\sigma = \sigma_\alpha$.

□

1.2.3 The space $\mathrm{Stab}(X)$

This section contains the full classification of all stability conditions on $\mathrm{D^b}(X)$.

Lemma 1.2.8. *The subspace $U(X) \subseteq \mathrm{Stab}(X)$ is connected.*

Proof. Version 1: Since $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ acts continuously on $\mathrm{Stab}(X)$, $U^+(X)$ and $U^-(X)$ are connected. We show that every neighbourhood of σ_α meets $U^+(U)$ and $U^-(U)$, and the assertion follows. Since the central charge of a stability condition depends only on the Mukai vector, the set $\{k(x) \mid x \in X\}$ of sheaves has bounded mass. See [7] Definition 9.1 for the definition of a set with bounded mass. Hence, we can apply Corollary 9.4 of [7] because $k(x)$ has primitive Mukai vector. Since all the sheaves $k(x)$ are stable in σ_α, they must be stable in some neighbourhood of σ_α in $\mathrm{Stab}(X)$ (use Corollary 9.4 in [7]). The slicing underlying σ_α is independent of α and the central charge Z_α depends continuously on α. Thus, $(\sigma_\alpha)_{\alpha>1}$ is a continuous family of stability conditions, and since the two-dimensional $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbits are connected, the set $U^0(X)$ is connected as well and of dimension three. Since $\mathrm{Stab}(X)$ has an even real dimension, there must be stability conditions $\sigma_1 = (Z_1, \mathcal{P}_1)$ and $\sigma_2 = (Z_2, \mathcal{P}_2)$ in this neighbourhood of σ_α with $\det Z_1 > 0$ and $\det Z_2 < 0$. Therefore, each neighbourhood meets $U^+(X)$ and $U^-(X)$ by Proposition 1.2.5.

Version 2: For a fixed $\alpha > 1$ we construct a continuous path in $U(X)$ from σ^- over σ_α to σ^+. Since all $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbits are connected, the assertion follows from Proposition 1.2.5. Let us consider

$$
\begin{aligned}
\sigma(t) &= \begin{cases} \Big(Z^{(t)}(r,s) = -s + \alpha(1-|t|)r + itr, \mathrm{Coh}^\sharp(X)\Big) & \text{for } t \in [-1,0], \\ \Big(Z^{(t)}(r,s) = -s + \alpha(1-|t|)r + itr, \mathrm{Coh}(X)\Big) & \text{for } t \in (0,1] \end{cases} \\
&= \begin{cases} \Big(-s - ir, \mathrm{Coh}^\sharp(X)\Big)\left(\begin{pmatrix} 1 & -\alpha(1-|t|) \\ 0 & -t \end{pmatrix}^{-1}, f_t^-\right) & \text{for } t \in [-1,0), \\ \Big(-s + \alpha r, \mathrm{Coh}^\sharp(X)\Big) & \text{for } t = 0, \\ \Big(-s + ir, \mathrm{Coh}(X)\Big)\left(\begin{pmatrix} 1 & \alpha(1-|t|) \\ 0 & t \end{pmatrix}^{-1}, f_t^+\right) & \text{for } t \in (0,1]. \end{cases}
\end{aligned}
$$

for suitable continuous functions $f^- : \mathbb{R} \times [-1,0) \ni (s,t) \mapsto f_t^-(s) \in \mathbb{R}$ and $f^+ : \mathbb{R} \times (0,1] \ni (s,t) \mapsto f_t^+(s) \in \mathbb{R}$, both increasing and 1-periodic with respect to s. Using the second description, we see that $\sigma(t)$ is in $\mathrm{Stab}(X)$ for all $t \in [-1,1]$, and $\sigma(\cdot)$ is continuous at least on $[-1,0) \cup (0,1]$. Let us now consider the case $t \nearrow 0$. We have $\mathcal{P}_\alpha(1) = \mathrm{Coh}^\sharp(X) = \mathcal{P}^-((0,1]) = \mathcal{P}^-([1/4,1])$, and for $\mathcal{O}_X[1] \in \mathcal{P}^-(1/4)$ we get $Z^{(t)}(\mathcal{O}_X[1]) = 1 - \alpha(1-|t|) - it \longrightarrow 1 - \alpha$ for $t \nearrow 0$. Hence, $\phi^{(t)}(\mathcal{O}_X[1]) \nearrow 1$, and of course $\phi^{(t)}(k(x)) = 1$. Because of $\phi^{(t)}(\mathcal{O}_X[1]) \leq$

$\phi^{(t)}(E) \leq \phi^{(t)}(k(x))$ for every semistable $E \in \mathrm{Coh}^\sharp(X) = \mathcal{P}^-([1/4,1])$, there exists for all $\epsilon > 0$ a real number $\delta > 0$ such that for all $t \in (-\delta, 0]$

$$\mathcal{P}_\alpha(1) = \mathcal{P}^-([1/4,1]) \subseteq \mathcal{P}^{(t)}([1-\epsilon,1]) \subseteq \mathcal{P}^{(t)}([1-\epsilon,1+\epsilon])$$

which is equivalent to $d(\mathcal{P}_\alpha, \mathcal{P}^{(t)}) \leq \epsilon$, where d is the metric on the space of slicings introduced by Bridgeland in Lemma 6.1 of [8]. Since $\lim_{t\to 0} Z^{(t)} = Z_\alpha$, we obtain $\lim_{t\nearrow 0}\sigma(t) = \sigma_\alpha$. Similar, $\mathcal{P}_\alpha(1) = \mathcal{P}^+([1,2)) = \mathcal{P}^+([1,7/4])$, and for $\mathcal{O}_X[1] \in \mathcal{P}^+(7/4)$ we get $Z^{(t)}(\mathcal{O}_X[1]) = 1-\alpha(1-|t|) - it \longrightarrow 1-\alpha$ for $t \searrow 0$. Hence, $\phi^{(t)}(\mathcal{O}_X[1]) \searrow 1$, and of course $\phi^{(t)}(k(x)) = 1$. Since $\phi^{(t)}(k(x)) \leq \phi^{(t)}(E) \leq \phi^{(t)}(\mathcal{O}_X[1])$ for every semistable $E \in \mathrm{Coh}^\sharp(X) = \mathcal{P}^+([1,7/4])$, there exists for all $\epsilon > 0$ a real number $\delta > 0$ such that for all $t \in (-\delta, 0]$

$$\mathcal{P}_\alpha(1) = \mathcal{P}^+([1,7/4]) \subseteq \mathcal{P}^{(t)}([1,1+\epsilon]) \subseteq \mathcal{P}^{(t)}([1-\epsilon,1+\epsilon])$$

which is equivalent to $d(\mathcal{P}_\alpha, \mathcal{P}^{(t)}) \leq \epsilon$. Since $\lim_{t\to 0} Z^{(t)} = Z_\alpha$, we obtain $\lim_{t\searrow 0}\sigma(t) = \sigma_\alpha$. Thus, $\sigma(\cdot) : [-1,1] \longrightarrow U(X)$ is a continuous path and the assertion follows. $\square$

Theorem 1.2.9. *The topological space* $\mathrm{Stab}(X)$ *is connected and given by*

$$\begin{aligned}\mathrm{Stab}(X) &= \bigcup_{k\in\mathbb{Z}} T^k\, U(X) \\ &= \bigcup_{k\in\mathbb{Z}} T^k \left(\sigma^+ \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \;\cup\; \cup_{\alpha>1}\sigma_\alpha \widetilde{\mathrm{GL}}^+(2,\mathbb{R})\right),\end{aligned}$$

and all $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$*-orbits in the last equation are disjoint.*

Proof. The first equation follows from Lemma 1.2.4 and Proposition 1.2.5. Since $U^-(X) = T^{-1}U^+(X)$, the second equation is a consequence of the first equation and the description of $U^+(X)$ and $U^0(X)$. Using the first equation and $T^kU(X) \cap T^{k-1}U(X) = T^k(U(X) \cap T^{-1}U(X)) \supseteq T^kU^-(X) \neq \emptyset$ for all $k \in \mathbb{Z}$, we obtain that the space $\mathrm{Stab}(X)$ is connected because $U(X)$ is connected by the previous lemma. For the last statement of the theorem we consider the case

$$T^p\sigma_{\alpha'}\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \cap T^q\sigma_\alpha\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \;\neq\; \emptyset$$

with $\alpha, \alpha' > 1$. After applying T^{-p} and some $(g', f') \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ we may assume $\sigma_{\alpha'} = T^m\sigma_\alpha(g,f)$. Since the orbits of the corresponding central charges are parametrized by $\alpha > 1$, and T maps the orbit of Z_α onto the orbit of $Z_{1/\alpha}$, we conclude $m = 2k$ and $\alpha = \alpha'$. Hence, $g^{-1}Z_\alpha = Z_\alpha$, i.e. g is contained in the stabilizer of Z_α. Using the covering map $\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \ni (g,f) \longmapsto g \in \mathrm{GL}^+(2,\mathbb{R})$, the group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ acts on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X),\mathbb{C})$ as well, and the stabilizer of every stability condition σ is a normal subgroup of the stabilizer of its central charge Z. The quotient group of these stabilizers is freely generated by the pair $(id, f(t) = t+2)$ which acts like the double shift functor [2] on $\mathrm{Stab}(X)$. Therefore, we can substitute the action of (g,f), which is in the stabilizer of Z_α, by the action of some shift functor $[2l]$. The case $\sigma_\alpha = T^{2k} \circ [2l]\, \sigma_\alpha$ can only occur for $k = l = 0$. Indeed, due to $\sigma_\alpha = T^{2k} \circ [2l]\, \sigma_\alpha$, the object $T^{2k}(k(x))[2l]$ is stable in σ_α of phase 1. But $\mathcal{P}_\alpha(1) = \mathrm{Coh}^\sharp(X)$ contains only $k(x)$ and $T(k(x))$, as a direct calculation of $T^m(k(x))$ shows, and $T(k(x))$ is not stable in $\mathcal{P}_\alpha(1)$. Thus, the orbits of $T^k\sigma_\alpha$ are disjoint for different k and different α. In the same way one shows that the orbits of $T^k\sigma^+$ are disjoint for different k. $\square$

Let us denote the image of the map $\pi : \mathrm{Stab}(X) \ni \sigma = (Z, \mathcal{P}) \longmapsto Z \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}, \mathbb{C})$ by $\mathcal{W}$. Since π is a local homeomorphism, $\mathcal{W}$ is open. We aim to show that $\pi : \mathrm{Stab}(X) \to \mathcal{W}$ is the universal cover of $\mathcal{W}$.

Proposition 1.2.10. *The map $\pi : \mathrm{Stab}(X) \to \mathcal{W}$ is a covering.*

Proof. Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$ and $W \subseteq \mathcal{W}$ be a sufficiently small neighbourhood of Z in $\mathcal{W}$. Since π is a local homeomorphism, there is a small neighbourhood $V \subseteq \mathrm{Stab}(X)$ of σ in $\mathrm{Stab}(X)$ such that $\pi : V \to W$ is an isomorphism. By the definition of the topology on $\mathrm{Stab}(X)$, we can choose V so small that

$$|\phi^-_{\sigma_1}(k(x)) - \phi^-_{\sigma_2}(k(x))| < 1/2 \quad \text{and} \quad |\phi^+_{\sigma_1}(k(x)) - \phi^+_{\sigma_2}(k(x))| < 1/2$$

for all $\sigma_1, \sigma_2 \in V$, where $\phi^+_{\sigma_i}(k(x))$ and $\phi^-_{\sigma_i}(k(x))$ denote the biggest and the smallest phase of a semistable factor of $k(x)$ with respect to the stability condition σ_i. See section 6 in [8] for more details.
For every stability condition $\sigma = (Z, \mathcal{P})$ we get

$$\pi^{-1}(Z) \cap \big(\sigma\, \widetilde{\mathrm{GL}}^+(2,\mathbb{R})\big) = \sigma \cdot \left(\widetilde{\mathrm{GL}}^+(2,\mathbb{R})_\sigma \setminus \widetilde{\mathrm{GL}}^+(2,\mathbb{R})_Z\right) = \{[2l]\sigma \mid l \in \mathbb{Z}\}$$

as we have seen in the proof of the previous theorem. Using this and the same theorem, we obtain

$$\pi^{-1}W = \bigcup_{k,l\in\mathbb{Z}} T^{2k} \circ [2l]\, V \xrightarrow{\pi} W,$$

and $\pi : T^{2k}\circ[2l]\, V \to W$ is an isomorphism. We have to show that the union over k and l is a disjoint union. Let us take some $\sigma' = T^{2k}\circ[2l](\sigma'') \in V \cap (T^{2k}\circ[2l]\, V)$. Since the intersection of the two open sets is open, we can assume $\det Z_{\sigma'} \neq 0$. Replacing V by $T^m \cdot V \cdot (g, f)$ for suitable $m \in \mathbb{Z}$ and $(g, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and correspondingly for W, we can assume $\sigma^+ = \sigma' = T^{2k} \circ [2l](\sigma'')$ by Theorem 1.2.9. Since σ'' is contained in the neighbourhood V of σ^+, we can assume $\sigma'' \in \sigma^+\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ because the orbit is open. Therefore, $k(x)$ is stable in σ'' and because of $\phi_{\sigma^+}(k(x)) = 1$, the phase $\phi_{\sigma''}(k(x))$ is contained in the interval $(1/2, 3/2)$ due to the upper inequalities. The central charges of σ'' and σ^+ are the same and we conclude $\phi_{\sigma''}(k(x)) = 1$. Using $\sigma^+ = T^{2k} \circ [2l](\sigma'')$, we conclude that $T^{2k} \circ [2l](k(x))$ is stable in σ^+ of phase one. This is only possible for $k = l = 0$ because $T^{2k} \circ [2l](k(x))$ is not a sheaf for $(k, l) \neq (0, 0)$. Thus, the upper union is a disjoint union. $\square$

Lemma 1.2.11. *The fundamental group* $\pi_1(\mathcal{W})$ *of* $\mathcal{W}$ *is* $\mathbb{Z} \oplus \mathbb{Z}$.

We will prove this lemma in the next subsection by giving a nice model for $\mathcal{W}$.

Corollary 1.2.12. *The projection* $\pi : \mathrm{Stab}(X) \to \mathcal{W}$ *is the universal cover of* $\mathcal{W}$ *and the group of deck transformations is* $\mathbb{Z}T^2 \oplus \mathbb{Z}[2] \subseteq \mathrm{Aut}(\mathrm{D^b}(X))$.

Proof. As we have seen in the proof of Proposition 1.2.10, the group $G := \mathbb{Z}T^2 \oplus \mathbb{Z}[2] \subseteq \mathrm{Aut}(\mathrm{D^b}(X))$ is the group of deck transformations of the covering $\pi : \mathrm{Stab}(X) \to \mathcal{W}$. Furthermore, there is the short exact sequence

$$0 \longrightarrow \pi_*(\pi_1(\mathrm{Stab}(X))) \longrightarrow \mathrm{N}\,\pi_*(\pi_1(\mathrm{Stab}(X))) \longrightarrow G \longrightarrow 0,$$

where $\mathrm{N}\,\pi_*(\pi_1(\mathrm{Stab}(X)))$ is the normalizer of $\pi_*(\pi_1(\mathrm{Stab}(X)))$ in $\pi_1(\mathcal{W})$ (see [18], Proposition 1.39). Since the group $\pi_1(\mathcal{W}) = \mathbb{Z} \oplus \mathbb{Z}$ is abelian, we get $\mathrm{N}\,\pi_*(\pi_1(\mathrm{Stab}(X))) = \mathbb{Z} \oplus \mathbb{Z}$, and because of $G \cong \mathbb{Z} \oplus \mathbb{Z}$, we conclude that $\pi_*(\pi_1(\mathrm{Stab}(X))) = 0$. The map $\pi_* : \pi_1(\mathrm{Stab}(X)) \to \pi_1(\mathcal{W})$ is injective and $\pi_1(\mathrm{Stab}(X)) = 0$ follows. □

1.2.4 Autoequivalences of a generic K3 surface

This subsection is devoted to the calculation of the group of autoequivalences $\mathrm{Aut}(\mathrm{D^b}(X))$ of Fourier–Mukai type. In contrast to the algebraic case, not every autoequivalence of $\mathrm{D^b}(X)$ is a Fourier–Mukai transformation. We denote by $O^+(\mathrm{H}^*(X,\mathbb{Z}))$ the group of Hodge isometries preserving the orientation of the positive 4-plane

$$P = \mathrm{span}_{\mathbb{R}}(\mathrm{Re}[\sigma], \mathrm{Im}[\sigma], [\omega], (1,0,1)) \subseteq \mathrm{H}^*(X,\mathbb{Z}),$$

where $[\omega]$ is a fixed Kähler class and σ is a fixed holomorphic symplectic 2-form on X. Similarly, $O^+(\mathrm{H}^2(X,\mathbb{Z}))$ is the group of Hodge isometries fixing $\mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})$ and preserving the orientation of the positive 3-plane

$$P \cap \mathrm{H}^2(X,\mathbb{Z}) = \mathrm{span}_{\mathbb{R}}(\mathrm{Re}[\sigma], \mathrm{Im}[\sigma], [\omega]),$$

and $O(\mathcal{N}(X))$ is the orthogonal group of $\mathcal{N}(X) = \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})$.

Lemma 1.2.13. *Using the notation* $[\hat{1}] := -id \in O^+(\mathrm{H}^*(X,\mathbb{Z}))$ *and* $\hat{T} = s_\delta \in O^+(\mathrm{H}^*(X,\mathbb{Z}))$, *where* $s_\delta(x) = x + \langle x, \delta\rangle\delta$ *is the reflection in the plane* $\delta^\perp \supseteq P$ *with* $\delta = v(\mathcal{O}_X) = (1,0,1)$, *we obtain*

$$\begin{aligned} O^+(\mathrm{H}^*(X,\mathbb{Z})) &= O^+(\mathrm{H}^2(X,\mathbb{Z})) \oplus \mathbb{Z}/2\mathbb{Z}\cdot\hat{T} \oplus \mathbb{Z}/2\mathbb{Z}\cdot[\hat{1}] \\ &\cong O^+(\mathrm{H}^2(X,\mathbb{Z})) \oplus O(\mathcal{N}(X)). \end{aligned}$$

Proof. Since every Hodge isometry preserves $\mathbb{C}\cdot[\sigma]$, it also preserves the extended Neron–Severi group $\mathrm{H}^*(X,\mathbb{Z}) \cap [\sigma]^\perp = \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z}) = \mathcal{N}(X)$. Restriction to $\mathcal{N}(X)$ yields the following short exact sequence

$$0 \longrightarrow O^+(\mathrm{H}^2(X,\mathbb{Z})) \longrightarrow O^+(\mathrm{H}^*(X,\mathbb{Z})) \longrightarrow O(\mathcal{N}(X))\,.$$

Since the orthogonal group $O(\mathcal{N}(X))$ consists of the following four transformations

1. $id : (r, s) \mapsto (r, s)$,
2. $\rho_1 : (r, s) \mapsto (-r, -s)$,
3. $\rho_2 : (r, s) \mapsto (-s, -r)$,
4. $\rho_1 \circ \rho_2 : (r, s) \mapsto (s, r)$,

it is generated by the images of $[\hat{1}]$ and $\hat{T}$, Thus, the last map of the exact sequence is surjective. A splitting of the sequence is given by $\rho_1 \longmapsto [\hat{1}]$ and $\rho_2 \longmapsto \hat{T}$. □

There is a natural map $\mathrm{Aut}(X) \longrightarrow O^+(\mathrm{H}^2(X, \mathbb{Z}))$ for every K3 surface X because for every $f \in \mathrm{Aut}(X)$ the induced map f_* on $\mathrm{H}^2(X, \mathbb{Z})$ is a Hodge isometry preserving the orientation of the 3-plane. Due to the Strong Global Torelli Theorem (see e.g. [10]), this map is injective. The following lemma shows that the subgroup $\mathrm{Aut}(X)$ of $\mathrm{Aut}(\mathrm{D^b}(X))$ is as big as possible. In contrast to this, the subgroup $\mathrm{Pic}(X)$ of $\mathrm{Aut}(\mathrm{D^b}(X))$ is as small as possible.

Lemma 1.2.14. *The natural map* $\mathrm{Aut}(X) \longrightarrow O^+(\mathrm{H}^2(X, \mathbb{Z}))$ *is an isomorphism, i.e.* $\mathrm{Aut}(X) \cong O^+(\mathrm{H}^2(X, \mathbb{Z}))$.

Proof. It remains to show the surjectivity of the map. Take $\Psi \in O^+(\mathrm{H}^2(X, \mathbb{Z}))$. Since for a generic K3 surface the Kähler cone coincides with one connected component of the positive cone $\{c \,|\, c^2 > 0, c \cdot [\sigma] = 0\} \subseteq \mathrm{H}^{1,1}(X, \mathbb{R})$, the transformation Ψ maps a Kähler class $[\omega]$ onto a Kähler class or its negative. The latter case cannot occur since Ψ preserves the orientation of the 3-plane $\mathrm{span}_{\mathbb{R}}(\mathrm{Re}[\sigma], \mathrm{Im}[\sigma], [\omega])$. Due to the Strong Global Torelli Theorem, a Hodge isometry $\Psi \in O^+(\mathrm{H}^2(X, \mathbb{Z}))$ which maps a Kähler class onto a Kähler class is of the form $\Psi = f_*$ for an unique isomorphism $f : X \cong X$. □

Proposition 1.2.15. $\mathrm{Aut}(\mathrm{D^b}(X)) = \mathbb{Z}T \oplus \mathbb{Z}[1] \oplus \mathrm{Aut}(X)$.

Proof. Let Φ be an autoequivalence of Fourier–Mukai type and let us consider $\Phi(\sigma^+)$. By Lemma 1.2.4 there are integers m and n such that for every point $x \in X$ the sheaf $k(x)$ is stable in $\sigma' := T^m \circ [n] \circ \Phi\,(\sigma^+)$ of the same phase $\phi'(k(x)) \in (0, 1]$. Using $\det Z' \neq 0$ for the central charge Z' of σ' and Proposition 1.2.5, we can assume $\det Z' > 0$ for a suitable choice of m. The autoequivalence $\Psi := T^m \circ [n] \circ \Phi$ is also of Fourier–Mukai type. Due to Mukai [30] adapted to the non-projective case, every Fourier–Mukai autoequivalence induces a Hodge isometry on $\mathrm{H}^*(X, \mathbb{Z})$ and, therefore, an isometry on the extended Neron–Severi group $\mathrm{H}^*(X, \mathbb{Z}) \cap [\sigma]^\perp = \mathrm{H}^0(X, \mathbb{Z}) \oplus \mathrm{H}^4(X, \mathbb{Z}) = \mathcal{N}(X)$. The four group elements of $O(\mathcal{N}(X))$ are given in the proof of Lemma 1.2.13, and we conclude that $Z' := Z_{\Psi(\sigma^+)} = Z^+ \circ \Psi^{-1}$ is one of the following four expressions

$$Z'(r, s) = -s + ir \ , \ Z'(r, s) = s - ir \ , \ Z'(r, s) = r - is \ , \ Z'(r, s) = -r + is.$$

Because of $\phi'(k(x)) \in (0, 1]$ and $\det Z' > 0$, we can exclude the last three cases. In the first case we get $\phi'(k(x)) = 1$, and since $\Psi(k(x))$ is also stable in σ' of phase one, we see that $\Psi(k(x)) = k(f(x))$ for a suitable bijective map f on the set X. It can be shown (see e.g. [20], Corollary 5.22) that $f : X \longrightarrow X$ is an automorphism of the complex manifold X and Ψ is the composition of f_* and a line bundle twist. But $\mathcal{O}_X$ is the only line bundle on X. Thus, $\Psi = f_*$. $\square$

It can be shown that the group $\mathrm{Aut}(X)$ is either trivial or isomorphic to $\mathbb{Z}$ (see [27], Theorem 3.4 or [33], Theorem 1.5). Combining the previous two lemmas with the proposition, we obtain the following corollary.

Corollary 1.2.16. *On a generic K3 surface there is the short exact sequence of groups*

$$0 \longrightarrow \mathbb{Z}T^2 \oplus \mathbb{Z}[2] \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow O^+(\mathrm{H}^*(X, \mathbb{Z})) \longrightarrow 0.$$

1.2.5 A generalization of $\mathrm{Stab}(X)$

In this subsection we follow a proposal of D. Huybrechts in order to generalize the notion of a stability condition. The reason for this is to connect the space $\mathrm{Stab}(X)$ of stability conditions with the complexified Kähler cone as expected from physics. The notion of a generalized Calabi–Yau structure is very important in that approach.

Definition 1.2.17 ([19], Definition 1.1)**.** *A generalized Calabi–Yau structure on an oriented four-dimensional manifold M is a closed even form $\varphi \in \mathcal{A}^{2*}_{\mathbb{C}}(M)$ such that*

$$\langle \varphi, \varphi \rangle = 0 \quad \text{and} \quad \langle \varphi, \bar{\varphi} \rangle > 0\,.$$

In this definition we extended the Mukai pairing to the level of forms. So $\langle \varphi, \bar{\varphi} \rangle > 0$ means that $\langle \varphi, \bar{\varphi} \rangle \in \mathcal{A}^4_{\mathbb{R}}(M)$ defines the given orientation of M. If X is a K3 surface, every holomorphic symplectic 2-form $\sigma \in \mathcal{A}^{2,0}_{\mathbb{C}}(X)$ defines a generalized Calabi–Yau structure $\varphi = \sigma$.

Definition 1.2.18 ([19], Definition 2.1)**.** *Let ψ be a generalized Calabi–Yau structure on a four-dimensional manifold M. A generalized Calabi–Yau structure φ on M is called a generalized Kähler structure for ψ if*

$$\langle \psi, \varphi \rangle = \langle \bar{\psi}, \varphi \rangle = 0.$$

If such a generalized Kähler structure exists, we call ψ Kähler and (M, ψ) a generalized Kähler manifold. The cohomology class $[\varphi] \in \mathrm{H}^0(M, \mathbb{C}) \oplus \mathrm{H}^2(M, \mathbb{C}) \oplus \mathrm{H}^4(M, \mathbb{C})$ associated to φ is called the generalized Kähler class of φ.

In the following (M, ψ) should be a K3 surface (X, σ). The form σ is only determined up to scalars, but the notion of a generalized Kähler structure for σ is independent of the choice. Due to this, we can speak about generalized Kähler structures on a K3 surface X. Each generalized Kähler structure on X has the form $\varphi = \lambda \exp(B + i\omega)$ with $\lambda \in \mathbb{C}^*$ and closed real $(1,1)$-forms B and ω with $\omega^2 = \omega \wedge \omega > 0$. See [19], Example 2.2.i for details. We denote by $\mathcal{K}^g_X$ the set of all generalized Kähler classes on X. Note that $\mathcal{K}^g_X$ is an open subset of the quadric $[\varphi]^2 = 0$ in the complex vector subspace $\{[\sigma], [\bar{\sigma}]\}^\perp \in \mathrm{H}^{2*}(X, \mathbb{C})$. The quadric is non-singular because the Mukai pairing is non-degenerate. Hence, $\mathcal{K}_X^g$ is a complex manifold. Following a proposal of D. Huybrechts we give a new definition of a stability condition.

Definition 1.2.19. *A generalized stability condition on X is a pair $(\varphi, \mathcal{P})$, where*

1. *$\varphi \in \mathrm{H}^0(M, \mathbb{C}) \oplus \mathrm{H}^2(M, \mathbb{C}) \oplus \mathrm{H}^4(M, \mathbb{C})$ is a generalized Kähler class on X, and*
2. *the pair $(Z_{(\varphi)}, \mathcal{P})$ with $Z_{(\varphi)}(v(E)) = \langle \varphi, v(E) \rangle \; \forall E \in \mathrm{D}^{\mathrm{b}}(X)$ is an ordinary stability condition on X.*

The set of all generalized stability conditions is denoted by $\mathrm{Stab}(X)^g$ *and* $\pi^g : \mathrm{Stab}(X)^g \ni (\varphi, \mathcal{P}) \longmapsto \varphi \in \mathcal{K}_X^g$ *is the natural forgetful map.*

Note that φ is from now on a cohomology class and not a closed form. We have the following holomorphic map

$$\Theta : \mathcal{K}_X^g \ni \varphi \longmapsto Z_{(\varphi)} = \langle \varphi, \cdot \rangle|_{\mathcal{N}(X)} \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{N}(X) \otimes \mathbb{C}, \mathbb{C}) = (\mathcal{N}(X) \otimes \mathbb{C})^\vee.$$

It is easy to see that $\mathrm{Stab}(X)^g$ is the fibre product of $\pi : \mathrm{Stab}(X) \to (\mathcal{N}(X) \otimes \mathbb{C})^\vee$ and $\Theta : \mathcal{K}_X^g \to (\mathcal{N}(X) \otimes \mathbb{C})^\vee$. To be precise, there is the following fibre product diagram.

$$\begin{tikzcd} \mathrm{Stab}(X)^g \arrow[r] \arrow[d, "\pi^g"'] & \mathrm{Stab}(X) \arrow[d, "\pi"] \\ \mathcal{K}_X^g \arrow[r, "\Theta"] & (\mathcal{N}(X) \otimes \mathbb{C})^\vee \end{tikzcd}$$

Since π is a local homeomorphism, π^g is also a local homeomorphism. Hence, $\mathrm{Stab}(X)^g$ has a natural structure of a complex manifold such that all maps are holomorphic. Since $\dim_{\mathbb{C}} \mathcal{K}_X^g = 1 + 20 = 21$ is independent of the given complex structure on X, the dimension of $\mathrm{Stab}(X)^g$ might also be independent and, perhaps, of the 'right' dimension 21. In general, it is not clear whether each central charge of a stability condition is of the form $Z_{(\varphi)}$ for a suitable $\varphi \in \mathcal{K}_X^g$. Therefore, one can loose some stability conditions by the step $\mathrm{Stab}(X) \rightsquigarrow \mathrm{Stab}(X)^g$. On the other hand, it may happen that for $\varphi \in \mathcal{K}_X^g$ there is no stability condition $\sigma \in \mathrm{Stab}(X)$ with central charge $\Theta(\varphi) = Z_{(\varphi)}$. In the case of a generic K3 the latter question will be answered by the next proposition.

Proposition 1.2.20. *The map $\pi^g : \mathrm{Stab}(X)^g \to \mathcal{K}_X^g$ has the image $\mathcal{K}_X^g \setminus \delta^\perp$, where $\delta = v(\mathcal{O}_X) \in \mathcal{N}(X)$ is one of the two root vectors in the lattice $\mathcal{N}(X)$. In other words, for every $\varphi \in \mathcal{K}_X^g$ with $\langle \varphi, \delta \rangle \neq 0$ there is a stability condition in* $\mathrm{Stab}(X)$ *with central charge $Z_{(\varphi)}$.*

The other root vector is $-\delta$ which gives no further restrictions.

Proof. Due to a remark given earlier, a class $\varphi \in \mathcal{K}_X^g$ is of the form $\lambda \exp(B+i\omega)$ with $\lambda \in \mathbb{C}^*$ and $B, \omega \in \mathrm{H}^{1,1}(X, \mathbb{R})$ with $\omega^2 > 0$. If $\sigma \in \mathrm{Stab}(X)$ has the central charge $\Theta(\exp(B+i\omega))$, then $\sigma \cdot (\lambda^{-1}, f(t) = t - \arg(\lambda)/\pi)$ has the central charge $\Theta(\varphi)$. Hence, we can assume $\lambda = 1$. In that case $Z_{(\varphi)}$ is given by the matrix

$$Z_{(\varphi)} \simeq \begin{pmatrix} \frac{\omega^2 - B^2}{2} & -1 \\ -\langle B, \omega\rangle & 0 \end{pmatrix}.$$

There are three cases:

1. $\langle B, \omega \rangle < 0$, i.e. $\det Z_{(\varphi)} > 0$. In that case $Z_{(\varphi)}$ is contained in the $\mathrm{GL}^+(2, \mathbb{R})$-orbit of Z^+ and $Z_{(\varphi)}$ is the central charge of a stability condition in $U^+(X)$.

2. $\langle B, \omega \rangle > 0$, i.e. $\det Z_{(\varphi)} < 0$. Similar to the previous case we can find a stability condition in $U^-(X)$ with central charge $Z_{(\varphi)}$.

3. $\langle B, \omega \rangle = 0$, i.e. $\det Z_{(\varphi)} = 0$. Since there is only one positive direction in $\mathrm{H}^{1,1}(X, \mathbb{R})$ and $\omega^2 > 0$, we get $B^2 \leq 0$. Therefore, $\alpha := \frac{\omega^2 - B^2}{2} > 0$. If $\alpha > 1$, $Z_{(\varphi)} = Z_\alpha$ and σ_α has the central charge $Z_{(\varphi)}$. If $\alpha \in (0,1)$, the function $Z_{(\varphi)}(r,s) = -s + \alpha r = \alpha(-\frac{s}{\alpha} + r)$ is contained in the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of $T(Z_{1/\alpha})(r,s) = -(-r) + (1/\alpha)(-s) = -s/\alpha + r$ since $\alpha \in \mathbb{R}_{>0} \cong \mathbb{R}_{>0} \cdot id \subseteq \mathrm{GL}^+(2, \mathbb{R})$. Thus, $Z_{(\varphi)}$ is a central charge of a stability condition in the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit of $T\sigma_{1/\alpha}$. The case $\alpha = 1$ was excluded by the assumption $Z_{(\varphi)}(\mathcal{O}_X) = \langle \varphi, v(\mathcal{O}_X)\rangle \neq 0$. This assumption is necessary because of Lemma 1.2.2, $\mathcal{O}_X$ is stable in any stability condition $\sigma = (Z, \mathcal{P})$ and, therefore, $Z(\mathcal{O}_X) \neq 0$.

□

In the case of a generic K3 surface we do not loose any stability condition as the following lemma shows.

Lemma 1.2.21. *For every $\omega_0 \in \mathrm{H}^{1,1}(X, \mathbb{R})$ with $\omega_0^2 = 1$ the map*

$$\vartheta : \mathbb{C}^* \times H^* \ni (\lambda, z) \longmapsto \langle \lambda \exp(z\omega_0), \cdot \rangle|_{\mathcal{N}(X)} \in (\mathcal{N}(X) \otimes \mathbb{C})^\vee$$

is a biholomorphic map onto its image which is $\mathcal{W} = \pi(\mathrm{Stab}(X))$. Here $H^* := \{z \in \mathbb{C} |\ \mathrm{Im}\, z > 0\} \setminus \{\sqrt{2}i\}$ is the upper half plane without $\sqrt{2}i$.

If we write $z\omega_0 = B + i\omega$, we see that $\varphi = \lambda \exp(z\omega_0)$ is indeed a generalized Kähler class. The assumption $z \neq \sqrt{2}i$ is necessary for $\varphi \not\perp \delta$ and the restriction $\mathrm{Im}\, z > 0$ is important for injectivity as we will see in the proof.

Proof. The proof of the last proposition shows $\mathrm{im}\, \vartheta \subseteq \mathcal{W}$. Therefore, $\vartheta(\lambda, z) = Z$ is the central charge of some stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$. An easy calculation shows $\lambda = -Z(k(x))$ and $\lambda z^2/2 = Z(T(k(x)))$, and since $\arg(z) \in (0, \pi)$, the pair (λ, z) is completely determined by its image Z. This proves the injectivity of ϑ. In order to show the surjectivity $\operatorname{im}\vartheta = \mathcal{W}$, we take a stability condition $\sigma = (Z, \mathcal{P}) \in \operatorname{Stab}(X)$ and consider the pair

$$\left(-Z(k(x)), \sqrt{-\frac{2Z(T(k(x)))}{Z(k(x))}}\right),$$

where we use the root with nonnegative imaginary part. We claim that this pair is in $\mathbb{C}^* \times H^*$. Indeed, $Z(k(x)) \neq 0$ since $T^m(k(x))$ is stable in σ for a suitable $m \in \mathbb{Z}$, and T acts as an isomorphism on $\mathcal{N}(X)$. First of all, the root is not real. Otherwise, $Z(T(k(x))) = \alpha Z(k(x))$ with $\alpha < 0$. Because of $v(k(x)) = (0,1)$ and $v(T(k(x))) = (-1,0)$, we get

$$Z(r,s) = Z\big(s\, v(k(x)) - r\, v(T(k(x)))\big) = Z(k(x))(s - \alpha r)\,.$$

After applying some $(g,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, we can assume $Z(k(x)) = -1$ and $Z(r,s) = -s + \alpha r$ with $\alpha < 0$. The only $\mathrm{GL}^+(2,\mathbb{R})$-orbits in $\mathcal{W}$ with $\det Z = 0$ are those with $\alpha \in (0,1) \cup (1,\infty)$ due to Theorem 1.2.9 and the calculation of $T(Z_{1/\alpha})$ above. This is a contradiction, and the root must lie in the upper half plane. The root is not $\sqrt{-2}$. Otherwise, $Z(T(k(x))) = Z(k(x))$ and the previous calculation would give a stability condition with central charge $Z(r,s) = -s + r$ which is also impossible since $\alpha \neq 1$. Hence, the pair is indeed contained in the set $\mathbb{C}^* \times H^*$. Obviously, the map ϑ is holomorphic. Since every holomorphic bijective map is biholomorphic, the assertion follows. $\square$

Corollary 1.2.22 (see Lemma 1.2.11). *The fundamental group of $\mathcal{W}$ is $\mathbb{Z} \oplus \mathbb{Z}$.*

Since the set $\{\omega \in \mathrm{H}^{1,1}(X,\mathbb{R}) \,|\, \omega^2 > 0\}$ consists of two components, the space $\mathcal{K}_X^g$ is not connected. One can fix an oriented positive two-plane P in $\mathrm{H}^{2*}(X,\mathbb{R})$ perpendicular to the real positive oriented plane $(\mathrm{Re}[\sigma], \mathrm{Im}[\sigma])$ spanned by the real and the imaginary part of the class $[\sigma] \in \mathrm{H}^{2,0}(X,\mathbb{C})$ of a holomorphic symplectic form. For any generalized Kähler class φ the orthogonal projection of the oriented positive plane $(\mathrm{Re}\,\varphi, \mathrm{Im}\,\varphi)$ onto P is an isomorphism since there are only 4 positive directions in $\mathrm{H}^{2*}(X,\mathbb{R})$. Let us denote by ${\mathcal{K}_X^g}^+$ the open subset in $\mathcal{K}_X^g$ consisting of those φ for which the orthogonal projection onto P preserves the orientation. The set ${\mathcal{K}_X^g}^+$ is connected and the restriction $\Theta : {\mathcal{K}_X^g}^+ \setminus \delta^\perp \to \mathcal{W}$ is still surjective. This follows from the previous lemma after replacing ω_0 by $-\omega_0$ if necessary. Finally, we obtain the following description of the space $\operatorname{Stab}(X)^{g+} = (\pi^g)^{-1}({\mathcal{K}_X^g}^+)$ similar to Theorem 1.1.5.

Proposition 1.2.23. *In the diagram*

$$\begin{tikzcd}
\operatorname{Stab}(X)^{g+} \arrow[r] \arrow[d, "\pi^g"'] & \operatorname{Stab}(X) \arrow[d, "\pi"] \\
{\mathcal{K}_X^g}^+ \setminus \delta^\perp \arrow[r, "\Theta"] & \mathcal{W}
\end{tikzcd}$$

all maps are surjective and all spaces are connected. Furthermore, π *and* π^g *are universal covering maps and* $\dim_{\mathbb{C}} \operatorname{Stab}(X)^{g+} = 21$ *as well as* $\dim_{\mathbb{C}} \operatorname{Stab}(X) = 2$.

Proof. The universal lifting property of a universal cover is stable under base change. Thus, if $\operatorname{Stab}(X)^{g+}$ is connected, π^g is a universal covering map. To see the connectivity of $\operatorname{Stab}(X)^{g+}$, we choose two points $(\varphi, \mathcal{P})$ and $(\varphi', \mathcal{P}')$ in $\operatorname{Stab}(X)^{g+}$. Using the universal lifting property of π^g and the connectivity of ${\mathcal{K}_X^g}^+ \setminus \delta^\perp$, we can construct two paths in $\operatorname{Stab}(X)^{g+}$ starting in $(\varphi, \mathcal{P})$ respectively $(\varphi', \mathcal{P}')$ with endpoint in the fibre over $\exp(i\omega_0) \in {\mathcal{K}_X^g}^+ \setminus \delta^\perp$ for a suitable $\omega_0 \in \mathrm{H}^{1,1}(X, \mathbb{R})$ with $\omega_0^2 = 1$. Using Lemma 1.2.21, we see that $(\pi^g)^{-1}\big(\{\lambda \exp(z\omega_0) \mid \lambda \in \mathbb{C}^*, z \in H^*\}\big)$ is isomorphic to the connected space $\operatorname{Stab}(X)$. Since the endpoints of the two paths are contained in this connected subspace, they can be connected by a third path. Thus, $\operatorname{Stab}(X)^{g+}$ is connected. The other assertions follow from the remarks before. $\square$

Note that the dimension of the 'stringy Kähler moduli space'

$$\big(\operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \backslash \operatorname{Stab}(X)^{g^+}\big)/\mathbb{C}^*$$

is $20 = \dim_{\mathbb{C}} \mathrm{H}^{1,1}(X, \mathbb{C})$ which is the dimension of the complexified classical Kähler cone.

1.3 Stability conditions on generic complex tori

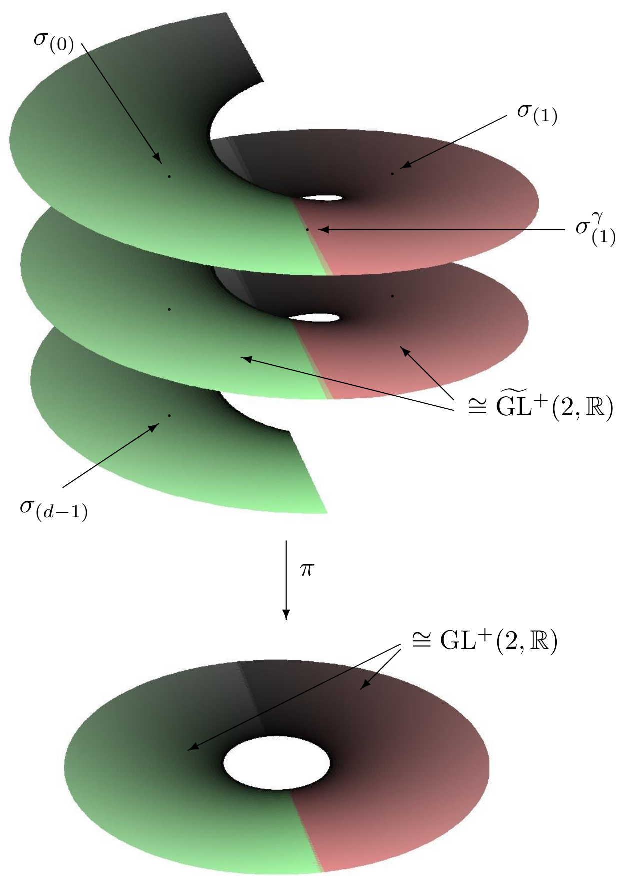
In this section we construct stability conditions on generic complex tori of any dimension d. A complex torus is called generic if

$$\mathrm{H}^{p,p}(X) \cap \mathrm{H}^{2p}(X, \mathbb{Z}) = 0 \qquad \forall\, 0 < p < \dim X.$$

Let $U(X)$ be the set of all numerical locally-finite stability conditions $\sigma = (Z, \mathcal{P})$ such that there exist certain real numbers ϕ and ψ such that $k(y) \in \mathcal{P}(\phi)$ for all $y \in X$ and $L \in \mathcal{P}(\psi)$ for all $L \in \mathrm{Pic}^0(X)$. We will show that $U(X)$ is a simply connected component of $\mathrm{Stab}(X)$. Furthermore, $U(X)$ can be written as a disjoint union of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbits

$$U(X) = \bigcup_{0 \le p < d} \sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \quad \cup \bigcup_{\substack{1 \le p < d \\ \gamma \in (0,1/2)}} \sigma^{\gamma}_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$$

with explicitly given stability conditions $\sigma_{(p)}$ and $\sigma^{\gamma}_{(p)}$. The picture below illustrates $U(X)$ and $\pi(U(X))$ of a generic complex torus of dimension $d = 5$.



Note that a point in the helix represents a simply connected 2-dimensional subspace in the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit of some stability condition, whereas a point in the annulus below represents a 2-dimensional subspace in the $\mathrm{GL}^+(2,\mathbb{R})$-orbit with the fundamental group $\mathbb{Z}$.
Since the case $\dim X = 1$ has already been studied by Macrì in [25] and the case $\dim(X) = 2$ by D. Huybrechts, P. Stellari and E. Macrì in [22], we restrict ourselves to tori of dimension $d \geq 3$. In contrast to the case $d \leq 2$ the space $U(X)$ is no longer a covering of its image under the map $\pi : U(X) \ni \sigma = (Z, \mathcal{P}) \longmapsto Z \in \pi(U(X)) \subseteq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X), \mathbb{C})$. Moreover, in the case $d \geq 3$ it is still open whether or not $\mathrm{Stab}(X)$ is connected.
Note that the characterizing condition of $U(X)$ is invariant under the Fourier–Mukai transform with respect to the Poincaré bundle. Hence, there is a natural isomorphism $U(X) \cong U(\hat{X})$, where $\hat{X} = \mathrm{Pic}^0(X)$ is the dual torus.

1.3.1 Sheaves on generic tori

In this subsection we study sheaves on a generic torus X of dimension $d \geq 3$. The following facts and arguments are well known (see e.g. [43] or [44]). The main result states that on such a torus every reflexive sheaf is locally free and possesses a filtration whose quotients are line bundles in $\mathrm{Pic}^0(X)$.

Definition 1.3.1. *A compact complex torus X of dimension d is called generic, if*

$$\mathrm{H}^{p,p}(X) \cap \mathrm{H}^{2p}(X, \mathbb{Z}) = 0 \qquad \forall\, 0 < p < d.$$

As an immediate consequence of the definition we get

- $\mathrm{Pic}(X) = \mathrm{Pic}^0(X)$,
- the support of any torsion sheaf is a finite set of points in X.

The last observation leads to the simple but frequently used formula

$$\mathrm{Ext}^i(T, F) = \mathrm{Ext}^{d-i}(F, T)^\vee = \mathrm{H}^{d-i}(X, T \otimes F^\vee)^\vee = 0 \qquad \forall\, i < d \tag{1.4}$$

for a torsion sheaf T and a locally free sheaf F on X. We begin our investigation of reflexive sheaves with the following lemma.

Lemma 1.3.2. *On a generic complex torus X of dimension $d \geq 2$ the following conditions for a coherent sheaf G on X are equivalent.*

(a) G is reflexive,

(b) $\mathrm{Hom}(T, G) = \mathrm{Ext}^1(T, G) = 0$ for all torsion sheaves T.

Proof. (a) $\Longrightarrow$ (b) For any extension $0 \to G \to F \to T \to 0$ of a torsion sheaf T by G we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r, "i"] \arrow[d, "\gamma"', "\wr"] & F \arrow[r] \arrow[d, "\alpha"] & T \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G^{\vee\vee} \arrow[r, "i^{\vee\vee}"] & F^{\vee\vee} \arrow[r] & T' \arrow[r] & 0 \end{tikzcd}$$

with exact rows and a suitable torsion sheaf T'. Since $G^{\vee\vee}$ and $F^{\vee\vee}$ are reflexive, the morphism $i^{\vee\vee}$ is determined on a complement of a Zariski-closed subset Z of codimension ≥ 2. If we take $Z = \mathrm{supp}(T')$, we see that $i^{\vee\vee} : G^{\vee\vee} \longrightarrow F^{\vee\vee}$ is an isomorphism. The morphism $\pi := \gamma^{-1} \circ (i^{\vee\vee})^{-1} \circ \alpha$ splits our extension. The vanishing $\mathrm{Hom}(T, G) = 0$ is obvious because G is torsionfree.
(b) $\Longrightarrow$ (a) Like every coherent sheaf, G fits into a short exact sequence

$$0 \longrightarrow S \longrightarrow G \longrightarrow G^{\vee\vee} \longrightarrow T \longrightarrow 0$$

with torsion sheaves S and T. Due to our assumption, $S = 0$ and the resulting short exact sequence splits. But the reflexive sheaf $G^{\vee\vee}$ has no torsion subsheaves, hence $T = 0$ and G is reflexive. □

Corollary 1.3.3. *Assume X is a generic complex torus of dimension $d \geq 3$. If $0 \to F_1 \to F_2 \to F_3 \to 0$ is a short exact sequence in $\mathrm{Coh}(X)$ with a locally free sheaf F_1 and a reflexive sheaf F_2, then the sheaf F_3 is also reflexive.*

Proof. Apply $\mathrm{Hom}(T, -)$ to the short exact sequence and use (1.4) and Lemma 1.3.3. □

Let ω be a Kähler class and denote as usual the slope

$$\frac{\int_X \mathrm{c}_1(E) \wedge \omega^{d-1}}{\mathrm{rk}(E)}$$

of a torsionfree sheaf E with $\mu_\omega(E)$. There is the notion of μ_ω-(semi)stability, and on a generic torus X of dimension $d \geq 2$ every torsionfree sheaf is semistable with slope $\mu_\omega(E) = 0$.
The following important proposition is a special case of a theorem by Bando and Siu [2].

Proposition 1.3.4. *Let X be a generic complex torus of dimension $d \geq 3$ with fixed Kähler metric ω. Then every μ_ω-stable reflexive sheaf F is a line bundle in $\mathrm{Pic}^0(X)$.*

Proof. (see [2] for more details) Bando and Siu construct a canonical Hermite–Einstein connection on the restriction of F to an open set on which F is locally free and whose complement consists of finitely many points. The curvature is L^2-integrable and satisfies the Bogomolov–Lübke inequality on X. Since $\mathrm{c}_1(F) = \mathrm{ch}_2(F) = 0$, this connection is flat outside this finite set of points.

As points have codimension ≥ 2, this flat connection has no local monodromy and one can extend the flat bundle to a flat bundle on X. Since F is reflexive, it coincides with this flat bundle up to isomorphism. The connection on the stable bundle F corresponds to an irreducible representation of the abelian fundamental group of X. Thus, F is a line bundle. □

Proposition 1.3.5. *On a generic complex torus X of dimension $d \geq 3$ every reflexive sheaf is locally free and admits a locally free filtration with quotients in* $\mathrm{Pic}^0(X)$.

Proof. Since all sheaves have trivial first Chern class, every reflexive sheaf F is μ_ω-semistable and admits a Jordan–Hölder filtration

$$0 \subset F_0 \subset F_1 \subset \dots F_n = F$$

with stable quotients. We may assume that F_i is reflexive for all $0 \leq i \leq n$. Due to the previous proposition $F_0 \in \mathrm{Pic}^0(X)$. Furthermore, F_1/F_0 is reflexive by Corollary 1.3.3. Hence, $F_1/F_0 \in \mathrm{Pic}^0(X)$. Since F_0 and F_1/F_0 are locally free, F_1 is also locally free. Now we proceed in this way and obtain the assertion. □

Remark 1.3.6. Note that Proposition 1.3.5 implies that for a reflexive sheaf F there are nontrivial morphisms $L_1 \to F$ and $F \to L_2$ with $L_1, L_2 \in \mathrm{Pic}^0(X)$.

In order to use these results, we will assume $\dim X \geq 3$ in the following.

1.3.2 Some stability conditions on generic tori

In this subsection we construct and characterize certain stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$. Recall, a stability condition on X consists of a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ and an additive function on the K-group of its heart satisfying certain properties.
On $\mathrm{D}^{\mathrm{b}}(X)$ there is the standard t-structure with heart $\mathrm{Coh}(X) =: \mathrm{Coh}_{(0)}(X)$. For the construction of other t-structures we follow the method of Happel, Reiten, and Smalø using torsion pairs.

Definition 1.3.7. *A torsion pair in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ with the property $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Furthermore, every object $E \in \mathcal{A}$ fits into a short exact sequence*

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

for some objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

For the abelian category $\mathrm{Coh}(X)$ the two subcategories $\mathcal{T} := \{\text{torsion sheaves}\}$ and $\mathcal{F} := \{\text{torsionfree sheaves}\}$ form a torsion pair. The following lemma illustrates the importance of this notion.

Lemma 1.3.8 ([16], Proposition 2.1). *Suppose $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$ and let us denote by $H : \mathcal{D} \to \mathcal{A}$ the cohomology functor with respect to this t-structure. For every torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ the full subcategory*

$$\mathcal{A}^\sharp = \{E \in \mathcal{D} \mid H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T}\}$$

is the heart of a bounded t-structure on $\mathcal{D}$.

Using our torsion pair on $\mathrm{Coh}(X)$, we obtain a new t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ whose heart $\mathrm{Coh}(X)^\sharp =: \mathrm{Coh}_{(1)}(X)$ consists of complexes E of length two with a torsion sheaf $H^0(E)$ and a torsionfree sheaf $H^{-1}(E)$.
We claim that on a generic torus X of dimension $d \geq 3$ the pair $\mathcal{T}_{(1)} = \mathcal{T} = \{\text{torsion sheaves}\}$ and $\mathcal{F}_{(1)} = \{\text{locally free sheaves}\}[1]$ is a torsion pair in $\mathrm{Coh}_{(1)}(X)$. For a torsion sheaf T and a locally free sheaf F we have $\mathrm{Ext}^1(T, F) = 0$ due to (1.4). Hence it remains to show the existence of a short exact sequence as in the definition of a torsion pair. For any $E \in \mathrm{Coh}_{(1)}(X)$ there is a triangle

$$T \longrightarrow H^{-1}(E)[1] \longrightarrow F[1] \longrightarrow T[1]$$

with locally free $F := H^{-1}(E)^{\vee\vee}$ (use Proposition 1.3.5) and $T := F/H^{-1}(E) \in \mathcal{T}_{(1)}$. We denote by C the cone of the composition $T \to H^{-1}(E)[1] \to E$. From the octahedron axiom we get the triangle

$$F[1] \longrightarrow C \longrightarrow H^0(E) \longrightarrow F[2]$$

and conclude $C = H^0(E) \oplus F[1]$ since $\mathrm{Ext}^1(H^0(E), F[1]) = \mathrm{Ext}^2(H^0(E), F) = 0$. If we define K as the cone of the composition $E \to C \to F[1]$, we get the triangle

$$K[-1] \longrightarrow E \longrightarrow F[1] \longrightarrow K.$$

Using the associated long exact cohomology sequence in $\mathrm{Coh}(X)$ and the definition of F we see $K[-1] \in \mathcal{T}_{(1)}$ and we are done. By definition the heart $\mathrm{Coh}_{(1)}(X)^\sharp =: \mathrm{Coh}_{(2)}(X)$ of the new t-structure consists of objects E which fit into a triangle

$$(F[1])[1] = F[2] \longrightarrow E \longrightarrow T$$

with some torsion sheaf T and some locally free sheaf F. For $\dim(X) = d > 3$ any such triangle splits and we get $E = T \oplus F[2]$. It is easy to check that $\mathcal{T}_{(2)} = \mathcal{T} = \{\text{torsion sheaves}\}$ and $\mathcal{F}_{(2)} = \{\text{locally free sheaves}\}[2]$ define a torsion pair on $\mathrm{Coh}_{(2)}(X)$. For every object E of the new abelian category $\mathrm{Coh}_{(2)}(X)^\sharp =: \mathrm{Coh}_{(3)}(X)$ one has a triangle

$$(F[2])[1] = F[3] \longrightarrow E \longrightarrow T$$

with a torsion sheaf T and some locally free sheaf F. For $d > 4$ we proceed in this way. Eventually one has d bounded t-structures with hearts $\mathrm{Coh}_{(p)}(X), 0 \leq p < d$. In the case $0 < p$ every object $E \in \mathrm{Coh}_{(p)}(X)$ fits into a unique triangle

$$F[p] \longrightarrow E \longrightarrow T$$

with some torsion sheaf $T = H^0(E)$. The sheaf $F = H^{-p}(E)$ is torsionfree and, moreover, locally free for $p \geq 2$. In the case $2 \leq p < d-1$ the extension is trivial.

Lemma 1.3.9. *For every $0 \leq p < d$ the category $\mathcal{T}$ of torsion sheaves is an abelian subcategory of* $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$*, i.e. if $f : S \to T$ is a morphism in $\mathcal{T}$ and if we denote the kernel of f in $\mathcal{T}$ and in* $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ *by* $\ker f$ *resp.* $\ker_{(p)} f$*, then* $\ker f = \ker_{(p)} f$ *and similar for the cokernels.*

Proof. Let us denote by $H^i_{(p)}$ the i-th cohomology functor of the t-structure corresponding to $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. We assume $p \geq 1$ and form the triangle $S \xrightarrow{f} T \to M \to S[1]$. Then we have $H^0(M) = \operatorname{coker} f$ and $H^{-1}(M) = \ker f$ as well as $H^0_{(p)}(M) = \operatorname{coker}_{(p)} f =: C$ and $H^{-1}_{(p)}(M) = \ker_{(p)} f =: K$. We form the long exact cohomology sequence in $\mathrm{Coh}(X)$ of the triangle $K[1] \to M \to C \to K[2]$ and use $K, C \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$.

$$
\begin{gathered}
0 \longrightarrow H^{-p}(K) \longrightarrow \underbrace{H^{-p-1}(M)}_{=0} \longrightarrow 0 \longrightarrow H^{1-p}(K) \longrightarrow \underbrace{H^{-p}(M)}_{\text{torsion}} \longrightarrow \\
\underbrace{H^{-p}(C)}_{\text{torsionfree}} \longrightarrow \underbrace{H^{2-p}(K)}_{\text{torsion}} \longrightarrow \ldots \longrightarrow \underbrace{H^{-2}(M)}_{=0} \longrightarrow H^{-2}(C) \longrightarrow \underbrace{H^{0}(K)}_{\text{torsion}} \\
\longrightarrow \underbrace{H^{-1}(M)}_{=\ker f, \text{torsion}} \longrightarrow H^{-1}(C) \longrightarrow 0 \longrightarrow \underbrace{H^{0}(M)}_{=\operatorname{coker} f} \longrightarrow H^0(C) \longrightarrow 0
\end{gathered}
$$

From this sequence we deduce $H^{-p}(K) = 0$ and $H^{-p}(C) = 0$. Hence $K \cong H^0(K)$ and $C \cong H^0(C)$ and, therefore, $K = \ker f$ as well as $C = \operatorname{coker} f$. □

Lemma 1.3.10. *Any morphism $f \in \mathrm{Hom}(F, G)$ between torsionfree sheaves F and G defines a morphism $f[1] : F[1] \longrightarrow G[1]$ in $\mathrm{Coh}_{(1)}(X)$ and if we denote by $\Gamma(E)$ the torsion subsheaf of a sheaf E, we get* $H^{-1}(\ker_{(1)}(f[1])) = \ker f$, $H^0(\ker_{(1)}(f[1])) = \Gamma(\operatorname{coker} f)$, $H^{-1}(\operatorname{coker}_{(1)}(f[1])) = \operatorname{coker} f / \Gamma(\operatorname{coker} f)$ *as well as* $H^0(\operatorname{coker}_{(1)}(f[1])) = 0$.

Proof. We imitate the proof of the previous lemma. Let M be defined by the triangle $F[1] \xrightarrow{f} G[1] \to M \to F[2]$. Thus, $H^{-1}(M) = \operatorname{coker} f$ and $H^{-2}(M) = \ker f$ are the only nontrivial cohomology sheaves. The rest of the proof is straight forward. □

Proposition 1.3.11. *For $1 \leq p < d$ the abelian category $\mathrm{Coh}_{(p)}(X)$ is of finite length, i.e. noetherian and artinian.*

Proof. We show that $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ is noetherian. The proof for $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ being artinian is similar.
Take an infinite sequence $E = E_0 \twoheadrightarrow E_1 \twoheadrightarrow E_2 \twoheadrightarrow \ldots$ of quotients. We obtain

the commutative diagram

$$\begin{tikzcd} E_n \arrow[r, two heads] \arrow[d, two heads] & E_{n+1} \arrow[d, two heads] \\ H^0(E_n) \arrow[r] & H^0(E_{n+1}) \end{tikzcd}$$

which shows that $H^0(E_n) \to H^0(E_{n+1})$ is an epimorphism in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ for all $n \geq 0$. Since there are only finitely many quotients of the torsion sheaf $H^0(E)$ in $\mathrm{Coh}(X)$ and by Lemma 1.3.9 also in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$, we get $H^0(E_n) \cong H^0(E_{n+1})$ for all $n \gg 0$. Then we apply the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & H^{-p}(E_n)[p] \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d, two heads] & H^0(E_n) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & H^{-p}(E_{n+1})[p] \arrow[r] & E_{n+1} \arrow[r] & H^0(E_{n+1}) \arrow[r] & 0 \end{tikzcd} \tag{1.5}$$

which yields that $H^{-p}(E_n)[p] \twoheadrightarrow H^{-p}(E_{n+1})[p]$ is an epimorphism. Since the rank function rk is additive, the sequence $(\operatorname{rk} H^{-p}(E_n))_{n\in\mathbb{N}} = ((-1)^p \operatorname{rk}(E_n))_{n\in\mathbb{N}}$ of natural numbers decreases. Thus, without loss of generality we can assume $\operatorname{rk} H^{-p}(E_n) = \operatorname{rk} H^{-p}(E_{n+1})$ for all $n \gg 0$. Hence the kernel $K_n \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ of the epimorphism $H^{-p}(E_n)[p] \twoheadrightarrow H^{-p}(E_{n+1})[p]$ has rank zero and is, therefore, a torsion sheaf. In the case $2 \leq p < d$ there is no triangle

$$H^{-p}(E_{n+1})[p-1] \longrightarrow K_n \longrightarrow H^{-p}(E_n)[p]$$

with $K_n \neq 0$. Hence $H^{-p}(E_n)[p] \cong H^{-p}(E_{n+1})[p]$ and (1.5) yields $E_n \xrightarrow{\sim} E_{n+1}$ for all $n \gg 0$.

In the case $p = 1$ set $T_n = H^{-1}(E_n)^{\vee\vee}/H^{-1}(E_n)$ and consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_n \arrow[r] \arrow[d, "\alpha"] & H^{-1}(E_n)[1] \arrow[r] \arrow[d, two heads] & H^{-1}(E_n)^{\vee\vee}[1] \arrow[r] \arrow[d, "\beta[1]"] & 0 \\ 0 \arrow[r] & T_{n+1} \arrow[r] & H^{-1}(E_{n+1})[1] \arrow[r] & H^{-1}(E_{n+1})^{\vee\vee}[1] \arrow[r] & 0 \end{tikzcd} \tag{1.6}$$

of exact sequences in $\mathrm{Coh}_{(1)}(X)$. Hence $\beta[1]$ is an epimorphism in $\mathrm{Coh}_{(1)}(X)$ and due to Lemma 1.3.10 we get $\operatorname{coker}\beta \in \mathcal{T}$. Together with $\operatorname{rk} H^{-1}(E_n) = \operatorname{rk} H^{-1}(E_{n+1})$ this shows $\ker\beta = 0$. Since the sheaves are locally free and all divisors are trivial, $\operatorname{coker}\beta = 0$. Using Lemma 1.3.10 we conclude $\ker_{(1)}(\beta[1]) = \operatorname{coker}_{(1)}(\beta[1]) = 0$ and $\beta[1]$ is an isomorphism. Hence α is an epimorphism in $\mathrm{Coh}_{(1)}(X)$ and due to Lemma 1.3.9 also in $\mathrm{Coh}(X)$. Since T_n has only finitely many quotients, $T_n \xrightarrow{\sim} T_{n+1}$ for all $n \gg 0$. If we first apply the snake lemma to (1.6) and then to (1.5), we obtain isomorphisms $E_n \xrightarrow{\sim} E_{n+1}$ for all $n \gg 0$. $\square$

Corollary 1.3.12. *For $0 \leq p < d$ the additive function $Z_{(p)}(E) = -\operatorname{ch}_d(E) + (-1)^p \operatorname{rk}(E) \cdot i$ is a stability function on $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$, where $\operatorname{ch}_d(E)$ is (the integral over) the d-th Chern character of E. Furthermore, the pair $\sigma_{(p)} := (Z_{(p)}, \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}))$ is a numerical locally-finite stability condition on $D^b(X)$.*

Proof. For the first part we remark that any $0 \neq E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ with $\operatorname{rk}(E) = 0$ is a torsion sheaf supported on a finite set. For those sheaves $\operatorname{ch}_d(E) > 0$. The second assertion is clear for $0 < p < d$ due to the fact that $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ is of finite length. For $p = 0$ we only have to consider the case of an infinite decreasing sequence of subsheaves

$$\ldots \subseteq G_{n+1} \subseteq G_n \subseteq \ldots \subseteq G_0 = G,$$

because $\mathrm{Coh}(X)$ is noetherian. For $n \gg 0$ we have $\operatorname{rk}(G_{n+1}) = \operatorname{rk}(G_n)$ and, therefore, $Z_{(0)}(G_{n+1}) = Z_{(0)}(G_n) - Z_{(0)}(G_n/G_{n+1}) = Z_{(0)}(G_n) + \operatorname{ch}_d(T_n)$ with the torsion sheaf $T_n := G_n/G_{n+1}$. Hence, the sequence of phases does not increase for $n \gg 0$. This shows that $Z_{(0)}$ satisfies the Harder–Narasimhan property on $\mathrm{Coh}(X)$.
The condition of local finiteness is automatically fulfilled since the values of $Z_{(p)}$ form a discrete set. □

Remark. After suitable modifications of the definition of $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ all the previous statements of this subsection remain true for compact complex Kähler manifolds without nontrivial subvarieties like generic complex tori or general deformations of Hilbert schemes of K3 surfaces. See section 1.4 for the details.

The next proposition gives a rough classification of the objects E in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ which are stable with respect to $\sigma_{(p)}$.

Proposition 1.3.13. *In $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ the sheaf $k(y)$ is stable of phase 1 for any $y \in X$ and $L[p]$ is stable of phase $1/2$ for any $L \in \operatorname{Pic}^0(X)$. For $0 < p < d-1$ these are the only stable objects in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. The phases of all stable objects in $\mathrm{Coh}(X)$ are contained in $(0, 1/2] \cup \{1\}$ and the phases of all stable objects in $\mathrm{Coh}_{(d-1)}(X)$ are contained in $[1/2, 1]$.*

Proof. The case $p = 0$: It is an easy calculation to check the stability of L for any $L \in \operatorname{Pic}^0(X)$ and of $k(y)$ for any $y \in X$. If $E \in \mathrm{Coh}(X)$ is stable but not torsion, it must be torsionfree. Otherwise there is a nontrivial morphism $k(y) \to E$ which cannot exist. Furthermore, there is a nontrivial morphism $E \to E^{\vee\vee} \to L$ for some $L \in \operatorname{Pic}^0(X)$ (see Remark 1.3.6). Hence $\phi(E) \leq \phi(L) = 1/2$.

The case $0 < p < d-1$: For $1 < p < d$ we know that $H^{-p}(E)$ is locally free for any $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. This also holds for every stable object $E \in \mathrm{Coh}_{(1)}(E)$ which is not a torsion sheaf. Indeed, if $H^{-1}(E)$ is not locally free, there is a nonzero morphism $T \to H^{-1}(E)[1] \to E$ coming from the extension $0 \to H^{-1}(E) \to H^{-1}(E)^{\vee\vee} \to T \to 0$ with $T \in \mathcal{T}$. This contradicts the stability of E. Hence $H^{-p}(E)$ is locally free for any stable

$E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}), 0 < p < d, E \notin \mathcal{T}$. Due to formula (1.4)

$$\mathrm{Ext}^1(H^0(E), H^{-p}(E)[p]) = \mathrm{Ext}^{1+p}(H^0(E), H^{-p}(E)) = 0$$

and, therefore, $E \cong H^0(E) \oplus H^{-p}(E)[p]$. Hence $E \cong H^{-p}(E)[p]$ and the only stable objects are of the form $k(y)$ with phase 1 or $F[p]$ with F being locally free of phase $1/2$. For any $L \in \mathrm{Pic}^0(X)$ the complex $L[p]$ has phase $1/2$. Thus, the stable factors of $L[p]$ are of the form $F[p]$ with F being locally free. Since $\mathrm{rk}(L[p]) = (-1)^p$, the complex $L[p]$ is already stable. Conversely, due to the existence of nontrivial morphisms $L[p] \to F[p]$ for any locally free F and some line bundle L, any stable object has rank $(-1)^p$ and the assertion follows.

The case $p = d-1$: One has $Z_{(d-1)}(E) = -\mathrm{ch}_d(H^0(E)) + \mathrm{rk}(H^{1-d}(E)) \cdot i$ for any $E \in \mathrm{Coh}_{(d-1)}(X)$. Hence $\phi(E) \in [1/2, 1]$ for all $E \in \mathrm{Coh}_{(d-1)}(X)$. Since the phases of $k(y)$ and of $L[d-1]$ are in the boundary of the interval $[1/2, 1]$ for any $y \in X$ and $L \in \mathrm{Pic}^0(X)$, these objects have to be semistable. They are also stable, because their Chern character is primitive. □

Note that any ideal sheaf $\mathcal{I}_{\{p_1,\dots,p_n\}}$ is also stable in $\mathrm{Coh}(X)$. Hence there is no positive lower bound for the phases of stable objects in $\mathrm{Coh}(X)$. Similarly, there is a sequence of stable objects in $\mathrm{Coh}_{(d-1)}(X)$ whose phases form a strictly increasing sequence converging to 1.

Corollary 1.3.14. *For any $0 < p \leq d-1$ and any $\gamma \in (0, 1/2)$ the pair*

$$\sigma^{\gamma}_{(p)} := \Big(Z^{\gamma}_{(p)}(\cdot) = -\mathrm{ch}_d(\cdot) - (-1)^p \cot(\pi\gamma)\,\mathrm{rk}(\cdot), \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}) \Big)$$

is a numerical locally-finite stability condition.

Proof. Since $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ is of finite type, we only have to show $Z^{\gamma}_{(p)}(E) < 0$ for all $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. It is enough to check this for those objects in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ which are stable with respect to $\sigma_{(p)}$. Using Proposition 1.3.13 this is an easy calculation. □

Next, consider the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbits through the stability conditions $\sigma_{(p)} = (Z_{(p)}, \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}))$ and $\sigma^{\gamma}_{(p)} = (Z^{\gamma}_{(p)}, \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}))$ in $\mathrm{Stab}(X)$. It is an easy exercise to check that they are disjoint.
At the end of this subsection we will characterize the set

$$U(X) := \bigcup_{0 \leq p < d} \sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \quad \cup \bigcup_{\substack{1 \leq p < d \\ \gamma \in (0,1/2)}} \sigma^{\gamma}_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$$

of our stability conditions.

Proposition 1.3.15. *Assume X is a generic complex torus of dimension $d \geq 3$. If $\mathcal{P}$ is a slicing on $\mathrm{D^b}(X)$ such that $k(y) \in \mathcal{P}(1)$ for all $y \in X$ and $L \in \mathcal{P}(\psi)$ for all $L \in \mathrm{Pic}^0(X)$ for a fixed $\psi \in \mathbb{R}$, then $\mathcal{A} := \mathcal{P}((0,1]) = \mathrm{Coh}_{(p)}(X)$, where $p \in \mathbb{N}$ is the unique number with $\psi + p \in (0,1]$.*

Proof. Since $\mathrm{Hom}(\mathcal{O}_X, k(y)) \neq 0$ and $\mathrm{Hom}(k(y), \mathcal{O}_X[d]) \neq 0$, we conclude

$$\psi \in (1-d, 1) \qquad \text{and, therefore,} \qquad 0 \leq p < d.$$

The case $p = 0$: In this case $k(y) \in \mathcal{A}$ for all $y \in X$ and $L \in \mathcal{A}$ for all $L \in \mathrm{Pic}^0(X)$. Furthermore, $E \in \mathcal{P}([0,1))$ for all $\sigma_{(0)}$-stable torsionfree $E \in \mathrm{Coh}(X)$. Indeed, for such E there is a triangle

$$(E^{\vee\vee}/E)[-1] \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow E^{\vee\vee}/E$$

with the locally free sheaf $E^{\vee\vee} \in \mathcal{P}(\psi)$ and the torsion sheaf $E^{\vee\vee}/E \in \mathcal{P}(1)$. This shows $E \in \mathcal{P}([0,1))$ If $E \notin \mathcal{P}((0,1))$, we find a nontrivial morphism $E \to T[-1]$ with stable $T \in \mathcal{P}(1)$. We show $T \cong k(y)$ for some $y \in X$ which contradicts $\mathrm{Hom}(E, k(y)[-1]) = 0$.
In order to show $T \cong k(y)$, assume $H^m(T) \neq 0$ and $H^n(T) \neq 0$ but $H^k(T) = 0$ $\forall\, k \notin [m,n]$ for two integers $m \leq n$. If $H^m(T)$ is not torsionfree, there are nontrivial compositions

$$k(y)[-m] \longrightarrow H^m(T)[-m] \longrightarrow T \quad \text{and} \quad T \longrightarrow H^n(T)[-n] \longrightarrow k(z)[-n]$$

for suitable $y, z \in X$. If $H^m(T)$ is torsionfree but not reflexive, we replace the first composition by

$$k(y)[-1-m] \longrightarrow H^m(T)[-m] \longrightarrow T$$

and if $H^m(T)$ is reflexive, we take

$$L[-m] \longrightarrow H^m(T)[-m] \longrightarrow T$$

for a suitable $L \in \mathrm{Pic}^0(X)$. If $T \in \mathcal{P}(1)$ is not isomorphic to $k(y)$, we get in all cases $-m \leq \psi - m < 1 < 1 - n$ and, therefore, $n < 0 \leq m$, a contradiction to $m \leq n$.
Thus, any $\sigma_{(0)}$-stable sheaf is contained in $\mathcal{A} = \mathcal{P}((0,1])$ and we get $\mathrm{Coh}(X) \subseteq \mathcal{A}$. By standard arguments $\mathrm{Coh}(X) = \mathcal{A}$.

The case $0 < p < d$: From the proof of Proposition 1.3.13 we know that any $\sigma_{(p)}$-stable object $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ fits into a triangle

$$H^{-p}(E)[p] \longrightarrow E \longrightarrow H^0(E)$$

with locally free $H^{-p}(E) \in \mathcal{P}(\psi)$ and the torsion sheaf $H^0(E) \in \mathcal{P}(1)$. Since $\mathcal{P}(\psi + p) \subseteq \mathcal{A}$ and $\mathcal{P}(1) \subseteq \mathcal{A}$, we see $E \in \mathcal{A}$ and, therefore, $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}) \subseteq \mathcal{A}$. Again we can conclude $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}) = \mathcal{A}$. □

Proposition 1.3.16. *Assume $(Z', \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}))$ is a locally-finite numerical stability condition on X with $0 \leq p \leq d-1$ and $\phi'(k(y)) = 1$ for all $y \in X$. Then there is a matrix $G \in \mathrm{GL}^+(2,\mathbb{R})$ with $G \cdot Z_{(p)} = Z'$ or $G \cdot Z^{\gamma}_{(p)} = Z'$ for a unique $\gamma \in (0, 1/2)$.*

Proof. Since Z' is numerical, we get $Z'(E) = -e\,\mathrm{ch}_d(E) - (-1)^p f\,\mathrm{rk}(E) + (g\,\mathrm{ch}_d(E) + (-1)^p h\,\mathrm{rk}(E)) \cdot i$ for a suitable matrix

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{R})\,.$$

Since $\phi'(k(y)) = 1$, we obtain $g = 0$ and $e > 0$. If Z' takes values in $(-\infty, 0)$, then $h = 0$, $f > 0$ and

$$\begin{pmatrix} \mathrm{Re}\,Z' \\ \mathrm{Im}\,Z' \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\mathrm{ch}_d - (-1)^p \cot(\pi\gamma)\,\mathrm{rk} \\ 0 \end{pmatrix} \quad \text{with } \cot(\pi\gamma) = f/e.$$

This can only occur for $0 < p \leq d-1$ since $\mathrm{Coh}(X)$ is not of finite type. If the image of Z' is not contained in $(-\infty, 0)$,then $h > 0$ and

$$\begin{pmatrix} \mathrm{Re}\,Z' \\ \mathrm{Im}\,Z' \end{pmatrix} = \begin{pmatrix} e & -f \\ 0 & h \end{pmatrix} \cdot \begin{pmatrix} -\mathrm{ch}_d \\ (-1)^p\,\mathrm{rk} \end{pmatrix}.$$

□

Using these two propositions we get the main result of this subsection which characterizes the set $U(X)$ of stability conditions.

Theorem 1.3.17. *Assume X is a generic complex torus of dimension d. The set*

$$U(X) = \bigcup_{0 \leq p < d} \sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \;\cup\; \bigcup_{\substack{1 \leq p < d \\ \gamma \in (0,1/2)}} \sigma^{\gamma}_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$$

is the set of all numerical locally-finite stability conditions $\sigma = (Z, \mathcal{P})$ such that there exist certain real numbers ϕ and ψ such that $k(y) \in \mathcal{P}(\phi)$ for all $y \in X$ and $L \in \mathcal{P}(\psi)$ for all $L \in \mathrm{Pic}^0(X)$.

Proof. Choose some stability condition $\sigma = (Z, \mathcal{P})$ with the property described in the theorem. After applying some $G \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ we can assume $k(y) \in \mathcal{P}(1)$ $\forall\, y \in X$. Using Proposition 1.3.15 and Proposition 1.3.16 we get $\sigma \in U(X)$. Of course, every stability condition in $U(X)$ has the characterizing property. □

1.3.3 The topology of $U(X)$

In this subsection we study the topology of $U(X)$. As we will see, $U(X)$ is a simply connected component of $\mathrm{Stab}(X)$.
The first part of this subsection is a more general consideration of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbits in the space $\mathrm{Stab}(\mathcal{D})$ of locally-finite stability conditions on a $\mathbb{C}$-linear triangulated category $\mathcal{D}$. In the second part we come back to the case $\mathcal{D} = \mathrm{D^b}(X)$.

Let $\Sigma \subseteq \mathrm{Stab}(\mathcal{D})$ be a connected component and let us denote by $V(\Sigma)$ the linear subspace in $\mathrm{Hom}(K(\mathcal{D}),\mathbb{C})$ such that the forgetful map

$$\pi : \mathrm{Stab}(\mathcal{D}) \supseteq \Sigma \ni \sigma = (Z, \mathcal{P}) \longmapsto Z \in V(\Sigma) \subseteq \mathrm{Hom}(K(\mathcal{D}),\mathbb{C})$$

is a local homeomorphism. Given a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$ the space $V(\Sigma)$ is characterized by

$$V(\Sigma) = \{U \in \operatorname{Hom}(K(\mathcal{D}), \mathbb{C}) \mid \|U\|_\sigma < \infty\},$$

where

$$\|U\|_\sigma := \sup\left\{ \frac{|U(E)|}{|Z(E)|} \,\middle|\, E \text{ semistable in } \sigma \right\}$$

and $\|\cdot\|_\sigma$ can be used to define the topology on $V(\Sigma)$ [8]. It follows that the evaluation map $V(\Sigma) \ni U \mapsto U(E) \in \mathbb{C}$ is continuous for a fixed $E \in \mathcal{D}$.

The universal cover $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ of $\mathrm{GL}^+(2, \mathbb{R})$ acts on $\operatorname{Stab}(\mathcal{D})$ from the left by $g \cdot \sigma := \sigma \cdot g^{-1}$, where the latter action is the one considered in section 1.1. Furthermore, there is an action from the left of the ring $\operatorname{Mat}(2, \mathbb{R})$ on $\operatorname{Hom}(K(\mathcal{D}), \mathbb{C})$ and the map π commutes with these actions. Let us consider a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$ such that the image of the central charge is not contained in a real line in $\mathbb{C}$ and $\mathcal{P}(1) \neq \{0\}$. We are interested in the boundary points of the orbit

$$\sigma \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) = (\widetilde{\mathrm{GL}}^+(2, \mathbb{R}))^{-1} \cdot \sigma = \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \cdot \sigma \subseteq \Sigma.$$

The orbit is a real submanifold of Σ of real dimension four. It follows from the definition that the central charges of all stability conditions of this orbit factorize over the quotient by the linear subspace $K(\mathcal{D})^{\perp}_{\mathbb{R},\sigma} := \{e \in K(\mathcal{D})_{\mathbb{R}} \mid Z(e) = 0\}$ of real codimension two, i.e. they are contained in the closed real four-dimensional subspace $V(\Sigma)_\sigma := \operatorname{Hom}_{\mathbb{R}}(K(\mathcal{D})_{\mathbb{R}}/K(\mathcal{D})^{\perp}_{\mathbb{R},\sigma}, \mathbb{C})$. The map $\operatorname{Mat}(2, \mathbb{R}) \ni M \mapsto M \circ Z \in V(\Sigma)$ is an $\mathbb{R}$-linear isomorphism onto $V(\Sigma)_\sigma$. This isomorphism identifies $\mathrm{GL}^+(2, \mathbb{R})$ with $\pi(\widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \cdot \sigma)$. We write $Z(E) = \Re(E) + i \cdot \Im(E)$ with linear independent $\Re$ and $\Im \in \operatorname{Hom}(K(\mathcal{D}), \mathbb{R})$.

Let us denote by $\sigma' = (Z', \mathcal{P}')$ a boundary point of the orbit $\sigma \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. Since the evaluation map is continuous, Z' still factorizes over $K(\mathcal{D})^{\perp}_{\mathbb{R},\sigma}$. After applying some element of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ to σ', we can, therefore, assume $Z' = \Re - \cot(\pi\gamma)\Im$ with a suitable $\gamma \in (0, 1)$, because semistability is a closed property and, therefore, $Z'(E) \neq 0 \;\forall E \in \mathcal{P}(1)$. The line $Z' = 0$ in $\mathbb{C} \cong \mathbb{R}\Re \oplus \mathbb{R}\Im$ is given by the equation $\Re = \cot(\pi\gamma)\Im$ and since $Z'(E) \neq 0 \;\forall\, E$ semistable in σ, we have

$$\gamma \neq \phi(E) \quad \forall\, E \text{ stable in } \sigma. \tag{1.7}$$

The following result was already known to the experts (see for example [7] and [4]).

Proposition 1.3.18. *The heart $\mathcal{P}'((0, 1]) = \mathcal{P}'(1)$ of σ' is the tilt $\mathcal{A}^\sigma_\gamma$ of $\mathcal{A} := \mathcal{P}((0, 1])$ with respect to the torsion theory $(\mathcal{P}((\gamma, 1]), \mathcal{P}((0, \gamma)))$, i.e.*

$$\mathcal{P}'(1) = \{E \in \mathcal{D} \mid H^0(E) \in \mathcal{P}((\gamma, 1]), H^{-1}(E) \in \mathcal{P}((0, \gamma)), H^k(E) = 0 \textit{ else } \},$$

where H denotes the cohomology functor associated to the bounded t-structure with heart $\mathcal{A}$.

Proof. Due to (1.7), the pair $(\mathcal{P}((\gamma,1]),\mathcal{P}((0,\gamma)))$ is indeed a torsion theory in $\mathcal{A}=\mathcal{P}((0,1])$ and since $Z'(E)\neq 0\;\forall E$ semistable in σ, we obtain $E\in\mathcal{P}'(0)\;\forall\, E$ semistable in σ with $\phi(E)\in(0,\gamma)$. Therefore, $\mathcal{P}((0,\gamma))\subseteq\mathcal{P}'(0)$ and, similar, $\mathcal{P}((\gamma,1])\subseteq\mathcal{P}'(1)$. Hence $\mathcal{P}'(1)$ contains the tilt of $\mathcal{P}((0,1])$ with respect to the upper torsion theory. By standard arguments one concludes equality. □

In order to show the nonexistence of boundary points σ', we introduce the following two phases for our stability condition $\sigma=(Z,\mathcal{P})$ and the real number $\gamma\in(0,1)$.

$$\begin{aligned}\gamma^+ &:= \inf\{\phi(E)\mid E\in\mathcal{D}\text{ stable in }\sigma,\phi(E)>\gamma\},\\ \gamma^- &:= \sup\{\phi(E)\mid E\in\mathcal{D}\text{ stable in }\sigma,\phi(E)<\gamma\}.\end{aligned}$$

Clearly $\gamma^-\leq\gamma\leq\gamma^+$ and there is no $E\in\mathcal{D}$, stable in σ, with $\phi(E)\in(\gamma^-,\gamma^+)$. Hence for all $\gamma'\in[\gamma^-,\gamma^+]$ satisfying (1.7) we obtain $\gamma^+=\gamma'^+$ and $\gamma^-=\gamma'^-$. Note that for $\gamma\in(\gamma^-,\gamma^+)$ the condition (1.7) is always fulfilled.

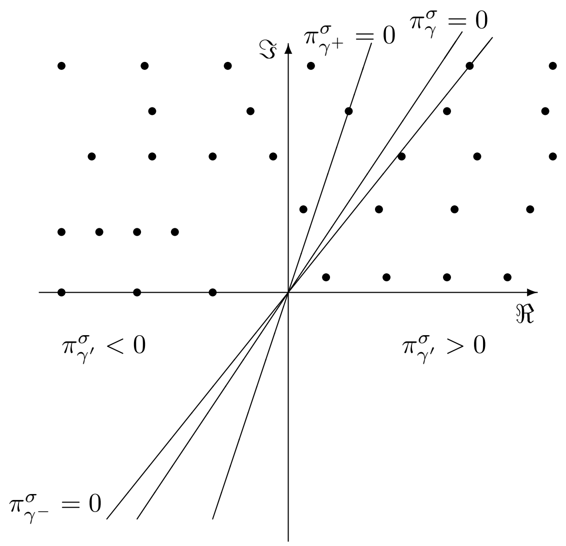
Proposition 1.3.19. *If $\mathcal{P}(\gamma^+)=\{0\}$ or $\mathcal{P}(\gamma^-)=\{0\}$ and $\gamma'\in[\gamma^-,\gamma^+]$ satisfying (1.7), there is no boundary point of $\sigma\cdot\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ with central charge $Z':=Z^\sigma_{\gamma'}:=\Re-\cot(\pi\gamma')\Im$.*

Proof. We consider the case $\mathcal{P}(\gamma^+)=\{0\}$. The second case is similar. If there is a boundary point $\sigma'=(Z'=Z^\sigma_{\gamma'},\mathcal{P}')$, we can replace $Z^\sigma_{\gamma'}$ and assume $\gamma'=\gamma'^+=\gamma^+$. Indeed, since $\mathcal{P}'(1)$ is of finite length and γ^+ satisfies (1.7), the pair $\sigma^+:=(Z^\sigma_{\gamma^+},\mathcal{P}')$ is a locally-finite stability condition. It is easy to see that σ^+ is still in the boundary of $\sigma\cdot\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. Since π is a local homeomorphism, there is an open neighbourhood of σ^+ in Σ which is isomorphic to an open ball in $V(\Sigma)$. The intersection of this ball with $V(\Sigma)_\sigma$ can be identified with an open ball in $\mathrm{Mat}(2,\mathbb{R})$ with center

$$Z^\sigma_{\gamma^+}\cong\begin{pmatrix}1 & -\cot(\pi\gamma^+)\\ 0 & 0\end{pmatrix}.$$

Such a ball contains the central charge $Z^\sigma_{\gamma''}=\Re-\cot(\pi\gamma'')\Im$ with $\gamma''\in(\gamma^+,\gamma^++\varepsilon)$ and $\varepsilon>0$ sufficiently small. By definition of γ^+ we can assume without loss of generality $Z^\sigma_{\gamma''}(E)=0$ for some σ-stable $0\neq E\in\mathcal{D}$. As $Z^\sigma_{\gamma''}$ is a boundary point of the orbit $\mathrm{GL}^+(2,\mathbb{R})\cdot Z=(\mathrm{GL}^+(2,\mathbb{R}))^{-1}\cdot Z=\pi(\sigma\cdot\widetilde{\mathrm{GL}}^+(2,\mathbb{R}))$ and semistability is a closed property, this E is still semistable in the stability condition lying in the neighbourhood of σ^+ and mapped by π onto $Z^\sigma_{\gamma''}$. This contradicts $Z^\sigma_{\gamma''}(E)=0$. □

Due to Proposition 1.3.19, we have to assume $\mathcal{P}(\gamma^+)\neq\{0\}$ and $\mathcal{P}(\gamma^-)\neq\{0\}$ in order to obtain stability conditions with central charges Z^σ_γ in the boundary of the orbit $\sigma\cdot\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.



The dots are the central charges of the σ-semistable objects in $\mathcal{A}$.

As in the end we want to avoid boundary points, we need a criterion that excludes the cases $\mathcal{P}(\gamma^+) \neq \{0\}$ and $\mathcal{P}(\gamma^-) \neq \{0\}$. This is only possible in special situations and the following will be enough in the geometric context we are interested in.

Lemma 1.3.20. *Suppose there exists a sequence $E_n \in \mathcal{P}(\gamma^+), n \in \mathbb{N}$, of non isomorphic simple objects. Then there is no object $I \in \mathcal{P}((0, \gamma^-])$ with $\mathrm{Ext}^1(E_n, I) \neq 0$ for all $n \in \mathbb{N}$.*

Proof. If such an object I exists, we construct by induction a sequence of non-trivial extensions

$$0 \longrightarrow I_n \longrightarrow I_{n+1} \longrightarrow E_n \longrightarrow 0$$

in $\mathcal{A} = \mathcal{P}((0,1])$ with $I_n \in \mathcal{P}((0, \gamma^-])$ and the additional property $\mathrm{Ext}^1(E_k, I_n) \neq 0$ for all $k \geq n$ and $n \in \mathbb{N}$. Since $Z(I_{n+1}) = Z(I_n) + Z(E_n)$, we get $\phi(I_n) > \gamma^-$ for $n \gg 0$ which contradicts $I_n \in \mathcal{P}((0, \gamma^-])$.
The construction of I_n starts with $I_0 = I$. Due to our assumption this is possible. Assume we have constructed $I_n \in \mathcal{P}((0, \gamma^-])$. Choose an element $0 \neq e \in \mathrm{Ext}^1(E_n, I_n)$ and consider the corresponding nontrivial extension in $\mathcal{A}$

$$0 \longrightarrow I_n \longrightarrow I_{n+1} \longrightarrow E_n \longrightarrow 0.$$

For any $0 \neq F \in \mathcal{P}(\gamma^-, 1] = \mathcal{P}[\gamma^+, 1]$ stable in σ we get the following long exact sequence

$$0 \longrightarrow \mathrm{Hom}(F, I_{n+1}) \longrightarrow \mathrm{Hom}(F, E_n) \longrightarrow \mathrm{Ext}^1(F, I_n) \longrightarrow \mathrm{Ext}^1(F, I_{n+1}).$$

Now, $\mathrm{Hom}(F, E_n) = 0$ unless $F = E_n$ and in the latter case $\mathrm{Hom}(E_n, E_n) = \mathbb{C} \cdot \mathrm{Id}_{E_n}$. But Id_{E_n} is mapped to $0 \neq e \in \mathrm{Ext}^1(E_n, I_n)$. Therefore, $\mathrm{Hom}(F, I_{n+1}) =$

0 for all $F \in \mathcal{P}((\gamma^-, 1])$ and we conclude $I_{n+1} \in \mathcal{P}((0, \gamma^-])$. Furthermore, the map $\mathrm{Ext}^1(E_k, I_n) \longrightarrow \mathrm{Ext}^1(E_k, I_{n+1})$ is an injection for $k \geq n+1$. Hence $\mathrm{Ext}^1(E_k, I_{n+1}) \neq 0$ for all $k \geq n+1$ by the induction hypothesis and we are done. □

Using this we get our main result of this subsection.

Theorem 1.3.21. *Assume X is a generic complex torus of dimension $d \geq 3$. Then*

$$U(X) := \bigcup_{0 \leq p < d} \sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \quad \cup \bigcup_{\substack{1 \leq p < d \\ \gamma \in (0,1/2)}} \sigma^{\gamma}_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$$

is a simply connected component of $\mathrm{Stab}(X)$*, but $\pi : U(X) \longrightarrow \pi(U(X))$ is not a covering.*

Proof. On a generic complex torus of dimension d any stability function of a numerical stability condition is a complex linear combination of $\mathrm{ch}_0 = \mathrm{rk}$ and ch_d. Since the orbits $\sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ are of real dimension four, they are open in $\mathrm{Stab}(X)$. We describe the closure of these open orbits beginning with that of $\sigma_{(0)} = (Z_{(0)}, \mathrm{Coh}(X))$.
We want to exclude boundary points with $\gamma \in (0, 1/2]$. In order to apply Proposition 1.3.19, we show $\mathcal{P}(\gamma^+) = \{0\}$. Indeed, if $0 \neq E \in \mathcal{P}(\gamma^+)$ is a stable sheaf, then it is torsionfree, because $\gamma^+ \leq 1/2$. Now, choose a sequence of numerically trivial line bundles $L_n \in \mathrm{Pic}^0(X)$ with $L_m^{\mathrm{rk}(E)} \neq L_n^{\mathrm{rk}(E)}$ for all $m \neq n$. Hence $E \otimes L_m \neq E \otimes L_n$ for $m \neq n$, because of $\det(E^{\vee\vee} \otimes L) = \det(E^{\vee\vee}) \otimes L^{\mathrm{rk}(E)}$ for every $L \in \mathrm{Pic}(X)$. Furthermore, the sheaves $E_n := E \otimes L_n$ are also $\sigma_{(0)}$-stable of phase γ^+. We introduce the sheaf $P := k(y)$ for some $y \in X$. Choose an epimorphism $f : E_0 \twoheadrightarrow P$ and denote the kernel by I. We prove $I \in \mathcal{P}((0, \gamma^-])$ and $\mathrm{Ext}^1(E_n, I) \neq 0$ for all $n \in \mathbb{N}$ which contradicts Lemma 1.3.20. Thus, $\mathcal{P}(\gamma^+) = \{0\}$.
In order to show $I \in \mathcal{P}((0, \gamma^-])$, we take a $\sigma_{(0)}$-stable sheaf $F \in \mathcal{P}((\gamma^-, 1]) = \mathcal{P}([\gamma^+, 1])$ and consider the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(F, I) \longrightarrow \mathrm{Hom}(F, E_0) \longrightarrow \mathrm{Hom}(F, P) \longrightarrow \mathrm{Ext}^1(F, I).$$

Now, $\mathrm{Hom}(F, E_0) = 0$ unless $F = E_0$ and in the latter case $\mathrm{Hom}(E_0, E_0) = \mathbb{C} \cdot \mathrm{Id}_{E_0}$. But Id_{E_0} is mapped to $0 \neq f \in \mathrm{Hom}(E_0, P)$. Therefore, $\mathrm{Hom}(F, I) = 0$ for all $F \in \mathcal{P}((\gamma^-, 1])$ and $I \in \mathcal{P}((0, \gamma^-])$ follows.
For the second property of I we consider the inclusion $\mathrm{Hom}(E_n, P) \hookrightarrow \mathrm{Ext}^1(E_n, I)$ and note that the former set contains $f \otimes id_{L_n} \neq 0$ for all $n \in \mathbb{N}$.
On the other hand, for every $\gamma \in (1/2, 1)$ we obtain $\sigma^{1-\gamma}_{(1)}$ as a boundary point. In the case $0 < p < d-1$ the situation is very easy. There are two regions of boundary points of the orbit $\sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. For $\gamma \in (0, 1/2)$ the boundary points are given by $\sigma^{\gamma}_{(p)}$ and for $\gamma \in (1/2, 1)$ the boundary points are $\sigma^{1-\gamma}_{(p+1)}$.
The case $p = d-1$ is similar to the case $p = 0$. First of all $E \otimes L \in \mathrm{Coh}_{(d-1)}(X)$ for all $E \in \mathrm{Coh}_{(d-1)}(X)$ and $L \in \mathrm{Pic}^0(X)$. Indeed, this is true for $E \cong k(y)$ and $E \cong H^{1-d}(E)[d-1]$ locally free. But any $E \in \mathrm{Coh}_{(d-1)}(X)$ is an extension of

such special objects and tensoring with L maps extensions to extensions. Furthermore, $E \otimes L \ncong E$ for all $E \in \mathrm{Coh}_{(d-1)}(X) \setminus \mathcal{T}$ and all $L \in \mathrm{Pic}^0(X)$ with $L^{\mathrm{rk}(E)} \ncong \mathcal{O}_X$, because $H^{1-d}(E \otimes L) = H^{1-d}(E) \otimes L \ncong H^{1-d}(E)$.

Now, we can exclude boundary points with $\gamma \in (1, 1/2)$ in the same way as for $\sigma_{(0)}$. Note that $\gamma^+ < 1$, because there are $\sigma_{(d-1)}$-stable objects with phases sufficiently close to 1. The definitions of P, f and I are given by the short exact sequence

$$0 \longrightarrow \underbrace{H^{1-d}(E_0)[d-1]}_{:=I} \longrightarrow E_0 \xrightarrow{=:f} \underbrace{H^0(E_0)}_{:=P} \longrightarrow 0.$$

Since $H^{1-d}(E_0)[d-1]$ has phase $1/2$ (see Proposition 1.3.13) and $\gamma^- \geq 1/2$, the property $I \in \mathcal{P}((0, \gamma^-])$ is obvious in this case.

On the other hand, for every $\gamma \in (0, 1/2)$ we obtain $\sigma^{\gamma}_{(d-1)}$ as a boundary point.

Hence the open four-dimensional orbits $\sigma_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ are successive connected by the three dimensional 'walls' $\cup_{\gamma \in (0,1/2)} \sigma^{\gamma}_{(p)} \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. Furthermore, the connected set $U(X)$ is closed in $\mathrm{Stab}(X)$. The image of $U(X)$ under $\pi : \mathrm{Stab}(X) \longrightarrow (H^0(X, \mathbb{C}) \oplus H^d(X, \mathbb{C}))^{\vee} \cong \mathrm{Mat}(2, \mathbb{R})$ is an open subset. Since π is a local homeomorphism, $U(X)$ is also open and hence a connected component of $\mathrm{Stab}(X)$.

It is easy to see that $(\{\sigma_{(p-1)}, \sigma_{(p)}\} \cup \{\sigma^{\gamma}_{(p)} \mid \gamma \in (0, 1/2)\}) \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ is the universal cover of its image for all $0 < p \leq d-1$. This image has fundamental group $\mathbb{Z}$ which is 'resolved' by the shift functor [2]. Using the Seifert–van Kampen Theorem (see [18], Theorem 1.20) one concludes $\pi_1(U(X)) = 0$. Since the number of preimages of a stability function is not constant, $U(X)$ is not the universal cover of its image. $\square$

1.4 Stability conditions for simple manifolds

The aim of this section is to generalize the stability conditions $\sigma_{(0)}, \ldots, \sigma_{(d-1)}$ of the previous section to the case of compact complex manifolds without nontrivial subvarieties. In addition to curves, generic K3 surfaces and generic tori, the general deformations of a Hilbert scheme of a K3 surface form another class of examples.
Remember that a stability condition on a complex manifold X is a bounded t-structure on the bounded derived category $\mathrm{D^b}(X)$ of coherent sheaves together with a central charge on the heart of the t-structure. In order to produce interesting stability conditions on a complex manifold X we have to look for interesting bounded t-structures on $\mathrm{D^b}(X)$. We follow the method of Kashiwara [23] to construct t-structures.

Definition 1.4.1 (family of supports)**.** *Let X be a complex manifold. A set Φ of Zariski-closed subsets of X is called a family of supports if the following conditions hold:*

1. $Z \in \Phi, Z' \subseteq Z$ *Zariski-closed* $\implies Z' \in \Phi$,
2. $Z, Z' \in \Phi \implies Z \cup Z' \in \Phi$,
3. $\emptyset \in \Phi$.

For a family of supports Φ we define the functor Γ_Φ on the abelian category $\mathcal{M}od(X)$ of $\mathcal{O}_X$-modules G by

$$\Gamma_\Phi(G) = \varinjlim_{\mathrm{supp}(\mathcal{O}_Z)\in\Phi} \mathcal{H}om(\mathcal{O}_Z, G),$$

where the inductive limit is taken over all (possibly non-reduced) structure sheaves $\mathcal{O}_Z$ of complex analytic subspaces with support $Z \in \Phi$. The $\mathcal{O}_X$-module $\Gamma_\Phi(G)$ is the sheaf associated to the presheaf

$$U \longmapsto \{s \in \Gamma(U, G) \mid \overline{\mathrm{supp}(s)}^{Zar} \in \Phi\}.$$

It is obvious that Γ_Φ is leftexact and we denote by $R\Gamma_\Phi$ its right derived functor defined on $\mathrm{D}^+(\mathcal{M}od(X))$. For $E \in \mathrm{D}^{\geq n}(\mathcal{M}od(X))$ one has $H^n R\Gamma_\Phi(E) = \Gamma_\Phi H^n(E)$. Furthermore, there is a natural transformation $R\Gamma_\Phi \longrightarrow \mathrm{Id}$ induced by the surjection $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ and $\mathcal{H}om(\mathcal{O}_X, G) \cong G$.

Definition 1.4.2 (support datum)**.** *A support datum is a decreasing sequence $\Phi := (\Phi^n)_{n\in\mathbb{Z}}$ of families of supports such that*

1. Φ^n *is the set of all closed subsets of X for $n \ll 0$ and*
2. $\Phi^n = \{\emptyset\}$ *for $n \gg 0$.*

For a support datum $\Phi = (\Phi^n)_{n\in\mathbb{Z}}$ we define

$$\mathrm{D}^{\leq n}_\Phi(\mathcal{M}od(X)) := \{E \in \mathrm{D^b}(\mathcal{M}od(X)) \mid \mathrm{supp}(H^k(E)) \subseteq \Phi^{k-n}\ \forall\, k \in \mathbb{Z}\},$$

$$\mathrm{D}^{\geq n}_\Phi(\mathcal{M}od(X)) := \{E \in \mathrm{D^b}(\mathcal{M}od(X)) \mid R\Gamma_{\Phi^k}(E) \in \mathrm{D}^{\geq k+n}(\mathcal{M}od(X))\ \forall\, k \in \mathbb{Z}\}.$$

Theorem 1.4.3 ([23], Theorem 3.5.). *Let $\Phi = (\Phi^n)_{n\in\mathbb{Z}}$ be a support datum on a compact complex manifold. The pair $(\mathrm{D}^{\leq 0}_{\Phi}(\mathcal{M}od(X)), \mathrm{D}^{\geq 0}_{\Phi}(\mathcal{M}od(X)))$ is a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(\mathcal{M}od(X))$.*

In this section we are interested in compact complex manifolds such that the points are the only irreducible proper Zariski-closed subsets. If we are looking for a support datum Φ such that all points belong to the same family, we obtain the following list of possibilities for Φ indexed by two integers $p, q \in \mathbb{Z}$ with $-p \leq q$

$$\Phi^n_{(p,q)} := \begin{cases} \{\text{all Zariski-closed subsets}\} & \text{for } n \leq -p, \\ \{\text{finite subsets}\} & \text{for } -p < n \leq q, \\ \{\emptyset\} & \text{for } n > q. \end{cases}$$

Note that the t-structure associated to the support datum $\Phi_{(p,q)}$ is the t-structure associated to $\Phi_{(p+1,q-1)}$ shifted by [1]. Hence, it suffices to consider the pairs $(p,0)$ for $p \in \mathbb{N}$. We denote the corresponding t-structures by $(\mathrm{D}^{\leq 0}_{(p)}(\mathcal{M}od(X)), \mathrm{D}^{\geq 0}_{(p)}(\mathcal{M}od(X)))$.
Actually, we are interested in t-structures on the full subcategory $\mathrm{D}^{\mathrm{b}}(X) \subseteq \mathrm{D}^{\mathrm{b}}(\mathcal{M}od(X))$ of bounded complexes with coherent cohomology sheaves. The restriction

$$(\mathrm{D}^{\leq 0}_{\Phi}(X), \mathrm{D}^{\geq 0}_{\Phi}(X)) := (\mathrm{D}^{\leq 0}_{\Phi}(\mathcal{M}od(X)) \cap \mathrm{D}^{\mathrm{b}}(X), \mathrm{D}^{\geq 0}_{\Phi}(\mathcal{M}od(X)) \cap \mathrm{D}^{\mathrm{b}}(X))$$

of our t-structures gives a t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ if and only if a technical condition is fulfilled.

Theorem 1.4.4 ([23], Theorem 5.9.). *Let $\Phi = (\Phi^n)_{n\in\mathbb{Z}}$ be a support datum on a compact complex manifold. The pair $(\mathrm{D}^{\leq 0}_{\Phi}(X), \mathrm{D}^{\geq 0}_{\Phi}(X))$ is a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ if and only if for any irreducible Zariski-closed subsets Z and S such that $S \subseteq Z$ and $S \in \Phi^n$, one has $Z \in \Phi^{n+\operatorname{codim}(Z)-\operatorname{codim}(S)}$.*

For our t-structures $(\mathrm{D}^{\leq 0}_{(p)}(\mathcal{M}od(X)), \mathrm{D}^{\geq 0}_{(p)}(\mathcal{M}od(X)))$ this condition leads to the inequality $p \leq \dim(X)$. Finally, we obtain $d+1$ bounded t-structures on $\mathrm{D}^{\mathrm{b}}(X)$ indexed by $0 \leq p \leq d$, where d is the dimension of X. We denote the heart of these t-structures by $\mathrm{Coh_{(p)}(X)}$. Moreover, $\mathrm{Coh_{(p)}(X)}$ is the abelian category of perverse sheaves with the constant perversity function $-p$. Bounded t-structures of perverse sheaves on schemes has been investigated by M. Kashiwara [23] and R. Bezrukavnikov [3]. Note that for $p = 0$ we obtain the standard t-structure on $\mathrm{D}^{\mathrm{b}}(X)$ and we write $\mathrm{Coh}(X)$ instead of $\mathrm{Coh}_0(X)$.
We aim to show that the heart $\mathrm{Coh_{(p)}(X)}$ of these t-structures carries a central charge for $0 \leq p < d$. If we denote the set of coherent torsion sheaves by $\mathcal{T}$, we can rewrite the upper definition of our t-structures.

$$\mathrm{D}^{\leq 0}_{(p)}(X) = \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid H^k(E) \in \mathcal{T}\ \forall -p < k \leq 0\ ,\ H^k(E) = 0\ \forall k > 0\},$$
$$\mathrm{D}^{\geq 0}_{(p)}(X) = \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid H^k(E) = 0\ \forall\, k < -p\ ,\ R\Gamma(E) \in \mathrm{D}^{\geq 0}(\mathcal{M}od(X))\},$$

where we denote by Γ the functor which associates to each coherent sheaf its torsion subsheaf. Due to our assumptions on X, the structure sheaf of a proper closed subspace is the direct sum of its stalks, and we get

$$R^k\Gamma(E) = \bigoplus_{x \in X} \mathrm{H}^k_x(X, E)$$

for every sheaf or complex E on X, where $\mathrm{H}^k_x(X,E)$ is the local cohomology of E in $x \in X$. In this formula we regard $\mathrm{H}^k_x(X,E)$ as a skyscraper sheaf concentrated in x. See [15] or [9] for more information about local cohomology. If we use the classical Godement resolution in order to calculate the local cohomology of a coherent sheaf, we obtain $\mathrm{H}^k_x(X,T) = 0$ for all $k > 0$ for every coherent sheaf T of zero-dimensional support, i.e. an isomorphism $R\Gamma(T) \xrightarrow{\sim} T$. Using the exactness of $R\Gamma$, we can generalize this result to $R\Gamma(E) \xrightarrow{\sim} E$ for any complex with coherent cohomology sheaves of zero-dimensional support. On the other hand, if the sheaf E is locally free in x, we get $\mathrm{H}^k_x(X,E) = 0$ for all $k < \dim(X)$ because $\mathrm{depth}_x(E_x) = \dim(X)$ due to the Auslander–Buchsbaum formula (see Prop. 18.4 in [13] or Ex. 3.4 in Chap. III of [17]). Since every coherent sheaf E is locally free outside a proper closed subset, we see that $R^k\Gamma(E)$ is a torsion sheaf with zero-dimensional support for $k < \dim(X)$. Using the fact $\dim \mathrm{supp}(\mathcal{E}xt^k(E, \mathcal{O}_X)) = \dim \mathrm{supp}(\mathcal{E}xt^k(E, \omega_X)) = 0$ (use Prop. 1.1.6.ii) in [21] and the assumptions on X) and the following criterion for a coherent sheaf E (see [38] or [23])

$$\begin{aligned} &\mathrm{H}^k_x(X,E) \text{ is coherent for any } k < n \\ &\Longleftrightarrow \mathrm{codim}\left(\{x\} \cap \overline{\mathrm{supp}(\mathcal{E}xt^k(E, \mathcal{O}_X)) \cap (X \setminus \{x\})}\right) \geq k + n \text{ for all } k, \end{aligned}$$

we obtain that $R^k\Gamma(E)$ is a coherent sheaf for all $k < \dim(X)$.

Let E be a complex in the heart $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ of our new t-structures. Using the definition of $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$, we get a triangle

$$H^{-p}(E)[p] \longrightarrow E \longrightarrow \tau^{>-p}(E) \xrightarrow{\alpha} H^{-p}(E)[p+1]$$

with a complex $\tau^{>-p}(E) \in \mathrm{D}^{[1-p,0]}(X)$ of zero-dimensional support. If we apply $R\Gamma$ to the previous triangle and use $R\Gamma\tau^{>-p}(E) \cong \tau^{>-p}(E)$, we obtain the following triangle

$$\underbrace{R\Gamma(E)}_{\in \mathrm{D}^{\geq 0}(\mathcal{M}od(X))} \longrightarrow \tau^{>-p}(E) \longrightarrow R\Gamma H^{-p}(E)[p+1] \longrightarrow \underbrace{R\Gamma(E)[1]}_{\in \mathrm{D}^{\geq -1}(\mathcal{M}od(X))}.$$

This induces a natural isomorphism

$$\tau^{[1-p,-2]}(E) \xrightarrow[\sim]{\tau^{\leq -2}R\Gamma(\alpha)} \tau^{\leq -2}(R\Gamma H^{-p}(E)[p+1]) \tag{1.8}$$

as well as the following exact sequence for $p \geq 2$

$$0 \longrightarrow H^{-1}(E) \xrightarrow{R^{-1}\Gamma(\alpha)} R^p\Gamma H^{-p}(E) \longrightarrow R^0\Gamma(E) \longrightarrow H^0(E). \tag{1.9}$$

Conversely, any complex $E \in \mathrm{D}^{[-p,0]}(X)$ such that $\tau^{>-p}(E)$ has zero-dimensional support and such that $\alpha : \tau^{-p}(E) \longrightarrow H^{-p}(E)[p+1]$ satisfies (1.8) and (1.9), the latter for $p \geq 2$, is contained in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$.
For $p = 1$ equation (1.8) reduces to $R^0\Gamma H^{-1}(E) = \Gamma H^{-1}(E) = 0$, i.e. $H^{-1}(E)$ needs to be torsionfree. Thus,

$$\mathrm{Coh}_{(1)}(X) = \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid H^{-1}(E) \text{ is torsionfree}, H^0(E) \in \mathcal{T}, H^k(E) = 0 \text{ else}\}.$$

For $p \geq 2$ we get in addition to $\Gamma H^{-p}(E) = 0$ the condition $R^1\Gamma H^{-p}(E) = 0$. This is equivalent to the fact that $H^{-p}(E)$ is a reflexive sheaf due to the following lemma which generalizes Lemma 1.3.2.

Lemma 1.4.5. *Assume $d = \dim(X) \geq 2$ and that all proper nonempty Zariski-closed subsets of X are of dimension zero. In that case the following conditions for a coherent sheaf G on X are equivalent.*

(a) *G is reflexive,*

(b) $\mathrm{Hom}(T, G) = \mathrm{Ext}^1(T, G) = 0 \qquad \forall\, T \in \mathcal{T}$,

(c) $\mathcal{H}om(T, G) = \mathcal{E}xt^1(T, G) = 0 \qquad \forall\, T \in \mathcal{T}$,

(d) $\Gamma(G) = R^1\Gamma(G) = 0$.

Proof. (a) $\Longrightarrow$ (b) See the proof of Lemma 1.3.2.
(b) $\Longrightarrow$ (c) For $T \in \mathcal{T}$ we have

$$\mathcal{E}xt^1(T, G)_x = \mathcal{E}xt^1_{\mathcal{O}_{X,x}}(T_x, G_x) = 0 \quad \forall\, x \notin \mathrm{supp}(T).$$

Thus, the support of $\mathcal{E}xt^1(T, G)$ is a finite set of points and similar for $\mathcal{H}om(T, G)$. From the spectral sequence $\mathrm{H}^m(X, \mathcal{E}xt^n(T, G)) \implies \mathrm{Ext}^{m+n}(T, G)$ we conclude $\mathrm{H}^0(X, \mathcal{H}om(T, G)) = \mathrm{Hom}(T, G) = 0$ and $\mathrm{H}^0(X, \mathcal{E}xt^1(T, G)) = \mathrm{Ext}^1(T, G) = 0$ and the assertion follows.
(c) $\Longrightarrow$ (d) This is an easy consequence of

$$\begin{aligned} \Gamma(G) &= \varinjlim_{\mathcal{O}_Z \in \mathcal{T}} \mathcal{H}om(\mathcal{O}_Z, G) \quad \text{and} \\ R^1\Gamma(G) &= \varinjlim_{\mathcal{O}_Z \in \mathcal{T}} \mathcal{E}xt^1(\mathcal{O}_Z, G). \end{aligned}$$

(d) $\Longrightarrow$ (a) G is torsionfree because of $\Gamma(G) = 0$. Consider the short exact sequence $0 \to G \to G^{\vee\vee} \to T \to 0$ with $T \in \mathcal{T}$. By looking at the associated long exact $R\Gamma$-sequence we conclude from $\Gamma(G^{\vee\vee}) = R^1\Gamma(G) = 0$ the equality $T = \Gamma(T) = 0$ and G is reflexive. $\square$

Remember that a central charge on an abelian category $\mathcal{A}$ is an additive function Z on $\mathcal{A}$ with values $Z(E)$ in the strict upper half-plane $H = \{r\exp(i\pi\phi) \mid r > 0 \text{ and } 0 < \phi \leq 1\} \subset \mathbb{C}$ for every object $0 \neq E \in \mathcal{A}$ such that every nonzero object of $\mathcal{A}$ has a Harder–Narasimhan filtration. Using the sequences (1.8) and (1.9), we obtain immediately the following Corollary.

Corollary 1.4.6. *For $0 \leq p \leq d$ we have*

(a) $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$, $H^{-p}(E) = 0 \implies E \cong H^0(E) \in \mathcal{T}$,

(b) $Z_{(p)}(E) = -\mathrm{ch}_d(E) + (-1)^p \,\mathrm{rk}(E) \cdot i \in H$ *for all* $0 \neq E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$, *where* $\mathrm{ch}_d(E)$ *is (the integral over) the d-th Cherncharacter of* E.

In order to show that $(Z_{(p)}, \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}))$ is a stability condition on X, we have to check the Harder–Narasimhan property. This will follow for $0 \leq p < d$ from the fact that $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ is noetherian for $0 \leq p < d$ and artinian for $0 < p \leq d$. The idea of the proof of these two properties is to consider two full subcategories in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ which span $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ in a very nice way. The first subcategory is the category of torsion sheaves, while the second subcategory is equivalent to the category of reflexive sheaves.

Let us illustrate in the case $p = 0$, i.e. $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}) = \mathrm{Coh}(X)$, how these two categories span $\mathrm{Coh}(X)$. As before we denote by $\mathcal{T}$ the full abelian subcategory of torsion sheaves, and in addition to this we introduce the full subcategory $\mathrm{Refl}(\mathrm{X})$ of $\mathrm{Coh}(X)$ consisting of reflexive sheaves. For $f : F \longrightarrow G$ in $\mathrm{Refl}(\mathrm{X})$ we define a kernel via $\ker^r f := \ker f \longrightarrow F$ and a cokernel via the composition $G \longrightarrow \mathrm{coker}\, f \longrightarrow (\mathrm{coker}\, f)^{\vee\vee} =: \mathrm{coker}^r f$. With these definitions $\mathrm{Refl}(\mathrm{X})$ becomes an abelian category, but in contrast to $\mathcal{T}$ the inclusion functor to $\mathrm{Coh}(X)$ is not exact. Note that the image and the coimage of f in $\mathrm{Refl}(\mathrm{X})$ are isomorphic to $(\mathrm{im}\, f)^{\vee\vee} \hookrightarrow G$. For every sheaf F there is a functorial 4-term exact sequence in $\mathrm{Coh}(X)$

$$0 \longrightarrow \Gamma(F) \longrightarrow F \longrightarrow F^{\vee\vee} \longrightarrow F^{\vee\vee}/F \longrightarrow 0$$

with $F^{\vee\vee} \in \mathrm{Refl}(\mathrm{X})$ and $\Gamma(F) \in \mathcal{T}$ as well as $F^{\vee\vee}/F \in \mathcal{T}$. If $\dim(X) \geq 2$, we can write this sequence as follows

$$0 \longrightarrow R^0\Gamma(F) \longrightarrow F \longrightarrow F^{\vee\vee} \longrightarrow R^1\Gamma(F) \longrightarrow 0. \tag{1.10}$$

Indeed, applying $R\Gamma$ to $0 \longrightarrow \Gamma(F) \longrightarrow F \longrightarrow F/\Gamma(F) \longrightarrow 0$ yields $R^1\Gamma(F) \cong R^1\Gamma(F/\Gamma(F))$, and the same argument applied to the short exact sequence $0 \longrightarrow F/\Gamma(F) \longrightarrow F^{\vee\vee} \longrightarrow F^{\vee\vee}/F \longrightarrow 0$ proves the assertion.

Surprisingly there is a similar sequence for any object $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ for $p \geq 2$. But in contrast to the upper sequence, the arrows have the reverse direction. Furthermore, we have to find the appropriate two subcategories. For the first category we can choose the abelian category $\mathcal{T}$ again and the inclusion functor is exact due to the following lemma which generalizes Lemma 1.3.9 and is proved in the same way.

Lemma 1.4.7. *For every* $0 \leq p \leq d$ *the category* $\mathcal{T}$ *of torsion sheaves is an abelian subcategory of* $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$, *i.e. if* $f : S \to T$ *is a morphism in* $\mathcal{T}$ *and if we denote the kernel of* f *in* $\mathcal{T}$ *and in* $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ *by* $\ker f$ *resp.* $\ker_{(p)} f$, *then* $\ker f = \ker_{(p)} f$ *and similar for the cokernels.*

For the second subcategory we choose the full subcategory $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ of those complexes $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ with $H^0(E) = H^{-1}(E) = 0$. For $f : F \longrightarrow G$ in $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ we introduce the notation $\ker^r_{(p)} f := \tau^{\leq -2}(\ker_{(p)} f) \longrightarrow \ker_{(p)} f \longrightarrow F$ and $G \longrightarrow \mathrm{coker}_{(p)} f =: \mathrm{coker}^r_{(p)} f$. Using $\mathrm{Hom}(F, T) = \mathrm{Hom}(F, T[1]) = 0$ for any $F \in \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ and $T \in \mathcal{T}$ and the proof of the next proposition, it is easy to see that $\ker^r_{(p)} f$ and $\mathrm{coker}^r_{(p)} f$ belong to $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ and that $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ becomes an exact category with these kernels and cokernels. The suggestive notation is motivated by the following proposition.

Proposition 1.4.8. *The functor $H^{-p} : \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X}) \longrightarrow \mathrm{Refl}(\mathrm{X})$ is exact and induces an equivalence of categories. Hence, $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ is also abelian.*

Proof. In order to show the exactness of H^{-p} we consider a morphism $f : F \longrightarrow G$ in $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ and the corresponding triangle $F \xrightarrow{f} G \longrightarrow M \longrightarrow F[1]$. We form the associated long exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow H^{-p-1}(M) \longrightarrow H^{-p}(F) &\xrightarrow{H^{-p}(f)} H^{-p}(G) \longrightarrow H^{-p}(M) \\ \longrightarrow H^{1-p}(F) &\xrightarrow{H^{1-p}(f)} H^{1-p}(G) \longrightarrow H^{1-p}(M) \longrightarrow \dots \end{aligned}$$

and get $H^{-p-1}(M) = \ker H^{-p}(f)$ and $M \in \mathrm{D}^{\leq -2}(X)$ as well as the short exact sequence

$$0 \longrightarrow \mathrm{coker}\, H^{-p}(f) \longrightarrow H^{-p}(M) \longrightarrow \underbrace{\ker H^{1-p}(f)}_{\text{torsion}} \longrightarrow 0. \tag{1.11}$$

As in the proof of Lemma 1.3.9, we look at the long exact cohomology sequence of the triangle $K[1] := (H^{-1}_{(p)} M)[1] \longrightarrow M \longrightarrow H^0_{(p)} M =: C \longrightarrow K[2]$, where $H^i_{(p)}$ denotes the i-th cohomology functor of the t-structure with heart $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. Note that $K = \ker_{(p)} f, \tau^{\leq -2}(K) = \ker^r_{(p)} f$ and $C = \mathrm{coker}_{(p)} f = \mathrm{coker}^r_{(p)} f$.

$$\begin{aligned} &0 \longrightarrow H^{-p}(K) \longrightarrow \underbrace{H^{-p-1}(M)}_{=\ker H^{-p}(f)} \longrightarrow 0 \longrightarrow \underbrace{H^{1-p}(K)}_{\text{torsion}} \\ &\longrightarrow H^{-p}(M) \xrightarrow{\alpha} \underbrace{H^{-p}(C)}_{\text{reflexive}} \longrightarrow \underbrace{H^{2-p}(K)}_{\text{torsion}} \longrightarrow \dots \longrightarrow H^{-2}(C) \longrightarrow \\ &H^0(K) \longrightarrow \underbrace{H^{-1}(M)}_{=0} \longrightarrow H^{-1}(C) \longrightarrow 0 \longrightarrow \underbrace{H^0(M)}_{=0} \longrightarrow H^0(C) \longrightarrow 0 \end{aligned} \tag{1.12}$$

This sequence shows $C \in \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ and

$$H^{-p}(\ker^r_{(p)} f) = H^{-p}(\tau^{\leq -2}(K)) = H^{-p}(K) = \ker H^{-p}(f) = \ker^r H^{-p}(f)$$

as well as $H^{-p}(M)^{\vee\vee} \cong (\mathrm{im}\, \alpha)^{\vee\vee} = H^{-p}(C) = H^{-p}(\mathrm{coker}^r_{(p)} f)$. But from (1.11) we obtain

$$H^{-p}(\mathrm{coker}^r_{(p)} f) \cong H^{-p}(M)^{\vee\vee} \cong (\mathrm{coker}\, H^{-p}(f))^{\vee\vee} = \mathrm{coker}^r H^{-p}(f).$$

Thus, we have proved the exactness of H^{-p}. The fact that H^{-p} is an equivalence is not used in the sequel and we will postpone the proof to appendix A. We only need the fact that $H^{-p}(E) = 0$ implies $E = 0$ for any $E \in \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$. This follows directly from the isomorphism (1.8). $\square$

Having these two categories at hand, we try to produce an exact sequence similar to (1.10). To do this, we start with the the triangle

$$\tau^{\leq -1}(E) \longrightarrow E \longrightarrow H^0(E) \longrightarrow \tau^{\leq -1}(E)[1]$$

for $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. The triangle

$$\underbrace{R\Gamma H^0(E)[-1]}_{\in \mathrm{D}^{\geq 1}(\mathcal{M}od(X))} \longrightarrow R\Gamma\tau^{\leq -1}(E) \longrightarrow \underbrace{R\Gamma(E)}_{\in \mathrm{D}^{\geq 0}(\mathcal{M}od(X))} \longrightarrow R\Gamma H^0(E)$$

shows $R\Gamma\tau^{\leq -1}(E) \in \mathrm{D}^{\geq 0}(\mathcal{M}od(X))$. Furthermore, $H^k\tau^{\leq -1}(E) = H^k(E) \in \mathcal{T}$ for all $-p < k \leq -1$ and $H^k\tau^{\leq -1}(E) = 0$ otherwise. Thus, $\tau^{\leq -1}(E)$ is an object of $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. Hence, we get the following short exact sequence in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$

$$0 \longrightarrow \tau^{\leq -1}(E) \longrightarrow E \longrightarrow H^0(E) \longrightarrow 0. \tag{1.13}$$

There is another triangle

$$H^{-1}(E) \longrightarrow \tau^{\leq -2}(E) \longrightarrow \tau^{\leq -1}(E) \longrightarrow H^{-1}(E)[1],$$

and because of

$$\underbrace{R\Gamma H^{-1}(E)}_{\in \mathrm{D}^{\geq 0}(\mathcal{M}od(X))} \longrightarrow R\Gamma\tau^{\leq -2}(E) \longrightarrow \underbrace{R\Gamma\tau^{\leq -1}(E)}_{\in \mathrm{D}^{\geq 0}(\mathcal{M}od(X))} \longrightarrow R\Gamma H^{-1}(E)[1]$$

and $H^k\tau^{\leq -2}(E) = H^k(E) \in \mathcal{T}$ for all $-p < k \leq -2$ as well as $H^k\tau^{\leq -2}(E) = 0$ otherwise, $\tau^{\leq -2}(E)$ is also contained in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. Thus, we get a second short exact sequence

$$0 \longrightarrow H^{-1}(E) \longrightarrow \tau^{\leq -2}(E) \longrightarrow \tau^{\leq -1}(E) \longrightarrow 0$$

in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ for $p \geq 2$. Using the definition of $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ and combining both short exact sequences we obtain a 4-term exact sequence in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ for $p \geq 2$

$$0 \longrightarrow H^{-1}(E) \longrightarrow \tau^{\leq -2}(E) \longrightarrow E \longrightarrow H^0(E) \longrightarrow 0$$

with $\tau^{\leq -2}(E) \in \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ and $H^0(E), H^{-1}(E) \in \mathcal{T}$ similar to the 4-term sequence (1.10) in $\mathrm{Coh}(X)$.

In the case $p = 1$ we have only the short exact sequence (1.13) and instead of Proposition 1.4.8 we will use the following lemma generalizing Lemma 1.3.10 and which is proved in the same way.

Lemma 1.4.9. *Any morphism $f \in \mathrm{Hom}(F,G)$ between torsionfree sheaves F and G defines a morphism $f[1] : F[1] \longrightarrow G[1]$ in $\mathrm{Coh}_{(1)}(X)$, and we get $H^{-1}(\ker_{(1)} f[1]) = \ker f$, $H^0(\ker_{(1)} f[1]) = \Gamma(\mathrm{coker}\, f)$, $H^{-1}(\mathrm{coker}_{(1)} f[1]) = \mathrm{coker}\, f/\Gamma(\mathrm{coker}\, f)$ as well as $H^0(\mathrm{coker}_{(1)} f[1]) = 0$.*

Using the two subcategories $\mathcal{T}$ and $\mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ as well as the corresponding Propositions 1.4.7 and 1.4.8, we can prove now the crucial results of this section.

Proposition 1.4.10. *The category $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ is noetherian for all $0 \le p < d$.*

Proof. For $p = 0$ this is a well-known fact. Assume $2 \le p < d$ and take an infinite sequence $E = E_0 \twoheadrightarrow E_1 \twoheadrightarrow E_2 \twoheadrightarrow \dots$ of quotients. The commutative diagram

$$\begin{tikzcd} E_n \arrow[r, two heads] \arrow[d, two heads] & E_{n+1} \arrow[d, two heads] \\ H^0(E_n) \arrow[r] & H^0(E_{n+1}) \end{tikzcd}$$

shows that $H^0(E_n) \longrightarrow H^0(E_{n+1})$ is an epimorphism in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ for all $n \ge 0$. Since there are only finitely many quotients of the torsion sheaf $H^0(E_0)$ in $\mathrm{Coh}(X)$ and by Lemma 1.4.7 also in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$, we get $H^0(E_n) \cong H^0(E_{n+1})$ for all $n \gg 0$. Then, we apply the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & \tau^{\le -1}(E_n) \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d, two heads] & H^0(E_n) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \tau^{\le -1}(E_{n+1}) \arrow[r] & E_{n+1} \arrow[r] & H^0(E_{n+1}) \arrow[r] & 0 \end{tikzcd} \tag{1.14}$$

which yields that $\tau^{\le -1}(E_n) \twoheadrightarrow \tau^{\le -1}(E_{n+1})$ is an epimorphism in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. Since the rank function rk is additive on $\mathrm{D^b}(X)$, the sequence $(\mathrm{rk}\, H^{-p}(E_n))_{n\in\mathbb{N}} = ((-1)^p\, \mathrm{rk}(E_n))_{n\in\mathbb{N}}$ of natural numbers decreases. Without loss of generality we can assume $\mathrm{rk}\, H^{-p}(E_n) = \mathrm{rk}\, H^{-p}(E_{n+1})$ for all $n \gg 0$. Applying the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & H^{-1}(E_n) \arrow[r] \arrow[d, "\alpha"] & \tau^{\le -2}(E_n) \arrow[r] \arrow[d, "\beta"] & \tau^{\le -1}(E_n) \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & H^{-1}(E_{n+1}) \arrow[r] & \tau^{\le -2}(E_{n+1}) \arrow[r] & \tau^{\le -1}(E_{n+1}) \arrow[r] & 0 \end{tikzcd} \tag{1.15}$$

yields the following short exact sequence with an appropriate complex $K_n \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$

$$0 \longrightarrow K_n \longrightarrow \mathrm{coker}\, \alpha = \mathrm{coker}_{(p)}\, \alpha \longrightarrow \mathrm{coker}_{(p)}\, \beta = \mathrm{coker}^r_{(p)}\, \beta \longrightarrow 0\,.$$

From the corresponding cohomology sequence

$$\underbrace{H^{-p}(\mathrm{coker}\, \alpha)}_{=0} \longrightarrow \underbrace{H^{-p}(\mathrm{coker}^r_{(p)}\, \beta)}_{\mathrm{coker}^r\, H^{-p}(\beta)} \longrightarrow \underbrace{H^{-p}(K_n[1]) = H^{1-p}(K_n)}_{\text{torsion}}$$

we conclude $\operatorname{coker}^r H^{-p}(\beta) = 0$ and by Proposition 1.4.8 $\operatorname{coker}^r_{(p)} \beta = \operatorname{coker}_{(p)} \beta = 0$. Since $\operatorname{rk} H^{-p}(E_n) = \operatorname{rk} H^{-p}(E_{n+1})$, we get $\operatorname{rk} \ker H^{-p}(\beta) = \operatorname{rk} \operatorname{coker} H^{-p}(\beta) = \operatorname{rk} \operatorname{coker}^r H^{-p}(\beta) = 0$ and, therefore, $\ker H^{-p}(\beta) = \ker^r H^{-p}(\beta) = 0$. Due to Proposition 1.4.8, we obtain $\ker^r_{(p)} \beta = 0$, hence $\ker_{(p)} \beta = H^0(\ker_{(p)} \beta)$. From the sequence (1.12) we deduce $\ker_{(p)} \beta = H^0(\ker_{(p)} \beta) = H^0(K) = 0$ because $C = \operatorname{coker}_{(p)} \beta = 0$. Hence, β is an isomorphism and α must be a monomorphism. The commutative diagram (see (1.9))

$$\begin{tikzcd}
H^{-1}(E_n) \arrow[r, hook] \arrow[d, hook] & R^p\Gamma H^{-p}(E_n) \arrow[d, "R^p\Gamma H^{-p}(\beta)", "\wr"'] \\
H^{-1}(E_{n+1}) \arrow[r, hook] & R^p\Gamma H^{-p}(E_{n+1})
\end{tikzcd}$$

leads to an infinite increasing sequence of torsion subsheaves for $n \geq N \gg 0$

$$H^{-1}(E_N) \hookrightarrow \ldots \hookrightarrow H^{-1}(E_n) \hookrightarrow H^{-1}(E_{n+1}) \hookrightarrow \ldots \hookrightarrow R^p\Gamma H^{-p}(E_N).$$

For $p < d$ the sheaf $R^p\Gamma H^{-p}(E_N)$ is coherent and the sequence becomes stationary. Thus, $\alpha : H^{-1}(E_n) \longrightarrow H^{-1}(E_{n+1})$ is an isomorphism for all $n \gg 0$. If we apply the snake lemma to (1.15) and then to (1.14), we obtain isomorphisms $E_n \xrightarrow{\sim} E_{n+1}$ for all $n \gg 0$.
In the case $p = 1$ we start with the same arguments but replace the diagram (1.15) by the following diagram. Note that $\tau^{\leq -1}(E) = H^{-1}(E)[1]$ for $E \in \operatorname{Coh}_{(1)}(X)$.

$$\begin{tikzcd}
0 \arrow[r] & R^1\Gamma H^{-1}(E_n) \arrow[r] \arrow[d, "\alpha"] & H^{-1}(E_n)[1] \arrow[r] \arrow[d, two heads] & H^{-1}(E_n)^{\vee\vee}[1] \arrow[r] \arrow[d, "\beta[1]"] & 0 \\
0 \arrow[r] & R^1\Gamma H^{-1}(E_{n+1}) \arrow[r] & H^{-1}(E_{n+1})[1] \arrow[r] & H^{-1}(E_{n+1})^{\vee\vee}[1] \arrow[r] & 0
\end{tikzcd}$$

Thus, $\beta[1]$ is an epimorphism and due to Lemma 1.4.9 we get $\operatorname{coker} \beta \in \mathcal{T}$. Using $\operatorname{rk} H^{-1}(E_n) = \operatorname{rk} H^{-1}(E_{n+1})$, this proves $\ker \beta = 0$ and we obtain the short exact sequence

$$0 \longrightarrow H^{-1}(E_n)^{\vee\vee} \longrightarrow H^{-1}(E_{n+1})^{\vee\vee} \longrightarrow \operatorname{coker} \beta \longrightarrow 0$$

with $\operatorname{coker} \beta \in \mathcal{T}$. The long exact $R\Gamma$-sequence shows $\operatorname{coker} \beta = 0$ because of Lemma 1.4.5. Using Lemma 1.4.9, we conclude $\ker_{(1)}(\beta[1]) = \operatorname{coker}_{(1)}(\beta[1]) = 0$ and $\beta[1]$ is an isomorphism. Hence, α is an epimorphism in $\operatorname{Coh}_{(1)}(X)$ and due to Lemma 1.4.7 also in $\operatorname{Coh}(X)$. But $R^1\Gamma H^{-1}(E_n)$ has only finitely many quotients and, therefore, $R^1\Gamma H^{-1}(E_n) \xrightarrow{\sim} R^1\Gamma H^{-1}(E_{n+1})$ for all $n \gg 0$. As in the case $p \geq 2$ we apply the snake lemma twice and get the desired result. $\square$

Proposition 1.4.11. *The category* $\operatorname{Coh_{(p)}(X)}$ *is artinian for all* $0 < p \leq d$.

Proof. Assume $2 \leq p \leq d$ and take an infinite sequence $\ldots \hookrightarrow E_{n+1} \hookrightarrow E_n \hookrightarrow \ldots \hookrightarrow E_0 = E$ of subobjects in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$. Without loss of generality we can assume $\operatorname{rk} H^{-p}(E_{n+1}) = \operatorname{rk} H^{-p}(E_n)$ for all $n \in \mathbb{N}$. The application of the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & \tau^{\leq -1}(E_{n+1}) \arrow[r] \arrow[d, "\gamma"] & E_{n+1} \arrow[r] \arrow[d, hook] & H^0(E_{n+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau^{\leq -1}(E_n) \arrow[r] & E_n \arrow[r] & H^0(E_n) \arrow[r] & 0
\end{tikzcd} \tag{1.16}$$

shows that γ is a monomorphism. Applying the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}(E_{n+1}) \arrow[r] \arrow[d, "\alpha"] & \tau^{\leq -2}(E_{n+1}) \arrow[r] \arrow[d, "\beta"] & \tau^{\leq -1}(E_{n+1}) \arrow[r] \arrow[d, hook, "\gamma"] & 0 \\
0 \arrow[r] & H^{-1}(E_n) \arrow[r] & \tau^{\leq -2}(E_n) \arrow[r] & \tau^{\leq -1}(E_n) \arrow[r] & 0
\end{tikzcd} \tag{1.17}$$

yields $\ker_{(p)} \alpha = \ker_{(p)} \beta$. Due to Lemma 1.4.7 and Proposition 1.4.8, we get $\ker H^{-p}(\beta) = H^{-p}(\ker^r_{(p)} \beta) = H^{-p}(\ker_{(p)} \beta) = H^{-p}(\ker_{(p)} \alpha) = 0$ and, therefore, $\ker_{(p)} \beta = H^0(\ker_{(p)} \beta)$. Because of $\operatorname{rk} H^{-p}(E_{n+1}) = \operatorname{rk} H^{-p}(E_n)$, we conclude $\operatorname{rk} \operatorname{coker} H^{-p}(\beta) = 0$, i.e. $\operatorname{coker}^r H^{-p}(\beta) = 0$ and, therefore, $\operatorname{coker}_{(p)} \beta = 0$ due to Proposition 1.4.8. Using (1.12) with $K = \ker_{(p)} \beta$ and $C = \operatorname{coker}_{(p)} \beta = 0$, this shows $\ker_{(p)} \beta = H^0(\ker_{(p)} \beta) = 0$ and β is an isomorphism. Applying the snake lemma to (1.17), we get isomorphisms $\gamma : \tau^{\leq -1}(E_{n+1}) \xrightarrow{\sim} \tau^{\leq -1}(E_n)$. From this observation together with (1.16) we obtain an infinite increasing sequence of torsion sheaves.

$$\ldots \hookrightarrow H^0(E_{n+1}) \hookrightarrow H^0(E_n) \hookrightarrow \ldots \hookrightarrow H^0(E_N) \qquad \text{for } \; n \geq N \gg 0.$$

Hence $H^0(E_{n+1}) \xrightarrow{\sim} H^0(E_n)$ and by (1.16) $E_{n+1} \xrightarrow{\sim} E_n$ for all $n \gg 0$.
In the case $p = 1$ we start with the same arguments to show that $\gamma = \delta[1]$ is a monomorphism and we assume $\operatorname{rk} H^{-1}(E_n) = \operatorname{rk} H^{-1}(E_{n+1})$ for all $n \in \mathbb{N}$. Using Lemma 1.4.9, we see $\ker \delta = 0$ and $\Gamma(\operatorname{coker} \delta) = 0$. From the first equation we conclude $\operatorname{rk}(\operatorname{coker} \delta) = 0$ and we obtain $\operatorname{coker} \delta = \Gamma(\operatorname{coker} \delta) = 0$. Due to Lemma 1.4.9, $\operatorname{coker}_{(1)} \gamma = 0$ and γ is an isomorphism. Now we proceed as in the case $p \geq 2$ to get the desired result. □

The following corollary is a generalization of Corollary 1.3.11 and is proved in the same way.

Corollary 1.4.12. *For $0 \leq p < d$ the pair $(\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}), Z_{(p)})$ is a locally-finite stability condition on $\mathrm{D}^{\mathrm{b}}(X)$.*

Remarks.

1. $\mathrm{Coh}(X)$ is not artinian since there is an infinite decreasing sequence of ideal sheaves
$$\ldots \subset \mathcal{I}_{p_1,\ldots,p_{n+1}} \subset \mathcal{I}_{p_1,\ldots,p_n} \subset \ldots \subset \mathcal{O}_X.$$
2. $\mathrm{Coh}_{(d)}(X)$ is not noetherian since $[d] \circ R\mathcal{H}om(\cdot, \mathcal{O}_X) : \mathrm{Coh}(X)^{op} \xrightarrow{\sim} \mathrm{Coh}_{(d)}(X)$ is an equivalence of abelian categories due to [23], Theorem 5.9. Furthermore, $Z_{(d)} = -\mathrm{ch}_d + (-1)^d \,\mathrm{rk} \cdot i$ does not satisfy the Harder–Narasimhan property. For example, the phases of the infinite sequence of quotients
$$R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X)[d] = \mathcal{O}_X[d] \twoheadrightarrow \ldots \twoheadrightarrow R\mathcal{H}om(\mathcal{I}_{p_1,\ldots,p_n}, \mathcal{O}_X)[d] \twoheadrightarrow$$
$$R\mathcal{H}om(\mathcal{I}_{p_1,\ldots,p_{n+1}}, \mathcal{O}_X)[d] \twoheadrightarrow \ldots$$
form a strictly decreasing sequence.

In the last part of this section we aim to show that $\mathrm{Coh}_{(p+1)}(X)$ is the tilt of $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ with respect to the torsion theory $\mathcal{T} \subseteq \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ and
$$\mathcal{F}_{(p)} := \{E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}) \mid R\Gamma(E) \in \mathrm{D}^{\geq 1}(\mathcal{M}od(X))\}$$
as the full category of 'free' objects.

Lemma 1.4.13. *$(\mathcal{T}, \mathcal{F}_{(p)})$ is a torsion theory in $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ for all $0 \leq p \leq d-1$.*

Proof. For $T \in \mathcal{T}$ and $F \in \mathcal{F}_{(p)}$ we have the following commutative diagram
$$\begin{tikzcd} R\Gamma(T) \arrow[r, "\sim"] \arrow[d, "0"] & T \arrow[d, "f"] \\ R\Gamma(F) \arrow[r] & F \end{tikzcd}$$
and, therefore, $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$.
Using $R\Gamma(E) \in \mathrm{D}^{\geq 0}(\mathcal{M}od(X))$, we have for any $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ the composition $R^0\Gamma(E) \longrightarrow R\Gamma(E) \longrightarrow E$ and the associated triangle
$$R^0\Gamma(E) \longrightarrow E \longrightarrow F \longrightarrow R^0\Gamma(E)[1]. \tag{1.18}$$
We aim to show $F \in \mathcal{F}_{(p)}$. The cohomology sequence of (1.18) yields $H^k(F) \in \mathcal{T}\ \forall k \in \{1-p, \ldots, 0\}$ and $H^k(F) = 0\ \forall k \notin \{-p, \ldots, 0\}$. We still have to show $R\Gamma(F) \in \mathrm{D}^{\geq 1}(\mathcal{M}od(X))$. Using the exact sequence (1.9) and the fact that $R^p\Gamma H^{-p}(E)$ is a coherent sheaf for $0 \leq p \leq d-1$, we obtain $R^0\Gamma(E) \in \mathcal{T}$. Applying $R\Gamma$ to (1.18) yields the exact sequence
$$\underbrace{R^{-1}\Gamma(E)}_{=0} \longrightarrow R^{-1}\Gamma(F) \longrightarrow \underbrace{R^0\Gamma(R^0\Gamma(E))}_{=R^0\Gamma(E)} \xrightarrow{\sim} R^0\Gamma(E) \longrightarrow R^0\Gamma(F) \longrightarrow$$
$$\longrightarrow \underbrace{R^1\Gamma(R^0\Gamma(E))}_{=0}$$

and $R^{-1}\Gamma(F) = R^0\Gamma(F) = 0$ follows. Using the triangle (1.18), we see $R\Gamma(F) \in \mathrm{D}^{\geq -1}(\mathcal{M}od(X))$ and, therefore, $R\Gamma(F) \in \mathrm{D}^{\geq 1}(\mathcal{M}od(X))$. From (1.18) we get the desired short exact sequence

$$0 \longrightarrow R^0\Gamma(E) \longrightarrow E \longrightarrow F \longrightarrow 0$$

with $R^0\Gamma(E) \in \mathcal{T}$ and $F \in \mathcal{F}_{(p)}$. □

Proposition 1.4.14. *The tilt of the the torsion theory $(\mathcal{T}, \mathcal{F}_{(p)})$ is the category $\mathrm{Coh}_{(p+1)}(X)$ for $0 \leq p \leq d-1$.*

Proof. We introduce

$$\begin{aligned} \mathrm{D}^{\sharp\leq 0} &:= \{E \in \mathrm{D}^{\leq 0}_{(p)}(X) \mid H^0_{(p)}(E) \in \mathcal{T}\} \\ \mathrm{D}^{\sharp\geq 0} &:= \{E \in \mathrm{D}^{\geq -1}_{(p)}(X) \mid H^{-1}_{(p)}(E) \in \mathcal{F}_{(p)}\} \end{aligned}$$

and have to show $\mathrm{D}^{\sharp\leq 0} = \mathrm{D}^{\leq 0}_{(p+1)}(X)$ and $\mathrm{D}^{\sharp\geq 0} = \mathrm{D}^{\geq 0}_{(p+1)}(X)$. Remember the definitions

$$\begin{aligned} \mathrm{D}^{\leq 0}_{(p+1)} &= \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid H^k(E) = 0\ \forall\, k > 0,\ H^k(E) \in \mathcal{T}\ \forall\, k \in \{-p, \ldots, 0\}\}, \\ \mathrm{D}^{\leq 0}_{(p)} &= \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid H^k(E) = 0\ \forall\, k > 0,\ H^k(E) \in \mathcal{T}\ \forall\, k \in \{1-p, \ldots, 0\}\}. \end{aligned}$$

For $E \in \mathrm{D}^{\leq 0}_{(p+1)}(X)$ there is a triangle

$$\underbrace{\tau^{\leq -1}(E)}_{\in \mathrm{D}^{\leq -1}_{(p)}(X)} \longrightarrow E \longrightarrow H^0(E) \longrightarrow \underbrace{\tau^{\leq -1}(E)[1]}_{\in \mathrm{D}^{\leq -2}_{(p)}(X)}.$$

Hence, $H^0_{(p)}(E) = H^0_{(p)}(H^0(E)) = H^0(E) \in \mathcal{T}$. Since $E \in \mathrm{D}^{\leq 0}_{(p)}$, this proves $E \in \mathrm{D}^{\sharp\leq 0}$ for all $E \in \mathrm{D}^{\leq 0}_{(p+1)}(X)$. For the reverse inclusion we take an element $E \in \mathrm{D}^{\sharp\leq 0}$, and in order to show $E \in \mathrm{D}^{\leq 0}_{(p+1)}(X)$ we still have to prove $H^{-p}(E) \in \mathcal{T}$. The cohomology sequence of the triangle

$$\underbrace{\tau^{\leq -1}_{(p)}(E)}_{\in \mathrm{D}^{\leq -1}_{(p)}(X)} \longrightarrow E \longrightarrow H^0_{(p)}(E) \longrightarrow \underbrace{\tau^{\leq -1}_{(p)}(E)[1]}_{\in \mathrm{D}^{\leq -2}_{(p)}(X)}$$

gives

$$\underbrace{H^{-p}(\tau^{\leq -1}_{(p)}(E))}_{\in \mathcal{T}} \longrightarrow H^{-p}(E) \longrightarrow \underbrace{H^{-p}(H^0_{(p)}(E))}_{\in \mathcal{T}}$$

and the assertion $H^{-p}(E) \in \mathcal{T}$ follows. The equation $\mathrm{D}^{\sharp\leq 0} = \mathrm{D}^{\leq 0}_{(p+1)}(X)$ implies $\mathrm{D}^{\sharp\geq 0} = \mathrm{D}^{\geq 0}_{(p+1)}(X)$ because of

$$\begin{aligned} \mathrm{D}^{\sharp\geq 0} &= \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid \mathrm{Hom}(F, E) = 0\ \forall\, F \in \mathrm{D}^{\sharp\leq 0}\}[1], \\ \mathrm{D}^{\geq 0}_{(p+1)}(X) &= \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid \mathrm{Hom}(F, E) = 0\ \forall\, F \in \mathrm{D}^{\geq 0}_{(p+1)}(X)\}[1]. \end{aligned}$$

□

Note that $\mathcal{F}_{(p)}$ is determined by $\mathcal{T}$ due to

$$\mathcal{F}_{(p)} = \{E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X}) \mid \mathrm{Hom}(F, E) = 0 \, \forall \, F \in \mathcal{T}\}.$$

This shows that our torsion theories coincide with the torsion theories of section 1.2 and 1.3 in the case of generic K3 surfaces and generic complex tori. Thus, our abelian categories $\mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ and our stability conditions $\sigma_{(p)}$ generalize those of section 1.2 and 1.3.

Chapter 2

Quotient categories and stability

2.1 The derived category modulo codimension ≥ 2

It is known that the quotient of the derived category of coherent sheaves on an irreducible projective variety X of dimension d by the full subcategory of complexes whose support has codimension ≥ 1, is the derived category of $K(X)$-vector spaces, where $K(X)$ is the function field of X (see Corollary 2.1.10). In particular, this quotient category has homological dimension zero. In this section we consider the quotient of the derived category of coherent sheaves on the variety X by the full subcategory of complexes of codimension ≥ 2. We aim to show that this quotient category has homological dimension at most one. Although, one might expect this result for every dimension d, we prove this only for $d \leq 2$. Furthermore, we compute the K-group of this category.

2.1.1 The equivalence of the different approaches

Let X be an irreducible smooth projective variety of dimension d. We introduce several full subcategories of $\mathrm{D^b}(X)$ and $\mathrm{Coh}(X)$.

Definition 2.1.1. *For any natural number $0 \leq d' \leq d$ we define $\mathrm{D}^b_{d'}(X)$ to be the full subcategory of $\mathrm{D^b}(X)$ consisting of complexes whose support has dimension $\leq d'$. Note that the support of a complex is the union of the supports of its cohomology sheaves. Similar, we denote by $\mathrm{Coh}_{d'}(X)$ the full subcategory of $\mathrm{Coh}(X)$ consisting of sheaves whose support has dimension $\leq d'$.*

The following lemma is an easy consequence of the fact that for every short exact sequence of coherent sheaves on X

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

the supports satisfy $\mathrm{supp}(E) = \mathrm{supp}(E') \cup \mathrm{supp}(E'')$.

Lemma 2.1.2. *The categories $\mathrm{D}^b_{d'}(X)$ are thick triangulated subcategories of $\mathrm{D}^{\mathrm{b}}(X)$. Similar, the categories $\mathrm{Coh}_{d'}(X)$ are Serre subcategories of $\mathrm{Coh}(X)$.*

Recall that a full subcategory $\mathcal{D}'$ of a triangulated category $\mathcal{D}$ is called thick if $E' \oplus E'' \in \mathcal{D}'$ implies $E', E'' \in \mathcal{D}'$. A full subcategory $\mathcal{A}'$ of an abelian category $\mathcal{A}$ is a Serre subcategory if for every short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

in $\mathcal{A}$ the object E is in $\mathcal{A}'$ if and only if E' and E'' are in $\mathcal{A}'$. In particular, $\mathcal{A}'$ is an abelian subcategory.
The next propositions are special cases of results of Serre respectively Verdier.

Proposition 2.1.3 ([32], Lemma A.2.3)**.** *For any $0 < d' \leq d$ there is an abelian category $\mathrm{Coh}_{d,d'}(X)$ and an exact functor $P : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}_{d,d'}(X)$ which is zero on the subcategory $\mathrm{Coh}_{d'-1}(X)$ and which is universal among all exact functors $\tilde{P} : \mathrm{Coh}(X) \longrightarrow \mathcal{A}$ between abelian categories vanishing on $\mathrm{Coh}_{d'-1}(X)$. Furthermore, the kernel of P is exactly the category $\mathrm{Coh}_{d'-1}(X)$. The category $\mathrm{Coh}_{d,d'}(X)$ is called the quotient category of $\mathrm{Coh}(X)$ by $\mathrm{Coh}_{d'-1}(X)$. The objects of $\mathrm{Coh}_{d,d'}(X)$ are those of $\mathrm{Coh}(X)$, and a morphism between two objects E and F is described by a roof*

$$\begin{tikzcd} & E' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ E & & F \end{tikzcd},$$

where s and f are morphisms in $\mathrm{Coh}(X)$ with $\ker(s), \mathrm{coker}(s) \in \mathrm{Coh}_{d'-1}(X)$.

Proposition 2.1.4 ([32], Theorem 2.1.8)**.** *For any $0 < d' \leq d$ there is a triangulated category $\mathrm{D}^b_{d,d'}(X)$ and an exact functor $Q : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^b_{d,d'}(X)$ which is zero on the subcategory $\mathrm{D}^b_{d'-1}(X)$ and which is universal among all exact functors $\tilde{Q} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathcal{T}$ between triangulated categories vanishing on $\mathrm{D}^b_{d'-1}(X)$. Furthermore, the kernel of Q is exactly the category $\mathrm{D}^b_{d'-1}(X)$. The category $\mathrm{D}^b_{d,d'}(X)$ is called the quotient category of $\mathrm{D}^{\mathrm{b}}(X)$ by $\mathrm{D}^b_{d'-1}(X)$. The objects of $\mathrm{D}^b_{d,d'}(X)$ are those of $\mathrm{D}^{\mathrm{b}}(X)$, and a morphism between two objects E and F is described by a roof*

$$\begin{tikzcd} & E' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ E & & F \end{tikzcd},$$

where s and f are morphisms in $\mathrm{D}^{\mathrm{b}}(X)$ with $C(s) \in \mathrm{D}^b_{d'-1}(X)$.

For the quotient $\mathrm{D}^b_{d,d'}(X)$ as well as for $\mathrm{Coh}_{d,d'}(X)$ two roofs

$$\begin{tikzcd} & E' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ E & & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & E'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ E & & F \end{tikzcd}$$

describe the same morphism if there is commutative diagram

$$\begin{tikzcd} & & E''' \arrow[dl, "u"'] \arrow[dr, "v"] & & \\ & E' \arrow[dl, "s"'] \arrow[drrr, "f"] & & E'' \arrow[dlll, "t"'] \arrow[dr, "g"] & \\ E & & & & F \end{tikzcd}$$

with $C(su) \in \mathrm{D}^b_{d'-1}(X)$ respectively $\ker(su), \operatorname{coker}(su) \in \mathrm{Coh}_{d'-1}(X)$.
There is a similar description of the morphisms using right roof

$$\begin{tikzcd} & F' & \\ E \arrow[ur, "f"] & & F \arrow[ul, "s"'] \end{tikzcd}$$

with $C(s) \in \mathrm{D}^b_{d'-1}(X)$ resp. $\ker(s), \operatorname{coker}(s) \in \mathrm{Coh}_{d'-1}(X)$, instead of the upper left roofs.
The definition of $\mathrm{D}^b_{d,d'}(X)$ and Q also shows that $Q : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^b_{d,d'}(X)$ is the localization functor with respect to the set of morphisms $f : E \longrightarrow F$ in $\mathrm{D}^{\mathrm{b}}(X)$ with the property $C(f) \in \mathrm{D}^b_{d'-1}(X)$ which is equivalent to $\ker H^i(f), \operatorname{coker} H^i(f) \in \mathrm{Coh}_{d'-1}(X)$ for all $i \in \mathbb{Z}$. Let us denote by $S_{d'-1}$ the set of complex homomorphisms $g : E \longrightarrow F$ in $\mathrm{Kom}^{\mathrm{b}}(X)$ with $\ker H^i(g), \operatorname{coker} H^i(g) \in \mathrm{Coh}_{d'-1}(X)$ for all $i \in \mathbb{Z}$. This set contains the set of quasi-isomorphisms. If we represent $f \in \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(E, F)$ by the roof

$$\begin{tikzcd} & E' \arrow[dl, "s"'] \arrow[dr, "g"] & \\ E & & F \end{tikzcd},$$

we see that $C(f) \in \mathrm{D}^b_{d'-1}(X)$ if and only if $g \in S_{d'-1}$. Using this and the definition of $\mathrm{D}^{\mathrm{b}}(X)$ as the localization of $\mathrm{Kom}^{\mathrm{b}}(X)$ with respect to the set of quasi-isomorphisms, we obtain that $\mathrm{D}^b_{d,d'}(X)$ is (isomorphic to) the localization of $\mathrm{Kom}^{\mathrm{b}}(X)$ by $S_{d'-1}$. In particular, a morphism in $\mathrm{D}^b_{d,d'}(X)$ can be represented by a roof

$$\begin{tikzcd} & E' \arrow[dl, "t"'] \arrow[dr, "h"] & \\ E & & F \end{tikzcd}$$

with t and h complex homomorphisms and $t \in S_{d'-1}$. We will use this equivalent description of $\mathrm{D}^b_{d,d'}(X)$ to verify the following proposition.

Proposition 2.1.5. *For any $0 < d' \leq d$ the naturally induced exact functor $\mathrm{D}^{\mathrm{b}}(P) : \mathrm{D}^{\mathrm{b}}(X) = \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X)) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$ factorizes over $Q : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^b_{d,d'}(X)$ and the resulting functor $T : \mathrm{D}^b_{d,d'}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$ is an equivalence of triangulated categories.*

$$\begin{tikzcd} & \mathrm{D^b}(X) = \mathrm{D^b}(\mathrm{Coh}(X)) \arrow[dl, "Q"'] \arrow[dr, "\mathrm{D^b}(P)"] & \\ \mathrm{D}^b_{d,d'}(X) \arrow[rr, "\sim", "T"'] & & \mathrm{D^b}(\mathrm{Coh}_{d,d'}(X)) \end{tikzcd}$$

Proof. We prove the proposition in several steps. First of all we show that T is well defined. This is an easy consequence of the fact that $\mathrm{D^b}(P)$ commutes with the cohomology functors.

$$\begin{tikzcd} \mathrm{D^b}(X) \arrow[r, "\mathrm{D^b}(P)"] \arrow[d, "H^i"'] & \mathrm{D^b}(\mathrm{Coh}_{d,d'}(X)) \arrow[d, "H^i"] \\ \mathrm{Coh}(X) \arrow[r, "P"] & \mathrm{Coh}_{d,d'}(X) \end{tikzcd}$$

For $E \in \mathrm{D}^b_{d'-1}(X)$ we have $H^i(\mathrm{D^b}(P)(E)) = P(H^i(E)) = 0$ for all $i \in \mathbb{Z}$ and, therefore, $\mathrm{D^b}(P)(E) = 0$. The existence of T follows now by the universal property of $Q : \mathrm{D^b}(X) \longrightarrow \mathrm{D}^b_{d,d'}(X)$. We need to show that T is fully faithful and that every object of $\mathrm{D^b}(\mathrm{Coh}_{d,d'}(X))$ is isomorphic to some object in the image of T. We formulate these assertions as three lemmas following the definition. □

Definition 2.1.6. *We fix $0 < d' \leq d$. For every coherent sheaf F let $\mathrm{t}(F)$ be the biggest subsheaf of F whose support has dimension $\leq d' - 1$. For every complex E of coherent sheaves we define $\mathrm{t}(E)$ to be the subcomplex of E with components $\mathrm{t}(E_n)$, where the E_n are the components of E. It is the biggest subcomplex of E which is a complex in $\mathrm{Coh}_{d'-1}(X)$.*

Lemma 2.1.7. *For every bounded complex E in $\mathrm{Coh}_{d,d'}(X)$ there is a bounded complex E' in $\mathrm{Coh}(X)$ with $\mathrm{t}(E') = 0$ and an isomorphism $s : T(E') \longrightarrow E$ of complexes, where $T(E')$ is the complex E' regarded as a complex in $\mathrm{Coh}_{d,d'}(X)$. In particular, every object of $\mathrm{D^b}(\mathrm{Coh}_{d,d'}(X))$ is isomorphic to some object in the image of T.*

Proof. Let us write the complex E as $E_1 \xrightarrow{d_1} \ldots \longrightarrow E_{n-1} \xrightarrow{d_{n-1}} E_n \xrightarrow{d_n} E_{n+1}$. We represent d_n by a roof in $\mathrm{Coh}(X)$

$$\begin{tikzcd} & E'_n \arrow[dl, "s_n"'] \arrow[dr, "\tilde{d}_n"] & \\ E_n & & E_{n+1} \end{tikzcd}$$

with $\ker(s_n), \mathrm{coker}(s) \in \mathrm{Coh}_{d'-1}(X)$ and obtain an isomorphism of complexes in $\mathrm{Coh}_{d,d'}(X)$

$$\begin{array}{ccccccccccc}
E^{(1)} & : & E_1 & \xrightarrow{d_1} & \cdots & \xrightarrow{d_{n-2}} & E_{n-1} & \xrightarrow{s_n^{-1}d_{n-1}} & E'_n & \xrightarrow{\tilde{d}_n} & E_{n+1} \\
\downarrow{\scriptstyle s^{(1)}} & & \| & & & & \| & & \downarrow{\scriptstyle s_n} & & \| \\
E & : & E_1 & \xrightarrow{d_1} & \cdots & \xrightarrow{d_{n-2}} & E_{n-1} & \xrightarrow{d_{n-1}} & E_n & \xrightarrow{d_n} & E_{n+1}\,.
\end{array}$$

The advantage of $E^{(1)}$ is that $\tilde{d}_n$ is a morphism in $\mathrm{Coh}(X)$ instead of $\mathrm{Coh}_{d,d'}(X)$. We repeat this procedure with the differential $s_n^{-1}d_{n-1}$ and obtain an isomorphism $s^{(2)} : E^{(2)} \longrightarrow E^{(1)}$ and the last two differentials of $E^{(2)}$ are morphisms of sheaves. Progressing in this way, we get an isomorphism $s^{(1)} \circ \ldots \circ s^{(n)} : E^{(n)} \longrightarrow E$ of complexes in $\mathrm{Coh}_{d,d'}(X)$ and all differentials of $E^{(n)}$ are morphisms of sheaves. However, $E^{(n)}$ does not need to be a complex in $\mathrm{Coh}(X)$. It is a complex in $\mathrm{Coh}_{d,d'}(X)$, i.e. the image of the composition of two successive differentials is contained in $\mathrm{Coh}_{d'-1}(X)$. We obtain another isomorphism of complexes

$$\begin{array}{ccccccc}
E^{(n)} & : & E_1^{(n)} & \xrightarrow{d_1^{(n)}} & \cdots & \xrightarrow{d_n^{(n)}} & E_{n+1}^{(n)} \\
\downarrow{\scriptstyle s'} & & \downarrow & & & & \downarrow \\
E' = E^{(n)}/\,\mathrm{t}(E^{(n)}) & : & E_1^{(n)}/\,\mathrm{t}(E_1^{(n)}) & \xrightarrow{d'_1} & \cdots & \xrightarrow{d'_n} & E_{n+1}^{(n)}/\,\mathrm{t}(E_{n+1}^{(n)})
\end{array}$$

in $\mathrm{Coh}_{d,d'}(X)$. The composition of two successive differentials in the complex below is zero by the construction of E'. Thus, E' can be regarded as a complex in $\mathrm{Coh}(X)$ with $\mathrm{t}(E') = 0$ which is mapped by T onto itself as an object of $\mathrm{Kom}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$. The requested isomorphism is $s^{(1)} \circ \ldots \circ s^{(n)} \circ s'^{-1} : T(E') \longrightarrow E$. □

Remark. Note that the isomorphism in the lemma is an isomorphism of complexes in $\mathrm{Coh}_{d,d'}(X)$ and not just a quasi-isomorphism which would be enough for the second statement of the lemma. Furthermore, for any complex E in $\mathrm{Coh}(X)$ the complex homomorphism $E \twoheadrightarrow E/\,\mathrm{t}(E)$ is contained in $S_{d'-1}$ and, thus, an isomorphism in $\mathrm{D}^b_{d,d'}(X)$. In other words, we can replace any object in $\mathrm{D}^b_{d,d'}(X)$ by an isomorphic complex without nontrivial subcomplexes in $\mathrm{Coh}_{d'-1}(X)$.

Lemma 2.1.8. *The functor T is full, i.e. for two arbitrary objects $E, F \in \mathrm{D}^b_{d,d'}(X)$ the map*

$$\mathrm{Hom}_{\mathrm{D}^b_{d,d'}(X)}(E,F) \xrightarrow{T} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))}(T(E),T(F))$$

is surjective.

Proof. Due to the upper remark, we can assume $\mathrm{t}(E) = \mathrm{t}(F) = 0$. We represent a morphism $f : T(E) \longrightarrow T(F)$ by a roof

$$\begin{tikzcd} & G \arrow[dl, "s"'] \arrow[dr, "\tilde{f}"] & \\ T(E) & & T(F) \end{tikzcd}$$

with complex homomorphisms $\tilde{f}$ and s in $\mathrm{Coh}_{d,d'}(X)$ and s is a quasi-isomorphism. Due to the previous lemma, we have an isomorphism $s' : T(G') \longrightarrow G$ of complexes and we can replace the roof by the equivalent roof

$$\begin{tikzcd} & T(G') \arrow[dl, "ss'"'] \arrow[dr, "\tilde{f}s'"] & \\ T(E) & & T(F) \end{tikzcd} \qquad \text{with } \mathrm{t}(E) = \mathrm{t}(F) = \mathrm{t}(G') = 0.$$

Thus, it is enough to show that we can 'lift' every morphism represented by a complex homomorphism $f : T(E) \longrightarrow T(F)$ in $\mathrm{Coh}_{d,d'}(X)$ to a morphism $\hat{f} : E \longrightarrow F$ in $\mathrm{D}^b_{d,d'}(X)$ under the assumption $\mathrm{t}(E) = \mathrm{t}(F) = 0$. Furthermore, $\hat{f}$ needs to be an isomorphism if f is a quasi-isomorphism.
If we represent every component of f by a roof in $\mathrm{Coh}(X)$, we get the following diagram

$$\begin{tikzcd} E_1 \arrow[r, "d_1"] & \cdots \arrow[r, "d_{n-1}"] & E_n \arrow[r, "d_n"] & E_{n+1} \\ E'_1 \arrow[u, "s_1"'] \arrow[d, "f_1"] & \cdots & E'_n \arrow[u, "s_n"'] \arrow[d, "f_n"] & E'_{n+1} \arrow[u, "s_{n+1}"'] \arrow[d, "f_{n+1}"] \\ F_1 \arrow[r, "h_1"] & \cdots \arrow[r, "h_{n-1}"] & F_n \arrow[r, "h_n"] & F_{n+1} \end{tikzcd}$$

with $\ker(s_1), \ldots, \ker(s_{n+1}), \mathrm{coker}(s_1), \ldots, \mathrm{coker}(s_{n+1}) \in \mathrm{Coh}_{d'-1}(X)$. Since $\mathrm{t}(E_k) = \mathrm{t}(F_k) = 0$, the morphisms s_k and f_k factorize over the quotient map $E'_k \twoheadrightarrow E'_k/\,\mathrm{t}(E'_k)$. The commutativity of the upper diagram in $\mathrm{Coh}_{d,d'}(X)$ is still valid if we replace E'_k by the quotient $E'_k/\,\mathrm{t}(E'_k)$. Thus, we can assume $\mathrm{t}(E'_k) = 0$. In this case, s_k is injective and we can regard E'_k as a subsheaf of E_k with the inclusions s_k. We consider now the subsheaves $E''_k := E'_k \cap d_k^{-1}(E'_{k+1})$ of E_k. Since $d_{k+1} \circ d_k = 0$, we get $d_k(E''_k) \subseteq E''_{k+1}$ and obtain a subcomplex E'' of E with components E''_k. Using $E_k/E'_k, E_{k+1}/E'_{k+1} \in \mathrm{Coh}_{d'-1}(X)$, it is an easy exercise to show that E_k/E''_k is contained in $\mathrm{Coh}_{d'-1}(X)$. Thus, the inclusion $s'' : E'' \longrightarrow E$ is in $S_{d'-1}$ and, therefore, an isomorphism in $\mathrm{D}^b_{d,d'}(X)$. The restrictions f''_k of the f_k to E''_k form a complex homomorphism in $\mathrm{Coh}_{d,d'}(X)$, i.e. the image of $h_k f''_k - f''_{k+1} d_k$ is contained in $\mathrm{Coh}_{d'-1}(X)$. Due to our assumption on F, there are no nontrivial subsheaves of F_{k+1} of this kind. This

shows that the f''_k form a complex homomorphism $f'' : E'' \longrightarrow F$ in $\mathrm{Coh}(X)$. The composition $\hat{f} := f'' \circ s''^{-1} : E \longrightarrow F$ which is well defined in $\mathrm{D}^b_{d,d'}(X)$ is the desired lift of $f : T(E) \longrightarrow T(F)$.
Finally, we have to show that for a quasi-isomorphism f the lift $\hat{f}$ is an isomorphism. If f is a quasi-isomorphism in $\mathrm{Kom}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$, its cone $C(f)$ must be zero in $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$. On the other hand, we have $\mathrm{D}^{\mathrm{b}}(P)(C(f'')) \cong T(C(\hat{f})) \cong C(f) = 0$. Since $\mathrm{D}^{\mathrm{b}}(P)$ commutes with cohomology, the cohomology sheaves of $C(f'')$ are contained in the kernel of P which is $\mathrm{Coh}_{d'-1}(X)$. Therefore, $f'' \in S_{d'-1}$ and $\hat{f} = f'' \circ s''^{-1}$ is an isomorphism in $\mathrm{D}^b_{d,d'}(X)$. □

Lemma 2.1.9. *The functor T is faithful, i.e. for two arbitrary objects $E, F \in \mathrm{D}^b_{d,d'}(X)$ the map*

$$\mathrm{Hom}_{\mathrm{D}^b_{d,d'}(X)}(E,F) \xrightarrow{T} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))}(T(E),T(F))$$

is injective.

Proof. Let $\tilde{f} : E \longrightarrow F$ be a morphism with $T(\tilde{f}) = 0$. We represent $\tilde{f}$ by a roof

$$\begin{tikzcd} & E' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ E & & F \end{tikzcd}$$

of complex homomorphisms in $\mathrm{Coh}(X)$ with $s \in S_{d'-1}$. Replacing E and F if necessary, we can assume $\mathrm{t}(E) = \mathrm{t}(F) = 0$. In this case f and s factorize over the quotient map $E' \twoheadrightarrow E'/\,\mathrm{t}(E')$. Thus, we can assume $\mathrm{t}(E') = 0$ as well. It is enough to find a complex homomorphism $t'' : G'' \longrightarrow E'$ in $S_{d'-1}$ with $f \circ t'' = 0$.
Since $T(s)$ is an isomorphism, we have $T(f) = 0$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$. The latter is equivalent to the existence of a quasi-isomorphism $u : G \longrightarrow T(E')$ in $\mathrm{Kom}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$ with $f \circ u = 0$ as a complex homomorphism. The Lemma 2.1.7 provides us with a complex isomorphism $u' : T(G') \longrightarrow G$ with $\mathrm{t}(G') = 0$. We denote the composition $u \circ u'$ by $t : T(G') \longrightarrow T(E')$. Due to the proof of the previous lemma, we can lift t to an isomorphism $\hat{t} : G' \longrightarrow E'$ in $\mathrm{D}^b_{d,d'}(X)$ represented by the roof

$$\begin{tikzcd} & G'' \arrow[dl, "s''"'] \arrow[dr, "t''"] & \\ G' & & E' \end{tikzcd}$$

with $s'', t'' \in S_{d'-1}$ and, moreover, s'' is an isomorphism regarded as a complex homomorphism in $\mathrm{Coh}_{d,d'}(X)$. Since $f \circ t'' \circ s''^{-1} = 0$ in $\mathrm{Kom}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$, we get $f \circ t'' = 0$ in $\mathrm{Kom}^{\mathrm{b}}(\mathrm{Coh}_{d,d'}(X))$. This means that the image of $f \circ t'' : G'' \longrightarrow F$ is a subcomplex of F in $\mathrm{Coh}_{d'-1}(X)$. Due to our assumption on F, we get $f \circ t'' = 0$ in $\mathrm{Kom}^{\mathrm{b}}(\mathrm{Coh}(X))$ and we are done. □

As an application of the Proposition 2.1.5 we give a proof the following well known statement.

Corollary 2.1.10. *The quotient category $\mathrm{D}^b_{d,d}(X)$ is equivalent to the bounded derived category of finite-dimensional vector spaces over the function field $K(X)$ of the irreducible smooth projective variety X.*

Proof. Using Proposition 2.1.5, it suffices to prove that $\mathrm{Coh}_{d,d}(X)$ is equivalent to the abelian category of finite-dimensional vector spaces over the function field $K(X)$. By Serre's theorem, there is for every coherent sheaf E a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D)^{\oplus \operatorname{rk}(E)} \longrightarrow E \longrightarrow T \longrightarrow 0$$

for some very ample divisor D and a torsion sheaf T. Using this and the sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0\,,$$

we get $E \cong \mathcal{O}_X^{\oplus \operatorname{rk}(E)}$ in $\mathrm{Coh}_{d,d}(X)$. If a morphism $\tilde{f} \in \mathrm{Hom}_{\mathrm{Coh}_{d,d}(X)}(\mathcal{O}_X, \mathcal{O}_X)$ is represented by the roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ \mathcal{O}_X & & \mathcal{O}_X \end{tikzcd}$$

with $\ker(s), \operatorname{coker}(s) \in \mathrm{D}^b_{d-1}(X)$, we can assume $E = \mathcal{O}_X(-D)$ for some (ample) divisor D by Serre's theorem. Otherwise, we replace E by $\mathcal{O}_X(-D) \hookrightarrow E$ which leads to an equivalent roof. Since $s \neq 0$, we can form the rational function $f^\vee/s^\vee \in K(X)$ by taking the quotient of the sections $f^\vee, s^\vee \in \mathrm{H}^0(X, \mathcal{O}_X(D))$. If the morphism $\tilde{f}$ is represented by another roof, we get the following commutative diagram

$$\begin{tikzcd} & & E'' \arrow[dl, "t"'] \arrow[dr, "t'"] & & \\ & \mathcal{O}_X(-D) \arrow[dl, "f"'] \arrow[drrr, "s" near end] & & \mathcal{O}_X(-D') \arrow[dlll, "f'"' near end] \arrow[dr, "s'"] & \\ \mathcal{O}_X & & & & \mathcal{O}_X\,. \end{tikzcd}$$

We can apply the same simplifications to E'' as before. Doing this, we can assume $E'' = \mathcal{O}_X(-D'')$ and we obtain the following equations of rational functions $f^\vee/s^\vee = t^\vee f^\vee/t^\vee s^\vee = t'^\vee f'^\vee/t'^\vee s'^\vee = f'^\vee/s'^\vee$ which proves the independence of the rational function from the chosen representation of the morphism $\tilde{f}$.

Conversely, every nontrivial rational function $g : X \longrightarrow \mathbb{P}^1$ is the quotient of the two sections $s^\vee := g^*x$ and $f^\vee := g^*y$ in $E := g^*\mathcal{O}_{\mathbb{P}^1}(1)$, where x and y are the canonical sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ vanishing in ∞ respectively in $0 \in \mathbb{P}^1$. Thus, g comes from a roof. This proves $\mathrm{Hom}_{\mathrm{Coh}_{d,d}(X)}(\mathcal{O}_X, \mathcal{O}_X) \cong K(X)$ and since every object of $\mathrm{Coh}_{d,d}(X)$ is a direct sum of copies of $\mathcal{O}_X$, the category $\mathrm{Coh}_{d,d}(X)$ is equivalent as an abelian category to the category of vector spaces. Indeed, using arguments from linear algebra one can show, that every short exact sequence in $\mathrm{Coh}_{d,d}(X)$ splits. See the proof of Theorem 2.2.6 for similar arguments. $\square$

2.1.2 Properties of the quotient category

We know from the previous section that the triangulated quotient category $\mathrm{D}^b_{d,d-1}(X)$ is equivalent to the bounded derived category of the abelian quotient category $\mathrm{Coh}_{d,d-1}(X)$. As before, X is an irreducible smooth projective variety of dimension d. Note that in the case of curves these quotient categories coincide with the usual categories $\mathrm{D^b}(X)$ and $\mathrm{Coh}(X)$. In the case of arbitrary dimensions the quotient categories still possess some properties of the corresponding categories for curves. Nevertheless, there are also significant differences between the case of curves and the general case. For example, it is not known whether $\mathrm{D}^b_{d,d-1}(X)$ has a Serre functor for $d \geq 2$. Furthermore, the Hom-groups may have infinite dimensions over the base field k.
The first two propositions of this subsection have already been proved by Holger Partsch in his diploma thesis [35].

Proposition 2.1.11 ([35], Proposition 2.1)**.** *The K-group of $\mathrm{D}^b_{d,d-1}(X) \cong \mathrm{D^b}(\mathrm{Coh}_{d,d-1}(X))$ is*

$$K(\mathrm{D}^b_{d,d-1}(X)) \cong K(\mathrm{Coh}_{d,d-1}(X)) \cong \operatorname{Pic} X \oplus \mathbb{Z}.$$

Proof. If we associate to any Weil divisor $D = \sum_{i=1}^p n_i D_i$ with irreducible divisors D_i the class $\sum_{i=1}^p n_i \operatorname{cl} \mathcal{O}_{D_i}$ in the K-group $K(\mathrm{Coh}_{d,d-1}(X))$, we obtain a group homomorphism from the Weil group of X into this K-group. The short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D)|_D \cong \mathcal{O}_D \longrightarrow 0$$

in $\mathrm{Coh}_{d,d-1}(X)$ shows $\operatorname{cl} \mathcal{O}_D = \operatorname{cl} \mathcal{O}_X(D) - \operatorname{cl} \mathcal{O}_X$ for every effective divisor D. Using this and the fact that every divisor is the difference of two effective divisors, we conclude $\operatorname{cl}(\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)) - \operatorname{cl} \mathcal{O}_X = \operatorname{cl} \mathcal{O}_X(D_1) - \operatorname{cl} \mathcal{O}_X + \operatorname{cl} \mathcal{O}_X(D_2) - \operatorname{cl} \mathcal{O}_X$ for arbitrary divisors D_1 and D_2. Thus, we obtain a group homomorphism $\Psi : \operatorname{Pic}(X) \longrightarrow K(\mathrm{Coh}_{d,d-1}(X))$ mapping a line bundle L onto $\operatorname{cl} L - \operatorname{cl} \mathcal{O}_X$. The morphism Ψ maps $\operatorname{Pic}(X)$ onto a direct summand of $K(\mathrm{Coh}_{d,d-1}(X))$ because $\det : K(\mathrm{Coh}_{d,d-1}(X)) \longrightarrow \operatorname{Pic}(X)$ is a left inverse of Ψ. The image of Ψ is contained in the kernel of the rank homomorphism $\operatorname{rk} : K(\mathrm{Coh}_{d,d-1}(X)) \longrightarrow \mathbb{Z}$.
Due to Serre's theorem, every coherent sheaf G on X fits into a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(mH)^{\oplus \operatorname{rk}(G)} \longrightarrow G \longrightarrow T \longrightarrow 0, \tag{2.1}$$

where H is some fixed ample divisor, m some sufficiently small integer and T is a torsion sheaf. If we regard T as an object in $\mathrm{Coh}_{d,d-1}(X)$, we can assume that T is a successive extension of torsionfree sheaves T_i on irreducible divisors D_i. Repeating the argument with the short exact sequence (2.1) with $H|_{D_i}$, we see that T_i is a direct sum of line bundles $\mathcal{O}_{D_i}(m_i H|_{D_i})$ in $\mathrm{Coh}_{d,d-1}(X)$. The latter are isomorphic to $\mathcal{O}_{D_i}$ in $\mathrm{Coh}_{d,d-1}(X)$ and we see that $\operatorname{cl} T$ is a sum of classes $\operatorname{cl} \mathcal{O}_{D_i}$ in $K(\mathrm{Coh}_{d,d-1}(X))$, i.e. contained in the image of Ψ. If some object $\operatorname{cl} E - \operatorname{cl} F$ of $K(\mathrm{Coh}_{d,d-1}(X))$ has rank zero, we get

$$\operatorname{cl} E - \operatorname{cl} F = \big(\operatorname{cl} \mathcal{O}_X(mH)^r + \operatorname{cl} T_E\big) - \big(\operatorname{cl} \mathcal{O}_X(mH)^r + \operatorname{cl} T_F\big) = \operatorname{cl} T_E - \operatorname{cl} T_F \in \operatorname{im} \Psi.$$

Thus, the following short sequence is exact and splits

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\Psi} K(\mathrm{Coh}_{d,d-1}(X)) \xrightarrow{\mathrm{rk}} \mathbb{Z} \longrightarrow 0.$$

□

Proposition 2.1.12 ([35], Proposition 2.2). *The abelian category* $\mathrm{Coh}_{d,d-1}(X)$ *is noetherian.*

Proof. Consider a chain of epimorphisms

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \ldots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \ldots$$

We have to show $E_j \cong E_{j+1}$ for sufficiently large j. Let us denote by T_j the torsion subsheaf of E_j and by F_j the torsionfree quotient E_j/T_j. Consider the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & \ker f_j \arrow[d] & & \ker g_j \arrow[d] & \\
0 \arrow[r] & T_j \arrow[r] \arrow[d, "f_j"] & E_j \arrow[r] \arrow[d] & F_j \arrow[r] \arrow[d, "g_j"] & 0 \\
0 \arrow[r] & T_{j+1} \arrow[r] \arrow[d] & E_{j+1} \arrow[r] \arrow[d] & F_{j+1} \arrow[r] \arrow[d] & 0 \\
 & \mathrm{coker}\, f_j \arrow[r] & 0 \arrow[r] & \mathrm{coker}\, g_j \arrow[r] & 0
\end{tikzcd}$$

We conclude $\mathrm{coker}\, g_j = 0$, i.e. g_j is an epimorphism. Furthermore, the sequence $\mathrm{rk}(F_k), k \in \mathbb{N}$, decreases and $\mathrm{rk}(F_j) = \mathrm{rk}(F_{j+1})$ for $j \gg 0$. Thus, $\mathrm{rk}(\ker g_j) = 0$ for $j \gg 0$ and $\ker g_j = 0$, i.e. $F_j \cong F_{j+1}$ follows. Due to the snake lemma, f_j must be an epimorphism for $j \gg 0$. Thus, the sequence of degrees $\deg T_j$ with respect to some fixed ample divisor H decreases. Hence, $\deg T_j = \deg T_{j+1}$, i.e. $\deg \ker f_j = 0$ for all $j \gg 0$. If the torsion sheaf $\ker f_j$ has degree zero, it vanishes in $\mathrm{Coh}_{d,d-1}(X)$ and we conclude $T_j \cong T_{j+1}$ for all $j \gg 0$. Due to the five lemma, $E_j \cong E_{j+1}$ for all $j \gg 0$. □

The following statement is the main result of this section. Although the proof of the theorem works only for $\dim X \leq 2$, one might expect that the statement is true for arbitrary dimensions.

Theorem 2.1.13. *Let X be an irreducible smooth projective variety of dimension $d \leq 2$. Then the abelian category* $\mathrm{Coh}_{d,d-1}(X)$ *has at most homological dimension one, i.e. for all coherent sheaves on X we have*

$$\mathrm{Ext}^i_{\mathrm{Coh}_{d,d-1}(X)}(E,F) = 0 \qquad \textit{for all } i \geq 2.$$

Proof. The theorem is well known for $d \leq 1$ and we restrict ourselves to the case of surfaces. In the proof of Proposition 2.1.11 we saw that every coherent sheaf on X as an object in $\mathrm{Coh}_{2,1}(X)$ is a successive extension of sheaves $\mathcal{O}_X(mH)$ with m sufficiently small and $\mathcal{O}_{C_i}$ for some integral curves $C_i \subset X$. Using this and the long exact $\mathrm{Ext}_{\mathrm{Coh}_{2,1}(X)}$-sequence, we can assume that E and F are either $\mathcal{O}_X(mH)$ or $\mathcal{O}_C$. Furthermore, it is enough to show

$$\mathrm{Ext}^i_{\mathrm{Coh}_{d,d-1}(X)}(\mathcal{O}_X(mH), \mathcal{O}_X(nH)) = 0 \qquad \text{for all } i \geq 2$$

in the case $m \ll n$, because we can choose m arbitrary small. The proof of the remaining equations

$$\begin{aligned}
\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_C) &= 0, \\
\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_X(mH), \mathcal{O}_C) &= 0, \\
\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_X(nH)) &= 0, \\
\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_X(mH), \mathcal{O}_X(nH)) &= 0
\end{aligned}$$

for $i \geq 2$ is very technical and we postpone it to the next subsection. □

We will obtain the following assertion as a corollary of the proof of the previous four equations in the next subsection. It generalizes the corresponding statements for curves. On the other hand, the last part shows that $\mathrm{Coh}_{2,1}(X)$ is not of finite type over the base field k.

Proposition 2.1.14. *Let X be an irreducible smooth projective surface, E a torsionfree sheaf and T, T' two torsion sheaves on X whose supports have no common curves. Then,*

1. $\mathrm{Ext}^1_{\mathrm{Coh}_{2,1}(X)}(E, T) = 0$,
2. $\mathrm{Hom}_{\mathrm{Coh}_{2,1}(X)}(T', T) = 0$ *and*
3. $\mathrm{End}_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_C) = K(C)$, *where $C \subset X$ is an integral curve of X with function field $K(C)$.*

2.1.3 The proof of the Theorem

In this subsection we prove the equations

$$\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_C) = 0, \tag{2.2}$$
$$\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_X(mH), \mathcal{O}_C) = 0, \tag{2.3}$$
$$\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_X(nH)) = 0, \tag{2.4}$$
$$\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_X(mH), \mathcal{O}_X(nH)) = 0 \tag{2.5}$$

for all $i \geq 2$, all integral curves $C, C' \subseteq X$ and all integers $m, n \in \mathbb{Z}$ with $m \ll n$. In order to prove the last two equations, we need some techniques from the proof of the first two equations for a more general curve C in X. Hence, we allow C and C' to be any closed curve in X not necessarily reduced or irreducible. Due to this, all divisors on the curves C and C' have to be Cartier divisors. In this subsection we use right roofs (see below) instead of the left roofs of subsection 2.1.1. All triangles appearing in the text are assumed to be distinguished triangles.

We start with the proof of the first two equations. Let G be a coherent sheaf on X. We aim at computing $\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_C) = \mathrm{Hom}_{\mathrm{D}^b_{2,1}(X)}(G[-i], \mathcal{O}_C)$. For $i < 0$ these Ext^i-groups vanish due to the existence of a t-structure on $\mathrm{D}^b_{2,1}(X)$ with heart $\mathrm{Coh}_{2,1}(X)$. Therefore, we will restrict ourselves to the case $i \geq 0$. Let us fix an element $\tilde{f} \in \mathrm{Hom}_{\mathrm{D}^b_{2,1}(X)}(G[-i], \mathcal{O}_C)$ and represent it by some roof

$$\begin{tikzcd} & E & \\ G[-i] \arrow[ur, "f"] & & \mathcal{O}_C \arrow[ul, "s"'] \end{tikzcd} \tag{2.6}$$

such that the cone $C(s)$ belongs to $\mathrm{D}^b_0(X)$. Looking at the long exact cohomology sequence associated to the triangle $\mathcal{O}_C \longrightarrow E \longrightarrow C(s) \longrightarrow \mathcal{O}_C[1]$, we obtain

$$H^l(E) \in \mathrm{Coh}_0(X) \text{ for all } l \neq 0 \text{ and there is a short exact sequence}$$

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{H^0(s)} H^0(E) \longrightarrow T \longrightarrow 0$$

for some torsion sheaf T with zero-dimensional support. Conversely, every morphism $\mathcal{O}_C \xrightarrow{s} E$ having these properties satisfies $C(s) \in \mathrm{D}^b_0(X)$, i.e. becomes an isomorphism in the quotient category $\mathrm{D}^b_{2,1}(X)$. Using this, we see that we can replace the upper roof by the roof

$$\begin{tikzcd} & \tau^{\geq 0}(E) & \\ G[-i] \arrow[ur, "tf"] & & \mathcal{O}_C \arrow[ul, "ts"'] \end{tikzcd} ,$$

where $t : E \longrightarrow \tau^{\geq 0}(E)$ is the usual morphism. Due to this, we will assume $E \in \mathrm{D}^{\geq 0}(X)$. On the other hand, f and s lift over the morphism $\tau^{\leq i}(E) \longrightarrow E$ and we can replace the upper roof by the lower roof in the following diagram.

$$\begin{tikzcd} & E & \\ & \tau^{\leq i}(E) \arrow[u] & \\ G[-i] \arrow[uur, "f"] \arrow[ur, "\hat{f}"'] & & \mathcal{O}_C \arrow[uul, "s"'] \arrow[ul, "\hat{s}"] \end{tikzcd}$$

Due to this observation, we may assume $E \in \mathrm{D}^{[0,i]}(X)$. Furthermore, we will assume that $H^0(E)$ contains no torsion subsheaf with zero-dimensional support. Otherwise, we replace our roof by the equivalent roof

$$\begin{tikzcd} & E/T' & \\ G[-i] \arrow[ur, "uf"] & & \mathcal{O}_C \arrow[ul, "us"'] \end{tikzcd} ,$$

where T' is the biggest subsheaf of $H^0(E)$ with zero-dimensional support and E/T' as well as u are defined by the triangle $T' \longrightarrow E \xrightarrow{u} E/T' \longrightarrow T'[1]$. In general, the ideal sheaf $\mathcal{I}_C$ of C does not annihilate $H^0(E)$ because the torsion sheaf $T = H^0(E)/\mathcal{O}_C$ might have 'directions transverse to C'. If we take the tensor product of the short exact sequence

$$0 \longrightarrow \mathcal{I}_C / \operatorname{Ann}(H^0(E)) \longrightarrow \mathcal{O}_X / \operatorname{Ann}(H^0(E)) \longrightarrow \mathcal{O}_C \longrightarrow 0$$

with $H^0(E)$, we obtain the exact sequence

$$\mathcal{I}_C \otimes H^0(E) \longrightarrow H^0(E) \longrightarrow H^0(E)|_C \longrightarrow 0.$$

Since $H^0(E) \cong \mathcal{O}_C$ outside the finite set of points $\operatorname{supp}(T)$, we conclude $H^0(E) \cong H^0(E)|_C$ because $H^0(E)$ has no subsheaves of zero-dimensional support. Thus, $H^0(E)$ can be regarded as a torsionfree sheaf on C. If the curve is not reduced and smooth, it might happen that $H^0(E)$ is not locally free. In this case, we embed $H^0(E)$ into some locally free sheaf $\mathcal{O}_C(D)$ of rank one. Indeed, due to Serre's theorem and $H^0(E) \cong \mathcal{O}_C$ outside $\operatorname{supp}(T)$, there is a monomorphism $\mathcal{O}_C(-D) \xrightarrow{i} H^0(E)^\vee$ for some ample divisor D on C such that the cokernel of i has zero-dimensional support. The composition $H^0(E) \hookrightarrow H^0(E)^{\vee\vee} \xrightarrow{i^\vee} \mathcal{O}_C(D)$ is the desired embedding.
There is a morphism v from E to the cone E' of the composition $\tau^{\geq 1}(E)[-1] \longrightarrow H^0(E) \hookrightarrow \mathcal{O}_C(D)$ which makes the following diagram commutative

$$\begin{tikzcd} \tau^{\geq 1}(E)[-1] \arrow[r] \arrow[d, equal] & H^0(E) \arrow[r] \arrow[d, hook, "v"] & E \arrow[r] \arrow[d] & \tau^{\geq 1}(E) \arrow[d, equal] \\ \tau^{\geq 1}(E)[-1] \arrow[r] & \mathcal{O}_C(D) \arrow[r] & E' \arrow[r] & \tau^{\geq 1}(E) \end{tikzcd} .$$

Using v and E' we can replace our previous roof by the equivalent roof

$$\begin{tikzcd} & E' & \\ G[-i] \arrow[ur, "vf"] & & \mathcal{O}_C \arrow[ul, "vs"'] \end{tikzcd}$$

with $H^0(E') = \mathcal{O}_C(D)$ for some effective divisor D defined by the nontrivial 'section' $H^0(vs)$.

As an immediate consequence of these simplifications we obtain the following corollary.

Corollary 2.1.15. *For two torsion sheaves T, T' whose supports have no common curves we get*

$$\mathrm{Hom}_{\mathrm{Coh}_{2,1}(X)}(T', T) = 0.$$

Furthermore, for an integral curve C the endomorphism ring $\mathrm{Hom}_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_C, \mathcal{O}_C)$ is the function field $K(C)$ of the curve.

Proof. In $\mathrm{Coh}_{2,1}(X)$ every torsion sheaf is a successive extension of structure sheaves $\mathcal{O}_{C_i}$ of integral curves C_i. Using the exact $\mathrm{Hom}_{\mathrm{Coh}_{2,1}(X)}$-sequences and induction with respect to the number of necessary extensions, we can restrict ourselves to the case $T = \mathcal{O}_C$ and $T' = \mathcal{O}_{C'}$ for two distinct integral curves C and C'. We choose a morphism $\tilde{f}$ and represent it by some roof as in (2.6). Due to our simplifications, we can assume that E is a locally free sheaf $\mathcal{O}_C(D)$ of rank one. The divisor D is defined by the 'section' $s : \mathcal{O}_C \longrightarrow E$. The first statement follows from the simple fact that there is no nonzero morphism f between torsionfree sheaves which are supported on two curves without common components. For the second assertion we regard the morphisms f and s of the roof as sections of $E = \mathcal{O}_C(D)$. We associate to our roof the rational function $f/s \in K(C)$. If the morphism $\tilde{f}$ is represented by another roof, we get the following commutative diagram

$$\begin{tikzcd}
 & & E'' & & \\
 & \mathcal{O}_C(D) \arrow[ur, "t"] & & \mathcal{O}_C(D') \arrow[ul, "t'"'] & \\
\mathcal{O}_C \arrow[ur, "f"] \arrow[urrr, "f'"'] & & & & \mathcal{O}_C \, . \arrow[ulll, "s"] \arrow[ul, "s'"']
\end{tikzcd}$$

We can apply the same simplifications to E'' as before. Doing this, we can assume $E'' = \mathcal{O}_C(D'')$ and we obtain the following equations of rational functions $f/s = tf/ts = t'f'/t's' = f'/s'$ which proves the independence of the rational function from the chosen representation of the morphism $\tilde{f}$.
Conversely, every nontrivial rational function $g : C \longrightarrow \mathbb{P}^1$ is the quotient of the two sections $s := g^*x$ and $f := g^*y$ in $E := g^*\mathcal{O}_{\mathbb{P}^1}(1)$, where x and y are the canonical sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ vanishing in ∞ respectively in $0 \in \mathbb{P}^1$. □

We come back to the roof (2.6) and we may assume $i \geq 0$, $E \in \mathrm{D}^{[0,i]}(X)$ and $H^0(E) = \mathcal{O}_C(D)$ for some effective divisor D on C. Assume, there is some morphism $t : E \longrightarrow E'$ with $C(t) \in \mathrm{D}^b_0(X)$ and $H^1(E') = 0$. Then, we can replace the roof (2.6) by

$$\begin{tikzcd}
 & E' & \\
G[-i] \arrow[ur, "tf"] & & \mathcal{O}_C \, . \arrow[ul, "ts"']
\end{tikzcd}$$

If we simplify this as before, we may assume $E' \in \mathrm{D}^{[0,i]}(X)$ and $H^0(E') = \mathcal{O}_C(D')$ for some effective divisor D' on C. These simplifications do not effect the equation $H^1(E') = 0$ which implies $E' \cong H^0(E') \oplus \tau^{\geq 2}(E')$. Thus, we get a 'projection' $p : E' \longrightarrow H^0(E') = \mathcal{O}_C(D')$ and we can replace our original roof

$$\begin{tikzcd} & E & \\ G[-i] \arrow[ur, "f"] & & \mathcal{O}_C \arrow[ul, "s"'] \end{tikzcd} \quad \text{by} \quad \begin{tikzcd} & \mathcal{O}_C(D') & \\ G[-i] \arrow[ur, "ptf"] & & \mathcal{O}_C \arrow[ul, "pts"'] \end{tikzcd} .$$

If such a morphism $t : E \longrightarrow E'$ always exists, we can conclude the equations (2.2) and (2.3) from the following lemma.

Lemma 2.1.16. *If every morphism $\tilde{f} \in \mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_C)$ is representable by a roof of the following shape*

$$\begin{tikzcd} & \mathcal{O}_C(D) & \\ \mathcal{O}_{C'}[-i] \arrow[ur, "f"] & & \mathcal{O}_C \arrow[ul, "s"'] \end{tikzcd} ,$$

then $\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_C) = 0$ for $i \geq 2$.. The same is true if we replace $\mathcal{O}_{C'}$ by $\mathcal{O}_X(mH)$ for any m.

Proof. Since $\dim X = 2$, the only nontrivial case is $i = 2$. After embedding $\mathcal{O}_C(D)$ into $\mathcal{O}_C(D')$ for $D' \gg D$ and replacing the roof, we can assume $\deg \mathcal{O}_C(D) > \deg K_X|_C$. Using this, the assertion follows from Serre duality

$$\mathrm{Ext}^2(\mathcal{O}_{C'}, \mathcal{O}_C(D)) = \mathrm{Hom}(\mathcal{O}_C(D), K_X|_{C'})^\vee = 0$$

and

$$\mathrm{Ext}^2(\mathcal{O}_X(mH), \mathcal{O}_C(D)) = \mathrm{Hom}(\mathcal{O}_C(D), K_X(mH))^\vee = 0.$$

□

What remains to show is the existence of $t : E \longrightarrow E'$ with $C(t) \in \mathrm{D}^b_0(X)$ and $H^1(E') = 0$. We prove this by induction on the length $\ell(H^1(E))$ of $H^1(E)$. For $\ell(H^1(E)) = 0$ there is nothing to do. For $\ell(H^1(E)) \geq 1$ we choose a subsheaf $k(x)$ of $H^1(E)$ and consider the composition

$$\sigma : k(x)[-1] \hookrightarrow H^1(E)[-1] \longrightarrow \tau^{\geq 1}(E) \longrightarrow H^0(E)[1] = \mathcal{O}_C(D)[1].$$

If $\sigma = 0$, there is a morphism $\tilde{\sigma} : k(x)[-1] \longrightarrow E$ inducing the inclusion on the first cohomology. Define $u : E \longrightarrow E^{(1)}$ by the triangle

$$k(x)[-1] \xrightarrow{\tilde{\sigma}} E \xrightarrow{u} E^{(1)} \longrightarrow k(x).$$

The cone of u is $k(x)$ and $\ell(H^1(E^{(1)})) = \ell(H^1(E)) - 1$. Thus, we can apply the induction hypothesis to $E^{(1)}$ and obtain a morphism $t^{(1)} : E^{(1)} \longrightarrow E'$ with $C(t^{(1)}) \in \mathrm{D}^b_0(X)$ and $H^1(E') = 0$. The composition $t := t^{(1)}u : E \longrightarrow E'$ is the desired morphism.
If $\sigma \neq 0$, we choose a very ample divisor A on C with $x \in \mathrm{supp}(A)$ and embed

$\mathcal{O}_C(D)$ into $\mathcal{O}_C(D+A)$ with quotient sheaf $\mathcal{O}_A(D+A)$.
Define $\hat{E}$ and $v : E \longrightarrow \hat{E}$ by the requirement that the following diagram is commutative and that the rows are distinguished triangles

$$\begin{tikzcd}
\tau^{\geq 1}(E)[-1] \arrow[r] \arrow[d, equal] & \mathcal{O}_C(D) \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, "v"] & \tau^{\geq 1}(E) \arrow[d, equal] \\
\tau^{\geq 1}(E)[-1] \arrow[r] & \mathcal{O}_C(D+A) \arrow[r] & \hat{E} \arrow[r] & \tau^{\geq 1}(E) .
\end{tikzcd}$$

By the 3×3-lemma, we can choose v in such a way that the cone of v is isomorphic to $\mathcal{O}_C(D+A)/\mathcal{O}_C(D) = \mathcal{O}_A(D+A)$ and, therefore, in $\mathrm{D}^b_0(X)$. Since $H^1(E) = H^1(\hat{E})$, $k(x)$ is also a subsheaf of $H^1(\hat{E})$ and the composition

$$\hat{\sigma} : k(x)[-1] \hookrightarrow H^1(\hat{E})[-1] \longrightarrow \tau^{\geq 1}(\hat{E}) \longrightarrow H^0(\hat{E})[1] = \mathcal{O}_C(D+A)[1],$$

which is the composition $k(x)[-1] \xrightarrow{\sigma} \mathcal{O}_C(D)[1] \hookrightarrow \mathcal{O}_C(D+A)[1]$, vanishes. To see this, we look at the long exact Ext-sequence

$$\mathrm{Ext}^2(k(x), \mathcal{O}_C(D)) \xrightarrow{\alpha} \mathrm{Ext}^2(k(x), \mathcal{O}_C(D+A)) \longrightarrow \mathrm{Ext}^2(k(x), \mathcal{O}_A(D+A)) \longrightarrow 0.$$

Due to Serre duality, all the Ext^2-groups have dimension one and, therefore, $\alpha = 0$. The left Ext^2-group contains σ which is mapped by α onto $\hat{\sigma}$ and the assertion $\hat{\sigma} = 0$ follows. Due to this, we can lift $\hat{\sigma}$ to some morphism $\tilde{\sigma} : k(x)[-1] \longrightarrow \hat{E}$ and define $u : \hat{E} \longrightarrow E^{(1)}$ to be the cone of $\tilde{\sigma}$ as before. By induction hypothesis, there is a morphism $t^{(1)} : E^{(1)} \longrightarrow E'$ with $C(t^{(1)}) \in \mathrm{D}^b_0(X)$ and $H^1(E') = 0$. Finally, $t := t^{(1)}uv : E \longrightarrow E'$ is the desired morphism.

The existence of $t : E \longrightarrow E'$ as above implies also the following corollary.

Corollary 2.1.17. *For a torsionfree sheaf G and a torsion sheaf T on X we have*

$$\mathrm{Ext}^1_{\mathrm{Coh}_{2,1}(X)}(G, T) = 0.$$

Proof. Every torsionfree sheaf G is isomorphic in $\mathrm{Coh}_{2,1}(X)$ to its reflexive hull $G^{\vee\vee}$. The latter is locally free on a smooth surface and we can assume that G is locally free. Furthermore, every torsion sheaf is a successive extension of structure sheaves $\mathcal{O}_{C_i}$ in $\mathrm{Coh}_{2,1}(X)$, where the C_i are integral curves. Using the long exact $\mathrm{Ext}_{\mathrm{Coh}_{2,1}(X)}$-sequence and induction with respect to the number of necessary extensions, we can restrict ourselves to the case $T = \mathcal{O}_C$. We represent $\tilde{f} \in \mathrm{Ext}^1_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_C)$ by a roof of the following shape

$$\begin{tikzcd}
& \mathcal{O}_C(D) & \\
G[-1] \arrow[ur, "f"] & & \mathcal{O}_C . \arrow[ul, "s"']
\end{tikzcd}$$

Then replace D by $D + mH|_C$ with $m \gg 0$ and embed $\mathcal{O}_C(D)$ into $\mathcal{O}_C(D + mH|_C)$. Since H is ample, we use Serre's theorem to conclude

$$\mathrm{Ext}^1(G, \mathcal{O}_C(D+mH|_C)) = \mathrm{H}^1(X, \mathcal{O}_C(D) \otimes G^\vee \otimes \mathcal{O}_X(mH)) = 0 \qquad \text{for } m \gg 0.$$

Thus, $f = 0$ and the assertion follows. $\square$

Remark. The preceding arguments of this subsection are also valid if we replace the sheaf $\mathcal{O}_C$ by the sheaf $\mathcal{O}_C(D)$ for some divisor D on C. In particular, every morphism $\tilde{f} \in \mathrm{Hom}_{\mathrm{D}^b_{2,1}(X)}(G[-i], \mathcal{O}_C(D))$ can be represented by a roof

$$\begin{tikzcd} & \mathcal{O}_C(D') & \\ G[-i] \arrow[ur, "f"] & & \mathcal{O}_C(D) \arrow[ul, "s"'] \end{tikzcd}$$

with an effective divisor $D' \geq D$ and some section $s \in \mathrm{H}^0(C, \mathcal{O}_C(D' - D))$.

The remaining part of this subsection is devoted to the proof of the equations (2.4) and (2.5). As before, we denote by G the coherent sheaf $\mathcal{O}_{C'}$ or $\mathcal{O}_X(mH)$ with $m \ll n$ and we represent a morphism $\tilde{f} \in \mathrm{Ext}^i(G, \mathcal{O}_X(nH))$ by a roof

$$\begin{tikzcd} & E & \\ G[-i] \arrow[ur, "f"] & & \mathcal{O}_X(nH) \arrow[ul, "s"'] \end{tikzcd}$$

with $C(s) \in \mathrm{D}^b_0(X)$. The property $C(s) \in \mathrm{D}^b_0(X)$ is equivalent to $H^l(E) \in \mathrm{Coh}_0(X)$ for all $l \neq 0$ and $H^0(E) = \mathcal{O}_X(nH) \oplus T$ with $T \in \mathrm{Coh}_0(X)$. Indeed, the short exact sequence $0 \longrightarrow \mathcal{O}_X(nH) \longrightarrow H^0(E) \longrightarrow T \longrightarrow 0$ splits by Serre duality. The only interesting case is $i \geq 0$ and we can assume $E \in \mathrm{D}^{[0,i]}(X)$ as before. After dividing out $T \longrightarrow E$, we can assume $H^0(E) = \mathcal{O}_X(nH)$.
If there is a morphism $t : E \longrightarrow E'$ with $C(t) \in \mathrm{D}^b_0(X)$ and $H^1(E') = 0$, we can replace our roof by the equivalent roof

$$\begin{tikzcd} & \mathcal{O}_X(nH) & \\ G[-i] \arrow[ur, "f'"] & & \mathcal{O}_X(nH) \arrow[ul, "id"'] \end{tikzcd}$$

similar to the case of $\mathcal{O}_C$.

Lemma 2.1.18. *If a morphism $\tilde{f} \in \mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_X(nH))$ is representable by a roof of the following shape*

$$\begin{tikzcd} & \mathcal{O}_X(nH) & \\ \mathcal{O}_{C'}[-i] \arrow[ur, "f"] & & \mathcal{O}_X(nH) \arrow[ul, "id"'] \end{tikzcd},$$

then $\tilde{f} = 0$ for $i \geq 2$. The same is true if we replace $\mathcal{O}_{C'}$ by $\mathcal{O}_X(mH)$ with $m \ll n$.

Proof. As $\dim X = 2$, the only nontrivial case is $i = 2$. The second assertion for $i = 2$ follows from Serre duality

$$\mathrm{Ext}^2(\mathcal{O}_X(mH), \mathcal{O}_X(nH)) = \mathrm{Hom}(\mathcal{O}_X(nH), K_X(mH))^\vee = 0 \text{ and } m \ll n.$$

For the first assertion we consider an inclusion $\iota : \mathcal{O}_{C'}(-D) \hookrightarrow \mathcal{O}_{C'}$ with some effective divisor D on C' of degree $\deg(D) > \deg(K_X(-nH)|_{C'})$. Since $\mathcal{O}_{C'} \cong \mathcal{O}_{C'}(-D)$ in $\mathrm{Coh}_{2,1}(X)$, this induces an isomorphism

$$\mathrm{Ext}^2_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}, \mathcal{O}_X(nH)) \xrightarrow{\sim} \mathrm{Ext}^2_{\mathrm{Coh}_{2,1}(X)}(\mathcal{O}_{C'}(-D), \mathcal{O}_X(nH)).$$

The image of $\tilde{f}$ under this isomorphism is represented by the following roof.

$$\begin{tikzcd} & \mathcal{O}_X(nH) & \\ \mathcal{O}_{C'}(-D)[-2] \arrow[ur, "f\iota"] & & \mathcal{O}_X(nH) \arrow[ul, "id"'] \end{tikzcd}$$

Due to our assumptions on D, we get

$$\begin{aligned} \mathrm{Ext}^2(\mathcal{O}_{C'}(-D), \mathcal{O}_X(nH)) &= \mathrm{Hom}(\mathcal{O}_X(nH), K_X|_{C'}(-D))^\vee \\ &= \mathrm{H}^0(C', K_X(-nH)|_{C'}(-D))^\vee = 0 \end{aligned}$$

and $\tilde{f} = 0$ follows. $\square$

In contrast to the case $\mathcal{O}_C$, such a morphism $t : E \longrightarrow E'$ with $C(t) \in \mathrm{D}^b_0(X)$ and $H^1(E') = 0$ does not need to exist. Instead of that, we will construct a morphism $\mathcal{O}_X(nH) \xrightarrow{\beta} \mathcal{O}_X(pH)$ below with $p > n$ such that $\beta \circ \tilde{f}$ satisfies the assumptions of the previous lemma. Thus, $\beta \circ \tilde{f} = 0$ for $i \geq 2$ and there is some morphism $\tilde{g} \in \mathrm{Ext}^{i-1}_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_X(pH)|_C)$ with $C \in |(p-n)H|$ which is mapped onto $\tilde{f} \in \mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_X(nH))$ in the following long exact $\mathrm{Ext}^i_{\mathrm{Coh}_{2,1}(X)}$-sequence

$$\begin{aligned} \mathrm{Ext}^{i-1}_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_X(pH)) \longrightarrow \mathrm{Ext}^{i-1}_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_X(pH)|_C) \\ \longrightarrow \mathrm{Ext}^{i}_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_X(nH)). \end{aligned}$$

If $i > 2$ we get $\tilde{g} = 0$ and, therefore, $\tilde{f} = 0$ because of

$$0 = \mathrm{Ext}^{i-1}_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_C) \xrightarrow{\sim} \mathrm{Ext}^{i-1}_{\mathrm{Coh}_{2,1}(X)}(G, \mathcal{O}_X(pH)|_C)$$

and the results of the first part of this subsection. Using Corollary 2.1.17, we deduce in the same way $\tilde{f} = 0$ in the case $G = \mathcal{O}_X(mH)$ and $i = 2$. The remaining case is $G = \mathcal{O}_{C'}$ and $i = 2$. By the results of the first part of this subsection, we can represent $\tilde{g}$ by the following roof

$$\begin{tikzcd} & \mathcal{O}_C(D) & \\ \mathcal{O}_{C'}[-1] \arrow[ur, "g"] & & \mathcal{O}_X(pH)|_C \arrow[ul, "t"'] \end{tikzcd}$$

for some effective divisor $D \geq pH|_C$. After increasing D, we can assume $\mathcal{O}_C(D) = \mathcal{O}_C((p+l)H|_C) = \mathcal{O}_X((p+l)H)|_C$ for some sufficiently large number $l \in \mathbb{N}$ because $H|_C$ is ample. By the same reason, $t \in \mathrm{H}^0(C, \mathcal{O}_C(lH|_C))$ is the restriction of some section $\bar{t} \in \mathrm{H}^0(X, \mathcal{O}_X(lH))$ if we choose l large enough. Due to the isomorphism

$$\mathrm{Ext}^1(\mathcal{O}_{C'}, \mathcal{O}_C((p+l)H|_C)) \cong \mathrm{Ext}^1(\mathcal{O}_{C'}(-lH|_{C'}), \mathcal{O}_C(pH|_C)),$$

there is some $\hat{g} \in \mathrm{Ext}^1(\mathcal{O}_{C'}(-lH|_{C'}), \mathcal{O}_C(pH|_C))$ making the following diagram commutative.

$$\begin{tikzcd} \mathcal{O}_{C'}[-1] \arrow[r, "g"] & \mathcal{O}_C(D) \\ \mathcal{O}_{C'}(-lH|_{C'})[-1] \arrow[u, "\hat{t}[-1]"] \arrow[r, "\hat{g}"] & \mathcal{O}_C(pH|_C) \arrow[u, "t"'] \end{tikzcd}$$

The morphism $\hat{t} \in \mathrm{H}^0(C', \mathcal{O}_{C'}(lH|_{C'}))$ is the restriction of $\bar{t}$ to C'. If l is sufficiently large, the composition

$$\mathcal{O}_{C'}(-lH|_{C'})[-2] \xrightarrow{\hat{t}[-2]} \mathcal{O}_{C'}[-2] \xrightarrow{\tilde{g}[-1]} \mathcal{O}_X(pH)|_C[-1] \longrightarrow \mathcal{O}_X(nH)\ ,$$

which is $\mathcal{O}_{C'}(-lH|_{C'})[-2] \xrightarrow{\hat{g}[-1]} \mathcal{O}_X(pH)|_C[-1] \longrightarrow \mathcal{O}_X(nH)$ and, therefore, in $\mathrm{Ext}^2(\mathcal{O}_{C'}(-lH|_{C'}), \mathcal{O}_X(nH))$, vanishes by Serre duality. On the other hand, this composition is $\mathcal{O}_{C'}(-lH|_{C'})[-2] \xrightarrow{\hat{t}[-2]} \mathcal{O}_{C'}[-2] \xrightarrow{\tilde{f}} \mathcal{O}_X(nH)$, and we conclude $\tilde{f} = 0$ because $\mathcal{O}_{C'}(-lH|_{C'})[-2] \xrightarrow{\hat{t}[-2]} \mathcal{O}_{C'}[-2]$ is an isomorphism in $\mathrm{D}^b_{2,1}(X)$.

Finally, we have to construct the morphism $\beta : \mathcal{O}_X(nH) \longrightarrow \mathcal{O}_X(pH)$ depending on $\tilde{f}$ represented by the roof

$$\begin{tikzcd} & E & \\ G[-i] \arrow[ur, "f"] & & \mathcal{O}_X(nH) \arrow[ul, "s"'] \end{tikzcd}\ .$$

In order to do this, we choose some curve $C \in |(p-n)H|$ for a sufficiently large $p > n$ such that $H^1(E)$ is a torsion subsheaf on C. Note that C might be nonreduced and reducible. This curve defines a morphism $\beta : \mathcal{O}_X(nH) \hookrightarrow \mathcal{O}_X(pH)$ and it remains to show that $\beta \circ \tilde{f}$ is representable by a roof of the following form.

$$\begin{tikzcd} & \mathcal{O}_X(pH) & \\ G[-i] \arrow[ur, "h"] & & \mathcal{O}_X(pH) \arrow[ul, "id"'] \end{tikzcd}$$

First of all, the composition $\beta \circ \tilde{f}$ is representable by the roof

$$\begin{tikzcd} & E' & \\ G[-i] \arrow[ur, "tf"] & & \mathcal{O}_X(pH) \arrow[ul, "s'"'] \end{tikzcd} ,$$

where $t : E \longrightarrow E'$ and s' are defined by the following commutative diagram with distinguished rows

$$\begin{tikzcd} \tau^{\geq 1}(E)[-1] \arrow[r] \arrow[d, equal] & \mathcal{O}_X(nH) \arrow[r] \arrow[d, hook, "\beta"] & E \arrow[r] \arrow[d, "t"] & \tau^{\geq 1}(E) \arrow[d, equal] \\ \tau^{\geq 1}(E)[-1] \arrow[r] & \mathcal{O}_X(pH) \arrow[r, "s'"] & E' \arrow[r] & \tau^{\geq 1}(E) \end{tikzcd} .$$

The nontriviality of $\sigma : H^1(E)[-1] \longrightarrow \tau^{\geq 1}(E) \longrightarrow \mathcal{O}_X(nH)[1]$ is the obstruction to lift $H^1(E)[-1] \longrightarrow \tau^{\geq 1}(E)$ over $E \longrightarrow \tau^{\geq 1}(E)$ to some morphism $H^1(E)[-1] \longrightarrow E$ which induces the identity on the first cohomology. The obstruction for E' is the composition

$$\sigma' : H^1(E')[-1] = H^1(E)[-1] \xrightarrow{\sigma} \mathcal{O}_X(nH)[1] \xrightarrow{\beta[1]} \mathcal{O}_X(pH)[1]$$

which vanishes due to the following short exact sequence

$$\underbrace{\mathrm{Ext}^2(H^1(E), \mathcal{O}_X(nH))}_{\ni \sigma} \longrightarrow \underbrace{\mathrm{Ext}^2(H^1(E), \mathcal{O}_X(pH))}_{\dim(\ldots) = \ell(H^1(E))} \twoheadrightarrow \underbrace{\mathrm{Ext}^2(H^1(E), \mathcal{O}_X(pH)|_C)}_{\dim(\ldots) = \ell(H^1(E))} .$$

For the last dimension we use that $H^1(E)$ is a sheaf on C. Thus, there is a morphism $H^1(E')[-1] \longrightarrow E'$ which induces the identity on the first cohomology. Using this, we see that the first cohomology of $E'/H^1(E')[-1] := C(H^1(E')[-1] \longrightarrow E')$ is zero, while the zeroth cohomology is still $\mathcal{O}_X(pH)$. Thus, we can project onto the zeroth cohomology. If we denote the composition $E' \longrightarrow E'/H^1(E')[-1] \xrightarrow{pr} \mathcal{O}_X(pH)$ by p, we can represent $\beta \circ \tilde{f}$ by the equivalent roof

$$\begin{tikzcd} & \mathcal{O}_X(pH) & \\ G[-i] \arrow[ur, "h := ptf"] & & \mathcal{O}_X(pH) \arrow[ul, "id"'] \end{tikzcd}$$

and we are done.

2.2 Quotients modulo stable objects of degree zero

In this section we compute the quotient of the bounded derived category $\mathrm{D^b}(X)$ by the full subcategory generated by μ-stable sheaves of degree zero. It turns out that under suitable conditions on the geometry of X this quotient category is equivalent to the bounded derived category of vector spaces of finite dimension over a division algebra. We will give some examples in the last subsection. As explained in the introduction (see page vi) this quotient category was motivated by the attempt to compare the bounded derived category of a generic K3 surface with the corresponding category of a projective K3 surface. As before, all triangles appearing in the text are distinguished triangles.

2.2.1 General results

Let X be an irreducible smooth projective variety of dimension $d \geq 1$. As before, we denote by $\mathrm{D}^b_{d,d-1}(X)$ the triangulated quotient category of $\mathrm{D^b}(X)$ by the subcategory of all complexes whose support has dimension $\leq d-2$. We saw that this category is equivalent to the bounded derived category of the abelian quotient category $\mathrm{Coh}_{d,d-1}(X)$. Assume that the following two assumptions are true.

1. *The quotient category* $\mathrm{Coh}_{d,d-1}(X)$ *has homological dimension one.*

2. *For every torsionfree sheaf E and every torsion sheaf T on X we have*

$$\mathrm{Ext}^1_{\mathrm{Coh}_{d,d-1}(X)}(E,T) = 0.$$

Due to the preceding section, these assumptions are valid for $d \leq 2$. If we fix an ample divisor, we have the notion of μ-stability on $\mathrm{D}^b_{d,d-1}(X)$. This defines a stability condition in the sense of Bridgeland [8],[35]. The stability function is $Z = -\deg + i\,\mathrm{rk}$. We denote by $\mathcal{P}(I)$ the full subcategory of all complexes whose semistable factors have phases in the set I. Let us denote by $\mathcal{D}$ the full triangulated subcategory of $\mathrm{D}^b_{d,d-1}(X)$ generated by the stable sheaves of phase $1/2$, i.e. of degree zero.

Lemma 2.2.1. *The category $\mathcal{D} \subset \mathrm{D}^b_{d,d-1}(X)$ is the full subcategory $\mathcal{P}(1/2+\mathbb{Z})$ of all complexes whose cohomology sheaves are semistable of phase* $1/2$.

Proof. Since every semistable sheaf of phase $1/2$ has a filtration with stable quotients of phase $1/2$ and due to the existence of the cohomology filtration for every complex, the subcategory $\mathcal{P}(1/2+\mathbb{Z})$ is contained in $\mathcal{D}$. It remains to show that for every exact triangle

$$E' \xrightarrow{f} E'' \longrightarrow E \longrightarrow E'[1]$$

with E' and E'' in $\mathcal{P}(1/2+\mathbb{Z})$, the object E belongs to $\mathcal{P}(1/2+\mathbb{Z})$. Consider the associated long exact cohomology sequence in $\mathrm{Coh}_{d,d-1}(X)$

$$H^i(E') \xrightarrow{H^i(f)} H^i(E'') \longrightarrow H^i(E) \longrightarrow H^{i+1}(E') \xrightarrow{H^{i+1}(f)} H^{i+1}(E'')$$

and assume $E', E'' \in \mathcal{P}(1/2+\mathbb{Z})$. Since the category $\mathcal{P}(1/2)$ of all semistable sheaves of phase 1/2 is abelian and closed under extension, the object $H^i(E)$ is semistable of phase 1/2 because it is an extension of the semistable sheaves $\operatorname{coker} H^i(f)$ and $\ker H^{i+1}(f)$ of phase 1/2. This shows $E \in \mathcal{P}(1/2+\mathbb{Z})$, and with the definition of $\mathcal{D}$ we get $\mathcal{D} = \mathcal{P}(1/2+\mathbb{Z})$. □

The lemma shows that $\mathcal{D}$ is a thick triangulated subcategory and we denote by $\mathcal{Q}$ the triangulated quotient category $\mathrm{D}^b_{d,d-1}(X)/\mathcal{D}$. The aim of this section is to describe this category $\mathcal{Q}$. It is a standard result that $\mathcal{Q}$ is the localization of $\mathrm{D}^b_{d,d-1}(X)$ with respect to the class S of morphisms with cone in $\mathcal{D}$, and the class S is localizing (see [32], Chapter 2). Due to this, every morphism in $\mathcal{Q}$ can be represented by a roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ F & & G \end{tikzcd} \qquad \text{with } s \in S, \text{ i.e. } C(s) \in \mathcal{D}.$$

Let F be a stable sheaf of phase $\phi(F) > 1/2$ and consider a morphism $s : E \longrightarrow F$ with cone $C(s)$ in $\mathcal{D}$, i.e. $s \in S$. Because of our first assumption on $\mathrm{Coh}_{d,d-1}(X)$, the complex E is isomorphic to $\oplus_{i\in\mathbb{Z}} H^i(E)[-i]$ ([20]). In order to compute the cohomology sheaves, we look at the long exact cohomology sequence in $\mathrm{Coh}_{d,d-1}(X)$ corresponding to the exact triangle of s

$$0 \longrightarrow H^{-1}(C(s)) \longrightarrow H^0(E) \longrightarrow F \xrightarrow{\alpha} H^0(C(s)) \longrightarrow H^1(E) \longrightarrow 0\,, \quad (2.7)$$
$$0 \longrightarrow H^k(C(s)) \longrightarrow H^{k+1}(E) \longrightarrow 0 \qquad \text{for all } k \in \mathbb{Z} \text{ with } k \neq -1, 0\,. \quad (2.8)$$

Since the phase of F is greater than the phase of the semistable sheaf $H^0(C(s))$, we get $\alpha = 0$ and we see that $H^i(E)$ is a semistable sheaf of phase 1/2 for $i \neq 0$. The zeroth cohomology of E is an extension of F by some semistable sheaf of phase 1/2. Conversely, any complex E with these cohomology sheaves and any morphism $s : E \longrightarrow F$ in $\mathrm{D}^b_{d,d-1}(X)$, whose restriction to $H^0(E)$ is the epimorphism from the extension, has a cone in $\mathcal{D}$. Since E is the direct sum of its cohomology sheaves, we can replace the original roof by the equivalent roof

$$\begin{tikzcd} & H^0(E) \arrow[dl, "s\imath"'] \arrow[dr, "f\imath"] & \\ F & & G\,, \end{tikzcd}$$

where $\imath : H^0(E) \longrightarrow E$ is the natural 'inclusion'. Thus, we can assume that E is a sheaf.

Corollary 2.2.2. *For any sheaf $F \in \mathcal{P}((1/2, 1])$ and any torsion sheaf T on X we have*

$$\mathrm{Hom}_{\mathcal{Q}}(F, T[n]) = 0 \qquad \text{for all } n \neq 0.$$

In particular, $\mathrm{Hom}_{\mathcal{Q}}(\mathcal{O}_{D'}, \mathcal{O}_D[n]) = 0$ *for all $n \neq 0$ and all effective divisors D, D' on X.*

Proof. It is enough to prove the assertion for a stable sheaf F of phase $1/2 < \phi(F) \leq 1$. We represent a morphism $\tilde{f} \in \mathrm{Hom}_{\mathcal{Q}}(F, T[n])$ by a roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ F & & T[n] \end{tikzcd}$$

with a coherent sheaf E as before. If F is torsionfree, E is also torsionfree because the kernel of s is semistable of phase $1/2$ and, therefore, torsionfree. In this case the assertion follows easily from the two assumptions on $\mathrm{Coh}_{d,d-1}(X)$. If F is not torsionfree, it must be a torsion sheaf. Since F is stable, we have $F \cong \mathcal{O}_{D'}$ in $\mathrm{Coh}_{d,d-1}(X)$ for some effective divisor D' on X. Due to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D') \longrightarrow \mathcal{O}_{D'} \longrightarrow 0$$

in $\mathrm{Coh}_{d,d-1}(X)$, we get $F \cong \mathcal{O}_X(D')$ in $\mathcal{Q}$ and we can use the result for torsionfree sheaves. □

In a similar way one proves the following statement.

Corollary 2.2.3. *For any sheaf F and any torsion sheaf T on X we have*

$$\mathrm{Hom}_{\mathcal{Q}}(F, T[n]) = 0 \qquad \text{for all } n > 0.$$

Proof. Due to the previous corollary, we can restrict ourselves to stable torsionfree sheaves F of phase $0 < \phi(F) \leq 1/2$. Let us represent a morphism $\tilde{f} \in \mathrm{Hom}_{\mathcal{Q}}(F, T[n])$ by a roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ F & & T[n] \end{tikzcd}$$

with a complex E. We can replace E by the length two subcomplex $\tau^{[0,1]}(E) \cong H^0(E) \oplus H^1(E)[-1]$. Looking at the short exact sequence (2.7), we see that $H^0(E)$ is torsionfree because $\ker \alpha$ is torsionfree. Using $n > 0$, the assertion follows easily from our two assumptions on $\mathrm{Coh}_{d,d-1}(X)$. □

Remark. The statement of the last corollary is wrong if we relax the assumption $n > 0$ to $n \neq 0$ as in the previous corollary. Indeed, due to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we obtain $\mathcal{O}_D[-1] \cong \mathcal{O}_X(-D)$ in $\mathcal{Q}$ and, therefore, $\mathrm{Hom}_{\mathcal{Q}}(\mathcal{O}_X(-D), \mathcal{O}_D[-1]) \neq 0$ for every effective divisor D on X.

We will assume now that F is stable sheaf of positive degree and $\deg(F)$ generates the subgroup $\{\deg(E) \mid E \in \mathrm{Coh}_{d,d-1}(X)\}$ of $\mathbb{Z}$. Due to this assumption, the degree of every sheaf E of phase $\phi(E) > 1/2$ is a positive multiple of $\deg(F)$. We will investigate the endomorphism ring of F in $\mathcal{Q}$.

Lemma 2.2.4. *Any roof*

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ F & & G \end{tikzcd}$$

with $E, G \in \mathrm{D}^b_{d,d-1}(X)$ *is equivalent to a roof*

$$\begin{tikzcd} & E' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ F & & G \end{tikzcd}$$

with a stable sheaf E' *of degree* $\deg(E') = \deg(F)$.

Proof. Due to our previous remarks, we can assume that E is a sheaf of phase $\phi(E) > 1/2$. Using the Harder–Narasimhan filtration of E, we find a maximal semistable subsheaf $0 \neq E' \subseteq E$ of phase $\phi(E') > 1/2$ with maximal phase among all subsheaves. Let us denote by s' and f' the restrictions of s resp. f to E' and let us write C and C' for the kernel of s resp. s'. Then $s' \neq 0$. Otherwise, E' is a subsheaf of the semistable sheaf C of phase $1/2$ which is impossible. Thus, we have an extension $0 \longrightarrow C' \longrightarrow E' \longrightarrow F' \longrightarrow 0$ with subsheaves C' of C and F' of F. Because of $\mathrm{rk}(F') \leq \mathrm{rk}(F)$ and the stability of F, we get $\deg(F') < \deg(F)$ if $F' \neq F$. Due to the assumptions on $\deg(F)$, we conclude $\deg(F') \leq 0$ in this case and we obtain $\deg(C') \geq \deg(E') > 0$ which contradicts $C' \subseteq C$. Thus, $F' = F$ and $\deg(C') = \deg(E') - \deg(F) \geq 0$. If C' is not semistable of degree zero, C' and, therefore, C contain a semistable subsheaf of degree greater than zero, which is impossible. Hence, C' is semistable of degree zero and

$$\begin{tikzcd} & E' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ F & & G \end{tikzcd}$$

is a roof which is equivalent to the previous one due to the following diagram.

$$\begin{tikzcd} & & E' \arrow[dl, hook'] \arrow[dr, equal] & & \\ & E \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & E' \arrow[dlll, "s'"' near end] \arrow[dr, "f'"] & \\ F & & & & G \end{tikzcd}$$

Furthermore, E' has degree $\deg(E') = \deg(F)$ and is, therefore, stable due to the choice of $\deg(F)$. $\square$

Corollary 2.2.5. *Any endomorphism of F is either zero or invertible, i.e. $\mathrm{Hom}_{\mathcal{Q}}(F,F)$ is a division algebra.*

Proof. Using the previous lemma, we represent this endomorphism by a roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ F & & F \end{tikzcd}$$

with a stable sheaf E of degree $\deg(E) = \deg(F)$. If f is not zero, we have to show $f \in S$, i.e. that f is an epimorphism and that the kernel C of the epimorphism $f : E \twoheadrightarrow F$ is semistable. If the image of $f \neq 0$ is a proper subsheaf F' of F, we get $\deg(C) \geq \deg(E)$ as in the proof of the previous lemma. Using $\mathrm{rk}(C) \leq \mathrm{rk}(E)$, this contradicts the stability of E. Thus, $F' = F$ and f is an epimorphism. If C is not semistable, there must be a proper semistable subsheaf E' of C and, therefore, of E of degree $\deg(E') > 0$ because C has degree zero. Due to the assumption on $\deg(F) = \deg(E)$, we obtain $\deg(E') \geq \deg(E)$ which contradicts the stability of E as before. $\square$

If we combine the Corollaries 2.2.2 and 2.2.5, we obtain the following theorem.

Theorem 2.2.6. *Assume there is an effective divisor D such that $\deg(D) > 0$ generates the subgroup $\{\deg(E) \mid E \in \mathrm{Coh}(X)\}$ of $\mathbb{Z}$. In this case the category $\mathcal{Q}$ is equivalent to the bounded derived category of vector spaces of finite dimension over the division algebra $K := \mathrm{Hom}_{\mathcal{Q}}(\mathcal{O}_D, \mathcal{O}_D)$.*

Proof. First of all, we show that $\mathcal{O}_D$ generates $\mathcal{Q}$ as a triangulated category. For this let us denote by $\mathcal{Q}'$ the full triangulated subcategory generated by $\mathcal{O}_D$. Due to our first assumption on $\mathrm{Coh}_{d,d-1}(X)$, every complex is isomorphic to the direct sum of its cohomology sheaves and it suffices to show $E \in \mathcal{Q}'$ for every coherent sheaf. We start with the case $E = \mathcal{O}_X(D')$ for some effective divisor D' on X. Due to our assumption on D, there is a positive integer m such that $\deg(D') = m\deg(D)$. The short exact sequence

$$0 \longrightarrow \mathcal{O}_X(D' - mD) \longrightarrow \mathcal{O}_X(D') \longrightarrow \mathcal{O}_X(D')|_{mD} \longrightarrow 0$$

and $\deg(\mathcal{O}_X(D' - mD)) = 0$ imply $\mathcal{O}_X(D') \cong \mathcal{O}_X(D')|_{mD} \cong \mathcal{O}_{mD}$ in $\mathcal{Q}$. The sheaf $\mathcal{O}_{mD}$ is a successive extension of $\mathcal{O}_D$ and, therefore, in $\mathcal{Q}'$. Using this and the relation $\mathcal{O}_X(D') \cong \mathcal{O}_{D'}$ as well as $\mathcal{O}_X(-D') \cong \mathcal{O}_{D'}[1]$ in $\mathcal{Q}$, we get $\mathcal{O}_{D'}, \mathcal{O}_X(D'), \mathcal{O}_X(-D') \in \mathcal{Q}'$. Since every torsion sheaf is a successive extension of structure sheaves $\mathcal{O}_{D'}$ in $\mathrm{Coh}_{d,d-1}(X)$, we obtain immediately $T \in \mathcal{Q}'$ for every torsion sheaf T on X. Furthermore, every sheaf E is an extension of a torsion sheaf by a vector bundle $\mathcal{O}_X(-H)^{\oplus \mathrm{rk}(E)}$ for some ample divisor H and we finally conclude $E \in \mathcal{Q}'$. Thus, $\mathcal{Q}' = \mathcal{Q}$.

The second part of the proof is more or less linear algebra. Using the definition of

K and Corollary 2.2.2, we obtain a fully faithful functor $\Phi : \mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(K)) \longrightarrow \mathcal{Q}$ mapping $\bigoplus_{i\in\mathbb{Z}} K^{n_i}[i]$ to $\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_D^{\oplus n_i}[i]$. If Φ is an exact functor, the image is a full triangulated subcategory of $\mathcal{Q}$ containing $\mathcal{O}_D$. Due to the first part, it coincides with $\mathcal{Q}$ up to equivalence and we are done. What remains to show is the exactness of Φ. For this, let

$$\bigoplus_i M_i[i] : \bigoplus_i K^{m_i}[i] \longrightarrow \bigoplus_i K^{n_i}[i]$$

be a morphism in $\mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(K))$ represented by (n_i, m_i)-matrices M_i. Denote by C_i the cone of M_i in $\mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(K))$. Since the cone of direct sum of morphisms is the direct sum of their cones, we have the exact triangle in $\mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(K))$

$$\bigoplus_i K^{m_i}[i] \longrightarrow \bigoplus_i K^{n_i}[i] \longrightarrow \bigoplus_i C_i[i] \longrightarrow \bigoplus_i K^{m_i}[i+1].$$

We have to show that this triangle remains exact after applying Φ. Since Φ is additive and commutes with the shift functor, we can assume $m_i = n_i = 0$ for all $i \in \mathbb{Z}, i \neq 0$ and we will suppress the index 0 from the notion. Then, there are isomorphisms $\alpha : K^m \longrightarrow K^m$ and $\beta : K^n \longrightarrow K^n$, such that

$$\beta \circ M \circ \alpha^{-1} = \begin{pmatrix} id & 0 \\ 0 & 0 \end{pmatrix} : K^r \oplus K^{m-r} \longrightarrow K^r \oplus K^{n-r},$$

where r is the rank of M. For such a morphism the cone is just the sum of the cone of id, which is zero, and the trivial extension of $K^{m-r}[1]$ by K^{n-r}

$$K^r \oplus K^{n-r} \xrightarrow{\begin{pmatrix} 0 & id \\ 0 & 0 \end{pmatrix}} K^{n-r} \oplus K^{m-r}[1] \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & id \end{pmatrix}} (K^r \oplus K^{m-r})[1].$$

Since Φ maps trivial extensions to trivial extensions and isomorphisms to isomorphisms, the image $\Phi(C)$ of the cone of M is the cone of $\Phi(M)$ and we are done. $\square$

Remark. There is a natural factorization of the degree map $\deg : \mathrm{D}^b_{d,d-1}(X) \longrightarrow \mathbb{Z}$ over the quotient functor $\mathrm{D}^b_{d,d-1}(X) \longrightarrow \mathcal{Q}$ because all sheaves of $\mathcal{D}$ have degree zero. We see immediately that the composition of this degree map $d : \mathcal{Q} \longrightarrow \mathbb{Z}$ with $\Phi : \mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(K)) \longrightarrow \mathcal{Q}$ is the dimension map $\dim : \mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(K)) \longrightarrow \mathbb{Z}$.

2.2.2 Examples

In the last part of this section we consider examples to illustrate the Theorem 2.2.6. Note that the assumptions on $\mathrm{Coh}_{d,d-1}(X)$ of the previous section are satisfied in the case of irreducible smooth projective curves and surfaces.

The projective space $\mathbb{P}^1$

In the case of curves, every point x is a good choice for the divisor in the theorem. We write $k(x)$ instead of $\mathcal{O}_x$. Let us consider a morphism $\tilde{f} : k(x) \longrightarrow k(x)$ in $\mathcal{Q}$. We represent this morphism by a roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ k(x) & & k(x) \end{tikzcd}$$

with E a stable sheaf of degree one. In the case of $\mathbb{P}^1$, the only stable sheaves of degree one are $\mathcal{O}_{\mathbb{P}^1}(1)$ and $k(y)$ for $y \in \mathbb{P}^1$. Therefore, $f = r \cdot s$ for some scalar $r \in k$ and we conclude $\tilde{f} = r \cdot id_{k(x)}$. Thus, the division algebra K is just the base field k and $\mathcal{Q}$ is equivalent to the bounded derived category of k-vector spaces of finite dimension

$$\mathcal{Q} \cong \mathrm{D^b}(\mathrm{Spec}(k)).$$

The elliptic curve X

As before we can choose $D = x$ and conclude that $\mathcal{Q}$ is equivalent to the bounded derived category of K-vector spaces of finite dimension, where K is the endomorphism ring of $k(x)$ in $\mathcal{Q}$.

Proposition 2.2.7. *The division algebra is naturally isomorphic to the quotient field $K(\hat{X})$ of the dual curve $\hat{X} = \mathrm{Pic}^0(X)$. The latter is isomorphic to X, but this isomorphism depends on the choice of a polarisation.*

Proof. We consider the usual Fourier–Mukai equivalence $F : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(\hat{X})$ with respect to the Poincaré-bundle (see [29]). Due to a classical result of Atiyah [1], every stable sheaf of degree zero is a line bundle L in $\mathrm{Pic}^0(X) = \hat{X}$ and F maps this line bundle onto the shifted structure sheaf $k(L)[1]$ of the point $L \in \hat{X}$. Thus, F maps every complex with semistable cohomology sheaves of degree zero onto a complex with zero-dimensional support. Conversely, F^{-1} maps every complex with zero-dimensional support onto some complex which is a successive extension of shifted line bundles of degree zero. Because of this, F identifies the subcategory $\mathcal{D}$ with the subcategory $\mathrm{D}_0^b(\hat{X})$ of complexes in $\mathrm{D^b}(\hat{X})$ with zero-dimensional support. Using Corollary 2.1.10, we see that F induces an equivalence

$$\mathcal{Q} \cong \mathrm{D^b}(\mathrm{Spec}(K(\hat{X}))).$$

□

Note that the quotient category $\mathcal{Q}$ still determines the isomorphism class of X.

The curve X of genus $\mathbf{g(X) \geq 2}$

If we choose $D = x$, we can apply the theorem and conclude that $\mathcal{Q}$ is equivalent to the bounded derived category of K-vector spaces, where K is the endomorphism ring of $k(x)$ in $\mathcal{Q}$. We will show in appendix B that the k-dimension of

K is infinite as in the case of the elliptic curve. This is related to the existence of positive dimensional moduli spaces of stable sheaves of degree zero. But in contrast to the elliptic case, there are stable sheaves of degree zero and rank greater than one. This may cause the noncommutativity of K.

The projective space $\mathbb{P}^2$

We choose the standard polarisation $\mathcal{O}_{\mathbb{P}^2}(H) = \mathcal{O}_{\mathbb{P}^2}(1)$ and see that any hyperplane $D = H$ satisfies the assumption of the theorem. Thus, $\mathcal{Q}$ is equivalent to the bounded derived category of K-vector spaces. Note that there is a ring homomorphism

$$\mathrm{End}_{\mathrm{Coh}_{2,1}(\mathbb{P}^2)}(\mathcal{O}_H) \longrightarrow \mathrm{End}_{\mathcal{Q}}(\mathcal{O}_H) = K$$

and since the first endomorphism ring is the function field of H (see Prop. 2.1.13), which is the function field $k(Z)$ of one variable Z, we see that $k(Z)$ is a subfield of K.

Ruled surfaces

An ample divisor on a ruled surface over a curve C is given by $H := C_0 + mF$ with $m \gg 0$, where C_0 is some section of $\pi : \mathbb{P}(E) \longrightarrow C$ and $F \cong \mathbb{P}^1$ is a fibre of π. Using the intersection products $C_0 \cdot F = 1$ and $F \cdot F = 0$, we see that the degree of $D := F$ with respect to H is one and we can apply the Theorem 2.2.6. The same holds for elliptic surfaces with a section.

Appendix A

$H^{-p} : \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X}) \longrightarrow \mathrm{Refl}(\mathrm{X})$ is an equivalence

We prove the following statement to complete the proof of Proposition 1.4.8. See section 1.4 for the notation.

Proposition A.1. *For any $p \geq 2$ the functor $H^{-p} : \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X}) \longrightarrow \mathrm{Refl}(\mathrm{X})$ is an equivalence of categories.*

Proof. For any complex $E \in \mathrm{Coh}_{(\mathrm{p})}(\mathrm{X})$ we consider the triangle

$$H^{-p}(E)[p] \longrightarrow E \longrightarrow \tau^{>-p}(E) \xrightarrow{\alpha} H^{-p}(E)[p+1]$$

and we already know that the natural transformation $\tau^{\leq -2}R\Gamma(\alpha) : \tau^{[1-p,-2]}(E) \longrightarrow \tau^{\leq -2}(R\Gamma H^{-p}(E)[p+1])$ is an isomorphism making the following diagram commutative.

$$\begin{tikzcd}
\tau^{[1-p,-2]}(E)[-1] \arrow[r, "\tau^{\leq -2}(R\Gamma(\alpha))[-1]", "\sim"'] \arrow[d, "\delta[-1]"] & \tau^{\leq -1}(R\Gamma H^{-p}(E)[p]) \arrow[d, "\varepsilon"] \\
\tau^{>-p}(E)[-1] \arrow[r, "\alpha[-1]"] & H^{-p}(E)[p]
\end{tikzcd}$$

For $E \in \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$ we have $H^0(E) = H^{-1}(E) = 0$ and the morphism δ is an isomorphism. Looking at the following diagram

$$\begin{tikzcd}
\tau^{\leq -1}(R\Gamma H^{-p}(E)[p]) \arrow[r, "\varepsilon"] \arrow[d, "\wr"] & H^{-p}(E)[p] \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, dotted] & \tau^{\leq -1}(R\Gamma H^{-p}(E)[p])[1] \arrow[d, "\wr"] \\
\tau^{>-p}(E)[-1] \arrow[r, "\alpha[-1]"] & H^{-p}(E)[p] \arrow[r] & E \arrow[r] & \tau^{>-p}(E) ,
\end{tikzcd}$$

we see that E is isomorphic to the cone of $\varepsilon : \tau^{\leq -1}(R\Gamma H^{-p}(E)[p]) \longrightarrow H^{-p}(E)[p]$. Using this construction, we conclude that every reflexive sheaf is up to isomorphism of the form $H^{-p}(E)$ for some complex $E \in \mathrm{Refl}_{(\mathrm{p})}(\mathrm{X})$.

We still need to show that H^{-p} is fully faithful. Since ε is a natural transformation, we can always 'extend' a morphism $f : H^{-p}(E) \longrightarrow H^{-p}(E')$ to a morphism $g : E \longrightarrow E'$ making the following diagram commutative.

$$\begin{tikzcd}
\tau^{\leq -1}(R\Gamma H^{-p}(E)[p]) \arrow[r, "\varepsilon"] \arrow[d, "\tau^{\leq -1}(R\Gamma(f)[p])"] & H^{-p}(E)[p] \arrow[r] \arrow[d, "f[p]"] & E \arrow[r] \arrow[d, dotted, "g"] & \tau^{\leq -1}(R\Gamma H^{-p}(E)[p])[1] \arrow[d, "\tau^{\leq -1}(R\Gamma(f)[p])[1]"] \\
\tau^{\leq -1}(R\Gamma H^{-p}(E')[p]) \arrow[r, "\varepsilon'"] & H^{-p}(E')[p] \arrow[r] & E' \arrow[r] & \tau^{\leq -1}(R\Gamma H^{-p}(E')[p])[1]
\end{tikzcd}$$

Therefore, H^{-p} is a full functor.

For the faithfulness let us consider a morphism $g : E \longrightarrow E'$ with $H^{-p}(g) = 0$. Using the fact that $\tau^{\leq -2}R\Gamma(\alpha) \circ \delta^{-1} : \tau^{>-p}(E) \longrightarrow \tau^{\leq -2}(R\Gamma H^{-p}(E)[p+1])$ is a natural isomorphism, we conclude $\tau^{>-p}(g) = 0$ and obtain the following commutative diagram with distinguished rows.

$$\begin{tikzcd}
\tau^{>-p}(E)[-1] \arrow[r, "\alpha[-1]"] \arrow[d, "0"] & H^{-p}(E)[p] \arrow[r, "i"] \arrow[d, "0"] & E \arrow[r, "\pi"] \arrow[d, "g"] & \tau^{>-p}(E) \arrow[d, "0"] \\
\tau^{>-p}(E')[-1] \arrow[r, "\alpha'[-1]"] & H^{-p}(E')[p] \arrow[r, "i'"] & E' \arrow[r, "\pi'"] & \tau^{>-p}(E')
\end{tikzcd}$$

Because of $\pi' g = 0$, there is a morphism $g' : E \longrightarrow H^{-p}(E')[p]$ with $i'g' = g$. From the equation $i'g'i = gi = 0$ we conclude the existence of a morphism $h : H^{-p}(E)[p] \longrightarrow \tau^{>-p}(E')[-1]$ such that $(\alpha'[-1])h = g'i : H^{-p}(E)[p] \longrightarrow H^{-p}(E')[p]$. Since $\tau^{>-p}(E')[-1] \in \mathrm{D}^{>-p}(X)$ and $H^{-p}(E)[p] \in \mathrm{D}^{\leq -p}(X)$, such a morphism h must be zero. Thus, $g'i = 0$ and there exists a morphism $f : \tau^{>-p}(E) \longrightarrow H^{-p}(E')[p]$ with $f\pi = g'$. Applying $\tau^{\leq -1}R\Gamma$ to f and using $\tau^{>-p}(E) \cong \tau^{\leq -1}R\Gamma\tau^{>-p}(E)$, we obtain another morphism $f' : \tau^{>-p}(E) \longrightarrow \tau^{\leq -1}(R\Gamma H^{-p}(E')[p])$ with $\varepsilon' f' = f$.

$$\begin{tikzcd}
\tau^{\leq -1}R\Gamma\tau^{>-p}(E) \arrow[r, "\tau^{\leq -1}R\Gamma(f)"] \arrow[d, "\wr"] & \tau^{\leq -1}(R\Gamma H^{-p}(E')[p]) \arrow[r, "\sim"] \arrow[d, "\varepsilon'"] & \tau^{>-p}(E')[-1] \arrow[dl, "\alpha'[-1]"'] \\
\tau^{>-p}(E) \arrow[ur, "f'"] \arrow[r, "f"] & H^{-p}(E')[p] &
\end{tikzcd}$$

Using the isomorphism $\tau^{\leq -1}(R\Gamma H^{-p}(E')[p]) \cong \tau^{>-p}(E')[-1]$, we can regard f' as a morphism from $\tau^{>-p}(E)$ to $\tau^{>-p}(E')[-1]$ with the property $(\alpha'[-1])f' = f$. Finally, we conclude $g = i'g' = i'f\pi = i'(\alpha'[-1])f'\pi = 0$ because of $i'(\alpha'[-1]) = 0$. Thus, H^{-p} is a faithful functor. $\square$

Appendix B

On the dimension of the division algebra

Let X be a smooth projective curve of genus $g \geq 1$. We show that the division algebra $\mathrm{Hom}_{\mathcal{Q}}(k(x), k(x))$ is infinite-dimensional over the base field k. For the notation see subsection 2.2.1. Due to Lemma 2.2.4, every roof representing a morphism $\tilde{f} \in \mathrm{Hom}_{\mathcal{Q}}(k(x), k(x))$ is equivalent to a roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ k(x) & & k(x) \end{tikzcd}$$

with a stable sheaf E of degree one. If $E = k(x)$, we can replace E by $\mathcal{O}_X(x)$ and compose s and f with the surjection $\mathcal{O}_X(x) \twoheadrightarrow k(x)$. Thus, we can assume without loss of generality that E is a stable locally free sheaf of degree one and we call these roofs stable.

Lemma B.1. *Assume we have two stable roofs*

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ k(x) & & k(x) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & E' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ k(x) & & k(x) \end{tikzcd}$$

with locally free sheaves E and E', and let us denote the kernels of s and s' by C resp. C'. If $f = r \cdot s$ and $f' = r \cdot s'$ for a scalar $r \in k$, the two roofs are equivalent. If the semistable sheaves C and C' of degree zero have no common stable factor, this is the only possibility for the roofs to become equivalent.

Proof. The direct sum of the two short exact sequences corresponding to s and s' defines an extension of $k(x) \oplus k(x)$ by $C \oplus C'$. The pull back of this extension by the map $\binom{1}{1} : k(x) \longrightarrow k(x) \oplus k(x)$ defines an extension of $k(x)$ by $C \oplus C'$

and we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
0 \arrow[r] & C \oplus C' \arrow[r] \arrow[d, equal] & \widetilde{E} \arrow[r, "\sigma"] \arrow[d, hook, "\begin{pmatrix} p \\ p' \end{pmatrix}"'] & k(x) \arrow[r] \arrow[d, hook, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] & 0 \\
0 \arrow[r] & C \oplus C' \arrow[r] & E \oplus E' \arrow[r, "\begin{pmatrix} s & 0 \\ 0 & s' \end{pmatrix}"] \arrow[d, "(s\ \ -s')"] & k(x) \oplus k(x) \arrow[r] \arrow[d, "(1\ \ -1)"] & 0 \\
 & & k(x) \arrow[r, equal] & k(x) &
\end{tikzcd}$$

Using the assumption $f = r \cdot s$ and $f' = r \cdot s'$, we obtain the following commutative diagram

$$\begin{tikzcd}
 & & \widetilde{E} \arrow[dl, "p"'] \arrow[dr, "p'"] & & \\
 & E \arrow[dl, "s"'] \arrow[drrr, "r \cdot s" near end] & & E' \arrow[dlll, "s'" near end] \arrow[dr, "r \cdot s'"] & \\
k(x) & & & & k(x)
\end{tikzcd}$$

which shows the equivalence of our roofs.

Let us assume now that the roofs are equivalent and the set of stable factors of C is disjoint from the one of C'. Thus, we have a commutative diagram

$$\begin{tikzcd}
 & & E'' \arrow[dl, "q"'] \arrow[dr, "q'"] & & \\
 & E \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & E' \arrow[dlll, "s'" near end] \arrow[dr, "f'"] & \\
k(x) & & & & k(x)
\end{tikzcd}$$

and we denote the semistable kernel of the map $sq = s'q' : E'' \longrightarrow k(x)$ by C''. Since $sq - s'q' = 0$, there is a map $\pi : E'' \longrightarrow \widetilde{E}$ with $p\pi = q$ and $p'\pi = q'$ and we can extend the upper diagram to

$$
\begin{tikzcd}
0 \arrow[r] & C'' \arrow[r] \arrow[d, "\theta"] & E'' \arrow[r] \arrow[d, "\pi"] & k(x) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \oplus C' \arrow[r] \arrow[d, equal] & \widetilde{E} \arrow[r, "\sigma"] \arrow[d, hook, "\begin{pmatrix} p \\ p' \end{pmatrix}"'] & k(x) \arrow[r] \arrow[d, hook, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] & 0 \\
0 \arrow[r] & C \oplus C' \arrow[r] & E \oplus E' \arrow[r, "\begin{pmatrix} s & 0 \\ 0 & s' \end{pmatrix}"] \arrow[d, "(s\ \ -s')"] & k(x) \oplus k(x) \arrow[r] \arrow[d, "(1\ \ -1)"] & 0 \\
 & & k(x) \arrow[r, equal] & k(x)\ . &
\end{tikzcd}
$$

We claim that θ is an epimorphism. If this is not the case, there is an epimorphism $\eta : C \oplus C' \longrightarrow C^\sharp$ onto some stable factor of $C \oplus C'$ with $\eta\theta = 0$ because C, C' and C'' are contained in the abelian category of semistable sheaves of degree zero. The stable factor $C^\sharp$ is either a stable factor of C or a stable factor of C' due to our assumption. Let us assume for simplicity that $C^\sharp$ is a stable factor of C. As $\eta|_{C'} = 0$, we can write η as a composition $C \oplus C' \xrightarrow{pr_1} C \xrightarrow{\delta} C^\sharp$ with some epimorphism δ. The image of the extension $0 \longrightarrow C'' \longrightarrow E'' \longrightarrow k(x) \longrightarrow 0$ under the map $pr_1 \circ \theta : \mathrm{Ext}^1(k(x), C'') \longrightarrow \mathrm{Ext}^1(k(x), C)$ is just the extension $0 \longrightarrow C \longrightarrow E \longrightarrow k(x) \longrightarrow 0$ and the image of the latter extension under $\delta : \mathrm{Ext}^1(k(x), C) \longrightarrow \mathrm{Ext}^1(k(x), C^\sharp)$ has to be zero because of $\delta pr_1 \theta = 0$. Thus, we find an epimorphism $E \longrightarrow k(x) \oplus C^\sharp$ and the projection onto $C^\sharp$ leads to a contradiction because E is stable of degree one and $C^\sharp$ is stable of degree zero. This proves that θ is an epimorphism and using the snake lemma, we conclude that π is an epimorphism. Due to this and $0 = fq - f'q' = fp\pi - f'q'\pi = (fp - f'p')\pi$ we conclude $fp - f'p' = 0$. Since $\binom{p}{p'} : \widetilde{E} \longrightarrow E \oplus E'$ is the kernel of $(s \ \ -s) : E \oplus E' \longrightarrow k(x)$, the map $(f \ \ -f') : E \oplus E' \longrightarrow k(x)$ factorizes over $(s \ \ -s')$ and we obtain a scalar $r \in k$ with $(f \ \ -f') = r \cdot (s \ \ -s')$. Restriction to E and E' leads to the desired formulas. $\square$

Remark B.2. The proof shows that the sheaf $\widetilde{E}$ in the sum

$$0 \longrightarrow C \oplus C' \longrightarrow \widetilde{E} \xrightarrow{\sigma} k(x) \longrightarrow 0$$

of the two extensions

$$0 \longrightarrow C \longrightarrow E \longrightarrow k(x) \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow C' \longrightarrow E' \longrightarrow k(x) \longrightarrow 0$$

under the natural map $\mathrm{Ext}^1(k(x), C) \times \mathrm{Ext}^1(k(x), C') \xrightarrow{+} \mathrm{Ext}^1(k(x), C \oplus C')$ is stable of degree one if E and E' are stable of degree one and C and C' have no common stable factors. To see this, we define E'' to be the stable subobject of

$\widetilde{E}$ of degree one and denote by C'' the kernel of the map $\sigma|_{E''} : E'' \longrightarrow k(x)$. The morphisms π and θ in the upper diagram are just the inclusions. As θ and π has been shown to be surjective, they are isomorphism.
This result can be used to constract stable sheaves of degree one and arbitrary rank r on any curve X of genus greater than zero. To do this, we choose r line bundles C_i of degree zero which are pairwise non-isomorphic. Then we add the extensions $0 \longrightarrow C_i \longrightarrow C_i(x) \longrightarrow k(x) \longrightarrow 0$ successively to obtain the sequence of extensions

$$0 \longrightarrow \bigoplus_{i=1}^{l} C_i \longrightarrow \widetilde{E}_l \longrightarrow k(x) \longrightarrow 0$$

with stable vector bundles $\widetilde{E}_l$ of degree one and rank l.

Proposition B.3. *If the genus of the curve X is greater than zero, the k-dimension of K is uncountable.*

Proof. Let us assume for simplicity that the dimension of $\mathrm{Hom}_{\mathcal{Q}}(k(x), k(x))$ is countable infinite. Thus, there is a sequence of stable roofs

$$\begin{tikzcd} & E_i \arrow[dl, "s_i"'] \arrow[dr, "f_i"] & \\ k(x) & & k(x) \end{tikzcd} \qquad \text{with } i \in \mathbb{N}$$

which form a basis of $\mathrm{Hom}_{\mathcal{Q}}(k(x), k(x))$. Then, every stable roof must be equivalent to the sum of finitely many stable roofs

$$\begin{tikzcd} & E_{i_1} \arrow[dl, "s_{i_1}"'] \arrow[dr, "r_1 \cdot f_{i_1}"] & \\ k(x) & & k(x) \end{tikzcd} \quad + \quad \dots \quad + \quad \begin{tikzcd} & E_{i_n} \arrow[dl, "s_{i_n}"'] \arrow[dr, "r_n \cdot f_{i_n}"] & \\ k(x) & & k(x) \end{tikzcd}$$

with $n \in \mathbb{N}$ and some scalars $r_1, \dots, r_n \in k$. In order to compute such a sum, one introduces the equalizer of $s_{i_1}, \dots, s_{i_n}$ which is a locally free sheaf $\widetilde{E}$ defined by the commutative diagram

$$\begin{tikzcd} \bigoplus_{l=1}^{n} C_{i_l} \arrow[r, hook] \arrow[d, equal] & \widetilde{E} \arrow[r, two heads, "\sigma"] \arrow[d, hook, "\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}"'] & k(x) \arrow[d, hook, "\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}"] \\ \bigoplus_{l=1}^{n} C_{i_l} \arrow[r, hook] & \bigoplus_{l=1}^{n} E_{i_l} \arrow[r, two heads, "\begin{pmatrix} s_{i_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & s_{i_n} \end{pmatrix}"] & \bigoplus_{l=1}^{n} k(x) \end{tikzcd}$$

together with some maps $p_l : \widetilde{E} \longrightarrow E_{i_l}$, such that $s_{i_l} \circ p_l = s_{i_m} \circ p_m$ for all $1 \leq l, m \leq n$. The sum of the roofs is then given by the following roof.

$$\begin{tikzcd} & \widetilde{E} \arrow[dl, "\sigma"'] \arrow[dr, "\sum_{l=1}^{n} r_l f_{i_l} p_l"] & \\ k(x) & & k(x) \end{tikzcd} \tag{B.1}$$

Note that the morphism $\sigma : \widetilde{E} \longrightarrow k(x)$ depends only on the finite set of morphisms $s_{i_1}, \ldots, s_{i_n}$. The number of all such finite sets is countable and we obtain only countable many morphisms $\sigma : \widetilde{E} \longrightarrow k(x)$ in all possible 'linear combinations' (B.1) of our basis.
In order to compare the roof (B.1) of the sum with any stable roof, we restrict σ and $\sum_{l=1}^{n} r_l f_{i_l} p_l$ to the stable locally free subsheaf $\widetilde{E}'$ of $\widetilde{E}$ of degree one as described in the proof of Lemma 2.2.4. This subsheaf is uniquely determined by the sheaf $\widetilde{E}$ and the finite set of stable factors of the semistable kernel $\ker(\widetilde{E}' \xrightarrow{\sigma} k(x))$ is, therefore, also uniquely determined by σ. We see that the set of all stable vector bundles of degree zero occurring as factors of this kernel for all possible σ in (B.1) is a countable set.
On the other hand, for a curve of genus $g(X) \geq 1$ there is a moduli space of stable vector bundles of degree zero whose connected components have positive dimension. Using the construction of Remark B.2, we obtain uncountable many stable roofs

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "f"] & \\ k(x) & & k(x) \end{tikzcd}$$

such that $f \notin k \cdot s$ and the sets of stable factors of $\ker(s)$ are disjoint. By Lemma B.1, these roofs are not equivalent to each other. Therefore, not all stable roofs are equivalent to a 'linear combination' of our countable basis, and we obtain a contradiction. Thus, the k-dimension of $K = \mathrm{Hom}_{\mathcal{Q}}(k(x), k(x))$ is uncountable. $\square$

Bibliography

[1] M. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc.*, 7:414–452, 1957.

[2] S. Bando and Y. T. Siu. *Stable sheaves and Einstein–Hermitian metrics, in Geometry and analysis on complex manifolds.* Mabuchi, T. (ed.) et al., Festschrift for Professor S. Kobayashi's 60th birthday. World Scientific, Singapore, 39–59, 1994.

[3] R. Bezrukavnikov. Perverse coherent sheaves (after Deligne). math.AG/0005152.

[4] T. Bridgeland. Spaces of stability conditions. *preprint 2006.* math.AG/0611510.

[5] T. Bridgeland. Stability conditions and Kleinian singulaties. *preprint 2005.* math.AG/0508257.

[6] T. Bridgeland. Stability conditions on a non-compact Calabi–Yau threefold. *preprint 2005.* math.AG/0509048.

[7] T. Bridgeland. Stability conditions on K3 surfaces. *to appear in: Duke Math. J.* math.AG/0307164.

[8] T. Bridgeland. Stability conditions on triangulated categories. *to appear in: Annals of Math.*, 2007. math.AG/0212237.

[9] M.P. Brodmann and R.Y. Sharp. *Local cohomology: an algebraic introduction with geometric applications.* Cambridge University Press, Cambridge.

[10] D. Burns and M. Rapoport. On the Torelli Problem for Kählerian K3 Surfaces. *Ann. scient. Éc. Norm. Sub.*

[11] M. Douglas. D-branes, categories and $n = 1$ supersymmetry. Strings, branes, and M-theory. *J. Math. Phys.*, 42, no. 7:2818–2843, 2001.

[12] M. Douglas. Dirichlet branes, homological mirror symmetry, and stability. In *Proceedings of the International Congress of Mathematicians*, volume Vol. III, pages 395–408, Beijing, 2002. Higher Ed. Press.

[13] D. Eisenbud. *Commutative Algebra: with a view toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, New York.

[14] S.I. Gelfand and Y.I. Manin. *Methods of Homological Algebra*. Springer, Berlin, 2007.

[15] A. Grothendieck. *Local Cohomology*, volume 41 of *Lecture Notes in Math. 41*. Springer, New York, 1967.

[16] D. Happel, I. Reiten, and S.O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc. 120*, 575, 1996.

[17] R. Hartshorne. *Algebraic Geometry*. Springer, 1977.

[18] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[19] D. Huybrechts. Generalized Calabi-Yau structures, K3 surfaces, and B-fields. *Int. J. Math.*, 16:13–36, 2005. math.AG/0306162.

[20] D. Huybrechts. *Fourier–Mukai transforms in Algebraic Geometry*. Oxford Mathematical Monographs, 2006.

[21] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of shaves*. Vieweg, 1997.

[22] D. Huybrechts, P. Stellari, and E. Macrì. Stability conditions for generic K3 categories. *to appear in Compositio Math.* math.AG/0608430.

[23] M. Kashiwara. T-structures on the derived categories of holonomic D-modules and coherent O-modules. *preprint 2003*. math.AG/0302086.

[24] E. Macrì. Some examples of spaces of stability conditions on derived categories. *preprint 2004*. math.AG/0411613.

[25] E. Macrì. Stability conditions on curves. *to appear in Math. Res. Lett.* arXiv:0705.3794.

[26] E. Macrì, S. Mehrotra, and P. Stellari. Inducing stability conditions. *preprint 2007*. arXiv:0705.3752.

[27] E. Macrì and P. Stellari. Automorphisms and autoequivalences of generic analytic K3 surfaces. *to appear in J. Geom. Phys.* math.AG/0702848.

[28] S. Meinhardt. Stability conditions on generic tori. *preprint 2007, submitted*. arXiv:0708.3053.

[29] S. Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its applications to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981.

[30] S. Mukai. *On the moduli space of bundles on K3 surfaces*. Bombay, 1984.

[31] D. Mumford and J. Fogarty. *Geometric Invariant Theory*, volume Erg. Math. 34, 2nd ed. Springer, Berlin, Heidelberg, 1982.

[32] A. Neeman. *Triangulated categories.* Princeton University Press, 2001.

[33] K. Oguiso. Bimeromorphic automorphism groups of non-projective hyperkähler manifolds - a note inspired by C.T. McMullen. *to appear in J. Diff. Geom.* math.AG/0312515.

[34] S. Okada. Stability Manifold of P^1. *J. Algebraic Geometry 15 (3)*, 487-505, 2006.

[35] H. Partsch. Stabilitätsbedingungen auf derivierten Kategorien algebraischer Varietäten modulo Kodimension ≥ 2. *Diploma thesis*, 2007. Universität Bonn.

[36] H.W. Schuster. Locally free resolutions of coherent sheaves on surfaces. *J. Reine Angew. Math.*, 337:159–165, 1982.

[37] P. Seidel and R. Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108:37–108, 2001.

[38] Y. Siu and G. Trautmann. Gap-Sheaves and Extension of Coherent Analytic Sheaves. 172:v+172 pp., 1971.

[39] F. Takemoto. Stable vector bundles on algebraic surfaces I. *I. Nagoya Math. J.*, 47:29–43, 1972.

[40] F. Takemoto. Stable vector bundles on algebraic surfaces II. *I. Nagoya Math. J.*, 52:173–195, 1973.

[41] M. Verbitsky. Coherent sheaves on general K3 surfaces and tori. *preprint 2002.* math.AG/0205210.

[42] M. Verbitsky. Hyperholomorphic bundles over a hyperkähler manifold. *Journal of Alg. Geom.*, 5 no. 4:633–669, 1996. math.AG/9307008.

[43] M. Verbitsky. Coherent sheaves on generic compact tori. *CRM Proc. and Lecture Notes*, 38:229–249, 2004. math.AG/0310329.

[44] C. Voisin. A counterexample to the Hodge conjecture extended to Kähler varieties. *Int. Math. Res. Not. 20*, 1057-1075, 2002.