# Centralisers of polynomially growing automorphisms of free groups

DISSERTATION

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Moritz Rodenhausen

aus

Kiel

Bonn, März 2013

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Carl-Friedrich Bödigheimer
2. Gutachter: Prof. Dr. Martin Bridson

Tag der Promotion: 02. Juli 2013

Erscheinungsjahr: 2013

## Zusammenfassung


Zentralisatoren in Gruppen sind wichtig für das Konjugationsproblem, eines der bekanntesten algorithmischen Probleme der Gruppentheorie. Ziel dieser Arbeit ist Theorem 13.21, dass viele Zentralisatoren in den Automorphismengruppen $\mathrm{Aut}(F_n)$ und $\mathrm{Out}(F_n)$ der freien Gruppe $F_n$ die Endlichkeitseigenschaft VF erfüllen, also eine Untergruppe $G$ mit endlichem Index besitzen, die einen endlichen CW-Komplex als $K(G,1)$-Raum hat.

Ein *Graph von Gruppen* $\mathcal{G}$ besteht aus einem endlichen Graph $\Gamma$ mit Basispunkt $v$, Eckengruppen $G_w$ für jede Ecke $w$ in $\Gamma$, Kantengruppen $G_e$ für jede orientierte Kante $e$ in $\Gamma$ und injektiven Gruppenhomomorphismen $f_e : G_e \to G_{\tau(e)}$, wobei $\tau(e)$ der Endpunkt der Kante $e$ ist. Wir bezeichnen mit $\overline{e}$ die Kante $e$ mit umgekehrter Orientierung und mit $\iota(e) = \tau(\overline{e})$ den Anfangspunkt von $e$. Die Kantengruppen erfüllen $G_{\overline{e}} = G_e$, sind also geometrischen (oder unorientierten) Kanten zugeordnet.

Das Fundamentalgruppoid $\pi_1(\mathcal{G})$ ist gegeben durch die Ecken von $\Gamma$ als Objekte. Für jede Kante $e$ haben wir einen Morphismus $t_e$ von $\iota(e)$ nach $\tau(e)$. Jedes Element $g$ in der Eckengruppe $G_w$ definiert einen Morphismus von $w$ nach $w$. Ein allgemeiner Morphismus im Gruppoid $\pi_1(\mathcal{G})$ ist eine formale Komposition dieser erzeugenden Morphismen mit den Relationen $t_{\overline{e}} = t_e^{-1}$ und $t_e f_e(a) t_e^{-1} = f_{\overline{e}}(a)$ für $a \in G_e$. Wir bezeichnen mit $\pi_1(\mathcal{G}, v, w)$ die Menge der Morphismen von $v$ nach $w$. Ferner schreiben wir $\pi_1(\mathcal{G}, v) = \pi_1(\mathcal{G}, v, v)$ für die *Fundamentalgruppe* von $\mathcal{G}$.

Ein Morphismus $H : \mathcal{G} \to \mathcal{G}'$ von Graphen von Gruppen ist ein Tupel aus einem Graphmorphismus $H_\Gamma$, Eckengruppenhomomorphismen $H_w : G_w \to G'_{H_\Gamma(w)}$, Kantengruppenhomomorphismen $H_e : G_e \to G'_{H_\Gamma(e)}$ und Elementen $\delta_H(e) \in G_{\tau(H_\Gamma(e))}$. Ein Morphismus $H$ induziert einen Gruppoidmorphismus (eine natürliche Transformation) $H_* : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}')$. Auf den erzeugenden Morphismen ist $H_*$ definiert durch $H_*(g) = H_w(g)$ für $g \in G_w$ und $H_*(t_e) = \delta_H(\overline{e}) t_{H_\Gamma(e)} \delta_H(e)^{-1}$. Durch Einschränken erhalten wir einen Gruppenhomomorphismus $\pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', H_\Gamma(v))$ auf den Fundamentalgruppen.

Wir schreiben $\mathrm{Aut}(\mathcal{G})$ für die Automorphismengruppe von $\mathcal{G}$ und $\mathrm{Aut}^0(\mathcal{G})$ für die Untergruppe aller Automorphismen $H$ mit $H_\Gamma = 1$. Ein *Dehn-Twist* auf $\mathcal{G}$ ist ein Automorphismus $D \in \mathrm{Aut}^0(\mathcal{G})$, so dass alle $D_w = 1$, alle $D_e = 1$ und $\delta_D(e) = f_e(\gamma_e)$ für Elemente $\gamma_e$ im Zentrum von $G_e$.

Wenn $\Gamma$ der Graph mit zwei Ecken $v$ und $w$ sowie einer geometrischen Kante $e$ von $v$ nach $w$ ist, dann ist $\pi_1(\mathcal{G}, v)$ isomorph zum amalgamierten freien Produkt $G_v *_{G_e} G_w$ bezüglich der Abbildungen $f_{\overline{e}} : G_e \to G_v$ und $f_e : G_e \to G_w$. Sei $D$ der durch $\gamma_e$ und $\gamma_{\overline{e}}$ definierte Dehn-Twist auf $\mathcal{G}$. Der Gruppenautomorphismus $D_{*v}$ entspricht dann dem Automorphismus von $G_v *_{G_e} G_w$, der $G_v$ punktweise fest lässt und $g \in G_w$ auf $z_e g z_e^{-1}$ abbildet, wobei $z_e := \gamma_e \gamma_{\overline{e}}^{-1} \in G_e$.

Ein endliches Erzeugendensystem einer Gruppe $G$ definiert eine Längenfunktion $l : G \to \mathbb{N}_0$. Ein Automorphismus $\alpha \in \mathrm{Aut}(G)$ wächst höchstens polynomiell vom Grad $d$, wenn für jedes $x \in G$ die Länge $l(\alpha^j(x))$ von oben durch ein Polynom vom Grad $d$ in $j \geq 0$ beschränkt ist. Gibt es ein $x \in G$, so dass die Länge $l(\alpha^j(x))$ auch von unten

durch ein solches Polynom beschränkt ist, so heißt $\alpha$ polynomiell wachsend vom Grad $d$. Für äußere Automorphismenklassen definieren wir einen ähnlichen Wachstumsbegriff mit Hilfe der zyklischen Länge von Konjugationsklassen. Das Wachstum von $\alpha$ ist polynomiell vom Grad $d$ genau dann, wenn das Wachstum jeder nicht-trivialen Potenz von $\alpha$ polynomiell vom Grad $d$ ist.

Wenn $D$ ein Dehn-Twist auf $\mathcal{G}$ ist, dann wachsen der Automorphismus $D_{*v}$ und seine äußere Automorphismenklasse polynomiell vom Grad 1, also linear. Umgekehrt hat jeder linear wachsende Automorphismus einer freien Gruppe eine Potenz, die durch einen Dehn-Twist gegeben ist. In dieser Arbeit verallgemeinern wir dies für polynomiell wachsende Automorphismen höheren Grades wie folgt.

Ein *höherer* Graph von Gruppen $\mathbb{G}$ ist ein Paar aus einem (gewöhnlichen) Graph von Gruppen $\mathcal{G}$ mit einer Gradfunktion deg, die jeder Kante des unterliegenden Graph $\Gamma$ einen Grad zuordnet. Der Grad $d$ von $\mathbb{G}$ ist der maximale Grad einer Kante. Wir bezeichnen mit $\Gamma^{(m)}$ den Untergraph von $\Gamma$ mit den gleichen Ecken wie $\Gamma$. Eine Kante $e$ von $\Gamma$ gehört zu $\Gamma^{(m)}$ genau dann, wenn ihr Grad höchstens $m$ ist. Durch Einschränken erhalten wir eine Filtrierung

$$\mathbb{G}^{(0)} \subset \mathbb{G}^{(1)} \subset \ldots \subset \mathbb{G}^{(d-1)} \subset \mathbb{G}^{(d)} = \mathbb{G},$$

wobei $\mathbb{G}^{(m)}$ durch Einschränken von $\mathbb{G}$ auf den Teilgraph $\Gamma^{(m)}$ entsteht.

Der wesentliche Unterschied zwischen gewöhnlichen und höheren Graphen von Gruppen liegt in der Definition der Morphismen. In einem Morphismus $H : \mathbb{G} \to \mathbb{G}'$ von höheren Graphen von Gruppen haben wir $\delta_H(e) \in \pi_1(\mathbb{G}^{(\deg(e)-1)})$. Wir erlauben also, dass $\delta_H(e)$ über Kanten mit kleinerem Grad als $e$ läuft. Jeder Morphismus $H : \mathbb{G} \to \mathbb{G}'$ induziert $H^{(m)} : \mathbb{G}^{(m)} \to \mathbb{G}'^{(m)}$ durch Einschränkung.

Ein höherer Dehn-Twist ist ein Automorphismus $D$ von $\mathbb{G}$ mit trivialer Operation auf dem unterliegenden Graph, so dass $D^{(1)}$ ein Dehn-Twist eines gewöhnlichen Graph von Gruppen ist. Höhere Dehn-Twists wachsen polynomiell, wobei der Grad des Polynoms höchstens der Grad des höheren Graph von Gruppen ist. Umgekehrt kann für jeden polynomiell wachsenden Automorphismus eine Potenz durch einen höheren Dehn-Twist beschrieben werden (s. Proposition 4.24 und 4.25). In Theorem 13.21 zeigen wir, dass der Zentralisator jedes höheren Dehn-Twist-Automorphismus die Endlichkeitseigenschaft VF hat.

Im Allgemeinen kann der Wachstumsgrad eines Dehn-Twist-Automorphismus kleiner sein als der Grad des höheren Graph von Gruppen. In den Kapiteln 6 und 7 definieren wir effiziente Dehn-Twists auf gewöhnlichen Graphen von Gruppen und normalisierte höhere Dehn-Twists auf höheren Graphen von Gruppen, bei denen der polynomielle Wachstumsgrad tatsächlich gleich dem maximalen Kantengrad ist. In Kapitel 8 zeigen wir, dass jeder Dehn-Twist-Automorphismus einer freien Gruppe durch einen normalisierten höheren Dehn-Twist repräsentiert werden kann.

Normalisierte höhere Dehn-Twists besitzen die wichtige Eigenschaft, dass jeder mit ihnen kommutierende Automorphismus der Fundamentalgruppe durch einen Automorphismus desselben höheren Graph von Gruppen repräsentiert wird. Auf diese Weise benutzen wir in Kapitel 13 die Automorphismengruppe des unterliegenden höheren

Graph von Gruppen, um Zentralisatoren in $\mathrm{Out}(F_n)$ und $\mathrm{Aut}(F_n)$ zu verstehen und die Endlichkeitseigenschaft VF zu zeigen.

Schließlich erklären wir in Kapitel 14, wie Informationen über Zentralisatoren die Translationslängen in isometrischen CAT(0)-Wirkungen bestimmen. Theorem 14.2 benutzt dabei die Abelianisierung des Zentralisators. Obwohl die Zentralisatoren oft algorithmisch berechenbare endliche Präsentationen haben, ist es schwierig, die Abelianisierung explizit auszurechnen. In Kapitel 15 diskutieren wir Vereinfachungen der Präsentationen im Spezialfall von Zentralisatoren von Rechts-Translationen $\rho_{a,w}$, die ein Basiselement $a$ von $F_n$ auf $aw$ für ein gegebenes Element $w \in F_n$ abbilden und alle anderen Basiselemente fest lassen.

# Contents

# 1 Introduction

## 1.1 Why studying centralisers?

Centralisers show up in many interesting aspects of geometric group theory. They are closely related to the conjugacy problem, one of the most basic algorithmic questions in group theory. For the outer automorphism group $\mathrm{Out}(F_n)$ of the free group $F_n$, a solution of the conjugacy problem has been outlined by Lustig [25].

When we want to classify homomorphisms from any group $G$ to another group $G'$, then centralisers may be an important tool. For a fixed element $g \in G$, a homomorphism $f : G \to G'$ induces a homomorphism $C_G(g) \to C_{G'}(f(g))$ on centralisers. A good understanding of centralisers sometimes gives information to what elements a given $g$ can be mapped by some homomorphism $f : G \to G'$.

Moreover, centralisers in $G$ can force a group element $g$ to have zero translation length in every isometric action on a $\mathrm{CAT}(0)$ space. Information about these translation lengths again gives rise to information about possible group homomorphisms $f : G \to G'$ for two given groups $G$ and $G'$ (cf. [2] for homomorphisms between mapping class groups of surfaces).

There is a construction of classifying spaces $\underline{\underline{E}}(G)$ for the family of virtually cyclic groups in terms of classifying spaces $\underline{E}(G)$ for the family of finite groups. Important groups in this construction are commensurators, which are closely related to centralisers (cf. Section 3 of [15], Section 4 of [23], and Section 2 of [24]).

Apart from the centralisers studied in this thesis, centralisers of elements of finite order are important to construct groups with finiteness property VF which do not admit a finite-type universal proper $G$-space (cf. [20]). Centralisers of finite subgroups in $\mathrm{Aut}(F_n)$ also show up in work by McCool [28] about automorphism groups of finite extensions of free groups.

## 1.2 Outline of this work

The main result of this thesis is Theorem 13.21 showing that certain centralisers in $\mathrm{Out}(F_n)$ or $\mathrm{Aut}(F_n)$ satisfy finiteness property VF, i.e. these centralisers have a finite index subgroup with a finite classifying space.

A graph of groups $\mathcal{G}$ consists of a finite graph $\Gamma$ with basepoint $v$, vertex groups $G_w$ for every vertex $w$ of $\Gamma$, edge groups $G_e$ for every oriented edge $e$ of $\Gamma$, and injective group homomorphisms $f_e : G_e \to G_{\tau(e)}$, where $\tau(e)$ denotes the terminal vertex of $e$. The edge groups are required to satisfy $G_{\overline{e}} = G_e$, where $\overline{e}$ is the edge $e$ with reversed orientation. Thus edge groups can be thought of as being assigned to geometric (or unoriented) edges.

The *fundamental groupoid* $\pi_1(\mathcal{G})$ of $\mathcal{G}$ is the groupoid with objects being the vertices of $\Gamma$. Every edge $e$ defines a morphism $t_e$ from the initial vertex $\iota(e) = \tau(\overline{e})$ to the terminal vertex $\tau(e)$. An element $g$ in the vertex group $G_w$ is a morphism from $w$ to $w$. In general, a morphism in $\pi_1(\mathcal{G})$ is a formal composition of symbols $t_e$ and elements in vertex groups subject to the relations $t_{\overline{e}} = t_e^{-1}$ and $t_e f_e(a) t_e^{-1} = f_{\overline{e}}(a)$ for

$a \in G_e$. We denote the set of morphisms in $\pi_1(\mathcal{G})$ from $v$ to $w$ by $\pi_1(\mathcal{G}, v, w)$. We write $\pi_1(\mathcal{G}, v, v) = \pi_1(\mathcal{G}, v)$ and refer to it as the *fundamental group* of the graph of groups $\mathcal{G}$.

The terminology "fundamental group" is motivated by the following topological situation of a graph of spaces shown in Figure 1. Consider vertex spaces $X_w$ and edge spaces $X_e$ having the given groups $G_w$ and $G_e$ as fundamental groups. Let $X$ be the space obtained from the disjoint union of all $X_w$ and cylinders over the $X_e$ by attaching the end of each cylinder over $X_e$ to $X_{\tau(e)}$ by means of a map inducing $f_e$ on fundamental groups. Then the fundamental group of the topological space $X$ is naturally isomorphic to the (combinatorial) fundamental group of $\mathcal{G}$.

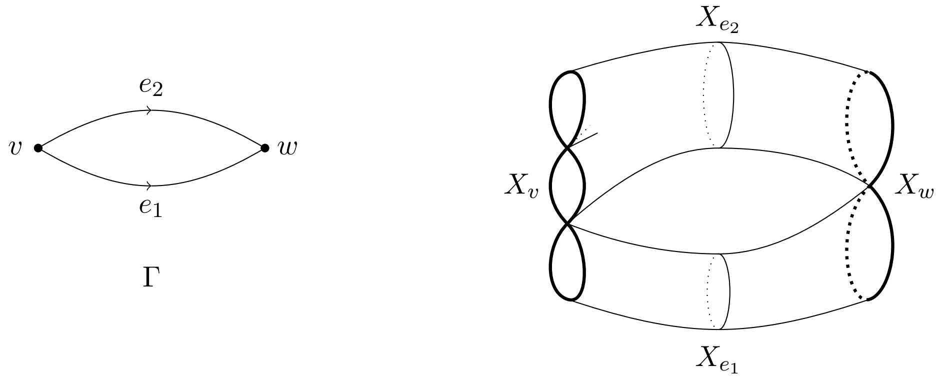


Figure 1: A graph of spaces.

A morphism $H : \mathcal{G} \to \mathcal{G}'$ of graphs of groups is a tuple of a graph morphism $H_\Gamma$, vertex group homomorphisms $H_w : G_w \to G'_{H_\Gamma(w)}$, edge group homomorphisms $H_e : G_e \to G'_{H_\Gamma(e)}$, and elements $\delta_H(e) \in G_{\tau(H_\Gamma(e))}$ satisfying certain compatibility conditions. The morphism $H$ induces a morphism (or natural transformation) $H_* : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}')$. On the generating morphisms it is given by $H_*(g) = H_w(g)$ for $g \in G_w$ and $H_*(t_e) = \delta_H(\overline{e}) t_{H_\Gamma(e)} \delta_H(e)^{-1}$. The morphism $H_*$ of groupoids restricts to a group homomorphism $H_{*v} : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', H_\Gamma(v))$ of fundamental groups.

We denote the automorphism group of $\mathcal{G}$ by $\mathrm{Aut}(\mathcal{G})$, and we denote by $\mathrm{Aut}^0(\mathcal{G})$ the subgroup of all $H \in \mathrm{Aut}(\Gamma)$ such that $H_\Gamma = 1_\Gamma$. A *Dehn twist* on $\mathcal{G}$ is an automorphism $D \in \mathrm{Aut}^0(\mathcal{G})$ such that all $D_w = 1$, $D_e = 1$, and $\delta_D(e) = f_e(\gamma_e)$ for elements $\gamma_e$ in the centre of $G_e$. When all $G_e \cong Z$, and we take ordinary cylinders in the above picture of a graph of spaces, then $D_{*v}$ can be regarded as the automorphism induced on $\pi_1(X)$ by a topological (multiple) Dehn twist around the core curves of the cylinders.

For a better understanding, we now consider the following example. Let $\Gamma$ be a graph with two vertices $v$ and $w$ and two edges $e$ and $\overline{e}$ (determining one geometric edge) with $\iota(e) = v$ and $\tau(e) = w$. For a graph of groups $\mathcal{G}$ with underlying graph $\Gamma$, the fundamental group $\pi_1(\mathcal{G}, v)$ is isomorphic to an amalgamated free product $G_v *_{G_e} G_w$ with respect to $f_{\overline{e}} : G_e \to G_v$ and $f_e : G_e \to G_w$. If $D$ is a Dehn twist on $\mathcal{G}$ given by central elements $\gamma_e$ and $\gamma_{\overline{e}}$ in $G_e$, then $D_{*v}$ corresponds to the automorphism $\alpha$ of $G_v *_{G_e} G_w$ fixing $G_v$ pointwise and acting by $\alpha(g) = z_e g z_e^{-1}$ on $g \in G_w$, where

$z_e = \gamma_e \gamma_{\overline{e}}^{-1} \in G_e$.

A finite generating set of any group $G$ determines a length function $l : G \to \mathbb{N}_0$. An automorphism $\alpha \in \mathrm{Aut}(G)$ grows at most polynomially of degree $d$ if the length $l(\alpha^j(x))$ can be bounded above by a polynomial of degree $d$ in $j$ for $j \geq 0$. The automorphism $\alpha$ grows polynomially of degree $d$, if there is additionally an element $x \in G$ such that $l(\alpha^j(x))$ is bounded below by a polynomial of degree $d$. Using cyclic lengths of conjugacy classes, there is a similar notion of growth for the outer automorphism class $\widehat{\alpha} \in \mathrm{Out}(F_n)$ represented by $\alpha$. Moreover, it is easily verified that the growth of $\alpha$ is polynomial of degree $d$ if and only if the growth of some non-trivial power is polynomial of degree $d$.

When $D$ is a Dehn twist on a graph of groups $\mathcal{G}$, then $D_{*v}$ and its outer automorphism class $\widehat{D}$ grow at most polynomially of degree one, i.e. linearly. Conversely, it is known that every linearly growing automorphism $\alpha \in \mathrm{Aut}(F_n)$ or $\widehat{\alpha} \in \mathrm{Out}(F_n)$ has a power which is represented by a Dehn twist. In this thesis we extend this to polynomially growing automorphisms of higher degree as follows.

A *higher* graph of groups $\mathbb{G}$ is a pair of an (ordinary) graph of groups $\mathcal{G}$ together with a “degree function” deg assigning a positive integer called the degree to each edge of the underlying graph. The degree $d$ of $\mathbb{G}$ is the maximal value of its degree function. A higher graph of groups $\mathbb{G}$ comes with a filtration

$$\mathbb{G}^{(0)} \subset \mathbb{G}^{(1)} \subset \ldots \subset \mathbb{G}^{(d-1)} \subset \mathbb{G}^{(d)} = \mathbb{G},$$

where the underlying graph $\Gamma^{(m)}$ of $\mathbb{G}^{(m)}$ is the subgraph consisting of all vertices of $\Gamma$, but only those edges whose degree is at most $m$. The other structure of $\mathbb{G}^{(m)}$ is obtained from that of $\mathbb{G}$ by restriction.

The main difference between ordinary and higher graphs of groups lies in the definition of morphisms. In a morphism $H : \mathbb{G} \to \mathbb{G}'$ of higher graphs of groups, the $\delta$-terms are not forced to lie in single vertex groups, but $\delta_H(e)$ lies in $\pi_1(\mathbb{G}^{(\deg(e)-1)})$, so it is allowed to go across edges of degree strictly less than $\deg(e)$ in the target graph of groups. Every morphism $H : \mathbb{G} \to \mathbb{G}'$ induces morphisms $H^{(m)} : \mathbb{G}^{(m)} \to \mathbb{G}'^{(m)}$ by restriction.

A *higher Dehn twist* is an automorphism $D$ of $\mathbb{G}$ acting trivially on the underlying graph such that $D^{(1)}$ is an (ordinary) Dehn twist of graphs of groups. Higher Dehn twist automorphisms grow polynomially, and conversely, every polynomially growing automorphism of a finitely generated free group has a power which can be represented by a higher Dehn twist (cf. Proposition 4.24 for $\mathrm{Out}(F_n)$ and Proposition 4.25 for $\mathrm{Aut}(F_n)$). Our main theorem is

**Theorem 13.21.** *Whenever $D$ is a higher Dehn twist on a higher graph of groups $\mathbb{G}$ with finitely generated free fundamental group, then the centralisers $C(D_{*v})$ and $C(\widehat{D})$ satisfy property VF.*

This thesis is structured as follows. In Chapter 2 we make the definitions of higher graphs of groups and the fundamental groupoid more precise. Chapter 3 discusses how to compare higher graphs of groups with ordinary graphs of groups whose vertex groups again have a graph of groups decomposition. Chapter 4 defines several variants of the growth of an automorphism of a finitely generated group. We then discuss that higher Dehn twists always grow polynomially (cf. Proposition 4.21). Given an arbitrary polynomially growing automorphism $\widehat{\alpha} \in \mathrm{Out}(F_n)$, upper triangular relative train track maps in the sense of Bestvina, Feighn, and Handel ([6], [7], and [8]) will allow us to construct a Dehn twist representing a power of $\widehat{\alpha}$.

Given an automorphism $L$ of a higher graph of groups and elements $\eta, \eta'$ in the fundamental group of $\mathbb{G}$, then we call $\eta$ and $\eta'$ $L$-conjugate if there is a $\delta$ such that $\eta' = L_*(\delta)\eta\delta^{-1}$. In particular, 1-conjugate means conjugate in the ordinary sense. To understand the symmetries of these $L$-conjugacy classes, we look at periodicity of the bi-infinite expression

$$\dots L_*^2(\eta)L_*(\eta)\eta L_*^{-1}(\eta)L_*^{-2}(\eta)\dots$$

in Chapter 5.

When $D$ is a higher Dehn twist of a higher graph of groups of degree $d$, then $D_{*v}$ and $\widehat{D}$ grow at most polynomially of degree $d$. In Chapters 6 and 7, we discuss how to bound the growth of automorphisms from below. We will define the class of *normalised* higher Dehn twists in Section 7.4, and we show that they grow indeed polynomially of the maximal possible degree.

Chapter 8 shows that, for every higher Dehn twist $D \in \mathrm{Aut}(\mathbb{G})$ with free fundamental group $\pi_1(\mathbb{G}, v)$, there is a normalised higher Dehn twist $D' \in \mathrm{Aut}(\mathbb{G}')$ representing a conjugate automorphism on fundamental groups. This is done by introducing a list of moves (M1) to (M10) successively improving Dehn twist representatives which are not normalised. In particular, this reduces the study of centralisers of higher Dehn twists to the study of centralisers of normalised higher Dehn twists. This notion also generalises the notion of efficient Dehn twists in [13], which will be the special case of normalised higher Dehn twists in degree one.

The main advantage of studying centralisers of normalised higher Dehn twist automorphisms lies in the fact that every element in the centraliser is indeed represented by an automorphism of the same higher graph of groups. This is shown in Chapter 10, and it allows us to study the centralisers by looking at the structure of the automorphism group of the higher graph of groups $\mathbb{G}$. We will also have to understand which automorphisms of a higher graph of groups act trivially on the fundamental group. We study this in Chapter 9.

In Chapter 11 we discuss subgroups $\mathrm{Aut}(F_n, \mathcal{C})$ of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n, \mathcal{C})$ of $\mathrm{Out}(F_n)$ fixing a given set $\mathcal{C}$ of conjugacy classes in $F_n$. These groups have already been studied by McCool [26], [27], and we recall their basic definitions, which are needed in our description of centralisers in Chapters 12 and 13.

In Chapter 14 we discuss aspects of CAT(0) geometry. The connection to centralisers is given by Theorem 14.2, which gives information about translation lengths of isometric actions of a group on a CAT(0) space. To apply it, we need a good understanding of the abelianisations of centralisers.

We know that many centralisers have finiteness property VF, so they are finitely presented. In [31], it is discussed that these finite presentations can even be computed algorithmically in the case of (linearly growing) ordinary Dehn twist automorphisms. Nevertheless, it is very hard to read off the abelianisation. In Chapter 15 we discuss how this can be done by hand in the special case of a right translation $\rho_{a,w}$. It requires a simplification of the presentations given by McCool's algorithm.

## 1.3 Acknowledgement

The author has benefited very much from his visit at the University of Oxford in 2011. Without this visit, the preprint [31] joint with Ric Wade would never have appeared. From that preprint, some parts have been taken over as a starting point for the present thesis. Therefore the author thanks his PhD advisor Carl-Friedrich Bödigheimer for organising this visit and for giving the author the opportunity to continue the topic to this thesis. Many thanks go also to Martin Bridson, Ulrike Tillmann, and Ric Wade for a lot of mathematical and organisatorial help in Oxford. The author is also grateful for the funding provided by the International Max Planck Research School for Moduli Spaces (IMPRS) in Bonn.

Furthermore, the author thanks Henrik Rüping for proof-reading and Viktoriya Ozornova for helpful comments about the formatting and the figures, which have been drawn by means of the latex package TIKZ.

Moreover, thanks go to Carl-Friedrich Bödigheimer, Martin Bridson, Dieter Degrijse, Giovanni Gandini, Vincent Guirardel, Sebastian Hensel, Arnaud Hilion, Dawid Kielak, Martin Lustig, Sebastian Meinert, and Ric Wade for helpful discussions about the contents and the exposition of this thesis as well as pointing out interesting references to related topics.

# 2 Higher graphs of groups

In this chapter we introduce higher graphs of groups. They generalise the well-known graphs of groups already defined in [3], [4], [13], [32], and others.

## 2.1 Definition of graphs of groups

**Definition 2.1.** A *graph of groups* is a tuple

$$\mathcal{G} = (\Gamma, (G_w)_{w \in V(\Gamma)}, (G_e)_{e \in E(\Gamma)}, (f_e)_{e \in E(\Gamma)})$$

where

- $\Gamma$ is a finite graph in the sense of Serre (cf. I §2.1 in [32]) with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$,
- all $G_w$ and $G_e$ are groups,
- for every edge $e$ we have an injective group homomorphism $f_e : G_e \to G_{\tau(e)}$, where $\tau(e)$ denotes the terminal vertex of $e$,
- $G_e = G_{\overline{e}}$ for every edge $e$, where $\overline{e}$ is the edge $e$ with reversed orientation.

We refer to the $G_w$ and $G_e$ as vertex and edge groups respectively. We call $f_e$ the edge maps or attaching maps.

A *pointed* graph of groups is a pair $(\mathcal{G}, v)$ where $v$ is a vertex of the underlying graph $\Gamma$ of $\mathcal{G}$.

A subset $E^+$ of $E(\Gamma)$ is called *orientation* of $\Gamma$ if, for every $e \in E(\Gamma)$, exactly one of $e$ and $\overline{e}$ belongs to $E^+$.

The initial vertex of an edge $e$ is denoted by $\iota(e) = \tau(\overline{e})$.

## 2.2 Degree functions and the subgraphs $\Gamma^{(m)}$

**Definition 2.2.** A *higher graph of groups* is a pair $\mathbb{G} = (\mathcal{G}, \deg)$ of a graph of groups $\mathcal{G} = (\Gamma, (G_w)_w, (G_e)_e, (f_e)_e)$ together with a function $\deg : E(\Gamma) \to \mathbb{N} \smallsetminus \{0\}$ such that $\deg(\overline{e}) = \deg(e)$, and $G_e$ is trivial whenever $\deg(e) \geq 2$.

We call deg the *degree function*. Its value on an edge $e$ is referred to as the *degree* of the edge, and its maximal value

$$d = \deg(\mathbb{G}) := \max\{\deg(e) | e \in E(\Gamma)\}$$

is called the degree of $\mathbb{G}$. If $E(\Gamma) = \varnothing$, then we define $\deg(\mathbb{G}) = 0$.

For $m \geq 0$ let $\Gamma^{(m)}$ denote the subgraph of $\Gamma$ with $V(\Gamma^{(m)}) = V(\Gamma)$ and $E(\Gamma^{(m)}) = \{e \in E(\Gamma) | \deg(e) \leq m\}$. Let $\mathcal{G}^{(m)}$ be the graph of groups with underlying graph $\Gamma^{(m)}$, and the same vertex groups as in $\mathcal{G}$. The edge groups are those $G_e$ such that $e \in \Gamma^{(m)}$, and for those edges the maps $f_e$ are the same in both $\mathcal{G}$ and $\mathcal{G}^{(m)}$. We denote by $\mathbb{G}^{(m)}$ the higher graph of groups given by $\mathcal{G}^{(m)}$ together with the degree function $\deg|_{E(\Gamma^{(m)})} : E(\Gamma^{(m)}) \to \mathbb{N} \smallsetminus \{0\}$.

## 2.3 The fundamental groupoid $\pi_1(\mathcal{G})$

Let $\mathcal{G}$ be a graph of groups, and $F$ the group freely generated by symbols $t_e$, $e \in E(\Gamma)$. The *path group* $\Pi(\mathcal{G})$ of $\mathcal{G}$ is the quotient of the free product $(*_{w\in V(\Gamma)}G_w) * F$ by the relations

$$t_{\bar{e}} = t_e^{-1},$$
$$t_e f_e(a) t_e^{-1} = f_{\bar{e}}(a)$$

for all $e \in E(\Gamma)$ and $a \in G_e$.

We write elements in $(*_{w\in V(\Gamma)}G_w) * F$ as tuples, which we refer to as *words*. A *connected* word is a word of the form

$$W = (g_0, t_{e_1}, g_1, \ldots, t_{e_{k-1}}, g_{k-1}, t_{e_k}, g_k),$$

where the edges $e_1, \ldots, e_k$ form a connected path, $g_0 \in G_{\iota(e_1)}$, $g_j \in G_{\tau(e_j)}$ for $1 \leq j \leq k$. We often write $t_j$ instead of $t_{e_j}$.

The element in $\Pi(\mathcal{G})$ represented by $W$ is denoted by

$$|W| = g_0 t_1 g_1 \ldots t_k g_k.$$

We denote by $\pi_1(\mathcal{G}, v, w)$ the set of elements in $\Pi(\mathcal{G})$ represented by connected words whose underlying path initiates at $v$ and terminates at $w$. We consider elements in a vertex group $G_w$ as connected words (with $k = 0$ in the above notation), and they represent elements in $\pi_1(\mathcal{G}, w, w)$.

There are obvious concatenation maps

$$\pi_1(\mathcal{G}, u, v) \times \pi_1(\mathcal{G}, v, w) \to \pi_1(\mathcal{G}, u, w),$$

which are clearly associative and have identity elements and inverses. The *fundamental groupoid* $\pi_1(\mathcal{G})$ of $\mathcal{G}$ is the groupoid with object set $V(\Gamma)$ and morphism sets $\pi_1(\mathcal{G}, v, w)$. For simplicity, we write $\pi_1(\mathcal{G}, v)$ for $\pi_1(\mathcal{G}, v, v)$ and refer to it as the *fundamental group* of $\mathcal{G}$ at the basepoint $v$.

*Remark* 2.3. The terminology "fundamental group" is motivated by the following geometric picture of a graph of spaces. For every vertex $w$ of $\Gamma$ we take a space $X_w$ with a fixed isomorphism $\pi_1(X_w) \cong G_w$. If $G_w$ is free, then we may for instance take graphs as vertex spaces such as those drawn by bold lines in Figure 1. Similarly, we take edge spaces $X_e$, which are circles in the example of Figure 1. We define the realisation $X$ of the graph of spaces to be the space obtained by attaching cylinders over the edge spaces to the disjoint union of vertex spaces such that each attaching map $X_e \to X_{\tau(e)}$ induces the given $f_e$ on (topological) fundamental groups. Then the (combinatorial) fundamental group of $\mathcal{G}$ coincides with the (topological) fundamental group of $X$.

For higher graphs of groups $\mathbb{G} = (\mathcal{G}, \deg)$, we define $\Pi(\mathbb{G}) = \Pi(\mathcal{G})$, $\pi_1(\mathbb{G}, v, w) = \pi_1(\mathcal{G}, v, w)$, and $\pi_1(\mathbb{G}, v) = \pi_1(\mathcal{G}, v)$, so we do not take the degree function into account.

Whenever $\Lambda$ is a subgraph of $\Gamma$, we can restrict the structure of the graph of groups $\mathcal{G}$ on $\Gamma$ to $\Lambda$ by disregarding the data outside $\Lambda$. We denote this new graph of groups over

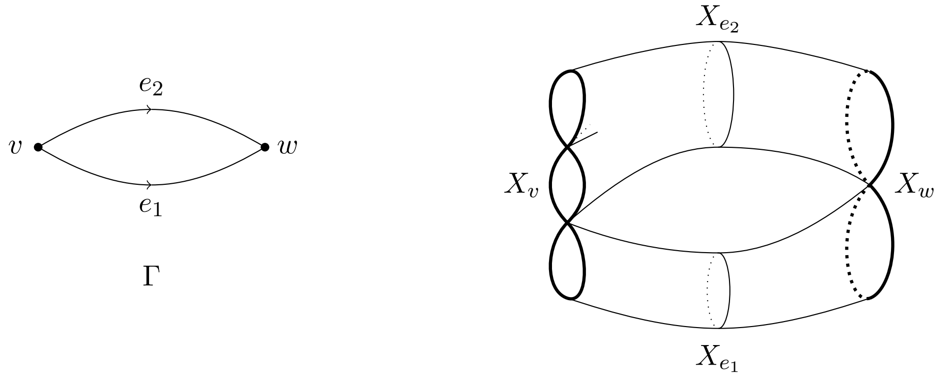


Figure 1: A graph of spaces.

$\Lambda$ by $\mathcal{G}|_\Lambda$. Given two vertices $v$ and $w$ of $\Lambda$, there is an obvious injection $\pi_1(\mathcal{G}|_\Lambda, v, w) \to \pi_1(\mathcal{G}, v, w)$. We usually consider $\pi_1(\mathcal{G}|_\Lambda, v, w)$ as a subset of $\pi_1(\mathcal{G}, v, w)$. In particular, we identify $\pi_1(\mathbb{G}^{(m)}, v, w)$ with a subset of $\pi_1(\mathbb{G}, v, w)$ for a higher graph of groups $\mathbb{G}$.

## 2.4 Morphisms of higher graphs of groups

Let $\mathbb{G}$ be as above and $\mathbb{G}' = (\mathcal{G}', \deg')$, where $\mathcal{G}'$ is a graph of groups with underlying graph $\Gamma'$, vertex groups $G'_w$, edge groups $G'_e$, and edge maps $f'_e$.

**Definition 2.4.** A *morphism* $H : \mathbb{G} \to \mathbb{G}'$ is a tuple

$$H = (H_V, H_E, (H_w)_{w \in V(\Gamma)}, (H_e)_{e \in E(\Gamma)}, (\delta_H(e))_{e \in E(\Gamma)})$$

such that

(1) $H_V : V(\Gamma) \to V(\Gamma')$ and $H_E : E(\Gamma) \to E(\Gamma')$ are functions,

(2) $H_E(\overline{e}) = \overline{H_E(e)}$ for every edge $e$ of $\Gamma$,

(3) $\deg'(H_E(e)) = \deg(e)$ for every edge $e$ of $\Gamma$,

(4) every $H_w : G_w \to G'_{H_V(w)}$ is a group homomorphism,

(5) every $H_e = H_{\overline{e}} : G_e \to G'_{H_E(e)}$ is a group homomorphism,

(6) $\delta_H(e) \in \pi_1(\mathcal{G}'^{(\deg(e)-1)}, H_V(\tau(e)), \tau(H_E(e)))$,

(7) $H_{\tau(e)}(f_e(a)) = \delta_H(e) f'_{H_E(e)}(H_e(a)) \delta_H(e)^{-1}$ for every edge $e \in E(\Gamma)$ with $\deg(e) = 1$ and $a \in G_e$.

We denote the set of morphisms $\mathbb{G} \to \mathbb{G}'$ of higher graphs of groups by $\mathrm{Hom}(\mathbb{G}, \mathbb{G}')$.

If $\deg(e) = 1$, then $\delta_H(e) \in \pi_1(\mathcal{G}'^{(0)}, H_V(\tau(e)), \tau(H_E(e)))$. Since the underlying graph $\Gamma'^{(0)}$ of $\mathcal{G}'^{(0)}$ is discrete, this set is non-empty only if $H_V(\tau(e)) = \tau(H_E(e))$. If all edges of $\Gamma'$ have degree 1, then by (3), all edges of $\Gamma$ have degree 1 as well. In this case, our argument shows that $H_V$ and $H_E$ form a graph morphism $H_\Gamma : \Gamma \to \Gamma'$, and every $\delta_H(e)$ lies in the single vertex group $G'_{\tau(H_\Gamma(e))}$. We then call $\mathrm{Hom}(\mathbb{G}, \mathbb{G}') = \mathrm{Hom}(\mathcal{G}, \mathcal{G}')$ the set of *morphisms of (ordinary) graphs of groups* from $\mathcal{G}$ to $\mathcal{G}'$. This coincides with the morphisms called $\delta\Phi$ in Section 2.9 of [3] and Section 3.4 of [4].

We often write $H$ instead of $H_V$ or $H_E$ when there is no risk of confusion.

We shall sometimes be looking at *pointed* higher graphs of groups $(\mathbb{G}, v)$, where $v$ is a specified base vertex in the underlying graph $\Gamma$ of $\mathbb{G}$. Given two pointed higher graphs of groups $(\mathbb{G}, v)$ and $(\mathbb{G}, v')$, we define the pointed morphism set $\mathrm{Hom}(\mathbb{G}, v, \mathbb{G}', v')$ to be the set of all $H \in \mathrm{Hom}(\mathbb{G}, \mathbb{G}')$ such that $H(v) = v'$.

A morphism $H : \mathbb{G} \to \mathbb{G}'$ induces a map $H_* : \Pi(\mathbb{G}) \to \Pi(\mathbb{G}')$ by

$$H_*(g) = H_w(g) \text{ for } g \in G_w,$$
$$H_*(t_e) = \delta_H(\overline{e}) t_{H(e)} \delta_H(e)^{-1}.$$

It is left to the reader to verify that the defining relators $t_e t_{\overline{e}}$ and $t_e f_e(a) t_e^{-1} f_{\overline{e}}(a)^{-1}$ of $\Pi(\mathbb{G})$ are respected, so $H_*$ is well-defined.

Note that $\delta_H(\overline{e}) \in \pi_1(\mathbb{G}', H(\iota(e)), \iota(H(e)))$, $t_{H(e)} \in \pi_1(\mathbb{G}', \iota(H(e)), \tau(H(e)))$, and $\delta_H(e)^{-1} \in \pi_1(\mathbb{G}', \tau(H(e)), H(\tau(e)))$, so we have $H_*(t_e) \in \pi_1(\mathbb{G}', H(\iota(e)), H(\tau(e)))$. Since we also have $H_*(g) \in G'_{H(w)}$ for $g \in G_w$, it follows that $H_*$ maps the set $\pi_1(\mathbb{G}, v, w)$ represented by connected words to the set $\pi_1(\mathbb{G}', H(v), H(w))$.

We get maps

$$\pi_1 : \mathrm{Hom}(\mathbb{G}, v, \mathbb{G}', v') \to \mathrm{Hom}(\pi_1(\mathbb{G}, v), \pi_1(\mathbb{G}', v'))$$

by sending $H \in \mathrm{Hom}(\mathbb{G}, v, \mathbb{G}', v')$ to the restriction $\pi_1(H) = H_{*v}$ of $H_*$ to the fundamental group $\pi_1(\mathbb{G}, v)$.

## 2.5 The category of higher graphs of groups

Let $\mathbb{G}$, $\mathbb{G}'$, and $\mathbb{G}''$ be three higher graphs of groups. We define a composition

$$\mathrm{Hom}(\mathbb{G}, \mathbb{G}') \times \mathrm{Hom}(\mathbb{G}', \mathbb{G}'') \to \mathrm{Hom}(\mathbb{G}, \mathbb{G}'')$$

as follows: Given two morphisms $H : \mathbb{G} \to \mathbb{G}'$ and $H' : \mathbb{G}' \to \mathbb{G}''$, the *composition* $H'H : \mathbb{G} \to \mathbb{G}''$ is defined by

$$(H'H)_V = H'_V H_V,$$
$$(H'H)_E = H'_E H_E,$$
$$(H'H)_w = H'_{H(w)} H_w,$$
$$(H'H)_e = H'_{H(e)} H_e,$$
$$\delta_{H'H}(e) = H'_*(\delta_H(e)) \delta_{H'}(H_E(e)).$$

This composition is associative. It also has identity elements $\mathbb{1} : \mathbb{G} \to \mathbb{G}$ given by $\mathbb{1}_V = 1$, $\mathbb{1}_E = 1$, $\mathbb{1}_w = 1_{G_w}$, $\mathbb{1}_e = 1_{G_e}$, and $\delta_{\mathbb{1}}(e) = 1$.

A morphism has an inverse if and only if $H_V$ and $H_E$ are bijections, and all $H_w$ and $H_e$ are isomorphisms. In this case we say that $H$ is an *equivalence* of higher graphs of groups, and $\mathbb{G}$ is equivalent to $\mathbb{G}'$. If in addition $\mathbb{G} = \mathbb{G}'$, we refer to $H$ as an *automorphism* of $\mathbb{G}$, and we denote the group of automorphisms by $\mathrm{Aut}(\mathbb{G})$.

The composition map restricts to a composition

$$\mathrm{Hom}(\mathbb{G}, v, \mathbb{G}', v') \times \mathrm{Hom}(\mathbb{G}', v', \mathbb{G}'', v'') \to \mathrm{Hom}(\mathbb{G}, v, \mathbb{G}'', v'')$$

for pointed higher graphs of groups. An isomorphism (or equivalence) in the category of pointed higher graphs of groups is the same as an equivalence of higher graphs of groups which respects the basepoints. This way the automorphism group $\mathrm{Aut}(\mathbb{G}, v)$ becomes the subgroup of $\mathrm{Aut}(\mathbb{G})$ given by automorphisms $H$ such that $H_V(v) = v$.

The forgetful functor $H \mapsto (H_V, H_E)$ induces a group homomorphism

$$\mathrm{Aut}(\mathbb{G}) \to \mathrm{Aut}(V(\Gamma)) \times \mathrm{Aut}(E(\Gamma)),$$

whose kernel we denote by $\mathrm{Aut}^0(\mathbb{G})$. Since $\Gamma$ is finite, $\mathrm{Aut}^0(\mathbb{G})$ has finite index in $\mathrm{Aut}(\mathbb{G})$.

If the degree of $\mathbb{G}$ is 1, then the automorphism group $\mathrm{Aut}(\mathbb{G})$ in the present sense is also denoted by $\mathrm{Aut}(\mathcal{G})$ and referred to as the automorphism group of the ordinary graph of groups $\mathcal{G}$.

## 2.6 Outer homomorphism classes

Let $G$ and $H$ be any groups. Two homomorphisms $f, f' : G \to H$ determine the same *outer homomorphism class* $\widehat{f} = \widehat{f'}$ if there is $h \in H$ such that $f' = \mathrm{ad}_h \circ f$, that is $f'(x) = hf(x)h^{-1}$ for all $x \in G$. We denote the set of equivalence classes of $\mathrm{Hom}(G, H)$ by $O\mathrm{Hom}(G, H)$. Given a further group $K$, there is a well-defined composition

$$O\mathrm{Hom}(G, H) \times O\mathrm{Hom}(H, K) \to O\mathrm{Hom}(G, K).$$

We refer to an outer homomorphism class represented by an isomorphism as an *outer isomorphism class*. The set of outer isomorphisms from $G$ to itself is then the well-known outer automorphism group

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

We now return to higher graphs of groups. Given $H \in \mathrm{Hom}(\mathbb{G}, \mathbb{G}')$ and fixed basepoints $v$ and $v'$ of $\mathbb{G}$ and $\mathbb{G}'$ respectively, we have

$$H_{*v} : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', H(v))$$

with possibly $H(v) \neq v'$. For every $\epsilon \in \pi_1(\mathbb{G}', v', H(v))$, we have $\mathrm{ad}_\epsilon H_{*v} : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', v')$ mapping $\eta \in \pi_1(\mathbb{G}, v)$ to $\epsilon H_*(\eta)\epsilon^{-1}$. For two choices $\epsilon, \epsilon' \in \pi_1(\mathbb{G}', v', H(v))$, the group homomorphisms $\mathrm{ad}_\epsilon H_{*v}$ and $\mathrm{ad}_{\epsilon'} H_{*v}$ differ by the inner automorphism $\mathrm{ad}_{\epsilon'\epsilon^{-1}}$, so we get a well-defined outer homomorphism class

$$\widehat{H} \in O\mathrm{Hom}(\pi_1(\mathbb{G}, v), \pi_1(\mathbb{G}', v'))$$

whenever the underlying graph $\Gamma'$ is connected. It can be checked that $\widehat{1} = 1$ and $\widehat{H'H} = \widehat{H'}\widehat{H}$ whenever the composition $H'H$ is defined.

On the automorphism groups, we now have group homomorphisms $U : \mathrm{Aut}(\mathbb{G}) \to \mathrm{Out}(\pi_1(\mathbb{G}, v))$ and $V : \mathrm{Aut}(\mathbb{G}, v) \to \mathrm{Aut}(\pi_1(\mathbb{G}, v))$ given by $H \mapsto \widehat{H}$ and $H \mapsto H_{*v}$ respectively. They are in general neither injective nor surjective. In Chapter 9 we will see important elements in the kernels of these homomorphisms.

## 2.7 Reduced words

We now introduce some terminology for words in a graph of groups $\mathcal{G}$.

**Definition 2.5.** A word $W = (x, t_1, g_1, \ldots, g_{k-1}, t_k, y)$ is called reduced if $g_j \notin f_{e_j}(G_{e_j})$ whenever $e_{j+1} = \overline{e_j}$ for $j$, $1 \le j \le k-1$.

When there is no risk of confusion, we sometimes say that $xt_1g_1 \ldots t_ky$ is a *reduced expression* although we mean the word $(x, t_1, g_1, \ldots, t_k, y)$ by that.

**Proposition 2.6** ([13], Proposition 3.6)**.** *If two reduced words $W = (g_0, t_1, g_1 \ldots, t_k, g_k)$ and $W' = (g'_0, t'_1, g'_1, \ldots, t'_{k'}, g'_{k'})$ represent the same element of $\Pi(\mathcal{G})$, then:*

- *$k = k'$ and $t_i = t'_i$ for all $i = 1, \ldots, k$,*
- *there are $h_i \in G_{e_i}$ for all $i$, $1 \le i \le k$, such that*

$$\begin{aligned} g'_0 &= g_0 f_{\overline{e_1}}(h_1)^{-1}, \\ g'_i &= f_{e_i}(h_i) g_i f_{\overline{e_{i+1}}}(h_{i+1})^{-1} \text{ for } 1 \le i \le k-1, \\ g'_k &= f_{e_k}(h_k) g_k. \end{aligned}$$

Every word $W$ can be transformed to a reduced word representing the same element in $\Pi(\mathcal{G})$ using the defining relations of this group. During this procedure, the number of $t$-symbols strictly decreases in each step. Connected words are then transformed to connected, reduced words.

From now on, we assume throughout that all words are connected.

**Definition 2.7.** For $\epsilon \in \pi_1(\mathcal{G}, u, w)$ we define the *path length* $pl(\epsilon)$ to be the length $k$ of a reduced word $W = (x, t_1, g_1, \ldots, g_{k-1}, t_k, y)$ such that $|W| = \epsilon$. If $u = w$, then we define the *cyclic path length* as

$$pl_c(\epsilon) = \min\{pl(\delta\epsilon\delta^{-1}) | \delta \in \pi_1(\mathcal{G}, u', u) \text{ for some } u' \in V(\Gamma)\}.$$

Different reduced representatives $W$ and $W'$ of $\epsilon$ have the same underlying path by Proposition 2.6, so $pl(\epsilon)$ is well-defined.

We sometimes write

$$\epsilon * \epsilon'$$

for the concatenation $\epsilon\epsilon'$ to emphasize that $pl(\epsilon\epsilon') = pl(\epsilon) + pl(\epsilon')$. For two words $W = (x, t_1, g_1, \ldots, t_k, y)$ and $W' = (x', t'_1, g'_1, \ldots, t'_{k'}, y')$ with $\tau(e_k) = \iota(e'_1)$, we define

$$W * W' = (x, t_1, g_1, \ldots, t_{k-1}, g_{k-1}, t_k, yx', t'_1, g'_1, \ldots, t'_{k'-1}, g'_{k'-1}, t_{k'}, y').$$

Thus we concatenate $W$ and $W'$ and multiply (only) the entries in $G_{\tau(e_k)} = G_{\iota(e'_1)}$ together. We call $W$ an *initial segment* and $W'$ a *terminal segment* of $W * W'$. The lengths of different terminal segments are related as follows.

**Lemma 2.8.** *Let $W$ and $W'$ be reduced words with $|W| = |W'|$, $V$ a terminal segment of $W$, and $V'$ a terminal segment of $W'$. Then*

$$pl(|V'V^{-1}|) = \big|pl(|V|) - pl(|V'|)\big|.$$

*Proof.* Let $W = (g_0, t_1, g_1, \ldots, t_k, g_k)$ and $W' = (g'_0, t_1, g'_1, \ldots, t_k, g'_k)$. We denote the lengths of the underlying paths of $V$ and $V'$ by $r$ and $r'$ respectively. We may w.l.o.g. assume $r' \geq r$. Then there are $x \in G_{\tau(e_{k-r})}$ and $x' \in G_{\tau(e_{k-r'})}$ such that

$$\begin{aligned} V &= (x, t_{k-r+1}, g_{k-r+1}, \ldots, t_k, g_k), \\ V' &= (x', t_{k-r'+1}, g'_{k-r'+1}, \ldots, t_k, g'_k). \end{aligned}$$

Let $h_1, \ldots, h_k$ be as in Proposition 2.6. Then

$$\begin{aligned} |V'| &= x't_{k-r'+1}\big(f_{e_{k-r'+1}}(h_{k-r'+1})g_{k-r'+1}f_{\overline{e_{k-r'+2}}}(h_{k-r'+2})^{-1}\big)t_{k-r'+2}\ldots t_k(f_{e_k}(h_k)g_k) \\ &= x'f_{\overline{e_{k-r'+1}}}(h_{k-r'+1})t_{k-r'+1}g_{k-r'+1}\ldots t_k g_k, \end{aligned}$$

so $|V'||V|^{-1} = x'f_{\overline{e_{k-r'+1}}}(h_{k-r'+1})t_{k-r'+1}g_{k-r'+1}\ldots t_{k-r}g_{k-r}x^{-1}$ is a reduced expression with underlying path of length $r' - r$. $\square$

# 3 Truncations and tree actions

## 3.1 Truncation of higher graphs of groups

Let $\mathbb{G} = (\mathcal{G}, \deg)$. Recall that the subgraph $\Gamma^{(m)}$ of the underlying graph $\Gamma$ of $\mathcal{G}$ consists of all edges of degree at most $m$. By restriction we obtain a higher graph of groups $\mathbb{G}^{(m)} = (\mathcal{G}^{(m)}, \deg)$. Every automorphism $H$ of $\mathbb{G}$ restricts to an automorphism $H^{(m)} := H|_{\Gamma^{(m)}}$ of $\mathbb{G}^{(m)}$.

**Definition 3.1.** A pointed higher graph of groups $(\mathbb{G}, v)$ is *truncatable at degree $m$* if there is a subset $V_m \subset V(\Gamma)$ such that

- every connected component of $\Gamma^{(m)}$ contains exactly one vertex in $V_m$,
- every edge $e \in E(\Gamma)$ with $\deg(e) > m$ has its terminal vertex $\tau(e) \in V_m$,
- the basepoint $v \in V_m$.

If the underlying graph $\Gamma$ is connected, then the set $V_m$ is unique: Suppose $m < d = \deg(\mathbb{G})$. For every component of $\Gamma^{(m)}$, there is an edge $e$ of degree bigger than $m$ with $\tau(e)$ in this component. This is then the unique vertex in $V_m$ of this connected component of $\Gamma^{(m)}$.

We denote by $\Gamma/\Gamma^{(m)}$ the graph obtained from $\Gamma$ by collapsing each connected component of $\Gamma^{(m)}$ to a vertex. Since $\Gamma^{(m)}$ contains all vertices of $\Gamma$, the vertices of $\Gamma/\Gamma^{(m)}$ are in bijection with the connected components of $\Gamma^{(m)}$, so we may identify $V(\Gamma/\Gamma^{(m)}) = V_m$.

If $\mathbb{G}$ is truncatable at degree $m$, we define its *truncation* $T^m\mathbb{G}$ to be the higher graph of groups with underlying graph $\Gamma/\Gamma^{(m)}$ and vertex groups $\pi_1(\mathbb{G}^{(m)}, w)$, where $w \in V_m = V(\Gamma/\Gamma^{(m)})$. The edge groups of $T^m\mathbb{G}$ are those edge groups $G_e$ of $\mathbb{G}$ such that $\deg(e) \geq m + 1$. The attaching maps are the compositions

$$G_e \xrightarrow{f_e} G_{\tau(e)} \to \pi_1(\mathbb{G}^{(m)}, \tau(e))$$

of the attaching maps of $\mathbb{G}$ and the inclusion of a basepoint vertex group of $\mathbb{G}^{(m)}$ to the fundamental group. Finally, we define a new degree function $\deg_{T^m\mathbb{G}} : E(\Gamma/\Gamma^{(m)}) \to \mathbb{N}$ by $\deg_{T^m\mathbb{G}}(e) = \deg_{\mathbb{G}}(e) - m$.

If $\mathbb{G}$ is a graph of groups of degree $d$, then $T^m\mathbb{G}$ is a graph of groups of degree $d - m$, and there is an obvious identification

$$\pi_1(\mathbb{G}, v) \cong \pi_1(T^m\mathbb{G}, v).$$

Figure 2 shows the graphs of spaces according to two higher graphs of groups $\mathbb{G}$ and $\mathbb{G}'$. The cylinders are the vertex spaces together with the edge spaces in degree one. In both $\mathbb{G}$ and $\mathbb{G}'$, there are two edges of degree two. $\mathbb{G}$ is not truncatable at degree 1, but $\mathbb{G}'$ is.

Given a morphism $H : \mathbb{G} \to \mathbb{G}'$ of higher graphs of groups truncatable at degree $m$, there is an induced morphism $T^mH : T^m\mathbb{G} \to T^m\mathbb{G}'$. This assignment is functorial in

Figure 2: $\mathbb{G}$ is not truncatable, but $\mathbb{G}'$ is.

the sense that $T^m 1 = 1$ and $T^m(H'H) = (T^m H')(T^m H)$. Hence $T^m H$ is an equivalence (or automorphism respectively) if $H$ is.

We are often interested in higher graphs of groups whose truncations can be defined in every degree:

**Definition 3.2.** A higher graph of groups $(\mathbb{G}, v)$ of degree $d$ is called *fully truncatable* if it is truncatable at every degree $m$ with $1 \leq m \leq d-1$.

A fully truncatable higher graph of groups $\mathbb{G}$ of degree $d \geq 2$ may be regarded as an ordinary graph of groups with trivial edge groups such that all its vertex groups have the structure of a fully truncatable higher graph of groups of degree $d-1$. This may be called a "graph of graph of ... graph of groups".

## 3.2 Truncation of words

Given a word $W = (x, t_1, g_1, \dots, t_{k-1}, g_{k-1}, t_k, y)$ representing an element in $\pi_1(\mathbb{G})$ in a higher graph of groups of degree $d$, we define a truncation $T^{d-1}W$ as follows. Let $E_1 = e_{i_1}, \dots, E_l = e_{i_l}$ be the edges of degree $d$ among $e_1, \dots, e_k$ such that $1 \leq i_1 < \dots < i_l \leq k$. Write

$$\begin{aligned}
\theta_0 &= x t_1 g_1 \dots t_{i_1-1} g_{i_1-1}, \\
\theta_j &= g_{i_j} t_{i_j+1} g_{i_j+1} \dots t_{i_{j+1}-1} g_{i_{j+1}-1} \text{ for } 1 \leq j \leq l-1, \\
\theta_l &= g_{i_l} t_{i_l+1} g_{i_l+1} \dots t_k y.
\end{aligned}$$

Then we define

$$T^{d-1}W = (\theta_0, t_{E_1}, \theta_1, \dots, t_{E_l}, \theta_l).$$

If $W$ is reduced, then $T^{d-1}W$ is reduced in the following sense: Whenever $E_{i+1} = \overline{E_i}$ for some $i$, then $\theta_i \neq 1$. We say that $T^{d-1}W$ is *reduced in the truncated sense*.

If $\mathbb{G}$ is truncatable at degree $d-1$, then every $\theta_j$ is a (closed) element in the fundamental group of a connected component of $\mathbb{G}^{(d-1)}$. It can be checked that $|W| = |T^{d-1}W|$ under the canonical identification of $\pi_1(\mathbb{G})$ with $\pi_1(T^{d-1}\mathbb{G})$.

We note that a truncated reduced word is uniquely determined by the element it represents if $d \geq 2$. This follows from Proposition 2.6 together with the fact that edge groups of degree $d$ are trivial.

### 3.3 Edge slide equivalences

To study automorphisms of higher graphs of groups and their centralisers, it will be convenient to pass to truncations as defined in the last section. To build them, the higher graph of groups must be truncatable. This turns out to be achieved easily: Proposition 3.3 will show that every higher graph of groups is in fact equivalent to a fully truncatable one. The basic piece of building such an equivalence will be the following construction.

Let $\mathbb{G}$ be a higher graph of groups with chosen edge $e$ such that $G_e = 1$. Define the underlying graph $\Gamma'$ for a new higher graph of groups $\mathbb{G}'$ as follows: The vertices and edges of $\Gamma'$ are the same as those of $\Gamma$ with $e$ and $\overline{e}$ replaced by new $e'$ and $\overline{e'}$. We set $\deg(e') = \deg(e)$, $\iota(e') = \iota(e)$, but $\tau(e')$ is only required to lie in the same component of $\Gamma^{(\deg(e)-1)} = \Gamma'^{(\deg(e)-1)}$ as $\tau(e)$. We put $G'_{e'} = 1$, and all other edge groups, vertex groups, and attaching maps for $\mathbb{G}'$ are the same as those for $\mathbb{G}$.

Let $\delta \in \pi_1(\mathbb{G}^{(\deg(e)-1)}, \tau(e), \tau(e'))$. There is a morphism $H : \mathbb{G} \to \mathbb{G}'$ defined as follows: On the underlying graph, $e$ and $\overline{e}$ are mapped to $e'$ and $\overline{e'}$ respectively, whereas all vertices and other edges of $\Gamma$ are mapped to those of $\Gamma'$ called by the same name. Define all vertex and edge group homomorphisms $H_w$ and $H_{\tilde{e}}$ to be the identity on the respective group. Let $\delta_H(e) = \delta$ and $\delta_H(\tilde{e}) = 1$ for $\tilde{e} \neq e$. This finishes the definition of $H$, which is clearly an equivalence.

### 3.4 Building truncatable representatives

**Proposition 3.3.** *Let $(\mathbb{G}, v)$ be a pointed higher graph of groups. Then there is an equivalence $(\mathbb{G}, v) \to (\mathbb{G}', v')$ such that $(\mathbb{G}', v')$ is fully truncatable.*

*Proof.* Choose a filtration $V(\Gamma) = V_0 \supset V_1 \supset \ldots \supset V_{d-1} \supset V_d = \{v\}$ such that every connected component of $\Gamma^{(m)}$ contains exactly one vertex of $V_m$.

The proof will now be by induction on the number of (oriented) edges $e$ such that $\tau(e) \notin V_{\deg(e)-1}$. We refer to these edges as unfitting edges. If there is no unfitting edge, then $\mathbb{G}$ is truncatable at every degree, and we take $\mathbb{G}' = \mathbb{G}$.

Fix now an unfitting edge $e$, and define $\mathbb{G}'$ as in Section 3.3 with $\tau(e')$ being the unique vertex in $V_{\deg(e)-1}$ lying in the same component of $\Gamma^{(\deg(e)-1)}$ as $\tau(e)$. Then we construct an equivalence from $\mathbb{G}'$ to $\mathbb{G}$ as in Section 3.3, and $\mathbb{G}'$ has fewer unfitting edges. This finishes the induction. □

We now use this to detect equivalences among morphisms inducing an isomorphism on fundamental groups.

**Lemma 3.4.** *If $H : \mathbb{G} \to \mathbb{G}'$ is a morphism of connected higher graphs of groups such that $H_V$, $H_E$, all $H_e$, and $H_{*v}$ are isomorphisms, then $H$ is an equivalence.*

*Proof.* Let $d$ be the degree of $\mathbb{G}$. As we may pre- and postcompose $H$ with equivalences of higher graphs of groups, there is no loss of generality in assuming that $\mathbb{G}$ and $\mathbb{G}'$ are fully truncatable.

If $d \geq 2$ and we know the assertion for higher graphs of groups of degree at most $d-1$, the truncation $T^{d-1}H : T^{d-1}\mathbb{G} \to T^{d-1}\mathbb{G}'$ and the restriction $H^{(d-1)} = H|_{\Gamma^{(d-1)}} : \mathbb{G}^{(d-1)} \to \mathbb{G}'^{(d-1)}$ are equivalences on each connected component, and so $H$ is an equivalence. Therefore we assume $d = 1$.

We have to show that all vertex group homomorphisms $H_w : G_w \to G'_{H(w)}$ are isomorphisms. Let $w$ be any vertex of $\Gamma$. For an arbitrary $\epsilon \in \pi_1(\mathcal{G}, v, w)$, we have $H_{*w} = \mathrm{ad}_{H_*(\epsilon)}^{-1} H_{*v} \mathrm{ad}_\epsilon$. As $H_{*v}$ is an isomorphism, it follows that $H_{*w}$ is an isomorphism as well. Therefore $H_w = H_{*w}|_{G_w}$ is injective.

To show surjectivity of $H_w$, pick $g' \in G'_{H(w)}$. We now choose a reduced expression $(H_{*w})^{-1}(g') = g_0 t_1 g_1 \dots t_k g_k \in \pi_1(\mathbb{G}, w)$. By Lemma 3.5 of [4], we have $pl(g') = pl((H_*)^{-1}(g'))$, so $k = 0$. This shows that $(H_{*w})^{-1}(g') = g$ for some $g \in G_w$. But then $H_w(g) = g'$, so $H_w$ is surjective. □

### 3.5 Bass-Serre Trees

Let $\mathcal{G}$ be an ordinary graph of groups with basepoint $v$. We now define the Bass-Serre tree $T = \widetilde{(\mathcal{G}, v)}$ of $\mathcal{G}$ is the following graph, which is called the universal cover in [3]: A vertex of $T$ is a coset of the form $\delta G_w$, where $\delta \in \pi_1(\mathcal{G}, v, w)$. Two vertices $\delta G_w$ and $\delta' G_{w'}$ are connected by an edge if the path length $pl(\delta^{-1}\delta') = 1$, which is clearly independent of the coset representatives $\delta$ and $\delta'$. In general, the distance of $\delta G_w$ and $\delta' G_{w'}$ is $d_T(\delta G_w, \delta' G_{w'}) = pl(\delta^{-1}\delta')$. It is not hard to check that this graph is indeed a tree (cf. Theorem 1.17 of [3]).

The fundamental group $\pi_1(\mathcal{G}, u)$ acts on this tree, where the action of $\eta \in \pi_1(\mathcal{G}, v)$ is given by mapping $\delta G_w$ to $\eta\delta G_w$. The quotient of this action can be identified with the underlying graph $\Gamma$ of $\mathcal{G}$. The "translation lengths" in these action are related to path lengths as follows.

**Lemma 3.5.** *Let $\eta \in \pi_1(\mathcal{G}, v)$ and $T = \widetilde{(\mathcal{G}, v)}$ with distance function $d_T$ on the vertex set $V(T)$. Then*

$$\begin{aligned} pl(\eta) &= d_T(G_v, \eta G_v), \\ pl_c(\eta) &= \min_{\delta G_w \in V(T)} d_T(\delta G_w, \eta\delta G_w). \end{aligned}$$ □

A morphism $H : \mathcal{G} \to \mathcal{G}'$ induces an equivariant map of the Bass-Serre trees. If $H$ respects the basepoints, then the associated map on Bass-Serre trees maps the vertex $G_v$ to the vertex $G'_{v'}$.

When $\mathcal{G}$ carries a degree function, then this function can be naturally lifted to the edges of the Bass Serre tree. A morphism $H : \mathbb{G} \to \mathbb{G}'$ then induces a (topological) map on Bass-Serre trees sending vertices to vertices and each edge of degree $m$ across exactly one edge of degree $m$ and possibly edges of lower degree.

*Remark* 3.6. Collapsing each edge of degree at most a fixed $m$ in the Bass Serre tree corresponds to taking the Bass-Serre tree of the truncation $T^m\mathbb{G}$, if it exists. Even if $\mathbb{G}$ is not truncatable of degree $m$, we will sometimes abuse the notation and refer to the Bass-Serre tree of the truncation $T^m\mathbb{G}$ when we mean the tree for $\mathbb{G}$ with edges of degree at most $m$ collapsed. The vertices of this tree then correspond to elements $\delta \in \pi_1(\mathbb{G}, v, u)$ subject to the equivalence relation identifying $\delta \in \pi_1(\mathbb{G}, v, u)$ with $\delta' \in \pi_1(\mathbb{G}, v, u')$ if and only if $\delta^{-1}\delta' \in \pi_1(\mathbb{G}^{(m)}, u, u')$. We sometimes denote such an equivalence class by $\delta\pi_1(\mathbb{G}^{(m)}, u, \bullet)$ or simply $\delta\pi_1(\mathbb{G}^{(m)})$.

*Remark* 3.7. When $H$ is a normalised higher Dehn twist $D \in \mathrm{Aut}^0(\mathbb{G})$ on a higher graph of groups $\mathbb{G}$ of degree $d$ as defined in Section 7.4 below, then the Bass-Serre tree $T$ of the truncation $T^{d-1}\mathbb{G}$ has the following interpretation. When we view Outer space as the space of isometric actions on metric simplicial trees endowed with the topology given by length functions, then there is a boundary given by length functions which are allowed to attain the value zero on non-trivial group elements. The action of $D$ on $T$ extends to this compactification, and it has limit points at the boundary. They correspond to equivariant metrics on the Bass-Serre tree $T$ of $T^{d-1}\mathbb{G}$. In the case of ordinary graphs of groups in degree one, Cohen and Lustig [12] have studied this in terms of metric graphs of groups.

# 4 Growth types

## 4.1 Definition of growth in $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$

Let $G$ be any group with finite generating set $\{x_1, \ldots, x_n\}$. We denote by $l_x$ its length function, that is

$$l_x(g) := \min\{k \geq 0 | g = x_{j_1}^{\epsilon_1} \ldots x_{j_k}^{\epsilon_k} \text{ for some } j_i \text{ and } \epsilon_i \in \{\pm 1\}\}. \tag{1}$$

For conjugacy classes we have the cyclic length

$$l_{x,c}(g) := \min\{l_x(ygy^{-1}) | y \in G\}.$$

If $\alpha \in \mathrm{Aut}(G)$ and $g \in G$, we define

$$\mathrm{gr}(\alpha, g) : \mathbb{N}_0 \to \mathbb{N}_0, \qquad \mathrm{gr}(\alpha, g)(j) = l_x(\alpha^j(g)).$$

For the cyclic length we similarly have

$$\mathrm{gr}_c(\alpha, g) : \mathbb{N}_0 \to \mathbb{N}_0, \qquad \mathrm{gr}_c(\alpha, g)(j) = l_{x,c}(\alpha^j(g)),$$

which also makes sense for $\widehat{\alpha} \in \mathrm{Out}(G)$.

Considering two sets of generators $\{x_1, \ldots, x_n\}$ and $\{x'_1, \ldots, x'_{n'}\}$ of $G$, there is a constant $C > 0$ such that

$$\frac{1}{C} l_x(g) \leq l_{x'}(g) \leq C l_x(g)$$

for all $g \in G$. We also say $l_x$ and $l_{x'}$ are equivalent up to a multiplicative constant. Hence $\mathrm{gr}_x(\alpha, g)$ and $\mathrm{gr}_{x'}(\alpha, g)$ can be mutually estimated up to a multiplicative constant. The same argument applies to the cyclic growth functions $\mathrm{gr}_{x,c}$ and $\mathrm{gr}_{x',c}$. Thus the following definition is independent of the generating set of $G$.

**Definition 4.1.** A group element $g \in G$ *grows at most polynomially of degree $d$* under iteration of $\alpha \in \mathrm{Aut}(G)$ if $\mathrm{gr}(\alpha, g)$ is bounded above by a polynomial of degree $d$. The conjugacy class $[g]$ grows at most polynomially of degree $d$ under iteration of $\alpha \in \mathrm{Aut}(G)$ (or $\widehat{\alpha} \in \mathrm{Out}(G)$) if $\mathrm{gr}_c(\alpha, g)$ is bounded above by a polynomial of degree $d$.

$g$ (or $[g]$) grows *at least polynomially* of degree $d$ if $\mathrm{gr}(\alpha, g)$ (or $\mathrm{gr}_c(\widehat{\alpha}, g)$ respectively) is bounded below by a polynomial of degree $d$ with positive leading coefficient. It grows polynomially of degree $d$ if it grows both at most and at least polynomially of degree $d$.

$\alpha \in \mathrm{Aut}(G)$ is called polynomially growing of degree $d$, if every $g \in G$ grows at most polynomially of degree $d$ under iteration of $\phi$, and there is a $g$ growing polynomially of degree $d$. We similarly define polynomial growth for $\widehat{\alpha} \in \mathrm{Out}(G)$ using the growth of conjugacy classes.

We say that a function $\mathbb{N}_0 \to \mathbb{N}_0$ grows polynomially of degree zero if it is bounded. This does *not* agree with the above definition if it assumes the value 0 infinitely many times. However, this turns out to be more practical for later purposes. Often we can mutually estimate two growth functions up to an additive constant, and this new definition of polynomial growth of degree zero will then be invariant under addition of a bounded function.

**Lemma 4.2.** *Let $g \in G$. If $\alpha, \beta \in \operatorname{Aut}(G)$ commute, then $\operatorname{gr}(\alpha, g)$ and $\operatorname{gr}(\alpha, \beta(g))$ are equivalent. Likewise, if $\widehat{\alpha}, \widehat{\beta} \in \operatorname{Out}(G)$ commute, then $\operatorname{gr}_c(\widehat{\alpha}, [g])$ and $\operatorname{gr}_c(\widehat{\alpha}, \widehat{\beta}([g]))$ are equivalent up to a multiplicative constant.* □

We sometimes refer to the present notion of length as *basis length*. We warn the reader that this is *not* equivalent to the notion of path length introduced in Definition 2.7.

## 4.2 Compatible groupoid generating sets

When we study basis lengths in the fundamental group $\pi_1(\mathcal{G}, v)$ of a graph of groups, the following type of generating set will be convenient: We take a generating set consisting of finite generating sets for all vertex groups $G_w$ together with the set of all symbols $t_e$ for the edges $e \in E(\Gamma)$. We call such a generating set *compatible*.

However, the generators are not elements of the fundamental group $\pi_1(\mathcal{G}, v)$, but rather in the fundamental groupoid, that is their underlying paths are not necessarily closed loops at the basepoint. The definition of the basis length in (1) on page 26 also makes sense for such groupoid generating sets when we only allow compositions defined in the groupoid, i.e. those such that the terminal vertex of one factor coincides with the initial vertex of the next factor.

It therefore makes sense to define the growth type of an element $\epsilon \in \pi_1(\mathcal{G}, u, w)$, and, up to an estimate by a multiplicative constant, it does not depend on the (finite) groupoid generating set.

In particular, when we choose a generating set consisting of a finite *group* generating set for $\pi_1(\mathcal{G}, v)$ together with one element $\epsilon \in \pi_1(\mathcal{G}, v, w)$ for every vertex $w \neq v$, then it is clear that the two notions of basis length in $\pi_1(\mathcal{G}, v)$ in the group and the groupoid setting coincide (up to a multiplicative constant). Indeed, the only composable expressions in the groupoid $\pi_1(\mathcal{G})$ giving rise to an element in $\pi_1(\mathcal{G}, v)$ consist of generators in $\pi(\mathcal{G}, v)$ only.

## 4.3 Basis lengths of cosets

Let $\delta \in \pi_1(\mathcal{G}, u, w)$, and let $e$ and $e'$ be edges with $\tau(e) = u$ and $\iota(e') = w$. Then we define the *basis length of the double coset* $f_e(G_e)\delta f_{\overline{e'}}(G_{e'})$ to be the minimum of the basis lengths of its representatives:

$$l(f_e(G_e)\delta f_{\overline{e'}}(G_{e'})) = \min\left\{l(f_e(h)\delta f_{\overline{e'}}(h'))\middle| h \in G_e, h' \in G_{e'}\right\}.$$

We now define the basis length $l$ on $\pi_1(\mathcal{G})$ using a compatible finite groupoid generating set as in Section 4.2. If $e$, $e'$, and $e''$ are three edges as well as $\delta \in \pi_1(\mathcal{G}, \tau(e), \iota(e'))$ and $\delta' \in \pi_1(\mathcal{G}, \tau(e'), \iota(e'')))$, then it is checked easily that

$$l(f_e(G_e)(\delta * t_{e'} * \delta')f_{\overline{e''}}(G_{e''})) \geq l(f_e(G_e)\delta f_{\overline{e'}}(G_{e'})) + l(f_{e'}(G'_e)\delta' f_{\overline{e''}}(G_{e''})) + 1,$$

where $*$ denotes that the direct concatenation of reduced words for the elements will be reduced.

The following lemma follows directly from the definitions.

**Lemma 4.3.** *Let $\mathcal{G}$ be a graph of groups with finitely generated vertex groups and a compatible groupoid generating set for $\pi_1(\mathcal{G})$ such that the length of $f_{\overline{e}}(a)$ in $G_{\iota(e)}$ is not more than two bigger than the length of $f_e(a)$ in $G_{\tau(e)}$ whenever $a \in G_e$. Then:*

*(i) If $\delta = g_0 t_1 g_1 \dots t_k g_k$ is a reduced expression with $k \geq 1$, then*

$$l(\delta) = \min_{h_\bullet} \Big( k + l(g_0 f_{\overline{e_1}}(h_1)^{-1}) + \sum_{j=1}^{k-1} l(f_{e_j}(h_j) g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}) + l(f_{e_k}(h_k) g_k) \Big),$$

*where the minimum denoted $\min_{h_\bullet}$ is taken over all tuples $(h_1, \dots, h_k)$ such that $h_j \in G_{e_j}$.*

*(ii) If $\eta = t_1 g_1 \dots t_k g_k$ is a cyclically reduced expression, then*

$$l_c(\eta) = \min_{h_\bullet} \Big( k + \sum_{j=1}^{k} l(f_{e_j}(h_j) g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}) \Big),$$

*where the minimum denoted $\min_{h_\bullet}$ is taken over all tuples $(h_1, \dots, h_{k+1})$ with $h_j \in G_{e_j}$ and $h_{k+1} = h_1$.*

*(iii) If $(t_0, g_0, t_1, g_1, \dots, t_k, g_k, t_{k+1})$ is a reduced word and $\delta = g_0 t_1 g_1 \dots t_k g_k$, then*

$$l(f_{e_0}(G_{e_0}) \delta f_{\overline{e_{k+1}}}(G_{e_{k+1}})) = \min_{h_\bullet} \Big( k + \sum_{j=0}^{k} l(f_{e_j}(h_j) g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}) \Big),$$

*where the minimum denoted $\min_{h_\bullet}$ is taken over all tuples $(h_0, \dots, h_{k+1})$ with $h_j \in G_{e_j}$.* □

## 4.4 Cyclic and twisted reduction

**Definition 4.4.** For given $L \in \mathrm{Aut}^0(\mathcal{G})$, a word $W = (x, t_1, g_1, \dots, g_{k-1}, t_k, y)$ is called *$L$-twistedly reduced* if it is reduced, and at least one of the following holds true:

- $k = 0$, or
- $e_k \neq \overline{e_1}$, or
- $\delta_L(\overline{e_1})^{-1} L_*(y) x \notin f_{\overline{e_1}}(G_{e_1})$.

In the case $L = 1$, we say that $W$ is *cyclically reduced.* The last bullet point then simplifies to $yx \notin f_{\overline{e_1}}(G_{e_1})$.

The following characterisation of $L$-twistedly reduced words follows immediately from the definition:

**Lemma 4.5.** *Let $W$ be a closed reduced word. Let $Z$ be a reduced word such that $|Z| = L_*(|W|)$. Then $W$ is $L$-twistedly reduced if and only if $Z * W$ is reduced.* □

Given $\eta \in \pi_1(\mathcal{G}, u)$, a reduced word $W$ representing $\eta$ is $L$-twistedly reduced if and only if the path length of $L_*(\eta)\eta$ is exactly twice the path length of $\eta$. Since this is independent of the representative $W$ of $\eta$, the following definition makes sense:

**Definition 4.6.** An element $\eta \in \pi_1(\mathcal{G}, u)$ is called *$L$-twistedly reduced* if some (or equivalently every) reduced word representing $\eta$ is $L$-twistedly reduced. If $L = 1$, then we also call $\eta$ cyclically reduced.

We often study words of the form

$$W = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_{k-1}, g_{k-1}, t_k, g_k) \tag{2}$$

with $t_j := t_{e_j}$, where $k \geq 1$. If the right hand side is reduced, then it is $L$-twistedly reduced if and only if $e_k \neq \overline{e_1}$ or $\delta_L(\overline{e_1})^{-1} L_*(g_k) \delta_L(\overline{e_1}) \notin f_{\overline{e_1}}(G_{e_1})$. Using Definition 2.4(7), this is equivalent to saying that $(t_k, g_k, t_1)$ is reduced.

**Definition 4.7.** Two elements $\eta \in \pi_1(\mathcal{G}, u)$ and $\eta' \in \pi_1(\mathcal{G}, u')$ are called *$L$-conjugate* if there is an $\epsilon \in \pi_1(\mathcal{G}, u', u)$ such that $\eta = L_*(\epsilon)\eta'\epsilon^{-1}$.

Let us now introduce an action of a morphism $L : \mathcal{G} \to \mathcal{G}'$ on *words* rather than elements in the fundamental groupoid of $\mathcal{G}$. For $W = (x, t_{e_1}, g_1, \ldots, t_{e_{k-1}}, g_{k-1}, t_{e_k}, y)$ we define

$$L_*(W) = (\tilde{x}, t_{L(e_1)}, \tilde{g}_1, \ldots, t_{L(e_{k-1})}, \tilde{g}_{k-1}, t_{L(e_k)}, \tilde{y}),$$

where

$$\begin{aligned}
\tilde{x} &= L_{\iota(e_1)}(x)\delta_L(\overline{e_1}),\\
\tilde{g}_j &= \delta_L(e_j)^{-1} L_{\tau(e_j)}(g_j)\delta_L(\overline{e_{j+1}}) \text{ for } 1 \leq j \leq k-1,\\
\tilde{y} &= \delta_L(e_k)^{-1} L_{\tau(e_k)}(y).
\end{aligned}$$

Moreover, we write

$$W^{-1} = (y^{-1}, t_{\overline{e_k}}, g_{k-1}^{-1}, t_{\overline{e_{k-1}}}, \ldots, g_1^{-1}, t_{\overline{e_1}}, x^{-1}).$$

We clearly have $|L_*(W)| = L_*(|W|)$ and $|W^{-1}| = |W|^{-1}$.

Returning to $L \in \mathrm{Aut}^0(\mathcal{G})$, the following proposition shows that every closed element in the fundamental groupoid of $\mathcal{G}$ is $L$-conjugate to an $L$-twistedly reduced one in the form (2) or to an element in a single vertex group. The special case $L = 1$ is cyclic reduction within an ordinary conjugacy class, and it corresponds to Lemma 2.7 in [4].

**Proposition 4.8.** *Let $W = (x, t_1, g_1, \ldots, t_{k-1}, g_{k-1}, t_k, y)$ be a reduced word from $v$ to $v$. Then:*

*(i) There are reduced words $V$, $W'$, and $V'$ such that*

- $W = V * W' * V'$,
- $|V| = L_*(|V'|)^{-1}$,

- *$W'$ is $L$-twistedly reduced,*
- *if $k' := pl(|W'|) \geq 1$, then $W' = (\delta_L(\overline{e'_1}), t'_1, g'_1, \ldots, t'_{k'}, g'_{k'})$ for some edges $e'_1, \ldots, e'_{k'}$ and some elements $g'_j \in G_{\tau(e'_j)}$.*

*(ii) $k'$ is unique, and the decomposition is unique in the case that $k' \geq 1$.*

*(iii) The word $W$ is $L$-twistedly reduced if and only if $pl(|V|) = 0$ for some (or equivalently every) such decomposition.*

*(iv) If $\tilde{W}$ is another reduced word such that $|\tilde{W}| = |W|$ and $\tilde{W} = \tilde{V} * \tilde{W}' * \tilde{V}'$ as in (i), then*

$$\begin{aligned}|\tilde{V}| &= |V| L_*(z)^{-1}, \\ |\tilde{W}'| &= L_*(z)|W'|z^{-1}, \\ |\tilde{V}'| &= z|V'|\end{aligned}$$

*for some $z \in G_u$, where $u$ is the initial vertex of the underlying paths of $\tilde{W}'$ and $W'$.*

*Proof.* Given any decomposition as in (i), the element $L_*(|W|)|W|$ is represented by the reduced word $L_*(V) * L_*(W') * W' * V'$. Hence

$$\begin{aligned}pl(L_*(|W|)|W|) &= pl(|V|) + 2pl(|W'|) + pl(|V'|) \\ &= pl(|W|) + pl(|W'|), \end{aligned} \tag{3}$$

so the number $k'$ is uniquely defined by $W$. By Lemma 4.5 we see that $W$ is $L$-twistedly reduced if and only if $pl(|V|) = 0$ and $k' = k$, so we see (iii).

We now show (i) and (ii). Suppose first that $W$ is $L$-twistedly reduced. If $k = pl(|W|) = 0$, the choice $V = V' = 1$ and $W' = W$ works. If $k \geq 1$, then we have to achieve that the leftmost entry of the word is $\delta_L(\overline{e_1})$. This is the case if and only if we choose $V = x\delta_L(\overline{e_1})^{-1}$, and $V' = L_*^{-1}(\delta_L(\overline{e_1})x^{-1})$. This is the desired uniqueness in (ii).

The proof will now work by induction on $k$, and we can assume that $W$ is not $L$-twistedly reduced. By definition this means that $k \geq 1$, $e_k = \overline{e_1}$, and $\delta_L(\overline{e_1})^{-1} L_*(y)x = f_{\overline{e_1}}(h)$ for some $h \in G_{e_1}$. We have a reduced decomposition $W = U * W'' * U'$, where

$$\begin{aligned}U &= (x, t_1, f_{e_1}(h)^{-1}\delta_L(e_1)^{-1}), \\ U' &= (t_k, y), \\ W'' &= (\delta_L(e_1) f_{e_1}(h) g_1, t_2, g_2, \ldots, t_{k-1}, g_{k-1}).\end{aligned}$$

These words also satisfy

$$\begin{aligned}L_*(|U'|)^{-1} &= L_*(y^{-1}t_k^{-1}) = L_*(y)^{-1}\delta_L(\overline{e_1})t_1\delta_L(e_1)^{-1} \\ &= xf_{\overline{e_1}}(h)^{-1}t_1\delta_L(e_1)^{-1} = xt_1f_{e_1}(h)^{-1}\delta_L(e_1)^{-1} = |U|.\end{aligned}$$

By induction we find reduced words $Z$, $Z'$, and $W'$ such that $W'' = Z * W' * Z'$ and $|Z| = L_*(|Z'|)^{-1}$, and $W'$ is as desired. This fits together to a decomposition

$$W = U * Z * W' * Z' * U'.$$

We now define $V = U*Z$ and $V' = Z'*U'$. By construction, they satisfy $W = V*W'*V'$ and $|V| = L_*(|V'|)^{-1}$. This finishes the verification of (i) and (ii).

It remains to check (iv). The above calculation (3) shows that the lengths of the underlying paths of $\tilde{W}'$ and $W'$ coincide. Thus we have $pl(|\tilde{V}'|) = pl(|V'|)$ as well. As $\tilde{V}'$ and $V'$ are terminal segments of $\tilde{W}$ and $W$, we have $pl(|\tilde{V}'V'^{-1}|) = 0$ by Lemma 2.8, so we put $z = |\tilde{V}'V'^{-1}|$. Then clearly

$$\begin{aligned}
|\tilde{V}'| &= z|V'|, \\
|\tilde{V}| &= L_*(|\tilde{V}'|)^{-1} = L_*(z|V'|)^{-1} = |V|L_*(z)^{-1}, \\
|\tilde{W}'| &= |\tilde{V}|^{-1}|\tilde{W}||\tilde{V}'|^{-1} = L_*(z)|V|^{-1}|W||V'|^{-1}z^{-1} = L_*(z)|W'|z^{-1},
\end{aligned}$$

as asserted. □

**Definition 4.9.** An element $\eta \in \pi_1(\mathcal{G})$ is called *$L$-local* if there is a vertex $u \in V(\Gamma)$, an element $\epsilon \in \pi_1(\mathcal{G}, v, u)$, and $x \in G_u$ such that $\eta = L_*(\epsilon)x\epsilon^{-1}$. Otherwise, we say that $\eta$ is *$L$-cyclic*.

If $\eta$ is $L$-local of positive path length, then it cannot be $L$-twistedly reduced. In Proposition 4.8(i), the length $k'$ of the underlying path of $W'$ is zero if and only if $|W|$ is $L$-local.

We now investigate the path lengths of powers of a fixed element in $\pi_1(\mathcal{G}, v)$.

**Lemma 4.10.** *Let $\mathcal{G}$ be any graph of groups and $\eta \in \pi_1(\mathcal{G}, v)$.*

*(i) If $j \in \mathbb{Z} \setminus \{0\}$, then $pl(\eta^j) = (|j| - 1)pl(\eta^2) - (|j| - 2)pl(\eta)$.*

*(ii) If $\eta$ and $\eta'$ lie in a common cyclic subgroup of $\pi_1(\mathcal{G}, v)$ as well as $pl(\eta) \geq 1$ and $pl(\eta') = 0$, then $\eta' = 1$.*

*Proof.* We may w.l.o.g. assume $j \geq 1$. Let $W$ be a reduced word representing $\eta$, and write $W = V*W'*V'$ as in Proposition 4.8(i) with $L = 1$. As $|V| = |V'|^{-1}$, the reduced word $V * W' * V^{-1}$ also represents $\eta$. Then $\eta^j$ is represented by te reduced word

$$V * \underbrace{W' * \ldots * W'}_{j \text{ times}} * V^{-1},$$

whose underlying path has length

$$\begin{aligned}
pl(\eta^j) &= j \cdot pl(|W'|) + 2pl(|V|) \\
&= (j-1)(2pl(|W'|) + 2pl(|V|)) - (j-2)(pl(|W'|) + 2pl(|V|)) \\
&= (j-1)pl(\eta^2) - (j-2)pl(\eta).
\end{aligned}$$

This proves (i).

To see (ii), let $\zeta$ be a generator of the common cyclic subgroup of $\eta$ and $\eta'$. As $pl(\zeta^2) \geq pl(\zeta)$, part (i) implies that $pl(\zeta^j)$ grows monotonously with respect to $|j|$. If we had $\eta' \neq 1$, then $pl(\eta') = 0$ shows $pl(\zeta) = 0$. But then all powers of $\zeta$ also have path length zero. As $\eta$ is a power of $\zeta$ having positive path length, this is impossible. Hence $\eta' = 1$. $\square$

**Lemma 4.11.** *If $\eta \in \pi_1(\mathcal{G}, v)$ is $L$-twistedly reduced and $H \in \mathrm{Aut}(\mathcal{G})$ then $H_*(\eta)$ is $HLH^{-1}$-twistedly reduced.*

*Proof.* As $H_*$ preserves the path lengths of elements in $\pi_1(\mathcal{G})$, we have

$$\begin{aligned} pl((HLH^{-1})_*(H_*(\eta))H_*(\eta)) &= pl(H_*L_*(\eta)H_*(\eta)) = pl(L_*(\eta)\eta) = 2pl(\eta) \\ &= 2pl(H_*(\eta)). \end{aligned}$$

$\square$

## 4.5 Twisted reduction for higher graphs of groups

Given a word $W$ representing an element in $\pi_1(\mathbb{G}, u)$, we have a truncated word

$$T^{d-1}W = (\epsilon, t_1, \theta_1, \ldots, t_{k-1}, \theta_{k-1}, t_k, \epsilon')$$

as in Section 3.2. Recall that $T^{d-1}W$ is reduced in the truncated sense if all $\theta_j \neq 1$.

**Definition 4.12.** The truncated word $T^{d-1}W$ is called *$L$-twistedly reduced in the truncated sense* if it is closed, reduced in the truncated sense, and

- $k = 0$, or
- $e_k \neq \overline{e_1}$, or
- $\delta_L(\overline{e_1})^{-1}L_*(\epsilon')\epsilon \neq 1$.

If the higher graph of groups $\mathbb{G}$ is truncatable at degree $d-1$, then $T^{d-1}W$ is $L$-twistedly reduced in the truncated sense if and only if it is $T^{d-1}L$-twistedly reduced when considered as a word in $T^{d-1}\mathbb{G}$. Note that, in this case, all $\theta_j$ are closed elements in $\pi_1(\mathbb{G}^{(d-1)})$, which define elements in the vertex groups of the truncation $T^{d-1}\mathbb{G}$.

We call an element $\eta \in \pi_1(\mathbb{G}, u)$ $L$-twistedly reduced if $T^{d-1}W$ is $L$-twistedly reduced in the truncated sense for any word $W$ representing $\eta$. Here it is important to remember the degree $d$ of the higher graph of groups. If $W$ is a word only involving edges of degree at most $d-1$, then $T^{d-1}W$ is a truncated word of length zero, so it is always $L$-twistedly reduced in the truncated sense as above. However, this does not say anything about whether $T^{d-2}W$ is $L^{(d-1)}$-twistedly reduced in the truncated sense when we view $W$ as a word in $\mathbb{G}^{(d-1)}$.

## 4.6 The sequence $A_j(x,\alpha)$

Let $\alpha \in \mathrm{Aut}(G)$, $j \in \mathbb{Z}$, and $x \in G$. We define $A_j(x,\alpha)$ by the recursion formula $A_0(x,\alpha) = 1$ and

$$A_{j+1}(x,\alpha) = \alpha^j(x)A_j(x,\alpha). \tag{4}$$

Moreover, $A'_j(x,\alpha) := \alpha^{-j+1}(A_j(x,\alpha))$. We can explicitly write

$$A_j(x,\alpha) = \begin{cases} \alpha^{j-1}(x)\alpha^{j-2}(x)\dots\alpha(x)x, & \text{if } j \geq 1, \\ 1, & \text{if } j = 0, \\ \alpha^j(x)^{-1}\dots\alpha^{-2}(x)^{-1}\alpha^{-1}(x)^{-1}, & \text{if } j \leq -1, \end{cases} \tag{5}$$

$$A'_j(x,\alpha) = \begin{cases} x\alpha^{-1}(x)\dots\alpha^{-j+1}(x), & \text{if } j \geq 1, \\ 1, & \text{if } j = 0, \\ \alpha(x)^{-1}\alpha^2(x)^{-1}\dots\alpha^{-j}(x)^{-1}, & \text{if } j \leq -1. \end{cases} \tag{6}$$

When $\alpha$ is understood, we sometimes write $A_j(x)$ and $A'_j(x)$.

**Lemma 4.13.**

*(i)* $A_{j+1}(x,\alpha) = \alpha(A_j(x,\alpha))x$ *for all* $j \in \mathbb{Z}$*,*

*(ii)* $A_j(x,\alpha) = \alpha^{-1}(A_{-j}(x^{-1},\alpha^{-1}))$.

*Proof.* If $j = 0$, then $A_1(x) = x$ and $A_0(x) = 1$ prove (i).

Multiplying both sides of the assertion for $j$ from the left by $\alpha^{j+1}(x)$, we obtain

$$\alpha^{j+1}(x)A_{j+1}(x) = \alpha(\alpha^j(x)A_j(x))x,$$

which can be equivalently rewritten as (i) for $j+1$ using the recursive formula (4). This proves (i).

For (ii) we again use induction on $j$. If $j = 0$, then the assertion is trivial. Formula (ii) for $j$ is equivalent to

$$\alpha^j(x)A_j(x,\alpha) = \alpha^{-1}(\alpha^{j+1}(x)A_{-j}(x^{-1},\alpha^{-1})),$$

which is equivalent to (ii) for $j+1$ by (4). $\square$

*Remark* 4.14. We sometimes use the following notational convention for composable morphisms $\dots, g_{-2}, g_{-1}, g_0, g_1, g_2, \dots$ in any groupoid: When writing down a composition $g_j g_{j+1}\dots g_k$, then we mean 1 if $k = j-1$, and we mean $g_{j-1}^{-1}\dots g_{k+2}^{-1}g_{k+1}^{-1}$ if $k \leq j-2$. This convention ensures that

$$(g_{i+1}g_{i+2}\dots g_j)(g_{j+1}g_{j+2}\dots g_k) = g_{i+1}g_{i+2}\dots g_k$$

independently of the relative order of $i, j, k \in \mathbb{Z}$.

Using Remark 4.14, the formula for $A_j(x,\alpha)$ can be written simply as $A_j(x,\alpha) = \alpha^{-j-1}(x)\alpha^{j-2}(x)\dots\alpha(x)x$, regardless whether $j > 0$, $j = 0$, or $j < 0$.

**Lemma 4.15.** *If $G$ is any finitely generated group, and $x \in G$ grows at most polynomially of degree $d-1$ under iteration of $\alpha \in \operatorname{Aut}(G)$ and $\alpha^{-1}$, then the basis lengths of $A_j(x,\alpha)$ and $A'_j(x,\alpha)$ grow at most polynomially of degree $d$ when $j \to \infty$ or $j \to -\infty$.* □

We are mostly interested in the case of free groups, and it follows from train track theory that the growth types of an automorphism and its inverse coincide. However, we will not use this fact, and the author does not know whether this holds true for automorphisms of arbitrary finitely generated groups.

**Lemma 4.16.** *Let $\mathbb{G}$ be any higher graph of groups, $e$ an edge, and $L \in \operatorname{Aut}^0(\mathbb{G})$. Then*

$$L_*^j(t_e) = A_j(\delta_L(\overline{e}), L_*) t_e A_j(\delta_L(e), L_*)^{-1}$$

*for every $j \in \mathbb{Z}$.*

*Proof.* The lemma is trivial for $j = 0$ because $A_0(x) = 1$ for every $x$. We now show that the assertion for $j$ is equivalent to the assertion for $j+1$. Applying $L_*$ to the assertion for $j$, we obtain

$$L_*^{j+1}(t_e) = L_*(A_j(\delta_L(\overline{e})))\delta_L(\overline{e}) t_e \delta_L(e)^{-1} L_*(A_j(\delta_L(e)))^{-1}.$$

Using Lemma 4.13(i), this is the assertion for $j+1$. □

In the following lemma, we define basis lengths in the fundamental group of $\mathcal{G}$ again by a finite groupoid generating set which is compatible with the graph of groups structure.

**Lemma 4.17.** *Let $\mathcal{G}$ be a graph of groups with finitely generated vertex groups and trivial edge groups.*

*(i) Let $W = (x, t_1, g_1, \dots, t_{k-1}, g_{k-1}, t_k, y)$ be a reduced word with $k \geq 1$. Then*

$$\begin{aligned} l(L_*^j(|W|)) = k &+ l(L_*^j(x) A_j(\delta(\overline{e_1}))) + l(A_j(\delta(e_k))^{-1} L_*^j(y)) + \\ &+ \sum_{i=1}^{k-1} l\big(A_j(\delta(e_i))^{-1} L_*^j(g_i) A_j(\delta(\overline{e_{i+1}}))\big). \end{aligned}$$

*(ii) Assume that $W = (x, t_1, g_1, \dots, g_{k-1}, t_k, y)$ is cyclically reduced with $k \geq 1$ and $\tau(e_k) = \iota(e_1)$. Then*

$$\begin{aligned} l_c(L_*^j(|W|)) = k &+ l\big(A_j(\delta(e_k), L_*)^{-1} L_*^j(yx) A_j(\delta(\overline{e_1}), L_*)\big) + \\ &+ \sum_{i=1}^{k-1} l\big(A_j(\delta(e_i), L_*)^{-1} L_*^j(g_i) A_j(\delta(\overline{e_{i+1}}), L_*)\big). \end{aligned}$$

*Proof.* By Lemma 4.16 we can easily compute reduced (or cyclically reduced) expressions for $L_*^j(|W|)$ in both (i) and (ii). Writing an element (or conjugacy class) as a minimal product of generators in a compatible generating set is the same as finding a reduced (or cyclically reduced) expression together with a decomposition for each vertex group element in terms of a minimal number of generators of that vertex group. This proves the assertion. □

*Remark* 4.18. Estimates similar to Lemma 4.17 can also be obtained for higher graphs of groups $\mathbb{G}$ and words $(t_1, \theta_1, \dots, t_l, \theta_l)$ reduced (or cyclically reduced) in the truncated sense. If $\mathbb{G}$ is not truncatable at degree $d-1$, then $\theta_i \in \pi_1(\mathbb{G}^{(d-1)}, \tau(e_i), \iota(e_{i+1}))$ is then not necessarily closed. In this more general situation, the conclusions of Lemma 4.17 are still true.

## 4.7 Higher Dehn twists

We first define Dehn twists $D \in \mathrm{Aut}(\mathcal{G})$ for an ordinary graph of groups $\mathcal{G}$. The terminology is motivated by the correspondence to graphs of spaces. If all edge groups are infinite cyclic, the corresponding automorphism $D_{*v}$ of $\pi_1(\mathcal{G}, v)$ is induced by a (multiple) Dehn twist around the core curves of the edge cylinders in Figure 1 on page 16.

For a group $G$, let $Z(G)$ denote its centre.

**Definition 4.19.** Let $\mathcal{G}$ be a graph of groups. An automorphism $D \in \mathrm{Aut}(\mathcal{G})$ is called *Dehn twist* if $D_V = 1$, $D_E = 1$, all $D_w = 1$, all $D_e = 1$, and there are elements $\gamma_e \in Z(G_e)$ such that $\delta_D(e) = f_e(\gamma_e)$ for all edges $e \in E(\Gamma)$.

In fact, every collection of $\gamma_e \in Z(G_e)$ defines a Dehn twist. We often use the notation $z_e = \gamma_e \gamma_{\overline{e}}^{-1}$, and we refer to it as the *twistor* of the edge $e$. The set of Dehn twists forms a subgroup of $\mathrm{Aut}^0(\mathcal{G})$.

**Definition 4.20.** An automorphism $D \in \mathrm{Aut}(\mathbb{G})$ is called *higher Dehn twist* if $D_V = 1$, $D_E = 1$, all $D_w = 1$, all $D_e = 1$, and for every non-trivial edge group $G_e$ there is $\gamma_e \in Z(G_e)$ such that $\delta_D(e) = f_e(\gamma_e)$.

Note that we do not require anything about $\delta_D(e)$ for trivial $G_e$. However, if $\mathbb{G}$ has no trivial edge groups in degree 1, then $D \in \mathrm{Aut}^0(\mathbb{G})$ is a higher Dehn twist if and only if the restriction $D|_{\Gamma^{(1)}}$ is a Dehn twist (of ordinary graphs of groups) on each connected component.

We now verify that higher Dehn twists have polynomial growth. Let $\mathbb{G}$ be a higher graph of groups with basepoint $v$.

**Proposition 4.21.** *Let $D \in \mathrm{Aut}(\mathbb{G})$ be a higher Dehn twist. Then every element $\epsilon \in \pi_1(\mathbb{G}, u, w)$ grows at most polynomial of degree* $\deg(\mathbb{G})$ *under iteration of $D_*$. Moreover, $\widehat{D}$ grows at most polynomially of degree* $\deg(\mathbb{G})$*.*

We need the following lemma:

**Lemma 4.22.** *Let $\mathcal{G}$ be a connected graph of groups with finitely generated vertex groups, and let $H \in \mathrm{Aut}^0(\mathcal{G})$. Suppose all vertex automorphisms $H_w$ grow at most polynomially of degree $d-1$. Then $\widehat{H}$ grows at most polynomially of degree $d$, and every element in $\pi_1(\mathcal{G}, u, w)$ grows at most polynomially of degree $d$ under iteration of $H_*$.*

*Proof.* Every element in $\pi_1(\mathcal{G}, u, w)$ can be written as a composition of elements in vertex groups and letters $t_e$. Elements in vertex groups grow at most polynomially of degree $d-1$ under iteration of $H_*$ by requirement. The growth of $t_e$ is bounded above by a polynomial of degree $d$ by Lemma 4.15 and Lemma 4.16. This proves the statement about $H_*$.

The growth of $\widehat{H}$ is bounded above by the growth of $H_{*v}$, which is a special case of that of $H_*$. □

*Proof of Proposition 4.21.* As before, it suffices to verify the case of $D_*$ on $\pi_1(\mathbb{G}, u, w)$.

The proof is by induction on $d = \deg(\mathbb{G})$. If $d = 0$, then the underlying graph $\Gamma$ is a point, and $D = 1$. Then the growth is clearly polynomial of degree zero. If $d = 1$, then we have an ordinary Dehn twist with possibly trivial edges added. In particular all $D_w = 1$, and Lemma 4.22 shows that the growth is at most linear.

Let now $d \geq 2$. If $H : (\mathbb{G}, v) \to (\mathbb{G}', v')$ is an equivalence such that $\mathbb{G}'$ is fully truncatable, then the growth types of $D_*$ and $(HDH^{-1})_*$ agree. By induction, all vertex group homomorphisms of $T^{d-1}(HDH^{-1})$ grow at most polynomially of degree $d-1$. Then Lemma 4.22 proves that $(HDH^{-1})_*$ and hence $D_*$ grows at most polynomially of degree $d$. □

We shall sometimes fix a group $G$ and consider different higher graphs of groups $\mathbb{G}$ with fundamental group isomorphic to $G$.

**Definition 4.23.** If $G$ is any group, then a *higher Dehn twist representative* for $\alpha \in \mathrm{Aut}(G)$ (or $\widehat{\alpha} \in \mathrm{Out}(G)$) is a triple $(\mathbb{G}, D, \rho)$ where $\mathbb{G}$ is a higher graph of groups, $D \in \mathrm{Aut}^0(\mathbb{G})$ is a higher Dehn twist, and $\rho : G \to \pi_1(\mathbb{G}, v)$ is an isomorphism such that $\alpha = \rho^{-1} D_{*v} \rho$ (or $\widehat{\alpha} = \widehat{\rho}^{-1} \widehat{D} \widehat{\rho}$ respectively). $\alpha$ (or $\widehat{\alpha}$) is called a *higher Dehn twist automorphism* if it has a higher Dehn twist representative.

## 4.8 Train track representatives

Here we discuss how the notion of higher graph of groups automorphisms fits into the setting of topological representatives and train tracks for $\mathrm{Out}(F_n)$ developed by Bestvina, Feighn, and Handel (cf. [6], [7], and [8] for example).

Fix the standard rose $R_n$ with $n$ petals, and identify its fundamental group with $F_n$. An outer automorphism class $\widehat{\alpha} \in \mathrm{Out}(F_n)$ is represented by a homotopy equivalence $h : R_n \to R_n$. A *marked graph* is a (finite) graph $\Gamma$ together with a (homotopy class of a) homotopy equivalence $g : R_n \to \Gamma$ referred to as the *marking*. $\widehat{\alpha}$ is said to be represented by the homotopy equivalence $f : \Gamma \to \Gamma$ if the following diagram commutes up to homotopy:

$$\begin{tikzcd} \Gamma \arrow[r, "f"] & \Gamma \\ R_n \arrow[u, "g"] \arrow[r, "h"] & R_n \arrow[u, "g"'] \end{tikzcd}$$

The homotopy equivalence $f$ is called a *topological representative* for $\widehat{\alpha}$ if $f$ maps vertices to vertices, and if the restriction of $f$ to any edge is locally injective.

We now compare this to higher graphs of groups $\mathbb{G}$ all of whose vertex and edge groups are trivial. We can identify the fundamental groupoid of $\mathbb{G}$ with the fundamental groupoid of the underlying graph $\Gamma$ here, so we do not distinguish between $e$ and $t_e$. An automorphism of $\mathbb{G}$ corresponds to a (topological) map $f : \Gamma \to \Gamma$ mapping each edge $e$ of degree $i$ to an edge path $f(e)$ of the form $\delta_H(\overline{e})H_E(e)\delta_H(e)^{-1}$ such that $\delta_H(\overline{e})$ and $\delta_H(e)$ are edge paths in $\Gamma^{(i-1)}$ and $H_E(e)$ has degree $i$. It follows from Theorem 3.7 of [7] that every polynomially growing $\widehat{\alpha} \in \mathrm{Out}(F_n)$ has a topological representative $f : \Gamma \to \Gamma$ of this form (sometimes called *upper triangular*).

As all vertex and edge groups of $\mathbb{G}$ are trivial, every $H \in \mathrm{Aut}^0(\mathbb{G})$ is a higher Dehn twist. For an arbitrary choice of a basepoint $u \in \Gamma$, this shows the following:

**Proposition 4.24.** *For every polynomially growing $\widehat{\alpha} \in \mathrm{Out}(F_n)$ there is a higher graph of groups $\mathbb{G}$, an automorphism $H \in \mathrm{Aut}(\mathbb{G})$, and an outer isomorphism class $\widehat{\rho} : F_n \to \pi_1(\mathbb{G}, u)$ such that $\widehat{\alpha} = \widehat{\rho}^{-1}\widehat{H}\widehat{\rho}$ and $H^N$ is a higher Dehn twist for some $N \geq 1$.* □

The situation is completely analogous in the Aut case, where we have

**Proposition 4.25.** *For every polynomially growing $\alpha \in \mathrm{Aut}(F_n)$ there is a pointed higher graph of groups $(\mathbb{G}, v)$, an automorphism $H \in \mathrm{Aut}(\mathbb{G}, v)$, and an isomorphism $\rho : F_n \to \pi_1(\mathbb{G}, v)$ such that $\alpha = \rho^{-1}H_{*v}\rho$ and $H^N$ is a higher Dehn twist for some $N \geq 1$.*

*Proof.* By Proposition 4.24, there are a higher graph of groups $\mathbb{G}'$, $H' \in \mathrm{Aut}(\mathbb{G}')$, and an isomorphism $\rho' : F_n \to \pi_1(\mathbb{G}', u)$ such that the outer automorphism class $\widehat{\alpha} = \widehat{\rho'}^{-1}\widehat{H'}\widehat{\rho'}$, and some power $H'^N$ with $N \geq 1$ is a higher Dehn twist.

Let $\Gamma$ be the graph obtained from $\Gamma'$ by adding a new basepoint vertex $v$ and two oriented edges $e_0$ and $\overline{e_0}$ with $\iota(e_0) = v$ and $\tau(e_0) = u$. Let $\mathcal{G}$ be the graph of groups with underlying graph $\Gamma$ obtained from $\mathcal{G}'$ by adding $G_v = 1$ and $G_e = 1$. Let $d$ be the degree of $\mathbb{G}'$, and define $\deg(e_0) = d + 1$. Then $\mathbb{G} = (\mathcal{G}, \deg)$ becomes a higher graph of groups of degree $d + 1$. The situation is illustrated by Figure 3.

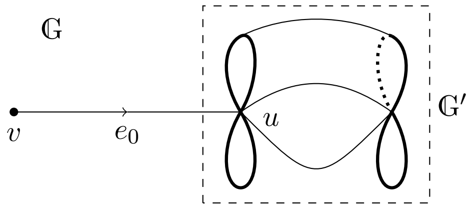


Figure 3: $\mathbb{G}$ and $\mathbb{G}'$ in the proof of Proposition 4.25.

We now take $\tilde{H} \in \mathrm{Aut}(\mathbb{G}, v)$ given by the same data as the automorphism $H'$ along with the following additional data: Let $\tilde{H}_V(v) = v$, $\tilde{H}_E(e_0) = e_0$, $\delta_{\tilde{H}}(\overline{e_0}) = 1$, and $\delta_{\tilde{H}}(e_0)$ an arbitrary element in $\pi_1(\mathbb{G}', H_V(u), u)$.

Let $\rho : F_n \to \pi_1(\mathbb{G}, v)$ be the isomorphism $\rho'$ composed with the isomorphism $\mathrm{ad}_{t_{e_0}} : \pi_1(\mathbb{G}', u) \to \pi_1(\mathbb{G}, v)$. Since the underlying outer automorphism class represented by $H'$ is not affected by adding $v$ and $e_0$, we have that $\alpha = \rho^{-1}\mathrm{ad}_g\tilde{H}_{*v}\rho$ for some $g \in \pi_1(\mathbb{G}, v)$.

Let the automorphism $H \in \mathrm{Aut}(\mathbb{G}, v)$ be given by the same data as $\tilde{H}$ except that $\delta_H(e_0) = t_{e_0}^{-1} g^{-1} t_{e_0} \delta_{\tilde{H}}(e_0)$. Note that $t_{e_0}^{-1} g^{-1} t_{e_0} \in \pi_1(\mathbb{G}, u)$ can be viewed as an element in $\pi_1(\mathbb{G}', u)$ by canceling out all occurrences of $t_{e_0}^{-1} t_{e_0}$. Then $H_{*v} = \mathrm{ad}_g \tilde{H}_{*v} = \rho\alpha\rho^{-1}$.

As $H|_{\mathbb{G}'} = H'$ and $H'^N$ is a higher Dehn twist, the automorphism $H^N$ is a higher Dehn twist as well. $\square$

# 5 Periods

## 5.1 Asymptotic equivalence of sequences

Let $\mathcal{G}$ be an arbitrary graph of groups. Given elements $\epsilon, \zeta \in \pi_1(\mathcal{G}, v, w)$, the expression $pl(\epsilon) + pl(\zeta) - pl(\epsilon^{-1}\zeta)$ is twice the path length of a common initial segment of $\epsilon$ and $\zeta$.

**Definition 5.1.** Let $\epsilon_1, \epsilon_2, \dots$ and $\zeta_1, \zeta_2, \dots$ be two sequences in $\pi_1(\mathcal{G}, v, w)$. We say that $(\epsilon_j)_j$ and $(\zeta_j)_j$ are *asymptotically equivalent* if

$$pl(\epsilon_j) + pl(\zeta_j) - pl(\epsilon_j^{-1}\zeta_j) \to \infty$$

for $j \to \infty$. They are *partially asymptotically equivalent* if there is a sequence $(j_m)_m$ with $j_m \geq 1$ such that the subsequences $(\epsilon_{j_m})_m$ and $(\zeta_{j_m})_m$ are asymptotically equivalent.

Any two sequences $(\epsilon_j)_j$ and $(\zeta_j)_j$ in $\pi_1(\mathcal{G}, v, w)$ are either partially asymptotically equivalent, or the sequence

$$\big(pl(\epsilon_j) + pl(\zeta_j) - pl(\epsilon_j^{-1}\zeta_j)\big)_{j \geq 1}$$

is bounded.

**Lemma 5.2.** *Let $(\epsilon_j)_j$ and $(\zeta_j)_j$ be two sequences in $\pi_1(\mathcal{G}, v, w)$ and $\delta_j \in \pi_1(\mathcal{G}, v', v)$, $\theta_j \in \pi_1(\mathcal{G}, w, w')$ such that $pl(\delta_j)$ and $pl(\theta_j)$ are bounded. Then $(\epsilon_j)_j$ and $(\zeta_j)_j$ are asymptotically equivalent if and only if $(\delta_j\epsilon_j\theta_j)_j$ and $(\delta_j\zeta_j\theta_j)_j$ are asymptotically equivalent.*

*Proof.* There is the following estimate by the triangle inequality:

$$\begin{aligned}\Big|\Big(pl(\delta_j\epsilon_j\theta_j) + pl(\delta_j\zeta_j\theta_j) - pl(\theta_j^{-1}\epsilon_j^{-1}\zeta_j\theta_j)\Big) - \Big(pl(\epsilon_j) + pl(\zeta_j) - pl(\epsilon_j^{-1}\zeta_j)\Big)\Big| \\ \leq 2pl(\delta_j) + 4pl(\theta_j).\end{aligned}$$

Since the right hand side is a bounded sequence in $j$, we obtain the assertion. □

Recall the notation

$$A_k(x) = A_k(x, L_*) = L_*^{k-1}(x)L_*^{k-2}(x)\dots L_*(x)x$$

given by (5) on page 33.

**Corollary 5.3.** *Let $\delta \in \pi_1(\mathcal{G}, w, v)$ and $\eta, \eta' \in \pi_1(\mathcal{G}, v)$. Then $(A_j(\eta))_j$ and $(A_j(\eta'))_j$ are asymptotically equivalent if and only if $(A_j(L_*(\delta)\eta\delta^{-1}))_j$ and $(A_j(L_*(\delta)\eta'\delta^{-1}))_j$ are asymptotically equivalent.*

*Proof.* Since $A_j(L_*(\delta)\theta\delta^{-1}) = L_*^j(\delta)A_j(\theta)\delta^{-1}$ for $\theta \in \pi_1(\mathcal{G}, v)$, this is an immediate consequence of Lemma 5.2. □

The following lemma follows easily because graph of groups automorphisms preserve path lengths of elements in the fundamental groupoid.

**Lemma 5.4.** *Suppose $(\epsilon_j)_j$ and $(\zeta_j)_j$ are sequences in $\pi_1(\mathcal{G}, v, w)$, and $H_j \in \mathrm{Aut}^0(\mathcal{G})$. Then $(\epsilon_j)_j$ and $(\zeta_j)_j$ are asymptotically equivalent (respectively partially asymptotically equivalent) if and only if $(H_{j*}(\epsilon_j))_j$ and $(H_{j*}(\zeta_j))_j$ are.* □

## 5.2 Extending $L$-cyclic elements to bi-infinite sequences

We now fix an automorphism $L \in \mathrm{Aut}^0(\mathcal{G})$ of an ordinary graph of groups $\mathcal{G}$. When

$$W = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$$

is a (connected) word with $k \geq 1$ and $\tau(e_k) = \iota(e_1)$, then it will be convenient to write $g_j$ for any $j \in \mathbb{Z}$, possibly $j \leq 0$ or $j > k$. They are uniquely defined by the formula

$$g_{j+k} = L_{\tau(e_j)}^{-1}(\delta_L(e_j) g_j \delta_L(\overline{e_{j+1}})^{-1}) \tag{7}$$

and $e_{j+k} = e_j$. This recursion is motivated by the following lemma.

**Lemma 5.5.**

*(i)* $(t_j, g_j, t_{j+1})$ *is reduced if and only if* $(t_{j+k}, g_{j+k}, t_{j+k+1})$ *is.*

*(ii) If* $W = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$ *is an* $L$*-twistedly reduced word with* $k \geq 1$*, then* $W^{-1} = (g_k^{-1}, t_k^{-1}, \ldots, g_1^{-1}, t_1^{-1}, \delta_L(\overline{e_1})^{-1})$ *is* $L^{-1}$*-twistedly reduced, and the sequence* $(g_{-j}^{-1})_{j\in\mathbb{Z}}$ *satisfies the recursion* (7) *for the automorphism* $L^{-1}$*.*

*Proof.* The word $(t_{j+k}, g_{j+k}, t_{j+k+1})$ is not reduced if and only if $e_{j+k} = \overline{e_{j+k+1}}$ and $g_{j+k} = f_{e_{j+k}}(h)$ for some $h \in G_{e_{j+k}}$. This condition is equivalent to $e_j = \overline{e_{j+1}}$ and $L_{\tau(e_j)}^{-1}(\delta_L(e_j) g_j \delta_L(\overline{e_{j+1}})) = f_{e_j}(h)$ for some $h \in G_{e_j}$. This can be equivalently rewritten as $e_j = \overline{e_{j+1}}$ and $g_j = f_{e_j}(L_{e_j}(h))$. Thus we obtain (i).

In (ii), the word $W^{-1}$ is clearly reduced. Lemma 4.5 shows that the concatenation $L_*(W) * W$ is reduced. As $L_*^{-1}$ maps reduced words to reduced words, $W * L_*^{-1}(W)$ is also reduced. Then $L_*^{-1}(W^{-1}) * W^{-1}$ is reduced, so Lemma 4.5 implies that $W^{-1}$ is $L^{-1}$-twistedly reduced.

As $\delta_{L^{-1}}(e) = L_*^{-1}(\delta_L(e))^{-1}$, the formula (7) can be written as

$$g_{j+k} = \delta_{L^{-1}}(e_j)^{-1} L_*^{-1}(g_j) \delta_{L^{-1}}(\overline{e_{j+1}}),$$

so

$$g_j^{-1} = L_*(\delta_{L^{-1}}(\overline{e_{j+1}}) g_{j+k}^{-1} \delta_{L^{-1}}(e_j)^{-1}),$$

which is the desired recursion formula for $(g_{-j}^{-1})_{j\in\mathbb{Z}}$ with respect to $L^{-1}$. $\square$

## 5.3 Periodicity of $L$-cyclic elements

Let $W$ be a representing word for $\eta \in \pi_1(\mathcal{G}, v)$. Recall that, by Proposition 4.8, there is a decomposition $W = V * W' * V'$ such that $|V| = L_*(|V'|)^{-1}$ and $W'$ is $L$-twistedly reduced. Here we will be interested in the case that $\eta$ is $L$-cyclic, so $pl(\eta') \geq 1$. In this case, Proposition 4.8 allows us to achieve $W' = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$.

In the following definition, we again use the convention (7) to define $g_j$ for $j \leq 0$ and $j > k$.

**Definition 5.6.** Suppose $W = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$ is an $L$-twistedly reduced word with $k \geq 1$. An integer $p$ is called *period* of $W$ if there are $h_j \in G_{e_j}$ satisfying

- $e_{j+p} = e_j$ and
- $g_{j+p} = f_{e_j}(h_j) g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}$

for all $j \in \mathbb{Z}$. We call $W$ *non-periodic* if zero is its only period.

**Lemma 5.7.** *The set of all periods of $W$ forms a subgroup of $\mathbb{Z}$.*

*Proof.* If $p$ is a period, then $-p$ is a period as well.

If $p$ and $p'$ are periods, then $e_{j+p+p'} = e_{j+p} = e_j$. If $g_{j+p} = f_{e_j}(h_j) g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}$ and $g_{j+p'} = f_{e_j}(h'_j) g_j f_{\overline{e_{j+1}}}(h'_{j+1})^{-1}$, then

$$\begin{aligned} g_{j+p+p'} &= f_{e_j}(h'_{j+p}) g_{j+p} f_{\overline{e_{j+1}}}(h'_{j+p+1})^{-1} \\ &= f_{e_j}(h'_{j+p} h_j) g_j f_{\overline{e_{j+1}}}(h'_{j+p+1} h_{j+1})^{-1}. \end{aligned}$$

This shows that $p + p'$ is a period as well. □

**Definition 5.8.** $W = V * (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k) * V'$ is called *non-bonding* if there is $N \geq 0$ such that, whenever we have an integer $q$ and elements $h_j \in G_{e_j}$ satisfying $g_j = f_{e_j}(h_j) g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}$ for all $j$ with $q \leq j \leq q + N$, then all $h_j = 1$.

If all edge groups are trivial, then every $L$-cyclic element is clearly non-bonding.

**Lemma 5.9.** *Let $W = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$ be non-bonding and $\tilde{g}_j \in G_{\tau(e_j)}$ another bi-infinite sequence satisfying the recursion formula* (7)*. Let $x_j \in G_{e_j}$ for $j \geq 1$. Assume that*

$$\tilde{g}_j = f_{e_j}(x_j) g_j f_{\overline{e_{j+1}}}(x_{j+1})^{-1} \tag{8}$$

*holds for every $j \geq 1$. Then there are $x_j$ for $j \leq 0$ such that* (8) *holds true for all $j \in \mathbb{Z}$. Moreover, $x_{j+k} = L_{e_j}^{-1}(x_j)$ for all $j \in \mathbb{Z}$.*

*Proof.* Since $\tilde{g}_{j+k} = L_*^{-1}(\delta_L(e_j) \tilde{g}_j \delta_L(\overline{e_{j+1}})^{-1})$, the formula (8) for $j$ can be equivalently written as

$$\tilde{g}_{j+k} = L_*^{-1}(\delta_L(e_j) f_{e_j}(x_j) g_j f_{\overline{e_{j+1}}}(x_{j+1})^{-1} \delta_L(\overline{e_{j+1}})^{-1}). \tag{9}$$

On the other hand, since $g_{j+k} = L_*^{-1}(\delta_L(e_j) g_j \delta_L(\overline{e_{j+1}})^{-1})$, equation (8) for $j + k$ is equivalent to

$$\tilde{g}_{j+k} = f_{e_j}(x_{j+k}) L_*^{-1}(\delta_L(e_j) g_j \delta_L(\overline{e_{j+1}})^{-1}) f_{\overline{e_{j+1}}}(x_{j+k+1})^{-1}. \tag{10}$$

The expressions (9) and (10) for $j \geq 1$ show that

$$\begin{aligned} &\delta_L(e_j) f_{e_j}(x_j) g_j f_{\overline{e_{j+1}}}(x_{j+1})^{-1} \delta_L(\overline{e_{j+1}})^{-1} \\ &= L_*(f_{e_j}(x_{j+k})) \delta_L(e_j) g_j \delta_L(\overline{e_{j+1}})^{-1} L_*(f_{\overline{e_{j+1}}}(x_{j+k+1}))^{-1} \\ &= \delta_L(e_j) f_{e_j}(L_{e_j}(x_{j+k})) g_j f_{\overline{e_{j+1}}}(L_{e_{j+1}}(x_{j+k+1}))^{-1} \delta_L(\overline{e_{j+1}})^{-1}, \end{aligned}$$

so

$$g_j = f_{e_j}(x_j^{-1} L_{e_j}(x_{j+k})) g_j f_{\overline{e_{j+1}}}(x_{j+1}^{-1} L_{e_{j+1}}(x_{j+k+1}))^{-1}$$

for all $j \geq 1$. Since $W$ is non-bonding, we conclude $x_j^{-1} L_{e_j}(x_{j+k}) = 1$ for all these $j$. This shows that we can define $x_j \in G_{e_j}$ for all $j \in \mathbb{Z}$ such that $x_{j+k} = L_{e_j}^{-1}(x_j)$ for all $j \in \mathbb{Z}$. Then the same calculation shows that (9) and (10) are equivalent. Hence (8) for $j$ is equivalent to (8) for $j + k$. This proves the lemma. □

## 5.4 Iterating $L_*$ on $L$-twistedly reduced elements

Throughout this section, $L \in \mathrm{Aut}^0(\mathcal{G})$ is a fixed automorphism of an ordinary graph of groups.

**Lemma 5.10.** *Let $\eta = \delta_L(\overline{e_1})t_1 g_1 \dots t_k g_k$ be an $L$-twistedly reduced expression and $i, j \in \mathbb{Z}$. Then*

*(i)* $g_{i-jk} = A_j(\delta_L(e_i))^{-1} L_*^j(g_i) A_j(\delta_L(\overline{e_{i+1}}))$,

*(ii)* $L_*^{-1}(t_i g_i) = L_*^{-1}\delta_L(\overline{e_i})^{-1} t_{k+i} g_{k+i} L_*^{-1}\delta_L(\overline{e_{i+1}})$,

*(iii)* $L_*^j(t_i g_i) = A_j(\delta_L(\overline{e_i})) t_{-jk+i} g_{-jk+i} A_j(\delta_L(\overline{e_{i+1}}))^{-1}$,

*(iv)* $L_*^j(\eta) = A_{j+1}(\delta_L(\overline{e_1})) t_{-jk+1} g_{-jk+1} \dots t_{-jk+k} g_{-jk+k} A_j(\delta_L(\overline{e_1}))^{-1}$,

*(v)* $A_j'(\eta) = \delta_L(\overline{e_1}) t_1 g_1 \dots t_{jk} g_{jk} A_{-j+1}(\delta_L(\overline{e_1}))^{-1}$,

*(vi)* $A_j(\eta) = A_j(\delta_L(\overline{e_1})) t_{-jk+k+1} g_{-jk+k+1} \dots t_k g_k$.

*Proof.* (i) is proved by induction. The case $j = 0$ is clear because $A_0(x) = 1$ for every $x$ by (5) on page 33. We now multiply (i) for $j$ on the left by $\delta_L(e_i)$, on the right by $\delta_L(\overline{e_{i+1}})^{-1}$ and apply $L_*^{-1}$ to obtain

$$L_*^{-1}\big(\delta_L(e_i) g_{i-jk} \delta_L(\overline{e_{i+1}})^{-1}\big) = L_*^{-1}\big(\delta_L(e_i) A_j(\delta_L(e_i))^{-1} L_*^j(g_i) A_j(\delta_L(\overline{e_{i+1}})) \delta_L(\overline{e_{i+1}})^{-1}\big).$$

By (7) on page 40, the left hand side equals $g_{i-(j-1)k}$. As $L_*^{-1}(x A_j(x)^{-1}) = A_{j-1}(x)^{-1}$ by Lemma 4.13(i), we see that (i) for $j$ is equivalent to (i) for $j-1$. Then (i) follows by induction.

To prove (iii), we isolate an expression for $L_*^j(g_i)$ in (i). When we multiply this on the left with the formula for $L_*^j(t_i)$ given by Lemma 4.16, we obtain (iii). Assertion (ii) is the special case $j = -1$ of (iii).

Formula (iv) follows from (iii) because $L_*^j(\delta_L(\overline{e_1})) A_j(\delta_L(\overline{e_1})) = A_{j+1}(\delta_L(\overline{e_1}))$ by the definition in (4) on page 33.

When we insert (iv) in each factor of $A_j'(\eta) = \eta L_*^{-1}(\eta) \dots L_*^{-j+1}(\eta)$ and $A_j(\eta) = L_*^{j-1}(\eta) \dots L_*(\eta)\eta$, we obtain (v) and (vi). For $j < 0$ these expressions have to be read as described in Remark 4.14. □

For later reference, we now record an immediate consequence for the growth of the basis length of the $g_j$.

**Lemma 5.11.** *Let all vertex automorphisms $L_w$ and their inverses grow at most polynomially of degree $d-1$. Then the basis length of $g_j$ grows at most polynomially of degree $d$ when $j \to \pm\infty$.*

*Proof.* By Lemma 5.10(i), we have

$$g_{i+jk} = A_{-j}(\delta_L(e_i))^{-1} L_*^{-j}(g_i) A_{-j}(\delta_L(\overline{e_{i+1}})).$$

Since the vertex automorphism $L_{\tau(e_i)}^{\pm 1}$ grows at most polynomially of degree $d-1$, the basis lengths of $A_{-j}(\delta_L(e_i))$ and $A_{-j}(\delta_L(\overline{e_{i+1}}))$ grow at most polynomially of degree $d$ by Lemma 4.15, and $L_*^{-j}(g_i)$ grows at most polynomially of degree $d-1$ when $j \to \pm\infty$. Thus the basis length of $g_{i+jk}$ grows at most polynomially of degree $d$. Since we have this upper bound for the growth for all $i$ with $1 \le i \le k$, we get the desired upper bound for the growth of $g_j$. $\square$

The following lemma will be needed in Section 5.6.

**Lemma 5.12.** *Let $e$ be any edge of $\Gamma$, $h \in G_e$, and $j \in \mathbb{Z}$. Then*

$$A_{-j}(\delta_L(e))^{-1} L_*^{-j}(f_e(L_e^j(h))) = f_e(h) A_{-j}(\delta_L(e))^{-1}.$$

*Proof.* Since $A_0(\delta_L(e)) = 1$, the case $j = 0$ is clear.

Multiplying on the left by $\delta_L(e)$, applying $L_*^{-1}$, and replacing $h$ by $L_e(h)$, the assertion for $j-1$ can be equivalently written as

$$\begin{aligned} &L_*^{-1}\big(\delta_L(e) A_{-j+1}(\delta_L(e))^{-1} L_*^{-j+1}(f_e(L_e^j(h)))\big) \\ &= L_*^{-1}\big(\delta_L(e) f_e(L_e(h)) A_{-j+1}(\delta_L(e))^{-1}\big). \end{aligned}$$

By Lemma 4.13(i), this is equivalent to

$$A_{-j}(\delta_L(e))^{-1} L_*^{-j}(f_e(L_e^j(h))) = L_*^{-1}\big(L_*(f_e(h)) \delta_L(e) A_{-j+1}(\delta_L(e))^{-1}\big)$$

and hence to the assertion for $j$. This proves the lemma inductively for all $j \in \mathbb{Z}$. $\square$

## 5.5 Period fitting segments

**Definition 5.13.** Assume $W = V * W' * V'$ such that $|V| = L_*|V'|^{-1}$ and $W' = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$ is $L$-twistedly reduced with $k \ge 1$ (cf. Proposition 4.8). A *period fitting segment* of $W$ is an element

$$\eta' = |V| \cdot \delta_L(\overline{e_1}) t_1 g_1 \ldots t_{k'} g_{k'} f_{\overline{e_1}}(h_1) \cdot |V'|$$

such that

- $k' \ge 1$,
- $k' - k$ is a period of $W'$,
- $h_j \in G_{e_j}$ are such that $g_{j+k'} = f_{e_j}(h_j) g_{j+k} f_{\overline{e_{j+1}}}(h_{j+1})^{-1}$ for all $j \in \mathbb{Z}$.

**Lemma 5.14.** *Let $W = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$ be an $L$-twistedly reduced word, and let $x_j \in G_{e_j}$ such that $x_{j+k} = L_{e_j}^{-1}(x_j)$ for all $j \in \mathbb{Z}$. Assume that the elements $\tilde{g}_j := f_{e_j}(x_j) g_j f_{\overline{e_{j+1}}}(x_{j+1})^{-1}$ also satisfy (7) on page 40. Then a period fitting segment of $W$ is the same as a period fitting segment of*

$$\tilde{W} = (\delta_L(\overline{e_1}) f_{\overline{e_1}}(x_1)^{-1}, t_1, \tilde{g}_1, \ldots, t_k, \tilde{g}_k f_{\overline{e_1}}(x_{k+1})).$$

*Proof.* We rewrite $\tilde{W}$ as

$$\tilde{W} = L_*(f_{\overline{e_1}}(L_{e_1}^{-1}(x_1)))^{-1} * (\delta_L(\overline{e_1}), t_1, \tilde{g}_1, \ldots, t_k, \tilde{g}_k) * f_{\overline{e_1}}(x_{k+1}).$$

Let $k' \in \mathbb{Z}$ and $\tilde{h}_j \in G_j$ satisfy $\tilde{g}_{j+k'} = f_{e_j}(\tilde{h}_j)\tilde{g}_{j+k} f_{\overline{e_{j+1}}}(\tilde{h}_{j+1})^{-1}$ for all $j$. From this we get

$$\begin{aligned}
g_{j+k'} &= f_{e_j}(x_{j+k'})^{-1}\tilde{g}_{j+k'} f_{\overline{e_{j+1}}}(x_{j+k'+1}) \\
&= f_{e_j}(x_{j+k'}^{-1}\tilde{h}_j)\tilde{g}_{j+k} f_{\overline{e_{j+1}}}(\tilde{h}_{j+1}^{-1} x_{j+k'+1}) \\
&= f_{e_j}(x_{j+k'}^{-1}\tilde{h}_j x_{j+k}) g_{j+k} f_{\overline{e_{j+1}}}(x_{j+k+1}^{-1}\tilde{h}_{j+1}^{-1} x_{j+k'+1}) \\
&= f_{e_j}(h_j) g_{j+k} f_{\overline{e_{j+1}}}(h_{j+1})^{-1},
\end{aligned}$$

where $h_j := x_{j+k'}^{-1}\tilde{h}_j x_{j+k}$. A period fitting segment of $\tilde{W}$ is now an element of the form

$$\begin{aligned}
&L_*(f_{\overline{e_1}}(x_{k+1}))^{-1}\delta_L(\overline{e_1}) t_1\tilde{g}_1 \ldots t_{k'}\tilde{g}_{k'} f_{\overline{e_1}}(\tilde{h}_1) f_{\overline{e_1}}(x_{k+1}) \\
&= \delta_L(\overline{e_1}) f_{\overline{e_1}}(x_1)^{-1} t_1\tilde{g}_1 \ldots t_{k'}\tilde{g}_{k'} f_{\overline{e_1}}(x_{k'+1}h_1) \\
&= \delta_L(\overline{e_1}) t_1 g_1 \ldots t_{k'} g_{k'} f_{\overline{e_{k'+1}}}(x_{k'+1})^{-1} f_{\overline{e_1}}(x_{k'+1}h_1) \\
&= \delta_L(\overline{e_1}) t_1 g_1 \ldots t_{k'} g_{k'} f_{\overline{e_1}}(h_1).
\end{aligned}$$

This is a period fitting segment of $W$, as claimed. Conversely, every period fitting segment of $W$ is of this form, and hence a period fitting segment of $\tilde{W}$ by the same calculation backwards. ☐

**Proposition 5.15.** *Let $W$ and $\tilde{W}$ be two reduced words representing the same $L$-cyclic element in $\pi_1(\mathcal{G})$. Suppose $W$ is non-bonding. Then a period fitting segment of $W$ is the same as a period fitting segment of $\tilde{W}$.*

*Proof.* We write $W = V * W' * V'$ and $\tilde{W} = \tilde{V} * \tilde{W}' * \tilde{V}'$ as in Proposition 4.8. Let $z \in G_{\tau(e_k)}$ be as in part (iv) of that proposition. We will show that period fitting segments of $W'$ and $L_*(z)^{-1} * \tilde{W}' * z$ are the same. Then the general case follows by multiplying the segments on the left by $|V|$ and on the right by $|V'|$. Therefore we may assume that $W = (\delta_L(\overline{e_1}), t_1, g_1, \ldots, t_k, g_k)$ is $L$-twistedly reduced.

Since $\tilde{W}$ represents the same element of $\pi_1(\mathcal{G})$, Proposition 2.6 provides $x_j \in G_{e_j}$ such that, writing

$$\tilde{g}_j := f_{e_j}(x_j) g_j f_{\overline{e_{j+1}}}(x_{j+1})^{-1} \tag{11}$$

for $1 \leq j \leq k$, we have

$$\tilde{W} = (\delta_L(\overline{e_1}) f_{\overline{e_1}}(x_1)^{-1}, t_1, \tilde{g}_1, \ldots, t_k, \tilde{g}_k f_{\overline{e_1}}(x_{k+1})).$$

Note that $x_1, \ldots, x_k$ are uniquely determined by these formulas, and we may arrange $x_{k+1} = L_{e_1}^{-1}(x_1)$.

By Lemma 5.10(v), there are $y_J, \tilde{y}_J \in G_{\iota(e_1)}$ for every $J \geq 1$ such that

$$\begin{aligned}
&\delta_L(\overline{e_1}) t_1 g_1 \ldots t_{Jk} g_{Jk} y_J = A'_J(|W|, L_*) = A'_J(|\tilde{W}|, L_*) \\
&= A'_J\big(L_*(f_{\overline{e_1}}(x_{k+1}))^{-1}\delta_L(\overline{e_1}) t_1\tilde{g}_1 \ldots t_k\tilde{g}_k f_{\overline{e_1}}(x_{k+1})\big)
\end{aligned}$$

$$= L_*(f_{\overline{e_1}}(x_{k+1}))^{-1}A'_J(\delta_L(\overline{e_1})t_1\tilde{g}_1\ldots t_k\tilde{g}_k)L_*^{-J+1}(f_{\overline{e_1}}(x_{k+1}))$$
$$= \delta_L(\overline{e_1})f_{\overline{e_1}}(x_1)^{-1}t_1\tilde{g}_1\ldots t_{Jk}\tilde{g}_{Jk}\tilde{y}_J,$$

where $\tilde{g}_j$ for $j > k$ is defined by the convention (7) on page 40. This way we obtain $x_j$ for all $j \geq 1$ such that they satisfy (11) for all $j \geq 1$.

Since $W$ is non-bonding, Lemma 5.9 allows us to define $x_j$ for all $j \in \mathbb{Z}$ such that they satisfy $x_{j+k} = L_{e_j}^{-1}(x_j)$ and (11) for all $j \in \mathbb{Z}$. Lemma 5.14 now proves the assertion. □

Sometimes we will refer to period fitting segments of an element $\eta$ in the fundamental group of $\mathcal{G}$ when there are no bonding cyclic elements. By this we will mean a period fitting segment of any reduced representative of $\eta$. Proposition 5.15 shows that this definition does not depend on the choice of the representative.

## 5.6 Iterating $L_*$ on period fitting segments

Throughout this section, $k'$ denotes an integer such that the difference $k'-k$ is a period of $\eta = \delta_L(\overline{e_1})t_1g_1\ldots t_kg_k$. We write

$$\eta' = \delta_L(\overline{e_1})t_1g_1\ldots t_{k'}g_{k'}f_{\overline{e_1}}(h_1)$$

denote a period fitting segment, where the $h_i \in G_{e_i}$ satisfy

$$g_{i+k'} = f_{e_i}(h_i)g_{i+k}f_{\overline{e_{i+1}}}(h_{i+1})^{-1}. \tag{12}$$

Let $H_j \in G_{e_1}$ be defined recursively by $H_0 = 1$ and

$$H_{j+1} = L_{e_1}^j(h_{jk'+1})H_j$$

for all $j \in \mathbb{Z}$. Recall the notation $A_j(x)$ introduced on page 33. We define

$$B_j = A_{-j+1}(\delta_L(\overline{e_1}))^{-1}L_*^{-j+1}(f_{\overline{e_1}}(H_j)). \tag{13}$$

**Lemma 5.16.** *In the above notation, we have $B_0 = \delta_L(\overline{e_1})^{-1}$, $B_1 = f_{\overline{e_1}}(h_1)$, and*

$$B_{j+1} = f_{\overline{e_1}}(h_{jk'+1})L_*^{-1}(\delta_L(\overline{e_1})B_j)$$

*for all $j \in \mathbb{Z}$.*

*Proof.* Since $H_0 = 1$, we have $B_0 = A_1(\delta_L(\overline{e_1}))^{-1} = \delta_L(\overline{e_1})^{-1}$. Also, $A_0(\delta_L(\overline{e_1})) = 1$ shows that $B_1 = f_{\overline{e_1}}(H_1) = f_{\overline{e_1}}(h_1)$.

To show the recursive formula, we compute

$$\begin{aligned} B_{j+1} &= A_{-j}(\delta_L(\overline{e_1}))^{-1}L_*^{-j}(f_{\overline{e_1}}(H_{j+1})) \\ &= A_{-j}(\delta_L(\overline{e_1}))^{-1}L_*^{-j}(f_{\overline{e_1}}(L_{e_1}^j(h_{jk'+1})))L_*^{-j}(f_{\overline{e_1}}(H_j)) \\ &= f_{\overline{e_1}}(h_{jk'+1})A_{-j}(\delta_L(\overline{e_1}))^{-1}L_*^{-j}(f_{\overline{e_1}}(H_j)) \\ &= f_{\overline{e_1}}(h_{jk'+1})L_*^{-1}\big(\delta_L(\overline{e_1})A_{-j+1}(\delta_L(\overline{e_1}))^{-1}L_*^{-j+1}(f_{\overline{e_1}}(H_j))\big) \\ &= f_{\overline{e_1}}(h_{jk'+1})L_*^{-1}(\delta_L(\overline{e_1})B_j), \end{aligned}$$

where the third equality is by Lemma 5.12. This finishes the proof of the lemma. □

**Lemma 5.17.**

*(i)* $L_*^{-j}(\eta') = B_j^{-1} t_{jk'+1} g_{jk'+1} \dots t_{jk'+k'} g_{jk'+k'} B_{j+1}$,

*(ii)* $A'_j(\eta') = \delta_L(\overline{e_1}) t_1 g_1 \dots t_{jk'} g_{jk'} B_j$,

*(iii)* $A_j(\eta') = A_j(\delta_L(\overline{e_1})) t_{-jk+k+1} g_{-jk+k+1} \dots$
$\dots t_{j(k'-k)+k} g_{j(k'-k)+k} A_{j-1}(\delta_L(\overline{e_1}))^{-1} L_*^{j-1}(B_j)$.

*Proof.* (i) is clear for $j = 0$ because $B_0 = \delta_L(\overline{e_1})^{-1}$ and $B_1 = f_{\overline{e_1}}(h_1)$ by Lemma 5.16. Lemma 5.10(ii) shows that

$$\begin{aligned} L_*^{-1}(t_i g_i) &= L_*^{-1}(\delta_L(\overline{e_i}))^{-1} t_{i+k} g_{i+k} L_*^{-1}(\delta_L(\overline{e_{i+1}})) \\ &\overset{(12)}{=} L_*^{-1}(\delta_L(\overline{e_i}))^{-1} f_{\overline{e_i}}(h_i)^{-1} t_{i+k'} g_{i+k'} f_{\overline{e_{i+1}}}(h_{i+1}) L_*^{-1}(\delta_L(\overline{e_{i+1}})). \end{aligned} \tag{14}$$

We now have the following chain of equivalences:

$$\begin{aligned} &\text{(i) for } j-1 \\ \Leftrightarrow\ & L_*^{-j}(\eta') = L_*^{-1}(B_{j-1})^{-1} L_*^{-1}(t_{jk'-k'+1} g_{jk'-k'+1} \dots t_{jk'} g_{jk'}) L_*^{-1}(B_j) \\ \overset{(14)}{\Leftrightarrow}\ & L_*^{-j}(\eta') = L_*^{-1}(B_{j-1}^{-1} \delta_L(\overline{e_1})^{-1}) f_{\overline{e_1}}(h_{jk'-k'+1})^{-1} t_{jk'+1} g_{jk'+1} \dots \\ & \qquad \dots t_{jk'+k'} g_{jk'+k'} f_{\overline{e_1}}(h_{jk'+1}) L_*^{-1}(\delta_L(\overline{e_1}) B_j). \end{aligned}$$

By Lemma 5.16 this is equivalent to (i) for $j$. This shows (i) for all integers $j$.

(ii) follows directly from (i), the formula (6) on page 33, and $B_0 = \delta_L(\overline{e_1})^{-1}$.

Finally we show (iii). We compute

$$A_j(\eta') = L_*^{j-1}(A'_j(\eta')) \overset{(ii)}{=} L_*^{j-1}(\delta_L(\overline{e_1}) t_1 g_1 \dots t_{jk'} g_{jk'} B_j).$$

Using Lemma 5.10(iii), this can be rewritten as

$$\begin{aligned} A_j(\eta') &= L_*^{j-1}(\delta_L(\overline{e_1})) A_{j-1}(\delta_L(\overline{e_1})) t_{-jk+k+1} g_{-jk+k+1} \dots \\ &\quad \dots t_{j(k'-k)+k} g_{j(k'-k)+k} A_{j-1}(\delta_L(\overline{e_1}))^{-1} L_*^{j-1}(B_j) \\ &= A_j(\delta_L(\overline{e_1})) t_{-jk+k+1} g_{-jk+k+1} \dots \\ &\quad \dots t_{j(k'-k)+k} g_{j(k'-k)+k} A_{j-1}(\delta_L(\overline{e_1}))^{-1} L_*^{j-1}(B_j). \end{aligned}$$

This finishes the proof of (iii). □

Later we will have to study how the basis length of $B_j$ grows when $j \to \infty$. This is the content of the following lemma.

**Lemma 5.18.** *Let $B_j$ be defined as in* (13). *Suppose that $L_{e_1} = 1$, and that the vertex automorphisms and their inverses grow at most polynomially of degree $d-1$. Then the basis length of $B_j$ grows at most polynomially of degree $d$ when $j \to \infty$ or $j \to -\infty$.*

*Proof.* For varying $j \in \mathbb{Z}$, the element $h_{jk'+1}$ can at most assume $k$ different values because $h_{k+i} = L_{e_i}^{-1}(h_i) = h_i$ for all $i \in \mathbb{Z}$ such that $e_i = e_1$. Hence there is a polynomial $P$ of degree $d-1$ such that the basis length of $L_*^{-j+1}(f_{\overline{e_1}}(h_{ik'+1}))$ is at most $P(j)$ for every $i \in \mathbb{Z}$.

Assume first $j \geq 0$. Since $H_j = h_{(j-1)k'+1} \dots h_{k'+1}h_1$, we can use the triangle inequality for the basis length $l$ to see

$$l(L_*^{-j+1}(f_{\overline{e_1}}(H_j))) \leq \sum_{i=0}^{j-1} l(L_*^{-j+1}(f_{\overline{e_1}}(h_{ik'+1}))) \leq jP(j).$$

Since the polynomial $P$ has degree $d-1$, this is an upper bound for $l(L_*^{-j+1}(f_{\overline{e_1}}(H_j)))$ by a polynomial of degree $d$. If $j < 0$, then we use $H_j = h_{jk'+1}^{-1} \dots h_{-2k'+1}^{-1} h_{-k'+1}^{-1}$ to obtain a similar estimate.

By Lemma 4.15 the basis length of $A_{-j+1}(\delta_L(\overline{e_1}))$ grows at most polynomially of degree $d$, so the assertion follows by the definition (13) and the triangle inequality. $\square$

## 5.7 Asymptotic equivalence of $A_j(\eta)$ and $A_j(\eta')$

**Proposition 5.19.** *Let $\eta \in \pi_1(\mathcal{G}, v)$ be $L$-cyclic and non-bonding. Then the following are equivalent:*

*(i) $A_j(\eta, L_*)$ and $A_j(\eta', L_*)$ are asymptotically equivalent,*

*(ii) $A_j(\eta, L_*)$ and $A_j(\eta', L_*)$ are partially asymptotically equivalent,*

*(iii) $\eta'$ is a period fitting segment of $\eta$.*

*Proof.* By Corollary 5.3 we may replace $\eta$ and $\eta'$ with $L_*(\delta)\eta\delta^{-1}$ and $L_*(\delta)\eta'\delta^{-1}$ respectively. Therefore it is no loss of generality to assume an $L$-twistedly reduced expression $\eta = \delta_L(\overline{e_1})t_1g_1 \dots t_kg_k$.

The implication (i)⇒(ii) is trivial.

We now show (ii)⇒(iii). By Lemma 5.4, the sequences $A_j(\eta)$ and $A_j(\eta')$ are partially asymptotically equivalent if and only if the sequences $A'_j(\eta) = L_*^{-j+1}(A_j(\eta))$ and $A'_j(\eta') = L_*^{-j+1}(A_j(\eta'))$ are.

By Proposition 4.8(i), we may write $\eta' = \epsilon * \bar{\eta} * L_*^{-1}(\epsilon)$ such that $\bar{\eta}$ is $L$-twistedly reduced. If we had $pl(\bar{\eta}) = 0$, then the path length of $A'_j(\eta') = \epsilon A'_j(\bar{\eta})L_*^{-j}(\epsilon)$ would be bounded, and $A'_j(\eta)$ and $A'_j(\eta')$ would not be partially asymptotically equivalent. Hence $k' := pl(\bar{\eta}) \geq 1$, and Proposition 4.8 allows us to write

$$\eta' = \epsilon * \delta_L(\overline{e'_1})t'_1g'_1 \dots t'_{k'}g'_{k'} * L_*^{-1}(\epsilon)^{-1}. \tag{15}$$

Lemma 5.10(v) yields reduced expressions

$$A'_j(\eta, L_*) = \delta_L(\overline{e_1})t_1g_1 \dots t_{jk}g_{jk}A_{-j+1}(\delta_L(\overline{e_1}))^{-1},$$
$$A'_j(\eta', L_*)L_*^{-j}(\epsilon) = \epsilon * \delta_L(\overline{e'_1})t'_1g'_1 \dots t'_{jk'}g'_{jk'}A_{-j+1}(\delta_L(\overline{e'_1}))^{-1}.$$

They are partially asymptotically equivalent, so the underlying paths agree on initial segments of unbounded length, which is only possible as follows: There are $q \geq 0$ and $h_i \in G_{e_i}$ such that $e'_i = e_{i+q}$ and $t'_i = t_{i+q}$ for all $i \geq 1$. Furthermore, there are $h_i \in G_{e_i}$ with

$$g'_i = f_{e_{i+q}}(h_{i+q})g_{i+q}f_{\overline{e_{i+q+1}}}(h_{i+q+1})^{-1} \tag{16}$$

for all $i \geq 1$, and

$$\epsilon = \delta_L(\overline{e_1})t_1g_1 \dots t_qg_qf_{\overline{e_{q+1}}}(h_{q+1})^{-1}\delta_L(\overline{e'_1})^{-1}. \tag{17}$$

For $i \geq q+1$ we have

$$e_{i+k'} = e'_{i-q+k'} = e'_{i-q} = e_i = e_{i+k},$$

and

$$\begin{aligned}
&f_{e_{i+k'}}(h_{i+k'})g_{i+k'}f_{\overline{e_{i+k'+1}}}(h_{i+k'+1})^{-1}\\
&\overset{(16)}{=} g'_{i-q+k'} = L_*^{-1}(\delta_L(e'_{i-q})g'_{i-q}\delta_L(\overline{e'_{i-q+1}})^{-1})\\
&\overset{(16)}{=} L_*^{-1}\big(\delta_L(e_i)f_{e_i}(h_i)g_if_{\overline{e_{i+1}}}(h_{i+1})^{-1}\delta_L(\overline{e_{i+1}})^{-1}\big)\\
&= f_{e_i}(L_{e_i}^{-1}(h_i))L_*^{-1}(\delta_L(e_i)g_i\delta_L(\overline{e_{i+1}})^{-1})f_{\overline{e_{i+1}}}(L_{e_{i+1}}^{-1}(h_{i+1}))^{-1}.
\end{aligned}$$

Thus

$$g_{i+k'} = f_{e_i}(h_{i+k'}^{-1}L_{e_i}^{-1}(h_i))g_{i+k}f_{\overline{e_{i+1}}}(L_{e_{i+1}}^{-1}(h_{i+1})^{-1}h_{i+k'+1}) \tag{18}$$

for all $i \geq q+1$. By Lemma 5.9 we can find $h_i$ for all $i \in \mathbb{Z}$ such that (18) and $h_{i+k} = L_{e_i}^{-1}(h_i)$ for all $i \in \mathbb{Z}$. Therefore $k'-k$ is a period of $\eta$.

Lemma 5.10(ii) and (17) allow us to write

$$\begin{aligned}
L_*^{-1}(\epsilon) &= t_{k+1}g_{k+1}\dots t_{k+q}g_{k+q}L_*^{-1}(\delta_L(\overline{e_{q+1}}))L_*^{-1}(f_{\overline{e_{q+1}}}(h_{q+1})^{-1}\delta_L(\overline{e'_1})^{-1})\\
&= t_{k+1}g_{k+1}\dots t_{k+q}g_{k+q}f_{\overline{e_{q+1}}}(L_{e_{q+1}}^{-1}(h_{q+1}))^{-1}.
\end{aligned} \tag{19}$$

Note that

$$\begin{aligned}
t'_ig'_i &\overset{(16)}{=} t_{i+q}f_{e_{i+q}}(h_{i+q})g_{i+q}f_{\overline{e_{i+q+1}}}(h_{i+q+1})^{-1}\\
&= f_{\overline{e_{i+q}}}(h_{i+q})t_{i+q}g_{i+q}f_{\overline{e_{i+q+1}}}(h_{i+q+1})^{-1},
\end{aligned}$$

which implies

$$t'_1g'_1\dots t'_{k'}g'_{k'} = f_{\overline{e_{q+1}}}(h_{q+1})t_{1+q}g_{1+q}\dots t_{k'+q}g_{k'+q}f_{\overline{e_{k'+q+1}}}(h_{k'+q+1})^{-1}. \tag{20}$$

We now conclude that

$$\begin{aligned}
\eta' &\overset{(15)}{=} \epsilon * \delta_L(\overline{e'_1})t'_1g'_1\dots t'_{k'}g'_{k'} * L_*^{-1}(\epsilon)^{-1}\\
&\overset{(17),(19)}{=} \big(\delta_L(\overline{e_1})t_1g_1\dots t_qg_qf_{\overline{e_{q+1}}}(h_{q+1})^{-1}\delta_L(\overline{e'_1})^{-1}\big)\delta_L(\overline{e'_1})t'_1g'_1\dots\\
&\qquad \dots t'_{k'}g'_{k'}\big(f_{\overline{e_{q+1}}}(L_{e_{q+1}}^{-1}(h_{q+1}))g_{k+q}^{-1}t_{k+q}^{-1}\dots g_{k+1}^{-1}t_{k+1}^{-1}\big)\\
&\overset{(20)}{=} \delta_L(\overline{e_1})t_1g_1\dots t_{k'+q}g_{k'+q}f_{\overline{e_{q+1}}}(h_{k'+q+1}^{-1}L_{e_{q+1}}^{-1}(h_{q+1}))g_{k+q}^{-1}t_{k+q}^{-1}\dots g_{k+1}^{-1}t_{k+1}^{-1}\\
&\overset{(18)}{=} \delta_L(\overline{e_1})t_1g_1\dots t_{k'}g_{k'}f_{\overline{e_1}}(h_{k'+1}^{-1}L_{e_1}^{-1}(h_1)).
\end{aligned}$$

Comparing this to (18), we see that $\eta'$ is a period fitting segment of $\eta$.

Finally we check (iii)$\Rightarrow$(i). If $\eta'$ is a period fitting segment of $\eta$, we are in the situation of Lemma 5.17, which shows $A'_j(\eta') = \delta_L(\overline{e_1})t_1 g_1 \dots t_{jk'} g_{jk'} B_j$. We compare this to the (reduced) expression for $A'_j(\eta)$ in Lemma 5.10(v), and we read off that $A'_j(\eta)$ and $A'_j(\eta')$ are asymptotically equivalent for $j \to \infty$. $\square$

# 6 Pre-efficient Dehn twists

## 6.1 Efficient and pointedly efficient Dehn twists

Recall from Definition 4.19 that a Dehn twist $D \in \mathrm{Aut}^0(\mathcal{G})$ is defined by elements $\gamma_e \in Z(G_e)$ such that $\delta_D(e) = f_e(\gamma_e)$. We also recall the twistor $z_e = \gamma_e \gamma_{\bar{e}}^{-1}$.

Cohen and Lustig give a notion of when a Dehn twist is *efficient*. This may be thought of as when the graph of groups $\mathcal{G}$ that the Dehn twist is defined on is, in a certain sense, optimal.

**Definition 6.1.** Let $D$ be the Dehn twist given by $(\gamma_e)_{e \in E(\Gamma)}$. Two edges $e'$ and $e''$ with common terminal vertex $w$ are called

- *positively bonded*, if there are $m, n \geq 1$ such that $f_{e'}(z_{e'}^m)$ and $f_{e''}(z_{e''}^n)$ are conjugate in $G_w$,
- *negatively bonded*, if there are $m \geq 1$ and $n \leq -1$ such that $f_{e'}(z_{e'}^m)$ and $f_{e''}(z_{e''}^n)$ are conjugate in $G_w$.

**Definition 6.2** (cf. Definition 6.2 in [13])**.** A Dehn twist $D$ given by $(\gamma_e)_{e \in E(\Gamma)}$ is called *efficient* if

(1) $\Gamma$ is *minimal:* If $w$ has valence one and $w = \tau(e)$, then the edge map $f_e\colon G_e \to G_w$ is not surjective.

(2) There is *no invisible vertex:* There is no 2-valent vertex $w$ such that both edge maps $f_{e_i}\colon G_{e_i} \to G_w$ are surjective, where $\tau(e_1) = \tau(e_2) = w$ and $e_1 \neq e_2$.

(3) *No unused edges:* For every edge $e$, we have $z_e \neq 1$, or equivalently $\gamma_{\bar{e}} \neq \gamma_e$.

(4) *No proper powers:* If $r^p \in f_e(G_e)$ for some $p \neq 0$, then $r \in f_e(G_e)$.

(5) Whenever $w = \tau(e_1) = \tau(e_2)$, then $e_1$ and $e_2$ are not *positively bonded.*

A Dehn twist $D$ satisfying properties (3)–(5) is called *pre-efficient.*

**Definition 6.3** ([31], Definition 6.3)**.** Let $D$ be a Dehn twist on a graph of groups $\mathcal{G}$ with a chosen basepoint $v$. It is called *pointedly efficient* if

(1*) $\Gamma$ is *minimal away from the basepoint:* if $w \neq v$ has valence one and $w = \tau(e)$, then the edge map $f_e$ is not surjective,

(2*) there is *no invisible vertex away from the basepoint:* There is no 2-valent vertex $w \neq v$ such that both edge maps $f_{e_i}$ are surjective, where $\tau(e_1) = \tau(e_2) = w$ and $e_1 \neq e_2$,

and $D$ is pre-efficient (conditions (3)–(5) of Definition 6.2 are satisfied).

Suppose that $\pi_1(\mathcal{G}, v)$ is isomorphic to the free group $F_n$, and $D$ is an efficient Dehn twist on $\mathcal{G}$. Then each edge and vertex group is a subgroup of $F_n$, so is also free. Condition (3) implies that each edge group has non-trivial centre, as $z_e \in Z(G_e)$, therefore each edge group is infinite cyclic. Conditions (1) and (4) imply that if $w$ is a valence one vertex, then $G_w$ is free of rank at least two. Similarly, conditions (2) and (4) imply that if $w$ is of valence two, then $G_w$ is free of rank at least two, and conditions (3) and (5) imply that if $w$ is a vertex of valence at least three then $G_w$ is also free of rank at least two (cf. Lemma 6.4 of [13] for more detail).

It follows from Section 7.3 of [13] that all vertex groups are finitely generated.

**Definition 6.4.** A subgroup $H$ of a group $G$ is called *malnormal* if, whenever $g \in G$ and $H \cap gHg^{-1} \neq 1$, then $g \in H$.

For a pre-efficient Dehn twist on a graph of groups $\mathcal{G}$ with free fundamental group, Definition 6.2(4) implies that $f_e(G_e)$ is malnormal in $G_{\tau(e)}$ for every edge $e$.

## 6.2 Non-periodicity for pre-efficient Dehn twists

Let $D$ be a pre-efficient Dehn twist on $\mathcal{G}$ given by $(\gamma_e)_{e\in E(\Gamma)}$, i.e. $D$ satisfies properties (3)–(5) of Definition 6.2. We assume throughout the section that $\pi_1(\mathcal{G}, v) \cong F_n$. Since all $\delta_D(e) = f_e(\gamma_e)$ lie in the image of the corresponding edge group and all $D_{\tau(e)} = 1$, Definition 4.4 shows that $\eta = t_1 g_1 \dots t_k g_k$ is $D$-twistedly reduced if and only if it is cyclically reduced. We now assume this to be given.

We again use the notation $t_j = t_{e_j}$ and $g_j$ for every $j \in \mathbb{Z}$, possibly for $j \leq 0$ or $j > k$. Here (7) on page 40 becomes

$$
\begin{aligned}
t_{j+k} &= t_j, \\
g_{j+k} &= f_{e_j}(\gamma_{e_j}) g_j f_{\overline{e_{j+1}}}(\gamma_{\overline{e_{j+1}}})^{-1}.
\end{aligned} \tag{21}
$$

Recall that every $z_e = \gamma_e \gamma_{\overline{e}}^{-1}$ is non-trivial by (3) of Definition 6.2. Let $a_e \in G_e \cong \mathbb{Z}$ be the unique generator such that $z_e = a_e^{n_e}$ with $n_e > 0$. As $z_{\overline{e}} = z_e^{-1}$, this convention implies $a_{\overline{e}} = a_e^{-1}$ and $n_{\overline{e}} = n_e$.

Moreover, we define $n'_e$ to be the unique integer such that $\gamma_e = a_e^{n'_e}$. Since

$$z_e = \gamma_e \gamma_{\overline{e}}^{-1} = a_e^{n'_e} a_{\overline{e}}^{-n'_{\overline{e}}} = a_e^{n'_e + n'_{\overline{e}}},$$

we have

$$n_e = n'_e + n'_{\overline{e}} \tag{22}$$

for all edges $e$. The formula (21) becomes

$$g_{j+k} = f_{e_j}(a_{e_j})^{n'_{e_j}} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{n'_{\overline{e_{j+1}}}}. \tag{23}$$

Recall furthermore the notion of bonded edges given by Definition 6.1.

**Lemma 6.5.** *Let $D$ be a pre-efficient Dehn twist. Suppose $e$ and $e'$ are edges with $\tau(e) = \iota(e') = w$. Let $g \in G_w$ and $r, r', s, s' \in \mathbb{Z}$ such that*

$$f_e(a_e)^r g f_{\overline{e'}}(a_{e'})^s = f_e(a_e)^{r'} g f_{\overline{e'}}(a_{e'})^{s'}. \tag{24}$$

*If $e \neq \overline{e'}$, then $r + s = r' + s'$. If $e$ and $\overline{e'}$ are not bonded, then $r = r'$ and $s = s'$. If $e = \overline{e'}$, then $g \in f_e(G_e)$ or $(r, s) = (r', s')$.*

*Proof.* We can w.l.o.g. assume that $r' = s' = 0$. Using $a_{\overline{e'}} = a_{e'}^{-1}$, we rewrite (24) as

$$f_e(a_e)^r = g f_{\overline{e'}}(a_{\overline{e'}})^s g^{-1}. \tag{25}$$

If $r = s = 0$, then we are done. If one of $r$ and $s$ is non-zero, then both are non-zero, and $e, \overline{e'}$ are bonded (either positively or negatively). If $e \neq \overline{e'}$, they have to be negatively bonded by Definition 6.2(5). Neither $f_e(a_e)$ nor $f_{\overline{e'}}(a_{\overline{e'}})$ is a proper power. Thus we have $r + s = 0$, as asserted.

If $e = \overline{e'}$ and $(r, s) \neq (0, 0)$, then (25) and malnormality of $f_e(G_e)$ in $G_w$ show $g \in f_e(G_e)$. □

**Lemma 6.6.** *Let $D$ be a pre-efficient Dehn twist. Suppose $e_1, \ldots, e_k$ is a closed edge path, $k \geq 1$, and $e_{k+1} = e_1$. Assume that $e_j$ and $\overline{e_{j+1}}$ are bonded for every $j$, $1 \leq j \leq k$. Then there are $i \neq j$ with $1 \leq i, j \leq k$ such that $e_i = \overline{e_{i+1}}$, $e_j = \overline{e_{j+1}}$.*

*Proof.* Suppose first $e_j \neq \overline{e_{j+1}}$ for every $j$, $1 \leq j \leq k$. Since we assume that each such edge pair is bonded, but there are no positively bonded edges, $e_j$ and $\overline{e_{j+1}}$ are negatively bonded for every $j$. As there are no proper powers, there exist $g_1, \ldots, g_k$ such that $g_j \in G_{\tau(e_j)}$ and $g_j f_{\overline{e_{j+1}}}(a_{\overline{e_{j+1}}}) g_j^{-1} = f_{e_j}(a_{e_j})^{-1}$. Since $a_{\overline{e}} = a_e^{-1}$, this means

$$g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}}) g_j^{-1} = f_{e_j}(a_{e_j}).$$

We obtain

$$t_j g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}}) g_j^{-1} t_j^{-1} = t_j f_{e_j}(a_{e_j}) t_j^{-1} = f_{\overline{e_j}}(a_{e_j}),$$

and hence

$$\eta f_{\overline{e_1}}(a_{e_1}) \eta^{-1} = f_{\overline{e_1}}(a_{e_1}),$$

where $\eta := t_1 g_1 \ldots t_k g_k$. Thus $\eta$ lies in a common cyclic subgroup with $f_{\overline{e_1}}(a_{e_1})$. Lemma 4.10(ii) implies $f_{\overline{e_1}}(a_{e_1}) = 1$, which is a contradiction.

We are left to rule out the case that $e_j = \overline{e_{j+1}}$ for exactly one $j$. By cyclic rotation of the indices, we may assume that $e_k = \overline{e_1}$, and that $e_j$ and $\overline{e_{j+1}}$ are negatively bonded for every $j$, $1 \leq j \leq k - 1$.

Assume $e_{k-j+1} = \overline{e_j}$ for some $j$, $1 \leq j < k$, as in Figure 4. Then $e_{k-j}$ and $\overline{e_{k-j+1}} = e_j$ are negatively bonded, and $e_j$ and $\overline{e_{j+1}}$ are negatively bonded. This shows that $e_{k-j}$ and $\overline{e_{j+1}}$ are positively bonded, hence equal. As $e_k = \overline{e_1}$, we see by induction on $j$ that $e_{k-j+1} = \overline{e_j}$ for all $j$, $1 \leq j \leq k$.

If $k$ is odd, then we obtain $e_{\frac{k+1}{2}} = \overline{e_{\frac{k+1}{2}}}$, which is impossible. If $k$ is even, then $e_{\frac{k}{2}} = \overline{e_{\frac{k}{2}+1}}$, which is a contradiction to the assumption that there is exactly one $j$ with $e_j = \overline{e_{j+1}}$. This finishes the proof of the assertion. □

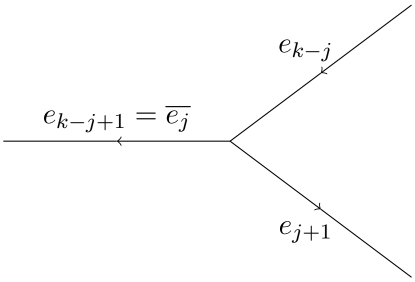


Figure 4: Constellation of the edges in the proof of Lemma 6.6.

**Lemma 6.7.** *Assume that* $\eta = f_{\overline{e_1}}(\gamma_{\overline{e_1}})t_1 g_1 \ldots t_k g_k$ *is a cyclically reduced expression, and* $q, Q \in \mathbb{Z}$ *with* $Q - q \geq k$*. If we have* $m_q, m_{q+1}, \ldots, m_Q \in \mathbb{Z}$ *with*

$$g_j = f_{e_j}(a_{e_j})^{m_j} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{-m_{j+1}} \tag{26}$$

*for all* $j$*,* $q \leq j \leq Q-1$*, then* $m_q = m_{q+1} = \ldots = m_Q = 0$*.*

*Proof.* As $Q - q \geq k$, Lemma 6.6 provides some $j_0$ with $q \leq j_0 \leq Q-1$ such that $e_{j_0}$ and $\overline{e_{j_0+1}}$ are either equal or not bonded. If they are equal, we have $g_{j_0} \notin f_{e_{j_0}}(G_{e_{j_0}})$ because $(t_{j_0}, g_{j_0}, t_{j_0+1})$ is reduced. In either case, Lemma 6.5 shows that $m_{j_0} = m_{j_0+1} = 0$. The formula (26) now implies $m_j = 0$ for all $j$ with $q \leq j \leq Q$. □

**Lemma 6.8.** *Suppose* $\eta = f_{\overline{e_1}}(\gamma_{\overline{e_1}})t_1 g_1 \ldots t_k g_k$ *is a cyclically reduced expression for* $\eta \in \pi_1(\mathcal{G}, u)$*. Assume that*

$$t_q g_q t_{q+1} g_{q+1} \ldots t_Q g_Q = x t_{q-p} g_{q-p} \ldots t_{Q-p} g_{Q-p} y \tag{27}$$

*for some* $p, q, Q \in \mathbb{Z}$ *with* $Q - q \geq 2k$*. Then* $p = 0$ *and* $x = y = 1$*.*

*Proof.* We assume w.l.o.g. $p \geq 0$ in (27). Otherwise replace $x$, $p$, $q$, and $Q$ with $x^{-1}$, $-p$, $q-p$, and $Q-p$ respectively.

Since all $G_e \cong \mathbb{Z}$ generated by $a_e$, Proposition 2.6 provides $m_q, m_{q+1}, \ldots, m_Q \in \mathbb{Z}$ such that

$$x = f_{\overline{e_q}}(a_{e_1})^{-m_q}, \tag{28}$$

$$g_{j-p} = f_{e_j}(a_{e_j})^{m_j} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{-m_{j+1}} \text{ for } q \leq j \leq Q-1, \tag{29}$$

$$g_{Q-p} y = f_{e_k}(a_{e_k})^{m_Q} g_Q. \tag{30}$$

We multiply (29) on the left by $f_{e_j}(a_{e_j})^{n'_{e_j}}$ and on the right by $f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{n'_{\overline{e_{j+1}}}}$. Using (23) and that $e_{j-p} = e_j$, we get

$$g_{j+k-p} = f_{e_{j+k}}(a_{e_{j+k}})^{m_j} g_{j+k} f_{\overline{e_{j+k+1}}}(a_{e_{j+k+1}})^{-m_{j+1}}. \tag{31}$$

After replacing $j$ by $j-k$, this reads

$$g_{j-p} = f_{e_j}(a_{e_j})^{m_{j-k}} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{-m_{j-k+1}}$$

for $q+k \leq j \leq Q-1$. Together with (29), this becomes

$$g_j = f_{e_j}(a_{e_j})^{m_j - m_{j-k}} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{-m_{j+1}+m_{j-k+1}}$$

for all $j$ with $q+k \leq j \leq Q-1$. As $Q-q \geq 2k$, Lemma 6.7 implies that $m_j = m_{j-k}$ for all $j$ with $q+k \leq j \leq Q$.

We now use the formula $m_j = m_{j-k}$ to define $m_j$ for all $j \in \mathbb{Z}$. The above discussion shows that this is consistent with the $m_j$ already defined. In the same way as we proved (31), we now see that (29) for $j$ is equivalent to (29) for $j \pm k$. It now follows that the $m_j$ satisfy (29) for every $j \in \mathbb{Z}$.

Observe that

$$f_{e_j}(a_{e_j})^{-pn'_{e_j}} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{-pn'_{\overline{e_{j+1}}}} \overset{(23)}{=} g_{j-kp} \overset{(29)}{=} f_{e_j}(a_{e_j})^{M_j} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{-M_{j+1}} \qquad (32)$$

for all $j \in \mathbb{Z}$, where

$$M_j := m_j + m_{j-p} + m_{j-2p} + \ldots + m_{j-(k-1)p}.$$

Note that $\overline{e_{j+1}} \neq e_j$ or $g_j \notin f_{e_j}(G_{e_j})$ because $t_j g_j t_{j+1}$ is reduced. Comparing the exponents in (32) with Lemma 6.5, we obtain

$$-p(n'_{e_j} + n'_{\overline{e_{j+1}}}) = M_j - M_{j+1}.$$

Taking the sum over all $j$, $1 \leq j \leq p$, and using (22) on page 51 as well as $e_{p+1} = e_1$, we conclude

$$\begin{aligned}
&-p(n_{e_1} + n_{e_2} + \ldots + n_{e_p}) = M_1 - M_{p+1} \\
&= (m_1 + m_{1-p} + \ldots + m_{1-(k-1)p}) - (m_{1+p} + m_1 + \ldots + m_{1-(k-2)p}) \\
&= m_{1+p-kp} - m_{1+p} = 0.
\end{aligned}$$

Since all $n_{e_j}$ are positive, we conclude $p = 0$. We now apply Lemma 6.7 to (29) and conclude $m_j = 0$ for all $j \in \mathbb{Z}$. By (28) and (30), we conclude $x = 1$ and $y = 1$. □

Recall the notion of a non-bonding cyclic element in Definition 5.8. We now show the following.

**Proposition 6.9.** *With respect to a pre-efficient Dehn twist $D$, every $D$-cyclic element is non-periodic and non-bonding.*

*Proof.* Let $Q - q \geq 2k$. Suppose $e_{j+p} = e_j$ and $g_{j+p} = f_{e_j}(h_j) g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}$ for all $j$ with $q \leq j \leq Q$. Then

$$t_{j+p} g_{j+p} = f_{\overline{e_j}}(h_j) t_j g_j f_{\overline{e_{j+1}}}(h_{j+1})^{-1}$$

for these $j$, and hence

$$t_{q+p} g_{q+p} \ldots t_{Q+p} g_{Q+p} = f_{\overline{e_q}}(h_q) t_q g_q \ldots t_Q g_Q f_{\overline{e_{Q+1}}}(h_{Q+1})^{-1}.$$

By Lemma 6.8 we have $p = 0$ and $h_q = 1$. This implies that all $h_j$ are trivial. □

## 6.3 Cancellation bound at vertex groups with unbonded edges

**Lemma 6.10.** *Let $x$, $x'$, and $g$ be elements in any finitely generated free group.*

*(i) If no positive power of $x$ is a positive power of $gx'g^{-1}$, then there are constants $A, B > 0$ such that*

$$l(x^a g x'^{-b}) \geq A(a+b) - B$$

*for all $a, b \geq 0$.*

*(ii) If $x$ and $gx'g^{-1}$ do not lie in a common cyclic subgroup, then there are $A, B > 0$ such that*

$$l(x^a g x'^{-b}) \geq Aa - B,$$
$$l(x^a g x'^{-b}) \geq Ab - B.$$

*Proof.* Assertion (i) follows roughly because high powers of $x$ and $gx'g^{-1}$ can only have common initial segments of bounded length. More details are left to the reader.

In the situation of (ii), we can apply (i) to $x$ and $gx'^{-1}g^{-1}$ as well as to $x^{-1}$ and $gx'g^{-1}$ to obtain, after possibly redefining $A$ and $B$,

$$l(x^a g(x'^{-1})^b) \geq A(a-b) - B,$$
$$l((x^{-1})^{-a} g x'^{-b}) \geq A(-a+b) - B.$$

Taking the sum of each of these estimates with (i), we obtain the two inequalities in (ii). □

## 6.4 Improving groupoid generating sets

The goal of this section is Corollary 6.13 that, for every pre-efficient Dehn twist, we can find a compatible finite groupoid generating set satisfying the requirement of Lemma 4.3. We need the following lemma, which involves the (basis) length with respect to an infinite group generating set. Its proof is left to the reader.

**Lemma 6.11.** *Let $F$ be a free group with (not necessarily free) generators $x_1, \ldots, x_m$, $y_1, \ldots, y_n$. Denote by $l'$ the length with respect to the (infinite) generating set consisting of $x_1, \ldots, x_m$ and all powers of $y_1, \ldots, y_n$. If no power of $z \in F$ is conjugate to a non-trivial power of some $y_i$, then $l'(z^j)$ grows linearly in $j$.* □

For $y \in \mathbb{R}$ let $\lfloor y \rfloor$ and $\lceil y \rceil$ be the unique integers such that

$$y - 1 < \lfloor y \rfloor \leq y \leq \lceil y \rceil < y + 1.$$

**Proposition 6.12.** *If $x_1, \ldots, x_m$ generate the free group $F$ (not necessarily freely) and $y_1, \ldots, y_n \in F$ are pairwise non-conjugate and no proper powers, then we have for sufficiently large $N$ that $l_N(y_i^j) = \lceil \frac{|j|}{N} \rceil$ for $1 \leq i \leq n$ and $j \in \mathbb{Z}$, where $l_N$ is the length with respect to $x_1, \ldots, x_m$, and all $y_i^j$ with $1 \leq i \leq n$ and $1 \leq j \leq N$.*

*Proof.* It is clear that $l_N(y_i^j) \leq \lceil \frac{|j|}{N} \rceil$.

Fix $i$ with $1 \leq i \leq n$, and let $l'_i$ denote the length on $F$ with respect to $x_1, \ldots, x_m$ and all powers of all $y_{i'}$ with $i' \neq i$. According to Lemma 6.11, choose $N$ sufficiently large such that $l'_i(y_i^j) \geq \frac{|j|}{N}$. Then no power $y_i^j$ can be written as a product of fewer than $\lceil \frac{|j|}{N} \rceil$ elements of the form $x_1^{\pm 1}, \ldots, x_m^{\pm 1}$ or $y_{i'}^k$ with $i' \neq i$, $|k| \leq N$. Hence $l_N(y_i^j) \geq \lceil \frac{|j|}{N} \rceil$. □

**Corollary 6.13.** *Let $D \in \mathrm{Aut}^0(\mathcal{G})$ be a pre-efficient Dehn twist such that $\pi_1(\mathcal{G}, v)$ is finitely generated free. Then there are finite generating sets of the vertex groups $G_w$ with length function $l_w$ such that*

$$\left| l_{\iota(e)}(f_{\overline{e}}(a)) - l_{\tau(e)}(f_e(a)) \right| \leq 2$$

*for all $a \in G_e$.*

*Proof.* As three distinct edges with common terminal vertex are never pairwise bonded, we can construct an orientation $E^+ \subset E(\Gamma)$ such that exactly one of $e$ and $e'$ is in $E^+$ whenever $e$ and $e'$ are negatively bonded. We enlarge $E^+$ to a subset $E' \subset E(\Gamma)$ such that

- for every $e \in E(\Gamma)$ there is a unique $e' \in E'$ such that $e$ and $e'$ are bonded,
- for every $e \in E(\Gamma)$, at least one of $e$ and $\overline{e}$ is in $E'$.

Let now $X_w$ be finite generating sets of the vertex groups $G_w$ such that there is $g \in X_{\tau(e)}$ with $g f_e(a_e) g^{-1} = f_{e'}(a_{e'})^{-1}$ whenever $e$ and $e'$ are negatively bonded. We then apply Proposition 6.12 to find $N \geq 1$ such that $l_{\tau(e)}(f_e(a_e)^j) = \lceil \frac{j}{N} \rceil$ for $e \in E'$, where the $l_w$ are the length functions with respect to

$$X'_w = X_w \cup \left\{ f_e(a_e)^k \middle| 1 \leq k \leq N, e \in E', \tau(e) = w \right\}.$$

This implies by construction that

$$\left\lceil \frac{j}{N} \right\rceil - 2 \leq l_{\tau(e)}(f_e(a_e)^j) \leq \left\lceil \frac{j}{N} \right\rceil + 2$$

for $e \in E(\Gamma) \setminus E'$ and $j \geq 0$. As $e$ or $\overline{e}$ is in $E'$ for every $e \in E(\Gamma)$, this proves the assertion. □

Corollary 6.13 allows us to define a compatible groupoid generating set for $D$ satisfying the requirement of Lemma 4.3. We now assume this for the rest of this chapter.

## 6.5 Lower growth bounds for pre-efficient Dehn twists

Throughout this section we fix a pre-efficient Dehn twist $D \in \mathrm{Aut}^0(\mathcal{G})$. We have already seen in Proposition 4.21 that elements in $\pi_1(\mathcal{G}, u, w)$ grow at most linearly under iteration of $D_*$. In this section we discuss several variants of the growth of $D$ which are bounded *below* by a linear function. All basis lengths are defined using a compatible groupoid generating set as described in the last section.

We shall look at reduced expressions of the form $g_0 t_1 g_1 \ldots t_{k'} g_{k'}$. Recall that, for a pre-efficient Dehn twist, we have $\delta_D(e) = f_e(\gamma_e) = f_e(a_e)^{n'_e}$ with $n_e = n'_e + n'_{\overline{e}}$. Thus

$$D_*^j(t_i) = f_{\overline{e_i}}(a_{e_i})^{-jn'_{\overline{e_i}}} t_i f_{e_i}(a_{e_i})^{-jn'_{e_i}}.$$

We fix $j \in \mathbb{Z}$. For integers $a$, $b$, and $s$, we will abbreviate

$$l_s(a,b) := l\big(f_{e_s}(a_{e_s})^{a-jn'_{e_s}} g_s f_{\overline{e_{s+1}}}(a_{e_{s+1}})^{-b-jn'_{\overline{e_{s+1}}}}\big).$$

Note that $f_{e_s}(a_{e_s})$ and $g_s f_{\overline{e_{s+1}}}(a_{e_{s+1}})^{-1} g_s^{-1}$ do not have positive powers in common because $D$ has no positively bonded edges. Then Lemma 6.10(i) shows that there are $A_s, B_s > 0$ for every $s \in \mathbb{Z}$ such that

$$l_s(a,b) \geq A_s(-a + jn'_{e_s} + b + jn'_{\overline{e_{s+1}}}) - B_s \tag{33}$$

for all $a, b \in \mathbb{Z}$.

**Lemma 6.14.** *If $g_0 t_1 g_1 \ldots t_k g_k$ is a reduced expression and $k \geq 1$, then the basis length $l(D_*^j(g_0 t_1 g_1 \ldots t_k g_k))$ grows linearly when $j \to \pm\infty$.*

*Proof.* There are $A, B > 0$ such that

$$l(g_0 f_{\overline{e_1}}(a_{e_1})^{-m}) \geq Am - B, \tag{34}$$

$$l(f_{e_k}(a_{e_k})^m g_k) \geq -Am - B, \tag{35}$$

and such that (33) holds true with $A_s = A$ and $B_s = B$ when $1 \leq s \leq k-1$. Lemma 4.3(i) allows us to write

$$l(D_*^j(g_0 t_1 g_1 \ldots t_k g_k)) = \min_{m_s} \Big(k + l(g_0 f_{\overline{e_1}}(a_{e_1})^{-m_1 - jn'_{\overline{e_1}}}) + \\ + l(f_{e_k}(a_{e_k})^{m_k - jn'_{e_k}} g_k) + \sum_{s=1}^{k-1} l_s(m_s, m_{s+1})\Big)$$

We insert (33) for $1 \leq s \leq k-1$, (34), and (35) to obtain

$$\begin{aligned} l(D_*^j(g_0 t_1 g_1 \ldots t_k g_k)) &\geq \min_{m_s} \Big(k + A(m_1 + jn'_{\overline{e_1}}) - B + A(-m_k + jn'_{e_k}) - B + \\ &\quad + \sum_{s=1}^{k-1} (A(-m_s + jn'_{e_s} + m_{s+1} + jn'_{\overline{e_{s+1}}}) - B)\Big) \\ &= \min_{m_s} \Big(Aj \sum_{j=1}^{k} (n'_{e_s} + n'_{\overline{e_s}}) + k - (k+1)B\Big) \\ &= Aj(n_{e_1} + \ldots + n_{e_k}) + k - (k+1)B. \end{aligned}$$

As $k \geq 1$ and all $n_e > 0$, this is a lower bound for the growth by a linear function in $j$. This is the desired behaviour for $j \to +\infty$. For $j \to -\infty$, we apply the same arguments to the pre-efficient Dehn twist $D^{-1}$. $\square$

**Lemma 6.15.** *If $t_1g_1\dots t_kg_k$ is a cyclically reduced expression with $k \geq 1$, then the cyclic basis length $l_c(D_*^j(t_1g_1\dots t_kg_k))$ grows linearly when $j \to \pm\infty$.*

*Proof.* For suitable $A, B > 0$ we may assume that $A_s = A$ and $B_s = B$ in (33) for $1 \leq s \leq k$. Lemma 4.3(ii) yields

$$\begin{aligned} l_c(D_*^j(t_1g_1\dots t_kg_k)) &= \min_{m_\bullet}\Big(k + \sum_{s=1}^{k} l_s(m_s, m_{s+1})\Big) \\ &\overset{(33)}{\geq} \min_{m_\bullet}\Big((-B+1)k + A\sum_{s=1}^{k}(-m_s + jn'_{e_s} + m_{s+1} + jn'_{\overline{e_{s+1}}})\Big) \end{aligned}$$

The minimum denoted by $\min_{m_\bullet}$ is always taken over all tuples $(m_1, \dots, m_{k+1}) \in \mathbb{Z}^{k+1}$ such that $m_{k+1} = m_1$. In particular, all $m_s$ in the last sum cancel, and we obtain

$$l_c(D_*^j(t_1g_1\dots t_kg_k)) \geq (-B+1)k + Aj\sum_{s=1}^{k}(n'_{e_s} + n'_{\overline{e_{s+1}}})$$

We now use the formula (22) on page 51 and $e_{k+1} = e_1$. Then the estimate simplifies to

$$l_c(D_*^j(t_1g_1\dots t_kg_k)) \geq (-B+1)k + Aj(n_{e_1} + \dots + n_{e_k})$$

As $D$ is pre-efficient, all $n_e$ are positive. Since $k \geq 1$, this is a lower bound for the growth of the cyclic length by a linear function of positive slope with respect to $j$. This together with the same argument for $D^{-1}$ proves the assertion. □

**Lemma 6.16.** *If $\delta_D(\overline{e_1})t_1g_1\dots t_kg_k$ is a cyclically reduced expression with $k \geq 1$ and $k' \geq 2k+1$, then the basis length of the double coset*

$$f_{e_0}(G_{e_0})D_*^j(g_0t_1g_1\dots t_{k'}g_{k'})f_{\overline{e_{k'+1}}}(G_{e_{k'+1}})$$

*grows linearly when $j \to \pm\infty$.*

We assume that the $g_j$ satisfy (7) on page 40.

*Proof.* By Lemma 6.6, there is $r$ with $1 \leq r \leq k$ such that $e_r$ and $\overline{e_{r+1}}$ are equal or not bonded. For this $r$, neither $f_{e_r}(a_{e_r})$ nor its inverse has a positive power in common with $g_rf_{\overline{e_{r+1}}}(a_{e_{r+1}})g_r^{-1}$. Lemma 6.10(ii) shows that there are $A, B > 0$ such that

$$l_r(a,b) \geq A(b + jn'_{\overline{e_{r+1}}}) - B, \tag{36}$$

$$l_{k+r}(a,b) \geq A(-a + jn'_{e_{k+r}}) - B \tag{37}$$

for $a, b \in \mathbb{Z}$. After possibly redefining $A$ and $B$, we may assume that $A_s = A$ and $B_s = B$ in (33) when $r+1 \leq s \leq r+k-1$. We write

$$\begin{aligned} L(j) &:= l\big(f_{e_0}(G_{e_0})D_*^j(g_0t_1g_1\dots t_{k'}g_{k'})f_{\overline{e_{k'+1}}}(G_{e_{k'+1}})\big) \\ &= l\big(f_{e_0}(G_{e_0})g_0f_{\overline{e_1}}(a_{e_1})^{-jn'_{\overline{e_1}}}t_1f_{e_1}(a_{e_1})^{-jn'_{e_1}}g_1\dots \\ &\quad \dots f_{e_{k'}}(a_{e_{k'}})^{-jn'_{\overline{e_{k'}}}}t_{k'}f_{e_{k'}}(a_{e_{k'}})^{-jn'_{e_{k'}}}g_{k'}f_{\overline{e_{k'+1}}}(G_{e_{k'+1}})\big). \end{aligned}$$

Using Lemma 4.3(iii), we estimate

$$\begin{aligned}L(j) = \min_{m_1,\ldots,m_{k'}\in\mathbb{Z}} &\Big( l\big(f_{e_0}(G_{e_0})g_0 f_{\overline{e_1}}(a_{e_1})^{-jn'_{\overline{e_1}}-m_1}\big) + \sum_{s=1}^{k'-1} l_s(m_s,m_{s+1}) + \\ &+ l\big(f_{e_{k'}}(a_{e_{k'}})^{-jn'_{e_{k'}}+m_{k'}} g_{k'} f_{\overline{e_{k'+1}}}(G_{e_{k'+1}})\big)\Big) + k' \\ \geq \min_{m_s} &\sum_{s=1}^{k'-1} l_s(m_s,m_{s+1}).\end{aligned} \tag{38}$$

Recall that $k' \geq 2k+1 \geq k+r+1$, and we have arranged $A_s = A$ and $B_s = B$ in (33) for $r+1 \leq s \leq r+k-1$. We insert this together with (36) and (37) into (38) to obtain

$$\begin{aligned}L(j) &\overset{(38)}{\geq} \min_{m_s} \Big( \sum_{s=r}^{k+r} l_s(m_s,m_{s+1})\Big) \\ &\geq \min_{m_s} \Big( A(m_{r+1} + jn'_{\overline{e_{r+1}}}) - B + A(-m_{k+r} + jn'_{e_r}) - B + \\ &\quad + \sum_{s=r+1}^{k+r-1} \big(A(-m_s + jn'_{e_s} + m_{s+1} + jn'_{\overline{e_{s+1}}}) - B\big)\Big) \\ &= \min_{m_s} \Big( Aj \sum_{s=r}^{k+r-1} (n'_{e_s} + n'_{\overline{e_s}}) - (k+1)B\Big) \\ &= Aj(n_{e_1} + \ldots + n_{e_k}) - (k+1)B.\end{aligned}$$

Since all $n_{e_s}$ are positive and $A > 0$, this is the desired lower bound for $L(j)$ by a linear function in $j$.

The element $\eta^{-1}$ is $D^{-1}$-twistedly reduced by Lemma 5.5(ii). We can now use a similar argument to bound the growth of $f_{\overline{e_{k'+1}}}(G_{e_{k'+1}})D_*^{-j}(g_{k'}^{-1}t_{k'}^{-1}\ldots g_1^{-1}t_1^{-1}g_0^{-1})f_{e_0}(G_{e_0})$ below by a linear function in $j$ when $j \to \infty$. This leads to a linear lower bound for the coset basis length in the assertion for $j \to -\infty$. □

## 6.6 $A_j(\eta, D_*)$ for pre-efficient Dehn twists

**Lemma 6.17.** *Suppose we are given integers $k' < k''$ and $a, b \in \mathbb{Z}$. Let $D$ be a pre-efficient Dehn twist, and let $\eta = f_{\overline{e_1}}(\gamma_{\overline{e_1}})t_1g_1\ldots t_kg_k$ be cyclically reduced. Then the basis length of*

$$f_{e_{jk'+a+1}}(G_{e_{jk'+a+1}})g_{jk'+a+1}t_{jk'+a+2}g_{jk'+a+2}\ldots t_{jk''+b}g_{jk''+b}f_{\overline{e_{jk''+b+1}}}(G_{e_{jk''+b+1}})$$

*grows quadratically when $j \to \infty$.*

*Proof.* The upper bound for the growth follows easily using the fact that $l(g_j)$ grows linearly when $j \to \pm\infty$ (cf. Lemma 5.11).

The length of the coset in consideration is bounded below by $\sum_{i=r(j)}^{s(j)} L'(i)$, where $r(j) := \lceil \frac{jk'+a}{4k} \rceil$, $s(j) := \lfloor \frac{jk''+b}{4k} \rfloor - 1$, and

$$L'(i) := l\big(f_{e_1}(G_{e_1})g_{4ik+1}t_{4ik+2}g_{4ik+2}\dots t_{4ik+4k}g_{4ik+4k}f_{\overline{e_1}}(G_{e_1})\big).$$

Since $r(j)$ and $s(j)$ can be estimated by functions linear in $j$ of different slopes, the desired quadratic growth will follow when we have shown that $L'(i)$ grows (at least) linearly for $i \to \pm\infty$.

We note that

$$t_{4ik+s}g_{4ik+s} = A_{-4i}(\delta_D(\overline{e_s}))^{-1}D_*^{-4i}(t_sg_s)A_{-4i}(\delta_D(\overline{e_{s+1}}))$$

by Lemma 5.10(iii) and

$$g_{4ik+1} = A_{-4i}(\delta_D(e_1))^{-1}g_1A_{-4i}(\delta_D(\overline{e_2}))$$

by Lemma 5.10(i). Therefore

$$\begin{aligned}&g_{4ik+1}t_{4ik+2}g_{4ik+2}\dots t_{4ik+4k}g_{4ik+4k}\\ &= A_{-4i}(\delta_D(e_1))^{-1}D_*^{-4i}(g_1t_2g_2\dots t_{4k}g_{4k})A_{-4i}(\delta_D(\overline{e_1})).\end{aligned}$$

As $D$ is a Dehn twist, we have $\delta_D(e) \in f_e(G_e)$, and we obtain

$$L'(i) = l\big(D_*^{-4i}(f_{e_1}(G_{e_1})g_1t_2g_2\dots t_{4k}g_{4k}f_{\overline{e_1}}(G_{e_1}))\big).$$

Lemma 6.16 now shows that $L'(i)$ grows at least linearly when $i \to \pm\infty$. □

**Lemma 6.18.** *Let $D$ be a pre-efficient Dehn twist on $\mathcal{G}$. If the cyclically reduced expression $\eta = f_{\overline{e_1}}(\gamma_{\overline{e_1}})t_1g_1\dots t_kg_k \in \pi_1(\mathcal{G},u)$ has positive path length, then the basis length $l(A_j(\eta, D_*))$ grows quadratically when $j \to \pm\infty$.*

*Proof.* It is clear by Lemma 4.15 that the growth can be at most quadratic, so we only have to verify a lower bound. By Lemma 5.10(vi), we have a reduced expression

$$A_j(\eta) = A_j(\delta_L(\overline{e_1}))t_{-jk+k+1}g_{-jk+k+1}\dots t_kg_k.$$

For $j \geq 1$ this implies

$$l(A_j(\eta)) \geq l\big(f_{e_1}(G_{e_1})g_{-jk+k+1}t_{-jk+k+2}g_{-jk+k+2}\dots t_kg_kf_{\overline{e_1}}(G_{e_1})\big)$$

By Lemma 6.17, this length grows quadratically when $j \to \infty$.

If $j \leq -1$, then Remark 4.14 leads to

$$A_j(\eta) = A_j(\delta_L(\overline{e_1}))g_{-jk+k}^{-1}t_{-jk+k}^{-1}\dots g_{k+1}^{-1}t_{k+1}^{-1}.$$

This yields

$$l(A_j(\eta)) \geq l(f_{e_k}(G_{e_k})t_{k+1}g_{k+1}\dots t_{|j|k}g_{|j|k}f_{\overline{e_1}}(G_{e_1})),$$

which grows quadratically by Lemma 6.17 when $j \to -\infty$. □

**Proposition 6.19.** *Let $D$ be a pre-efficient Dehn twist on $\mathcal{G}$, and let $\eta, \eta' \in \pi_1(\mathcal{G}, v)$. Then $A_j(\eta, D_*)^{-1} A_j(\eta', D_*)$ grows quadratically when $j \to \pm\infty$ if $\eta \neq \eta'$ and at least one of $\eta$ and $\eta'$ is $D$-cyclic. Otherwise it grows either linearly or is bounded.*

*Proof.* Since $A_j(\eta)$ and $A_j(\eta')$ grow at most quadratically, we get the upper bound for the growth.

We first assume that both $\eta$ and $\eta'$ are $D$-local, so there are vertices $u$ and $u'$ and $\epsilon \in \pi_1(\mathcal{G}, v, u)$, $\epsilon' \in \pi_1(\mathcal{G}, v, u')$, $x \in G_u$, and $x' \in G_{u'}$ such that $\eta = D_*(\epsilon) x \epsilon^{-1}$ and $\eta' = D_*(\epsilon') x' \epsilon'^{-1}$. Then $A_j(\eta)^{-1} A_j(\eta') = \epsilon x^{-j} D_*^j(\epsilon^{-1}\epsilon') x'^j \epsilon'$ either grows linearly or is bounded.

Assume now that exactly one of $\eta$ and $\eta'$ is $D$-local, $\eta'$ say. As

$$A_j(D_*(\epsilon)\eta\epsilon^{-1})^{-1} A_j(D_*(\epsilon)\eta'\epsilon^{-1}) = \epsilon A_j(\eta)^{-1} A_j(\eta')\epsilon^{-1},$$

we may replace $\eta$ and $\eta'$ with $D_*(\epsilon)\eta\epsilon^{-1}$ and $D_*(\epsilon)\eta'\epsilon^{-1}$ for some $\epsilon$. Therefore we can assume that $\eta = f_{\overline{e_1}}(\gamma_{\overline{e_1}}) t_1 g_1 \dots t_k g_k$ is cyclically reduced. The growth of $A_j(\eta')$ is at most linear, and the one of $A_j(\eta)$ is quadratic by Lemma 6.18. Thus the assertion is clear in that case.

We are left to show the proposition in the case that both $\eta$ and $\eta'$ are $D$-cyclic. If $\eta'$ is a period fitting segment of $\eta$, Proposition 6.9 shows $\eta = \eta'$, and everything is clear.

If $\eta'$ is not a period fitting segment of $\eta$, we may again assume a cyclically reduced expression $\eta = f_{\overline{e_1}}(\gamma_{\overline{e_1}}) t_1 g_1 \dots t_k g_k$. Since $A_j(\eta) = f_{\overline{e_1}}(\gamma_{\overline{e_1}})^j t_{-jk+k+1} g_{-jk+k+1} \dots t_k g_k$ by Lemma 5.10(vi), Proposition 5.19 shows that there is a constant $C$ such that a reduced expression for $A_j(\eta)^{-1} A_j(\eta')$ begins with $g_k^{-1} t_k^{-1} \dots g_{-jk+C}^{-1} t_{-jk+C}^{-1}$. Hence its basis length is bounded below by the one of the double coset

$$f_{e_C}(G_{e_C}) g_{-jk+C} t_{-jk+C+1} g_{-jk+C+1} \dots t_k g_k f_{\overline{e_1}}(G_{e_1}),$$

which grows quadratically by Lemma 6.17 when $j \to \infty$.

For $j \to -\infty$ we use Lemma 4.13(ii), which says $A_j(\zeta, D_*) = D_*^{-1}(A_{-j}(\zeta^{-1}, D_*^{-1}))$ for $\zeta = \eta$ or $\zeta = \eta'$. As $\eta^{-1}$ and $\eta'^{-1}$ are $D^{-1}$-cyclic, the same arguments apply. □

# 7 Prenormalised higher Dehn twists

## 7.1 Property (P)

**Definition 7.1.** An automorphism $\alpha$ of a finitely generated group $G$ satisfies *property (P)* if, for every $x, y \in G$, there is an integer $m(x,y)$ such that the element $A_j(x,\alpha)^{-1}A_j(y,\alpha)$ grows polynomially of degree $m(x,y)$ when $j \to \infty$ or $j \to -\infty$. An automorphism $L \in \mathrm{Aut}^0(\mathbb{G})$ of higher graphs of groups satisfies property (P) if $L_{*u}$ satisfies property (P) for every choice of basepoint $u$.

**Lemma 7.2.** *If $L \in \mathrm{Aut}^0(\mathbb{G})$ satisfies property (P), then all $A_j(\delta)^{-1}L_*^j(\epsilon)A_j(\delta')$ with $\delta, \delta', \epsilon \in \pi_1(\mathbb{G}, u)$ grow polynomially of some degree when $j \to \pm\infty$.*

*Proof.* We have $A_j(\delta)^{-1}L_*^j(\epsilon)A_j(\delta') = A_j(\delta)^{-1}A_j(L_*(\epsilon)\delta'\epsilon^{-1})\epsilon$, and the constant factor $\epsilon$ does not change the growth. □

When $G$ is any finitely generated group and $x \in G$ has infinite order such that the subgroup $\langle x \rangle$ of $G$ is distorted, then, by definition of subgroup distorsion, the growth of the length of $A_j(x,1) = x^j$ is slower then linear, but it is unbounded. If $G$ is a Heisenberg group for instance, then we may realise that the length of $x^j$ grows like $j^{\frac{1}{2}}$ when $j \to \infty$. Thus the identity automorphism of $G$ does not satisfy property (P).

It is well-known (cf. Theorem 6.2 of [22] for example) that automorphisms of free groups either grow polynomially or exponentially. For a polynomially growing $\widehat{\alpha} \in \mathrm{Out}(F_n)$, every conjugacy class in $F_n$ grows polynomially of some integer degree under iteration of $\widehat{\alpha}$. From this fact, it can be deduced that polynomially growing automorphisms of free groups always have property (P).

When building normalised higher Dehn twists of free groups, we will get a new proof of property (P) for higher Dehn twist automorphisms. As some of the ingredient lemmas will hold for more general graphs of groups, we will keep track of property (P) although this is not necessary in the free group case.

Using compatible groupoid generating sets, the inclusion $\pi_1(\mathbb{G}^{(m)}) \to \pi_1(\mathbb{G})$ is an isometric embedding for every $m \geq 1$. In particular:

**Lemma 7.3.** *If the automorphism $L \in \mathrm{Aut}^0(\mathbb{G})$ satisfies property (P), then so does its restriction $L^{(m)} \in \mathrm{Aut}^0(\mathbb{G}^{(m)})$ for every $m \geq 1$.* □

## 7.2 Lower growth bounds for trivial edge groups

**Definition 7.4.** Let $L \in \mathrm{Aut}^0(\mathbb{G})$. An element $\eta \in \pi_1(\mathbb{G}, u)$ grows *dominantly of degree $d$* with respect to $L_*$, if the basis length of $A_j(\eta, L_*)$ grows at least polynomially of degree $d+1$ when $j \to \infty$ or $j \to -\infty$.

In the remainder of this section, $\mathcal{G}$ denotes an ordinary graph of groups with trivial edge groups, and we fix $L \in \mathrm{Aut}^0(\mathcal{G})$.

By means of the following lemma, we can often reduce to the study of dominant growth of $L$-twistedly reduced elements.

**Lemma 7.5.** *If all vertex group automorphisms $L_w$ grow at most polynomially of degree $d-1$, and $\eta \in \pi_1(\mathcal{G}, u)$, $\epsilon \in \pi_1(\mathcal{G}, u', u)$, then $\eta$ grows dominantly of degree $d$ if and only if $L_*(\epsilon)\eta\epsilon^{-1}$ grows dominantly of degree $d$.* □

For the rest of this section we assume that all vertex group automorphisms $L_w$ and inverses $L_w^{-1}$ grow at most polynomially of degree $d-1$ and that they have property (P).

**Lemma 7.6.** *Let $\eta = \delta_L(\overline{e_1})t_1g_1 \dots t_kg_k$ be an $L$-twistedly reduced expression. Then:*

*(i) If there is an $i$ with $1 \leq i \leq k$ such that $g_{i-jk}$ grows at least polynomially of degree $d$ when $j \to \pm\infty$, then $\eta$ grows dominantly of degree $d$.*

*(ii) If all of $g_{1-jk}, g_{2-jk}, \dots, g_{k-jk}$ grow at most polynomially of degree $d-1$ when $j \to \pm\infty$, then $\eta$ does not grow dominantly of degree $d$.*

*Proof.* We will only study $j \to \infty$ because $j \to -\infty$ behaves similarly. Therefore we assume $j \geq 0$. We will use the formula for $A_j(\eta)$ in Lemma 5.10(vi). Note that the basis length of $A_j(\delta_L(\overline{e_1}))$ grows at most polynomially of degree $d$ by Lemma 4.15. Therefore it remains to investigate the growth of $t_{-jk+k+1}g_{-jk+k+1} \dots t_kg_k$. As all edge groups are trivial, using a compatible generating set reduces this to the growth of $g_{i-jk}$ for $j \to \infty$: If there is some $i$ such that $l(g_{i-jk})$ grows at least polynomially of degree $d$, then $\eta$ will grow dominantly. If $l(g_{i-jk})$ grows at most polynomially of degree $d-1$ for every $i$, then $\eta$ will not grow dominantly of degree $d$. □

*Remark* 7.7. The statement of Lemma 7.6 is also valid without the requirement of property (P) for the vertex group automorphisms. However, if all vertex group automorphisms $L_w$ satisfy property (P), then Lemma 7.6 decides in all cases whether $\eta$ grows dominantly of degree $d$. For, Lemma 5.10(i) shows

$$g_{i-jk} = A_j(\delta_L(e_i)))^{-1}L_*^j(g_i)A_j(\delta_L(\overline{e_{i+1}})).$$

Property (P) and Lemma 7.2 ensure that all $g_{i-jk}$ grow polynomially of some degree when $j \to \pm\infty$.

**Corollary 7.8.** *If $\eta = \delta_L(\overline{e_1})t_1g_1 \dots t_kg_k$ with $k \geq 1$ is $L$-twistedly reduced and does not grow dominantly of degree $d$, then $A_j(\eta, L_*)$ grows at most polynomially of degree $d$ when $j \to \pm\infty$.* □

**Lemma 7.9.** *Let $\eta = \delta_L(\overline{e_1})t_1g_1 \dots t_kg_k$ be $L$-twistedly reduced and have dominant growth of degree $d$. Given integers $a$, $b$, and $k' < k''$, the basis length of the element $t_{jk'+a+1}g_{jk'+a+1} \dots t_{jk''+b}g_{jk''+b}$ grows polynomially of degree $d+1$ when $j \to \pm\infty$.*

*Proof.* We first consider $j \to \infty$. It is clear by Lemma 5.11 and the linearly growing path length that the growth cannot be faster than a polynomial of degree $d+1$.

Since $\eta$ grows dominantly, Lemma 7.6 shows that there is some $i$ such that $g_{i-jk}$ grows polynomially of degree $d$ when $j \to \pm\infty$.

By Lemma 5.11, the basis lengths of the elements $t_{jk'+1}g_{jk'+1}\dots t_{jk'+a}g_{jk'+a}$ and $t_{jk''+1}g_{jk''+1}\dots t_{jk''+b}g_{jk''+b}$ grow at most polynomially of degree $d$. Therefore we may assume $a = b = 0$.

Let $j' := \lceil \frac{jk'}{k} \rceil$ and $j'' := \lfloor \frac{jk''}{k} \rfloor$. We estimate

$$l(t_{jk'+1}g_{jk'+1}\dots t_{jk''}g_{jk''}) \geq l(t_{j'k+1}g_{j'k+1}\dots t_{j''k}g_{j''k}) \geq \sum_{m=j'}^{j''-1} l(g_{i+mk}).$$

Since $l(g_{i+mk})$ grows polynomially of degree $d$ with $m \to \infty$, and the number of summands grows linearly in $j$, we see that the basis length of $t_{jk'+1}g_{jk'+1}\dots t_{jk''}g_{jk''}$ grows at least polynomially of degree $d+1$.

For $j \to -\infty$, the convention in Remark 4.14 asks us to study the basis length of $g_{jk'}^{-1}t_{jk'}^{-1}\dots g_{jk''+1}^{-1}t_{jk''+1}^{-1}$, and similar arguments apply. □

**Lemma 7.10.** *If $\eta \in \pi_1(\mathcal{G}, v)$ is $L$-local, then $A_j(\eta, L_*)$ grows at most polynomially of degree $d$ for $j \to \pm\infty$.*

*Proof.* Let $\eta = L_*(\epsilon)x\epsilon^{-1}$, where $x \in G_u$ and $\epsilon \in \pi_1(\mathcal{G}, v, u)$. Then

$$A_j(\eta, L_*) = L_*^j(\epsilon)A_j(x)\epsilon^{-1}.$$

Proposition 4.21 bounds the length of the factor $L_*^j(\epsilon)$ above by a polynomial of degree $d$. By Lemma 4.15, the length of the element $A_j(x)$ does not grow faster than a polynomial of degree $d$. Since the basis length $l(\epsilon^{-1})$ is independent of $j$, we see that $A_j(\eta)$ grows at most polynomially of degree $d$. □

**Proposition 7.11.** *Let $\eta, \eta' \in \pi_1(\mathcal{G}, v)$. Then the basis length of $A_j(\eta, L_*)^{-1}A_j(\eta', L_*)$ grows polynomially of degree $d+1$ if $\eta \neq \eta'$ and at least one of $\eta$ and $\eta'$ grows dominantly of degree $d$. Otherwise the basis length of $A_j(\eta, L_*)^{-1}A_j(\eta', L_*)$ grows at most polynomially of degree $d$ when $j \to \pm\infty$.*

For the proof we need the following lemma.

**Lemma 7.12.** *Suppose $\eta, \eta' \in \pi_1(\mathcal{G}, v)$. If either*

*(i) exactly one of $\eta$ and $\eta'$ grows dominantly of degree $d$, or*

*(ii) both $\eta$ and $\eta'$ grow dominantly and $\eta \neq \eta'$,*

*then $A_j(\eta)^{-1}A_j(\eta')$ grows polynomially of degree $d+1$ when $j \to \pm\infty$.*

*Proof.* Both $A_j(\eta)$ and $A_j(\eta')$ can grow at most polynomially of degree $d+1$, so the upper bound for the growth is clear.

If $\eta$ grows dominantly and $\eta'$ does not, then the basis length of $A_j(\eta)$ grows polynomially of degree $d+1$, and the one of $A_j(\eta')$ grows slower. This proves (i).

We now prove (ii). Lemma 7.5 and the relation

$$A_j(L_*(\epsilon)\eta\epsilon^{-1})^{-1}A_j(L_*(\epsilon)\eta'\epsilon^{-1}) = \epsilon A_j(\eta)^{-1}A_j(\eta')\epsilon^{-1}$$

allow us to $L$-conjugate $\eta$ and $\eta'$ simultaneously. We may therefore assume that we have an $L$-twistedly reduced expression $\eta = \delta_L(\overline{e_1})t_1g_1 \dots t_kg_k$.

Suppose first that $\eta'$ is not a period fitting segment of $\eta$. Then $A_j(\eta)$ and $A_j(\eta')$ are not partially asymptotically equivalent by Proposition 5.19. Using the reduced expression for $A_j(\eta)$ in Lemma 5.10(vi), there is a constant $C \in \mathbb{Z}$ such that a reduced expression for $A_j(\eta)^{-1}A_j(\eta')$ begins with $g_k^{-1}t_k^{-1}g_{k-1}^{-1}t_{k-1}^{-1} \dots g_{-jk+C}^{-1}t_{-jk+C}^{-1}$. By Lemma 7.9, this element grows polynomially of degree $d+1$. Thus $A_j(\eta)^{-1}A_j(\eta')$ grows polynomially of degree $d+1$ when $j \to \pm\infty$.

If $\eta'$ is a period fitting segment of $\eta$, we can write $\eta'$ in the form as in Section 5.6. If $k = k'$, then $\eta = \eta'$, and everything is clear. For notational convenience we now assume $k' > k$. Combining Lemma 5.10(vi) and Lemma 5.17(iii), we obtain

$$A_j(\eta)^{-1}A_j(\eta') = t_{k+1}g_{k+1} \dots t_{j(k'-k)+k}g_{j(k'-k)+k}A_{j-1}(\delta_L(\overline{e_1}))^{-1}L_*^{j-1}(B_j).$$

The basis length of this is bounded below by the one of $t_{k+1}g_{k+1} \dots t_{j(k'-k)}g_{j(k'-k)}$, which grows polynomially of degree $d+1$ by Lemma 7.9.

We are left to investigate $A_{-j}(\eta)^{-1}A_{-j}(\eta')$ when $j \to \infty$. By Lemma 4.13(ii), we have $A_{-j}(\eta, L_*) = L_*^{-1}(A_j(\eta^{-1}, L_*^{-1}))$. Moreover, $\eta$ grows dominantly with respect to $L_*$ if and only if $\eta^{-1}$ does with respect to $L_*^{-1}$. Then similar arguments work. □

*Proof of Proposition 7.11.* As usual, the upper growth bound is clear. Lemma 7.12 tackles all cases except the one that neither $\eta$ nor $\eta'$ grows dominantly of degree $d$.

If $\eta$ is $L$-local, Lemma 7.10 shows that $A_j(\eta)$ grows at most polynomially of degree $d$. If $\eta$ is $L$-cyclic, Corollary 7.8 together with property (P) for the vertex group automorphisms shows that $A_j(\eta)$ grows at most polynomially of degree $d$. As none of $\eta$ and $\eta'$ grows dominantly, this argument bounds the basis lengths of both $A_j(\eta)$ and $A_j(\eta')$ above by polynomials of degree $d$. □

**Proposition 7.13.** *If every $L$-cyclic element grows dominantly of degree $d$, then $L$ satisfies property (P).*

Recall that we assume property (P) for the vertex group automorphisms. This proposition will then help us to show property (P) inductively for higher Dehn twists.

*Proof of Proposition 7.13.* We have to show that $A_j(\eta)^{-1}A_j(\eta')$ always grows polynomially of some degree when $j \to \pm\infty$. In Lemma 7.12, we have seen this for all relevant cases except for the case of two $L$-local elements $\eta$ and $\eta'$, which we now focus on:

Let $\eta = L_*(\epsilon)x\epsilon^{-1}$ and $\eta' = L_*(\epsilon')x'\epsilon'^{-1}$ for some vertices $u$ and $u'$ with elements $\epsilon \in \pi_1(\mathcal{G}, v, u)$, $\epsilon' \in \pi_1(\mathcal{G}, v, u')$, $x \in G_u$, and $x' \in G_{u'}$. Then we have

$$A_j(\eta)^{-1}A_j(\eta') = \epsilon A_j(x)^{-1}L_*^j(\epsilon^{-1}\epsilon')A_j(x')\epsilon'^{-1}.$$

As the constant factors $\epsilon$ and $\epsilon'^{-1}$ do not affect the growth type, we have to show that $A_j(x)^{-1}L_*^j(\epsilon^{-1}\epsilon')A_j(x')$ grows polynomially of some (integer) degree. To this end, we fix a reduced expression

$$\epsilon^{-1}\epsilon' = g_0t_1g_1 \dots t_kg_k.$$

This leads to a reduced expression

$$A_j(x)^{-1}L_*^j(\epsilon^{-1}\epsilon')A_j(y) = \tilde{g}_0(j)t_1\tilde{g}_1(j)\dots t_k\tilde{g}_k(j)$$

with

$$\begin{aligned}
&\tilde{g}_0(j) := A_j(x)^{-1}L_*^j(g_0)A_j(\delta_L(\overline{e_1})),\\
&\tilde{g}_i(j) := A_j(\delta_L(e_i))^{-1}L_*^j(g_i)A_j(\delta_L(\overline{e_{i+1}})) \text{ for } 1 \le i \le k-1,\\
&\tilde{g}_k(j) := A_j(\delta_L(e_k)^{-1}L_*^j(g_k)A_j(y).
\end{aligned}$$

As all vertex group automorphisms have property (P), Lemma 7.2 shows that the basis lengths of $\tilde{g}_i(j)$ grow polynomially of some degree $d_i$ when $j \to \pm\infty$. Therefore the basis length of $A_j(\eta)^{-1}A_j(\eta')$ grows polynomially of degree $\max_i d_i$. □

## 7.3 Truncatable replacements

**Definition 7.14.** A *truncatable replacement* of $D \in \mathrm{Aut}(\mathbb{G})$ is an automorphism of the form $HDH^{-1} \in \mathrm{Aut}(\mathbb{G}')$, where $H : (\mathbb{G}, v) \to (\mathbb{G}', v')$ is an equivalence such that $(\mathbb{G}', v')$ is fully truncatable.

**Lemma 7.15.** *Let $H : \mathcal{G} \to \mathcal{G}'$ be an equivalence of ordinary graphs of groups and $D \in \mathrm{Aut}^0(\mathcal{G})$ a Dehn twist. Then:*

*(i) $HDH^{-1} \in \mathrm{Aut}(\mathcal{G}')$ is a Dehn twist. The terms $\gamma'_{e'}$ and $z'_{e'}$ of $HDH^{-1}$ for edges $e' \in E(\Gamma')$ are given by $\gamma'_{H(e)} = H_e(\gamma_e)$ and $z'_{H(e)} = H_e(z_e)$, where the $\gamma_e$ and $z_e$ are the data of $D$.*

*(ii) $HDH^{-1}$ is efficient (or pointedly efficient, or pre-efficient) if and only if $D$ is.*

*Proof.* It is clear that all vertex and edge group automorphisms of $HDH^{-1}$ are trivial. Furthermore, $\delta_D(e) = f_e(\gamma_e)$ leads to

$$\begin{aligned}
\delta_{HDH^{-1}}(H(e)) &= (HDH^{-1})_*(\delta_H(e))^{-1}H_*(\delta_D(e))\delta_H(e)\\
&= \delta_H(e)^{-1}H_*(f_e(\gamma_e))\delta_H(e) = f'_{H(e)}(H_e(\gamma_e))
\end{aligned}$$

for $e \in E(\Gamma)$. This finishes the proof of (i).

As $H$ is an equivalence of ordinary graphs of groups, $H_V$ and $H_E$ form an underlying graph isomorphism. It follows immediately from Definition 2.4(7) that the edge map $f'_{H(e)}$ of $\mathcal{G}'$ is surjective if and only if the edge map $f_e$ of $\mathcal{G}$ is.

The formula for the twistors in (i) shows that $HDH^{-1}$ has unused edges if and only if $D$ has.

Moreover, Definition 2.4(7) shows that there are proper powers in $HDH^{-1}$ if and only if there are some in $D$.

From the relation

$$H_*(f_e(z_e)) = \delta_H(e)f_{H(e)}(z'_{H(e)})\delta_H(e)^{-1},$$

we read off that $e, e' \in E(\Gamma)$ are positively bonded if and only if $H(e), H(e') \in E(\Gamma')$ are. □

**Lemma 7.16.** *If $H : \mathbb{G} \to \mathbb{G}'$ is an equivalence and $D \in \mathrm{Aut}^0(\mathbb{G})$ is a higher Dehn twist, then $HDH^{-1}$ is a higher Dehn twist on $\mathbb{G}'$.*

*Proof.* Let $\Gamma_0 \subset \Gamma$ be the subgraph with $V(\Gamma_0) = V(\Gamma)$ and $e \in E(\Gamma_0)$ if and only if $G_e \neq 1$. Define $\Gamma_0'$ similarly. Then $(HDH^{-1})|_{\Gamma_0'} = (H|_{\Gamma_0})(D|_{\Gamma_0})(H|_{\Gamma_0})^{-1}$. Since $D|_{\Gamma_0}$ is an ordinary Dehn twist and $H|_{\Gamma_0} : \mathbb{G}|_{\Gamma_0} \to \mathbb{G}'|_{\Gamma_0'}$ an equivalence, Lemma 7.15 shows that $(HDH^{-1})|_{\Gamma_0'}$ is a Dehn twist, so $HDH^{-1}$ is a higher Dehn twist. □

**Lemma 7.17.** *Let $H : \mathbb{G} \to \mathbb{G}'$ be an equivalence and $\eta \in \pi_1(\mathbb{G}', u, w)$. Then:*

*(i) The growth of $\eta$ under iteration of $(HDH^{-1})_*$ and the growth of $H_*^{-1}(\eta)$ under iteration of $D_*$ are equivalent up to a multiplicative constant.*

*(ii) If $u = w$, then $A_j(\eta, (HDH^{-1})_*) = H_*(A_j(H_*^{-1}(\eta), D_*))$.*

*(iii) If $u = w$, then the growth of the conjugacy class $[\eta]$ under iteration of $\widehat{HDH^{-1}}$ and the growth of $[H_*^{-1}(\eta)]$ under iteration of $\widehat{D}$ are equivalent.* □

*Remark* 7.18. The conclusion of Proposition 7.11 is also valid when $D$ is a higher Dehn twist on a higher graph of groups $\mathbb{G}$ of degree $d$ such that $D^{(d-1)}$ satisfies property (P). If $\mathbb{G}$ is truncatable at degree $d-1$, then the vertex automorphisms of $T^{d-1}D$ grow at most polynomially of degree $d-1$ by Proposition 4.21, and all requirements of Proposition 7.11 are clear. Otherwise we pick a truncatable replacement and apply Lemma 7.17.

## 7.4 Definition of prenormalised higher Dehn twists

Normalised higher Dehn twists will be a special case of prenormalised higher Dehn twists, which we define by induction on the degree. A prenormalised higher Dehn twist always comes with a set of vertices called clutching points. The precise definitions are as follows.

By the degree of a higher Dehn twist $D \in \mathrm{Aut}^0(\mathbb{G})$ we always mean the degree of the underlying higher graph of groups $\mathbb{G}$.

**Definition 7.19.** A higher Dehn twist $D \in \mathrm{Aut}^0(\mathbb{G})$ of degree at most one is *prenormalised* if

- it has no trivial edge groups, and
- it is pre-efficient (satisfies properties (3)–(5) of Definition 6.2).

A *clutching point* of $D$ is a vertex $w \in V(\Gamma)$ of valence one or two with only surjective edge maps.

Recall from Definition 4.20 that a higher Dehn twist $D$ of degree 1 is an ordinary Dehn twist together with edges $e$ such that $G_e = 1$ and possibly $\delta_D(e) \neq 1$. If $D$ is prenormalised, it has no trivial edge groups, and we see that $D$ is an ordinary Dehn twist. We now come to the inductive definition in higher degree.

**Definition 7.20.** A higher Dehn twist $D \in \mathrm{Aut}^0(\mathbb{G})$ of degree $d \geq 2$ is *prenormalised* if

(1) $D^{(d-1)} = D|_{\Gamma^{(d-1)}}$ is prenormalised,

(2) for every edge $e$ of degree $d$, exactly one of $\delta_D(e)$ and $\delta_D(\overline{e})$ is trivial,

(3) all $\delta_D(e)$ with $\deg(e) = d$ are $D^{(d-1)}$-twistedly reduced in the truncated sense,

(4) when $\deg(e) = d$ and $G_{\tau(e)} = 1$, then $\delta_D(e) = 1$,

(5) when $\deg(e) = d$, $e' \in E(\Gamma)$, and $\epsilon \in \pi_1(\mathbb{G}^{(\deg(e')-1)}, \tau(e), \tau(e'))$, then $\delta_D(e) \neq D_*(\epsilon)\delta_D(e')\epsilon^{-1}$, or $\delta_D(e) = \delta_D(e') = 1$, or $e = e'$,

(6) when $\deg(e) = d$ and $\delta_D(e) \neq 1$, then $A_j(\delta_D(e), D_*)$ grows polynomially of degree $d$ when $j \to \pm\infty$.

A *clutching point* of $D$ is a 1-valent vertex $w$ with $G_w = 1$ or a clutching point of $D^{(d-1)}$ which is not the terminal vertex of an edge $e$ of degree $d$ with $\delta_D(e) = 1$.

The most important special cases of prenormalised higher Dehn twists are those with at most one clutching point:

**Definition 7.21.** A higher Dehn twist $D \in \mathrm{Aut}(\mathbb{G})$ is called *pointedly normalised* if the underlying graph $\Gamma$ is connected, and it has no clutching points away from the basepoint $v$. The higher Dehn twist $D$ is called *normalised* if $\Gamma$ is connected, and $D$ has no clutching points.

Note that the identity of any group is an efficient Dehn twist whose underlying graph is a point, so Definition 7.19 allows us to have some connected components which are single points. This is exactly how clutching points in higher degree show up as 1-valent trivial vertex groups in Definition 7.20 when we connect such a point to other components for the first time when building up the Dehn twist by its strata.

By Proposition 4.21 the growth of the Dehn twist $D^{(m)}$ of degree $m$ is at most polynomial of degree $m$. In (6), it is then always clear by Lemma 4.15 that the growth of $A_j(\delta_D(e), D_*)$ is bounded *above* by a polynomial of degree $d$.

## 7.5 Growth of prenormalised higher Dehn twists

Let $\mathbb{G}$ be a higher graph of groups such that $\pi_1(\mathbb{G}, u)$ is finitely generated free for every $u \in V(\Gamma)$.

**Proposition 7.22.** *For a prenormalised higher Dehn twist $D$ on $\mathbb{G}$ we have:*

*(i) For every reduced word $W = (x, t_1, g_1, \dots, t_k, y)$ from $u$ to $w$, the element $|W|$ grows polynomially under iteration of $D_*$ with degree equal to the maximum of all $\deg(e_i)$.*

*(ii)* *For every reduced word $W \neq 1$ representing an element $D$-twistedly reduced in the truncated sense in $\mathbb{G}^{(m)}$ such that $\max_i \deg(e_i) = m$, the basis length of $A_j(|W|, D_*)$ grows polynomially of degree $m+1$.*

*(iii)* *For every cyclically reduced $W = (t_1, g_1, \ldots, t_k, g_k)$ from $u$ to $u$, the conjugacy class of $|W|$ grows polynomially under iteration of $\widehat{D}$ with degree equal to the maximum of all* $\deg(e_i)$*.*

*(iv)* *$D$ satisfies property (P).*

To prove Proposition 7.22 by a certain induction argument, we need this lemma:

**Lemma 7.23.** *Let $\mathbb{G}$ be a higher graph of groups of degree $d \geq 1$ such that Proposition 7.22(iii) holds true for $\mathbb{G}$ with the prenormalised higher Dehn twist $D$. Suppose $\eta, \zeta \in \pi_1(\mathbb{G}, u)$ such that $\eta$ grows dominantly of degree $d$ and $D_*(\zeta)\eta\zeta^{-1} = \eta$. Then $\zeta = 1$.*

*Proof.* Since $D_*(\zeta) = \eta\zeta\eta^{-1}$, the conjugacy class of $\zeta$ does not grow under iteration of $D_*$. Using Proposition 7.22(iii), we see that $\zeta = \epsilon x \epsilon^{-1}$, where $x \in G_{u'}$ for some vertex $u'$ and $\epsilon \in \pi_1(\mathbb{G}, u, u')$. Writing $\eta' = D_*(\epsilon)^{-1}\eta\epsilon$, we compute

$$\begin{aligned} \eta' x \eta'^{-1} &= (D_*(\epsilon)^{-1}\eta\epsilon)(\epsilon^{-1}\zeta\epsilon)(\epsilon^{-1}\eta^{-1}D_*(\epsilon)) = D_*(\epsilon)^{-1}\eta\zeta\eta^{-1}D_*(\epsilon) \\ &= D_*(\epsilon^{-1}\zeta\epsilon) = D_*(x) = x. \end{aligned}$$

Thus the elements $x$ and $\eta'$ commute. As $\pi_1(\mathbb{G}, u)$ is free, $x$ and $\eta'$ lie in a common cyclic subgroup.

As $\eta$ grows dominantly of degree $d$, Lemma 7.5 (applied to a truncatable replacement of $D$) shows that $\eta'$ does as well. In particular $\eta' \notin G_{u'}$, so $pl(\eta') \geq 1$. Lemma 4.10(ii) proves $x = 1$, whence $\zeta = \epsilon x \epsilon^{-1} = 1$. □

**Lemma 7.24.** *Let $D \in \mathrm{Aut}^0(\mathbb{G})$ be prenormalised of degree $d$ and satisfying Proposition 7.22. Suppose we are given $D$-twistedly reduced elements $\eta \in \pi_1(\mathbb{G}, u)$ and $\eta' \in \pi_1(\mathbb{G}, u')$ going across edges of degree $d$ (if $d \geq 1$) as well as $\zeta \in \pi_1(\mathbb{G}, u, u')$. If the basis length of $A_j(\eta, D_*)^{-1}D_*^j(\zeta)A_j(\eta', D_*)$ does not grow polynomially of degree $d+1$ when $j \to \infty$, then $\eta = D_*(\zeta)\eta'\zeta^{-1}$.*

*Proof.* Note that

$$A_j(\eta, D_*)^{-1}D_*^j(\zeta)A_j(\eta', D_*) = A_j(\eta, D_*)^{-1}A_j(D_*(\zeta)\eta'\zeta^{-1}, D_*) \cdot \zeta.$$

The constant factor $\zeta$ does not contribute to the growth degree. As we assume Proposition 7.22(iv), $D$ satisfies property (P). Then $D^{(d-1)}$ satisfies property (P) by Lemma 7.3 if $d \geq 2$. If $d = 1$, then property (P) for $D^{(0)} = 1$ can be checked using that all vertex groups are free. By Proposition 7.22(ii), both $\eta$ and $\eta'$ grow dominantly of degree $d$. Proposition 7.11 together with Remark 7.18 now shows $\eta = D_*(\zeta)\eta'\zeta^{-1}$. □

*Proof of Proposition 7.22.* We say that the proposition holds true in degree $d$ if (i)–(iii) are satisfied for words $W$ going only across edges $e_i$ of degree at most $d$, and if (iv) holds true in the case that $\mathbb{G}$ only has edges of degree at most $d$. It now suffices to show inductively that the proposition holds true for all degrees $d \geq 0$.

If $d = 0$, then $W$ lies in a single vertex group, and the growth is clearly polynomial of degree zero in both cases (i) and (iii), and it is linear in (ii). Part (iv) is clear because the identity automorphism of a free group satisfies property (P).

In the case $d = 1$ we have an ordinary pre-efficient Dehn twist. The growth in (i) is linear by Lemma 6.14. In (ii), we have quadratic growth by Lemma 6.18. The growth in (iii) is linear by Lemma 6.15. Moreover, property (P) is given by Proposition 6.19.

We now assume that $d \geq 2$, and that the assertion has already been proved for degree at most $d-1$. We write $T^{d-1}W =: (\theta_0, t_{E_1}, \theta_1, \ldots, t_{E_l}, \theta_l)$ for the truncation of $W$ (cf. Section 3.2). We now prove (i). By Lemma 4.17(i) and Remark 4.18, we have to investigate the length of $X_{0,j}, X_{1,j}, \ldots, X_{l,j}$ for $j \to \pm\infty$, where

$$\begin{aligned}
X_{0,j} &:= D_*^j(\theta_0) A_j(\delta_D(\overline{E_1})),\\
X_{i,j} &:= A_j(\delta_D(E_i))^{-1} D_*^j(\theta_i) A_j(\delta_D(\overline{E_{i+1}})) \text{ for } 1 \leq i \leq l-1,\\
X_{l,j} &:= A_j(\delta_D(E_l))^{-1} D_*^j(\theta_l).
\end{aligned}$$

As we know property (P) for $D^{(d-1)}$, all sequences $X_{0,j}, \ldots, X_{l,j}$ grow polynomially of some degrees. We have to show that at least one $X_{i,j}$ grows polynomially of degree $d$ for $j \to \pm\infty$. If $\delta_D(\overline{E_1})$ or $\delta_D(E_l)$ is non-trivial, it grows dominantly of degree $d-1$, and we are done. Assume now that $\delta_D(\overline{E_1}) = 1$ and $\delta_D(E_l) = 1$.

As exactly one of $\delta_D(e)$ and $\delta_D(\overline{e})$ is trivial for every edge $e$ of degree $d$, there is some $i$, $1 \leq i \leq l-1$, such that both $\delta_D(E_i)$ and $\delta_D(\overline{E_{i+1}})$ are non-trivial, so both grow dominantly of degree $d-1$. If the basis length of $X_{i,j}$ does not grow (at least) polynomially of degree $d$, Lemma 7.24 shows $\delta_D(E_i) = D_*(\theta_i)\delta_D(\overline{E_{i+1}})\theta_i^{-1}$. Definition 7.20(5) shows $\overline{E_{i+1}} = E_i$. As $\delta_D(E_i)$ grows dominantly of degree $d-1$, and $\delta_D(E_i) = D_*(\theta_i)\delta_D(E_i)\theta_i^{-1}$, Lemma 7.23 shows $\theta_i = 1$. But this is a contradiction to the fact that $T^{d-1}W$ is reduced in the truncated sense, whence (i).

Assertions (ii) and (iii) follow similarly. As every $D$-cyclic element going across edges of degree $d$ grows dominantly of degree $d$ by (ii), property (P) for $D$ follows from Proposition 7.13 (in a version for higher graphs of groups) and property (P) for $D^{(d-1)}$, so we also know (iv). $\square$

We remark that Proposition 7.22 as well as Lemmas 7.23 and 7.24 also hold true for truncatable replacements $HDH^{-1}$ of prenormalised higher Dehn twists $D$.

## 7.6 Dehn twists in rank at most one

The goal of this section is to discuss prenormalised higher Dehn twists with trivial or infinite cyclic fundamental group.

**Definition 7.25.** A *cylinder of length k* is a graph of groups $\mathcal{G}$ with underlying graph $\Gamma$ an interval with $k$ edges $e_1, \ldots, e_k$ such that $\tau(e_j) = \iota(e_{j+1})$ for $1 \leq j \leq k-1$, and such that all vertex groups are infinite cyclic and all edge maps $f_e$ are isomorphisms.

The terminology "cylinder" is motivated by the corresponding graph of spaces shown in Figure 5. If $\Gamma = \{v\}$ and $G_v = \mathbb{Z}$, then we call $\mathcal{G}$ a cylinder of length zero.

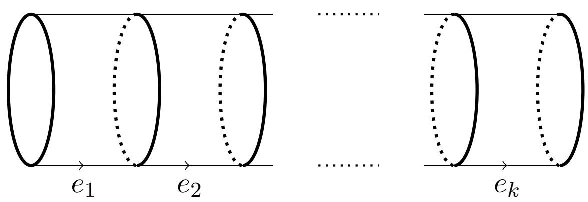


Figure 5: A cylinder of length $k$.

A higher graph of groups $\mathbb{G}$ is called cylinder of length $k$ if its underlying ordinary graph of groups is a cylinder of length $k$. This forces all edges to have degree one because there are no trivial edge groups.

Whenever $D$ is a (higher) Dehn twist on a cylinder $\mathcal{G}$ (or $\mathbb{G}$), then $D_{*v} = 1$ for every choice of basepoint $v$.

If $\Gamma$ is a point and $G_v = 1$, then we sometimes refer to $\mathcal{G}$ or $\mathbb{G}$ as a point.

**Definition 7.26.** A graph of groups $\mathcal{G}$ is called *mapping torus* if the underlying graph $\Gamma$ is a (possibly subdivided) circle and all edge maps $f_e$ are surjective.

**Lemma 7.27.** *Let $\mathcal{G}$ be a graph of groups with finitely generated vertex groups and free fundamental group. Assume $\mathcal{G}$ is a mapping torus. Then all its vertex groups are trivial.*

*Proof.* As $\mathcal{G}$ is a mapping torus with finitely generated $G_v$, the fundamental group is isomorphic to a semidirect product $G_v \rtimes \mathbb{Z}$. A $K(\pi_1(\mathcal{G}, v), 1)$-space can be built as a (topological) mapping torus over $K(G_v, 1)$. It has Euler characteristic zero. Therefore the rank of the free group $\pi_1(\mathcal{G}, v)$ has to be one. $\square$

**Proposition 7.28.** *Suppose that all vertex groups $G_w$ of $\mathbb{G}$ are either infinite cyclic or trivial. If $\mathbb{G}$ is connected and there is a prenormalised higher Dehn twist $D \in \mathrm{Aut}^0(\mathbb{G})$, then $\mathbb{G}$ is either a point or a cylinder of some length $k \geq 0$.*

*Proof.* The proof is by induction on the degree $d$ of $\mathbb{G}$. If $d \leq 1$, then $D$ is a pre-efficient Dehn twist on an ordinary graph of groups. If there is an edge $e$, then $G_e \cong \mathbb{Z}$. This forces $G_{\tau(e)}$ to be non-trivial, so $G_{\tau(e)} \cong \mathbb{Z}$. By Definition 6.2(4), we know that there are no proper powers, so every $f_e : G_e \to G_{\tau(e)}$ is an isomorphism. As there are no positively bonded edges, all vertices have valence at most two. By Lemma 7.27 the graph $\Gamma$ is not a circle, hence it is an interval. This shows that $\mathbb{G}$ is a cylinder.

If $d \geq 2$, then we know by induction that $\mathbb{G}^{(d-1)}$ is a disjoint union of points and cylinders. Whenever $e$ is an edge of degree $d$ with $\tau(e)$ lying in a cylinder component, then $D^{(d-1)}$ acts trivially on $\delta_D(e)$. This is also true if $G_{\tau(e)} = 1$ is an isolated trivial vertex group. Therefore $A_j(\delta_D(e)) = \delta_D(e)^j$. Since the basis length of this element cannot grow polynomially of degree $d$, Definition 7.20(6) shows $\delta_D(e) = 1$. Similarly $\delta_D(\overline{e}) = 1$, but his is a contradiction to Definition 7.20(2). Therefore $\mathbb{G}$ has no edge of degree $d$. Since it is connected, it is of the desired form. □

**Corollary 7.29.** *Let $D$ be a prenormalised higher Dehn twist or a truncatable replacement on a connected $\mathbb{G}$.*

*(i) If $\pi_1(\mathbb{G}, v)$ is trivial, then $\mathbb{G}$ is a point.*

*(ii) If $\pi_1(\mathbb{G}, v) \cong \mathbb{Z}$, then there is a $k \geq 0$ such that $\mathbb{G}$ is a cylinder of length $k$.*

*Proof.* Assume first that $D$ is a prenormalised higher Dehn twist. As all vertex groups inject into the fundamental group $\pi_1(\mathbb{G}, v)$, they have to be either trivial or infinite cyclic. Then Proposition 7.28 gives rise to the desired conclusion.

If $D$ is a truncatable replacement of a prenormalised higher Dehn twist, then $\mathbb{G}$ is an ordinary graph of groups equivalent to a point or a cylinder. Arguments similar to those in the proof of Lemma 7.15 show that $\mathbb{G}$ is a point or a cylinder. □

# 8 Normalising moves for higher Dehn twists

In Theorem 8.6 we will see that, for every higher Dehn twist automorphism of a free group, we can find a representative $(\mathbb{G}, D, \rho)$ as in Definition 4.23 such that $D$ is normalised. The idea of the proof will be to successively improve the representing higher graph of groups by moves which we define in the present chapter.

## 8.1 Building equivalences inductively by degree

For the underlying graph $\Gamma$ of a higher graph of groups of degree $d$, we write

$$E^d(\Gamma) := \{e \in E(\Gamma) | \deg(e) = d\}.$$

Let $\psi : \pi_1(\mathbb{G}^{(d-1)}) \to \pi_1(\mathbb{G}'^{(d-1)})$ be an equivalence of categories for higher graphs of groups $\mathbb{G}$ and $\mathbb{G}'$. More explicitly, this consists of the following data: First, we have a function denoted $\psi_V : V(\Gamma) \to V(\Gamma')$ such that every connected component of $\Gamma'^{(d-1)}$ contains at least one vertex in the image of $\psi_V$. Moreover, we have bijections

$$\psi_0 : \pi_1(\mathbb{G}, u, w) \to \pi_1(\mathbb{G}', \psi_V(u), \psi_V(w))$$

compatible with concatenation of paths. An important example will be the case of $H_*$ for an equivalence $H : \mathbb{G} \to \mathbb{G}'$ of higher graphs of groups. Other examples will appear in the subsequent sections.

Let $\psi_E : E^d(\Gamma) \to E^d(\Gamma')$ be a bijection such that $\psi_E(\bar{e}) = \overline{\psi_E(e)}$ and $\tau(\psi_E(e)) = \psi_V(\tau(e))$ for all edges $e$ with $\deg(e) = d$.

For every choice of basepoint $u \in V(\Gamma)$, the maps $\psi_0$ and $\psi_E$ induce isomorphisms of fundamental groups, which we denote by

$$\begin{aligned}\psi_u^{(d-1)}&: \pi_1(\mathbb{G}^{(d-1)}, u) \to \pi_1(\mathbb{G}'^{(d-1)}, \psi_V(u)),\\ \psi_u&: \pi_1(\mathbb{G}, u) \to \pi_1(\mathbb{G}', \psi_V(u)).\end{aligned}$$

They are given by $\psi_0$ on elements in vertex groups and on edges of degree at most $d-1$, and by $t_e \mapsto t_{\psi_E(e)}$ on edges $e$ of degree $d$.

Sometimes we write $\psi_u$ for $\psi_u^{(d-1)}$ when there is no risk of confusion.

**Lemma 8.1.** *In the above notation, let $D$ be a higher Dehn twist on $\mathbb{G}$ and $D''$ a higher Dehn twist on $\mathbb{G}'^{(d-1)}$ such that $\psi_u^{(d-1)} \circ D_{*u}^{(d-1)} = D''_* \circ \psi_u^{(d-1)}$ for every vertex $u$. Then there is a higher Dehn twist $D'$ on $\mathbb{G}'$ such that $\psi_u \circ D_{*u} = D'_{*u} \circ \psi_u$ and $D'^{(d-1)} = D''$.*

We visualise the equivalences of fundamental groupoids in this lemma in terms of the following diagrams:

$$\begin{tikzcd}
\pi_1(\mathbb{G}^{(d-1)}) \arrow[r, "\psi"] \arrow[d, "D_*"] & \pi_1(\mathbb{G}'^{(d-1)}) \arrow[d, "D''_*"] \\
\pi_1(\mathbb{G}^{(d-1)}) \arrow[r, "\psi"] & \pi_1(\mathbb{G}'^{(d-1)})
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(\mathbb{G}) \arrow[r, "\psi"] \arrow[d, "D_*"] & \pi_1(\mathbb{G}') \arrow[d, dashed, "D'_*"] \\
\pi_1(\mathbb{G}) \arrow[r, "\psi"] & \pi_1(\mathbb{G}')
\end{tikzcd}$$

*Proof of Lemma 8.1.* We define $\delta_{D'}(\psi_E(e)) = \psi_0(\delta_D(e))$ for edges $e$ of degree $d$. The other data is the same as for $D''$. Then $D'$ has the desired properties. $\square$

## 8.2 A mapping cylinder construction

In this section we replace a trivial edge group in degree 1 by an infinite cyclic edge group. Let $D \in \mathrm{Aut}^0(\mathbb{G})$ be a higher Dehn twist and $e$ an edge such that $G_e = 1$, $\deg(e) = 1$, $\delta_D(\overline{e}) = 1$, and $\delta_D(e) \neq 1$.

Let $\mathbb{G}'$ be the higher graph of groups with the same underlying graph $\Gamma$ as $\mathbb{G}$, vertex group $G'_{\iota(e)} = G_{\iota(e)} * \mathbb{Z}$, edge group $G'_e = \mathbb{Z}$. The attaching map $f'_{\overline{e}}$ is the inclusion of the new free factor $\mathbb{Z}$, and $f'_e$ maps the generator of $G'_e$ to $\delta_D(e)$. The other data the same as for $\mathbb{G}$. Geometrically, this corresponds to pulling a cylinder along the edge $e$ as shown in Figure 6. There is a morphism $H : \mathbb{G} \to \mathbb{G}'$ defined by the canonical inclusion on each piece.

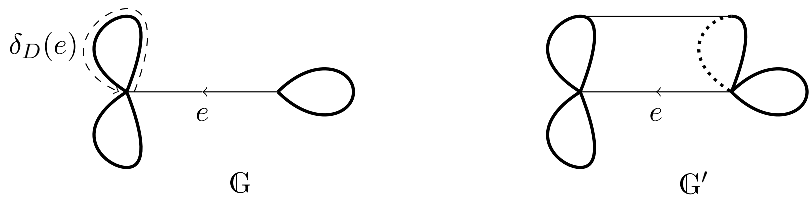


Figure 6: A mapping cylinder construction.

Define a higher Dehn twist $D'$ on $\mathbb{G}'$ as follows: We take $\delta_{D'}(\tilde{e}) = H_*(\delta_D(\tilde{e}))$ for every edge $\tilde{e} \in E(\Gamma)$ with $\deg(\tilde{e}) = 1$. We then use Lemma 8.1 inductively to extend the definition of $D'$ on $\mathbb{G}'^{(1)}$ to $\mathbb{G}'^{(2)}, \mathbb{G}'^{(3)}, \dots, \mathbb{G}'^{(d)} = \mathbb{G}'$ such that $D'_* H_* = H_* D_*$. In particular, for any choice of basepoint, we have $D'_{*v} = H_{*v} D_{*v} H_{*v}^{-1}$.

## 8.3 Sliding edges within lower strata

Let $\mathbb{G}$ be a higher graph of groups with a distinguished edge $e$ such that $G_e = 1$. In Section 3.3 we discussed how to modify the terminal vertex of this edge, and we defined an equivalence $H : \mathbb{G} \to \mathbb{G}'$. We re-use the notation of that section. Recall that we obtained the new underlying graph $\Gamma'$ by replacing the edge $e$ with another edge $e'$, and we had the flexibility to choose an arbitrary $\delta_H(e) \in \pi_1(\mathbb{G}^{(\deg(e)-1)}, \tau(e), \tau(e'))$, whereas the other $\delta_H(\tilde{e}) = 1$.

We investigate how higher Dehn twists behave under conjugation by this equivalence $H$.

**Lemma 8.2.** *Let $D \in \mathrm{Aut}^0(\mathbb{G})$ be a higher Dehn twist. Then $HDH^{-1} \in \mathrm{Aut}^0(\mathbb{G}')$ is a higher Dehn twist with*

$$\delta_{HDH^{-1}}(\tilde{e}) = \begin{cases} \delta_D(\tilde{e}), & \text{if } \tilde{e} \neq e, \deg(\tilde{e}) \leq \deg(e), \\ D_*(\delta_H(e))^{-1} \delta_D(e) \delta_H(e), & \text{if } \tilde{e} = e', \\ H_*(\delta_D(\tilde{e})), & \text{if } \deg(\tilde{e}) > \deg(e). \end{cases}$$

*Proof.* If $\deg(\tilde{e}) \le \deg(e)$, we have $\delta_D(\tilde{e}), \delta_H(\tilde{e}) \in \pi_1(\mathbb{G}^{(\deg(e)-1)})$, so $H_*$ acts trivially on them. If $\tilde{e} \ne e$, then $\delta_H(\tilde{e}) = 1$. Using this, the formulas for $\delta_{HDH^{-1}}(\tilde{e})$ follow by a straightforward calculation. □

## 8.4 Subdivision of edges

Fix an edge $e$ in the higher graph of groups $\mathbb{G}$. Let $\Gamma'$ be the graph obtained by removing the edges $e$ and $\overline{e}$, and adding new edges $e'$, $\overline{e'}$, $e''$, $\overline{e''}$ with $\iota(e') = \iota(e)$, $\tau(e'') = \tau(e)$, and a new vertex $\tau(e') = \iota(e'')$. Define $\deg(e') = \deg(e'') = \deg(e)$. Let $\mathbb{G}'$ be the higher graph of groups with underlying graph $\Gamma'$, the new vertex group $G_{\tau(e')} = G_e$, new edge groups $G_{e'} = G_{e''} = G_e$, the identity maps $f_{e'} : G_{e'} \to G_{\tau(e')}$ and $f_{\overline{e''}} : G_{e''} \to G_{\tau(e')}$. The other data of $\mathbb{G}'$ is chosen to agree with that of $\mathbb{G}$.

There are natural bijections $\psi : \pi_1(\mathbb{G}, u, w) \to \pi_1(\mathbb{G}', u, w)$ for all vertices $u$ and $w$. They are given by $t_e \mapsto t_{e'} t_{e''}$ and fixing the other $t_{\tilde{e}}$ with $\tilde{e} \notin \{e, \overline{e}\}$ as well as all elements in vertex groups.

Given a higher Dehn twist $D$ on $\mathbb{G}$, we can define a new higher Dehn twist $D'$ on $\mathbb{G}'$ by $\delta_{D'}(\overline{e'}) = \delta_D(\overline{e})$, $\delta_{D'}(e') = 1 = \delta_{D'}(\overline{e''})$, and $\delta_{D'}(e'') = \delta_D(e)$. The other $\delta$-terms of edges of degree at most $\deg(e)$ are the same as for $D$. This defines $D'$ on $\mathbb{G}'^{(\deg(e))}$, and the definition can be extended by Lemma 8.1 to all of $\mathbb{G}'$.

## 8.5 Folding edges with $D$-conjugate $\delta$-terms

Suppose we have $D$-conjugate non-trivial $\delta$-terms $\delta_D(e) = D_*(\epsilon)\delta_D(e')\epsilon^{-1}$, where the degree $\deg(e) = \deg(e') \ge 2$. Assume that $\delta_D(\overline{e}) = 1$ and $\delta_D(\overline{e'}) = 1$. Consider an edge slide equivalence $H : \mathbb{G} \to \mathbb{G}''$ with $\delta_H(e) = \epsilon^{-1}$. Then Lemma 8.2 shows that $\delta_{HDH^{-1}}(e) = \delta_{HDH^{-1}}(e')$. In particular, $\tau(e) = \tau(e')$ in $\mathbb{G}''$.

We now define a new higher graph of groups $\mathbb{G}'$. The underlying graph $\Gamma'$ is obtained from the underlying graph $\Gamma''$ of $\mathbb{G}''$ by folding the edges $e$ and $e'$ as follows: The vertices of $\Gamma'$ are the same as those of $\Gamma''$, but with $\iota(e)$ and $\iota(e')$ identified. We denote this new vertex of $\Gamma'$ by $u$. Instead of $e$ and $e'$, we have one new edge $e''$ from $u$ to $\tau(e) = \tau(e')$ in $\Gamma'$ having the same degree as $e$ and $e'$ in $\mathbb{G}''$. The vertex groups of $\mathbb{G}'$ are

$$G'_u = \begin{cases} G''_{\iota(e)} * G''_{\iota(e')}, & \text{if } \iota(e) \ne \iota(e') \text{ in } \Gamma'', \\ G''_{\iota(e)} * \langle c \rangle, & \text{if } \iota(e) = \iota(e') \text{ in } \Gamma''. \end{cases}$$

and $G'_w = G''_w$ for $w \ne u$. Here $c$ denotes a new formal generator with $\langle c \rangle \cong \mathbb{Z}$. The non-trivial edge groups of $\mathbb{G}'$ are the same as those of $\mathbb{G}''$. Note that $G''_e$ and $G''_{e'}$ are trivial because $e$ and $e'$ have degree at least two. Therefore we have to put $G'_{e''} = 1$. The attaching maps $f'_{\tilde{e}}$ of $\mathbb{G}'$ are those of $\mathbb{G}''$ composed with some obvious inclusions.

The higher graphs of groups $\mathbb{G}''$ and $\mathbb{G}'$ are illustrated in Figures 7 and 8. To simplify the picture, we depict the situation that $e$ and $e'$ are the only edges of $\mathbb{G}''$. The vertex spaces are indicated by the thick lines, where the additional loop at $\mathbb{G}'$ in Figure 8 corresponds to $c \in G'_u$.

There is a morphism $C : \mathbb{G}'' \to \mathbb{G}'$ given by the obvious folding map $\Gamma'' \to \Gamma'$ on underlying graphs. On vertex groups, we define $C_w$ to be the identity if $w$ is neither

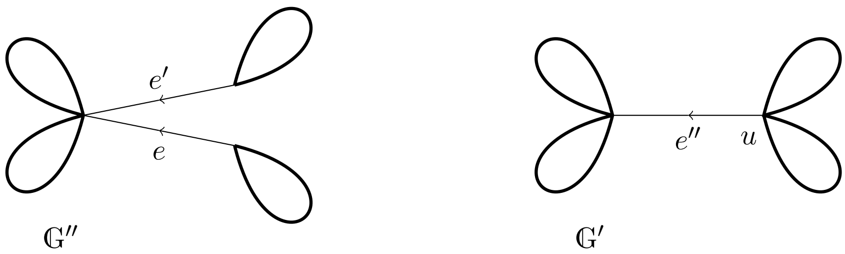


Figure 7: Folding $\mathbb{G}''$ to $\mathbb{G}'$ when $\iota(e) \neq \iota(e')$.


Figure 8: Folding $\mathbb{G}''$ to $\mathbb{G}'$ when $\iota(e) = \iota(e')$.

$\iota(e)$ nor $\iota(e')$. The vertex group morphisms $C_{\iota(e)} : G''_{\iota(e)} \to G'_u$ and $C_{\iota(e')} : G''_{\iota(e')} \to G'_u$ are the inclusions of the free factors. On non-trivial edge groups $G''_{\tilde{e}}$, the map $C_{\tilde{e}}$ is the identity. We put all $\delta_C(\tilde{e}) = 1$ except $\delta_C(\overline{e'}) = c$ in the case that $\iota(e) = \iota(e')$ in $\Gamma''$. This finishes the definition of the morphism $C$, and it can be checked that it induces an isomorphism on fundamental groups.

We now define a higher Dehn twist $D'$ on $\mathbb{G}'$ in the following way. If $\deg(\tilde{e}) \leq \deg(e)$ and $\tilde{e}$ is neither $e''$ nor $\overline{e''}$, then we define $\delta_{D'}(\tilde{e}) = C_*(\delta_{HDH^{-1}}(\tilde{e}))$. On the folded edge $e''$, we define $\delta_{D'}(\overline{e''}) = 1$ and $\delta_{D'}(e'') = C_*(\delta_{HDH^{-1}}(e))$. This finishes the definition of $D'$ on $\mathbb{G}'^{(\deg(e))}$. It can be checked that, when restricting to $\mathbb{G}''^{(\deg(e))}$, we have $C_*(HDH^{-1})_* = D'_* C_*$. We inductively use Lemma 8.2 to define $D'$ on all of $\mathbb{G}'$ such that $(CH)_* D_* = D'_*(CH)_*$.

## 8.6 Twisted reduction of $\delta$-terms

If the higher Dehn twist $D$ on $\mathbb{G}$ has some $\delta_D(e)$ with $\deg(e) \geq 2$ which is not $D$-twistedly reduced in the truncated sense, then we show how to define an equivalence $H : \mathbb{G} \to \mathbb{G}'$ such that the higher Dehn twist $D' := HDH^{-1}$ on $\mathbb{G}'$ has one more twistedly reduced $\delta$-term as $D$ has. Below we give the details of this construction together with some additional features. We need the following lemma.

**Lemma 8.3.** *Let $D \in \mathrm{Aut}^0(\mathbb{G})$ be a prenormalised higher Dehn twist and $\eta \in \pi_1(\mathbb{G}, u)$. Then $\eta$ is $D$-conjugate to some $\eta' \in \pi_1(\mathbb{G}, u')$ such that*

- *$\eta'$ is $D^{(m)}$-cyclic and $D^{(m)}$-twistedly reduced in the truncated sense for some $m$,*
- *$G_{u'} \neq 1$ or $\eta' = 1$,*
- *if $m = 1$, then $\eta' \in G_{u'}$ or $u'$ is not a clutching point of $D$.*

*Proof.* Proposition 4.8 (applied to the truncation $T^{d-1}$ of a truncatable replacement) provides a decomposition $\eta = D_*(\epsilon)\eta''\epsilon^{-1}$ such that $\eta''$ is $D$-twistedly reduced in the truncated sense. If $\eta'$ does not go across edges of degree $d$, then we repeat this construction to find $\eta''$ satisfying the first bullet point for some $m$, $1 \leq m \leq d$. If $\eta''$ is contained in a single vertex group of $\mathbb{G}$, then we take $\eta' = \eta''$, and we are done. Therefore we may assume a reduced expression $\eta'' = xt_1g_1 \dots t_{k-1}g_{k-1}t_ky$ with $k \geq 1$.

If $m = 1$, then the group $G_{\iota(e_1)}$ has to be non-trivial because $G_{e_1} \cong \mathbb{Z}$ and the map $f_{\overline{e_1}} : G_{e_1} \to G_{\iota(e_1)}$ is injective. If some $\tau(e_j)$ is not a clutching point of $D$, then we may take $\eta' = D_*(t_{j+1}g_{j+1} \dots t_ky)xt_1g_1 \dots t_jg_j$. If every $\tau(e_j)$ is a clutching point, then all $f_{e_j}$ and $f_{\overline{e_j}}$ are surjective. As $(t_j, g_j, t_{j+1})$ for $1 \leq j \leq k-1$ and $(t_k, yx, t_1)$ are reduced, we have $e_j \neq \overline{e_{j+1}}$ for $1 \leq j \leq k$, where $e_{k+1} := e_1$. We also observe that $e_j$ and $\overline{e_{j+1}}$ are bonded for every $j$ because $G_{\tau(e)}$ is infinite cyclic. This is a contradiction to Lemma 6.6. Therefore we may achieve all desired properties for $\eta'$ if it goes only across edges of degree one.

Assume now $m \geq 2$. If some $G_{\tau(e_j)} \neq 1$, then $\eta' = D_*(t_{j+1}g_{j+1} \dots t_ky)xt_1g_1 \dots t_jg_j$ works. If all $G_{\tau(e_j)} = 1$, then Definition 7.20(4) implies $\delta_D(\overline{e_1}) = 1 = \delta_D(e_1)$, which is a contradiction to Definition 7.20(2). $\square$

Given a higher Dehn twist $D \in \mathrm{Aut}^0(\mathbb{G})$ and an edge $e$ such that $D^{(\deg(e)-1)}$ is prenormalised, then Lemma 8.3 provides a decomposition $\delta_D(e) = D_*(\epsilon)\eta'\epsilon^{-1}$, where $\eta'$ is $D^{(\deg(e)-1)}$-twistedly reduced. If $\eta' \in \pi_1(\mathbb{G}^{(\deg(e)-1)}, u')$ is not contained in $\pi_1(\mathbb{G}^{(\deg(e)-2)}, u')$, then we arrange $G_{u'} \neq 1$. If further $\deg(e) = 2$, then we arrange that $u'$ is not a clutching point of $D^{(1)}$. Let now $H : \mathbb{G} \to \mathbb{G}'$ be the edge slide equivalence with $\delta_H(e) = \epsilon$. Lemma 8.2 shows that $\delta_{HDH^{-1}}(e) = \eta'$, whereas the other $\delta$-terms in degree at most $\deg(e)$ are unaffected.

## 8.7 The list of moves

The following moves will be the steps to build normalised higher Dehn twists in Theorem 8.6. Given a higher graph of groups $\mathbb{G}$ with a higher Dehn twist $D$, each of the moves will define a new higher graph of groups $\mathbb{G}'$ with a higher Dehn twist $D'$. We will always assume that $\mathbb{G}$ is connected, and this will ensure that $\mathbb{G}'$ in each move is connected.

Recall the concept of positively and negatively bonded edges in Definition 6.1 as well as invisible vertices and unused edges from Definition 6.2.

(M1) Remove a valence one vertex $w$ with surjective edge map (regardless whether $G_w = 1$ or not).

(M2) Delete an invisible vertex $w$ with $G_w \cong \mathbb{Z}$ and negatively bonded edges.

(M3) Fold positively bonded edges with infinite cyclic edge groups.

(M4) Contract unused edges (form either amalgam or HNN extension, either $G_e \neq 1$ with trivial twistor $z_e = 1$ or $G_e = 1$ with trivial $\delta$-terms).

(M5) Adjoin a formal root to an edge group.

(M6) Apply the mapping cylinder construction (cf. Section 8.2).

(M7) Subdivide $e$ with $G_e = 1$ when both $\delta_D(e)$ and $\delta_D(\overline{e})$ are non-trivial (cf. Section 8.4).

(M8) $D$-conjugate a $\delta$-term and fold $e$ with $e'$ when $\delta_D(e)$ and $\delta_D(e')$ are non-trivial and $D$-conjugate, and $\delta_D(\overline{e})$ and $\delta_D(\overline{e'})$ are trivial (cf. Section 8.5).

(M9) $D$-conjugate some $\delta_D(e)$ with $\deg(e) \geq 2$ to make it $D$-twistedly reduced (cf. Section 8.6). If $\delta_{D'}(e) \notin G'_{\tau(e)}$ and $D^{(\deg(e)-1)}$ is prenormalised, then we achieve that $G_{\tau(e)} \neq 1$, and that $\tau(e)$ is not a clutching point of $D'^{(1)}$ when $\deg(e) = 2$.

(M10) Lower the degree of an edge $e$ and $\overline{e}$ when $\deg(e) \geq 2$, $\delta_D(\overline{e}) = 1$, and $\delta_D(e) \in \pi_1(\mathbb{G}^{(\deg(e)-2)}, \tau(e))$.

The moves (M1) to (M5) specialise to those in Section 8.2 of [13] when we restrict our attention to Dehn twists of ordinary graphs of groups (with infinite cyclic edge groups only). In general, the description in [13] defines our new Dehn twist $D'$ on edges of degree 1, and we use Lemma 8.1 to define $D'$ on edges of higher degree.

For each of the moves (M1) through (M10), there is a distinguished outer isomorphism class $\widehat{\psi} : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', v')$ for any choice of basepoint $v'$ such that $\widehat{D'} = \widehat{\psi}\widehat{D}\widehat{\psi}^{-1}$. In (M3) through (M10) we may choose $v'$ such that we get an isomorphism $\psi : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', v')$ with $D'_{*v'} = \psi D_{*v} \psi^{-1}$. In (M1) and (M2) this is also the case if the removed vertex $w$ is different from the basepoint $v$. When we study higher Dehn twist automorphisms in $\mathrm{Aut}(\pi_1(\mathbb{G}, v))$, we will have to restrict the first two moves to:

(M1*) Perform (M1) at a vertex $w \neq v$.

(M2*) Perform (M2) at a vertex $w \neq v$.

**Proposition 8.4.** *If $D$ is a higher Dehn twist on a connected $\mathbb{G}$ with finitely generated free $\pi_1(\mathbb{G}, v)$ such that $D$ is not normalised, then at least one of the moves (M1)–(M10) is applicable. If $D$ is not pointedly normalised, then one of (M1*), (M2*), (M3)–(M10) is applicable.*

*Proof.* Recall that all vertex and edge groups are free because they inject into the fundamental group $\pi_1(\mathbb{G}, v)$, which is free. Since the twistors $z_e$ are central in $G_e$, they are always trivial at free edge groups of rank at least 2. We can apply (M4) to remove them. Therefore we may assume that all edge groups are either infinite cyclic or trivial.

Suppose first that $D^{(1)}$ is not prenormalised, so it violates one of the two bullet points of Definition 7.19. We first discuss how to get rid of trivial edge groups $G_e$ with $\deg(e) = 1$: If both $\delta_D(e)$ and $\delta_D(\overline{e})$ are trivial, then we contract the edge using (M4). If they are both non-trivial, we may apply (M7) to subdivide the edge. If exactly one of $\delta_D(e)$ and $\delta_D(\overline{e})$ is trivial, then we apply (M6). We can now assume that $D^{(1)}$ is an ordinary Dehn twist.

If $D^{(1)}$ is not pre-efficient, then $D^{(1)}$ fails to satisfy one of the properties (3)–(5) of Definition 6.2, and one of the moves (M3)–(M5) can be carried out. Therefore we may assume from now on that $D^{(1)}$ is prenormalised.

Suppose that $D$ is not prenormalised. Then there is a unique $m$ with $2 \leq m \leq d$ such that $D^{(m-1)}$ is prenormalised, but $D^{(m)}$ is not. We now check for each condition of Definition 7.20 that, whenever $D^{(m)}$ violates it, then we can apply a move.

If there is an edge $e$ of degree $m$ such that both $\delta_D(e)$ and $\delta_D(\overline{e})$ are trivial (or non-trivial), then we may apply (M4) (or (M7) respectively). If some $\delta_D(e)$ with $\deg(e) = m$ is not $D$-twistedly reduced in the truncated sense, then we may $D$-conjugate it using (M9). If $\deg(e) = m$, $G_{\tau(e)} = 1$, and $\delta_D(e) \neq 1$, then we may assume by (M10) that $\delta_D(e)$ is $D^{(m-1)}$-cyclic, and we can apply (M9) to move $\tau(e)$ to another vertex with non-trivial vertex group. If two non-trivial $\delta_D(e)$ and $\delta_D(e')$ are $D$-conjugate and violate Definition 7.20(5), then we assume that $\delta_D(\overline{e})$ and $\delta_D(\overline{e'})$ are trivial, and we can carry out (M8). If $e$ is an edge of degree $m$ such that $\delta_D(e) \neq 1$ and the growth of $A_j(\delta_D(e), D_*)$ is slower than polynomial of degree $m$, then Proposition 7.22(ii) proves $\delta_D(e) \in \pi_1(\mathbb{G}^{(m-2)})$, and we may apply move (M10).

We now assume that $D$ is prenormalised with connected underlying graph. We are left to verify that we can apply one of the moves when there is at least one clutching point $w$ (different from the basepoint $v$ in the pointed case).

Let $w$ be a clutching point of $D^{(1)}$. If there is an edge $e$ of degree at least 2 terminating at $w$ with $\delta_D(e) = 1$, then $w$ is not a clutching point of $D$. If $\delta_D(e) \in G_{\tau(e)}$, then we may apply (M10). If $\delta_D(e) \notin G_{\tau(e)}$, then we can apply move (M9) to move $\tau(e)$. If no edge of degree at least 2 terminates at $w$, then we can apply (M1) or (M2) at $w$. In the pointed case, this amounts to (M1*) or (M2*) because $w \neq v$.

Suppose now that $m \geq 2$, and $w$ is a clutching point of $D^{(m)}$ which is not a clutching point of $D^{(m-1)}$. Then $w$ is a 1-valent trivial vertex group in $\Gamma^{(m)}$. If there is an edge $e$ of degree $\deg(e) > m$ with $\tau(e) = w$, there are two cases: If $\delta_D(e) \neq 1$, we can move $\tau(e)$ by means of (M9). If $\delta_D(e) = 1$, then $w$ is not a clutching point of $D$, and there is nothing to show. If there is no such edge $e$ terminating at $w$, then $w$ a 1-valent trivial vertex group of all of $D$, and (M1) applies. In the pointed case we have assumed $w \neq v$, so (M1*) applies then. □

## 8.8 The semi-invariant $\Lambda$

To measure the failure of a higher Dehn twist $D$ being normalised, we define a tuple $\Lambda(D)$ of $2d+4$ non-negative integers.

The components of $\Lambda(D)$ are (in this order):

- the number of edges $e$ such that $G_e$ is free of rank at least 2,
- the number of edges $e$ with $G_e = 1$, $\delta_D(e) \neq 1$, and $\delta_D(\overline{e}) \neq 1$,
- the number of edges of degree $d, d-1, \dots, 2$ (i.e. $d-1$ numbers in this order),
- the number of edges $e$ of degree 1 with $G_e = 1$,
- the number of edges $e$ with $G_e \neq 1$,
- the sum of all exponents of $f_e(a_e)$ with $G_e = \langle a_e \rangle \cong \mathbb{Z}$ (counted *without* multiplicity when $f_{e'}(a_{e'}) = f_e(a_e)^{\pm 1}$), where the exponent of an element of a free group is the largest $p \geq 1$ such that it is a $p$-th power,
- the sum of all twist exponents $n_e$ taken over all $e$ with $G_e \cong \mathbb{Z}$,
- the number of edges with $\deg(e) = 2$ such that $\delta_D(e)$ is not $D^{(1)}$-twistedly reduced, or $\delta_D(e) \neq 1$ and $\tau(e)$ is a clutching point of $D^{(1)}$,
- the number of edges $e$ with $\deg(e) = 3, \dots, d$ such that $\delta_D(e)$ is not $D$-twistedly reduced in the truncated sense or based at a trivial vertex group $G_{\tau(e)}$ with $\delta_D(e) \neq 1$ (i.e. $d-2$ numbers in this order).

**Lemma 8.5.** *Let $D$ be a higher Dehn twist, and suppose $D'$ is obtained from $D$ by one of the moves (M1)–(M10). Then $\Lambda(D') < \Lambda(D)$ with respect to the lexicographic order.*

*Proof.* In moves (M1), (M2), (M4) an edge disappears, so one of the first $d+3$ components of $\Lambda(D)$ decreases strictly, and none of these components increases.

When we apply (M3), the first $d+4$ components of $\Lambda(D)$ and $\Lambda(D')$ agree, and the sum of all exponents $n_e$ decreases strictly. Note that the sum of all exponents of $f_e(a_e)$ is left invariant because we do *not* count multiplicities.

Move (M5) strictly decreases the sum of all exponents of $f_e(a_e)$, and it leaves the preceding components of $\Lambda(D)$ invariant.

The mapping cylinder construction (M6) replaces an edge of degree one with trivial edge group by an edge with infinite cyclic edge group, so the first $d+1$ components of $\Lambda(D)$ are left invariant, whereas the $(d+2)$-nd component strictly decreases.

The subdivision move (M7) strictly decreases the second component of $\Lambda(D)$, and it does not affect the first one.

In move (M8), the $D$-conjugation does not affect the numbers of edges of each degree, and the folding decreases one of these components of $\Lambda(D)$.

Move (M9) only lowers one component of $\Lambda(D)$ mentioned in the two last bullet points, and the other components are unaffected.

Lowering the degree of an edge reduces the number of edges of the old degree, and it leaves the numbers of edges in higher degree invariant. This finishes the verification that each of the moves (M1)–(M10) strictly decreases $\Lambda(D)$ with respect to the lexicographic order. $\square$

## 8.9 Normalising higher Dehn twists of free groups

**Theorem 8.6.** *Let $D \in \mathrm{Aut}^0(\mathbb{G}, v)$ be a higher Dehn twist, where $\mathbb{G}$ is connected and has a finitely generated free fundamental group. Then:*

*(i) There is a higher graph of groups $(\mathbb{G}', v')$ together with a higher Dehn twist $D'$ and an isomorphism $\rho : \pi_1(\mathbb{G}', v') \to \pi_1(\mathbb{G}, v)$ such that $D'$ is pointedly normalised and $D_{*v} = \rho D'_{*v'} \rho^{-1}$.*

*(ii) There is a higher Dehn twist $D''$ on some $(\mathbb{G}'', v'')$ and an isomorphism $\rho : \pi_1(\mathbb{G}'', v'') \to \pi_1(\mathbb{G}, v)$ such that $D''$ is normalised and $\widehat{D} = \widehat{\rho}\widehat{D''}\widehat{\rho}^{-1}$.*

*Proof.* We only prove (ii) because the arguments for (i) are similar.

If $D$ is normalised, then we take $\rho = 1$ and $D' = D$, and there is nothing to show. Otherwise we proceed by induction on $\Lambda(D)$, which can be done because the set of tuples of $2d + 4$ non-negative integers does not contain an infinite strictly descending chain in the lexicographic order.

Suppose now that $D$ is not normalised. Then Proposition 8.4 shows that there applies at least one of the moves (M1)–(M10). This provides a higher Dehn twist $D_1$ on some $\mathbb{G}_1$ such that $\widehat{D} = \widehat{\rho_1}\widehat{D_1}\widehat{\rho_1}^{-1}$ for some isomorphism $\rho_1$. We have $\Lambda(D_1) < \Lambda(D)$ by Lemma 8.5. We know by induction that there is a higher Dehn twist $D'$ on $\mathbb{G}'$ such that $\widehat{D_1} = \widehat{\rho'}\widehat{D'}\widehat{\rho'}^{-1}$ for some isomorphism $\rho'$. Then $\widehat{D} = \widehat{\rho_1\rho'}\widehat{D'}\widehat{\rho_1\rho'}^{-1}$ proves the assertion. $\square$

# 9 Automorphisms acting trivially on $\pi_1$

Let $\mathbb{G}$ be a higher graph of groups. We now study the groups

$$\begin{aligned}KA(\mathbb{G}) &= \{H \in \mathrm{Aut}^0(\mathbb{G}) | H_{*v} = 1\},\\ KO(\mathbb{G}) &= \{H \in \mathrm{Aut}^0(\mathbb{G}) | \widehat{H} = 1\}.\end{aligned}$$

For an ordinary graph of groups $\mathcal{G}$ we write $KA(\mathcal{G})$ and $KO(\mathcal{G})$. We first introduce notation for some elements in these groups.

## 9.1 The automorphisms $M(v,\gamma)$ and $K(e,h)$

For any vertex $w$ of the underlying graph $\Gamma$ of $\mathcal{G}$ and $\gamma \in G_w$, the automorphism $M := M(w,\gamma)$ is defined by

$$\begin{aligned}M_\Gamma &= 1,\\ M_e &= 1,\\ M_u &= \begin{cases}\mathrm{ad}_\gamma, & \text{if } u = w,\\ 1, & \text{if } u \neq w,\end{cases}\\ \delta_M(e) &= \begin{cases}\gamma, & \text{if } \tau(e) = w,\\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

If the vertex $w$ is equal to the basepoint $v$, then $M(v,\gamma)_{*v} = \mathrm{ad}_\gamma$, and otherwise $M(w,\gamma)_{*v} = 1$.

Next we fix some edge $e \in E(\Gamma)$ and $h \in G_e$. We define $K = K(e,h) \in \mathrm{Aut}(\mathcal{G})$ by

$$\begin{aligned}K_\Gamma &= 1,\\ K_{\tilde{e}} &= \begin{cases}\mathrm{ad}_h^{-1}, & \text{if } \tilde{e} = e \text{ or } \tilde{e} = \overline{e},\\ 1 & \text{otherwise,}\end{cases}\\ K_u &= 1,\\ \delta_K(\tilde{e}) &= \begin{cases}f_{\tilde{e}}(h), & \text{if } \tilde{e} = e \text{ or } \tilde{e} = \overline{e},\\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

It is easily verified that $(K(e,h))_{*v} = 1$.

**Lemma 9.1.** *Let $\mathcal{G}$ have no surjective edge map at the basepoint vertex group $G_v$ and $\eta \in \pi_1(\mathcal{G}, v)$. If $\eta G_v \eta^{-1} = G_v$, then $\eta \in G_v$.*

*Proof.* Choose a reduced word $W = (g_0, t_1, g_1, \ldots, t_k, g_k)$ representing $\eta$. We have to show $k = 0$.

As $f_{e_k}$ is not surjective, we find $g \in G_v$ such that $g_k g g_k^{-1} \notin f_{e_k}(G_{e_k})$. Then

$$(g_0, t_1, g_1, \ldots, t_k, g_k g g_k^{-1}, t_k^{-1}, \ldots, g_1^{-1}, t_1^{-1}, g_0^{-1})$$

is a reduced word going across $2k$ edges which represents $\eta g \eta^{-1} \in \eta G_v \eta^{-1} = G_v$. This shows $k = 0$. $\square$

We now show which automorphisms of an ordinary graph of groups $\mathcal{G}$ act trivially on the fundamental group. With slightly different requirements on $\mathcal{G}$, this corresponds to Proposition 3.4 of [31]. In a similar setting, the case of $KO(\mathcal{G})$ has also been investigated in Theorem 6.4 of [4].

**Lemma 9.2.** *Suppose that, for every edge $e$, the normaliser of $f_e(G_e)$ in $G_{\tau(e)}$ equals $f_e(G_e)$. Then:*

*(i) $KA(\mathcal{G})$ is generated by maps of the form $M(w,\gamma)$ with $w \neq v$ and $K(e,h)$.*

*(ii) If there is additionally a vertex $u$ such that no $f_e$ with $\tau(e) = u$ is surjective, then $KO(\mathcal{G})$ is generated by automorphisms of the form $M(w,\gamma)$ and $K(e,h)$.*

*Proof.* We first verify the asserted generating set for $KA(\mathcal{G})$. Let $\mathcal{A}$ be the subgroup of $\mathrm{Aut}^0(\mathcal{G})$ generated by all elements of the form $M(w,\gamma)$ (for $w \neq v$) and $K(e,h)$. We define a subgraph $\Delta(H)$ of $\Gamma$, which may be thought of as the subgraph of $\Gamma$ where $H$ acts trivially. We say that a vertex $w$ of $\Gamma$ belongs to $\Delta(H)$ if and only if $H_w = 1$. An edge $e$ is defined to belong to $\Delta(H)$ if and only if its initial and terminal vertices are in $\Delta(H)$, and $\delta_H(e) = 1 = \delta_H(\overline{e})$.

As $H_{*v} = 1$ the vertex group automorphism $H_v = 1$, so $v \in \Delta(H)$. We define $\Lambda(H)$ to be the connected component of $v$ in $\Delta(H)$.

<u>Claim:</u> If $\Lambda(H) \neq \Gamma$, then there is some $H' \in H.\mathcal{A}$ such that $\Lambda(H')$ strictly contains $\Lambda(H)$.

This claim will prove assertion (i): we can apply it inductively to find $H, H', H'', \ldots$ such that $\Lambda(H) \subsetneq \Lambda(H') \subsetneq \Lambda(H'') \subsetneq \ldots$. Since $\Gamma$ is finite, we will eventually get some $\tilde{H} \in H.\mathcal{A}$ such that $\Lambda(\tilde{H}) = \Gamma$.

By applying the compatibility condition in Definition 2.4(7), we see that if $e \in \Lambda(\tilde{H})$, then

$$f_e(a) = \tilde{H}_{\tau(e)}(f_e(a)) = \delta_{\tilde{H}}(e) f_e(\tilde{H}_e(a)) \delta_{\tilde{H}}(e)^{-1} = f_e(\tilde{H}_e(a))$$

for $a \in G_e$. As $f_e$ is injective, we have $\tilde{H}_e = 1$. Then $\Lambda(\tilde{H}) = \Gamma$ ensures $\tilde{H} = 1$ and $H \in \mathcal{A}$, as asserted.

To prove the claim, assume that $\Lambda(H) \neq \Gamma$. Then, as $\Gamma$ is connected, there is some edge $e \in \Gamma \setminus \Lambda(H)$ with initial vertex $u = \iota(e)$ lying in $\Lambda(H)$. Let $w = \tau(e)$.

Let $e_1, \ldots, e_k$ be an edge path from $v$ to $u$ in $\Lambda(H)$. Let $T := t_{e_1} \ldots t_{e_k}$. Recall that $H_*(t_{e_i}) = \delta_H(\overline{e_i}) t_{e_i} \delta_H(e_i)^{-1}$. Since each $e_i$ is an edge in $\Lambda(H) \subset \Delta(H)$, we have $\delta_H(e_i) = 1 = \delta_H(\overline{e_i})$. Hence $H_*(t_{e_i}) = t_{e_i}$ and $H_*(T) = T$.

By Definition 2.4(7), we have

$$f_{\overline{e}}(a) = \delta_H(\overline{e}) f_{\overline{e}}(H_e(a)) \delta_H(\overline{e})^{-1}$$

for $a \in G_e$. As $f_{\overline{e}}(G_e)$ is its own normaliser in $G_{\iota(e)}$, we have $\delta_H(\overline{e}) = f_{\overline{e}}(h)$ for some $h \in G_e$. By injectivity of $f_{\overline{e}}$ this implies $H_e = \mathrm{ad}_h^{-1}$. We define $H' = HK(e,h)^{-1}$. Then $\Lambda(H') \supset \Lambda(H)$, $H'_e = 1$, and $\delta_{H'}(\overline{e}) = 1$. We want to arrange $e \in \Lambda(H')$. We need to consider the case when $w \notin \Lambda(H)$ and the case when $w \in \Lambda(H)$.

Assume first $w \notin \Lambda(H)$. For every $g \in G_w$ we have

$$Tt_e\delta_{H'}(e)^{-1}H'_w(g)\delta_{H'}(e)t_e^{-1}T^{-1} = H'_{*v}(Tt_egt_e^{-1}T^{-1}) = Tt_egt_e^{-1}T^{-1},$$

so $H'_w(g) = \delta_{H'}(e)g\delta_{H'}(e)^{-1}$. We now define $H'' = H'M(w, \delta_{H'}(e))^{-1}$, and we see $\Lambda(H) \subset \Lambda(H'')$ and $e \in \Lambda(H'')$.

If $w \in \Lambda(H)$, we choose an edge path $e'_1, \ldots, e'_{k'}$ from $w$ to $v$ in the graph $\Lambda(H)$. Let $T' := t_{e'_1} \ldots t_{e'_{k'}}$. As for $T$, we have $H_*(T') = T'$. It follows that

$$Tt_e\delta_{H'}(e)^{-1}T' = H'_{*v}(Tt_eT') = Tt_eT',$$

so $\delta_{H'}(e) = 1$ and $e \in \Lambda(H')$. Therefore $\Lambda(H') \supsetneq \Lambda(H)$. This finishes the verification of the above claim, which proves (i).

To obtain (ii), let $H \in KO(\mathcal{G})$. Then $\widehat{H} = 1$ and $H_{*u} = \mathrm{ad}_\eta$ for some $\eta \in \pi_1(\mathcal{G}, u)$. Lemma 9.1 implies $\eta \in G_u$. Then $(HM(u, \eta)^{-1})_{*u} = 1$ and (i) prove assertion (ii). $\square$

## 9.2 The automorphisms $Z(\mathbb{F}, \gamma_\bullet)$

In the following sections, we want to extend Lemma 9.2 to higher graphs of groups $\mathbb{G}$. We need more generators $Z(\mathbb{F}, \gamma_\bullet)$ and $O(e, \delta)$, which we define now.

Let $m \geq 1$ and $\mathbb{F}$ a connected component of $\mathbb{G}^{(m)}$. Moreover, we need an element $\gamma_u$ in the centre of $\pi_1(\mathbb{G}^{(m)}, u)$. We write $\gamma_w := \epsilon_w\gamma_u\epsilon_w^{-1}$ for every vertex $w$ in $\mathbb{F}$, where $\epsilon_w \in \pi_1(\mathbb{G}^{(m)}, w, u)$. As $\gamma_u$ is central, $\gamma_w$ is independent of the choice of $\epsilon_w$.

Using this notation, we define $Z = Z(\mathbb{F}, \gamma_\bullet) \in \mathrm{Aut}^0(\mathbb{G})$ as follows:

$$\begin{aligned} Z_\Gamma &= 1, \\ Z_e &= 1, \\ Z_u &= 1, \\ \delta_Z(e) &= \begin{cases} \gamma_{\tau(e)}, & \text{if } \deg(e) > m \text{ and } \tau(e) \text{ lies in } \mathbb{F}, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

It is straightforward to verify that $Z(\mathbb{F}, \gamma_\bullet)_{*v} = \mathrm{ad}_{\gamma_v}$ if $v$ lies in the connected component $\mathbb{F}$, and otherwise $Z(\mathbb{F}, \gamma_\bullet)_{*v} = 1$. We sometimes write $Z(\mathbb{F}, \gamma_u)$ for $Z(\mathbb{F}, \gamma_\bullet)$.

## 9.3 The automorphisms $O(e, \delta)$

Suppose $e$ is an edge of $\mathbb{G}$ such that $G_{\iota(e)} = 1$ and $\iota(e)$ is 1-valent in the stratum $\mathbb{G}^{(\deg(e))}$. Given an arbitrary $\delta \in \pi_1(\mathbb{G}^{(\deg(e)-1)}, \tau(e))$, we define $O = O(e, \delta) \in \mathrm{Aut}^0(\mathbb{G})$ by $O_\Gamma = 1$, $O_{\tilde{e}} = 1$, $O_u = 1$, and

$$\delta_O(\tilde{e}) = \begin{cases} \delta, & \text{if } \tilde{e} = e, \\ t_e\delta^{-1}t_e^{-1}, & \text{if } \tilde{e} \neq \overline{e} \text{ and } \tau(\tilde{e}) = \iota(e), \\ 1 & \text{otherwise.} \end{cases}$$

The reader can check that $O(e, \delta)_{*v} = \mathrm{ad}_{t_e\delta^{-1}t_e^{-1}}$ if $v = \iota(e)$ and $O(e, \delta)_{*v} = 1$ otherwise.

## 9.4 The kernel of the restriction homomorphism

In this section $\mathbb{G}$ is any higher graph of groups of degree $d \geq 2$.

**Lemma 9.3.** *If $H \in KO(\mathbb{G})$, then $H|_{\mathbb{F}} \in KO(\mathbb{F})$ for every connected component $\mathbb{F}$ of $\mathbb{G}^{(d-1)}$. If $\pi_1(\mathbb{G}^{(d-1)}, v) \neq 1$, then $H_{*v} = \mathrm{ad}_\zeta$ for some $\zeta \in \pi_1(\mathbb{G}^{(d-1)}, v)$.*

*Proof.* Pick a vertex $u \in \mathbb{F}$. Since $\widehat{H} = 1$, we have $H_{*u} = \mathrm{ad}_\zeta$ for some $\zeta \in \pi_1(\mathbb{G}, u)$. If $\pi_1(\mathbb{F}, u)$ is trivial, then $H|_{\mathbb{F}} \in KO(\mathbb{F})$ is clear, so we assume $\pi_1(\mathbb{F}, u) \neq 1$. Then

$$\pi_1(\mathbb{F}, u) = H_{*u}(\pi_1(\mathbb{F}, u)) = \zeta \pi_1(\mathbb{F}, u)\zeta^{-1}$$

shows that $\zeta \in \pi_1(\mathbb{F}, u)$. Then $(H|_{\mathbb{F}})_{*u} = \mathrm{ad}_\zeta$ implies $\widehat{H|_{\mathbb{F}}} = 1$. □

**Lemma 9.4.** *Assume $H \in \mathrm{Aut}^0(\mathbb{G})$ with $\deg(\mathbb{G}) = d$ and $H^{(d-1)} = 1$. Let $e$ and $e'$ be edges of degree $d$ and $\epsilon \in \pi_1(\mathbb{G}^{(d-1)}, \tau(e), \tau(e'))$. If either*

*(i) $H \in KO(\mathbb{G})$, there is at least one vertex $u$ with $\pi_1(\mathbb{G}^{(d-1)}, u) \neq 1$, and none of $G_{\iota(e)}$ and $G_{\iota(e')}$ is a 1-valent trivial vertex group, or*

*(ii) $H \in KA(\mathbb{G})$ and $G_{\iota(e)}$, $G_{\iota(e')}$ are allowed to be 1-valent trivial only at the basepoint $v$,*

*then $\delta_H(e) = \epsilon \delta_H(e')\epsilon^{-1}$.*

*Proof.* As $\mathbb{G}$ is connected and has no 1-valent trivial non-basepoint vertex groups at $\iota(e)$ or $\iota(e')$, we can find a word $W = (\theta_0, t_1, \theta_1, \ldots, t_k, \theta_k)$ reduced in the truncated sense such that $(t_j, \theta_j, t_{j+1}) = (t_e, \epsilon, t_{\overline{e'}})$ for some $j$, $1 \leq j \leq k-1$. In case (i) we assume that $W$ goes from $u$ to $u$, where $\pi_1(\mathbb{G}^{(d-1)}, u)$ is non-trivial, and Lemma 9.3 shows $H_{*u} = \mathrm{ad}_\zeta$ for some $\zeta \in \pi_1(\mathbb{G}^{(d-1)}, u)$. In the situation (ii), we assume that $W$ goes from the basepoint $v$ to itself.

As $H_*(|W|) = \zeta|W|\zeta^{-1}$, uniqueness of expressions reduced in the truncated sense shows that

$$(\zeta\theta_0, t_1, \theta_1, \ldots, t_k, \theta_k\zeta^{-1}) = (\theta_0\delta_H(\overline{e_1}), t_1, \delta_H(e_1)^{-1}\theta_1\delta_H(\overline{e_2}), \ldots, t_k, \delta_H(e_k)^{-1}\theta_k),$$

where $\zeta = 1$ in (ii). We read off

$$\epsilon = \theta_j = \delta_H(e_j)^{-1}\theta_j\delta_H(\overline{e_{j+1}}) = \delta_H(e)^{-1}\epsilon\delta_H(e'),$$

which is the assertion. □

**Proposition 9.5.** *Every $H \in KO(\mathbb{G})$ with $H^{(d-1)} = 1$ is a composition of automorphisms $Z(\mathbb{F}, \gamma_\bullet)$, where $\mathbb{F}$ is a connected component of $\mathbb{G}^{(d-1)}$, and $O(e, \delta)$ with $\deg(e) = d$. If $H \in KA(\mathbb{G})$, then no $Z(\mathbb{F}, \gamma_\bullet)$ with $v \in \mathbb{F}$ and no $O(e, \delta)$ with $\iota(e) = v$ is needed.*

*Proof.* We pick an arbitrary $H \in KO(\mathbb{G})$. We may assume $\pi_1(\mathbb{G}^{(d-1)}, u) \neq 1$ for some vertex $u$ because $H = 1$ otherwise.

We denote by $\Gamma'$ the subgraph of $\Gamma$ obtained by removing $e$, $\overline{e}$, and $\iota(e)$ for a degree $d$ edge $e$ whenever $\iota(e)$ is 1-valent in $\Gamma$ and $G_{\iota(e)} = 1$. We write $\mathbb{G}' = \mathbb{G}|_{\Gamma'}$. As $H \in KO(\mathbb{G})$, we have $H|_{\Gamma'} \in KO(\mathbb{G}')$.

If $e \in E(\Gamma')$ has degree $d$, then Lemma 9.4(i) with $e = e'$ shows that $\delta_H(e)$ is central in $\pi_1(\mathbb{G}^{(d-1)}, \tau(e))$. Moreover, Lemma 9.4 shows $\delta_H(e') = \epsilon^{-1}\delta_H(e)\epsilon$ whenever the edge $e' \in E(\Gamma')$ also has degree $d$ and $\epsilon \in \pi_1(\mathbb{G}^{(d-1)}, \tau(e), \tau(e'))$. We now write $\tilde{H} = HZ(\mathbb{F}, \delta_H(e))^{-1}$, where $\mathbb{F}$ is the connected component of $\mathbb{G}^{(d-1)}$ containing $\tau(e)$. Then $\tilde{H}^{(d-1)} = H^{(d-1)} = 1$, and

$$\delta_{\tilde{H}}(e') = \begin{cases} \delta_H(e'), & \text{if } \deg(e') = d, \tau(e') \notin \mathbb{F}, \\ 1, & \text{if } \deg(e') = d, \tau(e') \in \mathbb{F}. \end{cases}$$

Repeating this construction on all components of $\mathbb{G}^{(d-1)}$, we find $H' \in \mathrm{Aut}^0(\mathbb{G})$ such that $H'|_{\Gamma'} = 1$ and $H'H^{-1}$ is a composition of automorphisms of type $Z(\mathbb{F}, \gamma_\bullet)$.

The only non-trivial data of $H'$ is now all $\delta_{H'}(e)$ such that $G_{\iota(e)}$ is a 1-valent trivial vertex group. Then $H'$ is the composition of automorphisms $O(e, \delta_{H'}(e))$. Therefore $H$ may be written in terms of the asserted generators. This finishes the proof of the case $KO(\mathbb{G})$.

The present argument also applies to the case $H \in KA(\mathbb{G})$ with a slightly different definition of $\Gamma'$: We keep the basepoint $v \in \Gamma'$ even if it carries a 1-valent trivial vertex group with an edge of degree $d$. The rest of the proof works in the same way. □

## 9.5 The group $\mathrm{Aut}^0_I(\mathbb{G})$

Suppose we are given a connected higher graph of groups $\mathbb{G}$ and a finite set $I$. For every element $i \in I$, we fix a vertex $v_i$. We define the group

$$\mathrm{Aut}^0_I(\mathbb{G}) = \{(H, (\delta_i)_{i\in I}) | H \in \mathrm{Aut}^0(\mathbb{G}), \delta_i \in \pi_1(\mathbb{G}, v_i)\},$$

where we suppress the $v_i$ from the notation. The group law is given by

$$(H, (\delta_i)_{i\in I})(H', (\delta'_i)_{i\in I}) = (HH', (H_*(\delta'_i)\delta_i)_{i\in I}).$$

We now define the "central conjugation subgroup" $Z_I(\mathbb{G}) \subset \mathrm{Aut}^0_I(\mathbb{G})$ to be the subgroup of all tuples $(1, (\delta_i)_i)$ such that $\delta_i$ is central in $\pi_1(\mathbb{G}, v_i)$ and $\delta_i = \epsilon_{i,j}\delta_j\epsilon_{i,j}^{-1}$ for all $i, j \in I$, where $\epsilon_{i,j} \in \pi_1(\mathbb{G}, v_i, v_j)$ is arbitrary. Then $Z_I(\mathbb{G})$ is a normal subgroup of $\mathrm{Aut}^0_I(\mathbb{G})$.

We further define

$$KA_I(\mathbb{G}) = \{(H, (\delta_i)_{i\in I}) | H_{*v} = 1, H_*(\epsilon_i) = \epsilon_i\delta_i^{-1} \text{ for } \epsilon_i \in \pi_1(\mathbb{G}, v, v_i)\}, \tag{39}$$

$$KO_I(\mathbb{G}) = \{(H, (\delta_i)_{i\in I}) | \widehat{H} = 1, H_*(\epsilon) = \delta_i\epsilon\delta_j^{-1} \text{ for all } \epsilon \in \pi_1(\mathbb{G}, v_i, v_j)\}. \tag{40}$$

A calculation using $H_{*v} = 1$ shows that, in the definition of the group $KA_I$, the condition $H_*(\epsilon_i) = \epsilon_i\delta_i^{-1}$ for *some* $\epsilon_i$ is equivalent to that condition for *every* element

$\epsilon_i \in \pi_1(\mathbb{G}, v, v_i)$. It can be checked that $KA_I := KA_I(\mathbb{G})$ and $KO_I := KO_I(\mathbb{G})$ are normal subgroups of $\mathrm{Aut}_I^0(\mathbb{G})$ and $KA_I \subset KO_I$.

*Remark* 9.6. If the underlying graph of $\mathbb{G}$ is a single point $v$ with $G_v = G$, then we write $\mathrm{Aut}_I(G)$ for $\mathrm{Aut}_I^0(\mathbb{G})/KA_I = \mathrm{Aut}_I^0(\mathbb{G})$ and $\mathrm{Out}_I(G)$ for $\mathrm{Aut}_I^0(\mathbb{G})/KO_I$. These groups also appear in other work. In [5] they appear as $\mathrm{Out}(n,k)$ in the construction of a bordification for outer space. In the context of automorphisms of hyperbolic groups by Levitt [21], they are called $\mathrm{Aut}_{\mathcal{H}}^{\partial}$ and $\mathrm{PMCG}^{\partial}$. Hatcher [17] investigates homological stability of $\mathrm{Aut}(F_n)$ using sphere systems in certain 3-manifolds, and our group $\mathrm{Out}_I(F_n)$ coincides with his $\Gamma_{n,|I|}$.

The map $KA_I \to \mathrm{Aut}^0(\mathbb{G})$ forgetting the $\delta_i$ induces an isomorphism

$$KA_I(\mathbb{G}) \cong KA(\mathbb{G}). \tag{41}$$

Similarly, forgetting the terms $\delta_i$ of elements in $KO_I(\mathbb{G})$ induces the surjection in a short exact sequence

$$1 \to Z_I(\mathbb{G}) \to KO_I(\mathbb{G}) \to KO(\mathbb{G}) \to 1. \tag{42}$$

The elements in $KO(\mathbb{G})$ introduced in the last sections lift to the following elements in $KO_I(\mathbb{G})$:

$$\begin{aligned}
M(w,\gamma)_I &= (M(w,\gamma), (\delta_i^M)), &\quad \text{where } \delta_i^M &= \begin{cases} \gamma, & \text{if } v_i = w, \\ 1 & \text{otherwise,} \end{cases} \\
K(e,h)_I &= (K(e,h), (1)), && \\
Z(\mathbb{F},\gamma_\bullet)_I &= (Z(\mathbb{F},\gamma_\bullet), (\delta_i^Z)), &\quad \text{where } \delta_i^Z &= \begin{cases} \gamma_{v_i}, & \text{if } v_i \in \mathbb{F}, \\ 1 & \text{otherwise,} \end{cases} \\
O(e,\delta)_I &= (O(e,\delta)_I, (\delta_i^O)), &\quad \text{where } \delta_i^O &= \begin{cases} t_e\delta^{-1}t_e^{-1}, & \text{if } v_i = \iota(e), \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

These automorphisms lie in $KA_I(\mathbb{G})$ if they are lifts of automorphisms in $KA(\mathbb{G})$.

If the degree of $\mathbb{G}$ is $d \geq 2$ and $\mathbb{F}_1, \dots, \mathbb{F}_l$ are the connected components of $\mathbb{G}^{(d-1)}$, we write

$$E_r = \{e \in E(\Gamma) | \deg(e) = d, \tau(e) \in \mathbb{F}_r\}$$

for $1 \leq r \leq l$. There is an isomorphism

$$\mathrm{Aut}^0(\mathbb{G}) \cong \bigoplus_{r=1}^{l} \mathrm{Aut}_{E_r}^0(\mathbb{F}_r) \tag{43}$$

given by restriction to the components: $H$ is mapped to $(H|_{\mathbb{F}_r}, (\delta_H(e))_{e \in E_r})$ in the $r$-th component. When $\mathbb{G}$ has no 1-valent trivial vertex groups with a degree $d$ edge (away from the basepoint in (45)), then this isomorphism restricts to isomorphisms

$$KO(\mathbb{G}) \cong \bigoplus_{r=1}^{l} KO_{E_r}(\mathbb{F}_r), \tag{44}$$

$$KA(\mathbb{G}) \cong KA_{E_1}(\mathbb{F}_1) \oplus \bigoplus_{r=2}^{l} KO_{E_r}(\mathbb{F}_r), \tag{45}$$

where we assume that the basepoint $v$ lies in $\mathbb{F}_1$.

We now study the groups $KA_I$ and $KO_I$ for a cylinder as in Definition 7.25.

**Lemma 9.7.** *Let $\mathbb{G}$ be a cylinder of some length $k \geq 0$ with an arbitrary basepoint $v$ and $I$ a finite set. Then:*

*(i) $KO_I(\mathbb{G})$ is generated by automorphisms of the type $M(w,\gamma)_I$ and $K(e,h)_I$.*

*(ii) $KA_I(\mathbb{G})$ is generated by automorphisms of the type $M(w,\gamma)_I$ with $w \neq v$ and $K(e,h)_I$.*

*Proof.* We first assume $I = \varnothing$. We have $\pi_1(\mathbb{G}, v) = G_v$ because there are no reduced words from $v$ to $v$ with non-trivial underlying path. If $H \in \mathrm{Aut}^0(\mathbb{G})$ with $\widehat{H} = 1$, then $H_{*v} = 1$ because the inner automorphism group of $\pi_1(\mathbb{G}, v)$ is trivial. Hence $KO(\mathbb{G}) = KA(\mathbb{G})$, and Lemma 9.2(i) proves that this group is generated by the desired elements.

If $I \neq \varnothing$, then (ii) follows directly using the isomorphism (41). For (i), it remains to show that all elements in the kernel $Z_I(\mathbb{G})$ in the short exact sequence (42) can be expressed in terms of the desired generators.

We denote the edges of the underlying graph $\Gamma$ by $e_1, \ldots, e_k$ and the vertices by $w_0, \ldots, w_k$ such that $\iota(e_j) = w_{j-1}$ and $\tau(e_j) = w_j$. Moreover, we denote by $b_j$ a generator of $G_{w_j} \cong \mathbb{Z}$ and by $h_j$ a generator of $G_{e_j} \cong \mathbb{Z}$ such that $f_{e_j}(h_j) = b_j$ and $f_{\overline{e_j}}(h_j) = b_{j-1}$. Then $Z_I(\mathbb{G})$ is infinite cyclic generated by the automorphism $(1, (b_{j(i)})_{i\in I})$ , where the index $j(i)$ is determined by $v_i = w_{j(i)}$. The relation

$$(1, (b_{j(i)})) = M(w_0, b_0)_I M(w_1, b_1)_I \ldots M(w_k, b_k)_I K(e_1, h_1)_I^{-1} \ldots K(e_k, h_k)_I^{-1}$$

finishes the proof. □

**Corollary 9.8.** *Assume $D \in \mathrm{Aut}^0(\mathbb{G})$ is a prenormalised higher Dehn twist or a truncatable replacement. Then every automorphism $Z(\mathbb{F}, \gamma_\bullet) \in \mathrm{Aut}^0(\mathbb{G})$ with $\mathbb{F}$ being a connected component of $\mathbb{G}^{(d-1)}$ is a composition of some $M(w,\gamma)$ and $K(e,h)$ with $w$ and $e$ lying in $\mathbb{F}$.*

*Proof.* $Z(\mathbb{F}, \gamma_\bullet)$ can be non-trivial only if the fundamental group of $\mathbb{F} = \mathbb{F}_r$ is infinite cyclic. Then $\mathbb{F}$ is a cylinder by Corollary 7.29(ii). Under the isomorphism (44), the automorphism $Z(\mathbb{F}_r, \gamma_\bullet)$ corresponds to an element in $Z_{E_r}(\mathbb{F}_r) \subset KO_{E_r}(\mathbb{F}_r)$. Lemma 9.7 allows us to decompose this in terms of $M(w,\gamma)_{E_r}$ and $K(e,h)_{E_r}$. On the left hand side of (44), this corresponds to the desired decomposition of $Z(\mathbb{F}_r, \gamma_\bullet)$. □

# 10 Preserving the graph of groups structure

## 10.1 Preserving cyclic path lengths

Bass and Jiang [4] use a slightly different notion of graph of groups morphisms. After a careful translation of the definitions, Section 1.7 of [4] has the following consequence.

**Proposition 10.1.** *Suppose all edge groups of $\mathcal{G}$ and $\mathcal{G}'$ are trivial, and there are no 1-valent trivial vertex groups. Let $\phi : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', v')$ be an isomorphism such that the cyclic path length $pl_c(\phi(\eta)) = pl_c(\eta)$ for every element $\eta \in \pi_1(\mathcal{G}, v)$. Then there is a graph of groups equivalence $H : \mathcal{G} \to \mathcal{G}'$ such that $\widehat{\phi} = \widehat{H}$ as outer isomorphism classes $\pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', v')$.* $\square$

We now want to rephrase Proposition 10.1 in terms of vertex group conjugates in the fundamental group of a graph of groups.

**Definition 10.2.** A subgroup $A$ of $\pi_1(\mathcal{G}, v)$ is a *vertex group* (or vertex group conjugate), if it is of the form $A = \theta G_u \theta^{-1}$ for some vertex $u$ and some $\theta \in \pi_1(\mathcal{G}, v, u)$.

In the following we assume that $\mathcal{G}$ has no surjective edge maps at non-trivial vertex groups, which will ensure that $u$ and the coset $\theta G_u$ are determined uniquely by $A$ when $A \neq 1$.

Non-trivial vertex groups $\theta G_u \theta^{-1}$ correspond bijectively to vertices $\theta G_u$ in the Bass-Serre tree of $\mathcal{G}$ with non-trivial $G_u$. These vertices span the whole Bass-Serre tree if there are no 1-valent vertices with surjective edge maps.

By the *distance* of two non-trivial vertex groups $\theta G_u \theta^{-1}$ and $\epsilon G_w \epsilon^{-1}$ we mean the path length of $\theta^{-1}\epsilon$, which equals the distance of the vertices $\theta G_u$ and $\epsilon G_w$ in the Bass-Serre tree.

**Corollary 10.3.** *Let the isomorphism $\phi : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', v')$ and its inverse $\phi^{-1}$ map non-trivial vertex groups to non-trivial vertex groups in a distance-preserving way. If $\mathcal{G}$ and $\mathcal{G}'$ have trivial edge groups, no 1-valent trivial vertex groups, and at least one non-trivial vertex group, then the outer isomorphism class $\widehat{\phi}$ is represented by an equivalence $H : \mathcal{G} \to \mathcal{G}'$ of graphs of groups.*

*Proof.* We denote by $T = \widetilde{(\mathcal{G}, v)}$ and $T' = \widetilde{(\mathcal{G}', v')}$ the Bass-Serre trees of $\mathcal{G}$ and $\mathcal{G}'$. Let $V'(T) := \{\delta G_u \in V(T) | G_u \neq 1\}$, and define $V'(T')$ similarly. Whenever $G_u \neq 1$ and $\delta \in \pi_1(\mathcal{G}, v, u)$, then $\phi(\delta G_u \delta^{-1}) = \delta' G_{u'} \delta'^{-1}$ for unique $u' \in V(\Gamma')$ and $\delta' \in \pi_1(\mathcal{G}', v', u')$. We define $\alpha : V'(T) \to V'(T')$ by $\alpha(\delta G_u) = \delta' G_{u'}$. The same construction for $\phi^{-1}$ leads to an inverse $\alpha^{-1}$, so $\alpha$ is bijective. As $\phi$ preserves the distances between vertex group conjugates in $\pi_1(\mathcal{G}, v)$, we have $d_{T'}(\alpha(\delta G_u), \alpha(\epsilon G_w)) = d_T(\delta G_u, \epsilon G_w)$ for all $\delta G_u, \epsilon G_w \in V'(T)$. Whenever $\eta \in \pi_1(\mathcal{G}, v)$, then $\alpha(\eta \delta G_u) = \phi(\eta)\alpha(\delta G_u)$. As the convex hull of $V'(T)$ in $T$ is all of $T$, the bijection $\alpha$ extends to a unique isometry $\alpha : T \to T'$ such that $\alpha(\eta.x) = \phi(\eta)\alpha(x)$ for every $x \in V(T)$ and $\eta \in \pi_1(\mathcal{G}, v)$. We compute by Lemma 3.5 that

$$
\begin{aligned}
pl_c(\phi(\eta)) &= \min_{x' \in V(T')} d_T(x', \phi(\eta)x') = \min_{x \in V(T)} d_{T'}(\alpha(x), \phi(\eta)\alpha(x)) \\
&= \min_{x \in V(T)} d_{T'}(\alpha(x), \alpha(\eta x)) = \min_{x \in V(T)} d_T(x, \eta x) = pl_c(\eta).
\end{aligned}
$$

Proposition 10.1 leads to the desired conclusion. □

We shall also look at the more general situation of vertex group conjugates for higher graphs of groups.

**Definition 10.4.** Given a higher graph of groups $\mathbb{G}$ of degree $d$, we call a subgroup $A$ of $\pi_1(\mathbb{G}, v)$ a vertex group (conjugate) of the truncation $T^{d-1}\mathbb{G}$ if it is of the form $A = \theta\pi_1(\mathbb{G}^{(d-1)}, u)\theta^{-1}$ for some vertex $u$ and some $\theta \in \pi_1(\mathbb{G}, v, u)$.

Here we abuse the notation, and we do *not* always require that $\mathbb{G}$ is truncatable at degree $d-1$. We usually assume that $\mathbb{G}$ has no surjective edge maps at non-trivial vertex groups. If $A$ is non-trivial, then it determines the coset $\theta\pi_1(\mathbb{G}^{(d-1)})$ uniquely.

By the distance of two non-trivial vertex group conjugates $\theta\pi_1(\mathbb{G}^{(d-1)}, u)\theta^{-1}$ and $\epsilon\pi_1(\mathbb{G}^{(d-1)}, w)\epsilon^{-1}$ we will mean the number of edges of degree $d$ in an expression for $\theta^{-1}\epsilon$ reduced in the truncated sense. It coincides with the distance of the vertices $\theta\pi_1(\mathbb{G}^{(d-1)}, u, \bullet)$ and $\epsilon\pi_1(\mathbb{G}^{(d-1)}, w, \bullet)$ in the Bass-Serre tree of the truncation $T^{d-1}\mathbb{G}$ (cf. Remark 3.6).

## 10.2 Abstract automorphism representatives

We now need a criterion when a given isomorphism $\phi : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', v')$ between fundamental groups of connected higher graphs of groups is induced by a morphism of higher graphs of groups. The goal of Proposition 10.5 is to reduce the higher graph of groups case to knowledge about ordinary graphs of groups.

Suppose that the higher graphs of groups $\mathbb{G}$ and $\mathbb{G}'$ both have degree $d$, and that both are truncatable at degree $d-1$. Let $V_{d-1} = \{v_1, \ldots, v_l\}$ and $V'_{d-1} = \{v'_1, \ldots, v'_l\}$ be the sets of basepoints for the connected components $\mathbb{F}_j$ and $\mathbb{F}'_j$ of $\mathbb{G}^{(d-1)}$ and $\mathbb{G}'^{(d-1)}$ respectively.

Recall that we denote by $\widehat{\alpha} \in O\mathrm{Hom}(G, G')$ the outer homomorphism class represented by a homomorphism $\alpha : G \to G'$ of groups.

**Proposition 10.5.** *Assume that both $\mathbb{G}$ and $\mathbb{G}'$ are truncatable at degree $d-1$. Let $\widehat{\phi} \in O\mathrm{Hom}(\pi_1(\mathbb{G}, v), \pi_1(\mathbb{G}', v'))$ be an outer isomorphism class such that*

*(i) $\widehat{\phi} = \widehat{H'}$ for some equivalence $H' : T^{d-1}\mathbb{G} \to T^{d-1}\mathbb{G}'$,*

*(ii) for each homomorphism $H'_{v_j} : \pi_1(\mathbb{G}^{(d-1)}, v_j) \to \pi_1(\mathbb{G}'^{(d-1)}, v'_j)$ (defined after renumbering $v'_1, \ldots, v'_l$), the outer isomorphism $\widehat{H'_{v_j}} = \widehat{H_j}$ for some equivalence $H_j : \mathbb{F}_j \to \mathbb{F}'_j$ of higher graphs of groups.*

*Then there is an equivalence $H : \mathbb{G} \to \mathbb{G}'$ such that $H' = T^{d-1}H$ and $H_j = H|_{\mathbb{F}_j}$. In particular $\widehat{\phi} = \widehat{H}$.*

*Proof.* Given $H'$ and $H_1, \dots, H_l$, we start building $\tilde{H} : \mathbb{G} \to \mathbb{G}'$ on $\mathbb{G}^{(d-1)}$ by taking the disjoint union of all $H_j$. This defines $\tilde{H}^{(d-1)}$. We define $\tilde{H}(e) = H'(e)$ for $\deg(e) = d$.

Since $H'$ is a morphism of ordinary graphs of groups, $H'(\tau(e))$ and $\tau(H'(e))$ coincide in $\Gamma'/\Gamma'^{(d-1)}$, the underlying graph of $T^{d-1}\mathbb{G}'$. Therefore the vertices $\tilde{H}(\tau(e))$ and $\tau(\tilde{H}(e))$ lie in the same connected component of the stratum $\Gamma'^{(d-1)}$, and we define $\delta_{\tilde{H}}(e) \in \pi_1(\mathbb{G}'^{(d-1)}, \tilde{H}(\tau(e)), \tau(\tilde{H}(e)))$ arbitrarily.

Replacing $\widehat{\phi}$ with the outer isomorphism class $\widehat{\tilde{H}}^{-1}\widehat{\phi}$, we may assume $\mathbb{G} = \mathbb{G}'$, all $H'_{v_j} = 1$, and $\widehat{\phi}$ is represented by $H' \in \mathrm{Aut}^0(T^{d-1}\mathbb{G})$. In this special case, we easily construct $H \in \mathrm{Aut}^0(\mathbb{G})$ as follows: We take $H^{(d-1)}$ to be the identity on $\mathbb{G}^{(d-1)}$. Whenever $e$ is an edge of degree $d$, we define $\delta_H(e)$ to agree with $\delta_{H'}(e)$ under the identification of $\pi_1(\mathbb{G}^{(d-1)}, \tau(e))$ with the vertex group of $T^{d-1}\mathbb{G}$ at $\tau(e)$. Then clearly $\widehat{\phi} = \widehat{H'} = \widehat{H}$. □

## 10.3 Clusters of vertex group conjugates

We now consider a normalised or pointedly normalised higher Dehn twist $D$. Let $A \subset \pi_1(\mathbb{G}, v)$ be a subgroup such that the conjugacy class of every element in $A$ grows at most polynomially of degree $d-1$ under iteration of $D$. Whenever $\widehat{\phi} \in \mathrm{Out}(\pi_1(\mathbb{G}, v))$ commutes with $\widehat{D}$, then by Lemma 4.2, the group $\phi(A)$ is again of this type.

We now develop more information for vertex group conjugates of $T^{d-1}\mathbb{G}$ which is invariant under automorphisms commuting with $\widehat{D}$.

Some of the following definitions make sense for arbitrary higher graphs of groups $\mathbb{G}$ with any $D \in \mathrm{Aut}^0(\mathbb{G})$, but we are mainly interested in the case of a prenormalised higher Dehn twist of a finitely generated free group.

For simplicity of notation, we shall sometimes write $\theta\pi_1\theta^{-1}$ for $\theta\pi_1(\mathbb{G}^{(d-1)}, u)\theta^{-1}$ when $\theta \in \pi_1(\mathbb{G}, v, u)$.

**Definition 10.6.** Two non-trivial vertex groups $\theta\pi_1\theta^{-1}$ and $\epsilon\pi_1\epsilon^{-1}$ are in a *common cluster* if there is a constant $C > 0$ such that for every $j \geq 0$ there are non-trivial $g_\theta(j) \in \pi_1(\mathbb{G}^{(d-1)}, u)$ and $g_\epsilon(j) \in \pi_1(\mathbb{G}^{(d-1)}, w)$ such that the cyclic length $l_c(D_*^j(g_\theta(j)\theta^{-1}\epsilon g_\epsilon(j)\epsilon^{-1}\theta)) \leq C$.

In other words, being in a common cluster means that the basis length of the coset $\pi_1(\mathbb{G}^{(d-1)})\theta^{-1}\epsilon\pi_1(\mathbb{G}^{(d-1)})$, which measures the distance (in basis length) between $\theta\pi_1\theta^{-1}$ and $\epsilon\pi_1\epsilon^{-1}$, does not grow under iteration of the automorphism $D_*$.

As the terminology suggests, a cluster $\mathcal{C}$ is a maximal collection of non-trivial vertex groups of $T^{d-1}\mathbb{G}$ which are pairwise in a common cluster.

**Lemma 10.7.** *Let $D$ be a prenormalised higher Dehn twist on $\mathbb{G}$ of degree $d \geq 2$. The vertex groups $\theta\pi_1\theta^{-1}$ and $\epsilon\pi_1\epsilon^{-1}$ of $T^{d-1}\mathbb{G}$ are in a common cluster if and only if either*

- *$\theta^{-1}\epsilon$ goes across at most one edge of degree $d$, or*

- $\theta^{-1}\epsilon = xt_e yt_{e'}^{-1}z$, *where* $x$ *and* $z$ *lie in* $\pi_1(\mathbb{G}^{(d-1)})$, *the element* $y \in G_{\tau(e)}$, *and* $\delta_D(e) = \delta_D(e') = 1$.

*Proof.* Suppose a reduced expression for $\theta^{-1}\epsilon$ contains a segment of the form $t_e\zeta t_{e'}$ with $\deg(e) = \deg(e') = d$ and $\zeta \in \pi_1(\mathbb{G}^{(d-1)}, \tau(e), \iota(e'))$. Then $D_*^j(\theta^{-1}\epsilon)$ contains a segment $(t_e, A_j(\delta_D(e))^{-1}D_*^j(\zeta)A_j(\delta_D(\overline{e'})), t_{e'})$ in a representing word reduced in the truncated sense. If $\theta\pi_1\theta^{-1}$ and $\epsilon\pi_1\epsilon^{-1}$ are in a common cluster, then the basis length of

$$X_j := A_j(\delta_D(e))^{-1}D_*^j(\zeta)A_j(\delta_D(\overline{e'}))$$

has to be bounded when $j \to \infty$. By Lemma 7.24 this is possible only if $\delta_D(e) = D_*(\zeta)\delta_D(\overline{e'})\zeta^{-1}$. Using Definition 7.20(5), we have $\delta_D(e) = 1 = \delta_D(\overline{e'})$ or $e = e'$. If we had $e = e'$, then Lemma 7.23 would show $\zeta = 1$, so the expression $t_e\zeta t_{e'}$ would not be reduced. Therefore $\delta_D(e)$ and $\delta_D(\overline{e'})$ have to be trivial. Moreover, as the length of $X_j$ is bounded, Proposition 7.22(i) shows that $\zeta$ has to lie in the vertex group $G_{\tau(e)}$.

If a reduced expression for $\theta^{-1}\epsilon$ contains at least three edges of degree $d$, a segment $(t_e, \zeta, t_{e'}, \zeta', t_{e''})$ say, then this argument leads to the contradiction $\delta_D(\overline{e'}) = 1 = \delta_D(e')$. □

Given a vertex $u$ and $\theta \in \pi_1(\mathbb{G}, v, u)$, we write

$$\begin{aligned}\mathcal{C}_\theta := \{\theta g t_e^{-1}\pi_1 t_e g^{-1}\theta^{-1} | g \in G_u, \deg(e) = d, \tau(e) = u, \delta_D(e) = 1\} \cup \\ \cup \{\theta\pi_1\theta^{-1}\} \setminus \{1\}.\end{aligned} \tag{46}$$

Lemma 10.7 shows that $\mathcal{C}_\theta$ is contained in a cluster.

Whenever $\epsilon\pi_1\epsilon^{-1}$ and $\epsilon'\pi_1\epsilon'^{-1}$ are in a common cluster, then it can be checked by Lemma 10.7 that they are in some $\mathcal{C}_\theta$, and the coset $\theta G_u$ is uniquely determined. Furthermore, every degree $d$ edge $e$ with $\tau(e) = u$ and $\delta_D(e) = 1$ satisfies $\delta_D(\overline{e}) \neq 1$, so $\theta g t_e^{-1}\pi_1 t_e g^{-1}\theta^{-1} \neq 1$. If $\mathcal{C}_\theta$ has at most one non-trivial group, then $G_u = 1$ and $u$ has valence at most one (cf. Definition 7.20(4)). As this does not occur in normalised higher Dehn twists, we have:

**Proposition 10.8.** *If* $D \in \mathrm{Aut}^0(\mathbb{G})$ *is a normalised higher Dehn twist with free* $\pi_1(\mathbb{G}, v) \neq 1$, *then the clusters are exactly the sets* $\mathcal{C}_\theta$. □

## 10.4 Central vertex groups in clusters

**Definition 10.9.** $\theta\pi_1(\mathbb{G}^{(d-1)}, u)\theta^{-1}$ is *central in the cluster* $\mathcal{C}$ if there is a non-trivial $g \in \pi_1(\mathbb{G}^{(d-1)}, u)$ such that for every vertex group $\epsilon\pi_1\epsilon^{-1} \in \mathcal{C}$, the vertex group $\theta g\theta^{-1}\epsilon\pi_1\epsilon^{-1}\theta g^{-1}\theta^{-1} \in \mathcal{C}$.

**Lemma 10.10.** *If* $D$ *is a normalised higher Dehn twist, then* $\theta\pi_1\theta^{-1}$ *is central in the cluster* $\mathcal{C}$ *if and only if* $\theta^{-1}\epsilon$ *goes across (only) one edge of degree* $d$ *for every other* $\epsilon\pi_1\epsilon^{-1} \in \mathcal{C}$.

*Proof.* As every cluster is of the form $\mathcal{C}_\theta$ for some $u \in V(\Gamma)$ and $\theta \in \pi_1(\mathcal{G}, u)$, it suffices to show that $\theta\pi_1\theta^{-1}$ is the only central vertex group in the cluster $\mathcal{C}_\theta$ (cf. (46)).

Suppose first there is a non-trivial $g' \in G_u$. Then

$$(\theta g'\theta^{-1})(\theta g t_e^{-1}\pi_1 t_e g^{-1}\theta^{-1})(\theta g'^{-1}\theta^{-1}) = \theta(g'g)t_e^{-1}\pi_1 t_e(g'g)^{-1}\theta^{-1} \in \mathcal{C}_\theta$$

shows that $\theta\pi_1\theta^{-1}$ is central in the cluster $\mathcal{C}_\theta$.

If $\delta_D(e) = 1$, then $\delta_D(\overline{e}) \neq 1$, and we may pick any non-trivial $g' \in \pi_1(\mathbb{G}^{(d-1)}, \iota(e))$. Then

$$\begin{aligned}&(\theta g t_e^{-1} g' t_e g^{-1}\theta^{-1})(\theta\pi_1\theta^{-1})(\theta g t_e^{-1} g' t_e g^{-1}\theta^{-1})^{-1}\\ &= \theta g t_e^{-1} g' t_e \pi_1 t_e^{-1} g'^{-1} t_e g^{-1}\theta^{-1}\end{aligned}$$

does not belong to the cluster $\mathcal{C}_\theta$. Thus $\theta g t_e^{-1}\pi_1 t_e g^{-1}\theta^{-1}$ is not central in $\mathcal{C}_\theta$.

If $G_u = 1$, then $\delta_D(e) = 1$ for every degree $d$ edge $e$ terminating at $u$. Therefore $\delta_D(\overline{e}) \neq 1$ and $\pi_1(\mathbb{G}^{(d-1)}, \iota(e)) \neq 1$. Similar calculations show that the cluster

$$\mathcal{C}_\theta = \{\theta t_e^{-1}\pi_1 t_e \theta | \deg(e) = d, \tau(e) = u\}$$

has no central vertex group. $\square$

**Proposition 10.11.** *Let $D \in \mathrm{Aut}^0(\mathbb{G})$ and $D' \in \mathrm{Aut}^0(\mathbb{G}')$ be normalised higher Dehn twists of degree $d$ and $\phi : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', v')$ an isomorphism such that $\widehat{\phi}\widehat{D}\widehat{\phi}^{-1} = \widehat{D'}$. Then:*

- *(i) For every vertex group $\theta\pi_1\theta^{-1}$ in $T^{d-1}\mathbb{G}$ there is some $\theta' \in \pi_1(\mathbb{G}', v', u')$ such that $\phi(\theta\pi_1\theta^{-1}) = \theta'\pi_1\theta'^{-1}$.*
- *(ii) Two vertex groups $\theta_1\pi_1\theta_1^{-1}$ and $\theta_2\pi_1\theta_2^{-1}$ are in a common cluster if and only if their images $\phi(\theta_1\pi_1\theta_1^{-1})$ and $\phi(\theta_2\pi_1\theta_2^{-1})$ are. Thus every cluster $\mathcal{C}$ of $\mathbb{G}$ is mapped to a unique cluster of $\mathbb{G}'$, which we denote by $\phi(\mathcal{C})$.*
- *(iii) The group $\theta\pi_1\theta^{-1}$ is central in $\mathcal{C}$ if and only if $\phi(\theta\pi_1\theta^{-1})$ is central in $\phi(\mathcal{C})$.*
- *(iv) If $\phi(\theta_1\pi_1\theta_1^{-1}) = \theta_1'\pi_1\theta_1'^{-1}$ and $\phi(\theta_2\pi_1\theta_2^{-1}) = \theta_2'\pi_1\theta_2'^{-1}$, then reduced words for $\theta_1^{-1}\theta_2$ and $\theta_1'^{-1}\theta_2'$ contain the same number of edges of degree $d$.*

*Proof.* (i) follows from Lemma 4.2 because vertex group conjugates are the maximal subgroups $A$ such that the conjugacy class in $A$ grows at most polynomially of degree $d-1$.

To show (ii), let $g_1(j)$ and $g_2(j)$ be elements in $\pi_1(\mathbb{G}^{(d-1)})$ such that the cyclic basis length of $D_*^j(\theta_1 g_1(j)\theta_1^{-1}\theta_2 g_2(j)\theta_2^{-1})$ is bounded when $j \to \infty$. Then the cyclic basis length of $\phi(D_*^j(\theta_1 g_1(j)\theta_1^{-1}\theta_2 g_2(j)\theta_2^{-1}))$, which is equal to the cyclic length of $D_*'^j(\phi(\theta_1 g_1(j)\theta_1^{-1}\theta_2 g_2(j)\theta_2^{-1}))$, is also bounded. If we have $\phi(\theta_1\pi_1\theta_1^{-1}) = \theta_1'\pi_1\theta_1'^{-1}$ and $\phi(\theta_2\pi_1\theta_2^{-1}) = \theta_2'\pi_1\theta_2'^{-1}$, there are unique $g_1'(j), g_2'(j) \in \pi_1(\mathbb{G}'^{(d-1)}) \smallsetminus \{1\}$ such that

$$\begin{aligned}\phi(\theta_1 g_1(j)\theta_1^{-1}) &= \theta_1' g_1'(j)\theta_1'^{-1},\\ \phi(\theta_2 g_2(j)\theta_2^{-1}) &= \theta_2' g_2'(j)\theta_2'^{-1}.\end{aligned}$$

In particular, the cyclic length

$$l_c(D_*^{\prime j}(\theta_1' g_1'(j)\theta_1'^{-1}\theta_2' g_2'(j)\theta_2'^{-1}))$$

is bounded when $j \to \infty$. Thus $\theta_1'\pi_1\theta_1'^{-1}$ and $\theta_2'\pi_1\theta_2'^{-1}$ are in a common cluster, as claimed. The converse implication follows by the same argument applied to $\phi^{-1}$.

For the proof of (iii) we use the notation of Definition 10.9. Let $\phi(\theta\pi_1\theta^{-1}) = \theta'\pi_1\theta'^{-1}$. There is a unique non-trivial $g' \in \pi_1(\mathbb{G}'^{(d-1)})$ such that $\phi(\theta g\theta^{-1}) = \theta' g'\theta'^{-1}$. Whenever $\epsilon\pi_1\epsilon^{-1} \in \mathcal{C}$, then we write $\phi(\epsilon\pi_1\epsilon^{-1}) = \epsilon'\pi_1\epsilon'^{-1}$. By (ii), every vertex group of the cluster $\phi(\mathcal{C})$ is of this form. We have

$$\theta' g'\theta'^{-1}\epsilon'\pi_1\epsilon'^{-1}\theta' g'^{-1}\theta'^{-1} = \phi(\theta g\theta^{-1}\epsilon\pi_1\epsilon^{-1}\theta g^{-1}\theta^{-1}) \in \phi(\mathcal{C}).$$

Therefore $\theta'\pi_1\theta'^{-1}$ is central in $\phi(\mathcal{C})$. The converse follows by looking at $\phi^{-1}$.

The proof of (iv) requires some technical preparation. A *cluster sequence* from $\theta_1\pi_1\theta_1^{-1}$ to $\theta_2\pi_1\theta_2^{-1}$ is a sequence of non-trivial vertex group conjugates

$$\epsilon_0\pi_1\epsilon_0^{-1}, \epsilon_1\pi_1\epsilon_1^{-1}, \ldots, \epsilon_l\pi_1\epsilon_l^{-1}$$

such that $\epsilon_{i-1}\pi_1\epsilon_{i-1}^{-1}$ and $\epsilon_i\pi_1\epsilon_i^{-1}$ are in a common cluster for every $i$, $1 \leq i \leq l$. We let $\lambda_i = 1$ if at least one of $\epsilon_{i-1}\pi_1\epsilon_{i-1}^{-1}$ and $\epsilon_i\pi_1\epsilon_i^{-1}$ is central in their common cluster. Otherwise we set $\lambda_i = 2$. The *length* of the cluster sequence is now defined to be $\lambda_1 + \ldots + \lambda_l$.

Using Lemma 10.7 and Lemma 10.10, it can be checked that the number of edges of degree $d$ appearing in a reduced word for $\theta_1^{-1}\theta_2$ agrees with the minimal length of a cluster sequence from $\theta_1\pi_1\theta_1^{-1}$ to $\theta_2\pi_1\theta_2^{-1}$. As $\phi$ maps cluster sequences to cluster sequences of the same length by (i), (ii), and (iii), it follows that $\theta_1'^{-1}\theta_2'$ contains at most as many edges of degree $d$ as $\theta_1^{-1}\theta_2$ does. This together with the same argument for $\phi^{-1}$ proves (iv). $\square$

## 10.5 Stabilisation of clusters

Fix a normalised higher Dehn twist $D$ on $\mathbb{G}$ of degree $d \geq 2$ with finitely generated free $\pi_1(\mathbb{G}, v)$ throughout this section. All connected components of $D|_{\Gamma^{(d-1)}}$ are prenormalised, but not necessarily normalised. We now discuss how to modify $D$ to $\bar{D}$ on the same underlying graph such that all connected components of $\bar{D}|_{\Gamma^{(d-1)}}$ are indeed normalised.

The higher graph of groups $\bar{\mathbb{G}}$ for $\bar{D}$ is obtained as follows. We add new free factors at the vertex groups:

$$\bar{G}_w = G_w * \langle c_e | \deg(e) = d, \tau(e) = w, \delta_D(e) = 1\rangle.$$

Here $c_e$ is a new formal free generator. In other words, we add to each vertex group of $\mathbb{G}$ a free group of rank equal to the number of degree $d$ edges $e$ terminating there with trivial $\delta$-term.

The map $f_e$ for $\bar{\mathbb{G}}$ is the composition $\bar{G}_e = G_e \to G_{\tau(e)} \to \bar{G}_{\tau(e)}$ of the map $f_e$ for $\mathbb{G}$ with the obvious inclusion map. The other data of $\bar{\mathbb{G}}$ is the same as for $\mathbb{G}$.

There is an inclusion (or stabilisation) morphism $S : \mathbb{G} \to \bar{\mathbb{G}}$ and a projection morphism $P : \bar{\mathbb{G}} \to \mathbb{G}$ defined in the obvious way on all pieces. The higher Dehn twist $\bar{D}$ on $\bar{\mathbb{G}}$ is defined by the identity on the underlying graph, all vertex and edge groups, and with $\delta_{\overline{D}}(e) = S_*(\delta_D(e))$ for every edge $e$.

Suppose we are now given two normalised higher Dehn twists $D$ and $D'$ on $\mathbb{G}$ and $\mathbb{G}'$ respectively, and let $\phi$ be an isomorphism such that the following diagram commutes in the category of outer isomorphism classes:

$$\begin{tikzcd} \pi_1(\mathbb{G},v) \arrow[r, "\widehat{\phi}"] \arrow[d, "\widehat{D}"'] & \pi_1(\mathbb{G}',v') \arrow[d, "\widehat{D'}"] \\ \pi_1(\mathbb{G},v) \arrow[r, "\widehat{\phi}"'] & \pi_1(\mathbb{G}',v') \end{tikzcd} \tag{47}$$

Translating the diagram (47) to ordinary homomorphisms, there is $\zeta \in \pi_1(\mathbb{G}', v')$ such that the following diagram commutes.

$$\begin{tikzcd} \pi_1(\mathbb{G},v) \arrow[r, "\phi"] \arrow[d, "D_{*v}"'] & \pi_1(\mathbb{G}',v') \arrow[r, "D'_{*v'}"] & \pi_1(\mathbb{G}',v') \arrow[d, equal] \\ \pi_1(\mathbb{G},v) \arrow[r, "\phi"] & \pi_1(\mathbb{G}',v') \arrow[r, "\mathrm{ad}_\zeta"] & \pi_1(\mathbb{G}',v') \end{tikzcd} \tag{48}$$

By Proposition 10.8 every cluster $\mathcal{C}_\theta$ of vertex groups in $T^{d-1}\mathbb{G}$ defines a unique coset $\theta G_u$ with $\theta \in \pi_1(\mathbb{G}, v, u)$. For every edge $e$ with $\deg(e) = d$ and $\delta_D(e) = 1$, we pick some $\theta_e \in \pi_1(\mathbb{G}, v, \tau(e))$. As $\phi$ maps the cluster $\mathcal{C}_{\theta_e}$ to another cluster by Proposition 10.11(ii), every such $\theta_e$ uniquely determines a degree $d$ edge $e^\phi$ of $\Gamma'$ (only depending on $e$, but not on $\theta_e$) and $\theta'_e \in \pi_1(\mathbb{G}', v', \tau(e^\phi))$ such that

$$\begin{aligned} \phi(\theta_e \pi_1 \theta_e^{-1}) &= \theta'_e \pi_1 \theta_e'^{-1}, \\ \phi(\theta_e t_e^{-1} \pi_1 t_e \theta_e^{-1}) &= \theta'_e t_{e^\phi}^{-1} \pi_1 t_{e^\phi} \theta_e'^{-1}, \\ \delta_{D'}(e^\phi) &= 1. \end{aligned}$$

We now define the isomorphism $\overline{\phi} : \pi_1(\bar{\mathbb{G}}, v) \to \pi_1(\bar{\mathbb{G}}', v')$ to agree with $\phi$ on $\pi_1(\mathbb{G}, v)$ and to satisfy

$$\overline{\phi}(\theta_e c_e \theta_e^{-1}) = \theta'_e c_{e^\phi} \theta_e'^{-1}. \tag{49}$$

**Lemma 10.12.** *The following diagram commutes:*

$$\begin{tikzcd} \pi_1(\bar{\mathbb{G}},v) \arrow[r, "\bar{\phi}"] \arrow[d, "\bar{D}_{*v}"'] & \pi_1(\bar{\mathbb{G}}',v') \arrow[r, "\bar{D}'_{*v'}"] & \pi_1(\bar{\mathbb{G}}',v') \arrow[d, equal] \\ \pi_1(\bar{\mathbb{G}},v) \arrow[r, "\bar{\phi}"] & \pi_1(\bar{\mathbb{G}}',v') \arrow[r, "\mathrm{ad}_\zeta"] & \pi_1(\bar{\mathbb{G}}',v') \end{tikzcd} \tag{50}$$

*Proof.* As we require commutativity of (48), we already know commutativity of (50) when precomposing with the inclusion homomorphism $S_* : \pi_1(\mathbb{G}, v) \to \pi_1(\bar{\mathbb{G}}, v)$. It now suffices to verify that both compositions in (50) agree on all elements $\theta_e c_e \theta_e^{-1}$, which are defined whenever $\deg(e) = d$ and $\delta_D(e) = 1$. Recall that $\delta_{D'}(e^\phi) = 1$.

We now look at the images of the vertex group conjugates $\theta_e \pi_1 \theta_e^{-1}$ and $\theta_e t_e^{-1} \pi_1 t_e \theta_e^{-1}$ under both compositions in (48).

The vertex group $\theta_e t_e^{-1} \pi_1 t_e \theta_e^{-1}$ is mapped to $\theta'_e t_{e^\phi}^{-1} \pi_1 t_{e^\phi} \theta_e'^{-1}$ by $\phi$ and then to the vertex group $D'_*(\theta'_e)\delta_{D'}(e^\phi) t_{e^\phi}^{-1} \pi_1 t_{e^\phi} \delta_{D'}(e^\phi)^{-1} D'_*(\theta'_e)^{-1}$, which equals the vertex group $D'_*(\theta'_e) t_{e^\phi}^{-1} \pi_1 t_{e^\phi} D'_*(\theta'_e)^{-1}$, by $D'_{*v'}$.

Following the other composition, $\theta_e t_e^{-1} \pi_1 t_e \theta_e^{-1}$ is first mapped to the vertex group $D_*(\theta_e)\delta_D(e) t_e^{-1} \pi_1 t_e \delta_D(e)^{-1} D_*(\theta_e)^{-1}$ by $D_*$. This equals $D_*(\theta_e) t_e^{-1} \pi_1 t_e D_*(\theta_e)^{-1}$ and is mapped to $\zeta\phi(D_*(\theta_e)\theta_e^{-1})\theta'_e t_{e^\phi}^{-1} \pi_1 t_{e^\phi} \theta_e'^{-1} \phi(\theta_e D_*(\theta_e)^{-1})\zeta^{-1}$ by $\mathrm{ad}_\zeta \circ \phi$.

As (48) commutes, these two expressions agree:

$$D'_*(\theta'_e) t_{e^\phi}^{-1} \pi_1 t_{e^\phi} D'_*(\theta'_e)^{-1} = \zeta\phi(D_*(\theta_e)\theta_e^{-1})\theta'_e t_{e^\phi}^{-1} \pi_1 t_{e^\phi} \theta_e'^{-1} \phi(\theta_e D_*(\theta_e)^{-1})\zeta^{-1}. \tag{51}$$

Similarly, chasing the vertex group $\theta_e \pi_1 \theta_e^{-1}$ around (48) leads to

$$D'_*(\theta'_e) \pi_1 D'_*(\theta'_e)^{-1} = \zeta\phi(D_*(\theta_e)\theta_e^{-1})\theta'_e \pi_1 \theta_e'^{-1} \phi(\theta_e D_*(\theta_e)^{-1})\zeta^{-1}. \tag{52}$$

Comparing (51) and (52), we conclude

$$D'_*(\theta'_e) = \zeta\phi(D_*(\theta_e)\theta_e^{-1})\theta'_e$$

and hence

$$D'_*(\theta'_e) c_{e^\phi} D'_*(\theta'_e)^{-1} = \zeta\phi(D_*(\theta_e)\theta_e^{-1})\theta'_e c_{e^\phi} \theta_e'^{-1} \phi(\theta_e D_*(\theta_e)^{-1})\zeta^{-1}.$$

By definition of $\bar{D}'$ and $\overline{\phi}$, this can be rewritten as

$$\bar{D}'_*(\theta'_e c_{e^\phi} \theta_e'^{-1}) = \zeta\overline{\phi}(D_*(\theta_e) c_e D_*(\theta_e)^{-1})\zeta^{-1},$$

hence

$$\bar{D}'_*(\overline{\phi}(\theta_e c_e \theta_e^{-1})) = \zeta\overline{\phi}(\bar{D}_*(\theta_e c_e \theta_e^{-1}))\zeta^{-1}.$$

This finishes the verification that (50) commutes. $\square$

**Lemma 10.13.** *If* $\deg(e) = d$, $\delta_D(e) = 1$, *and* $\bar{H} : \bar{\mathbb{G}} \to \bar{\mathbb{G}}'$ *is an equivalence with* $\widehat{\bar{H}}\widehat{\bar{D}} = \widehat{\bar{D}'}\widehat{\bar{H}}$, *then* $\delta_{\bar{H}}(e) \in \bar{G}'_{\tau(\bar{H}(e))}$ *and* $\delta_{\bar{D}'}(\bar{H}(e)) = 1$.

*Proof.* For a given $e \in E(\Gamma)$ of degree $d$ with $\delta_D(e) = 1$, choose again $\theta \in \pi_1(\mathbb{G}, v, \tau(e))$. Then the vertex groups $\theta t_e^{-1} \pi_1 t_e \theta^{-1}$ and $\theta c_e t_e^{-1} \pi_1 t_e c_e^{-1} \theta^{-1}$ are in a common cluster. By Proposition 10.11(ii), their image vertex groups $\bar{H}_*(\theta)\bar{H}_*(t_e^{-1})\pi_1 \bar{H}_*(t_e)\bar{H}_*(\theta)^{-1}$ and $\bar{H}_*(\theta)\bar{H}_*(c_e t_e^{-1})\pi_1 \bar{H}_*(t_e c_e^{-1})\bar{H}_*(\theta)^{-1}$ are in a common cluster. As

$$\bar{H}_*(t_e c_e t_e^{-1}) = \delta_{\bar{H}}(\bar{e}) t_{\bar{H}(e)} \delta_{\bar{H}}(e)^{-1} \bar{H}_{\tau(e)}(c_e) \delta_{\bar{H}}(e) t_{\bar{H}(e)}^{-1} \delta_{\bar{H}}(\bar{e})^{-1},$$

Lemma 10.7 implies $\delta_{\bar{D}'}(\bar{H}(e)) = 1$ and $\delta_{\bar{H}}(e)^{-1}\bar{H}_{\tau(e)}(c_e)\delta_{\bar{H}}(e) \in G_{\bar{H}(\tau(e))}$, so

$$\bar{H}_*^{-1}(\delta_{\bar{H}}(e))^{-1}c_e\bar{H}_*^{-1}(\delta_{\bar{H}}(e)) \in \bar{G}_{\tau(e)}. \tag{53}$$

In the vertex group $\bar{G}_{\tau(e)}$, the element $c_e$ is not conjugate to an element in any $f_{e'}(\bar{G}_{e'})$ with $\tau(e') = \tau(e)$. Looking at a reduced word for $\bar{H}_*^{-1}(\delta_{\bar{H}}(e))$, we read off in (53) that $\bar{H}_*^{-1}(\delta_{\bar{H}}(e)) \in \bar{G}_{\tau(e)}$ and hence $\delta_{\bar{H}}(e) \in \bar{G}'_{\bar{H}(\tau(e))} = \bar{G}'_{\tau(\bar{H}(e))}$. □

Using the inclusion $S$ and the projection $P$, we have:

**Lemma 10.14.** *If the outer isomorphism class of $\overline{\phi}$ is represented by an equivalence $\bar{H} : \bar{\mathbb{G}} \to \bar{\mathbb{G}}'$, then $H := P\bar{H}S : \mathbb{G} \to \mathbb{G}'$ is an equivalence with $\widehat{H} = \widehat{\phi}$.*

*Proof.* Clearly $\widehat{H} = \widehat{\phi}$, and all edge group homomorphisms $H_e$ are isomorphisms. Then Lemma 3.4 shows that $H$ is indeed an equivalence of higher graphs of groups. □

## 10.6 Representing centralisers by abstract automorphisms

Let first $\mathcal{G}$ and $\mathcal{G}'$ be ordinary graphs of groups.

**Proposition 10.15** ([13], Theorem 6.9(a))**.** *If $D$ is an efficient Dehn twist on $\mathcal{G}$, $D'$ an efficient Dehn twist on $\mathcal{G}'$, and $\phi : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', v')$ an isomorphism such that $\widehat{D'} = \widehat{\phi}\widehat{D}\widehat{\phi}^{-1}$, then there is an equivalence $H : \mathcal{G} \to \mathcal{G}'$ such that $\widehat{\phi} = \widehat{H}$.* □

The goal of this chapter is to extend this theorem to the following analogue for higher graphs of groups.

**Theorem 10.16.** *Let $D \in \mathrm{Aut}^0(\mathbb{G})$ and $D' \in \mathrm{Aut}^0(\mathbb{G}')$ be normalised higher Dehn twists and $\phi : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', v')$ an isomorphism. If $\widehat{D'} = \widehat{\phi}\widehat{D}\widehat{\phi}^{-1}$, then there is an equivalence $H : \mathbb{G} \to \mathbb{G}'$ such that $\widehat{H} = \widehat{\phi}$. Moreover, $\delta_{D'}(H(e)) = 1$ and $\delta_H(e) \in G'_{\tau(H(e))}$ whenever $\deg(e) \geq 2$ and $\delta_D(e) = 1$.*

*Proof.* Note that $\mathbb{G}$ and $\mathbb{G}'$ have the same degree because $\widehat{D}$ and $\widehat{D'}$ are conjugate, $\widehat{D}$ grows polynomially of degree $\deg(\mathbb{G})$, and $\widehat{D'}$ grows polynomially of degree $\deg(\mathbb{G}')$. The proof is now by induction on this degree $d$. If $d \leq 1$, then we have efficient Dehn twists on ordinary graphs of groups, and the assertion is exactly Proposition 10.15.

If the degree $d \geq 2$, then we construct $\bar{D} \in \mathrm{Aut}^0(\bar{\mathbb{G}})$, $\bar{D}' \in \mathrm{Aut}^0(\bar{\mathbb{G}}')$, and $\overline{\phi} : \pi_1(\bar{\mathbb{G}}, v) \to \pi_1(\bar{\mathbb{G}}', v')$ as in Section 10.5. All connected components of the restrictions $\bar{D}^{(d-1)}$ and $\bar{D}'^{(d-1)}$ are now normalised higher Dehn twists of degree $d - 1$.

The restrictions of $\overline{\phi}$ to these components are well-defined outer isomorphism classes $\pi_1(\mathbb{F}, v_{\mathbb{F}}) \to \pi_1(\phi(\mathbb{F}), v_{\phi(\mathbb{F})})$ for any choice of basepoints $v_{\mathbb{F}}$ and $v_{\phi(\mathbb{F})}$. By induction, they are represented by equivalences $H_{\mathbb{F}} : \mathbb{F} \to \phi(\mathbb{F})$ of graphs of groups of degree $d-1$ such that $\delta_{D'}(H_{\mathbb{F}}(e)) = 1$ and $\delta_{H_{\mathbb{F}}}(e)$ lies in a vertex group of $\mathbb{G}'$ when $2 \leq \deg(e) \leq d - 1$ and $\delta_D(e) = 1$.

Let now $J : \bar{\mathbb{G}} \to \tilde{\mathbb{G}}$ and $J' : \bar{\mathbb{G}}' \to \tilde{\mathbb{G}}'$ be equivalences such that $\tilde{\mathbb{G}}$ and $\tilde{\mathbb{G}}'$ are truncatable at degree $d - 1$. Then the outer isomorphism class of the restriction of

$J'_*\overline{\phi}J_*^{-1}$ to the fundamental group of each connected component of $\tilde{\mathbb{G}}^{(d-1)}$ is represented by a higher graph of groups equivalence.

By Proposition 10.11(iv), the outer isomorphism class of $J'_*\overline{\phi}J_*^{-1}$ maps vertex groups of $T^{d-1}\tilde{\mathbb{G}}$ to vertex groups of $T^{d-1}\tilde{\mathbb{G}}'$ and preserves their distances (in terms of edges of degree $d$). Since every normalised higher Dehn twist has at least one non-trivial vertex group, Corollary 10.3 shows that $\widehat{J'}\overline{\phi}\widehat{J}^{-1}$ is represented by an equivalence $T^{d-1}\tilde{\mathbb{G}} \to T^{d-1}\tilde{\mathbb{G}}'$. Proposition 10.5 now shows that $\widehat{J'}\widehat{\overline{\phi}}\widehat{J}^{-1}$ is represented by an equivalence $\tilde{H} : \tilde{\mathbb{G}} \to \tilde{\mathbb{G}}'$. Then $\widehat{\overline{\phi}}$ is represented by the equivalence $\bar{H} = J'^{-1}\tilde{H}J : \bar{\mathbb{G}} \to \bar{\mathbb{G}}'$.

Chasing the definitions, we have $\bar{H}|_{\mathbb{F}} = H_{\mathbb{F}}$ for every connected component $\mathbb{F}$ of $\bar{\mathbb{G}}^{(d-1)}$. In particular, $\delta_{\bar{D}'}(\bar{H}(e)) = 1$ and $\delta_{\bar{H}}(e)$ lies in the vertex group $G_{\tau(\bar{H}(e))}$ whenever $\delta_D(e) = 1$ and $2 \leq \deg(e) \leq d-1$. By Lemma 10.13 we also have this conclusion for edges of degree $d$.

Lemma 10.14 gives rise to the desired equivalence $H = P\bar{H}S : \mathbb{G} \to \mathbb{G}'$. As $\delta_S(e) = 1$ and $\delta_P(e) = 1$ for every edge $e$, we have $\delta_H(e) = P_*(\delta_{\bar{H}}(e))$, so it lies in $G_{\tau(H(e))}$ whenever $\delta_D(e) = 1$. Since $D' = P\bar{D}'S$, we also have $\delta_{D'}(\bar{H}(e)) = P_*(\delta_D(H(e))) = 1$ for these edges $e$. □

There is the following analogue of Theorem 10.16 in the pointed case.

**Theorem 10.17.** *Let $D \in \mathrm{Aut}^0(\mathbb{G})$ and $D' \in \mathrm{Aut}^0(\mathbb{G}')$ be pointedly normalised higher Dehn twists and $\phi : \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}', v')$ an isomorphism. If $D'_{*v'} = \phi D_{*v}\phi^{-1}$, then there is an equivalence $H : \mathbb{G} \to \mathbb{G}'$ with $H(v) = v'$, $H_{*v} = \phi$. Moreover, $\delta_{D'}(H(e)) = 1$ and $\delta_H(e) \in G'_{\tau(H(e))}$ whenever $\deg(e) \geq 2$ and $\delta_D(e) = 1$.*

*Proof.* We define $\bar{\mathbb{G}}$ to be obtained from $\mathbb{G}$ by attaching an additional free factor $\langle c\rangle \cong \mathbb{Z}$ in the base vertex group $\bar{G}_v := G_v * \langle c\rangle$. The pointedly normalised higher Dehn twist $D$ now extends to a normalised higher Dehn twist $\bar{D}$ on $\bar{\mathbb{G}}$ by defining $\bar{D}_v(c) = c$ and leaving all other data unchanged. Similarly, we attach a free factor $\langle c'\rangle \cong \mathbb{Z}$ to $\mathbb{G}'$ to obtain $\bar{\mathbb{G}}'$ along with a normalised higher Dehn twist $\bar{D}'$. We extend $\phi$ to $\overline{\phi} : \pi_1(\bar{\mathbb{G}}, v) \to \pi_1(\bar{\mathbb{G}}', v')$ by $\overline{\phi}(c) = c'$.

$\bar{D}$, $\bar{D}'$, and $\overline{\phi}$ satisfy the requirements of Theorem 10.16, and we conclude that the outer isomorphism class of $\overline{\phi}$ is represented by an equivalence $\bar{H} : \bar{\mathbb{G}} \to \bar{\mathbb{G}}'$ with $\delta_{\bar{D}'}(\bar{H}(e)) = 1$ and $\delta_{\bar{H}}(e) \in G'_{\tau(\bar{H}(e))}$ whenever $\deg(e) \geq 2$ and $\delta_D(e) = 1$. There is $\zeta \in \pi_1(\mathbb{G}', v', \bar{H}(v))$ such that $\overline{\phi}(\eta) = \zeta\bar{H}_*(\eta)\zeta^{-1}$ for all $\eta \in \pi_1(\mathcal{G}, v)$.

Within $\bar{G}'_{v'}$, the element $c'$ is not conjugate to an element in the image of some $f_{e'}$ with $\tau(e') = v'$. Since we have $\bar{H}_v(c) = \zeta^{-1}\overline{\phi}(c)\zeta = \zeta * c' * \zeta^{-1}$, we conclude $2pl(\zeta) = pl(\zeta c'\zeta^{-1}) = pl(\bar{H}_v(c)) = 0$, so $\zeta \in \bar{G}'_{v'}$ and $\bar{H}(v) = v'$. Replacing $\bar{H}$ with $M(v', \zeta)^{-1}\bar{H}$, we can assume that $\zeta = 1$ and $\bar{H}_{*v} = \overline{\phi}$.

We now define $H : \mathbb{G} \to \mathbb{G}'$ as the composition

$$\mathbb{G} \to \bar{\mathbb{G}} \to \bar{\mathbb{G}}' \to \mathbb{G}'$$

of the inclusion morphism, $\bar{H}$, and the projection homomorphism. It can be checked that $H_{*v} = \phi$, and Lemma 3.4 ensures that $H$ is an equivalence of higher graphs of groups. The terms $\delta_{D'}(H(e))$ and $\delta_H(e)$ are the images of $\delta_{\bar{D}'}(\bar{H}(e))$ and $\delta_{\bar{H}}(e)$ under the projection morphism, so they are of the desired form whenever $\delta_D(e) = 1$. □

## 10.7 Centralisers in $\mathrm{Out}(\pi_1(\mathbb{G}, v))$ versus $\mathrm{Aut}^0(\mathbb{G})$

When $D$ is a prenormalised higher Dehn twist on $\mathbb{G}$ and $H$ represents an element in the centraliser of $\widehat{D}$, then it is a priori not clear whether $H$ and $D$ actually commute in the abstract automorphism group $\mathrm{Aut}(\mathbb{G})$. This can indeed often be arranged.

**Proposition 10.18.** *Suppose that $D, H \in \mathrm{Aut}^0(\mathbb{G})$, where $D$ is a normalised higher Dehn twist, and $H_e = 1$ for every edge $e$. If $\widehat{D}$ and $\widehat{H}$ commute and $\delta_H(e) \in G_{\tau(e)}$ for $\delta_D(e) = 1$, then there is $H' \in \mathrm{Aut}^0(\mathbb{G})$ with coinciding with $H$ of $\mathbb{G}^{(1)}$ such that $H'D = DH'$ and $\widehat{H'} = \widehat{H}$. If $D_{*v}$ and $H_{*v}$ commute and $D$ is only required to be pointedly normalised, then we may arrange $H'_{*v} = H_{*v}$.*

*Proof.* If the degree $d$ of $\mathbb{G}$ is at most one, then we may take $H' = H$. As $H_e = 1$, Lemma 7.15 shows that $HD = DH$. This finishes the proof for $d \leq 1$.

We now proceed by induction on $d \geq 2$. We apply the stabilisation construction in Section 10.5 to find a higher graph of groups $\bar{\mathbb{G}}$ obtained by adding new free factors freely generated by symbols $c_e$ to the vertex groups. Recall that the Dehn twist $\bar{D}$ is given by the same data as $D$.

If we put $\phi = H_{*v}$, then the definition of $\overline{\phi}$ in (49) on page 95 leads to $\overline{\phi} = \bar{H}_{*v}$, where $\bar{H} \in \mathrm{Aut}^0(\bar{\mathbb{G}})$ is given by the same data as $H$ on $\mathbb{G}$ and by $\bar{H}_{\tau(e)}(c_e) = \delta_H(e) c_e \delta_H(e)^{-1}$ on the new symbols $c_e$ in the vertex groups of $\bar{\mathbb{G}}$. Lemma 10.12 then implies that the outer automorphisms $\widehat{\bar{H}}$ and $\widehat{\bar{D}}$ commute.

By Lemma 9.3, we conclude that $(\bar{H}\bar{D}\bar{H}^{-1}\bar{D}^{-1})|_{\mathbb{F}} \in KO(\mathbb{F})$ for every connected component $\mathbb{F}$ of $\bar{\mathbb{G}}^{(d-1)}$. As each $\bar{D}|_{\mathbb{F}}$ is a normalised higher Dehn twist, the induction hypothesis allows us to define $\bar{H}'' \in \mathrm{Aut}^0(\bar{\mathbb{G}}^{(d-1)})$ such that $(\bar{H}''\bar{H}^{-1})|_{\mathbb{F}} \in KO(\mathbb{F})$ and $\bar{H}''$ commutes with $\bar{D}^{(d-1)}$ on every component $\mathbb{F}$ of $\mathbb{G}^{(d-1)}$.

We now look at the isomorphism (44) on page 88. Since each forgetful homomorphism $KO_I(\mathbb{F}) \to KO(\mathbb{F})$ is surjective, we may construct $\bar{H}' \in \mathrm{Aut}^0(\bar{\mathbb{G}})$ with $\bar{H}'^{(d-1)} = \bar{H}''$ and $\bar{H}'\bar{H}^{-1} \in KO(\bar{\mathbb{G}})$. As $\bar{H}'$ and $\bar{D}$ commute on $\mathbb{G}^{(d-1)}$, Proposition 9.5 implies that the commutator $\bar{H}'\bar{D}\bar{H}'^{-1}\bar{D}^{-1}$ is a composition of automorphisms $Z(\mathbb{F}, \gamma_\bullet)$ with $\pi_1(\mathbb{F}, v_{\mathbb{F}}) \cong \mathbb{Z}$. Whenever $\deg(e) = d$ and $\tau(e) \in \mathbb{F}$, then $\bar{D}_{*\tau(e)} = 1$, so the basis length of $A_j(\delta_{\bar{D}}(e), D_*) = \delta_{\bar{D}}(e)^j$ cannot grow dominantly of degree $d$. Thus $\delta_{\bar{D}}(e) = 1$ and

$$\delta_{\bar{H}'\bar{D}\bar{H}'^{-1}\bar{D}^{-1}}(e) = (\bar{H}'\bar{D}\bar{H}'^{-1})_*(\delta_{\bar{H}'}(e))^{-1}\delta_{\bar{H}'}(e) = 1.$$

Hence $\bar{H}'$ and $\bar{D}$ commute.

Denote by $H' \in \mathrm{Aut}^0(\mathbb{G})$ the composition $P\bar{H}'S$. Chasing the definitions, we see $\widehat{H'} = \widehat{H}$. Moreover, $H'P = P\bar{H}'$ and $DP = P\bar{D}$, so

$$H'D = H'DPS = P\bar{H}'\bar{D}S = P\bar{D}\bar{H}'S = DH'PS = DH'.$$

The pointed version is similar and left to the reader. $\square$

# 11 Relative automorphism groups

## 11.1 Whitehead automorphisms

Let $F_n$ be the free group with basis $a_1, \ldots, a_n$. There are two types of "standard" elements in $\mathrm{Aut}(F_n)$ called *Whitehead automorphisms* of type I and II.

A Whitehead automorphism of type I is an automorphism $P \in \mathrm{Aut}(F_n)$ permuting the set

$$L = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$$

of basis elements and inverses.

A Whitehead automorphism of type II is written as a symbol $(A; a)$ where $A$ is a subset of $L$ such that $a \in A$ and $a^{-1} \notin A$. On elements $x \in L$, it is given by

$$(A; a)(x) = \begin{cases} x, & \text{if } x, x^{-1} \notin A \text{ or } x = a^{\pm 1}, \\ xa, & \text{if } x \in A, x^{-1} \notin A, x \neq a, \\ a^{-1}x, & \text{if } x \notin A, x^{-1} \in A, x \neq a^{-1}, \\ a^{-1}xa, & \text{if } x, x^{-1} \notin A. \end{cases}$$

Sometimes we refer to $a$ as the *operative factor* of $(A; a)$.

The next proposition shows how to successively reduce the length of elements in a free group by applying Whitehead automorphisms. For a tuple $W = (w_1, \ldots, w_p)$ of elements in $F_n$, we write

$$\begin{aligned} l(W) &= l(w_1) + \ldots + l(w_m), \\ l_c(W) &= l_c(w_1) + \ldots + l_c(w_m). \end{aligned}$$

For $\alpha \in \mathrm{Aut}(F_n)$ we write $\alpha(W) = (\alpha(w_1), \ldots, \alpha(w_p))$. We call $W$ *minimal* if $l(\alpha(W)) \geq l(W)$ for every $\alpha \in \mathrm{Aut}(F_n)$. This $\mathrm{Aut}(F_n)$-action on $p$-tuples induces an action of $\mathrm{Out}(F_n)$ on the set of $p$-tuples of conjugacy classes in $F_n$. We call a tuple $\mathcal{C} = ([w_1], \ldots, [w_p])$ minimal if $l_c(\widehat{\alpha}(\mathcal{C})) \geq l_c(\mathcal{C})$ for all $\widehat{\alpha} \in \mathrm{Out}(F_n)$.

**Proposition 11.1.** *Let $|\bullet|$ denote either the cyclic length $l_c$ or the linear length $l$ with respect to the standard basis of $F_n$. Let $W = (w_1, \ldots, w_p) \in (F_n)^p$ and $\alpha \in \mathrm{Aut}(F_n)$ such that $|\alpha(W)|$ is minimal. Then there are Whitehead automorphisms $T_1, \ldots, T_m$ such that $W_m = \alpha(W)$ and*

$$|W| = |W_0| > |W_1| > |W_2| > \ldots > |W_k| = \ldots = |W_m| = |\alpha(W)|$$

*for some $k$, where $W_i := T_i T_{i-1} \ldots T_1(W)$ for $0 \leq i \leq m$.*

*Proof.* This is shown in [18] for the cyclic length and $p = 1$. It is remarked in [18] that the same argument also works for the linear length as well as for arbitrary $p$. □

## 11.2 The McCool complex

Fix a tuple $\mathcal{C} = ([w_1], \ldots, [w_p])$. We will frequently be interested in the subgroups $\mathrm{Aut}(F_n, \mathcal{C})$ and $\mathrm{Out}(F_n, \mathcal{C})$ of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ given by those automorphisms fixing each of $w_1, \ldots, w_p$ up to conjugacy. These two groups are related by a short exact sequence

$$1 \to \mathrm{Inn}(F_n) \to \mathrm{Aut}(F_n, \mathcal{C}) \to \mathrm{Out}(F_n, \mathcal{C}) \to 1$$

because inner automorphisms are clearly a normal subgroup in $\mathrm{Aut}(F_n, \mathcal{C})$.

We now recall a construction by McCool [27] how to algorithmically construct a finite 2-dimensional CW-complex $K$ having $\mathrm{Out}(F_n, \mathcal{C})$ as fundamental group. In particular, these groups have algorithmically computable finite presentations. We refer to $K$ as the *McCool complex* for $\mathrm{Out}(F_n, \mathcal{C})$.

The vertices of $K$ are given by minimal tuples $\alpha(\mathcal{C})$, where $\alpha \in \mathrm{Out}(F_n)$. As all of them have the same total cyclic length, there are only finitely many of them. There is an oriented edge $(V_1, V_2, P)$ from $V_1$ to $V_2$ whenever $P \in \mathrm{Out}(F_n)$ is a Whitehead automorphism such that $P(V_1) = V_2$. For every such edge, there is also an oriented edge $(V_2, V_1, P^{-1})$, and these two oriented edges determine a 1-cell of $K$. We can now assign to each combinatorial edge path a label by multiplying together the labels of the edges. Along a loop of length at most 6 we attach a 2-cell in $K$ whenever the labeling function is 1 on this loop. This finishes the definition of the 2-complex $K$.

By construction, there is a homomorphism $\pi_1(K, \mathcal{C}) \to \mathrm{Out}(F_n, \mathcal{C})$ given by evaluating a combinatorial loop representing an element in the fundamental group at the product of all its edge labels. It is shown in [27] that this is an isomorphism.

There is a similar complex for groups of the form $\mathrm{Aut}(F_n, w_1, \ldots, w_p)$, the group of automorphisms of $F_n$ fixing the elements $w_1, \ldots, w_p$ genuinely (cf. Section 4(2) of [27]).

## 11.3 Rigid elements in free factors

We now construct elements $w$ in a free group $F$ such that group $\mathrm{Aut}(F, [w])$ is as small as possible, namely the inner automorphism group only. This is roughly done by taking sufficiently irregular words in the basis elements, which we call rigid elements. They are defined as follows, and we will see their main property in Corollary 11.6.

**Definition 11.2.** An element $w$ in a free group $F$ is called *rigid* if there is an isomorphism $\rho : F \to F_n$ such that no Whitehead automorphism of $F_n$ of type I (except the identity) fixes the conjugacy class of $\rho(w)$, and every Whitehead automorphism of type II (apart from the identity and inner automorphisms) strictly increases the cyclic length of $\rho(w)$.

When we want to emphasize $\rho$, we will sometimes say that $w$ is rigid with respect to the basis $\rho^{-1}(a_1), \ldots, \rho^{-1}(a_n)$ of $F$, where $a_1, \ldots, a_n$ is the standard basis of $F_n$.

**Proposition 11.3.** *Every finitely generated free group $F$ contains a rigid element $w$.*

*Proof.* Without loss of generality, we can assume that we have the free group $F_n$ with basis $a_1, \dots, a_n$. We claim that

$$w_N := a_1^{N+1} a_2^N \dots a_n^N a_1^N \dots a_n^N (a_1 a_2)^N \dots (a_1 a_n)^N (a_2 a_3)^N \dots (a_2 a_n)^N \dots (a_{n-1} a_n)^N$$

is rigid with respect to the standard basis for sufficiently large $N$.

Lemma 2 in [27] states that the tuple

$$Z := (a_1, \dots, a_n, a_1, \dots, a_n, a_1 a_2, \dots, a_1 a_n, a_2 a_3, \dots, a_2 a_n, \dots, a_{n-1} a_n)$$

of cyclic words is minimal, and its (total) cyclic length is left minimal only by inner automorphisms and permutations of generators.

Fix a Whitehead automorphism $P$ of type II which is neither the identity nor an inner automorphism. We now apply $P$ to each letter in $w$. At each point in $w_N$ where two consecutive $N$-th powers hit, at most two letters cancel. The remaining cancellation corresponds to cyclic reduction of the pieces. Then

$$\begin{aligned} l_c(P(w_N)) &\geq l_c(w_N) + N(l_c(P(Z)) - l_c(Z)) - 2(2n + \frac{1}{2}(n^2 - n)) \\ &\geq l_c(w_N) + N - n^2 - 3n > l_c(w_N), \end{aligned}$$

if we choose $N > n^2 + 3n$.

It can also be checked that no Whitehead automorphism $P \neq 1$ of type I fixes the conjugacy class of $w_N$. $\square$

**Definition 11.4.** Let $A$ be a subgroup of an arbitrary group $G$. An automorphism $\alpha \in \mathrm{Aut}(G)$ (or $\widehat{\alpha} \in \mathrm{Out}(G)$) fixes $A$ up to *uniform conjugacy* if there is $g \in G$ such that $\alpha(x) = gxg^{-1}$ for all $x \in A$.

Fixing a rigid element in a free factor up to conjugacy is the same as fixing the free factor up to uniform conjugacy. More precisely, given $k < n$, we identify $F_k$ with the free factor of $F_n$ generated by $a_1, \dots, a_k$, and we have:

**Proposition 11.5.** *Let $k \leq n$ and $w \in F_k$ be rigid. If $\alpha \in \mathrm{Aut}(F_n)$ is an automorphism fixing the conjugacy class of $w$, then $\alpha$ fixes $F_k$ up to uniform conjugacy.*

The special case $m = n$ of this proposition is the following:

**Corollary 11.6.** *If $w$ is a rigid element in $F_n$, then* $\mathrm{Aut}(F_n, [w]) = \mathrm{Inn}(F_n)$ *and* $\mathrm{Out}(F_n, [w]) = 1$. $\square$

*Proof of Proposition 11.5.* We assume that the element $w$ is rigid with respect to the basis $\phi(a_1), \dots, \phi(a_k)$ for $\phi \in \mathrm{Aut}(F_k)$. Let $\tilde{\phi} := \phi * 1 \in \mathrm{Aut}(F_n)$. Then $\tilde{\phi}^{-1}(w)$ is rigid with respect to the standard basis. Moreover fixing $F_k$ up to uniform conjugacy is independent of a basis of $F_k$, so we may assume that $w$ is rigid with respect to the standard basis.

An automorphism fixing $w$ up to conjugacy can be written as a composition of edge labels along a loop in the McCool complex based at the vertex $[w]$. By means of

$(A;a) \circ T = T \circ (T^{-1}(A); T^{-1}(a))$, we can always move factors $T$ of type I to the left. Over there, they can be multiplied, so we will assume that the loop contains at most one factor of type I, and that this is at the very left. Therefore we have to investigate the Whitehead automorphisms of type II for $F_n$ which do not increase the cyclic length of $w$.

By definition of being rigid, Whitehead automorphisms with operative factor $a_i^\epsilon$ for some $i \leq k$ preserve the cyclic length of $w$ if and only if they either fix $a_1, \dots, a_k$ or conjugate all of these basis elements by $a_i^\epsilon$.

An operative factor $a_i^\epsilon$ with $i > k$ preserves the cyclic length if it either fixes $a_1, \dots, a_k$ or conjugates all of them. Otherwise there would be letters $a_i^{\pm 1}$ inserted in the image of $w$ which are not canceled out by cyclic reduction, and they would increase the cyclic length of $w$.

In any case, Whitehead automorphisms of type II preserve $w$ up to conjugacy, whenever they do not increase the cyclic length. This means that the loop based at $[w]$ in the McCool complex can be decomposed into loops of length 1 because it does not go across other vertices. The automorphism of type I is therefore the identity. Thus $\mathrm{Aut}(F_n, [w])$ is generated by type II automorphisms which either fix $a_1, \dots, a_k$ or conjugate all of them by the same element. $\square$

## 11.4 Simultaneous conjugacy classes of partitioned tuples

Recall that $\alpha \in \mathrm{Aut}(F_n, [w_1], \dots, [w_m])$ means that there are $x_1, \dots, x_m \in F_n$ such that $\alpha(w_i) = x_i w_i x_i^{-1}$ for $1 \leq i \leq m$. We are often interested in the situation that some of the $x_i$ are required to be equal. More precisely, for a tuple $W = (w_1, \dots, w_l)$ and $x \in F_n$, we write

$$xWx^{-1} = (xw_1x^{-1}, \dots, xw_lx^{-1}).$$

Let $W_1, \dots, W_p$ be tuples of elements in $F_n$, say $W_i = (w_{i,1}, \dots, w_{i,l_i})$. We are interested in

$$\begin{aligned} G &= \mathrm{Aut}(F_n, [W_1], \dots, [W_p]) \\ &:= \big\{\alpha \in \mathrm{Aut}(F_n) \big| \exists x_1, \dots, x_L \in F_n : \alpha(W_i) = x_i W_i x_i^{-1}\big\}. \end{aligned}$$

If $p = 1$, this is the same as the group of automorphisms $\alpha$ which are the composition of an inner automorphism and an automorphism in $\mathrm{Aut}(F_n, w_{1,1}, \dots, w_{1,l_1})$, the genuine stabiliser of elements in $F_n$. On the other hand, if each $W_i$ only contains one element, $W_i = (w_{i,1})$ with $l_i = 1$ for every $i$, then this group $G$ is equal to $\mathrm{Aut}(F_n, [w_{1,1}], \dots, [w_{p,1}])$. In this sense, $G$ is an interpolation between the stabilisers of conjugacy classes respectively elements of a free group in McCool's work.

The goal of this section is to show that these groups can "virtually" be reduced to stabilisers of single conjugacy classes. More precisely:

**Proposition 11.7.** *Let $W_1, \dots, W_p$ be tuples of elements in $F_n$. Then there are elements $y_1, \dots, y_N \in F_n$ such that the group $\mathrm{Aut}(F_n, [W_1], \dots, [W_p])$ is a subgroup of $\mathrm{Aut}(F_n, [y_1], \dots, [y_N])$ of finite index.*

Guirardel has announced a proof for this statement more generally in toral relatively hyperbolic groups instead of $F_n$, and his proof even shows that we need not pass to a finite index subgroup. As it is not written down yet, we give another proof here. We use the following well-known theorem.

**Theorem 11.8** (Marshall Hall's theorem)**.** *If $A$ is a finitely generated subgroup of $F_n$, then there is a subgroup $B$ of $F_n$ of finite index such that $A$ is a free factor of $B$.* □

The following lemma is an easy exercise left to the reader:

**Lemma 11.9.** *Let $G$ be a group and $H_1$, $H_2$ subgroups of $G$. Let $H_1'$ be a finite index subgroup of $H_1$ and $H_2'$ a finite index subgroup of $H_2$. Then $H_1' \cap H_2'$ is a finite index subgroup of $H_1 \cap H_2$.* □

*Proof of Proposition 11.7.* As

$$\mathrm{Aut}(F_n, [W_1], \dots, [W_p]) = \bigcap_{i=1}^{p} \mathrm{Aut}(F_n, [W_i]),$$

Lemma 11.9 reduces the statement to the special case $p = 1$. Let $W = (x_1, \dots, x_l)$ and $A$ the subgroup of $F_n$ generated by $x_1, \dots, x_l$. Theorem 11.8 provides a finite index subgroup $B \subset F_n$ such that $A$ is a free factor in $B$. According to Proposition 11.3, we now pick a rigid element $w \in A$ for the free factor $A$ in $B$.

Since the group of all $\alpha \in \mathrm{Aut}(F_n)$ with $\alpha(B) = B$ has finite index in $\mathrm{Aut}(F_n)$, the group $\mathrm{Aut}(F_n, [w])$ contains the finite index subgroup

$$\big\{\alpha \in \mathrm{Aut}(F_n) \big| \alpha(B) = B \text{ and } \alpha(w) = xwx^{-1} \text{ for some } x \in F_n\big\},$$

and this contains the finite index subgroup

$$\big\{\alpha \in \mathrm{Aut}(F_n) \big| \alpha(B) = B \text{ and } \alpha(w) = xwx^{-1} \text{ for some } x \in B\big\}.$$

By Proposition 11.5, the latter group equals

$$\big\{\alpha \in \mathrm{Aut}(F_n) \big| \alpha(B) = B \text{ and } \alpha|_A = \mathrm{ad}_x \text{ for some } x \in B\big\},$$

which is contained in

$$\mathrm{Aut}(F_n, [W]) = \big\{\alpha \in \mathrm{Aut}(F_n) \big| \alpha|_A = \mathrm{ad}_x \text{ for some } x \in F_n\big\}.$$

The last group, which is clearly contained in $\mathrm{Aut}(F_n, [w])$, has therefore finite index in $\mathrm{Aut}(F_n, [w])$. □

## 11.5 Natural free factors

Given a basis $a_1, \dots, a_n \in F_n$ and a tuple $W = (w_1, \dots, w_p)$ of elements in $F_n$, we denote by

$$G(W) = G(w_1, \dots, w_p)$$

the free factor of $F_n$ generated by all basis elements occurring in reduced words for $w_1, \dots, w_p$.

**Lemma 11.10.** *Let $W = (w_1, \dots, w_p)$ with $w_i \in F_n$ be minimal, and let $k$ be the number of basis elements involved in $w_1, \dots, w_m$. Then, for every $\alpha \in \mathrm{Aut}(F_n)$, the number of basis elements involved in $\alpha(w_1), \dots, \alpha(w_p)$ is at least $k$. If it is exactly $k$, then $\alpha(G(W)) = G(\alpha(W))$.*

*Proof.* If for some Whitehead automorphism $(A; a)$, the tuple $(A; a)(V)$ involves more basis elements than $V = (v_1, \dots, v_p)$, then the new basis element has to be $a^{\pm 1}$. But then the length of $(A; a)(V)$ is strictly greater than that of $V$. Given any (not necessarily minimal) tuple $V$, none of the steps in the length reducing sequence in Proposition 11.1 increases the number of basis elements involved. Therefore $\alpha(W)$ involves at least $k$ basis elements.

Assume now that $W$ is minimal. If the numbers of basis elements in $\alpha(W)$ and $W$ agree, then every Whitehead automorphism in Proposition 11.1 applied to $\alpha(W)$ has to preserve the number of involved basis elements. It now suffices to show that $\alpha(G(W)) = G(\alpha(W))$ in the case that $\alpha$ is a single Whitehead automorphism, and $W$ and $\alpha(W)$ involve the same number of basis elements. We shall look at several cases for $\alpha$ separately.

If $\alpha$ is of type I, then it only permutes the letters, and the claim is clear.

We now assume that $\alpha = (A; a)$ is a type II Whitehead automorphism. If the letters $a^{\pm 1}$ do not appear in $w_1, \dots, w_p$, then each $\alpha(w_j)$ is obtained by inserting letters $a^{\pm 1}$ into a reduced word for $w_j$. But, as $\alpha(W)$ is minimal, we actually have $\alpha(w_j) = w_j$, and none of the basis elements and inverses involved in the $w_j$ belongs to $A \setminus \{a\}$. Then $\alpha(G(W)) = G(\alpha(W))$ is evident.

We are left to verify the claim in the case that $\alpha = (A; a)$, and $a$ is involved in at least one of the words $w_j$. Here every letter occurring in some $\alpha(w_j)$ also occurs in some $w_{j'}$. We therefore obtain

$$G(\alpha(W)) \subset G(W) = \alpha(G(W)).$$

Applying the same argument to $\alpha^{-1}$, we obtain the opposite inclusion. □

**Proposition 11.11.** *Let $W = (w_1, \dots, w_p)$ be a tuple of elements in a finitely generated free group $F$. Then:*

*(i) There is a unique free factor $B(W)$ of minimal rank containing $w_1, \dots, w_p$.*

*(ii) For $\alpha \in \mathrm{Aut}(F_n)$ we have $B(\alpha(W)) = \alpha(B(W))$.*

We refer to $B(W)$ as the *natural free factor* of the tuple $W$.

*Proof of Proposition 11.11.* We may w.l.o.g. assume that $F = F_n$. Note that (ii) follows from (i): Both sides of (ii) contain $\alpha(w_1), \dots, \alpha(w_p)$, and both these free factors have minimal rank subject to this condition. By the uniqueness part of (i) we conclude (ii).

To show (i), we first have to construct a candidate for a free factor $B(W)$. If $W$ is minimal, we define $B(W) = G(W)$. It clearly contains all $w_i$, and it has the desired properties by Lemma 11.10.

If $W$ is not necessarily minimal, then we choose $\beta \in \mathrm{Aut}(F_n)$ such that $\beta(W)$ is minimal. We then define $B(W) = \beta^{-1}(G(\beta(W)))$.

Whenever both $W$ and $\alpha(W)$ are minimal, then we have $G(\alpha(W)) = \alpha(G(W))$ by Lemma 11.10. Thus, the definition of $B(W)$ does not depend on the choice of $\beta$. This finishes the verification of the proposition. $\square$

## 11.6 Relative stabilisation maps

Suppose we are given $w \in F_k$. Given $n > k$, we can identify $F_k$ with a free factor of $F_n$, so $w \in F_n$. Later we will be interested in the relationship between $\mathrm{Aut}(F_k, [w])$ and $\mathrm{Aut}(F_n, [w])$, i.e. the stabiliser of the conjugacy class of $w$, once viewed as element in $F_k$ and once as element in $F_n$. Using natural free factors as constructed in the last section, we define a map

$$\pi : \mathrm{Aut}(F_n, [w]) \to \mathrm{Out}(B(w), [w])$$

as follows: Given $\alpha \in \mathrm{Aut}(F_n, [w])$, there is $v \in F_n$ such that $\alpha(w) = vwv^{-1}$. We define $\pi(\alpha) \in \mathrm{Out}(B(w))$ to be the outer automorphism class of $x \mapsto v^{-1}\alpha(x)v$.

If we use $v'$ with $\alpha(w) = v'wv'^{-1}$ instead, then the two definitions of $\pi(\alpha)$ differ by the inner automorphism $\mathrm{ad}_{v^{-1}v'}$. Note that $v^{-1}v'$ commutes with $w$, so it lies in the free factor $B(w)$. This shows that $\mathrm{ad}_{v^{-1}v'}$ is an inner automorphism of $B(w)$, and $\pi(\alpha) \in \mathrm{Out}(B(w))$ does not depend on the choice of $v$.

It is clear that $\pi(\alpha)$ fixes the conjugacy class of $w$ and that $\pi$ is a group homomorphism.

We now show that $\pi$ is surjective. This is the special case $m = 0$ of the following more general statement:

**Proposition 11.12.** *Let $\mathcal{C} = ([u_1], \dots, [u_m])$, where $u_1, \dots, u_m \in F_n$ lie in a complementary free factor of $w \in F_n$. Then $\pi$ restricts to a surjection (called by the same name) fitting into the short exact sequence*

$$1 \to \mathrm{Aut}(F_n, \mathcal{C}, [B(w)]) \to \mathrm{Aut}(F_n, \mathcal{C}, [w]) \xrightarrow{\pi} \mathrm{Out}(B(w), [w]) \to 1.$$

Here $\mathrm{Aut}(F_n, \mathcal{C}, [B(w)])$ is the group of all $f \in \mathrm{Aut}(F_n)$ fixing the conjugacy classes of $u_1, \dots, u_m$ and fixing $B(w)$ up to uniform conjugacy.

*Proof.* It is immediate from the definition that the kernel of $\pi$ is as asserted. It remains to show that $\pi$ is surjective. To see this, choose any $\beta \in \mathrm{Aut}(B(w), [w])$. Define $\alpha \in \mathrm{Aut}(F_n)$ by taking $\beta$ on $B(w)$ and the identity on a complementary free factor containing $u_1, \dots, u_m$. Then $\pi(\alpha)$ is the outer automorphism class $\widehat{\beta}$. $\square$

## 11.7 Conjugacy classes of labeled graphs

Fix a group $G$. Let $\Lambda$ be a graph together with a labeling function $\lambda : E(\Lambda) \to G$ such that $\lambda(\overline{e}) = \lambda(e)^{-1}$ for every edge $e$. By multiplying the edge labels along a given edge path, the labeling function $\lambda$ uniquely extends to functions

$$\lambda : \pi_1(\Lambda, u, w) \to G$$

for all vertices $u$ and $w$, which we call by the same name. So these give a morphism of groupoids. Assume $V_0 \subset V(\Lambda)$ such that every connected component of $\Lambda$ contains exactly one vertex in $V_0$.

**Proposition 11.13.** *For an automorphism $\alpha \in \mathrm{Aut}(G)$ we have:*

*(i) If we have $\delta_w \in G$ for $w \in V(\Gamma)$ such that $\alpha(\lambda(e)) = \delta_{\iota(e)}\lambda(e)\delta^{-1}_{\tau(e)}$ for every edge $e$ of $\Gamma$, then $\alpha$ coincides with $\mathrm{ad}_{\delta_w}$ on $\lambda(\pi_1(\Lambda, w))$ for every $w \in V(\Lambda)$.*

*(ii) Given $\kappa_w$, $w \in V_0$, such that $\alpha$ agrees with $\mathrm{ad}_{\kappa_w}$ on $\lambda(\pi_1(\Lambda(w)))$ for each $w \in V_0$, there are unique $\delta_w \in G$ for all $w \in V(\Lambda)$ such that*

$$\alpha(\lambda(p)) = \delta_u \lambda(p) \delta_w^{-1} \tag{54}$$

*for $p \in \pi_1(\Lambda, u, w)$ and $\delta_w = \kappa_w$ for $w \in V_0$.*

*Proof.* Since the edges of $\Lambda$ generate the fundamental groupoid $\pi_1(\Lambda)$, we conclude $\alpha(\lambda(p)) = \delta_u\lambda(p)\delta_w^{-1}$ in (i) whenever $p \in \pi_1(\Lambda, u, w)$. Taking $u = w$, we get (i).

In (ii), we have to define $\delta_w = \kappa_w$ if $w \in V_0$. Fix now a maximal forest $F$ in $\Lambda$. If there is a vertex $w'$ of $\Lambda$ such that $\delta_{w'}$ has not yet been defined, then we find an edge $e \in E(\Lambda)$ such that $\delta_{\iota(e)}$ is defined, but $\delta_{\tau(e)}$ is not. (54) then forces us to define $\delta_{\tau(e)} = \alpha(\lambda(e))^{-1}\delta_{\iota(e)}\lambda(e)$. This finishes the definition of all $\delta_w$.

We now know that (54) is satisfied whenever $p$ is a single edge of the forest $F$ or $p \in \pi_1(\Lambda, w)$ for some $w \in V_0$. Since every path in the fundamental groupoid of $\Lambda$ is a composition of such $p$, we conclude (54) on the entire fundamental groupoid. $\square$

We summarise Proposition 11.13 as follows. Given $\alpha \in \mathrm{Aut}(G)$, it is equivalent whether we prescribe $\delta_w$, $w \in V(\Lambda)$ such that $\alpha(\lambda(e)) = \delta_{\iota(e)}\lambda(e)\delta^{-1}_{\tau(e)}$ or $\kappa_w$, $w \in V_0$, such that $\alpha$ conjugates $\lambda(\pi_1(\Lambda(w)))$ uniformly by $\kappa_w$. We will use this extensively in Chapter 13 when we decompose centralisers of normalised higher Dehn twists into "smaller" pieces.

## 11.8 Primitive elements

Recall that an element $w$ in a finitely generated free group $F$ is called *primitive* if there is a basis $a_1, \ldots, a_n \in F$ such that $w = a_1$. The following lemma will be needed in Chapter 15.

**Lemma 11.14.** *Let $A$ be a free factor of the finitely generated free group $F$ and $w \in A$. Then $w$ is primitive in $A$ if and only if it is primitive in $F$.*

*Proof.* We may assume w.l.o.g. that $F = F_n$ with basis $a_1, \ldots, a_n$ and $A = F_k = \langle a_1, \ldots, a_k \rangle$ for some $k \leq n$. If $w$ is primitive in $F_k$, then it is also primitive in $F_n$ because every basis of $F_k$ extends to a basis of $F_n$ by adding $a_{k+1}, \ldots, a_n$.

Assume conversely that $w$ is primitive in $F_n$. In particular, $w$ is not a proper power in $F_k$. Replacing $w$ with $\beta(w)$ for some $\beta \in \mathrm{Aut}(F_k)$ if necessary, we may assume that $w$ is minimal, so the length $l(\alpha(w)) \geq l(w)$ for every $\alpha \in \mathrm{Aut}(F_k)$.

If $w$ is not primitive in $F_k$, then $k \geq 2$. As $w$ is primitive in $F_n$, Proposition 11.1 shows that there is a Whitehead automorphism $(A; a_i^\epsilon)$ with $1 \leq i \leq n$ and $\epsilon \in \{\pm 1\}$ such that $l((A; a_i^\epsilon)(w)) < l(w)$. This is possible only if the operative factor $a_i$ is involved in $w$, so $1 \leq i \leq k$. But then the Whitehead automorphism $(A \cap \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}, a_i^\epsilon) \in \mathrm{Aut}(F_k)$ also reduces the length of $w$ strictly. This is a contradiction to the assumption that $w$ is minimal in $F_k$ under the action of $\mathrm{Aut}(F_k)$. $\square$

# 12 Automorphisms fixing $L$-conjugacy classes

## 12.1 $L$-conjugacy classes of cyclic elements

Denote by $\mathcal{G}$ an ordinary graph of groups with a fixed automorphism $L \in \mathrm{Aut}^0(\mathcal{G})$.

**Proposition 12.1.** *Let $\eta = \delta_L(\overline{e_1})t_1g_1 \dots t_{k-1}g_{k-1}t_kg_k \in \pi_1(\mathcal{G}, u)$ be an $L$-twistedly reduced expression, and $k \geq 1$. Suppose $\delta \in \pi_1(\mathcal{G}, w, u)$ such that the element $L_*(\delta)\eta\delta^{-1}$ is $L$-twistedly reduced as well. Then are unique $r \in \mathbb{Z}$ and $x \in G_{\tau(e_{k-r})}$ such that*

$$\delta = \begin{cases} xt_{k-r+1}g_{k-r+1}\dots t_kg_k, & \text{if } r \geq 1, \\ x, & \text{if } r = 0, \\ xg_{k-r}^{-1}t_{k-r}^{-1}g_{k-r-1}^{-1}t_{k-r-1}^{-1}\dots g_{k+1}^{-1}t_{k+1}^{-1}, & \text{if } r \leq -1, \end{cases}$$

*and $L_*(\delta)\eta\delta^{-1} = L_*(x)\delta_L(\overline{e_{1-r}})t_{1-r}g_{1-r}\dots t_{k-r}g_{k-r}x^{-1}$.*

Using the convention in Remark 4.14, we can write $\delta = xt_{k-r+1}g_{k-r+1}\dots t_kg_k$ independently of the sign of $r$.

*Proof of Proposition 12.1.* Suppose $\delta = g_0't_1'g_1't_2'\dots g_{l-1}'t_l'y$ is a reduced expression satisfying the assumptions of the proposition. We argue by induction on $l$. For $l = 0$ we have $y = x$, and everything is clear.

For $l \geq 1$ we have

$$\begin{aligned} L_*(\delta)\eta\delta^{-1} = L_*(g_0')\delta_L(\overline{e_1'})t_1'\delta_L(e_1')^{-1}\dots \\ \dots t_{l-1}'\delta_L(e_{l-1}')^{-1}L_*(g_{l-1}')\delta_L(\overline{e_l'})t_l'\delta_L(e_l')^{-1}L_*(y)\delta_L(\overline{e_1})t_1g_1t_2\dots \\ \dots t_{k-1}g_{k-1}t_kg_ky^{-1}t_l'^{-1}g_{l-1}'^{-1}t_{l-1}'^{-1}\dots t_1'^{-1}g_0'^{-1}. \end{aligned} \tag{55}$$

As the right hand side begins with $L_*(g_0')\delta_L(\overline{e_1'})t_1'\dots$ and terminates by $\dots t_1'^{-1}g_0'^{-1}$, it is not an $L$-twistedly reduced expression. But it represents an $L$-twistedly reduced element, so it cannot be reduced. There are the following two cases:

<u>Case 1:</u> $e_l' = \overline{e_1}$ and $\delta_L(e_l')^{-1}L_*(y)\delta_L(\overline{e_1}) = f_{\overline{e_1}}(h)$ for some $h \in G_{e_1}$,

<u>Case 2:</u> $e_l' = e_k$ and $g_ky^{-1} = f_{e_k}(h)$ for some $h \in G_{e_k}$.

Note that Case 1 is equivalent to $e_l' = \overline{e_1}$ and $y = f_{\overline{e_1}}(L_{e_1}^{-1}(h))$. In that case we write $\tilde{y} := g_{l-1}'f_{e_1}(L_{e_1}^{-1}(h))g_{k+1}$, which implies

$$\begin{aligned} L_*(\tilde{y}) &= L_*(g_{l-1}')(\delta_L(e_1)f_{e_1}(h)\delta_L(e_1)^{-1})L_*(g_{k+1}) \\ &= L_*(g_{l-1}')\delta_L(e_1)f_{e_1}(h)g_1\delta_L(\overline{e_2})^{-1}, \end{aligned}$$

where the last equality uses (7) on page 40. To simplify (55), we observe

$$\begin{aligned} &L_*(g_{l-1}')\delta_L(\overline{e_l'})t_l'\delta_L(e_l')^{-1}L_*(y)\delta_L(\overline{e_1})t_1g_1t_2 \\ &= L_*(g_{l-1}')\delta_L(e_1)f_{e_1}(h)g_1t_2 = L_*(\tilde{y})\delta_L(\overline{e_2})t_2 \end{aligned}$$

and

$$t_kg_ky^{-1}t_l'^{-1}g_{l-1}'^{-1}t_{l-1}'^{-1} = t_kg_kt_{k+1}f_{e_1}(L_{e_1}^{-1}(h))^{-1}g_{l-1}'^{-1}t_{l-1}'^{-1} = t_kg_kt_{k+1}g_{k+1}\tilde{y}^{-1}t_{l-1}'^{-1}.$$

Then we read off in (55) that $L_*(\delta)\eta\delta^{-1} = L_*(\tilde{\delta})\tilde{\eta}\tilde{\delta}^{-1}$, where $\tilde{\delta} := g_0' t_1' g_1' t_2' \dots g_{l-2}' t_{l-1}' \tilde{y}$ and $\tilde{\eta} := \delta_L(\overline{e_2})t_2 g_2 \dots t_{k+1} g_{k+1}$. By induction on $l$, we have an expression of the form $\tilde{\delta} = x t_{k-r+1} g_{k-r+1} \dots t_{k+1} g_{k+1}$ for some $r \in \mathbb{Z}$. Since we have

$$\delta = \tilde{\delta}\tilde{y}^{-1} g_{l-1}' t_l' y = \tilde{\delta} g_{k+1}^{-1} f_{e_1}(L_{e_1}^{-1}(h))^{-1} t_1^{-1} f_{\overline{e_1}}(L_{e_1}^{-1}(h)) = \tilde{\delta} g_{k+1}^{-1} t_{k+1}^{-1},$$

we obtain the desired formula for $\delta$.

In Case 2 we again want to rewrite (55) to define $\tilde{\delta}$ and $\tilde{\eta}$ (which are different from those in Case 1). We have $e_l' = e_k$ and $y = f_{e_k}(h)^{-1} g_k$. We define $\tilde{y} = g_{l-1}' f_{\overline{e_k}}(h)^{-1}$, and we see

$$\begin{aligned}
&L_*(g_{l-1}')\delta_L(\overline{e_l'})t_l'\delta_L(e_l')^{-1}L_*(y)\delta_L(\overline{e_1})t_1 g_1\\
&= L_*(\tilde{y})L_*(f_{\overline{e_k}}(h))\delta_L(\overline{e_k})t_k\delta_L(e_k)^{-1}L_*(f_{e_k}(h))^{-1}L_*(g_k)\delta_L(\overline{e_1})t_1 g_1\\
&= L_*(\tilde{y})\delta_L(\overline{e_k})f_{\overline{e_k}}(L_{e_k}(h))t_k f_{e_k}(L_{e_k}(h))^{-1}\delta_L(e_k)^{-1}L_*(g_k)\delta_L(\overline{e_1})t_1 g_1\\
&= L_*(\tilde{y})\delta_L(\overline{e_k})t_k\delta_L(e_k)^{-1}L_*(g_k)\delta_L(\overline{e_1})t_1 g_1\\
&= L_*(\tilde{y})\delta_L(\overline{e_0})t_0 g_0 t_1 g_1,
\end{aligned}$$

where the last equality uses the definition of $g_0$ by (7) on page 40.

Moreover, we have

$$t_{k-1}g_{k-1}t_k g_k y^{-1} t_l'^{-1} g_{l-1}'^{-1} t_{l-1}'^{-1} = t_{k-1}g_{k-1}f_{\overline{e_k}}(h) g_{l-1}'^{-1} t_{l-1}'^{-1} = t_{k-1}g_{k-1}\tilde{y}^{-1}t_{l-1}'^{-1}.$$

This allows to simplify (55) to $L_*(\delta)\eta\delta^{-1} = L_*(\tilde{\delta})\tilde{\eta}\tilde{\delta}^{-1}$ with $\tilde{\delta} = g_0' t_1' g_1' \dots g_{l-2}' t_{l-1}' \tilde{y}$ and $\tilde{\eta} = \delta_L(\overline{e_0})t_0 g_0 \dots t_{k-1} g_{k-1}$. We see by induction that $\tilde{\delta} = x t_{k-r+1} g_{k-r+1} \dots t_{k-1} g_{k-1}$ for some $r \in \mathbb{Z}$ and some $x$. Since $\delta = \tilde{\delta}\tilde{y}^{-1} g_{l-1}' t_l' y = \tilde{\delta} f_{\overline{e_k}}(h) t_k f_{e_k}(h)^{-1} g_k = \tilde{\delta} t_k g_k$, this yields the asserted formula for $\delta$. □

**Proposition 12.2.** *Let $\eta = \delta_L(\overline{e_1})t_1 g_1 \dots t_k g_k \in \pi_1(\mathcal{G}, u)$ be an $L$-twistedly reduced expression, $\delta \in \pi_1(\mathcal{G}, u)$, and let $H \in \mathrm{Aut}^0(\mathcal{G})$ commute with $L$. Then:*

*(i) $H_*(\eta) = L_*(\delta)\eta\delta^{-1}$ if and only if there are $r \in \mathbb{Z}$ and $h_j \in G_{e_j}$ such that $e_{j+r} = e_j$, $L_{e_j}^{-1}(h_j) = h_{j+k}$ for all $j \in \mathbb{Z}$, and*

$$\begin{aligned}
H_{\tau(e_j)}(g_j) &= \delta_H(e_j) f_{e_j}(h_j) g_{j-r} f_{\overline{e_{j+1}}}(h_{j+1})^{-1}\delta_H(\overline{e_{j+1}})^{-1}, \\
\delta &= \delta_H(\overline{e_1}) f_{\overline{e_1}}(h_{k+1}) t_{k-r+1} g_{k-r+1} \dots t_k g_k.
\end{aligned} \tag{56}$$

*(ii) Equation (56) for $1 \le j \le k$ implies (56) for all $j \in \mathbb{Z}$.*

*Proof.* Suppose $H_*(\eta) = L_*(\delta)\eta\delta^{-1}$. By Lemma 4.11, $H_*(\eta)$ is again $L$-twistedly reduced, so Proposition 12.1 provides $r \in \mathbb{Z}$ and $x \in G_{\tau(e_{k-r})}$ such that

$$\delta = x t_{k-r+1} g_{k-r+1} \dots t_k g_k \tag{57}$$

$$L_*(\delta)\eta\delta^{-1} = L_*(x)\delta_L(\overline{e_{1-r}})t_{1-r} g_{1-r} \dots t_{k-r} g_{k-r} x^{-1}. \tag{58}$$

Note that

$$H_*(\delta_L(e))\delta_H(e) = \delta_{HL}(e) = \delta_{LH}(e) = L_*(\delta_H(e))\delta_L(e) \tag{59}$$

for every edge $e$. We define

$$g'_j = \delta_H(e_j)^{-1} H_*(g_j)\delta_H(\overline{e_{j+1}}) \tag{60}$$

and obtain

$$\begin{aligned} H_*(\eta) &= H_*(\delta_L(\overline{e_1}))\delta_H(\overline{e_1})t_1 g'_1 \dots t_k g'_k \delta_H(\overline{e_{k+1}})^{-1} \\ &= L_*(\delta_H(\overline{e_1}))\delta_L(\overline{e_1})t_1 g'_1 \dots t_k g'_k \delta_H(\overline{e_1})^{-1}. \end{aligned} \tag{61}$$

Since $H_*(\eta) = L_*(\delta)\eta\delta^{-1}$, we can compare the right hand sides of (58) and (61) by means of Proposition 2.6: We see that $e_j = e_{j-r}$, and there are $h_1, \dots, h_k$ such that $h_j \in G_{e_j}$ and

$$L_*(\delta_H(\overline{e_1}))\delta_L(\overline{e_1}) = L_*(x)\delta_L(\overline{e_1}) f_{\overline{e_1}}(h_1)^{-1}, \tag{62}$$

$$g'_j = f_{e_j}(h_j) g_{j-r} f_{\overline{e_{j+1}}}(h_{j+1})^{-1} \text{ for } 1 \le j \le k-1, \tag{63}$$

$$g'_k \delta_H(\overline{e_1})^{-1} = f_{e_k}(h_k) g_{k-r} x^{-1}. \tag{64}$$

This leads to

$$\begin{aligned} g'_k &\overset{(64)}{=} f_{e_k}(h_k) g_{k-r} x^{-1} \delta_H(\overline{e_1}) \\ &\overset{(62)}{=} f_{e_k}(h_k) g_{k-r} L_*^{-1}(L_*(\delta_H(\overline{e_1}))\delta_L(\overline{e_1}) f_{\overline{e_1}}(h_1)\delta_L(\overline{e_1})^{-1})^{-1} \delta_H(\overline{e_1}) \\ &= f_{e_k}(h_k) g_{k-r} L_*^{-1}(\delta_L(\overline{e_1}) f_{\overline{e_1}}(h_1)^{-1} \delta_L(\overline{e_1})^{-1}) \\ &= f_{e_k}(h_k) g_{k-r} f_{\overline{e_1}}(L_{e_1}^{-1}(h_1))^{-1}. \end{aligned}$$

This is (63) for $j = k$, where we use $h_{j+k} = L_{e_j}^{-1}(h_j)$ to define $h_j$ for $j > k$ or $j \le 0$.

Next we check that the $g'_j$ satisfy the same recursion formula as the $g_j$ in (7) on page 40:

$$\begin{aligned} &L_*^{-1}(\delta_L(e_j) g'_j \delta_L(\overline{e_{j+1}})^{-1}) \\ &\overset{(60)}{=} L_*^{-1}\big(\delta_L(e_j)\delta_H(e_j)^{-1} H_*(g_j)\delta_H(\overline{e_{j+1}})\delta_L(\overline{e_{j+1}})^{-1}\big) \\ &\overset{(59)}{=} L_*^{-1}\big(L_*(\delta_H(e_j))^{-1} H_*(\delta_L(e_j)) H_*(g_j) H_*(\delta_L(\overline{e_{j+1}}))^{-1} L_*(\delta_H(\overline{e_{j+1}}))\big) \\ &= \delta_H(e_j)^{-1} H_* L_*^{-1}(\delta_L(e_j) g_j \delta_L(\overline{e_{j+1}})^{-1})\delta_H(\overline{e_{j+1}}) \\ &= \delta_H(e_j)^{-1} H_*(g_{j+k})\delta_H(\overline{e_{j+1}}) = g'_{j+k}. \end{aligned}$$

We equivalently rewrite (63) as

$$L_*^{-1}\big(\delta_L(e_j) g'_j \delta_L(\overline{e_{j+1}})^{-1}\big) = L_*^{-1}\big(\delta_L(e_j) f_{e_j}(h_j) g_{j-r} f_{\overline{e_{j+1}}}(h_{j+1})^{-1} \delta_L(\overline{e_{j+1}})^{-1}\big),$$

which is in turn equivalent to

$$\begin{aligned} g'_{j+k} &= f_{e_j}(L_{e_j}^{-1}(h_j))L_*^{-1}(\delta_L(e_{j-r})g_{j-r}\delta_L(\overline{e_{j-r+1}})^{-1})f_{\overline{e_{j+1}}}(L_{e_{j+1}}^{-1}(h_{j+1}))^{-1} \\ &= f_{e_{j+k}}(h_{j+k})g_{j+k-r}f_{\overline{e_{j+k+1}}}(h_{j+k+1})^{-1}. \end{aligned}$$

This way we obtain (63) for every $j \in \mathbb{Z}$. We conclude

$$\begin{aligned} H_*(g_j) &\overset{(60)}{=} \delta_H(e_j)g'_j\delta_H(\overline{e_{j+1}})^{-1} \\ &\overset{(63)}{=} \delta_H(e_j)f_{e_j}(h_j)g_{j-r}f_{\overline{e_{j+1}}}(h_{j+1})^{-1}\delta_H(\overline{e_{j+1}})^{-1}, \end{aligned}$$

which is the desired formula for $H_*(g_j)$ for every $j \in \mathbb{Z}$. From (62) we get

$$x = L_*^{-1}(L_*(\delta_H(\overline{e_1}))\delta_L(\overline{e_1})f_{\overline{e_1}}(h_1)\delta_L(\overline{e_1})^{-1}) = \delta_H(\overline{e_1})f_{\overline{e_1}}(L_{e_1}^{-1}(h_1)).$$

Together with (57), this leads to the asserted formula for $\delta$.

We now show the "if" part of (i). Suppose $\delta$ and $H_*(g_j)$ for all $j \in \mathbb{Z}$ are as in the statement of the proposition. Since $t_j = t_{j-r}$, we have

$$H_*(t_jg_j) = \delta_H(\overline{e_j})f_{\overline{e_j}}(h_j)t_{j-r}g_{j-r}f_{\overline{e_{j+1}}}(h_{j+1})^{-1}\delta_H(\overline{e_{j+1}})^{-1},$$

and hence

$$\begin{aligned} H_*(\eta) = H_*(\delta_L(\overline{e_1}))&\delta_H(\overline{e_1})f_{\overline{e_1}}(h_1)t_{1-r}g_{1-r}\dots \\ &\dots t_{k-r}g_{k-r}f_{\overline{e_{k+1}}}(h_{k+1})^{-1}\delta_H(\overline{e_{k+1}})^{-1} \\ = \delta_{HL}&(\overline{e_1})f_{\overline{e_1}}(h_1)t_{1-r}g_{1-r}\dots t_{k-r}g_{k-r}f_{\overline{e_1}}(h_{k+1})^{-1}\delta_H(\overline{e_1})^{-1}. \end{aligned}$$

By Lemma 5.10(iii), we have

$$L_*(t_jg_j) = \delta_L(\overline{e_j})t_{j-k}g_{j-k}\delta_L(\overline{e_{j+1}})^{-1},$$

so we see

$$\begin{aligned} L_*(\delta) &= L_*(\delta_H(\overline{e_1}))\delta_L(\overline{e_1})f_{\overline{e_1}}(L_{e_1}(h_{k+1}))t_{1-r}g_{1-r}\dots t_0g_0\delta_L(\overline{e_{k+1}})^{-1} \\ &= \delta_{LH}(\overline{e_1})f_{\overline{e_1}}(h_1)t_{1-r}g_{1-r}\dots t_0g_0\delta_L(\overline{e_1})^{-1}. \end{aligned}$$

This shows $H_*(\eta) = L_*(\delta)\eta\delta^{-1}$, as claimed.

We are left to prove (ii). Equation (56) for a fixed value of $j$ is equivalent to

$$\begin{aligned} &L_*^{-1}H_*((\delta_L(e_j)g_j\delta_L(\overline{e_{j+1}}))) \\ &= L_*^{-1}\big(H_*(\delta_L(e_j))\delta_H(e_j)f_{e_j}(h_j)g_{j-r}f_{\overline{e_{j+1}}}(h_{j+1})^{-1}\delta_H(\overline{e_{j+1}})^{-1}H_*(\delta_L(\overline{e_{j+1}}))^{-1}\big). \end{aligned}$$

Using (59), we rewrite this as

$$\begin{aligned} H_*(g_{j+k}) &= \delta_H(e_j)L_*^{-1}\big(\delta_L(e_j)f_{e_j}(h_j)g_{j-r}f_{\overline{e_{j+1}}}(h_{j+1})^{-1}\big)\delta_H(\overline{e_{j+1}})^{-1} \\ &= \delta_H(e_j)f_{e_j}(L_{e_j}^{-1}(h_j))L_*^{-1}(\delta_L(e_j)g_{j-r}\delta_L(\overline{e_{j+1}})^{-1})\cdot \\ &\quad \cdot f_{\overline{e_{j+1}}}(L_{e_{j+1}}^{-1}(h_{j+1}))^{-1}\delta_H(\overline{e_{j+1}})^{-1} \\ &= \delta_H(e_{j+k})f_{e_{j+k}}(h_{j+k})g_{j+k-r}f_{\overline{e_{j+k+1}}}(h_{j+k+1})^{-1}\delta_H(\overline{e_{j+k+1}})^{-1}. \end{aligned}$$

This is (56) for $j + k$. By induction we conclude (ii). □

## 12.2 $L$-conjugacy classes of local elements

**Proposition 12.3.** *Assume $D$ is a pre-efficient Dehn twist. Fix $\eta, \eta' \in G_u$ for some $u$. Let $\delta \in \pi_1(\mathcal{G}, u)$ and $H \in \mathrm{Aut}^0(\mathcal{G})$. If*

*(i) $H_*(\eta) = D_*(\delta)\eta\delta^{-1}$, and $\eta = 1$ or $\eta$ is not $D$-conjugate to 1, or*

*(ii) $H_*(\eta) = \delta\eta\delta^{-1}$ and $\eta \neq 1$, or*

*(iii) $H_*(\eta) = D_*(\delta)\eta\delta^{-1}$, $H_*(\eta') = D_*(\delta)\eta'\delta^{-1}$, and $\eta \neq \eta'$,*

*then $\delta \in G_u$.*

In the special case $\eta = 1$, statement (i) reduces to the study of the fixed subgroup of the Dehn twist automorphism $D_{*u}$. This has been done in Lemma 6.5 of [13], but we reprove this here in a more general setting.

*Proof of Proposition 12.3.* We start with proving (i) and (ii) simultaneously. Assume $H_*(\eta) = D_*^q(\delta)\eta\delta^{-1}$, where $q = 1$ for (i) and $q = 0$ for (ii). Clearly $pl(D_*(\delta)\eta\delta^{-1}) = pl(H_*(\eta)) = 0$. Let $\delta = g_0 t_1 g_1 \dots t_{k-1} g_{k-1} t_k y$ be a reduced expression. We have to show $k = pl(\delta) = 0$.

We have

$$\begin{aligned} &H_*(\eta) g_0 t_1 g_1 \dots t_{k-1} g_{k-1} t_k y \eta^{-1} \\ &= g_0 t_1 f_{e_1}(a_{e_1})^{qn_{e_1}} g_1 \dots t_{k-1} f_{e_{k-1}}(a_{e_{k-1}})^{qn_{e_{k-1}}} g_{k-1} t_k f_{e_k}(a_{e_k})^{qn_{e_k}} y, \end{aligned}$$

an equality of two reduced expressions. If $k \geq 1$, then Proposition 2.6 provides $m_1, \dots, m_k \in \mathbb{Z}$ such that

$$g_0 = H_*(\eta) g_0 f_{\overline{e_1}}(a_{e_1})^{m_1} \tag{65}$$

$$f_{e_j}(a_{e_j})^{qn_{e_j}+m_j} g_j = g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{m_{j+1}} \text{ for } 1 \leq j \leq k-1, \tag{66}$$

$$f_{e_k}(a_{e_k})^{qn_{e_k}+m_k} y = y\eta^{-1}. \tag{67}$$

Writing $m_{k+1} := m_k + qn_{e_k}$, we combine this to

$$\begin{aligned} g_0 f_{\overline{e_1}}(a_{e_1})^{-m_1} g_0^{-1} &\overset{(65)}{=} H_u(\eta) \overset{(67)}{=} H_u(y^{-1} f_{e_k}(a_{e_k})^{-m_{k+1}} y) \\ &= H_u(y)^{-1} \delta_H(e_k) f_{e_k}(H_{e_k}(a_{e_k}))^{-m_{k+1}} \delta_H(e_k)^{-1} H_u(y). \end{aligned} \tag{68}$$

Lemma 6.5 applied to (66) shows

$$m_{j+1} = m_j + qn_{e_j} \tag{69}$$

for $1 \leq j \leq k-1$ and by definition also for $j = k$. Using that no $f_e(a_e)$ is a proper power and $H_{e_k}(a_{e_k}) = a_{e_k}^{\pm 1}$, we compare the exponents of both sides in (68) to get $m_1 = \pm m_{k+1}$.

If none of $m_1, \dots, m_k$ is zero, then (66) shows that $e_j$ and $\overline{e_{j+1}}$ are negatively bonded for $1 \leq j \leq k-1$, and (68) shows that $e_k$ and $\overline{e_1}$ are bonded. This is a contradiction to Lemma 6.6. Hence $m_i = 0$ for some $i$, $1 \leq i \leq k$.

For $1 \leq j \leq k-1$ we compute

$$\begin{aligned} D_*^q(t_j g_j) &\overset{(66)}{=} t_j f_{e_j}(a_{e_j})^{qn_{e_j}} (f_{e_j}(a_{e_j})^{-qn_{e_j}-m_j} g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{m_{j+1}}) \\ &= f_{\overline{e_j}}(a_{e_j})^{-m_j} t_j g_j f_{\overline{e_{j+1}}}(a_{e_{j+1}})^{m_{j+1}}. \end{aligned}$$

Using $m_i = 0$, this leads to

$$\begin{aligned} D_*^q(t_i g_i \dots t_k y) &= t_i g_i \dots t_{k-1} g_{k-1} f_{\overline{e_k}}(a_{e_k})^{m_k} t_k f_{e_k}(a_{e_k})^{qn_{e_k}} y \\ &\overset{(67)}{=} t_i g_i \dots t_{k-1} g_{k-1} t_k y \eta^{-1}, \end{aligned}$$

so $\eta = D_*^q(t_i g_i \dots t_k y)^{-1}(t_i g_i \dots t_k y)$ is $D^q$-conjugate to 1. This finishes the proof of (ii), and in (i) we only need $\eta \neq 1$. As all $n_e > 0$ and $q = 1$, formula (69) shows

$$m_1 < m_2 < \ldots < m_{k+1} = \pm m_1,$$

so $m_1 \neq 0$. Then $\eta \neq 1$ follows from (65), and we know (i).

Assertion (iii) follows from (ii) because $H_*(\eta^{-1}\eta') = \delta\eta^{-1}\eta'\delta^{-1}$. □

**Lemma 12.4.** *Let* $\mathbb{G}$ *have degree* $d \geq 2$. *If* $\eta \in \pi_1(\mathbb{G}^{(d-1)}, u) \smallsetminus \{1\}$, $\delta \in \pi_1(\mathbb{G}, u', u)$, *and* $\delta\eta\delta^{-1} \in \pi_1(\mathbb{G}^{(d-1)}, u')$, *then* $\delta \in \pi_1(\mathbb{G}^{(d-1)}, u', u)$.

*Proof.* If a reduced word representing $\delta$ goes across edges of degree $d$, then a reduced word for $\delta\eta\delta^{-1}$ does as well. □

**Lemma 12.5.** *Suppose we are given* $\mathbb{G}$ *of degree* $d \geq 2$ *and a prenormalised higher Dehn twist* $D \in \mathrm{Aut}^0(\mathbb{G})$. *Assume that* $\eta \in \pi_1(\mathbb{G}^{(d-1)}, u)$ *and* $\delta \in \pi_1(\mathbb{G}, u)$ *satisfy* $D_*(\delta)\eta\delta^{-1} \in \pi_1(\mathbb{G}^{(d-1)}, u)$. *Suppose that either* $\eta = 1$ *or* $\eta$ *is not* $D$*-conjugate to 1. Then* $\delta \in \pi_1(\mathbb{G}^{(d-1)}, u)$.

*Proof.* If $\eta = 1 \in G_u$, then the requirements are $D_{*u}(\delta) = \delta$. In particular, the basis length of $\delta$ does not grow under iteration of $D_{*u}$. By Proposition 7.22(i), we conclude $\delta \in G_u \subset \pi_1(\mathbb{G}^{(d-1)}, u)$.

We now assume that $\eta$ is not $D$-conjugate to 1. Let $\delta = \theta_0 t_1 \theta_1 \dots t_k \theta_k$ be an expression reduced in the truncated sense (cf. Section 3.2). Then

$$D_*(\delta)\eta\delta^{-1} = D_*(\theta_0)\delta_D(\overline{e_1})t_1 \dots t_k \delta_D(e_k)^{-1} D_*(\theta_k)\eta\theta_k^{-1}t_k^{-1} \dots \theta_1^{-1}t_1^{-1}\theta_0^{-1}. \tag{70}$$

We have to show $k = 0$. If $k \geq 1$, then the right hand side of (70) is not reduced in the truncated sense, so $D_*(\theta_k)\eta\theta_k^{-1} = \delta_D(e_k)$. Since $\eta$ is not $D$-conjugate to 1, we have $\delta_D(e_k) \neq 1$. Definition 7.20(2) shows $\delta_D(\overline{e_k}) = 1$. But then $D_*(t_k\theta_k)\eta\theta_k^{-1}t_k^{-1} = 1$ leads to a contradiction. □

## 12.3 Simultaneous $D$-conjugacy classes

In this section we collect some lemmas needed in the next chapter.

**Lemma 12.6.** *Fix an automorphism $L \in \mathrm{Aut}^0(\mathbb{G})$ of a higher graph of groups $\mathbb{G}$ of degree $d \geq 2$. Assume that $\eta \neq \eta' \in \pi_1(\mathbb{G}^{(d-1)}, u)$ and $\delta \in \pi_1(\mathbb{G}, u', u)$. If both $L_*(\delta)\eta\delta^{-1}$ and $L_*(\delta)\eta'\delta^{-1}$ lie in $\pi_1(\mathbb{G}^{(d-1)}, u')$, then $\delta \in \pi_1(\mathbb{G}^{(d-1)}, u', u)$.*

*Proof.* Since $\delta\eta^{-1}\eta'\delta^{-1} \in \pi_1(\mathbb{G}^{(d-1)}, u')$ and $\eta^{-1}\eta' \in \pi_1(\mathbb{G}^{(d-1)}, u) \smallsetminus \{1\}$, this follows immediately from Lemma 12.4. □

**Lemma 12.7.** *Let $A \subset \pi_1(\mathcal{G}, v)$ be a finitely generated subgroup of the fundamental group of any graph of groups $\mathcal{G}$. Assume that, for every $\zeta \in A$ there is a vertex $u(\zeta)$ and $\epsilon(\zeta) \in \pi_1(\mathcal{G}, v, u(\zeta))$ such that $\zeta \in \epsilon(\zeta)G_{u(\zeta)}\epsilon(\zeta)^{-1}$. Then there is $\epsilon \in \pi_1(\mathcal{G}, v, u)$ for some vertex $u$ such that $A \subset \epsilon G_u \epsilon^{-1}$.*

*Proof.* Suppose $A$ is generated by $\mu_1, \ldots, \mu_N$. If $N = 0$ or $N = 1$, then we may take $\epsilon = 1$ or $\epsilon = \epsilon(\mu_1)$ respectively, and we are done. We now proceed by induction on $N$.

If $N \geq 2$, the induction hypothesis provides vertices $u$ and $u'$ together with $\epsilon \in \pi_1(\mathcal{G}, v, u)$ and $\epsilon' \in \pi_1(\mathcal{G}, v, u')$ such that $\mu'_1, \ldots, \mu'_{N-1} \in G_u$ and $\mu'_N \in G_{u'}$, where $\mu'_i := \epsilon^{-1}\mu_i\epsilon$ for $1 \leq i \leq N-1$ and $\mu'_N := \epsilon'^{-1}\mu_N\epsilon'$. Assume $\epsilon$ and $\epsilon'$ are chosen to satisfy this condition such that the path length of $\epsilon^{-1}\epsilon'$ is minimal. Let $\epsilon^{-1}\epsilon' = g_0t_1g_1\ldots t_kg_k$ be a reduced expression. We have to show $k = 0$.

By requirement, the element $\mu_i\mu_N$ is conjugate to an element in a vertex group for every $i$, $1 \leq i \leq N-1$. Its conjugacy class equals $[\mu'_i\epsilon^{-1}\epsilon'\mu'_N\epsilon'^{-1}\epsilon]$, which is represented by the word

$$(\mu'_ig_0, t_1, g_1, \ldots, t_k, g_k\mu'_Ng_k^{-1}, t_k^{-1}, \ldots, g_1^{-1}, t_1^{-1}, g_0^{-1}).$$

This cannot be cyclically reduced when $k \geq 1$. Thus either $g_k\mu'_Ng_k^{-1} \in f_{e_k}(G_{e_k})$ or $g_0^{-1}\mu'_ig_0 \in f_{\overline{e_1}}(G_{e_1})$ for every $i$, $1 \leq i \leq N-1$.

If $g_k\mu'_Ng_k^{-1} = f_{e_k}(h)$ for some $h \in G_{e_k}$, then $t_kg_k\mu'_Ng_k^{-1}t_k^{-1} \in G_{\iota(e_k)}$. If we had chosen $\tilde{\epsilon}' = \epsilon' g_k^{-1}t_k^{-1}$ instead of $\epsilon'$, the path length of $\epsilon^{-1}\tilde{\epsilon}' = g_0t_1g_1\ldots t_{k-1}g_{k-1}$ would be shorter than that of $\epsilon^{-1}\epsilon'$. This is a contradiction to the choice of $\epsilon$ and $\epsilon'$.

If $g_0^{-1}\mu'_ig_0 = f_{\overline{e_1}}(h_i)$ for some $h_1, \ldots, h_{N-1} \in G_{e_1}$, then

$$t_1^{-1}g_0^{-1}\epsilon^{-1}\mu_i\epsilon g_0t_1 = t_1^{-1}g_0^{-1}\mu'_ig_0t_1 \in G_{\tau(e_1)}.$$

If we had taken $\tilde{\epsilon} = \epsilon g_0t_1$ instead of $\epsilon$, then $\tilde{\epsilon}^{-1}\epsilon' = g_1t_2g_2\ldots t_kg_k$ would again have shorter path length than $\epsilon^{-1}\epsilon'$, which is a contradiction to the minimality condition in the choice of $\epsilon$ and $\epsilon'$. □

Recall truncatable replacements introduced in Definition 7.14.

**Proposition 12.8.** *Fix a truncatable replacement $D \in \mathrm{Aut}^0(\mathbb{G})$ of a prenormalised higher Dehn twist of degree $d \geq 1$ with free $\pi_1(\mathbb{G}, v)$. Let $\eta, \mu_1, \ldots, \mu_N \in \pi_1(\mathbb{G}, v)$. Then there are $\epsilon \in \pi_1(\mathbb{G}, v, u)$ for some $u$ and $\eta', \mu'_1, \ldots, \mu'_{N'} \in \pi_1(\mathbb{G}, u)$ such that*

*(i) The coset $D_*(\epsilon)\eta'\langle\mu'_1, \ldots, \mu'_{N'}\rangle\epsilon^{-1} = \eta\langle\mu_1, \ldots, \mu_N\rangle$,*

*(ii)* $\eta', \eta'\mu'_1, \ldots, \eta'\mu'_{N'}$ *are $D$-twistedly reduced in the truncated sense,*

*(iii)* *either* $\eta', \mu'_1, \ldots, \mu'_{N'} \in \pi_1(\mathbb{G}^{(d-1)}, u)$ *or, in reduced words for* $\eta', \eta'\mu'_1, \ldots, \eta'\mu'_{N'}$*, there is a common initial segment containing at least one edge of degree $d$.*

*Proof.* Suppose first that there is a 1-cyclic element $\zeta \in \langle \mu_1, \ldots, \mu_N \rangle$, i.e. a cyclically reduced expression for the conjugacy class $[\zeta]$ goes across edges of degree $d$. Replacing $\eta$ with $D_*(\epsilon')\eta\epsilon'^{-1}$ and every $\mu_i$ with $\epsilon'\mu_i\epsilon'^{-1}$, we replace $\zeta$ with $\epsilon'\zeta\epsilon'^{-1}$. This way we may assume that $\zeta$ is cyclically reduced itself and goes across edges of degree $d$.

For $m \geq 0$ we define $\eta''(m) = D_*(\zeta)^m\eta\zeta^m$ and $\mu''_i(m) = \zeta^{-m}\mu_i\zeta^m$. The cancellation in $D_*(\zeta)^m \cdot (\eta\zeta\eta^{-1})^m$ eliminates an unbounded number of edges of degree $d$ only if $D_*(\zeta)$ and $\eta\zeta\eta^{-1}$ lie in a common cyclic subgroup of $\pi_1(\mathbb{G}, v)$. If $k \geq 1$ is maximal such that $\zeta$ is a $k$-th power, then both $D_*(\zeta)$ and $\eta\zeta\eta^{-1}$ are $k$-th powers but no higher ones. Thus $D_*(\zeta) = \eta\zeta^{\pm 1}\eta^{-1}$ and $D^2_*(\zeta) = \eta D_*(\eta)\zeta D_*(\eta)^{-1}\eta^{-1}$, so the cyclic basis length of the conjugacy class $[\zeta]$ stays bounded under iteration of $D_*$. By Proposition 7.22(iii), $\zeta$ is conjugate to an element in some vertex group. But this is a contradiction to the assumption that $\zeta$ goes across edges of degree $d$. Therefore the cancellation of the three factors in $D_*(\zeta)^m \cdot \eta \cdot \zeta^m$ only affects a bounded number (independent of $m$) of edges of degree $d$. If we choose $m$ sufficiently large, $\eta' = \mu''(m)$ and $\mu'_i = \mu''_i(m)$ satisfy the conditions asserted in the proposition. This finishes the proof in the case that there is at least one 1-cyclic $\zeta$.

If every element in the subgroup $\langle \mu_1, \ldots, \mu_N \rangle$ is conjugate to an element in the stratum $\pi_1(\mathbb{G}^{(d-1)})$, then Lemma 12.7 shows that there is $\epsilon \in \pi_1(\mathbb{G}, v, u)$ such that

$$\mu_1, \ldots, \mu_N \in \epsilon\pi_1(\mathbb{G}^{(d-1)}, u)\epsilon^{-1}. \tag{71}$$

Suppose $\epsilon$ is chosen such that $\eta'' = \eta''(\epsilon) := D_*(\epsilon)^{-1}\eta\epsilon$ has the minimal number of edges of degree $d$ among all possible choices of $\epsilon$ satisfying (71). We further write $\mu''_i = \epsilon^{-1}\mu_i\epsilon \in \pi_1(\mathbb{G}^{(d-1)}, u)$.

If a reduced word for $\eta''$ does not involve edges of degree $d$, then we may take $\eta' = \eta''$, $\mu'_i = \mu''_i$, and we are done.

Otherwise, let $\iota : F_N \to \pi_1(\mathbb{G}^{(d-1)}, u)$ be the homomorphism sending the $i$-th basis element $a_i \in F_N$ to $\mu''_i$. Let

$$\eta'' = \theta_0 t_1 \theta_1 \ldots t_k \theta_k$$

with $k \geq 1$ be an expression reduced in the truncated sense. The element $\eta''\iota(x)$ is $D$-twistedly reduced if and only if $e_k \neq \overline{e_1}$ or $\delta_D(\overline{e_1})^{-1}D_*(\theta_k\iota(x))\theta_0 \notin f_{\overline{e_1}}(G_{e_1})$. Hence the set

$$R := \{x \in F_N | \eta''\iota(x) \text{ is not } D\text{-twistedly reduced.}\}$$

is either empty or a coset of a subgroup of $F_N$.

If $R \subsetneq F_N$, then we find $x_0 \in F_N$ such that $x_0, x_0a_1, \ldots, x_0a_N \notin R$. In this case we define $\eta' = \eta''\iota(x_0)$ and $\mu'_i = \mu''_i$. The elements $\eta', \eta'\mu'_1, \ldots, \eta'\mu'_N$ satisfy the desired properties by construction.

If $R = F_N$, then

$$\delta_D(\overline{e_1})^{-1}D_*(\theta_k\iota(x))\theta_0 \in f_{\overline{e_1}}(G_{e_1})$$

for all $x \in F_N$. This implies that $\theta_0^{-1} D_*(\iota(x))\theta_0 \in f_{\overline{e_1}}(G_{e_1})$ for all $x \in F_n$. We pick $h_0, h_1, \ldots, h_N \in G_{e_1}$ such that

$$\begin{aligned} \delta_D(\overline{e_1})^{-1} D_*(\theta_k)\theta_0 &= f_{\overline{e_1}}(h_0) \qquad (72) \\ \theta_0^{-1} D_*(\mu_i'')\theta_0 = \theta_0^{-1} D_*(\iota(a_i))\theta_0 &= f_{\overline{e_1}}(h_i) \text{ for } 1 \leq i \leq N. \end{aligned}$$

We compute

$$\begin{aligned} \mu_i''' &:= t_1^{-1} D_*^{-1}(\delta_D(\overline{e_1})\theta_0^{-1})\mu_i'' D_*^{-1}(\theta_0 \delta_D(\overline{e_1})^{-1})t_1 \\ &= t_1^{-1} D_*^{-1}(\delta_D(\overline{e_1}) f_{\overline{e_1}}(h_i)\delta_D(\overline{e_1})^{-1})t_1 \\ &= t_1^{-1} f_{\overline{e_1}}(D_{e_1}^{-1}(h_i))t_1 = f_{e_1}(h_i) \end{aligned}$$

and

$$\begin{aligned} \eta''' &:= D_*(t_1)^{-1}\delta_D(\overline{e_1})\theta_0^{-1}\eta'' D_*^{-1}(\theta_0\delta_D(\overline{e_1})^{-1})t_1 \\ &\overset{(72)}{=} \delta_D(e_1)t_1^{-1}\theta_0^{-1}\eta'' D_*^{-1}\big(D_*(\theta_k)^{-1}\delta_D(\overline{e_1})f_{\overline{e_1}}(h_0)\delta_D(\overline{e_1})^{-1}\big)t_1 \\ &= \delta_D(e_1)\theta_1 t_2\theta_2 \ldots t_{k-1}\theta_{k-1}t_k f_{\overline{e_1}}(D_{e_1}^{-1}(h_0))t_1 \\ &= \delta_D(e_1)\theta_1 t_2\theta_2 \ldots t_{k-1}\theta_{k-1} f_{e_1}(h_0). \end{aligned}$$

If we had chosen $\epsilon D_*^{-1}(\theta_0\delta_D(\overline{e_1})^{-1})t_1$ instead of $\epsilon$, we would have got fewer edges of degree $d$ in $\eta'''$ than in $\eta''$, which is a contradiction to the minimality assumption for the choice of $\epsilon$ with (71). $\square$

# 13 Description of centralisers

## 13.1 The groups $C_I^0$ and $S_{\mathcal{I}}^0$

Given a higher graph of groups $\mathbb{G}$ and $L \in \mathrm{Aut}^0(\mathbb{G})$, we define

$$\begin{aligned}
\mathrm{Aut}^{(E)}(\mathbb{G}) &= \{H \in \mathrm{Aut}^0(\mathbb{G}) | H_e = 1 \text{ for every edge } e\},\\
C^0(L) &= \{H \in \mathrm{Aut}^{(E)}(\mathbb{G}) | HL = LH\},\\
C^0(L_{*v}) &= \{H_{*v} | H \in C^0(L)\},\\
C^0(\widehat{L}) &= \{\widehat{H} | H \in C^0(L)\}.
\end{aligned}$$

**Lemma 13.1.**

*(i) If $D$ is a normalised higher Dehn twist on $\mathbb{G}$, then $C^0(\widehat{D})$ has finite index in $C(\widehat{D})$.*

*(ii) If $D$ is a pointedly normalised higher Dehn twist on $\mathbb{G}$, then $C^0(D_{*v})$ has finite index in $C(D_{*v})$.*

*Proof.* We write

$$\tilde{C} = \big\{H \in \mathrm{Aut}(\mathbb{G}) \big| \widehat{H}\widehat{D} = \widehat{D}\widehat{H}, \delta_D(H(e)) = 1 \text{ and } \delta_H(e) \in G'_{\tau(H(e))} \text{ when } \delta_D(e) = 1\big\}$$

and $U : \tilde{C} \to C(\widehat{D})$ for the homomorphism $H \mapsto \widehat{H}$. By Theorem 10.16 the map $U$ is surjective. As all $\mathrm{Aut}(G_e)$ are finite, $\mathrm{Aut}^{(E)}(\mathbb{G})$ has finite index in $\mathrm{Aut}^0(\mathbb{G})$ and hence in $\mathrm{Aut}(\mathbb{G})$. The subgroup $\tilde{C}^0 = \tilde{C} \cap \mathrm{Aut}^{(E)}(\mathbb{G})$ now has finite index in $\tilde{C}$, so $U(\tilde{C}^0)$ has finite index in $C(\widehat{D})$. But, by Proposition 10.18, we have

$$U(\tilde{C}^0) \subset \big\{\widehat{H} \big| H \in \mathrm{Aut}^{(E)}(\mathbb{G}), HD = DH\big\} = C^0(\widehat{D}).$$

This proves (i).

The proof of (ii) uses Theorem 10.17 instead of Theorem 10.16. The details are left to the reader. □

Lemma 13.1 also applies to truncatable replacements of (pointedly) normalised higher Dehn twists.

Given closed elements $\eta_i \in \pi_1(\mathbb{G}, v_i)$, we define

$$\begin{aligned}
C_I^0(L, (\eta_i)_{i \in I}) = \big\{(H, (\delta_i)) \in \mathrm{Aut}_I^0(\mathbb{G}) \big| & (H, (\delta_i)) \text{ commutes with } (L, (\eta_i))\\
& \text{and } H \in \mathrm{Aut}^{(E)}(\mathbb{G})\big\}.
\end{aligned}$$

We sometimes abbreviate this group by $C_I^0$.

Recall the groups $KA_I(\mathbb{G})$ and $KO_I(\mathbb{G})$ defined in Section 9.5. We have the quotient groups

$$\begin{aligned}
C_I^0(L_{*v}, (\eta_i)_i) &= C_I^0(L, (\eta_i))/(KA_I(\mathbb{G}) \cap C_I^0),\\
C_I^0(\widehat{L}, (\eta_i)_i) &= C_I^0(L, (\eta_i))/(KO_I(\mathbb{G}) \cap C_I^0).
\end{aligned}$$

We warn the reader about the following abuse of notation: When we replace $L$ with $L'$ such that $\widehat{L} = \widehat{L}'$, then we have in general $C_I^0(\widehat{L}, (\eta_i)) \neq C_I^0(\widehat{L}', (\eta_i))$, but rather

$$C_I^0(\widehat{L}, (\eta_i)) = C_I^0(\widehat{L}', (\eta_i')),$$

where $\eta_i' \in \pi_1(\mathbb{G}, v_i)$ is such that $(L, (\eta_i))^{-1}(L', (\eta_i')) \in KO_I(\mathbb{G})$. A similar transformation formula holds true for $C_I^0(L_{*v}, (\eta_i))$.

Suppose $\mathcal{I}$ is a partition of $I$, i.e. $\mathcal{I} = \{I_1, \ldots, I_M\}$ such that

$$I = I_1 \sqcup \ldots \sqcup I_M.$$

Assume $v_i = v_j$ whenever $i, j \in I_m$ for some $m$, $1 \leq m \leq M$. We define

$$S_{\mathcal{I}}^0(L, (\eta_i)) \subset C_I^0(L, (\eta_i))$$

to be the subgroup which consists of all tuples $(H, (\delta_i)_{i\in I})$ such that $\delta_i = \delta_j$ whenever $i, j \in I_m$ for some $m$. In other words, $\delta_i$ is constant when $i$ ranges over a set of the partition $\mathcal{I}$. When $\mathcal{I} = \{\{i\} | i \in I\}$ is the discrete partition, then $S_{\mathcal{I}}^0 = C_I^0$.

We define further

$$\begin{aligned} S_{\mathcal{I}}^0(L_{*v}, (\eta_i)) &= S_{\mathcal{I}}^0(L, (\eta_i))/(S_{\mathcal{I}}^0 \cap KA_I), \\ S_{\mathcal{I}}^0(\widehat{L}, (\eta_i)) &= S_{\mathcal{I}}^0(L, (\eta_i))/(S_{\mathcal{I}}^0 \cap KO_I). \end{aligned}$$

*Remark* 13.2. When $(H, (\delta_i)_{i\in I}) \in S_{\mathcal{I}}^0(L, (\eta_i))$, then we sometimes write $\delta_m$ for the value of all $\delta_i$ with $i \in I_m$. If we pick some $i_m \in I_m$, then we have $H_*(\zeta) = D_*(\delta_m)\zeta\delta_m^{-1}$ for all $\zeta \in \eta_{i_m}\langle \eta_{i_m}^{-1}\eta_i | i \in I_m\rangle$. If we have $(\eta_i')_{i\in I'}$ for a second index set $I'$ with partition $\mathcal{I}' = \{I_1', \ldots, I_M'\}$ such that the cosets

$$\eta_{i_m}\langle \eta_{i_m}^{-1}\eta_i | i \in I_m\rangle = \eta'_{i'_m}\langle \eta'^{-1}_{i'_m}\eta'_i | i \in I'_m\rangle$$

for every $m$, then we obtain an isomorphism

$$S_{\mathcal{I}}^0(L, (\eta_i)_{i\in I}) \cong S_{\mathcal{I}'}^0(L, (\eta_i')_{i\in I'})$$

sending $(H, (\delta_i)_{i\in I})$ to $(H, (\delta_i')_{i\in I'})$ where $\delta_i' = \delta_{i_m}$ when $i \in I_m'$.

## 13.2 Generating sets for $KO_I(\mathbb{G}) \cap C_I^0$ and $KA_I(\mathbb{G}) \cap C_I^0$

Recall the automorphisms $M(w, \gamma)$ and $K(e, h)$ introduced in Section 9.1 and their lifts $M(w, \gamma)_I$ and $K(e, h)_I$ defined in Section 9.5.

**Proposition 13.3.** *Let $D \in \mathrm{Aut}^0(\mathbb{G})$ be a prenormalised higher Dehn twist or truncatable replacement, and fix $(\eta_i)_{i\in I}$. Then:*

*(i) $KO_I(\mathbb{G}) \cap C_I^0(D, (\eta_i))$ is generated by automorphisms of the form $M(w, \gamma)_I$ and $K(e, h)_I$.*

(ii) *$KA_I(\mathbb{G}) \cap C_I^0(D, (\eta_i))$ is generated by automorphisms $M(w,\gamma)_I$ with $w \neq v$ and $K(e,h)_I$.*

*Proof.* It is easily verified that all asserted generators commute with $(D, (\eta_i))$, and they lie indeed in the respective group.

We first prove (i) in the special case $I = \varnothing$, so let $H \in KO(\mathbb{G}) \cap C^0(D)$. Assume that the degree of $\mathbb{G}$ is $d = 1$. If there is a vertex group of rank at least two, then Lemma 9.2 proves the assertion. If all vertex groups are infinite cyclic, then $\mathbb{G}$ is a cylinder by Proposition 7.28, and the assertion is proved by Lemma 9.7.

We now assume still $I = \varnothing$, but $d \geq 2$. As $\widehat{H} = 1$, Lemma 9.3 shows $\widehat{H|_{\mathbb{F}}} = 1$ whenever $\mathbb{F}$ is a connected component of $\mathbb{G}^{(d-1)}$. By induction, $H^{(d-1)}$ can be expressed in terms of the asserted generators on $\mathbb{G}^{(d-1)}$. Since all these generators extend from $\mathrm{Aut}^0(\mathbb{G}^{(d-1)})$ to all of $\mathrm{Aut}^0(\mathbb{G})$, we may assume $H^{(d-1)} = 1$. Proposition 9.5 shows that $H$ is a composition of automorphisms $Z(\mathbb{F}, \gamma_\bullet)$ and $O(e, \delta)$. By Corollary 9.8, the automorphisms $Z(\mathbb{F}, \gamma_\bullet)$ can be expressed in terms of the desired generators. Composing $H$ with the inverses of these $Z(\mathbb{F}, \gamma_\bullet)$, we may assume $\delta_H(e) = 1$ whenever $\deg(e) = d$ and $G_{\iota(e)}$ is not 1-valent trivial.

Let $e$ be an arbitrary edge of degree $d$ such that $G_{\iota(e)}$ is 1-valent trivial in $\mathbb{G}$. We have to show $\delta_H(e) = 1$. Since $HD = DH$ and $H^{(d-1)} = 1$, we have

$$\delta_D(e) = H_*(\delta_D(e)) = D_*(\delta_H(e))\delta_D(e)\delta_H(e)^{-1}.$$

As $\delta_D(\bar{e}) \in \pi_1(\mathbb{G}^{(d-1)}, \iota(e)) = 1$, parts (2) and (6) of Definition 7.20 imply that $\delta_D(e)$ grows dominantly of degree $d-1$. Lemma 7.23 now proves $\delta_H(e) = 1$. This finishes the verification of (i) in the special case $I = \varnothing$.

We now prove (i) for general $I$. The exact sequence (42) on page 87 restricts to an exact sequence

$$1 \to Z_I(\mathbb{G}) \cap C_I^0 \to KO_I(\mathbb{G}) \cap C_I^0 \to KO(\mathbb{G}) \cap C^0(D).$$

Since $KO(\mathbb{G}) \cap C^0(D)$ is generated by elements of the form $M(w,\gamma)$ and $K(e,h)$, the group $KO_I \cap C_I^0$ is generated by symbols $M(w,\gamma)_I$, $K(e,h)_I$, and a generating set for $Z_I(\mathbb{G}) \cap C_I^0$. By Lemma 9.7(i), the latter can be expressed in terms of $M(w,\gamma)_I$ and $K(e,h)_I$. This finishes the proof of (i).

Similar arguments apply to $KA_I$ in (ii). □

## 13.3 The rotation homomorphisms $\mathrm{ev}_r$

Let $\mathcal{G}$ be an ordinary graph of groups and $L \in \mathrm{Aut}^0(\mathcal{G})$. In the following, $I$ has only one element, and we write $C_1^0$ for $C_I^0$. Given an $L$-twistedly reduced expression $\eta = \delta_L(\overline{e_1})t_1 g_1 \dots t_k g_k$ and $(H, \delta) \in C_1^0(L, \eta)$, Proposition 12.2(i) provides $h_j \in G_{e_j}$ and $r \in \mathbb{Z}$ such that

$$\delta = \delta_H(\overline{e_1}) f_{\overline{e_1}}(h_{k+1}) t_{k-r+1} g_{k-r+1} \dots t_k g_k. \tag{73}$$

The integer $r$ is uniquely determined by $\delta$. For, if we had

$$\delta = x t_{k-r+1} g_{k-r+1} \dots t_k g_k = x' t_{k-r'+1} g_{k-r'+1} \dots t_k g_k$$

for some $r' < r$ and $x, x' \in G_{\tau(e_{k-r})}$, then

$$xt_{k-r+1}g_{k-r+1}\dots t_{k-r'}g_{k-r'}x'^{-1} = 1,$$

but the path length of the left hand side is $r - r' > 0$. This contradiction proves uniqueness of $r$.

There is a map

$$\mathrm{ev}_r : C_1^0(L, \eta) \to \mathbb{Z}$$

given by $(H, \delta) \mapsto r$, where $r$ is the unique integer satisfying (73).

**Lemma 13.4.** $\mathrm{ev}_r$ *is a group homomorphism.*

*Proof.* Let $(H, \delta), (H', \delta') \in C_1^0(L, \eta)$. If $H_*(g_j)$ is as stated in (56) in Proposition 12.2, we have

$$H_*(t_jg_j) = \delta_H(\overline{e_j})f_{\overline{e_j}}(h_j)t_{j-r}g_{j-r}f_{\overline{e_{j+1}}}(h_{j+1})^{-1}\delta_H(\overline{e_{j+1}})^{-1}.$$

Using

$$\delta' = \delta_{H'}(\overline{e_1})f_{\overline{e_1}}(h'_{k+1})t_{k-r'+1}g_{k-r'+1}\dots t_kg_k,$$

we obtain

$$\begin{aligned}H_*(\delta') &= H_*(\delta_{H'}(\overline{e_1}))\delta_H(\overline{e_1})f_{\overline{e_1}}(H_{e_1}(h'_{k+1}))\delta_H(\overline{e_1})^{-1}H_*(t_{k-r'+1}g_{k-r'+1}\dots t_kg_k)\\ &= \delta_{HH'}(\overline{e_1})f_{\overline{e_1}}(H_{e_1}(h'_{k+1})h_{k-r'+1})t_{k-r-r'+1}g_{k-r-r'+1}\dots\\ &\quad\dots t_{k-r}g_{k-r}f_{\overline{e_1}}(h_{k+1})^{-1}\delta_H(\overline{e_1})^{-1}.\end{aligned}$$

When we multiply this on the right by

$$\delta = \delta_H(\overline{e_1})f_{\overline{e_1}}(h_{k+1})t_{k-r+1}g_{k-r+1}\dots t_kg_k,$$

this leads to

$$H_*(\delta')\delta = \delta_{HH'}(\overline{e_1})f_{\overline{e_1}}(H_{e_1}(h'_{k+1})h_{k-r'+1})t_{k-r-r'+1}g_{k-r-r'+1}\dots t_kg_k.$$

We read off that $\mathrm{ev}_r((H,\delta)(H',\delta')) = r + r' = \mathrm{ev}_r(H,\delta) + \mathrm{ev}_r(H',\delta')$. □

Given $(\eta_i)_{i\in I}$, let $I^c$ denote the subset of $I$ of all $i$ such that $\eta_i$ is $L$-cyclic, i.e. not $L$-conjugate to an element in some vertex group. By $I^l = I \setminus I^c$ we denote the set of all $i$ such that $\eta_i$ is $L$-local. We assume that every $\eta_i$ is $L$-twistedly reduced.

For every $i \in I^c$, we have now defined a rotation homomorphism. All these fit together to a rotation homomorphism

$$\mathrm{ev}_r : C_I^0(L, (\eta_i)_{i\in I}) \to \mathbb{Z}^{I^c}.$$

If $I$ is endowed with a partition $\mathcal{I}$, then this restricts to a homomorphism

$$\mathrm{ev}_r : S_{\mathcal{I}}^0(L, (\eta_i)) \to \mathbb{Z}^{I^c}. \tag{74}$$

We denote the kernels of these homomorphisms by $C_I^1(L, (\eta_i))$ and $S_{\mathcal{I}}^1(L, (\eta_i))$ respectively.

## 13.4 Hypothesis (S)

Let $L \in \mathrm{Aut}^0(\mathbb{G})$ be an automorphism of a higher graph of groups of degree $d$.

**Definition 13.5.** A tuple $(L, \mathcal{I}, (\eta_i))$ satisfies *hypothesis (S)* if the following statements hold true:

- All $\eta_i \in \pi_1(\mathbb{G}, v_i)$ are $L$-twistedly reduced in the truncated sense. They are of the form $\delta_L(\overline{e_{i,1}})t_{i,1}g_{i,1} \dots t_{i,k_i}g_{i,k_i}$ whenever they go across edges of degree $d$.
- If $i, i' \in I_m \in \mathcal{I}$, then either both $\eta_i$ and $\eta_{i'}$ lie in $\pi_1(\mathbb{G}^{(d-1)}, v_i)$, or their underlying paths initiate with the same edge $e_{i,1} = e_{i',1}$ of degree $d$.
- Whenever $\eta_i$ is $L$-conjugate to 1 and $i \in I_m$, then $\eta_i = 1 \in G_{v_i}$ or there is $i' \in I_m$ such that $\eta_{i'} \neq \eta_i$.

**Lemma 13.6.** *Suppose $D \in \mathrm{Aut}^0(\mathbb{G})$ is a truncatable replacement of a prenormalised higher Dehn twist. For every $(D, \mathcal{I}, (\eta_i))$, there is $(D, \mathcal{I}', (\eta_i'))$ satisfying hypothesis (S) such that*

$$\begin{aligned} S^0_{\mathcal{I}}(D, (\eta_i)) &\cong S^0_{\mathcal{I}'}(D, (\eta_i')), \\ S^0_{\mathcal{I}}(D_{*v}, (\eta_i)) &\cong S^0_{\mathcal{I}'}(D_{*v}, (\eta_i')), \\ S^0_{\mathcal{I}}(\widehat{D}, (\eta_i)) &\cong S^0_{\mathcal{I}'}(\widehat{D}, (\eta_i')). \end{aligned}$$

*Proof.* Given $\epsilon_i \in \pi_1(\mathbb{G}, v_i', v_i)$ for every $i \in I$, there is an isomorphism

$$C^0_I(D, (\eta_i)_{i\in I}) \to C^0_I(D, (D_*(\epsilon_i)\eta_i\epsilon_i^{-1})_{i\in I})$$

defined by $(H, (\delta_i)_{i\in I}) \mapsto (H, (H_*(\epsilon_i)\delta_i\epsilon_i^{-1})_{i\in I})$. If $\epsilon_i = \epsilon_{i'}$ whenever $i, i' \in I_m \in \mathcal{I}$ are in the same partition set, then this isomorphism restricts to an isomorphism

$$S^0_{\mathcal{I}}(D, (\eta_i)_{i\in I}) \cong S^0_{\mathcal{I}}(D, (D_*(\epsilon_i)\eta_i\epsilon_i^{-1})_{i\in I}). \tag{75}$$

Fix a partition set $I_m$ and $i_m \in I_m$. When we replace $(\eta_i)_{i\in I_m}$ with $(\eta_i')_{i\in I_m'}$ such that the cosets $\eta_{i_m}\langle\eta_{i_m}^{-1}\eta_i | i \in I_m\rangle$ and $\eta'_{i'_m}\langle\eta'^{-1}_{i'_m}\eta'_i | i \in I'_m\rangle$ coincide, we only change $S^0_{\mathcal{I}}$ by isomorphism (cf. Remark 13.2). Proposition 12.8 and an appropriate choice of $\epsilon_i$ for $i \in I_m$ in (75) allow us to achieve that all $\eta_i'$, $i \in I_m'$, are $D$-twistedly reduced in the truncated sense and they initiate all with the same segment containing an edge of degree $d$ if they are not entirely in the stratum $\pi_1(\mathbb{G}^{(d-1)})$. Thus we can arrange the first and second bullet points of Definition 13.5.

If $\eta_i = \eta_{i'}$ for all $i, i' \in I_m$, and this element is $D$-conjugate to 1, then the isomorphism (75) allows us to achieve $\eta_i = 1$ for all $i \in I_m$.

This leads to the first asserted isomorphism. Using the definitions in (39) and (40) on page 86, the reader can check that both the isomorphism in Remark 13.2 and that in (75) map elements in $KA_I$ to elements in $KA_I$ and elements in $KO_I$ to elements in $KO_I$. Therefore the first isomorphism in the assertion induces the other two. □

## 13.5 Explicit description of $S^1_{\mathcal{I}}$

In this section we summarise explicitly the data determining an element in $S^1_{\mathcal{I}}(D,(\eta_i))$. We first need a lemma following directly from the definitions.

**Lemma 13.7.** *Let $\mathcal{G}$ be a graph of groups with infinite cyclic edge groups $G_e = \langle a_e \rangle$. A tuple $H = ((H_w)_{w\in V(\Gamma)}, (H_e)_{e\in E(\Gamma)}, (\delta_H(e))_{e\in E(\Gamma)})$ with $H_e = 1$ for all e defines an element in $\mathrm{Aut}^0(\mathcal{G})$ if and only if $H_{\tau(e)}(f_e(a_e)) = \delta_H(e) f_e(a_e)\delta_H(e)^{-1}$ for all edges $e$.* □

Assume now that $D$ is a pre-efficient Dehn twist on $\mathcal{G}$. Let $(D,\mathcal{I},(\eta_i))$ satisfy hypothesis (S). Fix cyclically reduced expressions $\eta_i = \delta_D(\overline{e_{i,1}})t_{i,1}g_{i,1}\dots t_{i,k_i}g_{i,k_i}$ for $i\in I^c$.

**Lemma 13.8.** *The group $S^1_{\mathcal{I}}(D,(\eta_i))$ for a pre-efficient Dehn twist $D$ consists of all tuples $(H,(\delta_i)_{i\in I})$ such that there are (automatically unique) $h_{i,j}\in G_{e_{i,j}}$ for all $i\in I^c$ with*

*(i) $H\in \mathrm{Aut}^{(E)}(\mathbb{G})$,*

*(ii) $H_{\tau(e_{i,j})}(g_{i,j}) = \delta_H(e_{i,j}) f_{e_{i,j}}(h_{i,j}) g_{i,j} f_{\overline{e_{i,j+1}}}(h_{i,j+1})^{-1}\delta_H(\overline{e_{i,j+1}})^{-1}$ for all $i\in I^c$ and all $j$ with $1\le j\le k_i$,*

*(iii) $h_{i,j+k_i} = h_{i,j}$ for all $i\in I^c$ and $j\in\mathbb{Z}$,*

*(iv) $\delta_i = \delta_H(\overline{e_{i,1}}) f_{\overline{e_{i,1}}}(h_{i,k_i+1})$ for all $i\in I^c$,*

*(v) $\delta_i\in G_{v_i}$ with $H_*(\eta_i) = \delta_i\eta_i\delta_i^{-1}$ for $i\in I^l$,*

*(vi) $\delta_i = \delta_{i'}$ for $i,i'\in I_m\in\mathcal{I}$.*

*Proof.* Conditions (ii)–(iv) follow from Proposition 12.2. In (ii) we only have to take $j$ with $1\le j\le k_i$ into account. Proposition 12.2(ii) then implies this condition for all $j\in\mathbb{Z}$.

For every $i\in I^c$, uniqueness of $h_{i,k_i+1}$ follows from (iv), and all other $h_{i,j}$ are then unique by (ii).

We can assume $\delta_i\in G_{v_i}$ for $i\in I^l$ in (v) because of Proposition 12.3(i),(iii). □

If $D$ is a truncatable replacement of a prenormalised higher Dehn twist of degree $d\ge 2$, then $C^1_I(D,(\eta_i))$ can be described as follows. In the given $D$-twistedly reduced expressions $\eta_i = \delta_D(\overline{e_{i,1}})t_{i,1}g_{i,1}\dots t_{i,k_i}g_{i,k_i}$, all elements $g_{i,j}$ are understood to be elements in $\pi_1(\mathbb{G}^{(d-1)})$. They are closed because we assume that $\mathbb{G}$ is truncatable at degree $d-1$. We again assume hypothesis (S). Note that all $h_{i,j}=1$ because edge groups in degree $d$ are always trivial.

We denote by $\mathbb{F}_1,\dots,\mathbb{F}_l$ the connected components of $\mathbb{G}^{(d-1)}$. We shall write $H_r$ and $D_r$ for the restrictions of $H$ and $D$ to $\mathbb{F}_r$.

**Lemma 13.9.** *$S^1_{\mathcal{I}}(D,(\eta_i))$ for a truncatable replacement $D$ of a prenormalised higher Dehn twist of degree $d\ge 2$ consists of all tuples $(H,(\delta_i)_i)$ such that* *(i)* $H \in \mathrm{Aut}^{(E)}(\mathbb{G})$,

*(ii)* *for every connected component* $\mathbb{F}_r$ *of* $\mathbb{G}^{(d-1)}$, *the automorphisms* $H_r$ *and* $D_r$ *commute,*

*(iii)* $(H_r)_*(\delta_D(e)) = (D_r)_*(\delta_H(e))\delta_D(e)\delta_H(e)^{-1}$ *whenever* $\deg(e) = d$ *and* $\tau(e) \in \mathbb{F}_r$,

*(iv)* $(H_r)_*(g_{i,j}) = \delta_H(e_{i,j})g_{i,j}\delta_H(\overline{e_{i,j+1}})^{-1}$ *for all* $i \in I^c$ *and all* $j$ *with* $1 \le j \le k_i$, *where* $r$ *is such that* $\tau(e_{i,j}) \in \mathbb{F}_r$,

*(v)* $\delta_i = \delta_H(\overline{e_{i,1}})$ *for all* $i \in I^c$,

*(vi)* $\delta_i \in \pi_1(\mathbb{F}_r)$ *with* $H_*(\eta_i) = D_*(\delta_i)\eta_i\delta_i^{-1}$ *for* $i \in I^l$, *where* $r$ *is determined by* $v_i \in \mathbb{F}_r$,

*(vii)* $\delta_{i'} = \delta_i$ *whenever* $i, i' \in I_m \subset I^l$.

*Proof.* (iv) and (v) follow from Proposition 12.2, which also reduces (iv) from all $j \in \mathbb{Z}$ to $1 \le j \le k_i$. By Lemmas 12.5 and 12.6, we can assume $\delta_i \in \pi_1(\mathbb{F}_r, v_i)$ for $i \in I^l$ in (vi). When $i, i' \in I_m \subset I^c$, then (v) implies $\delta_i = \delta_{i'}$. Therefore it suffices to require this in the case $I_m \subset I^l$ in (vii). □

## 13.6 The homomorphism $\mathrm{ev}_h$

We use the notation used in the previous sections. Assume that $L = D$ is a pre-efficient Dehn twist, so all edge groups are infinite cyclic. Fix an orientation $E^+ \subset E(\Gamma)$. Consider the group homomorphism

$$diag : \bigoplus_{e \in E^+} G_e \to \bigoplus_{\substack{i \in I^c, \\ 1 \le j \le k_i+1}} G_{e_{i,j}}$$

mapping the summand $G_e$ by the identity to the summand $G_{e_{i,j}}$ if $e_{i,j} = e$ or $e_{i,j} = \overline{e}$ and trivially otherwise. We denote the cokernel of *diag* by $P$, which is a free abelian group of finite rank.

Consider the function

$$\mathrm{ev}_h : C_I^1(D, (\eta_i)) \to P \tag{76}$$

sending $(H, (\delta_i)_i)$ to the collection of the $h_{i,j}$ given by Lemma 13.8 in the components of the target of *diag*.

**Lemma 13.10.** $\mathrm{ev}_h$ *is a homomorphism.*

*Proof.* Pick $(H, (\delta_i)), (H', (\delta_i')) \in C_I^1$. By Lemma 13.8(iv) applied to the discrete partition $\mathcal{I}$ of $I$, we have

$$\delta_i' = \delta_{H'}(\overline{e_{i,1}})f_{\overline{e_{i,1}}}(h'_{i,k_i+1})$$

for $i \in I^c$. Recall that all edge group automorphisms $H_e = 1$ by definition of $C_I^1 \subset C_I^0$. Hence

$$\begin{aligned} H_*(\delta_i')\delta_i &= H_*(\delta_{H'}(\overline{e_{i,1}}))\delta_H(\overline{e_{i,1}})f_{\overline{e_{i,1}}}(H_{e_{i,1}}(h'_{i,k_i+1}))\delta_H(\overline{e_{i,1}})^{-1}\delta_i \\ &= \delta_{HH'}(\overline{e_{i,1}})f_{\overline{e_{i,1}}}(h'_{i,k_i+1}h_{i,k_i+1}). \end{aligned}$$

This shows that $\mathrm{ev}_h((H,(\delta_i))(H',(\delta'_i)))$ and $\mathrm{ev}_h(H,(\delta_i)) + ev_h(H',(\delta'_i))$ are both $h'_{i,k_i+1}h_{i,k_i+1}$ on the component $G_{e_{i,k_i+1}}$. Similar calculations using Lemma 13.8(ii) show the same for the components $G_{e_{i,j}}$ with $j \neq 1$. □

*Remark* 13.11. Explicitly, $S^2_{\mathcal{I}}(D,(\eta_i))$ consists of all tuples $(H,(\delta_i)_{i\in I})$ such that there are $h_e \in G_e$ for all edges $e$ of $\Gamma$ such that $h_e = h_{\overline{e}}$ and the conditions in Lemma 13.8 are satisfied for $h_{i,j} = h_{e_{i,j}}$ and $h_{q,j} = h_{e_{q,j}}$. In other words, we have $h_{i,j} = h_{i',j'}$ in Lemma 13.8 whenever $e_{i,j} = e_{i',j'}$ or $e_{i,j} = \overline{e_{i',j'}}$.

By verification of the generators of $KO_I \cap C^0_I$ in Proposition 13.3, we know that $KO_I \cap C^0_I$ is contained in the kernel of $\mathrm{ev}_r$ and $\mathrm{ev}_h$. Therefore $\mathrm{ev}_r$ induces a homomorphism

$$\mathrm{ev}_r : C^0_I(\widehat{D},(\eta_i)) \to \mathbb{Z}^{I^c}, \tag{77}$$

whose kernel we denote by $C^1_I(\widehat{D},(\eta_i))$. On that kernel, we have a homomorphism

$$\mathrm{ev}_h : C^1_I(\widehat{D},(\eta_i)) \to P. \tag{78}$$

Let $C^2_I(D,(\eta_i))$ be the kernel of $\mathrm{ev}_h : C^1_I(D,(\eta_i)) \to P$, and define $C^2_I(D_{*v},(\eta_i))$ and $C^2_I(\widehat{D},(\eta_i))$ similarly. When $I$ is equipped with a partition $\mathcal{I}$, we also define corresponding versions of $S^1_{\mathcal{I}}$ and $S^2_{\mathcal{I}}$.

### 13.7 Relative centralisers in degree one

**Definition 13.12.** A *relative centraliser* is a group isomorphic to some $S^0_{\mathcal{I}}(D_{*v},(\eta_i))$ or $S^0_{\mathcal{I}}(\widehat{D},(\eta_i))$, where $D$ is (a truncatable replacement of) a prenormalised higher Dehn twist on a finitely generated free group.

When $J : \mathbb{G} \to \mathbb{G}'$ is an equivalence and $D \in \mathrm{Aut}^0(\mathbb{G})$, then there is an isomorphism

$$S^0_{\mathcal{I}}(D,(\eta_i)) \cong S_{\mathcal{I}}(JDJ^{-1},(J_*(\eta_i)))$$

given by $(H,(\delta_i)) \mapsto (JHJ^{-1},(J_*(\delta_i)))$. This induces isomorphisms

$$S^0_{\mathcal{I}}(D_{*v},(\eta_i)) \cong S_{\mathcal{I}}((JDJ^{-1})_{*v},(J_*(\eta_i))),$$
$$S^0_{\mathcal{I}}(\widehat{D},(\eta_i)) \cong S_{\mathcal{I}}(\widehat{JDJ^{-1}},(J_*(\eta_i)))$$

Therefore it is not important in Definition 13.12 whether we ask $D$ to be a prenormalised higher Dehn twist or a truncatable replacement.

In this section we assume $\deg(\mathbb{G}) = 1$. We have seen in Lemma 7.15 that truncatable replacements of pre-efficient Dehn twists are pre-efficient Dehn twists.

Important elements in a relative centraliser are given by the group

$$LSC_{\mathcal{I}}(\mathcal{G}) = \left\{(H,(\delta_i)_{i\in I}) \in S^0_{\mathcal{I}}(D,(\eta_i)) \middle| H = 1, \delta_i = 1 \text{ for } i \in I^c\right\}$$

of *local self-conjugations*.

The only non-trivial data of an element $(H,(\delta_i)) \in LSC_{\mathcal{I}}(\mathcal{G})$ are the terms $\delta_i$, $i \in I^l$. When $i \in I_m \subset I^l$, the element $\delta_m = \delta_i$ has to commute with all $\eta_{i'}$, $i' \in I_m$. Since centralisers of subgroups of finitely generated free groups are always finitely generated free, $LSC_{\mathcal{I}}(\mathcal{G})$ is a free product of finitely many such free groups.

**Proposition 13.13.** *Let $D \in \mathrm{Aut}^0(\mathcal{G})$ be a pre-efficient Dehn twist together with a partition $\mathcal{I}$ of $I$, and $\eta_i \in \pi_1(\mathcal{G}, v_i)$ for $i \in I$ such that $(D, \mathcal{I}, (\eta_i))$ satisfies hypothesis (S). For every vertex $w$ of $\Gamma$, there is then a finite set $\mathcal{C}_w$ of conjugacy classes in $G_w$ such that the image of each horizontal map in the following diagram has finite index in the target group.*

$$\begin{array}{ccc}
S^2_{\mathcal{I}}(D, (\eta_i)) & \xrightarrow{\quad A \quad} & \bigoplus_{w \in V(\Gamma)} \mathrm{Aut}(G_w, \mathcal{C}_w) \\
\downarrow & & \downarrow \\
S^2_{\mathcal{I}}(D_{*v}, (\eta_i)) & \xrightarrow{A_{*v}} & \mathrm{Aut}(G_v, \mathcal{C}_v) \oplus \bigoplus_{w \neq v} \mathrm{Out}(G_w, \mathcal{C}_w) \\
\downarrow & & \downarrow \\
S^2_{\mathcal{I}}(\widehat{D}, (\eta_i)) & \xrightarrow{\quad \widehat{A} \quad} & \bigoplus_{w \in V(\Gamma)} \mathrm{Out}(G_w, \mathcal{C}_w)
\end{array}$$

*Here $A$ is the homomorphism sending $(H, (\delta_i)_{i \in I})$ to $(H_w)_{w \in V(\Gamma)}$. The kernels of $A$, $A_{*v}$, and $\widehat{A}$ are direct products of finitely many finitely generated free groups. They are generated by local self-conjugations and Dehn twists.*

*Proof.* Recall that $I^c$ and $I^l$ are the sets of those $i \in I$ such that $\eta_i$ is $D$-cyclic or $D$-local respectively. Let $\mathcal{I} = \{I_1, \ldots, I_M\}$ be the underlying partition, and assume $I_m \subset I^l$ for $1 \leq m \leq M'$ and $I_m \subset I^c$ for $M' < m \leq M$.

We use the description of Remark 13.11 (cf. also Lemma 13.8). Together with Lemma 13.7, we see that an element of $S^2_{\mathcal{I}}(D, (\eta_i))$ is encoded by a tuple

$$\big((H_w)_{w \in V(\Gamma)}, (\delta_H(e))_{e \in E(\Gamma)}, (\delta_m)_{1 \leq m \leq M'}\big)$$

such that there are $h_e = h_{\overline{e}} \in G_e$ satisfying the conditions

- $\delta_H(e) \in G_{\tau(e)}$ for $e \in E(\Gamma)$,
- $\delta_i \in G_{v_i}$ for $i \in I^l$,
- $H_{\tau(e)}(f_e(a_e)) = \delta_H(e) f_e(a_e) \delta_H(e)^{-1}$,
- $H_{\tau(e_{i,j})}(g_{i,j}) = \delta_H(e_{i,j}) f_{e_{i,j}}(h_{e_{i,j}}) g_{i,j} f_{\overline{e_{i,j+1}}}(h_{e_{i,j+1}})^{-1} \delta_H(\overline{e_{i,j+1}})^{-1}$ for all $i \in I^c$ and all $j$ with $1 \leq j \leq k_i$,
- $H_*(\eta_i) = \delta_m \eta_i \delta_m^{-1}$ for $i \in I_m \subset I^l$.

We observe that, whenever $i, i' \in I_m \subset I^c$, then $e_{i,k_i+1} = e_{i',k_{i'}+1}$ by hypothesis (S), so $h_{i,k_i+1} = h_{i',k_{i'}+1}$, and Lemma 13.8(iv) implies $\delta_i = \delta_{i'}$.

Every $K(e_0, h)_I$ determines such a tuple, in which the only non-trivial components are $\delta_K(e_0) = f_{e_0}(h)$ and $\delta_K(\overline{e_0}) = f_{\overline{e_0}}(h)$. As its vertex group automorphisms are trivial, it lies in the kernel of

$$A : S^2_{\mathcal{I}}(D, (\eta_i)) \to \bigoplus_w \mathrm{Aut}(G_w).$$

To understand the image of $A$, we may compose any tuple with the composition of all $K(e, h_e^{-1})_I$, $e \in E^+$. Therefore we may assume that all $h_e = 1$. We have to find finite sets $\mathcal{C}_w$ of conjugacy classes such that $A$ corestricts to the top horizontal map in the diagram of the assertion, and such that the image of $A$ has finite index.

Given any vertex $w \in V(\Gamma)$, we define the *link graph* $\Lambda(w)$ to be the following labeled graph. The set of its vertices is $V(\Lambda(w)) = E_w$, the set of edges of $\Gamma$ terminating at the given vertex $w$. There are the following labeled edges: For every $e \in V(\Lambda(w))$ we take a loop at $e$ with label $f_e(a_e)$. For every pair $(i,j)$ such that $i \in I^c$, $1 \leq j \leq k_i$, and $\tau(e_{i,j}) = w$, we define an edge in $\Lambda(w)$ from $e_{i,j}$ to $\overline{e_{i,j+1}}$ with label $g_{i,j}$ (cf. Figure 9). Let $V_0(w)$ be a set of representative vertices for the connected components of $\Lambda(w)$, and write $V_0 = \bigsqcup_{w \in V(\Gamma)} V_0(w)$.

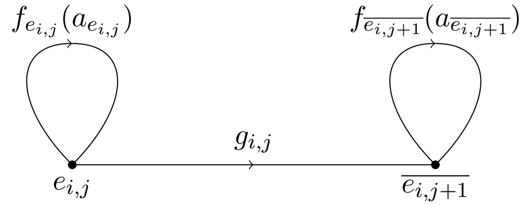


Figure 9: The link graph $\Lambda(w)$ in the proof of Proposition 13.13.

The third and fourth bullet points of the above list can be reformulated by Proposition 11.13 on the labeled graph $\Lambda(w)$: For every $e \in V_0(w)$, the automorphism $H_{\tau(e)} = H_w$ has to act as $\mathrm{ad}_{\delta_H(e)}$ on the finitely generated subgroup $\lambda(\pi_1(\Lambda(w), e))$ of $G_{\tau(e)}$, where $\lambda : \pi_1(\Lambda(w)) \to G_w$ denotes the labeling function. This determines all $\delta_H(e)$ with $e \notin V_0$ as well.

Hence $S^2_{\mathcal{I}}$ is given by tuples $((H_w)_w, (\delta_H(e))_{e \in V_0}, (\delta_m)_{1 \leq m \leq M'})$ with

- $H_w(\zeta) = \delta_H(e)\zeta\delta_H(e)^{-1}$ for $e \in V_0(w)$ and $\zeta \in \lambda(\pi_1(\Lambda(w), e))$,
- $H_{v_i}(\eta_i) = \delta_m \eta_i \delta_m^{-1}$ for $i \in I_m$, $1 \leq m \leq M'$.

By Proposition 11.7, there is a finite set $\mathcal{C}_w$ of conjugacy classes in each $G_w$ such that, if we corestrict $A$ to $\mathrm{Aut}(G_w, \mathcal{C}_w)$ in each summand of the target, then the image of $A$ has finite index.

The kernel of $A$ is generated by all $K(e_0, h)_I$ and tuples

$$((1_{G_w})_{w \in V(\Gamma)}, (\delta_H(e))_{e \in V_0}, (\delta_m)_{1 \leq m \leq M'})$$

such that

- $\delta_H(e)$ centralises $\lambda(\pi_1(\Lambda(w), e))$ for every $e \in V_0(w)$,
- $\delta_m$ commutes with $\eta_i$ for $i \in I_m$, $1 \leq m \leq M'$.

These elements define Dehn twists and local self-conjugations respectively.

We have to show that we can define $A_{*v}$ and $\widehat{A}$ fitting into the commutative diagram in the assertion. Then their images will also have finite index in the respective target group.

By Proposition 13.3(ii), the kernel $KA_I \cap C_I^0$ of the map $S_{\mathcal{I}}^2(D,(\eta_i)) \to S_{\mathcal{I}}^2(D_{*v},(\eta_i))$ is generated by the tuples corresponding to the $K(e_0,h)_I$, of which we know that they are in the kernel of $A$, and $M(w,\gamma)_I$ for $w \neq v$. The latter are mapped to $1 \in \mathrm{Aut}(G_u,\mathcal{C}_u)$ for $u \neq w$ and to $\mathrm{ad}_\gamma \in \mathrm{Aut}(G_w,\mathcal{C}_w)$ by $A$, so they are trivial in the target group for $A_{*v}$. This allows us to define $A_{*v}$ as indicated in the diagram. The kernel of $A_{*v}$ is the kernel of $A$ with the $K(e_0,h)_I$ divided out, so it is generated by Dehn twists and local self-conjugations.

Similar arguments apply to $\widehat{A}$ when we take automorphisms $M(v,\gamma)_I$ into account.

$\square$

## 13.8 Centralisers of efficient Dehn twists

Suppose that $D$ is an efficient (or pointedly efficient) Dehn twist on $\mathcal{G}$. The groups $C^0(\widehat{D})$ and $C^0(D_{*v})$ have finite index in $C(\widehat{D})$ and $C(D_{*v})$ by Lemma 13.1. We now discuss the special case $I = \varnothing$ in Proposition 13.13 and its proof in more detail, which will lead the to explicit short exact sequences (80) and (81) for $C^0(\widehat{D})$ and $C^0(D_{*v})$ respectively. These sequences have also been obtained in Theorems 5.10 and 6.7 of [31].

The rotation homomorphism $\mathrm{ev}_r$ in (74) on page 121 and the homomorphism $\mathrm{ev}_h$ in (76) on page 124 have trivial target groups. Thus we have $C_I^2(D) = C_I^0(D)$, $C_I^2(D_{*v}) = C_I^0(D_{*v})$, and $C_I^2(\widehat{D}) = C_I^0(\widehat{D})$.

The link graph $\Lambda(w)$ in the proof of Proposition 13.13 is a disjoint union of circles, one at each vertex $e \in E_w = V(\Lambda(w))$ with label $f_e(a_e)$. The sets of conjugacy classes $\mathcal{C}_w$ can be chosen as

$$\mathcal{C}_w = \big\{[f_e(a_e)] \big| e \in E(\Gamma), \tau(e) = w\big\} \tag{79}$$

because $\lambda(\pi_1(\Lambda(w),e)) = \langle f_e(a_e)\rangle$. We only appeal to Proposition 11.7 in the special case of tuples of length one, so we need not pass to a subgroup of finite index. Hence the maps $A_{*v}$ and $\widehat{A}$ are surjective. We therefore have short exact sequences

$$1 \to DO(\mathcal{G}) \to C^0(\widehat{D}) \to \bigoplus_{w \in V(\Gamma)} \mathrm{Out}(G_w,\mathcal{C}_w) \to 1, \tag{80}$$

$$1 \to DA(\mathcal{G}) \to C^0(D_{*v}) \to \mathrm{Aut}(G_v,\mathcal{C}_v) \oplus \Big(\bigoplus_{w \neq v} \mathrm{Out}(G_w,\mathcal{C}_w)\Big) \to 1 \tag{81}$$

for some kernels $DO(\mathcal{G})$ and $DA(\mathcal{G})$. By Proposition 13.13, these kernels are generated by Dehn twists because there are no local self-conjugations. Since $I = \varnothing$, items (ii) through (vi) in Lemma 13.8 are void, and the group of Dehn twists of $\mathcal{G}$ is entirely contained in $C_I^2$. Thus $DO(\mathcal{G})$ and $DA(\mathcal{G})$ are its images in $\mathrm{Out}(\pi_1(\mathcal{G},v))$ and $\mathrm{Aut}(\pi_1(\mathcal{G},v))$ respectively. After adding an additional free factor $\mathbb{Z}$ at the basepoint in the situation of $DA(\mathcal{G})$, it can be deduced from Proposition 5.4 of [13] that these kernels are free abelian of rank equal to the number of geometric edges of $\Gamma$.

## 13.9 Relative centralisers in higher degree

Let $D \in \mathrm{Aut}^0(\mathbb{G})$ be a truncatable replacement of a prenormalised higher Dehn twist. Assume that we are given $\eta_i \in \pi_1(\mathbb{G}, v_i)$ for $i \in I$ as well as a partition $\mathcal{I}$ of $I$. We assume that this data fulfills hypothesis (S) in Definition 13.5. Recall that $S^1_{\mathcal{I}}$ is the kernel of the rotation homomorphism $\mathrm{ev}_r : S^0_{\mathcal{I}} \to \mathbb{Z}^{I^c}$.

We denote by $\mathbb{F}_1, \ldots, \mathbb{F}_l$ the connected components of $\mathbb{G}^{(d-1)}$, and we assume that the basepoint $v$ lies in $\mathbb{F}_1$. For simplicity, we write $D_r$ for the restriction of $D$ to $\mathbb{F}_r$.

**Proposition 13.14.** *There are closed elements $\eta'_{i,r}$, $i \in I'_r$, with partitions $\mathcal{I}'_r$ of $I'_r$ in $\pi_1(\mathbb{F}_r)$ fitting into the commutative diagram*

$$\begin{tikzcd}
S^1_{\mathcal{I}}(D, (\eta_i)) \arrow[r, "B"] \arrow[d, two heads] & \bigoplus_{r=1}^{l} S^0_{\mathcal{I}'_r}(D_r, (\eta'_{i,r})) \arrow[d, two heads] \\
S^1_{\mathcal{I}}(D_{*v}, (\eta_i)) \arrow[r, "B_{*v}"] \arrow[d, two heads] & S^0_{\mathcal{I}'_1}((D_1)_{*v}, (\eta'_{i,r})) \oplus \bigoplus_{r=2}^{l} S^0_{\mathcal{I}'_r}(\widehat{D_r}, (\eta'_{i,r})) \arrow[d, two heads] \\
S^1_{\mathcal{I}}(\widehat{D}, (\eta_i)) \arrow[r, "\widehat{B}"] & \bigoplus_{r=1}^{l} S^0_{\mathcal{I}'_r}(\widehat{D_r}, (\eta'_{i,r}))
\end{tikzcd}$$

*such that the horizontal homomorphisms are isomorphisms.*

*Proof.* Roughly speaking, the homomorphism $B$ is obtained by “extending” the isomorphism in (43) on page 87 to additional $\eta_i$.

We use the description of $S^1_{\mathcal{I}}$ in Lemma 13.9. All $g_{i,j}$ are closed elements in $\pi_1(\mathbb{F}_r)$ for some $r$. In the partition $\mathcal{I}$, we assume that $I^l = I_1 \cup \ldots \cup I_{M'}$ and $I^c = I_{M'+1} \cup \ldots \cup I_M$. The elements in $S^1_{\mathcal{I}}$ are described by tuples $((H_r)_{1 \le r \le l}, (\delta_H(e))_{\deg(e)=d}, (\delta_m)_{1 \le m \le M'})$ such that

(1) $H_r, D_r \in \mathrm{Aut}^0(\mathbb{F}_r)$ commute,

(2) $(H_r)_*(\delta_D(e)) = (D_r)_*(\delta_H(e))\delta_D(e)\delta_H(e)^{-1}$ if $\deg(e) = d$, $\tau(e) \in \mathbb{F}_r$,

(3) $(H_r)_*(g_{i,j}) = \delta_H(e_{i,j})g_{i,j}\delta_H(\overline{e_{i,j+1}})^{-1}$ for $i \in I^c \sqcup Q^c$, $1 \le j \le k_i$,

(4) $\delta_m \in \pi_1(\mathbb{G}^{(d-1)})$ satisfies $H_*(\eta_i) = D_*(\delta_m)\eta_i\delta_m^{-1}$ for $i \in I_m$, $1 \le m \le M'$.

We define the link graph $\Lambda_r$ to be the graph with

$$V(\Lambda_r) = \left\{ e \in E(\Gamma) \middle| \deg(e) = d, \tau(e) \in \mathbb{F}_r \right\}$$

and an edge labeled $g_{i,j}$ from each $e_{i,j}$ to $\overline{e_{i,j+1}}$ when $\tau(e_{i,j}) \in \mathbb{F}_r$.

We claim that (2) for $e = e_{i,j}$ is equivalent to (2) for $e' = \overline{e_{i,j+1}}$. By Proposition 12.2(ii), item (3) of the above list for $1 \le j \le k_i$ implies (3) for all $j \in \mathbb{Z}$. By (7) on page 40, the condition

$$H_{r*}(\delta_D(e')) = D_{r*}(\delta_H(e'))\delta_D(e')\delta_H(e')^{-1}$$

is equivalent to

$$H_{r*}(D_{r*}(g_{i,j})^{-1}\delta_D(e)g_{i,j-k_i}) = D_{r*}(\delta_H(e'))D_{r*}(g_{i,j})^{-1}\delta_D(e)g_{i,j-k_i}\delta_H(e')^{-1}.$$

(1) and (3) allow us to equivalently rewrite this as

$$\begin{aligned}&D_{r*}H_{r*}(g_{i,j})^{-1}H_{r*}(\delta_D(e))H_{r*}(g_{i,j-k_i})\\ =\ &D_{r*}H_{r*}(g_{i,j})^{-1}D_{r*}(\delta_H(e))\delta_D(e)\delta_H(e)^{-1}H_{r*}(g_{i,j-k_i}),\end{aligned}$$

which simplifies to $H_{r*}(\delta_D(e)) = D_{r*}(\delta_H(e))\delta_D(e)\delta_H(e)^{-1}$. Thus (2) can be reduced to the special case $e \in V_0(\Lambda_r)$, where $V_0(\Lambda_r) \subset V(\Lambda_r)$ is a set of representative vertices for the connected components of $\Lambda_r$.

Proposition 11.13 shows that (3) reduces to $(H_r)_* = \mathrm{ad}_{\delta_H(e)}$ on $\lambda(\pi_1(\Lambda_r, e))$ for $e \in V_0(\Lambda_r)$.

On each component $\mathbb{F}_r$, the above data satisfying (1) through (4) therefore reduces to tuples $(H_r, (\delta_H(e))_{e\in V_0(\Lambda_r)}, (\delta_m)_m)$ such that

- $H_r$ and $D_r$ commute,
- $(H_r)_*(\delta_D(e)) = (D_r)_*(\delta_H(e))\delta_D(e)\delta_H(e)^{-1}$ if $e \in V_0(\Lambda_r)$,
- $(H_r)_*$ acts as $\mathrm{ad}_{\delta_H(e)}$ on $\lambda(\pi_1(\Lambda_r, e))$ if $e \in V_0(\Lambda_r)$,
- $(H_r)_*(\eta_i) = (D_r)_*(\delta_m)\eta_i\delta_m^{-1}$ if $i \in I_m$, $1 \le m \le M'$, and $\eta_i \in \pi_1(\mathbb{F}_r)$.

Here the second and third bullet points can be rephrased to the requirement that $(H_r)_*$ simultaneously $D_r$-conjugates

$$\delta_D(e), \delta_D(e)\mu_1(e), \ldots, \delta_D(e)\mu_{N(e)}(e)$$

by $\delta_H(e)$ for every $e \in V_0(\Lambda_r)$, where $\mu_1(e), \ldots, \mu_N(e)$ are generators for $\lambda(\pi_1(\Lambda_r, e))$. Hence the data on each component $\mathbb{F}_r$ is encoded by $S^0_{\mathcal{I}'_r}(D_r, (\eta'_{i,r}))$ for some elements $\eta'_{i,r} \in \pi_1(\mathbb{F}_r)$, $i \in I'_r$, and a partition $\mathcal{I}'_r$. The isomorphism $B$ is then given by restriction to $\mathbb{F}_r$ in the $r$-th summand.

To show that $B_{*v}$ is well-defined, we use Proposition 13.3. The kernel $KA_I(\mathbb{G}) \cap C^0_I$ of the map $S^1_{\mathcal{I}}(D, (\eta_i)) \to S^1_{\mathcal{I}}(D_{*v}, (\eta_i))$ is generated by automorphisms of the form $K(e_0, h)_I$ and $M(w, \gamma)_I$ with $w \ne v$. These elements are mapped to $K(e_0, h)_{I'_r}$ and $M(w, \gamma)_{I'_r}$ by $B$. These images generate the kernel of the right hand map. Similar arguments apply to $\widehat{B}$. □

## 13.10 Finiteness property VF

The goal of this section is Theorem 13.21, the main theorem of this thesis, which shows that the centraliser of every higher Dehn twist automorphism in $\mathrm{Out}(F_n)$ or $\mathrm{Aut}(F_n)$ satisfies finiteness property VF.

**Definition 13.15.** A group $G$ has *finiteness property F* if there is a finite CW complex which is a $K(G, 1)$-space. $G$ has *finiteness property VF* if it has a subgroup $H$ of finite index which has property F.

We first need some elementary preparation.

**Lemma 13.16.**

*(i) If $G$ satisfies property F, and $G^0$ is a subgroup of finite index in $G$, then $G^0$ satisfies property F.*

*(ii) If $G^0$ is a subgroup of $G$ of finite index, then $G$ satisfies VF if and only if $G^0$ does.*

*(iii) If $G_1, \ldots, G_l$ satisfy F (or VF respectively), then so does $\bigoplus_{i=1}^{l} G_i$.*

*(iv) All finitely generated abelian groups have finiteness property VF.*

*Proof.* In (i), a finite $K(G^0, 1)$ space can be obtained as a covering space of a finite $K(G, 1)$ space with finitely many sheets.

In (ii), if $H^0$ has finite index in $G^0$ and property F, it also has finite index in $G$, so $G$ has property VF. Conversely, if $H$ has finite index in $G$ and satisfies property F, then $H^0 := H \cap G^0$ has finite index in $G^0$. As $H^0$ also has finite index in $H$, part (i) shows that $H^0$ has property F, so $G^0$ satisfies VF. This proves (ii).

If each $X_i$ is a finite $K(G_i, 1)$ space in (iii), then $\prod_i X_i$ is a finite $K(G, 1)$ space. This proves (iii) for property F. For property VF, we use that $\bigoplus H_i$ has finite index in $\bigoplus G_i$ when each $H_i$ has finite index in $G_i$.

As every finitely generated abelian group is a finite direct sum of cyclic groups, (iii) reduces (iv) to the special case of cyclic groups. Using (ii), we only have to show finiteness property VF for the trivial group and for $\mathbb{Z}$. But these groups have finite classifying spaces, namely a point and a circle respectively. □

**Proposition 13.17.** *If $1 \to G' \to G \to G'' \to 1$ is a short exact sequence of groups, $G'$ has finiteness property F, and $G''$ has finiteness property VF, then $G$ has finiteness property VF.*

*Proof.* Let $H''$ be a finite index subgroup of $G''$ satisfying finiteness property F, and denote its preimage in $G$ by $H$. We then have a short exact sequence

$$1 \to G' \to H \to H'' \to 1.$$

By Theorem 7.1.10 in [16], we can construct a finite classifying space for $H$ from finite classifying spaces for $G'$ and $H''$. As $H$ has finite index in $G$, the group $G$ has property VF. □

**Proposition 13.18.** *Suppose that in the short exact sequence $1 \to G' \xrightarrow{\iota} G \xrightarrow{\pi} G'' \to 1$, the group $G'$ has finiteness property VF, and $G''$ is finitely generated abelian. Then $G$ has finiteness property VF.*

*Proof.* Suppose first that $G'' \cong \mathbb{Z}$. Let $H'$ be a subgroup of some finite index $d$ in $G'$ such that $H'$ satisfies property F. Then the intersection $K'$ of all (finitely many) index $d$ subgroups of $G'$ also satisfies property F by Lemma 13.16(i). Furthermore, the group $K'$ is a characteristic subgroup of $G'$, i.e. it is left invariant by every automorphism of

$G'$. Let now $t \in G$ be an element such that $\pi(t)$ generates $G''$. We denote by $K$ the subgroup of $G$ generated by $t$ and $\iota(K')$. It has finite index in $G$ and fits into a short exact sequence

$$1 \to K' \to K \to G'' \to 1.$$

Proposition 13.17 implies that $K$ and hence $G$ has property VF. This finishes the proof for $G'' \cong \mathbb{Z}$.

By induction on $m$, we now prove the assertion in the case that $G'' \cong \mathbb{Z}^m$. Let $A'' \cong \mathbb{Z}$ be a direct summand in $G''$, so $G''/A'' \cong \mathbb{Z}^{m-1}$. The short exact sequences

$$1 \to G' \to \pi^{-1}(A'') \to A'' \to 1,$$
$$1 \to \pi^{-1}(A'') \to G \to G''/A'' \to 1,$$

and the induction hypothesis now prove the assertion for free abelian $G''$.

We now come to the general case. Let $H''$ be a free abelian subgroup of $G''$ of finite index and $H = \pi^{-1}(H'') \subset G$. We obtain a short exact sequence

$$1 \to G' \to H \to H'' \to 1.$$

Hence $H$ satisfies VF, and Lemma 13.16(ii) implies VF for $G$ as well. $\square$

**Proposition 13.19.** *If $\mathcal{C}$ is a finite set of conjugacy classes in $F_n$, then* $\mathrm{Aut}(F_n, \mathcal{C})$ *and* $\mathrm{Out}(F_n, \mathcal{C})$ *satisfy finiteness property VF.*

*Proof.* This is shown in Corollary 6.1.4 of [14] for $\mathrm{Out}(F_n, \mathcal{C})$. As $\mathrm{Inn}(F_n) \cong F_n$ (or $\mathrm{Inn}(F_1) = 1$) has property $F$, the exact sequence

$$1 \to \mathrm{Inn}(F_n) \to \mathrm{Aut}(F_n, \mathcal{C}) \to \mathrm{Out}(F_n, \mathcal{C}) \to 1$$

and Proposition 13.17 show VF for $\mathrm{Aut}(F_n, \mathcal{C})$ as well. $\square$

**Theorem 13.20.** *All relative centralisers have finiteness property VF.*

*Proof.* We show by induction on $d := \deg(\mathbb{G})$ that $S^0_{\mathcal{I}}(D_{*v}, (\eta_i))$ and $S^0_{\mathcal{I}}(\widehat{D}, (\eta_i))$ have finiteness property VF whenever $D$ is (a truncatable replacement of) a prenormalised higher Dehn twist on $\mathbb{G}$.

Let first $d \leq 1$. The homomorphisms $A_{*v}$ and $\widehat{A}$ in Proposition 13.13 have kernels which are direct products of finitely many free groups, hence $\ker(A_{*v})$ and $\ker(\widehat{A})$ satisfy property F. The images of $A_{*v}$ and $\widehat{A}$ have finite index in the direct sum of finitely many groups of the form $\mathrm{Out}(G_w, \mathcal{C}_w)$ or $\mathrm{Aut}(G_v, \mathcal{C}_v)$. By Proposition 13.19 and Lemma 13.16, we conclude that the images of $A_{*v}$ and $\widehat{A}$ have finiteness property VF. Proposition 13.17 shows that $S^2_{\mathcal{I}}(D_{*v}, (\eta_i))$ and $S^2_{\mathcal{I}}(\widehat{D}, (\eta_i))$ satisfy VF whenever $(D, \mathcal{I}, (\eta_i))$ satisfies hypothesis (S) of Definition 13.5. In the following, we exhibit the arguments for $S^2_{\mathcal{I}}(\widehat{D}, (\eta_i))$, which we simply denote by $S^2_{\mathcal{I}}$. The proof works in the same way for $S^2_{\mathcal{I}}(D_{*v}, (\eta_i))$.

Recall that $C^2_I$ is defined to be the kernel of $\mathrm{ev}_h : C^1_I \to P$ in (78) on page 125, and $S^2_{\mathcal{I}} = C^2_I \cap S^0_{\mathcal{I}}$ is the kernel of the restricted homomorphism $\mathrm{ev}_h : S^1_{\mathcal{I}} \to P$. The group $P$

is finitely generated abelian. As any subgroup of a finitely generated abelian group is finitely generated abelian, $\mathrm{ev}_h(S^1_{\mathcal{I}}) \subset P$ is finitely generated abelian. Proposition 13.18 shows that $S^1_{\mathcal{I}}$ has property VF.

The target group of the rotation homomorphism $\mathrm{ev}_r$ in (77) on page 125 is finitely generated abelian, so the image of $\mathrm{ev}_r$ is. Proposition 13.18 shows that $S^0_{\mathcal{I}}$ satisfies VF. By Lemma 13.6 these groups $S^0_{\mathcal{I}}$ also satisfy VF if we do not require hypothesis (S). This finishes the proof for $d \leq 1$.

We now prove inductively the assertion for $d \geq 2$. Suppose first that hypothesis (S) is satisfied for $S^1_{\mathcal{I}}(\widehat{D}, (\eta_i))$ or $S^1_{\mathcal{I}}(D_{*v}, (\eta_i))$. By Proposition 13.14 this group is a direct sum of finitely many relative centralisers of degree $d-1$. By induction and Lemma 13.16(iii), we conclude that $S^1_{\mathcal{I}}(D_{*v}, (\eta_i))$ and $S^1_{\mathcal{I}}(\widehat{D}, (\eta_i))$ have property VF. As this is the kernel of $\mathrm{ev}_r$ in (77), and the image of that homomorphism is finitely generated abelian, Proposition 13.18 shows that $S^0_{\mathcal{I}}$ has property VF. By Lemma 13.6 this also holds true without assuming (S). □

We are now in a position to deduce the main theorem of this thesis.

**Theorem 13.21.** *Whenever $D$ is a higher Dehn twist on a higher graph of groups $\mathbb{G}$ with finitely generated free fundamental group, then the centralisers $C(D_{*v})$ and $C(\widehat{D})$ satisfy property VF.*

*Proof.* If we have $\widehat{D} = \widehat{\rho}\widehat{D'}\widehat{\rho}^{-1}$ for two higher Dehn twists $D \in \mathrm{Aut}^0(\mathbb{G})$ and $D' \in \mathrm{Aut}^0(\mathbb{G}')$, then $C(\widehat{D}) \cong C(\widehat{D'})$. Therefore Theorem 8.6(ii) reduces the statement to normalised $D$ for $C(\widehat{D})$. Similarly, we can reduce to pointedly normalised $D$ for $C(D_{*v})$ by Theorem 8.6(i).

Note that $C^0(\widehat{D})$ is the relative centraliser with $I = \varnothing$ and the empty partition $\mathcal{I}$ in Definition 13.12. Therefore Theorem 13.20 shows that it satisfies VF. By Lemma 13.1, it has finite index in $C(\widehat{D})$, so Lemma 13.16(ii) shows that $C(\widehat{D})$ satisfies property VF. Similar arguments apply to $C(D_{*v})$. □

*Remark* 13.22. When we have a finite set $\mathcal{C}$ of conjugacy classes in $\pi_1(\mathbb{G}, v)$, then we can also look at the group $G$ of those $(H, (\delta_i)_{i\in I}) \in S^0_{\mathcal{I}}(D, (\eta_i))$ such that $H_*$ fixes each class in $\mathcal{C}$. A refined version of the lemmas in this chapter can be used to show that groups of the form $G/(G \cap KA_I)$ or $G/(G \cap KO_I)$ satisfy property VF when $D$ is (a truncatable replacement of) a prenormalised higher Dehn twist. This specialises to Theorem 13.20 when $\mathcal{C} = \varnothing$.

When $I = \varnothing$, then we obtain that the intersections $C(D_{*v}) \cap \mathrm{Aut}(\pi_1(\mathbb{G}, v), \mathcal{C})$ and $C(\widehat{D}) \cap \mathrm{Out}(\pi_1(\mathbb{G}, v), \mathcal{C})$ satisfy VF for every higher Dehn twist $D$ on $\mathbb{G}$ with finitely generated free fundamental group.

# 14 Isometric CAT(0) actions

In this chapter we discuss the translation length of an element $g$ in a group $G$ acting by isometries on a CAT(0) space. We first recall some definitions.

A CAT(0) space is a geodesic metric space $X$ satisfying the *CAT(0)-inequality* for all $p, q, r \in X$: That is, for all geodesic triangles with vertices $p, q, r \in X$ and all $x \in [p, q]$ and $y \in [p, r]$, the euclidean comparison triangle with $d(\bar{p}, \bar{q}) = d(p, q)$, $d(\bar{p}, \bar{r}) = d(p, r)$, $d(\bar{q}, \bar{r}) = d(q, r)$, $d(\bar{p}, \bar{x}) = d(p, x)$, $d(\bar{p}, \bar{y}) = d(p, y)$, $\bar{x} \in [\bar{p}, \bar{q}]$, $\bar{y} \in [\bar{p}, \bar{r}]$ satisfies $d(x, y) \leq d(\bar{x}, \bar{y})$. The situation is depicted in Figure 10.


Figure 10: The CAT(0) inequality.

Taking $q = r$, it follows that there is a *unique* geodesic between $p$ and $q$, which we denote by $[p, q]$.

A more precise definition of CAT(0) spaces can be found in Chapter II.1 of [11].

Recall that $X$ is called *complete* if every Cauchy sequence converges. The space $X$ is called *proper* if all closed balls $\overline{B(x, r)}$ are compact.

## 14.1 Translation lengths and centralisers

**Definition 14.1.** Let $\gamma : X \to X$ be an isometry. Its *translation length* is

$$|\gamma| := \inf_{x \in X} d(x, \gamma(x)).$$

$\gamma$ is called

- *elliptic*, if $|\gamma| = 0$ and the infimum is attained at some $x \in X$, i.e. $x$ is a fixed point of $\gamma$,
- *hyperbolic*, if $|\gamma| > 0$ and the infimum is attained at some $x \in X$,
- *parabolic*, if the infimum is not attained,
- *semisimple*, if it is either elliptic or hyperbolic.

Translation lengths are invariant under conjugation, i.e. $|\alpha| = |\beta\alpha\beta^{-1}|$. Moreover, $\alpha$ is elliptic, parabolic, or hyperbolic respectively, if and only if $\beta\alpha\beta^{-1}$ is. Similarly $|\alpha| = |\alpha^{-1}|$.

If $Q$ is any group and $g \in Q$, then we denote by $[\![g]\!]$ the class in the abelianisation $H_1(Q) = Q/[Q,Q]$ represented by $g$.

There is the following criterion for translation lengths using centralisers: Suppose $g \in G$ has positive translation length for an action on a proper CAT(0) space $X$. Then it can be shown that it fixes a point in the boundary of $X$. The action of the centraliser $C(g)$ preserves the set of horospheres at this point, but it might shift them by a certain length. This gives rise to a group homomorphism $C(g) \to \mathbb{R}$ sending $g$ to the translation length $|g| > 0$. Since $\mathbb{R}$ is abelian and torsion-free, this is possible only if $[\![g]\!] \in H_1(C(g))$ has infinite order. This gives rise to the following theorem, which has also appeared implicitly in the proof of Theorem 1 in [9].

**Theorem 14.2** (Bridson, Karlsson, Margulis)**.** *Let $G$ be any group and $g \in G$. Assume that $[\![g]\!]$ has finite order in $H_1(C(g))$. Then $|g| = 0$ whenever $G$ acts by isometries on a proper CAT(0) space.* □

## 14.2 Orthogonal projections

A subset $C$ of a geodesic metric space $X$ is called *convex* if, for all $x, y \in C$, the geodesic segment $[x, y] \subset C$. The following proposition describes an orthogonal projection onto convex subsets and is illustrated in Figure 11.

**Proposition 14.3.** *Let $C$ be a closed convex subset of a complete CAT(0)-space $X$. Then:*

*(i) For every point $x \in X$, there is a unique point $\pi_C(x) \in C$ such that the distance $d(x, \pi_C(x)) = d(x, C) := \inf_{y\in C} d(x, y)$.*

*(ii) The map $\pi_C : X \to X$ is 1-Lipschitz, i.e. $d(\pi_C(x), \pi_C(y)) \leq d(x, y)$.*

*(iii) If $\gamma$ is an isometry of $X$, then $\pi_{\gamma(C)} \circ \gamma = \gamma \circ \pi_C$.*

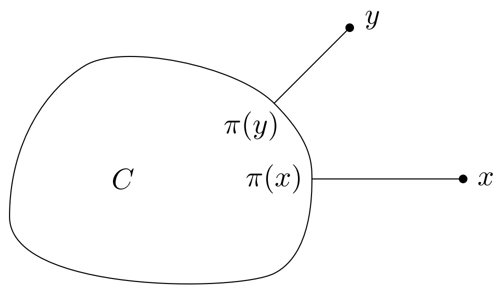


Figure 11: Orthogonal projection onto a convex set.

*Proof.* (i) and (ii) are Proposition II2.4(1) and Corollary II2.5(2) of [11]. Part (iii) is easy and left to the reader. □

In the following we are interested in the sets

$$A_\epsilon(\gamma) := \{x \in X \big| d(x, \gamma(x)) \le |\gamma| + \epsilon\},$$

where $\gamma$ is an isometry of a complete CAT(0) space $X$ and $\epsilon > 0$. Note that $A_\epsilon(\gamma) \neq \varnothing$ by definition of $|\gamma|$.

**Lemma 14.4.** *$A_\epsilon(\gamma)$ is convex.*

*Proof.* For arbitrary points $x, y \in A_\epsilon(\gamma)$, let $c : [0,1] \to X$ be the unique geodesic segment (parametrised proportionally to arclength) joining $c(0) = x$ and $c(1) = y$. Then $\gamma \circ c$ is the geodesic segment from $\gamma(x)$ to $\gamma(y)$. It is well-known (cf. Proposition II.2.2 in [11]) that the function $t \mapsto f(t) := d(c(t), \gamma c(t))$ is convex. Since $f(0), f(1) \le |\gamma| + \epsilon$, we have $f(t) \le |\gamma| + \epsilon$ for all $t \in [0,1]$. Hence $c(t) \in A_\epsilon(\gamma)$. □

**Lemma 14.5.** *Let $\alpha$ and $\beta$ be two commuting isometries of a complete CAT(0) space $X$. Then the intersection $A_\epsilon(\alpha) \cap A_\epsilon(\beta) \neq \varnothing$.*

*Proof.* Pick $x \in A_\epsilon(\alpha)$. It is easy to calculate that $\alpha(A_\epsilon(\beta)) = A_\epsilon(\beta)$. The desired intersection point will be $y := \pi_{A_\epsilon(\beta)}(x) \in A_\epsilon(\alpha) \cap A_\epsilon(\beta)$. It is in $A_\epsilon(\beta)$ by definition. By Proposition 14.3 we have

$$\begin{aligned} d(y, \alpha y) = d(\pi_{A_\epsilon(\beta)}(x), \alpha\pi_{A_\epsilon(\beta)}(x)) &= d(\pi_{A_\epsilon(\beta)}(x), \pi_{A_\epsilon(\beta)}(\alpha x)) \\ &\le d(x, \alpha x) \le |\alpha| + \epsilon, \end{aligned}$$

hence $y \in A_\epsilon(\alpha)$. This proves the lemma. □

## 14.3 A parallelogram law

In this section we prove the following formula for translation lengths, which will also restrict translation lengths in isometric CAT(0) actions.

**Theorem 14.6** (Parallelogram Law)**.** *Any two commuting isometries $\alpha$ and $\beta$ of a complete CAT(0)-space $X$ satisfy the formula*

$$|\alpha\beta|^2 + |\alpha\beta^{-1}|^2 = 2(|\alpha|^2 + |\beta|^2).$$

In the case of semisimple isometries, the flat torus theorem provides an isometrically embedded euclidean plane on which $\alpha$ and $\beta$ act by translation. Then this formula is simply the well-known parallelogram formula in euclidean space. As we don't require the isometries to be semisimple in our proof, we have to find an approximate version of the parallelogram of translation vectors.

We first need the following lemma.

**Lemma 14.7.** *Whenever $\gamma$ is an isometry of a CAT(0) space $X$, then $|\gamma^2| = 2|\gamma|$.*

*Proof.* For $x \in X$, we write $m(x)$ for the midpoint of the geodesic segment $[x, \gamma(x)]$. Then $m(\gamma(x)) = \gamma(m(x))$. The triangle spanned by $x$, $\gamma(x)$, and $\gamma^2(x)$ is illustrated by Figure 12. Using the CAT(0) inequality, we have

$$\frac{1}{2}d(x, \gamma^2(x)) \geq d(m(x), \gamma(m(x))) \geq |\gamma|$$

for every $x \in X$. Hence $\frac{1}{2}|\gamma^2| \geq |\gamma|$.

On the other hand,

$$d(x, \gamma^2(x)) \leq d(x, \gamma(x)) + d(\gamma(x), \gamma^2(x)) = 2d(x, \gamma(x)).$$

Taking the infimum over all $x \in X$, we see $|\gamma^2| \leq 2|\gamma|$. □

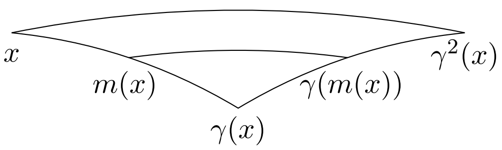


Figure 12: The points $x$, $m(x)$, $\gamma(x)$, $\gamma(m(x))$, and $\gamma^2(x)$ in the proof of Lemma 14.7.

*Proof of Theorem 14.6.* Let $\epsilon > 0$, and pick some $x \in A_\epsilon(\alpha) \cap A_\epsilon(\beta)$, which exists by Lemma 14.5. The two comparison triangles of $x$, $\alpha x$, $\alpha\beta x$ and $x$, $\beta x$, $\alpha\beta x$ fit together to a parallelogram, which is illustrated in Figure 13. Let $y \in [x, \alpha\beta x] \subset X$ be the midpoint and $\overline{y}$ the corresponding comparison point.

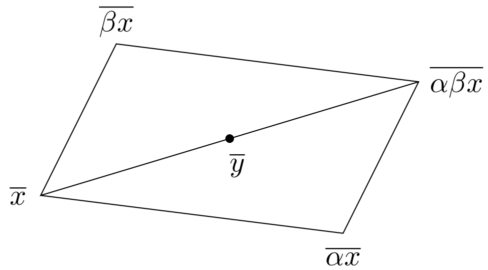


Figure 13: A comparison parallelogram.

Using the CAT(0) inequality and elementary euclidean geometry, there is the following estimate:

$$\begin{aligned}|\alpha\beta^{-1}| &\leq d(\alpha x, \beta x) \leq d(y, \alpha x) + d(y, \beta x) \leq d(\overline{y}, \overline{\alpha x}) + d(\overline{y}, \overline{\beta x}) \\ &= d(\overline{\alpha x}, \overline{\beta x}) = (2d(\overline{x}, \overline{\alpha x})^2 + 2d(\overline{x}, \overline{\beta x})^2 - d(\overline{x}, \overline{\alpha\beta x})^2)^{\frac{1}{2}} \\ &\leq (2(|\alpha| + \epsilon)^2 + 2(|\beta| + \epsilon)^2 - |\alpha\beta|)^{\frac{1}{2}}.\end{aligned}$$

Tending $\epsilon \to 0$, this is one inequality of the theorem. The other one follows after replacing $\alpha$, $\beta$ by $\alpha\beta$, $\alpha\beta^{-1}$ and using Lemma 14.7. □

An infinite cyclic subgroup $\langle\gamma\rangle \subset G$ of a finitely generated group $G$ is called *distorted* if $\frac{1}{n}l(\gamma^n) \to 0$ for $n \to \infty$, where $l$ is the length in $G$ with respect to a fixed finite generating set. It is known that $\gamma$ then acts by translation length zero whenever $G$ acts by isometries on any metric space.

Therefore it is interesting to look for examples of groups $G$ and $\gamma \in G$ such that $\mathbb{Z} \cong \langle\gamma\rangle \subset G$ is *un*distorted, but for every isometric action of $G$ on a complete CAT(0)-space $X$, the translation length $|\gamma|$ is zero. An example of this is the automorphism group of the free group $F_n$ on $n$ generators:

**Theorem 14.8** ([1], Theorem 1.1)**.** *In* $\mathrm{Aut}(F_n)$ *and* $\mathrm{Out}(F_n)$*, every infinite cyclic subgroup is undistorted.* □

Fix a basis $a, b_1, \ldots, b_k, c_1, \ldots, c_n$ of the free group $F_{n+k+1}$ and $w \in \langle b_1, \ldots, b_k\rangle$. We denote by $\rho_{a,w} \in \mathrm{Aut}(F_{n+k+1})$ the automorphism sending $a$ to $aw$ and fixing all $b_i$ and $c_i$.

**Theorem 14.9.** *If $n \geq 2$, then $|\rho_{a,w}| = 0$ whenever* $\mathrm{Aut}(F_{n+k+1})$ *acts by isometries on a complete CAT(0) space.*

In the special case $w = b_1$, Theorem 14.9 was shown by Bridson [10] under the additional assumption that the action is by *semisimple* isometries. It was also shown by Bridson for $w = b_1$ without using semisimplicity in the case $n + k + 1 \geq 6$. We drop here the semisimplicity assumption.

We will compute the abelianisation of the centraliser for some values of $w$ explicitly in Chapter 15, and we will see $[\![\rho_{a,w}]\!] = 0 \in H_1(C(\rho_{a,w}))$. Then Theorem 14.9 will follow from Theorem 14.2. There is another proof using the parallelogram law, which we give now:

*Proof of Theorem 14.9:* We define the automorphisms $\alpha$ and $\beta$ of $F_{n+k+1}$ by

$$\begin{aligned} \alpha\colon\ a &\mapsto a, & \beta = \rho_{a,w}\colon\ a &\mapsto aw,\\ b_i &\mapsto b_i, & b_i &\mapsto b_i,\\ c_1 &\mapsto c_1 w, & c_1 &\mapsto c_1,\\ c_2 &\mapsto c_2 w^{-1}, & c_2 &\mapsto c_2 \end{aligned}$$

and fixing $c_3, \ldots, c_n$. It can be computed that

$$\begin{aligned} \alpha\beta &= \beta\alpha,\\ \rho_{a,c_1}^{-1}\alpha\rho_{a,c_1} &= \alpha\beta,\\ \rho_{a,c_2}^{-1}\alpha\rho_{a,c_2} &= \alpha\beta^{-1}. \end{aligned}$$

Hence $\alpha$, $\alpha\beta$, and $\alpha\beta^{-1}$ act by isometries of the same translation length. Theorem 14.6 proves $|\beta| = 0$. □

## 14.4 Application to group homomorphisms

The following is stated in [10]: Let $\Sigma_g$ be an orientable surface of genus $g \geq 2$ without boundary. There is an action of the mapping class group $MCG(\Sigma_g)$ on a complete CAT(0) space (the completion of the Teichmüller space endowed with the Weil-Petersson metric) such that the elements acting with zero translation length are exactly the roots of multitwists. Here $\gamma$ is called a root of multitwist if some power $\gamma^n$ is a product of Dehn twists about disjoint simple closed curves. Since we can compose this action with homomorphisms $\mathrm{Aut}(F_n) \to MCG(\Sigma_g)$, Theorem 14.9 has the following as an immediate consequence:

**Corollary 14.10.** *If $n \geq 2$ and $\rho_{a,w}$ are as above, then every group homomorphism $\mathrm{Aut}(F_{n+k+1}) \to Mod(\Sigma_g)$ maps $\rho_{a,w}$ to a root of a multitwist.* □

This corollary has already been known by Bridson using the fact that the above action of $Mod(\Sigma_g)$ is by semisimple isometries.

We now show how Theorem 14.9 gives information about linear representations of $\mathrm{Aut}(F_{n+k+1})$. Let $P(m, \mathbb{R})$ be the space of all symmetric, positive definite $m \times m$-matrices with real entries. In Chapter II.10 of [11], there is a construction of a Riemannian metric of non-positive sectional curvature on the manifold $P(m, \mathbb{R})$. In particular, $P(m, \mathbb{R})$ is a CAT(0) space. The general linear group $GL(m, \mathbb{R})$ acts on $P(m, \mathbb{R})$ by isometries according to the formula $g.A := gAg^\top$ for $g \in GL(m, \mathbb{R})$ and $A \in P(m, \mathbb{R})$. It can be checked that every $g \in \mathrm{GL}(m, \mathbb{R})$ with translation length $|g| = 0$ in this action only has (complex) eigenvalues on the unit circle. As we can compose any homomorphism $\mathrm{Aut}(F_{n+k+1}) \to \mathrm{GL}(m, \mathbb{R})$, Theorem 14.9 implies:

**Corollary 14.11.** *If $n \geq 2$, then in every linear representation $\mathrm{Aut}(F_{n+k+1}) \to \mathrm{GL}(m, \mathbb{R})$, all eigenvalues of the element $\rho_{a,w}$ lie on the complex unit circle.* □

## 14.5 Construction of positive translation lengths

The goal of this section is to show that $\rho_{a,b}$, $\rho_{a,bcb^{-1}c^{-1}}$, and $\rho_{a,b^2c^2}$ can act hyperbolically when $\mathrm{Aut}(F_3)$ acts by isometries on a complete CAT(0) space. We know by Theorem 14.9 that $\rho_{a,b}$ has to act by zero translation length in (not necessarily semisimple) actions of $\mathrm{Aut}(F_n)$, $n \geq 4$. The elements $\rho_{a,bcb^{-1}c^{-1}}$ and $\rho_{a,b^2c^2}$ have to act by zero translation length for $n \geq 5$, but the author does not know whether there are actions of $\mathrm{Aut}(F_4)$ with positive translation length.

If $G$ is any group and $H$ a subgroup of finite index $d$, then an isometric action of $H$ on a metric space $X$ can be "induced up" to an action of $G$ on the cartesian power $X^d$ (cf. Section 2.1 of [10]). An element $g \in G$ acts by positive translation length (or hyperbolically) on $X^d$ if some (or equivalently every) non-trivial power $g^m \in H$ acts by positive translation length (or hyperbolically respectively) on $X$.

We now construct actions of $\mathrm{Aut}(F_3)$ with positive translation length, where the one for $\rho_{a,b_1}$ is closely related to that in Section 6 of [10].

Write $F_3 = \langle a, b, c\rangle$. Consider the index 2 subgroup freely generated by $a$, $b$, $c^2$, $cac^{-1}$, and $cbc^{-1}$. We denote by $H$ its stabiliser in $\mathrm{Aut}(F_3)$, which is a finite index subgroup because there are only finitely many index two subgroups in $F_3$. The non-trivial Deck transformation of the two-sheeted covering of the standard rose has a three dimensional eigenspace of eigenvalue $+1$ in first homology with $\mathbb{Q}$-coefficients, and a two dimensional eigenspace of eigenvalue $-1$ with basis $a - cac^{-1}$, $b - cbc^{-1}$. Every element of $H$ now acts on this $-1$-eigenspace, giving rise to a homomorphism $H \to \mathrm{GL}(2, \mathbb{Z})$. It can be checked that $\rho_{a,b}$, $\rho_{a,bcb^{-1}c^{-1}}$, and $\rho_{a,b^2c^2}$ belong to $H$, and that their images in $\mathrm{GL}(2, \mathbb{Z})$ have infinite order. Since $\mathrm{GL}(2, \mathbb{Z})$ is virtually free, we can construct an isometric $\mathrm{CAT}(0)$ action such that every non-torsion element of $\mathrm{GL}(2, \mathbb{Z})$ acts hyperbolically. Hence we get an action of $H$ such that any of the three aforementioned $\rho_{a,w}$ acts hyperbolically. Inducing the action of $H$ to one of $\mathrm{Aut}(F_3)$, we obtain the desired action.

# 15 Simplifications of presentations

To apply Theorem 14.2 to the translation lengths in some isometric CAT(0) action of $\mathrm{Aut}(F_n)$ or $\mathrm{Out}(F_n)$, we have to compute the abelianisation of the centraliser. Although finite presentations can often be computed algorithmically (cf. [31] for the ordinary Dehn twist case), it is far from being obvious to check the conditions of Theorem 14.2. We will discuss the necessary simplifications of the presentations in this chapter for the special case of right translations $\rho_{a,w}$, which includes the case of standard Nielsen automorphisms.

## 15.1 Right translations as Dehn twists

Let $\Gamma$ be the graph with one vertex $v$ and one loop $e$ (that is two oriented edges $e$ and $\overline{e}$). We define $G_v$ to be the free group with basis $W, b_1, \ldots, b_k, c_1, \ldots, c_n$. Let $w$ be any non-trivial element in the free group $B$ generated by $b_1, \ldots, b_k$ such that $w$ is not a proper power. We take $G_e$ to be infinite cyclic with generator $r$, and we define the attaching maps by $f_e(r) = w$ and $f_{\overline{e}}(r) = W$. We denote this graph of groups by $\mathcal{G}$.

For the first glance, a notation $b_{k+1}, \ldots, b_{k+n}$ might seem more convenient for the basis elements $c_1, \ldots, c_n$. However, we will often assume that $B$ is the natural free factor $B(w)$, and this will only make sense in the present notation.

Since we only have one vertex here, every word representing an element in $\Pi(\mathcal{G})$ is connected. Thus the fundamental group is $\pi_1(\mathcal{G}, v) = \Pi(\mathcal{G})$. It is generated by $t_e =: a$, $W$, and all $b_i$ and $c_j$, subject to the relation $awa^{-1} = W$. Hence it is free with basis $a, b_1, \ldots, b_k, c_1, \ldots, c_n$. Let $p \geq 1$. We define a Dehn twist $D$ by $\gamma_e = r^{-p}$ and $\gamma_{\overline{e}} = 1$. Then $\delta_D(e) = f_e(\gamma_e) = w^{-p}$ and $\delta_D(\overline{e}) = f_{\overline{e}}(\gamma_{\overline{e}}) = 1$. Then $D_{*v}$ is the right translation $\rho := \rho_{a,w^p}$ sending $a$ to $aw^p$ and fixing the other basis elements of $F_{n+k+1}$.

It is easily checked that $D$ is efficient, in particular pointedly efficient. The short exact sequence (81) on page 128 for the centraliser now simplifies to

$$1 \to \langle \rho \rangle \to C^0(\rho) \to \mathrm{Aut}(G_v, \mathcal{C}_v) \to 1, \tag{82}$$

where $\mathcal{C}_v = \{[W], [w]\}$ by (79).

Note that this sequence is completely independent of $p$. From now on we always assume $p = 1$, that is we are only considering right translations $\rho_{a,w}$ such that $w$ is not a proper power.

To study the finite index subgroup $C^0(\rho) = C^0(D_{*v})$ of $C(\rho)$, we need the following lemma.

**Lemma 15.1.** $C^0(\rho) = \{H_{*v} | H \in \mathrm{Aut}^0(\mathcal{G}), H_{*v}\rho = \rho H_{*v}\}$.

*Proof.* By definition of $C^0(D_{*v})$ in Section 13.1, we have

$$C^0(\rho) = \big\{H_{*v} \big| H \in \mathrm{Aut}^0(\mathcal{G}), HD = DH, H_e = 1\big\}.$$

The inclusion "$\subset$" of the assertion is clear. Conversely, let $H \in \mathrm{Aut}^0(\mathcal{G})$ such that $H_{*v}$ and $\rho = D_{*v}$ commute. By Lemma 7.15(i), the elements $\gamma'_e$ and $\gamma'_{\overline{e}}$ defining the Dehn twist $HDH^{-1}$ satisfy $\gamma'_e = H_e(\gamma_e)$ and $\gamma'_{\overline{e}} = H_e(\gamma_{\overline{e}}) = 1$, so we obtain $z'_e := \gamma'^{-1}_{\overline{e}} = H_e(\gamma_{\overline{e}})^{-1} = H_e(z_e)$. As $D_{*v} = (HDH^{-1})_{*v}$, Proposition 5.4 of [13] shows that the twistors of these Dehn twists coincide, so $H_e(z_e) = z'_e = z_e$ and $H_e = 1$. Then Lemma 7.15 shows $D = HDH^{-1}$, so $H \in C^0(\rho)$. This proves "$\supset$". □

Recall that an element of a free group is called *primitive* if it is part of a basis.

**Lemma 15.2.** *If $w$ is primitive in $B = \langle b_1, \ldots, b_k \rangle$, then $C^0(\rho)$ is a subgroup of index 2 in $C(\rho)$. Otherwise $C^0(\rho) = C(\rho)$.*

*Proof.* As $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}/2$ generated by the graph automorphism swapping $e$ with $\overline{e}$, it follows that $\mathrm{Aut}^0(\mathcal{G})$ has index at most two in $\mathrm{Aut}(\mathcal{G})$. Using the description in Lemma 15.1, it follows from Theorem 10.17 that $C^0(\rho)$ has index at most two in $C(\rho)$.

Any $\alpha \in \mathrm{Aut}(B)$ extends to $\overline{\alpha} \in \mathrm{Aut}(F_{n+k+1})$ by the identity on $a$ and $c_1, \ldots, c_n$. As $\overline{\alpha}\rho_{a,w}\overline{\alpha}^{-1} = \rho_{a,\alpha(w)}$, the centralisers of $\rho_{a,w}$ and $\rho_{a,\alpha(w)}$ are isomorphic.

If $w$ is primitive, then we have w.l.o.g. $w = b_1 =: b$ and $k = 1$. Consider the automorphism $\theta \in C(\rho_{a,b})$ given by

$$\theta(x) = \begin{cases} a^{-1}, & \text{if } x = a, \\ ab^{-1}a^{-1}, & \text{if } x = b, \\ c_i, & \text{if } x = c_i. \end{cases}$$

It is not in $C^0(\rho_{a,b})$ because $\theta(t_e) = \theta(a) \notin G_v t_e G_v$.

Conversely, suppose that $C^0(\rho) \subsetneq C(\rho)$. Then there is $H \in \mathrm{Aut}(\mathcal{G})$ such that the underlying graph automorphism swaps $e$ with $\overline{e}$. By Definition 2.4(7),

$$H_v(w) = H_v(f_e(r)) = \delta_H(e) f_{\overline{e}}(H_e(r))\delta_H(e)^{-1} = \delta_H(e) W^{\pm 1} \delta_H(e)^{-1}.$$

The right hand side is primitive, which is possible only if the left hand side is. Therefore $w$ is primitive in $G_v$. Lemma 11.14 shows that $w$ is also primitive in the free factor $B$ of $G_v$. □

## 15.2 Short exact sequences and presentations

Suppose we are given a short exact sequence

$$1 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 1$$

of groups. Assume we have a finite presentation of the group $A$ with generating set $X = \{x_1, \ldots, x_k\}$ and set of relations $R$ and a finite presentation of $C$ with generating set $Z = \{z_1, \ldots, z_m\}$ and relations $S$. We fix $\tilde{z}_1, \ldots, \tilde{z}_m$ such that $\pi(\tilde{z}_j) = z_j$. Taking the union of $\iota(X) = \{\iota(x_1), \ldots, \iota(x_n)\}$ and $\tilde{Z} = \{\tilde{z}_1, \ldots, \tilde{z}_m\}$ gives a generating set for $B$.

We need three types of relations. We first have the relations in $B$ given by $R$, the set of relations in $A$. We call these the *kernel relations*.

We may lift the set of relations $S$ to a set of words $\tilde{S}$ on $\tilde{Z}$ in the obvious way. Each element of $\tilde{S}$ is mapped to the identity under $\pi$, so lies in $\iota(A)$. For each element $\tilde{s} \in \tilde{S}$ there is a word $w_s$ in $\iota(A)$ such that $w_s = \tilde{s}$ in $B$. We call the set $\{w_s = \tilde{s} | s \in S\}$ the set of *lifted relations*.

Finally note that for $\epsilon \in \{1, -1\}$ the element $\tilde{z}_j^{\epsilon}\iota(x_i)\tilde{z}_j^{-\epsilon}$ is mapped to 1 by $\pi$, so $w_{i,j,\epsilon} = \tilde{z}_j^{\epsilon}\iota(x_i)\tilde{z}_j^{-\epsilon}$ in $B$ for some word $w_{i,j,\epsilon}$ in $\iota(A)$. We say the set of all such relations is called the set of *conjugation relations*. The following is an exercise in combinatorial group theory:

**Proposition 15.3.** *$B$ has a finite presentation given by the generating set $\iota(X) \sqcup \tilde{Z}$ with the kernel relations, lifted relations, and conjugation relations.* □

## 15.3 Centralisers of right translations

We will apply Proposition 15.3 to the short exact sequence (82) on page 141. The set $\mathcal{C}_v = \{[W], [w]\}$ consists of two conjugacy classes, one of which is the basis element $W$, and the other lies in a complementary free factor. To compute the abelianisation of $C(\rho)$, we are interested in presentations which are reasonably small, so we should avoid using the McCool complex for $\mathrm{Aut}(G_v, \mathcal{C}_v)$ directly.

We now discuss how to reduce the study of centralisers of arbitrary right translations to those by rigid elements in a free factor. Recall that $\rho_{a,w}$ and $\rho_{a,\alpha(w)}$ are conjugate for every automorphism $\alpha \in \mathrm{Aut}(B)$. If $w$ does not involve one of $b_1, \ldots, b_k$, we may rename that symbol to $c_{n+1}$. This way we can assume that $B$ is the natural free factor $B(w)$ described in Proposition 11.11.

By Proposition 11.12 we have a short exact sequence

$$1 \to \mathrm{Aut}(G_v, [W], [B]) \to \mathrm{Aut}(G_v, [W], [w]) \xrightarrow{\pi} \mathrm{Out}(B, [w]) \to 1 \tag{83}$$

where the left hand term denotes the group of automorphisms of $G_v$ fixing $W$ up to conjugacy and $B$ up to uniform conjugacy, i.e. acting on $B$ as $\mathrm{ad}_x$ for some $x \in G_v$.

In our examples, we will compute a presentation of $\mathrm{Out}(B, [w])$ by the McCool complex. We assume that $w$ is not a proper power and not primitive. Using the sequence (83), we only have to put this together with the computations for rigid elements, where we deal with fixing free factors up to uniform conjugacy. This will lead to a presentation for the middle term in (83) and hence for $C(\rho_{a,w}) = C^0(\rho_{a,w})$ by (82) on 141.

We can also compute $C(\rho_{a,w})$ more directly in terms of the special case of rigid $w$ and $\mathrm{Out}(B, [w])$. To this end, we put the short exact sequences together to the diagram

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] & & \\
1 \arrow[r] & C(\rho_{a,B}) \arrow[r] \arrow[d] & C(\rho_{a,w}) \arrow[r] \arrow[d] & \mathrm{Out}(B,[w]) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{Aut}(G_v,[W],[B]) \arrow[r] \arrow[d] & \mathrm{Aut}(G_v,[W],[w]) \arrow[r] \arrow[d] & \mathrm{Out}(B,[w]) \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}
\tag{84}
$$

with exact rows and columns. Here

$$C(\rho_{a,B}) := \big\{ f \in \mathrm{Aut}(F_{n+k+1}) \big| f \in C(\rho_{a,w}), f \text{ fixes } B \text{ up to uniform conjugacy.} \big\}.$$

Observe that the first two non-trivial columns coincide if $w$ is rigid in the free factor $B$.

Exactness of the middle column follows from (82) on page 141. The left hand column is then exact by construction. The bottom row is the exact sequence (83). Exactness of the middle row follows by an easy diagram chase.

## 15.4 Stabilisers of conjugacy classes of rigid elements

In this section we state an important presentation, whose case $k = 1$ coincides with $k = 2$ in Proposition 7.1 of [19]. Recall that, in our notation, $k$ denotes the rank of the free factor $B$.

In the following, we use the symbol $P_{i,j}$ to denote the automorphism of either $F_{n+k+1} = \langle a, b_1, \ldots, b_k, c_1, \ldots, c_n \rangle$ or $G_v = \langle W, b_1, \ldots, b_k, c_1, \ldots, c_n \rangle$ which permutes the basis elements $c_i$ and $c_j$. Similarly, $I_i$ denotes the automorphism mapping $c_i$ to $c_i^{-1}$ and fixing the other basis elements.

If $y$ and $z$ are elements of a fixed basis of a free group and $\epsilon \in \{\pm 1\}$, then $(y^\epsilon; z)$ is the automorphism fixing all basis elements different from $y$ and sending $y$ to $yz$ if $\epsilon = 1$, and to $z^{-1}y$ if $\epsilon = -1$. Moreover, $(y^\pm; z)$ is the automorphism $y \mapsto z^{-1}yz$ fixing the other basis elements. Furthermore

$$(\underline{b}^\pm; z) := (b_1^\pm; z) \ldots (b_k^\pm; z).$$

We sometimes identify the symbol $\underline{b}^\pm$ with the set $\{b_1, \ldots, b_k, b_1^{-1}, \ldots, b_k^{-1}\}$ and $W^\pm$ with $\{W, W^{-1}\}$. We warn the reader that this notation does *not* follow McCool's convention $z \in A$, $z^{-1} \notin A$ whenever denoting an automorphism by $(A; z)$.

**Theorem 15.4.** *The following is a presentation for* $\mathrm{Aut}(G_v, [W], [B])$*, the group of automorphisms of* $G_v$ *fixing* $W$ *up to conjugacy and the free factor* $B$ *up to uniform conjugacy:*

*Generators:* $P_{i,j}$, $I_i$, $(W^\pm; b_i)$, $(W^\pm; c_i)$, $(\underline{b}^\pm; W)$, $(\underline{b}^\pm; b_i)$, $(\underline{b}^\pm; c_i)$, $(c_i^\epsilon; W)$, $(c_i^\epsilon; b_j)$, $(c_i^\epsilon; c_j)$*, whenever these symbols make sense with* $\epsilon \in \{\pm 1\}$.

*Relations: For* $u_\bullet \in \{W^\pm, \underline{b}^\pm, c_1, \ldots, c_n, c_1^{-1}, \ldots, c_n^{-1}\}$, $z_\bullet \in \{W, b_1, \ldots, b_k, c_1, \ldots, c_n\}$

| | |
|---|---|
| *Z1.1* | *Relations in* $\mathrm{Aut}\langle c_1, \ldots, c_n\rangle$ |
| *Z1.2* | $(\underline{b}^\pm; b_1) = 1$ *if* $k = 1$ |
| *Z2* | $(u_1; z_1)(u_2; z_2) = (u_2; z_2)(u_1; z_1)$ *for* $z_1 = z_2$ *or* $(u_1 \cup \{z_1^{\pm 1}\}) \cap (u_2 \cup \{z_2^{\pm 1}\}) = \varnothing$ |
| *Z3.1* | $(W^\pm; c_j)P_{j,l} = P_{j,l}(W^\pm; c_l)$ |
| *Z3.2* | $(\underline{b}^\pm; c_j)P_{j,l} = P_{j,l}(\underline{b}^\pm; c_l)$ |
| *Z3.3* | $(W^\pm; c_j)I_j = I_j(W^\pm; c_j^{-1})$ |
| *Z3.4* | $(\underline{b}^\pm; c_j)I_j = I_j(\underline{b}^\pm; c_j^{-1})$ |
| *Z3.5* | $P_{j,l}$ *and* $I_j$ *commute with* $(\underline{b}^\pm; W)$, $(\underline{b}^\pm; b_i)$, $(W^\pm; b_i)$ |
| *Z3.6* | $(c_j^\epsilon; z)P_{j,l} = P_{j,l}(c_l^\epsilon; z)$ *for* $z = W$ *or* $z = b_i$ |
| *Z3.7* | $(c_j; z)I_j = I_j(c_j^{-1}; z)$ *for* $z = W$ *or* $z = b_i$ |
| *Z4.1* | $(u; c_j^{-\epsilon})(c_j^\epsilon; z)(u; c_j^\epsilon) = (u; z)(c_j^\epsilon; z)$ *for* $(u; z) \neq (\underline{b}^\pm; b_i^\eta)$ *such that all symbols denote generators or inverses* |
| *Z4.2* | $(W^\pm; z^{-\epsilon})(u; W)(W^\pm; z^\epsilon) = (u; z^\epsilon)(u; W)(u; z^{-\epsilon})$ *for* $(u; z) \neq (\underline{b}^\pm; b_i^\eta)$ |
| *Z4.3* | $(\underline{b}^\pm; z^{-\epsilon})(u; b_i)(\underline{b}^\pm; z^\epsilon) = (u; z^\epsilon)(u; b_i)(u; z^{-\epsilon})$ *for* $u \neq \underline{b}^\pm$ *such that all symbols denote generators or inverses* |
| *Z5.1* | $(\underline{b}^\pm; z)(\underline{b}^\pm; b_i) = (\underline{b}^\pm; b_i)(\underline{b}^\pm; z)$ *for* $z = c_j$ *or* $z = W$ |
| *Z5.2* | $(c_j^{-\eta}; W^\epsilon)(W^\pm; c_j^\eta) = (W^\pm; c_j^\eta)(c_j^\eta; W^{-\epsilon})$ |
| *Z5.3* | $(c_j^{-\eta}; b_i^\epsilon)(\underline{b}^\pm; c_j^\eta) = (\underline{b}^\pm; c_j^\eta)(\underline{b}^\pm; b_i^{-\epsilon})(c_j^\eta; b_i^{-\epsilon})$ |

Here we read compositions from right to left. We warn the reader that the articles [19], [26], and [27] use the opposite convention.

The proof of Proposition 15.4 is lengthy and not needed elsewhere in this work. Therefore we postpone it to Section 15.9. We first exhibit a presentation for $C^0(\rho_{a,w})$ in Corollary 15.5.

For $z \in F_{n+k+1}$ let $\gamma_z$ be defined by

$$\gamma_z(x) = \begin{cases} az, & \text{if } x = a, \\ z^{-1}bz, & \text{if } x = b, \\ c_i, & \text{if } x = c_i. \end{cases}$$

For notational convenience, we sometimes write $(\gamma_\bullet; z)$ instead of $\gamma_z$.

**Corollary 15.5.** *Let* $w \in F_{n+k+1}$ *be rigid in the free factor* $B$ *of rank* $k \geq 2$*. Then a presentation for* $C(\rho_{a,w})$ *is obtained in the following way: The generators are the same as in Theorem 15.4 with one additional generator* $\rho := \rho_{a,w}$*. When we write* $(a^{-1}; z)$ *instead of* $(W^\pm; z)$ *and* $\gamma_z$ *instead of* $(\underline{b}^\pm; z)$*, then the relations are the following:*

- *Relations in Theorem 15.4, except for adding $\rho^\epsilon$ in Z5.2:*

$$(c_j^{-\eta}; W^\epsilon)(a^{-1}; c_j^\eta) = (a^{-1}; c_j^\eta)(c_j^\eta; W^{-\epsilon}) \circ \rho^\epsilon,$$

- *$\rho$ commutes with all generators.*

*Proof.* We apply Proposition 15.3 to the central extension (82) on page 141. We first have to lift the generators of $\mathrm{Aut}(G_v, \mathcal{C}_v)$ to $C^0(\rho) = C(\rho)$. Since the surjection in the short exact sequence (82) is given by restriction to the vertex group $G_v$, lifting means extending automorphisms from $G_v$ to all of $F_{n+k+1}$. The generators $P_{i,j}$, $I_i$, $(c_i^\epsilon; W)$, $(c_i^\epsilon; b_j)$, and $(c_i^\epsilon; c_j)$ are lifted to the elements of $C(\rho)$ called by the same name. The generators $(W^\pm; z)$ and $(\underline{b}^\pm; z)$ can be lifted to $(a^{-1}; z)$ and $\gamma_z$ respectively.

A presentation for the right hand term $\mathrm{Aut}(G_v, [W], [w])$ of (82) on page 141 is given by Theorem 15.4. In the lifted relations for $C(\rho)$, the only possible changes after lifting the generators are that powers of $\rho$ could show up. A straightforward calculation shows that these powers are as claimed: The only non-vanishing one is in relation Z5.2.

Finally, we have to add the conjugation relations, which prescribe how to move $\rho$ along other generators. As the extension is central, this amounts to saying that $\rho$ commutes with all other generators. □

**Corollary 15.6.** *Let $w$ be a rigid element in the free factor $B$ of rank $k \geq 2$ in $F_{n+k+1}$. Then there are the following abelianisations:*

$$H_1(\mathrm{Aut}(G_v, [W], [w])) \cong \begin{cases} \mathbb{Z}^{2k+1}, & \text{if } n = 0, \\ \mathbb{Z}^k \oplus (\mathbb{Z}/2)^4, & \text{if } n = 1, \\ (\mathbb{Z}/2)^2, & \text{if } n = 2, \\ \mathbb{Z}/2, & \text{if } n \geq 3, \end{cases}$$

$$H_1(C(\rho_{a,w})) \cong \begin{cases} \mathbb{Z}^{2k+2}, & \text{if } n = 0, \\ \mathbb{Z}^{k+1} \oplus (\mathbb{Z}/2)^3, & \text{if } n = 1, \\ (\mathbb{Z}/2)^2, & \text{if } n = 2, \\ \mathbb{Z}/2, & \text{if } n \geq 3. \end{cases}$$

*The class $[\![\rho]\!] \in H_1(C(\rho_{a,w}))$ is one of the generators for $n = 0$, twice a generator of the free summand in the case $n = 1$, and $[\![\rho_{a,w}]\!] = 0$ when $n \geq 2$.*

*Remark* 15.7. For later reference, we record some of the facts which will be clear from the proof of Corollary 15.6:

(i) When $n = 0$, the abelianisation $H_1(C(\rho))$ is free abelian with basis $[\![a^{-1}; b_i]\!]$, $[\![\gamma_W]\!]$, $[\![\gamma_{b_i}]\!]$, $[\![\rho]\!]$.

(ii) For $n = 1$, a basis of the free abelian summand is given by $[\![c; W]\!]$ and $[\![c; b_i]\!]$. Important relations are $[\![\gamma_{b_i}]\!] = -2[\![c; b_i]\!]$, $[\![a^{-1}; b_i]\!] = 0$, and $[\![\rho]\!] = 2[\![c; W]\!]$.

(iii) In the case $n \geq 2$, we have $[\![a^{-1}; b_i]\!] = [\![\gamma_{b_i}]\!] = [\![c_j^\epsilon; b_i]\!] = 0$.

*Proof.* It suffices to verify the assertion about $H_1(C(\rho_{a,w}))$. The first abelianisation is then obtained by taking the additional relation $[\![\rho]\!] = 0$, for this is already true before taking abelianisations (cf. Theorem 15.4 and Corollary 15.5).

We have to abelianise the presentation in Corollary 15.5: We first consider the case $n = 0$, where there are no $c_i$. The only generators are $[\![a^{-1}; b_i]\!]$, $[\![\gamma_W]\!]$, $[\![\gamma_{b_i}]\!]$, and $[\![\rho]\!]$. All relations are void or trivial in the abelianisation, whence the assertion.

Next we consider $n = 1$: For simplicity we write $c := c_1$ and $I := I_1$. The generators of $C(\rho)$ are $I$, $(a^{-1}; b_i)$, $(a^{-1}; c)$, $\gamma_{awa^{-1}}$, $\gamma_{b_i}$, $\gamma_c$, $(c^\epsilon; awa^{-1})$, $(c^\epsilon; b_j)$, and $\rho$. The relations in Theorem 15.4 and Corollary 15.5 give rise to the following in the abelianisation:

| | | | |
|---|---|---|---|
| Z1: | $2[\![I]\!] = 0,$ | Z3.3: | $2[\![a^{-1}; c]\!] = 0,$ |
| Z3.4: | $2[\![\gamma_c]\!] = 0,$ | Z3.7: | $[\![c; awa^{-1}]\!] = [\![c^{-1}; awa^{-1}]\!],$ |
| Z3.7: | $[\![c; b_i]\!] = [\![c^{-1}; b_i]\!],$ | Z4.1: | $[\![a^{-1}; b_i]\!] = 0,$ |
| Z4.1: | $[\![\gamma_{awa^{-1}}]\!] = 0,$ | Z5.2: | $2[\![c; awa^{-1}]\!] = [\![\rho]\!],$ |
| Z5.3: | $2[\![c; b_i]\!] + [\![\gamma_{b_i}]\!] = 0.$ | | |

All other relations are either trivial in the abelianisation, or they can be deduced from this table. This proves the assertion in the case $n = 1$.

From now on we tackle $n \geq 2$: Recall that $C(\rho)$ has the generators $P_{i,j}$, $I_i$, $(a^{-1}; b_i)$, $(a^{-1}; c_i)$, $\gamma_{awa^{-1}}$, $\gamma_{b_i}$, $\gamma_{c_i}$, $(c_i^\epsilon; awa^{-1})$, $(c_i^\epsilon; b_j)$, $(c_i^\epsilon; c_j)$, and $\rho$. Checking all relations, it follows that there is a group homomorphism

$$H_1(C(\rho)) \to H_1(\mathrm{Aut}(F_n))$$

sending the generators $[\![P_{i,j}]\!]$, $[\![I_i]\!]$, and $[\![c_i^\epsilon; c_j]\!]$ to the corresponding generators in $H_1(\mathrm{Aut}(F_n))$, and all other generators of $H_1(C(\rho))$ are mapped to zero. We claim that this map is an isomorphism. To see this, we have to verify that all generators mapped to zero already vanish in $H_1(C(\rho))$. Relation Z4.1 shows that all $[\![u; z]\!] = 0$ for $(u; z) \neq \gamma_{b_i}^\eta, (c_i^\epsilon; c_j)$. Moreover, $[\![\gamma_{b_i}]\!] = 0$ by Z5.3 and $[\![\rho]\!] = 0$ by Z5.2. This proves $H_1(C(\rho)) \cong H_1(\mathrm{Aut}(F_n))$ for $n \geq 2$. By abelianising a presentation of $\mathrm{Aut}(F_n)$ (from [26], [29], or [30] for instance), the reader can check that this is as asserted. □

## 15.5 Computation strategy for centralisers of right translations

We now discuss how to obtain a presentation for $C^0(\rho_{a,w})$. As input, we need a finite presentation for $\mathrm{Out}(B, [w])$, where $B = B(w)$ is the natural free factor of $w$. We assume that the rank $k$ of $B$ is at least 2.

Before we spell out the strategy, we need some notational preparation: Some of the generators will be $(a^{-1}; b_i)$, $\gamma_{b_i}$, and $(c_j^\epsilon; b_i)$. Sometimes we will write down a symbol $(u; x)$ for some $x \in B$. This is an actual generator only if $x = b_i$ for some $i$. Otherwise, this is meant to be an abbreviation for a product of some $(u; b_i)$ and their inverses, according to the following

**Lemma 15.8.** *For $x, y \in B$ we have:*

*(i)* $(a^{-1}; x)(a^{-1}; y) = (a^{-1}; xy)$,

*(ii)* $(c_m^\epsilon; x)(c_m^\epsilon; y) = (c_m^\epsilon; xy)$,

*(iii)* $\gamma_x \gamma_y = \gamma_{yx}$,

*(iv)* $(u; x^{-1}) = (u; x)^{-1}$, *where* $u = a^{-1}$, $u = c_m^\epsilon$, *or* $u = \gamma_\bullet$. □

Recall that we read compositions from right to left. Note that the composition order of $x$ and $y$ is preserved in (i) and (ii), but reversed in (iii).

**Strategy:** We need a presentation of $\mathrm{Out}(B, [w])$ as input. Using the presentation in Corollary 15.5, we apply Proposition 15.3 to the middle row in the diagram (84) on page 144. We lift the given generators of $\mathrm{Out}(B, [w])$ to $\alpha_1, \ldots, \alpha_p \in \mathrm{Aut}(B)$ fixing $w$ genuinely, i.e. $\alpha_i(w) = w$. We extend each $\alpha_i$ to $\mathrm{Aut}(F_{n+k+1})$ by setting $\alpha_i(a) = a$ and $\alpha_i(c_j) = c_j$. Then these $\alpha_i$ commute with $\rho_{a,w}$. The presentation for $C^0(\rho_{a,w})$ is then obtained as follows.

<u>Generators:</u> $P_{i,j}$, $I_i$, $(a^{-1}; b_i)$, $(a^{-1}; c_i)$, $\gamma_{awa^{-1}}$, $\gamma_{b_i}$, $\gamma_{c_i}$, $(c_i^\epsilon; awa^{-1})$, $(c_i^\epsilon; b_j)$, $(c_i^\epsilon; c_j)$, $\rho_{a,w}$, $\alpha_i$.

<u>Relations:</u>

(i) Relations for $C(\rho_{a,B})$ (cf. Corollary 15.5),

(ii) Lifted relations: for every relator $\alpha_{i_1}^{\epsilon_1} \ldots \alpha_{i_r}^{\epsilon_r}$ in the given presentation of the group $\mathrm{Out}(B, [w])$, there is a relation $\alpha_{i_1}^{\epsilon_1} \ldots \alpha_{i_r}^{\epsilon_r} f = 1$ for a unique $f \in C(\rho_{a,B})$,

(iii) $\alpha_l \rho = \rho \alpha_l$,

(iv) $\alpha_l$ commutes with $P_{i,j}, I_i$, $(a^{-1}; c_i)$, $\gamma_{c_i}$, $\gamma_W$, $(c_i^\epsilon; W)$, $(c_i^\epsilon; c_j)$,

(v) $\alpha_l \circ (u; b_i) = (u; \alpha_l(b_i))\alpha_l$, where $u = a^{-1}$ or $u = \gamma_\bullet$ or $u = c_j^\epsilon$. (The meaning of the abbreviation $(u; \alpha_l(b_i))$ is made precise by Lemma 15.8 above.)

*Example* 15.9. Let us consider the above strategy in the special case that $w$ is rigid in its natural free factor $B$. By Corollary 11.6 the group $\mathrm{Out}(B, [w])$ is trivial. We use the empty presentation for it. Then the above strategy does not lead to any generators $\alpha_i$, so the generating set is the same as in Corollary 15.5. The relations given by the strategy are no more than the ones in that corollary because all relations involving some $\alpha_i$ are void. Hence we recover exactly the presentation in Corollary 15.5.

*Remark* 15.10. The element $f \in C(\rho_{a,B})$ showing up in (ii) above is actually a power of $\gamma_w \rho^{-1}$. To see this, recall that $\beta := \alpha_{i_1}^{\epsilon_1} \ldots \alpha_{i_r}^{\epsilon_r}$ is an inner automorphism on $B$ and fixes $a$ and all $c_j$. By our choice of the representatives $\alpha_i$ of outer automorphism classes, we know that all of them fix $w \in B$. Hence $\beta(w) = w$ as well. Therefore $\beta|_B$ is conjugation by an element in the centraliser of $w \in B$, hence a power of $w$. Note that conjugation by $w$ on the free factor $B$ is exactly the map $\gamma_w \rho^{-1}$.

## 15.6 Centraliser of a Nielsen automorphism

We now discuss the centraliser $C(\rho_{a,b})$ of the standard Nielsen automorphism $\rho = \rho_{a,b} : a \mapsto ab$. By Lemma 15.2, the subgroup $C^0(\rho)$ has index two in $C(\rho)$. An element in $C(\rho) \setminus C^0(\rho)$ is given by

$$\theta(x) = \begin{cases} a^{-1}, & \text{if } x = a, \\ ab^{-1}a^{-1}, & \text{if } x = b, \\ c_i, & \text{if } x = c_i. \end{cases}$$

**Theorem 15.11** ([31], Theorem 8.2)**.** *The centraliser of the Nielsen automorphism $\rho \in \mathrm{Aut}(F_n)$ has the following presentation:*
*Generators:*

| | |
|---|---|
| $P_{i,j}$ | *for* $1 \le i,j \le n-2$ *and* $i \neq j$, |
| $I_i$ | *for* $1 \le i \le n$, |
| $(c_i^\epsilon; z)$ | *for* $1 \le i \le n-2$, $\epsilon = \pm 1$, *and* $c_i \neq z \in \{aba^{-1}, b, c_1, \ldots, c_{n-2}\}$, |
| $\gamma_z$ | *for* $z \in \{aba^{-1}, c_1, \ldots, c_{n-2}\}$, |
| $(a^{-1}; z)$ | *for* $z \in \{b, c_1, \ldots, c_{n-2}\}$, |
| $\rho$, | |
| $\theta$. | |

*Relations: For every $z, z_i \in \{aba^{-1}, b, c_1, \ldots, c_{n-2}\}$ and $u, u_i \in \{c_1^{\pm 1}, \ldots, c_{n-2}^{\pm 1}, a^{-1}, \gamma_\bullet\}$:*

| | |
|---|---|
| *R1* | *Relations in* $\mathrm{Aut}(F_{n-2})$ *for* $\{(c_i^\epsilon; c_j), P_{i,j}, I_j\}$, |
| *R2* | $(u_1; z_1)(u_2; z_2) = (u_2; z_2)(u_1; z_1)$ *for* $u_1 \neq u_2$ *and* $z_i^{\pm 1} \notin \{u_1, u_2\}$, |
| *R3.1* | $(a^{-1}; c_j)P_{j,l} = P_{j,l}(a^{-1}; c_l)$, |
| *R3.2* | $(a^{-1}; c_j)I_j = I_j(a^{-1}; c_j^{-1})$, |
| *R3.3* | $\gamma_{c_j} \circ P_{j,l} = P_{j,l} \circ \gamma_{c_l}$, |
| *R3.4* | $\gamma_{c_j} \circ I_j = I_j \circ \gamma_{c_j}^{-1}$, |
| *R3.5* | $P_{j,l}$, $I_j$ *commute with* $(a^{-1}; b)$ *and* $\gamma_{aba^{-1}}$, |
| *R3.6* | $(c_j^\epsilon; z)P_{j,l} = P_{j,l}(c_l^\epsilon; z)$ *for* $z = aba^{-1}$ *or* $z = b$, |
| *R3.7* | $(c_j; z)I_j = I_j(c_j^{-1}; z)$ *for* $z = aba^{-1}$ *or* $z = b$, |
| *R4.1* | $(u; c_j^{-\eta})(c_j^\eta; z)(u; c_j^\eta) = (u; z)(c_j^\eta; z)$, |
| *R4.2* | $(a^{-1}; z^{-\epsilon})(u; aba^{-1})(a^{-1}; z^\epsilon) = (u; z^\epsilon)(u; aba^{-1})(u; z^{-\epsilon})$, |
| *R4.3* | $\gamma_z^{-\epsilon}(u; b)\gamma_z^\epsilon = (u; z^\epsilon)(u; b)(u; z^{-\epsilon})$, |
| *R5.1* | $(c_j^{-\eta}; ab^\epsilon a^{-1})(a^{-1}; c_j^\eta) = (a^{-1}; c_j^\eta)(c_j^\eta; ab^{-\epsilon}a^{-1})\rho^\epsilon$, |
| *R5.2* | $(c_j^{-\eta}; b)\gamma_{c_j}^\eta \circ \rho = \gamma_{c_j}^\eta(c_j^\eta; b^{-1})$, |
| *R6* | $\rho$ *commutes with all generators,* |
| *R7* | $\theta^2 = 1$, |
| *R8.1* | $\theta \circ P_{i,j} = P_{i,j} \circ \theta$, |
| *R8.2* | $\theta \circ I_i = I_i \circ \theta$, |
| *R8.3* | $\theta \circ (c_i^\epsilon; c_j) = (c_i^\epsilon; c_j) \circ \theta$, |
| *R8.4* | $\theta \circ (c_i^\epsilon; aba^{-1}) = (c_i^\epsilon; b^{-1}) \circ \theta$, |
| *R8.5* | $\theta \circ \gamma_{c_i} = (a^{-1}; c_i) \circ \theta$, |
| *R8.6* | $\theta \circ \gamma_{aba^{-1}} = (a^{-1}; b^{-1}) \circ \theta$, |

*whenever the symbols involved denote generators or inverses.*

*Proof.* Recall the short exact sequence (82) on page 141:

$$1 \to \langle\rho\rangle \to C^0(\rho) \to \mathrm{Aut}(G_v, \mathcal{C}_v) \to 1.$$

To get a presentation for $C^0(\rho)$, we now use Proposition 15.3. We have a generating set for $C^0(\rho)$ consisting of these lifted generators together with $\rho$, which is as asserted in the present theorem. The lifted relations are R1 through R5, which have the same numbers as the corresponding relations in Proposition 15.4. A direct calculation shows that $\rho$ only shows up in R5. Since $\rho$ is central, the conjugation relations are simply the commutation rules R6. As the left hand term in this exact sequence is the infinite cyclic group generated by $\rho$, there are no kernel relations.

To get a presentation for $C(\rho)$, we apply Proposition 15.3 to the short exact sequence

$$1 \to C^0(\rho) \to C(\rho) \to \mathbb{Z}/2 \to 1,$$

arising from the fact that $C^0(\rho)$ has index 2 in $C(\rho)$ and is therefore normal.

A generating set for $C(\rho)$ is then given by $\theta$ and our chosen generators of $C^0(\rho)$. The relations are again R1 through R6 along with the lifted relation R7 and the conjugation relations given by R8. $\square$

We now study the abelianisation of $C(\rho)$.

**Corollary 15.12** ([31], Corollary 8.3)**.** *Let $\rho \in \mathrm{Aut}(F_n)$ be a Nielsen automorphism. Then*

$$H_1(C(\rho)) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/2, & \text{if } n = 2, \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^3, & \text{if } n = 3, \\ (\mathbb{Z}/2)^3, & \text{if } n = 4, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & \text{if } n \geq 5. \end{cases}$$

*When $n = 2$, the class $[\![\rho]\!]$ is a generator of $\mathbb{Z}^2$, when $n = 3$ it is twice a generator of $\mathbb{Z}$, and otherwise $[\![\rho]\!] = 0$.*

*Proof.* We abelianise the presentation in Theorem 15.11. We first restrict to the case $n = 2$: the generators of $C(\rho)$ in this case are $\gamma_{aba^{-1}}$, $(a^{-1}; b)$, $\rho$ and $\theta$. The only relations which occur and are non-trivial in the abelianisation are R7 and R8.6, which become $2[\![\theta]\!] = 0$ and $[\![\gamma_{aba^{-1}}]\!] + [\![(a^{-1}; b)]\!] = 0$. This finishes the proof of the assertion for $n = 2$.

Next we consider $n = 3$. For simplicity we write $c := c_1$. Here the generators of $C(\rho)$ are $I := I_1$, $(c^\epsilon; aba^{-1})$, $(c^\epsilon; b)$, $\gamma_{aba^{-1}}$, $\gamma_c$, $(a^{-1}; b)$, $(a^{-1}; c)$, $\rho$, and $\theta$. By Theorem 15.11 we obtain the following relations:

| | | | |
|---|---|---|---|
| R1: | $2[\![I]\!] = 0$, | R3.2: | $2[\![a^{-1}; c]\!] = 0$, |
| R3.4: | $2[\![\gamma_c]\!] = 0$, | R3.7: | $[\![c; aba^{-1}]\!] = [\![c^{-1}; aba^{-1}]\!]$, |
| R3.7: | $[\![c; b]\!] = [\![c^{-1}; b]\!]$, | R4.1: | $[\![\gamma_{aba^{-1}}]\!] = 0$, |

| | | | |
|---|---|---|---|
| R4.1: | $[\![a^{-1};b]\!]=0,$ | R5.1: | $[\![\rho]\!]=2[\![c;aba^{-1}]\!],$ |
| R5.2: | $-[\![\rho]\!]=2[\![c;b]\!],$ | R7: | $2[\![\theta]\!]=0,$ |
| R8.4: | $[\![c;b]\!]=-[\![c;aba^{-1}]\!],$ | R8.5: | $[\![\gamma_c]\!]=[\![a^{-1};c]\!].$ |

All other relations in $H_1(C(\rho))$ either follow from the ones above or are trivial. It follows that $H_1(C(\rho))\cong\mathbb{Z}\oplus(\mathbb{Z}/2)^3$ with the torsion part generated by $[\![I]\!]$, $[\![\gamma_c]\!]$, and $[\![\theta]\!]$ and the torsion-free part generated by $[\![(c;b)]\!]$ with $[\![\rho]\!]=-2[\![(c;b)]\!]$.

For $n\geq 4$, by checking the relations R1–R8 one finds that there is a homomorphism

$$C(\rho)\to H_1(\mathrm{Aut}(F_{n-2}))$$

given by sending the elements $(c_i^\epsilon;aba^{-1})$, $(c_i^\epsilon;b)$, $\gamma_z$, $(a^{-1};z)$, $\rho$ and $\theta$ to 0 and letting the remaining generators of $C(\rho)$ act on $F_{n-2}=\langle c_1,\dots,c_{n-2}\rangle$. We also have the homomorphism

$$C(\rho)\to C(\rho)/C^0(\rho)\cong\mathbb{Z}/2$$

that takes every generator except $\theta$ to 0. Combining these gives a surjective homomorphism

$$f:C(\rho)\to H_1(\mathrm{Aut}(F_{n-2}))\oplus C(\rho)/C^0(\rho).$$

The relation R4.1 implies that $[\![(u;z)]\!]=0$ if there is a symbol $c_j$ different from both $u,z$ and their inverses. Hence any generator $(u;z)$ not of the form $(c_i^\epsilon,c_j)$ is trivial in $H_1(C(\rho))$. Furthermore, as $[\![(c_i;aba^{-1})]\!]=0$, we have $[\![\rho]\!]=0$ by R5.1. It follows that any non-trivial element of $H_1(C(\rho))$ is non-trivial under the map $f$, and as $\mathrm{im}(f)$ is abelian, we have $\mathrm{im}(f)\cong H_1(C(\rho))$. The observation

$$H_1(\mathrm{Aut}(F_{n-2}))=\begin{cases}\mathbb{Z}/2\oplus\mathbb{Z}/2, & \text{if } n=4,\\ \mathbb{Z}/2, & \text{if } n\geq 5\end{cases}$$

finishes the proof. □

## 15.7 Right translation by a commutator

Here we discuss the case $w=b_1b_2b_1^{-1}b_2^{-1}$ in the strategy on page 148. The natural free factor $B=\langle b_1,b_2\rangle$ is of rank 2. For simplicity we write $x:=b_1$ and $y:=b_2$. In view of (83) on page 143, we are interested in $\mathrm{Out}(F_2,[xyx^{-1}y^{-1}])$. It is well-known and easy to verify with the McCool complex that this group is equal to $\mathrm{SOut}(F_2)$, the index 2 subgroup of automorphisms whose abelianisation has determinant $+1$. The abelianisation induces a homomorphism

$$H_1:\mathrm{SOut}(F_2)\to\mathrm{SL}(2,\mathbb{Z})$$

In order to introduce some notation important for our computations, we now verify the well-known fact that $H_1$ is an isomorphism. In [32], there is a presentation for $\mathrm{SL}(2,\mathbb{Z})$: There is an isomorphism

$$\mathrm{SL}(2,\mathbb{Z})\cong\mathbb{Z}/4*_{\mathbb{Z}/2}\mathbb{Z}/6,$$

where the generators of $\mathbb{Z}/4$ and $\mathbb{Z}/6$ are given by the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ respectively. Both these matrices have preimages in $\mathrm{SOut}(F_2)$, hence the above map $H_1$ is surjective. We choose the preimages $\begin{pmatrix} x & y \\ y^{-1} & x \end{pmatrix}$ and $\begin{pmatrix} x & y \\ y^{-1} & x \end{pmatrix} \circ (x; y)$. It is easily calculated that, in $\mathrm{Out}(F_2)$, they satisfy all relations of $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$. Moreover, using the McCool complex, $\mathrm{SOut}(F_2)$ is generated by these elements. Hence the abelianisation map $H_1$ is an isomorphism.

We now apply the strategy on page 148 with this presentation of $\mathrm{Out}(B, [b_1 b_2 b_1^{-1} b_2^{-1}])$. We lift the generators to $\mathrm{Aut}(B)$ by

$$\alpha_4\colon \begin{aligned} b_1 &\mapsto b_2^{-1}, \\ b_2 &\mapsto b_2 b_1 b_2^{-1}, \end{aligned} \qquad\qquad \alpha_6\colon \begin{aligned} b_1 &\mapsto b_1 b_2^{-1}, \\ b_2 &\mapsto b_2 b_1 b_2^{-1}. \end{aligned}$$

It is readily checked that $\alpha_4$ and $\alpha_6$ fix the element $b_1 b_2 b_1^{-1} b_2^{-1}$. We extend these maps by the identity on $a, c_1, \dots, c_n$ to automorphisms of the free group $F_{n+k+1} = F_{n+3} = \langle a, b_1, b_2, c_1, \dots, c_n \rangle$, which we call by the same names.

We now compute the relations in (ii) of the strategy on page 148: In the group $\mathrm{Out}(B, [b_1 b_2 b_1^{-1} b_2^{-1}])$, we can reduce to the relators $\alpha_4^4$ and $\alpha_4^2 \alpha_6^{-3}$ because the relation $\alpha_6^6 = 1$ follows from them. A calculation shows that these relations become

$$\begin{aligned} \alpha_4^4 &= \gamma_{b_2}^{-1} \gamma_{b_1}^{-1} \gamma_{b_2} \gamma_{b_1} \rho^{-1}, \\ \alpha_4^2 &= \alpha_6^3 \end{aligned} \tag{85}$$

in $C(\rho_{a, b_1 b_2 b_1^{-1} b_2^{-1}})$.

Since the complete list of relations for this centraliser is quite long, we do not spell it out completely. At this place, we only compute its abelianisation and describe the class $[\![\rho]\!]$ in it.

**Proposition 15.13.** *For $\rho_{a, b_1 b_2 b_1^{-1} b_2^{-1}} \in \mathrm{Aut}(F_{n+3})$ we have*

$$H_1(C(\rho_{a, b_1 b_2 b_1^{-1} b_2^{-1}})) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^4, & \text{if } n = 1, \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/12, & \text{if } n = 2, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/12, & \text{if } n \geq 3. \end{cases}$$

*The class $[\![\rho]\!]$ is 12 times a generator of the torsion-free summand when $n = 0$ or $n = 1$, and it is zero when $n \geq 2$.*

*Proof.* By the strategy, the generators are those for $H_1(C(\rho_{a,B}))$ in Corollary 15.6 with $[\![\alpha_4]\!]$ and $[\![\alpha_6]\!]$ added.

The relations (i) in the strategy on page 148 lead to the same as in Corollary 15.6. The set of relations (ii) is given by (85), whose abelianised forms can be read as

$$\begin{aligned} [\![\rho]\!] &= -4[\![\alpha_4]\!], \\ 2[\![\alpha_4]\!] &= 3[\![\alpha_6]\!]. \end{aligned} \tag{86}$$

The relations of type (iii) and (iv) are irrelevant here because they are only commutators.

The relations of (v) of the strategy for $\alpha_6$ lead to $[\![u; b_1]\!] = [\![u; b_1]\!] - [\![u; b_2]\!]$ and $[\![u; b_2]\!] = [\![u; b_1]\!]$, which annihilate all classes $[\![u; b_i]\!]$. After that, the relations (v) for $\alpha_4$ become trivial in the abelianisation.

We change the basis elements $[\![\alpha_4]\!]$ and $[\![\alpha_6]\!]$ to

$$\begin{aligned} A_1 &:= [\![\alpha_4]\!] - [\![\alpha_6]\!], \\ A_2 &:= 2[\![\alpha_4]\!] - 3[\![\alpha_6]\!]. \end{aligned}$$

Then the above relations (86) show $A_2 = 0$ and $[\![\rho]\!] = -12A_1$.

Summarising, we have

$$H_1(C(\rho_{a,w})) = \frac{H_1(C(\rho_{a,B})) \oplus \langle A_1 \rangle}{[\![\rho]\!] + 12A_1 = [\![a^{-1}; b_i]\!] = [\![\gamma_{b_i}]\!] = [\![c_j^{\epsilon}; b_i]\!] = 0}. \tag{87}$$

To understand the additional relations, we refer to Remark 15.7.

If $n = 0$, the generators are $[\![a^{-1}; b_i]\!]$, $[\![\gamma_W]\!]$, $[\![\gamma_{b_i}]\!]$, $[\![\rho]\!]$, and $A_1$ subject to the relations $[\![\rho]\!] = -12A_1$ and $[\![a^{-1}; b_i]\!] = [\![\gamma_{b_i}]\!] = 0$, whence the asserted abelianisation.

In the case $n = 1$, Remark 15.7(ii) shows

$$H_1(C(\rho_{a,B})) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3,$$

where a basis of the free summand consists of $[\![c; W]\!]$ and $[\![c; b_i]\!]$ with $[\![\rho]\!] = 2[\![c; W]\!]$, $[\![\gamma_{b_i}]\!] = -2[\![c; b_i]\!]$, $[\![a^{-1}; b_i]\!] = 0$. Thus

$$\begin{aligned} H_1(C(\rho)) &\cong (\mathbb{Z}/2)^3 \oplus \frac{\mathbb{Z}[[\![c; W]\!], [\![c; b_1]\!], [\![c; b_2]\!], A_1]}{2[\![c; W]\!] + 12A_1 = [\![c; b_i]\!] = 0} \\ &= (\mathbb{Z}/2)^3 \oplus \mathbb{Z}[[\![c; W]\!], A_1]/(2[\![c; W]\!] + 12A_1 = 0). \end{aligned}$$

We now write $B_1 = A_1$ and $B_2 = [\![c; W]\!] + 6A_1$, so we have

$$H_1(C(\rho)) \cong ((\mathbb{Z}/2)^3 \oplus \mathbb{Z}[B_1, B_2])/(2B_2 = 0) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^4.$$

Since $[\![\rho]\!] = -12B_1$, it is 12 times a generator of the summand $\mathbb{Z}$.

In the case $n \geq 2$, we know by Corollary 15.6 and Remark 15.7 that the classes $[\![a^{-1}; b_i]\!] = [\![\gamma_{b_i}]\!] = [\![c_j^{\epsilon}; b_i]\!] = [\![\rho]\!] = 0$, so (87) shows

$$\begin{aligned} H_1(C(\rho)) &= (H_1(C(\rho_{a,B})) \oplus \langle A_1 \rangle)/(12A_1 = 0) \\ &= H_1(C(\rho_{a,B})) \oplus \mathbb{Z}/12. \end{aligned}$$

This is the asserted abelianisation. $\square$

## 15.8 Right translation by a Klein bottle relator

Another example for the strategy on page 148 is about the right translation by $w = b_1^2 b_2^2$. Determining its centraliser is equivalent to finding the centraliser of right translations by the more popular Klein bottle relator $w' = b_1 b_2 b_1 b_2^{-1}$ because these two elements differ by an automorphism of $B = \langle b_1, b_2 \rangle$. Note that this free factor is again the natural free factor of $w$.

We first need a presentation for $\operatorname{Out}(B, [b_1^2 b_2^2])$:

**Lemma 15.14.** *A presentation of* $\operatorname{Out}(F_2, [x^2y^2])$ *is the infinite dihedral group with generators* $\widehat{\beta}_1$ *and* $\widehat{\beta}_2$ *given by*

$$\begin{aligned} \beta_1\colon\ x &\mapsto xy^2, & \qquad\qquad \beta_2\colon\ x &\mapsto y, \\ y &\mapsto y^{-1}, & y &\mapsto x \end{aligned}$$

*subject to the relations* $\widehat{\beta}_1^{\,2} = 1$ *and* $\widehat{\beta}_2^{\,2} = 1$.

*Proof.* Using the McCool complex, we prove that $\beta_1$ and $\beta_2$ generate $\operatorname{Out}(F_2, [x^2y^2])$. Every loop at the basepoint $[x^2y^2]$ can be homotoped to one containing only one edge label which is a Whitehead automorphism of type one. We can assume that this edge is the very last one of the loop.

In $\operatorname{Out}(F_2)$, the only non-trivial type II Whitehead automorphisms are $(x;y)^{\pm 1}$ and $(y;x)^{\pm 1}$. As the permutation $P_{x,y}$ swapping $x$ with $y$ is a one edge loop at the basepoint, conjugation by this reduces us to finding all edge paths from the basepoint $[x^2y^2]$ starting with a label $(x;y)^{\pm 1}$. They are depicted in the following diagram.

$$\begin{tikzcd}
{[x^2y^2]} \arrow[loop above, "P_{x,y}"] \arrow[d, leftrightarrow, "I_y"'] & {[xyxy^{-1}]} \arrow[l, "(x;y)"'] \arrow[loop above, "(y;x)"] \\
{[x^2y^{-2}]} \arrow[ur, "(x;y)"'] &
\end{tikzcd}$$

We read off that $\operatorname{Out}(F_2, [x^2y^2])$ is generated by

$$\begin{aligned} P_{x,y} &= \beta_2, \\ (x;y)(y;x)(x;y)^{-1} &= \operatorname{ad}_y \beta_1 \beta_2, \\ I_y \circ (x;y)^{-2} &= \beta_1. \end{aligned}$$

Therefore $\operatorname{Out}(F_2, [x^2y^2])$ is generated by $\widehat{\beta}_1$ and $\widehat{\beta}_2$.

It remains to check the relations. Clearly $\beta_1$ and $\beta_2$ have order two. The reader can check that $\widehat{\beta_1\beta_2}$ has infinite order, so there are no more relations. $\square$

We now use Lemma 15.14 as input for the strategy on page 148. According to (ii) of that strategy, we have to find $\alpha_1, \alpha_2 \in \operatorname{Aut}(B)$ that fix $w = b_1^2 b_2^2$ and differ from $\beta_1$

and $\beta_2$ in the lemma only by inner automorphisms. We choose

$$\begin{aligned} \alpha_1 \colon\ b_1 &\mapsto b_1^2 b_2^2 b_1^{-1}, & \alpha_2 \colon\ b_1 &\mapsto b_1^2 b_2 b_1^{-2}, \\ b_2 &\mapsto b_1 b_2^{-1} b_1^{-1}, & b_2 &\mapsto b_1. \end{aligned}$$

We again have to compute the relations (ii) of the algorithm. They are

$$\alpha_1^2 = \alpha_2^2 = \gamma_{b_1}^{-2}\gamma_{b_2}^{-2}\rho. \tag{88}$$

As in Section 15.7 for commutators, we restrict to an explicit calculation of the abelianisation $H_1(C(\rho_{a,b_1^2 b_2^2}))$.

**Proposition 15.15.** *Consider the right translation $\rho_{a,b_1^2 b_2^2} \in \mathrm{Aut}(F_{n+3})$. Then*

$$H_1(C(\rho_{a,b_1^2 b_2^2})) \cong \begin{cases} \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^3, & \text{if } n = 0, \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^6, & \text{if } n = 1, \\ (\mathbb{Z}/2)^4, & \text{if } n = 2, \\ (\mathbb{Z}/2)^3, & \text{if } n \geq 3. \end{cases}$$

*The class $[\![\rho]\!]$ is twice a generator of the torsion-free summand when $n \leq 1$. Otherwise $[\![\rho]\!] = 0$.*

*Proof.* Since we compute the abelianisation, we only have to take relation types (i), (ii), and (v) of the strategy into account. As usual, relations (i) leave us with a direct sum $H_1(C(\rho_{a,B})) \oplus \mathbb{Z}[[\![\alpha_1]\!], [\![\alpha_2]\!]]$, i.e. we add to the abelianisation of Corollary 15.6 the summand $\mathbb{Z}^2$ generated by the classes $[\![\alpha_i]\!]$. Relations (ii), which are (88) in the present situation, become

$$2[\![\alpha_1]\!] = 2[\![\alpha_2]\!] = -2[\![\gamma_{b_1}]\!] - 2[\![\gamma_{b_2}]\!] + [\![\rho]\!].$$

The abelianised relations of type (v) corresponding to $\alpha_1$ in the strategy reduce to $2[\![u; b_2]\!] = 0$. The map $\alpha_2$ produces the relation $[\![u; b_1]\!] = [\![u; b_2]\!]$. Therefore $H_1(C(\rho_{a,b_1^2 b_2^2}))$ is obtained from $H_1(C(\rho_{a,B})) \oplus \mathbb{Z}[[\![\alpha_1]\!], [\![\alpha_2]\!]]$ by dividing out the relations

$$\begin{aligned} 2[\![\alpha_1]\!] = 2[\![\alpha_2]\!] &= -2[\![\gamma_{b_1}]\!] - 2[\![\gamma_{b_2}]\!] + [\![\rho]\!], \\ [\![u; b_1]\!] &= [\![u; b_2]\!], \\ 2[\![u; b_2]\!] &= 0. \end{aligned}$$

We transform the basis as follows: We introduce $A_1 := [\![\alpha_1]\!] - [\![\alpha_2]\!]$. Then we get

$$H_1(C(\rho_{a,w})) \cong \frac{H_1(C(\rho_{a,B})) \oplus \mathbb{Z}[[\![\alpha_1]\!], A_1]}{2A_1 = 0, [\![\rho]\!] = 2[\![\alpha_1]\!], [\![u; b_1]\!] = [\![u; b_2]\!], 2[\![u; b_1]\!] = 0}.$$

We can use Corollary 15.6 and Remark 15.7 to obtain the desired abelianisation. □

## 15.9 Proof of Theorem 15.4

We now derive the presentation for $\mathrm{Aut}(G_v, [W], [B])$ in Proposition 15.4. Recall that $G_v$ is freely generated by $W, b_1, \ldots, b_k, c_1, \ldots, c_n$, and $B$ is the free factor $\langle b_1, \ldots, b_k \rangle$.

We use the McCool complex $K$. By Proposition 11.5, fixing the free factor $B$ up to uniform conjugacy is the same as fixing the conjugacy class of a rigid element $w$ in this factor.

The vertex set of the McCool complex is given by

$$\begin{aligned} K_0 &:= \big\{ V \text{ minimal tuple} \big| V = (P(W), P(w)) \text{ for some } P \in \mathrm{Aut}(F_{n+k+1}) \big\} \\ &= \big\{ (\gamma(W), \gamma(w)) \big| \gamma \in \Omega_{n+k+1} \big\}. \end{aligned}$$

Here $\Omega_{n+k+1}$ denotes the group of permutations of $\{W, b_1, \ldots, b_k, c_1, \ldots, c_n\}$ and their inverses. It follows from rigidity of $w$ in $B$ that we only obtain vertices by using these permutations $\gamma \in \Omega_{n+k+1}$.

In the above description of $K_0$, the vertices $(\gamma(W), \gamma(w))$ and $(\gamma'(W), \gamma'(w))$ are the same if and only if the permutations $\gamma$ and $\gamma'$ coincide on $W$ and all $b_i$.

The edges and 2-cells of the complex $K$ are given as in Section 11.2. The edges give rise to the generators of our presentation: The $P_{i,j}$ generate the type I Whitehead automorphisms which are a loop at the basepoint. The type II Whitehead automorphisms can be written as a composition of the generators in the assertion, or they do not belong to $K$ because they increase the cyclic length of $(W, w)$.

The relations stated in the theorem are verified by calculation, which is a good exercise for the reader.

In Section 4(1) of [27], it is remarked that the relations given by the 2-cells of $K$ follow from the relations R1 through R10 of [26]. Therefore it remains to show that our relations Z1 to Z5 imply R1 to R10 of [26], which we will recall below. As in the proof of Theorem 7.1 of [19], we only have to take loops at the basepoint $(W, w)$ into account. Therefore we only have to show that our relations imply R1 to R10 of [26] when interpreted as relations at the basepoint $(W, w) \in K_0$.

We remind the reader that McCool [26] reads compositions from left to right, whereas we read them from right to left. Therefore the generators appear in the opposite order in the relations.

When $A$ and $B$ are sets, then we write $A + B$ for $A \cup B$ only if $A$ and $B$ are disjoint, and we write $A - B$ for $A \setminus B$ only if $B \subset A$. We sometimes identify a letter $z$ with the set $\{z\}$.

We now start to verify R1 through R10. Relation R1 is used to identify inverses. R2 is already used to decompose more general symbols $(A; a)$ as a product of generators. In order to make this work, we need that $(u_1; z)$ and $(u_2; z)$ commute, which is included in Z2 of Proposition 15.4.

R3 states that $(A; a)$ and $(B; b)$ commute if $A + a^{-1}$ and $B + b^{-1}$ are disjoint. This relation follows from Z2.

R4 is $(B; b)(A; a)(B; b)^{-1} = (A + B - b; a)$ for $A \cap B = \varnothing$, $a^{-1} \notin B$ and $b^{-1} \in A$. This follows from Z4.1 as in [19]: In our situation $b = c_i^{\epsilon}$ because $b \in A$, $b^{-1} \notin A$, and

$a \neq b$. We proceed by induction on the cardinality of $B$. The minimal case $B = \{b\}$ is trivial. Otherwise we choose some $u \in B - c_i^\epsilon$ and calculate

$$\begin{aligned}
&(B; c_i^\epsilon)(A; a)(B; c_i^{-\epsilon}) \\
&\overset{Z2}{=} (B - u; c_i^\epsilon)(A - c_i^{-\epsilon}; a)(u; c_i^\epsilon)(c_i^{-\epsilon}; a)(u; c_i^{-\epsilon})(B - u; c_i^{-\epsilon}) \\
&\overset{Z4.1}{=} (B - u; c_i^\epsilon)(A - c_i^{-\epsilon}; a)(u; a)(c_i^{-\epsilon}; a)(B - u; c_i^{-\epsilon}) \\
&= (B - u; c_i^\epsilon)(A + u; a)(B - u; c_i^{-\epsilon}) \\
&= (A + B - c_i^\epsilon; a),
\end{aligned}$$

where the last equality follows by induction. Note that this argument still works for $u = \underline{b}^\pm$ or $u = W^\pm$.

R5 is the relation $(A - a + a^{-1}; b)(A; a) = (A - b + b^{-1}; a) \begin{pmatrix} a & b \\ b^{-1} & a \end{pmatrix}$ for $b \in A$, $b^{-1} \notin A$ and $a \neq b$. This relation can only appear when $a$ and $b$ are $c$-symbols. In the minimal case $A = \{a, b\}$, this relation R5 is included in Z1. In the general case we compute

$$\begin{aligned}
&(A - a + a^{-1}; b)(A; a) \\
&= (A - a; b)(a^{-1}; b)(b; a)(A - b; a) \\
&\overset{Z1}{=} (A - a; b)(b^{-1}; a) \begin{pmatrix} a & b \\ b^{-1} & a \end{pmatrix} (A - b; a) \\
&\overset{Z3}{=} (A - a; b)(b^{-1}; a)(A - a - b + b^{-1}; b^{-1}) \begin{pmatrix} a & b \\ b^{-1} & a \end{pmatrix} \\
&\overset{R4}{=} (A - b + b^{-1}; a)(A - a; b)(A - a - b + b^{-1}; b^{-1}) \begin{pmatrix} a & b \\ b^{-1} & a \end{pmatrix} \\
&= (A - b + b^{-1}; a) \begin{pmatrix} a & b \\ b^{-1} & a \end{pmatrix}.
\end{aligned}$$

R6 is a commutation rule for edges of type I, which asserts $T(A; a)T^{-1} = (T(A); T(a))$ for $T$ of type I. This has already been eliminated during the reduction to edges at the basepoint $(W, w)$. The remaining relations are encoded in Z1 and Z3.

R7 is a set of relations for $\Omega_{n+k+1}$. It has also been reduced to the basepoint, so it is included in Z1.

R8 states that $(A; a) = (L - a^{-1}; a)(L - A; a^{-1}) = (L - A; a^{-1})(L - a^{-1}; a)$, where $L := \{W^{\pm 1}, b_1^{\pm 1}, \ldots, b_k^{\pm 1}, c_1^{\pm 1}, \ldots, c_n^{\pm 1}\}$ denotes the set of all letters. This follows from R1 and R2.

R9 asserts that $(L - b^{-1}; b)(A; a)(L - b; b^{-1}) = (A; a)$, if $b, b^{-1} \notin A$. Here we have $a \neq b^{\pm 1}$. We have several cases, depending on what $a$ is.

First we assume $a = c_j^\delta$:

$$\begin{aligned}
&(L - b^{-1}; b)(A; c_j^\delta)(L - b; b^{-1}) \\
&= (L - b^{-1}; b)(A; c_j^\delta)(c_j^{-\delta}; b^{-1})(L - b - c_j^{-\delta}; b^{-1})
\end{aligned}$$

$$
\begin{aligned}
&\overset{Z4.1}{=} (L - b^{-1}; b)(A - c_j^{\delta} + b^{-1}; b^{-1})(c_j^{-\delta}; b^{-1})(A; c_j^{\delta})(L - b - c_j^{-\delta}; b^{-1})\\
&= (L - A - b^{-1} - c_j^{-\delta}; b)(c_j^{\delta}; b)(A; c_j^{\delta})(L - b - c_j^{-\delta}; b^{-1})\\
&\overset{Z4.1}{=} (L - A - b^{-1} - c_j^{-\delta}; b)(A; c_j^{\delta})(A - c_j^{\delta} + b; b)(c_j^{\delta}; b)(L - b - c_j^{-\delta}; b^{-1})\\
&= (L - A - b^{-1} - c_j^{-\delta}; b)(A; c_j^{\delta})(L - A - b - c_j^{-\delta}; b^{-1})\\
&\overset{R3}{=} (A; c_j^{\delta}).
\end{aligned}
$$

In the case $a = W^{\epsilon}$ we calculate

$$
\begin{aligned}
&(L - b^{-1}; b)(A; W^{\epsilon})(L - b; b^{-1})\\
&= (L - b^{-1}; b)(A; W^{\epsilon})(W^{\pm}; b^{-1})(L - W - W^{-1} - b; b^{-1})\\
&\overset{Z4.2}{=} (L - b^{-1}; b)(W^{\pm}; b^{-1})(A - W^{\epsilon} + b^{-1}; b^{-1})(A; W^{\epsilon})(A - W^{\epsilon} + b; b)\cdot\\
&\quad \cdot(L - W - W^{-1} - b; b^{-1})\\
&= (L - A - W^{-\epsilon} - b^{-1}; b)(A; W^{\epsilon})(L - A - W^{-\epsilon} - b; b^{-1})\\
&\overset{R3}{=} (A; W^{\epsilon}).
\end{aligned}
$$

Next we tackle the case $a = b_i^{\epsilon}$, $b = b_j$ for some $i \neq j$. Note that the intersection $\{b_1^{\pm 1}, \dots, b_k^{\pm 1}\} \cap A = \{b_i^{\epsilon}\}$, as $b^{\pm 1} = b_j^{\pm 1} \notin A$. Then

$$
\begin{aligned}
&(L - b_j^{-1}; b_j)(A; b_i^{\epsilon})(L - b_j; b_j^{-1})\\
&= (L - b_j^{-1}; b_j)(A; b_i^{\epsilon})(\underline{b}^{\pm}; b_j^{-1})(L - \underline{b}^{\pm} + b_j^{-1}; b_j^{-1})\\
&\overset{Z4.3}{=} (L - b_j^{-1}; b_j)(\underline{b}^{\pm}; b_j^{-1})(A - b_i^{\epsilon} + b_j^{-1}; b_j^{-1})(A; b_i^{\epsilon})(A - b_i^{\epsilon} + b_j; b_j)(L - \underline{b}^{\pm} + b_j^{-1}; b_j^{-1})\\
&= (L - A + b_i^{\epsilon} - \underline{b}^{\pm} + b_j; b_j)(A; b_i^{\epsilon})(L - A + b_i^{\epsilon} - \underline{b}^{\pm} + b_j^{-1}; b_j^{-1})\\
&\overset{R3}{=} (A; b_i^{\epsilon}).
\end{aligned}
$$

For $a = b_i^{\epsilon}$ and $b = W$, we have $W^{\pm 1} \notin A$. In the following calculation we assume $A \cap \underline{b}^{\pm} = \{b_i^{\epsilon}\}$:

$$
\begin{aligned}
&(L - W^{-1}; W)(A; b_i^{\epsilon})(L - W; W^{-1})\\
&= (L - W^{-1}; W)(A; b_i^{\epsilon})(\underline{b}^{\pm}; W^{-1})(L - \underline{b}^{\pm} - W; W^{-1})\\
&\overset{Z4.3}{=} (L - W^{-1}; W)(\underline{b}^{\pm}; W^{-1})(A - b_i^{\epsilon} + W^{-1}; W^{-1})(A; b_i^{\epsilon})(A - b_i^{\epsilon} + W; W)\cdot\\
&\quad \cdot(L - \underline{b}^{\pm} - W; W^{-1})\\
&= (L - A + b_i^{\epsilon} - \underline{b}^{\pm} - W^{-1}; W)(A; b_i^{\epsilon})(L - A + b_i^{\epsilon} - \underline{b}^{\pm} - W; W^{-1})\\
&= (A; b_i^{\epsilon}).
\end{aligned}
$$

However, if $A \cap \underline{b}^{\pm} = \underline{b}^{\pm} - b_i^{\epsilon}$, then $(A; b_i^{\epsilon}) = (A + b_i^{\pm} - \underline{b}^{\pm}; b_i)(\underline{b}^{\pm}; b_i)$. For relation R9 we then only need:

$$
\begin{aligned}
&(L - W^{-1}; W)(\underline{b}^{\pm}; b_i^{\epsilon})(L - W; W^{-1})\\
&= (L - W^{-1}; W)(\underline{b}^{\pm}; b_i^{\epsilon})(\underline{b}^{\pm}; W^{-1})(L - \underline{b}^{\pm} - W; W^{-1})
\end{aligned}
$$

$$
\begin{aligned}
&\overset{Z5.1}{=} (L-W^{-1};W)(\underline{b}^{\pm};W^{-1})(\underline{b}^{\pm};b_i^{\epsilon})(L-\underline{b}^{\pm}-W;W^{-1})\\
&=(L-\underline{b}^{\pm}-W^{-1};W)(\underline{b}^{\pm};b_i^{\epsilon})(L-\underline{b}^{\pm}-W;W^{-1})\\
&\overset{R3}{=} (\underline{b}^{\pm};b_i^{\epsilon}).
\end{aligned}
$$

To finish the verification of R9, we have to investigate the case $a = b_i^{\epsilon}$ and $b = c_j$. Recall $c_j^{\pm 1} = b^{\pm 1} \notin A$. We again carry out the calculation separately for $A \cap \underline{b}^{\pm} = \{b_i^{\epsilon}\}$ and $A \cap \underline{b}^{\pm} = \underline{b}^{\pm} - b_i^{-\epsilon}$. We only spell out the former one because the latter is very similar to the previous paragraph.

$$
\begin{aligned}
&(L-c_j^{-1};c_j)(A;b_i^{\epsilon})(L-c_j;c_j^{-1})\\
&=(L-c_j^{-1};c_j)(A;b_i^{\epsilon})(\underline{b}^{\pm};c_j^{-1})(L-\underline{b}^{\pm}-c_j;c_j^{-1})\\
&\overset{Z4.3}{=} (L-c_j^{-1};c_j)(\underline{b}^{\pm};c_j^{-1})(A-b_i^{\epsilon}+c_j^{-1};c_j^{-1})(A;b_i^{\epsilon})(A-b_i^{\epsilon}+c_j;c_j)\cdot\\
&\quad\cdot(L-\underline{b}^{\pm}-c_j;c_j^{-1})\\
&=(L-A+b_i^{\epsilon}-\underline{b}^{\pm}-c_j^{-1};c_j)(A;b_i^{\epsilon})(L-A+b_i^{\epsilon}-\underline{b}^{\pm}-c_j;c_j^{-1})\\
&=(A;b_i^{\epsilon}).
\end{aligned}
$$

We finally deduce R10, which asserts that $(L-b^{-1};b)(A;a)(L-b;b^{-1}) = (L-A;a^{-1})$ if $b \neq a$, $b \in A$, $b^{-1} \notin A$. The only occurrence of this relation is when $b = c_i^{\epsilon}$.
For $a = c_j^{\eta}$, this follows as in [26]. For convenience, we repeat this argument in our notation:

$$
\begin{aligned}
&(L-b^{-1};b)(A;a)(L-b;b^{-1})(L-A;a^{-1})^{-1}\\
&=(L-b^{-1};b)(b;a)(A-b;a)(L-b;b^{-1})(L-A-a^{-1}+a;a)\\
&\overset{R9}{=} (L-b^{-1};b)(b;a)(L-b;b^{-1})(A-b;a)(L-A-a^{-1}+a;a)\\
&=(L-b^{-1};b)(b;a)(a;b^{-1})(L-b-a;b^{-1})(L-a^{-1}-b;a)\\
&\overset{R5}{=} (L-b^{-1};b)(b;a)(a;b^{-1})(L-a^{-1}-b^{-1};a)\begin{pmatrix} a & b\\ b & a^{-1}\end{pmatrix}\\
&\overset{R6}{=} (L-b^{-1};b)(b;a)(a;b^{-1})\begin{pmatrix} a & b\\ b & a^{-1}\end{pmatrix}(L-a^{-1}-b;b^{-1})\\
&=(L-b^{-1};b)(b;a)(a;b^{-1})\begin{pmatrix} a & b\\ b & a^{-1}\end{pmatrix}(a^{-1};b)(L-b;b^{-1})\\
&\overset{Z1}{=} (L-b^{-1};b)(L-b;b^{-1}) = 1.
\end{aligned}
$$

We next deal with $a = W^{\eta}$:

$$
\begin{aligned}
&(L-c_i^{-\epsilon};c_i^{\epsilon})(A;W^{\eta})(L-c_i^{\epsilon};c_i^{-\epsilon})\\
&=(L-c_i^{-\epsilon};c_i^{\epsilon})(A-c_i^{\epsilon};W^{\eta})(c_i^{\epsilon};W^{\eta})(W^{\pm};c_i^{-\epsilon})(L-W^{\pm}-c_i^{\epsilon};c_i^{-\epsilon})\\
&\overset{Z5.2}{=} (L-c_i^{-\epsilon};c_i^{\epsilon})(A-c_i^{\epsilon};W^{\eta})(W^{\pm};c_i^{-\epsilon})(c_i^{-\epsilon};W^{-\eta})(L-W^{\pm}-c_i^{\epsilon};c_i^{-\epsilon})
\end{aligned}
$$

$$
\begin{aligned}
&\overset{Z4.2,Z4.1}{=} (L - c_i^{-\epsilon}; c_i^{\epsilon})\cdot \\
&\quad \cdot(W^{\pm}; c_i^{-\epsilon})(A - c_i^{\epsilon} + c_i^{-\epsilon} - W^{\eta}; c_i^{-\epsilon})(A - c_i^{\epsilon}; W^{\eta})(A - W^{\eta}; c_i^{\epsilon})\cdot \\
&\quad \cdot(L - W^{\pm} - c_i^{\epsilon}; c_i^{-\epsilon})(L - W^{\eta} - c_i^{\pm 1}; W^{-\eta})(c_i^{-\epsilon}; W^{-\eta}) \\
&= (L - A - W^{-\eta} - c_i^{-\epsilon} + c_i^{\epsilon}; c_i^{\epsilon})(A - c_i^{\epsilon}; W^{\eta})(L - A - W^{-\eta}; c_i^{-\epsilon})\cdot \\
&\quad \cdot(L - W^{\eta} - c_i^{\epsilon}; W^{-\eta}) \\
&\overset{R3}{=} (A - c_i^{\epsilon}; W^{\eta})(L - W^{\eta} - c_i^{\epsilon}; W^{-\eta}) \\
&= (L - A; W^{-\eta}).
\end{aligned}
$$

Next we assume $a = b_j^{\eta}$, $b = c_i^{\epsilon}$ and $A \cap \underline{b}^{\pm} = \{b_j^{\eta}\}$.

$$
\begin{aligned}
&(L - c_i^{-\epsilon}; c_i^{\epsilon})(A; b_j^{\eta})(L - c_i^{\epsilon}; c_i^{-\epsilon}) \\
&= (L - c_i^{-\epsilon}; c_i^{\epsilon})(A - c_i^{\epsilon}; b_j^{\eta})(c_i^{\epsilon}; b_j^{\eta})(\underline{b}^{\pm}; c_i^{-\epsilon})(L - \underline{b}^{\pm} - c_i^{\epsilon}; c_i^{-\epsilon}) \\
&\overset{Z5.3}{=} (L - c_i^{-\epsilon}; c_i^{\epsilon})(A - c_i^{\epsilon}; b_j^{\eta})(\underline{b}^{\pm}; c_i^{-\epsilon})(\underline{b}^{\pm}; b_j^{-\eta})(c_i^{-\epsilon}; b_j^{-\eta})(L - \underline{b}^{\pm} - c_i^{\epsilon}; c_i^{-\epsilon}) \\
&\overset{Z4.3,Z4.1}{=} (L - c_i^{-\epsilon}; c_i^{\epsilon})(\underline{b}^{\pm}; c_i^{-\epsilon})(A - c_i^{\epsilon} + c_i^{-\epsilon} - b_j^{\eta}; c_i^{-\epsilon})(A - c_i^{\epsilon}; b_j^{\eta})(A - b_j^{\eta}; c_i^{\epsilon})\cdot \\
&\quad \cdot(\underline{b}^{\pm}; b_j^{-\eta})(L - \underline{b}^{\pm} - c_i^{\epsilon}; c_i^{-\epsilon})(L - \underline{b}^{\pm} - c_i^{\pm 1} + b_j^{-\eta}; b_j^{-\eta})(c_i^{-\epsilon}; b_j^{-\eta}) \\
&\overset{R3}{=} (L - A + c_i^{\epsilon} - c_i^{-\epsilon} + b_j^{\eta} - \underline{b}^{\pm}; c_i^{\epsilon})(A - c_i^{\epsilon}; b_j^{\eta})(L - A + b_j^{\eta} - \underline{b}^{\pm}; c_i^{-\epsilon})\cdot \\
&\quad \cdot(L - b_j^{\eta} - c_i^{\epsilon}; b_j^{-\eta}) \\
&\overset{R3}{=} (L - A; b_j^{-\eta}).
\end{aligned}
$$

If, however, $A \cap \underline{b}^{\pm} = \underline{b}^{\pm} - b_j^{-\eta}$, then we observe

$$
\begin{aligned}
&(L - c_i^{-\epsilon}; c_i^{\epsilon})(A; b_j^{\eta})(L - c_i^{\epsilon}; c_i^{-\epsilon}) \\
&= (L - c_i^{-\epsilon}; c_i^{\epsilon})(A - \underline{b}^{\pm} + b_j^{\pm}; b_j^{\eta})(\underline{b}^{\pm}; b_j^{\eta})(L - c_i^{\epsilon}; c_i^{-\epsilon}) \\
&\overset{Z2,Z5.1}{=} (L - c_i^{-\epsilon}; c_i^{\epsilon})(A - \underline{b}^{\pm} + b_j^{\pm}; b_j^{\eta})(L - c_i^{\epsilon}; c_i^{-\epsilon})(\underline{b}^{\pm}; b_j^{\eta}) \\
&= (L - A + \underline{b}^{\pm} - b_j^{\pm}; b_j^{-\eta})(\underline{b}^{\pm}; b_j^{\eta}) \\
&= (L - A; b_j^{-\eta}),
\end{aligned}
$$

where the next to last equality follows from the previous paragraph. This finishes the proof of Theorem 15.4.

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