Limiting Properties of a Continuous Local Mean-Field Interacting Spin System

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Hydrodynamic Limit, Propagation of Chaos, Energy Landscape and Large Deviations

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Summary

A key interest in the study of interacting spin systems is the rigorous analysis of the macroscopic dynamical behaviour of systems that are described by their microscopic evolution. In this dissertation, we investigate unbounded spin systems where the microscopic evolution is modelled by stochastic differential equations (SDE). To each site of the discrete d-dimensional torus a spin is associated. The spins are distributed on the whole real line and evolve randomly according to the SDEs. The interaction between the spins is of local mean-field type, a long-range spatially variable interaction. The strength of the interaction between two spins depends on the difference of their positions on the torus. We aim to understand rigorously the time evolution of random variables as the size of the system increases.

We prove in Chapter I the convergence of (space and spin dependent) empirical processes under proper rescaling to the classical solution of a nonlinear partial differential equation (PDE). This PDE is called hydrodynamic equation. We use the relative entropy method, to show this *hydrodynamic limit* result. To apply this method, we need to prove the existence of a classical solution of the hydrodynamic equation, which is non-linear and non-elliptic.

In Chapter II we prove the *propagation of chaos* property of the system. We show that finitely many tagged spins are in the limit mutually independent. They evolve in the limit according to stochastic differential equations, without an interaction term. Instead (compared to the original SDEs), there is a term involving the solution of the hydrodynamic equation.

In Chapter III we derive *large deviation principles* for the corresponding equilibrium system. We look at random variables that are distributed according to the invariant measure of the stochastic differential equation. For the empirical measure, defined by these random variables, we derive large deviation principles. We use a generalisation of Varadhan's lemma that is stated and proven in Appendix C.

In Chapter IV we analyse the *landscape* of the rate function of one of the equilibrium large deviation principles. We interpret this rate function as energy of the system in the limit. This is motivated by the fact that the hydrodynamic equation is the Wasserstein gradient flow of this rate function. We determine minima, critical values, bifurcation properties and lowest paths between minima.

Finally in Chapter V we prove a *dynamical large deviation principle* for the empirical processes and the empirical measures. We derive different representations of the rate functions. By one of these representations it becomes obvious that it is exponentially unlikely that empirical processes deviate from the deterministic flow. In this chapter we allow the system to be more general, e.g. it can contain a random environment and a more general diffusion coefficient.

The main distinctive features of the spin system considered in this dissertation are the relevance of the spatially fixed positions of the spins and the possibility of unbounded spins. The spatial positions of the spins affect the interaction and the initial distributions. Therefore new approaches in the proofs are necessary, in particular compared to mean field models. All these results can be used in the future to study long time phenomena like tunnelling and metastability.

Acknowledgement

> “ If you are deeply immersed and committed to a topic, day after day after day, your subconscious has nothing to do but work on your problem.
>
> Richard Hamming, 7 March 1986

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Chapter 0

Introduction

As a branch of theoretical physics, statistical mechanics uses probabilistic methods to study systems that consist microscopically of a large number of components, typically of the order 10^{23}. The microscopic systems are modelled such that the state or evolution of the system is random or uncertain. Therefore, statistical mechanics uses probabilistic methods. One of the goals of statistical mechanics is to infer from the microscopic description macroscopically observable properties of a sample. In this dissertation we analyse rigorously the macroscopic dynamical behaviour of systems that evolve microscopically according to local mean-field interacting diffusions.

0.1 The model

We study d-dimensional lattice spin systems with continuous spins in $\mathbb{R}$. For each $N \in \mathbb{N}$ we consider a system of N^d spins $\underline{\theta}^N$. Each spin is associated to a site of $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$, the periodic d-dimensional lattice of length N. The spins evolve over time according to a Langevin dynamics, given by the following system of N^d stochastic differential equations

$$\begin{aligned} &\mathrm{d}\theta_t^{i,N} = -\Psi'\left(\theta_t^{i,N}\right)\mathrm{d}t + \frac{1}{N^d}\sum_{j\in\mathbb{T}_N^d} J\left(\frac{i-j}{N}\right)\theta_t^{j,N}\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{i,N} \quad \text{and} \\ &\theta_0^{i,N} \sim \nu_i^N \in \mathbb{M}_1(\mathbb{R})\,. \end{aligned} \tag{0.1.1}$$

This stochastic differential equation contains three different influences on the spins:

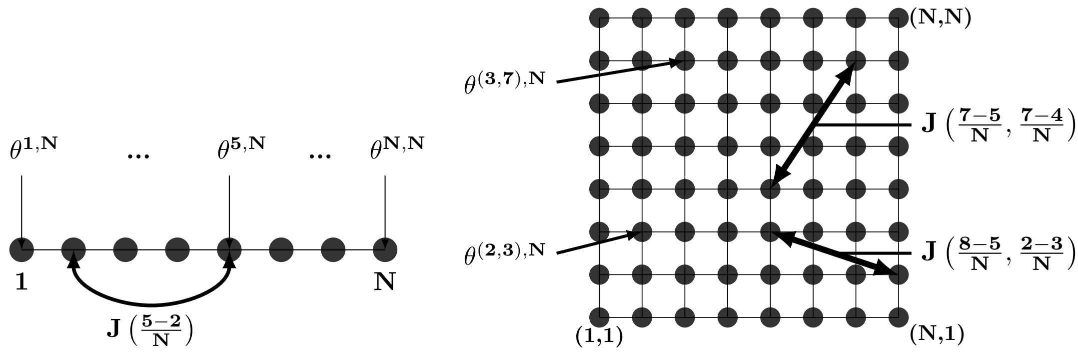
1. There is a local, single-spin potential $\Psi : \mathbb{R} \to \mathbb{R}$, e.g. $\Psi(\theta) = \theta^4$ or $\theta^4 - \theta^2$. It can represent, for example, the influence of the underlying structure of the material and stands for the anisotropy energy. Mathematically, this potential ensures that the spins stay with high probability within a compact set.

2. All spins affect the evolutions of the other spins through the interaction. We consider a two-body interaction. Two spins interact with each other weighted by $J : \mathbb{T}^d \to \mathbb{R}$ according to the difference of their fixed spatial positions.

3. Each spin is stochastically perturbed by an independent Brownian motion $W^{i,N}$ on $\mathbb{R}$.

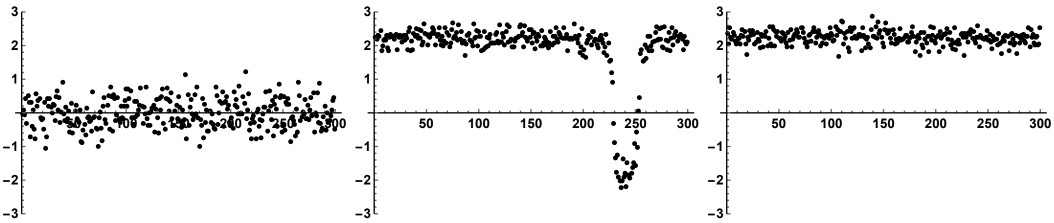
Systems similar to (0.1.1) appear in many different contexts. This includes, for example, spatial versions of the Kuramoto model ([RW96], [MdV02], [GPR12]), neuronal science ([BFFT12], [LS14] and reference therein), chemical kinetics ([Sch86]) or finance ([GPY13]).

Visualisation of the model. We give in Figure 0.1 a schematic picture of the spatial positions and values of the spins, and of the interaction. In this figure one sees boxes of side length N in the



(a) Box of length N in the 1d lattice (b) Box of length N in the 2d lattice

Figure 0.1: There is a spin $\theta^{i,N}$ at each site in the box. The arrows with the two heads are examples of the mutual interaction between the spins.



(a) Start: $t = 0$ (b) $t = 10$ (c) $t = 100$

Figure 0.2: Simulation with $N = 300$, Interaction: $J\left(\frac{i-j}{N}\right) = 5\frac{N}{2\frac{N}{30}+1}\mathbb{1}_{|i-j|\leq\frac{N}{30}}$, $\Psi(\theta) = \theta^4$.

one and the two dimensions lattice. To each site in this boxes a spin is attached which is indexed by its position $i \in \mathbb{T}_N^d$ on the lattice, e.g. $i = 1, ..., N$ respectively $i = (1,1), ..., (N,N)$. In Figure 0.1 we show some examples of the random variables that represent the values of the spins, $\theta^{1,N}, ..., \theta^{N,N}$ in the one dimensional case and $\theta^{(2,3),N}$,$\theta^{(3,7),N}$ in the two dimensional case. Moreover, all spins interact mutually with each other. The strength of the interaction between two spins is weighted by J according to their spatial distance. We indicate this interaction by the arrows with two heads in Figure 0.1. Note that these arrows are just examples, because between all pairs of spins there is a weighted interaction.

Next, we visualise the time evolution of the one dimensional system. Fix a realisation $\underline{\theta}^N_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{R}^{N^d}\right)$ of the system evolving according to (0.1.1). Let us visualise this realisation at a given time $t \in [0,T]$. For each $i \in \mathbb{T}_N$, draw a dot at $(i, \theta_t^{i,N})$ in a two dimensional coordinate system (see for example Figure 0.2a). Then one gets as x-axis the fixed spatial position of the spins and as y-axis the value of the spin at time t. The arising picture in Figure 0.2a represents the state of the spin system at a given time $t = 0$. Over time the spins evolve but not the spatial positions is fixed. Hence, the dots move up or down, but not left or right.

Simulation. We have simulated this time evolution for $N = 300$ spins with Mathematica. In Figure 0.2 we show the state of the system for three chosen time points. At first, the spins are randomly distributed with values close to zero (see Figure 0.2a). After a while (see Figure 0.2b) the spins concentrate near a positive and a negative stable point. Finally, all spins are close to one of the two stable points (Figure 0.2c) for a long time. On a longer time scale transitions between the two stable points occur. Note that the parameters in the simulation are chosen in such a way that there are two stable points. For a weaker interaction, zero might be the only stable point.

Increasing the size of the box. We are in particular interested in the time evolution when the number of spins, N^d, tends to infinity. While increasing the number of spins, we decrease the distance between neighbouring spins, such that the mesh width equals $\frac{1}{N}$. In the one dimensional lattice (in Figure 0.1) the distance between the site i and $i+1$ equals $\frac{1}{N}$.

This increase of the box size while decreasing the mesh width can be interpreted as zooming out with a microscope. Then one sees more particles while the distances seem to be smaller. Mathematically, this scaling appears as normalisation in the interaction term in (0.1.1). The normalisation is chosen in such a way that the number of spins that interact with a fixed spin is of the order N^d, while the strength of each of the interactions decreases like $\frac{1}{N^d}$.

This introduction is organised as follows. At first, we introduce and motivate in Section 0.2 the five topics that we investigate in this dissertation. To analyse these five topics, it is necessary to consider locally averaging random variables. This is motivated and introduced in Section 0.3. To give a first glimpse into the five topics, we consider in Section 0.4 a simple system of independent spins evolving according to a Ornstein-Uhlenbeck processes instead of (0.1.1). In Sections 0.5 - 0.9 we explain separately the five topics for the interacting system (0.1.1). Each section includes an historical overview and the main results that we derive in this dissertation. Finally, in Section 0.10, we list interesting, future challenges for the model (0.1.1) and similar ones. The most important notations, that are used throughout the remainder of this dissertation, are listed in Section 0.11.

0.2 The topics

We are interested in the macroscopic behaviour of the microscopic system evolving according to (0.1.1). Therefore, we derive in this dissertation results for the following five topics, that we explain later in more details. The five topics have in common, that the behaviour of the system, when the number N^d of spins tends to infinity, is investigated.

I The hydrodynamic limit and the hydrodynamic equation.

II Propagation of chaos.

III The equilibrium large deviation principle corresponding to particles distributed according the invariant measure of (0.1.1).

IV Properties of the rate function of the equilibrium large deviation principle like minima, saddle points and optimal paths.

V Dynamical large deviations from the hydrodynamic limit.

We interpret the rate function of the equilibrium large deviation principle as energy of the macroscopic system. This is motivated by the fact that the hydrodynamic equation is the Wasserstein gradient flow of this rate function (see Section 0.8 for more details). Hence, we get insight into the landscape given by the hydrodynamic equation, by investigating the landscape of the rate function.

The five topics are motivated by the overall desire to understand long time phenomena, like metastability, in models like (0.1.1). For an introduction to metastability, we refer to [OV05] and [BdH15]. By the Freidlin-Wentzell theory (see the introduction of [DG87], [FW98] Section 10.5, [FJL82] for a generalisation to infinite dimensional system), these long time phenomena are related to large deviation principles and the hydrodynamic equation (see [FW98]). The heuristic underlying idea is, that for large N, the time evolution of the system is governed by the deterministic flow described by the hydrodynamic equation. Then the system converges towards a globally attracting, stable point. Therefore we want to understand the hydrodynamic limit and the hydrodynamic equation (Topic I) and the stable points (Topic IV). However, for finite N, the noise lets the system deviate from the deterministic flow. Even a transition from one stable point to another occur on a long time scale. Hence, it is relevant to know the lowest paths between the stable points and the saddle points (Topic IV). These deviations from the deterministic flow are exponentially unlikely for large N. The dynamic large deviation principle (Topic V) is an appropriate result to determine the probability of such an event.

The system (0.1.1) has two features that distinguishes it from most of the existing literature concerning the five Topics I-V for interacting diffusions. On the one hand the position of the spins is highly relevant. The interaction depends on the fixed spatial distance between two spins. Also the initial distribution of the spins depend on the spatial positions of the spins. On the other hand we do not assume the boundedness of the spins. This implies in particular that the contribution from the interaction term in (0.1.1) might not be bounded.

0.3 Locally averaging random variables

One observes an interesting phenomena, in Figure 0.2c, the state at time $t = 100$ of the simulation of the system (0.1.1). Despite the interaction, neighbouring spins are still somehow randomly distributed around the stable point. This becomes even more obvious when increasing the number of particles, because through the normalisation of the interaction the influence of each particle on each other particle decreases like $\frac{1}{N^d}$. We can interpret this phenomena as entropy. It comes into play as an additional factor when considering large number of spins and in particular when taking the limit $N \to \infty$.

An heuristic argument against an approach that uses linear interpolation of spins Let us now heuristically explain how this observation excludes an approach that uses linear interpolation of spins. Fix an arbitrary realisation $\underline{\theta}^N_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{R}^{N^d}\right)$ of the system evolving according to (0.1.1). Define a function $u^N \in \mathsf{C}\left([0,T] \times \mathbb{T}^d\right)$ that has for each $i \in \mathbb{T}^d_N$ the value

$$u^N\left(t, \frac{i}{N}\right) := \theta^{i,N}_t \tag{0.3.1}$$

and that interpolates linear between these values. Then the entropy would lead (when ignoring other smoothing effects) to high oscillations of u^N when increasing N. In particular, if a limit of the sequence $\left\{u^N\right\}_N$ exists, it would not be continuous.

When considering nearest neighbour interaction instead of the local mean-field interaction, then a Laplace operator arises in the limit. The eigenvalues (with respect to the Fourier base) of this operator are tending to infinity. Hence, highly oscillating functions are punished the more the Laplace operator is weighted compared to the local potential (see [BFG07b] in particular Figure 1 therein). Hence, the Laplace operator smooths the paths defined in (0.3.1) such that an approach that uses linear interpolation of spins is possible for models with nearest neighbour interaction.

For the system (0.1.1), we get the convolution operator $J * \cdot$ and not a Laplace operator in the limit. The convolution operator $J * \cdot$ is self adjoint and compact (we show this in Section IV.1.1.5 for continuous J). Hence, its eigenvalues tend to zero and the interaction operator $J * \cdot$ does not smooth the paths defined in (0.3.1), but favours high oscillations. This implies that an approach that uses linear interpolation is not applicable for the local mean-field interaction considered here.

The random variables. Instead of a linear interpolation of spins, we consider random variables, that average out locally these oscillation. One of these random variables is the empirical process, defined for a given realisation $\underline{\theta}^N_{[0,T]} = \left\{t \mapsto \underline{\theta}^N_t\right\} \in \left(\mathsf{C}([0,T])\right)^{N^d}$ of (0.1.1) by

$$\mu^N_{[0,T]} := \left\{t \mapsto \mu^N_t := \frac{1}{N^d} \sum_{k \in \mathbb{T}^d_N} \delta_{\left(\frac{k}{N}, \theta^{k,N}_t\right)}\right\} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right), \tag{0.3.2}$$

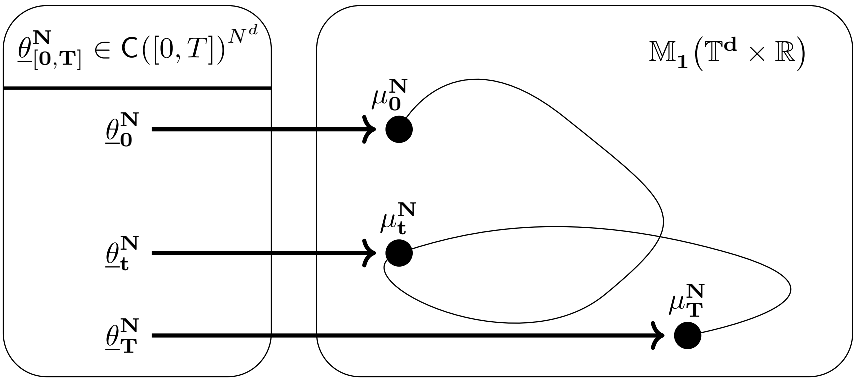


Figure 0.3: The construction of the empirical process $\mu^N_{[0,T]}$. The black line in the right box represents the empirical process as continuous path in $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$.

i.e. the time evolution of the paths of the (space-spin) empirical measures. $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ denotes the space of probability measures on $\mathbb{T}^d \times \mathbb{R}$, equipped with the topology of weak convergence. Moreover, we look at the empirical measures L^N on $\mathbb{T}^d \times \mathsf{C}([0,T])$, given by

$$L^N = L^N\left(\underline{\theta}_{[0,T]}\right) := \frac{1}{N^d} \sum_{k \in \mathbb{T}^d_N} \delta_{\left(\frac{k}{N}, \theta^{k,N}_{[0,T]}\right)} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathsf{C}([0,T])\right). \tag{0.3.3}$$

By the normalisation in the space variable, the empirical process $\mu^N_{[0,T]}$ and the empirical measure L^N contain in the limit at each space point a local average. These local averages smooth the oscillations arising from the entropy. Nevertheless, we see crucial differences in the landscape of the model (0.1.1) and the nearest neighbour model (see the discussion in Section 0.8.1).

In Figure 0.3 we visualise the construction of $\mu^N_{[0,T]}$. For each $t \in [0,T]$, the vector $\underline{\theta}^N_t \in \mathbb{R}^{N^d}$ is mapped to $\mu^N_t \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$. The arrows from the left box to the right box in Figure 0.3 sketch this map for $t = 0$, $t = T$ and a $t \in (0,T)$. All μ^N_t together form a connected line in the space $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ (see the black line in the box on the right hand side in Figure 0.3). The map $[0,T] \ni t \mapsto \mu^N \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ is continuous. Hence, the black line in Figure 0.3 represents an element of $\mathsf{C}([0,T], \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}))$.

0.4 The Topics I - V for a system of independent spins

We explain and motivate the Topics I - V, by analysing a simpler system than (0.1.1). When neglecting the interaction between the spins, i.e. $J \equiv 0$, the model simplifies dramatically. In this section, let the spins evolve according to an Ornstein–Uhlenbeck process, defined by

$$\begin{aligned} \mathrm{d}\theta^{i,N}_t &= -\theta^{i,N}_t \mathrm{d}t + \sqrt{2}\mathrm{d}W^{i,N}_t \quad \text{and} \\ \theta^{i,N}_0 &\sim \nu_{\frac{i}{N}} \in \mathbb{M}_1(\mathbb{R}). \end{aligned} \tag{0.4.1}$$

The spins evolve mutually independently with a quadratic single spin potential Ψ, i.e. $\Psi(\theta) = \frac{1}{2}\theta^2$. However, they are not identically distributed due to spatial differences in the initial distributions. The reasoning given in Section 0.3, why we consider the empirical process $\mu^N_{[0,T]}$ and the empirical measure L^N, is a fortiori true without any interaction.

Assume that there is a continuous and compactly supported $\xi^0 \in \mathsf{C}_c(\mathbb{T}^d \times \mathbb{R})$, such that $\nu_x(\mathrm{d}\theta) = \xi^0(x,\theta)\,\mathrm{d}\theta$ for all $x \in \mathbb{T}^d$. Due to the independence of the evolutions and of the initial distributions of the spins, we can analyse the Topics I-V in an easy way.

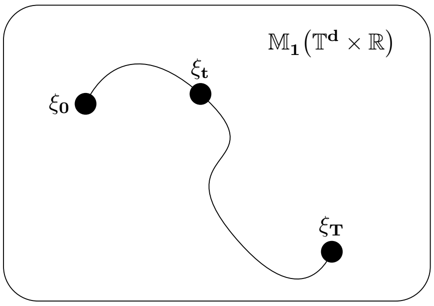


Figure 0.4: The deterministic path of the hydrodynamic limit, described as the solution to the hydrodynamic equation.

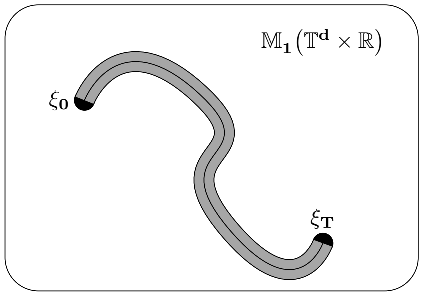


Figure 0.5: The probability of $\mu^N_{[0,T]}$ being not in the grey tube around the hydrodynamic limit decays exponentially fast.

Hydrodynamic limit (Topic I). To study the time evolution of the whole macroscopic system one can use the hydrodynamic limit approach. The idea of this approach is to average out (locally) rapid small variations of single spins by a law of large numbers. The macroscopic time evolution is described by a PDE, the so called *hydrodynamic equation* (see Section 0.5 for more details). As explained before, the empirical process $\mu^N_{[0,T]}$ contains such a local average in the limit. Indeed, the random variables $\left\{\mu^N_{[0,T]}\right\} \subset \mathsf{C}\big([0,T], \mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)\big)$ converge to a deterministic trajectory on $\mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)$. Each measure on this trajectory has a density ρ_t w.r.t. $e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}x$. Moreover, $\xi := \rho e^{-\Psi} \in \mathsf{C}_b\big([0,T] \times \mathbb{T}^d \times \mathbb{R}\big) \cap \mathsf{C}^{1,0,2}\big((0,T) \times \mathbb{T}^d \times \mathbb{R}\big)$ is the classical solution of the following partial differential equation

$$\begin{aligned} \partial_t \xi_t(x,\theta) &= \partial_\theta \left(\Psi'(\theta)\, \xi_t(x,\theta)\right) + \partial_\theta^2 \xi_t(x,\theta) && \text{for all } (t,x,\theta) \in (0,T) \times \mathbb{T}^d \times \mathbb{R} \\ \xi_0(x,\theta) &= \xi^0(x,\theta) && \text{for all } (x,\theta) \in \mathbb{T}^d \times \mathbb{R}. \end{aligned} \tag{0.4.2}$$

We visualise this deterministic path in the space $\mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)$ by the black line in Figure 0.4. We state a simple proof of this hydrodynamic limit result after discussing the dynamical large deviation.

Propagation of chaos (Topic II). A system has the propagation of chaos property, if finitely many spins evolve independently (at least as N tends to infinity), provided the initial values are independently distributed (see Section 0.6 for more details). Obviously this is satisfied for spins that evolve according to (0.4.1) already for finite N. Moreover, each spin evolves also in the limit according to an Ornstein–Uhlenbeck process.

Equilibrium large deviation principle (Topic III). Next, we consider the equilibrium large deviation principle for empirical measures of random variables distributed according to the invariant measure of (0.4.1). This is the Gaussian distribution with mean 0 and variance 1. Let Θ^i be random variables distributed according to this invariant measure, i.e. $\theta^i \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2}\mathrm{d}\theta$. Define the (space and spin) empirical measures by $\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N} \delta_{\left(\frac{i}{N},\Theta^i\right)} \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)$. These empirical measures satisfy the large deviation principle with good rate function

$$\mathcal{H}(\nu) := \int_{\mathbb{T}^d} \mathsf{H}\left(\rho(x,\cdot) \middle| \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2}\right) \mathrm{d}x, \tag{0.4.3}$$

if $\nu = \rho(x,\theta)\,\mathrm{d}x\mathrm{d}\theta$ and $\rho(x,\theta)\,\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$ for almost all $x \in \mathbb{T}^d$. Otherwise $\mathcal{H}(\nu) = \infty$. This follows for example by a generalisation of Sanov's theorem (see Lemma V.2.7 in Chapter V).

Energy landscape (Topic IV). We interpret the rate function $\mathcal{H}$ as energy of the system (0.4.1). Heuristically, this can be justified as follows: The hydrodynamic equation (0.4.2) can be written as the following Wasserstein gradient flow w.r.t. $\mathcal{H}$ (in the Monge-Kantonovich gradient flow sense, see [Vil03] Section 8.3)

$$\partial_t \rho = \partial_\theta \left(\rho \partial_\theta \left(\frac{\delta \mathcal{H}(\rho)}{\delta \rho} \right) \right). \tag{0.4.4}$$

However, the energy landscape is not very interesting for the system (0.4.1). The invariant measure, $\frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\theta^2} \mathrm{d}\theta$, is the unique critical value of $\mathcal{H}$ on suitable subset of $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$. Hence, interesting effects, like transitions between minima, cannot occur in this model.

Dynamical large deviation principle (Topic V). For large N, the system does not have to follow the macroscopic evolution described by (0.4.2). Indeed, the noise lets the system deviate from this deterministic evolution. It arises the question, how probable it is to see an empirical process that is not close to the solution of the hydrodynamic equation. For example, we want to estimate the probability of being not in the grey tube in Figure 0.5, where the black line in the centre of this tube is the hydrodynamic limit. With the theory of large deviation one studies the exponential decay of the probability of such tail events. Therefore, we derive the dynamical large deviation principle for the empirical process $\left\{\mu^N_{[0,T]}\right\}$.

One gets the large deviation principle for the empirical measures $L^N \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathsf{C}([0,T])\right)$ (defined in (0.3.3)), by an extension of the Sanov theorem to space dependent, but mutually independent random variables (see Lemma V.2.7 in Chapter V). The good rate function L_ν is for $Q \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathsf{C}([0,T])\right)$ given by the space integral of the relative entropy

$$L_\nu(Q) := \int_{\mathbb{T}^d} \mathsf{H}(Q_x | P_x)\, \mathrm{d}x, \tag{0.4.5}$$

if $Q = \mathrm{d}x \otimes Q_x$ and $L(Q) = \infty$ otherwise. Here $P_x \in \mathbb{M}_1(\mathsf{C}([0,T]))$ is the distribution of the time evolution of a spin, that evolves according to (0.4.1) with initial distribution ν_x. The relative entropy is defined as

$$\mathsf{H}(Q_x|P_x) := \begin{cases} \int_{\mathbb{R}} \log\left(\frac{\mathrm{d}Q_x}{\mathrm{d}P_x}\right) Q_x & \text{if } Q_x << P_x, \\ \infty & \text{otherwise.} \end{cases} \tag{0.4.6}$$

By the contraction principle, we infer that $\left\{\mu^N_{[0,T]}\right\} \subset \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$ satisfies the large deviation principle with good rate function

$$S_\nu\left(\mu_{[0,T]}\right) := \inf_{Q \in \mathbb{M}_1(\mathbb{T}^d \times \mathsf{C}([0,T])) : \Pi(Q) = \mu_{[0,T]}} L_\nu(Q), \tag{0.4.7}$$

for $\mu_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$, where $\Pi : \mathbb{M}_1\left(\mathbb{T}^d \times \mathsf{C}([0,T])\right) \to \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$ is the canonical map such that $\Pi\left(L^N\right) = \mu^N_{[0,T]}$. Moreover, we can show (by the same arguments used in Section V.3.1) that the rate function S has also the following representation

$$S_\nu\left(\mu_{[0,T]}\right) = \int_0^T \left|\partial_t \mu_t - \partial_\theta\left(\Psi' \mu_t\right) - \partial_\theta^2 \mu_t\right|^2_{\mu_t} \mathrm{d}t + \mathsf{H}(\mu_0 | \mathrm{d}x \otimes \nu_x), \tag{0.4.8}$$

for suitable $\mu_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$. The explicit form of the norm $|.|_{\mu_t}$ (see Definition V.1.10) is at this stage not relevant. But it is important to see, that the rate function S_ν measures somehow the deviation from the hydrodynamic equation. Applied to Figure 0.5, this result tells us, that the probability of μ^N being outside of the grey tube decays like $e^{-N^d a}$, for an $a > 0$.

Proof of the hydrodynamic limit. We prove the hydrodynamic limit by showing that S has a unique minimizer and that this minimizer has a density ζ, that satisfies the hydrodynamic equation (0.4.2). If $S\left(\mu_{[0,T]}\right)=0$, then there is a sequence $\left\{Q^{(n)}\right\}_n\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathsf{C}([0,T])\right)$ with $\Pi\left(Q^{(n)}\right)=\mu_{[0,T]}$, such that $L\left(Q^{(n)}\right)\to 0$. Hence, $Q^{(n)}\to \mathrm{d}x\otimes P_x$ (by the Pinsker inequality) and $\Pi\left(\mathrm{d}x\otimes P_x\right)=\mu_{[0,T]}$ (by the continuity of Π). But for each $x\in\mathbb{T}^d$, P_x is the distribution of Ornstein-Uhlenbeck process (0.4.1), hence its time marginal is the classical solution of (0.4.2).

0.5 Hydrodynamic Limit (Topic I)

The aim of the hydrodynamic limit approach is to derive rigorously results about macroscopic quantities from a microscopic system that consists of a large number of spins (or particles). By this large number of spins, rapid small variations of single spins are locally averaged out on the macroscopic scale by a law of large numbers. One takes advantage of this behaviour of the system, such that only the evolution of the macroscopic quantities stays visible in the limit. This limiting time evolution is described by the so called hydrodynamic equation. We refer to [KL99] (in particular Chapter 1) and [Var00] for an introduction.

There are the following four main strategies for proving hydrodynamic limits for empirical processes of interacting particle/spin systems. In each of these approaches one requires different (non probabilistic) results concerning the hydrodynamic equation.

(S.1) The first strategy consists of two steps.

1. Show that the sequence of the laws of the empirical process is relatively compact.
2. Prove that the accumulation points of subsequences is unique.

For the second step one proves usually that the limit points of each subsequence is concentrated on weak solutions of the hydrodynamic equation and that there is exactly one weak solutions of the hydrodynamic equation.

By this approach the hydrodynamic limit is shown for various models with mean-field interaction ([Daw83],[Gär88],[Oel84],[L8́6], [Luç11]) and for models with moderate interaction ([Oel85]). Also for discrete spin systems this method is used to prove the hydrodynamic limit, e.g. for the symmetric simple exclusion process (see [KL99], in particular Chapter 4, and references therein) and for the symmetric zero range process ([KL99] Chapter 5).

(S.2) The hydrodynamic limit result also follows from the dynamical large deviation principle, if one can show that the rate function has a unique minimizer. We used this strategy in Section 0.4 for the independent system. To show that the rate function has only one minimizer, a uniqueness results of the hydrodynamic equation is often required (see for example [DPdH96] and also the rate function derived in [DG87] is only zero on the weak solution of the hydrodynamic equation).

These two approaches require usually uniqueness of (weak, probability measure valued) solutions of the hydrodynamic equation. This is not the case in the following two strategies. In both strategies one has to show at first the existence of a (classical) solution of the hydrodynamic equation. Then one defines a system of N^d independent SDEs, by replacing the interaction in (0.1.1) with a given effective field. This effective field is defined via the solution of the hydrodynamic equation, in such a way that the law of the SDE is the solution of the hydrodynamic limit (see (0.5.4)).

(S.3) One strategy is used in [LS14] where it is called "*propagator method*". The authors bound the expected (in a suitable way defined) distance between the empirical process $\mu^N_{[0,T]}$ and the solution of the hydrodynamic limit. To get this bound, they use the semigroup of the system of SDEs with fixed effective field. The main step is to derive separately bounds on the propagated difference of the initial distributions, on the influence of the differences of the interactions and on a martingale term arising from the propagated influence of the randomness. These estimates require regularity properties of the semigroup.

(S.4) The last strategy is the "*relative entropy method*". Transferred to the diffusion setting considered here, one looks at the relative entropy of the law of the solution of (0.1.1) w.r.t. the law of the system of N^d independent SDEs, defined with the effective field as explained above. The main step is to show that this relative entropy is or order $o\left(N^d\right)$. A general inequality concerning the relative entropy, in combination with a large deviation result, proves the hydrodynamic limit. We refer to [Yau91] and Chapter 6 in [KL99] and references therein for further and more general information on this approach. In Chapter 4 in [Var00] this method is used for an asymmetric simple exclusion process.

To prove an hydrodynamic limit result for the system (0.1.1), we use the *relative entropy method* (Strategy (S.4)). Beside the elegance of this approach, let us shortly explain why we do not use the other strategies. The main reason that excludes the Strategy (S.1) and (S.2) is that we are not able to show a priori the uniqueness of (weak) solutions of the hydrodynamic equation. More precisely these strategies require at least uniqueness of those weak solutions that might be attained in the limit. Although we expect this uniqueness, we are not able to show it in general, because of the space dependency of the hydrodynamic equation and of the initial distributions (see Section I.3 for uniqueness in the smaller class of classical solutions).

We give now a short historical overview with focus on the differences between the considered models and (0.1.1). In Section 0.5.2, we state the hydrodynamic limit result, that we derive in Chapter I, and explain its proof in more details.

0.5.1 Historical overview

In several papers the hydrodynamic limit, for models of interacting diffusions that are similar to (0.1.1), is investigated. These models differ from (0.1.1) either in the interaction, the local (single spin) potential, the possible range of the values of a spin or the dynamic itself. Due to these differences these results do not include the model we consider as we sketch in the following.

In [KPT05] the hydrodynamic limit is heuristically studied for the model (0.1.1) by the Strategy (S.1), however for spins with restricted values to the range $[0, 2\pi)$ instead of $\mathbb{R}$. We call it heuristic for three reasons. First of all, the authors do not discuss why the spins take values within the claimed range. Moreover, they state only the ideas of a proof, that relies on the bounded range. Last but not least for the required uniqueness of weak solutions of the hydrodynamic equation, they refer to a proof in [CL97] where only linear potentials like $\Psi(\theta) = a\theta - b$ are studied. We are interested in more general potentials, in the possibility of unbounded spins and a rigorous proof. These demands are not satisfied by the result of [KPT05], although some of their ideas are useful.

The hydrodynamic limit for a space dependent system similar to (0.1.1) with additional random environment is derived in [LS14], using Strategy (S.3). In this model, the contribution of the spins to the interaction is bounded and not linear as in (0.1.1). Moreover, their approach shows the convergence of the empirical process to the solution of the hydrodynamic equation in a distance that is adapted to the interaction weight J. For this specific distance the authors get a rate of convergence. However, we want to get convergence in the topology of weak convergence (which is in general stronger than the topology considered in [LS14]). Even if one could generalise the approach in [LS14] to unbounded contribution of the spins, this last requirement would not be satisfied. Moreover, we show more regularity of the hydrodynamic limit compared to [LS14].

Many authors investigated a spin system that evolves according to Langevin dynamics with a non-space dependent interaction. These are so called mean-field model. For constant interaction weight J and initial distributions that are not space-dependent, the system (0.1.1) is a mean-field model. For example, in [Daw83], [Oel84], [L8́6], [Gär88], the hydrodynamic limit is proven for mean-field models under various assumptions on the drift and diffusion coefficients. In all these papers the Strategy (S.1) is used. In particular, the authors are able to prove the uniqueness of the limit points. This is possible, because the considered empirical measures are without the spatial position and consequently also the hydrodynamic equation is not space dependent. This has among others the advantage that the operator in this PDE is elliptic. For the space dependent system (0.1.1), we are not able to prove uniqueness. However, our result covers the hydrodynamic limit results for most of these mean-field models as long as the diffusion coefficient does not depend on the empirical measure (as considered in [Gär88]).

In [DPdH96], [Luç11] a mean-field interacting Langevin dynamic with an additional random medium at each spin, is studied. In [Luç11] also Strategy (S.1) is used, that has the above mentioned obstacles. In contrast, [DPdH96] uses Strategy (S.2). At first, the authors show the large deviation principle for empirical measures like L^N (defined in (0.3.3), but with random medium dependency instead of space dependency). Then they prove uniqueness of the minimum of the rate function. This proof requires that solutions of a non interacting system of SDEs has a density that is a classical solution to the corresponding Fokker-Planck equation. They infer from the uniqueness of such classical solutions, the uniqueness of the minimum. This method cannot be generalised to the system (0.1.1). On the one hand only bounded drift coefficients are considered in [DPdH96] (whereas the coefficients in (0.1.1) are in general unbounded). On the other hand the statements concerning the Fokker-Planck equation used in [DPdH96] are not any more obvious for the space dependent system (0.1.1).

Systems like (0.1.1) and the mean-field models mentioned above are examples of models with weak interaction. In these systems, the number of spins that interact with each other are of order N^d and the influence of one spin on another is of order $\frac{1}{N^d}$. In contrast, in [Oel85] and [JM98] the law of large numbers for so called moderate interaction are derived. There, a single spin interacts with less spins (e.g. of order $N^{\beta d}$ for a $\beta \in (0,1)$), but each spin has more weight compared to weak interaction. Hence, the interaction gets (from a macroscopic view) more and more local when N tends to infinity. This implies in particular a different limiting macroscopic dynamic compared to systems with weak-interaction. For example, no integral term appears in the PDE that describes the macroscopic evolution.

In [CE88] and [Com87] a model with space dependent interaction is investigated. However, the spins evolve according to a jump dynamic instead of Langevin dynamic. For this model the authors prove in [CE88] a law of large numbers for $\gamma^N := \left\{t \mapsto \frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d} \theta_t^i \delta_{\frac{i}{N}}\right\}$ and for the empirical process $\mu^N_{[0,T]}$. For the empirical process they prove moreover a propagation of chaos result and a dynamical central limit theorem. The proofs of these results are highly dependent on the particular jump dynamic and cannot be generalised to models with Langevin dynamics like (0.1.1). For a discussion of the hydrodynamic limit result for γ^N see Section I.2.1.2.

0.5.2 The new hydrodynamic limit result

In Chapter I we prove the hydrodynamic limit result for the spin system (0.1.1). Let us explain this result in the following. For the proof, we assume some conditions on the drift coefficient and the initial distributions in (0.1.1). Let us sketch these assumptions, that we state detailed in Chapter I.

Assumption (sketched) 0.5.1. a.) *J is in $L^1\left(\mathbb{T}^d\right)$ and satisfies some convergence properties, e.g. J is continuous or $J(x) = x^{-1+\epsilon}$.*

b.) *Ψ is an even polynomial that grows faster at infinity than $\|J\|_{L^1}\theta^2$, e.g. $\Psi(\theta) = \theta^4 - \frac{1}{2}\theta^2$.*

c.) *The initial distributions ν^N satisfy some integrability conditions. Moreover, these measures converge in a suitable sense to a measure with density $\rho^0 e^{-\Psi}$.*

Under these assumptions, we prove in Chapter I the following hydrodynamic limit result.

Result I. *The (random) empirical process $\mu^N_{[0,T]}$ (defined in (0.3.2)), for a spin system that evolves according to (0.1.1), converges to a (deterministic) element of $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$. Each of the* *measures on this limiting trajectory has a density* ρ_t *w.r.t.* $e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}x$*, i.e.*

$$\mu^N_{[0,T]} \quad \to \quad \left\{t \mapsto \rho_t\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}x\right\} \qquad \text{as } N \to \infty. \tag{0.5.1}$$

Set $\xi := \rho e^{-\Psi}$*. Then* $\xi \in \mathsf{C}_b\big([0,T]\times\mathbb{T}^d\times\mathbb{R}\big)\cap\mathsf{C}^{1,0,2}\big((0,T)\times\mathbb{T}^d\times\mathbb{R}\big)$ *and* ξ *is the unique classical solution of the following PDE*

$$\partial_t\xi_t\left(x,\theta\right) = \partial_\theta\left(\left(\Psi'\left(\theta\right) - \int_{\mathbb{T}^d\times\mathbb{R}} J\left(x'-x\right)\theta'\xi_t\left(x',\theta'\right)\mathrm{d}\theta'\mathrm{d}x'\right)\xi_t\left(x,\theta\right)\right) + \partial^2_\theta\xi_t\left(x,\theta\right), \tag{0.5.2}$$

with initial condition $\xi_0 = \rho^0 e^{-\Psi(\theta)}$ *or equivalently*

$$\begin{aligned}\partial_t\rho_t\left(x,\theta\right) &= e^{\Psi}\partial_\theta\left(e^{-\Psi}\partial_\theta\rho_t\left(x,\theta\right)\right) - e^{\Psi}\partial_\theta\left(J * h^\rho\left(t,x\right)\ e^{-\Psi}\rho_t\left(x,\theta\right)\right) \quad \text{where}\\ h^\rho\left(t,x\right) &= \int_{\mathbb{T}^d\times\mathbb{R}} J\left(x'-x\right)\theta' e^{-\Psi}\rho_t\left(x',\theta'\right)\mathrm{d}\theta'\mathrm{d}x'.\end{aligned} \tag{0.5.3}$$

Interpretation of the result. By Result I, we can describe mathematically the macroscopic evolution of a system defined by the underlying microscopic dynamics. The macroscopic system evolves temporally according to the PDE (0.5.2), which is hence called the hydrodynamic equation. By the hydrodynamic limit approach, the randomness of the finite systems is averaged out, such that the macroscopic evolution is deterministic. This averaging happens locally at each spatial position $x \in \mathbb{T}^d$ by the form of the empirical process (see (0.3.2)). This can be seen by the following heuristic argument: The macroscopic dynamic at a point $x \in \mathbb{T}^d$ is the limit of the behaviour of the spins in an arbitrary small neighbourhood A_x of x. For finite N, there are $N^d\left|A_x\right|$ spins in A_x. When ignoring the influences of the interaction, the sum of the randomness of these $N^d\left|A_x\right|$ spins is of order $\sqrt{N^d\left|A_x\right|}$ (by the noise). Hence, in the macroscopic limit (by the prefactor $\frac{1}{N^d}$ in the empirical process), the influence of the randomness vanishes.

We call the hydrodynamic equation (0.5.2) also *local mean-field McKean-Vlasov equation.* This has the reason that the hydrodynamic equation for the mean-field models is also called McKean-Vlasov equation (see [Gär88]). The latter equation has a similar form as (0.5.2), of course without the space variable $x \in \mathbb{T}^d$.

Idea of proof. We prove the hydrodynamic limit result (Result I) by the *relative entropy method* (Strategy (S.4)). In particular, we show the following two statements:

1. We prove the existence of a function ρ^*, that is at the same time

   - a classical solution of (0.5.2) and
   - $\rho^*_t\left(\frac{i}{N},.\right)$ is for each $t \in [0,T]$, $i \in \mathbb{T}^d_N$, $N \in \mathbb{N}$ a probability density and the law of

   $$\mathrm{d}\widehat{\theta}^{i,N}_t = -\Psi'\left(\widehat{\theta}^{i,N}_t\right)\mathrm{d}t + h^*\left(t,\frac{i}{N}\right)\mathrm{d}t + \sqrt{2}\mathrm{d}\widehat{W}^{i,N}_t. \tag{0.5.4}$$

   Here $h^* = h^{\rho^*}$ is the effective field defined as in the second line of (0.5.3) by ρ^*.

   Note that (0.5.2) is the non-linear Fokker-Planck equation corresponding to (0.5.4).

   The challenges of this existence proof are in particular due to

   - the required relation of the hydrodynamic equation (0.5.2) to the solution of the SDE (0.5.4),
   - the non linearity, unboundedness and non Hölder -continuity of the drift coefficient and
   - the non-ellipticity of the right hand side of the hydrodynamic equation.

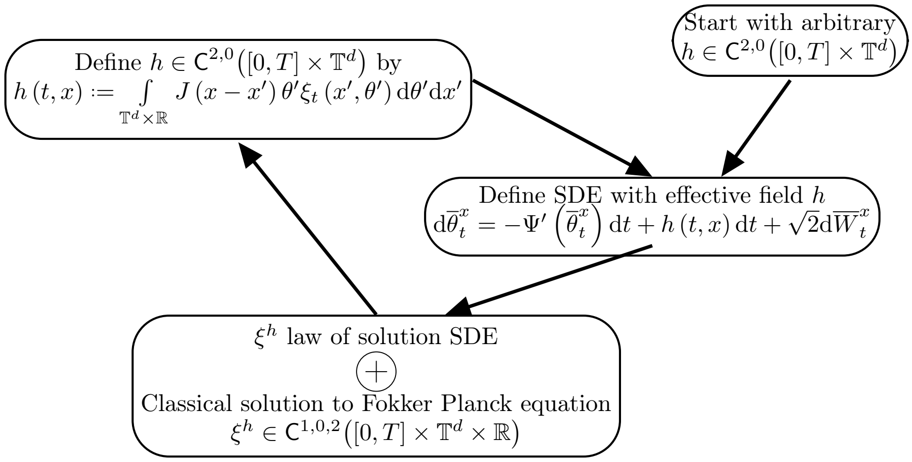


Figure 0.6: Idea how to define the (converging) sequence of effective fields in the proof of the existence of a classical solution of the hydrodynamic equation (0.5.2).

We structure the proof of the existence of such a solution as follows: At first, we show that the linearised problem, that arises by replacing the interaction by an arbitrary effective field h, has a classical solution. With this solution we define a new effective field $h^{(2)}$ and a new SDE, and continue this procedure (see the circle in Figure 0.6). Then, a fixed point argument implies the convergence to a solution of (0.5.2) (see Section I.1.1 for more details).

2. We show that the relative entropy of the law $\mathbb{P}^N_{[0,T]}$ of the solution of (0.1.1) w.r.t. the law $\widehat{\mathbb{P}}^N_{[0,T]}$ of the solution of (0.5.4) (with h^* defined by a classical solution) is of order $o\left(N^d\right)$, i.e.

$$\frac{1}{N^d}\mathsf{H}\left(\mathbb{P}^N_{[0,T]}\middle|\widehat{\mathbb{P}}^N_{[0,T]}\right) \to 0 \quad \text{as } N \to \infty. \tag{0.5.5}$$

To prove this convergence, we replace the Radon-Nikodym derivative in the relative entropy using Girsanov's theorem. Thus, we have to show that the expected quadratic difference of the interaction term in (0.1.1) and the fixed effective field h^* vanishes. Finally, we use the variational formula for the relative entropy and a large deviation result, to prove that this difference vanishes.

We infer from these two results the convergence of $\mu^N_{[0,T]}$ to the classical solution of (0.5.2), by a general inequality concerning the relative entropy, in combination with a large deviation result for independent spins distributed according to the solution of the hydrodynamic equation.

From the proof of the hydrodynamic limit, we infer in Section I.2 also uniqueness of classical, probability density valued solutions of (0.5.2). We get even uniqueness on a larger set of functions. By a different approach, we show in Section I.3 the uniqueness of classical, probability density valued solutions of (0.5.2). This approach uses two energy estimates instead of the hydrodynamic limit result.

□

We discuss in Section I.2.1.2 some reasons why we derive the hydrodynamic limit result for the empirical process $\mu^N_{[0,T]}$ and not for a different random element.

0.6 Propagation of chaos (Topic II)

Propagation of chaos is another concept to investigate spin system (or particle systems) when the number of spins tends to infinity. The idea of this approach is to investigate the behaviour of finitely many spins in the limit. To be more precise one assumes that finitely many (say r) spins are (stochastically) independent at time zero. Then, the stochastic spin system is said to have the propagation of chaos property if the (stochastic) independence of these r spins is preserved in time at least asymptotically, as the number of spins tends to infinity. The concept of propagation of chaos was mathematically first formulated by Kac in [Kac56]. McKean began to study propagation of chaos for interacting diffusions (see [McK66]). For an overview on propagation of chaos we refer to [Szn91], [Mél96] and references therein.

For exchangeable systems, i.e. when any perturbation of the spins has the same distribution, the propagation fo chaos property is equivalent to the hydrodynamic limit result (see e.g. [Szn91] Proposition 2.2). In these system, the law of the fixed r spins in the limit is given by a product measure $\mathbb{P}^{\otimes r} \in \mathbb{M}_1(\mathsf{C}([0,T]))$. The measure $\mathbb{P}$ is the measure one also in the hydrodynamic limit, i.e. the time marginal of $\mathbb{P}$ is the weak solution of the hydrodynamic equation. This is, for example, the case for diffusion models with mean-field interaction, i.e. when the space dependency is not relevant. Hence, all the hydrodynamic limit results of mean-field models, that we referenced in Section 0.5, imply the propagation of chaos property for these systems. Explicitly the propagation of chaos property is shown, for example, in [Gär88] (for mean-field interaction) and in [NT86] (for particles of two different types). In [Szn84] propagation of chaos is proven for a mean-field model where the spins stay within a bounded set by reflection at the boundary. For moderately interacting, exchangeable diffusions the propagation of chaos property is derived in [MRC87] and [JM98].

The system (0.1.1) is not exchangeable. The spatial position of spins is highly relevant, due to the dependency of the interaction weight and of the initial distribution on the spatial position. For such a system it is not clear, if having the propagation of chaos property implies the law of large number (hydrodynamic limit) and the other way round. Also the limiting distribution of the fixed r spins is not a product of the same measure. It has to be a product of measures that depend somehow on the spatial positions of the r spins. Nevertheless, we are able to show that the system (0.1.1) has the propagation of chaos property if the (sketched) Assumption 0.5.1 is satisfied. Indeed, we prove in Chapter II the following propagation of chaos result.

Result II. *Fix $r \in \mathbb{N}$ and spatial positions $x_1, ..., x_r \in \mathbb{T}^d$. We denote by $\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]}$ the distribution of the time evolution of the r spins at $x_1, ..., x_r$, when $\underline{\theta}^N_{[0,T]}$ evolves according to* (0.1.1)*.*

If the initial distributions $\nu^N_{x_k}$ converge in a suitable sense to $\rho_0(x_k, \theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta$, then $\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]}$ converges in the total variation distance on $\mathbb{M}_1(\mathsf{C}([0,T], \mathbb{R}^r))$ to the measure $\prod_{k=1,\dots,r} \overline{\mathbb{P}}^{x_k}$, where $\overline{\mathbb{P}}^{x_k}$ is for each $k \in \{1, ..., r\}$ the solution to the martingale problem of the SDE

$$\begin{aligned} \mathrm{d}\overline{\theta}^{x_k}_t &= \left(-\Psi'\left(\overline{\theta}^{x_k}_t\right) + h^*(t, x_k)\right)\mathrm{d}t + \sqrt{2}\mathrm{d}W^{x_k}_t \\ \overline{\theta}^{x_k}_0 &\sim \rho_0(x_k, \theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta. \end{aligned} \tag{0.6.1}$$

Here h^* is the effective field defined by the hydrodynamic limit density ρ^* as in the second line of (0.5.3)*. Moreover, the time marginal of $\overline{\mathbb{P}}^{x_k}$ is the hydrodynamic limit density $\rho^*(x_k, \theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta$, i.e. the solution of* (0.5.2) *at the spatial position x_k.*

Interpretation of the result. This results implies that a single spin evolves in the limit independently of finitely many other spins. Each of the fixed spin experiences as interaction force a deterministic fixed field. This field depends only on the classical solution of the hydrodynamic equation (0.5.2) and in particular not on the behaviour of finitely many other spins. The field represents the averaged interaction of the infinitely many other particles around the fixed spatial position. Heuristically, the normalisation implies that the influence of a single spin vanishes in the limit (with speed $\frac{1}{N^d}$) and the interaction is averaged over N^d spins. However, the average is spatially weighted by the function J. Therefore, also the deterministic field varies according to the spatial position. Hence, the r spins evolve in the limit independently, but not identically.

Idea of proof. To prove Result II, we show that the relative entropy of $\mathbb{P}^{N,\{x_1,\ldots,x_r\}}_{[0,T]}$ w.r.t. the product measure $\prod_{k=1,\ldots,r}\overline{\mathbb{P}}^{x_k}$ vanishes. To this end, we use again the result (0.5.5). Moreover, we compare, by an application of the Girsanov theorem, the SDE (0.1.1) and a second SDE, that equals the SDE (0.1.1) everywhere except for the spins a the positions $x_1,\ldots,x_r$. These spins evolve according to the SDE (0.6.1). Finally, Pinsker's inequality implies the convergence in the total variation norm of $\mathbb{P}^{N,\{x_1,\ldots,x_r\}}_{[0,T]}$ to $\prod_{k=1,\ldots,r}\overline{\mathbb{P}}^{x_k}$. □

0.7 Equilibrium large deviation (Topic III)

The stochastic system (0.1.1) is a Markov process that has as a gradient flow a unique invariant measure. Let $\underline{\Theta}^N$ be a vector of random variables that is distributed according to this invariant measure, i.e.

$$\underline{\Theta}^N := \{\Theta_i\}_{i\in\mathbb{T}^d_N} \sim \frac{1}{Z_N} e^{\frac{1}{2N^d}\sum_{i,j\in\mathbb{T}^d_N} J\left(\frac{j-i}{N}\right)\theta_i\theta_j} \prod_{i\in\mathbb{T}^d_N} e^{-\Psi(\theta_i)}\mathrm{d}\theta_i. \tag{0.7.1}$$

We would like to derive the large deviation principle for sequences of random elements such as empirical measures defined via $\underline{\Theta}^N$. The large deviation theory is on the one hand a suitable method to understand the asymptotic behaviour of tail events of these sequences. On the other hand, and by far more important in this dissertation, is that the arising rate function can be interpreted as energy of a macroscopic state (see Section 0.8). Therefore, we derive in Chapter III the following equilibrium large deviation principles, provided that the (sketched) Assumption 0.5.1 holds and that J is continuous.

Result III. (i) *The family* $\left\{\xi^N = \frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\Theta^i\delta_{\frac{i}{N}}\right\}$ *satisfies the large deviation principle on the space of signed finite measures* $\mathbb{M}\left(\mathbb{T}^d\right)$*, with good rate function* L_J*. For* $\nu\in\mathbb{M}\left(\mathbb{T}^d\right)$ *that has a local magnetisation profile* $m\in L^2\left(\mathbb{T}^d\right)$ *(i.e.* $\nu(\mathrm{d}x) = m(x)\,\mathrm{d}x$*),*

$$L_J(\nu) = \int_{\mathbb{T}^d} I(m(x))\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-y)\,m(x)\,m(y)\,\mathrm{d}x\mathrm{d}y + \mathrm{Const}, \tag{0.7.2}$$

where I *is the rate function of* $\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\widehat{\Theta}_i$*, when the* $\widehat{\Theta}_i$ *are mutually independently distributed according to* $e^{-\Psi(\theta)}\mathrm{d}\theta$*. Otherwise* $L_J(\nu)=\infty$*.*

(ii) *The family* $\left\{\xi^N = \frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\delta_{\left(\frac{i}{N},\Theta^i\right)}\right\}$ *satisfies the large deviation principle on* $\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)$ *with good rate function* Λ_J*. For* $\nu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)$ *with* $\int_{\mathbb{T}^d\times\mathbb{R}}\theta^2\nu<\infty$ *and that has a density* $\rho\in L^1\left(\mathbb{T}^d\times\mathbb{R}\right)$ *such that* $\rho(x,\theta)\,\mathrm{d}\theta\in\mathbb{M}_1(\mathbb{R})$ *for almost all* $x\in\mathbb{T}^d$*,*

$$\begin{aligned}\Lambda_J(\rho) = &\int_{\mathbb{T}^d}\mathsf{H}\left(\rho(x,\cdot)\middle|e^{-\Psi}\right)\mathrm{d}x\\ &-\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{R}} J(x-x')\,\theta'\theta\rho(x,\theta)\,\rho(x',\theta')\,\mathrm{d}x\mathrm{d}\theta\mathrm{d}x'\mathrm{d}\theta' + \mathrm{Const},\end{aligned} \tag{0.7.3}$$

where $\mathsf{H}(.|.)$ *is the relative entropy. Otherwise* $\Lambda_J(\nu)=\infty$*.*

Both rate functions depend on the entropy (first summand) and on the interaction between the spins (second summand). We finish this section with a short discussion of the proof of Result III.

Idea of proof. The proof of the Result III is organized as follows:

1. We consider spins that are independently and identically distributed according to the measure $e^{-\Psi(\theta)}\mathrm{d}\theta$. For these independent spins one can easily derive the LDPs for both families.

2. We infer the large deviation principles for interacting spins from the large deviation principles for independent spins, using a generalisation of Varadhan's lemma (see Theorem C.1.1 in the Appendix C). □

The usual strategy to prove the second step would be to transfer the LDP of the i.i.d. spins, by using the Varadhan's lemma and Bryc's inverse Varadhan's lemma (see [DZ98] Section 4.3 and 4.4). This approach is used, for example, by [BET00] (see also [BET99]) to achieve a LDP for a similar model with the crucial differences that the spins take only values in a bounded set. We refer also to [KPT05] Section 2.2 for an equilibrium LDP for bounded spins with Kac interaction. However, we can not use the original Varadhan's lemma, due to the unboundedness of the spin values.

0.8 The energy landscape (Topic IV)

We interpret the rate function Λ_J as energy of the macroscopic system. As in the independent system in Section 0.4 (see (0.4.4)), this is motived by the hydrodynamic equation (given in (0.5.2)) being the Wasserstein gradient flow w.r.t. Λ_J (in the Monge-Kantorovich gradient flow sense, see [Vil03] Section 8.3), i.e.

$$\partial_t \rho = \partial_\theta \left(\rho \partial_\theta \left(\frac{\delta \Lambda_J(\rho)}{\delta \rho} \right) \right). \tag{0.8.1}$$

Hence, understanding the landscape of the rate functions helps to understand the temporal behaviour of the macroscopic system. We investigate the landscape of the two rate functions L_J and Λ_J based on the following properties:

- Global minimizer.
- Critical points, in particular saddle points.
- Lowest paths between minima.
- Bifurcations of critical values from the trivial branch and properties of the bifurcating branches.

We are in particular interested in the changes of these properties when the total strength of the interaction is varied. Therefore, we introduce an interaction intensity parameter $\lambda > 0$, i.e. we investigate instead of (0.7.2), for $m \in L^2\left(\mathbb{T}^d\right)$,

$$\int_{\mathbb{T}^d} I(m(x))\,\mathrm{d}x - \lambda \frac{1}{2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} J(x-y)\, m(x)\, m(y)\,\mathrm{d}x\mathrm{d}y, \tag{0.8.2}$$

and instead of (0.7.3), for $\rho \in L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$,

$$\int_{\mathbb{T}^d} \mathsf{H}\left(\rho(x,\cdot)\middle| e^{-\Psi}\right) \mathrm{d}x - \lambda \frac{1}{2} \int_{\mathbb{T}^d\times\mathbb{R}} \int_{\mathbb{T}^d\times\mathbb{R}} J(x-x')\, \theta' \theta \rho(x,\theta)\, \rho(x',\theta')\,\mathrm{d}x\mathrm{d}\theta\mathrm{d}x'\mathrm{d}\theta'. \tag{0.8.3}$$

Moreover, we prove that all results concerning the four properties of the landscape of (0.8.2) can be transferred to results for (0.8.3). Before we explaining these results in more detail, we give an historical overview.

0.8.1 Historical overview

There are several results, concerning the four properties mentioned above, on functionals similar to (0.8.2). These functionals differ from (0.8.2) either by the assumption that the spins only take values in a compact set (e.g. [CES86], [EE83] ,[AN02]) or by assumptions on the local energy contribution to the energy ($\int_{\mathbb{T}^d} I\left(.\right) \mathrm{d}x$ in our case) (e.g. [Per06] and [BCC05] and references therein) or that the interaction is not of local mean-field type (e.g. [BET00], [BFG07a], [BFG07b]). For these different functionals, the authors investigate the existence of minima (e.g. [EE83],[BET00],[BCC05]), other critical points and bifurcations (e.g. [AN02], [CES86], [BFG07a], [BFG07b]). Let us explain in more details how the functionals and results in these papers differ from (0.8.2).

For *compactly supported spins*, the existence of *minimisers* is shown in [EE83] and *critical values* of (0.8.2) are analysed in [AN02] and [CES86]. For that purpose, the authors look for zero points of a nonlinear convolution equations

$$u = \left(I'\right)^{-1} \left(J * u\right). \tag{0.8.4}$$

This equation is similar to the condition of a vanishing (formal) Gâteaux derivative of (0.8.2) (see Section IV.1.1.6). In [AN02] the existence of a non-trivial solution to (0.8.4) is shown for J defined on the whole real line (instead on the torus). Bifurcations from the trivial solution of (0.8.4) at the lowest eigenvalue of J are shown in [CES86]. We show and extend (for other bifurcation points) this result for the model (0.1.1).

A class of models similar to (0.8.2) are *generalisations of the Lebowitz-Penrose functional*. In these models usually compactly supported spins are assumed. We refer to [Pre09] for an overview on the thermodynamic limit among other for ± 1 spins with Kac interaction and a study of the arising Lebowitz-Penrose functional. In Section 6.1.4 of [Pre09], Presutti states results concerning generalisations of the Lebowitz-Penrose functional for functions that take values only in the interval $[-1, 1]$. In these functionals, usually other local energy contributions than the I in (0.8.2) (e.g. the mean-field free energy instead of I) are considered.

With different assumptions on the interaction weight J and with the spins fixed on the whole real line instead on the torus, monotone global minimizer, also called *instantons* (i.e. functions that increase from the negative global minimum to the positive global minimum of the functional), of these functionals are derived in [BFRW97], [AB98b], [BC99], [BCC05]. The instantons serve as description of the optimal profile for an interface between separated phases close to one of the minimizer (see e.g. [AB98b]). These results are related to Γ*-convergence*. It can be shown (see Section 8.6 in [Pre09]) that the free energy of the instanton equals the surface tension in the Γ-limit. For this Γ-convergence, one replaces the interaction weight J by $L^d J\left(L\left(x-y\right)\right)$ and considers the behaviour of

$$L \int_{\mathbb{T}^d} I\left(m\left(x\right)\right) \mathrm{d}x - L\lambda \frac{1}{2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} L^d J\left(L\left(x-y\right)\right) m\left(x\right) m\left(y\right) \mathrm{d}x\mathrm{d}y, \tag{0.8.5}$$

when L tends to infinity. For local energy contributions different from the I considered in (0.8.2), the Γ-limit is derived in [ABCP96], [AB98a] (see also [Pre09] Section 7.1).

Properties of the functionals when L is large, i.e. close to the Γ-limit, critical values and also lowest paths between global minima are investigated in [BDMP05], [Man07] and [BDMDP07] (see also references in the latter paper).

The last difference in the investigated functionals are *other types of interactions*. For example, the Green function as interaction weight, i.e. $J\left(x\right) = \sum_{z \in \mathbb{Z}^2} \left|2\pi z\right|^{-2} e^{2\pi i z x}$, and spins that take values from a compact set are investigated in [BET00]. For this model the authors show an equilibrium LDP for coarse-grained measures (Theorem 3.1 in [BET00]) and characterise minima of the rate function (Section 3.2 in [BET00]).

Another type of interaction are *nearest neighbour interactions*. N-dimensional SDEs with this type of interaction are studied, for example, in [BFG07a], [BFG07b]. The authors investigate the

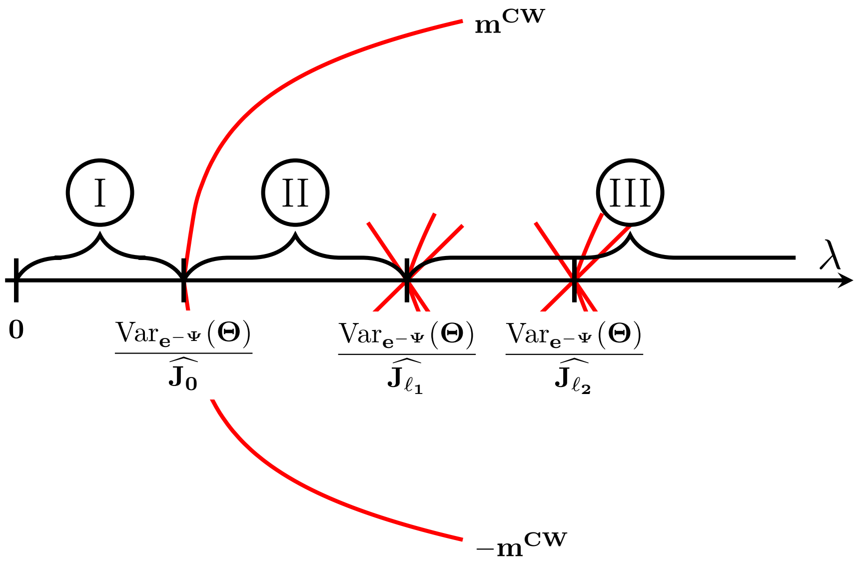


Figure 0.7: Landscape under Assumption 0.8.1. In (I), there is only one minimum. In (II) and (III) there are two constant minima $\pm m^{CW}$. The lowest path between these minima follows in (II) the constants but not any more in (III).

landscape of the Hamiltonian. This includes among others an analyse of the critical points and bifurcations of critical points when changing the strength of the interaction for finite N (small N in [BFG07a] and large N in [BFG07b]). They conclude for the stochastic case, metastability results when the intensity of the noise vanishes. The authors use linear interpolation of spins (see the discussion around (0.3.1)) and investigated the landscape of the Hamiltonian (in [BFG07b]) . We compare this landscape to the landscape of the rate function at the end of the next section.

0.8.2 Summary of the new energy landscape results

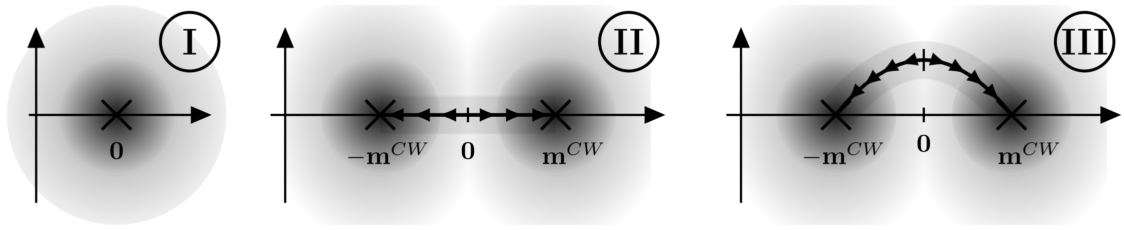
We derive in Chapter IV some results concerning the energy landscape of the functionals (0.8.2) and (0.8.3). The landscapes differ significantly for different interaction intensities $\lambda > 0$. We sketch in Figure 0.7 three regions of λ with interesting differences in the landscape. There $\widehat{J}_0 = \|J\|_{L^1}$ and $\widehat{J}_{\ell_1}$ is the second highest eigenvalue of $J * \cdot$ and $\widehat{J}_{\ell_2}$ the third highest. We need the following additional assumption to show the three regions sketched in Figure 0.7.

Assumption 0.8.1. *h' is strictly concave on $[0, \infty)$. This ism for example, satisfied if $\Psi(\theta) = \theta^{2k}$ for $k \geq 2$ or if $\Psi(\theta) = \theta^4 - \theta^2$ (see Assumption IV.0.3 and the subsequent discussion in Chapter IV).*

Let us now summarize the results that we derive in Chapter IV and that are sketched in Figure 0.7, provided that the (sketched) Assumption 0.5.1 holds, that J is continuous and that Assumption 0.8.1 holds:

Result IV. *The functional* (0.8.2) *has the following properties:*

- *At first, there is only one global minimiser $m^* \equiv 0$ in (I). At the end of this interval, two new global minimiser $\pm m^{CW}$ bifurcate form the trivial branch (Corollary IV.1.70, Theorem IV.1.36).*



(a) One minimum. (b) Two minima, lowest path follows the constants. (c) Two minima, lowest path not any more on the constants.

Figure 0.8: A sketch of the different situations of the energy landscape depending on the intensity of the interaction. Each plot represents the space $L^2(\mathbb{T}^d)$, where the x-axis stands for the subspace spanned by the constants and the y-axis for the rest of the space. A darker colour represents a lower energy.

- *For λ in (II), the previous minimiser $m^* \equiv 0$ is a saddle point. Moreover, a lowest path that connects the two minimiser, passes necessarily through $m^* \equiv 0$ (Theorem IV.1.52 (i)).*
- *At the end of the interval (II), further critical values bifurcate (Theorem IV.1.65) from the trivial branch. In the next interval (III), lowest pass between the two minimisers, do not any more pass through $m^* \equiv 0$ (Theorem IV.1.52 (ii)).*
- *A mountain pass result holds in (II) and (III). Hence, at least one critical value is still a saddle point and lowest paths necessarily pass arbitrary close by the set of critical values (Theorem IV.1.39 and Lemma IV.1.48) also in (III).*

The landscape of the functional (0.8.3) *has comparable properties.*

Some of these results are still valid *without Assumption 0.8.1.* For example, we prove that at the same values of λ bifurcation from the trivial branch occur (Section IV.1.4.3). Also the results from the mountain pass theorem (Theorem IV.1.39) still hold, at least if L_J has only finitely many global minimisers. However, there are differences in the landscape if the Assumption 0.8.1 is not satisfied. For example, there might be more than two minima and even infinitely many, or it might be that the first bifurcation curve consists not any more of global minima. We give in Section IV.1.2.4 an example of a setting when Assumption 0.8.1 is not satisfied. In this setting, global minimizer arise somewhere in space, away from the trivial curve.

Note that each critical value can be shifted on the torus $\mathbb{T}^d$. Then a non constant, critical value represents an (up to d-dimensional) manifold. This has the consequence, that at each saddle point, the energy in the shift directions is constant.

Visualisation of the results. We sketch the energy landscapes in Figure 0.8 for the three different regimes of the interaction intensity parameter λ. In each of the three figures each point in the two dimensional coordinate system stands for a function in $L^2(\mathbb{T}^d)$ and the brightness represents the energy of this function (the darker the spot, the lower the energy). The x-axis contains the subspace of the constant functions, expressed by a value $\mathbb{R}$ (i.e. the point 0 is the function that is everywhere zero). The y-axis represents the rest of the space $L^2(\mathbb{T}^d)$.

In Figure 0.9 we visualise the most important macroscopic states on the lowest paths between minima depending on the intensity parameter. Each of the boxes represents such a macroscopic state. The box stands for the two dimensional torus and the colour represents the value of a spin at a particular site (again a darker colour means a lower value). Hence, a box in a unique colour is a state that can be described by a constant function in $L^2(\mathbb{T}^d)$.

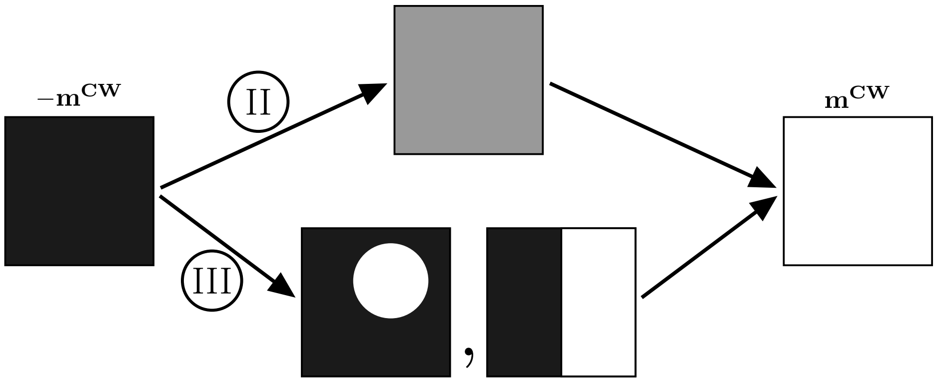


Figure 0.9: The (sketched) lowest paths from one minimum two another, for the interaction intensity parameter λ in (II) and (III).

Let us now explain the energy landscapes of the three different regimes of the interaction intensity parameter λ with help of these two figures. In Figure 0.8a one sees one constant, unique global minimum of the landscape. This represents the landscape for a low interaction intensity (i.e. the case (I)). On the two dimensional macroscopic torus, this constant minimum is represented by the grey box in the top part of Figure 0.9.

For a higher interaction intensity there are two constant minima $\pm m^{CW}$ In Figure 0.9, the left black box stands for the constant function $-m^{CW}$, whereas the white box on the right stands for the constant function m^{CW}. In Figure 0.8b we visualise the lowest path in $L^2\mathbb{T}^d$ between the minima, for λ in (II). The lowest path follows the constant functions, i.e. it stays on the x-axis. Hence, for each macroscopic state on this path, the magnetisation is everywhere on the torus the same. This is sketched in Figure 0.9 by the upper sequence.

For high interaction intensities, i.e. for λ in (III), the lowest path does not follow any more the constant functions (see Figure 0.8c). Therefore, one sees different shapes in the macroscopic states on this path. For example, (see the lower sequence in Figure 0.9), there might be (blurred versions of) droplets or a half filled torus. Hence, one might see areas on the torus with values close to $-m^{CW}$ and areas with values close to m^{CW}.

Comparison to landscape of the nearest neighbour model. To end this section, we compare these landscape results for (0.8.2) to the landscape of the nearest neighbour model described in [BFG07b]. The first difference is that in [BFG07b], the landscape of the Hamiltonian is considered, whereas we consider the landscape of the equilibrium rate function. For the local mean-field system (0.1.1), we consider the (locally averaging) empirical process (due to J being non smoothing, as explained in Section 0.3). Hence the landscape of the Hamiltonian does not provide us with useful information. Nevertheless, it is interesting to compare these two landscapes and how variations of the interaction intensity parameter λ have opposite effects.

For the system (0.1.1), we explained above, that for λ in (II), the lowest path between two minima follows the constant functions. Whereas for higher λ, this is not any more the case, but one sees on the lowest paths spatially varying states (see Figure 0.9). We interpret this as the influence of the entropy. By the increase of the interaction intensity, the higher Fourier modes (generated by the entropy) become more relevant. In the nearest neighbour model one sees the inverse influence of the intensity of the interaction (see [BFG07b] Section 2.3). In this model the spatially varying states are only relevant for a small intensity parameter. In both systems the source of the different behaviour for different interaction intensities is the entropy. For the system (0.1.1) the smoothing comes from the local averages in the empirical measure and the interaction favours oscillations. Whereas in the nearest neighbour models the interaction itself is smoothing. Consequently, a change in the interaction intensity λ has an inverse influence.

0.9 Dynamical large deviations (Topic V)

For large but finite N, the noise allows the system to deviate from the deterministic flow described by the hydrodynamic equation (0.5.2). In particular, if the system has two stable minima (see the regimes (II) and (III) in Section 0.8.2), the randomness allows even a transition from one minimum to another. Such a path is, until it reaches the basin of attraction of the second minimum, inverse to the deterministic flow. These deviations are exponentially unlikely for large N. A suitable description of the asymptotic behaviour of these tail events provides the theory of large deviation. Therefore, we derive a dynamical large deviation principle for random elements that contain in a suitable way the time evolution of each spin.

In Chapter V, we investigate a more general system of SDEs than (0.1.1). In particular, to each site $k \in \mathbb{T}_N^d$, we attach in addition to the spin, a random environment variable $w^{k,N}$ with values in $\mathcal{W} \subset \mathbb{R}^m$. $w^{k,N}$ is distributed according to $\zeta_{\frac{k}{N}} \in \mathbb{M}_1(\mathcal{W})$ and is frozen over time. We include the random environment also in the empirical process, i.e. (compare to the original definition (0.3.2)) we set

$$\mu^N_{[0,T]} := \left\{ t \mapsto \mu^N_t := \frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \delta_{\left(\frac{k}{N}, w^{k,N}, \theta^{k,N}_t\right)} \right\} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right), \tag{0.9.1}$$

and in the empirical measure L^N (compare to the original definition (0.3.3))

$$L^N = L^N\left(\underline{w}^N, \underline{\theta}_{[0,T]}\right) := \frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \delta_{\left(\frac{k}{N}, w^{k,N}, \theta^{k,N}_{[0,T]}\right)} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right). \tag{0.9.2}$$

In Chapter V, we consider (instead of (0.1.1)) a Langevin dynamics, with a very general drift coefficient $b\left(\frac{k}{N}, w^{k,N}, \theta^{k,N}_t, \mu^N_t\right)$ on $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$. For simplicity, we consider in this introduction not the most general case, but the following system that resembles (0.1.1)

$$\mathrm{d}\theta^{k,N}_t = -\left(\partial_\theta \Psi\left(\theta^{k,N}_t, w^{k,N}\right)\right)\mathrm{d}t + \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} J\left(\frac{k-j}{N}, w^{k,N}, w^{j,N}\right)\theta^{j,N}_t \mathrm{d}t + \mathrm{d}B^{k,N}_t, \tag{0.9.3}$$

where $\Psi(\theta, w) = \overline{\Psi}(\theta) + w_1$.

For the families of random elements $\left\{\mu^N_{[0,T]}\right\}$ and $\left\{L^N\right\}$ defined as in (0.9.1) and (0.9.2), by $\left\{\underline{\theta}^N_{[0,T]}\right\}$ evolving according to (0.9.3), we prove the dynamical large deviation principles. Moreover, we derive different representation of the rate functions. In particular, we prove that the rate function $S_{\nu,\zeta}$ of the family $\left\{\mu^N_{[0,T]}\right\}$ has the following expression

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \int_0^T \left|\partial_t \mu_t - \left(\mathbb{L}^{\mathrm{LMF}}_{\mu_t,\cdot,\cdot}\right)^* \mu_t\right|^2_{\mu_t} \mathrm{d}t + \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right), \tag{0.9.4}$$

for suitable $\mu_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$. The explicit form of the norm $|.|_{\mu_t}$ (see Definition V.1.10) is at this stage not relevant. The operator $\left(\mathbb{L}^{\mathrm{LMF}}_{\mu,x,w}\right)^*$ is for each $\mu \in \mathbb{M}_{\varphi,\infty}$ and $(x,w) \in \mathbb{T}^d \times \mathcal{W}$ the formal adjoint of the following operator, acting on $f \in \mathsf{C}_b^2(\mathbb{R})$,

$$\mathbb{L}^{\mathrm{LMF}}_{\mu,x,w} f(\theta) := \frac{\sigma^2}{2} \partial^2_{\theta^2} f(\theta) + b(x,w,\theta,\mu)\,\partial_\theta f(\theta). \tag{0.9.5}$$

For the system (0.1.1), $\left(\mathbb{L}^{\mathrm{LMF}}\right)$ is the operator given by the right hand side of the hydrodynamic equation (0.5.2). It is important to observe, that the rate function $S_{\nu,\zeta}$ measures somehow the deviation from the hydrodynamic equation.

Before stating and explaining this large deviation results in more details, we give a historical overview on dynamical large deviation principles for models similar to (0.1.1) or (0.9.3).

0.9.1 Historical overview

Dynamical large deviation principles for models similar to (0.1.1) and (0.9.3) are considered by many authors. These models differ however in one or more of the following three properties:

1. Various authors consider models with mean-field interaction (like the Curie-Weiss model), e.g. [Tan84], [DG87], [Bru93], [DPdH96], [FK06]. In these models the spatial structure of the spins is not relevant.
2. Some authors attach a random environment variable to each site (e.g. [DPdH96]) or to each pair interaction (e.g. [Cab16]).
3. The third difference is a different dynamic of the spins instead of the Langevin dynamic. For example, in [Com87] the spins evolve according to a Glauber dynamic with values ± 1. The proof of the large deviation result depends crucial on their jump dynamic.

For these different models, the following four strategies are used to prove the large deviation principle. In Chapter V, we generalize the following Strategy (S.1) and Strategy (S.2) to be applicable to the system (0.9.3) and emphasise in the following list the necessary changes and difficulties.

(S.1) For a model with irrelevance of the spatial structure and without random environment, the dynamical large deviation principle for empirical processes is derived in [DG87]. This principle is used in [DG89] to connect the quasi potential with the free energy function. The idea of the approach in [DG87] is to fix an empirical process in the drift coefficient to get a system of N^d independent, time inhomogeneous diffusions. For this independent system the large deviation principle is derived. Finally, this LDP is transferred to the LDP for the interacting system. The main difficulty is to show that the rate function has the particular form (similar to (0.9.4)). In Chapter V we generalize the approach of [DG87] to the space and random environment dependent empirical processes $\left\{\mu^N_{[0,T]}\right\}$ and to the empirical measures $\left\{L^N\right\}$. Changes in the proof are in particular required due to these dependencies of the drift coefficient, of the empirical process and of the initial data. Moreover, we consider the space of continuous functions on the usual space of probability measures $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$, equipped with the usual topologies (the uniform topology and the weak convergence) and not, as in [DG87], a subset of this space with a stronger topology. The diffusion coefficient in [DG87] can depend on the spin. We restrict the system to constant diffusion coefficient, to simplifies the notation. However, the large deviation results in Chapter V should hold for non constant diffusion coefficient.

(S.2) In [Tan84] the large deviation principle for the empirical measure defined by $\frac{1}{N^d}\sum_{k\in\mathbb{T}^d_N}\delta_{\theta^{k,N}_{[0,T]}}$ in $\mathbb{M}_1(\mathsf{C}([0,T]))$ is derived. In [DPdH96], a mean-field interaction with random environment is considered. In both models, the authors assume that the drift coefficient b is bounded and does not depend on the spatial positions of the spins. Due to this boundedness, it is possible to transfer the large deviation principle for N^d independent Wiener process to the large deviation principle for the interacting system, by an application of Varadhan's lemma. With the contraction principle, the authors easily infer the large deviation principle for the empirical process. However, the rate function does not have an expression like (0.9.4). In [DPdH96], the authors try to show that the rate function has such an expression. Unfortunately, there is a

circular reasoning in the proof of this result (in Step 4 of the proof of Theorem 3). Therefore, only for some trajectories of measures the equality of these two expressions is proven. For the same subsets of the trajectories this equality is also proven in [Bru93].

We generalise the approach of [DPdH96] to be applicable also for the unbounded, local mean-field model (0.9.3). This leads to one of the two proofs in Chapter V, of the large deviation principle for the empirical measure $\{L^N\}$. We infer a large deviation principle for the empirical process $\mu^N_{[0,T]}$. However, we are not able to correct the mentioned circular reasoning in [DPdH96] to show that the rate function equals (0.9.4). We get this equality of the rate functions by the uniqueness of the rate function and the proof of the dynamical large deviation principle that uses the Strategy (S.2).

In [Cab16] a model is considered, that differs from (0.9.3), by a bounded contribution of the interaction term, and by random i.i.d. interaction weights instead of interaction weights that dependent on the random environment attached to the spins. Moreover the model is not space dependent. The authors prove the large deviation principle for the empirical measures $\{L^N\}$ and characterise the minima of the rate function.

(S.3) In [FK06] a third strategy is used to prove the LDP for non space dependent, interacting system like in [DG87]. The authors connect the LDP with a variational problem arising from control theory (see Example 1.14, Section 13.3 and Theorem 13.37 of [FK06]). We decided to generalise the approach of [DG87] to the space and random environment dependent system considered here, because the assumptions in [FK06] are more restrictive than [DG87] and (at least from a probabilist's point of view) the approach of [DG87] gives more inside in the underlying structure.

(S.4) A direct approach to derive the large deviation principle for the empirical process is used in [KO90] for independent Brownian motions. This approach can be generalised to models with mean-field interaction. But it cannot be applied for the space dependent model we consider. This approach requires that the hydrodynamic limit has a unique weak solution (see also [Gui04] page 40), which we cannot prove (see the discussion in Section 0.5).

0.9.2 Summary of the new dynamical large deviation results

The results of this chapter can also be found in [Mül16]. Besides the (sketched) Assumption 0.5.1, let the functions in system (0.9.3) satisfy the following sketched conditions (see Chapter V Assumption V.1.4 for details).

Assumption (sketched) 0.9.1. *J is in $L^2\left(\mathbb{T}^d, \mathsf{C}(\mathcal{W}\times\mathcal{W})\right)$ and satisfies some convergence properties, e.g. J is continuous or $J(x,w,w') = x^{-\frac{1}{2}+\epsilon} + ww'$.*

In Chapter V we prove the following results.

Result V. (i) *The family $\left\{\mu^N_{[0,T]}\right\}$ satisfies on $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\right)$ the large deviation principle with good rate function $S_{\nu,\zeta}$, defined in* (0.9.4) *(see Section V.3).*

(ii) *We derive different representations of the rate function $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ (see Section V.4).*

(iii) *The family $\{L^N\}$ satisfies on $\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$ the large deviation principle with a good rate function. We derive two different representation of the rate function (see Theorem V.5.3 and Theorem V.5.12).*

(iv) *We show (in Section V.6) a one-to-one relation between the minimizers of the rate functions of $\left\{\mu^N_{[0,T]}\right\}$ and $\{L^N\}$.*

Interpretation of the result. By the large deviation principle of the empirical process with rate function $S_{\nu,\zeta}$, we get a better understanding of landscape of the paths on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ for large N. Indeed, it is exponentially unlikely to actually see an empirical process $\mu^N_{[0,T]}$ that deviates a lot (in a suitable sense) from a solution to the hydrodynamic equation (0.5.2).

The probability to observe a path from one minimum to another is hence (on this exponential scale) determined by the path until it reaches the basin of attraction of the second minimum. From the border of the basin of attraction it can follow the deterministic flow to the second minimum. Looking alone at this part of the trajectory, the rate function would be zero.

The different expressions of the rate function $S_{\nu,\zeta}$ (see Result V (ii)) can be useful when investigating the long time behaviour of the system (see also [DG89] in the mean-field case), in particular when the model is not reversible.

Idea of proof. We prove the large deviation principle for the empirical process $\left\{\mu^N_{[0,T]}\right\}$ by generalising the strategy of [DG87] to the space and random environment dependent model we consider here. This strategy consists of two steps:

1. We derive the LDP for the empirical process for systems of independent diffusions, that arise by fixing the empirical process in the interaction, i.e. by replacing the interaction by a time dependent external field (see Section V.3.1).

   The existence of the LDP for the independent system is a direct consequence of a generalisation of Sanovs theorem (see Section V.2.2). The better part of this proof is dedicated to showing that the rate function actually has a form like (0.9.4).

2. We prove the LDP for the interacting system by showing exponential tightness and a local version of the LDP (Section V.3.2). For this local version of the LDP, we use the LDP for the independent system.

For the family of empirical measures $\left\{L^N\right\}$, we show that the same strategy can be used to derive the large deviation principle. Then we prove this principle by the following second strategy:

1. In this second strategy, we derive the large deviation principle (by the Sanov type result of Section V.2.2) for the model without interaction term.

2. Wee transfer this LDP for the model without interaction to the LDP for the model with interaction. We use at first the Girsanov transformation and then a generalisation of Varadhan's lemma (given in Appendix C). We need this generalisation, because the exponent in the Girsanov transform is nowhere continuous on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$.

□

There are two reasons why we give a second proof. On the one hand the idea of investigating separately the entropy and adding then the interaction, seems to be more straightforward than the first strategy. On the other hand this approach gives a nice example how the generalisation of the Varadhan's lemma can be applied.

0.10 Future challenges

We analyse, for the spin system that evolves microscopically according to (0.1.1), the Topics I-V under different assumptions. In this context there are future challenges and open questions:

- Instead of (0.1.1), one could investigate the hydrodynamic limit and propagation of chaos for more general models, e.g. by looking at a general drift coefficient $b : \mathbb{T}^d \times \mathbb{R} \times \mathbb{M}_1 \left(\mathbb{T}^d \times \mathbb{R}\right) \to \mathbb{R}$ or by considering an additional random environment (see Chapter V where we look at the dynamical large deviation principle for such a more general system).

- As explained in Section 0.6, an exchangeable systems that has the hydrodynamic limit property also has the propagation of chaos property and vice versa (see [Szn91] Proposition 2.2). For the (non exchangeable) local mean-field system, we do not know if there is equivalence between these two properties. Therefore, we prove both properties separately. However, it might be interesting to see under which assumptions on a local mean-field system, there is also a general equality between the hydrodynamic limit property and (a variation of) the propagation of chaos property.

- In Section 0.8.1, we refer to papers where the Γ-limit and properties close to this limit are derived for a functional that differs slightly from the functional (0.8.2) (see the discussion near (0.8.5)). These differences are in particular other single spin contributions in (0.8.2) and $\mathbb{R}$ as underlying space instead of the torus. Note when the torus is more than one dimensional, it is not rotationally symmetric. Then rearrangement results (see e.g. [LL01] Chapter 3), that are often applied in this context (see [AB98b]) do not hold. It would be nice to derive the properties close to the Γ-limit also for the functional (0.8.2).

- Another interesting topic is the central limit theorem for the empirical process $\left\{\mu^N_{[0,T]}\right\}$. Different authors derived this property for diffusion models with mean-field interaction (see for example [TH81], [Tan84], [Daw83], [Luç11], [FM97]) and with moderately interaction (e.g. [Oel87], [JM98]). In [LS15] for a model similar to (0.1.1), a central limit theorem is shown. As (spatial dependent) interaction weight, the authors consider $J(x) := x^{-\alpha}$, $\alpha < 1$. A central limit theorem for the more general interaction weights that we consider in the hydrodynamic limit might be interesting.

- We show the dynamical large deviation principle amongst other for interaction kernels $J(x) = x^{-\alpha}$ with $\alpha < \frac{1}{2}$, whereas the hydrodynamic limit holds for $\alpha < 1$. Also the central limit theorems in [LS15] for $\alpha < \frac{1}{2}$ and $\alpha > \frac{1}{2}$ are significantly different. The question arises, if one could expect a large deviation principle or at least a variation of such a principle also for $\alpha \in \left[\frac{1}{2}, 1\right)$.

0.11 Notation

Throughout this dissertation we use the following notation:

The spaces:

- We use the symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ for the spaces of the natural numbers, the integers and the real numbers.
- We denote by $\mathbb{T}^d$ the d-dimensional torus, defined as the quotient of $\mathbb{R}^d$ under integral shifts. Then $\int_{\mathbb{T}^d} 1\mathrm{d}x = 1$.
- With $\mathbb{T}_N^d$ we denote the periodic d-dimensional lattice of length N.
- The random environment takes values in $\mathcal{W} \subset \mathbb{R}^m$ for a $m \in \mathbb{N}$.
- We denote the space of continuous functions from X to Y by $\mathsf{C}(X,Y)$. This space is equipped with the topology of uniform convergence. To shorten the notation we often skip Y if $Y = \mathbb{R}$, i.e. $\mathsf{C}(X) = \mathsf{C}(X,\mathbb{R})$.
- For the subset of $\mathsf{C}(X,Y)$ that consists of bounded functions we use the notation $\mathsf{C}_b(X,Y)$, of functions that vanish at the boundary $\mathsf{C}_0(X,Y)$, and of functions with compact support $\mathsf{C}_c(X,Y)$.
- We use the norm $\|g\|_{\mathsf{C}(X)} = |g|_\infty = \sup_{x\in X} |g(x)|$, for $g \in \mathsf{C}(X)$. This norm is finite for all functions in $\mathsf{C}_b(X,Y)$.
- With $\mathsf{C}^k(X,Y)$ we denote the set of k-times continuously differentiable functions.
- We denote by $\mathbb{M}_1(X)$ the space of probability measures on X, equipped with the topology of weak convergence.
- We denote by $\mathbb{M}(\mathbb{T}^d)$ the space of signed, finite, real-valued Borel measures on the torus endowed with the total variation norm and with the weak-(*)-topology (see also Appendix A).
- Let $(X, \mathcal{X}, \mu)$ be a measure space. We denote the L^p space by $L^p(X,\mu)$. With abuse of notation we write instead of μ sometimes only a density with respect to the Lebesgue measure, e.g. $L^p\left(\mathbb{R}, e^{-\Psi}\right)$. If μ is the Lebesgue measure, we use the shorter notation $L^p(X)$.

The variables:

- N is the side length $\mathbb{T}_N^d$, hence N^d is the number of the spins.
- With x, y, z we usually denote macroscopic coordinates, i.e. positions on the torus $\mathbb{T}^d$. Whereas by i, j, k we denote microscopic coordinates, i.e. sites on the discrete torus $\mathbb{T}_N^d$. These two coordinate systems are related by $x = \frac{i}{N}$.
- As time variables we use the letters s, t, u. These take values in the interval $[0,T]$ for a $T > 0$.
- We use the letters θ, η for the spin. With $\theta_{[0,T]}$ we denote the whole path of the spin, i.e. an element of $\mathsf{C}([0,T])$. With $\theta_t \in \mathbb{R}$ we denote the spin at time $t \in [0,T]$.
- For a N^d-dimensional vector of spins, we use the symbol $\underline{\theta}^N$ and analogue $\underline{\theta}^N_{[0,T]}$, $\underline{\theta}^N_t$. We write $\theta^{k,N}$ for the element at position $k \in \mathbb{T}_N^d$ in this vector.
- We use the letter w for a value of the random environment. Again $\underline{w}^N$ is the N^d-dimensional vector of the environment and $w^{k,N}$ the specific value of the environment associate with the site $k \in \mathbb{T}_N^d$.

- We use lower-case letters, mostly μ, ν, π for measures on $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, $\mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)$ or $\mathbb{M}_1(\mathbb{R})$ (ν is usually the distribution of the initial values). We write $\mu_{[0,T]}$, for a path on the probability measures, i.e. for an element in $\mathsf{C}([0,T], \mathbb{M}_1(X))$. For the measure at time $t \in [0,T]$ of the path $\mu_{[0,T]}$, we use the symbol μ_t.
- We use upper-case letters, in most cases Q or Γ, for measures on $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$.

Other notation:

- With $\langle \nu, f \rangle$ or $\langle g, f \rangle$ we shorten the notation of the integral of a function f with respect to a measure ν or a distribution or a second function g, according to the context over the spaces $\mathbb{T}^d \times \mathbb{R}$, $\mathbb{R}$ or $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$.
- With $\mathsf{H}\left(\mu | \nu\right)$ we denote the relative entropy between $\mu, \nu \in \mathbb{M}_1(X)$ that is defined as

$$\mathsf{H}\left(\mu | \nu\right) := \int_X \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x)\right) \mu(\mathrm{d}x), \tag{0.11.1}$$

  if μ is absolutely continuous w.r.t. ν and infinity otherwise.

Mathematical abbreviations:

- LDP: Large deviation principle.
- SDE: Stochastic differential equation.
- PDE: Partial differential equation.
- i.i.d.: Independent and identically distributed.

Chapter I

Hydrodynamic limit

In this chapter we prove the hydrodynamic limit result, sketched in Result I of Section 0.5.2, of the empirical process $\mu^N_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$, defined in (0.3.2). The spins $\underline{\theta}^N$ evolve according to the coupled system of SDEs

$$\begin{aligned} \mathrm{d}\theta^{i,N}_t &= -\Psi'\left(\theta^{i,N}_t\right)\mathrm{d}t + \frac{1}{N^d}\sum_{j\in\mathbb{T}^d_N} J\left(\frac{i-j}{N}\right)\theta^{j,N}_t\mathrm{d}t + \sqrt{2}\mathrm{d}W^{i,N}_t \quad \text{and} \\ \underline{\theta}^N_0 &\sim \nu^N = f^N_0\left(\underline{\theta}^N\right)\prod_{i\in\mathbb{T}^d_N} e^{-\Psi\left(\theta^{i,N}\right)}\mathrm{d}\theta^{i,N} \in \mathbb{M}_1\left(\mathbb{R}^{N^d}\right). \end{aligned} \tag{I.0.1}$$

We show that the (random) empirical process $\mu^N_{[0,T]}$ converges in the limit $N\to\infty$ to a deterministic time evolution of measures. At each time $t\in[0,T]$, this measure has a density ρ_t with respect to $e^{-\Psi(\theta)}\mathrm{d}\theta$. The time evolution of these densities ρ is the unique classical solution of the non linear partial differential equation, that we call *hydrodynamic equation*

$$\begin{aligned} \partial_t\rho_t(x,\theta) &= e^{\Psi}\partial_\theta\left(e^{-\Psi}\partial_\theta\rho_t(x,\theta)\right) - e^{\Psi}\partial_\theta\left(J * h^\rho(t,x)\ e^{-\Psi}\rho_t(x,\theta)\right) \quad &\text{with} \\ h^\rho(t,x) &:= \int_{\mathbb{T}^d\times\mathbb{R}} J(x'-x)\,\theta' e^{-\Psi(\theta')}\rho_t(x',\theta')\,\mathrm{d}\theta'\mathrm{d}x' \quad &\text{and} \\ \rho_0(x,\theta) &= \rho^0(x,\theta). \end{aligned} \tag{I.0.2}$$

For another expression of this PDE see (0.5.2). For each $x\in\mathbb{T}^d$ and $t\in[0,T]$, $\rho_t(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta$ is the probability density of $\widehat{\theta}^x_t := \widehat{\theta}^{x,\rho}_t$ that evolves according to the SDE

$$\begin{aligned} \mathrm{d}\widehat{\theta}^x_t &= -\Psi'\left(\widehat{\theta}^x_t\right)\mathrm{d}t + h^\rho(t,x)\,\mathrm{d}t + \sqrt{2}\mathrm{d}\widehat{W}^x_t, \\ \widehat{\theta}^x_0 &\sim \rho^0(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R}), \end{aligned} \tag{I.0.3}$$

where $\widehat{W}^x$ is a Brownian motion. The effective field h^ρ is defined by ρ as in (I.0.2). We call the SDE (I.0.3) *hydrodynamic SDE*. Note the existence of such a density $\rho e^{-\Psi}$ is a priori not clear, because $\rho e^{-\Psi}$ is the probability density corresponding to (I.0.3) and is used in the drift coefficient.

As explained in the introduction, the proof of this hydrodynamic limit result is based on the relative entropy method. For this method, we require a priori the existence of a probability density $\rho e^{-\Psi}$ corresponding to the SDE (I.0.3) (with effective field $h = h^\rho$ defined by this density). We prove in Section I.1 (see Theorem I.1.2), the existence of such a probability density, that is at the same time a classical solution of the hydrodynamic equation (I.0.2). Moreover, we show (see Theorem I.1.4) that each probability density valued, classical solution of the hydrodynamic equation (I.0.2) is a probability density corresponding to (I.0.3).

We prove in Section I.2 (Theorem I.2.3) the convergence of the empirical process by the relative entropy method. Its limit is an arbitrary time evolution of probability densities corresponding to (I.0.3). We infer from the uniqueness of the limit point, the uniqueness of these time evolutions of probability densities (see Corollary I.2.4). Hence, the hydrodynamic limit of the empirical process is the classical solution of the hydrodynamic equation that we derive in Section I.1. Moreover, this approach implies uniqueness of classical solutions of the hydrodynamic equation (I.0.2).

In addition, we show in Section I.3 the uniqueness of probability density valued classical solutions of the hydrodynamic equation (I.0.2) by a different approach.

First, we make the sketched Assumption 0.5.1, that are require in this chapter, precise. The following assumptions are not as general as possible to keep the notation simple (see also Section I.2.1.1).

Assumption I.0.1. *$J \in L^1(\mathbb{T}^d)$. Moreover*

$$\sum_{i \in \mathbb{T}_N^d} \int_{\Delta_{i,N}} \left| J\left(\frac{i}{N}\right) - J(x) \right| \mathrm{d}x \to 0 \qquad \text{when } N \to \infty, \tag{I.0.4}$$

with $\Delta_{i,N} := \left\{ x \in \mathbb{T}^d : \left| x - \frac{i}{N} \right| < \frac{1}{2N} \right\}$.

Remark I.0.2. (i) *This assumption is for example satisfied if J is continuous or Riemann-integrable, e.g. $J(x) = \mathbb{1}_{x \in A}$ with a rectangle $A \subset \mathbb{T}^d$ or $J(x) = |x|^{-\alpha}$ for $\alpha \in (0,1)$.*

(ii) *The convergence condition* (I.0.4) *is crucial in the proof of the hydrodynamic limit (in Section I.2) to establish a connection between the interaction contribution of the finite dimensional microscopic systems (described by* (I.0.1)*) and of the macroscopic system (described by the PDE* (I.0.2)*). It excludes for example the interaction weight J which is* 0 *on the rational numbers and* 1 *on the remaining real numbers. Obviously such an interaction weight would lead to no interaction in each finite system. However, one would see contribution from the interaction in the PDE* (I.0.2)*.*

(iii) *In Section I.1 we do not need* (I.0.4)*, because we investigate the system in the hydrodynamic limit and not the finite dimensional systems.*

Assumption I.0.3. *Ψ is a polynomial of even degree ≥ 2, with positive coefficient of that degree. If the degree is equal to two, we denote by c_Ψ the leading coefficient (i.e. $\Psi(\theta) = c_\Psi \theta^2 +$* lower order terms*). Otherwise set $c_\Psi = \infty$. We assume that*

$$c_\Psi > \|J\|_{L^1}. \tag{I.0.5}$$

For example Ψ can be chosen as a convex function like $\Psi(\theta) = (\|J\|_{L^1} + \epsilon)\,\theta^2$ or $\Psi(\theta) = \theta^{2k}$ for a $k \in \mathbb{N}$ or a double well potential like $\Psi(\theta) = \theta^4 - \theta^2$.

We assume on the initial distribution ρ^0 of the PDE (I.0.2) and of the SDE (I.0.3):

Assumption I.0.4. a.) *$\rho^0 \in \mathsf{C}(\mathbb{T}^d \times \mathbb{R})$ and $\rho^0(x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$, for each $x \in \mathbb{T}^d$.*

b.) *The variance of ρ^0 is uniformly (in x) bounded from below, i.e.*

$$\inf_{x \in \mathbb{T}^d} \left\{ \int_{\mathbb{R}} \theta^2 e^{-\Psi(\theta)} \rho^0(x,\theta)\, \mathrm{d}\theta - \left(\int_{\mathbb{R}} \theta e^{-\Psi(\theta)} \rho^0(x,\theta)\, \mathrm{d}\theta \right)^2 \right\} \geq C_0 > 0. \tag{I.0.6}$$

For the relative entropy method we need a vanishing relative entropy of the initial conditions of the N^d-dimensional SDEs described by (I.0.1) and (I.0.3).

Assumption I.0.5. a.)

$$\lim_{N\to\infty} \frac{1}{N^d} \mathsf{H}\left(f_0^N\left(\underline{\theta}^N\right) \prod_{i\in\mathbb{T}_N^d} e^{-\Psi\left(\theta^{i,N}\right)}\mathrm{d}\theta^{i,N} \middle| \prod_{i\in\mathbb{T}_N^d} \rho_0\left(\frac{i}{N},\theta^{i,N}\right) e^{-\Psi\left(\theta^{i,N}\right)}\mathrm{d}\theta^{i,N}\right) = 0. \quad \text{(I.0.7)}$$

b.) *There is a $C>0$ and a $\kappa>0$ such that for all $N\in\mathbb{N}$*

$$\int_{\mathbb{R}^{N^d}} e^{\kappa \sum_{i\in\mathbb{T}_N^d}\left(\theta^{i,N}\right)^2} f_0^N\left(\underline{\theta}^N\right) \prod_{i\in\mathbb{T}_N^d} e^{-\Psi\left(\theta^{i,N}\right)}\mathrm{d}\theta^{i,N} \le C^{N^d}. \quad \text{(I.0.8)}$$

For the proof of the existence of a classical solution of the hydrodynamic equation (I.0.2), we need the following integrability condition on the initial distribution.

Assumption I.0.6. *For an arbitrary $\epsilon>0$, there is a constant $C>0$ such that for each $x\in\mathbb{T}^d$*

$$\int_{\mathbb{R}} e^{\left(-\frac{1}{8}+\epsilon\right)\Psi(\theta)} \rho^0\left(x,\theta\right)\mathrm{d}\theta \le C < \infty. \quad \text{(I.0.9)}$$

Remark I.0.7. *For example ρ^0 with compact support, satisfies the Assumption I.0.6.*

I.1 Existence: Density of the hydrodynamic SDE and classical solution of the hydrodynamic equation

We define a classical solution of the hydrodynamic equation (I.0.2) in the following way

Definition I.1.1.

- *We call a function $\rho:[0,T]\times\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$ a classical solution of* (I.0.2) *with initial condition ρ^0, if the following conditions are satisfied:*
  (i) $\rho e^{-\Psi}\in\mathsf{C}_b\big([0,T]\times\mathbb{T}^d\times\mathbb{R}\big)\cap\mathsf{C}^{1,0,2}\big((0,T)\times\mathbb{T}^d\times\mathbb{R}\big)$.
  (ii) *The effective field $h=h^\rho$ defined by ρ as in the second line of* (I.0.2) *is continuous, i.e. $h\in\mathsf{C}\big([0,T]\times\mathbb{T}^d\big)$.*
  (iii) *ρ satisfies the hydrodynamic equation* (I.0.2) *with initial condition ρ^0.*
- *We say a classical solution ρ of* (I.0.2) *is probability density valued, if $\rho_t\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta\in\mathbb{M}_1(\mathbb{R})$ for all $(t,x)\in[0,T]\times\mathbb{T}^d$.*

In this section we prove (see Theorem I.1.2) the existence a function ρ^*, that is the probability density of $\widehat{\theta}_t$ evolving according to (I.0.3) and a classical solution of the hydrodynamic equation (I.0.2). Moreover, we show (see Theorem I.1.4) that there is a one to one correspondence between probability density valued, classical solution ρ of (I.0.2) and the time evolution of the (sufficient regular) probability densities of $\widehat{\theta}_t$, that are evolving according to (I.0.3), when the effective field $h=h^\rho$ is defined by ρ as in the second line of (I.0.2).

Before we state these results let us introduce the following notation of one site diffusion operators, that simplifies the notation in the following sections. We define the one site diffusion operator without effective field by

$$\mathbb{L}_0 \coloneqq e^{\Psi(\theta)}\partial_\theta\left(e^{-\Psi(\theta)}\partial_\theta\right) = -\Psi'\left(\theta\right)\partial_\theta+\partial_\theta^2. \quad \text{(I.1.1)}$$

Let h be a bounded and continuous function on $[0,T]$. We consider the following diffusion operator

$$\mathbb{L}_{h(t)} \coloneqq e^{\Psi(\theta)}\partial_\theta\left(e^{-\Psi(\theta)}\partial_\theta\right)+h\left(t\right)\partial_\theta = \mathbb{L}_0+h_t\partial_\theta = \left(-\Psi'\left(\theta\right)+h\left(t\right)\right)\partial_\theta+\partial_\theta^2. \quad \text{(I.1.2)}$$

The adjoint of $\mathbb{L}_{h(t)}$ on $L^2\left(\mathbb{R}, e^{-\Psi}\right)$ is

$$\mathbb{L}_{h(t)}^{*}\rho = \mathbb{L}_0\rho - h\left(t\right) e^{\Psi}\partial_\theta\left(e^{-\Psi}\rho\right) = \left(-\Psi'\left(\theta\right) - h\left(t\right)\right)\partial_\theta\rho + h\left(t\right)\Psi'\rho + \partial_\theta^2\rho. \tag{I.1.3}$$

Both operators $\mathbb{L}_{h(t)}$ and $\mathbb{L}_{h(t)}^{*}$ are linear and contain unbounded and time dependent drift coefficients. Using this notation the hydrodynamic equation (I.0.2) equals

$$\begin{aligned} \partial_t\rho_t\left(x,\cdot\right) &= \mathbb{L}_{h(t,x)}^{*}\rho_t\left(x,.\right) && \text{for all } \left(t,x,\theta\right) \in (0,T] \times \mathbb{T}^d \times \mathbb{R}, \text{ with} \\ \rho_0\left(x,\theta\right) &= \rho^0\left(x,\theta\right) && \text{for all } \left(x,\theta\right) \in \mathbb{T}^d \times \mathbb{R}. \end{aligned} \tag{I.1.4}$$

The rest of this section is organised as follows: At first, we state in Section I.1.1 the existence result and the result concerning the relation between classical solutions of the hydrodynamic equation (I.0.2) and the hydrodynamic SDE (I.0.3). Then we explain the main ideas and the structure of the existence proof. In Section I.1.2 we state preparatory lemmata. To prove the existence result, we consider in Section I.1.3, one dimensional SDEs by fixing an effective field, i.e. an SDE like (I.0.3) for a $x \in \mathbb{T}^d$, when h does not depend on ρ. In Section I.1.4, we use these result for the one dimensional SDEs to infer by a fixed point argument the existence of a classical solution of the hydrodynamic equation (I.0.2). Finally, we prove in Section I.1.5 the relation between the densities of the hydrodynamic SDE (I.0.3) and solutions of the hydrodynamic equation (I.0.2).

I.1.1 Main results and idea of the proofs

The following existence theorem is the main result of this Section I.1.

Theorem I.1.2. *Let the Assumption I.0.1, Assumption I.0.3, Assumption I.0.4 a.) and Assumption I.0.6 be satisfied. Then there exists a function $\rho^* : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$, that has the following properties*

- *ρ^* is a classical solution of* (I.0.2) *with initial condition ρ^0 in the sense of Definition I.1.1.*
- *ρ^* is probability density valued, i.e. $\rho_t^*\left(x,\theta\right) e^{-\Psi(\theta)}\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$ for all $x \in \mathbb{T}^d$ and $t \in [0,T]$.*
- *$\rho_t^*\left(x,\theta\right) e^{-\Psi(\theta)}\mathrm{d}\theta$ is the law of $\widehat{\theta}_t^x = \widehat{\theta}_t^{x,\rho^*}$ that evolves according to* (I.0.3) *with continuous effective field $h = h^{\rho^*}$ defined in* (I.0.2).
- *There is a constant $C > 0$ such that for all $(t,x,\theta) \in [0,T] \times \mathbb{T}^d \times \mathbb{R}$*

$$\rho_t^*\left(x,\theta\right) \le Ce^{\frac{1}{8}\Psi(\theta)}. \tag{I.1.5}$$

- *For each $T_1 > 0$, there is a constant $C_{T_1} > 0$ such that for all $(t,x,\theta) \in [T_1,T] \times \mathbb{T}^d \times \mathbb{R}$*

$$e^{-C_{T_1}\left(1+\Psi(\theta)^2\right)} \le \rho_t^*\left(x,\theta\right) e^{-\Psi(\theta)}. \tag{I.1.6}$$

Remark I.1.3. *Further interesting properties of ρ^* are those properties listed in Theorem I.1.6 and in Lemma I.1.23. Moreover, one can show that $\rho^* \in \mathsf{C}^{\infty,0,\infty}\big([0,T] \times \mathbb{T}^d \times \mathbb{R}\big)$ (see Remark I.1.31). However, we do not need these properties for the relative entropy method in Section I.2.*

Beside the existence, we show the following equivalence between classical solutions of the hydrodynamic equation (I.0.2) and the hydrodynamic SDE (I.0.3) (for the proof see Section I.1.5).

Theorem I.1.4. *Assume $\rho^* : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ satisfies the regularity condition (i) of Definition I.1.1 and defines a continuous effective field $h = h^{\rho^*}$ ((ii) of Definition I.1.1). Then the following statements are equivalent:*

- *ρ^* is a probability density valued, classical solution of* (I.0.2) *in the sense of Definition I.1.1.*
- *For each $(t,x) \in [0,T] \times \mathbb{T}^d$, $\rho_t^*\left(x,\theta\right) e^{-\Psi(\theta)}\mathrm{d}\theta$ is the law of $\widehat{\theta}_t^x$, that evolves according to the SDE* (I.0.3) *with effective field $h = h^{\rho^*}$.*

The proof of Theorem I.1.2 is organised in the following steps:

1.) **Fokker-Planck equation with fixed interaction:** At first, we linearise the problem by fixing an arbitrary effective field $h \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$. We investigate for each $x \in \mathbb{T}^d$ the SDE

$$\begin{aligned} \mathrm{d}\overline{\theta}_t^x &= -\Psi'\left(\overline{\theta}_t^x\right)\mathrm{d}t + h\left(t,x\right)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^x, \\ \overline{\theta}_0^x &\sim \rho^0\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})\,, \end{aligned} \tag{I.1.7}$$

and the corresponding Fokker-Planck equation

$$\begin{aligned} \partial_t \rho_t^{h,x}\left(\theta\right) &= \mathbb{L}_{h(t,x)}^* \rho_t^{h,x}\left(\theta\right), && \text{for all } (t,\theta) \in [0,T] \times \mathbb{R}, \\ \rho_0^{h,x}\left(\theta\right) &= \rho^0\left(x,\theta\right), && \text{for all } \theta \in \mathbb{R}. \end{aligned} \tag{I.1.8}$$

Due to the fixed effective field h, the SDE (I.1.7) and the Fokker-Planck equation (I.1.8) are linear problems, in contrast to the original SDE (I.0.3) and PDE (I.0.2).

In this first step, we want to derive a sufficient regular time evolution of densities of the SDE (I.1.7). In particular, we need continuity of this density in the $x \in \mathbb{T}^d$ coordinate. This density turns out to be also a solution of the Fokker-Planck equation (I.1.8). Indeed, we need the following two concepts of solutions.

Definition I.1.5. (i) *We call a function $\rho : [0,T] \times \mathbb{R} \to \mathbb{R}$ a classical solution of* (I.1.8) *at $x \in \mathbb{T}^d$, if $\rho e^{-\Psi} \in \mathsf{C}^{1,2}((0,T) \times \mathbb{R}) \cap \mathsf{C}_b([0,T] \times \mathbb{R})$ and if ρ solves* (I.1.8) *for this $x \in \mathbb{T}^d$.*

(ii) *We say that a function $\nu_{[0,T]} \in \mathsf{C}([0,T]\,,\mathbb{M}_1(\mathbb{R}))$ is a (probability measure valued) weak solution (in the sense of distributions) of* (I.1.8) *at $x \in \mathbb{T}^d$ with initial distribution $\nu_x^0\left(\mathrm{d}\theta\right) := \rho^0\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$, if*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \left(\mathbb{L}_{h(t,x)} - \partial_t\right) f\left(t,\theta\right)\nu_t\left(\mathrm{d}\theta\right)\mathrm{d}t = 0 && \text{for all } f \in \mathsf{C}_c^\infty((0,T) \times \mathbb{R}) \\ &\text{and } \lim_{t\to 0}\int_{\mathbb{R}} f\left(\theta\right)\nu_t\left(\mathrm{d}\theta\right) \to \int_{\mathbb{R}} f\left(\theta\right)\nu_x^0\left(\mathrm{d}\theta\right) && \text{for all } f \in \mathsf{C}_c^\infty(\mathbb{R})\,. \end{aligned} \tag{I.1.9}$$

We state in the following theorem the existence of sufficient regular time evolution of the density corresponding to (I.1.7), that is a classical solution of (I.1.8).

Theorem I.1.6. *Let the assumptions of Theorem I.1.2 hold. Fix a $h \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$. Then there exists a function $\rho^h : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ such that:*

- $\rho^h e^{-\Psi} \in \mathsf{C}^{1,0,2}\big((0,T) \times \mathbb{T}^d \times \mathbb{R}\big) \cap \mathsf{C}_b\big([0,T] \times \mathbb{T}^d \times \mathbb{R}\big)$.
- *For each $(t,x) \in [0,T] \times \mathbb{T}^d$, $\rho_t^h\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$.*
- *For each $t \in [0,T]$, $\rho_t^h\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta$ is the law of $\overline{\theta}_t^x$, that evolves according to* (I.1.7) *with the fixed h in the drift coefficient.*
- *For each $x \in \mathbb{T}^d$, $\rho_\cdot^h\left(x,\cdot\right) : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the unique classical solution of the Fokker-Planck equation* (I.1.8) *with the fixed h in the drift coefficient.*
- *ρ^h is, for each $x \in \mathbb{T}^d$, the unique weak solution of* (I.1.8) *with the fixed h in the drift coefficient.*
- *ρ^h satisfies* (I.1.5)*, with a constant $C = C\left(|h|_\infty\right) > 0$.*
- *ρ^h satisfies* (I.1.6)*, for a $T_1 > 0$ and a $C_{T_1} = C_{T_1}\left(|h|_\infty\right) > 0$.*

The strategy of proof of Theorem I.1.6 (given in Section I.1.3) is the following:

1.1.) **Explicit formula for the transition and probability density of time dependent SDEs and relation to Fokker-Planck equations:**

We consider at first a more general SDE than (I.1.7). For this one dimensional SDE, we prove in Section I.1.3.1 an explicit formula for the transition density ζ^x. Moreover, we show that ζ^x is smooth and solves the corresponding Fokker-Planck equation for a fixed initial value. From this we infer the smoothness of the probability density $\zeta^x\rho^0$ and that this probability density solves the Fokker-Planck equation with initial distribution ρ^0.

We apply in Section I.1.3.2 these general results to the SDE (I.1.7). Hence for each $x \in \mathbb{T}^d$, there is an explicitly given probability density ρ^x of (I.1.7), that is a classical solution ρ^x of the Fokker-Planck equation (I.1.8).

1.2.) **Properties of ρ^x:** Next, we infer from the explicit formula of ρ^x, the continuity of ρ^x in the space variable $x \in \mathbb{T}^d$ (in Section I.1.3.3). Moreover, we show that the classical solution is also a weak solution of (I.1.8) in the sense of Definition I.1.5 (ii).

For weak solutions of (I.1.8) we show in Section I.1.3.4 nice properties, including uniqueness and an upper bound. The reason why we use the concept of weak solutions here is, that we easily get these properties by applying general known results from the literature.

We turn to the second step in the proof of Theorem I.1.2.

2.) **Defining a sequence that converges to a classical solution of** (I.0.2) **(a fixed point argument):**

We know by the first step that, for each fixed effective field $h \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$, there exists a classical solution $\rho^{h(.,x)}$ of the Fokker-Planck equation (I.1.8). We define a new effective field $h^{(2)}$ as in the second line of (I.0.2) with this ρ^h. For this effective field $h^{(2)}$, we use again the result of first step, i.e. look at the linear Fokker-Planck equation with effective $h^{(2)}$. For this equation we get again a classical solution by Theorem I.1.6 and a new effective field $h^{(3)}$.

We define the operator G that maps an effective field $h^{(n)}$ to the effective field $h^{(n+1)}$ constructed by this procedure in the first step. Then a classical solution of (I.0.2) is a fixed point of the operator G.

For small time intervals $[0,\widehat{T}]$, we show the convergence of the sequence $h^{(n)}$ to a fixed point of this operator (Theorem I.1.25 (i) and Section I.1.4.1).

In Section I.1.4.2 (see also Theorem I.1.25 (ii)) we increase the length of the interval iteratively by $\widehat{T}$ until we have constructed a classical solution on the whole interval $[0,T]$.

3.) **Properties of the classical solution:** A classical solution of (I.0.2) is also a classical solution of (I.1.8) with the effective field $h = h^\rho$ (defined as in the second line of (I.0.2)). Therefore, the properties of the derived classical solution of (I.0.2) stated in Theorem I.1.2, carry over from the same properties stated in Theorem I.1.6.

Remark I.1.7. *This proof does not show the uniqueness of a classical solution of the nonlinear PDE* (I.0.2)*, because the map of the effective fields is not a contraction (see Remark I.1.35), although for each starting point the sequence $h^{(n)}$ converges. Nevertheless, we are able to show the uniqueness of a probability density valued classical solution (see Corollary I.2.4 and Section I.3)*

Let us end this section with a short discussion why we chose the above mentioned approach to prove Theorem I.1.2. One might ask why we do not apply usual results from the theory of PDEs to show the existence of a classical solution of the PDE (I.0.2). However, there are a couple reasons why we did not find an applicable result from this theory. First of all the PDE (I.0.2) is nonlinear and non elliptic due to the $\mathbb{T}^d$ space dependency. This leads to a degenerated diffusion operator (it has d times the eigenvalue 0). Moreover, we do not assume any growth conditions on the potential

Ψ, nor Hölder nor Lipschitz conditions. Finally, we look for a solution that consists of probability densities, hence solutions in a subspace of $L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$.

It is for the proof of the hydrodynamic limit even more important than the existence of a classical solution of (I.0.2), that we have a probability density of the SDE (I.0.1) (remember that the density influences also the drift coefficient). Even if there was a simpler proof for the existence of a classical solution of (I.0.2), it is a priori not obvious if this solution is a probability density of the SDE (I.0.1). Note that we require a main part of the proof of Theorem I.1.2 to prove the equivalence result (Theorem I.1.4) of classical solutions and such a probability density. Hence, the proof that we state in the next chapters has the advantage that we get not only a classical solution of the PDE (I.0.2), but we know that this solution is a probability density of the SDE (I.0.1).

Also for the existence of a classical solution of the one dimensional linear Fokker-Planck equation (I.1.8) we do not use an approach that uses pure techniques coming from the theory of PDE. This PDE is of course a lot simpler to study than the nonlinear PDE (I.0.2), because it is linear and elliptic, although some of the challenges, like the unbounded growth of the drift coefficient, are still present. Nevertheless, we need again not only a classical solution of the Fokker Planck equation (I.1.8) but a probability density corresponding to (I.1.7). Moreover, we get, by the approach sketched above, an explicit representation of the classical solution. From this representation, we infer easily the continuity of this solution in the space variable $x \in \mathbb{T}^d$ (see Corollary I.1.21).

I.1.2 Preliminaries and generalisation of the assumptions

In this section we first state some consequences of the assumptions. We need these results in the subsequent sections. At the end of this section, we discuss briefly the upper bound (I.1.5) on ρ and generalisation of the assumptions.

Lemma I.1.8. *The Assumption I.0.3 implies*

(i) $\Psi \in \mathsf{C}^4(\mathbb{R})$.

(ii) $e^{-\frac{1}{2}\Psi}, e^{-\frac{3}{4}\Psi}, \theta e^{-\frac{3}{4}\Psi}, \Psi' e^{-\frac{3}{4}\Psi}, \theta e^{-\frac{7}{8}\Psi} \in L^1(\mathbb{R})$.

(iii) $e^{-\frac{1}{8}\Psi} \in L^\infty(\mathbb{R})$.

(iv) $\Psi', \Psi'', \Psi''', \Psi'\Psi'' \in L^2\left(\mathbb{R}, e^{-\Psi}\right)$.

(v) *There is a constant $C_\Psi \in \mathbb{R}$ such that $-\Psi'(\theta)\,\theta \leq C_\Psi$.*

(vi) *For each $h \in \mathsf{C}_b\left([0,T] \times \mathbb{T}^d\right)$ and $\epsilon > 0$, there is a constant $C > 0$, that depends only on $|h|_\infty$ and ϵ, such that for all $(t,x,\theta) \in [0,T] \times \mathbb{T}^d \times \mathbb{R}$*

$$\mathbb{L}_{h(t,x)}\theta^2, \mathbb{L}_{h(t,x)} e^{\frac{7}{8}\Psi}, \mathbb{L}_{h(t,x)} e^{\left(\frac{7}{8}+\epsilon\right)\Psi} \leq C. \tag{I.1.10}$$

with $\mathbb{L}_{h(t,x)}$ defined in (I.1.2).

Moreover, also for all $(t,x,\theta) \in [0,T] \times \mathbb{T}^d \times \mathbb{R}$

$$\left|\mathbb{L}_{h(t,x)} e^{\frac{7}{8}\Psi}\right|, \left|\partial_\theta e^{\frac{7}{8}\Psi}\right| \leq C + e^{\left(\frac{7}{8}+\epsilon\right)\Psi}. \tag{I.1.11}$$

(vii) *There is a constant $C > 0$ such that for all $\theta \in \mathbb{R}$*

$$|\Psi'(\theta)|, |\Psi'(\theta)|^2 \leq C + e^{\frac{7}{8}\Psi(\theta)}. \tag{I.1.12}$$

Proof. All these claims follow directly from the Assumption I.0.3 on Ψ. For example (v) and (vi) follow because $\Psi(\theta) = c_k\theta^k +$ (lower order) for a $k \in 2\mathbb{N}$ and a $c_k > 0$. Indeed this implies $-\Psi'(\theta)\,\theta = -kc_k\theta^k +$ (lower order) $\leq C - \frac{1}{2}kc_k\theta^k$. □

In the following lemma we state simple consequences of the assumptions on the initial distribution.

Lemma I.1.9. (i) *The Assumption I.0.6 implies that $\rho^0(x,.)\,e^{-\Psi}$ integrates also $e^{\frac{7}{8}\Psi}$ uniformly (in $x \in \mathbb{T}^d$).*

(ii) *The continuity of Assumption I.0.4 a.) and the Assumption I.0.6 imply, that for all $(x,\theta) \in \mathbb{T}^d \times \mathbb{R}$*

$$\rho^0(x,\theta)\,e^{-\Psi(\theta)} \le Ce^{-\frac{7}{8}\Psi(\theta)}. \tag{I.1.13}$$

(iii) *From (i) and (ii) we infer that $\rho^0(x,.)\,e^{-\Psi} \in L^2(\mathbb{R})$, for all $x \in \mathbb{T}^d$.*

(iv) *By (ii) and Assumption I.0.3 the density $\rho^0(x,.)\,e^{-\Psi}$ has finite entropy for each $x \in \mathbb{T}^d$, i.e.*

$$\int_{\mathbb{R}} \rho^0(x,\theta)\,e^{-\Psi(\theta)} \log\left(\rho^0(x,\theta)\,e^{-\Psi(\theta)}\right) \mathrm{d}\theta < \infty. \tag{I.1.14}$$

Proof. To prove the claimed finite entropy in (iv), note that $\log\left(\rho^0(x,\theta)\,e^{-\Psi(\theta)}\right) < -C\Psi(\theta)$, by (ii). The right hand side is bounded by a constant by Assumption I.0.3. □

In the next remark we discuss briefly how the upper bound on ρ, which we derive in Theorem I.1.2 and Theorem I.1.6 can be optimised.

Remark I.1.10. *We claim in Theorem I.1.2 (in (I.1.5)) and in Theorem I.1.6, that the solution ρ^* is bounded from above by $e^{\frac{1}{8}\Psi}$. This upper bound is not optimal. Instead of the factor $\frac{1}{8}$ in the exponent of the upper bounds on ρ, there could be an arbitrary $q \in \left(0, \frac{1}{4}\right)$, provided that Assumption I.0.6 also holds with this exponent and that Lemma I.1.8 holds with $\frac{7}{8}$ replaced by $1-q$ everywhere (e.g. in (vi), $\mathbb{L}_{h(t,x)}e^{(1-q)\Psi} \le C$). The reason for the particular value $\frac{1}{8}$ in the bounds is only to simplify our notation.*

Let us finish this section with a short discussion about generalisations of the assumptions.

Remark I.1.11. *Some of the assumptions are stronger than actually necessary.*

- *As discussed in Remark I.1.10, when accepting a weaker upper bound on ρ in (I.1.5), the Assumption I.0.6 could be simplified. For example then ρ^0 has to integrate only $e^{\left(-\frac{1}{4}+\epsilon\right)\Psi}$ instead of $e^{\left(-\frac{1}{8}+\epsilon\right)\Psi}$.*
- *The proof of Theorem I.1.2, that we state in the following sections, still holds if Ψ is not a polynomial (i.e Assumption I.0.3 is not satisfied) provided that the following conditions are satisfied:*
  - a.) *Ψ is dominated by two polynomials in the sense, that there are two polynomials p_1 and p_2 of even degree greater or equal two, with positive coefficient of that degree such that $p_1(\theta) \le \Psi(\theta) \le p_2(\theta)$ for all $\theta \in \mathbb{R}$.*
  - b.) *There is a constant $C \in \mathbb{R}$, such that $\frac{1}{2}\left|\Psi'(\theta)\right|^2 \ge C + \Psi''(\theta)$ for all $\theta \in \mathbb{R}$ (this condition is needed in the proof of Theorem I.1.20).*
  - c.) *Lemma I.1.8 and Lemma I.1.9 still hold for the Ψ.*
- *Even if Ψ is not a polynomial and some statements of Lemma I.1.8 are not satisfied, the Theorem I.1.2 could still hold. For example for a general Ψ, the condition $\mathbb{L}_{h(t,x)}e^{\frac{7}{8}\Psi} \le C$ in Lemma I.1.8 (vi) can be replace by*

$$\mathbb{L}_{h(t,x)}e^{c|\theta|^{2k}} \le C, \tag{I.1.15}$$

for $C, c > 0$, $k \in \mathbb{N}$. If there are constants $C, \epsilon > 0$, $k \in \mathbb{N}$ and $c > 0$, such that $-\Psi'(\theta)\theta \leq C - (ck + \epsilon)|\theta|^{2k}$, then we know by Example 2.5 (iii) in [BDPR04] that

$$\mathbb{L}_{h(t,x)} e^{c|\theta|^{2k}} \leq C - \epsilon |\theta|^{2k} e^{c|\theta|^{2k}}, \tag{I.1.16}$$

and therefore (I.1.15) *holds.*

Remark I.1.12. *All the result of this chapter would also hold if we used a different constant diffusion coefficient $\sigma > 0$ in the SDE* (I.0.1) *instead of $\sqrt{2}$, provided that the assumptions are suitably adapted. Also the hydrodynamic equation* (I.0.2) *changes slightly.*

I.1.3 Proof of Theorem I.1.6: The one dimensional linearised system

In this section we prove Theorem I.1.6. We follow the strategy and structure of the proof explained in Section I.1.1. Hence, we show at first the existence of a smooth transition and probability densities for general one dimensional SDEs (Section I.1.3.1). Then we transfer this result to the SDE (I.1.7) (Section I.1.3.2). Finally in Section I.1.3.3 and Section I.1.3.4, we show properties of the probability density.

I.1.3.1 Existence of a smooth transition and probability densities for general one dimensional SDEs

As explained in Section I.1.1, we need regularity of the time evolution of the probability density of $\overline{\theta}_t^x$, that evolves according to (I.1.7). In this section, we prove this result in a more general setting, because it is interesting by itself. Hence, we consider the SDE

$$\mathrm{d}\theta_t = b(t, \theta_t)\,\mathrm{d}t + \sigma \mathrm{d}W_t \quad \text{with} \quad \theta_0 \sim \pi^0(\theta)\,\mathrm{d}\theta, \tag{I.1.17}$$

with $\sigma \in \mathbb{R}_+$ and the function $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

Assumption I.1.13. *Fix $r_1, r_2 \in \mathbb{N}$.*

a.) *There exists a $B(t, \theta) \in \mathsf{C}^{1+r_1, 2+r_2}([0, T] \times \mathbb{R})$ such that $\partial_\theta B(t, \theta) = b(t, \theta)$.*

b.) *There is a constant $C_F \in \mathbb{R}$ such that*

$$F(t, \theta) := \frac{1}{\sigma^2}(b(t, \theta))^2 + \sigma \partial_\theta b(t, \theta) + \frac{2}{\sigma}\partial_t B(t, \theta) \geq C_F, \tag{I.1.18}$$

for all $t \in [0, T]$ and $\theta \in \mathbb{R}$.

c.) *The martingale problem corresponding to* (I.1.17) *has a unique solution for each initial condition $\theta_0 \in \mathbb{R}$ and starting time $t_0 \in [0, T)$. We denote the solution by P_{t_0, θ_0}.*

On the initial condition $\pi^0 \in \mathsf{C}_0(\mathbb{R})$ we assume the following integrability assumption

$$\int_{\mathbb{R}} e^{-\frac{1}{\sigma}B(0,\theta_0)} \pi^0(\theta_0)\,\mathrm{d}\theta_0 < \infty. \tag{I.1.19}$$

We show in Theorem I.1.20, that the drift coefficient and the initial distribution of the SDE (I.1.7), satisfies these assumptions, provided that the assumptions of Theorem I.1.2 are satisfied.

We show at first (in Section I.1.3.1.1), that the SDE (I.1.17) has a sufficient regular transition density. Moreover, we show that this density also solves the corresponding Kolmogorov forward and backward equation. For both results we generalise the result of [Rog85] to time dependent and unbounded drifts. Then in Section I.1.3.1.2, we infer from the previous results, the smoothness of the probability density of the solution of the SDE (I.1.17) and that it solves the Fokker-Planck equation corresponding to (I.1.17).

This approach has the advantage, that we get explicit formulas for the probability density of the corresponding SDE. In particular, we infer easily from this explicit formula, the continuity of the probability density of the SDE (I.1.7) in the space variable $x \in \mathbb{T}^d$ (see Section I.1.3.3).

I.1.3.1.1 Existence of smooth transition density of SDE and FP equation (generalisation of [Rog85])

We show now that the SDE (I.1.17) has a smooth transition density, i.e. that there is a function ζ, such that $\theta_t \sim \zeta\,(t_0, \theta_0, t, \theta)\,\mathrm{d}\theta$, when $\theta_{[0,T]}$ evolves according to (I.1.17) and starts at $\theta_0 \in \mathbb{R}$ at time $t_0 \in [0, T]$, $t > t_0$. Moreover, we show that the transition density is a solution to the corresponding Kolmogorov forward (I.1.23) equation. To prove these statements, we generalise [Rog85] to time dependent and unbounded drift coefficients

Notation I.1.14. *We denote by $\widehat{P}_{t_0,\theta_0} \in \mathbb{M}_1\left(\mathsf{C}([t_0, T])\right)$ the law of $\sigma\widehat{W}_t$, where $\widehat{W}_t$ is a standard Brownian motion, and $\sigma\widehat{W}_t$ starts at $\theta_0 \in \mathbb{R}$ at time $0 \le t_0 < T$.*

Notation I.1.15. *To simplify the notation, we denote the spaces of possible domains of the transition density by*

$$\mathcal{S}_T := \{(t_0, \theta_0, t, \theta) \in [0, T) \times \mathbb{R} \times (0, T] \times \mathbb{R} : t_0 < t\} \tag{I.1.20}$$

$$\mathcal{S}_T^o := \left\{(t_0, \theta_0, t, \theta) \in ((0, T) \times \mathbb{R})^2 : t_0 < t\right\} \tag{I.1.21}$$

$$\overline{\mathcal{S}_T} := \left\{(t_0, \theta_0, t, \theta) \in ([0, T] \times \mathbb{R})^2 : t_0 \le t\right\}. \tag{I.1.22}$$

Theorem I.1.16. *If the Assumption I.1.13 holds, then the SDE* (I.1.17) *has a transition density $\zeta : \mathcal{S}_T \to (0, \infty)$ (w.r.t. the Lebesgue measure) such that $\zeta \in \mathsf{C}^{\min\{r_1,r_2\},r_2,\min\{r_1,r_2\},r_2}(\mathcal{S}_T^o) \cap \mathsf{C}^{0,r_2,0,r_2}(\mathcal{S}_T)$.*

If $r_1 \ge 1$ and $r_2 \ge 2$, then ζ solves the Kolmogorov forward equation (with fixed $t_0 \in [0, T)$ and fixed $\theta_0 \in \mathbb{R}$), i.e. for all $(t, \theta) \in (t_0, T] \times \mathbb{R}$

$$\partial_t \zeta\,(t_0, \theta_0, t, \theta) = \left[-\partial_\theta\,(b\zeta) + \frac{\sigma^2}{2}\partial_\theta^2 \zeta\right](t_0, \theta_0, t, \theta) \tag{I.1.23}$$

$$\text{with } \lim_{t \to t_0} \zeta\,(t_0, \theta_0, t, \theta) = \delta_{\theta_0 = \theta}, \tag{I.1.24}$$

where the convergence in the sense of distribution with test functions in $\mathsf{C}_c^{1,2}([0, T] \times \mathbb{R})$.

Proof. In this proof we generalisation [Rog85] to the setting of time dependent and unbounded drift coefficients. The approach of the following proof, therefore follows Rogers ideas and steps. We emphasise especially the parts where things are now more complicated and different, and we sketch the other parts only. We first prove a representation of the transition density (Step 1). Then we show that this transition density is sufficient regular (Step 2). Finally, we conclude that the transition density solves the Kolmogorov forward equations (Step 3).

Step 1: Representation of the transition density:

By Assumption I.1.13 a.), the generalised Girsanov formula (see Corollary B.2 in Appendix B), and the Itô's formula, we get for $f \in C_b(\mathbb{R})$

$$E_{P_{t_0,\theta_0}}\left[f(\theta_t)\right] = E_{\widehat{P}_{t_0,\theta_0}}\left[f(\theta_t)\, e^{\frac{1}{\sigma}B(t,\theta_t) - \frac{1}{\sigma}B(t_0,\theta_0) - \frac{1}{2}\int_{t_0}^t F(s,\theta_s)\mathrm{d}s}\right], \tag{I.1.25}$$

with F defined in (I.1.18). Note that in F, there is the additional (compared to [Rog85]) derivative of B w.r.t. time due to the time-inhomogeneous SDE.

As Rogers we condition on θ_t now and we use that the law of the process

$$\theta_0 + \frac{s - t_0}{t - t_0}(\eta - \theta_0) + \sigma\left(\overline{W}_{s-t_0} - \frac{s - t_0}{t - t_0}\overline{W}_{t-t_0}\right) \tag{I.1.26}$$

provides a conditional $\widehat{P}$-distribution for $\{\theta_s\}_{s\in[t_0,t]}$ given $\theta_t = \eta$ and $\theta_{t_0} = \theta_0$. Here $\overline{W}_t$ is a standard Brownian motion. Then (I.1.25) equals

$$\int_{\mathbb{R}} f(\eta) \underbrace{\gamma(t-t_0,\theta_0,\eta)\, e^{\frac{1}{\sigma}B(t,\eta)-\frac{1}{\sigma}B(t_0,\theta_0)}\phi(t_0,\theta_0,t,\eta)}_{=:\zeta(t_0,\theta_0,t,\eta)} \mathrm{d}\eta, \tag{I.1.27}$$

with $\gamma(.,.,.)$ the transition density of σB_t, i.e. $\gamma(t-t_0,\theta_0,\eta) = \frac{1}{\sigma\sqrt{(t-t_0)2\pi}} e^{-\frac{1}{2}\frac{(\theta_0-\eta)^2}{\sigma^2(t-t_0)}}$ and

$$\phi(t_0,\theta_0,t,\eta) := E_{\overline{W}}\left[e^{-\frac{1}{2}(t-t_0)\int_0^1 F\left(u(t-t_0)+t_0,\theta_0+u(\eta-\theta_0)+\sigma\sqrt{t-t_0}W_u^0\right)\mathrm{d}u}\right], \tag{I.1.28}$$

where $W_u^0 = \overline{W}_u - u\overline{W}_1$ is a Brownian bridge and $E_{\overline{W}}$ is the expectation w.r.t. the Brownian motion.

From (I.1.25), (I.1.27) and from P_{t_0,θ_0} being the solution to the martingale problem (Assumption I.1.13 c.)), we infer the convergence (I.1.24) at the starting points.

Step 2: Regularity of the transition density:
Now we prove the claimed regularity of the function ζ by showing that ϕ has the claimed regularity. For the other factors in ζ, this regularity is obvious by Assumption I.1.13 a.). To show the continuity and the continuously differentiability of ϕ we use multiple times in the sequel the following observation.

Lemma I.1.17. *Let X be a metric spaces, $I \subset \mathbb{R}^d$ be an open set and (Ω,μ) be a measure space. Fix a $k \in \mathbb{N}_0^d$. Let $g : I \times X \times \Omega \to \mathbb{R}$ be a function with the following properties:*

(i) *$g(z,x,.)$ is μ-integrable for all $x \in X$, $z \in I$.*

(ii) *$g(.,.,\omega)$ is $\mathsf{C}^{k,0}(I\times X)$ (k times continuously differentiable in I and continuous in X) for all $\omega \in \Omega$.*

(iii) *For each $(z,x) \in I \times X$ there is a neighbourhood $N_{z,x} \subset I \times X$ such that*

$$\sup_{k'\le k}\sup_{(z',x')\in N_{z,x}} \left|\partial^{|k'|}_{z^{k'}} g(z',x',.)\right| \in L^1(\Omega). \tag{I.1.29}$$

Then $G(z,x) = \int_\Omega g(z,x,\omega)\,\mu(\mathrm{d}\omega)$ is in $\mathsf{C}^{k,0}(I\times X)$.

This lemma is a consequence of the dominated convergence theorem (see also (10) in [Rog85] and [Bau01] Chapter 16).

Lemma I.1.18. *We define for $\left(t_0,\theta_0,t,\theta,W^0\right) \in \overline{\mathcal{S}_T} \times \mathsf{C}([0,1],\mathbb{R})$, the function*

$$\overline{F}\left(t_0,\theta_0,t,\theta,W^0\right) := \int_0^1 F\left(u(t-t_0)+t_0,\theta_0+u(\eta-\theta_0)+\sigma\sqrt{t-t_0}W_u^0\right)\mathrm{d}u. \tag{I.1.30}$$

This map is continuous on $\overline{\mathcal{S}_T} \times \mathsf{C}([0,1],\mathbb{R})$, and $\min\{r_1,r_2\}$ times continuously differentiable on $\mathcal{S}_T^o$ in the variables t_0 and t, and r_2 times in the variables θ_0 and θ.

Moreover, for all $(t_0,\theta_0,t,\theta) \in \overline{\mathcal{S}_T}$, $W^0 \in \mathsf{C}([0,1],\mathbb{R})$, there is a neighbourhood N_ϵ and a constant $C > 0$, such that

$$\sup_{\left(t_0',\theta_0',t',\theta',W^{0\prime}\right)\in N_\epsilon} \left|\partial_x \overline{F}\left(t_0',\theta_0',t',\theta',W^{0\prime}\right)\right| < C, \tag{I.1.31}$$

where ∂_x can be $\partial_t, .., \partial_t^{\min\{r_1,r_2\}}, \partial_\theta, ..., \partial_\theta^{r_2}$, the same for t_0 and θ_0 or omitted completely (i.e. the function $\overline{F}$ itself is bounded in N_ϵ).

Proof. We proof this lemma by an application of Lemma I.1.17. The map on $[0,1]\times\overline{\mathcal{S}_T}\times \mathsf{C}([0,1],\mathbb{R})$

$$\left(u,t_0,\theta_0,t,\theta,W^0\right)\mapsto F\left(u\left(t-t_0\right)+t_0,\theta_0+u\left(\eta-\theta_0\right)+\sigma\sqrt{t-t_0}W_u^0\right) \tag{I.1.32}$$

satisfies (i) and (ii) of Lemma I.1.17, as a composition of continuously differentiable functions (by Assumption I.1.13 a.)). The condition (iii) of Lemma I.1.17 is satisfied, because for each $\left(t_0,\theta_0,t,\theta,W^0\right)$, the neighbourhood N_ϵ can be chosen such that

$$\left|\theta_0'+u\left(\eta'-\theta_0'\right)+\sigma\sqrt{t'-t_0'}W_u^{0\prime}\right|\le R, \tag{I.1.33}$$

for all $u\in[0,1]$ and all $\left(t_0',\theta_0',t',\theta',W^{0\prime}\right)\in N_\epsilon$, where R is a constant.

The restriction of F to $[0,T]\times[-R,R]$ is a continuously differentiable function over a compact set and therefore uniformly bounded. This implies (iii) of Lemma I.1.17. Hence, Lemma I.1.17 implies that $\overline{F}$ is continuously differentiable as claimed in this Lemma.

The proof of condition (iii) of Lemma I.1.17 implies also that (I.1.31) is satisfied, because F and its derivatives are uniformly bounded for all elements in N_ϵ and for all $u\in[0,1]$. ☐

We conclude from Lemma I.1.18 and Lemma I.1.17, that ϕ has the claimed regularity. The conditions of Lemma I.1.17 are satisfied by the local boundedness and the continuity of $\overline{F}$ and of its derivatives (as shown in Lemma I.1.18).

Note that we need $B\in\mathsf{C}^{1+r,2+r}([0,T]\times\mathbb{R})$, for ϕ being r-times continuously differentiable in t or t_0. To differentiate ϕ continuously r-times in θ or in θ_0, we need $B\in\mathsf{C}^{1,2+r}([0,T]\times\mathbb{R})$ and no additional derivative in time.

Step 3: $\zeta\left(t_0,\theta_0,t,\theta\right)$ is the fundamental solution to Kolmogorov forward equation: This follows from Itô's formula because ζ is regular enough. Alternatively this can be shown by generalising the proof in [Rog85]. Then we would get that ζ is also the solution to the Kolmogorov forward and backward equation. Note, that these two equations can not deduced from each other in the time-inhomogeneous setting we consider here. ☐

I.1.3.1.2 Smooth probability density of SDE solves Fokker-Planck equation

We infer now from Theorem I.1.16, that also the probability density of (I.1.17) is smooth and satisfies the corresponding Fokker-Planck equation.

Theorem I.1.19. *Let the Assumption I.1.13 with $r_1\ge 1$ and $r_2\ge 2$ and let ζ be the transition density of* (I.1.17) *(derived in Theorem I.1.16). Moreover, let the integrability condition* (I.1.19) *on the initial distribution hold. Then*

$$\rho\left(t,\theta\right):=\int_{\mathbb{R}}\zeta\left(0,\theta_0,t,\theta\right)\pi^0\left(\theta_0\right)d\theta_0 \tag{I.1.34}$$

is in $\mathsf{C}^{1,2}((0,T)\times\mathbb{R})\cap\mathsf{C}([0,T]\times\mathbb{R})$ *and ρ is the probability density of the SDE* (I.1.17). *Moreover, there is a constant, such that*

$$\left|\rho\left(t,\theta\right)\right|\le Ce^{\frac{1}{\sigma}B(t,\theta)}, \tag{I.1.35}$$

and ρ solves

$$\begin{aligned}\partial_t\rho\left(t,\theta\right)&=\left[-\partial_\theta\left(b\rho\right)+\frac{\sigma^2}{2}\partial_\theta^2\rho\right]\left(t,\theta\right)\ \text{ for }\ (t,\theta)\in[0,T]\times\mathbb{R},\\ \rho\left(0,\theta\right)&=\pi^0\left(\theta\right)\ \text{ for }\theta\in\mathbb{R}.\end{aligned} \tag{I.1.36}$$

Proof. By its definition, ρ is obviously the probability density of the SDE (I.1.17). Let us now prove the smoothness of ρ.

Step 1: $\rho(t,\theta) \in \mathsf{C}^{1,2}((0,T)\times\mathbb{R})$: We infer the continuity of the derivatives from Theorem I.1.16. Hence, we have to justify that the derivatives w.r.t. t and θ can be interchanged with the integral in (I.1.34). To do this, we use Lemma I.1.17. We know by Theorem I.1.16, that $\zeta \in \mathsf{C}^{1,2}\left(((0,T)\times\mathbb{R})^2\right)$, i.e. that ζ and its derivatives are continuous. Moreover, for each $(t,\theta) \in (0,T)\times\mathbb{R}$, there is a neighbourhood $N_{t,\theta}$ of (t,θ) and a constant $C_{t,\theta} > 0$, such that for all $(s,\eta) \in N_{t,\theta}$

$$|\zeta(0,\theta_0,s,\eta)| \leq |\gamma(s,\theta_0,\eta)|\,|\phi(0,\theta_0,s,\eta)|\,e^{\frac{1}{\sigma}B(s,\eta)}e^{-\frac{1}{\sigma}B(0,\theta_0)} \leq C_{t,\theta}e^{-\frac{1}{\sigma}B(0,\theta_0)}, \tag{I.1.37}$$

because B is as a continuous function locally bounded, because $\gamma(s,\theta_0,\eta) \leq (s)^{-\frac{1}{2}}\,C$ and by (I.1.31). Hence, ζ is locally (for each $(t,\theta)\in(0,T)\times\mathbb{R}$) dominated by an integrable function. Therefore, ρ is continuous.

Similar we can show the existence of a local bound on the derivatives of ζ. This uses again (I.1.31) to bound the derivatives of ϕ.

Step 2: $\rho(t,\theta) \in \mathsf{C}([0,T]\times\mathbb{R})$: We get the continuity at time $t = T$ as in the previous step by the corresponding result in Theorem I.1.16. Hence, we only have to show the continuity at time $t = 0$. This is a direct consequence of (I.1.24), because $\pi^0 \in \mathsf{C}_0(\mathbb{R})$ can be uniformly approximated by $\mathsf{C}_c(\mathbb{R})$ functions.

Step 3: Upper bound on ρ: To prove the boundedness of ρ, insert at first the definition (I.1.27) of ζ into (I.1.34)

$$\rho(t,\theta) = \int_{\mathbb{R}} \gamma(t,\theta_0,\theta)\,e^{\frac{1}{\sigma}B(t,\theta)-\frac{1}{\sigma}B(0,\theta_0)}\phi(0,\theta_0,t,\theta)\,\pi^0(\theta_0)\,\mathrm{d}\theta_0. \tag{I.1.38}$$

Then there is a constant C independent of $t \in [0,T]$ and $\theta \in \mathbb{R}$ such that

$$|\rho(t,\theta)| \leq e^{\frac{1}{\sigma}B(t,\theta)} \int_{\mathbb{R}} \gamma(t,\theta_0,\theta)\,\mathrm{d}\theta_0\,|\phi|_\infty \left|e^{-\frac{1}{\sigma}B(0,.)}\pi^0(.)\right|_\infty \leq e^{\frac{1}{\sigma}B(t,\theta)}C, \tag{I.1.39}$$

because $|\phi|_\infty < C_\phi$ (by Assumption I.1.13 b.)), because γ is a probability density and because $e^{-\frac{1}{\sigma}B(.,0)}\pi^0(.) \in \mathsf{C}_0(\mathbb{R})$ (by (I.1.19) and Assumption I.1.13 a.)).

Step 4: ρ solves (I.1.36)**:** As shown in the first step we can interchange the integral and the derivatives hence the claim follows from (I.1.23). ☐

I.1.3.2 Existence of smooth probability density of (I.1.7) / classical solution of the Fokker-Planck equation (I.1.8)

In the following theorem we show the existence of a classical solution of the Fokker-Planck equation (I.1.8) that is at the same time the probability density of (I.1.7). To prove this theorem, we show that the general results of Section I.1.3.1 are applicable to the system described by the SDE (I.1.7).

Theorem I.1.20. *Let the same assumptions as in Theorem I.1.6 hold. For each $x \in \mathbb{T}^d$, there exists a function $\rho^x : [0,T]\times\mathbb{R} \to \mathbb{R}$, such that*

- *ρ^x is a classical solution (in the sense of Definition I.1.5 (i)) of the Fokker-Planck equation* (I.1.8) *at $x \in \mathbb{T}^d$.*
- *For each $t \in [0,T]$, $\rho^x_t(\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta$ is the law of $\underline{\theta}^x_t$ evolving according to* (I.1.7) *with fixed effective field $h(.,x)$.*

Proof. We show at first that the assumptions of Section I.1.3.1 are satisfied for a system described by the SDE (I.1.7) (Step 1 and Step 2). Then (see Step 3) Theorem I.1.16 and Theorem I.1.19 imply the existence of the function ρ^x.

Step 1: The Assumption I.1.13 holds:

- The drift coefficient of the SDE (I.1.7) and its integral are for a fixed $x \in \mathbb{T}^d$ given by

$$\begin{aligned} b^x(t,\theta) &= -\Psi'(\theta) + h(t,x), \\ B^x(t,\theta) &= \int_0^\theta b^x(t,\theta')\,\mathrm{d}\theta' = -\frac{1}{2}\Psi(\theta) + h(t,x)\,\theta + \frac{1}{2}\Psi(0). \end{aligned} \tag{I.1.40}$$

Hence, for $r_1 = 1$ and $r_2 = 2$, the Assumption I.1.13 a.) is satisfied (by Lemma I.1.8 (i) and because $h \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$).

- Moreover, for fixed $x \in \mathbb{T}^d$

$$F^x(s,\theta) = \frac{1}{2}\left(\Psi'(\theta)\right)^2 - \Psi'(\theta)\,h(t,x) + \frac{1}{2}\left(h(t,x)\right)^2 - \sqrt{2}\Psi''(\theta) + \sqrt{2}\partial_t h(t,x)\,\theta. \tag{I.1.41}$$

By Ψ being an even polynomial (Assumption I.0.3), we infer form this formula the Assumption I.1.13 b.).

- By the boundedness of h and by Lemma I.1.8 (vi) we know (by the same argument that leads to Notation I.2.6) that the corresponding martingale problem is well posed with measure $P^x_{t_0,\theta_0} \in \mathbb{M}_1\left(\mathsf{C}([0,T],\mathbb{R})\right)$. This implies Assumption I.1.13 c.).

Step 2: The assumption (I.1.19) **on** ρ^0 **holds:**
The assumptions on ρ^0 are satisfied, by Assumption I.0.4 a.) and Assumption I.0.6.

Step 3: Applying Theorem I.1.19 and Theorem I.1.16:
Hence, we know by Theorem I.1.16 that there is a transition density ζ^x of (I.1.7) for each $x \in \mathbb{T}^d$. Then the probability density with respect to the Lebesgue measure

$$\overline{\rho}^x(t,\theta) = \int_{\mathbb{R}} \zeta^x(0,\theta_0,t,\theta)\,e^{-\Psi(\theta_0)}\rho^0(x,\theta_0)\,d\theta_0, \tag{I.1.42}$$

has the desired properties by Theorem I.1.19. Note that $\overline{\rho}^x$ is bounded by (I.1.35), by h being bounded and by Assumption I.0.3.

Last but not least we set $\rho^x(t,\theta) := e^{\Psi(\theta)}\overline{\rho}^x(t,\theta)$ which is a classical solution of (I.1.8). □

By the proof of Theorem I.1.16 (in particular from the definition of the transition density in (I.1.28)) and by (I.1.34) we know the following explicit formula of ρ^x.

$$\begin{aligned} \rho^x(t,\theta) = e^{\Psi(\theta)} \int_{\mathbb{R}} & \gamma(t,\theta_0,\theta) e^{\frac{1}{\sqrt{2}}B^x(t,\theta) - \frac{1}{\sqrt{2}}B^x(0,\theta_0)} \\ & E_{\overline{W}}\left[e^{-\frac{1}{2}t\int_0^1 F^x\left(ut,\theta_0+u(\theta-\theta_0)+\sqrt{t}W_u^0\right)\mathrm{d}u}\right] e^{-\Psi(\theta_0)}\rho^0(x,\theta_0)\,\mathrm{d}\theta_0, \end{aligned} \tag{I.1.43}$$

with γ, $E_{\overline{W}}$ and W_u^0 defined next to (I.1.28).

I.1.3.3 Properties of the classical solution of the Fokker-Planck equation (I.1.8)

We know by Theorem I.1.20, that there is a classical solution ρ^x of the Fokker Planck equation (I.1.8) for each $x \in \mathbb{T}^d$. In the next corollary we state the regularity of ρ^x in the $x \in \mathbb{T}^d$ variable. The corollary follows by similar arguments used to show the continuity in the other variables in Theorem I.1.16 and Theorem I.1.19.

Corollary I.1.21. *The probability density $\rho^x e^{-\Psi}$ and their derivatives are also continuous in the space variable $x \in \mathbb{T}^d$.*

We show in the next lemma, that each classical solution of (I.1.8) is also a weak solution.

Lemma I.1.22. *Take an arbitrary $h \in \mathsf{C}\left([0,T] \times \mathbb{T}^d\right)$. For each $x \in \mathbb{T}^d$, assume there is a classical solution $\rho^{h,x}$ of* (I.1.8) *(in the sense of Definition I.1.5 (i)) with fixed effective field $h(.,x)$, such that $\rho_t^{h,x} e^{-\Psi} \mathrm{d}\theta$ is a probability measure for each $t \in [0,T]$. Then $\rho^{h(x,.)} e^{-\Psi} \mathrm{d}\theta$ is also a weak solution of* (I.1.8) *(in the sense of Definition I.1.5 (ii)).*

Proof. By partial integration, a classical solution solves the first line of (I.1.9). For the condition at the initial time of (I.1.9), note that for $f \in \mathsf{C}_c(\mathbb{R})$, there is a $\epsilon > 0$ such that

$$\begin{aligned}&\left|\int f(\theta)\,\rho_t^{h,x}(\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta - \int f(\theta)\,\rho^0(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\right| \\ &\leq |f|_\infty\,|\mathrm{supp}(f)| \sup_{\theta \in \mathrm{supp}(f)} \left|\rho^{h,x}(t,\theta)\,e^{-\Psi(\theta)} - \rho^0(x,\theta)\,e^{-\Psi(\theta)}\right| \leq |f|_\infty\,|\mathrm{supp}(f)|\,\epsilon,\end{aligned} \tag{I.1.44}$$

for t small enough, by the uniform continuity of $\rho^{h,x} e^{-\Psi}$ on $[0,T] \times [-R,R]$ for each $R > 0$. □

From this lemma we conclude that, for each $x \in \mathbb{T}^d$, the probability density $\rho^x e^{-\Psi}$ (constructed in Theorem I.1.20) is also a weak solution of (I.1.8) with initial distribution $\rho^0(x,\theta)\,e^{-\Psi}\mathrm{d}\theta$ and fixed effective field $h(.,x)$.

I.1.3.4 Properties of weak solutions of the Fokker Planck equation (I.1.8)

We infer in this section nice properties of weak solutions (in the sense of Definition I.1.5 (ii)) of the Fokker Planck equation (I.1.8) from [BDPRS07], [BDPR08], [BRS05], [BKR01] and [BDPR04]. These properties are listed in the following lemma.

Lemma I.1.23. *Fix a $h \in \mathsf{C}\left([0,T] \times \mathbb{T}^d\right)$. Let Ψ be a polynomial of even degree (first part of Assumption I.0.3). Moreover, let the Assumption I.0.4 a.) and Assumption I.0.6 hold. Then the following statements are true:*

(i) *For each $x \in \mathbb{T}^d$, a unique weak solution ν^x of* (I.1.8) *with $h(.,x)$ as effective field (in the sense of Definition I.1.5 (ii)) exists.*

   Moreover, there is a strictly positive function $\xi^x : (0,T) \times \mathbb{R} \to \mathbb{R}_+$, such that for each $t \in (0,T)$, $\nu_t^x(\mathrm{d}\theta) = \xi^x(t,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta$. The function ξ^x is locally Hölder continuous.

(ii) *There is a constant $C = C(|h|_\infty)$, such that for all $t \in [0,T)$, $x \in \mathbb{T}^d$ and $\theta \in \mathbb{R}$*

$$\xi^x(t,\theta)\,e^{-\Psi(\theta)} \leq C e^{-\frac{7}{8}\Psi(\theta)}. \tag{I.1.45}$$

(iii) $\xi^x e^{-\Psi} \in H^{(0,1),2}((0,T) \times \mathbb{R})$ *(the Sobolev space with generalised derivatives in L^2 in the $\mathbb{R}$ direction up to order* 1*).*

(iv) *For each $t_1 < t_2 \in (0,T)$, there is a constant $C_{t_1,t_2} = C(t_1,t_2,|h|_\infty,\Psi) \geq 0$, such that for all $(x,\theta) \in \mathbb{T}^d \times \mathbb{R}$ and $t \in [t_1,t_2]$*

$$e^{-C_{t_1,t_2}\left(1+\Psi(\theta)^2\right)} \leq \xi^x(t,\theta)\,e^{-\Psi(\theta)}. \tag{I.1.46}$$

Proof. Before we prove the statements of the lemma, we show that two conditions, that we need in the sequel, are satisfied.

(1.) For each bounded open set $B \subset \mathbb{R}$ the $\sup_{t\in(0,T)} \|-\Psi' + h(t,x)\|_{L^p(B)}$ is bounded, because h is bounded and $\Psi' \in L^\infty_{\mathrm{loc}}$ (Lemma I.1.8 (i)). The same is true for the derivatives w.r.t. $\theta \in \mathbb{R}$ of the drift coefficient. Moreover, the diffusion coefficient, we consider here, is constant.

(2.) By [BDPR08] Lemma 2.2 we get (due to Assumption I.0.6 and Lemma I.1.8 (vi)) for each weak solution ν^x of (I.1.8) that

$$\operatorname*{ess\,sup}_{t\in[0,T)} \int_{\mathbb{R}} e^{\frac{7}{8}\Psi(\theta)} \nu_t^x (\mathrm{d}\theta) \leq TC + \int_{\mathbb{R}} e^{\frac{7}{8}\Psi(\theta)} e^{-\Psi(\theta)} \rho^0 (x,\theta) \,\mathrm{d}\theta < \infty. \tag{I.1.47}$$

This implies that $|-\Psi' + h(.,x)|, \log\max(|\theta|, 1)$ are both in $L^2([0,T]\times\mathbb{R}, \nu_t^x(\mathrm{d}\theta)\otimes \mathrm{d}t)$ by Lemma I.1.8 (vii). By the same arguments as for (I.1.47), we get that

$$\operatorname*{ess\,sup}_{t\in[0,T)} \int_{\mathbb{R}} e^{\left(\frac{7}{8}+\epsilon\right)\Psi(\theta)} \nu_t^x (\mathrm{d}\theta) \leq TC + \int_{\mathbb{R}} e^{\left(\frac{7}{8}+\epsilon\right)\Psi(\theta)} e^{-\Psi(\theta)} \rho^0 (x,\theta) \,\mathrm{d}\theta < \infty. \tag{I.1.48}$$

We prove now the claimed statements of this lemma:

(i) By our assumptions we can apply [BDPRS07] Theorem 3.3 to get the uniqueness and the existence of a weak solution. The proof of this theorem uses [BDPR08] Theorem 3.1 for the *existence* and [BDPRS07] Proposition 3.1 for the *uniqueness* of a solution.

The Theorem 3.1 in [BDPRS07] is applicable by the condition (1.), by Lemma I.1.8 (vi). Hence, there exists a weak solution $\nu^x \in \mathsf{C}([0,T], \mathbb{M}_1(\mathbb{R}))$.

This solution has also a probability density. Indeed, define the measure $\overline{\nu}^x(\mathrm{d}t, \mathrm{d}\theta) := \nu_t^x(\mathrm{d}\theta)\otimes \mathrm{d}t \in \mathbb{M}_1([0,T]\times\mathbb{R})$. Then there is a strictly positive function $\xi^x : (0,T)\times\mathbb{R}\to\mathbb{R}$, such that $\overline{\nu}^x(\mathrm{d}t,\mathrm{d}\theta) = \xi^x(t,\theta)\, e^{-\Psi(\theta)}\mathrm{d}t\mathrm{d}\theta$, by [BDPRS07] Theorem 2.5 or [BKR01] Corollary 3.9 (which are applicable due to the condition (1.)). Hence, $\nu_t^x(\mathrm{d}\theta) = \xi^x(t,\theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta$.

To apply [BDPRS07] Proposition 3.1, we need [BDPRS07] $\frac{|\partial_\theta \xi^x e^{-\Psi}|}{\xi^x e^{-\Psi}} \in L^1\left((0,T)\times\mathbb{R}, \xi^x e^{-\Psi}\right)$. This follows from [BRS05] Theorem 2.1, which is applicable because $\rho^0(x,.)\,e^{-\Psi}$ has finite entropy (Lemma I.1.9 (iv)) and because of condition (2.).

Moreover, we get from [BDPRS07] Theorem 2.5 that ξ^x is locally Hölder continuous.

(ii) The upper bound on the solution is as consequence of [BRS05] Theorem 3.3. The conditions of this theorem on the initial distribution are satisfied by Lemma I.1.9 (i) and Lemma I.1.9 (ii). Moreover, the conditions of this theorem on the integrability of the upper bound w.r.t. $\nu_t^x\otimes \mathrm{d}t$, are satisfied, by Lemma I.1.8 (vi) and (I.1.48). Therefore, [BRS05] Theorem 3.3. implies the upper bound for almost all $t,\theta\in[0,T]\times\mathbb{R}$. By the local Hölder continuity shown in part (i) of this lemma it holds for all $t,\theta\in[0,T]\times\mathbb{R}$.

Moreover, the constant C is independent of x. Indeed, the constant depends only on an upper bound on $\operatorname{ess\,sup}_{t\in[0,T]} \|-\Psi'(.) + h(t,x)\|_{L^1(\xi^x(t,.)e^{-\Psi})}$ (see the definition of this constant in the proofs of [BRS05] Lemma 3.2 and of [BRS05] Theorem 3.2). This upper bound can be chosen independent of $x\in\mathbb{T}^d$. Indeed

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|-\Psi'(.) + h(t,x)\|_{L^1(\xi^x(t,.)e^{-\Psi})} \leq C + \operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}} e^{\frac{7}{8}\Psi(\theta)} \xi^x(t,\theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta < C, \tag{I.1.49}$$

where we use Lemma I.1.8 (vii), that h is bounded and finally the condition (2.).

(iii) By [BDPR08] Corollary 3.5 and [BDPR08] Lemma 2.6

$$\begin{aligned} &\left\|\xi^x(t,.)\,e^{-\Psi}\right\|_{L^2(\mathbb{R})} + \left\|\partial_\theta\left(\xi^x(.,.)\,e^{-\Psi}\right)\right\|_{L^2([0,T]\times\mathbb{R})} \\ &\leq \left\|\rho^0(x,.)\,e^{-\Psi}\right\|_{L^2(\mathbb{R})} + \|-\Psi'(.) + h(.,x)\|_{L^2([0,T]\times\mathbb{R},\xi^x(.,.)e^{-\Psi})}. \end{aligned} \tag{I.1.50}$$

The right hand side is finite by Lemma I.1.9 (iii) and by condition (2.).

(iv) We want to apply Corollary 3.3 of [BRS08]. This implies the desired lower bounds on ξ^x. The necessary conditions of this corollary are satisfied by [BDPRS07] Theorem 2.5. Hence, we know by Corollary 3.3 of [BRS08] that there is a constant $K(t_1, t_2) \geq 0$ and a continuous increasing function $V : \mathbb{R} \to \mathbb{R}$ with $V(0) > 0$, such that for each $(x, \theta) \in \mathbb{T}^d \times \mathbb{R}$ and $t \in \{t_1, t_2\}$

$$e^{-K(t_1,t_2)\left(1+V(|\theta|)^2+|\theta|^2\right)} \leq \xi^x(t,\theta)\, e^{-\Psi(\theta)} \tag{I.1.51}$$

and $\sup_{t\in[0,T]} |-\Psi'(\theta) + h(t,x)| \leq V(|\theta|)$. This corollary is applicable due to Ψ being a polynomial (Assumption I.0.3). Moreover, the function V can be chosen such that for all $\theta \in \mathbb{R}$, $(V(|\theta|))^2 + |\theta|^2 \leq C\left(\Psi(\theta)^2 + 1\right)$, with a constant $C = C(|h|_\infty, \Psi) \in \mathbb{R}$. □

Remark I.1.24. • *By the uniqueness of weak solutions of the Fokker-Planck equation* (I.1.8) *(shown in Lemma I.1.23 (i)) we get that $\rho^x e^{-\Psi}$ constructed in Theorem I.1.20 is the unique weak solution. Consequently this solution has all the properties stated in Lemma I.1.23.*

- *Some of the statements of Lemma I.1.23 can be proven easier, if we used the explicit formula* (I.1.43) *of the classical solution. For example, we show in the proof of the lower bound on ρ (Lemma I.1.23 (iv)), that Corollary 3.3 of [BRS08] is applicable. These conditions follow directly from the explicit formula. Indeed, by the construction of the solution, we know that it is strictly positive. Moreover, it belongs to the necessary Sobolev spaces because it is continuous as well as continuously differentiable and bounded. Therefore, we would not need the detour via [BDPRS07] Theorem 2.5. However, we want that the results of Lemma I.1.23 also hold for more general settings, without the previous knowledge of the explicit form of the classical solution.*

I.1.4 Proof of Theorem I.1.2: A fixed point argument

In this section we prove Theorem I.1.2, hence the existence of a classical solution (in the sense of Definition I.1.1) of the nonlinear equation (I.0.2) that is at the same time a probability density of the hydrodynamic SDE (I.0.3).

For the proof we use the method of induction over the length of the time interval, i.e. we show for a small time interval the existence of a classical solution (base case) and then we show that we can extend this interval (induction step). These results are stated in the next theorem.

Theorem I.1.25. *There is a $0 < T_1 = T_1(\rho^0) \leq T$ and a $C_{bd} > 0$ such that the following two statements hold:*

(i) *There is a classical solution ρ^* of* (I.0.2) *(in the sense of Definition I.1.1) on the time interval $[0, T_1]$. Moreover, the corresponding effective field $h^* \in \mathsf{C}^{2,0}\left([0,T_1] \times \mathbb{T}^d\right)$ (i.e. $h^* = h^{\rho^*}$ defined by ρ^* via the second line of* (I.0.2)*) satisfies*

$$\|h^*\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \leq C_{bd}. \tag{I.1.52}$$

(ii) *Fix an arbitrary ϵ with $T_1 > \epsilon > 0$. Let ρ^* be a classical solution of* (I.0.2) *on the time interval $\left[0, \overline{T}\right]$, with an arbitrary $\overline{T} > \epsilon$, such that the corresponding effective field h^* satisfies* (I.1.52) *with T_1 replaced by $\overline{T}$.*
Then there is also a classical solution $\rho^{*,2}$ of (I.0.2) *on the time interval $\left[0, \overline{T} + T_1 - \epsilon\right]$. Moreover, the corresponding effective field $h^{*,2} \in \mathsf{C}^{2,0}\left([0, \overline{T} + T_1 - \epsilon] \times \mathbb{T}^d\right)$ satisfies* (I.1.52) *on the time interval $[0, \overline{T} + T_1 - \epsilon]$.*

Repeating the induction step (ii), implies the existence of a classical solution of (I.0.2) on the time interval $[0, T]$. As explained in Step 3.) in Section I.1.1, this solution inherits all the properties stated in Theorem I.1.2 from the same properties of the classical solution of the linear equation

(I.1.8) derived in Theorem I.1.6. In particular the classical solution ρ^* is also the probability density of the hydrodynamic SDE (I.0.3).

We state the proofs and the precise ideas of the proofs of (i) and (ii) of Theorem I.1.25 in Section I.1.4.1 and I.1.4.2 respectively. To end this section, let us sketch the main ideas of both proofs.

The main step in the proof of (i) is to define a suitable operator that maps an effective field to another effective field, such that a fixed point of this operator corresponds to a classical solution of (I.0.2). Then we define a sequence, by applying iteratively this operator. Finally, we show that this sequence converges to such a fixed point on a small time interval.

For the proof of the induction step (ii) of Theorem I.1.25, we show the existence of a classical solution of (I.0.2) on the time interval $\left[\overline{T}-\epsilon, \overline{T}+T_1-\epsilon\right]$ and glue this solution to the classical solution up to time $\overline{T}$. For the existence on $\left[\overline{T}-\epsilon, \overline{T}+T_1-\epsilon\right]$ we can reuse parts of the proof that leads to (i). Relying on the overlap of length ϵ of the two time intervals, we show the claimed regularity of the solution that is glued together.

I.1.4.1 Existence of a classical solution of (I.0.2) on a small time interval (proof of Theorem I.1.25 (i))

In this section we prove Theorem I.1.25 (i), i.e. the existence of a classical solution of (I.0.2) on the small time interval $[0, T_1]$. Define the following operator

$$\begin{aligned} G = G_{[0,T_1],\rho^0} &: \mathsf{C}^{2,0}\big([0,T_1]\times\mathbb{T}^d\big) \to \mathsf{C}^{2,0}\big([0,T_1]\times\mathbb{T}^d\big) \\ G\,(h)\,(t,x) &:= \int_{\mathbb{R}}\int_{\mathbb{T}^d} J\,(y-x)\,\rho_t^h\,(y,\eta)\,\eta e^{-\Psi(\eta)}\mathrm{d}y\mathrm{d}\eta, \end{aligned} \tag{I.1.53}$$

where $\rho_\cdot^h\,(y,.) \in \mathsf{C}^{1,2}([0,T]\times\mathbb{R})$ is for each $y \in \mathbb{T}^d$ the classical solution of the linear PDE (I.1.8) with fixed effective field $h\,(.,y)$ and with initial condition $\rho^0\,(y,.)$. We know that such a ρ^h exists, by Theorem I.1.6. This operator maps thus an effective field h to a new effective field $G\,(h)$.

If h^* is a fixed point of G, then the linear PDE (I.1.8) with effective field h^* has a classical solution that defines (via the second line of (I.0.2)) again the effective field h^*. But this is exactly the Definition I.1.1 of a classical solution of (I.0.2) on the time interval $[0, T_1]$. Hence, proving that G has a fixed point, implies the existence of a classical solution of (I.0.2) as claimed in Theorem I.1.25 (i).

To show the existence of a fixed point of G, we define a sequence of effective fields through applying iteratively this operator. Fix an arbitrary effective field $h \in \mathsf{C}^{2,0}\big([0,T_1]\times\mathbb{T}^d\big)$ and define the sequence

$$h^{(n)} := G^n\,(h) = G\left(G^{n-1}\,(h)\right) = G\left(h^{(n-1)}\right). \tag{I.1.54}$$

We claim the convergence of $h^{(n)}$ to a fixed point of G:

Theorem I.1.26. *Let the assumptions of Theorem I.1.2, in particular the Assumption I.0.6 on ρ^0, be satisfied. Let $C_T > 0$ be a constant, such that*

$$\sup_{x\in\mathbb{T}^d}\int_{\mathbb{R}} \theta^2\rho^0\,(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta \le C_T. \tag{I.1.55}$$

For each $h \in \mathsf{C}^{2,0}\big([0,T]\times\mathbb{T}^d\big)$, there is a constant $T_1 = T_1\,(C_T, |h|_\infty) > 0$ (decreasing in C_T and $|h|_\infty$), such that the sequence $h^{(n)}$, defined by (I.1.54)*, converges in $\mathsf{C}^{2,0}\big([0,T_1]\times\mathbb{T}^d\big)$ to a h^*, with $G_{[0,T_1]}\,(h^*) = h^*$. Moreover, h^* satisfies*

$$\|h^*\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \le \|J\|_{L^1} + \frac{1}{2}\,\|J\|_{L^1}\,C_T := C_{bd} \in \mathbb{R}. \tag{I.1.56}$$

Note that this theorem implies the existence of a classical solution only on the time interval $[0, T_1]$, that is in general shorter than the time interval $[0, T]$. In the proof of this theorem, we need an uniform upper bound on the sup norm of the effective fields $h^{(n)}$. But we are only able to show this on the shorter time interval.

The rest of this section is organised as follows. We show at first an energy estimate for classical solutions of the linear Fokker-Planck equation (I.1.8) (in Section I.1.4.1.1). Then we use this result to show that the operator G is well defined and continuous (Section I.1.4.1.2). Finally, we apply the results of the previous two sections to prove Theorem I.1.26 in Section I.1.4.1.3 by showing that $h^{(n)}$ is a Cauchy sequence.

I.1.4.1.1 Energy estimate for the difference of two classical solutions

Let $\rho^{(1)}$ and $\rho^{(2)}$ be two probability density-valued, classical solutions derived in Theorem I.1.6 of the linear Fokker-Planck equation (I.1.8) corresponding respectively to the effective fields $h^{(1)}$ and $h^{(2)}$, both in $\mathsf{C}^{2,0}\left([0,T] \times \mathbb{T}^d\right)$. We denote their difference at time $t \in [0, T]$ by

$$D_t\left(x, \theta\right) := \left(\rho_t^{(1)} - \rho_t^{(2)}\right)\left(x, \theta\right), \tag{I.1.57}$$

and the difference of the corresponding effective field by

$$\Delta_t\left(x\right) := h_t^{(1)}\left(x\right) - h_t^{(2)}\left(x\right). \tag{I.1.58}$$

In the following energy estimate, we use the following norm for suitable functions $g : \mathbb{R} \to \mathbb{R}$

$$\|g\|_{L^2(e^{-\Psi})} := \left\|g e^{-\frac{1}{2}\Psi}\right\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(g\left(\theta\right)\right)^2 e^{-\Psi(\theta)} \mathrm{d}\theta\right)^{\frac{1}{2}}. \tag{I.1.59}$$

Lemma I.1.27. *There is a constant* $C = C\left(\|h^{(1)}\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)}, \|h^{(2)}\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)}\right) < \infty$*, such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{T}^d} \|D_t\left(x,.\right)\|_{L^2(e^{-\Psi})} < \infty, \tag{I.1.60}$$

and for each $(t, x) \in [0, T] \times \mathbb{T}^d$*,*

$$\|D_t\left(x,.\right)\|_{L^2(e^{-\Psi})}^2 \leq \|D_0\left(x,.\right)\|_{L^2(e^{-\Psi})}^2 + C\int_0^t \|D_s\left(x,.\right)\|_{L^2(e^{-\Psi})}^2 + \|\Delta_s\left(x,.\right)\|_{L^2(\mathbb{T}^d)}^2 \mathrm{d}s. \tag{I.1.61}$$

Proof. We start with a short proof, that the $\|D_t\left(x,.\right)\|_{L^2(e^{-\Psi})}^2$ are uniformly (in t and x) bounded. Note that

$$\|D_t\left(x,.\right)\|_{L^2(e^{-\Psi})}^2 \leq 2\left\|\rho_t^{(1)}\left(x,.\right)\right\|_{L^2(e^{-\Psi})}^2 + 2\left\|\rho_t^{(2)}\left(x,.\right)\right\|_{L^2(e^{-\Psi})}^2, \tag{I.1.62}$$

and by the upper bound on ρ in Theorem I.1.6 and Lemma I.1.8 (ii) we have

$$\left\|\rho_t^{(1)}\left(x,.\right)\right\|_{L^2(e^{-\Psi})}^2 \leq C\left(\left\|h^{(1)}\right\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)}\right)\int_{\mathbb{R}} e^{-\Psi} e^{\frac{1}{4}\Psi} \mathrm{d}\theta \mathrm{d}x < \infty. \tag{I.1.63}$$

This implies the uniform upper bound on $\|D_t\left(x,.\right)\|_{L^2(e^{-\Psi})}$.

Next we prove the inequality (I.1.61). Fix $(t, x) \in [0, T] \times \mathbb{T}^d$. Using that $\rho^{(i)}$ is a classical solution of (I.1.8), we see that

$$\begin{aligned}\|D_t(x,.)\|^2_{L^2(e^{-\Psi})} - \|D_0(x,.)\|^2_{L^2(e^{-\Psi})} &= 2\int_0^t \int_{\mathbb{R}} D_s(x,\theta)\, e^{-\Psi(\theta)} \partial_t D_s(x,\theta)\,\mathrm{d}\theta\mathrm{d}s \\ = 2\int_0^t \int_{\mathbb{R}} D_s(x,\theta)\, e^{-\Psi(\theta)} \mathbb{L}_0 D_s(x,\theta)\,\mathrm{d}\theta \\ &\quad + \int_{\mathbb{R}} \partial_\theta D_s(x,\theta)\, e^{-\Psi(\theta)} \left(h_s^{(1)}\rho_s^{(1)} - h_s^{(2)}\rho_s^{(2)}\right)(x,\theta)\,\mathrm{d}\theta\mathrm{d}s \\ = 2\int_0^t - \|\partial_\theta D_s(x,.)\|^2_{L^2(e^{-\Psi})} + \int_{\mathbb{R}} D_s(x,\theta)\, e^{-\Psi(\theta)} h_s^{(1)}(x,\theta)\,\partial_\theta D_s(x,\theta)\,\mathrm{d}\theta \\ &\quad + \int_{\mathbb{R}} \rho_s^{(2)}(x,\theta)\, e^{-\Psi(\theta)} \Delta_s(x,\theta)\,\partial_\theta D_s(x,\theta)\,\mathrm{d}\theta\mathrm{d}s.\end{aligned} \tag{I.1.64}$$

Now we derive suitable bounds for the two scalar products on the right hand side of (I.1.64). First of all we have

$$\begin{aligned}&\int_{\mathbb{R}} D_s(x,\theta)\, e^{-\Psi(\theta)} h_s^{(1)}(x,\theta)\,\partial_\theta D_s(x,\theta)\,\mathrm{d}\theta \\ &\le \left\|h^{(1)}\right\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)} \|D_s(x,.)\|_{L^2(e^{-\Psi})} \|\partial_\theta(D_s)(x,.)\|_{L^2(e^{-\Psi})}\,\mathrm{d}s \\ &\le \left\|h^{(1)}\right\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)} \frac{1}{2}\left(\frac{1}{\delta^2}\|D_s(x,.)\|^2_{L^2(e^{-\Psi})} + \delta^2\|\partial_\theta D_s(x,.)\|^2_{L^2(e^{-\Psi})}\right),\end{aligned} \tag{I.1.65}$$

by using $\delta a \frac{1}{\delta} b \le \frac{1}{2}\left(\delta^2 a^2 + \frac{1}{\delta^2} b^2\right)$ for $\delta > 0$. Moreover

$$\begin{aligned}&\int_{\mathbb{R}} \rho_s^{(2)}(x,\theta)\, e^{-\Psi(\theta)} \Delta_s(x,\theta)\,\partial_\theta D_s(x,\theta)\,\mathrm{d}\theta \\ &\le \left\|\rho^{(2)} e^{-\frac{1}{4}\Psi}\right\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)} \left\|e^{-\frac{1}{2}\Psi}\right\|^{\frac{1}{2}}_{L^1(\mathbb{R})} \|\Delta_s(x,.)\|_{L^2(\mathbb{T}^d)} \|\partial_\theta D_s(x,.)\|_{L^2(e^{-\Psi})} \\ &\le \left\|\rho^{(2)} e^{-\frac{1}{4}\Psi}\right\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)} \left\|e^{-\frac{1}{2}\Psi}\right\|^{\frac{1}{2}}_{L^1(\mathbb{R})} \frac{1}{2}\left(\frac{1}{\delta^2}\|\Delta_s(x,.)\|^2_{L^2(\mathbb{T}^d)} + \delta^2\|\partial_\theta D_s(x,.)\|^2_{L^2(e^{-\Psi})}\right).\end{aligned} \tag{I.1.66}$$

Finally, we choose a $0 < \delta$ such that

$$\delta \le \sqrt{2}\left(\left\|h^{(1)}\right\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)} + \left\|\rho^{(2)} e^{-\frac{1}{4}\Psi}\right\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)} \left\|e^{-\frac{1}{2}\Psi}\right\|^{\frac{1}{2}}_{L^1(\mathbb{R})}\right)^{-\frac{1}{2}}. \tag{I.1.67}$$

Finding such a δ is possible, because there is a constant $C\left(\|h^{(1)}\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)}, \|h^{(2)}\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)}\right) < \infty$ that bounds each of the norms on the right hand side of (I.1.67), by the boundedness of the effective fields, upper bound on ρ in Theorem I.1.6, Lemma I.1.8 (iii) and Lemma I.1.8 (ii).

Now we insert (I.1.65) and (I.1.66) with this δ into (I.1.64) and we get the claimed energy estimate (I.1.61). □

By applying the Gronwall inequality in (I.1.61) we get a further bound on D_t that depends not on $\{D_s\}_{s\in[0,t]}$ but still on $\{\Delta_s\}_{s\in[0,t]}$. Note that we derive, in the uniqueness proof in Section I.3, another bound that is independent of Δ_s.

Corollary I.1.28. *With the same constant $C = C\left(\|h^{(1)}\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)}, \|h^{(2)}\|_{\mathsf{C}([0,T]\times\mathbb{T}^d)}\right) < \infty$ as in Lemma I.1.27, the following inequalities hold for all $(t,x) \in [0,T]\times\mathbb{T}^d$*

$$\|D_t(x,.)\|^2_{L^2(e^{-\Psi})} \le \left(\|D_0(x,.)\|^2_{L^2(e^{-\Psi})} + C\int_0^t \|\Delta_s(x,.)\|^2_{L^2(\mathbb{T}^d)}\,\mathrm{d}s\right) e^{CT}. \tag{I.1.68}$$

Proof. The map $t \to \|D_t(x,.)\|^2_{L^2(e^{-\Psi})}$ is continuous, by the continuity of $\rho^{(i)}$ and by the uniform upper bound on ρ in Theorem I.1.6. Therefore, the Gronwall inequality can be applied to (I.1.61). □

I.1.4.1.2 Properties of G

In this section we prove that the map G is well defined (Lemma I.1.29) and that G is continuous (Lemma I.1.30).

Lemma I.1.29. *For all* $0 < T_1 \leq T$, $G = G_{[0,T_1]}$ *defined in* (I.1.53) *is well defined.*

Proof. Fix an arbitrary $h \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$. We need to show that $G(h) \in \mathsf{C}^{2,0}\big([0,T_1] \times \mathbb{T}^d\big)$. Let ρ^h be the function that we derive in Theorem I.1.6, which is in particular for each $x \in \mathbb{T}^d$ a classical solution of the PDE (I.1.8) with fixed effective field h.
Step 1: $G(h)$ **is continuous:** Define the function on $[0,T] \times \mathbb{T}^d$

$$H(t,y) := \int_{\mathbb{R}} \theta \rho^h_t(y,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta, \tag{I.1.69}$$

which is continuous and bounded by the upper bound on ρ in Theorem I.1.6 and Lemma I.1.8 (ii). With this function we can write $G(h)(t,x) = \int_{\mathbb{T}^d} J(y)\, H(t, x-y)\, \mathrm{d}y$. Then we conclude the continuity of $G(h)$ by Assumption I.0.1 and [Bau01] Lemma 16.1.

Step 2: $G(h)$ **is continuously differentiable:** The $\rho^{h(.,y)}$ is a classical solution of (I.1.8) for each $y \in \mathbb{T}^d$, hence

$$\begin{aligned}
\partial_t G(h)(t,x) &= \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(x-y)\, \theta \partial_t \rho^h_t(y,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \mathrm{d}y \\
&= \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(x-y)\, \theta \mathbb{L}^*_{h(t,y)} \rho^h_t(y,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \mathrm{d}y \\
&= -\int_{\mathbb{T}^d} \int_{\mathbb{R}} J(x-y)\, \Psi'(\theta)\, \rho^h_t(y,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \mathrm{d}y + \int_{\mathbb{T}^d} J(x-y)\, h(t,y)\, \mathrm{d}y,
\end{aligned} \tag{I.1.70}$$

which is a continuous function on $(0,T) \times \mathbb{T}^d$ (by the same arguments as in Step 1). Note that the derivative and the integrals can be interchanged because for each $a \in \mathbb{R}$ small enough, $t > 0$ and $y \in \mathbb{T}^d$

$$\int_{\mathbb{R}} \theta \left[\frac{\rho^h_{t+a}(y,\theta) - \rho^h_t(y,\theta)}{a} - \partial_t \rho^h_t(y,\theta) \right] \mathrm{d}\theta = \frac{1}{a} \int_0^a \int_{\mathbb{R}} \theta \left[\partial_t \rho^h_{t+u}(y,\theta) - \partial_t \rho^h_t(y,\theta) \right] \mathrm{d}\theta \mathrm{d}u, \tag{I.1.71}$$

continuity of $\partial_t \rho^h_t$. Relying on ρ^h being a classical solution of (I.1.8), the right hand side of (I.1.71) vanishes when $a \to 0$.

By the same argument, we get the following expression for the second derivative

$$\begin{aligned}
&\partial^2_{tt} G(h)(t,x) \\
&= -\int_{\mathbb{T}^d} \int_{\mathbb{R}} J(x-y)\, \mathbb{L}_{h(t,y)} (\Psi'(\theta))\, \rho^h_t(y,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \mathrm{d}y + \int_{\mathbb{T}^d} J(x-y)\, \partial_t h(t,y)\, \mathrm{d}y,
\end{aligned} \tag{I.1.72}$$

and that $\partial^2_{tt} G(h)(t,x)$ is a continuous function on $[0,T] \times \mathbb{T}^d$ (this follows for example by [Bau01] Lemma 16.2). □

Lemma I.1.30. *For all* $0 < T_1 \leq T$, $G = G_{[0,T_1]}$ *is a continuous map. Moreover, for all* $g, f \in \mathsf{C}^{1,0}\big([0,T_1] \times \mathbb{T}^d\big)$ *with* $g(0,.) = f(0,.)$ *there is a constant* $0 < C(|g|_\infty, |f|_\infty) < \infty$ *such that*

$$\|G(g) - G(f)\|_{\mathsf{C}^{2,0}([0,T_1]\times\mathbb{T}^d)} \leq C(|g|_\infty, |f|_\infty)\, \|g - f\|_{\mathsf{C}^{1,0}([0,T_1]\times\mathbb{T}^d)}\,. \tag{I.1.73}$$

Proof. We fix arbitrary $g, f \in \mathsf{C}^{2,0}\big([0,T_1] \times \mathbb{T}^d\big)$. In the following three steps we show the inequality (I.1.73) separately for the first derivative w.r.t. time, no derivative and the second derivative of $G(g) - G(f)$. We use thereby the energy estimate derived in Corollary I.1.28 and the formulas (I.1.70) and (I.1.72) of the derivatives of $G(g)$.

Step 1: First derivative:

We show in this step that there is a constant $C_{(\partial_t)}\left(|g|_\infty, |f|_\infty\right) > 0$ that depends on $|g|_\infty$ and $|f|_\infty$ such that

$$\left\|\partial_t \left(G(g) - G(f)\right)\right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \leq C_{(\partial_t)}\left(|g|_\infty, |f|_\infty\right) \|g - f\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}. \tag{I.1.74}$$

By (I.1.70) we know that

$$\begin{aligned} \partial_t \left(G(g) - G(f)\right)(t,x) = &\int_{\mathbb{T}^d}\int_{\mathbb{R}} J(x-y)\,\Psi'(\theta)\left(\rho_t^g - \rho_t^f\right)(y,\theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}y \\ &+ \int_{\mathbb{T}^d} J(x-y)\left(g(t,y) - f(t,y)\right)\mathrm{d}y. \end{aligned} \tag{I.1.75}$$

Now we apply the Cauchy-Schwartz inequality

$$\begin{aligned} &\left\|\partial_t \left(G(g)(t,.) - G(f)(t,.)\right)\right\|_{\mathsf{C}(\mathbb{T}^d)} \\ &\leq \|J\|_{L^1}\left(\|\Psi'(\theta)\|_{L^2(e^{-\Psi})}\sup_{x\in\mathbb{T}^d}\left(\int_{\mathbb{R}}\left(\rho_t^g - \rho_t^f\right)^2(x,\theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta\right)^{\frac{1}{2}} + \|g - f\|_{\mathsf{C}([0,t]\times\mathbb{T}^d)}\right) \\ &\leq \|J\|_{L^1}\left(C\left(|g|_\infty, |f|_\infty\right)\left(\int_0^t \|g(s,.) - f(s,.)\|^2_{\mathsf{C}(\mathbb{T}^d)}\,\mathrm{d}s\right)^{\frac{1}{2}} + \|g - f\|_{\mathsf{C}([0,t]\times\mathbb{T}^d)}\right) \\ &\leq C_{(\partial_t)}\left(|g|_\infty, |f|_\infty\right)\|g - f\|_{\mathsf{C}([0,t]\times\mathbb{T}^d)}, \end{aligned} \tag{I.1.76}$$

where we use Corollary I.1.28 and Lemma I.1.8 (iv). This proves (I.1.74).

Step 2: No derivative:

With

$$G(g)(t,x) = G(g)(0,x) + \int_0^t \partial_t G(g)(s,x)\,\mathrm{d}s, \tag{I.1.77}$$

we get by (I.1.76)

$$\begin{aligned} \left\|G(g)(t,.) - G(f)(t,.)\right\|_{\mathsf{C}(\mathbb{T}^d)} &\leq \int_0^t \left\|\partial_t \left(G(g)(s,.) - G(f)(s,.)\right)\right\|_{\mathsf{C}(\mathbb{T}^d)}\mathrm{d}s \\ &\leq C_{(\partial_t)}\left(|g|_\infty, |f|_\infty\right)\int_0^t \|g - f\|_{\mathsf{C}([0,s]\times\mathbb{T}^d)}\,\mathrm{d}s. \end{aligned} \tag{I.1.78}$$

Now we take the supremum over $t \in [0,T_1]$ and conclude

$$\left\|G(g) - G(f)\right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \leq T C_{(\partial_t)}\left(|g|_\infty, |f|_\infty\right)\|g - f\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}. \tag{I.1.79}$$

Step 3: Second derivative:

The difference of the second time derivatives of G can be written (by (I.1.72)) as

$$\begin{aligned} &\partial^2_{tt}\left(G(g) - G(f)\right)(t,x) \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} J(x-y)\left(\Psi'(\theta)\,\Psi''(\theta) - \Psi'''(\theta)\right)\left(\rho_t^g(y,\theta) - \rho_t^f(y,\theta)\right) e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}y \\ &- \int_{\mathbb{T}^d}\int_{\mathbb{R}} J(x-y)\,\Psi''(\theta)\, e^{-\Psi(\theta)}\left(g(t,y)\,\rho_t^g(y,\theta) - f(t,y)\,\rho_t^f(y,\theta)\right)\mathrm{d}\theta\mathrm{d}y \\ &+ \int_{\mathbb{T}^d} J(x-y)\,\partial_t\left(g(t,y) - f(t,y)\right)\mathrm{d}y. \end{aligned} \tag{I.1.80}$$

On the right hand side, there are the products $g\rho^g$ and $f\rho^f$. We rewrite this term as

$$g(t,y)\,\rho_t^g(y,\theta) - f(t,y)\,\rho_t^f(y,\theta) = \Big(g(t,y) - f(t,y)\Big)\rho_t^g(y,\theta) + f(t,y)\left(\rho_t^g - \rho_t^f\right)(y,\theta). \quad \text{(I.1.81)}$$

We get the following bound on the supremum over $(t,x) \in [0,T] \times \mathbb{T}^d$ in (I.1.80)

$$\begin{aligned}
&\left\|\partial_{tt}^2\left(G(g) - G(f)\right)\right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \le \|J\|_{L^1}\,\|\partial_t(g-f)\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}\\
&\quad + \|J\|_{L^1}\left(C_{\Psi,3} + \|\Psi''\|_{L^2(e^{-\Psi})}\,|f|_\infty\right)\sup_{x\in\mathbb{T}^d}\left|\int_{\mathbb{R}}\left(\rho_t^g - \rho_t^f\right)^2(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\right|^{\frac{1}{2}}\\
&\quad + \|J\|_{L^1}\,\|\Psi''\|_{L^2(e^{-\Psi})}\,\|g-f\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}\sup_{x\in\mathbb{T}^d}\|\rho_t^g(x,.)\|_{L^2(e^{-\Psi})},
\end{aligned} \quad \text{(I.1.82)}$$

where we use that $\|\Psi'\Psi'' - \Psi'''\|_{L^2(e^{-\Psi})} \le C_{\Psi,3} < \infty$. With $\|\Psi''\|_{L^2(e^{-\Psi})} < \infty$ (Lemma I.1.8 (iv)) and $\sup_{x\in\mathbb{T}^d}\|\rho_t^g(x,.)\|_{L^2(e^{-\Psi})} \le C(|g|_\infty)$ (by (I.1.63)) and (I.1.68) we conclude

$$\begin{aligned}
&\left\|\partial_{tt}^2\left(G(g) - G(f)\right)\right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}\\
&\le C(|g|_\infty, |f|_\infty)\left(\|g-f\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} + \|\partial_t(g-f)\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}\right).
\end{aligned} \quad \text{(I.1.83)}$$

Hence, we have shown the continuity of G on $\mathsf{C}^{2,0}\big([0,T_1]\times\mathbb{T}^d\big)$. □

Remark I.1.31. *In the expression* (I.1.72) *of the second derivative of $G(h)$ and in the proof of the continuity of this second derivative in Lemma I.1.30, only the first derivative of h appears. Analogue we would need for continuity of the third derivative of $G(h)$ only the second derivative of h. Repeating this approach, one sees that actually G maps $\mathsf{C}^{k,0}\big([0,T_1]\times\mathbb{T}^d\big)$ to $\mathsf{C}^{k+1,0}\big([0,T_1]\times\mathbb{T}^d\big)$ as long as the arising sums of derivatives of Ψ and g are suitably bounded. This is the case due to Assumption I.0.3 such that for each $n\in\mathbb{N}$, $G^n(h)\in\mathsf{C}^{n,0}\big([0,T_1]\times\mathbb{T}^d\big)$. In particular if h^* is a fixed point of G then $h^*\in\mathsf{C}^{\infty,0}\big([0,T_1]\times\mathbb{T}^d\big)$.*

I.1.4.1.3 Proof of Theorem I.1.26

Fix an arbitrary $h \in \mathsf{C}^{2,0}\big([0,T]\times\mathbb{T}^d\big)$ for the rest of this section. We prove now Theorem I.1.26 by showing that the sequence $h^{(n)}$, defined by (I.1.54) for the fixed h, is a Cauchy sequence. The convergence follows by apply repeatedly the inequalities (I.1.74), (I.1.78) and (I.1.83) (that we derived in the proof of Lemma I.1.30). But this is only possible, if the constants in these inequalities stay the same for all $h^{(n)}$ (or are at least suitably bounded). However, these constants depend on the sup norms of the effective fields $h^{(n)}$. Therefore, we show at first that the sup norms of the $h^{(n)}$ are uniformly bounded (Lemma I.1.33) and then the Cauchy property of $h^{(n)}$ (Lemma I.1.34).

Both lemmas are in general only valid for time intervals shorter than $[0,T]$. Let us fix now a length T_1 for which both lemmas are valid. To do this and to simplify the notation in the following, we define first some constants.

The first constant is the constant $C_{max} > 0$, that is defined by

$$C_{max} = C_{max}(|h|_\infty, C_T) := \max\{C_{bd}+1, |h|_\infty\}, \quad \text{(I.1.84)}$$

with C_{bd} is defined in (I.1.56).

Remark I.1.32. *That we added a "1" to C_{bd} in the definition of the constant C_{max} is not necessary in this section. However, it is helpful to simplify our notation in Section I.1.4.2, in particular in* (Step 2) *in the proof of Theorem I.1.25 (ii).*

Moreover, we define the constant $C_{\mathbb{L}}(C_{max})$ that depends on C_{max}, such that for all $(t,x) \in [0,T] \times \mathbb{T}^d$ and all $g \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$ with $|g|_\infty \leq C_{max}$, the generator $\mathbb{L}_{h(t,.)}$ applied to θ^2 satisfies

$$\mathbb{L}_{g(t,x)}\theta^2 = 2\left(-\Psi'(\theta)\,\theta + g(t,x)\,\theta + 1\right) \leq C_{\mathbb{L}}(C_{max})\,. \tag{I.1.85}$$

It is possible to find such a constant due to Lemma I.1.8 (vi).

Finally, we define the length of the time interval, for which we show the Cauchy property of $h^{(n)}$, by

$$T_1 := T_1(C_{max}) := \min\left\{\frac{1}{C_{\mathbb{L}}(C_{max})}, \frac{1}{2C_{(\partial_t)}(C_{max})}, T\right\}, \tag{I.1.86}$$

with $C_{(\partial_t)}(C_{max})$ defined in (I.1.74). We need the first element of this minimum for Lemma I.1.33 and the second element for Lemma I.1.34.

Now we are prepared to state and prove the uniform boundedness of the sequence $h^{(n)}$ on the time interval $[0,T_1]$. The main step of the proof is to use a uniform upper bound on the second moments for classical solutions of the linear equation (I.1.8) (see (I.1.88)).

Lemma I.1.33. *If* (I.1.55) *holds, then the* $h^{(n)}$*, defined by* (I.1.54) *for the fixed* h*, are uniformly bounded on* $[0,T_1]$ *by*

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T_1]}\left|h^{(n)}(t,.)\right|_\infty \leq C_{bd}, \tag{I.1.87}$$

with C_{bd} *defined in* (I.1.56).

Proof. Let ρ^h be the classical solution, derived in Theorem I.1.6, of the linear equation (I.1.8) with effective field h. Then by Lemma 2.2 of [BDPR08] and (I.1.85) (as in the proof of Lemma I.1.23 in item (2.)), for all $y \in \mathbb{T}^d$,

$$\sup_{t\in[0,T_1]}\int_{\mathbb{R}}\theta^2\rho^h(t,y,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta \leq C_{\mathbb{L}}(C_{max})\,T_1 + \int_{\mathbb{R}}\theta^2\rho^0(y,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta. \tag{I.1.88}$$

Using this inequality we get for each $t \in [0,T_1]$

$$\begin{aligned}\left|h^{(1)}(t,x)\right| &= |G(h)(t,x)| = \left|\int_{\mathbb{R}} J(x-y)\int_{\mathbb{T}^d}\theta\rho^h(t,y,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}y\right| \\ &\leq \frac{1}{2}\int_{\mathbb{T}^d}|J(x-y)|\int_{\mathbb{R}}\left(1+\theta^2\right)\rho^h(t,y,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}y \\ &\leq \frac{1}{2}\|J\|_{L^1}\left(1 + C_{\mathbb{L}}(C_{max})\,T_1 + \sup_{y\in\mathbb{T}^d}\int_{\mathbb{R}}\theta^2\rho^0(y,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\right) \leq C_{bd},\end{aligned} \tag{I.1.89}$$

by the definition of C_{bd} and T_1. This implies the claimed bound on $h^{(1)}$.

By the same arguments we infer from the bound C_{bd} of $\left|h^{(n)}\right|_\infty$ the same bound for $\left|h^{(n+1)}\right|$. This is true because the constants in the inequalities (I.1.88) and (I.1.85) depend only on C_{max} which bounds $\left|h^{(n)}\right|_\infty$. Moreover, for each $n \in \mathbb{N}$, the classical solution of (I.1.8) with effective field $h^{(n)}$ has the same initial distribution ρ^0. □

Knowing this uniform bound on the sup norm of $h^{(n)}$, we are able to show that $h^{(n)}$ is a Cauchy sequence.

Lemma I.1.34. *If* (I.1.55) *holds, then the sequence* $h^{(n)}$ *is a Cauchy sequence on* $\mathsf{C}^{2,0}\big([0,T_1] \times \mathbb{T}^d\big)$.

Proof. We know by Lemma I.1.33 that the sequence $h^{(n)}$ is uniformly (in n) bounded by C_{bd} on $[0, T_1]$. This implies by (I.1.78) that for all $n > 1$

$$\begin{aligned} \left\| h^{(n+1)} - h^{(n)} \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} &\leq C_{(\partial_t)} \left(C_{max}\right) T_1 \left\| h^{(n)} - h^{(n-1)} \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \\ &\leq \left(\frac{1}{2}\right)^{n-1} \left\| h^{(2)} - h^{(1)} \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}, \end{aligned} \tag{I.1.90}$$

by the definition of T_1. By this uniform boundedness of $h^{(n)}$ and by (I.1.74)

$$\left\| \partial_t \left(h^{(n+1)} - h^{(n)} \right) \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \leq C_{(\partial_t)} \left(C_{max}\right) \left\| h^{(n)} - h^{(n-1)} \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}, \tag{I.1.91}$$

for all $n > 1$ and by (I.1.83)

$$\left\| \partial_{tt}^2 \left(h^{(n+1)} - h^{(n)} \right) \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \leq C \left(C_{max}\right) \left\| h^{(n-1)} - h^{(n-2)} \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}. \tag{I.1.92}$$

Combining (I.1.90), (I.1.91) and (I.1.92), we get

$$\begin{aligned} \left\| h^{(n+1)} - h^{(n)} \right\|_{\mathsf{C}^{2,0}([0,T]\times\mathbb{T}^d)} &\leq C \left(C_{max}\right) \left\| h^{(n-1)} - h^{(n-2)} \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)} \\ &\leq C \left(C_{max}\right) \left(\frac{1}{2}\right)^{n-3} \left\| h^{(2)} - h^{(1)} \right\|_{\mathsf{C}([0,T_1]\times\mathbb{T}^d)}. \end{aligned} \tag{I.1.93}$$

This implies that $h^{(n)}$ is a Cauchy sequence. □

Remark I.1.35. *Note that this lemma does not yield that G is a contraction. The proof depends highly on the uniform boundedness of the sup norm of $h^{(n)}$. We can not find a time T_1 such that the $C_{(\partial_t)} \left(|g|_\infty, |f|_\infty\right) T_1$ in (I.1.90) is lower than 1 for all $f, g \in \mathsf{C}^{2,0}\big([0, T_1] \times \mathbb{T}^d\big)$.*

Proof of Theorem I.1.26. Using the previous lemmas, we prove the Theorem I.1.26. We know by Lemma I.1.34 that $h^{(n)}$ is a Cauchy sequence. Because the space $\mathsf{C}^{2,0}\big([0, T_1] \times \mathbb{T}^d\big)$ with the norm $|h|_\infty + |\partial_t h|_\infty + \left|\partial_{tt}^2 h\right|_\infty$ is complete, the sequence $h^{(n)}$ converges to a $h^* \in \mathsf{C}^{2,0}\big([0, T_1] \times \mathbb{T}^d\big)$. The continuity of $G(.)$ (Lemma I.1.30) implies that $G\left(h^{(n)}\right) \to G(h^*)$. From this we infer that $h^* = G(h^*)$. Hence, we found a classical solution of (I.0.2) in the sense of Definition I.1.1 on the time interval $[0, T_1]$. Finally, (I.1.56) is a consequence of the bound derived in Lemma I.1.33. □

I.1.4.2 Induction step: Proof of Theorem I.1.25 (ii)

Proof. In this section we prove the induction step Theorem I.1.25 (ii). Fix the constant C_T used in (I.1.55) to be equal to

$$C_T = e^{2\|J\|_{L^1} T} \left(\sup_{x\in\mathbb{T}^d} \int_{\mathbb{R}} \theta^2 \rho_0^* (x, \theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta + C_\Psi + 2 \right), \tag{I.1.94}$$

with C_Ψ the constant of Lemma I.1.8 (v) that depends only on Ψ. This choice of C_T also fixes the constants C_{bd} (via (I.1.56)), the C_{max} (via (I.1.84), where we use in this definition the h fixed in Section I.1.4.1.3) and the time T_1 (via (I.1.86)).

Fix an arbitrary classical solution ρ^* of (I.0.2) on the time interval $\left[0, \overline{T}\right] \subset [0, T]$, that has a corresponding effective field $h^* \in \mathsf{C}^{2,0}\big(\left[0, \overline{T}\right] \times \mathbb{T}^d\big)$, that satisfies (I.1.52) (with C_{bd} defined above and T_1 replaced by $\overline{T}$).

The proof that there exists a classical solution on the time interval $\left[0, \overline{T} + T_1 - \epsilon\right]$ (i.e. the proof of Theorem I.1.25 (ii)) is organised as follows:

Step 1: We extend h^* to a $\overline{h} \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$, such that $\overline{h} = h^*$ on $\big[0, \overline{T}\big]$ and $\big|\overline{h}\big|_\infty < C_{bd}+1 \leq C_{max}$.

Step 2: We show that the condition (I.1.55) is satisfied with ρ^0 replaced by $\rho^*_{\overline{T}-\epsilon}$ with the constant C_T given in (I.1.94).

Step 3: Moreover, we show that $\rho^*_{\overline{T}-\epsilon}$ integrates $e^{\left(-\frac{1}{8}+\epsilon\right)\Psi(\theta)}$ as the initial distribution (Assumption I.0.6).

Step 4: The previous steps let us apply Theorem I.1.26 for the time interval $\big[\overline{T}-\epsilon, \overline{T}+T_1-\epsilon\big]$ with initial condition $\rho^*_{\overline{T}-\epsilon}$ and effective field $\overline{h}$. This provides us with a classical solution ρ^{**} of (I.0.2) on this time interval.

Step 5: We compose ρ^* and ρ^{**} and we show that this composition is a classical solution of (I.0.2) on the time interval $\big[0, \overline{T}+T_1-\epsilon\big]$.

Remark I.1.36. *Before we show these steps, let us discuss why we chose exactly these steps.*

(i) *The Step 1, Step 2 and Step 3 are the conditions that we need to apply Theorem I.1.26 in Step 4.*

(ii) *The overarching goal is to keep the additional length of the time interval fixed to be* $T_1-\epsilon$*, independent of the classical solution up to time* $\overline{T}$ *and independent of the time* $\overline{T}$ *itself. If the additional length were not fixed, it might happen that the additional length of the interval decreases in each induction step. This might have the consequence that the total length of the time intervals converges to something smaller than* T*. Then this proof by induction would not show the existence of a classical solution of* (I.0.2) *on the entire interval* $[0,T]$ *but only of a subset of it.*

(iii) *However,* T_1 *defined in* (I.1.86) *depends (in a nonlinear way through* C_{max} *and* C_{bd}*) on the bound* C_T *on the second moment of the initial distribution. But the initial distribution density is for the next time interval not any more* ρ^0 *but the classical solution* ρ^* *at time* $\overline{T}-\epsilon$*. This is the reason why we make the particular choice* (I.1.94) *of the constant* C_T *(that appears first in* (I.1.55)*). This* C_T *is not only a bound for the second moment of* ρ^0*, but, as we show in Step 2, also a bound on the second moment at each time point of each classical solution of* (I.0.2)*. This choice of* C_T *ensures, in combination with the bound on* $\big|\overline{h}\big|_\infty$ *in Step 1, that the constant* C_{max} *and thus* T_1 *does not change.*

(iv) *With the* C_T *defined by* (I.1.94)*, the condition* (I.1.55) *is obviously satisfied for the initial distribution* ρ^0*. Therefore, we can use the same constants for the base condition and for the induction step. In particular* C_{bd} *and* T_1 *are the same for the base case and for each induction step, as claimed in Theorem I.1.25.*

(v) *Although we get from Theorem I.1.26 a classical solution on a time interval of length* T_1*, we claim, in Theorem I.1.25 (ii) and in this proof, to gain only* $T_1-\epsilon$*. Reason for this is, that we can not glue the solutions on the intervals* $\big[0,\overline{T}\big]$ *and* $\big[\overline{T}, \overline{T}+T_1\big]$ *together, to get a classical solution on the interval* $\big[0, \overline{T}+T_1\big]$*. The problem is the necessary regularity at time* $\overline{T}$*. Therefore, we compare the classical solution on the interval* $\big[0,\overline{T}\big]$ *with the classical solution on the interval* $\big[\overline{T}-\epsilon, \overline{T}+T_1-\epsilon\big]$*. By the overlap of these two intervals and the uniqueness of solutions of the linear PDE* (I.1.8) *(shown in Theorem I.1.6), we get the required regularity (in Step 5).*

Step 1: We need a new effective field $\overline{h} \in \mathsf{C}^{2,0}\big([0,T] \times \mathbb{T}^d\big)$, to have a new starting field for the sequence $\overline{h}^{(n)}$. This effective field should be equal to h^* on $\big[0,\overline{T}\big]$. The reason for this condition becomes obvious in Step 5. Moreover, $\overline{h}$ should be everywhere smaller than $C_{max} \geq C_{bd} + 1$. This condition is necessary to use the same constant in Lemma I.1.33.

A function $\overline{h}$ that satisfies all these conditions can be easily defined, because h^* can be extended to a $\mathsf{C}^{2,0}\left([0,T] \times \mathbb{T}^d\right)$ function by the compactness of $[0,T_1]$. This extension respects the condition on the sup norm because $|h^*|_\infty \leq C_{bd}$ (by (I.1.56)).

Step 2: We show in the following lemma that the choice (I.1.94) of C_T is an upper bound on the second moment of all classical solution of (I.0.2) on each arbitrary interval $[0,\overline{T}]$.

Lemma I.1.37. *If for* $0 < \overline{T} \leq T$*, the function* $\rho^* \in \mathsf{C}^{2,0}\left([0,\overline{T}] \times \mathbb{T}^d\right)$ *is a classical solution of* (I.0.2) *on the time interval* $[0,\overline{T}]$*, then for all* $t \in [0,\overline{T}]$

$$\sup_{x\in\mathbb{T}^d} \int_{\mathbb{R}} \theta^2 \rho_t^* (x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \leq C_T. \tag{I.1.95}$$

Proof.We call h^* the effective field corresponding to ρ^*. The ρ^* is a classical solution of (I.0.2), hence for each $x \in \mathbb{T}^d$

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \theta^2 \rho_t^* (x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}x\mathrm{d}\theta = \int_{\mathbb{R}} 2 \left(-\Psi'(\theta)\,\theta + h^*(t,x)\,\theta + 1\right) \rho_t^* (x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \\
&\leq C_\Psi + 2 \int_{\mathbb{R}} \int_{\mathbb{T}^d\times\mathbb{R}} J(x-x')\,\theta'\theta \rho_t^*(x',\theta')\, \rho_t^*(x,\theta)\, e^{-\Psi(\theta)} e^{-\Psi(\theta')} \mathrm{d}x'\mathrm{d}\theta'\mathrm{d}\theta + 2 \\
&\leq C_\Psi + 2\, \|J\|_{L^1} \sup_{x\in\mathbb{T}^d} \int_{\mathbb{R}} \theta^2 \rho_t^*(x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta + 2,
\end{aligned} \tag{I.1.96}$$

where we use that $\theta\theta' \leq \frac{1}{2}\left(\theta^2 + \theta'^2\right)$. Here C_Ψ is the constant of Lemma I.1.8 (v), that bounds $-\Psi'(\theta)\,\theta$.

We conclude the claimed upper bound on the second moment by applying the Gronwall inequality. This is applicable because $\sup_{x\in\mathbb{T}^d} \int_{\mathbb{R}} \theta^2 \rho_t^*(x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta$ is integrable with respect to $t \in [0,T]$ (by a similar calculation as in (I.1.88) and by Assumption I.0.6). □

We infer from this lemma in particular, that the second moment of $\rho^*_{\overline{T}-\epsilon}$ is also bounded by C_T.

Step 3: Now we show that $\rho^*_{\overline{T}-\epsilon}$ satisfies Assumption I.0.6. For each $x \in \mathbb{T}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}} e^{\left(\frac{7}{8}+\epsilon\right)\Psi} \rho^*_{\overline{T}-\epsilon}(x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta \\
&= \int_{\mathbb{R}} e^{\left(\frac{7}{8}+\epsilon\right)\Psi} \rho_0^*(x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta + \int_0^{\overline{T}-\epsilon} \int_{\mathbb{R}} \mathbb{L}_{h^*(t,x)} e^{\left(\frac{7}{8}+\epsilon\right)\Psi} \rho_t^*(x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta\mathrm{d}t \\
&\leq C + TC,
\end{aligned} \tag{I.1.97}$$

by Assumption I.0.6 for ρ^0 and by Lemma I.1.8 (vi). The constants on the right hand side are independent of $x \in \mathbb{T}^d$.

Step 4: We use now the restriction of $\overline{h}$ to $\left[\overline{T}-\epsilon, \overline{T}+T_1-\epsilon\right]$ and apply Theorem I.1.26 on this time interval. As initial condition we take $\rho^*_{\overline{T}-\epsilon}$. By Step 2, this initial distribution satisfies the condition (I.1.55) with $A = C_T$. Thus there is a sequence $\left\{\overline{h}^{(n)}\right\} \subset \mathsf{C}^{2,0}\left([0,T] \times \mathbb{T}^d\right)$, that converges to a fixed point of $G_{\left[\overline{T}-\epsilon,\overline{T}+T_1-\epsilon\right],\rho^*_{\overline{T}-\epsilon}}$ (see Theorem I.1.26). As in the base case, this implies the existence of a classical solution ρ^{**} of (I.0.2) on this time interval, with initial distribution $\rho^*_{\overline{T}-\epsilon}$. Moreover, h^{**} is also the effective field defined by ρ^{**} via the second line of (I.0.2).

Step 5: Let us compose the two classical solutions that we have so far, i.e.

$$\rho^{*,2} := \begin{cases} \rho^* & \text{on } \left[0, \overline{T} - \epsilon\right], \\ \rho^{**} & \text{on } \left(\overline{T} - \epsilon, \overline{T} + T_1 - \epsilon\right]. \end{cases} \tag{I.1.98}$$

We show in the rest of the proof that $\rho^{*,2}$ is a classical solution of (I.0.2) the sense of Definition I.1.1. We know already by the regularity of ρ^* and ρ^{**}, that $\rho^{*,2}$ has necessary regularity of a classical solution everywhere with a possible exception at time $\overline{T} - \epsilon$. However, $\rho^{*,2}$ has this property also at this time, if

$$\rho^* = \rho^{**} \text{ on } \left[\overline{T} - \epsilon, \overline{T} + T_1 - \epsilon\right]. \tag{I.1.99}$$

Moreover, we know from ρ^* and ρ^{**}, that for each time $t \in \left(0, \overline{T} + T_1 - \epsilon\right]$, the function $\rho_t^{*,2}$ solves (I.0.2). Therefore, we have constructed a classical solution of (I.0.2) on the time interval $\left[0, \overline{T} + T_1 - \epsilon\right]$, as soon as we have shown (I.1.99).

Let us prove now (I.1.99), which turns out to be a consequence of the overlap of the intervals $\left[0, \overline{T}\right]$ and $\left[\overline{T} - \epsilon, \overline{T} + T_1 - \epsilon\right]$. Due to the chosen initial distribution of $\rho^{*,2}$, we know already that $\rho^*_{\overline{T}-\epsilon} = \rho^{**}_{\overline{T}-\epsilon}$. We claim that also

$$h^{**} = h^* \text{ on } \left[\overline{T} - \epsilon, \overline{T}\right]. \tag{I.1.100}$$

If this is holds, then the uniqueness of solutions of (I.1.8) (shown in Theorem I.1.6) implies (I.1.99) (because ρ^* and ρ^{**} are two solutions of (I.1.8) with the same effective fields and the same initial distribution).

In the rest of this proof, we show (I.1.100). As stated already in Step 4, h^{**} is the limit of the sequence $\overline{h}^{(n)}$, which starts at $\overline{h}$. We show now that $\overline{h}^{(n)}$ equals h^* on $\left[\overline{T} - \epsilon, \overline{T}\right]$ for each $n \in \mathbb{N}$. This is true for $\overline{h}$ by its definition. Let this also be true for an arbitrary $\overline{h}^{(n)}$. Then $\overline{h}^{(n+1)} = G\left(\overline{h}^{(n)}\right)$ satisfies

$$\overline{h}^{(n+1)} = h^* \text{ on } \left[\overline{T} - \epsilon, \overline{T}\right], \tag{I.1.101}$$

because of the uniqueness of solutions of (I.1.8) (shown in Theorem I.1.6). To be more precise, let $\rho^{\overline{h}^{(n)}}$ be the classical solutions of the linear PDE (I.1.8) on $\left[\overline{T} - \epsilon, \overline{T} + T_1 - \epsilon\right]$ with effective field $\overline{h}^{(n)}$ and initial distribution $\rho^*_{\overline{T}-\epsilon}$. Then $\rho^{\overline{h}^{(n)}}$ and ρ^* are classical solutions of the same linear PDE (I.1.8) on $\left[\overline{T} - \epsilon, \overline{T}\right]$. By the uniqueness of Theorem I.1.6 these solutions have to be the same on this interval. Therefore, also the effective fields are the same, what proves the claimed equation (I.1.101).

We have thus shown that $\rho^{*,2}$ is a classical solution of (I.0.2) on the time interval $\left[0, \overline{T} + T_1 - \epsilon\right]$, i.e. the claim of Theorem I.1.25 (ii). □

I.1.5 Proof of Theorem I.1.4: Relation between probability densities of the hydrodynamic SDE and classical solutions of the hydrodynamic equation

In Theorem I.1.2, we have shown that there exists a smooth probability densities of the hydrodynamic SDE (I.0.3), that is also a classical solutions of the hydrodynamic equation (I.0.2). Let us prove now that there is a one to one relation between smooth probability densities of (I.0.3) and classical solutions of the hydrodynamic equation (I.0.2), i.e. Theorem I.1.4.

Proof of Theorem I.1.4. Let $\rho^* : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ be a function that satisfies the regularity condition (i) of Definition I.1.1. Moreover, assume that the effective field h^*, defined from ρ^* as in the second line of (I.0.2), is continuous (Definition I.1.1 (ii)).

"⇐": Assume that $\rho^*_t(x,\theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta$ is law of $\widehat{\theta}^x_t$ evolving according to the SDE (I.0.3). Then ρ^* is the solution to the hydrodynamic equation (I.0.2) by Itô's formula.

"⇒": Assume that ρ^* is a classical solution of the hydrodynamic equation (I.0.2) in the sense of Definition I.1.1. Then ρ^* is also a weak solution of (I.1.8) with effective field h^* by Lemma I.1.22. Hence, we get upper bounds on ρ by Lemma I.1.23 (ii). From this bound we infer that h^* is in $\mathsf{C}^{2,0}\left([0,T]\times\mathbb{T}^d\right)$ (as in the proof of Lemma I.1.29). Finally, Theorem I.1.6 implies that there is a ρ^{h^*} that is the law of $\overline{\theta}^x_t$ defined by (I.1.7) and the unique weak solution of (I.1.8) with effective field h^*. Hence, $\rho^{h^*} = \rho^*$. Therefore, ρ^* is also the law of $\widehat{\theta}^x_t$ evolving according to the SDE (I.0.3). □

I.2 Proof of the hydrodynamic limit using the relative entropy method

In this section we prove the hydrodynamic limit result by using the relative entropy method. The main idea of this approach is to compare the probability densities of the system of N^d interacting SDEs (I.0.1) and of the following system of N^d independent SDEs

$$\begin{aligned} \mathrm{d}\widehat{\theta}^{i,N}_t &= -\Psi'\left(\widehat{\theta}^{i,N}_t\right)\mathrm{d}t + h\left(t,\frac{i}{N}\right)\mathrm{d}t + \sqrt{2}\mathrm{d}W^{i,N}_t \qquad &&\text{for } i\in\mathbb{T}^d_N,\\ \widehat{\theta}^{i,N}_0 &\sim \rho^0\left(\frac{i}{N},\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta &&\text{for } i\in\mathbb{T}^d_N, \end{aligned} \tag{I.2.1}$$

where the $h = h^\rho$ is defined as in the second line of (I.0.2) for a given function ρ. The same ρ is the time evolution of the probability density of $\underline{\widehat{\theta}}^N$ (see Definition I.2.2).

Remark I.2.1. *For $i = xN \in \mathbb{T}^d_N$ for a $x\in\mathbb{T}^d$,* (I.0.3) *equals the i-th coordinate of the SDE* (I.2.1)*. Hence, the time evolution of the probability density corresponding to* (I.0.3) *derived in Section I.1 is a candidate for this ρ.*

One of the main results of this section is, that the relative entropy between the laws of the SDE (I.0.1) and of the SDE (I.2.1) is of order $o\left(N^d\right)$ (see Theorem I.2.8). We prove this result in Section I.2.2. Then we infer from the order of the relative entropy the hydrodynamic limit result (Theorem I.2.3) in Section I.2.3. We explain the approach of the proofs in more details at the beginning of both sections.

We state the hydrodynamic limit result and the result concerning the relative entropy in Section I.2.1. Moreover, we infer from these results uniqueness of probability densities corresponding to the SDE (I.0.3). This implies in particular the uniqueness of classical solutions of the hydrodynamic equation (I.0.2).

I.2.1 The main results

In this and the following sections we assume, without further mentioning it, that the Assumptions I.0.1, I.0.3, I.0.4 and I.0.5 hold.

To state the main result of this section, we use the following set of probability densities.

Definition I.2.2. *We call a function $\rho : [0,T]\times\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$ a continuous probability density of* (I.0.3)*, if*

(i) $[0,T]\times\mathbb{T}^d\times\mathbb{R} \ni (t,x,\theta) \to \rho_t(x,\theta)$ *is continuous,*

(ii) *The effective field* $h = h^{\rho}$ *defined as in the second line of* (I.0.2) *is continuous, and*

(iii) *for each* $x \in \mathbb{T}^d$ *and* $t \in [0,T]$, $\rho_t(x,\theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta$ *is the time marginal of the law of the SDE* (I.0.3) *with effective field* $h = h^{\rho}$.

We denote the set of all continuous probability density of (I.0.3) *by* $\mathbb{S}$.

A sufficiently regular $\rho \in \mathbb{S}$ is a classical solution to the nonlinear hydrodynamic equation (I.0.2) (by Theorem I.1.4). We know already by Theorem I.1.2 that there exists at least one continuous probability density of (I.0.3).

We show in the following theorem that each element in $\mathbb{S}$ is a density of the hydrodynamic limit of the empirical process $\mu^N_{[0,T]}$.

Theorem I.2.3. *Fix an arbitrary* $\rho \in \mathbb{S}$. *Then*

$$\mu^N_{[0,T]} \xrightarrow{p} \left\{ t \to \rho_t(x,\theta)\, e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}x \right\} \tag{I.2.2}$$

in probability as random variables in $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$, *where the empirical process* $\mu^N_{[0,T]}$ *is defined in* (0.3.2) *with* $\underline{\theta}^N$ *evolving according to* (I.0.1).

Corollary I.2.4.
- *There is exactly one continuous probability density of* (I.0.3) *in the sense of Definition I.2.2, i.e. there is one element in* $\mathbb{S}$.
- *This element is the classical solution of the hydrodynamic equation* (I.0.2) *that we derive in Section I.1.*
- *IThis implies that there is exactly one probability valued, classical solution of the hydrodynamic equation* (I.0.2).

Proof of Corollary I.2.4. The limit point of $\mu^N_{[0,T]}$ in Theorem I.2.3 is unique. Hence the corollary follows from Theorem I.2.3, Theorem I.1.2 and Theorem I.1.4. □

Remark I.2.5. *We prove the uniqueness of probability density valued classical solutions of* (I.0.2) *in Section I.3 by using different techniques.*

Notation I.2.6. (i) *We denote by* $\mathbb{P}^N_{[0,T]}$ *the solutions to the martingale problem corresponding to* (I.0.1) *with initial densities* $f_0^N e^{-\Psi}$.

(ii) *We denote by* $\widehat{\mathbb{P}}^N_{[0,T]} \in \mathbb{M}_1\left(\mathsf{C}\left([0,T], \mathbb{R}^{N^d}\right)\right)$ *the solutions to the martingale problem corresponding to* (I.2.1), *with* $h = h^{\rho}$ *defined (by the second line of* (I.0.2)*) via the* $\rho \in \mathbb{S}$ *that we fixed in Theorem I.2.3 .*

(iii) *For the expectations w.r.t.* $\mathbb{P}^N_{[0,T]}$ *and* $\widehat{\mathbb{P}}^N_{[0,T]}$ *we use the symbols* $\mathbb{E}^N_{[0,T]}$ *respectively* $\widehat{\mathbb{E}}^N_{[0,T]}$.

The two solutions to the martingale problems exist and are unique by Assumption I.0.1, Assumption I.0.3 and by applying the Theorem 7.2.2., Theorem 10.1.2 and Theorem 10.2.1 of [SV79]. Note that $\prod_{i\in\mathbb{T}^d_N} \rho_t\left(\frac{i}{N}, .\right) e^{-\Psi(.)}$ is the density of the one dimensional time marginal at time $t \in [0,T]$ of $\widehat{\mathbb{P}}^N_{[0,T]}$ (due to Definition I.2.2 (iii)).

Remark I.2.7. *Using this notation, the result of Theorem I.2.3 is equivalent to the weak convergence on* $\mathbb{M}_1\left(\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)\right)$ *of the law of* $\mu^N_{[0,T]}$ *(e.g. defined as an image of* $\mathbb{P}^N_{[0,T]}$*) to the measure* $\delta_{\rho_.(x,\theta)e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}x}$.

We state in the following theorem that the relative entropy of the laws of the solutions of the coupled (I.0.1) and of the independent (I.2.1) SDEs is of order $o\left(N^d\right)$. This result is needed in the proof of Theorem I.2.3.

Theorem I.2.8. *For any* $T \geq 0$,

$$\lim_{N\to\infty} \frac{1}{N^d} \mathsf{H}\left(\mathbb{P}^N_{[0,T]} \middle| \widehat{\mathbb{P}}^N_{[0,T]}\right) = 0. \tag{I.2.3}$$

I.2.1.1 Generalisation of the assumptions

The results of Theorem I.2.3, Corollary I.2.4 and Theorem I.2.8 hold under more general assumptions as well. For example, we could replace the assumption of Ψ being a polynomial (Assumption I.0.3) by the assumption that it complies with the condition (I.0.5) (now c_Ψ is the speed of the quadratic growth at infinity), that it is differentiable and that it implies the existence and uniqueness of the two martingale problems (see Notation I.2.6 and the subsequent discussion). For example, a similar assumption as Assumption (B) in [Gär88] would imply this existence and uniqueness without Ψ being a polynomial. Nevertheless, we use the polynomial assumption to simplify the notation.

I.2.1.2 Discussion of the empirical process

We discuss in this section, why we derive the hydrodynamic limit result for the empirical process $\mu^N_{[0,T]}$. As explained in Section 0.3, we cannot use a pathwise approach, because the operator $J * \cdot$ is not smoothing. Hence, we need a random object that locally averages in the limit. This is the case for the empirical process, but also in the following evolution of spin configurations

$$\gamma^N_t \left(\mathrm{d}x\right) := \frac{1}{N^d} \sum_{i \in \mathbb{T}^d_N} \delta_{\frac{i}{N}} \left(\mathrm{d}x\right) \theta^{i,N}_t . \tag{I.2.4}$$

Each γ^N_t is an element of $\mathbb{M}\left(\mathbb{T}^d\right)$, the space of signed finite measures (see Appendix A). For these random elements a hydrodynamic limit result is, for example, derived in [CE88] for a different model.

Let us explain the obstacles to get a hydrodynamic limit result for γ^N. To prove the hydrodynamic limit for γ^N_t we would need to get a closed form (in terms of γ^N_t) equation for $\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{N^d} \sum_{i \in \mathbb{T}^d_N} g\left(\frac{i}{N}\right) \theta^{i,N}_t$ ($g \in \mathsf{C}^\infty\left(\mathbb{T}^d\right)$) at least in the limit $N \to \infty$. When looking at the empirical process $\mu^N_{[0,T]}$, a closed form equation can be derived easily for all $N \in \mathbb{N}$. However, for γ^N_t there is no closed form equation for finite N in the model we consider here. Also in the limit we do not know a suitable closed form equation, as we explain in the following. If ρ^* is the limit of $\mu^N_{[0,T]}$ and a solution of (0.5.3), then we expect to see as limit of γ^N the time evolution m^* of the magnetisation profile, defined by

$$m^* \left(t, x\right) := \int_{\mathbb{R}} \theta e^{-\Psi(\theta)} \rho^*_t \left(x, \theta\right) \mathrm{d}\theta. \tag{I.2.5}$$

Then (0.5.3) tells us that

$$\partial_t m^* \left(t, x\right) = \int_{\mathbb{R}} \theta e^{-\Psi} \partial_t \rho^*_t \left(x, \theta\right) \mathrm{d}\theta = - \int_{\mathbb{R}} e^{-\Psi} \partial_\theta \rho^*_t \left(x, \theta\right) \mathrm{d}\theta + J * m^* \left(t, x\right), \tag{I.2.6}$$

or equivalently

$$\partial_t m^* \left(t, x\right) = - \int_{\mathbb{R}} \Psi' \left(\theta\right) e^{-\Psi} \rho_t \left(x, \theta\right) \mathrm{d}\theta + J * m^* \left(t, x\right). \tag{I.2.7}$$

Hence, to have at least a chance to derive the hydrodynamic limit for the random elements γ^N instead of the empirical process, we would need to write (I.2.6) in a closed form. This requires that we rewrite the right side of (I.2.7) as a function of m^*. If $\Psi\left(\theta\right) = c\theta^2$, this is trivial. However, for more interesting Ψ, we are not able to not able to derive a closed form equation for m^*. Therefore, we cannot show a hydrodynamic limit result for γ^N.

In contrast to this, a hydrodynamic limit result as well for the magnetisation profile γ^N and the empirical process $\mu^N_{[0,T]}$ is proven in [CE88]. The authors consider also a local mean field model, but with a jump dynamic. Due to this dynamic, the hydrodynamic equation for the empirical process, does not contain terms of higher order of the spins (i.e. there is nothing similar to the Ψ term in the limit). By the procedure sketched above, one easily gets a closed form PDE for the time evolution of the local magnetisation profile m^*. It seems to be not possible to generalise their approach, that depends highly on the particular jump dynamic, to more general dynamics such as Langevin dynamics.

Let us see if we could use results from the comparison of the energy landscape of the magnetisation profile m and of the density ρ (derived in Chapter IV, for an introduction see Section 0.8), to get a closed form equation for (I.2.6). Fix a local magnetisation profile m. We derive in Chapter IV (see (IV.2.12)) that the density ρ with the lowest energy that has this magnetisation profile, has the following representation

$$\rho_t(x,\theta) = e^{\theta I'(m(t,x)) - C(m(t,x))}, \tag{I.2.8}$$

where only the first term depends on θ. The explicit formulas of the functions I' and C are at this stage not relevant. If we had a solution ρ^* of (0.5.3) that has the form (I.2.8), then we would get by (I.2.6) the following closed form equation for m^*

$$\partial_t m^*(t,x) = -I'(m^*(t,x)) + J * m^*(t,x). \tag{I.2.9}$$

Let us assume we had a m^* that is a solution to (I.2.9). Define ρ by (I.2.8) with this m^*. Then ρ is in general not a solution to (0.5.3). Hence, although the ansatz (I.2.8) leads to a closed form equation (I.2.9) for m, a solution of (I.2.9) does not describe in general the evolution in the hydrodynamic limit. Moreover, the above means that the hydrodynamic limit evolution does not respect the landscape correspondence that we derive in Chapter IV (see Lemma IV.2.5). Indeed, the energy of a solution ρ^* of (0.5.3) is in general larger than the energy of the corresponding magnetisation profile m^*.

I.2.2 Proof of Theorem I.2.8: Order of the relative entropy

Proof. We prove in this section the Theorem I.2.8. Let us define for each $t \in [0,T]$, the difference of a $\theta \in \mathbb{R}$ and the expected value of the measure $\rho(t,x,\theta)\, e^{-\Psi(\theta)} \mathrm{d}\theta$ at position $x \in \mathbb{T}^d$ by

$$\eta_t^x := \theta - \int_{\mathbb{R}} \theta' \rho(t,x,\theta')\, e^{-\Psi(\theta')} \mathrm{d}\theta'. \tag{I.2.10}$$

For $t \in [0,T]$, $N \in \mathbb{N}$, $\underline{\theta}_t^N \in \mathbb{R}^{N^d}$ and $i \in \mathbb{T}_N^d$, we define

$$\eta_t^{i,N} := \theta_t^{i,N} - \widehat{\mathbb{E}}_{[0,T]}^N \left[\widehat{\theta}_t^{i,N} \right] \quad \text{and} \quad X_t^N := \frac{1}{N^{2d}} \sum_{i,j \in \mathbb{T}_N^d} K^N\left(\frac{i}{N}, \frac{j}{N}\right) \eta_t^{i,N} \eta_t^{j,N}, \tag{I.2.11}$$

where

$$K^N\left(\frac{i}{N}, \frac{j}{N}\right) := \frac{1}{N^d} \sum_{\ell \in \mathbb{T}_N^d} J\left(\frac{i-\ell}{N}\right) J\left(\frac{j-\ell}{N}\right). \tag{I.2.12}$$

Hence, for each $i \in \mathbb{T}_N^d$, $\eta_t^{i,N}$ defined in (I.2.11) is equal to $\eta_t^{\frac{i}{N}}$ defined in (I.2.10) with $\theta = \theta_t^{i,N}$. Using these definitions we show the following upper bound of the relative entropy between $\mathbb{P}_{[0,t]}^N$ and $\widehat{\mathbb{P}}_{[0,t]}^N$.

Lemma I.2.9. *For all $N \in \mathbb{N}$*

$$\mathsf{H}\left(\mathbb{P}^N_{[0,T]}\middle|\widehat{\mathbb{P}}^N_{[0,T]}\right) \leq e^{T\frac{1}{4\delta}}\left(\mathsf{H}\left(f_0^N e^{-\Psi}\middle|\prod_{i\in\mathbb{T}^d_N}\rho_0\left(\frac{i}{N},\cdot\right)e^{-\Psi}\right)\right.$$
$$\left.+\frac{1}{4\delta}\int_0^T \log\widehat{\mathbb{E}}^N_{[0,s]}\left[e^{\delta N^d X^N_s}\right]\mathrm{d}s + N^d\mathrm{o}(1)\right). \tag{I.2.13}$$

Moreover, we show that the time integral of the logarithmic moment generating function vanishes.

Lemma I.2.10. *Let Assumption I.0.4 b.) be satisfied. For sufficiently small δ*

$$\lim_{N\to\infty}\int_0^T \frac{1}{N^d}\log\widehat{\mathbb{E}}^N_{[0,t]}\left[e^{\delta N^d X^N_t}\right]\mathrm{d}t = 0. \tag{I.2.14}$$

These two lemmas and Assumption I.0.5 a.) imply Theorem I.2.8. □

We prove the two lemmas and the ideas of the proofs in Section I.2.2.2 and Section I.2.2.3. Moreover, we list preliminary results in Section I.2.2.1.

I.2.2.1 Preliminaries

In this section we give at first an uniform upper bound on the second moment of $\theta^{i,N}$ under $\mathbb{P}^N_{[0,T]}$.

Lemma I.2.11. *There is a constant $C > 0$ and a $\overline{N} \in \mathbb{N}$ such that*

$$\sup_{N\geq\overline{N},t\in[0,T],i\in\mathbb{T}^d_N}\mathbb{E}^N_{[0,T]}\left[\left(\theta^{i,N}_t\right)^2\right] \leq C(1+T). \tag{I.2.15}$$

Proof. Fix a $\delta > 0$ and $\overline{N} \in \mathbb{N}$, such that $\left|\frac{1}{N^d}\sum_{j\in\mathbb{T}^d_N}\left|J\left(\frac{j}{N}\right)\right| - \|J\|_{L^1}\right| < \delta$ for all $N > \overline{N}$ (by Assumption I.0.1) and $\|J\|_{L^1} + \delta < c_\Psi$ (by Assumption I.0.3). For $N \geq \overline{N}, t \in [0,T], i \in \mathbb{T}^d_N$,

$$\begin{aligned}\mathbb{E}^N_{[0,T]}\left[\left(\theta^{i,N}_t\right)^2\right] &\leq \mathbb{E}^N_{[0,T]}\left[\left(\theta^{i,N}_0\right)^2\right]\\ &\quad+\mathbb{E}^N_{[0,T]}\left[\int_0^t \frac{2}{N^d}\sum_{i\in\mathbb{T}^d_N}\underbrace{-\Psi'\left(\theta^{i,N}_s\right)\theta^{i,N}_s + \left(\|J\|_{L^1}+\delta\right)\left(\theta^{i,N}_s\right)^2}_{\leq C}\mathrm{d}s\right] + 2\\ &\leq C(1+T),\end{aligned} \tag{I.2.16}$$

by Assumption I.0.3 and Assumption I.0.5 b.) and because $\mathbb{P}^N_{[0,T]}$ is a solution to the martingale problem (see Notation I.2.6). □

Next we show uniform upper and lower bounds on ρ_t.

Lemma I.2.12. (i) *There is a $F \in L^1(\mathbb{R})$ and a $\kappa > 0$, such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{T}^d}\left\{\left(|\theta| + e^{\kappa\theta^2}\right)e^{-\Psi(\theta)}\rho_t(x,\theta)\right\} < F(\theta). \tag{I.2.17}$$

(ii) *There is a $\delta > 0$ and a function $V = \alpha\theta^{2k} +$* (lower order terms) *with $\alpha > 0$ and $k \in \mathbb{N}$, such that for all $t \in [\delta, T]$ and all $(x,\theta) \in \mathbb{T}^d \times \mathbb{R}$*

$$e^{-V(\theta)} \leq \rho_t(x,\theta)e^{-\Psi(\theta)}. \tag{I.2.18}$$

Proof. The function ρ is as continuous probability density of (I.0.3) (in the sense of Definition I.2.2) a weak solution of the linear Fokker-Planck equation (I.1.8) with fixed effective field $h = h^\rho$. Indeed, this follows by Itô's formula. By the continuity of the corresponding effective field h^ρ (Definition I.2.2 (ii)), Lemma I.1.23 is applicable. Hence, the upper and lower bounds follow from Lemma I.1.23 (ii) and (iv). □

I.2.2.2 Proof of Lemma I.2.9

The proof of Lemma I.2.9 is organized as follows.

1.) The starting point of the proof is to replace the derivative $\frac{\mathrm{d}\mathbb{P}^N_{[0,t]}}{\mathrm{d}\widehat{\mathbb{P}}^N_{[0,t]}}$ in the relative entropy $\mathsf{H}\left(\mathbb{P}^N_{[0,t]}\middle|\widehat{\mathbb{P}}^N_{[0,t]}\right)$ with the help of the generalised Girsanov formula (see (I.2.20) and (I.2.21)).

2.) Using this arising formula for the relative entropy, we show that the relative entropy equals the expected value of X_t^N (defined in (I.2.11)) with an error of order N^d (see Lemma I.2.13).

3.) Finally, the variation formula of the relative entropy (in (I.2.26)) and the Gronwall inequality, lead to the inequality in Lemma I.2.9.

To shorten the notation, we denote, for given spins $\underline{\theta}^N \in \mathbb{R}^{N^d}$, the strength of the interaction that acts on the spin at position $i \in \mathbb{T}_N^d$ by

$$h^{i,N}\left(\underline{\theta}^N\right) = \frac{1}{N^d} \sum_{j\in\mathbb{T}_N^d} J\left(\frac{j-i}{N}\right)\theta^{j,N}. \tag{I.2.19}$$

The martingale problems associated to $\mathbb{P}^N_{[0,t]}$ and $\widehat{\mathbb{P}}^N_{[0,t]}$ are well posed (see Notation I.2.6 and the subsequent discussion). Hence, by the generalised Girsanov formula Corollary B.2 in Appendix B,

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}^N_{[0,t]}}{\mathrm{d}\widehat{\mathbb{P}}^N_{[0,t]}} = \frac{\mathrm{d}\left(f_0^N e^{-\Psi}\right)}{\mathrm{d}\left(\prod_{i\in\mathbb{T}_N^d}\rho_0\left(\frac{i}{N},.\right)e^{-\Psi}\right)} \exp\Bigg\{ &\frac{1}{\sqrt{2}}\sum_{i\in\mathbb{T}_N^d}\int_0^t \left(h^{i,N}\left(\underline{\theta}_s^N\right) - h\left(s,\frac{i}{N}\right)\right)\mathrm{d}\widehat{B}_s^i \\ &-\frac{1}{4}\sum_{i\in\mathbb{T}_N^d}\int_0^t\left(h^{i,N}\left(\underline{\theta}_s^N\right) - h\left(s,\frac{i}{N}\right)\right)^2\mathrm{d}s\Bigg\},\end{aligned} \tag{I.2.20}$$

for all $t\in[0,T]$. Then

$$\mathsf{H}\left(\mathbb{P}^N_{[0,t]}\middle|\widehat{\mathbb{P}}^N_{[0,t]}\right) = \frac{1}{4}\int_0^t \mathbb{E}^N_{[0,T]}\left[\sum_{i\in\mathbb{T}_N^d}\left(h^{i,N}\left(\underline{\theta}_s^N\right) - h\left(s,\frac{i}{N}\right)\right)^2\right]\mathrm{d}s. \tag{I.2.21}$$

Lemma I.2.13. *With X_t^N defined in* (I.2.11)*,*

$$\mathsf{H}\left(\mathbb{P}^N_{[0,t]}\middle|\widehat{\mathbb{P}}^N_{[0,t]}\right) = \frac{1}{4}\int_0^t \mathbb{E}^N_{[0,s]}\left[N^d X_t^N\right]\mathrm{d}s + N^d o\left(1\right). \tag{I.2.22}$$

The convergence of $o(1)$ to zero is uniform in $t \in [0,T]$.

Proof. The function $[0,T]\times\mathbb{T}^d \ni (t,x') \to H(t,x') := \int_R \theta'\rho_t(x',\theta')\,e^{-\Psi(\theta')}\mathrm{d}\theta'$ is continuous and bounded. This follows from the continuity of ρ (Definition I.2.2 (i)) and the integrable upper bound

(Lemma I.2.12) (e.g. by [Bau01] Lemma 16.1). Hence

$$\begin{aligned}
&\left|h\left(t,\frac{i}{N}\right)-\frac{1}{N^d}\sum_{j\in\mathbb{T}_N^d}J\left(\frac{j-i}{N}\right)H\left(t,\frac{j}{N}\right)\right| \\
&\leq |H|_\infty \sum_{j\in\mathbb{T}_N^d}\int_{\Delta_{j,N}}\left|J\left(\frac{j}{N}\right)-J(x)\right|\mathrm{d}x \qquad \text{(I.2.23)}\\
&\quad+\sup_{j\in\mathbb{T}_N^d}\left\{H\left(t,\frac{j}{N}\right)-N^d\int_{\Delta_{j,N}}H(t,x)\,\mathrm{d}x\right\}\frac{1}{N^d}\sum_{j\in\mathbb{T}_N^d}\left|J\left(\frac{j}{N}\right)\right|.
\end{aligned}$$

The right hand side of (I.2.23) vanishes when N tends to infinity by Assumption I.0.1 and the uniform continuity of H. This convergence is even uniform in $t\in[0,T]$ and $i\in\mathbb{T}_N^d$.

By (I.2.23) and (I.2.21) we get for each $i\in\mathbb{T}_N^d$

$$\begin{aligned}
&\mathbb{E}_{[0,t]}^N\left[\left(h^{i,N}\left(\underline{\theta}_t^N\right)-h\left(t,\frac{i}{N}\right)\right)^2\right]\\
&=\mathbb{E}_{[0,t]}^N\left[\left(\frac{1}{N^d}\sum_{j\in\mathbb{T}_N^d}J\left(\frac{j-i}{N}\right)\left(\theta_t^{j,N}-\widehat{\mathbb{E}}_{[0,T]}^N\left[\theta_t^{j,N}\right]\right)+o(1)\right)^2\right]\\
&=\mathbb{E}_{[0,t]}^N\left[\left(\frac{1}{N^d}\sum_{j\in\mathbb{T}_N^d}J\left(\frac{j-i}{N}\right)\left(\theta_t^{j,N}-\widehat{\mathbb{E}}_{[0,T]}^N\left[\theta_t^{j,N}\right]\right)\right)^2\right] \qquad \text{(I.2.24)}\\
&\quad+\underbrace{\frac{2}{N^d}\sum_{j\in\mathbb{T}_N^d}J\left(\frac{j-i}{N}\right)\left(\mathbb{E}_{[0,t]}^N\left[\theta_t^{j,N}\right]-\widehat{\mathbb{E}}_{[0,T]}^N\left[\theta_t^{j,N}\right]\right)o(1)}_{=o(1)}+o(1).
\end{aligned}$$

Here we use Assumption I.0.1 and

$$\left|\mathbb{E}_{[0,t]}^N\left[\theta_t^{j,N}\right]-\widehat{\mathbb{E}}_{[0,T]}^N\left[\theta_t^{j,N}\right]\right|\leq\left|\mathbb{E}_{[0,t]}^N\left[\theta_t^{j,N}\right]\right|+\left|\widehat{\mathbb{E}}_{[0,T]}^N\left[\theta_t^{j,N}\right]\right|\leq C, \qquad \text{(I.2.25)}$$

where the constant $C>0$ is independent of $t\in[0,T]$, $j\in\mathbb{T}_N^d$ and $N\in\mathbb{N}$, by Lemma I.2.11 and Lemma I.2.12.

The equality (I.2.24) implies (I.2.22), when using the notation given in (I.2.11). □

By the variational formula of the relative entropy we infer from (I.2.22) that for any $\delta>0$,

$$\mathsf{H}\left(\mathbb{P}_{[0,t]}^N\middle|\widehat{\mathbb{P}}_{[0,t]}^N\right)\leq\frac{1}{4\delta}\int_0^t\mathsf{H}\left(\mathbb{P}_{[0,s]}^N\middle|\widehat{\mathbb{P}}_{[0,s]}^N\right)\mathrm{d}s+\frac{1}{4\delta}\int_0^t\log\widehat{\mathbb{E}}_{[0,t]}^N\left[e^{\delta N^dX_t^N}\right]\mathrm{d}s+N^d\mathrm{o}(1). \qquad \text{(I.2.26)}$$

The map $t\to\mathsf{H}\left(\mathbb{P}_{[0,t]}^N\middle|\widehat{\mathbb{P}}_{[0,t]}^N\right)$ is continuous for N large enough, because for all $s<t$, $s,t\in[0,T]$

$$\begin{aligned}
\left|\mathsf{H}\left(\mathbb{P}_{[0,t]}^N\middle|\widehat{\mathbb{P}}_{[0,t]}^N\right)-\mathsf{H}\left(\mathbb{P}_{[0,s]}^N\middle|\widehat{\mathbb{P}}_{[0,s]}^N\right)\right|&=\frac{1}{4}\int_s^t\sum_{i\in\mathbb{T}_N^d}\mathbb{E}_{[0,T]}^N\left[\left(h^{i,N}\left(\underline{\theta}_u^N\right)-h\left(u,\frac{i}{N}\right)\right)^2\right]\mathrm{d}u\\
&\leq\int_s^t\sum_{i\in\mathbb{T}_N^d}\mathbb{E}_{[0,T]}^N\left[\left(h^{i,N}\left(\underline{\theta}_u^N\right)\right)^2\right]\mathrm{d}u+(t-s)N^d|h(.,.)|_\infty^2 \qquad \text{(I.2.27)}\\
&\leq\left(\|J\|_{L^1}+\delta\right)^2\int_s^t\sum_{i\in\mathbb{T}_N^d}\mathbb{E}_{[0,T]}^N\left[\left(\theta_u^{i,N}\right)^2\right]\mathrm{d}u+(t-s)N^d|h(.,.)|_\infty^2\leq CN^d(t-s),
\end{aligned}$$

by (I.2.21), Lemma I.2.11 and the Assumption I.0.1 for a $\delta > 0$. Therefore, we can apply the Gronwall inequality to (I.2.26) and we get the claimed inequality (I.2.13) of Lemma I.2.9.

I.2.2.3 Proof of Lemma I.2.10

The Lemma I.2.10 follows by the dominated convergence theorem, as soon as we know that its integrand vanishes pointwise and that it is dominated by a constant. Note that the integrand is for each $N \in \mathbb{N}$ measurable by the continuity of ρ_t in t. We derive these properties in the following two lemmas. In the first lemma we show that the logarithmic moment generating function of δX_t under $\widehat{\mathbb{P}}^N$ vanishes.

Lemma I.2.14. *There exists a $\delta > 0$ such that for each $t \in (0, T]$*

$$\lim_{N\to\infty} \frac{1}{N^d} \log \widehat{\mathbb{E}}^N_{[0,t]} \left[e^{\delta N^d X^N_t} \right] = 0. \tag{I.2.28}$$

In the following lemma we show that the logarithmic moment generating function is uniformly bounded.

Lemma I.2.15. *There is a constant $C > 0$ such that*

$$0 \leq \frac{1}{N^d} \log \widehat{\mathbb{E}}^N_{[0,t]} \left[e^{\delta N^d X^N_t} \right] \leq C. \tag{I.2.29}$$

To prove these two lemmas, we derive at first some properties of the random variables $\eta^{i,N}_t$ (in Section I.2.2.3.1). Then in Section I.2.2.3.2, we infer from these properties the two lemmas.

I.2.2.3.1 Properties of the random variables η^x_t

In this section we state some properties of the random variables η^x_t defined in (I.2.10). This includes uniform lower bound on the variance (Lemma I.2.17) and a uniform upper bound on the quadratic exponential moments of the random variables η^x_t defined in (I.2.10) (Lemma I.2.18). Moreover, we show a general upper bound on exponential moments for specific measures, like the laws of η^x_t. (Lemma I.2.19).

Remark I.2.16. *The random variables $\left\{\eta^{i,N}_t\right\}_{i\in\mathbb{T}^d_N}$ defined in* (I.2.11) *are under $\widehat{\mathbb{P}}^N_{[0,T]}$ independent and centred.*

We define the variance of η^x_t in the usual way by

$$\widehat{\mathrm{Var}}^x_{[0,T]} [\eta^x_t] := \int_{\mathbb{R}} \left(\theta - \int_{\mathbb{R}} \theta \rho_t (x, \theta) \, e^{-\Psi(\theta)} \mathrm{d}\theta \right)^2 \rho_t (x, \theta) \, e^{-\Psi(\theta)} \mathrm{d}\theta. \tag{I.2.30}$$

We use, as usually, the hat over the symbol for the variance to indicate that the underlying measure is defined by ρ. This variance is uniformly bounded from below by the following lemma.

Lemma I.2.17. *The variances of η^x_t under $\rho_t(x, \theta) \, e^{-\Psi(\theta)} \mathrm{d}\theta$ are uniformly (and independent of N) bounded from below, i.e. there is a constant $C > 0$ such that for all $x \in \mathbb{T}^d$*

$$\inf_{(t,x)\in[0,T]\times\mathbb{T}^d} \widehat{\mathrm{Var}}^x_{[0,T]} [\eta^x_t] > C. \tag{I.2.31}$$

Proof. By Lemma I.2.12 we know that there is a constant $C_E > 0$ such that for all $x \in \mathbb{T}^d$

$$\left| \int_{\mathbb{R}} \theta \rho_t (x, \theta) \, e^{-\Psi(\theta)} \mathrm{d}\theta \right| \leq C_E. \tag{I.2.32}$$

Moreover, we get by Assumption I.0.4 b.) that

$$\inf_{x\in\mathbb{T}^d} \widehat{\mathrm{Var}}^x_{[0,T]}\left[\eta^x_0\right] = \inf_{x\in\mathbb{T}^d} \int_{\mathbb{R}} \left(\theta - \int_{\mathbb{R}} \theta\rho^0\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta\right)^2 \rho^0\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \geq C_0 > 0. \qquad \text{(I.2.33)}$$

The function $[0,T]\times\mathbb{T}^d \ni (t,x) \to \widehat{\mathrm{Var}}^x_{[0,T]}\left[\eta^x_t\right]$ is continuous by Definition I.2.2 (i) and Lemma I.2.12. This implies that for all $\epsilon > 0$, there is a $\delta > 0$, such that for all $0 \leq t \leq \delta$

$$\widehat{\mathrm{Var}}^x_{[0,T]}\left[\eta^x_t\right] \geq C_0 - \epsilon, \qquad \text{(I.2.34)}$$

by the continuity in time for finitely many $x \in \mathbb{T}^d$ and by the compactness of $\mathbb{T}^d$ and the continuity in x for all $x \in \mathbb{T}^d$. For $t \in [\delta, T]$ we know by Lemma I.2.12 that

$$\begin{aligned}\widehat{\mathrm{Var}}^x_{[0,T]}\left[\eta^x_t\right] &\geq \int_{\mathbb{R}} \left(\theta - \int_{\mathbb{R}} \theta\rho_t\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta\right)^2 e^{-\Psi(\theta)}\rho_t\left(x,\theta\right)\mathrm{d}\theta \\ &\geq \int_{C_E}^{\infty} \left(\theta - C_E\right)^2 e^{-\Psi(\theta)}\rho_t\left(x,\theta\right)\mathrm{d}\theta \geq \int_{C_E}^{\infty} \left(\theta - C_E\right)^2 e^{-V(\theta)}\mathrm{d}\theta \geq C > 0.\end{aligned} \qquad \text{(I.2.35)}$$

Hence

$$\inf_{t,i\in[0,T]\times\mathbb{T}^d_N} \widehat{\mathrm{Var}}^x_{[0,T]}\left[\eta^x_t\right] \geq \min\left\{C, C_0 - \epsilon\right\} > 0, \qquad \text{(I.2.36)}$$

the claimed uniform lower bound. □

In the next lemma we state an upper bound on the quadratic exponential moments of η^x_t.

Lemma I.2.18. *The quadratic exponential moments of η^x_t under $\rho_t\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta$ are uniformly (in $x \in \mathbb{T}^d$, $t \in [0,T]$) bounded, i.e. there is are constants $C_{qe} > 0$ and $\kappa > 0$ such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{T}^d} \int_{\mathbb{R}} e^{\kappa(\eta^x_t)^2}\rho_t\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \leq C_{qe}. \qquad \text{(I.2.37)}$$

Proof. By Lemma I.2.12 and (I.2.32) we have independent of $t \in [0,T]$ and $i \in \mathbb{T}^d_N$

$$\int_{\mathbb{R}} e^{\kappa(\eta^x_t)^2}\rho_t\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \leq Ce^{\kappa 2C_E^2}\int_{\mathbb{R}} e^{2\kappa\theta^2}\rho_t\left(x,\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \leq C_{qe}, \qquad \text{(I.2.38)}$$

because κ can be chosen arbitrary small. □

In the following lemma we infer an upper bound on the exponential moments (of a random variable multiplied by a constant). The upper bound depends quadratic on the constant. Note that the laws of η^x_t satisfy the condition (I.2.39) of this lemma by Lemma I.2.18.

Lemma I.2.19. *Take a $\pi \in \mathbb{M}_1(\mathbb{R})$ with $\int_{\mathbb{R}} \theta\pi\left(\mathrm{d}\theta\right) = 0$. Assume that there are two constants $C_{ex} > 1$ and $\kappa > 0$ such that*

$$\int_{\mathbb{R}} \max\left[1,\theta^2\right]e^{\kappa|\theta|^2}\pi\left(\mathrm{d}\theta\right) \leq C_{ex}. \qquad \text{(I.2.39)}$$

Then there is a constant $C = C\left(C_{ex},\kappa\right) > 0$, that depends only on C_{ex} and κ, such that for all $\gamma \geq 0$

$$\int e^{\gamma\theta}\pi\left(\mathrm{d}\theta\right) \leq e^{C\gamma^2}. \qquad \text{(I.2.40)}$$

Proof. We divide the analysis into large and small γ:

- $\gamma \geq \left(\kappa 4\log\left(C_{ex}\right)\right)^{\frac{1}{2}} = \gamma^*$ (**γ large**)**:** Here we have

$$\int_{\mathbb{R}} e^{\gamma\theta}\pi\left(\mathrm{d}\theta\right) \leq e^{\frac{1}{4}\frac{1}{\kappa}\gamma^2}\int_{\mathbb{R}} e^{\kappa\theta^2}\pi\left(\mathrm{d}\theta\right) \leq e^{\frac{1}{4}\frac{1}{\kappa}\gamma^2}e^{\log(C_{ex})} \leq e^{\gamma^2\frac{1}{2\kappa}}. \qquad \text{(I.2.41)}$$

- $\gamma \leq \gamma^*$ (**γ small):** By the Taylor expansion of $h(\gamma) = e^{\gamma\theta}$ around $\gamma = 0$, we get

$$\begin{aligned} e^{\gamma\theta} &= 1 + \gamma\theta + \frac{1}{2}\gamma^2\theta^2 e^{s(\theta)\theta} \leq 1 + \gamma\theta + \frac{1}{2}\gamma^2\theta^2 e^{\gamma|\theta|} \\ &\leq 1 + \gamma\theta + \frac{1}{2}\gamma^2 e^{\frac{1}{4\kappa}\gamma^2}\theta^2 e^{\kappa|\theta|^2}, \end{aligned} \tag{I.2.42}$$

where $s(\theta) \in (0, \gamma)$. By assumption $\int \theta\pi(\mathrm{d}\theta) = 0$. This implies

$$\int_{\mathbb{R}} e^{\gamma\theta}\pi(\mathrm{d}\theta) \leq 1 + \frac{1}{2}\gamma^2 e^{\frac{1}{4\kappa}\gamma^2} C_{ex} \leq 1 + \frac{1}{2}\gamma^2 e^{\frac{1}{2}(\gamma^*)^2} C_{ex} \leq 1 + \frac{C_{ex}^2}{2}\gamma^2 \leq e^{C\gamma^2}. \tag{I.2.43}$$

□

I.2.2.3.2 Proof of Lemma I.2.14 and Lemma I.2.15

Proof of Lemma I.2.14. The proof of this lemma is organised as follows. We study at first (Step 1) a large deviation principles for the random variables $\eta_t^{i,N}$. To show these large deviation principles we need some properties of $\eta_t^{i,N}$, more precisely Lemma I.2.17, Lemma I.2.18 and Lemma I.2.19. Then in Step 2 we apply an extended version of Varadhan's Lemma. Last but not least we conclude from this in Step 3 the claim of Lemma I.2.14.

Step 1: LDP for $\xi_t^N := \frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d} \eta_t^{i,N}\delta_i$ on $\mathbb{M}(\mathbb{T}^d)$:

Similar as for the i.i.d. random variables in Section III.2.4, we get that the $\{\xi^N\}$ satisfy for each $t \in [0, T]$ a large deviation principle with rate function

$$L_t(\nu) = \begin{cases} \int_{\mathbb{T}^d} I_t(x, m(x))\,\mathrm{d}x, & \text{if } \nu = m(x)\,\mathrm{d}x, \\ \infty, & \text{otherwise,} \end{cases} \tag{I.2.44}$$

where for $x \in \mathbb{T}^d$ and $u \in \mathbb{R}$

$$I_t(x, u) := \sup_{z\in\mathbb{R}} \{uz - h(x, z)\}, \text{ with } h_t(x, z) := \log \int_{\mathbb{R}} e^{z\eta}\lambda_t^x(\mathrm{d}\eta). \tag{I.2.45}$$

For each $x \in \mathbb{T}^d$ and $t \in [0, T]$, λ_t^x is a probability measure on $\mathbb{R}$ defined by

$$\lambda_t^x(\mathrm{d}\eta) := \rho_t\left(x, \eta + \int_{\mathbb{R}} \theta\rho_t(x, \theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\right) e^{-\Psi\left(\eta + \int_{\mathbb{R}}\theta\rho_t(x,\theta)e^{-\Psi(\theta)}\mathrm{d}\theta\right)}\mathrm{d}\eta. \tag{I.2.46}$$

For $x = \frac{i}{N}$, λ_t^x is the law of $\eta_t^{i,N}$, because $\widehat{\theta}_t^{i,N}$ has under $\widehat{\mathbb{P}}_{[0,T]}^N$ the law $\rho_t(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta$ (Definition I.2.2 (iii)).

Let us state some properties of the functions appearing in the rate function L_t. By Lemma I.2.17, $\inf_{x\in\mathbb{T}^d} h_t(x, 0)'' \geq C > 0$. Moreover, we get the following bounds on h and I:

Step 1.1: Upper Bound on h: By Lemma I.2.18 the condition of Lemma I.2.19 are satisfied for $\eta \sim \lambda_t^x$ for all $x \in \mathbb{T}^d$. Hence, there is a constant $C > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{T}^d$ and $z \in \mathbb{R}$

$$h_t(x, z) \leq \log\left(e^{Cz^2}\right) = Cz^2. \tag{I.2.47}$$

Step 1.2: Lower Bound on I: From the upper bound on h we infer for each $(t, x) \in [0, T] \times \mathbb{T}^d$ and $u \in \mathbb{R}$

$$I_t(x, u) \geq \sup_{z\in\mathbb{R}} \{uz - Cz^2\} \geq u^2\frac{1}{4C}, \tag{I.2.48}$$

by choosing $z = \frac{u}{2C}$ for the last inequality.

Step 2: Varadhan's lemma:
With the generalisation of Varadhan's lemma given in Theorem C.1.1 in Appendix C (see also Remark C.1.6), we get similar as in Section III.3.1

$$\lim_{N\to\infty} \frac{1}{N^d} \log \widehat{\mathbb{E}}^N_{[0,t]} \left[e^{\delta N^d X^N_t} \right] = \sup_{\mu\in\mathbb{M}(\mathbb{T}^d), L_t(\mu)<\infty} \left\{ \delta \left\langle K * \mu, \mu \right\rangle - L_t(\mu) \right\}, \tag{I.2.49}$$

with $K(x,x') := \int_{\mathbb{T}^d} J(x-y) J(x'-y) \, \mathrm{d}y$. Remember that $\left\langle K^N * \xi^N, \xi^N \right\rangle = X^N_t$.

Step 3: The right hand side of (I.2.49) **is** 0**:**
Step 3.1: RHS ≥ 0**:** We know that $I_t(0) = 0$, because $h_t(z) \geq 0$ for all $z \in \mathbb{R}$ (by Jensen and because η is centred) and $h_t(0) = 0$. Therefore, $L_t(0\mathrm{d}x) = 0$ and $0\mathrm{d}x \in \mathbb{M}(\mathbb{T}^d)$.
Step 3.2: RHS ≤ 0**:** We only have to consider μ such that $L_t(\mu) < \infty$. In particular it is enough to consider $\mu \in \mathbb{M}(\mathbb{T}^d)$ that have a density with respect to the Lebesgue measure. By (I.2.48) we get for $\mu = m(x)\,\mathrm{d}x \in \mathbb{M}(\mathbb{T}^d)$

$$\delta \left\langle K * m, m \right\rangle - L_t(m) \leq \left(\delta \|J\|^2_{L^1} - \frac{1}{4C} \right) \int_{\mathbb{T}^d} m(x)^2 \, \mathrm{d}x \leq 0, \tag{I.2.50}$$

for $\delta < \left(4C \|J\|^2_{L^1} \right)^{-1}$.
Hence, we have shown that the right hand side of (I.2.49) is zero, i.e. the claimed convergence of Lemma I.2.14. □

Proof of Lemma I.2.15. For all $t \in [0,T]$ and N large enough, there is a $\epsilon > 0$, such that the following uniform upper bound holds

$$\frac{1}{N^d} \log \widehat{\mathbb{E}}^N_{[0,t]} \left[e^{\delta N^d X^N_t} \right] \leq \frac{1}{N^d} \sum_{i\in\mathbb{T}^d_N} \log \widehat{\mathbb{E}}^N_{[0,t]} \left[e^{\delta \left(\|J\|_{L^1} + \epsilon \right)^2 \left(\eta^{i,N}_t \right)^2} \right] \leq \log C_{qe}, \tag{I.2.51}$$

by Lemma I.2.18 and the definition of X^N_t in (I.2.11). Moreover, we have the lower bound, by the Jensen inequality

$$\frac{1}{N^d} \log \widehat{\mathbb{E}}^N_{[0,t]} \left[e^{\delta N^d X^N_t} \right] \geq \widehat{\mathbb{E}}^N_{[0,t]} \left[\delta X^N_t \right] \geq 0, \tag{I.2.52}$$

because by the definition of X^N_t

$$X^N_t = \frac{1}{N^d} \sum_{\ell\in\mathbb{T}^d_N} \left(\frac{1}{N^d} \sum_{i\in\mathbb{T}^d_N} J\left(\frac{i-\ell}{N} \right) \eta^{i,N}_t \right)^2 \geq 0. \tag{I.2.53}$$

□

I.2.3 Proof of Theorem I.2.3: Convergence of the empirical process

In this section we prove the hydrodynamic limit Theorem I.2.3.

Notation I.2.20. *By an abuse of the notation we denote by $\mathbb{P}^N_{[0,T]}$ in this section also the distribution of the empirical process $\mu^N_{[0,T]}$, i.e. for a measurable set $A \subset \mathsf{C}([0,T], \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}))$*

$$\mathbb{P}^N_{[0,T]}[A] = \mathbb{P}^N_{[0,T]} \left[\underline{\theta}^N : \mu^N_{[0,T]} \in A \right]. \tag{I.2.54}$$

The claim of Theorem I.2.3 follows from the following two theorems

The first theorem states the relative weak compactness of the family $\left\{ \mathbb{P}^N_{[0,T]} \right\}$ (by Prokhorov's theorem and $\mathsf{C}([0,T], \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}))$ being metrisable).

Theorem I.2.21. *The family* $\left\{\mathbb{P}^N_{[0,T]}\right\} \subset \mathbb{M}_1\big(\mathsf{C}\big([0,T]\,,\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)\big)$ *is tight.*

By the next theorem, each limit of a weakly convergent subsequence of $\mathbb{P}^N_{[0,T]}$ is concentrated on the continuous probability density ρ of (I.0.3) that we fixed in Theorem I.2.3.

Theorem I.2.22. *Let* $\mathbb{P}_{[0,T]} \in \mathbb{M}_1\left(\mathsf{C}\big([0,T]\,,\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)\right)$ *be a limit of an arbitrary weakly convergent subsequence* $\mathbb{P}^{N_k}_{[0,T]}$*. Then*

$$\mathbb{P}_{[0,T]} = \delta_{\left\{t\to\rho_t(x,\theta)e^{-\Psi(\theta)}\mathrm{d}x\mathrm{d}\theta\right\}}. \tag{I.2.55}$$

We prove Theorem I.2.21 in Section I.2.3.2 and Theorem I.2.22 in Section I.2.3.1. Let us give the main idea of both proofs. For Theorem I.2.21, we transfer at first a sufficient condition of [Gär88] for the sequential weak compactness to the setting we consider here. Finally, we show that this condition is satisfied. For Theorem I.2.22 we use a suitable large deviation principle in combination with the order of the relative entropy shown in Theorem I.2.8. At the beginning of the two sections we state the ideas of the proofs in more details.

Remark I.2.23. *To prove Theorem I.2.3 we could also use the dynamical large deviation principle for the empirical process when the spins evolve according to the SDE* (I.2.1) *with fixed effective field. Hence, we could consider random variables on* $\mathsf{C}\big([0,T]\,,\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$*. We show this large deviation result in Chapter V. That approach would have the advantage, that we do not need Theorem I.2.21, i.e. the sequential weak compactness of* $\mathbb{P}^N_{[0,T]}$*. In the approach we use here, we need the sequential weak compactness. Indeed, from the large deviation principle for single time points, we infer only the convergence of finite dimensional (time) marginals (see the proof of Theorem I.2.22). Therefore, we use the sequential weak compactness to get the convergence result on the whole time interval* $[0,T]$*. By using directly the dynamical, pathwise large deviation principle, we would not need to look at finite dimensional distributions. We could infer directly from the dynamical large deviation principle the convergence of the empirical process. This could be done in a similar way as in the proof of Theorem I.2.22 where we infer the convergence of the finite dimensional distributions (see Section I.2.3.1). More precisely, as in Lemma I.2.25, we would get the weak convergence of the process* $\mu^N_{[0,T]}$ *to* $\rho_{[0,T]}e^{-\Psi}$ *under* $\widehat{\mathbb{P}}^N_{[0,T]}$*. We would use that the minima of the rate function is attained at weak solution of* (I.1.9) *with fixed effective field corresponding to* ρ*. But these weak solutions are unique by Lemma I.1.23 (i). Finally, similar to Theorem I.2.26, we would get the claimed weak convergence under* $\mathbb{P}^N_{[0,T]}$*. However, we want to avoid at this stage the more complicated and more technical proof of the dynamical large deviation principle. Therefore, we use the simpler large deviation principle at single time points.*

I.2.3.1 Uniqueness of the limit points (Proof of Theorem I.2.22)

In this section we prove Theorem I.2.22, i.e. we show that the limit point of each converging subsequence of $\mathbb{P}^N_{[0,T]}$ equals $\delta_{\{t\to\rho_t e^{-\Psi}\mathrm{d}x\mathrm{d}\theta\}}$. Thus we show that the limit points are unique. The following proof of Theorem I.2.22 is structured as follows:

1.) At first, we show the convergence of the empirical measures μ^N_t with law $\widehat{\mathbb{P}}^N_{[0,T]}$ to $\rho_t e^{-\Psi}$ for fixed time $t\in[0,T]$. Moreover, we prove that we are only with exponential small (in N) probability far away from ρ_t (Lemma I.2.25).

2.) We infer from this in Theorem I.2.26 different convergence results of μ^N_t to ρ_t for a fixed $t\in[0,T]$ now under $\mathbb{P}^N_{[0,T]}$. In this step we use that the relative entropy is of order $o\left(N^d\right)$ (Theorem I.2.8) in combination with a suitable inequality concerning the relative entropy.

3.) Finally, we conclude the uniqueness of the limit points by transferring the uniqueness for each $t\in[0,T]$ to the uniqueness on the whole time interval $[0,T]$

Notation I.2.24. *We denote the Levy-Prokhorov metric on* $\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)$ *by* β_{P} .

We state in the following lemma the convergence and the speed of convergence of the random variables μ_t^N to $\rho_t e^{-\Psi}$ under the measure $\widehat{\mathbb{P}}^N_{[0,T]}$. In its proof we use the large deviation principle of these variables and the uniqueness of the minimiser of the rate function.

Lemma I.2.25. *For each* $t\in[0,T]$

$$\mu_t^N \xrightarrow{p} \rho_t e^{-\Psi}\mathrm{d}x\mathrm{d}\theta, \qquad \text{when} \quad N\to\infty, \tag{I.2.56}$$

in probability under $\widehat{\mathbb{P}}^N_{[0,T]}$. *Moreover, the convergence is exponentially fast, i.e. for each* $\delta>0$ *there is a constant* $c(\delta)>0$ *such that for all* $N\in\mathbb{N}$

$$\widehat{\mathbb{P}}^N_{[0,T]}\left[\mu_t^N\notin B_\delta\left(\rho_t e^{-\Psi}\right)\right]\le e^{-N^d(c(\delta)+o(1))}, \tag{I.2.57}$$

where $B_\delta(\nu):=\{\mu:\beta_{\mathrm{P}}(\mu,\nu)<\delta\}\subset\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)$.

Proof. For fixed $t\in[0,T]$, the family $\{\mu_t^N\}\subset\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)$ satisfies under $\Big\{\widehat{\mathbb{P}}^N_{[0,T]}\Big\}$ a large deviation principle with good rate function (by the properties of ρ_t of Definition I.2.2)

$$\mathcal{H}_{\rho_t}(\mu):=\int_{\mathbb{T}^d}\mathsf{H}\left(\xi(x,.)\middle|\rho_t(x,.)\,e^{-\Psi}\right)\mathrm{d}x \quad \text{when } \mu=\xi(x,\theta)\,\mathrm{d}x\mathrm{d}\theta. \tag{I.2.58}$$

By the properties of $\mathsf{H}(.|.)$, we know that $\mathcal{H}_{\rho_t}$ is only zero when $m(x,.)=\rho_t(x,.)\,e^{-\Psi(\cdot)}$ for almost all x. Hence, the rate function has a unique minimiser in $\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)$. This implies that for each $\delta>0$

$$\widehat{\mathbb{P}}^N_{[0,T]}\left[\mu_t^N\notin B_\delta\left(\rho_t e^{-\Psi}\right)\right]\le e^{-N^d\inf_{\mu\notin B_\delta(\rho_t e^{-\Psi})}\mathcal{H}_{\rho_t}(\mu)+N^d o(1)}\le e^{-N^d(c(\delta)+o(1))}, \tag{I.2.59}$$

because B_δ is open and

$$\inf_{\mu\notin B_\delta(\rho_t)}\mathcal{H}_{\rho_t}(\mu)\ge c(\delta)>0. \tag{I.2.60}$$

Indeed, assume (I.2.60) were not true. Then we would find a sequence $\mu_n\in B_\delta^c$ such that $\mathcal{H}_{\rho_t}(\mu_n)\to 0$. Hence, there is a $\alpha>0$ such that for all $n\in\mathbb{N}$, $\mu_n\in\mathcal{L}^{\le\alpha}(\mathcal{H}_{\rho_t}):=\{\nu:\mathcal{H}_{\rho_t}(\nu)\le\alpha\}$. Moreover, $\mathcal{L}^{\le\alpha}(\mathcal{H}_{\rho_t})$ is compact because $\mathcal{H}_{\rho_t}$ is a good rate function. Therefore, there is a $\mu\in\mathcal{L}^{\le\alpha}(\mathcal{H}_{\rho_t})\cap B_\delta^c$ such that $\mu_{n_k}\to\mu$ and hence $\mathcal{H}_{\rho_t}(\mu)=0$. But this is a contradiction.

We have hence shown (I.2.57), which implies the convergence of μ_t^N. □

In the following theorem we show different kinds of convergence of the empirical measures μ_t^N to $\rho_t e^{-\Psi}$. Note that all the results are for fixed $t\in[0,T]$. The main idea of its proof is to apply the relative entropy result of Theorem I.2.8 in combination with a relative entropy inequality and the (exponential) convergence result for $\widehat{\mathbb{P}}^N_{[0,T]}$ of Lemma I.2.25.

Theorem I.2.26. (i) *For each* $t\in[0,T]$, $\mu_t^N\xrightarrow{p}\rho_t e^{-\Psi}$ *in probability under* $\mathbb{P}^N_{[0,T]}$, *i.e. for each* $\delta>0$

$$\mathbb{P}^N_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu_t^N,\rho_t e^{-\Psi}\right)>\delta\right]\to 0, \qquad \text{when} \quad N\to\infty. \tag{I.2.61}$$

(ii) *This implies for each* $t\in[0,T]$

$$\pi\left(t,\mathbb{P}^N_{[0,T]}\right)\xrightarrow{w*}\delta_{\rho_t e^{-\Psi}}, \qquad \text{when} \quad N\to\infty, \tag{I.2.62}$$

where $\pi\left(t,\mathbb{P}^N_{[0,T]}\right)\in\mathbb{M}_1\big(\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$ *is the one-dimensional distribution at time* t *of* $\mathbb{P}^N_{[0,T]}$, *i.e. the law of* μ_t^N.

(iii) *Moreover, for each $f \in \mathsf{C}_b\left(\mathbb{T}^d \times \mathbb{R}\right)$ that is Lipschitz continuous and each $t \in [0,T]$,*

$$\lim_{N\to\infty} \mathbb{E}^N_{[0,T]}\left[\left|\left\langle f,\mu^N_t\right\rangle - \left\langle f,\rho_t e^{-\Psi}\right\rangle\right|\right] = 0. \tag{I.2.63}$$

Proof of Theorem I.2.26. ***(i)*** To prove the convergence claimed here, we use inequality (2.18) in [Yau91] (see also [GPV88] proof of Lemma 6.1) and we get

$$\begin{aligned}
\mathbb{P}^N_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu^N_t,\rho_t e^{-\Psi}\right) > \delta\right] &\le \frac{\log(2) + \mathsf{H}\left(\mathbb{P}^N_{[0,T]}\middle|\widehat{\mathbb{P}}^N_{[0,T]}\right)}{\log\left(1 + \left(\widehat{\mathbb{P}}^N\left[\beta_{\mathrm{P}}\left(\mu^N_t,\rho_t e^{-\Psi}\right) > \delta\right]\right)^{-1}\right)} \\
&\le \frac{\log(2) + \mathsf{H}\left(\mathbb{P}^N_{[0,T]}\middle|\widehat{\mathbb{P}}^N_{[0,T]}\right)}{\log\left(1 + e^{N^d(c(\delta)+o(1))}\right)} \le \frac{\log(2) + \mathsf{H}\left(\mathbb{P}^N_{[0,T]}\middle|\widehat{\mathbb{P}}^N_{[0,T]}\right)}{N^d\left(c(\delta) + o(1)\right)} \to 0,
\end{aligned} \tag{I.2.64}$$

by Theorem I.2.8. Here we used the exponentially fast convergence of $\widehat{\mathbb{P}}^N_{[0,T]}$ shown in (I.2.57) in Lemma I.2.25.

(ii) To show the weak convergence we choose an arbitrary $F \in \mathsf{C}\left(\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$ that is Lipschitz continuous with constant L_F (it is enough to consider these F by [Dud02] Theorem 11.3.3). Then

$$\begin{aligned}
\mathbb{E}^N_{[0,T]}\left[\left|F\left(\mu^N_t\right) - F\left(\rho_t e^{-\Psi}\right)\right|\right] &\le L_F \mathbb{E}^N_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu^N_t,\rho_t e^{-\Psi}\right)\right] \\
&\le L_F \mathbb{P}^N_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu^N_t,\rho_t e^{-\Psi}\right) > \epsilon\right] + L_F\epsilon \le C\epsilon,
\end{aligned} \tag{I.2.65}$$

for all $N > N(\epsilon)$.

(iii) Let $f \in \mathsf{C}\left(\mathbb{T}^d \times \mathbb{R}\right)$ be a Lipschitz continuous function with Lipschitz constant L_f. We can assume that $|f|_\infty + L_f \le 1$. Then for δ small enough and N large enough

$$\begin{aligned}
\mathbb{E}^N_{[0,T]}\left[\left|\left\langle f,\mu^N_t\right\rangle - \left\langle f,\rho_t e^{-\Psi}\right\rangle\right|\right] &\le \mathbb{E}^N_{[0,T]}\left[\beta_{\mathrm{LIP}}\left(\mu^N_t,\rho_t e^{-\Psi}\right)\right] \le 2\mathbb{E}^N_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu^N_t,\rho_t e^{-\Psi}\right)\right] \\
&\le 2\mathbb{P}^N_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu^N_t,\rho_t e^{-\Psi}\right) > \delta\right] + 2\delta \le \epsilon,
\end{aligned} \tag{I.2.66}$$

by (i) and [Bog07] Theorem 8.10.43 (for the inequalities of the metrics). Here $\beta_{\mathrm{LIP}}(.,.)$ is the bounded Lipschitz distance (see [Dud02] page 394 or [Bog07] Section 8.3)

□

Remark I.2.27. *As usually, the convergence of the one dimensional distributions of $\mathbb{P}^N_{[0,T]}$, shown in Theorem I.2.26 (ii), (or the convergence of the finite dimensional distributions) does not imply the convergence of the measures $\mathbb{P}^N_{[0,T]}$.*

Proof of Theorem I.2.22. Fix an arbitrary convergence sequence $\left\{\mathbb{P}^{N_k}_{[0,T]}\right\}_k$ with limit $\mathbb{P}_{[0,T]} \in \mathbb{M}_1\left(\mathsf{C}\left([0,T],\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)\right)$. For each $\delta > 0$ and each $t \in [0,T]$ we know by the Portmanteau theorem and Theorem I.2.26 that

$$\mathbb{P}_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu_t,\rho_t e^{-\Psi}\right) \le \delta\right] \ge \limsup_{k\to\infty} \mathbb{P}^{N_k}_{[0,T]}\left[\beta_{\mathrm{P}}\left(\mu^N_t,\rho_t e^{-\Psi}\right) \le \delta\right] = 1, \tag{I.2.67}$$

because the set $\left\{\mu \in \mathsf{C}\left([0,T],\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right) : \beta_{\mathrm{P}}\left(\mu_t,\rho_t e^{-\Psi}\right) \le \delta\right\}$ is closed. Hence, $\mathbb{P}_{[0,T]}$-a.s.,$\mu_t = \rho_t e^{\Psi}\mathrm{d}x\mathrm{d}\theta$ for each $t \in [0,T]$. Finally apply Fubini's theorem

$$T = \int_0^T \mathbb{E}_{[0,T]}\left[\mathbb{1}_{\mu_t=\rho_t e^{\Psi}}\right]\mathrm{d}t = \mathbb{E}_{[0,T]}\left[\int_0^T \mathbb{1}_{\mu_t=\rho_t e^{\Psi}}\mathrm{d}t\right]. \tag{I.2.68}$$

This implies Theorem I.2.22. □

I.2.3.2 Tightness of $\left\{\mathbb{P}^N_{[0,T]}\right\}$ (Proof of Theorem I.2.21)

In this section we prove Theorem I.2.21, i.e. we show that the family $\left\{\mathbb{P}^N_{[0,T]}\right\}$ is sequentially weak compact as measures on $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$. The proof is organised as follows:

1.) In Section I.2.3.2.1 we state a sufficient condition for measures on $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$ being tight (Lemma I.2.28). This requires a general characterisation of relative compact subsets of $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$ (Lemma I.2.30).

2.) Finally, we show in Section I.2.3.2.1 that this condition is satisfied by the measures $\left\{\mathbb{P}^N_{[0,T]}\right\}$ (Lemma I.2.31).

This implies the claimed relative compactness of the measures $\left\{\mathbb{P}^N_{[0,T]}\right\}$ as claimed in Theorem I.2.21.

I.2.3.2.1 A sufficient condition for tightness

In this section we state a sufficient condition for tightness of measures on $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$. This condition was shown in [Gär88] in Lemma 1.4 for a subspace of $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$ with a different topology. To state the condition, we define the exit times from the sets

$$\mathbb{M}_R := \left\{\nu\in\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big) : \left\langle \nu,\theta^2\right\rangle \le R\right\}, \tag{I.2.69}$$

by

$$\tau_R\left(\mu_{[0,T]}\right) := \inf\left\{t\in[0,T] : \mu_t\notin\mathbb{M}_R\right\}. \tag{I.2.70}$$

Lemma I.2.28. *Let $\{Q_N\}$ be a sequence of probability measures on $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$. For $f\in\mathsf{C}^\infty_c\big(\mathbb{T}^d\times\mathbb{R}\big)$, and a $R>0$, we denote by $Q^{f,R}_N$ the sequence of measures obtained by projecting Q_N with f and stopped at τ_R, i.e. for a measurable set $A\subset\mathsf{C}([0,T],\mathbb{R})$*

$$Q^{f,R}_N[A] = Q_N\left[\left\{\mu\in\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big) : \left\langle \mu(.\wedge\tau_R\left(\mu_{[0,T]}\right),f\right\rangle\in A\right\}\right]. \tag{I.2.71}$$

If

(i) $\lim_{R\to\infty}\sup_N Q_N\left[\tau_R\le T\right]=0$ *and*

(ii) *for each $f\in\mathsf{C}^\infty_c\big(\mathbb{T}^d\times\mathbb{R}\big)$ and $R>0$ the sequence $\left\{Q^{f,R}_N\right\}$ is tight,*

then $\{Q_N\}$ is tight in $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$.

Remark I.2.29. *In [KL99] Proposition 4.1.7, there is a similar result for càdlàg processes on the space of finite positive measures over $\mathbb{T}^d$.*

In the proof of this lemma the following characterisation of the relative compact subsets of $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$ is used.

Lemma I.2.30 (Lemma 1.3 in [Gär88])**.** *Let $\{f_n\}_n$ be a countable dense subset of $\mathsf{C}_c\big(\mathbb{T}^d\times\mathbb{R}\big)$. A set $\mathcal{K}$ is relatively compact in $\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big)$ if and only if*

$$\mathcal{K}\subset\mathcal{K}_K\cap\bigcap\mathcal{K}_n, \tag{I.2.72}$$

with

$$\mathcal{K}_K = \left\{\mu_{[0,T]}\in\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big) : \mu_t\in K \text{ for all } t\in[0,T]\right\}, \tag{I.2.73}$$

$$\mathcal{K}_n = \left\{\mu_{[0,T]}\in\mathsf{C}\big([0,T],\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)\big) : \left\langle\mu_{[0,T]},f_n\right\rangle\in K_n\right\}, \tag{I.2.74}$$

with $K\subset\mathbb{M}_1\big(\mathbb{T}^d\times\mathbb{R}\big)$ compact and $K_n\subset\mathsf{C}([0,T],\mathbb{R})$ compact.

The proof of this lemma is given in [Gär88] for general domains $D \subset \mathbb{R}^n$. Therefore, it includes the case, $D = \mathbb{T}^d \times \mathbb{R}$, we are interested in.

Proof of Lemma I.2.28. In [Gär88] Lemma 1.4 this result is shown for measures on a subset of $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$ with a different topology. On the whole space $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$, the proof in [Gär88] shortens to the following arguments, that we state here to have a complete exposition.

It suffices to show that for arbitrary ϵ, there is a compact subset $\overline{K}_\epsilon$ of $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$ which satisfies $\sup_N Q_N \left[\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right) \setminus \overline{K}_\epsilon\right] \leq \epsilon$. Lemma I.2.30 characterises the compact sets, such that we choose in the following suitable components of (I.2.72). Fixing $K = \mathbb{M}_R$ for R big enough, then condition (i) implies that

$$\sup_N Q_N \left[\mathcal{K}_K^c\right] \leq \epsilon \frac{1}{2}. \tag{I.2.75}$$

The set $\mathbb{M}_R$ is compact in $\mathbb{M}_1$. This follows with Prokhorov's theorem, because $\mathbb{M}_R$ is tight in $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$. Indeed, for all $\mu \in \mathbb{M}_R$

$$\mu\left[\left(\mathbb{T}^d \times \mathbb{R}\right) \setminus \left\{(x,\theta) : \theta^2 \leq c\right\}\right] = \mu\left[\left\{(x,\theta) : \theta^2 > c\right\}\right] \leq \frac{1}{c}\left\langle \mu, \theta^2 \right\rangle \leq \frac{R}{c} . \tag{I.2.76}$$

Therefore, we have a suitable set $\mathcal{K}_K$. Finally, we choose for each $n \in \mathbb{N}$ a set $\setminus$ in the following way: For each element f_n of a countable subset of $\mathsf{C}_c\left(\mathbb{T}^d \times \mathbb{R}\right)$, the set K_n (in (I.2.74)) is chosen such that

$$\inf_N Q_N^{f_n, R_0}\left[K_n\right] > 1 - \epsilon 2^{-(r+1)}, \tag{I.2.77}$$

which is possible by the assumption (ii) of this lemma. □

I.2.3.2.2 The sufficient condition is satisfied by $\left\{\mathbb{P}^N_{[0,T]}\right\}$

Lemma I.2.31. *The measures* $\left\{\mathbb{P}^N_{[0,T]}\right\}$ *satisfy the condition (i) and (ii) of Lemma I.2.28.*

Proof. We proof at first (ii) and then (i).
(ii) Let us fix a $f \in \mathsf{C}_c^\infty\left(\mathbb{T}^d \times \mathbb{R}\right)$ and a $R > 0$. To show the tightness of $\left\{\mathbb{P}^{N,f,R}_{[0,T]}\right\}$, we show the conditions of Theorem 7.3 in [Bil99]. Note first that

$$\mathbb{P}^N_{[0,T]}\left[\left|\left\langle \mu_0^N, f\right\rangle\right| \geq a\right] \leq \mathbb{P}^N_{[0,T]}\left[|f|_\infty \geq a\right] = 0, \tag{I.2.78}$$

when a is large enough. Moreover, for $\mu_t = \mu_t^N$, we have by the Itô formula

$$\begin{aligned}
&\left\langle \mu_t^N, f\right\rangle - \left\langle \mu_s^N, f\right\rangle \\
&= \frac{1}{N^d} \sum_{i \in \mathbb{T}_N^d} \left\{ \int_s^t \partial_\theta f\left(\frac{i}{N}, \theta_u^{i,N}\right) \left(-\Psi'\left(\theta_u^{i,N}\right) + \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} J\left(\frac{i-j}{N}\right) \theta_u^{j,N}\right) \mathrm{d}u \right. \\
&\left. + \frac{1}{2} \int_s^t \partial_\theta^2 f\left(\frac{i}{N}, \theta_u^{i,N}\right) \mathrm{d}u + \int_s^t \partial_\theta f\left(\frac{i}{N}, \theta_u^{i,N}\right) \mathrm{d}W_u^{i,N} \right\}.
\end{aligned} \tag{I.2.79}$$

By the Assumption I.0.1, there is a $\delta > 0$, such that for $t, s \in [0,T]$ and N large enough

$$\begin{aligned}
&\left|\left\langle \mu^N_{t\wedge\tau_R\left(\mu^N_{[0,T]}\right)}, f\right\rangle - \left\langle \mu^N_{s\wedge\tau_R\left(\mu^N_{[0,T]}\right)}, f\right\rangle\right| \\
&\leq (t-s)\left(\left|\partial_\theta f\right|_\infty \left(\sup_{\theta\in\operatorname{supp}(f)} \left|\Psi'(\theta)\right| + \left(\|J\|_{L^1} + \delta\right)\sup_{\theta\in\operatorname{supp}(f)}|\theta|\right) + \left|\partial_\theta^2 f\right|_\infty\right) \\
&+ \left|\partial_\theta f\right|_\infty \frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\left|W^{i,N}_t - W^{i,N}_s\right| \\
&\leq (t-s)\, C_{f,R} + C_{f,R}\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\left|W^{i,N}_t - W^{i,N}_s\right|,
\end{aligned} \tag{I.2.80}$$

where $C_{f,R} > 0$ is a suitable constant. Here we used that $\mu^N_{t\wedge\tau_R\left(\mu^N_{[0,T]}\right)} \in \mathbb{M}_R$ for all $t \in [0,T]$ and that $f \in \mathsf{C}^\infty_c\left(\mathbb{T}^d \times \mathbb{R}\right)$. With this, we get

$$\begin{aligned}
&\mathbb{P}^N_{[0,T]}\left[\sup_{t,s:|t-s|<\delta}\left|\left\langle \mu^N_{t\wedge\tau_R\left(\mu^N_{[0,T]}\right)}, f\right\rangle - \left\langle \mu^N_{s\wedge\tau_R\left(\mu^N_{[0,T]}\right)}, f\right\rangle\right| \geq \epsilon\right] \\
&\leq \mathbb{P}^N_{[0,T]}\left[\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\sup_{t,s:|t-s|<\delta}\left|W^{i,N}_t - W^{i,N}_s\right| \geq \epsilon - \delta\right] \\
&\leq \frac{1}{\epsilon-\delta}\mathbb{E}^N_{[0,T]}\left[\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\sup_{t,s:|t-s|<\delta}\left|W^{i,N}_t - W^{i,N}_s\right|\right] \\
&= \frac{1}{\epsilon-\delta}\mathbb{E}^N_{[0,T]}\left[\sup_{t<\delta}\left|W^{i,N}_t\right|\right] = \frac{1}{\epsilon-\delta}2\sqrt{\frac{2\delta}{\pi}}.
\end{aligned} \tag{I.2.81}$$

For δ small enough, Theorem 7.3 in [Bil99] implies the tightness of $P^{N,f,R}_{[0,T]}$ i.e. the validity of condition (ii).

(i) To prove the condition (i), we have to show that

$$\mathbb{P}^N_{[0,T]}\left[\tau_R \leq T\right] = \mathbb{P}^N_{[0,T]}\left[\sup_{t\in[0,T]}\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\left(\theta^{i,N}_t\right)^2 > R\right] \tag{I.2.82}$$

vanishes uniformly in N, when R tends to infinity. For each finite N, (I.2.82) vanishes by [SV79] Corollary 10.1.2 and $\mathbb{P}^N_{[0,T]}$ being the by solution to the martingale problem (see Notation I.2.6). Hence, we can consider in the rest of the proof that N is large. This has the advantage that for N large enough, there is a constant $C_L > 0$ such that

$$\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N} -\Psi'\left(\theta^{i,N}\right)\theta^{i,N} + \frac{1}{N^{2d}}\sum_{j\in\mathbb{T}^d_N} J\left(\frac{i-j}{N}\right)\theta^{i,N}\theta^{j,N} + 1 \leq C_L, \tag{I.2.83}$$

by Assumption I.0.1 and Assumption I.0.3. By (I.2.79) with $f(x,\theta) = \theta^2$, we get

$$0 \leq \frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\left(\theta^{i,N}_t\right)^2 \leq \frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\left(\theta^{i,N}_0\right)^2 + TC_L + M_t, \tag{I.2.84}$$

where M_t is a continuous local $\mathbb{P}^N_{[0,T]}$ martingale, with $M_0 = 0$. Then

$$S_t^R := \min\left\{R, \frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\left(\theta_0^{i,N}\right)^2 + TC_L + M_t\right\} \tag{I.2.85}$$

is a non-negative $\mathbb{P}^N_{[0,T]}$ supermartingale. Therefore, we can apply the Doob supermartingale inequality and we get

$$\begin{aligned}
&\mathbb{P}^N_{[0,T]}\left[\sup_{t\in[0,T]}\frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\left(\theta_t^{i,N}\right)^2 > R\right] \le \mathbb{P}^N_{[0,T]}\left[\sup_{t\in[0,T]} S_t^R > R\right] \le \frac{1}{R}\mathbb{E}^N_{[0,T]}\left[S_0^R\right] \\
&\le \frac{1}{R}\int_{\mathbb{R}^{N^d}}\min\left\{R, \frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\left(\theta_0^{i,N}\right)^2 + TC_L\right\} f_0^N\left(\mathrm{d}\underline{\theta}^N\right) \qquad \text{(I.2.86)} \\
&\le R^{-\frac{1}{2}} + f_0^N\left[\frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\left(\theta^{i,N}\right)^2 > R^{\frac{1}{2}} - TC_L\right].
\end{aligned}$$

The right hand side vanishes because by exponential Chebyshev inequality

$$f_0^N\left[\sum_{i\in\mathbb{T}_N^d}\left(\theta^{i,N}\right)^2 > N^d A\right] \le e^{-N^d A\kappa}\int e^{\kappa\sum_{i\in\mathbb{T}_N^d}\left(\theta^{i,N}\right)^2} f_0^N(\mathrm{d}\theta) \le e^{-N^d A\kappa}C^{N^d}, \tag{I.2.87}$$

by Assumption I.0.5 b.). Therefore, (I.2.82) vanishes uniformly in N, when R tends to infinity.

□

Remark I.2.32. (i) *Note that the method we use to prove condition (i) is the same as in the proof of Theorem 1.5 in [Gär88] for this condition. The proof in [Gär88] is more general (and is in particular also applicable for the model we consider here). However, it has the disadvantage that the proof uses the martingale problem and Itô formula on the space of the empirical processes $\mu^N_{[0,T]}$. That is the reason why we wrote it anyway down, because in the model we consider here, we can directly work with the process $\underline{\theta}^N$ without this detour.*

(ii) *We could also use proof of Theorem 1.5 in [Gär88] for the condition (ii). However, the proof we give here, by using Theorem 7.3 in [Bil99], is more straightforward and easier.*

I.3 Uniqueness of the classical solution of the hydrodynamic equation

In this section we give an alternative proof of the uniqueness of probability density valued classical solutions of the hydrodynamic equation (I.0.2), in the sense of Definition I.1.1. The hydrodynamic limit result also implies uniqueness of the probability density valued classical solutions of the PDE (I.0.2) (see Corollary I.2.4). In this proof we do not require the hydrodynamic limit result. Instead we use two energy estimates together with results that we derive in Section I.1.

Theorem I.3.1. *Let the same assumptions as in Theorem I.1.2 hold. Then there exists at most one probability density valued classical solution (in the sense of Definition I.1.1) of the PDE* (I.0.2) *with initial distribution ρ^0.*

Proof. Fix two probability density valued, classical solutions $\rho^{(1)}, \rho^{(2)}$ of (I.0.2). We define the two corresponding effective fields $h^{(1)}, h^{(2)}$, as in the second line of (I.0.2). Both effective fields are continuous by Definition I.1.1 (ii). Hence, $\rho^{(1)}$ respectively $\rho^{(2)}$ are the unique weak solutions of (I.1.8) with the corresponding given continuous effective field (by Lemma I.1.23 (i)). This implies in particular that $\rho^{(1)}$ and $\rho^{(2)}$ are bounded by Lemma I.1.23 (ii). Using this bound and $\rho^{(1)}, \rho^{(2)}$ being classical solutions, we can show that $h^{(1)}$ and $h^{(2)}$ are $\mathsf{C}^{2,0}\left([0,T] \times \mathbb{T}^d\right)$ (as in the proof of Lemma I.1.29).

We denote the two difference of the solutions by D_t and of the effective fields by Δ_t (defined (I.1.57) and (I.1.58)). In the following we use two purely deterministic energy estimates for D_t and for Δ_t. For D_t, we know the energy estimate already by Lemma I.1.27. This energy estimate requires the boundedness of $\rho^{(i)}$ that we get by Lemma I.1.23 (ii) (e.g. in (I.1.63) and (I.1.67)). For Δ_t, we show the energy estimate in the following lemma.

Lemma I.3.2. *There is a constant $C = C(\Psi, J) > 0$ such that for all $t \in [0,T]$*

$$\frac{1}{2} \left\|\Delta_t\right\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2} \left\|\Delta_0\right\|^2_{L^2(\mathbb{T}^d)} \leq C \int_0^t \left\|\Delta_s\right\|^2_{L^2(\mathbb{T}^d)} + \left\|D_s e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} \mathrm{d}s, \tag{I.3.1}$$

where the norm $\|.\|_{L^2(e^{-\Psi})}$ is defined in (I.1.59).

Before we prove this lemma, we finish the proof of Theorem I.3.1. Summing up the two energy estimates (I.3.1) and (I.1.61), gives

$$\begin{aligned} &\left\|D_t e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} + \left\|\Delta_t\right\|^2_{L^2(\mathbb{T}^d)} \\ &\leq \left\|D_0 e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} + \left\|\Delta_0\right\|^2_{L^2(\mathbb{T}^d)} + C\int_0^t \left\|\Delta_s\right\|^2_{L^2(\mathbb{T}^d)} + \left\|D_s e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} \mathrm{d}s. \end{aligned} \tag{I.3.2}$$

We conclude with the Gronwall inequality

$$\left\|D_t e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} + \left\|\Delta_t\right\|^2_{L^2(\mathbb{T}^d)} \leq \left(\left\|D_0 e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} + \left\|\Delta_0\right\|^2_{L^2(\mathbb{T}^d)}\right) e^{ct} = 0. \tag{I.3.3}$$

The right hand side of (I.3.3) is zero because $\rho^{(1)}$ and $\rho^{(2)}$ have the same initial distribution. Hence, $\rho^{(1)}$ is equal to $\rho^{(2)}$. The two densities are chosen arbitrarily. Hence, there is at most one probability valued classical solution of (I.0.2) with the initial distribution ρ^0. ☐

Proof of Lemma I.3.2. We derive now the claimed upper bound on $\|\Delta_t\|^2_{L^2(\mathbb{T}^d)}$. By the definition of $h^{(i)}$ and the deduced differentiability of $h^{(i)}$

$$\begin{aligned} &\frac{1}{2} \left\|\Delta_t\right\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2} \left\|\Delta_0\right\|^2_{L^2(\mathbb{T}^d)} = \int_0^t \int_{\mathbb{T}^d} \Delta_s \partial_t \Delta_s \mathrm{d}x \mathrm{d}s \\ &= \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \Delta_s(x) J(x-x') \int_{\mathbb{R}} \theta' \partial_t D_s(x',\theta') e^{-\Psi(\theta')} \mathrm{d}\theta' \mathrm{d}x' \mathrm{d}x \mathrm{d}s. \end{aligned} \tag{I.3.4}$$

Relying on $\rho^{(i)}$ being a classical solutions of (I.0.2) and $\rho^{(i)}_t(x,.)\,e^{-\Psi}$ being a probability density, (I.3.4) equals

$$\begin{aligned} &= -\int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \int_{\mathbb{R}} \Psi'(\theta) \Delta_s(x) J(x'-x) D_s(x',\theta) e^{-\Psi(\theta')} \mathrm{d}\theta \mathrm{d}x' \mathrm{d}x \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{T}^d\times\mathbb{T}^d} \Delta_s(x) J(x-x') \left(h^{(1)}(s,x) - h^{(2)}(s,x')\right) \mathrm{d}x \mathrm{d}x' \mathrm{d}s. \end{aligned} \tag{I.3.5}$$

With the Cauchy-Schwartz inequality we conclude

$$\begin{aligned}
&\frac{1}{2}\left\|\Delta_t\right\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}\left\|\Delta_0\right\|^2_{L^2(\mathbb{T}^d)} \\
&\leq \|J\|_{L^1(\mathbb{T}^d)} \int_0^t \left\|\Delta_s\right\|^2_{L^2(\mathbb{T}^d)} + \left\|\Psi' e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{R})} \left\|D_s e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} + \left\|\Delta_s\right\|^2_{L^2(\mathbb{T}^d)} \,\mathrm{d}s \qquad \text{(I.3.6)} \\
&\leq C \int_0^t \left\|\Delta_s\right\|^2_{L^2(\mathbb{T}^d)} + \left\|D_s e^{-\frac{1}{2}\Psi}\right\|^2_{L^2(\mathbb{T}^d\times\mathbb{R})} \mathrm{d}s,
\end{aligned}$$

by $\left\|\Psi' e^{-\frac{1}{2}\Psi}\right\|_{L^2(\mathbb{R})}$ being finite (by Lemma I.1.8 (iv)). The inequality (I.3.6) is the claimed energy estimate. □

Chapter II

Propagation of chaos

In this chapter we prove that the local mean field interacting spin model evolving according to (I.0.1) has the propagation of chaos property, that we introduced in Section 0.6. Fix (for the rest of this chapter) $r \in \mathbb{N}$ positions $x_1, ..., x_r \in \mathbb{T}^d$. We denote by $[Nx_i]$ the position on the discrete torus $\mathbb{T}^d_N$, that is closest to Nx_i.

We require in addition to Assumption I.0.1 also a L^2-condition on J.

Assumption II.0.1. *$J \in L^2\left(\mathbb{T}^d\right)$. Moreover,*

$$\frac{1}{N^d} \sum_{i \in \mathbb{T}^d_N} \left| J\left(\frac{i}{N}\right)\right|^2 \to \|J\|^2_{L^2} \qquad \text{as } N \to \infty. \tag{II.0.1}$$

Before we state the main result, we define the projection of the solution $\mathbb{P}^N_{[0,T]}$ (defined in (I.2.6)) of the martingale problem of the N-dimensional SDE (I.0.1) to the measure of the time evolution of the r-spins at the position $x_1, ..., x_r$.

Definition II.0.2.

- *We denote by $\mathbb{P}^{N,\{x_1,...,x_r\}}_{[0,T]}$ the image of $\mathbb{P}^N_{[0,T]}$ under the projection that maps a $\underline{\theta}^N \in \mathsf{C}\left([0,T], \mathbb{R}^{N^d}\right)$ to $\left(\theta^{[Nx_i],N}\right)^r_{i=1} \in \mathsf{C}([0,T], \mathbb{R}^r)$.*
- *Similarly we denote by $f_0^{N,\{x_1,...,x_r\}}$ the image of the canonical projection $\mathbb{R}^{N^d} \to \mathbb{R}^r$ of f_0^N (the initial distributions of the SDE* (I.0.1)*).*
- *We use the symbol $\overline{\mathbb{P}}^{x_k}_{[0,T]}$ for the law of the spin $\widehat{\theta}^{x_k}$ evolving according to the SDE* (I.0.3) *at the position $x_k \in \mathbb{T}^d$.*

In addition to Assumption I.0.5 a.) on the order of convergence of the initial distributions of the whole N^d dimensional system, we need the following assumption on the initial distributions of the spins at the r fixed positions.

Assumption II.0.3. *With increasing N*

$$\mathsf{H}\left(f^{N,\{x_1,...,x_r\}} \middle| \prod_{k=1}^r \rho^0\left(x_k, .\right)\right) \to 0. \tag{II.0.2}$$

With these assumptions and notations, we state now the propagation of chaos result of the system evolving according to the SDE (I.0.1).

Theorem II.0.4. *Let Assumptions I.0.1, I.0.3, I.0.4, I.0.5, II.0.1 and II.0.3 hold. Denote by* $\rho^* \in \mathbb{S}$ *(see Definition I.2.2) the hydrodynamic limit element derived in Theorem I.2.3. Then* $\mathbb{P}_{[0,T]}^{N,\{x_1,\dots,x_r\}}$ *converges (when* $N \to \infty$*) in the total variation distance on* $\mathbb{M}_1(\mathsf{C}([0,T],\mathbb{R}^r))$ *to the product measure* $\overline{\mathbb{P}}_{[0,T]}^{\{x_1,\dots,x_r\}} := \prod_{i=1}^r \overline{\mathbb{P}}_{[0,T]}^{x_i}$.

Observe that the convergence in total variation distance implies the strong and weak convergence of measures.

Remark II.0.5. *Theorem II.0.4 implies:*

- *The spins at positions* $\{x_1,\dots,x_r\}$ *are, in the limit* $N \to \infty$*, mutually independent.*
- *A single spin at position* $x \in \mathbb{T}^d$ *evolves in the limit according to the hydrodynamic SDE* (I.0.3) *at* x.
- *Hence, the spin at position* $x \in \mathbb{T}^d$ *is in the limit distributed according to* $\rho_t^*(x,.)$*. This function is the classical solution of the hydrodynamic equation* (I.0.2) *and the limit element of the empirical process (Theorem I.1.2). Moreover,* $\rho_t^*(x,.)$ *is the time marginal of the measure* $\overline{\mathbb{P}}_{[0,T]}^x$ *at time* $t \in [0,T]$.

In the proof of Theorem II.0.4, we use the following SDE. For N large enough, define for each $i \in \mathbb{T}_N^d$,

$$
\begin{aligned}
\mathrm{d}\overline{\theta}_t^{i,N} &= \begin{cases} -\Psi'\left(\overline{\theta}_t^{i,N}\right)\mathrm{d}t + h^*(t,x_k)\,\mathrm{d}t + \sqrt{2}\mathrm{d}\overline{W}_t^{i,N} & \text{if } i = [Nx_k]\,, \\ -\Psi'\left(\overline{\theta}_t^{i,N}\right)\mathrm{d}t + \frac{1}{N^d}\sum_{j\in\mathbb{T}_N^d} J\left(\frac{i-j}{N}\right)\overline{\theta}_t^{N,j}\mathrm{d}t + \sqrt{2}\mathrm{d}\overline{W}_t^{i,N} & \text{else,} \end{cases} \\
\overline{\theta}_0^{i,N} &\sim \rho^0\left(\frac{i}{N},\theta\right)e^{-\Psi(\theta)}\mathrm{d}\theta \qquad \text{for } i \in \mathbb{T}_N^d.
\end{aligned}
\tag{II.0.3}
$$

In this SDE, the spins at (or better close to) the positions $x_1,\dots,x_r$ do not depend any more on the other $N^d - r$ spins. Indeed, the interaction in the drift coefficient is replaced by the fixed effective field $h^*(t,x_k)$. However, the other $N^d - r$ spins of this SDE still depend, via the local mean field interaction contribution, on the spins at the positions $x_1,\dots,x_r$. Hence, the SDE (II.0.3) differs form the SDE (I.2.1), where the interaction of all spins is replaced by a fixed effective field.

We know (see Notation I.2.6 and the subsequent discussion) that the martingale problems for the SDE (I.0.1) and the SDE (I.2.1) are well posed. By the same arguments, we get that the martingale problem for (II.0.3) is well posed (since Assumption I.0.3 and Assumption I.0.1 hold).

Notation II.0.6. *Extending the Notation I.2.6, denote by* $\overline{\mathbb{P}}_{[0,T]}^N$ *the solution to the martingale problem corresponding to* (II.0.3) *respectively, i.e. measures in* $\mathbb{M}_1\left(\mathsf{C}\left([0,T],\mathbb{R}^{N^d}\right)\right)$*. For the corresponding expectation, we use the symbols* $\overline{\mathbb{E}}_{[0,T]}^N$.

We refer to Notation I.2.6, for the definition of $\mathbb{P}_{[0,T]}^N, \widehat{\mathbb{P}}_{[0,T]}^N$, the solutions to the martingale problem corresponding to (I.0.1), (I.2.1) .

Remark II.0.7. *In the SDE* (II.0.3)*, the spins at the space positions* $\{x_1,\dots,x_r\}$ *evolve independent of the other* $N^d - r$ *spins. Therefore* $\overline{\mathbb{P}}_{[0,T]}^{\{x_1,\dots,x_r\}}$ *(defined in Theorem II.0.4) is the canonical projection on* $\mathbb{M}_1(\mathsf{C}([0,T],\mathbb{R}^r))$ *of* $\overline{\mathbb{P}}_{[0,T]}^N$*, for each* $N \in \mathbb{N}$.

With these notations, we state the proof of Theorem II.0.4.

Proof of Theorem II.0.4. The proof is organised as follows.

Step 1: We apply at first the Pinsker inequality, that bounds the total variation norm by the relative entropy. Then we show that the relative entropy between $\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]}$ and $\overline{\mathbb{P}}^{\{x_1,\dots,x_r\}}_{[0,T]}$ is bounded from above by the relative entropy between $\mathbb{P}^N_{[0,T]}$ and $\overline{\mathbb{P}}^N_{[0,T]}$.

Step 2: Next we derive another representation of the relative entropy between $\overline{\mathbb{P}}^N_{[0,T]}$ and $\mathbb{P}^N_{[0,T]}$ relying on the Girsanov theorem. This representation depends on the difference between the drift coefficients of the SDEs (I.0.1) and (II.0.3), i.e. only the evolution of the spins at the position $x_1, ..., x_r$ matter.

Step 3: Finally, we show (see Lemma II.0.8) that this representation vanishes when the number of spins tends to infinity. This follows by an application of Theorem I.2.8 (the vanishing change of the relative entropy) and a vanishing logarithmic moment generating function (similar to Lemma I.2.14). It is in this step crucial that the two SDEs differ only in finite many dimensions.

Step 1: The (Kullback-Csiysár-Kemperman-)Pinsker inequality bounds the total variation distance by the relative entropy (see [Tsy09] Lemma 2.5 and Section 2.8, see also [RW09] Appendix A for an historical overview on this inequality starting with [Pin64]):

$$\begin{aligned}\left\|\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]} - \overline{\mathbb{P}}^{\{x_1,\dots,x_r\}}_{[0,T]}\right\|_{TV} &= \sup_{A\in Sigma-Algebra}\left|\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]}[A] - \overline{\mathbb{P}}^{\{x_1,\dots,x_r\}}_{[0,T]}[A]\right| \\ &\leq \sqrt{\frac{\mathsf{H}\left(\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]}\middle|\overline{\mathbb{P}}^{\{x_1,\dots,x_r\}}_{[0,T]}\right)}{2}}\quad .\end{aligned} \tag{II.0.4}$$

Then we get by the variation formula of the relative entropy

$$\begin{aligned}&\mathsf{H}\left(\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]}\middle|\overline{\mathbb{P}}^{\{x_1,\dots,x_r\}}_{[0,T]}\right) - \mathsf{H}\left(f^{N,\{x_1,\dots,x_r\}}\middle|\prod_{k=1}^r \rho^0(x_k,.)\right) \\ &= \sup_{f\in\mathsf{C}_b(\mathsf{C}([0,T],\mathbb{R}^r))}\left\{\left\langle\mathbb{P}^{N,\{x_1,\dots,x_r\}}_{[0,T]}, f\right\rangle - \log\left\langle\overline{\mathbb{P}}^{\{x_1,\dots,x_r\}}_{[0,T]}, e^f\right\rangle\right\} \\ &\leq \sup_{F\in\mathsf{C}_b\left(\mathsf{C}\left([0,T],\mathbb{R}^{N^d}\right)\right)}\left\{\left\langle\mathbb{P}^N_{[0,T]}, F\right\rangle - \log\left\langle\overline{\mathbb{P}}^N_{[0,T]}, e^F\right\rangle\right\} \\ &= \mathsf{H}\left(\mathbb{P}^N_{[0,T]}\middle|\overline{\mathbb{P}}^N_{[0,T]}\right) - \mathsf{H}\left(f^N\middle|\prod_{k\in\mathbb{T}^d_N}\rho^0\left(\frac{k}{N},.\right)\right).\end{aligned} \tag{II.0.5}$$

In the next steps we show that the right hand side of (II.0.5) vanishes when N tends to infinity. Together with Assumption II.0.3 and (II.0.4), this implies the convergence claimed in Theorem II.0.4.

Step 2: Define for each $N \in \mathbb{N}$ and each $x_k \in \{x_1, ..., x_r\}$ a function $A^N_{x_k} : [0,T] \times \mathbb{R}^{N^d} \to \mathbb{R}$, to be the difference in the drift coefficients of the SDEs (I.0.1) and (II.0.3), i.e. for $i = 1, ...r$

$$A^N_{x_k}\left(t, \underline{\theta}^N_t\right) := h(t,x_k) - \frac{1}{N^d}\sum_{j\in\mathbb{T}^d_N} J\left(\frac{[Nx_k]-j}{N}\right)\theta^j_t. \tag{II.0.6}$$

With this notation, the right hand side of (II.0.5) equals

$$\frac{1}{4}\int_{\mathbb{R}^{N^d}}\int_0^T\sum_{k=1}^r\left(A^N_{x_k}\left(s,\underline{\theta}^N_s\right)\right)^2\mathrm{d}s\mathbb{P}^N_{[0,T]}\left(\mathrm{d}\underline{\theta}^N_{[0,T]}\right). \tag{II.0.7}$$

Indeed this follows because the generalised Girsanov formula (see Corollary B.2 in Appendix B) implies

$$\frac{\mathbb{P}^N_{[0,T]}}{\overline{\mathbb{P}}^N_{[0,T]}}=\frac{f^N}{\prod\limits_{i\in\mathbb{T}^d_N}\rho^0\left(\frac{i}{N},.\right)}\exp\left\{\frac{1}{\sqrt{2}}\sum_{k=1}^r\int_0^t A^N_{x_k}\left(s,\underline{\theta}^N_s\right)\mathrm{d}W^{x_k}_s+\frac{1}{4}\int_0^t\sum_{k=1}^r\left(A^N_{x_k}\left(s,\underline{\theta}^N_s\right)\right)^2\mathrm{d}s\right\}. \tag{II.0.8}$$

Step 3: We show in the following lemma that (II.0.7) vanishes when N tends to infinity. This implies convergence of the total variation norm (by (II.0.5) and (II.0.4)) as claimed in Theorem II.0.4. □

Lemma II.0.8. $\int_{\mathbb{R}^{N^d}}\int_0^T\sum_{k=1}^r\left(A^N_{x_k}\left(s,\underline{\theta}^N_s\right)\right)^2\mathrm{d}s\,\mathbb{P}^N_{[0,T]}\left(\mathrm{d}\underline{\theta}^N_{[0,T]}\right)\to 0$ *for* $N\to\infty$.

Proof of Lemma II.0.8. We prove this lemma by using parts of the proof of Theorem I.2.8. Let us rewrite $A^N_{x_k}$ by using the function $h^{[Nx_k],N}$, defined in (I.2.19),

$$A^N_{x_k}\left(t,\underline{\theta}^N_t\right)=h\left(t,x_k\right)-h^{[Nx_k],N}\left(\underline{\theta}^N_t\right). \tag{II.0.9}$$

We claim that

$$\int_{\mathbb{R}^{N^d}}\int_0^T\sum_{k=1}^r\left(A^N_{x_k}\left(s,\underline{\theta}^N_s\right)\right)^2\mathrm{d}s\mathbb{P}^N_{[0,T]}\left(\mathrm{d}\underline{\theta}^N_{[0,T]}\right)=\int_0^T\mathbb{E}^N_{[0,T]}\left[X^{r,N}_t\right]\mathrm{d}t+ro\left(1\right), \tag{II.0.10}$$

where

$$X^{r,N}_t:=\frac{1}{N^{2d}}\sum_{j,\ell\in\mathbb{T}^d_N}J^N_r\left(\frac{j}{N},\frac{\ell}{N}\right)\eta^{j,N}_t\eta^{\ell,N}_t, \tag{II.0.11}$$

with $J^N_r\left(y,y'\right):=\sum_{k=1}^r J\left(\frac{[Nx_k]}{N}-y\right)J\left(\frac{[Nx_k]}{N}-y'\right)$ and $\eta^{i,N}_t$ defined in (I.2.11). Note that we use in the definition of $\eta^{i,N}_t$ the measure $\widehat{\mathbb{P}}^N_{[0,T]}$ and not $\overline{\mathbb{P}}^N_{[0,T]}$.

The equation (II.0.10) can be proven similar to Lemma I.2.13.

As in the derivation of (I.2.26), we infer from (II.0.10), by the variation formula of the relative entropy,

$$\begin{aligned}&\int_{\mathbb{R}^{N^d}}\int_0^T\sum_{i=1}^r\left(A^N_{x_k}\left(s,\underline{\theta}^N_s\right)\right)^2\mathrm{d}s\,\mathbb{P}^N_{[0,T]}\left(\mathrm{d}\underline{\theta}^N_{[0,T]}\right)\\&\leq\underbrace{T\frac{1}{2\delta N^d}\mathsf{H}\left(\mathbb{P}^N_{[0,T]}\middle|\widehat{\mathbb{P}}^N_{[0,T]}\right)}_{①}+\int_0^T\underbrace{\frac{1}{2\delta N^d}\log\widehat{\mathbb{E}}^N_{[0,T]}\left[\mathrm{e}^{\delta N^d X^{r,N}_t}\right]}_{②}\mathrm{d}t+ro\left(1\right).\end{aligned} \tag{II.0.12}$$

There is the additional prefactor $\frac{1}{N^d}$ on the right hand side of (II.0.12), that we do not have in the inequality (I.2.26). This factor is hidden in (I.2.26) in the variable X (defined in (I.2.11)), more precisely in K^N defined in (I.2.12). The corresponding object in this chapter J_r, does not require this prefactor, because it sums only over r elements. This is the case because the SDEs (I.0.1) and (II.0.3) differ only in r dimensions.

Let us show that the right hand side of (II.0.12) vanishes when N tends to infinity. We know by Theorem I.2.8 that ① $\to 0$. Let us show that ② vanishes for each $t \in [0,T]$ and that ② is uniformly bounded. The convergence of ② $\to 0$ follows by almost the same proof that we use for Lemma I.2.10. However, we require now the stronger Assumption II.0.1 in a similar step as (I.2.50), i.e. to bound

$$\delta \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} J_r\left(x,y\right) m\left(x\right) m\left(y\right) \mathrm{d}x\mathrm{d}y \leq \delta r \left\|J\right\|_{L^2}^2 \left\|m\right\|_{L^2}^2 , \tag{II.0.13}$$

where $J_r\left(x,y\right) := \sum_{i=1}^{r} J\left(x_k - x\right) J\left(x_k - y\right)$. Moreover, ② is uniformly bounded by a similar argument as in the proof of Lemma I.2.15. For the upper bound (compare to (I.2.51)) we need again the Assumption II.0.1.

Hence, the right hand side of (II.0.12) vanishes when the number of spins tends to infinity. This implies the claimed convergence of Lemma II.0.8. □

Chapter III

Equilibrium large deviation

In this chapter we study the asymptotic behaviour of tail events for sequences of random elements such as empirical measures, that are defined as images of the random variables $\{\Theta_i | i \in \mathbb{T}_N^d\}$ with values in $\mathbb{R}^{N^d}$. A suitable method to determine the exponential decay of the probability of tail events is the theory of large deviation. Therefore, we derive the large deviation principles for the random elements of interest.

We interpret the random variables Θ_i as continuous spins . In Section III.2 we assume that these spins are independently and identically distributed according to $e^{-\Psi(\theta)}\mathrm{d}\theta$. Then we derive the usual large deviation principles for sums of these spins (Section III.2.1), sums of continuous images of these spins (Section III.2.2) and empirical measures (Section III.2.3). Moreover, we consider the weighted space empirical measures (Section III.2.4)

$$\frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\Theta_i\delta_{\frac{i}{N}}, \tag{III.0.1}$$

and the double (space-spin) empirical measures (Section III.2.5)

$$\frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\delta_{\left(\frac{i}{N},\Theta_i\right)}. \tag{III.0.2}$$

In Section III.3 we consider spinsm that are not independent, but that are distributed according to

$$\{\Theta_i\}_{i\in\mathbb{T}_N^d} \sim \frac{1}{Z_N} e^{\frac{1}{2N^d}\sum_{i,j\in\mathbb{T}_N^d} J\left(\frac{j-i}{N}\right)\theta_i\theta_j} \prod_{i\in\mathbb{T}_N^d} e^{-\Psi(\theta_i)}\mathrm{d}\theta_i. \tag{III.0.3}$$

Then we derive for the weighted space empirical measure and the double (space-spin) empirical measure defined as images of these random variables, the large deviation principles, as motivated in Section 0.7. We infer these principles from the large deviation principles for independent spins by applying the generalization of Varadhan's lemma given in Appendix C.

We see that the limit of the double empirical measures are concentrated on measures with Lebesgue measure as projection to the $\mathbb{T}^d$ coordinate. We define this set of measures now.

Definition III.0.1. *We denote by $\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R})$ the subspace of $\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$, that consists of those measures that have the Lebesgue measure as projection to the $\mathbb{T}^d$ coordinate, i.e.*

$$\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R}) := \left\{\mu\in\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R}) : \mu(\mathrm{d}x,\mathrm{d}\theta) = \mathrm{d}x\otimes\mu_x(\mathrm{d}\theta),\ \text{with } \mu_x\in\mathbb{M}_1(\mathbb{R})\right\}. \tag{III.0.4}$$

Before we derive the large deviation principles in Section III.2 and Section III.3, we give a short introduction to the theory of large deviation in the following section.

III.1 Short introduction to the theory of large deviation

With the theory of large deviation one can study the exponential decay of the probability of tail events. We repeat in this section the main definitions of this theory. For further information see for example the book [DZ98] or [DE97].

Let Ξ be a Hausdorff topological space with topology τ. Moreover, let $\{\xi_N\}$ be a family of random variables taking values in X and $\left\{P^N\right\}$ be the associated family of probability measures.

Definition III.1.1. *A lower semi-continuous function $I : X \to [0, \infty]$ is called a rate function.*

If I has furthermore compact level sets, i.e. $\{x \in X : I(x) \leq c\} \subset X$ is compact for each $c \in \mathbb{R}$, then I is a good rate function.

Definition III.1.2. *The family $\left\{\xi_N, \mathbb{P}^N\right\}$ satisfies the large deviation principle (LDP) on (X, τ) with rate function I and speed $\{a_N\}$ if*

a.) *For each closed set $F \subset X$, $\limsup_{N\to\infty} a_N \log \mathbb{P}^N (\xi_N \in F) \leq -\inf_{x\in F} I(x)$ and*

b.) *for each open set $O \subset X$, $\liminf_{N\to\infty} a_N \log \mathbb{P}^N (\xi_N \in O) \geq -\inf_{x\in O} I(x)$.*

In the following the speed of convergence is usually $a_N = \frac{1}{N^d}$ for $N \in \mathbb{N}$. The definition of large deviation principle can be further generalised, for example to the families of random variables might be uncountable.

Definition III.1.3. *We say that $\left(\xi_N, \mathbb{P}^N\right)$ is exponentially tight if for all $R > 0$, there exists a compact set $K_R \subseteq \Xi$ such that*

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N [\xi_N \notin K_R] \leq -R. \tag{III.1.1}$$

III.1.1 The contraction principles for the identity map with different topologies

Let $\widehat{\tau}$ be a second topology on Ξ. For example the topology induced by a metric $d(\cdot,\cdot)$ on Ξ. We explain now how the large deviation principles on $(\Xi, \widehat{\tau})$ and (Ξ, τ) are related.

To compare the topologies we use the following notation. We say that the topology τ is *weaker* than the topology $\widehat{\tau}$, when each element in τ is contained in $\widehat{\tau}$. Then we write $\tau \prec \widehat{\tau}$. If $\tau \prec \widehat{\tau}$, then the identity map $id : (\Xi, \widehat{\tau}) \to (\Xi, \tau)$ is continuous.

For continuous maps the large deviation principle can be transformed into a second large deviation principle, according to the *contraction principle* and the *inverse contraction principle*, if some conditions are satisfied (see [DZ98] Theorem 4.2.1 and Theorem 4.2.4). For the identity map, these two principles simplify as follows:

Corollary III.1.4 (of [DZ98] Theorem 4.2.1)**.** *Let $\tau \prec \widehat{\tau}$. Assume that $\left(\xi_N, \mathbb{P}^N\right)$ satisfies the LDP on $(\Xi, \widehat{\tau})$ with a good rate function I. Then $\left(\xi_N, \mathbb{P}^N\right)$ satisfies LDP on (Ξ, τ) (with the same rate function I).*

Corollary III.1.5 (of [DZ98] Theorem 4.2.4)**.** *Let $\tau \prec \widehat{\tau}$. Assume that $\left(\xi_N, \mathbb{P}^N\right)$ satisfies the LDP on (Ξ, τ) with a good rate function I, and that $\left(\xi_N, \mathbb{P}^N\right)$ is exponentially tight on $(\Xi, \widehat{\tau})$. Then $\left(\xi_N, \mathbb{P}^N\right)$ satisfies LDP on $(\Xi, \widehat{\tau})$ (with the same rate function I).*

III.1.2 Varadhan's lemma

Varadhan proved in Chapter 3 in [Var66] a generalisation of the Laplace method, that is referred to as Varadhan's lemma. The lemma is a consequence of the large deviation principle. It gives a precise description of the logarithmic asymptotic (for $N \to \infty$) of expectations like

$$\mathbb{E}\left[e^{N^d \phi(\xi_N)}\right]. \tag{III.1.2}$$

The following version of this theorem is taken from [DZ98] Theorem 4.3.1, see also [DE97] Theorem 1.2.1.

Theorem III.1.6 (Varadhan's lemma). *Assume that $\left(\xi_N, \mathbb{P}^N\right)$ satisfies the LDP on (Ξ, τ) with a good rate function I. Let ϕ be a continuous function such that the following tail condition holds*

$$\lim_{M\to\infty} \limsup_{\mathbb{N}\to\infty} \frac{1}{N^d} \log \mathbb{E}\left[e^{N^d \phi(\xi_N)} \mathbb{1}_{\{\phi(\xi_N)\geq M\}}\right] = -\infty. \tag{III.1.3}$$

Then

$$\lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}\left[e^{N^d \phi(\xi_N)}\right] = \sup_{x\in\Xi} \left\{\phi(x) - I(x)\right\}. \tag{III.1.4}$$

We give in the Appendix C a generalisation of this theorem. We use this generalisation in Section III.3 to infer from the large deviation principle for independent spins, the large deviation principle for interacting spins. The usual Varadhan's lemma is not applicable because the respective function ϕ is not continuous.

III.2 LDP for the non-interacting systems

We assume in this section, that the Θ_i for $i \in \mathbb{T}_N^d$ are i.i.d. $\mathbb{R}$-valued random variables with density $e^{-\Psi(\theta)}$. We prove for different random elements, that are defined as images of these random variables Θ_i, large deviation principles and further properties. These random elements are the sums of Θ_i (Section III.2.1), sums of continuous images of Θ_i (Section III.2.2), the usual empirical measures (Section III.2.3), weighted space empirical measures (Section III.2.4) and double empirical measures (Section III.2.5). In these sections we give exact definitions of these random elements.

Notation III.2.1.
- *$\underline{\Theta}^N = \{\Theta_i\}_{i\in\mathbb{T}_N^d}$ is a N^d-dimensional vector of the i.i.d. $\mathbb{R}$-valued random variables Θ_i.*
- *The probability measure and the expectation with respect to $e^{-\Psi(\theta)}\mathrm{d}\theta$ are denoted by $\mathbb{P}$ and $\mathbb{E}_{\mathbb{P}}$.*
- *$\mathbb{P}^N = \otimes_{i\in\mathbb{T}_N^d} \mathbb{P}$ is the product measure on $\mathbb{R}^{N^d}$.*

We assume the following properties of the function Ψ, that appears in the density of the random variables Θ_i,

Assumption III.2.2. a.) *Ψ is smooth and even,*

b.) *$\int_{\mathbb{R}} e^{\Psi(\theta)}\mathrm{d}\theta = 1$ and*

c.) *there is a constant $c_\Psi > 0$, possibly infinity, such that*

$$\liminf_{|\theta|\to\infty} \frac{\Psi(\theta)}{|\theta|^2} \geq c_\Psi\,. \tag{III.2.1}$$

Remark III.2.3. *This assumption is for example satisfied for $\Psi(\theta) = \theta^2 - C$ or for a double well function like $\Psi(\theta) = \theta^4 - \theta^2 - C$.*

III.2.1 Sums of spins

We consider in this section the random variables $\xi_N^{\mathrm{sum}} := \frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \Theta_i$ on the space $\mathbb{R}$.

Lemma III.2.4. *The family $\left\{\xi_N^{\mathrm{sum}}, \mathbb{P}^N\right\}$ satisfies the LDP with good rate function*

$$I(v) := \sup_{t\in\mathbb{R}} \left\{tv - h(t)\right\}, \quad \text{where} \quad h(t) = \log \mathbb{E}_{\mathbb{P}}\left[e^{t\Theta}\right]. \tag{III.2.2}$$

This large deviation result is the standard Cramer theorem (see e.g. [DZ98] Section 2.2). The rate function I and the moment generating function h have nice properties, that we state in the following lemma.

Lemma III.2.5. *The function h has the following properties:*

- $h(t)$ *is finite for all* $t \in \mathbb{R}$.
- *The second derivative of h is locally uniformly positive.*
- h *is strictly convex, continuous, even and increasing.*
- $|h'(t)| \to \infty$ *when* $|t| \to \infty$.
- h *is real analytic.*

The function I has the following properties:

- I *is strictly convex, lower semi-continuous and increasing.*
- I *is real analytic.*
- *The second derivative of I is locally uniformly positive.*
- *For any $v \in \mathbb{R}$, the* sup *in the definition of $I(v)$ is attained at a unique $t \in \mathbb{R}$. In particular the supremum is actually a maximum.*

Proof. To derive the claimed properties of h, we need the first and second derivative of h, which are given by

$$\begin{aligned} h'(t) &= \frac{\mathbb{E}_{\mathbb{P}}\left[\theta e^{t\theta}\right]}{\mathbb{E}_{\mathbb{P}}\left[e^{t\theta}\right]} = \mathbb{E}_{e^{t\theta-\Psi(\theta)}\mathrm{d}\theta}\left[\theta\right], \\ h''(t) &= \mathrm{Var}_{e^{t\theta-\Psi(\theta)}\mathrm{d}\theta}\left[\Theta\right]. \end{aligned} \tag{III.2.3}$$

- For each $t \in \mathbb{R}$ the log moment generating function $h(t)$ is *finite*, due to Ψ increasing at least quadratic at infinity (Assumption III.2.2).
- h is *strictly convex*: By the Hölder inequality

$$h(\lambda t + (1-\lambda)s) = \log \mathbb{E}_{\mathbb{P}^N}\left[e^{\lambda t\theta} e^{(1-\lambda)s\theta}\right] \le \log \mathbb{E}_{\mathbb{P}^N}\left[e^{t\theta}\right]^{\lambda} + \log \mathbb{E}_{\mathbb{P}^N}\left[e^{s\theta}\right]^{1-\lambda}, \tag{III.2.4}$$

  with equality if and only if $t = s$.
- h is *continuous* on $\mathbb{R}$ because it is convex on its domain $\mathbb{R}$.
- h'' is *locally uniformly positive*: For each interval $[-T, T]$ there exists a $c_T > 0$ such that $h''(t) = \mathrm{Var}_{e^{tx-\Psi(x)}\mathrm{d}x}(X) > c_T$ because h'' is continuous and greater than 0 for each t.
- h is *increasing*, because it is convex and even.
- By the unbounded support of $\mathbb{P}$

$$\lim_{|t|\to\infty} |h'(t)| = \infty. \tag{III.2.5}$$

  Indeed, this follows from the following standard argument. Let $t > 0$ and fix a $x \in \mathbb{R}$, then

$$\frac{h(t)}{t} \ge \frac{1}{t}\log\left(e^{tx}\mathbb{P}[\theta > x]\right) = x + \frac{1}{t}\log\left(\mathbb{P}[\theta > x]\right) \to x. \tag{III.2.6}$$

  Because the $x \in \mathbb{R}$ is arbitrary, we get that $\liminf_{t\to\infty} \frac{h(t)}{t} = \infty$ and by the convexity of h we infer (III.2.5) for positive t. By Ψ being even, also h is even. Therefore, (III.2.5) holds for all sequences $t \to -\infty$.

- *h is real analytic:* The function $t \to \mathbb{E}_{\mathbb{P}}\left[e^{t\Theta}\right]$ is analytic. Indeed, for an arbitrary $T > 0$, such that for all $t \in [-T, T]$ and for all $k \in \mathbb{N}$

$$\left|\int_{\mathbb{R}} \partial_t^k e^{t\theta} e^{-\Psi(\theta)} \mathrm{d}\theta\right| \leq C_T \left|\int_{\mathbb{R}} \theta^k e^{-\frac{1}{2}\Psi(\theta)} \mathrm{d}\theta\right| \leq C_T C \left|\int_{\mathbb{R}} \theta^{2k} e^{-\frac{1}{2}c_\Psi \theta^2} \mathrm{d}\theta\right| \leq C_T C^k 2^k k! \,, \quad \text{(III.2.7)}$$

by Assumption III.2.2 c.) to replace Ψ with θ^2. Moreover, $\int_{\mathbb{R}} e^{t\theta} e^{-\Psi(\theta)} \mathrm{d}\theta > \frac{1}{2}$ for all $t \in \mathbb{R}$ and $\log(.)$ is analytic on $\left(\frac{1}{2}, \infty\right)$. Hence h is analytic as a composition of analytic functions.

Properties of I:

- I is *convex* and *lower semi continuous*, because $I = h^*$ and h is finite.
- To show that the supremum in the definition of I is a *maximum*, note that range $(h') = \mathbb{R}$, by (III.2.5) and by the continuity of h'. Hence for each $v \in \mathbb{R}$ there exists a $t \in \mathbb{R}$ such that $v = h'(t)$. Moreover, the value t is unique, because h is strictly convex . Hence for each $v \in \mathbb{R}$ there exists exactly one maximising value $t(v) \in \mathbb{R}$.
- I is *strictly convex*: This is a dual property to the differentiability of h.
- I'' is *locally uniformly positive*: If v and t are conjugate points (i.e. $I(v) = tv - h(t)$ or equivalently $(I'(v) = t \leftrightarrow h'(t) = v)$), then $I'(h'(t)) = t$ and hence $I''(v) h''(t) = 1$. The second derivative of h is locally positive, hence also the second derivative of I. Because we know that h has locally uniformly positive second derivatives, the same has to be true for I.
- I is *analytic*: Set $f(t, v) = h'(v) - t$. Then $f(t, I'(t)) = 0$, $\partial_v f = h''(v) > 0$ and f is analytic in a neighbourhood of each $(t, I'(t))$. Hence by the analytic implicit function theorem (see Theorem 2.5.3 in [KP02]), $I'(t)$ is also analytic in a neighbourhood of each $t \in \mathbb{R}$. Because t was chosen arbitrary, I' is everywhere real analytic. This implies the real analyticity of its integral I.
- I is *smooth*, because it is analytic.
- I is *increasing*, because it is convex and even.

□

Lemma III.2.6. *By the Assumption III.2.2*

$$\limsup_{|t|\to\infty} \frac{h(t)}{|t|^2} \leq \frac{1}{4c_\Psi} \quad \text{and} \quad \liminf_{|v|\to\infty} \frac{I(v)}{|v|^2} \geq c_\Psi. \quad \text{(III.2.8)}$$

Proof. **Step 1: The bound on h:** We investigate the case when c_Ψ is finite and infinite separately.
Step 1.1: $c_\Psi < \infty$: Choose an arbitrary $\kappa \in (0,1)$. Fix a $q \in (1, \infty)$, such that $q\kappa < 1$. Set $p = \frac{q}{q-1}$. By the Hölder inequality

$$\begin{aligned} h(t) &= \int e^{t\theta - \Psi(\theta)} \mathrm{d}\theta = \int e^{t\theta - \kappa c_\Psi \theta^2} e^{\kappa c_\Psi \theta^2 - \Psi(\theta)} \mathrm{d}\theta \\ &\leq \left(\int e^{p\left(t\theta - \kappa c_\Psi \theta^2\right)} \mathrm{d}\theta\right)^{1/p} \left(\int e^{q\left(\kappa c_\Psi \theta^2 - \Psi(\theta)\right)} \mathrm{d}\theta\right)^{1/q}. \end{aligned} \quad \text{(III.2.9)}$$

The last integral is bounded by a finite constant $C^{\kappa,q}$ due to Assumption III.2.2 c.). With

$$p\left(t\theta - \kappa c_\Psi \theta^2\right) = p\kappa c_\Psi \left(\frac{t}{\kappa c_\Psi}\theta - \theta^2\right) = -p\kappa c_\Psi \left(\frac{t}{2\kappa c_\Psi} - \theta^2\right)^2 + p\frac{t^2}{4\kappa c_\Psi}, \quad \text{(III.2.10)}$$

we get

$$h(t) \le e^{\frac{t^2}{4\kappa c_\Psi}} \left(\int e^{-p\kappa c_\Psi \left(\frac{t}{2\kappa c_\Psi} - \theta\right)^2} \mathrm{d}\theta \right)^{1/p} C^{\kappa,q} = e^{\frac{t^2}{4\kappa c_\Psi}} \left(\frac{2\pi}{2c_\Psi p\kappa} \right)^{\frac{1}{2p}} C^{\kappa,q}. \tag{III.2.11}$$

From this we conclude

$$\limsup_{|t|\to\infty} \frac{h(t)}{t^2} \le \frac{1}{\kappa} \frac{1}{4c_\Psi}. \tag{III.2.12}$$

The $\kappa \in (0,1)$ is arbitrary, hence

$$\limsup_{|t|\to\infty} \frac{h(t)}{t^2} \le \frac{1}{4c_\Psi}. \tag{III.2.13}$$

Step 1.2: The case $c_\Psi = \infty$.: Fix a sequence $c_\Psi^n \in \mathbb{R}$ with $c_\Psi^n \to \infty$. We infer from the result above, that is valid for each c^n, that

$$\limsup_{|t|\to\infty} \frac{h(t)}{t^2} \le 0. \tag{III.2.14}$$

Step 2: Bound for I: By the Legendre-Fenchel transformation formula (III.2.2) of I

$$\frac{I(v)}{v^2} \ge \lambda - \lambda^2 \frac{h(\lambda v)}{\lambda^2 v^2}, \tag{III.2.15}$$

for each $\lambda \in \mathbb{R}$. With (III.2.13) this gives

$$\liminf_{|v|\to\infty} \frac{I(v)}{v^2} \ge \lambda - \lambda^2 \frac{1}{4c_\Psi}. \tag{III.2.16}$$

Maximizing the right hand side over λ leads to $\lambda = 2c_\Psi$, and

$$\liminf_{|v|\to\infty} \frac{I(v)}{v^2} \ge c_\Psi. \tag{III.2.17}$$

□

III.2.2 Sums of continuous images of spins

We consider in this section the vector of random variables

$$\xi_N^{\mathrm{img}} := \frac{1}{N^d} \left(\sum_{i\in\mathbb{T}_N^d} f_1(\Theta_i), \ldots, \sum_{i\in\mathbb{T}_N^d} f_n(\Theta_i) \right), \tag{III.2.18}$$

on the space $\mathbb{R}^n$. Here $\underline{f} = (f_1, \ldots, f_n)$ is a collection of n continuous functions on $\mathbb{R}$.

Lemma III.2.7. *The family $\left\{\xi_N^{\mathrm{img}}, \mathbb{P}^N\right\}$ satisfies the large deviation principle with rate function*

$$I_{\underline{f}}(\underline{v}) = \max_{\underline{t}} \left\{ \sum_1^n v_i t_i - H\left(\sum t_i f_i\right) \right\}, \quad \text{where} \quad H(f) = \log \mathbb{E}_{\mathbb{P}}\left[e^{f(\Theta)}\right], \tag{III.2.19}$$

for $\underline{v} \in \mathbb{R}^n$.

Let μ be a probability measure on $\mathbb{R}$. For $\underline{f}$ as above, we define with abuse of notation

$$I_{\underline{f}}(\mu) := I_{\underline{f}}(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle). \tag{III.2.20}$$

If $\underline{f}^2$ contains $\underline{f}^1$ and possibly also other functions, then we write $\underline{f}^1 \subseteq \underline{f}^2$.

Lemma III.2.8. (i) *If $\underline{f}^1 \subseteq \underline{f}^2$, then $I_{\underline{f}^1}(\mu) \le I_{\underline{f}^2}(\mu)$ for all $\mu \in \mathbb{M}_1(\mathbb{R})$.*

(ii) *For $\mu \in \mathbb{M}_1(\mathbb{R})$,*

$$\sup_{\underline{f}} I_{\underline{f}}(\mu) = \sup_{f \in \mathsf{C}(\mathbb{R})} \left\{ \langle \mu, f \rangle - \log \mathbb{E}_{\mathbb{P}} \left[e^{f(\Theta)} \right] \right\} = \mathsf{H}\left(\mu \middle| e^{-\Psi} \mathrm{d}\theta \right), \tag{III.2.21}$$

where $\mathsf{H}(\,|\,)$ is the relative entropy defined by

$$\mathsf{H}\left(\mu \middle| e^{-\Psi} \mathrm{d}\theta \right) := \begin{cases} \int m(\theta) \log\left(m(\theta) e^{\Psi(\theta)} \right) \mathrm{d}\theta, & \text{if } \mu = m(\theta)\,\mathrm{d}\theta, \\ \infty, & \text{otherwise.} \end{cases} \tag{III.2.22}$$

Proof. ***(i)*** Assume that $\underline{f}^1 = (f_1, ..., f_n) \subseteq \underline{f}^2$ and w.l.o.g. that $\underline{f}^2 = \left(\underline{f}^1, f_{n+1}, ..., f_{n+k} \right)$. Then

$$\begin{aligned} I_{\underline{f}^1}(\mu) &= \max_{\underline{t} \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \langle \mu, f_i \rangle t_i - H\left(\sum_{i=1}^n t_i f_i \right) \right\} \\ &= \max_{\underline{t} \in \mathbb{R}^{n+k}, t_{n+1} = \dots = t_{n+k} = 0} \left\{ \sum_{i=1}^{n+k} \langle \mu, f_i \rangle t_i - H\left(\sum_{i=1}^{n+k} t_i f_i \right) \right\} \le I_{\underline{f}^2}(\mu). \end{aligned} \tag{III.2.23}$$

(ii) Fix an arbitrary $\mu \in \mathbb{M}_1(\mathbb{R})$. Then

$$\begin{aligned} \sup_{\underline{f}} I_{\underline{f}}(\mu) &= \sup_{\underline{f}} \max_{\underline{t}} \left\{ \sum_i \langle \mu, f_i \rangle t_i - H\left(\sum_i t_i f_i \right) \right\} \\ &= \sup_{\underline{f}} \left\{ \left\langle \mu, \sum_i f_i \right\rangle - H\left(\sum_i f_i \right) \right\} = \sup_f \left\{ \langle \mu, f \rangle - H(f) \right\}. \end{aligned} \tag{III.2.24}$$

The right hand side is the variational formula of the relative entropy. □

III.2.3 Empirical measures

In this section we consider the empirical measures $\xi_N^{\text{emp}} = \frac{1}{N^d} \sum_{i \in \mathbb{T}_N^d} \delta_{\Theta_i}$ on the space of probability measures $\mathbb{M}_1(\mathbb{R})$. We denote with $d(\cdot, \cdot)$ the usual Prokhorov metric on this space.s

Lemma III.2.9. (i) $\left(\xi_N^{\text{emp}}, \mathbb{P}^N \right)$ *is exponentially tight on* $(\mathbb{M}_1(\mathbb{R}), d(\cdot, \cdot))$.

(ii) $\left(\xi_N^{\text{emp}}, \mathbb{P}^N \right)$ *satisfies the LDP on* $(\mathbb{M}_1(\mathbb{R}), d(\cdot, \cdot))$ *with rate function* $\mathsf{H}\left(. \middle| e^{-\Psi} \mathrm{d}\theta \right)$ *defined in* (III.2.22).

Proof. *(i)* The exponential tightness of $\left(\xi_N^{\text{emp}}, \mathbb{P}^N \right)$ is proven in Lemma 6.2.6 in [DZ98]. See also Possibility 1 in the proof of Lemma III.2.15 for a generalisation of this Lemma.

(ii) The large deviation principle for $\left(\xi_N^{\text{emp}}, \mathbb{P}^N \right)$ follows from Sanov's theorem (see [DZ98] Section 6.2 or [DE97] Chapter 2), because the Θ_i are independent and identically distributed.

We state here another, shorter proof that uses the projective limit approach. At first we look at the space $\mathbb{M}(\mathbb{R})$, which is the dual space to $W = \mathsf{C}_b(\mathbb{R})$. For this space we apply

[DZ98] Theorem 4.6.9 (where we use as system of fine sets all sets $\{(f_1,\dots,f_d) : f_i \in \mathsf{C}_b(\mathbb{R})\}$ with ordering $\subseteq$). Form this theorem we infer the large deviation principle for ξ_N^{emp} on $\mathbb{M}(\mathbb{R})$ (equipped with the projective limit topology) with rate function

$$I(\mu) = \sup_{d\in\mathbb{N}} \sup_{\underline{f}=(f_1,\dots,f_d)\in(\mathsf{C}_b(\mathbb{R}))^d} I_{\underline{f}}(\mu) = \mathsf{H}\left(\mu \middle| e^{-\Psi}\mathrm{d}\theta\right), \tag{III.2.25}$$

for $\mu \in \mathbb{M}_1(\mathbb{R})$, by (III.2.21).

The space $\mathbb{M}_1(\mathbb{R})$ is a subset of $\mathbb{M}(\mathbb{R})$. Moreover, $\mathbb{P}^N(\mathbb{M}_1(\mathbb{R})) = 1$ and the domain of the rate function $\mathsf{H}\left(.\middle|e^{-\Psi}\mathrm{d}\theta\right)$ is a subset of $\mathbb{M}_1(\mathbb{R})$. Therefore, we infer from the LDP on $\mathbb{M}(\mathbb{R})$ the LDP on $\mathbb{M}_1(\mathbb{R})$ (equipped with the projective limit topology) with the same rate function by [DZ98] Lemma 4.1.5.

The projective topology is weaker than the weak topology (induced by the Prokhorov metric). By (i) and the inverse contraction principle (see Theorem III.1.5), the family $\left(\xi_N^{\mathrm{emp}}, \mathbb{P}^N\right)$ satisfies the LDP on $\left(\mathbb{M}_1(\mathbb{R}), d(\cdot,\cdot)\right)$ with rate function $\mathsf{H}\left(.\middle|e^{-\Psi}\mathrm{d}\theta\right)$. □

III.2.4 Weighted space empirical measure

In this section we consider the weighted space empirical measures

$$\xi_N := \frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d} \Theta_i \delta_{\frac{i}{N}}, \tag{III.2.26}$$

that are elements of the space $\mathbb{M}(\mathbb{T}^d)$ defined as follows.

Definition III.2.10. *With $\mathbb{M}(\mathbb{T}^d)$ we denote the space of regular finite real-valued (signed) Borel measures, normed with the total variation norm $|.|(\mathbb{T}^d)$.*

We denote the closed ball of radius $R > 0$ in the total variation norm on $\mathbb{M}(\mathbb{T}^d)$ by

$$K_R(\mathbb{T}^d) := \left\{\mu \in \mathbb{M}(\mathbb{T}^d) : |\mu|(\mathbb{T}^d) \le R\right\}, \tag{III.2.27}$$

which is a bounded subset of $\mathbb{M}(\mathbb{T}^d)$. With this notation, we see that $\mathbb{M}(\mathbb{T}^d) = \bigcup_{R=1}^\infty K_R$.

Remark III.2.11. *The space $\mathbb{M}(\mathbb{T}^d)$ with the weak-*-topology is not metrisable (Appendix A (v)). However, on the bounded subsets $K_R(\mathbb{T}^d)$, this topology is metrisable (see Appendix A (viii)). To work on K_R, we could for example change the random variable ξ_N^{wei} by considering cutoffs of the random variables Θ_i, i.e. look at $\operatorname{sign}(\Theta_i)(|\Theta_i| \wedge R)$ instead of Θ_i. However, metrisability is not necessary for proving the LDP.*

Lemma III.2.12. (i) *The K_R-s are compact subsets of $\mathbb{M}(\mathbb{T}^d)$.*

(ii) *$\left(\xi_N^{\mathrm{wei}}, \mathbb{P}^N\right)$ is exponentially tight on $\left(\mathbb{M}(\mathbb{T}^d), weak-*-topology\right)$.*

(iii) *For any $f \in \mathsf{C}(\mathbb{T}^d)$*

$$\lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{N^d\langle \xi_N^{\mathrm{wei}}, f\rangle}\right] = \int_{\mathbb{T}^d} h(f(x))\,\mathrm{d}x, \tag{III.2.28}$$

where h is defined in (III.2.2).

(iv) *$\left(\xi_N^{\mathrm{wei}}, \mathbb{P}^N\right)$ satisfies the LDP on $\left(\mathbb{M}(\mathbb{T}^d), weak-*-topology\right)$ with good rate function*

$$\mathcal{I}(\nu) = \begin{cases} \int_{\mathbb{T}^d} I(m(x))\,\mathrm{d}x, & \text{if } \nu = m(x)\,\mathrm{d}x, \\ \infty, & \text{otherwise,} \end{cases} \tag{III.2.29}$$

where I is defined in (III.2.2).

Proof. ***(i)*** **Exponential tightness:**

By Appendix A property (ix) we know that K_R is weak-*-compact. Moreover

$$\begin{aligned}\mathbb{P}^N\left[\xi_N^{\text{wei}} \notin K_R\right] &= \mathbb{P}^N\left[\sum |\Theta_i| > N^d R\right] \leq \mathbb{P}^N\left[\sum |\Theta_i|^2 > N^d R^2\right] \\ &\leq \mathbb{E}_{\mathbb{P}^N}\left[e^{\kappa \sum |\Theta_i|^2}\right] e^{-\kappa N^d R^2} = \left(\mathbb{E}_{\mathbb{P}}\left[e^{\kappa |\Theta|^2}\right]\right)^N e^{-\kappa N^d R^2},\end{aligned} \tag{III.2.30}$$

by the exponential Chebyshev inequality. The expectation on the right hand side is bounded by a constant, when $\kappa < c_\Psi$ by the Assumption III.2.2 c.) (by a similar calculation as in (III.2.9)).

(ii) **The logarithmic moment generating function:** Fix a $f \in \mathsf{C}(\mathbb{T}^d)$. Then

$$\mathbb{E}_{\mathbb{P}^N}\left[e^{N^d \langle \xi_N^{\text{wei}}, f\rangle}\right] = \int_{\mathbb{R}^N} e^{\sum f\left(\frac{i}{N}\right)\theta_i - \Psi(\theta_i)} \mathrm{d}\underline{\theta}^N = \prod \int_{\mathbb{R}} e^{f\left(\frac{i}{N}\right)\theta - \Psi(\theta)} \mathrm{d}\theta. \tag{III.2.31}$$

By f, log and $t \to e^{t\theta}$ being continuous and $\theta \to e^{t\theta}$ being uniformly (for all $t \in [-|f|_\infty, |f|_\infty]$) integrable with respect to $\mathbb{P}$, we infer

$$\lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{N^d \langle \xi_N^{\text{wei}}, f\rangle}\right] = \int_{\mathbb{T}^d} h\left(f\left(x\right)\right) \mathrm{d}x. \tag{III.2.32}$$

(iii) **LDP and rate function:**

Step 1: LDP: We show the LDP by applying [DZ98] Corollary 4.6.14, which is based on the projective limit approach, that we use also in the proof of Lemma III.2.9. However, the necessary conditions are simplified. Indeed, the conditions of this corollary are satisfied, because $\mathbb{M}(\mathbb{T}^d)$ with weak-*-topology is a locally convex, Hausdorff topological vector space (see Appendix A (iii) and (iv)). Moreover, the log moment generating function is finite for every $f \in \mathsf{C}(\mathbb{T}^d)$ and it is Gâteaux differentiable. Hence [DZ98] Corollary 4.6.14 states, that the Legendre-Fenchel transform M^* of the log moment generating function is a good rate function on $\mathbb{M}(\mathbb{T}^d)$.

In the following steps we show that $M^*(\nu) = \mathcal{I}(\nu)$ for all $\nu \in \mathbb{M}(\mathbb{T}^d)$.

Step 2: $M^*(\nu) < \infty$ **implies that** ν **is absolutely continuous w.r.t. the Lebesgue measure:** We use now a similar idea as in [Var88] Theorem 4.1. Take $\nu \in \mathbb{M}(\mathbb{T}^d)$ with $M^*(\nu) < \infty$. Fix a $k > 0$ and a measurable set $A \subset \mathbb{T}^d$. Fix a sequence $f_n \in \mathsf{C}(\mathbb{T}^d)$ such that $f_n \to \mathbb{1}_A$ pointwise. Then

$$M^*(\nu) \geq k \int f_n \nu(\mathrm{d}x) - \int_{\mathbb{T}^d} \log \int e^{k f_n(x)\theta - \Psi(\theta)} \mathrm{d}\theta \mathrm{d}x, \tag{III.2.33}$$

for all n. By the dominated convergence theorem, the inequality also holds for f

$$M^*(\nu) \geq k\nu(A) - |A| \log \int e^{k\theta} \mathrm{d}\theta, \tag{III.2.34}$$

where $|A|$ denotes the Lebesgue measure of the set A. If $|A| = 0$, then $\nu(A)$ has to vanish, because $k > 0$ can be chosen arbitrary large. This implies $\nu << \mathrm{d}x$.

Step 3: We show that $M^*(\nu) \leq \mathcal{I}(\nu)$**:** Fix a $\nu \in \mathbb{M}(\mathbb{T}^d)$. We can assume that $\mathcal{I}(\nu) < \infty$. This implies $\nu(\mathrm{d}x) = m(x)\,\mathrm{d}x$ and therefore

$$\begin{aligned}M^*(\nu) &= \sup_{f \in \mathsf{C}(\mathbb{T}^d)} \left\{\int f(x)\, m(x) - h(f(x))\, \mathrm{d}x\right\} \\ &\leq \int_{\mathbb{T}^d} \sup_{t\in\mathbb{R}} \left\{t m(x) - h(t)\right\} \mathrm{d}x = \mathcal{I}(\nu).\end{aligned} \tag{III.2.35}$$

Step 4: We show that $M^*(\nu) \geq \mathcal{I}(\nu)$: Fix a $\nu \in \mathbb{M}(\mathbb{T}^d)$. We can assume $M^*(\nu) < \infty$ and hence as shown in Step 2, $\nu(\mathrm{d}x) = m(x)\,\mathrm{d}x$. For each $n \in \mathbb{N}$ define the function $f_n(x) = \min\{\max\{-n, I'(m(x))\}, n\}$. We claim that for each $x \in \mathbb{T}^d$

$$0 \leq f_n(x)\, m(x) - h(f_n(x)) \nearrow I(m(x)) \text{ as } n \to \infty. \tag{III.2.36}$$

Using this claim and the monotone convergence theorem, we conclude

$$\begin{aligned} M^*(\nu) &\geq \sup_n \left\{ \int f_n(x)\, m(x) - h(f_n(x))\,\mathrm{d}x \right\} \geq \lim_{n\to\infty} \int f_n(x)\, m(x) - h(f_n(x))\,\mathrm{d}x \\ &= \int \lim_{n\to\infty} \left(f_n(x)\, m(x) - h(f_n(x))\right)\mathrm{d}x = \int I(m(x))\,\mathrm{d}x = \mathcal{I}(m(.)) . \end{aligned} \tag{III.2.37}$$

In the rest of this proof we prove the claim (III.2.36). Fix a $x \in \mathbb{T}^d$, then for $t \in \mathbb{R}$

$$\partial_t (t m(x) - h(t)) \begin{cases} > 0 & \text{if} \quad t < I'(m(x)), \\ < 0 & \text{if} \quad t > I'(m(x)), \end{cases} \tag{III.2.38}$$

because $t = I'(m(x))$ is the unique maximizer of $tm(x) - h(t)$ (see Lemma III.2.5). The non negativity in (III.2.36), follows by $(tm(x) - h(t))\,|_{t=0} = 0$ and (III.2.38).

Take a $x \in \mathbb{T}^d$ and a $n' \in \mathbb{N}$, such that $f_{n'}(x) \leq I'(m(x))$. Then $f_n(x) \nearrow I'(m(x))$. Therefore, (III.2.38) implies that $f_n(x)\, m(x) - h(f_n(x)) \nearrow I(m(x))$. We can treat the case $f_n(x) = -n > I'(m(x))$ similarly. □

Remark III.2.13. *Alternatively we could also use the projective limit approach as in Lemma III.2.9 here.*

III.2.5 Double empirical measure

In this section we consider the double empirical measure

$$\xi_N^{\mathrm{db}} := \frac{1}{N^d} \sum_{i \in \mathbb{T}_N^d} \delta_{\left(\frac{i}{N}, \Theta_i\right)} \tag{III.2.39}$$

on the space of probability measures $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$, equipped with the Prokhorov metric $d(\cdot,\cdot)$.

Define for each $R > 0$ the following compact subset of $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$

$$\mathbb{M}_R := \left\{ \mu \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}) : \langle \mu, \theta^2 \rangle \leq R \right\}. \tag{III.2.40}$$

Lemma III.2.14. *$(\xi_N^{\mathrm{db}}, \mathbb{P}^N)$ satisfies LDP on $(\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}), d(\cdot,\cdot))$ with rate function*

$$\mathcal{H}(\mu) = \int_{\mathbb{T}^d} \mathsf{H}\left(m(x,\cdot) \middle| e^{-\Psi}\right) \mathrm{d}x, \tag{III.2.41}$$

if $\mu = m(x,\theta)\,\mathrm{d}x\mathrm{d}\theta$ and $\mu \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathbb{R})$ (see Definition III.0.1), i.e. $m(x,\theta)\,\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$ for almost all $x \in \mathbb{T}^d$. Otherwise $\mathcal{H}(\mu) = \infty$.

Lemma III.2.15. *The family $(\xi_N^{\mathrm{db}}, \mathbb{P}^N)$ is exponentially tight on $(\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}), d(\cdot,\cdot))$.*

Proof of Lemma III.2.15. We prove in the following the exponential tightness with two different families of compact sets in $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$. The first family is defined indirectly through the exponential tightness of $\mathbb{P}$. The second family is given by $\mathbb{M}_R$ defined in (III.2.40). For this family we show directly the exponential bound.

Possibility 1: For the first family of compact sets, we generalise Lemma 6.2.6 of [DZ98] to the space $\mathbb{T}^d \times \mathbb{R}$. The $\mathbb{P}$ is as a probability measure tight. Hence for each $a > 0$, that there are compact sets $\Gamma_a \subset \mathbb{R}$ such that $\mathbb{P}(\Gamma_a) \leq e^{-2a^2}(e^a - 1)$. Now define the set

$$K^a := \left\{ \pi \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right) : \pi\left(\mathbb{T}^d \times \Gamma_a\right) \geq 1 - \frac{1}{a} \right\}, \tag{III.2.42}$$

that are closed by the Portmanteau lemma. Then we get (by Prokhorov's theorem) for each $A \in \mathbb{N}$ that

$$K_A := \bigcap_{a=A}^{\infty} K^a \tag{III.2.43}$$

is compact. As in [DZ98] we conclude for these sets

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N \left[\theta^N : \frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \delta_{\left(\frac{i}{N},\theta_i\right)} \notin K_A \right] \leq -A. \tag{III.2.44}$$

Possibility 2: The compact sets $\mathbb{M}_R$ defined in (III.2.40), can be used for the exponential tightness. Indeed, by the exponential Chebyshev inequality

$$\mathbb{P}^N\left[\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right) \backslash \mathbb{M}_R\right] = \mathbb{P}^N\left[\sum_{i\in\mathbb{T}_N^d} (\theta_i)^2 > N^d R\right] \leq e^{-N^d R\kappa} \left(\int e^{\kappa\theta^2}\mathbb{P}(\mathrm{d}\theta)\right)^{N^d} \leq e^{-N^d R\kappa} C_\kappa^{N^d}. \tag{III.2.45}$$

In the last inequality we use that the integral with respect to $\mathbb{P}$ is uniformly bounded, when $\kappa < c_\Psi$ by Assumption III.2.2 c.) (by a similar calculation as in (III.2.9)). □

Proof of Lemma III.2.14. In the following proof we show the LDP of $\left(\xi_N^{\mathrm{db}}, \mathbb{P}^N\right)$ by the projective limit approach. This leads to the rate function (III.2.49). Finally, we show that this rate function equals $\mathcal{H}$.

It would also be possible to derive the LDP and the rate function (III.2.49) by applying Baldi's theorem ([DZ98] Theorem 4.5.20). For a similar setting this approach is used in [MR94] in Theorem 3.1.

Step 1: The logarithmic moment generating function: For $\phi \in \mathsf{C}_b\left(\mathbb{T}^d \times \mathbb{R}\right)$, the log moment generating function is

$$\begin{aligned} h(\phi) &:= \lim_{N\to\infty} \frac{1}{N^d} \log \int_{\mathbb{R}^{N^d}} \exp\left\{N\left\langle \phi, \mu^N \right\rangle\right\} \mathbb{P}^N\left(\mathrm{d}\theta^N\right) \\ &= \lim_{N\to\infty} \frac{1}{N^d} \log \int_{\mathbb{R}^{N^d}} \exp\left\{N\left\langle \phi, \mu^N \right\rangle\right\} \mathbb{P}^N\left(\mathrm{d}\theta^N\right) \\ &= \lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \log \int_{\mathbb{R}} e^{\phi(x,\theta)}\mathbb{P}(\mathrm{d}\theta) \\ &= \int_{\mathbb{T}^d} \log \int_{\mathbb{R}} e^{\phi(x,\theta)}\mathbb{P}(\mathrm{d}\theta)\,\mathrm{d}x. \end{aligned} \tag{III.2.46}$$

The last equality holds because $x \to \int_{\mathbb{R}} e^{\phi(x,\theta)}\mathbb{P}(\mathrm{d}\theta)$ is continuous as a parameter dependent integral for all $\phi \in \mathsf{C}_b\left(\mathbb{T}^d \times \mathbb{R}\right)$.

Step 2: LDP via the projective limit approach: Define the random variables

$$\widehat{\xi}_N := \frac{1}{N^d}\left(\sum_{i\in\mathbb{T}_N^d} f_1(i,\Theta_i),\ldots,\sum_{i\in\mathbb{T}_N^d} f_n(i,\Theta_i)\right)\in\mathbb{R}^n, \tag{III.2.47}$$

with $\underline{f} = (f_1,\ldots,f_n)$ a collection of n continuous bounded functions on $\mathbb{T}^d\times\mathbb{R}$. Similar to Lemma III.2.7, we can show that $\left(\widehat{\xi}, P_N\right)$ satisfies the large deviation principle with good rate function

$$I_{\underline{f}}(\underline{v}) = \max_{\underline{t}}\left\{\sum_{i=1}^n v_i t_i - h\left(\sum_{i=1}^n t_i f_i\right)\right\}. \tag{III.2.48}$$

As in Lemma III.2.8, we get for $\mu\in\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$ that $I_{\underline{f}^1}(\mu)\le I_{\underline{f}^2}(\mu)$ if $\underline{f}^1\subseteq\underline{f}^2$, and that

$$\sup_{\underline{f}} I_{\underline{f}}(\mu) = \sup_{f\in\mathsf{C}_b(\mathbb{T}^d\times\mathbb{R})}\left\{\langle\mu,f\rangle - \int_{\mathbb{T}^d}\log\mathbb{E}_{\mathbb{P}}\left[e^{f(x,\theta)}\right]\mathrm{d}x\right\}. \tag{III.2.49}$$

Hence the projective limit approach ([DZ98] Theorem 4.6.9) can be used as in Lemma III.2.9. This proves the LDP on the $\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$ with the projective limit topology and rate function (III.2.49). Then the exponential tightness shown in Lemma III.2.15 implies, as in Lemma III.2.9, the LDP with the Prokhorov metric.

Step 3: (III.2.49) **is equal to** $\mathcal{H}$**:** For a more detailed version of a similar proof see the proof of Lemma V.2.8 in Chapter V.
Step 3.1: (III.2.49) **is only finite if** $\mu\in\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R})$**:** Take the supremum on the right hand side of (III.2.49) only over $f\in\mathsf{C}_b(\mathbb{T}^d\times\mathbb{R})$ with $f(x,\theta)=f(x)$. This reduced supremum is already infinite if the projection of μ to the $\mathbb{T}^d$ coordinate is not the Lebesgue measure.
Step 3.2: (III.2.49) **is smaller or equal to** $\mathcal{H}$**:** If $\mathcal{H}(\mu)<0$, then there is a density ρ such that $\mu(\mathrm{d}x,\mathrm{d}\theta)=\rho(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\mathrm{d}x$. Moreover, by the previous step $\rho(x,\theta)\,e^{-\Psi(\theta)}\mathrm{d}\theta\in\mathbb{M}_1(\mathbb{R})$ for all $x\in\mathbb{T}^d$. Then as in the second point of the proof of Theorem 3.1 in [MR94] one can show the claimed inequality.
Step 3.3: (III.2.49) **is greater or equal to** $\mathcal{H}$**:** This follows by the Jensen inequality and the variation formula of the relative entropy. □

III.3 LDP for the interacting systems

In this section we derive the large deviation principles for the weighted space empirical measure (see Section III.3.1) and the double (space-spin) empirical measure (see Section III.3.2), when the underlying spins are not any more independent. More precisely we add to the distribution according to $\mathbb{P}^N$ (considered in Section III.2) a contribution coming from interaction between the spins. We infer the large deviation principles of the interacting system from the large deviation principles of the independent spins by applying the generalization of Varadhan's lemma given in Appendix C.

Assume that the spins $\{\Theta_i\}$ are distributed according to

$$\{\Theta_i\}_{i\in\mathbb{T}_N^d}\sim\mathbb{Q}^N := \frac{1}{Z_N}e^{\frac{1}{2N^d}\sum_{i,j\in\mathbb{T}_N^d} J\left(\frac{j-i}{N}\right)\theta_i\theta_j}\prod_{i\in\mathbb{T}_N^d} e^{-\Psi(\theta_i)}\mathrm{d}\theta_i, \tag{III.3.1}$$

where Z_N is the usual partition function and the interaction weight $J:\mathbb{T}^d\to\mathbb{R}$ satisfies the following assumptions.

Assumption III.3.1. a.) *J is non-negative, non trivial, symmetric and continuous.*

b.) *Moreover, the 0-th Fourier mode $\widehat{J_0} = \int_{\mathbb{T}^d} J(x)\,\mathrm{d}x$ satisfies*

$$\widehat{J_0} < 2c_\Psi, \tag{III.3.2}$$

where c_Ψ is the constant which shows up in Assumption III.2.2 c.).

Remark III.3.2. *These assumptions are by far not the most general assumptions possible. For example one could consider a J that attains also negative values or that is not continuous, provided some convergence properties still hold. We refer to Chapter V (in particular Section V.7) for a proof of a dynamical large deviation principle under more general assumptions. The ideas how the more general assumptions are handled in that proof, could also be used here. Nevertheless, we use the more restrictive Assumption III.3.1 here, because the proofs become significantly easier to understand, although the main problems still appear.*

The measures $\mathbb{Q}^N$ and $\mathbb{P}^N$ defined in Section III.2 are related by the following Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}^N} = \frac{1}{Z_N} \exp\left\{ \frac{1}{2N^d} \sum_{i,j\in\mathbb{T}_N^d} J\left(\frac{j-i}{N}\right) \theta_i\theta_j \right\}. \tag{III.3.3}$$

This derivative is well defined by Assumption III.2.2 c.) and Assumption III.3.1 b.) and because

$$\frac{1}{2N^d} \sum_{i,j\in\mathbb{T}_N^d} J\left(\frac{j-i}{N}\right) \theta_i\theta_j \leq \left(\frac{1}{2N^d} \sum_{j\in\mathbb{T}_N^d} J\left(\frac{j}{N}\right) \right) \sum_{i\in\mathbb{T}_N^d} \theta_i^2 \approx \frac{\widehat{J_0}}{2} \sum_{i\in\mathbb{T}_N^d} \theta_i^2. \tag{III.3.4}$$

Indeed, by a similar calculation as in (III.2.9)

$$Z_N = \int_{\mathbb{R}^N} \exp\left\{ \frac{1}{2N^d} \sum_{i,j\in\mathbb{T}_N^d} J\left(\frac{j-i}{N}\right) \theta_i\theta_j \right\} \mathbb{P}^N\left(\mathrm{d}\theta^N\right) \leq \left(\int_{\mathbb{R}} e^{\frac{1}{2}\widehat{J_0}\theta^2 - \Psi(\theta)} \mathrm{d}\theta \right)^N \leq C^N. \tag{III.3.5}$$

Discussion concerning the idea of the proofs. The canonical way to prove the LDPs when considering interacting spins distributed according to $\mathbb{Q}^N$, would be to transfer the LDP of i.i.d. spins under $\mathbb{P}^N$ by using the Varadhan's lemma and Bryc's inverse Varadhan lemma (see [DZ98] Section 4.3 and 4.4).

This approach is used for example by [BET00] (see also [BET99]) to achieve the LDP for a similar model with the important differences that their spins are within a bounded set. We refer also to [KPT05] Section 2.2 for an equilibrium LDP for bounded spins with Kac interaction.

However, in the setting we consider here, the spins are not bounded. This implies that the map (that would appear in Varadhan's lemma by (III.3.3))

$$\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right) \ni \mu \mapsto \int_{\mathbb{T}^d\times\mathbb{R}} \int_{\mathbb{T}^d\times\mathbb{R}} J\left(x-x'\right) \theta\theta' \mu\left(\mathrm{d}x, \mathrm{d}\theta\right) \mu\left(\mathrm{d}x', \mathrm{d}\theta'\right) \tag{III.3.6}$$

is not continuous and not even sequentially continuous. Hence the Varadhan's lemma is not directly applicable. Similarly the corresponding map on $\mathbb{M}\left(\mathbb{T}^d\right)$ is sequentially continuous but not continuous in general We overcome this difficulty by applying an extension of Varadhan's lemma (Theorem C.1.1 in Appendix C), that holds for functions in the exponent that are nowhere continuous provided some approximation conditions are satisfied.

Another approach could be to consider a large enough subset of $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$ with a different topology such that the map (III.3.6) is continuous. A possible subset are the measures $\mathbb{M}_\infty \subset$

$\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$ (defined in (III.2.40)), equipped with a stronger topology defined similar as in [Gär88] Appendix B.1.1. However, this approach would only work for the double empirical measure and requires topology considerations we want to avoid.

Last but not least we want to describe an approach, that we use in Chapter V in Section V.3 (and is based on [DG87]), and that could be transferred to the setting considered here. Indeed, we could consider at first independent spins, that are however not any more identically distributed. Under suitable assumptions on these distributions (in particular a continuity condition is necessary), we can also derive large deviation results for the random elements under consideration. Then similar as in Section V.3.2, we can show the large deviation principles for the interacting spins. The main idea of this step to fix an interaction, which leads to non interacting spins. For these we know the large deviation principle and from this we could derive a local large deviation principle. However, we do not use this approach, because investigating separately the entropy and adding then the interaction seems to respect the physical point of view better and seems to be easier to understand for the reader.

III.3.1 Weighted space empirical measure

As in Section III.2.4, we consider in this section the weighted space empirical measures

$$\xi_N^{\mathrm{wei}} := \frac{1}{N^d} \sum_{i \in \mathbb{T}_N^d} \theta_i \delta_{\frac{i}{N}} \in \mathbb{M}\left(\mathbb{T}^d\right). \tag{III.3.7}$$

We define the interaction energy function F_L on $\mathbb{M}\left(\mathbb{T}^d\right)$ by

$$\mathbb{M}\left(\mathbb{T}^d\right) \ni \mu \mapsto F_L(\mu) := \frac{1}{2} \langle J * \mu, \mu \rangle \in \mathbb{R}. \tag{III.3.8}$$

The exponent in (III.3.3) is equal to $N^d F_L\left(\xi_N^{\mathrm{wei}}\right)$, because

$$\frac{1}{2N^d} \sum_{i,j \in \mathbb{T}_N^d} J\left(\frac{j-i}{N}\right) \theta_i \theta_j = \frac{N^d}{2} \left\langle J * \xi_N^{\mathrm{wei}}, \xi_N^{\mathrm{wei}} \right\rangle = N^d F_L\left(\xi_N^{\mathrm{wei}}\right). \tag{III.3.9}$$

We need in this section the following assumption, which is more restrictive than the assumptions needed for the double empirical measure in Section III.3.2. We discuss this in Remark III.3.8.

Assumption III.3.3. $c_\Psi > \frac{1}{2} |J|_\infty$.

The next theorem is the main result of this section.

Theorem III.3.4. *If Assumption III.2.2, Assumption III.3.1 and Assumption III.3.3 holds, then* $\left(\xi_N^{\mathrm{wei}}, \mathbb{Q}^N\right)$ *satisfies on* $\mathbb{M}\left(\mathbb{T}^d\right)$ *the LDP with good rate function*

$$L_J(\nu) := \mathcal{I}(\nu) - F_L(\nu) - \min_{\nu \in \mathbb{M}(\mathbb{T}^d)} \left\{\mathcal{I}(\nu) - F_L(\nu)\right\}. \tag{III.3.10}$$

Moreover, we show the exponential tightness of $\left(\xi_N^{\mathrm{wei}}, \mathbb{Q}^N\right)$.

Lemma III.3.5. $\left(\xi_N^{\mathrm{wei}}, \mathbb{Q}^N\right)$ *is exponentially tight on* $\mathbb{M}\left(\mathbb{T}^d\right)$ *with corresponding compact sets* K_R *defined in* (III.2.27).

We state the proof of Lemma III.3.5 in Section III.3.1.4.

Proof of Theorem III.3.4. We know by Lemma III.2.12, that $\left(\xi_N^{\mathrm{wei}}, \mathbb{P}^N\right)$ satisfies the large deviation principle with rate function $\mathcal{I}$. To infer the LDP of $\left(\xi_N^{\mathrm{wei}}, \mathbb{Q}^N\right)$ from the LDP of $\left(\xi_N^{\mathrm{wei}}, \mathbb{P}^N\right)$, we need the on the one hand the validity of the Laplace principle (Lemma III.3.6). On the other hand we require that L_J is a good rate function (Lemma III.3.7).

Lemma III.3.6. *If Assumption III.3.3 holds, then for any $G \in \mathsf{C}_b\big(\mathbb{M}\big(\mathbb{T}^d\big)\big)$,*

$$\lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N} \left[e^{N^d (G+F_L)\left(\xi_N^{\mathrm{wei}}\right)} \right] = \max_{\nu\in\mathbb{M}(\mathbb{T}^d)} \left\{ \left(F_L + G - \mathcal{I}\right)(\nu) \right\}, \tag{III.3.11}$$

where $\mathcal{I}$ is defined in (III.2.29).

In particular

$$\lim_{N\to\infty} \frac{1}{N^d} \log Z_N = \max_{\nu\in\mathbb{M}(\mathbb{T}^d)} \left\{ F_L(\nu) - \mathcal{I}(\nu) \right\}. \tag{III.3.12}$$

We prove this lemma in Section III.3.1.1 by an application of the generalised Varadhan's lemma, Theorem C.1.1 in Appendix C. The usual Varadhan's lemma is not applicable because F_L is not continuous in general. Note that by Lemma A.2 F_L is sequentially continuous, but $\mathbb{M}\big(\mathbb{T}^d\big)$ with the weak-*-topology is not a sequential space (see Appendix A (vii)).

Lemma III.3.7. *L_J is a good rate function, in particular the level sets $\mathcal{L}^{\leq c}(L_J) := \{\mu : L_J(\mu) \leq c\}$ are compact for all $c \geq 0$.*

We prove this lemma in Section III.3.1.3.

By [DE97] Theorem 1.2.3 the validity of the Laplace principle for all $G \in \mathsf{C}\big(\mathbb{T}^d\big)$ shown in Lemma III.3.6 and L_J being a good rate function, implies the LDP of $\big(\xi_N^{\mathrm{wei}}, \mathbb{Q}^N\big)$ with good rate function L_J as stated in Theorem III.3.4. □

Remark III.3.8. *In Section III.3.2 we assume a weaker assumption on the relation between Ψ and J. This is not possible here. Indeed, for the proof of Lemma III.3.6 we need a sequence of compact sets in $\mathbb{M}\big(\mathbb{T}^d\big)$ that satisfies a exponential tightness condition. Moreover, we need that F_L is bounded on each set by a constant smaller than corresponding constant of the exponential decay. However, the only suitable compact sets on $\mathbb{M}\big(\mathbb{T}^d\big)$ we have, are the sets K_R defined in* (III.2.27)*. But on these sets $F_L|_{K_R} \leq \frac{1}{2}|F|_\infty R^2$. This bound is the best upper bound we could get. For example take $\theta_1 = N^d R$ and $\theta_i = 0$ for $i \in \mathbb{T}_N^d$, $i \neq 0$. Then $\xi_N^{\mathrm{wei}} = R\delta_{\frac{1}{N}} \in K_R$ and $F_L\left(\xi_N^{\mathrm{wei}}\right) = \frac{1}{2}J(0)R^2$.*

Heuristically if $J \in L^p$ for a $p \in [0,\infty]$, then we need that $\mu \in L^q$ for $2\frac{1}{q} = 2 - \frac{1}{p}$ (compare this to the Young inequality in [LL01] Theorem 4.2) and that $\frac{1}{2}\|J\|_{L^p} \leq c_\Psi$. In this sense we interpret the set K_R as L^1 functions, i.e $q = 1$ and hence $p = \infty$. Whereas when considering the space $\mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)$, we can use the set measures with bounded second moments $\mathbb{M}_R$. These can be interpreted as L^2 functions, i.e $q = 2$ and hence $p = 1$. Then $\widehat{J}_0$ appears in the assumption instead of $|J|_\infty$.

III.3.1.1 Proof of the Laplace principle (Lemma III.3.6)

To prove Lemma III.3.6, we apply the generalised Varadhan's Lemma (Theorem C.1.1 in Appendix C). We prove at the end of this section that the conditions of this generalisation are satisfied. This requires some results that we state now. The first of these results is that the probability of being outside of K_R, defined in (III.2.27), decays at least asymptotically exponentially fast under $Z_N\mathbb{Q}^N$.

Lemma III.3.9.

$$\limsup_{R\to\infty} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N} \left[e^{N^d F_L\left(\xi_N^{\mathrm{wei}}\right)} \mathbb{1}\left\{\xi_N^{\mathrm{wei}} \notin K_R\right\} \right] = -\infty. \tag{III.3.13}$$

Lemma III.3.10. *For all $G \in \mathsf{C}_b\big(\mathbb{M}\big(\mathbb{T}^d\big)\big)$,*

$$\lim_{R\to\infty} \sup_{\nu\notin K_R} \left\{ \left(G + F_L - \mathcal{I}\right)(\nu) \right\} = -\infty. \tag{III.3.14}$$

Proof of Lemma III.3.6. We show in Step 1 that the Theorem C.1.1 of the Appendix C is applicable. From this we theorem we infer (III.3.12) with a supremum on the right hand side instead of the maximum. Then in Step 2 we show that this supremum is actually a maximum.

Step 1: Application of the generalised Varadhan's Lemma: To apply the Theorem C.1.1 of the Appendix C we show that the model we consider here is within the class defined in the example in Section C.4.2 of the Appendix C. We consider as the increasing sets the K_R.

Step 1.1: (C.4.2.ii): The sets K_R are closed by Appendix A (ix).

Step 1.2: (C.4.2.iii): This condition is obviously satisfied because the whole space $\mathbb{M}(\mathbb{T}^d)$ is the union of the sets K_R.

Step 1.3: (C.4.2.iv): By Lemma A.2 and $J \in \mathsf{C}_b(\mathbb{T}^d)$ (Assumption III.3.1), we know that F_L is sequentially continuous on K_R (with weak-*-topology). Moreover, K_R is metrisable (see Appendix A (viii)), hence F_L is also continuous on K_R.

Step 1.4: (C.4.2.v): For an arbitrary measure $\mu \in K_R$, we get

$$F_L(\mu) \le \frac{1}{2} |J|_\infty R^2 =: \alpha(R). \tag{III.3.15}$$

Step 1.5: (C.4.2.vi): This follows from (III.2.30) with $\beta(R) := \kappa R^2 - C_\kappa$ for a $\kappa \in (0, c_\Psi)$ with C_κ a constant.

Step 1.6: (C.4.2.vii): $\alpha(R) - \beta(R) = \left(\frac{1}{2}|J|_\infty - \kappa\right) R^2 + C_\kappa \to -\infty$ by Assumption III.3.3 for κ close enough to c_Ψ.

Step 1.7: (C.4.2.viii): We show this in Lemma III.3.9.

Step 1.8: (C.4.2.ix): The sufficient moment condition is satisfied, because G is bounded and

$$\mathbb{E}_{\mathbb{P}^N}\left[e^{\gamma N^d F_L(\xi_N^{\mathrm{wei}})}\right] \le \mathbb{E}_{\mathbb{P}^N}\left[e^{\gamma \frac{\widehat{J_0}}{2}\theta^2}\right]^{N^d} \le C^{N^d}, \tag{III.3.16}$$

by Assumption III.3.1 b.) and by a similar estimate as in (III.3.5).

Hence the model we consider here is within the class defined in the example in Section C.4.2 of Appendix C. Therefore, the generalise Varadhan lemma, Theorem C.1.1 of the Appendix C implies, that for any $G \in \mathsf{C}_b(\mathbb{M}(\mathbb{T}^d))$,

$$\lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{N^d (G+F_L)(\xi_N^{\mathrm{wei}})}\right] = \sup_{\nu \in \mathbb{M}(\mathbb{T}^d)} \left\{(F_L + G - \mathcal{I})(\nu)\right\}. \tag{III.3.17}$$

This supremum is finite. Indeed, we get the lower bound $G(0)$, with $\nu \equiv 0\mathrm{d}x$. Moreover, we infer from (III.3.16) an upper bound on the left hand side of (III.3.17).

Step 2: There is a maximising object in (III.3.17)**:** By Lemma III.3.10 we can restrict the supremum in (III.3.17) to the set K_R for R large enough. On K_R the function F_L is continuous. Moreover, $\mathcal{I}$ is lower semi-continuous as a rate function. Hence $G + F_L - \mathcal{I}$ is upper semi-continuous on the compact set K_R. This implies that the maximum is attained. Indeed, fix a sequence $\mu_n \in K_R$ such that $(G + F_L - L)(\mu_n) \nearrow \sup\{G + F_L - L\}$. By the compactness of K_R we get that there is a subsequence μ_{n_k} that converges in K_R to μ^*. By the upper semi-continuity we get

$$\sup\{G + F_L - L\} = \lim_{k\to\infty} (G + F_L - L)(\mu_{n_k}) \le (G + F_L - L)(\mu^*) \le \sup\{G + F_L - L\}. \tag{III.3.18}$$

Hence μ^* is a maximising object. □

III.3.1.2 Proof of the auxiliary lemmas in Section III.3.1.1

Proof of Lemma III.3.9. By the Hölder inequality we get

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^N}\left[e^{\frac{N^d}{2}\langle J*\xi_N^{\text{wei}},\xi_N^{\text{wei}}\rangle}\mathbb{1}\{\xi_N^{\text{wei}}\notin K_R\}\right] &\leq \left(\mathbb{E}_{\mathbb{P}^N}\left[e^{p\frac{N^d}{2}\langle J*\xi_N^{\text{wei}},\xi_N^{\text{wei}}\rangle}\right]\right)^{\frac{1}{p}}\mathbb{P}^N\left[\xi_N^{\text{wei}}\notin K_R\right]^{\frac{1}{q}}\\ &\leq C^N\mathbb{P}^N\left[\xi_N^{\text{wei}}\notin K_R\right]^{\frac{1}{q}}.\end{aligned} \tag{III.3.19}$$

The expectation is bounded for p small enough by a constant $C > 0$ by the same estimate we use in (III.3.5). Hence by (III.2.30), for a small enough $\kappa > 0$

$$\frac{1}{N^d}\log\mathbb{E}_{\mathbb{P}^N}\left[e^{\frac{N^d}{2}\langle J*\xi_N^{\text{wei}},\xi_N^{\text{wei}}\rangle}\mathbb{1}\{\xi_N^{\text{wei}}\notin K_R\}\right] \leq 2\log(C) - tR^2\frac{1}{q}. \tag{III.3.20}$$

The right hand side tends to minus infinity for $R \to \infty$. □

Proof of Lemma III.3.10. Choose an arbitrary $\mu \in \mathbb{M}(\mathbb{T}^d)$. Then there is a $R > 0$ such that $\mu \in K_{R+\frac{1}{R}}\backslash K_R$. Therefore

$$-\mathcal{I}(\mu) \leq -\inf_{\mu\notin K_R}\mathcal{I}(\mu) \leq \liminf_{N\to\infty}\frac{1}{N^d}\log\mathbb{P}^N\left[\xi_N^{\text{wei}}\notin K_R\right] \leq -\kappa(R)^2, \tag{III.3.21}$$

where we use at first that $\mathcal{I}$ is the rate function of the large deviation principle for $\left(\xi_N^{\text{wei}},\mathbb{P}^N\right)$ (Lemma III.2.12). In the last inequality we use the exponential tightness derived in (III.2.30) for a $\kappa \in (0, c_\Psi)$. Moreover, $F_L(\mu) \leq \frac{1}{2}|J|_\infty\left(R+\frac{1}{R}\right)^2$. Hence by Assumption III.3.3

$$\sup_{\mu\notin K_R}(G + F_L - \mathcal{I})(\mu) \leq |G|_\infty + \frac{1}{2}|J|_\infty\left(R+\frac{1}{R}\right)^2 - \kappa R^2 \to -\infty, \tag{III.3.22}$$

when $R \to \infty$. This is the claimed statement of Lemma III.3.10. □

III.3.1.3 Proof of the good rate function (Lemma III.3.7)

Proof of Lemma III.3.7. We know that $L_J : \mathbb{M}(\mathbb{T}^d) \to [0,\infty]$, by the definition (III.3.10) of L_J. To prove Lemma III.3.7, we hence need to show that the level sets $\mathcal{L}^{\leq c}(L_J)$ are compact for each $c \geq 0$. By Lemma III.3.10 we know $\mathcal{L}^{\leq c}(L_J) \subset K_R$ for a $R = R(c)$ large enough. Hence it is enough to show that $\mathcal{L}^{\leq c}(L_J)$ is closed.

Fix a sequence $\mu_n \in \{\mu : L_J(\mu) \leq M\}$, that converges to a $\mu \in \mathbb{M}(\mathbb{T}^d)$. Due to K_R being compact, we know already that $\mu \in K_R$. Moreover, L_J is lower semi-continuous on K_R (see Step 2 in the proof of Lemma III.3.6). Hence

$$c \geq \liminf_{n\to\infty} L_J(\mu_n) \geq L_J(\mu). \tag{III.3.23}$$

Hence $\mathcal{L}^{\leq c}(L_J)$ is a closed subset of K_R and therefore in particular compact. This implies that L_J is a good rate function. □

III.3.1.4 Proof of the exponential tightness (Lemma III.3.5)

Proof of Lemma III.3.5. By the relation (III.3.3) of $\mathbb{Q}^N$ and $\mathbb{P}^N$, we see that for each $r > 0$ there exists a $R > 0$ such that

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{Q}\left[\xi_N^{\text{wei}}\notin K_R\right]\\ &= \limsup_{N\to\infty}\frac{1}{N^d}\log\left(\mathbb{E}_{\mathbb{P}^N}\left[e^{\frac{N^d}{2}\langle J*\xi_N^{\text{wei}},\xi_N^{\text{wei}}\rangle}\mathbb{1}\{\xi_N^{\text{wei}}\notin K_R\}\right]\right) - \limsup_{N\to\infty}\frac{1}{N^d}\log Z_N \leq -r,\end{aligned} \tag{III.3.24}$$

by Lemma III.3.9 and (III.3.5). Because the sets K_R-s are compact (see Appendix A (ix)), we have proven the exponential tightness. □

III.3.1.5 Discussion of the proof

Let us discuss an alternative way of proving Theorem III.3.4. Instead of the application of the generalised Varadhan's Lemma (Theorem C.1.1 of Appendix C), one could restrict the analysis somehow to the sets K_R. These sets are metrisable under the weak-*-topology (see Appendix A (viii)) and F_L is continuous on these sets (by Lemma A.2).

We could for example look at the measures $\bar{\xi}_N^R = \xi_N^{\text{wei}} 1\!\!1_{\xi_N^{\text{wei}} \in K_R}$ and use

$$\mathbb{E}_{\mathbb{P}^N}\left[e^{\frac{N^d}{2}\left\langle J*\xi_N^{\text{wei}},\xi_N^{\text{wei}}\right\rangle}\right] = \mathbb{E}_{\mathbb{P}^N}\left[e^{\frac{N^d}{2}\left\langle J*\bar{\xi}_N^R,\bar{\xi}_N^R\right\rangle}\right] + \mathbb{E}_{\mathbb{P}^N}\left[\left(e^{\frac{N^d}{2}\left\langle J*\xi_N^{\text{wei}},\xi_N^{\text{wei}}\right\rangle} - 1\right) 1\!\!1\left\{\xi_N^{\text{wei}} \notin K_R\right\}\right]. \tag{III.3.25}$$

The second term is under control by (III.3.13). For the first term one could first try to prove the LDP for $\bar{\xi}_N^R$ on K_R with rate function L_J^R and apply the usual Varadhan's lemma because F_L is continuous on K_R. Finally, one needs to show the convergence of the rate functions L_J^R to L_J. In this approach, one has to handle carefully the boundary of the sets K_R.

III.3.2 Double empirical measure

As in Section III.2.5, we consider in this section the double empirical measure

$$\xi_N^{\text{db}} := \frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \delta_{\left(\frac{i}{N},\theta_i\right)} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right). \tag{III.3.26}$$

Similar as in Section III.3.1, we define the interaction energy function $F_\Lambda : \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right) \to \mathbb{R} \cup \{-\infty, \infty\}$

$$\nu \mapsto F_\Lambda(\mu) := \frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{R}} J(x-x')\,\theta\theta'\nu(\mathrm{d}x,\mathrm{d}\theta)\,\nu(\mathrm{d}x',\mathrm{d}\theta'). \tag{III.3.27}$$

For a double empirical measure $\xi_N^{\text{db}} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$ defined by $\underline{\theta}^N \in \mathbb{R}^{N^d}$,

$$F_\Lambda\left(\xi_N^{\text{db}}\right) = \frac{1}{2N^d} \sum_{i,j\in\mathbb{T}_N^d} J\left(\frac{j-i}{N}\right)\theta_i\theta_j. \tag{III.3.28}$$

We use moreover the space of probability measures with bounded second moments

$$\mathbb{M}_\infty := \left\{\mu \in \mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right) : \left\langle\theta^2,\mu\right\rangle < \infty\right\} = \bigcup_{R=1}^{\infty} \mathbb{M}_R \subset \mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right), \tag{III.3.29}$$

where $\mathbb{M}_R$ is defined in (III.2.40).

Theorem III.3.11. *If Assumption III.2.2 and Assumption III.3.1 hold, then* $\left(\xi_N^{\text{db}}, \mathbb{Q}^N\right)$ *satisfies the LDP on* $\left(\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right), d(\cdot,\cdot)\right)$ *with good rate function*

$$\Lambda_J(\nu) = \begin{cases} (\mathcal{H} - F_\Lambda)(\nu) - \min\limits_{\nu\in\mathbb{M}_\infty\cap\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R})} \left\{(\mathcal{H} - F_\Lambda)(\nu)\right\} & \text{if } \nu \in \mathbb{M}_\infty \cap \mathbb{M}_1^L\left(\mathbb{T}^d\times\mathbb{R}\right), \\ \infty & \text{else.} \end{cases} \tag{III.3.30}$$

Before we start proving this theorem, we state the exponential tightness of $\left(\xi_N^{\text{db}}, \mathbb{Q}^N\right)$ in the next lemma.

Lemma III.3.12. $\left(\xi_N^{\text{db}}, \mathbb{Q}^N\right)$ *is exponentially tight on* $(\mathbb{M}_\infty, d(\cdot,\cdot))$, *with corresponding compact sets* $\mathbb{M}_R$.

This lemma follows from Lemma III.3.16 that we state in Section III.3.2.1 by a similar proof as Lemma III.3.5. Therefore, we do not state the proof here.

Proof of Theorem III.3.11. We use the same approach for the proof of this theorem that we use for the proof of Theorem III.3.4. Indeed, we know by Lemma III.2.14, that $\left(\xi_N^{\mathrm{db}}, \mathbb{P}^N\right)$ satisfies the large deviation principle with rate function $\mathcal{H}$. To infer the LDP of $\left(\xi_N^{\mathrm{db}}, \mathbb{Q}^N\right)$ from the LDP of $\left(\xi_N^{\mathrm{db}}, \mathbb{P}^N\right)$, we need the on the one hand the validity of the Laplace principle (Lemma III.3.13). On the other hand we require that L_J is a good rate function (Lemma III.3.7).

Lemma III.3.13. *If Assumption III.3.1 holds, then for any* $G \in \mathsf{C}_b\left(\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{N^d (G+F_\Lambda)(\xi_N^{\mathrm{db}})}\right] &= \max_{\nu\in\mathbb{M}_\infty\cap\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R})} \left\{(G + F_\Lambda - \mathcal{H})(\nu)\right\} \\ &= \max_{\nu\in\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})} \left\{(G+\Lambda_J)(\nu)\right\} + \min_{\nu\in\mathbb{M}_\infty\cap\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R})} \left\{(\mathcal{H} - F_\Lambda)(\nu)\right\}.\end{aligned} \tag{III.3.31}$$

This implies in particular

$$\lim_{N\to\infty} \frac{\log Z_N}{N^d} = \max_{\nu\in\mathbb{M}_\infty\cap\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R})} \left\{(F_\Lambda - \mathcal{H})(\nu)\right\}. \tag{III.3.32}$$

We prove this lemma in Section III.3.2.1 by an application of the generalised Varadhan's lemma, Theorem C.1.1 in Appendix C. The usual Varadhan's lemma is not applicable because we do not know if F_L is continuous. Note that by Lemma A.2 F_L is sequentially continuous, but $\mathbb{M}\left(\mathbb{T}^d\right)$ with the weak-*-topology is not a sequential space (see Appendix A (vii)).

Lemma III.3.14. Λ_J *is a good rate function. In particular, for all* $c \geq 0$, *the level sets* $\mathcal{L}^{\leq c}(\Lambda_J) := \{\mu : \Lambda_J(\mu) \leq c\}$ *are compact.*

We prove this lemma in Section III.3.1.3. By [DE97] Theorem 1.2.3 the validity of the Laplace principle for all $G \in \mathsf{C}\left(\mathbb{T}^d\right)$ shown in Lemma III.3.6 and L_J being a good rate function, implies the LDP of $\left(\xi_N^{\mathrm{db}}, \mathbb{Q}^N\right)$ with good rate function L_J as stated in Theorem III.3.4. □

Proof of Lemma III.3.7. We have to show that Λ_J is a good rate function. By the definition of Λ_J the level set $L^{\leq\alpha}(\Lambda_J)$ is a subset of $\mathbb{M}_\infty$ and by Lemma III.3.18 a subset of $\mathbb{M}_R$ for a R large enough. Then as in the proof of Theorem III.3.4, we can show the closeness (and hence the compactness) of the level set. □

Remark III.3.15. *We could also restrict our attention to the subspace* $\mathbb{M}_\infty \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$ *and transfer the LDP on this subset to an LDP* $\mathbb{Q}^N$. *This is possible because for all* $N \in \mathbb{N}$ *we have* $\mathbb{P}^N\left[\xi_N^{\mathrm{db}} \in \mathbb{M}_\infty\right] = 1$ *because as in the proof of Lemma III.3.17 we know that there is a constant* $C > 0$ *such that for all* R *large enough*

$$\mathbb{P}^N\left[\xi_N^{\mathrm{db}} \notin \mathbb{M}_\infty\right] \leq \mathbb{P}^N\left[\xi_N^{\mathrm{db}} \notin \mathbb{M}_R\right] \leq e^{-N^d C R}. \tag{III.3.33}$$

This and Lemma III.3.17 are the sufficient conditions (according to Lemma 4.1.5 (b) in [DZ98]), that Λ *is a rate function of the LDP on* $\mathbb{M}_\infty$ *of* $\left(\xi_N^{\mathrm{db}}, \mathbb{P}^N\right)$.

This would have the advantage that we would not need to define Λ_J *outside* $\mathbb{M}_\infty$. *Nevertheless, it simplifies the proof only marginally and the result for the whole space* $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$ *is stronger.*

III.3.2.1 Proof of the Laplace principle (Lemma III.3.13)

To prove Lemma III.3.13, we apply the generalised Varadhan's Lemma (Theorem C.1.1 in Appendix C) as in Section III.3.1.1 for the weighted space empirical measure. We prove at the end of this section that the conditions of this generalisation are satisfied. This requires some results that we state now. The first of these results is that the probability of being outside of $\mathbb{M}_R$, defined in (III.2.40), decays at least asymptotically exponentially fast under $Z_N \mathbb{Q}^N$.

Lemma III.3.16.
$$\limsup_{R\to\infty} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N} \left[e^{N^d F_\Lambda\left(\xi_N^{\mathrm{db}}\right)} \mathbb{1}\left\{\xi_N^{\mathrm{db}} \notin \mathbb{M}_R\right\}\right] = -\infty. \tag{III.3.34}$$

Next we show that Λ is infinite for probability measures with infinite second moment.

Lemma III.3.17. *$\mathcal{H}(\nu) = \infty$ if $\nu \notin \mathbb{M}_\infty$.*

Moreover, we prove that a measure with bounded but huge second moment can never be a maximiser in (III.3.31).

Lemma III.3.18. *For any $G \in \mathsf{C}_b\left(\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$*
$$\lim_{R\to\infty} \sup_{\nu\in\mathbb{M}_\infty \cap \mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R}):\nu\notin\mathbb{M}_R} \left\{(G + F_\Lambda - \mathcal{H})(\nu)\right\} = -\infty. \tag{III.3.35}$$

In the next lemma we show that the restriction of F_Λ to each $\mathbb{M}_R$ is continuous.

Lemma III.3.19. *F_Λ is continuous on $\mathbb{M}_R$ for each $R > 0$.*

The main step of the LDP proof is the following lemma. We proof the result of Varadhan's lemma also for our F_Λ, which is not continuous on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$ (what we show in Lemma III.3.20). Note that the maximum on the right hand side of (III.3.31) is over the set $\mathbb{M}_\infty \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$. Reason for this is that otherwise the maximum would not be well defined, because F is not necessarily finite outside $\mathbb{M}_\infty$, although $\mathcal{H}$ is minus infinity there.

Proof of Lemma III.3.13. As in the proof of Lemma III.3.6 we apply Theorem C.1.1 of the Appendix C to infer the Laplace principle. Therefore, we show in Step 1 that the Theorem C.1.1 of the Appendix C is applicable. From this we theorem we infer (III.3.12) with a supremum on the right hand side instead of the maximum. Then in Step 2 we show that this supremum is actually a maximum.

Step 1: Application of the generalised Varadhan's Lemma: To apply the Theorem C.1.1 of the Appendix C, we show that the model we consider here is within the class defined in the example in Section C.4.2 of Appendix C. We consider as the increasing sets the $\mathbb{M}_R$.

Step 1.1: (C.4.2.ii): The sets $\mathbb{M}_R$ are closed as compact subsets.

Step 1.2: (C.4.2.iii): We prove this condition in Lemma III.3.17.

Step 1.3: (C.4.2.iv): We show the continuity of F_Λ on $\mathbb{M}_R$ in Lemma III.3.19.

Step 1.4: (C.4.2.v): There is a $\delta > 0$ such that for N large enough and for an arbitrary empirical measure $\xi_N^{\mathrm{db}} \in \mathbb{M}_R$ defined by $\underline{\theta}^N \in \mathbb{R}^{N^d}$,
$$\left|F_\Lambda\left(\xi_N^{\mathrm{db}}\right)\right| \le \frac{1}{2} R \frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} J\left(\frac{i}{N}\right) \le \frac{1}{2} R \left(\widehat{J_0} + \delta\right), \tag{III.3.36}$$
by the uniform continuity of J.

Now fix an arbitrary $\mu \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$ with $\mathcal{H}(\mu) < \infty$. Hence $\mu \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathbb{R}\right)$. Therefore
$$\left|F_\Lambda(\mu)\right| \le \widehat{J_0} \frac{1}{2} \int_{\mathbb{T}^d\times\mathbb{R}} \theta^2 \mu(\mathrm{d}x, \mathrm{d}\theta) \le \frac{1}{2} \widehat{J_0} R. \tag{III.3.37}$$

Hence we have shown that wit $\alpha(R) = \frac{1}{2}\left(\widehat{J_0} + \delta\right) R$ the condition (C.4.2.v) is satisfied.

Step 1.5: (C.4.2.vi): This follows from (III.2.45) with $\beta(R) = \kappa R - C_\kappa$ for $\kappa \in (0, c_\Psi)$ with C_κ a constant.
Step 1.6: (C.4.2.vii): $\alpha(R) - \beta(R) = \left(\frac{1}{2}\widehat{J_0} + \frac{1}{2}\delta - \kappa\right) R + C_\kappa \to -\infty$ by Assumption III.3.1 b.) for κ close enough to c_Ψ.
Step 1.7: (C.4.2.viii): We show this in Lemma III.3.16.
Step 1.8: (C.4.2.ix): The sufficient moment condition is satisfied as shown in (III.3.16).

Hence the model we consider here is within the class defined in the example in Section C.4.2 of Appendix C. Therefore, the generalise Varadhan lemma, Theorem C.1.1 of the Appendix C implies, that for any $G \in \mathsf{C}_b\big(\mathbb{M}(\mathbb{T}^d)\big)$,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{N^d(G+F_\Lambda)(\xi_N^{\mathrm{db}})}\right] &= \sup_{\nu\in\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R}):\mathcal{H}(\nu)<\infty} \left\{(G + F_\Lambda - \mathcal{H})(\nu)\right\} \\ &= \sup_{\nu\in\mathbb{M}_\infty\cap\mathbb{M}_1^L(\mathbb{T}^d\times\mathbb{R})} \left\{(G + F_\Lambda - \mathcal{H})(\nu)\right\}.\end{aligned} \tag{III.3.38}$$

In the last equality we use Lemma III.3.17 and that F_Λ is finite for each $\nu \in \mathbb{M}_\infty \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathbb{R}\big)$ (see (III.3.37)).

The suprema in (III.3.38) are finite. Indeed, we get the lower bound with $\nu = e^{-\Psi}\mathrm{d}\theta\mathrm{d}x$. Moreover, we infer from (III.3.16) an upper bound on the left hand side of (III.3.38).

Step 2: There is a maximising object in (III.3.38) **:** By Lemma III.3.18 we can restrict the suprema in (III.3.38) to $\mathbb{M}_R$ for R large enough. As in Step 2 of the proof of Theorem III.3.4, we get that $G + F_\Lambda - \mathcal{H}$ is upper semi continuous on $\mathbb{M}_R$ (by Lemma III.3.19 and $\mathcal{H}$ being a rate function) and we conclude that there has to be a maximizer in $\mathbb{M}_R$ (by the weak-*-compactness of $\mathbb{M}_R$). □

III.3.2.2 Proof of the auxiliary lemmas in Section III.3.1.1

Proof of Lemma III.3.16. This claim can be shown similar to the proof of Lemma III.3.9 when we use Lemma III.2.15 for an upper bound on $\mathbb{P}^N\left[\xi_N^{\mathrm{db}} \notin \mathbb{M}_R\right]$. □

Proof of Lemma III.3.17. If $\mu \notin \mathbb{M}_\infty$, then $\mu \notin \mathbb{M}_R$ for all $R > 0$. Hence for each $\kappa \in (0, c_\Psi)$ and R large enough

$$-\mathcal{H}(\mu) \le -\inf_{\nu\notin\mathbb{M}_R} \mathcal{H}(\nu) \le \liminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}^N\left[\xi_N^{\mathrm{db}} \notin \mathbb{M}_R\right] \le -\kappa R, \tag{III.3.39}$$

where we use at first that $\left(\xi_N^{\mathrm{db}}, \mathbb{P}^N\right)$ satisfies the large deviation principle with rate function $\mathcal{H}$ (Lemma III.2.14) and that $\mathbb{M}_R$ is closed. In the last inequality we use the (III.2.45). □

Proof of Lemma III.3.18. The Lemma III.3.18 can be proven similar as Lemma III.3.10. Fix a $\mu \in \mathbb{M}_{R+\frac{1}{R}} \backslash \mathbb{M}_R$. Then $F_\Lambda(\mu) \le \frac{1}{2}\widehat{J_0}R$ (see (III.3.37)). We conclude with (III.3.39), that

$$(G + F_\Lambda - \mathcal{H})(\mu) \le |G|_\infty + \widehat{J_0}\frac{1}{2}\left(R + \frac{1}{R}\right) - \kappa R \to \infty, \tag{III.3.40}$$

by Assumption III.3.1 b.). □

Proof of Lemma III.3.19. Fix a $\mu \in \mathbb{M}_R$ and a sequence $\left\{\mu^{(n)}\right\} \subset \mathbb{M}_R$, such that $\mu^{(n)} \to \mu$ in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathbb{R}\big)$. To show the continuity of F_Λ we show that $F_\Lambda\left(\mu^{(n)}\right) \to F_\Lambda(\mu)$. To this end define the cut-off function for a $M > 0$,

$$\chi_M(\theta) := (\theta \wedge M) \vee -M. \tag{III.3.41}$$

Then

$$\begin{aligned}
&\left|F_\Lambda\left(\mu^{(n)}\right)-F_\Lambda(\mu)\right|=\left|\int J(x-x')\,\theta'\theta\left(\mu^{(n)}\otimes\mu^{(n)}-\mu\otimes\mu\right)\right|\\
&\le\left|\int J(x-x')\left(\theta'\theta-\chi_M(\theta')\,\chi_M(\theta)\right)\mu^{(n)}\otimes\mu^{(n)}\right|\\
&+\left|\int J(x-x')\left(\theta'\theta-\chi_M(\theta')\,\chi_M(\theta)\right)\mu\otimes\mu\right|\\
&+\left|\int J(x-x')\,\chi_M(\theta')\,\chi_M(\theta)\left(\mu^{(n)}\otimes\mu^{(n)}-\mu\otimes\mu\right)\right|=:①+②+③\,.
\end{aligned}\tag{III.3.42}$$

We bound ① by Cauchy-Schwartz inequality

$$①\le\left(\mu^{(n)}\otimes\mu^{(n)}\left[|\theta|>M\text{ or }|\theta'|>M\right]\right)^{\frac12}|J|_\infty\left(\int|\theta\theta'|^2\mu^{(n)}\otimes\mu^{(n)}\right)^{\frac12}\le\epsilon^{\frac12}|J|_\infty R,\tag{III.3.43}$$

independent of $n\in\mathbb{N}$, by the tightness of $\{\mu^{(n)}\otimes\mu^{(n)}\}$, for a fixed M large enough. The sequence of these product measures is tight, by the Prokhorov theorem as a converging sequence of measures (this is true by [Bil99] Theorem 2.8). By the same argument we bound ②.

For fixed M, the integrand of ③ is bounded and continuous. Hence by the convergence of $\mu^{(n)}\otimes\mu^{(n)}\to\mu\otimes\mu$, this integral vanishes when n tends to infinity.

Therefore, (III.3.42) vanishes, what implies the continuity of F_Λ on $\mathbb{M}_R$. □

III.3.2.3 Additional properties of F_Λ

In the following lemma we show that F_Λ is not (sequentially) continuous. Reason for this is that mass might escape to infinity in the weak-*-topology on $\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$. But the functional F_Λ does not ignore this mass, by its unbounded integrand.

Lemma III.3.20. *The function F_Λ is not (sequentially) continuous on $\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$ (or on $\mathbb{M}_\infty$), equipped with the topology of weak convergence.*

Proof. We show that F_Λ is not even sequentially (weak-*) continuous by constructing a converging sequence for which F_Λ does not converge.

Fix an arbitrary $\mu\in\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$. For $n\in\mathbb{N}$, define

$$\mu^{(n)}(\mathrm{d}x,\mathrm{d}\theta):=\left(1-\frac1n\right)\mu(\mathrm{d}x,\mathrm{d}\theta)+\frac1n\left(\mathrm{d}x\times U_{n-1,n}(\mathrm{d}\theta)\right)\in\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R}),\tag{III.3.44}$$

with $U_{n-1,n}$ the uniform distribution on $[n-1,n]\subset\mathbb{R}$.

Then $\mu^{(n)}\to\mu$, because for $f\in\mathsf{C}_b(\mathbb{T}^d\times\mathbb{R})$

$$\left\langle f,\mu^{(n)}\right\rangle=\left(1-\frac1n\right)\langle f,\mu\rangle+\frac1n\int_{n-1}^n\int_{\mathbb{T}^d}f(x,\theta)\,\mathrm{d}x\mathrm{d}\theta\to\langle f,\mu\rangle\,,\tag{III.3.45}$$

by the boundedness of f. But

$$\begin{aligned}
F_\Lambda\left(\mu^{(n)}\right)&=\left(1-\frac1n\right)^2\int_{\mathbb{T}^d\times\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{R}}J(x-x')\,\theta\theta'\mu(\mathrm{d}x,\mathrm{d}\theta)\,\mu(\mathrm{d}x',\mathrm{d}\theta')\\
&+\frac1{n^2}\widehat{J_0}\left(\int_{n-1}^n\theta\mathrm{d}\theta\right)^2+\frac1n\left(1-\frac1n\right)\widehat{J_0}\int_{n-1}^n\theta'\mathrm{d}\theta'\int_{\mathbb{T}^d\times\mathbb{R}}\theta\mu(\mathrm{d}x,\mathrm{d}\theta)\\
&=\left(1-\frac1n\right)^2F_\Lambda(\mu)+\widehat{J_0}\frac1{n^2}\left(n-\frac12\right)^2+\widehat{J_0}\frac{n-\frac12}{n}\left(1-\frac1n\right)\int_{\mathbb{T}^d\times\mathbb{R}}\theta\mu(\mathrm{d}x,\mathrm{d}\theta)\\
&\to F_\Lambda(\mu)+\widehat{J_0}+\widehat{J_0}\int_{\mathbb{T}^d\times\mathbb{R}}\theta\mu(\mathrm{d}x,\mathrm{d}\theta)\,.
\end{aligned}\tag{III.3.46}$$

Hence, if $\int_{\mathbb{T}^d\times\mathbb{R}} \theta\mu\left(\mathrm{d}x, \mathrm{d}\theta\right) \neq -1$, $F_\Lambda\left(\mu^{(n)}\right)$ does not converge to $F_\Lambda\left(\mu\right)$. Therefore, F_Λ is not continuous on $\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)$.

If $\mu\in\mathbb{M}_\infty$, then also $\mu^{(n)}\in\mathbb{M}_\infty$. Hence we have also shown that F_Λ is not continuous on $\mathbb{M}_\infty$. □

Next we show that F_Λ is neither upper nor lower semi-continuous.

Lemma III.3.21. *F_Λ is neither upper nor lower semi-continuous.*

Proof. Fix an arbitrary $\alpha\in\mathbb{R}$ and $\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)$ with

$$\int_{\mathbb{T}^d\times\mathbb{R}} \theta\mu\left(\mathrm{d}x, \mathrm{d}\theta\right) \geq 0 \quad \text{and} \quad F_\Lambda\left(\mu\right) \leq \alpha - \frac{1}{2}\widehat{J_0}. \tag{III.3.47}$$

We show now that $\mathcal{L}^{\geq\alpha}\left(F_\Lambda\right) := \left\{\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right) : F_\Lambda\left(\mu\right)\geq\alpha\right\}$ is not closed. The sequence $\left\{\mu^{(n)}\right\}$ defined in (III.3.44) converges to μ (see (III.3.45)). Moreover, $\mu^{(n)}\in\mathcal{L}^{\geq\alpha}\left(F_\Lambda\right)$ for n large enough, because $F_\Lambda\left(\mu^{(n)}\right)\geq\alpha+\frac{1}{4}\widehat{J_0}$ (see (III.3.46)). Hence $\mathcal{L}^{\geq\alpha}\left(F_\Lambda\right)$ is not closed.

By the same arguments $\mathcal{L}^{\leq\alpha}$ is not closed, when fixing $\alpha\in\mathbb{R}$ and $\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)$ with

$$\int_{\mathbb{T}^d\times\mathbb{R}} \theta\mu\left(\mathrm{d}x, \mathrm{d}\theta\right) \leq -2 \quad \text{and} \quad F_\Lambda\left(\mu\right) = \alpha + \frac{1}{2}\widehat{J_0}. \tag{III.3.48}$$

□

Chapter IV

Energy landscape

As motivated in the introduction in Section 0.8, we investigate in this chapter the landscape of the functional L_J defined for $\nu \in \mathbb{M}(\mathbb{T}^d)$ by

$$L_J(m) = \mathcal{I}(m) - F_L = \int_{\mathbb{T}^d} I(m(x))\,\mathrm{d}x - \lambda\frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-y)\,m(x)\,m(y)\,\mathrm{d}x\mathrm{d}y, \tag{IV.0.1}$$

if $\nu(\mathrm{d}x) = m(x)\mathrm{d}x$ and $L_J(\nu) = \infty$ otherwise, and of the functional Λ_J defined for $\mu \in \mathbb{M}_1(\mathbb{T}^d)$ by

$$\begin{aligned}\Lambda_J(\rho) &= \mathcal{H}(\rho) - F_\Lambda(\rho)\\ &= \int_{\mathbb{T}^d} \mathsf{H}\left(\rho(x,\cdot)\middle|e^{-\Psi}\right)\mathrm{d}x - \lambda\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{R}} J(x-x')\,\theta'\theta\rho(x,\theta)\,\rho(x',\theta')\,\mathrm{d}x\mathrm{d}\theta\mathrm{d}x'\mathrm{d}\theta',\end{aligned} \tag{IV.0.2}$$

if $\mu(\mathrm{d}x,\mathrm{d}\theta) = \rho(x,\theta)\,\mathrm{d}x\mathrm{d}\theta$ such that $\rho(x,\theta)\,\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$ and $\int_{\mathbb{T}^d\times\mathbb{R}}\theta^2\mu(\mathrm{d}x,\mathrm{d}\theta) < \infty$. Otherwise $\Lambda_J(\mu) = \infty$. These are the rate functions that we derived in Chapter IV in Theorem III.3.4 and Theorem III.3.11, with the additional parameter $\lambda \geq 0$ for the intensity of the interaction.

First, we investigate the functional L_J in Section IV.1. We characterise depending on the parameter λ, the minima, further critical points, bifurcations of critical points and lowest paths between the minima of L_J. From these results, we infer in Section IV.2 the same properties of Λ_J. We have sketched the differences in the landscape for different interaction intensities in Figure 0.7, Figure 0.8 and Figure 0.9 in the introduction, if Assumption IV.0.3 holds.

In this whole chapter, we assume that the assumptions of Chapter III on Ψ (Assumption III.2.2) and on J (Assumption III.3.1) hold. Moreover, let the following assumption be satisfied. We do not state the validity of these assumptions explicitly again.

Assumption IV.0.1. *The* 0*-th Fourier mode* $\widehat{J_0} = \int_{\mathbb{T}^d} J(x)\,\mathrm{d}x$ *satisfies*

$$\lambda\widehat{J_0} < c_\Psi, \tag{IV.0.3}$$

where c_Ψ *is the constant which shows up in Assumption III.2.2 c.).*

Note that we assume in addition in Section IV.1.4 (Assumption IV.1.57), that J depends on the distance of two points.

The Assumption IV.0.1 is more restrictive on the relation between J and Ψ as Assumption III.3.1. We need this stronger condition in Lemma IV.1.5 to show that the energy $L_J(m)$ can always be reduced by taking a smooth cutoff of a m outside a L^∞-ball. This property is a key element in the proofs in the subsequent sections.

Remark IV.0.2. *In contrast to the assumptions in [CES86] we do not assume that* h' *is bounded. This would imply that* $I(v) = \infty$ *for* $|v| > \sup|h'|$*, by the definition* (III.2.2) *of* I*. Then the law* $\mathbb{P}$ *of the random variables* Θ *has to have bounded support, a condition that is obviously not satisfied.*

We need an additional assumption for some of the results of the next sections (to be more precise in Section IV.1.2.3, Section IV.1.3.2, Section IV.1.3.3 and Section IV.1.4.4). We indicate explicitly, when this assumption has to hold, in contrast to the previously mentioned assumptions.

Assumption IV.0.3. *h' is strictly concave on $[0, \infty)$.*

This assumption is crucial for some results. For example global minimiser can be born somewhere away from 0 without this assumption (see the example in Section IV.1.2.4) in contrast to the situation with this assumption (Theorem IV.1.36 and Lemma IV.1.37).

Remark IV.0.4. *Let us assume that the GHS (Griffiths-Hurst-Sherman) inequality holds (see e.g. [EMN76] (1.4) or [GHS70]). This inequality implies that $h''' \leq 0$ on $[0, \infty)$, i.e. h' is concave on this set. If we assume further that $\mathbb{P}$ is not Gaussian (i.e. $\Psi \neq c\theta^2 + \log(C)$), then h' is not only concave, but also strictly concave on $[0, \infty)$ (by the real analyticity of h, as shown in [EE83] footnote 6). Hence, the GHS inequality and $\mathbb{P}$ being not a Gaussian measure, imply that the Assumption IV.0.3 is satisfied.*

Example IV.0.5. *Classes of measures, for which the GHS inequality, and hence the Assumption IV.0.3, are satisfied, are shown in [EMN76] in Theorem 1.2. In particular by [EMN76] Theorem 1.2.c), $\mathrm{e}^{-\Psi}$ satisfies the GHS inequality if Ψ' is convex on $[0, \infty)$. Easy examples for such a Ψ are*

(i) $\Psi(\theta) = \theta^{2k}$ *for all* $k \geq 2$, *or*

(ii) $\Psi(\theta) = \theta^4 - \theta^2$ *(double well).*

Note that each critical value can be shifted on the torus $\mathbb{T}^d$. Then a non-constant, critical value represents an (up to d-dimensional) manifold. This has the consequence, that at each saddle point, the energy level in these shift directions is constant.

Throughout this chapter we use the functions I, h defined in Lemma III.2.4, $\mathcal{I}$ in Lemma III.2.12 and $\mathcal{H}$ in Lemma III.2.14.

IV.1 Energy landscape of L_J

In this section we investigate the landscape of L_J by characterising its minima (Section IV.1.2), lowest paths between the minima (Section IV.1.3) and bifurcations of curves of critical values (Section IV.1.4). To simplify the notation we write in this section F instead of F_L.

The functional L_J is defined on the space of signed measures $\mathbb{M}(\mathbb{T}^d)$. This space has a couple disadvantages, mainly that it is not a Hilbert space. Therefore, we show in Section IV.1.1.1, that we can restrict ourselves on functions in $L^2(\mathbb{T}^d)$ instead of looking at measures in $\mathbb{M}(\mathbb{T}^d)$. On this Hilbert space, we would like to use that at a critical value of $\mathcal{I} - F$ the Gâteaux derivative has to vanish in each direction, what would be equivalent to

$$\lambda J * m^*(.) = I'(m^*(.)) \quad \text{in } L^2(\mathbb{T}^d). \tag{IV.1.1}$$

The problem is that even at points where $\mathcal{I}$ is well-defined, it is not Gâteaux differentiable in each direction of $L^2(\mathbb{T}^d)$. In particular, it is not Gâteaux differentiable in directions that are not in $L^\infty(\mathbb{T}^d)$. The functional $\mathcal{I}$ is even nowhere continuous on $L^2(\mathbb{T}^d)$.

To overcome this problem of differentiability, we show in Section IV.1.1.2 that for a R large enough, the value of L_J is always decreases when we cut the function at height $\pm R$ (Lemma IV.1.5). Therefore, all minima are inside the set $\left\{m \in L^2(\mathbb{T}^d) : \|m\|_{L^\infty} \leq R\right\}$ and also all lowest paths between these minima do not leave this set (see Section IV.1.1.3 and Section IV.1.1.4). Then we define functionals that are C^1 and equal $\mathcal{I} - F$ on this set of bounded L^∞ functions. This allows us to set up a variational problem, such that we get at least for these functionals the necessary condition (IV.1.1) for a critical value (Section IV.1.1.6).

Then in Section IV.1.2.2, we characterise the minima to be the solution to the one dimensional problem. In Section IV.1.2.3 we show under the Assumption IV.0.3 that the first important change of the system behaviour arises when increasing the intensity λ of the interaction because we have then two minima instead of one minimum.

To understand the lowest paths between two minima, we prove (in Section IV.1.3.1) a special type of the mountain pass theorem. From this theorem we infer abstract information about lowest paths and the existence of further critical points. Then we show in Section IV.1.3.3, under Assumption IV.0.3, that lowest paths follow the constant functions for some values of λ. For higher values of λ, this does not hold any more. This requires a profound study of the operator $J *.$ and of its eigenvalues (Section IV.1.1.5).

Finally, we want to understand bifurcations of curves that consists of critical values. The constant function that equals 0 is always a solution of (IV.1.1). Therefore, we characterise in Section IV.1.4.3 all bifurcation curves from the trivial solution by applying (local) bifurcation theory. Under the Assumption IV.0.3, we show in Section IV.1.4.4 further bifurcation results, for example that no bifurcation from the first bifurcating curve occurs.

IV.1.1 Preliminaries

IV.1.1.1 Reduction to L^2

Instead of searching critical values of L_J on the whole space $\mathbb{M}(\mathbb{T}^d)$, we restrict ourselves to elements in $L^2(\mathbb{T}^d)$. This is justified because each $m \in L^2(\mathbb{T}^d)$ is the density of a finite signed measure and L_J is only finite for measures that have such a density by the next Lemma IV.1.1. Therefore, we are investigating in the following the functional

$$L^2\left(\mathbb{T}^d\right) \ni m \to \mathcal{I}(m) - \lambda \frac{1}{2} \langle J * m, m \rangle \in \mathbb{R}. \tag{IV.1.2}$$

In [CES86] and [EE83], similar maximising problem are analysed. But their results do not cover the above problem, because the authors require that the support of $\mathbb{P}$ is bounded (see also Remark IV.0.2). Nevertheless, we transfer in the following some of their ideas to our situation.

Lemma IV.1.1. *If $\mathcal{I} - F(\nu) < \infty$ for a $\nu \in \mathbb{M}(\mathbb{T}^d)$, then ν has a density $m \in L^2(\mathbb{T}^d)$ w.r.t. the Lebesgue measure.*

Proof. Let $\nu \in \mathbb{M}(\mathbb{T}^d)$ such that $\mathcal{I} - F(\nu) < \infty$. We know by the boundedness of J that $F(\nu) < \infty$ and hence also $\mathcal{I}(\nu) < \infty$. This implies that ν has a density $m \in L^1(\mathbb{T}^d)$. This density is also in L^2, because

$$\mathcal{I}(m) = \int_{\mathbb{T}^d} I(m(x))\,\mathrm{d}x \geq \kappa \int_{\mathbb{T}^d} |m(x)|^2\,\mathrm{d}x - C_{I,\kappa}, \tag{IV.1.3}$$

by Lemma III.2.6 for an arbitrary $\kappa \in (0, c_\Psi)$ and a constant $C_{I,\kappa} > 0$ that is independent of m. □

IV.1.1.2 Restriction to bounded L^∞-norm

In this section we show that for each local magnetisation profile $m \in L^2(\mathbb{T}^d)$ the energy $\mathcal{I} - F$ decreases, if we decrease its absolute value outside some set $[-R, R]$ for R large enough (Lemma IV.1.5). If we choose the way how to decrease the absolute value suitably, then we gain regularity (compared to the unchanged functional $\mathcal{I} - F$) of the composition of $\mathcal{I} - F$ with the absolute value decreasing function (Lemma IV.1.9 and Lemma IV.1.10).

We denote the function that decreases the absolute value outside the set $[-R, R]$ by a_R. One example of such a function is a smoothed cutoff function, like the one we define in (IV.1.27). This would be enough in the following sections, because we just need one fixed function a_R, that leads to regularity of $\mathcal{I} - F$. However, it is not more complicated to prove the results for more general functions. Moreover, this has the advantage, that it becomes obvious which property of the function is needed in the proof of each particular result.

Definition IV.1.2. *We denote a set of function $\{a_R\}_{R>0} : \mathbb{R} \to \mathbb{R}$ a set of suitable cutoff-like functions, if for each $R \in \mathbb{R}_+$*

(i) *a_R is increasing,*

(ii) *$a_R(x) = x$ for $x \in [-R, R]$,*

(iii) *$|a_R(x)| \leq |x|$ for all $x \in \mathbb{R}$,*

(iv) *a_R is Lipschitz continuous with Lipschitz constant at most one,*

(v) *$a_R \in \mathsf{C}^2$,*

(vi) *a'_R and a''_R bounded and*

(vii) *$I(a_R(.))$ has a bounded second derivative, i.e. $\left|\partial_t^2 I(a_R(t))\right|_\infty < \infty$.*

Remark IV.1.3. *The conditions (vii) is for example satisfied, when a_R, a'_R, a''_R are bounded, because then $\partial_t^2 I(a_R(t)) = I''(a_R)(a'_R)^2 + I'(a_R)a''_R$ is a bounded function.*

Let us fix a function a_R that has these properties. Then we define the map $A_R : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ by

$$A_R(m)(x) := a_R(m(x)). \tag{IV.1.4}$$

Moreover, we use the notation

$$F_R := F \circ A_R, \quad \mathcal{I}_R := \mathcal{I} \circ A_R, \quad I_R := I \circ a_R. \tag{IV.1.5}$$

Proposition IV.1.4. *Let $\{a_R\}_{R>0}$ be a set of function that satisfies the Definition IV.1.2. Then there is a $R^* > 0$ such that for all $R \geq R^*$ the functionals $\mathcal{I}_R$ and F_R are C^1 with Fréchet derivatives*

$$F'_R(m) = \left\{g \to \lambda \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-y)\, a'_R(m(x))\, g(x)\, a_R(m(y))\,\mathrm{d}x\mathrm{d}y\right\} \tag{IV.1.6}$$

$$\mathcal{I}'_R(m) = \left\{g \to \int_{\mathbb{T}^d} I'_R(m(x))\, g(x)\,\mathrm{d}x\right\}, \tag{IV.1.7}$$

and for all $m \in L^2(\mathbb{T}^d)$ with $\|m\|_{L^\infty} > R$

$$(\mathcal{I} - F)(m) > (\mathcal{I}_R - F_R)(m). \tag{IV.1.8}$$

We prove this proposition by separating its claims into the Lemma IV.1.5, Lemma IV.1.9 and Lemma IV.1.10 and show them separately. This has the advantage that we can point out in each of the lemmas the conditions on a_R that are really necessary.

Lemma IV.1.5. *Let $\{a_R\}_{R>0}$ satisfy Definition IV.1.2 (i) - (iv). Fix an arbitrary $\kappa \in (0, c_\Psi]$, $\kappa < \infty$. Then there is a $R^* > 0$ such that for all $R \geq R^*$ and for each $m \in L^2(\mathbb{T}^d)$ with $\|m\|_{L^\infty} > R$*

$$\begin{aligned}(\mathcal{I} - F)(m) &> (\mathcal{I}_R - F_R)(m) + \underbrace{\frac{1}{2}\left(\kappa - \lambda\widehat{J_0}\right)\int_{\mathbb{T}^d} m(x)^2 - a_R(m(x))^2\,\mathrm{d}x}_{>0} \\ &> (\mathcal{I}_R - F_R)(m) = (\mathcal{I} - F)(A_R(m)).\end{aligned} \tag{IV.1.9}$$

For $\|m\|_{L^\infty} \leq R$ both sides are equal.

Remark IV.1.6. *The conditions on a_R of this Lemma are for example satisfied for the restriction to $\pm R$ outside $[-R,R]$: $a_R^{Cut}(t) := -R \vee (t \wedge R)$ (where we denote the corresponding functional by A_R^{Cut}). However, for this function, the other conditions of Definition IV.1.2 are not satisfied. Therefore, the other claims of Proposition IV.1.4, like L_R and F_R being C^1 are not valid in general.*

Proof of Lemma IV.1.5. If the set $\left\{y \in \mathbb{T}^d : |m(y)| > R\right\}$ has Lebesgue measure zero, then by definition both sides are equal. So we can restrict ourselves in the following on $m \in L^2\left(\mathbb{T}^d\right)$ with $\|m\|_{L^\infty} > R$. Relying on I' being increasing and on (III.2.8), we know that for all $\kappa \in (0, c_\Psi]$, $\kappa < \infty$ and t large enough

$$I'(t) \geq \kappa t. \tag{IV.1.10}$$

This implies that for t large enough

$$\begin{aligned} I(t) &= I(a_R(t)) + \int_{a_R(t)}^t I'(s)\,\mathrm{d}s \geq I(a_R(t)) + \int_{a_R(t)}^t \kappa s\ \mathrm{d}s \\ &\geq I(a_R(t)) + \kappa \frac{1}{2}\left(t^2 - a_R(t)^2\right). \end{aligned} \tag{IV.1.11}$$

For $t << 0$, we get the same bound by the symmetry of I. Moreover, we know that

$$\begin{aligned} &F(m) - F(A_R(m)) \\ &= \frac{1}{2}\lambda\widehat{J_0}\int_{\mathbb{T}^d} (m(x))^2 - a_R(m(x))^2\,\mathrm{d}x \\ &\quad + \frac{1}{4}\lambda \underbrace{\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-y)\left((a_R(m(x)) - a_R(m(y)))^2 - (m(x)-m(y))^2\right)\mathrm{d}x\mathrm{d}y}_{=:\textcircled{1}\leq 0}, \end{aligned} \tag{IV.1.12}$$

where we use Definition IV.1.2 (iv) to get the upper bound on ①. Then by (IV.1.11)

$$F(m) - F(A_R(m)) \leq \frac{1}{2}\left(\lambda\widehat{J_0} - \kappa\right)\int_{\mathbb{T}^d} (m(x))^2 - a_R(m(x))^2\,\mathrm{d}x + \mathcal{I}(m) - \mathcal{I}(A_R(m)). \tag{IV.1.13}$$

Then Assumption IV.0.1 implies the lemma. □

Remark IV.1.7. *Note that we need Assumption IV.0.1 in the proof, by the factor $\frac{1}{2}$ in the lower bound on $I(t)$ in* (IV.1.11).

In the next lemma we show that already F is C^1.

Lemma IV.1.8. *The functional F is C^1 (continuously Fréchet differentiable) on $L^2\left(\mathbb{T}^d\right)$ with derivative $F'(m)(g) = \lambda\langle J * m, g\rangle$.*

Proof. F is it is Fréchet differentiable because for arbitrary $m, g \in L^2\left(\mathbb{T}^d\right)$

$$|F(m+g) - F(m) - F'(m)(g)| = \frac{\lambda}{2}\left|\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-y)\,g(x)\,g(y)\,\mathrm{d}y\mathrm{d}x\right| \leq \frac{\lambda}{2}\widehat{J_0}\,\|g\|_{L^2}^2. \tag{IV.1.14}$$

The continuity of the Fréchet differentiability follows from

$$\begin{aligned} |F'(m_1)(g) - F'(m_2)(g)| &= \lambda\int_{\mathbb{T}^d} (J * (m_1 - m_2)(x))\,g(x)\,\mathrm{d}x \\ &\leq \lambda\,|J|_\infty\,\|m_1 - m_2\|_{L^2}\,\|g\|_{L^2}. \end{aligned} \tag{IV.1.15}$$

□

Although F is C^1 by Lemma IV.1.8, we do not get this property for F_R as a composition of C^1 functions because A_R is in general not $\mathsf{C}^1\left(L^2\left(\mathbb{T}^d\right), L^2\left(\mathbb{T}^d\right)\right)$ (it is in general not even Fréchet differentiable). However, under some boundedness assumptions on the function a_R and its derivatives, we derive in the next lemma that F_R is a C^1 functional.

Lemma IV.1.9. *Let $\{a_R\}_{R>0}$ satisfy Definition IV.1.2 (v). Moreover, if either*

(i) *Definition IV.1.2 (vi) holds or*

(ii) *a_R'' are bounded and $|a_R'(x)| \leq 1 + \alpha |x|$*

then $F_R(m)$ is C^1 (continuously Fréchet differentiable) functional, with Fréchet derivative (IV.1.6).

Proof. ***(i)*** **Step 1: F_R is Fréchet differentiable:** Using the formula for the Fréchet derivative we have

$$\begin{aligned}
&|F_R(m+g) - F_R(m) - F_R'(m)(g)| \\
&= \Big|\lambda \int\int J(x-y)\Big\{\frac{1}{2}a_R(m(x)+g(x))\,a_R(m(y)+g(y)) - \frac{1}{2}a_R(m(x))\,a_R(m(y)) \\
&\qquad -a_R'(m(x))\,g(x)\,a_R(m(y))\Big\}\mathrm{d}x\mathrm{d}y\Big| \\
&\leq \lambda \int\int J(x-y)\Big\{\Big|a_R(m(y))\Big|\Big|a_R(m(x)+g(x)) - a_R(m(x)) - a_R'(m(x))\,g(x)\Big| \\
&\qquad +\frac{1}{2}\Big|a_R(m(x)+g(x)) - a_R(m(x))\Big|\Big|a_R(m(y)+g(y)) - a_R(m(y))\Big|\Big\}\mathrm{d}x\mathrm{d}y
\end{aligned} \tag{IV.1.16}$$

$$\begin{aligned}
&\leq \lambda \frac{1}{2}|a_R''|_\infty |J|_\infty \int \Big|a_R(m(y))\Big|\mathrm{d}y \int |g(x)|^2\,\mathrm{d}x \\
&\quad + \lambda |J|_\infty \frac{1}{2}\left(\int \Big|a_R(m(x)+g(x)) - a_R(m(x))\Big|\mathrm{d}x\right)^2,
\end{aligned}$$

where we use the Taylor formula for the first summand

$$\Big|a_R(m+g(x)) - a_R(m(x)) - a_R'(m(x))\,g(x)\Big| \leq |a_R''(\theta)|\,g(x)^2\,\frac{1}{2}. \tag{IV.1.17}$$

By the assumptions on a_R we get that

$$\text{(IV.1.16)} \leq \frac{\lambda}{2}|a_R''|_\infty |J|_\infty |a_R'|_\infty \|m\|_{L^2}\|g\|_{L^2}^2 + \frac{\lambda}{2}|J|_\infty |a_R'|_\infty^2 \|g\|_{L^2}^2 \leq \lambda C\|g\|_{L^2}^2. \tag{IV.1.18}$$

Step 2: F_R' is continuous: We fix a sequence $m_n \to m \in L^2$ and an arbitrary $g \in L^2$ and we get

$$\begin{aligned}
&\Big|F_R'(m_n)(g) - F_R'(m)(g)\Big| \\
&\leq \lambda \int\int J(x-y)\Big|g(x)\Big|\Big|a_R'(m_n(x))\,a_R(m_n(y)) - a_R'(m(x))\,a_R(m(y))\Big|\mathrm{d}x\mathrm{d}y \\
&\leq \lambda \int\int J(x-y)\Big|g(x)\Big|\Big\{\Big|a_R'(m_n(x))\Big|\,|a_R(m_n(y)) - a_R(m(y))| \\
&\qquad\qquad + \Big|a_R'(m_n(x)) - a_R'(m(x))\Big|\Big|a_R(m(y))\Big|\Big\}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{IV.1.19}$$

Now we apply again the Taylor expansion such that

$$\begin{aligned}
\text{(IV.1.19)} \leq\; &\lambda |J|_\infty |a_R'|_\infty^2 \int \Big|g(x)\Big|\mathrm{d}x \int \Big|m_n(y) - m(y)\Big|\mathrm{d}y \\
&+ \lambda |J|_\infty |a_R''|_\infty \int \Big|g(x)\Big|\Big|m_n(x) - m(x)\Big|\mathrm{d}x \int \Big|a_R(m(y))\Big|\mathrm{d}y
\end{aligned} \tag{IV.1.20}$$

$$\leq \lambda |J|_\infty |a_R'|_\infty^2 \|g\|_{L^2}\|m_n - m\|_{L^2} + \lambda |J|_\infty |a_R''|_\infty |a_R'|_\infty \|g\|_{L^2}\|m_n - m\|_{L^2}^2 \|m\|_{L^2}.$$

Hence, we have shown the sequential continuity of F_R.

(ii) In the second case, we use the calculation of the first case.

Step 1: F_R is Fréchet differentiable: The assumptions on a_R imply

$$\begin{aligned}
\text{(IV.1.16)} &\leq \lambda \frac{1}{2} \left|a_R''\right|_\infty |J|_\infty \left(1+\alpha \|m\|_{L^2}\right) \|g\|_{L^2}^2 \\
&\quad + \lambda |J|_\infty \frac{1}{2} \left(\int 1 + \alpha \left| m(x)\, g(x) \right| \mathrm{d}x + \|g\|_{L^2}^2 \right)^2 \|g\|_{L^2}^2 \\
&\leq C \|g\|_{L^2}^2 + C \|m\|_{L^2}^2 \|g\|_{L^2}^2 + C \|g\|_{L^2}^4 .
\end{aligned} \tag{IV.1.21}$$

Step 2: F_R' is continuous:

$$\begin{aligned}
\text{(IV.1.19)} &\leq \lambda |J|_\infty \alpha \int \left| g(x) \right| \left| m_n(x) \right| \mathrm{d}x \alpha \int \left(\left| m(y) \right| + \left| m_n(y) \right| \right) \left| m_n(y) - m(y) \right| \mathrm{d}y \\
&\quad + \lambda |J|_\infty \left|a_R''\right|_\infty \int \left| g(x) \right| \left| m_n(x) - m(x) \right| \mathrm{d}x \int \left(1 + \alpha \left| m(y) \right| \right) \left| m(y) \right| \mathrm{d}y \\
&\leq C\left(\|m_n\|_{L^2}, \|m\|_{L^2} \right) \|m_n - m\|_{L^2} .
\end{aligned} \tag{IV.1.22}$$

By the convergence of m_n we know that there is a $N \in \mathbb{N}$ such that for all $n > N$, $\|m_n\|_{L^2} < \|m\|_{L^2} + \epsilon$. Therefore, the right hand side of (IV.1.19) vanishes. □

Finally, in the next lemma we show that $\mathcal{I}_R$ is also a C^1 functional if a_R satisfies some further assumptions.

Lemma IV.1.10. *If $\{a_R\}_{R>0}$ satisfies Definition IV.1.2 (v) and (vii), then $\mathcal{I}_R(m)$ is C^1 (continuously Fréchet differentiable) functionals, with Fréchet derivative* (IV.1.7).

Proof. **Step 1: $\mathcal{I}_R$ is Fréchet differentiable:** Let $m, g \in L^2\left(\mathbb{T}^d\right)$ be arbitrary, then by the boundedness of I_R''

$$\begin{aligned}
&\left| \int I_R\left(m(x) + g(x)\right) - I_R\left(m(x)\right) - I_R'\left(m(x)\right) g(x)\, \mathrm{d}x \right| \\
&\leq \frac{1}{2} \left| I_{A,R}'' \right|_\infty \int |g(x)|^2 \mathrm{d}x \leq C_R \|g\|_{L^2}^2 ,
\end{aligned} \tag{IV.1.23}$$

where we use that for all $b_1, b_2 \in \mathbb{R}$

$$\left| I_R(b_1 + b_2) - I_R(b_1) - I_R'(b_1)\, b_2 \right| \leq \frac{1}{2} \left|I_R''\right|_\infty b_2^2, \tag{IV.1.24}$$

by the Taylor formula because I_R is $\mathsf{C}^2(\mathbb{R},\mathbb{R})$ as a composition of C^2 functions.

Step 2: $\mathcal{I}_R'$ is continuous: Take $m_n, m \in L^2(\mathbb{R})$ such that $m_n \to m$ in L^2-norm. Then again we get with help of the Taylor formula

$$\begin{aligned}
\left| \mathcal{I}_R'(m_n)(g) - \mathcal{I}_R'(m)(g) \right| &\leq \left| I_{A,R}'' \right|_\infty \int |m_n(x) - m(x)|\, |g(x)|\, \mathrm{d}x \\
&\leq \left| I_{A,R}'' \right|_\infty \|m_n - m\|_{L^2} \|g\|_{L^2} .
\end{aligned} \tag{IV.1.25}$$

Hence, we conclude

$$\sup_{g \in L^2} \left| \mathcal{I}_R'(m_n)(g) - \mathcal{I}_R'(m)(g) \right| \frac{1}{\|g\|_{L^2}} \to 0. \tag{IV.1.26}$$

□

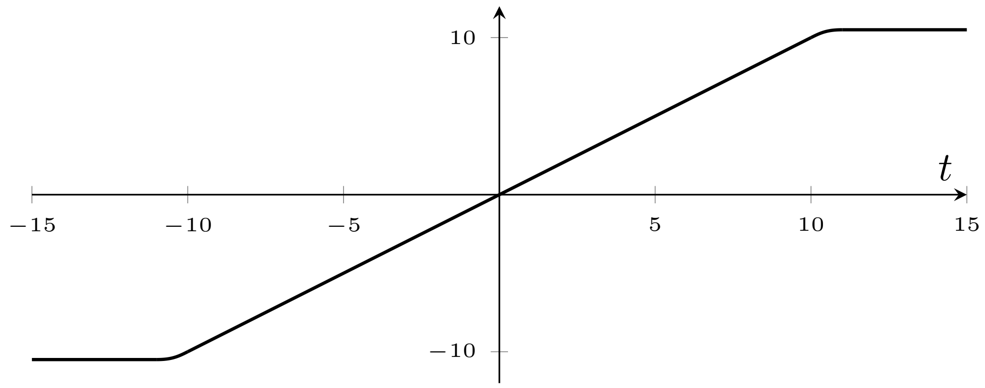


Figure IV.1: The function a_R of Example IV.1.11 1.)

We give two example of functions a_R that satisfies the Definition IV.1.2.

Example IV.1.11. *1.) The first example is a smoothed cutoff functional, that might look like*

$$a_R(t) = \begin{cases} t & \text{if } |t| \leq R, \\ \pm R + \frac{t \mp R}{(t \mp R)^2 + 1} & \text{if } t \in [\pm R, \pm(R+1)], \\ \pm R \pm 0.5 & \text{if } t \gtrless R + 1. \end{cases} \tag{IV.1.27}$$

This function satisfies obviously all the conditions of Definition IV.1.2.

2.) Although the previous example suffices for the restriction to bounded L^∞ functions in the following sections, we give a second example of a function a_R. This function is unbounded and it is interesting, because it leads to linear growth of I_R outside a compact set.

We denote by $I_+ : \mathbb{R}_+ \to [I(0), \infty)$ the restriction of I to the positive real numbers. This function is bijective, increasing and convex. Hence, it has an inverse $H_+ := I_+^{-1}$ which is concave and increasing.

Lemma IV.1.12. *For R large enough and $k \in \mathbb{R}_+$ is chosen such that $I'(R) > k$ for R large enough we define the function*

$$a_R(x) = \begin{cases} H_+(k(x - R - 2) + I(R+1)) & \text{if } x > R + 2, \\ , x & \text{if } x \in [-R, R]. \\ -H_+(k(-x - R - 2) + I(R+1)) & \text{if } x < -R - 2, \end{cases} \tag{IV.1.28}$$

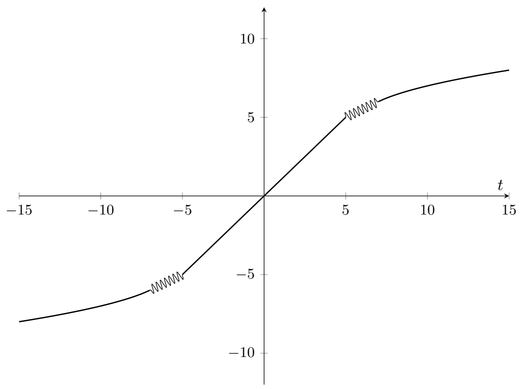
with some smoothing for $[-R-2, -R]$ and $[R, R+2]$ such that $a_R \in \mathsf{C}^2$.

This function satisfies the conditions of Proposition IV.1.4.

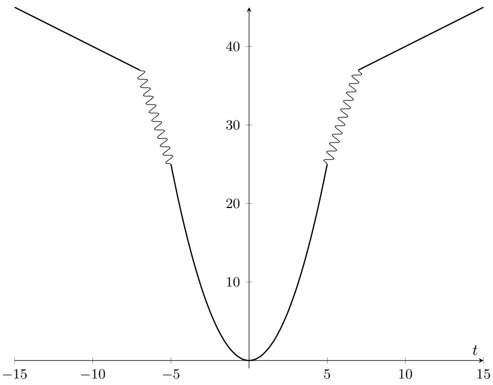
Proof. The first derivative of H_+ is given by $H_+'(x) = \left[I'\left(I^{-1}(x)\right)\right]^{-1}$ and its second derivative by $H_+''(x) = -\frac{I''\left(I^{-1}(x)\right)}{[I'(I^{-1}(x))]^3}$. Both are continuous as compositions of continuous functions.

Moreover, we have that $0 < H_+'(x) \leq H_+'(R)$ for $x \geq R$ by the concavity (upper bound) and the growth of H_+ (lower bound). In addition $H_+''(x)$ is bounded on $[R, \infty)$. Lets assume that this last property would not be true. Then $H_+''(x) \to -\infty$ for $x \to \infty$ and nowhere else by its continuity. But this would imply that $H_+' \to -\infty$, a contradiction.

By these properties and the definition of k, we know that $a_R'(x) < 1$ for $x > R + 2$.



(a) a_R of Example IV.1.11 2.) for $R = 5$



(b) I_R of Example IV.1.11 2.) for $R = 5$

Moreover, $a_R \in \mathsf{C}^2$ as composition of C^2 functions and a_R'' is bounded by the boundedness of H_+''.

Last but not least

$$I_R(x) = I(a_R(x)) = \begin{cases} k(x - R - 2) + I(R+1) & \text{if } x > R+2, \\ I(x) & \text{if } x \in [-R, R], \\ k(-x - R - 2) + I(R+1) & \text{if } x < -R-2, \end{cases} \tag{IV.1.29}$$

and

$$I_R'(x) \begin{cases} k & \text{if } x > R+2, \\ I'(x) & \text{if } x \in [-R, R], \\ -k & \text{if } x < -R-2, \end{cases} \quad \text{and} \quad I_R''(x) \begin{cases} 0 & \text{if } x > R+2, \\ I''(x) & \text{if } x \in [-R, R], \\ 0 & \text{if } x < -R-2, \end{cases} \tag{IV.1.30}$$

what implies the boundedness of the second derivative of I_R. □

Notation IV.1.13. *From now on when we write $\mathcal{I}_R - F_R$ (defined in* (IV.1.5)*), we assume that the underlying function a_R is the function defined in* (IV.1.27)*. However, all the results stay valid, if we fix as a_R an arbitrary function that satisfies all the conditions of Definition IV.1.2.*

IV.1.1.3 Critical value information through restriction to bounded L^∞-norm

In this section we compare the landscapes of $\mathcal{I} - F$ and $\mathcal{I}_R - F_R$ and infer from this already first information about critical values and the landscape. In particular, we see that the restriction to functions bounded in the L^∞-norm is not really troubling us. We start with the simple result that the global minimisers of $\mathcal{I} - F$ and $\mathcal{I}_R - F_R$ are the same.

Lemma IV.1.14. *$\mathcal{I} - F$ and $\mathcal{I}_R - F_R$ have the same global minimisers for R large enough.*

Proof. For $\|m\|_{L^\infty} > R$, we know that both $(\mathcal{I} - F)(m)$ and $(\mathcal{I}_R - F_R)(m)$ are larger than $(\mathcal{I} - F)\left(A_R^{Cut}(m)\right)$ and $(\mathcal{I}_R - F_R)\left(A_R^{Cut}(m)\right)$ respectively by Lemma IV.1.5 (with A_R^{Cut} defined in Remark IV.1.6). Hence, minimisers have to satisfy $\|m\|_{L^\infty} \leq R$. But for these functions both functionals have the same value. Therefore, they necessarily have the same global minimisers. □

In the following lemma, we show among other things that outside the set $\{L^\infty(.) \leq R\}$ there is no local minimum and inside that set no local maximum.

Lemma IV.1.15. *Let $R > R^*$ be arbitrary for R^* large enough and a_R satisfy the assumptions of Lemma IV.1.5.*

(i) *For $m \in L^2(\mathbb{T}^d)$ with $\|m\|_{L^\infty} \leq R$ and each $\epsilon > 0$, there is always a function $u \in L^2(\mathbb{T}^d)$ with $\|m-u\|_{L^2} < \epsilon$ such that*

$$(\mathcal{I}_R - F_R)(m) = (\mathcal{I} - F)(m) < (\mathcal{I} - F)(u). \tag{IV.1.31}$$

In particular, this implies that $\mathcal{I} - F$ has never a local maximum at such a function m.

(ii) *For $m \in L^2(\mathbb{T}^d)$ with $\|m\|_{L^\infty} > R^*$ and each $\epsilon > 0$, there is always a function $u \in L^2(\mathbb{T}^d)$ with $\|m-u\|_{L^2} < \epsilon$ such that*

$$(\mathcal{I} - F)(m) > (\mathcal{I} - F)(u), \tag{IV.1.32}$$

and the same holds for $\mathcal{I}_R - F_R$ when $R > \|m\|_{L^\infty} > R^*$.

In particular, this implies that $\mathcal{I} - F$ and $\mathcal{I}_R - F_R$ never attain local minima at $m \in L^2(\mathbb{T}^d)$ with $\|m\|_{L^\infty} > R^*$.

(iii) *Each $m \in L^2(\mathbb{T}^d)$ with $\infty > \|m\|_{L^\infty} > R^*$ is neither a local minimum nor a local maximum $\mathcal{I} - F$.*

Remark IV.1.16. *Note that (i) does not state anything about local maxima of the functional $\mathcal{I}_R - F_R$ or of local maxima within the set of bounded (by R) functions.*

Proof. ***(i)*** We set $f_n(x) = n\mathbb{1}_{[0,\frac{1}{n^4}]}(x)$ for $x \in \mathbb{T}^d$. Then $f_n \in L^2(\mathbb{T}^d)$ with $\|f_n\|_{L^2} = \frac{1}{n^2}$ and $f_n + m \to m$ in $L^2(\mathbb{T}^d)$. By Assumption IV.0.1, we know that for $n \in \mathbb{N}$ large enough

$$\sup_{\theta\in[-R,R]} I(\theta) - \lambda\widehat{J_0}\frac{1}{2}\theta^2 < I(n) - \lambda\widehat{J_0}\frac{1}{2}n^2, \tag{IV.1.33}$$

and for $n > 4R$

$$|m(x) - m(y)| \leq 2R \leq |m(x) - m(y) + n|. \tag{IV.1.34}$$

Therefore, for n large enough

$$\begin{aligned}(\mathcal{I} - F)(m + f_n) > \mathcal{I}(m) - \lambda\widehat{J_0}\frac{1}{2}\int_{\mathbb{T}^d} m(x)^2\,\mathrm{d}x \\ + \lambda\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-y)(m(x)-m(y))^2\,\mathrm{d}x\mathrm{d}y = (\mathcal{I} - F)(m).\end{aligned} \tag{IV.1.35}$$

Hence, for each ϵ we find a n large enough, such that $\|f_n + m - m\|_{L^2} < \epsilon$ and the claim is proven.

(ii) If $\infty > \|m\|_{L^\infty} = R > R^*$, then we define $u = A^{Cut}_{R-\epsilon}(m)$, such that $R - \epsilon > R^*$. Then by Lemma IV.1.5

$$(\mathcal{I} - F)\left(A^{Cut}_{R-\epsilon}(m)\right) < (\mathcal{I} - F)(m), \tag{IV.1.36}$$

and

$$\|m - u\|_{L^2} \leq \epsilon. \tag{IV.1.37}$$

If $\|m\|_{L^\infty} = \infty$, then $\left\|m - A^{Cut}_R(m)\right\|_{L^2} \to 0$ for $R \to \infty$ by the dominated convergence theorem and

$$(\mathcal{I} - F)\left(A^{Cut}_R(m)\right) < (\mathcal{I} - F)(m). \tag{IV.1.38}$$

□

Next we want to study the relation between critical values of $\mathcal{I} - F$ and $\mathcal{I}_R - F_R$. Therefore, we need the following two definitions of critical values depending on whether we are looking at the C^1 approximation $\mathcal{I}_R - F_R$ or at $\mathcal{I} - F$.

Definition IV.1.17. (i) *We say that a $m^* \in L^2\left(\mathbb{T}^d\right)$ is a critical value of $\mathcal{I}_R - F_R$ if for all $u \in L^2\left(\mathbb{T}^d\right)$*

$$\left(\mathcal{I}_R' - F_R'\right)\left(m^*\right)(u) = 0. \tag{IV.1.39}$$

(ii) *We denote a $m^* \in L^2\left(\mathbb{T}^d\right)$ as critical value of $\mathcal{I} - F$ if for all $u \in L^2\left(\mathbb{T}^d\right)$*

$$\mathcal{I}(u) - \mathcal{I}\left(m^*\right) - \left(F'\right)\left(m^*\right)(u) \geq 0. \tag{IV.1.40}$$

Remark IV.1.18. *The definition of critical values for $\mathcal{I}_R - F_R$ equals the general definition for C^1 functionals. When looking at $\mathcal{I} - F$ we cannot use this definition, because $\mathcal{I}$ is not a C^1 functional. But $\mathcal{I}$ is lower semi-continuous and convex, therefore we are in the setting of [Jab03] Chapter 14 for so called semismooth functions. As stated in [Jab03] Definition 14.1 our definition above is equivalent to $-F'$ being in the subdifferential of $\mathcal{I}$.*

When talking about saddle points we use the following definition.

Definition IV.1.19. *We say that $m^* \in L^2\left(\mathbb{T}^d\right)$ is a saddle point of a functional H, if it is a critical point (in one of the senses stated above) and for each neighbourhood N of m^*, there is a $u, v \in N$ such that*

$$H(u) < H\left(m^*\right) < H(v). \tag{IV.1.41}$$

In the next lemma we compare critical points and saddle points of $\mathcal{I} - F$ and $\mathcal{I}_R - F_R$. We see that for functions that are restricted by R in the L^∞-norm, these two properties are equivalent when looking only at L^∞ restricted directions.

Lemma IV.1.20. *Let $R \geq R^*$ and a_R be given as in Proposition IV.1.4.*

(i) *We take a $m \in L^2\left(\mathbb{T}^d\right)$ with $\|m\|_{L^\infty} \leq R$. Then the following are equivalent:*

- *m is a critical value in $\|.\|_{L^\infty} \leq R$ directions of $\mathcal{I}_R - F_R$, i.e. in the sense of Definition IV.1.17 (i), but only for $\|u\|_{L^\infty} \leq R$.*
- *m is a critical value in $\|.\|_{L^\infty} \leq R$ directions of $\mathcal{I} - F$, i.e. in the sense of Definition IV.1.17 (ii) but only for $\|u\|_{L^\infty} \leq R$.*

(ii) *Let $m \in L^2\left(\mathbb{T}^d\right)$ with $\|m\|_{L^\infty} \leq R$. Then*

- *For each neighbourhood $N \subset L^2\left(\mathbb{T}^d\right)$ of m, there are $u, v \in N$ such that $\left(\mathcal{I}_R - F_R\right)(u) < \left(\mathcal{I}_R - F_R\right)(m) < \left(\mathcal{I}_R - F_R\right)(v)$*

implies

- *For each neighbourhood $N \subset L^2\left(\mathbb{T}^d\right)$ of m, there are $u, v \in N$ such that $(\mathcal{I} - F)(u) < (\mathcal{I} - F)(m) < (\mathcal{I} - F)(v)$.*

Moreover, when we restrict the neighbourhoods to functions with $\|.\|_{L^\infty} \leq R$, then both these statements are equivalent.

In particular, these equivalences imply that m^* is a saddle point in $\|.\|_{L^\infty} \leq R$ directions of $\mathcal{I} - F$ if and only if it is the same for $\mathcal{I}_R - F_R$.

Proof. ***(i)*****"⇒"** Let m be a critical value of $\mathcal{I}_R - F_R$ and $u \in L^2(\mathbb{T}^d)$ with $\|u\|_{L^\infty} \leq R$, then

$$\begin{aligned} \mathcal{I}(u) - \mathcal{I}(m) - F'(m)(u-m) &= \mathcal{I}_R(u) - \mathcal{I}_R(m) - F'_R(m)(u-m) \\ &\geq (\mathcal{I}'_R - F'_R)(m)(u-m) = 0, \end{aligned} \tag{IV.1.42}$$

by the convexity of $\mathcal{I}$.

"⇐" Let m be a critical value of $\mathcal{I} - F$, then for all $u \in L^2(\mathbb{T}^d)$

$$\begin{aligned} 0 &\leq \mathcal{I}(A_R(u)) - \mathcal{I}(m) - F'(m)(A_R(u) - m) \\ &= \mathcal{I}(A_R(u)) - \mathcal{I}(m) - \mathcal{I}'_R(m)(A_R(u) - m) + (\mathcal{I}'_R(m) - F'(m))(A_R(u) - m) \\ &= \frac{1}{2}\mathcal{I}''_R(\zeta)(A_R(u) - m)^2 + (\mathcal{I}'_R(m) - F'(m))(A_R(u) - m), \end{aligned} \tag{IV.1.43}$$

for a ζ between m and $A_R(u)$. Therefore, $\|\zeta\|_{L^\infty} \leq R$, what implies that $I''_R(\zeta) = I''(\zeta) > 0$, hence $\mathcal{I}''_R(\zeta)(A_R(u) - m)^2 \leq |I''_R|_\infty \|A_R(u) - m\|^2_{L^2}$.
For an arbitrary function $k \in L^2(\mathbb{T}^d)$ with $\|k\|_{L^\infty} \leq R$ and a $c \in \mathbb{R}_+$ small enough (such that $c\|k\|_{L^\infty} \leq R$), we can choose u such that $A_R(u(.)) - m(.) = ck(.)$.
This implies

$$\frac{c}{c^2}(\mathcal{I}'_R(m) - F'(m))(k) \geq -\frac{1}{2}|I''_R|_\infty \|k\|^2_{L^2}. \tag{IV.1.44}$$

Repeating this with $-c$, we see that

$$\frac{1}{|c|}|(\mathcal{I}'_R(m) - F'(m))(k)| \leq \frac{1}{2}|I''_R|_\infty \|k\|^2_{L^2}. \tag{IV.1.45}$$

But c can be chosen arbitrary small, hence $(\mathcal{I}'_R - F')(m) = 0$.

(ii)**"⇒"** Let N be a ball around m and let $u, v \in N$ be the lower and upper values for $\mathcal{I}_R - F_R$ given above. Then we have

$$\begin{aligned} (\mathcal{I} - F)(A_R(u)) = (\mathcal{I}_R - F_R)(u) &< (\mathcal{I}_R - F_R)(m) = (\mathcal{I} - F)(m) \\ &< (\mathcal{I}_R - F_R)(v) \leq (\mathcal{I} - F)(v). \end{aligned} \tag{IV.1.46}$$

Moreover, we have that $A_R(u) \in N$, because

$$\|A_R(u) - m\|_{L^2} = \|A_R(u) - A_R(m)\|_{L^2} \leq |a'_R|_\infty \|u - m\|_{L^2} \leq \|u - m\|_{L^2}. \tag{IV.1.47}$$

Hence, $A_R(u)$ and v are the corresponding values in the neighbourhood for $\mathcal{I} - F$.

Last but not least, the equivalence on $\|.\|_{L^\infty} \leq R$ is due to the equality of $\mathcal{I} - F$ and $\mathcal{I}_R - F_R$ on this set. ☐

Remark IV.1.21. *Due to the missing differentiability, we can not hope to get from the criticality of $\mathcal{I}_R - F_R$ the criticality in all directions of $\mathcal{I} - F$ (in the two different senses given in Definition IV.1.17). Nevertheless, by the fact that we may restrict ourselves to L^∞ bounded functions by Section IV.1.1.2, this information in L^∞ bounded directions is enough to transfer results of critical values between $\mathcal{I}_R - F_R$ and $\mathcal{I} - F$.*

IV.1.1.4 Path information through restriction to bounded L^∞-norm

We are mainly interested in the lowest paths in the energy landscape between two point in $L^2(\mathbb{T}^d)$ and in particular between two minima of $\mathcal{I}_R - F_R$. We denote the set of all paths from m_1 to m_2 with $m_1, m_2 \in L^2(\mathbb{T}^d)$ by

$$\Gamma_{m_1,m_2} := \left\{\gamma \in \mathsf{C}\left([0,1], L^2(\mathbb{T}^d)\right), \gamma(0) = m_1, \gamma(1) = m_2\right\}. \tag{IV.1.48}$$

We show now that we can also use the L^∞-norm restriction procedure (of Section IV.1.1.2) for paths and we receive another path that is possibly lower in energy than the original path:

Lemma IV.1.22. *Let R^* and a_R be as in Lemma IV.1.5 and fix a $R \geq R^*$. Moreover, we fix $m_1, m_2 \in L^2(\mathbb{T}^d)$ with $\|m_1\|_{L^\infty}, \|m_2\|_{L^\infty} < R$. For an arbitrary $\gamma \in \Gamma_{m_1,m_2}$, we have $A_R(\gamma(.)) \in \Gamma_{m_1,m_2}$ and*

$$\max_{t\in[0,1]} (\mathcal{I} - F)(\gamma(t)) \geq \max_{t\in[0,1]} (\mathcal{I} - F)(A_R(\gamma(t))). \tag{IV.1.49}$$

Proof. We fix a path $\gamma \in \Gamma_{m_1,m_2}$. The inequality (IV.1.49) follows from Lemma IV.1.5. To show that $A_R(\gamma(.)) \in \Gamma_{m_1,m_2}$, we fix an arbitrary sequence $t_n \to t$ in $[0,1]$ and use

$$\|A_R(\gamma(t_n)) - A_R(\gamma(t))\|_{L^2} \leq 1 \|\gamma(t_n) - \gamma(t)\|_{L^2}, \tag{IV.1.50}$$

by the Lipschitz continuity of a_R with Lipschitz constant at most one. □

Finally, we can infer from this result, that the highest point on the lowest paths has the same energy independent whether we look at the restricted functional $\mathcal{I}_R - F_R$ or the original functional $\mathcal{I} - F$.

Lemma IV.1.23. *Let R^* and a_R be as in Lemma IV.1.5. Fix two functions $m_1, m_2 \in L^2(\mathbb{T}^d)$ with $\|m_1\|_{L^\infty}, \|m_2\|_{L^\infty} < R^*$. Define for all $R \in \mathbb{R}_+$*

$$\begin{aligned} c &:= \inf_{\gamma\in\Gamma_{m_1,m_2}} \max_{t\in[0,1]} (\mathcal{I} - F)(\gamma(t)) \\ c_R &:= \inf_{\gamma\in\Gamma_{m_1,m_2}} \max_{t\in[0,1]} (\mathcal{I}_R - F_R)(\gamma(t)). \end{aligned} \tag{IV.1.51}$$

Then $c_R = c$ for all $R \geq R^*$.

Proof. By Lemma IV.1.22 it is enough to take the infimum over paths that have L^∞-Norm lower than R for R large enough. But for these each function on these paths $\mathcal{I} - F$ equals $\mathcal{I}_R - F_R$. □

IV.1.1.5 Properties of the convolution operator $J*$.

We state now some general properties of the convolution operator $J*$. that we need in the subsequent sections. Fix as basis of $L^2(\mathbb{T}^d)$ (remember that we have defined the torus as the quotient of $\mathbb{R}^d$ under integral shifts) the d-dimensional sine and cosine (orthonormal) basis that are defined by

$$e^c_\ell(x) := \sqrt{2}\cos(2\pi\ell \cdot x) \text{ and } e^c_\ell(x) := \sqrt{2}\sin(2\pi\ell \cdot x), \tag{IV.1.52}$$

for $x \in \mathbb{T}^d$ and $\ell \in \Gamma^d$, with

$$\Gamma^d := \left\{\ell \in Z^d : \ell_i > 0 \text{ for } i = \operatorname{argmin}_{j=1,\ldots,d}\{\ell_j \neq 0\}\right\}. \tag{IV.1.53}$$

Remark IV.1.24. *We use the set Γ^d to not have the same basis function twice by the symmetries of* sin *and* cos, *because $e^c_\ell(x) = e^c_\ell(-x) = e^c_{-\ell}(x) = e^c_{-\ell}(-x)$ and that $e^s_\ell(x) = e^s_{-\ell}(-x) = -e^s_\ell(-x) = -e^s_{-\ell}(x)$.*

We define the d-dimensional sine and cosine transform of the real function J by (note that we dropped the factor 2 in order to use these coefficients as eigenvalues of $J*$.)

$$\widehat{J^c_\ell} = \int_{\mathbb{T}^d} J(x)\cos(2\pi\ell \cdot x)\,\mathrm{d}x \text{ and } \widehat{J^s_\ell} = \int_{\mathbb{T}^d} J(x)\sin(2\pi\ell \cdot x)\,\mathrm{d}x, \tag{IV.1.54}$$

and its Fourier transform by

$$\widehat{J}_\ell = \int_{\mathbb{T}^d} J(x)\,e^{-i2\pi\ell\cdot x}\mathrm{d}x = \widehat{J^c_\ell} + i\widehat{J^s_\ell}. \tag{IV.1.55}$$

Remark IV.1.25. *The Fourier transform defines the eigenvalues corresponding to the eigenfunctions* $e_\ell(x) := e^{i\ell\cdot x}$ *if we took as underlying field the complex functions, i.e.* $J * e_\ell = \widehat{J}_\ell e_\ell$*. This implies for example* $J * f(.) = \sum_{\ell\in\Gamma^d} \widehat{J}_\ell \sqrt{2}\widehat{f}_\ell e_\ell(.)$*. In the following we look however at* $\mathbb{R}$ *as underlying field and hence work with the sine and cosine basis as eigenfunctions.*

Lemma IV.1.26. *Let* J *be continuous.*

(i) $f \to J * f$ *is a Hilbert-Schmidt integral operator. Hence, it is in particular compact and continuous.*

(ii) *If* J *is even, then* $f \to J * f$ *is self-adjoint.*

(iii) *If* $J \geq 0$ *and even, then we have* $\widehat{J_0} > \max_{\ell\in\Gamma^d,\ell\neq 0} \left|\widehat{J_\ell}\right|$.

(iv) *If* J *is an even function, then* $\widehat{J_\ell} = \widehat{J_{-\ell}} = \widehat{J_\ell^c}$ *and* $\widehat{J_\ell^s} = 0$ *for all* $\ell \in \Gamma^d\setminus\{0\}$ *and*

$$J * e_\ell^c = \widehat{J_\ell^c} e_\ell^c \text{ and } J * e_\ell^s = \widehat{J_\ell^c} e_\ell^s. \tag{IV.1.56}$$

The set $\widehat{J_\ell^c}$ *are all the eigenvalues of the operator* J*. Each eigenspace for an eigenvalue different from zero is finite dimensional.*

(v) *If* J *depends only on the distance of two points, then we have additionally that*

$$\widehat{J_\ell} = \widehat{J_{\ell'}} \qquad \text{if } \ell'_i = \pm\ell_{\sigma(i)} \text{ for all } i = 1, ..., d, \tag{IV.1.57}$$

when σ *is a permutation of* $1, ..., d$.

Remark IV.1.27. *We state in the lemma which property of* J *are actually need to get the result, to make it obvious where we use which assumption. All the conditions on* J *in this lemma, except the condition on* J *in (v), are satisfied by the Assumption III.3.1. The condition on* J *in (v) is the additional Assumption IV.1.57, that we assume only in Section IV.1.4.*

Proof. ***(i)*** By J being continuity, we get that $\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} |J(x,y)|^2 \,\mathrm{d}x\mathrm{d}y \leq \left|J^2\right|_\infty < \infty$. Hence, J is a Hilbert-Schmidt kernel. Therefore, $J * f$ is a Hilbert-Schmidt integral operator and hence it is compact and continuous.

(ii) The operator $J * .$ is also symmetric because J is even. Moreover, $J * .$ is everywhere defined on $L^2(\mathbb{T}^d)$. Therefore, it is self-adjoint on $L^2(\mathbb{T}^d)$.

(iii) By the definition of the Fourier coefficients we know that for each $\ell \in \Gamma^d, \ell \neq 0$

$$\left|\widehat{J_\ell}\right| \leq \int_{\mathbb{T}^d} J(y)\,|cos(\ell y)|\,\mathrm{d}y < \int_{\mathbb{T}^d} J(y)\,\mathrm{d}y = \widehat{J_0}. \tag{IV.1.58}$$

This implies that

$$\widehat{J_0} \geq \sup_{\ell\in\Gamma^d,\ell\neq 0} \left|\widehat{J_\ell}\right|. \tag{IV.1.59}$$

The supremum on the right hand side of (IV.1.59) is actually a maximum and it is not equal to $\widehat{J_0}$. Indeed, the sequence of eigenvalues $\widehat{J_\ell^{c,s}}$ tends to 0 as $|\ell| = |\ell_1| + ... + |\ell_d| \to \infty$ by (i), (ii) and the spectral theory of self-adjoint compact operators (see for example [Mac09] Theorem 4.21). This implies that for each ϵ small enough there is a N such that for $|\ell| > N$, we have $\left|\widehat{J_\ell}\right| < \epsilon < \widehat{J_0}$. Because the $\ell \in \Gamma^d$ with $|\ell| < N$ are finitely many, $\widehat{J_0} > \max_{|\ell|<N} \widehat{J_\ell}$. This implies the claimed property.

(iv) This follows by the symmetry and antisymmetry of sinus and cosines and $J(x) = J(-x)$.

(v) This is a consequence of $J(x) = J\left(|x_i|_{i=1}^d\right) = J\left(|\overline{x}_i|_{i=1}^d\right) = J(\overline{x})$, if $\overline{x}_i = \pm x_{\sigma(i)}$ for all $i = 1, ..., d$ and σ a permutation of $1, ..., d$. □

IV.1.1.6 Variational problem

As shown in Proposition IV.1.4, the functional $L_{J,R}$ is a C^1 functional and in particular Gâteaux differentiable. Moreover, for functions $m \in L^2\left(\mathbb{T}^d\right)$ with $\|m\|_{L^\infty} \leq R$, we have the equality $\mathcal{I} - F(m) = \mathcal{I}_R - F_R(m)$ and also the critical values in L^∞-bounded directions of $\mathcal{I} - F$ and $\mathcal{I}_R - F_R$ equal each other as shown in Lemma IV.1.20 (i). At a critical point $m^* \in L^2\left(\mathbb{T}^d\right)$, $\mathcal{I}_R - F_R$ needs to have a vanishing Gâteaux derivative (see Definition IV.1.17 (i)), i.e.

$$\begin{aligned}&\partial_\epsilon \mathcal{I}_R - F_R\left(m^* + \epsilon g\right)\big|_{\epsilon=0}\\ &= \int\int \left(I_R'\left(m^*(x)\right) - \lambda J(x-y)\, a_R'\left(m(y)\right) a_R\left(m(x)\right)\right) g(y)\,\mathrm{d}x\mathrm{d}y = 0,\end{aligned} \tag{IV.1.60}$$

for all $g \in L^2\left(\mathbb{T}^d\right)$, where we use that J is even (Assumption III.3.1). For $\|m\|_{L^\infty} \leq R$ this is equivalent to

$$\int\int \left(I'\left(m^*(x)\right) - \lambda J(x-y)\, m(x)\right) g(y)\,\mathrm{d}x\mathrm{d}y = 0. \tag{IV.1.61}$$

By the du Bois-Reymond lemma, this is equivalent to (IV.1.1).

In the next lemma we show that each solution of (IV.1.1) (in particular each critical point of $\mathcal{I}_R - F_R$) is bounded and continuous.

Lemma IV.1.28. *Each function $m^* \in L^2\left(\mathbb{T}^d\right)$ that satisfies* (IV.1.1)*, is bounded in L^∞ and continuous. Moreover, it satisfies*

$$m^*(.) = I'^{-1}\left(\lambda J * m^*(.)\right) = h'\left(\lambda J * m^*(.)\right). \tag{IV.1.62}$$

Proof. The convolution $J * m^*$ is bounded and continuous, due to J and m^* being in $L^2\left(\mathbb{T}^d\right)$. Therefore, the strict convexity of I implies (IV.1.62). Finally, the continuity of h' implies the claimed boundedness and continuity of m^*. □

Note that the constant function $m \equiv 0$ is always a solutions of (IV.1.62) and hence a critical point. We show in the next sections that depending on the parameters, this function can be a global minimum of $\mathcal{I} - F$, a saddle point that lies on the lowest path between two global minima or of no special interest to us.

Moreover, each global minimiser of $\mathcal{I} - F$ is also a solution to (IV.1.62). We show in Section IV.1.2.1 that there exists a global minimiser of $\mathcal{I} - F$ in $\mathbb{M}\left(\mathbb{T}^d\right)$. This has a density in $L^2\left(\mathbb{T}^d\right)$ (Lemma IV.1.1). Moreover, we know by Lemma IV.1.5 that it has to be bounded in the L^∞-norm. Hence, this minimiser of $\mathcal{I} - F$ has to be a solution to (IV.1.62).

IV.1.2 Minimiser of $\mathcal{I} - F$

We show in this section at first the existence of minimiser (Section IV.1.2.1). Then we characterise the minimiser as solutions to a one dimensional problem (Section IV.1.2.2). Finally, we show that under Assumption IV.0.3 this implies that there is either one or two minima (Section IV.1.2.3).

IV.1.2.1 Existence of Minimiser

In this section we show that there exists always at least one minimiser of $\mathcal{I} - F$ in $L^2\left(\mathbb{T}^d\right)$ by using methods from functional analysis. By Lemma IV.1.1 and because L_J is a good rate function (Lemma III.3.4) we know the existence of a minimiser already. Nevertheless, we state now a proof that does not require whether L_J is a rate function or not. Therefore, the result is also applicable (and in particular useful) when the energy $\mathcal{I} - F$ is not a rate function. Moreover, we use some parts of the proof again when we prove Theorem IV.1.39.

Proposition IV.1.29. *There exists always a minimiser of $\mathcal{I}-F$ in $L^2\left(\mathbb{T}^d\right)$.*

Proof. To prove this claim, we use the idea of [EE83] Theorem 5.1 (ii) and transfer it to our situation. Due to the possibility of unbounded minimiser in the model we consider, we need now a new boundedness (Lemma IV.1.30) and a new continuity (Lemma IV.1.31) result

Fix a sequence $f_n \in L^2\left(\mathbb{T}^d\right)$ such that $(\mathcal{I}-F)(f_n) \to \inf_f \{(F-L)(f)\}$. We can assume that for all $n \in \mathbb{N}$

$$(\mathcal{I}-F)(f_n) \leq \inf_f \{(F-L)(f)\} + \epsilon. \tag{IV.1.63}$$

Lemma IV.1.30. *Let the Assumption IV.0.1 be satisfied. Then there is a $R>0$ such that $\{f_n\}_n \subset \left\{f \in L^2 : \|f\|_{L^2} \leq R\right\} =: B_{L^2}(R)$.*

Proof. Using (IV.1.3), we have for all $m \in L^2\left(\mathbb{T}^d\right)$

$$(\mathcal{I}-F)(m) \geq \underbrace{\left(\kappa - \frac{1}{2}\lambda\widehat{J_0}\right)}_{=:b>0} \int m(x)^2 \, \mathrm{d}x - C_{I,\kappa}, \tag{IV.1.64}$$

by Assumption IV.0.1 if κ close to c_Ψ. Hence

$$\|f_n\|_{L^2} \leq b^{-1}(\mathcal{I}-F)(f_n) + b^{-1}C_I. \tag{IV.1.65}$$

Now we take the supremum over all n

$$\begin{aligned} \sup_n \|f_n\|_{L^2} &\leq b^{-1}\sup_n (\mathcal{I}-F)(f_n) + b^{-1}C_I \\ &\leq b^{-1} \inf_{f \in L^2(\mathbb{T}^d)} (\mathcal{I}-F)(f) + b^{-1}(\epsilon + C_I). \end{aligned} \tag{IV.1.66}$$

The right hand side is the claimed upper bound on the L^2-norm of $\{f_n\}$. □

Lemma IV.1.31. (i) *$\mathcal{I}$ is weakly lower semi-continuous on $L^2\left(\mathbb{T}^d\right)$.*

(ii) *For each $R>0$, F is weakly continuous on $B_{L^2}(R)$.*

Proof. ***(i)*** $\mathcal{I}$ is as a rate function lower semi-continuous on $\mathbb{M}\left(\mathbb{T}^d\right)$. Moreover, a sequence $m_n \to m$ that convergence weakly in L^2, also (seen as densities of measures $m_n(x)\mathrm{d}x$) convergence in $\mathbb{M}\left(\mathbb{T}^d\right)$ (equipped with the weak-(*)-topology).

(ii) We fix $\{u_n\}_n, u \in B_{L^2}(R)$ such that $u_n \to u$ weakly in $L^2\left(\mathbb{T}^d\right)$. Then $J * u_n \to J * u$ in $\mathsf{C}\left(\mathbb{T}^d\right)$ and therefore also strongly in $L^2\left(\mathbb{T}^d\right)$. This implies

$$|F(u_n) - F(u)| \leq \lambda R \|J * (u_n - u)\|_{L^2} + \lambda |\langle J * u, u_n - u\rangle| \to 0. \tag{IV.1.67}$$

Another possibility to prove this claim is to use the same approach as [EE83] Lemma 3.3 by applying the Stone-Weierstrass theorem and at the end Lemma IV.1.30 instead of their uniform bound. □

We know from the Banach-Alaglou theorem that $B_{L^2}(R)$ is (weak and weak-* (the same by the reflexivity of L^2)) compact in L^2. Hence, there is a subsequence f_{n_k} that convergence weakly to a $f^* \in B_{L^2}(R)$. The weak lower semi-continuity of $\mathcal{I}-F$ on $B_{L^2}(R)$ implies

$$\inf_f \{(\mathcal{I}-F)(f)\} = \lim_{k\to\infty} (\mathcal{I}-F)(f_{n_k}) \geq (\mathcal{I}-F)(f^*) \geq \inf_f \{(\mathcal{I}-F)(f)\}. \tag{IV.1.68}$$

This implies that f^* is the wanted minimiser. □

Remark IV.1.32. *Another possibility to show the claim based more on probabilistic methods is to use that $\mathcal{I}$ is a good rate function with exponentially tight compact sets K_R (Lemma III.3.5) and that*

$$\inf_{m \notin K_R} (\mathcal{I} - F)(m) \to \infty, \tag{IV.1.69}$$

for $R \to \infty$.

IV.1.2.2 Characterisation of the minimiser

In this section we simplify the problem of finding the minimiser of $\mathcal{I} - F$ in $L^2(\mathbb{T}^d)$ to a one dimensional problem in Lemma IV.1.33 and finally we show that each L^∞ bounded local or global minimiser solves (IV.1.1) in Lemma IV.1.34.

Lemma IV.1.33. (i) *Each solution of the one dimensional variation problem*

$$\min_{m \in \mathbb{R}} \left\{ I(m) - \frac{1}{2} \lambda \widehat{J_0} m^2 \right\}, \tag{IV.1.70}$$

is a minimiser or $\mathcal{I} - F$.

(ii) *If $m(.) \in L^2(\mathbb{T}^d)$ is a global minimiser of $\mathcal{I} - F$, then $m(.)$ is constant and equals a solution of* (IV.1.70)*.*

(iii) *If m is a constant minimiser then also $-m$.*

Proof. The minimum in (IV.1.70) is attained. Indeed, we can restrict the minimum to a compact subset by Lemma III.2.6 and Assumption IV.0.1. Finally, we use the continuity of I (see Lemma III.2.5) to infer the existence of a minimiser.

Rewrite $\mathcal{I} - F$ as

$$\begin{aligned}&(\mathcal{I} - F)(m(.)) \\ &= \int_{\mathbb{T}^d} I(m(x)) - \lambda \frac{1}{2} \widehat{J_0} m(x)^2 \, \mathrm{d}x + \lambda \underbrace{\frac{1}{4} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} J(x-y)(m(x) - m(y))^2 \, \mathrm{d}x \mathrm{d}y}_{=: \textcircled{1}}.\end{aligned} \tag{IV.1.71}$$

Because $\textcircled{1} \geq 0$, we get for all $m(.) \in L^2(\mathbb{T}^d)$

$$(\mathcal{I} - F)(m(.)) \geq \min_{m \in \mathbb{R}} \left\{ I(m) - \frac{1}{2} \lambda \widehat{J_0} m^2 \right\}, \tag{IV.1.72}$$

with equality if and only if $m(.)$ is constant and equals a minimiser of (IV.1.70).

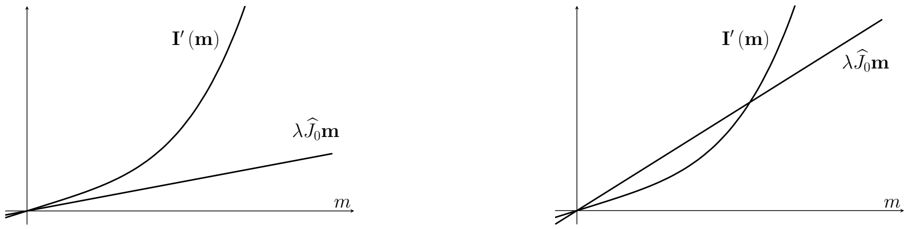
Finally (iii) is a direct consequence of $\mathcal{I}$ and F being even. □

All global minimiser satisfy (IV.1.1) because they are constant by Lemma IV.1.33 and (IV.1.70) is minimised (as a one dimensional problem) only at solutions of (IV.1.1). This is also true for all bounded local minimiser as we show in the next lemma.

Lemma IV.1.34. *If m^* is a local or global minimiser with $\|m^*\|_{L^\infty} < \infty$, then it satisfies* (IV.1.1).

Proof. Let m^* be a local or global minimiser of $\mathcal{I} - F$ with $\|m^*\|_{L^\infty} \leq R$. Then it is also a minimiser of the Gâteaux differentiable functional $\mathcal{I}_R - F_R$ and hence this derivative vanishes. Therefore, each bounded local or global minimiser satisfies (IV.1.1) as we have shown in Section IV.1.1.6. □

Remark IV.1.35. *Note that this result can also be shown without the cutoff procedure from Section IV.1.1.2 and the variation principle we derive in Section IV.1.1.6 by an adaptation of Theorem 5.1 (iii) in [EE83] (where they showed the claim for single spin measures with bounded support).*



(a) λ small, only one intersection at 0 (b) λ big, another intersection at m^{CW}

Figure IV.3: The functions $\lambda\widehat{J_0}m$ and $I'(m)$ for small and big values of λ.

IV.1.2.3 Minimisers under Assumption IV.0.3

In this section we characterise the minimisers of $\mathcal{I}-F$ if we have additionally the Assumption IV.0.3.

Theorem IV.1.36. *If the additional Assumption IV.0.3 holds, then the only minimiser of $\mathcal{I}-F$ are the constant functions*

$$\begin{aligned} m(.) &\equiv 0 && \text{if} \quad \lambda \leq \frac{1}{\widehat{J_0}h''(0)}, \\ m(.) &\equiv \pm m^{CW} && \text{if} \quad \lambda > \frac{1}{\widehat{J_0}h''(0)}, \end{aligned} \tag{IV.1.73}$$

where m^{CW} is defined in Lemma IV.1.37.

To prove this theorem we only need to characterise all minimiser of the one dimensional problem (IV.1.70) by Lemma IV.1.33. This is done in [EE83] Appendix B and we state their result in our notation in the next lemma. Moreover, the proof we give here is also an adaptation of the proof in [EE83].

Lemma IV.1.37 (see [EE83] Appendix B)**.** *If h' is strictly concave on $\mathbb{R}_+$ (additional Assumption IV.0.3), then the only minimisers of* (IV.1.70) *are*

$$\begin{aligned} m &= 0 && \text{if} \quad \lambda \leq \frac{1}{\widehat{J_0}h''(0)}, \\ m &= \pm m^{CW} && \text{if} \quad \lambda > \frac{1}{\widehat{J_0}h''(0)}. \end{aligned} \tag{IV.1.74}$$

Here $m^{CW} = m^{CW}\left(\Psi, \widehat{J_0}\right) \in (0,\infty)$ is the unique positive minimiser of (IV.1.70) *and the unique positive solution of*

$$\lambda\widehat{J_0}m = I'(m). \tag{IV.1.75}$$

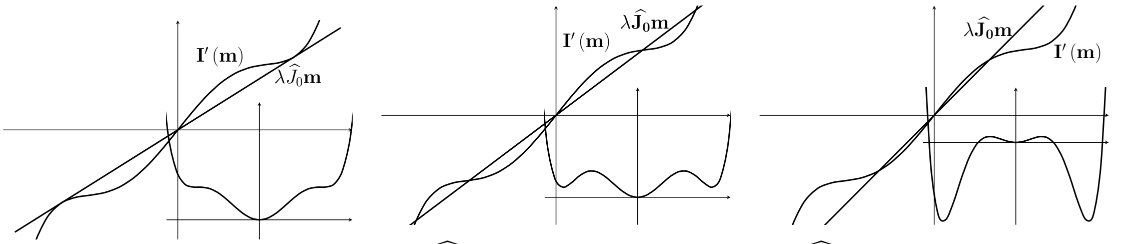
Moreover, m^{CW} is increasing for increasing λ.

Proof. Obviously a minimiser m^* of the one dimensional variation problem (IV.1.70) has to satisfy (IV.1.75).

(i) At first, let $\lambda \leq \frac{1}{\widehat{J_0}h''(0)}$. The strict concavity of h' implies for $u > 0$ that $h''(u) < h''(0) = \mathbb{E}\left(\theta^2\right)$. Hence, $I''(u) > I''(0) = \frac{1}{h''(0)}$ (by the duality). From this we get that for all $u > 0$

$$I'(u) > \frac{1}{h''(0)}u \leq \lambda\widehat{J_0}u. \tag{IV.1.76}$$

Then (IV.1.76) implies that (IV.1.75) is only satisfied if $m^* = 0$ (see also Figure IV.3a).



(a) $\lambda\widehat{J_0} = 1$, a saddle point appears

(b) $\lambda\widehat{J_0} = 1.2$, a local minimum away from zero

(c) $\lambda\widehat{J_0} = 1.6$, a global minimum away from zero

Figure IV.4: Without Assumption IV.0.3: Minimiser born somewhere away from zero. The coordinate systems in the right bottom corners show the respective value of $I(m) - \frac{1}{2}\lambda\widehat{J_0}m^2$.

(ii) Now let $\lambda > \frac{1}{\widehat{J_0}h''(0)}$. Then $\lambda\widehat{J_0}u > I'(u)$ for $u > 0$ small. By the strict concavity of h', there has to be a unique $m^* > 0$ such that (IV.1.75) holds (see Figure IV.3b). This also implies that m^* is increasing if λ is increasing.

Moreover the constant 0 still satisfies (IV.1.75), but it is now a local maximum of (IV.1.70). Indeed for this choice of λ

$$I''(0) - \lambda\widehat{J_0} < 0. \tag{IV.1.77}$$

□

IV.1.2.4 An example of minimisers (born away from zero) without Assumption IV.0.3

In this section we state an example of a function I such that the global minimisers of $\mathcal{I} - F$ do not bifurcate from the trivial branch, but appear somewhere else in space. By the previous section we know that this implies that I' does not satisfy Assumption IV.0.3. Set

$$I(m) = \frac{1}{6}m^6 - 2\frac{1}{4}m^4 + 2\frac{1}{2}m^2. \tag{IV.1.78}$$

Then is I strict convex, but $I'(m) = m^5 - 2m^3 + 2m$ is not convex on $[0, \infty)$. We have plotted this I' and $\lambda\widehat{J_0}$ in Figure IV.4. Let us explain with the help of this figure the different regimes of $I(m) - \frac{1}{2}\lambda\widehat{J_0}m^2$, dependent on the intensity λ of the interaction:

- $\lambda\widehat{J_0} < 1$: No critical values besides $m = 0$.
- $\lambda\widehat{J_0} = 1$: Now $m = \pm 1$ solves (IV.1.70). It is a saddle point of $I(m) - \lambda\frac{1}{2}\widehat{J_0}m^2$ (Figure IV.4a).
- $I''(0) = 2 > \lambda\widehat{J_0} > 1$: There is are two local maxima at $m^{max} = \pm\sqrt{1 - \sqrt{\lambda\widehat{J_0} - 1}}$ and two local minima at $m^{min} = \pm\sqrt{1 + \sqrt{\lambda\widehat{J_0} - 1}}$.
- $\frac{5}{4} > \lambda\widehat{J_0} > 1$: The local minima are not global minimiser (Figure IV.4b).
- $2 > \lambda\widehat{J_0} > \frac{5}{4}$: $\pm m^{min}$ are the global minimiser (Figure IV.4c).
- $\lambda\widehat{J_0} \geq 2 = I''(0)$: $I(m) - \lambda\frac{1}{2}\widehat{J_0}m^2$ has a double well structure.

Hence, at $\lambda^* = \frac{5}{4}\frac{1}{J_0}$ two global minimisers appear somewhere away from zero.

IV.1.3 Lowest paths

In this section we analyse the lowest paths in the energy landscape between two points. We start in Section IV.1.3.1 by showing general results about lowest paths, like critical values on or close to lowest paths (Theorem IV.1.39) by a mountain pass theorem, that we adapted to our specific setting. This result requires that there is a mountain ridge between the two points. In Section IV.1.3.2 we show that this is for example the case for the two minima when we have the additional Assumption IV.0.3. Also under this assumption, we prove in Section IV.1.3.3 that a lowest path between the two minima follows for a weaker interaction (compared to the single spin contribution) the constant functions and for a stronger interaction leaves the constant functions.

IV.1.3.1 Information about lowest paths by a modified mountain pass theorem

In this section we derive general information about the lowest paths (in the energy landscape of $\mathcal{I} - F$) between two minima. Under the assumption that there is a mountain ridge between two points (condition (IV.1.83)), we would like to know how to pass it such that the energy of a path on its highest point is as small as possible. We state in Lemma IV.1.48 that under Assumption IV.0.3 there is such a mountain ridge in the energy landscape between the minima.

Having a finite dimensional picture in mind we try to find saddle points on the mountain through which a lowest path has to go. In an infinite dimensional space, like the one we consider here, the existence of such a saddle point or of other critical points on the energy mountain ridge is not obvious any more. Nevertheless, under special compactness assumption on the functional (e.g. Palais-Smale conditions), there is an existence theorem of critical points, the mountain pass theorem. The mountain pass theorem states the existence of a critical value at the level set of the highest point on the lowest path connecting two points. Moreover, it can be extended, such that it implies that all lowest paths are somewhere close to the set of critical values. We refer to [Jab03] for an overview on the mountain pass theorem.

However, the classical mountain pass theorem as well as generalisations are not applicable to $\mathcal{I} - F$ as we explain in the Remark IV.1.40. Nevertheless, we are able to prove the same statements in the setting we consider here. To state the exact results, we define first the following sets.

Definition IV.1.38. (i) *Let us define the subset of $L^2\left(\mathbb{T}^d\right)$, that consists of functions that are bounded in the L^∞-norm by $R \in \mathbb{R}_+$, by*

$$M_R := \left\{m \in L^2\left(\mathbb{T}^d\right) : \|m\|_{L^\infty} \leq R\right\}. \tag{IV.1.79}$$

(ii) *Moreover, we denote the set of critical values of $\mathcal{I}_R - F_R$ in M_R at height $c \in \mathbb{R}$ by*

$$K_{c,R} := \{m \in M_R : (\mathcal{I}_R - F_R)(m) = c \text{ and } (\mathcal{I}'_R - F'_R)(m) = 0\}. \tag{IV.1.80}$$

(iii) *We define on Γ_{m_1,m_2} (the set of paths between m_1 and m_2 defined in (IV.1.48)) the norm*

$$\|\gamma\|_\Gamma := \max_{t\in[0,1]} \|\gamma(t)\|_{L^2}. \tag{IV.1.81}$$

Finally, we use the set of paths that are only on M_R

$$\Gamma_{R,m_1,m_2} := \{\gamma \in \mathsf{C}([0,1], M_R) : \gamma(0) = m_1, \gamma(1) = +m_2\}. \tag{IV.1.82}$$

To shorten the notation, we use in the following Γ and Γ_R, when two points m_1, m_2 are fixed.

Our main result of this section is the following theorem, where we show the existence of a critical value and that all lowest paths are close to the set of critical values. Note that mountain pass theorems do not tell us that the lowest path passes through a saddle point, let alone that the highest point of a lowest path is a critical value (see also discussion in [Eva10] after Theorem 8.5.2). To transfer the mountain pass result about critical values from $\mathcal{I}_R - F_R$ to $\mathcal{I} - F$ we refer to Lemma IV.1.20 (i).

Theorem IV.1.39. *Fix m_1, m_2 with $\|m_1\|_{L^\infty}, \|m_2\|_{L^\infty} \leq R^*$, with R^* defined as in Lemma IV.1.5. Assume that*

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} (\mathcal{I} - F)(\gamma(t)) > \max\{(\mathcal{I} - F)(m_1), (\mathcal{I} - F)(m_2)\} =: \alpha^*. \tag{IV.1.83}$$

Then for all $R \geq R^*$:

(i) *$K_{c,R}$ is not empty, i.e. there is at least one critical value $m \in M_R$ of $\mathcal{I}_R - F_R$ on the level set c.*

(ii) *The set $K_{c,R}$ is compact.*

(iii) *Let $\mathcal{N} \subset M_R \subset L^2(\mathbb{T}^d)$ be an arbitrary neighbourhood of $K_{c,R}$. Then there exists an $\epsilon = \epsilon(\mathcal{N}) > 0$ such that each path in Γ_R that is at its highest point of $\mathcal{I}_R - F_R$ lower than $c + \epsilon$ passes through $\mathcal{N}$, i.e. if $\gamma \in \Gamma_R$*

$$\max_{t\in[0,1]} (\mathcal{I}_R - F_R)(\gamma(t)) < c + \epsilon \qquad \Rightarrow \qquad \gamma([0,1]) \cap \mathcal{N} \neq \varnothing. \tag{IV.1.84}$$

Before we start with the proof of the theorem, we state some remarks, concerning why we have the result in this particular form and why usual mountain pass theorems are not applicable.

Remark IV.1.40. (i) *First of all, the classical mountain pass theorem is only applicable for C^1-Fréchet functions ([Jab03] Theorem 7.1, [dF89] Theorem 5.7, originally [AR73]). This is as pointed out above not satisfied by our functional $\mathcal{I} - F$.*

(ii) *There are generalisations for example to semismooth functionals (satisfied because $\mathcal{I}$ is convex, lower semi continuous and F is C^1) (see [Jab03] Theorem 14.7, originally [Szu86] and [Ma97]) and also to functionals that are convex after adding $c\|.\|_{L^2}^2$ (satisfied by $\mathcal{I} - F + \widehat{J_0}\|.\|_{L^2}^2$) (see [EL80] Theorem 2 and Lemma 7). These generalisations need special Palais-Smale conditions that we were not able to prove for our functional $\mathcal{I} - F$. For definitions and discussions of different Palais-Smale conditions, we refer to [Jab03] and references therein.*

By Lemma IV.1.22, we know that all lowest paths sit inside M_R (for R large enough). Therefore, we can look at the function $\mathcal{I}_R - F_R$, which equal $\mathcal{I} - F$ on this set. However, also for this functional, the proof of Theorem IV.1.39 is not a straight forward application of the mountain pass theorem. The problem is again the validity of the Palais-Smale condition.

Remark IV.1.41. *We know that $\mathcal{I}_R - F_R$ is a C^1 functional, hence this necessary condition of the classical mountain pass theorem is satisfied. Therefore, we know that a classical Palais-Smale sequence exists. However, to conclude from this the existence of a critical value, we need that the classical (local)Palais-Smale condition holds. But this is again a problem. For general functions a_R (in the definition of $\mathcal{I}_R - F_R$), it is not ensured that the boundedness of $\mathcal{I}_R - F_R$ implies an L^2 bound on the Palais-Smale sequence. To get this L^2 boundedness, we need to choose a_R such that $I(a_R)$ grows quadratic outside $[-R, R]$, but then other, necessary properties (e.g. always better to stay inside M_R (Lemma IV.1.5) and $\mathcal{I}_R - F_R$ is C^1 (Lemma IV.1.9)) are (for general I) not any more satisfied.*

Moreover, we need also a bound on the L^∞-norm of the Palais-Smale sequence (at least if there is no better proof for the validity of the Palais-Smale condition for $\mathcal{I}_R - F_R$ than the one we use for Lemma IV.1.44). But this also seems to be out of reach.

Therefore, we have to restrict the L^∞-norm of the Palais-Smale sequence that is constructed in the proof of the mountain pass theorem. This can be done either without changing the condition on the first derivative of $\mathcal{I}_R - F_R$ of the Palais-Smale condition or with changing it.

(i) *The first possibility is use the classical Palais-Smale condition, but to show that the Palais-Smale sequence that arises in the mountain pass theorem, can be chosen such that it is inside* M_R*. We use this approach in the proof of Theorem IV.1.39 by changing the classical proof of the mountain pass theorem that uses Ekeland's variational principle (see [Jab03] Theorem 7.1 second proof) and modify the Ekeland's variation principle such that the arising functions are inside* M_R*.*

(ii) *Let us change for now the differentiability condition of the Palais-Smale condition in the sense that the steepest slope within the convex set* M_R *(instead of the first derivative of* $\mathcal{I}_R - F_R$ *as in the classical Palais-Smale condition) vanishes. By this change, we are in the setting of generalisations of the mountain pass theorems to convex subsets (see [Jab03] Chapter 17 and references therein, for the exact definition of the modified Palais-Smale condition see [Jab03] Section 17.2). However, we are not able to show that this Palais-Smale condition holds. At least in a similar proof as the one of the subsequent Lemma IV.1.44, the vanishing steepest slope does not seem to be enough to show the weak compactness of the Palais-Smale sequence (here Step 3 of this Lemma IV.1.44 would fail). However, also here we can change the proof of the mountain pass theorem for convex sets, such that the arising Palais-Smale sequence is in* $M_{R-\epsilon}$*. The necessary changes in the proof of the mountain pass theorem for convex sets are almost identical to the changes described in the first possible way (and hence almost identical to the following proof of Theorem IV.1.39).*

We choose the first of these two possibilities, because in this approach the necessary changes in the proof of the mountain pass theorem seem to be easier to understand. Moreover, we know that the limiting object is a critical value of $\mathcal{I}_R - F_R$ *in the usual sense and not only a point with zero steepest slope in the* M_R *directions.*

The proof of Theorem IV.1.39 is organised as follows. At first, we define the explained Palais-Smale condition (Definition IV.1.42). Then we show that the set M_R is closed in $L^2\left(\mathbb{T}^d\right)$ and convex (Lemma IV.1.43). We need this result to prove afterwards, that the Palais-Smale condition is satisfied (Lemma IV.1.44). Next we prove a new version of the Ekeland's variational principle (see Lemma IV.1.45). With these two results, we are finally able to proof Theorem IV.1.39 by modifying the proof of the classical mountain pass theorem (the proof that uses Ekeland's variational principle see [Jab03] Theorem 7.1 second proof).

The arising Palais-Smale sequence in the proof is in M_R, by the new version of the Ekeland's variational principle. Consequently there exists a converging subsequence of the Palais-Smale condition. The limit of this subsequence is in the set$K_{c,R}$, i.e. Theorem IV.1.39 (i) is shown. Also for the proof of Theorem IV.1.39 (iii) we use new version of the Ekeland's variation principle (Lemma IV.1.45.

In the following definition we state the Palais-Smale condition that we use in the proof of Theorem IV.1.39. By the restriction to sequences that are in M_R it is weaker than classical local Palais-Smale condition.

Definition IV.1.42. *We call a sequence* $\{u_n\} \subset M_R$ *a Palais-Smale sequence, if*

$$(\mathcal{I} - F)(u_n) \to c \text{ and} \tag{IV.1.85}$$

$$(\mathcal{I}_R' - F_R')(u_n) \to 0, \tag{IV.1.86}$$

We say that $\mathcal{I}_R - F_R$ *satisfies the (local) Palais-Smale condition for sequences in* M_R*, if each Palais-Smale sequence* $\{u_n\} \subset M_R$ *has a subsequence* $\{u_{n_k}\}$ *that converges in* $L^2\left(\mathbb{T}^d\right)$*-norm to a* $u \in M_R$*.*

Lemma IV.1.43. M_R *is closed in* $L^2\left(\mathbb{T}^d\right)$ *and it is convex.*

Proof. The convexity of M_R is obvious by its definition. To show the closeness of M_R, we fix a sequence $f_n \to f \in L^2\left(\mathbb{T}^d\right)$ with $f_n \in M_R$. Then there is a subsequence f_{n_k} that convergence outside some set N of measure zero pointwise to f. Let N_{n_k} be the set of measure zero outside which f_{n_k} is at most R. Then $N \cup \bigcup N_{n_k}$ is a set of measure zero and off this set we know that f is at most R, hence $f \in M_R$. □

In the next lemma we show that the functional $\mathcal{I}_R - F_R$ satisfies the (local) Palais-Smale condition for sequences M_R (Definition IV.1.42).

Lemma IV.1.44. *Let R be large enough. Then $\mathcal{I}_R - F_R$ satisfies the (local) Palais-Smale condition for sequences M_R (Definition IV.1.42).*

Proof. Fix a Palais-Smale sequence $\{u_n\} \subset M_R$. We have to show that there is a L^2-convergent subsequence of u_n. Without loss of generality we assume that $(\mathcal{I} - F)(u_n) \leq c + 1$ for all n. Hence, by Lemma IV.1.30, there is a constant $0 < R_c < \infty$ such that $\|u_n\|_{L^2} \leq R_c$.

We show now at first that there is a weakly convergent subsequent of u_n to a $u \in L^2(\mathbb{T}^d)$ (Step 1) and we conclude from this in Step 2 that $J * u_n$ converges to $J * u$. In Step 3 we prove that $I'(u_n) - \lambda J * u_n$ vanishes what implies (Step 4) that $I'(u_n) \to \lambda J * u$. Last but not least we state in Step 5 that these results imply the claimed strong convergence.

Note that the function u is in M_R by the closeness of M_R (Lemma IV.1.43).

Step 1: Weakly convergent subsequence: As in the proof of Proposition IV.1.29 we know that $\{\|u\|_{L^2} \leq R_c\} \subset L^2(\mathbb{T}^d)$ is weak compact hence there is a subsequence $\{u_{n_k}\}$ that converges weakly in L^2 to a $u \in L^2(\mathbb{T}^d)$. To simplify our notation we assume that $\{u_n\}$ is already this subsequence. We have to show now that there is a subsequence of $\{u_n\}$ that converges strongly to a $u \in L^2$.

Step 2: Convergence of $J * u_n \to J * u$ in $L^2(\mathbb{T}^d)$-norm: We have $J \in L^2(\mathbb{T}^d)$, because J is continuous. Now we use this function as test function in the weak convergence of $\{u_n\}$ and we get the pointwise convergence for each $y \in \mathbb{T}^d$

$$J * u_n(y) = \int J(x-y)\, u_n(x)\, \mathrm{d}x \to \int J(x-y)\, u(x)\, \mathrm{d}x = J * u(y). \tag{IV.1.87}$$

Moreover, we know that $J * u_n(x) \leq \|J\|_{L^2} \|u_n\|_{L^2} \leq \|J\|_{L^2} R_c$, hence by the dominated convergence theorem we also see that $J * u_n \to J * u$ in L^2 norm.

Step 3: Convergence of $I'(u_n) - \lambda J * u_n \to 0$ in $L^2(\mathbb{T}^d)$-norm: By the second property of the Palais-Smale sequence we know that $(\mathcal{I}_R' - F_R')(u_n) \to 0$, i.e. for each $\epsilon > 0$ there is a N_ϵ such that for all $n > N_\epsilon$

$$\sup_{f \in L^2} \frac{\left|\int (\lambda J * u_n - I'(u_n))(x)\, f(x)\, \mathrm{d}x\right|}{\|f\|_{L^2}} \leq \epsilon, \tag{IV.1.88}$$

because $u_n \in M_R$. We further know that $\lambda J * u_n - I'(u_n) \in L^2(\mathbb{T}^d)$ because $u_n \in M_R$. Hence, we use $f = \lambda J * u_n - I'(u_n)$ in (IV.1.88), what implies that $\|\lambda J * u_n - I'(u_n)\|_{L^2} < \epsilon$ for all $n > N_\epsilon$.

Step 4: Convergence of $I'(u_n) \to \lambda J * u$ in $L^2(\mathbb{T}^d)$-norm: This is a direct consequence of the triangle inequality and the two convergence results in the previous steps.

Step 5: Convergence of $u_n \to u$ in $L^2(\mathbb{T}^d)$-norm: We know that I' is bijective on each compact set (by the strong convexity of I), with inverse h'. Because $u_n \in M_R$

$$\begin{aligned} \|u_n - h'(\lambda J * u)\|_{L^2} &= \|h'(I'(u_n)) - h'(\lambda J * u)\|_{L^2} \\ &\leq \sup_{z \in [-I'(R), I'(R)]} |h'(z)|\, \|I'(u_n) - \lambda J * u\|_{L^2} \to 0. \end{aligned} \tag{IV.1.89}$$

This shows the convergence $u_n \to h'(\lambda J * u)$ in $L^2(\mathbb{T}^d)$. But the weak and the strong limit has to be the same, hence we have even shown $h'(\lambda J * u) = u$.

Hence, we have shown that the Palais-Smale sequence $\{u_n\} \subset M_R$ has a $L^2(\mathbb{T}^d)$ strongly convergent subsequence. □

The following lemma shows a slightly changed Ekeland's variational principle that tells us that the approximating object is even in Γ_R and not only Γ (as in the classical principle). In the Ekeland's variational principle used for example for the mountain pass theorem on convex sets (see [Jab03] Section 17.2.2), the respective condition c.) in Lemma IV.1.45 does not have to hold for all Γ.

Define the following lower semi-continuous function $Y : \Gamma \to \mathbb{R}$ by

$$Y(\gamma) := \max_{t\in[0,1]} (\mathcal{I}_R - F_R)(\gamma(t)). \tag{IV.1.90}$$

Lemma IV.1.45 (Ekeland's variational principle on Γ with approximating object in Γ_R)**.** *If $\gamma_\epsilon \in \Gamma_R$ such that*

$$Y(\gamma_\epsilon) \leq \inf_{\gamma\in\Gamma_R} Y(\gamma) + \epsilon, \tag{IV.1.91}$$

then for each $\delta > 0$, there is a $\gamma_{\epsilon,\delta} \in \Gamma_R$ such that

a.) $Y(\gamma_{\epsilon,\delta}) \leq Y(\gamma_\epsilon)$,

b.) $\operatorname{dist}\{\gamma_\epsilon, \gamma_{\epsilon,\delta}\} \leq \delta$ *and*

c.) $Y(\gamma_{\epsilon,\delta}) < Y(\gamma) + \frac{\epsilon}{\delta} \operatorname{dist}\{\gamma, \gamma_{\epsilon,\delta}\}$ *for all* $\gamma \in \Gamma$.

Proof. To show the claim we slightly change the proof of the classical Ekeland's variation principle of [Jab03] (Theorem 3.1 in [Jab03]). Because there arises no new technical issues or necessary extensions, we sketch the proof here only. Our new idea is to restrict the shrinking sets in the original proof to subsets of the sets Γ_R. These are still closed in Γ by the closeness of M_R. Then the object in the limit $\gamma_{\epsilon,\delta}$ is necessarily in Γ_R and we get the conditions a.) and b.) as we stated them above and condition c.) for all $\gamma \in \Gamma_R$.

Because we want condition c.) for all $\gamma \in \Gamma$, we finally use the following observation: If $\gamma^* \in \Gamma_R$ and $\gamma \in \Gamma$, then

$$\operatorname{dist}\left\{\gamma^*, A_R^{cut}(\gamma)\right\} \leq \operatorname{dist}\{\gamma^*, \gamma\} \qquad \text{and} \tag{IV.1.92}$$

$$(\mathcal{I}_R - F_R)\left(A_R^{cut}(\gamma)\right) \leq (\mathcal{I}_R - F_R)(\gamma). \tag{IV.1.93}$$

This observation follows by Lemma IV.1.5, with A_R^{cut} defined in Remark IV.1.6.

Therefore, condition c.) holds for all $\gamma \in \Gamma$. □

Now we are ready to prove Theorem IV.1.39.

Proof of Theorem IV.1.39. (i) As explained we modify now the proof (that uses the Ekeland variational principle) of the classical mountain pass theorem (see [Jab03] Theorem 7.1 second proof) such that the arising Palais-Smale sequence is in M_R. Then the final application of Lemma IV.1.44 shows the claimed existence.

We know already that $\mathcal{I}_R - F_R$ is a C^1 functional by Lemma IV.1.9 and Lemma IV.1.10. The condition (IV.1.91) of Lemma IV.1.45 is satisfied by our choice of c and Lemma IV.1.23. Applying Lemma IV.1.45 (with $\delta = 1$ and abuse of notation), we know that for each $\epsilon > 0$ there is a path $\gamma_\epsilon \in \Gamma_R$ such that

$$\begin{aligned} &Y(\gamma_\epsilon) \leq c + \epsilon && \text{and} \\ &Y(\gamma) \geq Y(\gamma_\epsilon) - \epsilon \operatorname{dist}\{\gamma, \gamma_\epsilon\} && \text{for all } \gamma \in \Gamma. \end{aligned} \tag{IV.1.94}$$

The important difference to the original proof of the classical mountain pass theorem is, that $\gamma_\epsilon \in \Gamma_R$. However, that the second condition holds for all elements in Γ as in the original proof.

Then as in the proof of the classical mountain pass theorem we can conclude from (IV.1.94), that there is a sequence $f_n = \gamma_{\frac{1}{n}}\left(t_{\frac{1}{n}}\right) \in M_R$ that satisfies the Palais-Smale conditions defined in Definition IV.1.42. Hence, (i) is proven by Lemma IV.1.44.

(ii) We show in Lemma IV.1.44, that $\mathcal{I}_R - F_R$ satisfies the Palais-Smale condition (see Definition IV.1.42). Then each sequence of critical values within M_R is a Palais-Smale sequence. But then the Palais-Smale condition implies that there exists a convergent subsequence with limit in M_R. Hence, the set $K_{c,R}$ has to be compact.

(iii) We can restrict our attention to neighbourhoods $\mathcal{N}$ that are $\alpha > 0$ balls (in the $L^2\left(\mathbb{T}^d\right)$-norm) around $K_{c,R}$. For such a $\mathcal{N}$ we prove now the claim by a proof by contradiction. Assume that for each $\epsilon > 0$, there is a $\gamma_\epsilon \in \Gamma_R$ such that

$$Y\left(\gamma_\epsilon\right) < c + \epsilon \text{ and } \gamma_\epsilon \cap \mathcal{N} = \varnothing. \tag{IV.1.95}$$

By Lemma IV.1.45 (as in (i) with $\delta = \sqrt{\epsilon)}$ there is a $\gamma_{\epsilon,\sqrt{\epsilon}} \in \Gamma_R$, such that there is a sequence $t_\epsilon = t_{\epsilon,\sqrt{\epsilon}} \in [0,1]$, such that $\gamma_{\frac{1}{n},\frac{1}{\sqrt{n}}}\left(t_{\frac{1}{n}}\right) \to m^* \in K_{R,c}$. Then

$$\left\|\gamma_{\frac{1}{n}}\left(t_{\frac{1}{n}}\right) - m^*\right\|_{L^2} \leq \left\|\gamma_{\frac{1}{n}}\left(t_{\frac{1}{n}}\right) - \gamma_{\frac{1}{n},\frac{1}{\sqrt{n}}}\left(t_{\frac{1}{n}}\right)\right\|_{L^2} + \left\|\gamma_{\frac{1}{n},\frac{1}{\sqrt{n}}}\left(t_{\frac{1}{n}}\right) - m^*\right\|_{L^2} \leq ① + ②. \tag{IV.1.96}$$

The summand ① is bounded by $\delta = \sqrt{\epsilon}$ by property b.) of Lemma IV.1.45. The summand ② is bounded by a $\epsilon' > 0$ if $n > N(\epsilon')$. Hence, for n large enough (and hence ϵ' small) and ϵ small enough, the right hand is smaller than α and therefore $\gamma_\epsilon \cap \mathcal{N} \neq \varnothing$. But this is a contradiction. □

Remark IV.1.46. *If we had a deformation lemma (for convex closed sets see for example Lemma 17.3 in [Jab03]), a similar proof as the one of Theorem A in [PS84] would also show the claimed intersection of the paths with the neighbourhood (Theorem IV.1.39 (iii)).*

Remark IV.1.47. *Similar as in [PS84] Theorem 7, we could show that there exists at least one saddle point (in the sense of Definition IV.1.19) within the set of critical values at height c, because the set of critical values within M_R can not separate in M_R the two minima.*

IV.1.3.2 Mountain pass condition satisfied under Assumption IV.0.3

The Theorem IV.1.39 requires the condition (IV.1.83), i.e. that there is a mountain ridge in the energy landscape that each path from one minimum to another has to cross. We are in particular interested in lowest path between the global minimiser, therefore we assume in the following that the functions $m^{(1)}$ and $m^{(2)}$ in (IV.1.83) are global minimiser. We show in this section (see Lemma IV.1.48) that under some additional assumptions on the global minimiser, the condition (IV.1.83) is satisfied for paths between two global minimiser.

The conditions on the global minimiser of the following lemma are for example satisfied under Assumption IV.0.3. Therefore, the results of Theorem IV.1.39 are valid under Assumption IV.0.3.

Lemma IV.1.48. *Let $m^{(1)}, m^{(2)} \in L^2\left(\mathbb{T}^d\right)$ be a two global minimiser with $\widehat{m^{(1)}}_0 < \widehat{m^{(2)}}_0$. Assume that there is a $m^* \in \left(\widehat{m^{(1)}}_0, \widehat{m^{(2)}}_0\right) \subset \mathbb{R}$, such that $\mathcal{I} - F$ has no global minimiser with $\widehat{m_0} = m^*$. Then* (IV.1.83) *holds.*

To prove this lemma, we show that the following inequality holds

$$\inf_{m \in S_{m^*}} \left(\mathcal{I} - F\right)(m) > a > \max\left\{\left(\mathcal{I} - F\right)\left(m^{(1)}\right), \left(\mathcal{I} - F\right)\left(m^{(2)}\right)\right\} = a^*, \tag{IV.1.97}$$

where S_{m^*} is the subspace (of $L^2\left(\mathbb{T}^d\right)$)

$$S_{m^*} := \left\{m \in L^2\left(\mathbb{T}^d\right) : \widehat{m_0} = \int_{\mathbb{T}^d} m(x)\,\mathrm{d}x = m^*\right\}. \tag{IV.1.98}$$

The space S_{m^*} disconnects $L^2\left(\mathbb{T}^d\right)$ into two subspaces, each containing one of the points $m^{(1)}$ and $m^{(2)}$ (compare this to similar conditions for the mountain pass theorem, e.g. in Theorem 5.7 in [dF89]). Therefore, each path between the two minima has to pass through S_{m^*}. Therefore, (IV.1.97) implies (IV.1.83) .

We need in the proof of Lemma IV.1.48 the following property of S_{m^*}

Lemma IV.1.49. *The set S_{m^*} is closed in $L^2\left(\mathbb{T}^d\right)$.*

Proof. Assume that there is sequence $\{m^n\} \subset S_{m^*}$ that converges to $m \in L^2$. Then this sequence converges also weakly to m. Hence, $0 = \widehat{m_0^n} = \langle m^n, 1\rangle \to \langle m, 1\rangle = \widehat{m_0}$ and consequently $m \in S_{m^*}$. □

Now we are ready to prove Lemma IV.1.48 this section. As it becomes obvious in the proof, this is a consequence of the lower semi continuity of $\mathcal{I} - F$ on bounded subsets and of S_{m^*} being closed.

Proof of Lemma IV.1.48. We prove the lemma by a proof by contradiction. Assume that there is a sequence $m_n \subset S_{m^*}$ such that $(\mathcal{I} - F)(m_n) \searrow a^*$ and w.l.o.g $\|m_n\|_{L^\infty} \le R$ for all n (otherwise we can find another sequence that satisfies this bound by Lemma IV.1.5). This implies that $\|m_n\|_{L^2} \le R$ for all n, i.e. $\{m_n\}_n \subset B_{L^2\{\mathbb{T}^d\}}(R)$. This set is weakly compact by the Banach-Alaglou theorem (see also the proof of Proposition IV.1.29). Therefore, there is a converging subsequence $m_{n_k} \to m^* \in B_{L^2\{\mathbb{T}^d\}}(R)$.

Moreover, we know by Lemma IV.1.31 that $\mathcal{I} - F$ is weakly lower semi continuous on balls in $L^2\left(\mathbb{T}^d\right)$. Therefore

$$a^* = \liminf_{k\to\infty} (\mathcal{I} - F)(m_{n_k}) \ge (\mathcal{I} - F)(m^*)\,. \tag{IV.1.99}$$

The function m is in S_{m^*}, because S_{m^*} is closed. However, we have assumed that there is no global minimiser in S_{m^*}, hence this is a contradiction. □

IV.1.3.3 Explicit results under Assumption IV.0.3

In this section we show that under Assumption IV.0.3, the lowest path between the two minima $\pm m^{CW}$ (of Theorem IV.1.36) passes through 0 or goes around 0 depending on the parameters. We start with some general properties of I and h that we need several times.

Lemma IV.1.50. (i) *h is even with $h(0) = 0$.*

(ii) *I is even with $I(0) = 0$.*

(iii) *I', h' are odd and I'', h'' are even.*

Proof. This is a consequence of Lemma III.2.5 and of Ψ being even (Assumption III.2.2). □

Lemma IV.1.51. *Let the additional Assumption IV.0.3 be satisfied. Then:*

(i) *For $a, b \in \mathbb{R}$ with $|a| \ge |b|$*

$$I(a) - I(b) \ge I''(0)\,\frac{1}{2}\left(a^2 - b^2\right). \tag{IV.1.100}$$

(ii) *For $\lambda\widehat{J_0} \ge M > I''(0)$ there is a constant $0 < m_M \le m^{CW}$ such that for all $a \in [-m_M, m_M]$*

$$I(a) \le \frac{1}{2} M a^2. \tag{IV.1.101}$$

If $M = \widehat{J_0}$, then $m_M = m^{CW}$.

Proof. ***(i)*** $I'(.)$ is convex on $\mathbb{R}_+$. This implies $I'(n) \geq 0 + I''(0)\, n$ for $n \geq 0$. Hence

$$I(a) - I(b) = I(|a|) - I(|b|) = \int_{|b|}^{|a|} I'(y)\,\mathrm{d}y \geq I''(0)\,\frac{1}{2}\left(a^2 - b^2\right). \tag{IV.1.102}$$

(ii) By Assumption IV.0.3 and the assumption on M, we have $I'(n) < Mn$ for all $n \in (-m_M, m_M)$, $n \neq 0$. Hence,

$$I(a) = I(|a|) = \int_0^{|a|} I'(n)\,\mathrm{d}n \leq \int_0^{|a|} Mn\mathrm{d}n = \frac{1}{2} M a^2. \tag{IV.1.103}$$

□

In the following theorem we state the main results of this section. It shows the difference in the trajectory of a lowest path between the minima depending on the parameter of the model. We have defined Γ^d and $\widehat{J_\ell}$ in Section IV.1.1.5.

Theorem IV.1.52. *Let I' be convex on $\mathbb{R}_+$ (compare to the additional Assumption IV.0.3).*

(i) *If $\widehat{J_0} > \frac{I''(0)}{\lambda} > \max_{\ell \in \Gamma^d, \ell \neq 0} \left|\widehat{J_\ell}\right|$, then 0 is a saddle point and the path that follows the constant solutions is a lowest path between the two minima.*

(ii) *Take $\ell \in \Gamma^d \setminus \{0\}$ such that $\widehat{J_\ell} > \frac{I''(0)}{\lambda} > \max_{\ell' \in \Gamma^d \setminus \{0\} : \ell' \notin M_\ell} \left|\widehat{J_{\ell'}}\right|$, where $M_\ell = \left\{\ell' : \widehat{J_{\ell'}} = \widehat{J_\ell}\right\}$. Then there is a connected set U in the subspace spanned by $\{1\} \cup \{e^c_{\ell'}, e^s_{\ell'}\}_{\ell' \in M_\ell}$ (defined in* (IV.1.52)*), which is open in this subspace and contains 0, such that $\mathcal{I} - F < 0$ on $U \setminus \{0\}$.*

Moreover, there is a path $\gamma \in \mathsf{C}\left([0,1], L^2\left(\mathbb{T}^d\right)\right)$ from $-m^{CW}$ to m^{CW} that passes through U such that $\max_{t \in [0,1]} (\mathcal{I} - F)(\gamma(t)) < 0$. Hence, the lowest path between the minima does not any more pass through the constant function 0.

Remark IV.1.53. *The path we construct in (ii) is not necessary a lowest path between the two minima. However, it is lower than the path following the constants.*

Proof. ***(i)*** Fix a function $m \in L^2$ with $\widehat{m_0} = 0$ and $m \neq 0$, which can be written in its sine-cosine coefficients as $m = \sum_{\ell \in \Gamma^d \setminus \{0\}} \widehat{m^c_\ell} e^c_\ell + \widehat{m^s_\ell} e^s_\ell$. Then

$$\begin{aligned} \mathcal{I}(m) - F(m) &= \int_{\mathbb{T}^d} I(m(x))\,\mathrm{d}x - \lambda \frac{1}{2} \sum_\ell \widehat{J}_\ell \left(\left|\widehat{m^c_\ell}\right|^2 + \left|\widehat{m^s_\ell}\right|^2\right) \\ &\geq I''(0)\,\frac{1}{2} \int_{\mathbb{T}^d} (m(x))^2\,\mathrm{d}x - \lambda \frac{1}{2} \sum_\ell \widehat{J}_\ell \left(\left|\widehat{m^c_\ell}\right|^2 + \left|\widehat{m^s_\ell}\right|^2\right) \\ &\geq \frac{1}{2}\left(I''(0) - \lambda \max_{\ell \in \Gamma^d \setminus \{0\}} \left|\widehat{J}_\ell\right|\right) \sum_\ell \left(\left|\widehat{m^c_\ell}\right|^2 + \left|\widehat{m^s_\ell}\right|^2\right) > 0, \end{aligned} \tag{IV.1.104}$$

where we use Lemma IV.1.51 (i) with $b = 0$.

The subspace spanned by $\{e_\ell\}_{\ell \neq 0}$, disconnects the space L^2 and the two minima are in two separate components. Hence, each path has to cross this subspace and the lowest point there is the constant solution 0 with $\mathcal{I}(0) - F(0) = 0$.

Moreover, the path following the constant solutions from $-m^{CW}$ to m^{CW} is for $m \in \left(0, m^{CW}\right)$ always lower than 0, because by Lemma IV.1.51 (ii)

$$\mathcal{I}(m) - F(m) = I(m) - \lambda \widehat{J_0} m^2 \frac{1}{2} < 0. \tag{IV.1.105}$$

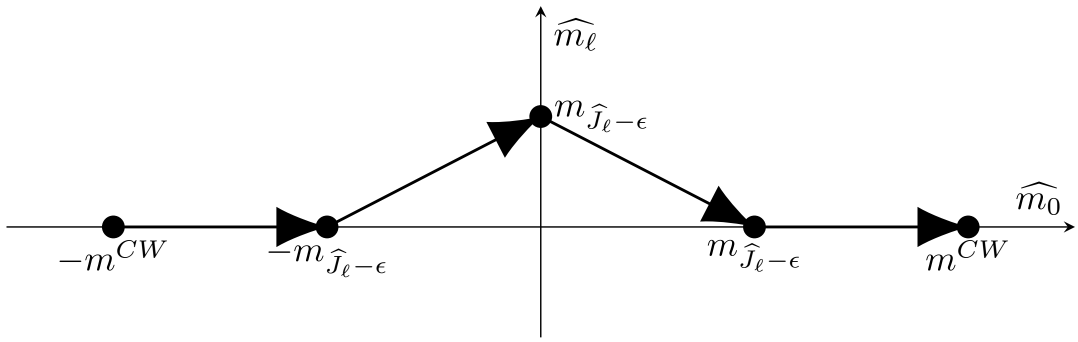


Figure IV.5: Path around 0 in Lemma IV.1.52 (ii)

(ii) Fix a $m(.) \in L^2(\mathbb{T}^d)$ that can be represented as $m(x) = \widehat{m_0^c} + \sum_{\ell' \in M_\ell} \widehat{m_{\ell'}^c} e_{\ell'}^c(x) + \widehat{m_{\ell'}^s} e_{\ell'}^s(x)$. Let $\epsilon > 0$ be such that $\lambda\left(\widehat{J_\ell} - \epsilon\right) \geq I''(0)$. By Lemma IV.1.51 (ii) and by the orthonormality of the basis functions

$$\begin{aligned}
&\mathcal{I}(m(.)) - F(m(.)) = \int_{\mathbb{T}^d} I(m(x))\,\mathrm{d}x - \frac{1}{2}\lambda\widehat{J_0^c}\widehat{m_0}^2 - \frac{1}{2}\lambda\widehat{J_\ell^c}\sum_{\ell' \in M_\ell} \widehat{m_{\ell'}^c}^2 + \widehat{m_{\ell'}^s}^2 \\
&\leq \frac{1}{2}\lambda\left(\widehat{J_\ell^c} - \epsilon\right)\left(\widehat{m_0}^2 + \sum_{\ell' \in M_\ell} \widehat{m_{\ell'}^c}^2 + \widehat{m_{\ell'}^s}^2\right) - \frac{1}{2}\lambda\widehat{J_0^c}\widehat{m_0}^2 - \frac{1}{2}\lambda\widehat{J_\ell^c}\sum_{\ell' \in M_\ell} \widehat{m_{\ell'}^c}^2 + \widehat{m_{\ell'}^s}^2 \qquad \text{(IV.1.106)} \\
&\leq -\frac{1}{2}\lambda\epsilon\left(\widehat{m_0}^2 + \sum_{\ell' \in M_\ell} \widehat{m_{\ell'}^c}^2 + \widehat{m_{\ell'}^s}^2\right) < 0,
\end{aligned}$$

for all $0 < |\widehat{m_0}| + \sum_{\ell' \in M_\ell} \left|\widehat{m_{\ell'}^c}\right| + \left|\widehat{m_{\ell'}^s}\right| \leq m_{\widehat{J}_\ell-\epsilon}$. Here $m_{\widehat{J}_\ell-\epsilon} \in \mathbb{R}_+$ is the constant defined in Lemma IV.1.51 (ii).

Define the following path (see also Figure IV.5):

- First the path follows the constant function from $-m^{CW}$ until $-m_{\widehat{J}_\ell-\epsilon}$.
- Then the path is the linear interpolation between $-m_{\widehat{J}_\ell-\epsilon}$ and $m_{\widehat{J}_\ell-\epsilon}e_\ell^c$.
- Finally, we reflect the paths, such that it goes to m^{CW}, by going first to $m_{\widehat{J}_\ell-\epsilon}$ (linearly) and then it follows the constants.

On this path we get by (IV.1.106), the claimed upper bound

$$\max_{t\in[0,1]} (\mathcal{I} - F)(\gamma(t)) \leq -\frac{1}{2}\epsilon\lambda\frac{1}{2}\left(m_{\widehat{J}_\ell-\epsilon}\right)^2 < 0. \qquad \text{(IV.1.107)}$$

□

IV.1.4 Bifurcation results

In this section we are looking for solutions of (IV.1.1), i.e. critical values of $\mathcal{I} - F$. In particular, we investigate bifurcations of critical values from the constant function $m \equiv 0$ and from the constant minimiser of $\mathcal{I} - F$. Define the function

$$\Phi(\lambda, m) := m - \lambda h'(J * m), \qquad \text{(IV.1.108)}$$

with $\lambda \in \mathbb{R}$ and $m \in L^2(\mathbb{T}^d)$. Then critical values of (IV.1.1) are tuples $(\lambda^*, m^*) \in \mathbb{R} \times L^2(\mathbb{T}^d)$ such that

$$\Phi(\lambda^*, m^*) = 0. \qquad \text{(IV.1.109)}$$

We know that $\Phi(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. We call $(\mathbb{R}, 0)$ the *trivial branch.* Now we ask if there is a value $\lambda \in \mathbb{R}_+$ at which there exist non-trivial solutions around $(\lambda, 0)$. To be precise we are trying to find bifurcation points of this trivial branch.

Definition IV.1.54. *We call a tuple* $(\lambda^*, 0) \in \mathbb{R} \times L^2(\mathbb{T}^d)$ *a bifurcation point of* $\Phi(\lambda^*, 0) = 0$ *(i.e. of the trivial branch), if every neighbourhood of* $(\lambda^*, 0)$ *in* $\mathbb{R} \times L^2(\mathbb{T}^d)$ *contains a non-zero solution.*

We refer to [AE84] Section 2.5 for this definition and further discussion (see also [Rab85], [Rab87]).

Definition IV.1.55. *We say a line through* $(\lambda^*, 0)$ *is a bifurcation curve, if there exists an* $\epsilon > 0$ *and two maps*

$$\lambda(.) : (-\epsilon, \epsilon) \to \mathbb{R} \text{ and } m_\epsilon(.) : (-\epsilon, \epsilon) \to L^2(\mathbb{T}^d), \tag{IV.1.110}$$

with $\lambda(0) = \lambda^*$, $m_\epsilon(0) = 0$, $m'_\epsilon(0) \neq 0$, *such that*

$$\Phi(\lambda(t), m_\epsilon(t)) = 0, \tag{IV.1.111}$$

for all $t \in (-\epsilon, \epsilon)$.

A bifurcation curve is C^k *for* $k \in \mathbb{N}$, *if* $t \to (\lambda(t), m_\epsilon(t))$ *is* C^k.

Remark IV.1.56. *The equation* (IV.1.111) *implies that* $m'_\epsilon(0) \in \operatorname{Ker}(\partial_m \Phi(\lambda^*, 0))$. *Indeed,* (IV.1.111) *implies that* $\partial_t|_{t=0} \Phi(\lambda(t), m_\epsilon(t)) = 0$. *Therefore,* $\partial_m \Phi(\lambda^*, 0) m'_\epsilon(0) = 0$.

We assume in this section that the following additional assumption on J holds besides the Assumption IV.0.1, Assumption III.2.2 and Assumption III.3.1.

Assumption IV.1.57. *We assume that* J *only depends on the distance, i.e.* $J(x - y) = J(|x - y|)$.

Remark IV.1.58. *In this section we need this additional assumption for Lemma IV.1.60 and Theorem IV.1.65. All the results in this section require besides this additional assumption, only that* J *satisfies Assumption III.3.1 and that* h' *is smooth.*

This subsequent sections are organised as follows. At first, we state properties of Φ (in Section IV.1.4.1) that we need in the subsequent sections. Then in Section IV.1.4.2, we show a necessary condition for bifurcations from a constant solution of (IV.1.109). Afterwards we investigate bifurcations from the trivial branch (in Section IV.1.4.3) in general. Last but not least we show in Section IV.1.4.4 further bifurcation results from this branch under the Assumption IV.0.3.

IV.1.4.1 Preliminary: Properties of Φ

We need the properties of Φ shown in the next lemma to prove our bifurcation theorems.

Lemma IV.1.59. (i) Φ *is a* C^∞ *functional (infinity many times continuously Fréchet differentiable) with derivatives*

$$\partial_m^1 \Phi(\lambda, m)(g_1) = g_1 - \lambda h''(J * m) J * g_1, \tag{IV.1.112}$$

$$\partial_m^n \Phi(\lambda, m)(g_1, ..., g_n) = \lambda h^{(n+1)}(J * m) J * g_1 ... J * g_n \quad \text{for } n \geq 2, \tag{IV.1.113}$$

$$\partial_{\lambda,m}^2 \Psi(\lambda, m)(g_1) = h''(J * m) J * g_1, \tag{IV.1.114}$$

$$\partial_{\lambda,\lambda}^2 \Phi(\lambda, m) = 0. \tag{IV.1.115}$$

(ii) *For* $\lambda \in \mathbb{R}$, *we define the set* $K_\lambda = \left\{\ell \in \Gamma^d : \lambda \widehat{J}_\ell = I''(0)\right\}$ *(see* (IV.1.53) *for the definition of* Γ^d*). Then*

$$\begin{aligned} X^1_{(\lambda)} &:= \operatorname{Ker}\left(\partial_m \Phi(\lambda,0)\right) = \operatorname{Ker}\left(I - \lambda h''(0)\, J * .\right) \\ &= \left\{\sum_{\ell \in K_\lambda} f^c_\ell e^c_\ell(.) + f^s_\ell e^s_\ell(.) : f^c_\ell, f^s_\ell \in \mathbb{R}\right\}, \end{aligned} \tag{IV.1.116}$$

$$X^2_{(\lambda)} := \operatorname{Im}\left(\partial_m \Phi(\lambda,0)\right) = \left\{\sum_{\ell \notin K_\lambda} f^c_\ell e^c_\ell(.) + f^s_\ell e^s_\ell(.) : f^c_\ell, f^s_\ell \in \mathbb{R}\right\}. \tag{IV.1.117}$$

(iii) *For each* $\lambda \in \mathbb{R}$, $L^2\left(\mathbb{T}^d\right) = \operatorname{Im}\left(\partial_m \Phi(\lambda,0)\right) \oplus \operatorname{Ker}\left(\partial_m \Phi(\lambda,0)\right)$.

(iv) $\dim\left(\operatorname{Ker}\left(\partial_m \Phi(\lambda,0)\right)\right) = \operatorname{codim}\left(\operatorname{Im}\left(\partial_m \Phi(\lambda,0)\right)\right) = 2\#\left\{K_\lambda\right\}$.

(v) *There is a* $\Gamma(\lambda,m) \in \mathsf{C}^\infty\left(\mathbb{R} \times L^2\left(\mathbb{T}^d\right)\right)$, *such that, for all* $(\lambda,m) \in \mathbb{R} \times L^2\left(\mathbb{T}^d\right)$

$$\Phi(\lambda,m) = m - \lambda h''(0)\, J * m + \Gamma(\lambda,m), \tag{IV.1.118}$$

with $\Gamma(\lambda,0) = 0$ *and* $\partial_m \Gamma(\lambda,0) = \partial_\lambda \Gamma(\lambda,0) = \partial^2_{\lambda,m}\Gamma(\lambda,0) = 0$ *for all* $\lambda \in \mathbb{R}$. *Moreover,* $\|\Gamma(\lambda,m)\|_{L^2} = o\left(\|m\|_{L^2}\right)$ *when* $\|m\|_{L^2} \to 0$.

(vi) $h'(J * .)$ *is a compact mapping of* $L^2\left(\mathbb{T}^d\right)$ *into* $L^2\left(\mathbb{T}^d\right)$.

(vii) *Let* $\lambda^* = \frac{1}{h''(0)\widehat{J}_\ell}$ *for a* $\ell \in \Gamma^d$ *with* $\widehat{J}_\ell \neq 0$, *then*

$$\Phi(\lambda^*,0) = 0 \quad \text{and} \quad \partial_\lambda \Phi(\lambda^*,0) = 0 \quad \text{and} \quad \partial^2_{m,m}\Phi(\lambda^*,0) = 0 \tag{IV.1.119}$$

$$\partial_m \Phi(\lambda^*,0)\Big|_{X^1_{(\lambda^*)}} \equiv 0 \quad \text{and} \quad \partial_m \Phi(\lambda^*,0)\Big|_{L^2(\mathbb{T}^d)\setminus X^1_{(\lambda^*)}} \subset L^2\left(\mathbb{T}^d\right)\setminus X^1_{(\lambda^*)} \tag{IV.1.120}$$

$$\partial^2_{\lambda,m}\Phi(\lambda^*,0)\Big|_{X^1_{(\lambda^*)}} = h''(0)\, J * \left(X^1_{(\lambda^*)}\right) = h''(0)\, \widehat{J^c_\ell}\, X^1_{(\lambda^*)} \not\equiv 0. \tag{IV.1.121}$$

Proof. ***(i)*** The formulas of the derivatives can be shown by the Taylor expansion and the local boundedness and continuity of the derivatives of h. Indeed, fix $m, g \in L^2\left(\mathbb{T}^d\right)$. For the first derivatives, there is a $s = s(m,g) : \mathbb{T}^d \to [0,1]$ such that

$$\begin{aligned} &\int \left|\Phi(m+g) - \Phi(m) - g + \lambda h''(J*m)\, J * g\right|^2 \mathrm{d}x \\ &= \int \frac{1}{4}\left|h'''(J*m + s(x)\, J*g)\,(J*g)^2\right|^2 \mathrm{d}x \\ &\leq \frac{1}{4} \sup_{\left\{s \in \mathbb{R} : |s| \leq \|J\|_{L^2}\left(\|m\|_{L^2} + \|g\|_{L^2}\right)\right\}} h'''(s)^2 \|J\|^4_{L^2} \|g\|^4_{L^2}. \end{aligned} \tag{IV.1.122}$$

The right hand side converges faster to 0 than $\|g\|^2_{L^2}$. The continuity of $m \to h''(J*m)\, J * .$ follows by a similar estimate.

(ii) Both results are a consequence of the chosen eigenfunction basis of $J * .$. The formula for the kernel follows direct from the eigenvalues and eigenfunctions of $J * .$ derived in Lemma IV.1.26 (iv). To show the formula of the image we choose an arbitrary $g \in X^2_{(\lambda)}$. Then we find a $f \in X^2_{(\lambda)}$ such that $\partial_m \Phi(\lambda,0)(f) = g$ because

$$f - \lambda h''(0)\, J * f = \sum_{\ell \notin K_\lambda} \left(1 - \lambda h''(0)\, \widehat{J^c_\ell}\right)\left(f^c_\ell e^c_\ell + f^s_\ell e^s_\ell\right) = \sum_{\ell \notin K_\lambda} \left(g^c_\ell e^c_\ell + g^s_\ell e^s_\ell\right), \tag{IV.1.123}$$

by setting $f_\ell^{s/c} = g_\ell^{s/c} \frac{1}{1-\lambda h''(0)\widehat{J_\ell^c}}$ for $\ell \notin K_\lambda$.

Moreover, by the definition of K_λ we can not find a f such that this procedure could work for a g with one ore more non zero coefficients in the directions in K_λ.

(iii) This follows directly from the definition of the kernel and the image in (ii).

(iv) This is a consequence of (ii).

(v) Define

$$\Gamma(\lambda, m) := \Phi(\lambda, m) - m + \lambda J * m. \tag{IV.1.124}$$

This function Γ is in C^∞ by (i). Then

$$\Gamma(\lambda, m) = \lambda \frac{1}{2} h'''\left(s(m(.))\, J * m\right)(J * m)^2, \tag{IV.1.125}$$

by the Taylor expansion $h'(J * m) = h'(0) + h''(0) J * m + \frac{1}{2} h'''(s(m(.)) J * m)(J * m)^2$ with a $s : \mathbb{R} \to [0, 1]$ and $s(0) = 0$.

This function satisfies obviously the claimed conditions at $m = 0$. Moreover

$$\left\| h'''(s(x) J * m)(J * m)^2 \right\|_{L^2}^2 \leq \sup_{|s| \leq \|J\|_{L^2} \|m\|_{L^2}} \left\{ h'''(s)^2 \right\} \|J\|_{L^2}^4 \|m\|_{L^2}^4 . \tag{IV.1.126}$$

Hence, $\left\| h'''(s(x) J * m)(J * m)^2 \right\|_{L^2} = o(\|m\|_{L^2})$ by the continuity of h'''.

(vi) We fix a bounded set $Y \subset L^2$. The operator $J *.$ is compact by Lemma IV.1.26 (i) hence $J * Y$ is relatively compact. Moreover, the function h' is continuous hence $h'\left(\overline{J * Y}\right)$ is compact.

Last but not least we have to show that $\overline{h'(J * Y)} = h'\left(\overline{J * Y}\right)$. To prove this we fix a sequence $u_n \in h'(J * Y)$, s.t. $u_n \to u$ and u is in the boundary of the set $h'(J * Y)$. There are $y_n \in J * Y$ such that $u_n = h'(y_n)$ or equivalently $I'(u_n) = y_n$. Because I' is continuous this implies that y_n converges to an $y = I'(u)$. This implies that $y \in \overline{J * Y}$. Hence, $u \in h'\left(\overline{J * Y}\right)$.

(vii) We use the formulas for the derivatives of Ψ derived in (i) and the special choice of λ^*. □

In the following lemma, we summarise symmetry properties of Φ, that we need in the subsequent proofs.

Lemma IV.1.60. *Let Assumption IV.1.57 hold.*

(i) *Φ has $\mathbb{Z}_2$-symmetry on $L^2(\mathbb{T}^d)$, i.e. reflection symmetry $\Phi(\lambda, -m) = -\Phi(\lambda, m)$.*

(ii) *Symmetries on $\mathbb{T}^d$: For the following maps $\gamma : \mathbb{T}^d \to \mathbb{T}^d$, we have*

$$\Phi(\lambda, m(.))(\gamma(x)) = \Phi(\lambda, m(\gamma(.)))(x). \tag{IV.1.127}$$

- *Shift-symmetry w.r.t. shifts on $\mathbb{T}^d$: $\gamma_z(x) = x + z$ for $x, z \in \mathbb{T}^d$.*
- *Reflection-symmetry in each of the d direction of $\mathbb{T}^d$: $\gamma_\beta(x) = (\beta_1 x_1, ..., \beta_d x_d)$ for $\beta \in \{-1, 1\}^d$.*
- *Permutation symmetry of the coordinates of $\mathbb{T}^d$: $\gamma_\sigma(x) = \left(x_{\sigma(1)}, ..., x_{\sigma(d)}\right)$ for σ a permutation of $\{1, ..., d\}$.*

Proof. *(i)* This is a consequence of h' being even.

(ii) Each of the symmetries follow from properties of the convolution $J * m$ and corresponding properties of J. For example the permutation and the reflection-symmetry follows because J only depends on $|x|$ (Assumption IV.1.57). □

IV.1.4.2 A necessary condition for a bifurcation at a constant function

In the following lemma we state a necessary condition for a bifurcation at constant functions.

Lemma IV.1.61. *Let $\lambda \in \mathbb{R}_+$ and $m(.) \in L^2(\mathbb{T}^d)$ be a constant function $m(.) \equiv m \in \mathbb{R}$. Then a necessary condition for a bifurcation at the tuple (λ, m) is that*

$$\lambda h''\left(\widehat{J_0}m\right)\widehat{J_\ell} = 1, \tag{IV.1.128}$$

for at least one $\ell \in \Gamma^d$.

Proof. By the implicit function theorem (see for example [KP13] Theorem 3.4.10) we know that bifurcations can only occur at points $m \in L^2$, when $\partial_m \Phi(\lambda, m)$ is not a Banach space isomorphism Equivalently there is only a bifurcation possible when 0 is in the spectrum of $\partial_m \Phi(\lambda, m)$.

If there is a $g \in L^2(\mathbb{T}^d)$, $g \neq 0$ such that $\partial_m \Phi(\lambda, m)(g) \equiv 0$, then 0 is in the spectrum. That can only happen if (IV.1.128) is satisfied for at least one $\ell \in \Gamma^d$ (see Lemma IV.1.59 (i)). If there is not such a g, then 0 is not in the spectrum. Indeed, we know that $h''\left(\widehat{J_0}m\right) > 0$, therefore the same calculation as in (IV.1.123) shows that $\partial_m \Phi(\lambda, m)$ is surjective. □

Remark IV.1.62. *When the function $m(.)$ is not a constant, then $\partial_m \Phi(\lambda, m(.))$ being surjective is in general not true. So even when* (IV.1.128) *is not satisfied for such a $m(.)$, there might still occur a bifurcation. However, if for such a $m(.)$, $\partial_m \Phi(\lambda, m(.))$ is surjective, then* (IV.1.128) *is a necessary condition for a bifurcation. At such a bifurcation point, $h''(J * m)$ is constant. When the additional Assumption IV.0.3 holds, this implies (by the continuity of $J * m$) that $m(.)$ has to be constant.*

IV.1.4.3 Bifurcation from the trivial branch

We know (by Lemma IV.1.61) the following necessary condition for bifurcations (from the trivial branch).

Corollary IV.1.63. *A bifurcation from the trivial branch can only occur when*

$$\lambda = \frac{1}{\widehat{J_\ell} h''(0)}. \tag{IV.1.129}$$

Thus bifurcations can only occur at points that are characterised by the eigenvalues of $J * .$.

In this section we show that this condition is also sufficient. We differentiate between the first bifurcation point (Theorem IV.1.64) and the other bifurcation points (Theorem IV.1.65). We see that at the first bifurcation point exactly one curve bifurcates. Moreover, all functions on this curve are constant. Whereas for the other bifurcation points we show that there bifurcate one curve, but there might be more. On the particular curve that we consider, we can characterise the functions. The symmetries of Φ are crucial in the proof of (Theorem IV.1.65).

In the first theorem, we show that at $\ell = 0$ there is a bifurcating curve from the trivial branch, that consists only of constant functions.

Theorem IV.1.64. *At $\lambda^* = \frac{1}{h''(0)\widehat{J_0}}$, the zero set close to the trivial branch consists beside the trivial branch of exactly one C^∞ curve $t \to (\lambda(t), m(t))$ that consists only of constant solutions ($m(t) \equiv m_t \in \mathbb{R}$) of*

$$m(t) = \lambda(t) h'\left(\widehat{J_0^c} m(t)\right). \tag{IV.1.130}$$

Theorem IV.1.65. *Let $\lambda^* = \frac{1}{h''(0)\widehat{J_{\ell^*}}}$ for a $\ell^* \in \Gamma^d \backslash \{0\}$ with $\widehat{J_{\ell^*}^c} \neq 0$, such that for all $\ell, \ell' \in K_{\lambda^*}$, there is a permutation σ, such that $\ell_i = \pm \ell'_{\sigma(i)}$ for all $i = 1, ..., d$. Then there is (modulo shifts)* *at least one* C^p *curve that bifurcates form the trivial branch at* $(\lambda^*, 0)$. *On this curve each function (parametrised by* $t \in (-\epsilon, \epsilon)$*) is of the form (modulo shifts)*

$$m(t)(.) = f(t)(.) + \alpha(t) \sum_{\ell \in K_{\lambda^*}} e^s_\ell(.), \tag{IV.1.131}$$

where $f(t) \in X^2_{(\lambda^*)}$ *is an odd function.*

In particular, each functions on this curve has zero mean, i.e. for all $t \in (-\epsilon, \epsilon)$,

$$\int_{\mathbb{T}^d} m(t)(x)\,\mathrm{d}x = 0. \tag{IV.1.132}$$

Remark IV.1.66. (i) *By Lemma IV.1.26 (v), we know that when* $\ell \in K_{\lambda^*}$, *all* $\ell' \in \Gamma^d$ *with* $\ell_i = \pm \ell'_{\sigma(i)}$ *for all* $i = 1, ..., d$ *for a permutation* σ, *are also in* K_{λ^*}. *Hence, the assumption on the elements in* K_{λ^*} *in Theorem IV.1.65 says, that exactly these* ℓ' *are in* K_{λ^*} *but no other. This excludes mainly the case that two elements are in* K_{λ^*}, *which can not be transferred into each other by permutations and reflections.*

(ii) *By the shift symmetry of* Φ, *each shift on* $\mathbb{T}^d$ *of each function on the curve is a solution of* (IV.1.109). *The bifurcating curve in Theorem IV.1.65 is hence not only a curve of functions, but a curve of manifolds.*

Proof of Theorem IV.1.64. For $\ell = 0$, $X^1_{(\lambda^*)}$ is one dimensional (Lemma IV.1.26 (iii)). Moreover, $\partial^2_{\lambda,m}(\lambda^*, 0)\left(X^1_{(\lambda^*)}\right) = X^1_{(\lambda^*)}$. Therefore, the usual procedure, consisting of the Lyapunov-Schmidt reduction in combination with the Morse lemma (see [AE84] Theorem 2.5.2 or [Rab85] Section 1.2 and 1.3, also the more general [Rab87] Theorem 2.2 implies this result) is applicable. From this we infer the claimed existence of exactly one bifurcation curve.

Each function on this curve is constant. Indeed, for m being constant, $\Phi(\lambda, m)$ is constant, i.e. $\Phi(\lambda, m) \in X^1_{(\lambda^*)}$. Therefore, the function in the Lyapunov-Schmidt reduction is equal to zero (by the uniqueness). Hence, each function on the bifurcation curve is constant. □

Remark IV.1.67. *We get the existence of a bifurcation point for* $\ell = 0$ *by the Krasnoselskii theorem (see [Nir74] Theorem 3.3.1, [Rab85] Theorem 1.2 or [Kra64] Chapter 4). The conditions of this theorem are satisfied because for* $\ell = 0$ *the algebraic multiplicity of the eigenvalue* $\frac{1}{h''(0)\widehat{J_0}}$ *is one (by Lemma IV.1.26 (iii)). Moreover, the other conditions of the Krasnoselskii theorem are satisfied by Lemma IV.1.59 (v) and (vi). However, this theorem does not imply the existence, let alone the regularity, of a bifurcation curve.*

Before we prove the Theorem IV.1.65, let us briefly explain why typical bifurcation theory results are not applicable in the situation that we consider here.

Remark IV.1.68. (i) *The Krasnoselskii theorem is not applicable for the other bifurcation points when* $\ell \neq 0$, *because the corresponding eigenvalues* $\widehat{J}_\ell$ *have an even algebraic multiplicity. The geometric multiplicity is even for the eigenvalues* $\widehat{J}_\ell$ *(by Lemma IV.1.59 (iv)) and equal to the algebraic multiplicity (by Lemma IV.1.59 (iii)).*

(ii) *Also results like Theorem 2.5.2 in [AE84] (see also [Rab85] Section 1.3) are only applicable when* $\operatorname{Ker}(\partial_m \Phi(\lambda^*, 0))$ *is one dimensional. However, this holds only if* $\ell = 0$ *(by Lemma IV.1.59 (iii) and Lemma IV.1.26 (iii)).*

There are generalisations of results like Theorem 2.5.2 in [AE84] for settings when the image still has codimension one but the kernel is higher dimensional (see for example [Nir74] Theorem 3.2.3 or references in Comment 1.3.3 in [Rab85]). These generalisations are not applicable in our situation, because by Lemma IV.1.59 (iv) both dimensions are equal.

(iii) *Moreover, there are generalisation to cases when the image has codimension and the kernel has dimension equal to* $n \in \mathbb{N}$ *(see Theorem 2.2 in [Rab87] which uses a generalised implicit function theorem (Theorem 3.2 in [Rab84], see also [BMS83] Theorem 2.7)). These generalisation state that the maximal amount of curves that bifurcate at a point* $(\lambda^*, 0)$ *(including the trivial one) is* $2^{2\#\{K_{\lambda^*}\}}$*. Moreover, we would even get that the number of these curves has to be even ([BMS83] Theorem 2.7). However, this generalisation requires that the so called* $\mathbb{R}$*-nondegeneracy condition holds. But this condition does not hold in our setting. We refer to the Remark IV.1.69 for more details.*

Although these usual bifurcation theory results are not applicable, we can still prove that a bifurcation occurs due to the symmetry properties of Φ (derived in Lemma IV.1.60). By these symmetry properties we can simplify the problem (after the Lyapunov-Schmidt reduction), by reducing the dimensions, such that we can finally apply the usual Morse lemma.

Proof of Theorem IV.1.65. We show now the existence of a bifurcation curve at $(\lambda^*, 0)$. The structure of the proof is as follows. We start (Step 1) the usual Lyapunov-Schmidt reduction. This reduces the problem (IV.1.109) to a $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ problem (see (IV.1.134)). Then in Step 2 we show symmetries of the function that arises during the Lyapunov-Schmidt reduction. These symmetries are a consequence of the same symmetries of Φ that we show in Lemma IV.1.60. We use these symmetries to reduce the dimension to the $\mathbb{R}^2 \to \mathbb{R}^2$ problem (IV.1.141) in Step 3. Thereby we restrict our attention to a one dimensional subspace of $X^1_{(\lambda^*)}$. Next (Step 4), we reduce the problem further to a $\mathbb{R}^2 \to \mathbb{R}$ we take advantage of the symmetries of Φ and the chosen subspace of $X^1_{(\lambda^*)}$. Finally, (Step 5), we use the Morse lemma to get the existence of the curve.

Step 1: Lyapunov-Schmidt reduction: By Lemma IV.1.59 (ii) and (vi) we know that $X^1_{(\lambda^*)}$ is spanned by the finite set $\{e^c_\ell, e^s_\ell : \ell \in K_{\lambda^*}\}$. To simplify the notation we use $n := \#\{K_{\lambda^*}\}$.

By Lemma IV.1.59 (ii) $\partial_m \Phi(\lambda^*, 0)(.) = . - \lambda^* h''(0) J * .$ is a bijective map from $X^2_{(\lambda^*)}$ to $X^2_{(\lambda^*)}$. Hence, the implicit function theorem implies that there is a $\epsilon > 0$ and a unique function $f^*(\lambda, \alpha) : B_\epsilon(\lambda^*) \times B^n_\epsilon(0) \to X^2_{(\lambda^*)}$ such that

$$\mathrm{Proj}_{X^2_{(\lambda^*)}} \Phi(\lambda, f^*(\lambda, v(.))(.) + v(.)) = 0, \tag{IV.1.133}$$

for all $\lambda \in B_\epsilon(\lambda^*) \subset \mathbb{R}$ and $v(.) \in B_\epsilon(0) \subset X^1_{(\lambda^*)}$. Hence, we have to find curves $t \to (\lambda(t), v(t)(.))$ such that

$$\mathrm{Proj}_{X^1_{(\lambda^*)}} \Phi(\lambda(t), f^*(\lambda(t), v(t)(.))(.) + v(t)(.)) = 0. \tag{IV.1.134}$$

As a result, we have reduced the problem (IV.1.109) to a finite dimensional problem $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$.

Step 2: Symmetries of f^* : f^* has the same symmetries of Φ, that we stated in Lemma IV.1.60.

This is a general result (see [GS85] Section VII.3 in (3.17) in the proof of Proposition VII.3.3.). However, for the sake of completeness we state the proof now. Let γ be one of these symmetries. Set $f^*_\gamma(\lambda, \alpha) := \gamma^{-1} f^*(\lambda, \gamma\alpha)$. Then

$$\begin{aligned} \mathrm{Proj}_{X^2_{(\lambda^*)}} \Phi\left(\lambda, \gamma^{-1} f^*(\lambda, \gamma v) + v\right) &= \mathrm{Proj}_{X^2_{(\lambda^*)}} \gamma^{-1} \Phi(\lambda, f^*(\lambda, \gamma v) + \gamma v) \\ &= \gamma^{-1} \mathrm{Proj}_{X^2_{(\lambda^*)}} \Phi(\lambda, f^*(\lambda, \gamma v) + \gamma v) = 0, \end{aligned} \tag{IV.1.135}$$

because $X^1_{(\lambda^*)}$ and consequently $X^2_{(\lambda^*)}$ are invariant w.r.t. γ. The uniqueness of f^* in the implicit function theorem implies the claim that $f^*(\lambda, \alpha) = f^*_\gamma(\lambda, \alpha)$.

Step 3: Reduction of the dimension of (IV.1.134) **to $\mathbb{R}^2 \to \mathbb{R}^2$ by the symmetries:** For $\alpha \in \mathbb{R}$, define the function

$$v_\alpha(.) := \alpha \sum_{\ell \in K_{\lambda^*}} e^s_\ell(.) \in X^1_{(\lambda^*)}. \tag{IV.1.136}$$

We show at first: If

$$\langle e_\ell^s, \Phi(\lambda, f^*(\lambda,\alpha), v_\alpha)\rangle = 0 \qquad \text{(IV.1.137)}$$

for one $\ell \in K_{\lambda^*}$, then this scalar product vanishes for all $\ell \in K_{\lambda^*}$.

Assume that (IV.1.137) holds for one $\ell \in K_{\lambda^*}$. Fix an arbitrary $\ell' \in K_{\lambda^*}$, with $\ell' \neq \ell$. By the assumptions, there is a permutation σ and a $\beta \in \{-1,1\}^d$ such that $\ell' = \gamma_\beta(\gamma_\sigma(\ell))$ and therefore

$$e_{\ell'}^s(.) = \gamma_\sigma^{-1}\gamma_\beta^{-1} e_\ell^s(.). \qquad \text{(IV.1.138)}$$

Let γ be one of these symmetry operators, then

$$\left\langle \gamma^{-1} e_\ell^s, \Phi(\lambda, f^*(\lambda, v_\alpha) + v_\alpha)\right\rangle = \langle e_\ell^s, \gamma\Phi(\lambda, f^*(\lambda, v_\alpha) + v_\alpha)\rangle = \langle e_\ell^s, \Phi(\lambda, f^*(\lambda, \gamma v_\alpha) + \gamma v_\alpha)\rangle, \qquad \text{(IV.1.139)}$$

where we use the symmetries of Φ (Lemma IV.1.60) and of f^* (Step 2). Finally, due to the definition of v_α and Lemma IV.1.26 (v), $\gamma v_\alpha = v_\alpha$ and consequently

$$\text{(IV.1.139)} = \langle e_\ell^s, \Phi(\lambda, f^*(\lambda, v_\alpha) + v_\alpha)\rangle = 0. \qquad \text{(IV.1.140)}$$

Hence, if (IV.1.137) holds for one $\ell \in K_{\lambda^*}$, then it holds for all $\ell \in K_{\lambda^*}$.

The same is holds with e_ℓ^s replaced by e_ℓ^c in (IV.1.137). Therefore, (instead of (IV.1.134)) we only have to show the existence of a curve $(-\epsilon,\epsilon) \ni t \to (\lambda(t), \alpha(t)) \in \mathbb{R}_+ \times \mathbb{R}$ such that

$$\begin{aligned} &\left\langle e_\ell^s, \Phi\left(\lambda(t), f^*\left(\lambda(t), v_{\alpha(t)}(.)\right)(.) + v_{\alpha(t)}(.)\right)\right\rangle = 0 \qquad \text{and} \\ &\left\langle e_\ell^c, \Phi\left(\lambda(t), f^*\left(\lambda(t), v_{\alpha(t)}(.)\right)(.) + v_{\alpha(t)}(.)\right)\right\rangle = 0. \end{aligned} \qquad \text{(IV.1.141)}$$

for one $\ell \in K_{\lambda^*}$. This is a $\mathbb{R}^2 \to \mathbb{R}^2$ problem.

Note that (IV.1.140) is only valid for the symmetry operators consisting of permutations and reflection and not for shifts, because we do not have a suitable shift symmetry in v_α. Therefore, the second scalar product in (IV.1.141) does not equal the first scalar product in general. However, we show in the next step, that we are even in a better situation due to the definition of v_α in (IV.1.136).

Step 4: The second scalar product in (IV.1.141) **is always zero:** We prove in this step that the second scalar product in (IV.1.141) vanishes for all $\alpha \in \mathbb{R}$. Thereby we show at first that $f^*(\lambda, v_\alpha)(.)$ is an odd function and finally we conclude from this the claim.

Step 4.1: $f^*(\lambda, v_\alpha)(.)$ **is an odd function:** This is a consequence of the previously derived symmetries of f^*:

$$f^*(\lambda, v_\alpha(.))(-x) = f^*(\lambda, v_\alpha(-.))(x) = f^*(\lambda, -v_\alpha)(.)(x) = -f^*(\lambda,\alpha)(x), \qquad \text{(IV.1.142)}$$

where we use at first the symmetry of f^* w.r.t. to γ_β with $\beta = (-1, ..., -1)$. The second equality is valid due to v_α being odd and finally we applied the $\mathbb{Z}_2$ symmetry of f^*.

Step 4.2: The second scalar product of (IV.1.141) **vanishes:** Because f^*, v_α and $h'(.)$ are odd, $\Phi(\lambda, f^*(\lambda, v_\alpha) + v_\alpha)$ is also odd. This implies that the integral with respect to even function has to vanish, i.e.

$$\langle e_\ell^c, \Phi(\lambda, f^*(\lambda, v_\alpha) + v_\alpha)\rangle = 0, \qquad \text{(IV.1.143)}$$

for all $\ell \in K_{\lambda^*}$.

Summing up we have reduced the problem (IV.1.134) via (IV.1.141) now to the following $\mathbb{R}^2 \to \mathbb{R}$ problem: Find a curve $(-\epsilon,\epsilon) \ni t \to (\lambda(t), \alpha(t)) \in \mathbb{R}_+ \times \mathbb{R}$ such that

$$\left\langle e_\ell^s, \Phi\left(\lambda(t), f^*\left(\lambda(t), v_{\alpha(t)}(.)\right)(.) + v_{\alpha(t)}(.)\right)\right\rangle = 0, \qquad \text{(IV.1.144)}$$

for one $\ell \in K_{\lambda^*}$.

Step 5: Existence of $\{s \to (\lambda(s), \alpha(s))\}$ **for** (IV.1.144)**:** Fix an $\ell \in K_{\lambda^*}$ and define the $\mathbb{R}^2 \to \mathbb{R}$ map of (IV.1.144) by

$$g(\lambda, \alpha) := \langle e_\ell^s, \Phi(\lambda, f^*(\lambda, v_\alpha(.)) + v_\alpha(.)) \rangle . \tag{IV.1.145}$$

The Fréchet -C^p differentiability of Φ transfers directly to $g \in \mathsf{C}^p(\mathbb{R}^2, \mathbb{R})$ with

$$D^i g = \left\langle e_\ell^s, D^i \Phi(\lambda, f^*(\lambda, v_\alpha) + v_\alpha) \right\rangle . \tag{IV.1.146}$$

By its construction, $g(\lambda^*, 0) = 0$ and

$$\begin{aligned} D_\lambda^1 g(\lambda^*, 0) &= \left\langle e_\ell^s, D_\lambda^1 \Phi(\lambda^*, f^*(\lambda^*, 0) + 0) \right\rangle \\ &= \langle e_\ell^s, -h'(0) + D_\lambda f^*(\lambda^*, 0) - \lambda^* h''(0) J * D_\lambda f^*(\lambda^*, 0) \rangle = 0, \end{aligned} \tag{IV.1.147}$$

because $f^* \in X^2_{(\lambda^*)}$ and consequently $D_\lambda f^*(\lambda^*, 0)$ is orthogonal to $e_\ell^s \in X^1_{(\lambda^*)}$.

In a similar way (by using the orthogonality of e_ℓ^s and $f^*(\lambda^*, 0)$ and its derivatives) we get $D_\alpha^1 g(\lambda^*, 0) = 0$, $D^2_{\lambda,\lambda} g(\lambda^*, 0) = 0$, $D^2_{\alpha,\alpha} g(\lambda^*, 0) = 0$ and $D^2_{\lambda,\alpha} g(\lambda^*, 0) = -h''(0) \neq 0$. Therefore,

$$\det\left(D^2 g(\lambda^*, 0)\right) = -\left(h''(0)\right)^2 < 0. \tag{IV.1.148}$$

Then the Morse lemma (see for example Theorem 1.3.1' in [Rab85] or implies the existence of a curve of zero-points that bifurcate at $(\lambda^*, 0)$ and that is different from the trivial curve.

Concluding we found a curve of zero points of Φ, that bifurcates at $(\lambda^*, 0)$. Moreover, by (IV.1.136) and (Step 4.1), each function on this curve is odd (modulo shifts). □

Remark IV.1.69. *In the Theorem IV.1.65 we have shown that at least one (modulo shifts) curve bifurcates at λ^*. However, there might be more. In the previous proof, we have restricted the projection to $X^1_{(\lambda^*)}$ of each function on this curve to be odd, by our choice of v_α in* (IV.1.136)*.*

(i) *This restriction is necessary to reduce the problem to a $\mathbb{R}^2 \to \mathbb{R}$ problem. Without this restriction, we can not apply the Morse lemma (see Step 5). There are generalisation of the Morse lemma (so called generalised implicit function theorems) to $\mathbb{R}^{n+1} \to \mathbb{R}^n$ problems (e.g. Theorem 3.2 in [Rab84], see also [BMS83] Theorem 2.7). These generalisation require that the so called $\mathbb{R}$-nondegeneracy condition (see Definition 1.1 in [Rab84]) is valid. But this condition is not satisfied in the setting we consider here. Indeed,*

$$\mathbb{R} \times X^1_{(\lambda^*)} \ni (\mu, m) \to D^2\Phi(\lambda^*, 0)(\mu, m)^2 = -\mu 2\lambda^* h''(0)\widehat{J_{\ell^*}} m + 0 \tag{IV.1.149}$$

vanishes on $\{0\} \times X^1_{(\lambda^*)}$ and for these values the derivative of this map is not onto. (We refer also to the discussion in [Rab87] after (3.29) concerning the invalidity of the $\mathbb{R}$-nondegeneracy condition for similar reasons in another model).

Therefore, we can not apply these generalisations of the Morse lemma and we have to reduce the problem first to a $\mathbb{R}^2 \to \mathbb{R}$ problem. However, beside the bifurcating curve, that we construct in the Theorem, there might be further curves that bifurcate, although we are not able to prove it. These curves might not be odd (modulo shifts on the torus), but they might have no symmetries at all.

This missing $\mathbb{R}$-nondegeneracy condition is also the reason why we assume the permutation/reflection symmetry on the elements in K_{λ^*} in this theorem. Without this assumption, we were not able to reduce the dimension of the problem to $\mathbb{R}^2 \to \mathbb{R}$ but only to $\mathbb{R}^{k+1} \to \mathbb{R}^k$, where k is the number of values in K_{λ^*} that can not be transferred into each others by permutations and reflection.

(ii) *Another option might be to define v_α differently. It must be chosen such that the problem can still be reduced to $\mathbb{R}^2 \to \mathbb{R}$. Then we would get another bifurcation curve. However, we do not know another choice of v_α such that the dimension reduction still works.*

Instead of (IV.1.136)*, one might come up with the idea to define v_α by*

$$v_\alpha := \alpha \sum_{\ell \in K_{\lambda^*}} \left(e_\ell^s + e_\ell^c\right) \in X^1_{(\lambda^*)} . \tag{IV.1.150}$$

Then we lose that v_α and f^* are odd. Therefore, we can not reduce the problem (IV.1.141) *to* (IV.1.144) *as we do it in Step 4. Also using the shift symmetry of the new v_α, does not help us. Indeed, we would want to use $e_\ell^s(.) = \gamma_{-t} e_\ell^c(.)$ with $t \cdot \ell = \frac{1}{2}\pi$ in combination with* (IV.1.139) *and* (IV.1.140)*. Already when the dimension of the torus is one, then we would need that v_α has period $t = \frac{1}{2\ell}\pi$, but it has period $\frac{2}{\ell}\pi$. Therefore, this choice of v_α is not helpful.*

IV.1.4.4 Further bifurcation results under Assumption IV.0.3

In this section we prove further bifurcation results under the additional Assumption IV.0.3. In Corollary IV.1.70 we conclude that all constant critical values of $\mathcal{I} - F$ are on the curve that bifurcates at the first bifurcation point. Moreover, this curve consists of the global minimisers of $\mathcal{I} - F$. Afterwards we show that this curve has no further bifurcation (Lemma IV.1.72).

As a corollary to Lemma IV.1.37, Theorem IV.1.36 and Theorem IV.1.64, we get the following results concerning the constant solutions of (IV.1.111), i.e. the constant critical values of $\mathcal{I} - F$.

Corollary IV.1.70. *Let h' be strictly concave on $\mathbb{R}_+$ (additional Assumption IV.0.3). Then*

- *The unique bifurcating curve that bifurcates at $\frac{1}{I''(0)\widehat{J_0}}$ (as shown in Theorem IV.1.64) consists for each λ (on that curve) of the two corresponding global minimisers m^{CW}.*
- *In particular, all constant critical values of $\mathcal{I} - F$ are on that curve.*
- *$s \to \lambda(s)$ is increasing on $\mathbb{R}_+$.*

Remark IV.1.71. *If the Assumption IV.0.3, then the statements of the corollary do not hold any more in general. Indeed, we give in Section IV.1.2.4 an example, where minima can be born somewhere away from zero. The curve that bifurcates at $\frac{1}{I''(0)\widehat{J_0}}$ consists nevertheless of constant functions. But in this example it consists of the local maxima (see Figure IV.4c). Moreover, $s \to \lambda(s)$ is decreasing on $\mathbb{R}_+$ i.e. opposite to the behaviour under Assumption IV.0.3.*

In the following lemma we show that the curve, that bifurcates at $\frac{1}{I''(0)\widehat{J_0}}$, does not further bifurcate. We refer to [CES86] Theorem 1 (ii) for the idea of the proof, that we state here.

Lemma IV.1.72. *Let h' be strictly concave on $\mathbb{R}_+$ (additional Assumption IV.0.3) and let $J \geq 0$. Then on the curve that bifurcates at $\frac{1}{I''(0)\widehat{J_0}}$, occurs no further bifurcation.*

Proof. Let $(-\epsilon, \epsilon) \ni s \to (\lambda_0(s), m_0(s)) \in \mathbb{R}_+ \times \mathbb{R}$ be the curve that bifurcates at $\frac{1}{I''(0)\widehat{J_0}}$. By Corollary IV.1.70 each function $m_0(s)$ is constant. Relying on Lemma IV.1.61, a bifurcation of $(\lambda(s), m_0(s))$ can only occur, when

$$\lambda(s) \in \bigcup_{\ell \in \Gamma^d} \left\{ \frac{1}{\widehat{J}_\ell h''\left(J * m_0(s)\right)} \right\} . \tag{IV.1.151}$$

Then we get by Lemma IV.1.26 (iii) that

$$\lambda(s)\widehat{J_\ell}h''\left(\widehat{J_0}m_0(s)\right) \leq \lambda(s)\widehat{J_0}h''\left(\widehat{J_0}m_0(s)\right) = \lambda(s)\,\partial_m h'\left(\widehat{J_0}m\right)\Big|_{m=m_0(s)} < 1, \qquad \text{(IV.1.152)}$$

where the last inequality is a consequence of Assumption IV.0.3 and $m_0(s)$ being a solution of (IV.1.109). Hence, $\lambda(s)$ can never satisfy (IV.1.151), i.e. no bifurcation can occur on this branch. □

IV.2 Energy landscape of Λ_J

In this section we transfer the results that we achieved in Section IV.1 for L_J to the functional Λ_J. Let us define a functional that maps a suitable probability measure in $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ to a measure in $\mathbb{M}(\mathbb{T}^d)$. For suitable $\nu \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$, we set

$$P(\nu)(A) = \int_{A\times\mathbb{R}} \theta\nu(\mathrm{d}x, \mathrm{d}\theta) \in \mathbb{M}\left(\mathbb{T}^d\right), \qquad \text{(IV.2.1)}$$

for all measurable sets $A \subset \mathbb{T}^d$. Obviously P is not well defined for all elements in $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$. However, it is well defined for all $\rho \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ with $\Lambda_J(\rho) < \infty$ (see Lemma IV.2.3 and Lemma IV.2.4). These are the measures for which we actually need the map P.

We use the map P in Section IV.2.2 to relate $\mathcal{H}$ and $\mathcal{I}$ by a minimisation problem and show that this problem has a unique minimiser. This relation allows us to infer properties of the landscape of Λ_J from the properties of L_J. We show a one to one relation between the minima in Section IV.2.3, a relation between lowest paths (Section IV.2.4). Last but not least we transfer the results concerning the critical values of L_J to Λ_J in Section IV.2.5.

We assume in this section without further mentioning, that the Assumption III.2.2, Assumption III.3.1 and Assumption IV.0.1 are satisfied.

Notation IV.2.1. *To keep the notation as obvious as possible, we mark in this section all functions in $L^2(\mathbb{T}^d)$ with a line over the function, i.e. $\overline{f}$.*

IV.2.1 Preliminary properties of Λ_J

In this section we state and prove some results that we need in the subsequent sections.

Lemma IV.2.2. *For each $f \in \mathsf{C}(\mathbb{R})$ with $\int e^{|f(\theta)|-\Psi(\theta)}\mathrm{d}\theta = C_f < \infty$ there is a constant $C > 0$ such that for all $\rho(x,\theta)\,\mathrm{d}x\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$*

$$\int_{\mathbb{T}^d\times\mathbb{R}} |f(\theta)|\,\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x \leq \mathcal{H}(\rho) + \log C_f. \qquad \text{(IV.2.2)}$$

Proof. Set $f_n(\theta) := \min\{|f(\theta)|, n\} \in \mathsf{C}_b(\mathbb{R})$. By its definition $f_n \nearrow |f|$ and $f_n \geq 0$. Hence, by the monotone convergence theorem, for each $x \in \mathbb{T}^d$

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}} f_n(\theta)\,\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x = \int_{\mathbb{T}^d\times\mathbb{R}} |f(\theta)|\,\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x \text{ and}\\ &\lim_{n\to\infty}\int_{\mathbb{R}} e^{f_n(\theta)-\Psi(\theta)}\mathrm{d}\theta = \int_{\mathbb{R}} e^{|f|-\Psi(\theta)}\mathrm{d}\theta.\end{aligned} \qquad \text{(IV.2.3)}$$

Therefore

$$\begin{aligned}
\mathcal{H}(\rho) &\geq \lim_{n\to\infty}\left\{\int_{\mathbb{T}^d\times\mathbb{R}} f_n(\theta)\,\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x - \log\left(\int e^{f_n(\theta)-\Psi(\theta)}\mathrm{d}\theta\right)\right\}\\
&= \int_{\mathbb{T}^d\times\mathbb{R}} |f(\theta)|\,\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x - \log\left(\int e^{|f(\theta)|-\Psi(\theta)}\mathrm{d}\theta\right) \qquad \text{(IV.2.4)}\\
&= \int_{\mathbb{T}^d\times\mathbb{R}} |f(\theta)|\,\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x - \log C_f.
\end{aligned}$$

□

Lemma IV.2.3. *There are two constants $C, C_\Psi > 0$, such that for all $\rho(x,\theta)\,\mathrm{d}x\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$*

$$\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\theta\rho(x,\theta)\,\mathrm{d}\theta\right)^2\mathrm{d}x \leq \mathcal{H}(\rho) + C_\Psi \text{ and} \qquad \text{(IV.2.5)}$$

$$\int_{\mathbb{T}^d}\left(\int_{\mathbb{R}}\theta\rho(x,\theta)\,\mathrm{d}\theta\right)^2\mathrm{d}x \leq C\Lambda_J(\rho) + C_\Psi. \qquad \text{(IV.2.6)}$$

In particular, $\mathcal{H}(\rho) < \infty$ or $\Lambda_J(\rho) < \infty$ implies $P(\rho) \in L^2(\mathbb{T}^d)$.

Proof. Fix a $\rho(x,\theta) \in \mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$ such that $\mathcal{H}(\rho) < \infty$. Choose a $\kappa \in (0, c_\Psi]$, $\kappa < \infty$. Applying Lemma IV.2.2 for $f(\theta) = \kappa\theta^2$ we get

$$\mathcal{H}(\rho) + C_\Psi \geq \kappa\int_{\mathbb{T}^d}\int_{\mathbb{R}}\theta^2\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x, \qquad \text{(IV.2.7)}$$

by Assumption III.2.2 c.). Hence

$$\Lambda_J(\rho) \geq \left(\kappa - \lambda\frac{1}{2}\widehat{J_0}\right)\int_{\mathbb{T}^d}\int_{\mathbb{R}}\theta^2\rho(x,\theta)\,\mathrm{d}\theta\mathrm{d}x - C_\Psi. \qquad \text{(IV.2.8)}$$

But $\left(\kappa - \lambda\frac{1}{2}\widehat{J_0}\right) > 0$ if κ is close to c_Ψ (by Assumption IV.0.1). Hence, the claim of the theorem follows by the Jensen inequality (applicable because $\rho(x,\theta)\,\mathrm{d}\theta \in \mathbb{M}_1(\mathbb{R})$ by the definition of $\mathcal{H}$ in (III.2.41)). □

From the definition (IV.0.2) of Λ_J, we infer the following property.

Lemma IV.2.4. *If $\mathcal{H}(\nu) = \infty$ for $\nu \in \mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})$, then $\Lambda_J(\nu) = \infty$.*

Proof. Assume that $\mathcal{H}(\nu) = \infty$, but $\Lambda_J(\nu) < \infty$. Then $\langle\nu,\theta^2\rangle < \infty$ by (IV.0.2). But this implies that $|F_\Lambda(\nu)| < \infty$ and consequently $(\mathcal{H} - F_\Lambda)(\nu) = \infty$, a contradiction. □

IV.2.2 Relation between $\mathcal{H}$ and $\mathcal{I}$

Obviously for each probability density $\rho \in L^1(\mathbb{T}^d\times\mathbb{R})$,

$$\begin{aligned}
F_\Lambda(\rho) &= \int_{\mathbb{T}^d\times\mathbb{R}}\int_{\mathbb{T}^d\times\mathbb{R}} J(x-x')\,\theta\theta'\rho(x,\theta)\,\rho(x',\theta')\,\mathrm{d}x\mathrm{d}\theta\mathrm{d}x'\mathrm{d}\theta'\\
&= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-x')\,P(\rho)(x)\,P(\rho)(x')\,\mathrm{d}x\mathrm{d}x' = F_L(P(\rho)).
\end{aligned} \qquad \text{(IV.2.9)}$$

Therefore

$$\Lambda_J(\rho(.,.)) = \mathcal{H}(\rho(.,.)) - \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J(x-x')\,P(\rho)(x)\,P(\rho)(x')\,\mathrm{d}x\mathrm{d}x'. \qquad \text{(IV.2.10)}$$

In the next lemma we show a connection between $\mathcal{I}$ and $\mathcal{H}$. The first part expresses $\mathcal{I}(\mu)$ as a infimum of $\mathcal{H}$ over a particular set of probability measures. This is similar to the contraction principle between rate functions. But to apply the latter we would need the continuity of P. This is not valid, due to the unboundedness of θ in the definition of P. Therefore, we show this connection by using the growth properties of Ψ to get $\mathcal{H}(\nu) \geq \mathcal{I}(P(\nu))$ (Step 2) and we show the equality (in Step 5) by constructing an explicit minimiser (Step 4).

Lemma IV.2.5. *For an arbitrary* $\overline{\mu} \in \mathbb{M}(\mathbb{T}^d)$,

$$\mathcal{I}(\overline{\mu}) = \inf_{\substack{\nu \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}) \\ \overline{\mu} = P(\nu)}} \mathcal{H}(\nu). \tag{IV.2.11}$$

If $\mathcal{I}(\overline{\mu})$ *is finite, then* $\overline{\mu} = \overline{m}(x)\,\mathrm{d}x$ *for a* $\overline{m} \in L^2(\mathbb{T}^d)$ *and the right hand side of* (IV.2.11) *has a unique minimiser* $\nu^*(\mathrm{d}x, \mathrm{d}\theta) = \rho^*(x,\theta)\,\mathrm{d}x\mathrm{d}\theta$ *where* $\rho^* \in L^1(\mathbb{T}^d \times \mathbb{R})$ *is given by*

$$\rho^*(x,\theta) = e^{\gamma_1(x) + \gamma_2(x)\theta - \Psi(\theta)}, \tag{IV.2.12}$$

with $\gamma_1(x) := -h(I'(\overline{m}(x)))$ *and* $\gamma_2(x) := I'(\overline{m}(x))$.

Proof. **Step 1: If** $\overline{\mu}(\mathrm{d}x) \neq \overline{m}(x)\,\mathrm{d}x$ **then both sides of** (IV.2.11) **are infinite:** Take a $\overline{\mu} \in \mathbb{M}(\mathbb{T}^d)$ that has no density w.r.t. the Lebesgue measure. Then $\mathcal{I}(\overline{\mu}) = \infty$.

Assume that there were a $\nu \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ such that $\overline{\mu} = P(\nu)$ with $\nu = \rho(x,\theta)\,\mathrm{d}x\mathrm{d}\theta$. Then $P(\nu)(x) = \int_{\mathbb{R}} \theta\rho(x,\theta)\,\mathrm{d}\theta =: \overline{u}(x)$ is well defined. Set $\overline{\mu}_2(\mathrm{d}x) := \overline{u}(x)\,\mathrm{d}x \in \mathbb{M}(\mathbb{T}^d)$. Hence, $\overline{\mu}_2 = \mu$, but this is a contradiction to the non existence of a density.

Hence, all $\nu \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ with $\overline{\mu} = P(\nu)$ have no density w.r.t. the Lebesgue measure and consequently $\mathcal{H}(\nu) = \infty$.

Step 2: If $\overline{m} = P(\rho)$**, then** $\mathcal{H}(\rho(.,.)) \geq \mathcal{I}(\overline{m})$**:** Assume $\mathcal{H}(\rho(.,.)) < \infty$ and set $f_n(\theta) = \mathrm{sign}(\theta)(|\theta| \cap n) \in \mathsf{C}_b(\mathbb{R})$. Then

$$\begin{aligned} \mathcal{H}(\rho(.,.)) &= \int_{\mathbb{T}^d} \sup_{f \in \mathsf{C}_b(\mathbb{R})} \left\{ \int_{\mathbb{R}} f(\theta)\rho(x,\theta)\,\mathrm{d}\theta - \log \int_{\mathbb{R}} e^{f(\theta) - \Psi(\theta)}\mathrm{d}\theta \right\} \mathrm{d}x \\ &\geq \int_{\mathbb{T}^d} \sup_{t,n} \left\{ \int_{\mathbb{R}} t f_n(\theta)\rho(x,\theta)\,\mathrm{d}\theta - \log \int_{\mathbb{R}} e^{t f_n(\theta) - \Psi(\theta)}\mathrm{d}\theta \right\} \mathrm{d}x \\ &\geq \int_{\mathbb{T}^d} \sup_{t} \lim_{n\to\infty} \left\{ \int_{\mathbb{R}} t f_n(\theta)\rho(x,\theta)\,\mathrm{d}\theta - \log \int_{\mathbb{R}} e^{t f_n(\theta) - \Psi(\theta)}\mathrm{d}\theta \right\} \mathrm{d}x. \end{aligned} \tag{IV.2.13}$$

The limit and the integral in both integrals can be exchanged by the dominated convergence theorem. Indeed, for the last integral we use the dominating function $e^{|\theta||t| - \Psi(\theta)}$ (which is integrable by Assumption III.2.2 c.)). For the first integral note that the function $|t|\,|\theta|\,\rho(x,\theta)$ is integrable w.r.t. $\mathbb{T}^d \times \mathbb{R}$ (by Lemma IV.2.2 and Assumption III.2.2 c.)). By Fubini's Theorem, there is at most a set $N \subset \mathbb{T}^d$ of measure zero, such that for all $x \in \mathbb{T}^d \backslash N$ the function $g_x(\theta) := t\,|\theta|\,\rho(x,\theta)$ is integrable w.r.t. $\mathrm{d}\theta$. Then g_x is the dominating function for the first integral.

Therefore, we conclude

$$\mathcal{H}(\rho(.,.)) \geq \int_{\mathbb{T}^d} \sup_{t \in \mathbb{R}} \left\{ t \int_{\mathbb{R}} \theta\rho(x,\theta)\,\mathrm{d}\theta - \log \int e^{-t\theta - \Psi(\theta)}\mathrm{d}\theta \right\} \mathrm{d}x = \mathcal{I}(\overline{m}). \tag{IV.2.14}$$

Step 3: $\nu^* = \rho^*\mathrm{d}x\mathrm{d}\theta$ **is in** $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ **with** $P(\nu^*) = \overline{\mu}$**:** Let ρ^* be defined by $\overline{\mu} = \overline{m}(x)\,\mathrm{d}x$ by (IV.2.12). Then $\nu^* = \rho^*\mathrm{d}x\mathrm{d}\theta$ is in $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ with $P(\nu^*) = \overline{\mu}$. Indeed

$$\rho^*(x,\theta) \geq 0, \tag{IV.2.15}$$

$$\int_{\mathbb{R}} \rho^*(x,\theta)\,\mathrm{d}\theta = e^{\gamma_1(x)} e^{\log\left(\int_{\mathbb{R}} e^{-\gamma_2(x)\theta} e^{-\Psi}\mathrm{d}\theta\right)} = e^{\gamma_1(x)} e^{h(\gamma_2(x))} = 1, \tag{IV.2.16}$$

$$P(\nu^*)(x) = \int_{\mathbb{R}} \theta\rho^*(x,\theta)\,\mathrm{d}\theta = e^{\gamma_1(x)} h'(\gamma_2(x))\, e^{h(\gamma_2(x))} = h'(I'(\overline{m}(x))) = \overline{m}(x). \tag{IV.2.17}$$

Step 4: $\nu^* = \rho^* \mathrm{d}x\mathrm{d}\theta$ **is the unique minimiser of** (IV.2.11)**:** Take a function $f \in L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$ ($f \neq \rho^*$ in L^1-sense) such that and $P(f) = \overline{\mu}$. We can assume that f is absolutely continuous w.r.t. $e^{-\Psi(\theta)}\mathrm{d}\theta$ and that $f(x,.)$ is a probability density for each $x \in \mathbb{T}^d$. Otherwise $\mathcal{H}(f)$ would be infinite (by the definition of $\mathcal{H}$ in (III.2.41)).

For all $x \in \mathbb{T}^d$ with $f(x,.) \neq \rho^*(x,.)$ (in the L^1-sense),

$$\begin{aligned}
\mathsf{H}\left(f(x,.)\middle|e^{-\Psi}\right) &= \int_{\mathbb{R}} f(x,\theta)\log\frac{f(x,\theta)}{e^{-\Psi(\theta)}}\mathrm{d}\theta \\
&= \int_{\mathbb{R}} f(x,\theta)\log\frac{f(x,\theta)}{\rho^*(x,\theta)}\mathrm{d}\theta + \int_{\mathbb{R}} f(x,\theta)\log\frac{\rho^*(x,\theta)}{e^{-\Psi(\theta)}}\mathrm{d}\theta \\
&> \int_{\mathbb{R}} f(x,\theta)\log\frac{\rho^*(x,\theta)}{e^{-\Psi(\theta)}}\mathrm{d}\theta = \int_{\mathbb{R}} f(x,\theta)\left(\gamma_1(x) + \gamma_2(x)\theta\right)\mathrm{d}\theta \qquad \text{(IV.2.18)} \\
&= \gamma_1(x) + \gamma_2(x)\overline{m}(x) = \int_{\mathbb{R}} \rho^*(x,\theta)\left(\gamma_1(x) + \gamma_2(x)\theta\right)\mathrm{d}\theta \\
&= \int_{\mathbb{R}} \rho^*(x,\theta)\log\frac{m^*(x,\theta)}{e^{-\Psi(\theta)}}\mathrm{d}\theta = \mathsf{H}\left(\rho^*(x,.)\middle|e^{-\Psi}\right),
\end{aligned}$$

where we use in the strict inequality that $\mathsf{H}\left(f(x,.)|\rho^*(x,.)\right) > 0 \Leftrightarrow f(x,.) \neq \rho^*(x,.)$. Hence, ρ^* is the unique minimiser of (IV.2.11).

Step 5: $\mathcal{H}(\rho^*) \leq \mathcal{I}(\overline{m})$**:** By (IV.2.18),

$$\begin{aligned}
\mathcal{H}(\rho^*) &= \int_{\mathbb{T}^d} \mathsf{H}\left(\rho^*(x,.)\middle|e^{-\Psi}\right)\mathrm{d}x = \int_{\mathbb{T}^d} \gamma_1(x) + \gamma_2(x)\overline{m}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{T}^d} -\log\left(\int_{\mathbb{R}} e^{\gamma_2(x)\theta}e^{-\Psi(\theta)}\mathrm{d}\theta\right) + \gamma_2(x)\overline{m}(x)\,\mathrm{d}x \leq \int_{\mathbb{T}^d} \sup_{t\in\mathbb{R}}\left\{t\overline{m}(x) - h(t)\right\}\mathrm{d}x.
\end{aligned} \qquad \text{(IV.2.19)}$$

The right hand side is the definition of $\mathcal{I}(\overline{m})$. Hence we have shown the upper bound on $\mathcal{H}(\rho^*)$.

□

IV.2.3 Minima

We show in the following theorem that Lemma IV.2.5 leads to a one to one relation between the minima of L_J and Λ_J. By Lemma IV.2.4 we can restrict ourselves to probability measures which have a density w.r.t. the Lebesgue measure.

Theorem IV.2.6. *If $\overline{m} \in L^2\left(\mathbb{T}^d\right)$ is a minimiser of L_J, then $\rho^* = \rho^*(\overline{m})$ defined by* (IV.2.12) *is a minimiser of Λ_J.*

If $\rho^* \in L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$ is a minimiser of Λ_J, then $\overline{m} = P(\rho^*)$ is a minimiser of L_J.

Proof. **Step 1: The values of $\mathcal{H} - F_\Lambda$ and $\mathcal{I} - F_L$ are the same at the minima:**

$$\begin{aligned}
&\inf_{\mu\in\mathbb{M}_1(\mathbb{T}^d\times\mathbb{R})} \left(\mathcal{H} - F_\Lambda\right)(\mu) = \inf_{\substack{\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)\\ \mu=\rho(x,\theta)\mathrm{d}x\mathrm{d}\theta}} \left(\mathcal{H} - F_\Lambda\right)(\rho) \\
&= \inf_{\overline{m}\in L^2(\mathbb{T}^d)} \inf_{\substack{\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)\\ \mu=\rho(x,\theta)\mathrm{d}x\mathrm{d}\theta\\ \overline{m}=P(\rho)}} \left(\mathcal{H} - F_\Lambda\right)(\rho) = \inf_{\overline{m}\in L^2(\mathbb{T}^d)} \inf_{\substack{\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)\\ \mu=\rho(x,\theta)\mathrm{d}x\mathrm{d}\theta\\ \overline{m}=P(\rho)}} \mathcal{H}(\rho) - F_L(\overline{m}) \qquad \text{(IV.2.20)} \\
&= \inf_{\overline{m}\in L^2(\mathbb{T}^d)} \left(\mathcal{I} - F_L\right)(\overline{m}),
\end{aligned}$$

where we use (IV.2.11) and (IV.2.9).

Step 2: For each minimiser $\overline{m}$ of $\mathcal{I} - F_L$, there is a unique $\rho^* = \rho^*(\overline{m})$ with $\left(\mathcal{H} - F_\Lambda\right)(\rho^*) = \left(\mathcal{I} - F_L\right)(\overline{m})$ and $P(\rho^*) = \overline{m}$ (Lemma IV.2.5). Hence, (IV.2.20) implies that ρ^* is also a minimiser of Λ_J.

Step 3: Let $\mu \in \mathbb{M}_1(\mathbb{T}^d)$ with $\mu = \rho(x,\theta)\,\mathrm{d}x\mathrm{d}\theta$, be a minimiser of Λ_J. Take a $\overline{m} \in L^2(\mathbb{T}^d)$, such that $P(\mu)(\mathrm{d}x) := \overline{m}(x)\,\mathrm{d}x$. Then $\rho = \rho^*(\overline{m})$, because $\rho^*(\overline{m})$ is a unique minimiser of Λ_J, by Lemma IV.2.5. By (IV.2.11) we get the equality $(\mathcal{H} - F_\Lambda)(\rho^*) = (\mathcal{I} - F_L)(\overline{m})$. Hence, we conclude by (IV.2.20) that $\overline{m}$ is a minimiser of $\mathcal{I} - F_L$. □

Remark IV.2.7. *We could show this last theorem also without using Lemma IV.2.4 and consequently without using that Λ_J is a rate function. To do this we could use the uniqueness of the minimiser constructed in Lemma IV.2.5 that has a density w.r.t. the Lebesgue measure and therefore we would get a contradiction if the minimiser of Λ_J had not a density w.r.t. the Lebesgue measure.*

We have proven in Theorem IV.1.36 that L_J has either one minimum 0 or the two minima $\pm m^{CW}$. Hence, the following corollary about the minima of Λ_J is a direct consequence of Theorem IV.2.6 and Theorem IV.1.36.

Corollary IV.2.8. *Let the additional Assumption IV.0.3 hold.*
For $\widehat{J_0}h''(0) \le 1$: Λ_J has only the minimum $\rho^*(x,\theta) = e^{-\Psi}$.
For $\widehat{J_0}h''(0) > 1$: Λ_J has only the two minima $\rho^\pm(x,\theta) = e^{\gamma_1(\pm m^{CW}) + \gamma_2(\pm m^{CW})\theta - \Psi(\theta)}$.

IV.2.4 Lowest path

Take a lowest path in $L^2(\mathbb{T}^d)$ which connects two minima of L_J. We show in this section that the path in $\mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$, that is related via (IV.2.12) to the first path, is a lowest path between the corresponding minima of Λ_J. The following theorem is a little bit more general by relating the paths connecting two arbitrary points.

Theorem IV.2.9. *Let $\overline{m}^*(.) \in \mathsf{C}([0,1], L^2(\mathbb{T}^d))$ be a lowest path that connects the points $\overline{m}_0, \overline{m}_1 \in L^2(\mathbb{T}^d)$ with $\sup_{t\in[0,1]} \|\overline{m}^*(t)\|_{L^\infty} \le R$ for a $R \in \mathbb{R}_+$.*
Then $\rho^*(.) = \rho^*(\overline{m}^*)(.) \in \mathsf{C}([0,1], \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R}))$, defined for each $t \in [0,1]$ by (IV.2.12), *is a lowest path connecting the two points $\rho^*(\overline{m}_0)$ and $\rho^*(\overline{m}_1) \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$.*

Proof. **Step 1: $t \to \rho^*(t)$ is a continuous path:** At first, we prove that $t \to \rho^*(t) \in \mathbb{M}_1(\mathbb{T}^d \times \mathbb{R})$ defined for each $t \in [0,1]$ by (IV.2.12) is a continuous map. Fix a sequence $t_n \in [0,1]$, $t_n \to t \in [0,1]$. For each $f \in \mathsf{C}_b(\mathbb{T}^d \times \mathbb{R})$,

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d\times\mathbb{R}} f(x,\theta)\,\rho^*(t_n)(x,\theta) - \rho^*(t)(x,\theta)\,\mathrm{d}x\mathrm{d}\theta \right| \\
&\le |f|_\infty \int_{\mathbb{T}^d\times\mathbb{R}} |\rho^*(t_n)(x,\theta) - \rho^*(t)(x,\theta)|\,\mathrm{d}x\mathrm{d}\theta \\
&\le |f|_\infty \sup_{\eta\in[-R,R]} e^{-h(I'(\eta))} \int_{\mathbb{T}^d\times\mathbb{R}} e^{-\Psi(\theta)} \left| e^{I'(\overline{m}^*(t_n)(x))\theta} - e^{I'(\overline{m}^*(t)(x))\theta} \right| \mathrm{d}x\mathrm{d}\theta \\
&\le C_1 \int_{\mathbb{T}^d\times\mathbb{R}} e^{-\Psi(\theta)} e^{R|\theta|}\,|\theta|\,|I'(\overline{m}^*(t_n)(x)) - I'(\overline{m}^*(t)(x))|\,\mathrm{d}x\mathrm{d}\theta,
\end{aligned} \tag{IV.2.21}$$

where we use the Taylor expansion and $\|\overline{m}^*(s)\|_{L^\infty(\mathbb{T}^d)} \le R$ for all $s \in [0,1]$. From the Lipschitz continuity of I' on $[-R,R]$ with Lipschitz constant $L_{I',R}$ (by Lemma III.2.5) we infer

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}} |\rho^*(t_n)(x,\theta) - \rho^*(t)(x,\theta)|\,\mathrm{d}x\mathrm{d}\theta \\
&\le C_1 \underbrace{\int_{\mathbb{R}} e^{-\frac{1}{2}\Psi(\theta)} e^{R|\theta|}\,|\theta|\,\mathrm{d}\theta}_{\le C} \quad L_{I',R} \int_{\mathbb{T}^d} |\overline{m}^*(t_n)(x) - \overline{m}^*(t)(x)|\,\mathrm{d}x \to 0,
\end{aligned} \tag{IV.2.22}$$

where we use Assumption III.2.2 c.) to bound the first integral. The convergence to zero is a consequence of the continuity of the path $\overline{m}^*$ on $[0,1] \to L^2\left(\mathbb{T}^d\right)$.
Step 2: $\rho^*(.)$ is a lowest path connecting these two points: Take an arbitrary path $\rho \in \mathsf{C}\left([0,1], \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)\right)$ that connects $\rho^*(\overline{m}_0)$ and $\rho^*(\overline{m}_1)$. Then we know by Lemma IV.2.5 and $\overline{m}^*$ being a lowest path, that

$$\sup_{t\in[0,1]} (\mathcal{H} - F_\Lambda)(\rho(t)) \geq \sup_{t\in[0,1]} (\mathcal{I} - F_L)(P(\rho(t))) \geq \sup_{t\in[0,1]} (\mathcal{I} - F_L)(\overline{m}^*(t)). \tag{IV.2.23}$$

Moreover, the highest energy on the path ρ^* equals $\sup_t (\mathcal{I} - F_L)(\overline{m}^*(t))$. Indeed, by the definition of ρ^*, we have for each t: $\mathcal{H}(\rho^*(t)) = \mathcal{I}(\overline{m}^*(t))$, hence

$$\sup_{t\in[0,1]} (\mathcal{H} - F_\Lambda)(\rho^*(t)) = \sup_{t\in[0,1]} (\mathcal{I} - F_L)(\overline{m}^*(t)). \tag{IV.2.24}$$

□

IV.2.5 Critical values

We finally show that also the critical values of L_J and Λ_J are related. This allows us to transfer also the bifurcation picture of L_J to Λ_J. To use the concept of critical values, we need that $\mathcal{H} - F_\Lambda$ is at least Gâteaux differentiable, what is obviously not true on $L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$. We first give an heuristic argument about the relation of critical points of $\mathcal{H} - F_\Lambda$ and $\mathcal{I} - F_L$ in Section IV.2.5.1, when ignoring this problem.

Then in Section IV.2.5.2 we establish this relation in a mathematical more precise way. We restrict our attention to a suitable subset of $L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$, such that all elements of interests (minima, paths) are inside this subset. This is possible due to Lemma IV.2.4 and Lemma IV.2.3 and the corresponding results for $\mathcal{I} - F_L$. We can define a new functional $\mathcal{H}_R - F_{\Lambda,R}$ that equals $\mathcal{H} - F_\Lambda$ on this subset, but that is nevertheless Gâteaux differentiable. Then in Section IV.2.5.3 we show that critical values of $\mathcal{I}_R - F_{L,R}$ are related to the critical values of $\mathcal{H}_R - F_{\Lambda,R}$. Finally, we can transfer all results concerning the critical values (like the bifurcation picture derived in Section IV.1.4 and the results from the mountain pass theorem in Section IV.1.3.1), from $\mathcal{I} - F_L$ to $\mathcal{H} - F_\Lambda$.

IV.2.5.1 An heuristic argument

Let us assume at first that $\mathcal{H} - F_\Lambda$ were differentiable, then we could look at least heuristically for critical values on $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$. Hence, we would use the method of Lagrange multipliers on Banach spaces to restrain the results to $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$. With Lagrange multiplier $\beta \in L^2\left(\mathbb{T}^d\right)$, we would have the condition for criticality of $\rho \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$

$$\log(\rho(x,\theta)) + 1 + \lambda\Psi(\theta) - \theta J * P(\rho)(x) - \beta(x) = 0, \tag{IV.2.25}$$

or equivalently

$$\rho(x,\theta) = e^{-1+\beta(x)-\lambda\Psi(\theta)+\theta J*P(\rho)(x)}. \tag{IV.2.26}$$

m should be in $\mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right)$, hence we would require $-1 + \beta(x) = h(\lambda J * P(\rho)(x))$. Then for almost all $x \in \mathbb{T}^d$

$$P(\rho)(x) = h'(\lambda J * P(\rho)(x)). \tag{IV.2.27}$$

This would imply that for each critical value m of $\mathcal{H} - F_\Lambda$ the corresponding $P(\rho)$ satisfies (IV.1.1), i.e. is a critical value of $\mathcal{I} - F_L$. Also for each critical value of $\overline{m}$ of $\mathcal{I} - F_L$, there is by Lemma IV.2.5 a $\rho^*(\overline{m})$ defined via (IV.2.12) that satisfies (IV.2.26), i.e. a critical value of $\mathcal{H} - F_\Lambda$.

IV.2.5.2 Restriction to Fréchet differentiable functions

We explain now mathematically precisely why the critical values are related. At first, we define the following subset of $L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$

$$\mathsf{S} := \left\{ \rho \in L^1\left(\mathbb{T}^d \times \mathbb{R}\right) : \int_{\mathbb{T}^d} P\left(\rho\right)\left(x\right)^2 \mathrm{d}x < \infty \right\}, \tag{IV.2.28}$$

with norm $\|\rho\|_{\mathsf{S}} := \|\rho\|_{L^1} + \left(\int_{\mathbb{T}^d} P\left(\rho\right)\left(x\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}$. We know by Lemma IV.2.4 and Lemma IV.2.3, that everything of interest happens on this space. This set has the following nice properties.

Lemma IV.2.10. *P is continuous on S.*

Proof. By the definition of $\|\rho\|_{\mathsf{S}}$, we get the continuity of P on this space. □

Lemma IV.2.11. *S with $\|\rho\|_{\mathsf{S}}$ is a Banach space.*

Proof. S is obviously a normed vector space. To show that it is complete, let $\{\rho_n\}$ be a Cauchy sequence. Then $\rho_n \to \rho \in L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$ because $L^1\left(\mathbb{T}^d \times \mathbb{R}\right)$ is a Banach space. Moreover, $P\left(\rho_n\right) \to m \in L^2\left(\mathbb{T}^d\right)$ because this is also a Banach space. Then by the continuity of $P\left(\rho\right)$ on S, we get that $P\left(\rho\right) = m$. Hence, the space S is complete. □

By Lemma IV.2.5, (IV.2.9), Theorem IV.2.9 and Lemma IV.1.5, we can restrain our attention to paths on

$$\widehat{\mathsf{S}}_R := \left\{ \rho \in L^1\left(\mathbb{T}^d \times \mathbb{R}\right) : \rho\left(x,\theta\right)\mathrm{d}x\mathrm{d}\theta \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathbb{R}\right), \|P\left(\rho\right)\|_{L^\infty} \leq R, \rho = \rho^*\left(P\left(\rho\right)\right) \right\}, \tag{IV.2.29}$$

for $R \in \mathbb{R}_+$ large enough, with $\rho^*\left(.\right)$ defined in (IV.2.12). Define the functional $\mathcal{H}_R - F_{\Lambda,R}$ on S by

$$\mathcal{H}_R\left(\rho\right) := \mathcal{I}_R\left(P\left(\rho\right)\right) \qquad \text{and} \qquad F_{\Lambda,R}\left(\rho\right) = F_{L,R}\left(P\left(\rho\right)\right), \tag{IV.2.30}$$

where $\mathcal{I}_R$ and $F_{L,R}$ are defined as in Section IV.1.1.2.

Lemma IV.2.12. *$\mathcal{H}_R - F_{\Lambda,R}$ is C^1-Fréchet differentiable as functions from S to $\mathbb{R}$ and $\mathcal{H}_R - F_{\Lambda,R}$ equals $\mathcal{H} - F_\Lambda$ on $\widehat{\mathsf{S}}_R$.*

Proof. By Lemma IV.1.9 and Lemma IV.1.10 we know already that $F_{L,R}$ and $\mathcal{I}_R$ are C^1. Moreover, $P : S \to L^2\left(\mathbb{T}^d\right)$ is C^1, by the norm that we defined on S, with derivative $P'\left(\rho\right)\left(g\right) = P\left(g\right)$. □

IV.2.5.3 Connection between critical values of Λ_J and L_J

By the previous section, the functional $\mathcal{H}_R - F_{\Lambda,R}$ is Fréchet differentiable on S and equals $\mathcal{H} - F_\Lambda$ on this space. We want to know which values in $\widehat{\mathsf{S}}_R$ are critical values of this functional. The following lemma shows a one to one relation between critical values of $\mathcal{I} - F_L$ and $\mathcal{H}_R - F_{\Lambda,R}$ on $\widehat{\mathsf{S}}_R$.

Lemma IV.2.13. *A function $\rho \in \widehat{\mathsf{S}}_R$ is a critical value of $\mathcal{H}_R - F_{\Lambda,R}$ if and only if $P\left(\rho\right)$ is a critical value of $\mathcal{I}_R - F_{L,R}$.*

Proof. A $\rho \in \widehat{\mathsf{S}}_R$ is a critical value if and only if for all $g \in \mathsf{S}$

$$\begin{aligned}&\left(\mathcal{H}'_R - F'_{\Lambda,R}\right)\left(\rho\right)\left(g\right)\\ &= \int_{\mathbb{T}^d} I'\left(P\left(\rho\right)\left(x\right)\right)P\left(g\right)\left(x\right)\mathrm{d}x - \lambda\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} J\left(x-y\right)P\left(\rho\right)\left(x\right)P\left(g\right)\left(y\right)\mathrm{d}x\mathrm{d}y = 0,\end{aligned} \tag{IV.2.31}$$

of equivalently for almost all $x \in \mathbb{T}^d$

$$I'\left(P\left(\rho\right)\left(x\right)\right) = \lambda J * P\left(\rho\right)\left(x\right). \tag{IV.2.32}$$

This implies the claimed relation of the critical values. □

Last but not least a critical value of $\mathcal{H}_R - F_{\Lambda,R}$ implies that the directional derivatives at least in $\widehat{\mathsf{S}}_R$ directions of $\mathcal{H} - F_\Lambda$ also vanish at this point. Note that the directional derivative between $\rho_1, \rho_2 \in \widehat{\mathsf{S}}_R$ can not be defined over the convex combination (because $\widehat{\mathsf{S}}_R$ is not convex). But we might use the image of ρ^* (defined in (IV.2.12)) of the linear combination of $P(\rho_1)$ and $P(\rho_2)$ (as in Theorem IV.2.9).

We have hence shown that there is a one to one relation between the critical values $\overline{m}$ of $\mathcal{I} - F_L$ with $\|\overline{m}\|_{L^\infty} \leq R$ and the critical values $\rho \in \widehat{\mathsf{S}}_R$ of $\mathcal{H} - F_\Lambda$ (in the respective senses).

By Lemma IV.2.5, (IV.2.9), Theorem IV.2.9, Lemma IV.1.5, Lemma IV.2.4 and Lemma IV.2.3, everything of interest happens inside the set $\widehat{\mathsf{S}}_R$. Therefore, all the relevant results concerning critical values (e.g. bifurcation results or the mountain pass result) of $\mathcal{I} - F_L$ transfer to corresponding results of $\mathcal{H} - F_\Lambda$.

Chapter V

Dynamical large deviation

The results and proofs of this chapter can also be found in [Mül16].

V.1 Introduction

In this chapter we consider the following more general system of N^d interacting spins

$$\begin{aligned} \mathrm{d}\theta_t^{k,N} &= b\left(\frac{k}{N}, w^{k,N}, \theta_t^{k,N}, \mu_t^N\right) \mathrm{d}t + \sigma \mathrm{d}W_t^{k,N}, \\ \theta_0^{k,N} &\sim \nu_{\frac{k}{N}} \in \mathbb{M}_1(\mathbb{R}), \end{aligned} \tag{V.1.1}$$

for $k \in \mathbb{T}_N^d$. Besides the random initial distribution and the independent stochastic fluctuations, there is a new (compared to (0.1.1)) source of randomness in this system. To each site a random environment $w^{k,N} \in \mathcal{W} \subset \mathbb{R}^m$ is attached. It expresses differences in the nature of the spins. The random environment is distributed according to $\zeta_{\frac{k}{N}} \in \mathbb{M}_1(\mathcal{W})$ and it is frozen over time.

We consider very general drift coefficients $b : \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}) \to \mathbb{R}$. This coefficient has to be continuous on a subset of the probability measures, but it might be unbounded. The drift coefficient depends on the fixed normalised spatial position $\frac{k}{N}$, on the random environment $w^{k,N}$ attached to the site $\frac{k}{N}$ and on the current spin $\theta_t^{k,N}$. Moreover, b depends through the empirical measure μ_t^N, defined (in this chapter) as

$$\mu_t^N := \frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \delta_{\left(\frac{k}{N}, w^{k,N}, \theta_t^{k,N}\right)} \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}), \tag{V.1.2}$$

also on the spatial positions, the random environments and the spins of the other sites. This dependency of the drift coefficient on the empirical measure μ_t^N models the interaction between the spins. By these dependencies, the geometric structure of the system, i.e. the spatial position of the spins, is highly relevant. Moreover, the initial distribution and the distribution of the random environment depend on the spatial position.

Given a realisation $\underline{\theta}_{[0,T]}^N = \left\{t \mapsto \underline{\theta}_t^N\right\}$ of the solution of (V.1.1) and a realisation of the random environment $\underline{w}^N$, let us denote by $\mu_{[0,T]}^N$ the empirical process, that is the time evolution of the empirical measures μ_t^N defined in (V.1.2), i.e.

$$\mu_{[0,T]}^N := \left\{t \mapsto \mu_t^N\right\} \in \mathsf{C}\left([0,T], \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})\right), \tag{V.1.3}$$

and by L^N the empirical measure on $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$

$$L^N = L^N\left(\underline{w}^N, \underline{\theta}_{[0,T]}\right) := \frac{1}{N^d}\sum_{k\in\mathbb{T}_N^d}\delta_{\left(\frac{k}{N}, w^{k,N}, \theta^{k,N}_{[0,T]}\right)} \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right). \tag{V.1.4}$$

We prove in the following that the families of random elements $\left\{\mu^N_{[0,T]}\right\}$ and $\left\{L^N\right\}$, satisfy large deviation principles. Moreover, we derive different representations of the rate functions and show relations between the two principles, the rate functions and the minimizer of the rate function. In particular we show that the rate function $S_{\nu,\zeta}$ corresponding to the family $\left\{\mu^N_{[0,T]}\right\}$ has the following expression

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \int_0^T \left|\partial_t\mu_t - \left(\mathbb{L}_{\mu_t,\cdot,\cdot}\right)^*\mu_t\right|^2_{\mu_t} \mathrm{d}t + \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right), \tag{V.1.5}$$

for suitable $\mu_{[0,T]} \in \mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$. We define the norm $|.|_{\mu_t}$ later. The operator $\left(\mathbb{L}_{\mu,x,w}\right)^*$ is for each $\mu \in \mathbb{M}_{\varphi,\infty}$ and $(x,w) \in \mathbb{T}^d \times \mathcal{W}$ the formal adjoint of the following operator

$$\mathbb{L}_{\mu,x,w} f(\theta) := \frac{\sigma^2}{2}\partial^2_{\theta^2} f(\theta) + b(x,w,\theta,\mu)\,\partial_\theta f(\theta), \tag{V.1.6}$$

acting on $f \in \mathsf{C}^2_b(\mathbb{R})$. Observe that the rate function $S_{\nu,\zeta}$ measures somehow the deviation from the hydrodynamic equation.

V.1.1 Results for the concrete example (0.9.3) of a local mean field model

We state in (0.9.3) a concrete example of a spin system with local mean field interaction, that is covered by the more general model (V.1.1). For this system we have sketched the results in Section 0.9.2. Let us now state the the result rigorously. We derive these principles also for the more general system of interacting SDEs (V.1.1) in the next sections. However, for the local mean field model (0.9.3), notations and assumptions are more comprehensible.

Let the spin system characterised by the model (0.9.3) satisfy the following assumptions. Here $\mathcal{W}$ is a compact subset of $\mathbb{R}^m$, for an $m > 0$.

Assumption V.1.1. *The family of initial distributions $\{\nu_x\}_{x\in\mathbb{T}^d} \subset \mathbb{M}_1(\mathbb{R})$ is Feller continuous, i.e. $\nu_{x^{(n)}}$ converges to ν_x when $x^{(n)} \to x$, or equivalently the map $x \mapsto \int_{\mathbb{R}} f(\theta)\,\nu_x(\mathrm{d}\theta)$ is continuous for all $f \in \mathsf{C}_b(\mathbb{R})$.*

See Section V.2.3.2 for examples of Feller continuous initial distributions.

Assumption V.1.2.

$$\sup_{x\in\mathbb{T}^d}\int_{\mathbb{R}} e^{2\overline{\Psi}(\theta)}\nu_x(\mathrm{d}\theta) < \infty. \tag{V.1.7}$$

Assumption V.1.3. *The family of distributions of the random environment $\{\zeta_x\}_{x\in\mathbb{T}^d} \subset \mathbb{M}_1(\mathcal{W})$ is Feller continuous.*

Assumption V.1.4. *The interaction weight J is in $L^2\left(\mathbb{T}^d, \mathsf{C}(\mathcal{W}\times\mathcal{W})\right)$ and satisfies the following conditions:*

- *There is a $\overline{J} \in L^2\left(\mathbb{T}^d\right)$, such that $\sup_{(w,w')\in\mathcal{W}\times\mathcal{W}} |J(x,w,w')| < \overline{J}(x)$ for all $x \in \mathbb{T}^d$.*
- *J is even on $\mathbb{T}^d$, i.e. $J(x,w,w') = J(-x,w,w')$ for all $x \in \mathbb{T}^d$ and $w,w' \in \mathcal{W}$.*
- *Moreover,*

$$\frac{1}{N^d}\sum_{i\in\mathbb{T}_N^d}\sup_{w,w'\in\mathcal{W}}\left|J\left(\frac{i}{N},w,w'\right) - N^d\int_{\Delta_{i,N}} J(x,w,w')\,\mathrm{d}x\right|^2 \to 0, \tag{V.1.8}$$

when $N \to \infty$, with $\Delta_{i,N} := \left\{x \in \mathbb{T}^d : \left|x - \frac{i}{N}\right| < \frac{1}{2N}\right\}$.

Example V.1.5. *This assumption is in particular satisfied in the following cases:*
•J *is continuous in all variables.*

•$J(x,w,w) = J_1(x) J_2(w,w')$ *or* $J(x,w,w) = J_1(x) + J_2(w,w')$*. In both situations:*
▪$J_2 \in \mathsf{C}(\mathcal{W} \times \mathcal{W})$*, for example* $J_2(w,w') = ww'$ *or* $J_2(w,w') = w - w'$*.*

▪$J_1 \in L^2(\mathbb{T}^d)$ *is even and*
- *either continuous, or*
- $J_1 = 1_A$ *for* $A \subset \mathbb{T}^d$ *a rectangle, or*
- J_1 *can even have a singularity like* $J_1(x) = |x|^{-\frac{1}{2}+\epsilon}$ *with* $J_1(0) = 0$*.*

Remark V.1.6. *We use the assumption that* J *is even, only in the proof of the large deviation principle for* $\{L^N\}$ *(Theorem V.1.11), but not in the proof of the large deviation principle for* $\left\{\mu^N_{[0,T]}\right\}$ *(Theorem V.1.12).*

Assumption V.1.7. $\Psi(\theta,w) = \overline{\Psi}(\theta) + w_1\theta$*, for* $(w,\theta) \in \mathcal{W} \times \mathbb{R}$*, where* $\overline{\Psi}$ *is a polynomial of even degree* ≥ 2*, with positive coefficient of that degree. Define*

$$c_\Psi := \liminf_{|\theta|\to\infty} \frac{\overline{\Psi}(\theta)}{|\theta|^2}, \tag{V.1.9}$$

with $c_\Psi = \infty$ *if the degree of* $\overline{\Psi}$ *is greater than two. Assume that*

$$c_\Psi > \left\|\overline{J}\right\|_{L^1}. \tag{V.1.10}$$

Example V.1.8. *For example* Ψ *can be chosen as* $\Psi(w,\theta) = \theta^4 + w_1\theta$ *or* $\theta^2 + w_1\theta$ *or* $\theta^4 - \theta^2 + w_1\theta$*. Also more general* Ψ *are covered by the approach we state. For example the randomness could merge into the single particle potential in a more general way than just as an additional chemical potential.*

We infer from these assumption that the corresponding martingale problem is well posed for each fixed $\underline{w}^N \in \mathcal{W}^{N^d}$ and each fixed initial values $\underline{\theta}^N \in \mathbb{R}^{N^d}$, i.e. that there is a unique weak solution to (0.9.3) (see Remark V.3.3). Hence there is a unique measure $P^N_{\underline{w}^N,\underline{\theta}^N} \in \mathbb{M}_1\left(\mathsf{C}([0,T])^{N^d}\right)$, which is the law of $\underline{\theta}^N_{[0,T]}$ evolving according to the SDE (0.9.3) with initial values $\underline{\theta}^N$ and with fixed environment $\underline{w}^N$.

Notation V.1.9. *We use the following notation:*

- *For each* $N \in \mathbb{N}$*, we denote by* $\nu^N := \bigotimes_{k\in\mathbb{T}^d_N} \nu_{\frac{k}{N}} \in \mathbb{M}_1\left(\mathbb{R}^{N^d}\right)$ *the initial distribution of the* N^d*-dimensional spin system.*

- *We define the product measure of the random environment* $\zeta^N := \bigotimes_{k\in\mathbb{T}^d_N} \zeta_{\frac{k}{N}} \in \mathbb{M}_1\left(\mathcal{W}^{N^d}\right)$*.*

- *We denote by* $P^N_{\underline{w}^N} := \int_{\mathbb{R}^{N^d}} P^N_{\underline{w}^N,\underline{\theta}^N} \nu^N\left(\mathrm{d}\underline{\theta}^N\right) \in \mathbb{M}_1\left(\mathsf{C}([0,T])^{N^d}\right)$*, the law of the paths of the* N^d*-dimensional spin system with a given environment* $\underline{w}^N \in \mathcal{W}$ *and with initial distribution* ν^N*.*

- *We use the symbol* $P^N = \zeta^N(\mathrm{d}w) \otimes P^N_{\underline{w}^N} \in \mathbb{M}_1\left(\mathcal{W}^{N^d} \times \mathsf{C}([0,T])^{N^d}\right)$ *for the joint distribution of the random environment and the paths of the spin system.*

$\mu^N_{[0,T]}$ and L^N are both defined as images of $\underline{w}^N$ and $\underline{\theta}^N_{[0,T]}$. Therefore, we consider $\mu^N_{[0,T]}$ and L^N as random elements under P^N.

The following norm appears in the rate function $S_{\nu,\zeta}$ in (V.1.5) (compare it to the -1 Sobolev norm).

Definition V.1.10. *For a measure $\pi \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ and ξ a distribution on the space of test functions $\mathsf{C}_c^\infty(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$, define*

$$
\begin{aligned}
|\xi|_\pi^2 &:= \frac{1}{2} \sup_{f \in \mathbb{D}_\pi} \frac{|\langle \xi, f \rangle|^2}{\sigma^2 \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} (\partial_\theta f(x,w,\theta))^2 \pi(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)} \\
&= \sup_{f \in \mathsf{C}_c^\infty(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})} \left\{ \langle \xi, f \rangle - \frac{\sigma^2}{2} \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} (\partial_\theta f(x,w,\theta))^2 \pi(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right\},
\end{aligned}
\tag{V.1.11}
$$

with $\mathbb{D}_\pi := \left\{ f \in \mathsf{C}_c^\infty(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}) : \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} (\partial_\theta f(x,w,\theta))^2 \pi(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \neq 0 \right\}$.

With abuse of notation we also use the symbol $|\xi|_\pi$ for $\pi \in \mathbb{M}_1(\mathbb{R})$ and ξ a distribution on the space of test functions $\mathsf{C}_c^\infty(\mathbb{R})$.

Theorem V.1.11 (For a more general version see Theorem V.5.3)**.** *Let the Assumptions V.1.1, V.1.2, V.1.3, V.1.4 and V.1.7 hold. Then the family $\{L^N, P^N\}$ satisfies on $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]))$ the large deviation principle with a good rate function. We derive two different representations of the rate function (see Theorem V.5.3 and Theorem V.5.12).*

Theorem V.1.12 (For a more general version see Theorem V.3.5)**.** *Under the same assumptions as Theorem V.1.11, the family $\left\{\mu_{[0,T]}^N, P^N\right\}$ satisfies on $\mathsf{C}([0,T], \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}))$ the large deviation principle with good rate function*

$$
S_{\nu,\zeta}(\mu_{[0,T]}) := \int_0^T \left| \partial_t \mu_t - \left(\mathbb{L}^{\mathrm{LMF}}_{\mu_t,\cdot,\cdot}\right)^* \mu_t \right|_{\mu_t}^2 \mathrm{d}t + \mathsf{H}(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x), \tag{V.1.12}
$$

when $\mu_{[0,T]} \in \mathsf{C}([0,T], \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}))$ is weakly differentiable, $\sup_{t\in[0,T]} \int \theta^2 \mu_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)$ is finite, and $\mu_t = \mathrm{d}x \otimes \mu_{t,x}$ with $\mu_{t,x} \in \mathbb{M}_1(\mathcal{W} \times \mathbb{R})$. Otherwise $S_{\nu,\zeta}(\mu_{[0,T]}) = \infty$.

Moreover, the integral with respect to $\mathbb{T}^d \times \mathcal{W}$ and the supremum in the norm in $S_{\nu,\zeta}$ can be interchanged, i.e. $S_{\nu,\zeta}(\mu_{[0,T]}) = S_{\nu,\zeta}^{\mathbb{T}^d \times \mathcal{W}}(\mu_{[0,T]})$ defined as

$$
\int_0^T \int_{\mathbb{T}^d} \int_{\mathcal{W}} \left| \partial_t \mu_{t,x,w} - \left(\mathbb{L}^{\mathrm{LMF}}_{\mu_t,x,w}\right)^* \mu_{t,x,w} \right|_{\mu_{t,x,w}}^2 \mu_{0,x,\mathcal{W}}(\mathrm{d}w)\, \mathrm{d}x \mathrm{d}t + \mathsf{H}(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x), \tag{V.1.13}
$$

with $\mu_{t,x,\mathcal{W}} \in \mathbb{M}_1(\mathcal{W})$ and $\mu_{t,x,w} \in \mathbb{M}_1(\mathbb{R})$ such that $\mu_{t,x} = \mu_{t,x,\mathcal{W}}(\mathrm{d}w) \otimes \mu_{t,x,w}$.

We state further representation of $S_{\nu,\zeta}$ in Section V.4.

V.1.2 Structure of this chapter

The rest of this chapter is organised as follows. We state in Section V.2 some preliminaries that are required in the subsequent sections. At first this comprises some definitions and notations (Section V.2.1). Then in Section V.2.2, we generalise Sanov's Theorem to vectors of space ($\mathbb{T}^d$) and random environment ($\mathcal{W}$) dependent empirical measures. This is also a generalisation of the Sanov type theorem in [DG87], because of the additional space and random environment dependency. Then we state a generalisation of the Arzelá Ascoli theorem for sets and measures on $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ (Section V.2.4), and we generalise the definitions and results on distribution-valued functions of the Section 4.1 of [DG87] to the space $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ (Section V.2.5). Finally (in Section V.2.6), we discuss how the spaces, on which L^N and $\mu_{[0,T]}^N$ are defined, are related.

We state and prove the large deviation principle for the empirical process $\left\{\mu_{[0,T]}^N\right\}$ in Section V.3. For the idea of the proof we also refer to Section 0.9.2 in the introduction. In Section V.4, we state different representations of the rate function for the empirical process. These expressions might be useful when working on the mentioned long time behaviour (see also [DG89] in the mean field case), in particular when the model is not reversible.

In Section V.5, we show that the same approach as in Section V.3 can be used to derive the large deviation principle for the family $\{L^N\}$, provided that this family is exponentially tight. We prove the exponential tightness for the concrete example (0.9.3) of the local mean field model in Section V.5.2. Moreover, we derive for this model a second representation of the rate function. In this second representation, the influence of the entropy and of the interaction becomes obvious.

In Section V.6, we show at first (Theorem V.6.1) a one-to-one relation between the minimizer of the rate functions for $\left\{\mu^N_{[0,T]}\right\}$ and $\{L^N\}$. An alternative approach to get a LDP for $\left\{\mu^N_{[0,T]}\right\}$ is to obtain it from the LDP of $\{L^N\}$ via the contraction principle. This approach is implemented in Section V.6.2. However, the arising rate function does not have the desired form $S_{\nu,\zeta}$ given in (V.1.5). In Section V.6.3 we show that the rate function is at least an upper bound on $S_{\nu,\zeta}$.

In Section V.7, we derive the large deviation principle for the empirical measure $\{L^N\}$ for the concrete example (0.9.3) of the local mean field model by a different approach than in Section V.5 (which is explained in Section 0.9.2 in the introduction).

V.2 Preliminaries

V.2.1 Definitions and notations

We use the following notation in this chapter in addition to the one listed in Section 0.11.

Notation V.2.1. *Let Y be a Polish space. We denote by $\mathbb{M}_1(Y)$ the space of probability measures on Y equipped with the topology of weak convergence.*

We write $\mathbb{M}_1^L\left(\mathbb{T}^d\times Y\right)$ for the subset of $\mathbb{M}_1\left(\mathbb{T}^d\times Y\right)$, that consists of those measures, that have the Lebesgue measure as projection to $\mathbb{T}^d$.

The measures in $\mathbb{M}_1^L\left(\mathbb{T}^d\times Y\right)$ are also called Young measures (see [ABM06] Definition 4.3.1).

Definition V.2.2. *We denote the space of continuous functions from $[0,T]$ into $\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$ by*

$$\mathscr{C} := \mathsf{C}\left([0,T],\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\right), \tag{V.2.1}$$

and its subspace with values in $\mathbb{M}_1^L\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$ by

$$\mathscr{C}^L := \mathsf{C}\left([0,T],\mathbb{M}_1^L\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\right). \tag{V.2.2}$$

For the rest of the chapter, fix a non-negative $\varphi\in\mathsf{C}^2(\mathbb{R})$, that satisfies $\lim_{|\theta|\to\infty}\varphi(\theta)=\infty$.

Definition V.2.3. *We denote the subset of $\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$ of measures, whose integral with respect to a $\varphi\in\mathsf{C}(\mathbb{R})$ is bounded by $R>0$ by*

$$\mathbb{M}_{\varphi,R} := \left\{\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right) : \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\varphi(\theta)\,\mu(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\le R\right\}. \tag{V.2.3}$$

Moreover, we denote the subset of $\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$, with finite integral with respect to φ by

$$\mathbb{M}_{\varphi,\infty} := \bigcup_{R>0}\mathbb{M}_{\varphi,R} = \left\{\mu\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right) : \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\varphi(\theta)\,\mu(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)<\infty\right\}. \tag{V.2.4}$$

With abuse of notation we use also the symbol $\mathbb{M}_{\varphi,R}$ for the appropriate subspace of $\mathbb{M}_1\left(\mathbb{T}^d\times\mathbb{R}\right)$.

Definition V.2.4. *We denote the subset of $\mathscr{C}$, that consists of the paths which are everywhere in $\mathbb{M}_{\varphi,R}$, for a $R>0$, by*

$$\mathscr{C}_{\varphi,R} := \left\{ \mu_{[0,T]} \in \mathscr{C} : \sup_{t\in[0,T]} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi(\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \le R \right\} \subset \mathscr{C}. \tag{V.2.5}$$

For the union of these sets we use the symbol

$$\mathscr{C}_{\varphi,\infty} := \bigcup_{R=1}^{\infty} \mathscr{C}_{\varphi,R} = \left\{ \mu_{[0,T]} \in \mathscr{C} : \sup_{t\in[0,T]} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi(\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) < \infty \right\}. \tag{V.2.6}$$

We endow $\mathbb{M}_{\varphi,R}, \mathbb{M}_{\varphi,\infty}, \mathscr{C}_{\varphi,R}$ and $\mathscr{C}_{\varphi,\infty}$ with the subspace topology of $\mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})$ and $\mathscr{C}$ respectively. By this property these spaces differ from the definition used in [Gär88] and [DG87]. There the authors equip the spaces with a stronger topology.

Definition V.2.5. *For a measure $\mu \in \mathbb{M}_1^L(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})$, we denote by $\mu_x \in \mathbb{M}_1(\mathcal{W}\times\mathbb{R})$ the regular conditional probability measures such that $\mu = \mathrm{d}x \otimes \mu_x$.*

For the projection of μ_x on the environment coordinate $\mathcal{W}$, we use the symbol $\mu_{x,\mathcal{W}}$ and for the corresponding regular conditional probability measures $\mu_{x,w} \in \mathbb{M}_1(\mathbb{R})$. Then $\mu = \mathrm{d}x \otimes \mu_{x,\mathcal{W}}(\mathrm{d}w) \otimes \mu_{x,w}$.

V.2.2 A Sanov type result

Let $Y_1, ..., Y_r$ be Polish spaces for $r \ge 1$ and let $\{Q_{x,w} : (x,w) \in \mathbb{T}^d\times\mathcal{W}\}$ be a family of probability measures on $Y = Y_1 \times ... \times Y_r$.

We generalise in this section the Sanov type Theorem 3.5 of [DG87] to the setting we consider here (Lemma V.2.7). More precisely we add the space dependency and the random environment in the vector of the empirical measure, i.e. for $(y^i)_{i\in\mathbb{T}_N^d} \in Y^{N^d}$ and $(w^{i,N}) \in \mathcal{W}^{N^d}$, we define the vector $L_r^N \in \mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times Y_1) \times ... \times \mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times Y_r)$ by

$$L_r^N := \left(N^{-d} \sum_{i\in\mathbb{T}_N^d} \delta_{\left(\frac{i}{N}, w^{i,N}, y_1^i\right)}, ..., N^{-d} \sum_{i\in\mathbb{T}_N^d} \delta_{\left(\frac{i}{N}, w^{i,N}, y_r^i\right)} \right). \tag{V.2.7}$$

Moreover, we prove (Lemma V.2.8), that the rate function can be expressed as a relative entropy.

The following assumption implies in particular that the integrals in Lemma V.2.7 are well defined and that we get a suitable convergence of the logarithmic moment generating function.

Assumption V.2.6. *$\{Q_{x,w} : (x,w) \in \mathbb{T}^d\times\mathcal{W}\} \subset \mathbb{M}_1(Y)$ is Feller continuous.*

With these $\{Q_{x,w}\}$, define the product measures $Q^N_{\underline{w}^N} := \bigotimes_{i\in\mathbb{T}_N^d} Q_{\frac{i}{N},w^{i,N}} \in \mathbb{M}_1\left(Y^{N^d}\right)$ and the joint measures $Q^N := \zeta^N(\mathrm{d}\underline{w}^N) \otimes Q^N_{\underline{w}^N} \in \mathbb{M}_1\left(\mathcal{W}^{N^d}\times Y^{N^d}\right)$ (compare this to Notation V.1.9).

Lemma V.2.7 (compare to [DG87] Theorem 3.5 for mean field LDP)**.** *If the Assumption V.1.3 and Assumption V.2.6 hold, then the family $\{L_r^N, Q^N\}$ satisfies the large deviation principle on the space $\mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times Y_1) \times ... \times \mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times Y_r)$ with good rate function*

$$\begin{aligned} L_{\nu,\zeta}\left(\Gamma^1, ..., \Gamma^r\right) = \sup_{\substack{f_1\in C_b(\mathbb{T}^d\times\mathcal{W}\times Y_1)\\ \cdots \\ f_r\in C_b(\mathbb{T}^d\times\mathcal{W}\times Y_r)}} &\left\{ \sum_{\ell=1}^r \int_{\mathbb{T}^d\times\mathcal{W}\times Y_\ell} f_\ell(x,w,y_\ell)\,\Gamma^\ell(\mathrm{d}x,\mathrm{d}w,\mathrm{d}y_\ell) \right. \\ &\left. - \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}} \int_Y e^{\sum_{\ell=1}^r f_\ell(x,w,y_\ell)} Q_{x,w}(\mathrm{d}y_1, ..., \mathrm{d}y_r)\,\zeta_x(\mathrm{d}w) \right) \mathrm{d}x \right\} \end{aligned} \tag{V.2.8}$$

for $\Gamma^\ell \in \mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times Y_\ell)$.

In the case when $r = 1$, i.e. $Y = Y_1$, we can express the rate function as a relative entropy.

Lemma V.2.8. *If $r = 1$ then for $\Gamma = \mathrm{d}x \otimes \Gamma_x \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$*

$$\begin{aligned} L_{\nu,\zeta}(\Gamma) &= \mathsf{H}\left(\Gamma | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}\right) = \int_{\mathbb{T}^d} \mathsf{H}\left(\Gamma_x | \zeta_x(\mathrm{d}w) \otimes Q_{x,w}\right) \mathrm{d}x \\ &= \int_{\mathbb{T}^d} \int_{\mathcal{W}} \mathsf{H}\left(\Gamma_{x,w} | Q_{x,w}\right) \Gamma_{x,\mathcal{W}}(\mathrm{d}w)\, \mathrm{d}x + \int_{\mathbb{T}^d} \mathsf{H}\left(\Gamma_{x,\mathcal{W}} | \zeta_x\right) \mathrm{d}x. \end{aligned} \tag{V.2.9}$$

Otherwise $L_{\nu,\zeta}(\Gamma) = \infty$. Here $\Gamma_{x,\mathcal{W}} \in \mathbb{M}_1(\mathcal{W})$ is defined as in Definition V.2.5.

Before we prove these two lemmas in Section V.2.2.2, we state in Section V.2.2.1 some immediate consequences of the assumptions. We need these consequences in the proofs of the Sanov type results.

V.2.2.1 Preliminaries for the proof of the Sanov type result

We infer from the Assumption V.2.6, the following stronger continuity result.

Lemma V.2.9. *The Assumption V.2.6 causes that the map $x, w \mapsto \int f(x,w,y)\, Q_{x,w}(\mathrm{d}y)$ is continuous for each $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$.*

Proof. Fix an arbitrary sequence $\big(x^{(n)}, w^{(n)}\big) \to (x,w) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. Then

$$\begin{aligned} &\left| \int f\left(x^{(n)}, w^{(n)}, y\right) Q_{x^{(n)},w^{(n)}}(\mathrm{d}y) - \int f(x,w,y)\, Q_{x,w}(\mathrm{d}y) \right| \\ &\leq \left| \int f\left(x^{(n)}, w^{(n)}, y\right) - f(x,w,y)\, Q_{x^{(n)},w^{(n)}}(\mathrm{d}y) \right| \\ &\quad + \left| \int f(x,w,y) \left(Q_{x^{(n)},w^{(n)}}(\mathrm{d}y) - Q_{x,w}(\mathrm{d}y)\right) \right| =: ① + ② . \end{aligned} \tag{V.2.10}$$

By the Feller continuity of $Q_{x,w}$ (Assumption V.2.6), the sequence $Q_{x^{(n)},w^{(n)}}$ is tight (Prokhorov 's theorem). Hence for each $\epsilon > 0$, there is a compact set $K^\epsilon \subset Y$, such that

$$① \leq \sup_{y \in K^\epsilon} \left| f\left(x^{(n)}, w^{(n)}, y\right) - f(x,w,y) \right| + 2\,|f|_\infty\, Q_{x^{(n)},w^{(n)}}\left(Y \backslash K^\epsilon\right) \leq \epsilon, \tag{V.2.11}$$

by the continuity of f and the compactness of K^ϵ for n large enough. From the Feller continuity (Assumption V.2.6), we infer moreover that ② is bounded by ϵ for n large enough. ☐

Now we show that Assumption V.1.3 and Assumption V.2.6 imply in particular a convergence, which we need to prove the large deviation result (in Lemma V.2.7).

Lemma V.2.10. *Let Assumption V.1.3 and Assumption V.2.6 be satisfied. Then for all $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$, that satisfy $f \geq c$ for an arbitrary $c > 0$,*

$$\begin{aligned} &\frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \log\left(\int_{\mathcal{W} \times Y} f\left(\frac{k}{N}, w, y\right) Q_{\frac{k}{N},w}(\mathrm{d}y)\, \zeta_{\frac{k}{N}}(\mathrm{d}w) \right) \\ &\quad \to \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W} \times Y} f(x,w,y)\, Q_{\frac{k}{N},w}(\mathrm{d}y)\, \zeta_x(\mathrm{d}w) \right) \mathrm{d}x. \end{aligned} \tag{V.2.12}$$

Proof. Fix an $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times Y\big)$, that satisfies $f \geq c$ for an arbitrary $c > 0$. By Lemma V.2.9 and the Feller continuity of ζ_x (Assumption V.1.3), the function

$$x \mapsto H_f(x) := \int_{\mathcal{W}} \int_Y f(x,w,y)\, Q_{\frac{k}{N},w}(\mathrm{d}y)\, \zeta_x(\mathrm{d}w) \tag{V.2.13}$$

is continuous. This can be shown by the same arguments, that we use to prove Lemma V.2.9. Then H_f is, as a continuous function, also Riemann integrable.

By the continuity of log on $[c, |f|_\infty] \subset \mathbb{R}$, also $x \mapsto \log H_f(x)$ is Riemann integrable. This Riemann integrability implies the convergence of the sums in Lemma V.2.10. □

Lemma V.2.11. *By Assumption V.2.6 and Assumption V.1.3,* $\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}$*, characterised by*

$$(\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})[A_1 \times A_2 \times A_3] = \int_{A_1} \int_{A_2} \int_{A_3} Q_{x,w}(\mathrm{d}y)\, \zeta_x(\mathrm{d}w)\, \mathrm{d}x, \tag{V.2.14}$$

for $A_1 \subset \mathbb{T}^d$*,* $A_2 \subset \mathcal{W}$ *and* $A_3 \subset Y$*, is a well defined probability measure in* $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y)$*.*

Proof. We show at first that $(\zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ is well defined for each $x \in \mathbb{T}^d$, by constructing a probability kernel. For each $f \in \mathsf{C}_b(Y)$,

$$\mathbb{T}^d \times \mathcal{W} \ni (x, w) \mapsto \overline{H}_f(x, w) := \int_Y f(y)\, Q_{x,w}(\mathrm{d}y) \tag{V.2.15}$$

is continuous by Assumption V.2.6. Therefore, $\overline{H}_f$ is also Borel-measurable, for all non negative $f \in \mathsf{C}_b(Y)$. Then for each open set $B \subset Y$, $\overline{H}_f$ is also Borel-measurable when $f = \mathbb{1}_B$, by a pointwise approximation of $\mathbb{1}_B$ with continuous function. Then $\overline{H}_{\mathbb{1}_B}$ is also Borel measurable for all Borel measurable $B \subset Y$ (as pointwise limits). Hence, $P(x, w, A) = \int_Y \mathbb{1}_A(y)\, Q_{x,w}(\mathrm{d}y)$ is a probability kernel. Therefore $(\zeta_x(\mathrm{d}w) \otimes Q_{x,w}) \in \mathbb{M}_1(\mathcal{W} \times Y)$ is well defined for all $x \in \mathbb{T}^d$.

By the same argument, also $P(x, B) = \int_{\mathcal{W} \times Y} \mathbb{1}_B(w, y)\, Q_{x,w}(\mathrm{d}y)\, \zeta_x(w)$ is a probability kernel. This requires the Assumption V.1.3. Therefore, $(\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ is well defined. □

V.2.2.2 Proof of Lemma V.2.7 and Lemma V.2.8

Proof of Lemma V.2.7. The log moment generating function can be calculated for each vector $f = (f_1, ..., f_r) \in \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times Y_1) \times ... \times \mathsf{C}_b(\mathbb{T}^d \times \mathcal{W} \times Y_r)$ by

$$\begin{aligned}
\Gamma_{\nu,\zeta}(f) &= \lim_{N\to\infty} N^{-d} \log \int_{\mathcal{W}^{N^d} \times Y^{N^d}} e^{N^d \langle L_r^N, f\rangle} \zeta^N(\mathrm{d}\underline{w}^N) \otimes Q^N_{\underline{w}^N}(\mathrm{d}\underline{y}) \\
&= \lim_{N\to\infty} N^{-d} \log \prod_{k\in\mathbb{T}^d_N} \int_{\mathcal{W}} \int_Y e^{\sum_{\ell=1}^r f_\ell\left(\frac{k}{N}, w, y_\ell\right)} Q_{\frac{k}{N},w}(\mathrm{d}y_1, ..., \mathrm{d}y_r)\, \zeta_{\frac{k}{N}}(\mathrm{d}w) \\
&= \lim_{N\to\infty} N^{-d} \sum_{k\in\mathbb{T}^d_N} \log \int_{\mathcal{W}} \int_Y e^{\sum_{\ell=1}^r f_\ell\left(\frac{k}{N}, w, y_\ell\right)} Q_{\frac{k}{N},w}(\mathrm{d}y_1, ..., \mathrm{d}y_r)\, \zeta_{\frac{k}{N}}(\mathrm{d}w) \\
&= \int_{\mathbb{T}^d} \log \left(\int_{\mathcal{W}} \int_Y e^{\sum_{\ell=1}^r f_\ell(x, w, y_\ell)} Q_{x,w}(\mathrm{d}y_1, ..., \mathrm{d}y_r)\, \zeta_x(\mathrm{d}w) \right) \mathrm{d}x.
\end{aligned} \tag{V.2.16}$$

In the last equality we use Lemma V.2.10. Note that by Lemma V.2.9 and by H_f (defined in (V.2.13)) being continuous, all integrals in (V.2.16) are well defined.

The right hand side of (V.2.16) is finite and Gateaux differentiable. Also as in [DG87] we can show if $L_{\nu,\zeta}(\Gamma^1, ..., \Gamma^r) < \infty$, then $\Gamma^i \in \mathbb{M}_1(\mathbb{T}^d \times Y_i)$. Therefore, all conditions of Theorem 3.4 in [DG87] are satisfied and the claims of Lemma V.2.7 are proven. □

Proof of Lemma V.2.8. By Lemma V.2.7, we know that $\{L_r^N\}$ satisfies under $\{Q^N_{v_N}\}$ a LDP with rate function $L_{\nu,\zeta}(\Gamma)$. Now we show that the rate function $L_{\nu,\zeta}$ has the claimed representation (V.2.9). The measure $(\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ in the relative entropy is well defined by Lemma V.2.11.

Step 1: If $L_{\nu,\zeta}(\Gamma) < \infty$ then $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$:
Fix $\Gamma \in \mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times Y)$ with $L_{\nu,\zeta}(\Gamma) < \infty$. Then $\int_{\mathbb{T}^d \times \mathcal{W} \times Y} f(x)\, \Gamma(\mathrm{d}x, \mathrm{d}w, \mathrm{d}y) = \int_{\mathbb{T}^d} f(x)\, \mathrm{d}x$ for all $f \in \mathsf{C}_b(\mathbb{T}^d)$. Indeed, assume there were a $f \in \mathsf{C}_b(\mathbb{T}^d)$ for which this is not satisfied. Then for all $\lambda \in \mathbb{R}$,

$$L_{\nu,\zeta}(\Gamma) \geq \lambda \int_{\mathbb{T}^d \times \mathcal{W} \times Y} f(x)\, \Gamma(\mathrm{d}x, \mathrm{d}w, \mathrm{d}y) - \lambda \int_{\mathbb{T}^d} f(x)\, \mathrm{d}x \neq 0. \tag{V.2.17}$$

Because λ is arbitrary, this is a contradiction to $L_{\nu,\zeta}(\Gamma) < \infty$.

For each open $A \subset \mathbb{T}^d$, we can find a sequence of $f_n \in \mathsf{C}_b(\mathbb{T}^d)$, such that $f_n \geq 0$, $f_n \nearrow \mathbb{1}_A$ (see e.g. [Ash72] A6). Therefore, we get by the dominant convergence theorem that the projection of Γ on $\mathbb{T}^d$ has to be the Lebesgue measure. The disintegration theorem for measures on a product space (see [ABM06] Theorem 4.2.4) states that $\Gamma = \mathrm{d}x \otimes \Gamma_x$ with $\Gamma_x \in \mathbb{M}_1(\mathcal{W} \times Y)$.

Step 2: $L_{\nu,\zeta}(\Gamma) \leq \mathsf{H}(\mathrm{d}x \otimes \Gamma_x | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ for $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$:
Fix $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$, such that $\mathsf{H}(\mathrm{d}x \otimes \Gamma_x | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}) < \infty$. Hence $\mathrm{d}x \otimes \Gamma_x$ is absolute continuous with respect to $\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}$ with density ρ:

$$\mathrm{d}x \otimes \Gamma_x(\mathrm{d}w, \mathrm{d}y) = \rho(x, w, y)\, \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w}(\mathrm{d}y). \tag{V.2.18}$$

Because $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$, $\int_{\mathcal{W}} \int_Y \rho(x, w, y)\, Q_{x,w}(\mathrm{d}y)\, \zeta_x(\mathrm{d}w) = 1$ for all $x \in \mathbb{T}^d$. The claimed upper bound on $L_{\nu,\zeta}(\Gamma)$, follows from finally by the same steps as in the second point of the proof of Theorem 3.1 in [MR94].

Step 3: $L_{\nu,\zeta}(\Gamma) \geq \mathsf{H}(\mathrm{d}x \otimes \Gamma_x | \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$ for $\Gamma \in \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times Y)$:
This is just an application of Jensen's inequality to the convex function $-\log$ in $L_{\nu,\zeta}$ and the variation formula of the relative entropy.

Step 4: Second representation of rate function:
The second representation of the rate function follows by [DE97] Theorem C.3.1. □

Remark V.2.12. *If r were larger than one in Lemma V.2.8 , also the Step 1, Step 2 and Step 4 of the proof of Lemma V.2.8 are true. However, the Step 2 is in general not true any more due to the larger set $\mathsf{C}(\mathbb{T}^d \times \mathcal{W} \times Y)$ used in the variation formula of $\mathsf{H}(\,.\,| \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes Q_{x,w})$, compared to the set of functions used in the supremum in $L_{\nu,\zeta}$.*

Remark V.2.13. *We could exchange the space $\mathbb{T}^d$ by an arbitrary compact Polish spaces X. If adjusted assumptions hold for X, then we would get the same large deviation result. We need the Lemma V.2.7 in the sequel only with the space $\mathbb{T}^d$. To simplify the comprehensibility, we state it here not in its most general form.*

V.2.3 Discussion of the Sanov type result for the spin system

We use in the next sections the Sanov type result (Lemma V.2.7), with the family of measures

$$Q_{x,w} := \int_{\mathbb{R}} Q_{x,w,\theta} \nu_x(\mathrm{d}\theta) \in \mathbb{M}_1(Y), \tag{V.2.19}$$

where $w \in \mathcal{W}$ and $x \in \mathbb{T}^d$. Here $\{Q_{x,w,\theta} : (x, w, \theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\}$ is a family of probability measures on a space $Y = Y_1 \times \ldots \times Y_r$ indexed by the spacial position $x \in \mathbb{T}^d$, the random environment $w \in \mathbb{W}$ and the initial spin θ.

We assume in this section the following assumption.

Assumption V.2.14. $\{Q_{x,w,\theta} : (x, w, \theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\} \subset \mathbb{M}_1(Y)$ *is Feller continuous.*

This assumption and Assumption V.1.1 imply that also the family $\{Q_{x,w}\}$ is Feller continuous, hence that Assumption V.2.6 is satisfied.

In the next section, we discuss weaker conditions on the initial distributions $\{\nu_x\}$ than Feller continuity (Assumption V.1.1), that are sufficient to imply the Sanov type result (Lemma V.2.7). Hence we show that weaker assumptions than the Assumption V.2.6 are sufficient. Finally in Section V.2.3.2, we state examples of initial distributions that satisfy Assumption V.1.1 or the weaker conditions.

V.2.3.1 Weaker assumptions on the initial distributions than Assumption V.1.1

We show in this section, that we do not need the Feller continuity of $\{\nu_x\}$ (Assumption V.1.1) to prove the Sanov type Lemma V.2.7 and Lemma V.2.8, i.e. that we do not need the Assumption V.2.6 for $Q_{x,w}$ (defined in (V.2.19)). We actually need only the results of Lemma V.2.10 and Lemma V.2.11. In the following lemma we show that these results hold under Assumptions V.2.14 and weaker conditions on the initial distributions than Assumption V.1.1.

Lemma V.2.15. *If we assume Assumption V.2.14 and Assumption V.1.3 and that*

(i) $x \mapsto \int_{\mathbb{R}} f(\theta)\, \nu_x(\mathrm{d}\theta)$ *is Riemann integrable for all* $f \in \mathsf{C}_b(\mathbb{R})$, $f \geq 0$,

(ii) *the set* $\{\nu_x\}_{x\in\mathbb{T}^d}$ *is tight and*

(iii) $x \mapsto \int_{\mathbb{R}} f(\theta)\, \nu_x(\mathrm{d}\theta)$ *is Borel-measurable for all* $f \in \mathsf{C}_b(\mathbb{R})$, $f \geq 0$,

then the statements of Lemma V.2.10 and of Lemma V.2.11 also hold.

Remark V.2.16. *The conditions (i), (ii) and (iii) are all implied by the Assumption V.1.1.*

Proof. To get the result of Lemma V.2.10, we only need to show that H_f (defined in (V.2.13)) is Riemann integrable. And for the result of Lemma V.2.11, we prove that $\overline{H}_f$ (defined in (V.2.15)) is Borel-measurable.

Step 1: H_f is Riemann integrable: By Assumption V.2.14 and the same argument used in the proof of Lemma V.2.9, it is enough to show that

$$x \mapsto \widehat{H}_f(x) := \int_{\mathcal{W}} \int_{\mathbb{R}} f(x,w,\theta)\, \nu_x(\mathrm{d}\theta)\, \zeta_x(\mathrm{d}w) \tag{V.2.20}$$

is Riemann integrable for all $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, with $f > c > 0$.

Fix an $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, with $f \geq c$ for an arbitrary $c > 0$. For each $\epsilon > 0$, we construct now a Riemann integrable function $\widehat{H}_{f,\epsilon} : \mathbb{T}^d \to \mathbb{R}$ which satisfies

$$\left|\widehat{H}_f(.) - \widehat{H}_{f,\epsilon}(.)\right|_\infty < \epsilon. \tag{V.2.21}$$

This implies the uniform convergence of Riemann integrable functions to $\widehat{H}_f$ and therefore also that $\widehat{H}_f$ is Riemann integrable.

For all $\epsilon > 0$, there is a $N_\epsilon \in \mathbb{N}$, such that

$$\left|\int_{\mathcal{W}\times\mathbb{R}} \left(f(x_1,w,\theta) - f(x_2,w,\theta)\right) \nu_{x_1}(\mathrm{d}\theta)\, \zeta_{x_1}(\mathrm{d}w)\right| \leq \epsilon, \tag{V.2.22}$$

for $x_1, x_2 \in \mathbb{T}^d$ with $|x_1 - x_2|_\infty \leq \frac{1}{N_\epsilon}$. This follows by the same calculation used in (V.2.11), by the tightness of ν_x (condition (ii)) and the tightness of ζ_x (Assumption V.1.3). Hence (V.2.21) is satisfied with

$$\widehat{H}_{f,\epsilon}(x) := \int_{\mathbb{R}} \int_{\mathcal{W}} f\left(\frac{i}{N_\epsilon}, w, \theta\right) \zeta_x(\mathrm{d}w)\, \nu_x(\mathrm{d}\theta), \tag{V.2.23}$$

where $i \in \mathbb{T}^d_{N_\epsilon}$ is chosen such that $\left|\frac{i}{N_\epsilon} - x\right|_\infty \leq \frac{1}{2N_\epsilon}$.

Moreover, $\widehat{H}_{f,\epsilon}$ is Riemann integrable. Indeed, for each $i \in \mathbb{T}^d_{N_\epsilon}$ and each $x \in \mathbb{T}^d$ with $\left|\frac{i}{N_\epsilon} - x\right|_\infty \leq \frac{1}{2N_\epsilon}$, the function in the integrand is always the function $f\left(\frac{i}{N_\epsilon}, w, \theta\right)$. Therefore, (i) implies that $\widehat{H}_{f,\epsilon}$ is Riemann integrable on this interval. There are only finitely many such rectangles and therefore $\widehat{H}_{f,\epsilon}$ is Riemann integrable on $\mathbb{T}^d$.

Step 2: $\overline{H}_f$ is Borel-measurable: The function $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \ni (x, w, \theta) \mapsto \int_Y f(y)\, Q_{x,w,\theta}(\mathrm{d}y)$ is continuous and bounded by Assumption V.2.14 for all $f \in \mathsf{C}_b(Y)$. Therefore, it suffices to prove that for all non negative $g \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, the function $x, w \mapsto \int_\mathbb{R} g(x, w, \theta)\, \nu_x(\mathrm{d}\theta)$ is Borel measurable. By the same argument as in Step 1, we can approximate $\overline{H}_g$ uniformly by $\overline{H}_{g,\epsilon}$. Then we only need to show that the $\overline{H}_{g,\epsilon}$ are Borel-measurable. This follows as in Step 1, but now by condition (iii) instead of (i). □

V.2.3.2 Examples of initial distributions

Example V.2.17. *We give now three easy examples of initial distributions that satisfy the Assumption V.1.1.*

(i) *All initial distributions equal each other, i.e.* $\nu_x = \nu_0 \in \mathbb{M}_1(\mathbb{R})$.

(ii) *There is a function* $g \in \mathsf{C}\big(\mathbb{T}^d\big)$ *such that* $\nu_x = \delta_{g(x)}$.

(iii) *There is a function* $g \in \mathsf{C}\big(\mathbb{T}^d\big)$ *such that* ν_x *is normal distributed with mean* $g(x)$ *and variance one, i.e.* $\nu_x \sim N(g(x), 1)$.

Example V.2.18. *Let us now state some examples of function, that satisfy the conditions of Lemma V.2.15 and are therefore also usable.*

- *Let* $A_i \subset \mathbb{T}^d$ *be measurable disjoint rectangles such that* $\mathbb{T}^d = \bigcup_{i=1}^n A_i$. *Let* $\nu_x = \nu_{A_i} \in \mathbb{M}_1(\mathbb{R})$ *if* $x \in A_i$. *Then* F_f *is a step function and therefore Borel measurable and Riemann integrable.*

  Moreover, the set $\{\nu_x\}$ *is a finite set of probability measures and therefore tight.*

  However, the stronger Assumption V.1.1 is in general not satisfied.

- *Explicit example of such measures are for example* $\nu_x = \nu^{\mathrm{UP/Down}}$ *respectively on the upper and lower half of the torus. The measures* $\nu^{\mathrm{UP/Down}}$ *could be for example* $\delta_{\pm 1}$ *or* $N(\pm 1, 1)$.

- *The conditions (i) and (iii) of Lemma V.2.15 are also satisfied if the map* $x \mapsto \int_\mathbb{R} f(\theta)\, \nu_x(\mathrm{d}\theta)$ *is a uniform limit of step functions (i.e. a d-dimensional regulated function). Therefore, even more general measures* ν_x *are possible, as long as these measures satisfy the tightness assumption.*

Note that by the same arguments that we use to prove Lemma V.2.11, we can also show that the probability measure $\nu(\mathrm{d}x, \mathrm{d}\theta) := \mathrm{d}x \otimes \nu_x(\mathrm{d}\theta) \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathbb{R}\big)$ is well defined.

Remark V.2.19. *We could choose the initial distribution of the* N^d *dimensional system more general than* ν^N *being the product measures of the* $\nu_{\frac{k}{N}}$, *and still get the results of Lemma V.2.7 and Lemma V.2.8.*

For example take measures $\left\{\nu_k^N\right\}_{k \in \mathbb{T}^d_N, N \in \mathbb{N}} \subset \mathbb{M}_1(\mathbb{R})$ *and define the product measures* ν^N *with these measures instead of* $\nu_{\frac{k}{N}}$. *If for each* $\epsilon > 0$ *and each positive* $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, *there is a* $N_{\epsilon,f} \in \mathbb{N}$ *such that*

$$\sup_{N > N_{\epsilon,f}} \sup_{w \in \mathcal{W}} \sup_{k \in \mathbb{T}^d_N} \left| \int_\mathbb{R} \int_Y f\left(\frac{k}{N}, w, y\right) Q_{\frac{k}{N}, w, \theta}(\mathrm{d}y) \left(\nu_k^N(\mathrm{d}\theta) - \nu_{\frac{k}{N}}(\mathrm{d}\theta)\right) \right| < \epsilon, \tag{V.2.24}$$

then (V.2.16) *would also hold for these measures.*

V.2.4 Extended Arzelá-Ascoli theorem

We give now a mild generalisation of the Arzelá-Ascoli theorem to subsets of $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$. By the compactness of $\mathbb{T}^d$ we basically only have to take care of the projections of a set $A \subset \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ to the $\mathcal{W}$ and the $\mathsf{C}([0,T])$ component. For the latter projection we can use the conditions of the original Arzelá-Ascoli theorem.

Lemma V.2.20 (Extended Arzelá-Ascoli Theorem)**.**

(i) *$A \subset \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ is relatively compact if and only if*

$$\mathrm{Proj}_{\mathsf{C}}[A] = \left\{\theta_{[0,T]} \in \mathsf{C}([0,T]) : \exists\,(x,w) \in \mathbb{T}^d \times \mathcal{W} : \left(x, w, \theta_{[0,T]}\right) \in A\right\} \tag{V.2.25}$$

is equibounded and equicontinuous and $\mathrm{Proj}_{\mathcal{W}}[A]$ *is relatively compact.*

(ii) *A sequence $\{Q^{(n)}\} \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ is tight if and only if*

1. *for each $\eta > 0$ there exists an $a > 0$ such that for all $n > 0$ and $t \in [0,T]$*

$$Q^{(n)}\left[\left(x, w, \theta_{[0,T]}\right) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : |\theta_0| \geq a\right] \leq \eta \qquad \text{and} \tag{V.2.26}$$

2. *for each $\kappa, \eta > 0$ there exists $\delta \in (0,1)$ such that for all $n > 0$*

$$Q^{(n)}\left[\left(x, w, \theta_{[0,T]}\right) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : \sup_{|t-s|\leq\delta} |\theta_t - \theta_s| \geq \kappa\right] \leq \eta \qquad \text{and} \tag{V.2.27}$$

3. *for each $\eta > 0$ there exists an $M > 0$ such that for all $n > 0$*

$$Q^{(n)}\left[\left(x, w, \theta_{[0,T]}\right) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : |w| \geq M\right] \leq \eta. \tag{V.2.28}$$

Proof. ***(i)*** We claim that the relative compactness of A is equivalent to the relative compactness of $\mathrm{Proj}_{\mathsf{C}}[A]$ and the relative compactness of $\mathrm{Proj}_{\mathcal{W}}[A]$.

Then (i) follows from the Arzelá-Ascoli theorem (see for example [Bil99] Theorem 7.2).

"$\Rightarrow$"If A is relatively compact, then, for each ϵ, there are $\left(x^{(\ell)}, w^{(\ell)}, \theta^{(\ell)}_{[0,T]}\right)_{\ell=1}^{n} \subset \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ for a $n = n(\epsilon) \in \mathbb{N}$, such that $A \subset \bigcup_{\ell=1}^{n} B_\epsilon\left(\left(x^{(\ell)}, w^{(\ell)}, \theta^{(\ell)}_{[0,T]}\right)\right)$. Then

$$\mathrm{Proj}_{\mathsf{C}}\left[B_\epsilon\left(\left(x^{(\ell)}, w^{(\ell)}, \theta^{(\ell)}_{[0,T]}\right)\right)\right] = B_\epsilon\left(\theta^{(\ell)}_{[0,T]}\right), \tag{V.2.29}$$

and therefore $\mathrm{Proj}_{\mathsf{C}}[A] \subset \bigcup_{i=1}^{n} B_\epsilon\left(\theta^{(\ell)}_{[0,T]}\right)$. Hence we found a finite open cover of $\mathrm{Proj}_{\mathsf{C}}[A]$, i.e. $\mathrm{Proj}_{\mathsf{C}}[A]$ is totally bounded and therefore relatively compact.

By the same argument there is a finite open cover for $\mathrm{Proj}_{\mathcal{W}}[A]$.

"$\Leftarrow$"If $\mathrm{Proj}_{\mathsf{C}}[A]$ is relatively compact, then $\mathrm{Proj}_{\mathsf{C}}[A] \subset \bigcup_{\ell=1}^{n} B_\epsilon\left(\theta^{(\ell)}_{[0,T]}\right)$. If $\mathrm{Proj}_{\mathcal{W}}[A]$ is relatively compact, then $\mathrm{Proj}_{\mathcal{W}}[A] \subset \bigcup_{i=1}^{n'} B_\epsilon\left(w^{(i)}\right)$. This implies that A is totally bounded with open cover $A \subset \bigcup_{\ell=1}^{n} \bigcup_{i=1}^{n'} \bigcup_{k \in \mathbb{T}^d_{\frac{1}{\epsilon}}} B_{4\epsilon}\left(\left(k\epsilon, w^{(i)}, \theta^{(\ell)}_{[0,T]}\right)\right)$.

(ii) This claim follows by applying part (i), as in the proof of [Bil99] Theorem 7.3. $\square$

V.2.5 Distribution-valued functions

In this section we state the definitions and results of Section 4.1 of [DG87] transferred to the space-dependent setting considered here.

Definition V.2.21. • *We denote by $\mathbb{D} = \mathsf{C}_c^\infty\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ the space of test functions having compact support and continuous derivatives of all orders with the usual inductive topology.*

- *For a compact set $K \subset \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, let $\mathbb{D}_K$ be the subset of $\mathbb{D}$ of functions with support in K.*
- *By $\mathbb{D}'$ and $\mathbb{D}'_K$, we denote the space of real distributions on $\mathbb{D}$ respectively on $\mathbb{D}_K$.*
- *Moreover, we write $\langle \xi, f\rangle$ for the application of $\xi \in \mathbb{D}'$ to $f \in \mathbb{D}$.*

Definition V.2.22 (Variation of Definition 4.1 in [DG87])**.** *A map $\xi : [0,T] \to \mathbb{D}'$ is called absolutely continuous if for each compact set $K \subset \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, there exist a neighbourhood U_K of 0 in $\mathbb{D}_K$ and a absolutely continuous function $H_K : [0,T] \to \mathbb{R}$ such that*

$$|\langle \xi(u), f\rangle - \langle \xi(v), f\rangle| \le |H_K(u) - H_K(v)|, \tag{V.2.30}$$

for all $u, v \in I$ and $f \in U_K$.

Lemma V.2.23 (Lemma 4.2 in [DG87])**.** *If $\xi : [0,T] \to \mathbb{D}'$ is absolutely continuous, then $\langle \xi(.), f\rangle : [0,T] \to \mathbb{R}$ is also absolutely continuous for each $f \in \mathbb{D}$.*

Moreover, the time derivative of ξ in the distributions sense

$$\partial_t \xi(t) = \lim_{h \to 0} h^{-1}\left(\xi(t+h) - \xi(t)\right) \tag{V.2.31}$$

exists for almost all $t \in [0,T]$.

Lemma V.2.24 (Lemma 4.3 in [DG87],integration by parts)**.** *For all absolutely continuous map $\xi : [0,T] \to \mathbb{D}'$ and each $f \in \mathsf{C}_c^\infty\big([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$,*

$$\langle \xi(t), f(t)\rangle - \langle \xi(s), f(s)\rangle = \int_s^t \langle \partial_t \xi(u), f(u)\rangle \,\mathrm{d}u + \int_s^t \langle \xi(u), \partial_t f(u)\rangle \,\mathrm{d}u\,. \tag{V.2.32}$$

The proofs of these two lemmas are analogue to the one of Lemma 4.2 in [DG87] respectively Lemma 4.3 in [DG87]. The crucial property of $\mathbb{D}$ and $\mathbb{D}_K$ for the proofs is their separability. This is the case for the spaces considered here as well as in [DG87].

Remark V.2.25. *We apply the results of this section later to probability measure valued functions in $\mathscr{C}$. This is possible because each measure in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ is a Radon measure and hence also an element of $\mathbb{D}'$.*

V.2.6 Relation between the spaces of the empirical measures and empirical processes

We are looking at two different levels of large deviation principles. The higher level are the empirical measures L^N in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. The second level are the empirical processes $\mu^N_{[0,T]}$ in $\mathscr{C}$. Both elements are defined (see (V.1.4) and (V.1.3)) as images of the paths of the spins on the space $\mathsf{C}([0,T])^{N^d}$ and of the random environment $\underline{w}^N \in \mathcal{W}^{N^d}$.

Let us now define a map $\Pi : \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big) \to \mathscr{C}$, which maps L^N to $\mu^N_{[0,T]}$ for each $N \in \mathbb{N}$.

Definition V.2.26. *For* $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ *we define* $\Pi\,(Q)_{[0,T]} \in \mathscr{C}$ *for each* $t \in [0,T]$ *by*

$$\begin{aligned}\Pi\,(Q)_t\,(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) &= Q\left[\big(y_x, y_w, y_{[0,T]}\big) \in \mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) : (y_x, y_w, y_t) \in \mathrm{d}x\mathrm{d}w\mathrm{d}\theta\right] \\ &= Q \circ (id_{\mathbb{T}^d}, id_{\mathcal{W}}, \theta_t)^{-1}\,(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\end{aligned} \tag{V.2.33}$$

for $(x, w, \theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$.

The measure $\Pi\,(Q)_t$ is the one-dimensional distribution at time $t \in [0,T]$ of the measure $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. Let us show that $\Pi\,(Q)_{[0,T]}$ of Definition V.2.26 is actually an element of the space $\mathscr{C}$.

Lemma V.2.27. *The function* Π *is well defined.*

Proof. Fix a $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. We have to show that $\Pi\,(Q)_{[0,T]}$ is in $\mathscr{C}$. By the definition of Π, we know already that $\Pi\,(Q)_t \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ for all $t \in [0,T]$. Now we prove the continuity in time. Take a bounded L_f-Lipschitz continuous function $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ and $s, t \in [0,T]$ with $|s-t| < \delta$, then

$$\begin{aligned}&\left|\int f\,(y, w, \theta)\,(\Pi\,(Q)_t - \Pi\,(Q)_s)\right| = \left|\int f\,(y, w, \theta_t) - f\,(y, w, \theta_s)\,Q\left(\mathrm{d}y, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right)\right| \\ &\le \int |f\,(y, w, \theta_t) - f\,(y, w, \theta_s)|\,\mathbb{1}_{|\theta_t - \theta_s| < \kappa} Q\left(\mathrm{d}y, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) + 2\,|f|_\infty\,Q\left[\hat{\theta} : |\theta_t - \theta_s| \ge \kappa\right] \\ &\le L_f \kappa + 2\,|f|_\infty\,Q\left[\sup_{|u-v|<\delta} |\theta_u - \theta_v| \ge \kappa\right] \le \epsilon,\end{aligned} \tag{V.2.34}$$

when $\kappa = \frac{\epsilon}{2L_f}$ and δ is small enough (by the extended Arzelá-Ascoli Lemma V.2.20 (ii)). Hence the Portmanteau theorem implies that $\Pi\,(Q)_{t_n} \to \Pi\,(Q)_t$ weakly in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ if $t_n \to t$. □

Moreover, we show now that Π is a continuous function.

Lemma V.2.28. *The function* Π *is continuous.*

Proof. The proof of this lemma follows the ideas in the proof of [DG87] Lemma 4.6 for the mean field model.

Take a sequence $Q^{(n)} \to Q$ in $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. This implies that for each $t \in [0,T]$ and each $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$, that is Lipschitz continuous,

$$\left|\int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\,(x, w, \theta)\left(\Pi\left(Q^{(n)}\right)_t - \Pi\,(Q)_t\right)(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\right| \to 0. \tag{V.2.35}$$

The topology on $\mathscr{C}$ is the topology of uniform convergence. Therefore, we have to show that the convergence (V.2.35) is uniform in t. The weak convergence of $Q^{(n)}$ implies tightness (Prokhorov's theorem), because $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ is a separable metric space. Moreover, we can split the absolute value in (V.2.35) into the following summands.

$$\begin{aligned}(\text{V.2.35}) \le &\left|\int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\,(x, w, \theta)\left(\Pi\left(Q^{(n)}\right)_s - \Pi\,(Q)_s\right)(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\right| \\ &+ \left|\int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\,(x, w, \theta)\left(\Pi\left(Q^{(n)}\right)_t - \Pi\left(Q^{(n)}\right)_s\right)(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\right| \\ &+ \left|\int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f\,(x, w, \theta)\,(\Pi\,(Q)_t - \Pi\,(Q)_s)\,(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\right| =: ① + ② + ③\,.\end{aligned} \tag{V.2.36}$$

The ② and ③ are bounded by ϵ for all $t, s \in [0,T]$ with $|t-s| < \delta$ for a δ small enough. This can be shown as in (V.2.34). Moreover, the δ is the same for all $n \in \mathbb{N}$, because the analogue of (V.2.34) is bounded uniformly in n by Lemma V.2.20 (ii).

For each $k \in \left\{1, ..., \frac{T}{\delta}\right\}$, there is a $N_k \in \mathbb{N}$, such that ① is bounded by ϵ for all $n > N_k$.

Therefore, we conclude that for all $n > \max_{k=0}^{\frac{T}{\delta}} N_k$

$$\sup_t \left| \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta) \left(\Pi\left(Q^{(n)}\right)_t - \Pi(Q)_t \right) (\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right| \leq 3\epsilon, \tag{V.2.37}$$

i.e. the uniform (in $t \in [0,T]$) convergence of (V.2.35). □

Notation V.2.29. *With abuse of notation, we use the symbol* Π *also for:*

- *The analogue defined function* $\mathbb{M}_1(\mathsf{C}([0,T])) \to \mathsf{C}([0,T], \mathbb{M}_1(\mathbb{R}))$. *Then* $\Pi(q)_{[0,T]}$ *takes values in* $\mathsf{C}([0,T], \mathbb{M}_1(\mathbb{R}))$ *for* $q \in \mathbb{M}_1(\mathsf{C}([0,T]))$.
- *The analogue defined function* $\mathbb{M}_1(\mathcal{W} \times \mathsf{C}([0,T])) \to \mathsf{C}([0,T], \mathbb{M}_1(\mathcal{W} \times \mathbb{R}))$.

In the following lemma we state that the projection of $\Pi(Q)$ to $\mathbb{T}^d$ is the Lebesgue measure, if this is the case for Q. Moreover, we show that the projection of Π to the environment coordinate is frozen over time.

Lemma V.2.30. *For* $Q \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$, $\Pi(Q)_t \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ *for all* $t \in [0,T]$. *Moreover,* $\Pi(Q)_{t,x,\mathcal{W}} = \Pi(Q)_{0,x,\mathcal{W}} = Q_{x,\mathcal{W}}$ *(see Definition V.2.5) for all* $t \in [0,T]$.

Proof. Fix a $Q \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ and a $t \in [0,T]$. Then $Q = \mathrm{d}x \otimes Q_x$ and it is easy to see that $\Pi(Q)_t = \mathrm{d}x \otimes \Pi(Q_x)_t$. Moreover, $Q = \mathrm{d}x \otimes Q_{x,\mathcal{W}}(\mathrm{d}w) \otimes Q_{x,w}$. Then for all $t \in [0,T]$

$$\begin{aligned} \int_{\mathbb{T}^d \times \mathcal{W}} f(x,w)\, Q_{x,\mathcal{W}}(\mathrm{d}w)\, \mathrm{d}x &= \int_{\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])} f(x,w)\, Q = \int_{\mathbb{T}^d \times \mathcal{W}} f(x,w)\, Q_{x,\mathcal{W}}(\mathrm{d}w)\, \mathrm{d}x \\ &= \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w)\, \Pi(Q)_t = \int_{\mathbb{T}^d \times \mathcal{W}} f(x,w)\, \Pi(Q)_{t,x,\mathcal{W}}(w)\, \mathrm{d}x, \end{aligned} \tag{V.2.38}$$

what we wanted to show. □

V.3 The LDP of the empirical process

In this section we state and prove the large deviation principle for the family of empirical processes $\left\{\mu^N_{[0,T]}\right\}$ define in (V.1.3). We investigate a more general setting than the model considered in Theorem V.1.12 (in Section V.1.1). Therefore, we state at first some notation and assumptions. We show in Section V.3.3, that the concrete example of a local mean field model considered in Section V.1.1 satisfies these assumptions.

We examine the N^d dimensional system of interacting spins defined by (V.1.1), with drift coefficient $b : \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right) \to \mathbb{R}$ and diffusion coefficient $\sigma > 0$. As explained in the introduction, the interaction between the spins is modelled as a dependency of the drift coefficient b on the empirical measure.

We define the N^d dimensional diffusion generator corresponding to (V.1.1) for fixed environment $\underline{w}^N$, acting on $f \in \mathsf{C}_b^2\left(\mathbb{R}^{N^d}\right)$ by

$$\mathbb{L}^N_{\underline{w}^N} f\left(\underline{\theta}^N\right) := \sum_{k \in \mathbb{T}^d_N} \mathbb{L}_{\mu^N, \frac{k}{N}, w^{k,N}}\, f\left(\underline{\theta}^N\right), \tag{V.3.1}$$

where $\mathbb{L}_{\mu^N,\frac{k}{N},w^{k,N}}$ is the operator defined in (V.1.6) with derivatives in the $\theta^{k,N}$ direction and with drift coefficient $b\left(\frac{k}{N}, w^{k,N}, ., \mu^N\right) : \mathbb{R} \to \mathbb{R}$. The $\mu^N \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ is the empirical measure defined as in (V.1.2) with $\underline{\theta}^N$ and $\underline{w}^N$.

For the proof of the large deviation principle, we require that the drift coefficient b is chosen in such a way that the following assumption is satisfied.

Assumption V.3.1. *There is a non-negative function $\varphi \in \mathsf{C}^2(\mathbb{R})$ with $\lim_{|\theta|\to\infty} \varphi(\theta) = \infty$, such that:*

a.) *The function $b : \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \mathbb{M}_{\varphi,\infty} \to \mathbb{R}$ satisfies:*

  a.a) *The restriction of b to $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \big(\mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)\big) \to \mathbb{R}$ is continuous for all $R > 0$.*

  a.b) *For all $N \in \mathbb{N}$ and all $\underline{w}^N \in \mathcal{W}^{N^d}$, $b^N : \mathbb{R}^{N^d} \to \mathbb{R}^{N^d}$, defined by*

$$b^N\left(\underline{\theta}^N\right) := \left(b\left(\frac{k}{N}, w^{k,N}, \theta_k, \mu^N\right)\right)_{k\in\mathbb{T}_N^d}, \tag{V.3.2}$$

  is a locally bounded measurable function.

b.) *There is a constant $\lambda > 0$ and a $\overline{N} \in \mathbb{N}$, such that for all $N > \overline{N}$ and all empirical measures μ^N (defined by $\underline{\theta}^N \in \mathbb{R}^{N^d}$ and $\underline{w}^N \in \mathcal{W}^{N^d}$),*

$$\int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \mathbb{L}_{\mu^N,x,w}\varphi(\theta) + \frac{\sigma^2}{2}\left|\partial_\theta\varphi(\theta)\right|^2 \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \le \lambda \int\limits_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \varphi(\theta)\,\mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\,. \tag{V.3.3}$$

c.) *For each $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$, there is a constant $\lambda\left(\mu_{[0,T]}\right) > 0$ such that*

$$\mathbb{L}_{\mu_t,x,w}\varphi(\theta) + \frac{\sigma^2}{2}\left|\partial_\theta\varphi(\theta)\right|^2 \le \lambda\left(\mu_{[0,T]}\right)\varphi(\theta)\,, \tag{V.3.4}$$

for all $(t, x, w, \theta) \in [0, T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$.

d.) *For each $R > 0$ and each $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$,*

$$\int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \sigma^2\left|b\left(x, w, \theta, \mu_t^{(n)}\right) - b\left(x, w, \theta, \overline{\mu}_t\right)\right|^2 \mu_t^{(n)}(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\,\mathrm{d}t \to 0, \tag{V.3.5}$$

for $n \to \infty$, when $\mu_{[0,T]}^{(n)} \to \overline{\mu}_{[0,T]}$, for a sequence $\left\{\mu_{[0,T]}^{(n)}\right\} \subset \left(\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L\right)$ or a sequence

$$\left\{\mu_{[0,T]}^{(n)}\right\} \subset \left\{\mu_{[0,T]} \in \mathscr{C}_{\varphi,R} : \mu_{[0,T]} = \mu_{[0,T]}^N \text{ is a empirical process for a } N \in \mathbb{N}\right\}. \tag{V.3.6}$$

Example V.3.2. *We show in Section V.3.3, that the concrete example* (0.9.3) *of a local mean field model, with the assumptions of Section V.1.1, satisfies the Assumption V.3.1.*

Remark V.3.3. *For each given environment $\underline{w}^N \in \mathcal{W}^{N^d}$, the martingale problem for the generator $\mathbb{L}_{\underline{w}^N}^N$ is well posed by the Assumption V.3.1 a.b) and b.). Indeed, from Theorem 10.1.2 of [SV79] and Theorem 7.2.1 of [SV79], we infer the uniqueness of the solution to the martingale problem, because the drift coefficient is locally bounded and measurable (Assumption V.3.1 a.b)). For the existence of a solution of the martingale Problem, we apply Theorem 10.2.1 of [SV79] with $\varphi\left(\underline{\theta}^N\right) := \frac{1}{N^d}\sum\varphi\left(\theta^{k,N}\right)$. The conditions of this theorem are satisfied by Assumption V.3.1 b.). We denote by $P_{\underline{w}^N,\underline{\theta}^N}^N \in \mathbb{M}_1\left(\mathsf{C}([0,T])^N\right)$ the unique solution of this martingale problem. For a short discussion of the other assumptions, see Remark V.3.6.*

With $P_{\underline{w}^N,\underline{\theta}^N}^N$, we define $P_{\underline{w}^N}^N$ and P^N as in Notation V.1.9.

Besides the Assumption V.1.1 on the Feller continuity of the initial distribution $\{\nu_x\}$, we require that these measures satisfy the following uniform integration condition.

Assumption V.3.4. *There is a $\ell > 1$ such that*

$$\sup_{x\in\mathbb{T}^d}\int_{\mathbb{R}} e^{\ell\varphi(\theta)}\nu_x(\mathrm{d}\theta) < C. \tag{V.3.7}$$

The following large deviation principle is the main result of this section.

Theorem V.3.5. *Let the Assumption V.1.1, Assumption V.1.3, Assumption V.3.1 and Assumption V.3.4 hold. Then the family $\left\{\mu^N_{[0,T]}, P^N\right\}$ satisfies on $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\right)$ the large deviation principle with good rate function*

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) := \begin{cases}\int_0^T \left|\partial_t\mu_t - \left(\mathbb{L}_{\mu_t,.,.}\right)^*\mu_t\right|^2_{\mu_t}\mathrm{d}t + \mathsf{H}\left(\mu_0|\mathrm{d}x\otimes\zeta_x\otimes\nu_x\right) & \text{if } \mu_{[0,T]}\in\mathbb{A}\cap\mathscr{C}_{\varphi,\infty},\\ \infty & \text{otherwise,}\end{cases} \tag{V.3.8}$$

where the norm $|.|_{\mu_t}$ is defined in Definition V.1.10 and where

$$\mathbb{A} := \left\{\mu\in\mathscr{C}^L : \mu_{[0,T]} \text{ is absolutely continuous in the sense of Definition V.2.22}\right\}. \tag{V.3.9}$$

Moreover, the integral with respect to $\mathbb{T}^d\times\mathcal{W}$ and the supremum in the norm in $S_{\nu,\zeta}$ can be interchanged, i.e. $S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$, defined by

$$\int_0^T\int_{\mathbb{T}^d}\int_{\mathcal{W}}\left|\partial_t\mu_{t,x,w} - \left(\mathbb{L}_{\mu_t,x,w}\right)^*\mu_{t,x,w}\right|^2_{\mu_{t,x,w}}\mu_{0,x,\mathcal{W}}(\mathrm{d}w)\,\mathrm{d}x\mathrm{d}t + \mathsf{H}\left(\mu_0|\mathrm{d}x\otimes\zeta_x\otimes\nu_x\right) \tag{V.3.10}$$

if $\mu_{[0,T]}\in\mathbb{A}\cap\mathscr{C}_{\varphi,\infty}\cap\mathscr{C}^L$ and $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \infty$ otherwise.

To prove this theorem, we generalise the proof of the large deviation principle for the mean field model of [DG87], to the space and random environment dependent setting we consider here. Therefore, the structure of the proof of Theorem V.3.5 is similar to the structure of the corresponding proof in [DG87]. However, there are three main differences to [DG87] in the model we consider here. The main difference is that the drift coefficient b and the empirical process $\mu^N_{[0,T]}$ depend on $x\in\mathbb{T}^d$ and on the random environment $w\in\mathcal{W}$. Moreover, in [DG87] the spins take fixed initial values, whereas in the model we consider, the spins are initially randomly distributed. Last but not least, we show the large deviation principle on the space $\mathscr{C}$ (and not, as in [DG87], on $\mathscr{C}_{\varphi,\infty}$ with another topology than the subspace topology).

Due to these differences, changes are necessary in the proofs (compared to the approach in [DG87]). Many of these changes are of technical nature. We point out at the beginning of each proof of the partial results, how the proof differs from the corresponding proof in [DG87]. Then we state the proofs with emphasis on these necessary modifications. Of course we explain proofs and parts of proofs, that are new, completely.

The proof of Theorem V.3.5 is organised as follows.

1.) At first (Section V.3.1), we prove the large deviation principle for a system of independent spins (see Theorem V.3.9) and show that the rate function has the representation $S^I_{\nu,\zeta}$ (defined in (V.3.13)), that is similar to $S_{\nu,\zeta}$. We infer this large deviation principle from the generalised Sanov-type large deviation result derived in Section V.2.2. The rest of this Section V.3.1 is dedicated to showing that the rate function has the representation $S^I_{\nu,\zeta}$.

1.1.) To show the form of the rate function, we derive at first two different representations $S_{\nu,\zeta}^{I,1}$ and $S_{\nu,\zeta}^{I,2}$ of the rate function (Section V.3.1.1). For both representation we use the Sanov-type large deviation result derived in Section V.2.2. These proofs are formally almost equal to the corresponding proofs in [DG87]. The space and random environment dependency only leads to formal changes in the notation. However, the applied results of Section V.2.2 are different from the Sanov-type results used in [DG87], due to these new dependencies. Moreover, to be able to apply the Sanov type result, we show that the measures corresponding to the independent SDEs are Feller continuous.

1.2.) Next we show that $S_{\nu,\zeta}^{I,1}$ ($S_{\nu,\zeta}^{I,2}$) is an upper (lower) bound on the claimed form $S_{\nu,\zeta}^{I,\mathbb{T}^d}$ ($S_{\nu,\zeta}^{I}$) of the rate function (Section V.3.1.1).

In the proof of the upper bound (Section V.3.1.2.1), we generalise an approach used in [DPdH96], which is partially based on approaches of [Föl88] and [Bru93]. In contrast to [DPdH96], we consider the space dependency $x \in \mathbb{T}^d$ in addition to the random environment $w \in \mathcal{W}$.

Note that the proof of the lower bound given in [DPdH96] unfortunately has a gap and cannot be used. We give a proof of the lower bound in (Section V.3.1.2.2) that generalises the ideas used in [DG87]. Besides the usual formal changes (due to the space dependency, compared to [DG87]), we have to handle a new problem here. The proof requires the existence of a solution to a boundary value partial differential equation, which has to be continuous in the space variable $x \in \mathbb{T}^d$ and the environment variable $w \in \mathcal{W}$. This condition is obviously not needed in [DG87]. Therefore, we show in Section V.3.1.2.3, that there exist such a solution. This section and the proof are new.

1.3.) Finally, we derive another formula for $S_{\nu,\zeta}^{I}$. This is (again modulo changes due to the space dependency) similar to the corresponding proof in [DG87]. However, in [DG87] this formula is used to derive the large deviation upper bound. We do not use it in the proof of the large deviation upper bound, because it only bounds $S_{\nu,\zeta}^{I}$ (see the beginning of Section V.3.1.2.1 for more details). However, we need this result in Section V.3.2 to show that the rate function $S_{\nu,\zeta}$ is actually lower semi-continuous.

2.) In Section V.3.2, we infer from this large deviation principle for independent spins, a local large deviation principle for the interacting spin system (Theorem V.3.27). To do this, we define the independent generator $\mathbb{L}_{t,x,w}^{I} := \mathbb{L}_{\overline{\mu}_t,x,w}$ for fixed $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$. For the empirical process defined by the spins that evolve according to the Langevin dynamics with this generator, we know by Section V.3.1 the large deviation principle. From this principle, we infer the local large deviation principle under $\{P^N\}$, with the help of exponential bounds (that we show in Section V.3.2.1.3). This is a again a generalisation of [DG87]. Moreover, we give a new proof of the local large deviation principle around $\overline{\mu}_{[0,T]}$ that are not in $\mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$ (see Section V.3.2.3). This is necessary because we assume the continuity of b only on a subset of $\mathbb{M}_{\varphi,R}$ (see Assumption V.3.1 a.a)). Also with the mentioned exponential bounds, we prove the exponential tightness of $\left\{\mu_{[0,T]}^N, P^N\right\}$ (Theorem V.3.28). Finally, we infer from the exponential tightness and the local large deviation principle, the Theorem V.3.5.

We explain the steps and proofs in more details in the respective sections. We finish this section with a short discussion how the Assumption V.3.1 enter into this approach.

Remark V.3.6. *As explained in Remark V.3.3, we use Assumption V.3.1 a.b) and b.), to infer that the martingale problem for the generator $\mathbb{L}_{w^N}^N$ is well defined. Moreover, the Assumption V.3.1 b.) implies the exponential bounds in Section V.3.2. We get analogue results for the independent system defined by the generator $L_{t,x,w}^I$ due to Assumption V.3.1 a.a) and c.). Finally, we require Assumption V.3.1 d.) to show that $S_{\nu,\zeta}$ is a good rate function (here we need the sequences in $\mathscr{C}^L$) and to connect the independent system with the interacting system when deriving the local large deviation principle in Section V.3.2 (here we need the sequences of empirical processes).*

V.3.1 Independent spins

In this section we investigate the large deviation principle for the empirical process for systems of independent spins. As explained, we derive such a system by fixing the interaction between the spins in the SDE (V.1.1). Therefore, we consider a drift coefficient $b^I : [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \to \mathbb{R}$ here that depends not any more on the empirical measure but on the time.

For each $x \in \mathbb{T}^d$, $w \in \mathcal{W}$ and $t \in [0,T]$, define the time-dependent diffusion generator

$$\mathbb{L}^I_{t,x,w} := \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial^2\theta} + b^I(t,x,w,.)\frac{\partial}{\partial\theta}, \tag{V.3.11}$$

that corresponds to the SDE

$$\mathrm{d}\theta^x_t = b^I(t,x,w,\theta^x_t)\,\mathrm{d}t + \sigma \mathrm{d}B^x_t. \tag{V.3.12}$$

Let us assume that b^I is chosen such that the following assumptions are satisfied.

Assumption V.3.7. a.) *b^I is continuous on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$.*

b.) *For each $x \in \mathbb{T}^d$ and each $w \in \mathcal{W}$, the martingale problem for $\mathbb{L}^I_{t,x,w}$ is well posed , with corresponding family of probability measures $\left\{P^I_{t,x,w,\theta} \in \mathbb{M}_1(\mathsf{C}([t,T])), (t\theta) \in [0,T] \times \mathbb{R}\right\}$.*

We interpret $P^I_{t,x,w,\theta}$ as the measure of the path of the spins at the position $x \in \mathbb{T}^d$ with initial value $\theta \in \mathbb{R}$ at time $s \in [0,T]$ and fixed environment $w \in \mathcal{W}$, that evolves according to (V.3.11). We use the shorter notation $P^I_{x,w,\theta}$, when $t = 0$. By (V.3.11), the spins at position $x, y \in \mathbb{T}^d$ evolve mutually independent for $x \neq y$.

Notation V.3.8. *We write $P^I_{x,w}$ for the distribution of the path of the spin at the position $x \in \mathbb{T}^d$ with fixed environment $w \in \mathcal{W}$ and with initial distribution ν_x at time 0, i.e. $P^I_{x,w} = \int_{\mathbb{R}} P^I_{x,w,\theta}\nu_x(\mathrm{d}\theta)$.*

Similar to Notation V.1.9, we define $P^{I,N}_{\underline{w}^N}$ and $P^{I,N}$ (now with $P^I_{x,w,\theta}$) .

The following large deviation principle with the particular form of the rate function is the main result of this section.

Theorem V.3.9. *Let the Assumption V.1.1, Assumption V.1.3 and Assumption V.3.7 hold. Then the family $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies on $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$ the large deviation principle with good rate function*

$$S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) := \begin{cases} \int_0^T \left|\partial_t\mu_t - \left(\mathbb{L}^I_{t,.,.}\right)^*\mu_t\right|^2_{\mu_t} \mathrm{d}t + \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) & \text{if } \mu_{[0,T]} \in \mathbb{A}, \\ \infty & \text{otherwises,} \end{cases} \tag{V.3.13}$$

with $\mathbb{A}$ defined in (V.3.9).

Moreover $S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) = S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$, defined by

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathcal{W}} \left|\partial_t\mu_{t,x,w} - \left(\mathbb{L}^I_{t,x,w}\right)^*\mu_{t,x,w}\right|^2_{\mu_{t,x,w}} \mu_{0,x,\mathcal{W}}(\mathrm{d}w)\,\mathrm{d}x\mathrm{d}t + \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) \tag{V.3.14}$$

if $\mu_{[0,T]} \in \mathbb{A}$ and $S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \infty$ otherwise.

Remark V.3.10. *The rate functions $S_{\nu,\zeta}$ (of Theorem V.3.5) and $S^I_{\nu,\zeta}$ (of Theorem V.3.9) are related to each other. Set $\mathbb{L}^I_{t,x,w} = \mathbb{L}_{\overline{\mu}_t,x,w}$ for a $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$. And let $S^I_{\nu,\zeta}$ be the rate function defined by* (V.3.13) *corresponding to this generator. Then $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = S^I_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)$. We use this relation in Section V.3.2.*

Proof of Theorem V.3.9. It is easy to see that the family $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies the large deviation principle by Lemma V.2.7 and the contraction principle (see the proof of Lemma V.3.11).

The main difficulty of the proof of Theorem V.3.9 is to show that the rate function $S_{\nu,\zeta}$ has the form (V.3.13). To prove this, we generalise the approach used to prove Theorem 4.5 in [DG87] to the setting we consider here.

As in [DG87], we derive two different representations, $S^{I,1}_{\nu,\zeta}$ and $S^{I,2}_{\nu,\zeta}$, of the rate function and show that these provide a lower bound on $S^I_{\nu,\zeta}$ and an upper bound on $S^{I,\mathbb{T}^d}_{\nu,\zeta}$, respectively

To get the first representation, we use the contraction principle and transfer the LDP for $\left\{L^N, P^{I,N}\right\}$, that we get by Lemma V.2.7, to the LDP for $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$.

Lemma V.3.11 (compare to [DG87] Lemma 4.6 for the mean field case)**.** *The family* $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ *satisfies on* $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\right)$ *the large deviation principle with rate function*

$$S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \inf_{Q\in\mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])):\Pi(Q)_{[0,T]}=\mu_{[0,T]}} L^1_{\nu,\zeta}\left(Q\right), \tag{V.3.15}$$

for $\mu_{[0,T]}\in\mathscr{C}$*, with*

$$\begin{aligned} L^1_{\nu,\zeta}\left(Q\right) &= \int_{\mathbb{T}^d}\int_{\mathcal{W}} \mathsf{H}\left(Q_{x,w}\middle|P^I_{x,w}\right) Q_{x,\mathcal{W}}\left(\mathrm{d}w\right)\mathrm{d}x + \int_{\mathbb{T}^d}\mathsf{H}\left(Q_{x,\mathcal{W}}|\zeta_x\right)\mathrm{d}x \\ &= \sup_{f\in\mathsf{C}_b(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T]))}\left\{\int_{\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])} f\left(x,w,\theta_{[0,T]}\right) Q\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta_{[0,T]}\right)\right. \\ &\qquad \left. - \int_{\mathbb{T}^d}\log\left(\int_{\mathcal{W}}\int_{\mathsf{C}([0,T])} e^{f\left(x,w,\theta_{[0,T]}\right)} P^I_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right)\zeta_x\left(w\right)\right)\mathrm{d}x\right\}, \end{aligned} \tag{V.3.16}$$

for $Q\in\mathbb{M}^L_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$ *and* $L^1_{\nu,\zeta}\left(Q\right)=\infty$ *otherwise.*

In particular, $S^{I,1}\left(\mu_{[0,T]}\right)$ *is only finite if* $\mu_t\in\mathbb{M}^L_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$ *for all* $t\in[0,T]$ *and if* $\mu_{t,x,\mathcal{W}}=\mu_{0,x,\mathcal{W}}$ *for all* $t\in[0,T]$ *and almost all* $x\in\mathbb{T}^d$*.*

To derive the second representation, we define for $0\leq s\leq t\leq T$ the operator acting on $f\in\mathsf{C}_b\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$ by

$$U_{s,t}f\left(x,w,\theta\right) := \int_{\mathsf{C}([s,T])} f\left(x,w,\theta_t\right) P^I_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right). \tag{V.3.17}$$

With this operator we get the following representation of the rate function.

Lemma V.3.12 (compare to [DG87] Lemma 4.7 for the mean field case)**.** *The family* $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ *satisfies on* $\mathsf{C}\left([0,T], \mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)\right)$ *the large deviation principle with rate function*

$$S^{I,2}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \sup_{r\in\mathbb{N},0\leq t_1<\ldots<t_r\leq T} L^{t_1,\ldots,t_r}_{\nu,\zeta}\left(\mu_{t_1},\ldots,\mu_{t_r}\right) \text{ for } \mu_{[0,T]}\in\mathscr{C}, \tag{V.3.18}$$

where for $\mu_i\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$*,* $L^{t_1,\ldots,t_r}_{\nu,\zeta}\left(\mu_1,\ldots,\mu_r\right)$ *is defined by*

$$\begin{aligned} &\sup_{f\in\mathsf{C}^\infty_c(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})}\left\{\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\left(x,w,\theta\right)\mu_1 - \int_{\mathbb{T}^d}\log\left(\int_{\mathcal{W}\times\mathbb{R}} U_{0,t_1}e^f\left(x,w,\theta\right)\nu_x\left(\mathrm{d}\theta\right)\zeta_x\left(\mathrm{d}w\right)\right)\mathrm{d}x\right\} \\ &+\sum_{i=2}^r \sup_{f\in\mathsf{C}^\infty_c(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})}\left\{\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\left(x,w,\theta\right)\mu_i - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\log U_{t_{i-1},t_i}e^f\left(x,w,\theta\right)\mu_{i-1}\right\}, \end{aligned} \tag{V.3.19}$$

where the μ_i *integrate with respect to the variables* $\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta$*.*

Finally, we show that $S^I_{\nu,\zeta}$, respectively $S^{I,\mathbb{T}^d}_{\nu,\zeta}$, is bounded by these two rate functions.

Lemma V.3.13. *For all* $\mu_{[0,T]} \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$

$$S^{I,2}_{\nu,\zeta}\left(\mu_{[0,T]}\right) \leq S^{I}_{\nu,\zeta}\left(\mu_{[0,T]}\right) \leq S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right) \leq S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right). \tag{V.3.20}$$

Moreover, $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$ *implies that* $\mu_{[0,T]}$ *is weakly differentiable.*

From these three lemmas, we conclude the Theorem V.3.9 by the uniqueness of the rate function of large deviation principles.

We prove the lemmas in the following sections. □

V.3.1.1 Proof of the two representation of the rate function (Proof of Lemma V.3.11 and Lemma V.3.12)

Proof of Lemma V.3.11. We apply the Sanov type Lemma V.2.7 with $r = 1$, $Y = \mathsf{C}([0,T])$ to conclude that the family $\left\{L^N, P^{I,N}\right\}$ satisfies on $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ the large deviation principle with rate function $L_{\nu,\zeta}$. The Lemma V.2.7 requires Assumption V.1.1, Assumption V.1.3 and the following Feller continuity (see also the discussion in Section V.2.3):

Lemma V.3.14. *The Assumption V.3.7 implies that the family* $\left\{P^I_{x,w,\theta} : (x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right\}$ *is Feller continuous.*

Before we prove this lemma, we finish the proof of Lemma V.3.11.

The map Π (defined in Definition V.2.26) is continuous (Lemma V.2.28). It maps each probability measure on $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$ to a continuous measure valued trajectories in $\mathscr{C}$. Moreover, for each fixed vector $\underline{\theta}^N_{[0,T]}$ and each $\underline{w}^N$, the image of the corresponding empirical path measure L^N under Π is the corresponding empirical process $\mu^N_{[0,T]}$. Therefore, the contraction principle implies the large deviation principle for $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ with the rate function $S^{I,1}_{\nu,\zeta}$.

The right hand side of (V.3.15) is only finite if there is a $Q \in \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ with $\Pi(Q)_{[0,T]} = \mu_{[0,T]}$. This implies that $\mu_t \in \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ for all $t \in [0,T]$, and that $\mu_{t,x,\mathcal{W}} = \mu_{0,x,\mathcal{W}}$ for all $t \in [0,T]$ and almost all $x \in \mathbb{T}^d$, by Lemma V.2.30. □

Proof of Lemma V.3.14. Fix an arbitrary convergent sequence $\left(x^{(n)}, w^{(n)}, \theta^{(n)}\right) \to (x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. We define $a_n := a \equiv \sigma$ and $b^{I,(n)}(t,\eta) := b^I\left(t, w^{(n)}, x^{(n)}, \eta\right)$, $b^I(t,\eta) = b^I(t,x,w,\eta)$ for $(t,\eta) \in [0,T] \times \mathbb{R}$. These functions are continuous by Assumption V.3.7 a.). Moreover, we know, by Assumption V.3.7 b.), that $P^I_{x^{(n)},w^{(n)},\theta^{(n)}}$ is the solution to the martingale problem corresponding to the drift coefficient $b^{I,(n)}$.

The Theorem 11.1.4 in [SV79] implies that the solutions to the martingale problem $P^I_{x^{(n)},w^{(n)},\theta^{(n)}}$ converge weakly to $P^I_{x,w,\theta}$. The conditions of Theorem 11.1.4 of [SV79] are satisfied by Assumption V.3.7. Therefore, $P^I_{x,w,\theta}$ is Feller continuous. □

Proof of Lemma V.3.12. This proof is a generalisation of the proof of [DG87] Lemma 4.6 and we use the ideas of this proof. At first we prove a LDP for the finite dimensional distributions of $\left\{\mu^N_{[0,T]}\right\}$ (i.e. the distribution of $\mu^N_{[0,T]}$ at a finite number of times) and in a second step we transfer this LDP to the LDP for $\left\{\mu^N_{[0,T]}\right\}$ by using the projective limit approach.

Step 1: LDP for the finite dimensional distributions of $P^{I,N}$:

Fix $N \geq 1$, $r \in \mathbb{N}$, $0 = t_0 \leq t_1 < .... < t_r \leq T$. We define the random elements

$$\mu^N_{t_1,...,t_r} := \left(\mu^N_{t_1}, ..., \mu^N_{t_r}\right) \in \left(\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)\right)^r. \tag{V.3.21}$$

Then $\mu^N_{t_1,...,t_r}$ depends only on the spins at the times $t_1, ..., t_r$, i.e. on $\underline{\theta}^N_{t_1}, ..., \underline{\theta}^N_{t_r}$ and not any more on the whole path.

By Lemma V.2.7 (with $Y_1 = ... = Y_r = \mathbb{R}$), the family $\{\mu^N_{t_1,...,t_r}, P^{I,N}\}$ satisfies the large deviation principle on $\left(\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)^r$ with rate function

$$L^{t_1,...,t_r}_{\mu_0}(\mu_1, ..., \mu_r) = \sup_{f_1,...,f_r \in \mathsf{C}_b(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} \left[\sum_{\ell=1}^r \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f_\ell(x, w, \theta)\, \mu_\ell(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) - H(f_1, ..., f_r)\right] \tag{V.3.22}$$

for $\mu_\ell \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$, where

$$H(f_1, ..., f_r) := \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}} \int_{\mathsf{C}([0,T])} e^{\sum_{\ell=1}^r f_\ell\left(x,w,\theta_{t_\ell}\right)} P^I_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right) \zeta_x(\mathrm{d}w)\right) \mathrm{d}x. \tag{V.3.23}$$

To show that this function coincides with (V.3.19), we first get by the Markov property of $\{P_{t,x,w,\theta}\}$ that

$$\begin{aligned} H(f_1, ..., f_r) &= \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}} \int_{\mathsf{C}([0,T])} \int_{\mathsf{C}([0,T])} e^{f_r(y,w,\theta_{t_r})} P^I_{t_{r-1},x,w,\theta_{t_{r-1}}}\left(\mathrm{d}\theta_{[0,T]}\right)\right. \\ &\qquad\qquad \left. e^{\sum_{\ell=1}^{r-1} f_\ell\left(y,w,\theta_{t_\ell}\right)} P^I_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right) \zeta_x(\mathrm{d}w)\right) \mathrm{d}x \\ &= \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}} \int_{\mathbb{R}} U_{t_0,t_1}\left(e^{f_1} ... U_{t_{r-1},t_r} e^{f_r}\right)(x, w, \theta)\, \nu_x(\mathrm{d}\theta)\, \zeta_x(\mathrm{d}w)\right) \mathrm{d}x. \end{aligned} \tag{V.3.24}$$

Now performing formally (by pushing through the space dependency) the same calculation as Dawson and Gärtner in [DG87] page 275, we can transfer the right hand side of (V.3.24) to the right hand side of (V.3.19) with the supremum taken over all $f \in \mathsf{C}_b\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$. But the operators $U_{s,t}$ are continuous linear operators, hence the supremum over $\mathsf{C}^\infty_c\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ equals the supremum over $\mathsf{C}_b\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$.

Step 2: Transfer of the LDP for $\{\mu^N_{t_1,...,t_r}\}$ to the LDP for $\left\{\mu^N_{[0,T]}\right\}$:

An LDP for $\left\{\mu^N_{[0,T]}\right\}$ follows from the LDP for the finite dimensional marginals of the first step, by the projective limit approach. In [DG87] on page 276 this is done for the mean field model. This proof can be almost directly used in the setting we consider here. To have a complete picture, we state nevertheless the idea here.

To have a projective system corresponding to $\left(\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)^r$ with order relation $\subseteq$ for $\{t_1, ..., t_r\}$, we embed the space $\mathscr{C}$ into $\mathbb{M}^{[0,T]}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right) := \{f : [0, T] \to \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\}$ furnished with the product topology.

We know by Lemma V.3.11 already that $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies the large deviation principle on $\mathscr{C}$. Then $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies also the large deviation principle on $\mathbb{M}^{[0,T]}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$, by the contraction principle. We denote its rate function by $\widehat{S}^2$. But this LDP can also be identified with the projective limit of the finite dimensional LDPs derived above. Hence by the projective limit theorem ([DZ98] Theorem 4.6.1, [DG87] Theorem 3.3) we see that $\widehat{S}^2$ has the desired form (V.3.18) on $\mathbb{M}^{[0,T]}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$.

Moreover, $\widehat{S}^2$ is infinite on $\mathbb{M}^{[0,T]}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right) \setminus \mathscr{C}$ and the random variables $\mu^N_{[0,T]}$ under $P^{I,N}$ are concentrated on $\mathscr{C}$. Hence we can reduce the LDP to an LDP on $\mathscr{C}$ by Lemma 4.1.5 (b) in [DZ98]. This finishes the proof of Lemma V.3.12. □

V.3.1.2 Coincidence of the two representations with $S^I_{\nu,\zeta}$ (proof of Lemma V.3.13)

In this section we prove Lemma V.3.13. Therefore, we show at first an upper bound on $S^{I,\mathbb{T}^d}_{\nu,\zeta}$ and then a lower bound on $S^I_{\nu,\zeta}$.

For the upper bound (Section V.3.1.2.1) we generalise an approach used in [DPdH96], which is partially based on approaches of [Föl88] and [Bru93]. In contrast to [DPdH96], we consider the space dependency $x \in \mathbb{T}^d$ in addition to the random environment $w \in \mathcal{W}$. Moreover, we look at an independent system, whereas in [DPdH96] an interacting system is considered (see also Lemma V.6.3, where we use this approach also for an interacting systems).

The proof that we give for the lower bound (see Section V.3.1.2.2) is a generalisation of Section 4.4 in [DG87] to the model we consider here. We require in the proof the existence and uniqueness of a solution to a PDE. In contrast to [DG87], this solution has to be continuous in the space variable $x \in \mathbb{T}^d$ and the environment variable $w \in \mathcal{W}$. We show the existence and this regularity of a solution in the completely new Section V.3.1.2.3. The rest of the proof in Section V.3.1.2.2 generalises the proof in [DG87]. Moreover, we correct minor mistakes of the proof in [DG87].

V.3.1.2.1 Upper bound on $S^{I,\mathbb{T}^d}_{\nu,\zeta}$

We show in this section that $S^{I,\mathbb{T}^d}_{\nu,\zeta} \le S^{I,1}_{\nu,\zeta}$. As mentioned, the proof we state here, is based on an approach in [DPdH96].

Lemma V.3.15. *If $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$ for a $\mu_{[0,T]} \in \mathscr{C}$, then*

$$S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = S^{I,\mathbb{T}^d}_{\nu,\zeta}\left(\mu_{[0,T]}\right), \tag{V.3.25}$$

and $t \mapsto \mu_{t,x,w}$ is weakly differentiable for almost all $(x,w) \in \mathbb{T}^d \times \mathcal{W}$.

In particular $S^{I,1}_{\nu,\zeta} \ge S^{I,\mathbb{T}^d}_{\nu,\zeta} \ge S^I_{\nu,\zeta}$.

Remark V.3.16. *Note that the lemma only states the equality of $S^{I,1}_{\nu,\zeta}$ and $S^{I,\mathbb{T}^d}_{\nu,\zeta}$, when $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ is finite, i.e. when there is a $Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ with $L^1_{\nu,\zeta}(Q) < \infty$ and $\Pi(Q)_{[0,T]} = \mu_{[0,T]}$. In [Föl88], $\mu_{[0,T]}$ that satisfy this condition are called admissible.*

Therefore, this result is not enough to show the claimed equality in Theorem V.3.9 and we are bound to prove also a lower bound (in Section V.3.1.2.2).

Proof of Lemma V.3.15. Fix a $\mu_{[0,T]} \in \mathscr{C}$ with $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$.

The idea of this proof is based on the steps 1-3 of the proof of Theorem 3 in [DPdH96], that are partly based on [Föl88] and [Bru93]. The proof is organised as follows. We show in Step 1, that there is a $\overline{Q} \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$, which is a minimizer of the right hand side of (V.3.15) for $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$. In Step 2 we derive another representation of $S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$, by applying a result of [Föl88]. Finally (in Step 3) we show, that the new representation of $S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$ equals $S^{I,\mathbb{T}^d}_{\nu,\zeta}$. We use that $\mu_{t,x,w}$ is the evolution of the time marginal of $\overline{Q}_{x,w}$ and a weak solution of a Fokker-Planck equation.

Step 1: There is a $\overline{Q}$ with $L^1_{\nu,\zeta}\left(\overline{Q}\right) = S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$ and nice properties:

We restrict the infimum in (V.3.15) to the set

$$A_{\mu,C} := \left\{Q : \Pi(Q)_{[0,T]} = \mu_{[0,T]}\right\} \cap \left\{Q : L^1_{\nu,\zeta}(Q) \le C\right\} \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big), \tag{V.3.26}$$

for a $C > 0$ large enough. This set is non empty and compact (the last set is compact because $L^1_{\nu,\zeta}$ is a good rate function and the first set is closed). Hence by the lower semi continuity of $L^1_{\nu,\zeta}$, there exists a $\overline{Q} \in A_{\mu,C}$ that is a minimiser of $L^1_{\nu,\zeta}$ in $A_{\mu,C}$. This implies that $L^1_{\nu,\zeta}\left(\overline{Q}\right) = S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$.

Therefore, $L^1_{\nu,\zeta}\left(\overline{Q}\right) = \mathsf{H}\left(\overline{Q}\middle|\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P^I_{x,w}\right) < \infty$ and $\overline{Q} \in \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. Let us write $\overline{Q} = \mathrm{d}x \otimes \overline{Q}_x$ for $\overline{Q}_x \in \mathbb{M}_1(\mathcal{W} \times \mathsf{C}([0,T]))$ and $Q_x = \overline{Q}_{x,\mathcal{W}} \otimes \overline{Q}_{x,w}$ for $\overline{Q}_{x,w} \in \mathbb{M}_1(\mathsf{C}([0,T]))$, $\overline{Q}_{x,w} \in \mathbb{M}_1(\mathcal{W})$. Then for almost all $x \in \mathbb{T}^d$ and $\overline{Q}_{x,\mathcal{W}}$-almost all $w \in \mathcal{W}$, $\mathsf{H}\left(\overline{Q}_{x,w}\middle|P^I_{x,w}\right) < \infty$, $\mathsf{H}\left(\overline{Q}_{x,\mathcal{W}}\middle|\zeta_x\right) < \infty$ and $\Pi\left(\overline{Q}_{x,w}\right)_t = \mu_{t,x,w}$. Moreover, $\Pi\left(\overline{Q}\right)_t = \mathrm{d}x \otimes \overline{Q}_{x,\mathcal{W}} \otimes \Pi\left(\overline{Q}_{x,w}\right)_t = \mu_t \in \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$.

Step 2: Another representation of $S^{I,1}_{\mu}\left(\mu_{[0,T]}\right)$:

By these properties, we get, for almost all $x \in \mathbb{T}^d$, as in [Föl88] Theorem II.1.31 and Remark II.1.3 (see also [LS01] Chapter 7 (in particular Theorem 7.11)), that there is a map $b^{x,w} : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $\overline{Q}_{x,w}$ is the law of $\theta^{x,w}_{[0,T]}$ described by the following SDE

$$\mathrm{d}\theta^{x,w}_t = \left(\sigma b^{x,w}\left(t, \theta^{x,w}_t\right) - b^I\left(t, x, w, \theta^{x,w}_t\right)\right)\mathrm{d}t + \sigma \mathrm{d}B^{\overline{Q}_{x,w}}_t, \tag{V.3.27}$$

with $\theta^{x,w}_0 \sim \mu_{0,x,w}$ and

$$\frac{\mathrm{d}\overline{Q}_{x,w}}{\mathrm{d}P^I_{x,w}} = e^{\int_0^T b^{x,w}(t,.)\mathrm{d}B^{\overline{Q}_{x,w}}_t + \frac{1}{2}\int_0^T b^{x,w}(t,.)^2 \mathrm{d}t} \frac{\mathrm{d}\mu_{0,x,w}}{\mathrm{d}\nu_x}. \tag{V.3.28}$$

Here $B^{\overline{Q}_{x,w}}_t$ is a Wiener process under $\overline{Q}_{x,w}$. Inserting this derivative in the relative entropy, we get

$$\begin{aligned}\mathsf{H}\left(\overline{Q}_{x,w}\middle|P^I_{x,w}\right) - \mathsf{H}\left(\mu_{0,x,w}|\nu_x\right) &= \frac{1}{2}\int_{\mathsf{C}([0,T])}\int_0^T \left(b^{x,w}\left(t,\theta_t\right)\right)^2 \mathrm{d}t\, Q_{x,w}\left(\mathrm{d}\theta_{[0,T]}\right)\\ &= \frac{1}{2}\int_0^T\int_{\mathbb{R}} \left(b^{x,w}\left(t,\theta_t\right)\right)^2 \mu_{t,x,w}\left(\mathrm{d}\theta\right)\, \mathrm{d}t.\end{aligned} \tag{V.3.29}$$

Integrating over $\mu_{0,x,\mathcal{W}} = \overline{Q}_{x,\mathcal{W}} \in \mathbb{M}_1(\mathcal{W})$ and then over $x \in \mathbb{T}^d$ implies that

$$\begin{aligned}S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right) = L^1_{\nu,\zeta}\left(\overline{Q}\right) =& \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathcal{W}}\int_0^T\int_{\mathbb{R}} \left(b^{x,w}\left(t,\theta\right)\right)^2 \mu_{t,x,w}\left(\mathrm{d}\theta\right)\mathrm{d}t\, \mu_{0,x,\mathcal{W}}\left(\mathrm{d}w\right)\mathrm{d}x\\ &+ \int_{\mathbb{T}^d}\int_{\mathcal{W}} \mathsf{H}\left(\mu_{0,x,w}|\nu_x\right)\mu_{0,x,\mathcal{W}}\left(\mathrm{d}w\right)\mathrm{d}x + \int_{\mathbb{T}^d} \mathsf{H}\left(\mu_{0,x,\mathcal{W}}|\zeta_x\right)\mathrm{d}x.\end{aligned} \tag{V.3.30}$$

Step 3: The new representation of $S^{I,1}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ equals $S^{I,\mathbb{T}^d}_{\nu,\zeta}$:

We show now that for almost all $t \in [0,T]$, almost all $x \in \mathbb{T}^d$ and $Q_{x,\mathcal{W}}$-almost all $w \in \mathcal{W}$

$$\frac{1}{2}\int_{\mathbb{R}} \left(b^{x,w}\left(t,\theta\right)\right)^2 \mu_{t,x,w}\left(\mathrm{d}\theta\right) = \left|\partial_t \mu_{t,x,w} - \left(\mathbb{L}^I_{t,x,w}\right)^* \mu_{t,x,w}\right|^2_{\mu_{t,x,w}}, \tag{V.3.31}$$

with $\mathbb{L}^I_{t,x,w}$ defined in (V.3.11).

The equation (V.3.31) can be shown as in the Steps 2 and 3 in the proof of Theorem 3 in [DPdH96]. Therefore, we sketch the proof here only.

The measure $\overline{Q}_{x,w}$ is the law of (V.3.27) and by construction $\mu_{t,x,w}$ is the evolution of the time marginal of this law. Hence $\mu_{t,x,w}$ is a weak solution of the Fokker-Plank equation

$$\partial_t \mu_{t,x,w} = -\partial_\theta\left(\left[\sigma b^{x,w}\left(t,.\right) - b^I\left(t,x,w,.\right)\right]\mu_{t,x,w}\right) + \frac{\sigma^2}{2}\partial^2_{\theta^2}\mu_{t,x,w}. \tag{V.3.32}$$

From this, we subtract now the generator $\left(\mathbb{L}^I_{t,x,w}\right)^*$

$$\partial_t \mu_{t,x,w} - \left(\mathbb{L}^I_{t,x,w}\right)^* \mu_{t,x,w} = -\partial_\theta\left(\sigma b^{x,w}\left(t,.\right)\mu_{t,x,w}\right), \tag{V.3.33}$$

what leads to

$$\begin{aligned}\left|\partial_t\mu_{t,x,w}-\left(\mathbb{L}^I_{t,x,w}\right)^*\mu_{t,x,w}\right|^2_{\mu_{t,x,w}}&=\frac{1}{2}\sup_{f\in\mathbb{D}_{\mu_{t,x,w}}}\frac{\left|\int_{\mathbb{R}}\sigma b^{x,w}\left(t,\theta\right)\partial_\theta f\left(\theta\right)\mu_{t,x,w}\left(\mathrm{d}\theta\right)\right|^2}{\sigma^2\int_{\mathbb{R}}\left(\partial_\theta f\left(\theta\right)\right)^2\mu_{t,x,w}\left(\mathrm{d}\theta\right)}\\&\leq\frac{1}{2}\int_{\mathbb{R}}\left(b^{x,w}\left(t,\theta\right)\right)^2\mu_{t,x,w}\left(\mathrm{d}\theta\right),\end{aligned}\tag{V.3.34}$$

with $\mathbb{D}_{\mu_{t,x,w}}:=\left\{f\in\mathsf{C}_c^\infty(\mathbb{R}):\int_{\mathbb{R}}\left(\partial_\theta f\left(\theta\right)\right)^2\mu_{t,x,w}\left(\mathrm{d}\theta\right)>0\right\}$.

To conclude (V.3.31), we have to show that the last inequality is actually an equality. This can be done as in Step 3 of the proof of Theorem 3 in [DPdH96], by showing that $\left\{\partial_\theta f:f\in\mathbb{D}_{\mu_{t,x,w}}\right\}$ is dense in $L^2\left(\mathbb{R},\mu_{t,x,w}\right)$. Then we take a approximating sequence $f_n\in\mathbb{D}_{\mu_{t,x,w}}$, $\partial_\theta f_n\to b_t^{x,w}$ and get the corresponding lower bound. □

Remark V.3.17. *Instead of Lemma V.3.15, we could also show similarly as in Lemma 4.9 in [DG87], that $S^{I,1}_{\nu,\zeta}\geq S^I_{\nu,\zeta}$, by using a representation of $S^I_{\nu,\zeta}$, that we derive in Lemma V.3.26. This would require some changes (compared to [DG87]), due to the space dependency and the initial distribution of the spins that we consider here. However, the advantage of Lemma V.3.15 is that it bounds also $S^{I,\mathbb{T}^d}_{\nu,\zeta}$. This could be archived also by a variation of Lemma 4.9 in [DG87] and a variation of Lemma V.3.26, i.e. by moving the integral with respect to $x\in\mathbb{T}^d$ out of the supremum in* (V.3.66)*. However, using this approach, one has to be careful whether functions are integrable with respect to $x\in\mathbb{T}^d$ and $w\in\mathcal{W}$.*

V.3.1.2.2 Lower bound on $S^I_{\nu,\zeta}$

We prove in this section the following lower bound on $S^I_{\nu,\zeta}$. The proof is a generalisation of the corresponding proof in [DG87]. The most important difference to the original proof is that we derive for solutions of the arising PDE (see the proof of Lemma V.3.19) also regularity in the space variable and the random environment variable.

Lemma V.3.18 (compare to Lemma 4.10 in [DG87] for the mean field case)**.** $S^{I,2}_{\nu,\zeta}\leq S^I_{\nu,\zeta}$.

Proof. It suffices to show, by (V.3.19), (V.3.13) and the second formula of the norm in Definition V.1.10, that

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}f\left(x,w,\theta\right)\mu_t\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)-\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\log U_{s,t}e^f\left(x,w,\theta\right)\mu_s\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\\&\leq\int_s^t\sup_{h\in\mathsf{C}_c^\infty(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})}\left(\left\langle\partial_u\mu_u,h\right\rangle-\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}}\mathbb{L}^I_{u,x}h\left(x,w,\theta\right)+\frac{\sigma^2}{2}\left(\partial_\theta h\left(x,w,\theta\right)\right)^2\mu_u\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\right)\mathrm{d}u\end{aligned}\tag{V.3.35}$$

for all $f\in\mathsf{C}_c^\infty\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$, $0\leq s<t\leq T$, $\nu\in\mathbb{M}_1^L\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$ and $\mu_{[0,T]}\in\mathscr{C}$ with $S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right)<\infty$. Indeed, by (V.3.35), we bound separately each summand of the sum on the right hand side of (V.3.19). For the first summand on the right hand side of (V.3.19), we have to differentiate between the cases $t_1=0$ and $t_1>0$ in the supremum in (V.3.18). If $t_1>0$, then apply first the Jensen inequality to the this summand of the right hand side of (V.3.19) before using (V.3.35). In the case $t_1=0$, the first summand on the right hand side of (V.3.19) equals to $\mathsf{H}\left(\mu_0|\mathrm{d}x\otimes\zeta_x\otimes\nu_x\right)$, which appears in formula (V.3.13) of $S^I_{\nu,\zeta}$ (by a similar estimate as used in the proof of Lemma V.2.8).

An easy heuristic proof for the mean-field counterpart to (V.3.35) is given in [DG87] on page 282. We refer to this heuristic to get an idea of the following proof. However, in particular due to the unbounded domain of the spins, problems arise such that the heuristic does not make sense.

However, we can prove (V.3.35) when restricting the analysis to compact sets (see Lemma V.3.19) and infer from this (V.3.35). Therefore, we define a new semi group corresponding to the diffusion processes which is killed when leaving the ball $B_R = \{(x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} : |\theta| < R\}$ by

$$U^R_{s,t} f(x,w,\theta) = \int_{\mathsf{C}([s,T])} f(x,w,\theta_t)\, \mathbb{1}_{\tau^s_R > t} P_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right), \tag{V.3.36}$$

for $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$, with $\tau^s_R\left(\theta_{[s,T]}\right) = \min\{t \in [s,T] : |\theta_t| \geq R\}$.

Lemma V.3.19 (compare to Lemma 4.11 in [DG87]). *Given a $\mu_{[0,T]} \in \mathscr{C}$ with $S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$, then for all $R > 0$, $0 \leq s < t \leq T$ and $f \in \mathsf{C}^\infty_c\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ with $f \leq 0$ and* supp $(f) \subset B_R$.

$$\begin{aligned}&\int f(x,w,\theta)\, \mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int \log\left[1 + U^R_{s,t}\left(e^f - 1\right)\right](x,w,\theta)\, \mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\\ &\leq \int_s^t \sup_{h \in \mathsf{C}^\infty_c(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})} \left(\langle \partial_u \mu_u, h\rangle - \int \mathbb{L}^I_{u,x,w} h(x,w,\theta) + \frac{\sigma^2}{2}\left(\partial_\theta h(x,w,\theta)\right)^2 \mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\right)\mathrm{d}u,\end{aligned} \tag{V.3.37}$$

where the integrals without bounds integrate over the space $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$.

This lemma implies (V.3.35) by the same approximation approach given after Lemma 4.11 in [DG87]. Hence once we prove Lemma V.3.19, the proof of Lemma V.3.18 is finished. □

Proof of Lemma V.3.19. In this proof we generalise the proof of Lemma 4.11 in [DG87] to the model considered here. In contrast to [DG87] we do not assume in Theorem V.3.9, that the drift coefficient b is locally Hölder continuous. However, we need this assumption to get the existence of a solution to a PDE (see Step 1.1). Therefore, we assume at first (Step 1), that b^I is Hölder continuous in time and spin. Finally, in Step 2, we show how to generalise this to general drift coefficients.

Fix an $R > 0$, an arbitrary $f \in \mathsf{C}^\infty_c\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ with $f \leq 0$ and supp $(f) \subset B_R$ and arbitrary $0 \leq s < t \leq T$. Let $\mathcal{W}_f \subset \mathcal{W}$ be a compact subset such that the projection on $\mathcal{W}$ of the support of f is contained in $\mathcal{W}_f$

Step 1: The drift coefficient is Hölder continuous:

Let us assume that b^I is $\frac{1}{4}$-Hölder continuous in time and $\frac{1}{2}$-Hölder continuous in $\theta \in B_R$ on the subset $[0,T] \times \mathbb{T}^d \times \mathcal{W}_f \times B_R$. Moreover, let b^I be continuous on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. To generalise the ideas of [DG87] to the space and random environment dependent model, we need in particular the existence of a unique solution to an initial boundary value problem. This solution has to be moreover continuous in the space variable $x \in \mathbb{T}^d$ and in the random environment variable. We prove the existence and uniqueness of such a solution in Theorem V.3.23. We follow the lines of the proof in [DG87] with focus on the extensions needed to treat the space and random environment dependency.

Step 1.1: Construction of a (non smooth) function that solves a PDE:

By Theorem V.3.23, there is a unique classical solution g^* to the terminal boundary value problem

$$\begin{aligned}\partial_s g(s,x,w,\theta) &= -\mathbb{L}^I_{s,x,w} g(s,x,w,\theta) && (s,x,w,\theta) \in [0,t) \times \mathbb{T}^d \times \mathcal{W}_f \times B_R,\\ g(t,x,w,\theta) &= e^{f(x,w,\theta)} - 1 && (x,w,\theta) \in \mathbb{T}^d \times \mathcal{W}_f \times B_R,\\ g(s,x,w,\theta) &= 0 && (s,x,w,\theta) \in [0,t) \times \mathbb{T}^d \times \mathcal{W}_f \times \partial B_R.\end{aligned} \tag{V.3.38}$$

This implies that $g^*(s,x,w,\theta) = 0$ for $(s,x,w,\theta) \in [0,t] \times \mathbb{T}^d \times \partial\mathcal{W}_f \times B_R$. We define g^* to be zero for $w \notin \mathcal{W}_f$ or $\theta \notin B_R$.

The g^* satisfies for $(s,x,w,\theta) \in [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$

$$\begin{aligned} g^*(s,x,w,\theta) &= \int_{\mathsf{C}([s,T])} g^*(t \wedge \tau_R, x, w, \theta_{t\wedge\tau_R}) P_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right) \\ &= \int_{\mathsf{C}([s,T])} \left(e^{f(x,w,\theta_t)} - 1\right) \mathbb{1}_{\tau_R > t} P_{s,x,w,\theta}\left(\mathrm{d}\theta_{[s,T]}\right) = U^R_{s,t}\left(e^f - 1\right)(x,w,\theta). \end{aligned} \tag{V.3.39}$$

The first equality is true because $g^*(t \wedge \tau_R, x, w, \theta(t \wedge \tau_R))$ is a $P_{s,x,w,\theta}$ martingale for all (s,x,w,θ) in $[0,t] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ by Assumption V.3.7 b.). The next equality is due to the boundary and the initial condition in (V.3.38), respectively the chosen continuation of g^*. Note that the equality of g^* and the third representation is the corresponding Feynman-Kac formula (for fixed $(x,w) \in \mathbb{T}^d \times \mathcal{W}_f$).

Define the function $h^* := \log(g^* + 1)$. This function solves

$$\begin{aligned} &\partial_t h = -\mathbb{L}^I_{t,x,w} h - \frac{\sigma^2}{2}(\partial_\theta h)^2 \qquad \text{on } [0,T] \times \mathbb{T}^d \times \mathcal{W}_f \times B_R \qquad \text{and} \\ &h(t,.,.,.)\big|_{\mathbb{T}^d \times \mathcal{W} \times B_R} = f(.,.,.) \qquad \text{and} \qquad h\big|_{\partial B_R} = 0. \end{aligned} \tag{V.3.40}$$

If we could use the function h on the right hand side of (V.3.37), then the integration by parts Lemma V.2.24 would prove Lemma V.3.19. Unfortunately h^* is not in $\mathsf{C}^\infty_c\left([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$. By its construction and the compactness of f, the support of g^* and thus of h^* is compact, but h^* is not smooth.

Step 1.2: Smoothing of g^*:

The last part of the proof consists of approaching g^* with smooth functions g_ϵ, defined by

$$g_\epsilon := k_\epsilon *_{x,w,\theta} g^*, \tag{V.3.41}$$

with $k_\epsilon(x,w,\theta) = k^1_\epsilon(x) k^2_\epsilon(w) k^3_\epsilon(\theta)$. Here k^1_ϵ is a Dirac sequence (approximation to the identity) in $\mathbb{T}^d$ such that $k^1_\epsilon(x) = \epsilon^{-d} k^1\left(\epsilon^{-1}x\right)$ and $k^1 \in \mathsf{C}^\infty_c\left(\mathbb{T}^d\right)$, $k^1 \geq 0$ and $\int_{\mathbb{T}^d} k^1(x)\,\mathrm{d}x = 1$. Analogue we define k^2_ϵ and k^3_ϵ as a Dirac sequence on $\mathcal{W}$ and $\mathbb{R}$ respectively.

Then $g_\epsilon \in \mathsf{C}^\infty_c\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ but it does not satisfy any more (V.3.38) and

$$h_\epsilon := \log(1 + g_\epsilon) \tag{V.3.42}$$

does not satisfy any more (V.3.40). Therefore, we can not use directly the integration by parts Lemma V.2.24 to show (V.3.37).

Step 1.3: Smoothed function almost satisfies (V.3.37)**:**

Nevertheless, we prove in the following that h_ϵ used on the right hand side of (V.3.37) (instead of the supremum) almost satisfies (V.3.37), with an error that vanishes as $\epsilon \to 0$.

Indeed, by the integration by parts Lemma V.2.24

$$\begin{aligned} \text{Ⓛ} &:= \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} h_\epsilon(t,x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} h_\epsilon(s,x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \\ &= \int_s^t \langle \partial_u \mu_u, h_\epsilon(u) \rangle + \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \partial_u h_\epsilon(u,x,w,\theta)\,\mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \quad \mathrm{d}u \\ &= \int_s^t \langle \partial_u \mu_u, h_\epsilon(u) \rangle - \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \mathbb{L}^I_{u,x,w} h_\epsilon(u,x,\theta) + \frac{\sigma^2}{2}\left|\partial_\theta h_\epsilon(u,x,w,\theta)\right|^2 \mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \\ &\quad + \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \frac{\left(\partial_u + \mathbb{L}^I_{u,x,w}\right) g_\epsilon(u,x,w,\theta)}{1 + g_\epsilon(u,x,w,\theta)} \mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \quad \mathrm{d}u =: \text{(R1)} - \text{(R2)} + \text{(R3)}, \end{aligned} \tag{V.3.43}$$

because $\partial_u h_\epsilon = \frac{\partial_u g_\epsilon}{1+g_\epsilon}$ and $\mathbb{L}^I_{u,x,w} h_\epsilon = \frac{\mathbb{L}^I_{u,x,w} g_\epsilon}{1+g_\epsilon} - \frac{\sigma^2}{2} |\partial_\theta h_\epsilon|^2$.

The Ⓛ converges to the left hand side of (V.3.37), because $g_\epsilon(s) \to g^*(s)$ uniformly on $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. Indeed

$$|g_\epsilon(s,x,w,\theta) - g^*(s,x,w,\theta)| \le \sup_{(y,\eta)\in \mathrm{supp}\{k_\epsilon\}} |g^*(s,x+y,w,\theta+\eta) - g^*(s,x,w,\theta)|, \qquad \text{(V.3.44)}$$

and $g^*(s)$ is uniformly continuous (as a continuous function with compact support). Therefore, $h_\epsilon(t) \to f$ and $h_\epsilon(s) \to \log(1+g^*(s))$ uniformly.

The integrals (R1) and (R2) are smaller or equal to the right hand side of (V.3.37). We interpret (R3) as an error and show in the next step that it can be bounded from a above by a vanishing function.

Step 1.4: A vanishing upper bound on (R3):

By the following lemma we get a vanishing upper bound on the last integral (R3) of (V.3.43).

Lemma V.3.20 (compare to Lemma 4.12 in [DG87]). *For $\epsilon > 0$ small enough, there exists a continuous function r_ϵ on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, such that*

$$\left(\partial_u + \mathbb{L}^I_{u,x,w}\right) g_\epsilon(u,x,w,\theta) \le r_\epsilon(u,x,w,\theta) \qquad \text{for } (u,x,w,\theta) \in [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \qquad \text{(V.3.45)}$$

and $r_\epsilon \to 0$ uniformly on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ for $\epsilon \to 0$.

We state the proof of this lemma after we have finished the proof of Lemma V.3.19. By Lemma V.3.20

$$\text{(R3)} \le \int_s^t \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \frac{r_\epsilon(u,x,w,\theta)}{1+g_\epsilon(u,x,w,\theta)} \mu_u(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\, \mathrm{d}u. \qquad \text{(V.3.46)}$$

The right hand side vanishes for $\epsilon \to 0$, because $r_\epsilon \to 0$ uniformly and $e^{-|f|_\infty} \le 1 + g_\epsilon \le 1$ (by (V.3.39)).

Hence we conclude that (V.3.37) holds for Hölder continuous drift coefficients.

Step 2: General drift coefficient b^I:

Last but not least we show now that Lemma V.3.19 also holds for general (non-Hölder continuous) drift coefficients provided that the Assumption V.3.7 is satisfied. Therefore, we approximate at first (Step 2.1) the drift coefficient b^I by a sequence of Hölder -continuous functions $b^{I,(n)}$, that converge to b^I on $\mathsf{C}\left([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$. Then we show that Step 1 can be applied for all $b^{I,(n)}$ (Step 2.2), i.e. that (V.3.37) holds for each $b^{I,(n)}$. Finally, we justify that we can take the limit on both sides of (V.3.37) such that this inequality also holds for b^I. To this end we only need to show that the left hand side of (V.3.37) for $b^{I,(n)}$ is in the limit greater than the corresponding one for b^I and an analogue result for the right hand side (Step 2.3 and Step 2.4). For that matter we follow the ideas of Dawson and Gärtner in Section 4.5 of [DG87] and generalise their proof to the setting we consider here. Along the way, we also fix a small issue of Dawson and Gärtner in their treatment of the left hand side (compare our Step 2.3 with their calculation on page 288).

Step 2.1: Approximation of b^I:

Denote by $\mathcal{W}_{f,2}$ the open set of all points in $\mathcal{W}$ with distance at most 1 from $\mathcal{W}_f$.

We approximate the continuous drift coefficient b^I by functions $b^{I,(n)} \in \mathsf{C}\left([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$. These functions are chosen such that $b^{I,(n)}$ is on $[0,T] \times \mathbb{T}^d \times \mathcal{W}_f \times B_R$ also $\frac{1}{4}$-Hölder continuous in time and $\frac{1}{2}$-Hölder continuous in B_R. Moreover, $b^{I,(n)} = b^I$ outside of $[0,T] \times \mathbb{T}^d \times \mathcal{W}_{f,2} \times B_{2R}$ and $b^{I,(n)} \to b^I$ uniformly. Finding such a sequence is for example possible by the Stone-Weierstrass Theorem (on the compact set $\overline{\mathcal{W}}_{f,2}$) and the Urysohn's Lemma (with $\mathcal{W}_f$ and $\mathcal{W}_{f,2}$).

Step 2.2: (V.3.37) **holds for each** $b^{I,(n)}$**:**

One has to prove, that the martingale problem for the generator $\mathbb{L}^{I,(n)}_{s,x,w}$ with drift coefficient $b^{I,(n)}$ is well posed. But this we get from the (Cameron-Martin-) Girsanov theorem ([SV79] Theorem 6.4.2) because the difference between $b^{I,(n)}$ and b^I is at most ϵ for n large enough by the uniform convergence. We call the corresponding solution $P^{I,(n)}_{s,x,w,\theta}$ and its semi-group $U^{R,(n)}_{s,t}$. Hence by Step 1, (V.3.37) holds with $U^{R,(n)}_{s,t}$ and $\mathbb{L}^{I,(n)}$.

Step 2.3: The LHS of (V.3.37) **for** $b^{I,(n)}$ **is in the limit greater than the LHS for** b^I**:**

Fix $(s,t,x,w,\theta) \in [0,T] \times [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. By [SV79] Theorem 11.1.4, $P^{I,(n)}_{s,x,w,\theta} \to P^I_{s,x,w,\theta}$, and by [SV79] Theorem 11.1.2, $\theta_{[s,T]} \mapsto \tau^s_R\left(\theta_{[s,T]}\right)$ is lower semi-continuous. Hence $\{\tau^s_R > t\}$ is an open set and $\mathbb{1}_{\tau^s_R > t}$ is lower semi-continuous. The function $\left(e^f - 1\right)$ is non positive and continuous, what implies that $\left(e^{f\left(x,w,\theta_{[0,T]}\right)} - 1\right) \mathbb{1}_{\tau^s_R > t}$ is upper semi continuous. By the Portmanteau theorem

$$\begin{aligned} \limsup_{n\to\infty} U^{R,(n)}_{s,t}\left(e^f - 1\right)(x,w,\theta) &= \limsup_{n\to\infty} \int_{\mathsf{C}([0,T])} \left(e^{f\left(x,w,\theta_{[0,T]}\right)} - 1\right) \mathbb{1}_{\tau^s_R > t} P^{I,(n)}_{s,x,w,\theta}\left(\mathrm{d}\theta_{[0,T]}\right) \\ &\leq \int_{\mathsf{C}([0,T])} \left(e^{f\left(x,w,\theta_{[0,T]}\right)} - 1\right) \mathbb{1}_{\tau^s_R > t} P^{I}_{s,x,w,\theta}\left(\mathrm{d}\theta_{[0,T]}\right) = U^R_{s,t}\left(e^f - 1\right)(x,w,\theta)\,. \end{aligned} \tag{V.3.47}$$

With the Fatou-Lebesgue theorem (possible because $-1 < U^{R,(n)}_{s,t}\left(e^f - 1\right)(x,w,\theta)$) we conclude

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \log\left[1 + U^{R,(n)}_{s,t}\left(e^f - 1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \\ &\leq \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \limsup_{n\to\infty} \log\left[1 + U^{R,(n)}_{s,t}\left(e^f - 1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \\ &\leq \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \log\left[1 + U^{R}_{s,t}\left(e^f - 1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,. \end{aligned} \tag{V.3.48}$$

Step 2.4: The RHS of (V.3.37) **for** $b^{I,(n)}$ **is in the limit smaller than the RHS for** b^I**:**

By the triangle inequality we get

$$\left|\partial_u \mu_u - \left(\mathbb{L}^{I,(n)}_{u,\cdot,\cdot}\right)^* \mu_u\right|^2_{\mu_u} \leq \left|\partial_u \mu_u - \left(\mathbb{L}^{I}_{u,\cdot,\cdot}\right)^* \mu_u\right|^2_{\mu_u} + \left|\left(\mathbb{L}^{I}_{u,\cdot,\cdot} - \mathbb{L}^{I,(n)}_{u,\cdot,\cdot}\right)^* \mu_u\right|^2_{\mu_u}. \tag{V.3.49}$$

The last summand is smaller or equal to $\frac{\sigma^2}{2} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left|b^{I,(n)}(x,w,\theta) - b^I(x,w,\theta)\right|^2 \mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)$, what vanishes when $n \to \infty$ by the uniform convergence.

Step 2.5: Conclusion: Hence we conclude

$$\begin{aligned} &\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \log\left[1 + U^R_{s,t}\left(e^f - 1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \\ &\leq \liminf_{n\to\infty} \left\{ \int f(x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int \log\left[1 + U^{R,(n)}_{s,t}\left(e^f - 1\right)\right](x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \right\} \\ &\leq \liminf_{n\to\infty} \int_s^t \left|\partial_u \mu_u - \left(\mathbb{L}^{I,(n)}_{u,\cdot}\right)^* \mu_u\right|^2_{\mu_u} \mathrm{d}u \leq \int_s^t \left|\partial_u \mu_u - \left(\mathbb{L}^{I}_{u,\cdot}\right)^* \mu_u\right|^2_{\mu_u} \mathrm{d}u. \end{aligned} \tag{V.3.50}$$

□

Proof of Lemma V.3.20. Fix $(s,x,w,\theta) \in [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$. We get by the integration by parts formula (and the same argument as in [DG87] in the proof of Lemma 4.12 to bound the derivatives at the boundary ∂B_R),

$$\begin{aligned}&\left(\partial_s+\mathbb{L}^I_{s,x,w}\right)g_\epsilon(s,x,w,\theta)\\&\leq\int k_\epsilon\left(x-x',w-w',\theta-\theta'\right)\bigg(\partial_s g\left(s,x',w',\theta'\right)+\frac{\sigma^2}{2}\partial^2_{\theta'^2}g\left(s,x',w',\theta'\right)\\&\qquad\qquad+b^I\left(s,x,w,\theta\right)\partial_{\theta'}g\left(s,x',w',\theta'\right)\bigg)\mathrm{d}\theta'\mathrm{d}w'\mathrm{d}x'\\&=\int k_\epsilon\left(x-x',w-w',\theta-\theta'\right)\left(b^I\left(s,x,w,\theta\right)-b^I\left(s,x',w',\theta'\right)\right)\partial_{\theta'}g\left(s,x',w',\theta'\right)\mathrm{d}\theta'\mathrm{d}w'\mathrm{d}x',\end{aligned}\tag{V.3.51}$$

where the two integrals are over the space $\mathbb{T}^d\times\mathcal{W}_f\times B_R$. In the last equality we use that g is a solution to (V.3.38). We denote the right hand side of (V.3.51) by $r_\epsilon(s,x,w,\theta)$.

For each ϵ, the integrand in r_ϵ is continuous and uniformly bounded, because b^I and $\partial_{\theta'}g$ are continuous and we consider a compact set. This implies that r_ϵ is continuous.

For all $(s,x,w,\theta)\in[0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}$

$$|r_\epsilon(s,x,w,\theta)|\leq\sup_{\substack{x',x''\in\mathbb{T}^d;w'\in\mathcal{W};w''\in\mathcal{W}_f;\theta',\theta''\in B_{2R}\\|x'-x''|<\epsilon,|w'-w''|<\epsilon,|\theta'-\theta''|<\epsilon}}\left|b^I\left(s,x',w',\theta'\right)-b^I\left(s,x'',w'',\theta''\right)\right|\left|\partial_{\theta'}g\right|_\infty,\tag{V.3.52}$$

for ϵ small enough. The derivative $\partial_{\theta'}g$ is bounded and b^I is uniform continuous on the compact set $[0,T]\times\mathbb{T}^d\times\overline{\mathcal{W}}_{f,2}\times B_{2R}$. Hence r_ϵ converges uniformly to 0. □

V.3.1.2.3 PDE preliminaries

In this section we prove (see Theorem V.3.23) the uniqueness and the existence of a Hölder continuous (in time and spin) solution of the terminal boundary value problem (V.3.38), that is moreover continuous on $\mathbb{T}^d$ and on a connected subset $\widehat{\mathcal{W}}\subset\mathcal{W}$. We did not find such a result in the literature due to the non-ellipticity in the $\mathbb{T}^d\times\mathcal{W}$-directions.

In the proof of this result, we look at first at the PDE (V.3.38) with fixed $(x,w)\in\mathbb{T}^d\times\widehat{\mathcal{W}}$. For each of these PDEs, we get by a result of [LSU68] (that we repeat in Theorem V.3.24) the existence and uniqueness of a solution $g_{x,w}$ on $[0,T]\times B_R$. The main part of the proof then consists of showing that these solutions are continuous in $x\in\mathbb{T}^d$ and $w\in\widehat{\mathcal{W}}$.

Now we define the Hölder space, on which we derive the solution. We refer to the page 7 in [LSU68] for this definition (without the dependency on $\mathbb{T}^d$).

Definition V.3.21. *We denote by $H^{\ell/2,0,0,\ell}\left([0,T]\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R\right)$ the Banach space of continuous functions on $[0,T]\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R$, which have continuous derivatives $\partial_t^r\partial_\theta^s$, with $2r+s\leq\ell$, and with finite norm*

$$|u|_{H^{\ell/2,0,\ell}}=\sum_{2r+s\leq\lfloor\ell\rfloor}|\partial_t^r\partial_\theta^s u|_\infty+\sum_{2r+s=\lfloor\ell\rfloor}|\partial_t^r\partial_\theta^s u|_{\ell-\lfloor\ell\rfloor,\theta}+\sum_{2r+s\in\{\lfloor\ell\rfloor-1,\lfloor\ell\rfloor\}}|\partial_t^r\partial_\theta^s u|_{\frac{2r+s}{2},t},\tag{V.3.53}$$

where $|.|_{\ell-\lfloor\ell\rfloor,\theta}$ and $|.|_{\ell-\lfloor\ell\rfloor,t}$ are the usual Hölder norms in $\theta\in\mathbb{R}$ and $t\in[0,T]$ respectively.

The space $H^{\ell/2,\ell}([0,T]\times B_R)$ is defined analogue, just without the dependency on $\mathbb{T}^d\times\widehat{\mathcal{W}}$.

Remark V.3.22. *For $\ell\in(0,1)$, the norm $|u|_{H^{\ell/2,0,\ell}}$ is simply $|u|_\infty+|u|_{\ell,\theta}+|u|_{\frac{\ell}{2},t}$.*

Theorem V.3.23. *Let $\ell>0$ be a non integer number. Assume that the drift coefficient of $\mathbb{L}^I$ (see (V.3.11)) $b^I\in H^{\ell/2,0,0,\ell}\left([0,T]\times\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R\right)$ and that $i\in H^{0,0,\ell+2}\left(\mathbb{T}^d\times\widehat{\mathcal{W}}\times B_R\right)$. Then for* *each $R \in \mathbb{R}$, there is a unique solution $g^* \in H^{\ell/2+1,0,0,\ell+2}\left([0,t] \times \mathbb{T}^d \times \widehat{\mathcal{W}} \times \overline{B_R}\right)$ of the following terminal boundary value problem*

$$\begin{aligned} \partial_s g(s,x,w,\theta) &= -\mathbb{L}^I_{t,x,w} g(s,x,w,\theta) && (s,x,w,\theta) \in [0,t) \times \mathbb{T}^d \times \widehat{\mathcal{W}} \times B_R, \\ g(t,x,w,\theta) &= i(x,w,\theta) && (x,w,\theta) \in \mathbb{T}^d \times \widehat{\mathcal{W}} \times B_R, \\ g(s,x,w,\theta) &= 0 && (s,x,w,\theta) \in [0,t) \times \mathbb{T}^d \times \widehat{\mathcal{W}} \times \partial B_R(0). \end{aligned} \tag{V.3.54}$$

In the proof of this theorem, we use the following version of the Theorem 5.2 in Chapter IV of [LSU68]. Because we need it only for a specific class of PDEs, it is not as general as the original version of the theorem.

Theorem V.3.24 ([LSU68] Chapter IV Theorem 5.2)**.** *Let $\ell > 0$ be a non integer number and $\bar{i} \in H^{\ell+2}(B_R)$ and $\bar{b}^I, w \in H^{\ell/2,\ell}([0,t] \times B_R)$. Then for each $R > 0$, there is a unique classical solution $g^* \in H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$ of the following terminal boundary value problem*

$$\begin{aligned} \partial_s g(s,\theta) &= -\left(\frac{\sigma^2}{2}\partial^2_{\theta^2} + \bar{b}^I(s,\theta)\partial_\theta\right) g(s,\theta) + w(\theta,s) && (s,\theta) \in [0,t) \times B_R, \\ g(t,\theta) &= \bar{i}(\theta) && \theta \in B_R, \\ g(s,\theta) &= 0 && (s,\theta) \in [0,t) \times \partial B_R(0). \end{aligned} \tag{V.3.55}$$

Moreover, the solution g^* satisfies

$$|g^*|_{H^{\ell/2+1,\ell+2}\left([0,t]\times\overline{B_R}\right)} \le C\left(|w|_{H^{\ell/2,\ell}\left([0,t]\times\overline{B_R}\right)} + |i|_{H^{\ell+2}(B_R)}\right), \tag{V.3.56}$$

for a constant $C > 0$ independent of w and i.

For a proof of this Theorem V.3.24 we refer to [LSU68]. Now we prove the Theorem V.3.23.

Proof of Theorem V.3.23. **Step 1: The existence and regularity:**
The PDE (V.3.54) corresponds for a fixed tuple $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$ to the PDE (V.3.55) with $w \equiv 0$, $\bar{i}(\theta) = i(x,w,\theta)$, $\bar{b}^I(s,\theta) = b^I(s,x,w,\theta)$, due to the independence in $x \in \mathbb{T}^d$ and $w \in \widehat{\mathcal{W}}$ of the operator $\mathbb{L}^I_{t,x,w}$. Therefore, we know by Theorem V.3.24, that there is unique solution $g^*_{x,w} \in H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$ of the corresponding PDE (V.3.55), for each $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$. Set $g^*(.,x,w,.) := g_{x,w}$. The function g^* is a solution of (V.3.54). To show the claimed regularity of this solution, we need to show that $(x,w) \mapsto g^*_{x,w}$ is a continuous map $\mathbb{T}^d \times \widehat{\mathcal{W}} \to H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$.

Fix an arbitrary tuple $(x_0,w_0) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$. The proof of the continuity at (x_0,w_0) is organised as follows:
Step 1.1: First we define an operator $I_{x,w} : H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right) \to H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$ for each $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$.
Step 1.2: Then we show that $I_{x,w}$ is a continuous contraction, when $|x - x_0|$ and $|w - w_0|$ are small enough.
Step 1.3: Next we show that the sequence $(I_{x,w})^n\left(g^*_{x_0,w_0}\right)$ converges to $g^*_{x,w}$ (also for $|x - x_0|$ and $|w - w_0|$ small enough).
Step 1.4: Finally, we conclude from the previous steps the continuity of $g^*_{x,w}$ at $(x_0,w_0) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$.

Let us carry out this program.

Step 1.1: Define the operator

$$T_{s,x,w} := \mathbb{L}^I_{s,x_0,w_0} - \mathbb{L}^I_{s,x,w} = \left(b^I(s,x_0,w_0,.) - b^I(s,x,w,.)\right)\partial_\theta. \tag{V.3.57}$$

With this operator, $\mathbb{L}^I_{s,x,w}$ can be seen as a perturbation of $\mathbb{L}^I_{s,x_0,w_0}$, by $\mathbb{L}^I_{s,x,w} = \mathbb{L}^I_{s,x_0,w_0} - T_{s,x,w}$. Moreover, we define the operator

$$I_{x,w} : H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right) \to H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right), \tag{V.3.58}$$

as the map that sends a function $v \in H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$ to the (unique) solution of

$$\begin{aligned} \partial_s g(s,\theta) &= -\mathbb{L}^I_{s,x_0,w_0} g(s,\theta) + T_{s,x,w} v && (s,\theta) \in [0,t) \times B_R, \\ g(t,\theta) &= i(x,w,\theta) && \theta \in B_R, \\ g(s,\theta) &= 0 && (s,\theta) \in [0,t) \times \partial B_R. \end{aligned} \tag{V.3.59}$$

We get the existence and the uniqueness of a solution to this PDE from Theorem V.3.24.

Step 1.2: We show now that $I_{x,w}$ is a continuous contraction.

Fix arbitrary $u_1, u_2 \in H^{\ell/2+1,\ell+2}\left([0,t] \times \overline{B_R}\right)$. By the definition, $I_{x,w}(u_1) - I_{x,w}(u_2)$ is the unique classical solution to

$$\begin{aligned} \left(\partial_s + \mathbb{L}^I_{s,x_0,w_0}\right)\left(I_{x,w}(u_1) - I_{x,w}(u_2)\right) = T_{s,x,w}(u_1 - u_2), \\ \text{with 0 terminal and 0 boundary condition.} \end{aligned} \tag{V.3.60}$$

Then by (V.3.56), for $|x_0 - x|$ and $|w_0 - w|$ small enough,

$$\begin{aligned} |I_{x,w}(u_1) - I_{x,w}(u_2)|_{H^{\ell/2+1,\ell+2}} &\le C\,|T_{.,x,w}(u_1 - u_2)|_{H^{\ell/2,\ell}} \\ &\le C\left|b^I(.,x_0,w_0,.) - b^I(.,x,w,.)\right|_{H^{\ell/2,\ell}} |\partial_\theta(u_1 - u_2)|_{H^{\ell/2,\ell}} \\ &\le \epsilon\,|u_1 - u_2|_{H^{\ell/2+1,\ell+2}}. \end{aligned} \tag{V.3.61}$$

In the last inequality we use that $b^I \in H^{\ell/2,0,0,\ell}\left([0,T] \times \mathbb{T}^d \times \widehat{\mathcal{W}} \times B_R\right)$. This implies that $I_{x,w}$ is a continuous contraction. Note that the ϵ is independent of $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$, as long as $|x_0 - x|$ and $|w_0 - w|$ are small enough, because the constant C depends only on $\mathbb{L}^I_{.,x_0,w_0}$.

Step 1.3: Define the sequence $\left\{(I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right\}_n$, where $g^*_{x_0,w_0}$ is the solution of (V.3.54) at (x_0,w_0). Then by (V.3.61)

$$\begin{aligned} &\left|(I_{x,w})^{n+1}\left(g^*_{x_0,w_0}\right) - (I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right|_{H^{\ell/2+1,\ell+2}} \\ &\le \epsilon\left|(I_{x,w})^n\left(g^*_{x_0,w_0}\right) - (I_{x,w})^{n-1}\left(g^*_{x_0,w_0}\right)\right|_{H^{\ell/2+1,\ell+2}} \\ &\le \epsilon^n\left|I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}}. \end{aligned} \tag{V.3.62}$$

Therefore, $\left\{(I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right\}_n$ is a Cauchy sequence. The Hölder spaces are complete, hence there is a $u^*_{x,w} \in H^{\ell/2+1,\ell+2}$ such that $(I_{x,w})^n\left(g^*_{x_0,w_0}\right) \to u^*_{x,w}$. The continuity of $I_{x,w}$ implies that also $I_{x,w}\left((I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right) \to I_{x,w}\left(u^*_{x,w}\right)$. Therefore, $u^*_{x,w} = I_{x,w}\left(u^*_{x,w}\right)$. By the definition of $I_{x,w}$ and the uniqueness of Theorem V.3.24, we conclude $u^*_{x,w} = g^*_{x,w}$.

Step 1.4: Then by (V.3.62)

$$\begin{aligned} \left|g^*_{x,w} - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} &\le \sum_{n=0}^{\infty}\left|(I_{x,w})^{n+1}\left(g^*_{x_0,w_0}\right) - (I_{x,w})^n\left(g^*_{x_0,w_0}\right)\right|_{H^{\ell/2+1,\ell+2}} \\ &\le \left|I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} \frac{1}{1-\epsilon}. \end{aligned} \tag{V.3.63}$$

We show now that the right hand side is bounded by a $\epsilon_1 > 0$ for $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$ with $|x_0 - x|$ and $|w_0 - w|$ small enough. By construction $I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}$ is the solution to the PDE

$\partial_s g = -\mathbb{L}^I_{s,x_0,w_0} g + T_{s,x,w} g^*_{x_0,w_0}$ with $i(x,w,.) - i(x_0,w_0,.)$ boundary condition. Hence by (V.3.56)

$$\begin{aligned}&\left|I_{x,w}\left(g^*_{x_0,w_0}\right) - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} \\ &\le C\left(\left|T_{t,x,w} g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} + \left|i(x,w,.) - i(x_0,w_0,.)\right|_{H^{\ell/2+1,\ell+2}}\right).\end{aligned} \tag{V.3.64}$$

Then as in (V.3.61) and finally by applying again (V.3.56) for $g^*_{x_0,w_0}$, we get that the right hand side of (V.3.64) is smaller or equal to

$$C\left(\epsilon\left|g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} + \epsilon\right) \le \epsilon C\left(\left|i(x_0,w_0,.)\right|_{H^{\ell+2}} + 1\right) \le \epsilon_1, \tag{V.3.65}$$

because $i(x_0,w_0,.) \in H^{\ell+2}$. Therefore, $\left|g^*_{x,w} - g^*_{x_0,w_0}\right|_{H^{\ell/2+1,\ell+2}} < \epsilon_1$ for $|x_0 - x|$ and $|w_0 - w|$ small enough, by (V.3.63).

This is the claimed regularity of the solution $g^*_{x,w}$ at $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$.

Step 2: The Uniqueness:

Let g^* be a solution of (V.3.54). Then, for each tuple $(x,w) \in \mathbb{T}^d \times \widehat{\mathcal{W}}$, $g^*_{x,w}$ has to be the unique solution of (V.3.55) with $w \equiv 0$, $\bar{i}(\theta) = i(x,w,\theta)$, $\bar{b}^I(s,\theta) = b^I(s,x,w,\theta)$. Therefore, there is at most one solution of (V.3.54) in $H^{\ell/2+1,0,0,\ell+2}\left([0,t] \times \mathbb{T}^d \times \widehat{\mathcal{W}} \times \overline{B_R}\right)$. □

Remark V.3.25. *Using the calculation in* (V.3.64) *and in* (V.3.65)*, we could show even higher regularity, than continuity, of the solution in* $\mathbb{T}^d \times \widehat{\mathcal{W}}$*, if we assume higher regularity of* b *and* i *in* $\mathbb{T}^d \times \widehat{\mathcal{W}}$*.*

V.3.1.3 Another representation of the rate function $S^I_{\nu,\zeta}$

We state in the next lemma another representation of the rate function $S^I_{\nu,\zeta}$. This representation is not used in the proof of Theorem V.3.9. As explained in Remark V.3.17 we could use it to show an upper bound on S^I. Nevertheless, we prove this lemma here, because we need it in Section V.3.2 when showing that the rate function of the interacting system is actually lower semi-continuous

Lemma V.3.26 (see [DG87] Lemma 4.8 for the mean field case)**.** *Take a* $\nu \in \mathbb{M}^L_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ *and a* $\mu \in \mathscr{C}$*. Then*

$$S^I_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) + \sup_{f \in \mathsf{C}^{1,0,2}_c([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} I\left(\mu_{[0,T]}, f\right), \tag{V.3.66}$$

where

$$\begin{aligned}I\left(\mu_{[0,T]}, f\right) = &\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(T,x,w,\theta)\,\mu_T(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(0,x,w,\theta)\,\mu_0(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) \\ &- \int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left(\frac{\partial}{\partial t} + \mathbb{L}^I_{t,x,w}\right) f(t,x,\theta) - \frac{\sigma^2}{2}\left(\partial_\theta f(t,x,\theta)\right)^2 \mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}t.\end{aligned} \tag{V.3.67}$$

Proof. Most parts of this proof are almost equal (modulo additional integrals with respect to $\mathbb{T}^d$ and $\mathcal{W}$) to the proof of Lemma 4.8 in [DG87]. Therefore, we only state the ideas and point out where things have to be changed due to the space and random environment dependency.

Fix a $\mu_{[0,T]} \in \mathscr{C}$ with $\mathsf{H}(\mu_0|\nu) < \infty$.

Step 1: We define for $f \in \mathsf{C}_c^{1,0,2}\big([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$

$$\begin{aligned}\ell_{s,t}(f) = &\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(t,x,w,\theta)\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta) - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f(s,x,w,\theta)\,\mu_s(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\\ &- \int_s^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \big(\partial_u + \mathbb{L}^I_{u,x,w}\big) f(u,x,w,\theta)\,\mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}t.\end{aligned} \tag{V.3.68}$$

Note that this is equal to $I(\mu,f)$ without the $(\partial_\theta f(t,.,.,.))^2$ part and with the restriction to the time interval $[s,t]$. Analogue to (4.26) of [DG87], we can prove that

$$\begin{aligned}&|\ell_{s,t}(f)|^2\\ &\quad\le \int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \sigma^2\,(\partial_\theta f(t,x,w,\theta))^2\,\mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}t \sup_{g\in\mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} I\big(\mu_{[0,T]},g\big).\end{aligned} \tag{V.3.69}$$

Step 2: As in the second step in [DG87] we can show that for each $g \in \mathsf{C}_c^{1,0,2}\big([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\big)$

$$I\big(\mu_{[0,T]},g\big) \le S^I_{\nu,\zeta}\big(\mu_{[0,T]}\big) - \mathsf{H}\left(\mu_0|\mathrm{d}x\otimes\zeta_x\otimes\nu_x\right), \tag{V.3.70}$$

by applying the integration by parts Lemma V.2.24.

Step 3: We may assume that $\sup_{g\in\mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} I\big(\mu_{[0,T]},g\big) < \infty$. Denote by $\widehat{L}^2_{\mu_{[0,T]}}(s,t)$ the Hilbert space of all measurable maps $h : [s.t]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\to\mathbb{R}$, with finite norm

$$|h|^2_{\mu_{[0,T]}} := \int_s^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \frac{\sigma^2}{2}\,(h(u,x,w,\theta))^2\,\mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}u. \tag{V.3.71}$$

Moreover, let $L^2_{\mu_{[0,T]}}(s,t)$ be the closure in $\widehat{L}^2_{\mu_{[0,T]}}(s,t)$ of the subset consisting of the maps $(t,x,\theta)\mapsto \partial_\theta h(t,x,\theta)$ with $h\in\mathsf{C}_c^{1,0,2}\big([s,t]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\big)$.

Similar as in the third step of the proof in [DG87] (but now with the additional dependency on the space $\mathbb{T}^d$), we can use this space to prove that there is a $h^{\mu_{[0,T]}}\in\widehat{L}^2_{\mu_{[0,T]}}(s,t)$, such that

$$\ell_{0,t}(f) = \int_0^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \sigma^2 h^{\mu_{[0,T]}}(u,x,w,\theta)\,\partial_\theta f(u,x,w,\theta)\,\mu_u(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}u. \tag{V.3.72}$$

The existence of such an $h^{\mu_{[0,T]}}$, origins from applying the Riesz representation theorem for ℓ. Then the same arguments as in [DG87] lead to

$$\sup_{f\in\mathsf{C}_c^{1,0,2}([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R})} I\big(\mu_{[0,T]},f\big) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \frac{\sigma^2}{2}\left(h^{\mu_{[0,T]}}(t,x,w,\theta)\right)^2 \mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}t. \tag{V.3.73}$$

Step 4: In this last part, one uses the right hand side of (V.3.73) to show the equation (V.3.66). This follows again from the same arguments as in [DG87], by showing that $\mu_{[0,T]}$ is absolutely continuous as a map from $[0,T]\to\mathbb{D}'$ and finally by applying the Lemma V.2.23. □

V.3.2 From independent to interacting spins

In this section, we finish the proof of Theorem V.3.5 by generalising the proofs given in Chapter 5 of [DG87]. As explained subsequent to the Theorem V.3.5, we use the following local version of an LDP (Theorem V.3.27) and exponential tightness result (Theorem V.3.28), to prove Theorem V.3.5.

Theorem V.3.27 (compare to Theorem 5.2 in [DG87] for the mean field version)**.** *Under the assumptions of Theorem V.3.5, the following statements are true for fixed $\overline{\mu}_{[0,T]} \in \mathscr{C}$.*

(i) *For all open neighbourhoods $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$*

$$\liminf_{N\to\infty} N^{-d} \log P^N \left[\mu^N_{[0,T]} \in V\right] \geq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right). \tag{V.3.74}$$

(ii) *For each $\gamma > 0$, there is an open neighbourhood $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$ such that*

$$\limsup_{N\to\infty} N^{-d} \log P^N \left[\mu^N_{[0,T]} \in V\right] \leq \begin{cases} -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma & \text{if } S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) < \infty, \\ -\gamma & \text{otherwise.} \end{cases} \tag{V.3.75}$$

Theorem V.3.28 (compare to Theorem 5.3 in [DG87] for the mean field version)**.** *Under the assumptions of Theorem V.3.5, there is, for all $s > 0$, a compact set $\mathcal{K}_s \subset \mathscr{C}$, with $\mathcal{K}_s \subset \mathscr{C}_{\varphi,R}$ for a R large enough, such that*

$$\limsup_{N\to\infty} N^{-d} \log P^N \left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathcal{K}_s\right] \leq -s. \tag{V.3.76}$$

We state the proofs of these two Theorems in Section V.3.2.3. and Section V.3.2.2.

Before we infer from these results the Theorem V.3.5, let us briefly state the idea of the proofs of these two theorems and explain how the rest of this section is organised.

1.) In Section V.3.2.1, we show some preliminary lemmas. At first (in Section V.3.2.1.1) we show that the operator $\mathbb{L}^I_{t,x,w} = \mathbb{L}_{\overline{\mu}_t,x,w}$ satisfies the assumptions of Section V.3.1, for all $\overline{\mu}_{[0,T]} \in \mathscr{C}_\varphi \cap \mathscr{C}^L$. This implies the validity of the results of Section V.3.1 for the independent system with fixed effective field $\overline{\mu}_{[0,T]}$.

   Then we show (in Section V.3.2.1.2), that $\mu^N_{[0,T]}$ is in $\mathscr{C}_{\varphi,\infty}$ almost surely under P^N, for all $N \in \mathbb{N}$.

   Finally (Section V.3.2.1.3), we derive exponential small bounds for P^N. For example we show that the probability of being outside of $\mathscr{C}_{\varphi,R}$ is exponentially small. The proofs of these result are for fixed initial data formally the same as the proofs in [DG87], at least after applying the result of Section V.3.2.1.2. However, due to the different initial distribution, some new estimates are required. Here we need the Assumption V.3.4.

2.) Next we prove in Section V.3.2.2 the Theorem V.3.28, by combining in a suitable way the exponential bounds. The approach of this proof does not differ from the corresponding proof in [DG87].

3.) Finally, we prove Theorem V.3.27 in Section V.3.2.3. Here we separate the proof in the cases when $\overline{\mu}_{[0,T]}$ is in $\mathscr{C}_{\varphi,\infty}$, in $\mathscr{C}^L$ and when it is not in these sets. The part of the proof when $\overline{\mu}_{[0,T]}$ is in $\mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$ is formally similar to the proof in [DG87]. We use, in this part, the exponential bounds derived in Section V.3.2.1.3 as well as the large deviation principle for independent spins (derived in Section V.3.1). The other case, i.e. when $\overline{\mu}_{[0,T]}$ not in $\mathscr{C}^L$ or not in $\mathscr{C}_{\varphi,\infty}$, are new here. When $\overline{\mu}_{[0,T]} \notin \mathscr{C}^L$, we show that in a small neighbourhood around $\overline{\mu}_{[0,T]}$, there is no empirical process for N large enough. From this we conclude the local large deviation result. For the case that $\overline{\mu}_{[0,T]}$ is not in $\mathscr{C}_{\varphi,\infty}$, we infer the local large deviation result from the exponential bounds.

   At the beginning of Section V.3.2.3, we explain the proof in more details.

Remark V.3.29. *All the results of this section can be transferred to hold also on $\mathscr{C}_{\varphi,\infty}$ with the stronger topology considered in [DG87]. The proofs would formally be the same.*

Proof of Theorem V.3.5. This proof of Theorem V.3.5 is similar to the proof of the corresponding mean field theorem in [DG87]. Despite these similarities we state the proof here, because it illustrates how the Theorem V.3.27 and Theorem V.3.28 are applied. Differences to [DG87] arise only in the proof that $S_{\nu,\zeta}$ is a good rate function. This is mainly due to the space and random environment dependency and because the spins do not start at fixed positions (as considered in [DG87]), but are initially distributed according to ν.

***(i)* The large deviation lower bound:**

Let $G \subset \mathscr{C}$ be a open set. The large deviation lower bound follows directly by applying Theorem V.3.27 (i) with $V = G$ for all $\mu_{[0,T]} \in G$.

***(ii)* The large deviation upper bound:**

Let $F \subset \mathscr{C}$ be a closed set. We assume that $\inf_{\mu \in F} S_{\nu,\zeta}(\mu) = \overline{s} < \infty$. The case when the infimum is not finite can be treated similarly.

By Theorem V.3.28 we know that there is compact set $\mathcal{K} \subset \mathscr{C}$ such that (V.3.76) is satisfied with $s = \overline{s}$. We further know by Theorem V.3.27 (ii) that for a fixed $\gamma > 0$ and for each $\mu_{[0,T]} \in F \cap \mathcal{K}$, there is an open neighbourhood $V_{\mu_{[0,T]}}$ of $\mu_{[0,T]}$ such that (V.3.75) is satisfied for $\mu_{[0,T]}$. Because $F \cap \mathcal{K}$ is compact, it is covered by a finite number of these neighbourhoods. Combining these results we get

$$\begin{aligned}&\limsup_{N\to\infty} N^{-d} \log P^N \left[\mu^N_{[0,T]} \in F\right]\\ &\le \max\left\{\limsup_{N\to\infty} N^{-d} \log P^N \left[\mu^N_{[0,T]} \in F \cap \mathcal{K}\right], \limsup_{N\to\infty} N^{-d} \log P^N \left[\mu^N_{[0,T]} \notin \mathcal{K}\right]\right\}\\ &\le -\overline{s} + \gamma.\end{aligned} \tag{V.3.77}$$

Because the parameter γ is arbitrary, this proves the large deviation upper bound.

***(iii)* $S_{\nu,\zeta}$ is a good rate function:**

To show that $S_{\nu,\zeta}$ is a good rate function, we have to show that the level sets

$$\mathcal{L}^{\le s}(S_{\nu,\zeta}) := \left\{\mu_{[0,T]} \in \mathscr{C} : S_{\nu,\zeta}\left(\mu_{[0,T]}\right) \le s\right\} \tag{V.3.78}$$

are compact in $\mathscr{C}$, for each $s \ge 0$. We show at first that the level set $\mathcal{L}^{\le s}(S_{\nu,\zeta})$ is relatively compact and then that it is closed.

Step 1: $\mathcal{L}^{\le s}(S_{\nu,\zeta})$ is relatively compact:

By Theorem V.3.28 we know that there is a compact set $\mathcal{K}_{s+\epsilon} \subset \mathscr{C}_{\varphi,R} \subset \mathscr{C}$, for $R > 0$ large enough, such that (V.3.76) holds for $s + \epsilon$. We claim that $\mathcal{L}^{\le s}(S_{\nu,\zeta}) \subset \mathcal{K}_{s+\epsilon}$. Let us assume that there is a $\mu_{[0,T]} \in \mathcal{L}^{\le s}(S_{\nu,\zeta})$ that is not in $\mathcal{K}_{s+\epsilon}$. Then we know by (V.3.76) and Theorem V.3.27 (i) (because $\mathscr{C} \backslash \mathcal{K}_{s+\epsilon}$ is an open neighbourhood of $\mu_{[0,T]}$), that $s + \epsilon \le S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$, a contradiction.

Step 2: $\mathcal{L}^{\le s}(S_{\nu,\zeta})$ is closed:

Let $I\left(\mu_{[0,T]}, f\right)$ be defined as in (V.3.67). By Lemma V.3.26 we know that

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) + \sup_{f \in \mathsf{C}^{1,0,2}_c([0,T]\times\mathbb{T}^d\times\mathbb{R})} I^{\mathbb{L}_{\mu_{[0,T]},\dots}}\left(\mu_{[0,T]}, f\right). \tag{V.3.79}$$

Moreover, we know by the previous step and the definition of $S_{\nu,\zeta}$ that $\mathcal{L}^{\le s}(S_{\nu,\zeta}) \subset \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$, for a R large enough. Therefore, $\mathcal{L}^{\le s}(S_{\nu,\zeta}) = \bigcap_{f \in \mathsf{C}^{1,0,2}_c([0,T]\times\mathbb{T}^d\times\mathbb{R})} \mathcal{L}^{\le s}_{f,R}(S_{\nu,\zeta})$ with

$$\mathcal{L}^{\le s}_{f,R}(S_{\nu,\zeta}) := \left\{\mu_{[0,T]} \in \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L : I^{\mathbb{L}_{\mu_{[0,T]},\dots}}\left(\mu_{[0,T]}, f\right) + \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right) \le s\right\}. \tag{V.3.80}$$

It is hence enough to show that $\mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right)$ is closed for each $f \in \mathsf{C}^{1,0,2}_c\big([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$.

The map $\mu_{[0,T]} \mapsto I^{\mathbb{L}_{\mu_{[0,T]}},\cdots}\left(\mu_{[0,T]}, f\right)$ is continuous as a function $\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L \to \mathbb{R}$ for all $R \in \mathbb{R}_+$ and for all $f \in \mathsf{C}^{1,0,2}_c\big([0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$. This follows from Assumption V.3.1 d.). Moreover, $\mu^{(n)} \to \mu$ implies that $\mu^{(n)}_0 \to \mu_0$, and $\mu_0 \mapsto \mathsf{H}\left(\mu_0 | \mathrm{d}x \otimes \zeta_x \otimes \nu_x\right)$ is lower semi continuous. From the continuity of $\mu_{[0,T]} \mapsto I^{\mathbb{L}_{\mu_{[0,T]}},\cdots}\left(\mu_{[0,T]}, f\right)$ and the lower semi continuity of $\mathsf{H}\left(.|\mathrm{d}x \otimes \zeta_x \otimes \nu_x\right)$ we infer, that the set $\mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right)$ is closed in $\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$. Due to $\mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$ being closed in $\mathscr{C}$, this implies that $\mathcal{L}^{\leq s}_{f,R}\left(S_{\nu,\zeta}\right)$ is also closed in $\mathscr{C}$. □

V.3.2.1 Preliminaries

V.3.2.1.1 The assumptions of the corresponding independent systems are satisfied

Fix a $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$. Define the function $b^I\left(t,x,w,\theta\right) := b\left(x,w,\theta,\overline{\mu}_t\right)$. We show now that the Assumption V.3.7 is satisfied for the independent spin system (given by (V.3.12)) with this drift coefficient b^I, i.e. $\mathbb{L}^I_{t,x,w} := \mathbb{L}_{\overline{\mu}_t,x,w}$.

a.) The Assumption V.3.7 a.) is satisfied because of Assumption V.3.1 a.a) and $\overline{\mu}_t \in \mathbb{M}_{\varphi,R}$ for all $t \in [0,T]$ and for a R large enough.

b.) We infer from Theorem 10.1.2 of [SV79] the uniqueness of the martingale problem for each tuple $(x,w) \in \mathbb{T}^d \times \mathcal{W}$, because the drift coefficient is continuous (by a.)). To apply this theorem, let G_n be a set with compact closure in R^{N^d} and define a continuous and bounded function $b^{I,(n)} : [0,T] \times \mathbb{R}$ to equal $b^I\left(.,x,.\right)$ on G_n. Then Theorem 7.2.1 of [SV79] gives that for each n the martingale problem corresponding to $b^{I,(n)}$ is well defined. To show the existence, we apply Theorem 10.2.1 of [SV79]. The conditions of this theorem are satisfied by Assumption V.3.1 c.), because $\mathbb{L}^I_{t,x,w} = \mathbb{L}_{\overline{\mu}_t,x,w}$.

Therefore, the martingale problem is well defined, i.e. Assumption V.3.7 b.) is satisfied.

V.3.2.1.2 The empirical process is with probability one in $\mathscr{C}_{\varphi,\infty}$

Lemma V.3.30. (i) *Let Assumption V.3.4 hold. Then for all* $N \in \mathbb{N}$,

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,\infty}\right] = 0. \tag{V.3.81}$$

(ii) *For any* $r > 0$ *and for all* $N \in \mathbb{N}$,

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d} : \mu^N_{\underline{\theta}^N} \in \mathbb{M}_{r,\varphi}} P^N_{\underline{w}^N,\underline{\theta}^N}\left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,\infty}\right] = 0, \tag{V.3.82}$$

where $P^N_{\underline{w}^N,\underline{\theta}^N} \in \mathbb{M}_1\left(\mathbb{R}^{N^d}\right)$ *is defined as* $P^N_{\underline{w}^N}$ *(see Notation V.1.9) with fixed initial values* $\underline{\theta}^N$.

Proof. ***(i)*** For all $R > 0$ and $\underline{w}^N \in \mathcal{W}^{N^d}$

$$\begin{aligned} P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,\infty}\right] &\leq P^N_{\underline{w}^N}\left[\underline{\theta}^N_{[0,T]} : \sup_{t \in [0,T]} \frac{1}{N^d} \sum_{k \in \mathbb{T}^d_N} \varphi\left(\theta^{k,N}_t\right) > R\right] \\ &= P^N_{\underline{w}^N}\left[\underline{\theta}^N_{[0,T]} : \sup_{t \in [0,T]} \log\left(1 + \frac{1}{N^d} \sum_{k \in \mathbb{T}^d_N} \varphi\left(\theta^{k,N}_t\right)\right) > \log\left(R+1\right)\right]. \end{aligned} \tag{V.3.83}$$

We want to show that the right hand side converge to zero when R tends to infinity. To do this, we use an approach that is for example used in the proof of Theorem 1.5 in [Gär88] and apply it to the setting we consider here.

Fix $\underline{w}^N \in \mathcal{W}^{N^d}$. Applying Itô's lemma to $h\left(\underline{\theta}_t^N\right) := \log\left(1 + \frac{1}{N^d}\sum_{k\in\mathbb{T}_N^d} \varphi\left(\theta_t^{k,N}\right)\right)$, we get

$$\begin{aligned} &h\left(\underline{\theta}_t^N\right) \\ &\leq h\left(\underline{\theta}_0\right) + \int_0^t \left(1 + \frac{1}{N^d}\sum_{k\in\mathbb{T}_N^d} \varphi\left(\theta_s^{k,N}\right)\right)^{-1} \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \mathbb{L}_{\mu_s^N,x,w}\varphi\left(\theta\right)\mu_s^N\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\mathrm{d}s + M_t \\ &\leq h\left(\underline{\theta}_0\right) + T + M_t, \end{aligned} \tag{V.3.84}$$

by Assumption V.3.1 b.), where μ_s^N is the empirical measure defined by $\underline{w}^N$ and $\underline{\theta}^N$. The M_t is a continuous local $P_{\underline{w}^N}^N$ martingale with $M_0 = 0$. Define the non negative $P_{\underline{w}^N}^N$ supermartingale

$$S_t^R := \min\left\{h\left(\underline{\theta}_0\right) + T + M_t, \log\left(R\right)\right\}. \tag{V.3.85}$$

By the Doob supermartingale inequality

$$\begin{aligned} &P_{\underline{w}^N}^N\left[\sup_{t\in[0,T]} h\left(\underline{\theta}_t^N\right) > \log\left(R+1\right)\right] \leq P_{\underline{w}^N}^N\left[\sup_{t\in[0,T]} S_t^R > \log\left(R+1\right)\right] \\ &\leq \frac{1}{\log\left(R+1\right)} E_{P_{\underline{w}^N}^N}\left[S_0^R\right] \leq \left(\log\left(R+1\right)\right)^{-\frac{1}{2}} + \nu^N\left[h\left(\underline{\theta}\right) > \left(\log\left(R+1\right)\right)^{\frac{1}{2}} - T\right]. \end{aligned} \tag{V.3.86}$$

To bound the probability, we apply the Chebyshev inequality,

$$\nu^N\left[h\left(\underline{\theta}\right) > \left(\log\left(R+1\right)\right)^{\frac{1}{2}} - T\right] \leq e^{-\kappa N^d\left(e^{\left(\log(R+1)\right)^{\frac{1}{2}} - T} - 1\right)} \prod_{i\in\mathbb{T}_N^d} \int_{\mathbb{R}} e^{\kappa\varphi(\theta)}\nu_{\frac{i}{N}}\left(\mathrm{d}\theta\right). \tag{V.3.87}$$

By Assumption V.3.4, the integral is bounded by a constant. Therefore, the right hand side of (V.3.86) converges to zero uniformly for all $\underline{w}^N$, when R tends to infinity. Combining this with (V.3.83), implies (i).

(ii) We get by the same arguments as in (i) ((V.3.83) to (V.3.86))

$$\sup_{\underline{\theta}^N\in\mathbb{R}^{N^d}:\mu_{\underline{\theta}^N}^N\in\mathbb{M}_{r,\varphi}} P_{\underline{w}^N,\underline{\theta}^N}^N\left[\mu_{[0,T]}^N \in \mathscr{C}\backslash\mathscr{C}_{\varphi,\infty}\right] \leq \left(\log\left(R+1\right)\right)^{-\frac{1}{2}}, \tag{V.3.88}$$

for all $R > 0$ large enough, when r is fixed. □

V.3.2.1.3 Exponential bounds

In the next two lemmas we show that it is exponentially unlikely that an empirical process leaves the sets $\mathscr{C}_{\varphi,R}$. At first we show it uniformly for fixed initial conditions in $\mathbb{M}_{r,\varphi}$ (Lemma V.3.31), then for initial conditions distributed according to ν (Lemma V.3.32).

Lemma V.3.31 (compare to Lemma 5.5 in [DG87] for the mean field case). *For any $r > 0$, $R > 0$ and for all $N \in \mathbb{N}$,*

$$\sup_{\underline{w}^N\in\mathcal{W}^{N^d}} \sup_{\underline{\theta}^N\in\mathbb{R}^{N^d}:\mu_{\underline{\theta}^N}^N\in\mathbb{M}_{r,\varphi}} P_{\underline{w}^N,\underline{\theta}^N}^N\left[\mu_{[0,T]}^N \in \mathscr{C}\backslash\mathscr{C}_{\varphi,R}\right] \leq e^{-N^d R_T}, \tag{V.3.89}$$

with $R_T = Re^{-\lambda T} - r$, where λ is defined in Assumption V.3.1 b.).

Proof. First note that by Lemma V.3.30 (ii), it is enough to show for each $\underline{w}^N \in \mathcal{W}^{N^d}$

$$\sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}: \mu^N \in \mathbb{M}_{r,\varphi}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{\underline{\theta}^N} \in \mathscr{C}_{\varphi,\infty} \backslash \mathscr{C}_{\varphi,R} \right] \leq e^{-N^d R_T}. \tag{V.3.90}$$

This bound can be proven (at least formally) exactly as the proof of Lemma 5.5 in [DG87]. Therefore, we do not state it here. Neither the different topology on $\mathscr{C}_{\varphi,\infty}$ considered in that paper nor the space dependency, is crucial in the proof. The proof requires the Assumption V.3.1 b.). □

Lemma V.3.32. *Let Assumption V.3.4 hold. For all $s > 0$, there is a $R = R_s > 0$, such that for all $N \in \mathbb{N}$*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} P^N_{\underline{w}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] \leq e^{-N^d s}. \tag{V.3.91}$$

Proof. For all $R > 0$, $\underline{w}^N \in \mathcal{W}^{N^d}$

$$\begin{aligned} P^N_{\underline{w}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] &= \int_{\mathbb{R}^{N^d}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] \nu^N \left(\mathrm{d}\underline{\theta}^N \right) \\ &\leq \sum_{k=0}^{\infty} e^{-N^d R e^{-\lambda T} + N^d (k+1)} \nu^N \left[\mathbb{M}_{k+1,\varphi} \backslash \mathbb{M}_{k,\varphi} \right], \end{aligned} \tag{V.3.92}$$

where we use Lemma V.3.31 in the inequality. For the probability of the right hand side we use the exponential Chebyshev inequality with $\ell > 1$

$$\begin{aligned} \nu^N \left[\mathbb{M}_{k+1,\varphi} \backslash \mathbb{M}_{k,\varphi} \right] &\leq \nu^N \left[\sum_{k \in \mathbb{T}^d_N} \varphi \left(\theta^{k,N} \right) > N^d k \right] \\ &\leq e^{-\ell N^d k} \prod_{i \in \mathbb{T}^d_N} \int_{\mathbb{R}} e^{\ell \varphi(\theta)} \nu_{\frac{i}{N}} (\mathrm{d}\theta) \leq e^{-\ell N^d k} C^{N^d}, \end{aligned} \tag{V.3.93}$$

by Assumption V.3.4. Then

$$\begin{aligned} P^N_{\underline{w}^N} \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] &\leq C^{N^d} e^{-N^d R e^{-\lambda T} + N^d} \sum_{k=0}^{\infty} e^{N^d k (1-\ell)} \\ &\leq C^{N^d} e^{-N^d R e^{-\lambda T} + N^d} \frac{1}{1 - e^{N^d (1-\ell)}} \leq e^{-N^d R e^{-\lambda T} \frac{1}{2}}, \end{aligned} \tag{V.3.94}$$

for R large enough. □

For the Theorem V.3.28, we need compact subsets of $\mathscr{C}$. These sets are characterised in the following lemma.

Lemma V.3.33 (Lemma 1.3 in [Gär88])**.** *Let $\{f_n\}_n$ be a countable dense subset of $\mathsf{C}_c\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$. A set $\mathcal{K}$ is relatively compact in $\mathscr{C}$ if and only if*

$$\mathcal{K} \subset \mathcal{K}_K \cap \bigcap \mathcal{K}_n, \tag{V.3.95}$$

with

$$\mathcal{K}_K = \left\{ \mu_{[0,T]} \in \mathscr{C} : \mu_t \in K \text{ for all } t \in [0,T] \right\}, \tag{V.3.96}$$

$$\mathcal{K}_n = \left\{ \mu_{[0,T]} \in \mathscr{C} : \left\{ t \mapsto \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f_n(x,w,\theta)\, \mu_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right\} \in K_n \right\}, \tag{V.3.97}$$

where $K \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ and $K_n \subset \mathsf{C}([0,T])$ are compact.

For a proof of this lemma, see Lemma 1.3 in [Gär88].

The next lemma states an exponential bound on the probability that the empirical process is outside of a subset of $\mathscr{C}$, that is defined via the projection to $\mathsf{C}([0,T])$. We use this set in Theorem V.3.28 as the set $\mathcal{K}_n$, defined in Lemma V.3.33 in the characterisation of relative compact subset of $\mathscr{C}$.

Lemma V.3.34 (compare to Lemma 5.6 in [DG87] for the mean field case)**.** *For all $R > 0, s > 0$ and $f \in \mathsf{C}_c^\infty\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$, there exists a compact set $K \subset \mathsf{C}([0,T])$, such that for all $N \in \mathbb{N}$*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R} \backslash \mathcal{K}_f\right] \leq e^{-N^d s}, \tag{V.3.98}$$

with $\mathcal{K}_f = \left\{\mu_{[0,T]} \in \mathscr{C} : \left\{t \mapsto \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta)\, \mu_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\right\} \in K\right\}$.

Proof. Also this proof is formally exactly the proof of Lemma 5.6 in [DG87] for each $\underline{w}^N \in \mathcal{W}^{N^d}$. Indeed, in the proof one only uses the function $\left\{t \mapsto \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta)\, \mu_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\right\}$, which is (here as in [DG87]) a function in $\mathsf{C}([0,T], \mathbb{R})$ and one does not have to care about the structure within the integral. Moreover, the topology of $\mathscr{C}$ is not relevant in the proof. The proof requires the Assumption V.3.1 a.b). □

V.3.2.2 Proof of Theorem V.3.28

Proof of Theorem V.3.28. This proof equals the proof of Theorem 5.3 in [DG87], besides formal changes due to the space dependency. The only generalisation is that we consider random initial data here.

By Lemma V.3.33 it is enough to define compact sets $K \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ and $K_n \subset \mathsf{C}([0,T])$ to get a compact set in $\mathscr{C}$. We set $K = \mathbb{M}_{\varphi,R}$ and therefore $\mathcal{K}_K = \mathscr{C}_{\varphi,R}$. Moreover, we choose by Lemma V.3.34 for each n a $K_n \subset \mathsf{C}([0,T])$, such that

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}} P^N_{\underline{w}^N, \underline{\theta}^N} \left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R} \backslash \mathcal{K}_n\right] \leq e^{-nN^d s}. \tag{V.3.99}$$

Define the compact set $\mathcal{K} := \overline{\mathscr{C}_{\varphi,R} \cap \bigcap \mathcal{K}_n}$. This is a subset of $\mathscr{C}_{\varphi,R}$, because $\mathscr{C}_{\varphi,R}$ is closed in $\mathscr{C}$.

By Lemma V.3.32 and (V.3.99) we conclude for all $N \in \mathbb{N}$ and R large enough

$$\begin{aligned} &P^N\left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathcal{K}\right] \\ &\leq P^N\left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R}\right] + \sum_{n=1}^\infty \sup_{\underline{w}^N \in \mathcal{W}^{N^d}} \sup_{\underline{\theta}^N \in \mathbb{R}^{N^d}} P^N_{\underline{w}^N, \underline{\theta}^N}\left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R} \backslash \mathcal{K}_n\right] \\ &\leq e^{-N^d s} + \sum_{n=1}^\infty e^{-nN^d s}. \end{aligned} \tag{V.3.100}$$

□

V.3.2.3 Proof of Theorem V.3.27

We prove in this section the Theorem V.3.27. In the proof, we investigate separately the cases, when $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ (Case 1 and Case 2), and when it is not in this space (Case 3). Moreover, we divide the first case in the subcases that $\overline{\mu}_t \in \mathscr{C}^L$ (Case 1), and when this is not true (Case 2). The ideas of the proofs of the three cases are as follows:

Case 1: For $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$, we reduce the claims of Theorem V.3.27 to large deviation upper and lower bounds for a system of independent SDEs. For this independent system we know these large deviation bounds by Theorem V.3.9. To reduce the claims we choose at first (Step 1.1), for each $N \in \mathbb{N}$, a system of spins, that evolve mutually independent, with the constraint that their empirical process should be close to $\overline{\mu}_{[0,T]}$ with high probability. Therefore, we choose the drift coefficient $\overline{b}^I(x, w, \theta, t) := b(x, w, \theta, \overline{\mu}_t)$. We regard the empirical process of interacting diffusions, in a small neighbourhood of $\overline{\mu}_{[0,T]}$, as a small perturbation of the empirical process for the independent diffusions with drift coefficient $\overline{b}^I$. Then, in Step 1.2, we apply the (Cameron-Martin-) Girsanov theorem and receive a density between the measures of the solution to the original SDE and the one of the SDE with drift coefficient $\overline{b}^I$. Using this density, we reduce in Step 1.3 and Step 1.4 the claims of Theorem V.3.27 to large deviation bounds for the independent system. We get these bounds by Theorem V.3.9, which is applicable by Section V.3.2.1.1.

The proof of this first case is very similar to the one in [DG87] in Section 5.4 for the mean-field setting. However, differences arise due to the space and random environment dependency. Moreover, we show the large deviation principle on the space $\mathscr{C}$ and not like in [DG87] on $\mathscr{C}_{\varphi,\infty}$ equipped even with another topology than the subspace topology.

Case 2: If we assumed in Assumption V.3.1 a.a) that the continuity of b holds on $\mathbb{M}_{\varphi,R}$ and not only on $\mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$, then we could handle the case of $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ with $\overline{\mu}_t \notin \mathbb{M}_1^L(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ for some $t \in [0,T]$, as in the previous step. However, to keep the assumption more general, we have to use a new approach. Indeed, we show that no empirical process is within an ϵ-ball around $\overline{\mu}_{[0,T]}$ for N large enough. From this we infer the claims of Theorem V.3.27.

Case 3: When $\overline{\mu}_{[0,T]}$ is not in $\mathscr{C}_{\varphi,\infty}$, then the first statement of Theorem V.3.27 is obviously satisfied and the second statement follows from Lemma V.3.32.

Proof. Fix an arbitrary $\overline{\mu}_{[0,T]} \in \mathscr{C}$.
Case 1: $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$:
Step 1.1: Definition of a system of diffusions with a fixed effective field:

We set $b^I(x, w, \theta, t) := b(x, w, \theta, \overline{\mu}_t)$ and use this function as drift coefficient to define the time dependent diffusion generator $\mathbb{L}^I_{t,x,w}$ (defined as in (V.3.11)). Then $\mathbb{L}^I_{t,x,w} = \mathbb{L}_{\overline{\mu}_t,x,w}$. Moreover, we define the measures $P^{I,N} \in \mathbb{M}_1\Big(\mathsf{C}([0,T])^{N^d}\Big)$ as in Notation V.3.8.

As shown in Section V.3.2.1.1, the Assumption V.3.1 implies the Assumptions V.3.7 for the generator $\mathbb{L}^I_{t,x,w}$. Therefore, the Theorem V.3.9 is applicable for $P^{I,N}$.

Step 1.2: Comparison of the two processes with help of the Girsanov theorem:

We claim that for each $\underline{w}^N \in \mathcal{W}^{N^d}$, $P^N_{\underline{w}^N}$ is absolutely continuous with respect to $P^{I,N}_{\underline{w}^N}$, with Radon-Nikodym derivative

$$\frac{\mathrm{d}P^N_{\underline{w}^N}}{\mathrm{d}P^{I,N}_{\underline{w}^N}} = e^{M^N_{\underline{w}^N,T} - \frac{1}{2}\langle\langle M^N_{\underline{w}^N}\rangle\rangle_T}, \tag{V.3.101}$$

for all $\underline{\theta}^N \in \mathbb{R}^{N^d}$. Here $M^N_{\underline{w}^N,t}$ is a continuous local $P^{I,N}_{\underline{w}^N}$ martingale with quadratic variation

$$\langle\langle M^N_{\underline{w}^N}\rangle\rangle_t\left(\underline{\theta}^N_{[0,T]}\right) = N^d \int_0^t \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \sigma^2 \left|b\left(x, w, \theta, \mu^N_u\right) - b\left(x, w, \theta, \overline{\mu}_u\right)\right|^2 \mu^N_u(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta)\,\mathrm{d}u, \tag{V.3.102}$$

where μ^N_u is the empirical measure defined by $\underline{\theta}^N_u$ and $\underline{w}^N$. This can be shown by a spatial localisation argument. The generators $\mathbb{L}^N_\cdot$ and $\mathbb{L}^{I,N}_\cdot$ only differ in their drift coefficients. The martingale problems corresponding to both generators are well defined. Moreover, b^N (defined in Assumption V.3.1 a.b)) and b^I (as continuous function) are both locally bounded. By spatial localisation (see [SV79] Theorem 10.1.1) it is hence enough to consider bounded drift coefficients. For bounded drift coefficients, we know by [SV79] Theorem 6.4.2 the claimed representation of the Radon-Nikodym formula.

Step 1.3: The proof of (i):

For $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = \infty$, (i) is obviously satisfied. Therefore, assume that $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) < \infty$. Fix an open neighbourhood $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$ and an arbitrary $\gamma > 0$.

The Lemma V.3.32 can also be applied to $P^{I,N}$ instead of P^N by Assumption V.3.1 c.). This lemma then states (with $s = S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma$), that there is a $R > 0$ such that

$$P^{I,N}\left[\mu^N_{[0,T]} \in \mathscr{C}\backslash\mathscr{C}_{\varphi,R}\right] \le e^{-N^d S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)} e^{-N^d\gamma}. \tag{V.3.103}$$

Assume that this R is so large that $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R}$. We choose now two constants $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and a $\delta > 0$ such that

$$\frac{1}{2}\left(1 + \frac{p}{q}\right)\delta + pS_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) \le S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma. \tag{V.3.104}$$

By Assumption V.3.1 d.) and (V.3.102), there is a open neighbourhood $W \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$ such that $W \cap \mathscr{C}_{\varphi,R} \subset V$ and $\langle\langle M^N_{\underline{w}^N}\rangle\rangle_T\left(\underline{\theta}^N_{[0,T]}\right) \le N^d\delta$ for $\underline{w}^N \in \mathcal{W}^{N^d}$ and $\underline{\theta}^N_{[0,T]} \in \mathsf{C}([0,T])^{N^d}$ when the corresponding empirical processes $\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}$. With the same arguments as in [DG87] we can show by using the Radon-Nikodym derivative (V.3.101) that for each $\underline{w}^N \in \mathcal{W}$

$$\begin{aligned} P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in V\right] &\ge P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right] \\ &\ge e^{-\frac{1}{2}\left(1+\frac{p}{q}\right)\delta N^d}\left(P^{I,N}_{\underline{w}^N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right]\right)^p. \end{aligned} \tag{V.3.105}$$

We integrate (V.3.105) with respect to ζ^N and apply the Jensen inequality,

$$P^N\left[\mu^N_{[0,T]} \in V\right] \ge e^{-\frac{1}{2}\left(1+\frac{p}{q}\right)\delta N^d}\left(P^{I,N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right]\right)^p. \tag{V.3.106}$$

Moreover

$$P^{I,N}\left[\mu^N_{[0,T]} \in W \cap \mathscr{C}_{\varphi,R}\right] \ge P^{I,N}\left[\mu^N_{[0,T]} \in W\right]\left(1 - e^{-N^d\frac{\gamma}{2}}\right), \tag{V.3.107}$$

for N large enough. Indeed, (V.3.107) holds, by the triangle inequality and

$$P^{I,N}\left[\mu^N_{[0,T]} \notin \mathscr{C}_{\varphi,R}\right] \le e^{-N^d S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)} e^{-N^d\gamma} \le e^{-N^d\frac{\gamma}{2}} P^{I,N}\left[\mu^N_{[0,T]} \in W\right], \tag{V.3.108}$$

by (V.3.103) and because W is an open set and $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ satisfies the large deviation principle (Theorem V.3.9).

Combine (V.3.106) and (V.3.107), we get

$$\begin{aligned} &\liminf_{N\to\infty} N^{-d}\log P^N\left[\mu^N_{[0,T]} \in V\right] \\ &\ge -\frac{1}{2}\left(1 + \frac{p}{q}\right)\delta + p\liminf_{N\to\infty} N^{-d}\log P^{I,N}\left[\mu^N_{[0,T]} \in W\right]. \end{aligned} \tag{V.3.109}$$

Finally, we conclude by the large deviation principle for $\left\{\mu^N_{[0,T]}, P^{I,N}\right\}$ (Theorem V.3.9) and (V.3.104)

$$(\text{V.3.109}) \geq -\frac{1}{2}\left(1+\frac{p}{q}\right)\delta - pS_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) \geq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) - \gamma. \tag{V.3.110}$$

This inequality holds for all $\gamma > 0$. Hence we have proven (i) for this case.

Step 1.4: The proof of (ii):

We assume $S_{\nu,\zeta}(\overline{\mu}) < \infty$. The case when it is not finite can be treated analogue. Fix a $\gamma > 0$. Due to Lemma V.3.32 it is sufficient to find for $R > 0$ large enough with $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R}$, an open neighbourhood $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$ such that

$$\limsup_{N\to\infty} N^{-d} \log P^N\left[\mu^N_{[0,T]} \in V \cap \mathscr{C}_{\varphi,R}\right] \leq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma. \tag{V.3.111}$$

Fix again $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and a $\delta > 0$, such that

$$\frac{p-1}{2}\delta + \frac{1}{q}\left(-S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \frac{\gamma}{2}\right) \leq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \gamma. \tag{V.3.112}$$

By Assumption V.3.1 d.) and (V.3.102) and by Theorem V.3.9, there is a small open neighbourhood $V \subset \mathscr{C}$ of $\overline{\mu}_{[0,T]}$, such that $\langle\langle M^N_{\underline{w}^N}\rangle\rangle_T\left(\underline{\theta}^N_{[0,T]}\right) \leq N^d\delta$ for $\underline{w}^N \in \mathcal{W}^{N^d}$ and $\underline{\theta}^N_{[0,T]} \in \mathsf{C}([0,T])^{N^d}$ when the corresponding empirical processes $\mu^N_{[0,T]} \in V \cap \mathscr{C}_{\varphi,R}$, and such that

$$\limsup_{N\to\infty} N^{-d} \log P^{I,N}\left[\mu^N_{[0,T]} \in V\right] \leq -S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) + \frac{\gamma}{2}. \tag{V.3.113}$$

In the last inequality we use that $S^I_{\nu,\zeta}$ is lower semi-continuous and $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = S^I_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right)$ As in [DG87], we can show, by using the Radon-Nikodym derivative (V.3.101), that for all $\underline{w}^N \in \mathcal{W}$

$$P^N_{\underline{w}^N}\left[\mu^N_{[0,T]} \in V \cap \mathscr{C}_{\varphi,R}\right] \leq e^{\frac{p-1}{2}\delta N}\left(P^{I,N}_{\underline{w}^N}\left[\mu^N_{[0,T]} \in V\right]\right)^{\frac{1}{q}}. \tag{V.3.114}$$

To conclude (V.3.111), integrate both sides with respect to ζ^N, apply the Jensen inequality and finally use (V.3.112) and (V.3.113). Hence we showed (ii) for this case.

Case 2: $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ and $\overline{\mu}_{[0,T]} \notin \mathscr{C}^L$:

Fix an arbitrary $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ with $\overline{\mu}_{[0,T]} \notin \mathscr{C}^L$. Then $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = \infty$, by the definition of the rate function. This implies that (i) of Theorem V.3.27 is obviously satisfied.

Now we prove that (ii) of Theorem V.3.27 holds. At first we fix an open ball around $\overline{\mu}_{[0,T]}$, that does not intersect $\mathscr{C}^L$ (Step 2.1). Then we show that in such an open ball there is no empirical process with N large enough (Step 2.2). From this we conclude (ii) (in Step 2.3).

Step 2.1: A open ball around $\overline{\mu}_{[0,T]}$: The set $\mathbb{M}^L_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ is closed in $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ (see e.g. [ABM06] Proposition 4.3.1). This implies that also the set $\mathscr{C}^L$ is closed in $\mathscr{C}$. Hence there is a $\epsilon > 0$ such that

$$\operatorname{dist}\left\{\overline{\mu}_{[0,T]}, \mathscr{C}^L\right\} = \inf_{\pi\in\mathscr{C}^L}\left\{\sup_{t\in[0,T]} \rho^{Lip}\left(\overline{\mu}_t, \pi_t\right)\right\} > 2\epsilon, \tag{V.3.115}$$

where ρ^{Lip} is the bounded Lipschitz norm on $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$.

Define the open ϵ ball $B_\epsilon\left(\overline{\mu}_{[0,T]}\right)$ around $\overline{\mu}_{[0,T]}$ in this norm.

Step 2.2: No empirical process in the open ball for N large enough: Assume that we could find a sequence $N_\ell \nearrow \infty$ in $\mathbb{N}$, such that for each N_ℓ there is an empirical process $\mu^{N_\ell}_{[0,T]} \in B_\epsilon(\overline{\mu})$, with a $\underline{\theta}^{N_\ell}_{[0,T]} \subset \mathsf{C}([0,T])^{N_\ell^d}$ and a $\underline{w}^N \in \mathcal{W}^{N^d}$. We claim that this leads to a contradiction. For each N_ℓ in the sequence, define $\mu^{(\ell)}_{t,x} = \delta_{w^{k,N_\ell}} \delta_{\theta^{k,N_\ell}_t}$ when $\left| x - \frac{k}{N_\ell} \right| < \frac{1}{2N_\ell}$. Then $\left\{ t \mapsto \mu^{(\ell)}_t := \mathrm{d}x \otimes \mu^{(\ell)}_{t,x} \right\} \in \mathscr{C}^L$. For each $f \in \mathsf{C}\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ that is Lipschitz continuous with $|f|_\infty + |f|_{Lip} \le 1$,

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta)\, \mu^{N_\ell}_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) - \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} f(x,w,\theta)\, \mu^{(\ell)}_t(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \right| \\
&= \sum_{k \in \mathbb{T}^d_{N_\ell}} \left| \frac{1}{N_\ell} f\left(\frac{k}{N_\ell}, w^{k,N_\ell}, \theta^{k,N_\ell}_t \right) - \int_{\Delta_{k,N_\ell}} f\left(x, w^{k,N_\ell}, \theta^{k,N_\ell}_t \right) \mathrm{d}x \right| \\
&\le \sum_{k \in \mathbb{T}^d_N} |f|_{Lip} \left(\frac{1}{N_\ell} \right)^2 \le \frac{1}{N_\ell},
\end{aligned} \tag{V.3.116}$$

with Δ_{k,N_ℓ} defined as in Assumption V.1.4. Hence the distance between $\mu^{N_\ell}_{[0,T]}$ and $\mathscr{C}^L$ vanishes, a contraction. Therefore, we can fix an $\overline{N} \in \mathbb{N}$, such that there is no empirical process $\mu^N_{[0,T]}$ in $B_\epsilon\left(\overline{\mu}_{[0,T]}\right)$ when $N > \overline{N}$.

Step 2.3: Conclusion of (ii): From the previous step we infer that for $N > \overline{N}$,

$$P^N \left[\mu^N_{[0,T]} \in B_\epsilon\left(\overline{\mu}_{[0,T]} \right) \right] = 0. \tag{V.3.117}$$

This implies (ii) of Theorem V.3.27 for this case.

Case 3: $\overline{\mu}_{[0,T]} \notin \mathscr{C}_{\varphi,\infty}$:

Because $\overline{\mu}_{[0,T]} \notin \mathscr{C}_{\varphi,\infty}$, $S_{\nu,\zeta}\left(\overline{\mu}_{[0,T]}\right) = \infty$. Therefore, the condition (i) of Theorem V.3.27 is obviously satisfied. To prove (ii) of Theorem V.3.27, note that for each $R > 0$, the open set $\mathscr{C} \backslash \mathscr{C}_{\varphi,R}$ is a neighbourhood of $\overline{\mu}_{[0,T]}$. By Lemma V.3.32, there is for each γ and R such that

$$P^N \left[\mu^N_{[0,T]} \in \mathscr{C} \backslash \mathscr{C}_{\varphi,R} \right] \le e^{-N^d \gamma}. \tag{V.3.118}$$

This implies the claimed condition (ii) of Theorem V.3.27 in this case. □

V.3.3 The concrete example (0.9.3) of a local mean field model

In this section we show that the concrete example (0.9.3) of a local mean field model, defined by $\sigma = 1$ and b being the drift coefficient of this SDE, with Assumption V.1.1, Assumption V.1.2, Assumption V.1.4 and Assumption V.1.7, satisfies the Assumptions V.3.1.

Proof. Fix $\varphi(\theta) := 1 + \theta^2$. We show now separately that each item of Assumption V.3.1 is satisfied.

Step 1: Assumption V.3.1 a.a):

The function $\partial_\theta \Psi$ is continuous by Assumption V.1.7. Hence the drift coefficient is continuous on $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \big(\mathbb{M}_{\varphi,R} \cap \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)\big)$ if the map

$$(x,w,\mu) \mapsto \beta(x,w,\mu) := \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} J(x - x', w, w')\, \theta' \mu(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta') \tag{V.3.119}$$

is continuous on this space. This holds if for $R > 0$ and each sequence $\left(x^{(n)}, w^{(n)}, \mu^{(n)}\right) \to (x, w, \mu)$ in $\mathbb{T}^d \times \mathcal{W} \times \left(\mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)\right)$, the following absolute value vanishes

$$\begin{aligned}\left|\beta\left(x^{(n)}, w^{(n)}, \mu^{(n)}\right) - \beta(x, w, \mu)\right| \leq &\left|\beta\left(x, w, \mu^{(n)}\right) - \beta\left(x^{(n)}, w^{(n)}, \mu^{(n)}\right)\right| \\ &+ \left|\beta\left(x, w, \mu^{(n)}\right) - \beta(x, w, \mu)\right| =: ① + ② .\end{aligned} \tag{V.3.120}$$

We show now that ① and ② vanish when n tends to infinity.
Step 1.1: ① : There is a sequence of continuous functions $J_\ell \in \mathsf{C}\left(\mathbb{T}^d \times \mathcal{W} \times \mathcal{W}\right)$, such that $J_\ell \to J$ in $L^2\left(\mathbb{T}^d, \mathsf{C}(\mathcal{W} \times \mathcal{W})\right)$, because $J \in L^2\left(\mathbb{T}^d, \mathsf{C}(\mathcal{W} \times \mathcal{W})\right)$. This implies that for all $\overline{x} \in \mathbb{T}^d$, $\overline{w} \in \mathcal{W}$ and $n \in \mathbb{N}$

$$\begin{aligned}&\left|\int_{\mathbb{T}^d \times \mathcal{W}} (J - J_\ell)(\overline{x} - x', \overline{w}, w') \int_{\mathbb{R}} \theta' \mu^{(n)}_{x',w'}(\mathrm{d}\theta')\, \mu^{(n)}_{x',\mathcal{W}}(\mathrm{d}w')\, \mathrm{d}x'\right| \\ &\leq \left(\int_{\mathbb{T}^d} \left(\sup_{w',w'' \in \mathcal{W}} \left|(J - J_\ell)(x, w'', w')\right|\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} R,\end{aligned} \tag{V.3.121}$$

because $\mu^{(n)} \in \mathbb{M}_{\varphi,R}$. Therefore ① is lesser or equal to

$$\sup_{x' \in \mathbb{T}^d, w' \in \mathcal{W}} \left|J_\ell\left(x^{(n)} - x', w^{(n)}, w'\right) - J_\ell(x - x', w, w')\right| (1 + R) + 2\left\|J - J_\ell\right\| R \leq \epsilon, \tag{V.3.122}$$

for $k \in \mathbb{N}$ and $n \in \mathbb{N}$ large enough, because J_ℓ is uniformly continuous on the compact set $\mathbb{T}^d \times \mathcal{W} \times \mathcal{W}$.

Step 1.2: ②: To bound ②, define the function $\chi_M(\theta) := (\theta \wedge M) \vee -M$ and approximate J by J_ℓ as in the previous step. Then ② is lesser or equal to

$$\begin{aligned}&\left|\int_{\mathbb{T}^d \times \mathcal{W}} (J - J_\ell)(x - x', w, w') \int_{\mathbb{R}} \theta' \mu^{(n)}_{x'}(\mathrm{d}\theta')\, \mu^{(n)}_{x',\mathcal{W}}(\mathrm{d}w')\, \mathrm{d}x'\right| + \text{ (this integral with } \mu) \\ &+ \left|\int J_\ell(x - x', w, w')(\theta' - \chi_M(\theta'))\, \mu^{(n)}(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta')\right| + \text{ (this integral with } \mu) \\ &+ \left|\int J_\ell(x - x', w, w')\, \chi_M(\theta')\left(\mu^{(n)} - \mu\right)(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta')\right| := Ⓐ + Ⓑ + Ⓒ + Ⓓ + Ⓔ .\end{aligned} \tag{V.3.123}$$

The Ⓐ and Ⓑ are bounded by ϵ, when k and n are large enough as shown in (V.3.121). We bound Ⓒ by

$$\begin{aligned}Ⓒ &\leq |J_\ell|_\infty \int |\theta'|\, \mathbb{1}_{|\theta'| > M}\, \mu^{(n)}(\mathrm{d}x', \mathrm{d}w, \mathrm{d}\theta') \\ &\leq |J_\ell|_\infty\, \mu^{(n)}\left[(x, w, \theta') : |\theta'| > M\right]^{\frac{1}{2}} \left(\int (\theta')^2 \mu^{(n)}(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta')\right)^{\frac{1}{2}} .\end{aligned} \tag{V.3.124}$$

For an arbitrary fixed $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$, the right hand side is bounded by ϵ for M large enough, because $\mu^{(n)} \in \mathbb{M}_{\varphi,R}$ and by the tightness of $\left\{\mu^{(n)}\right\}_n$ (as a converging sequence). The same arguments show Ⓓ is bounded by ϵ. The Ⓔ converges to zero when $n \to \infty$, for arbitrary fixed k and M, because the integrand is bounded and continuous.

Therefore, we fix at first a $k \in \mathbb{N}$, then an $M > 0$. Then for $n \in \mathbb{N}$ large enough, ② is bounded by ϵ.

We have hence shown that (V.3.120) vanishes when n tends to infinity, i.e. that β is continuous.

Step 2: Assumption V.3.1 a.b):

Fix an arbitrary $N \in \mathbb{N}$ and an arbitrary $\underline{w}^N \in \mathcal{W}$. The function $b^N : \mathbb{R}^{N^d} \to \mathbb{R}^{N^d}$ is continuous, by Assumption V.1.7 and because $\frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} J\left(\frac{i-j}{N}, w^{i,N}, w^{j,N}\right) \theta^{j,N}$ is continuous (because $J\left(\frac{i-j}{N}, w, w'\right)$ is finite for all $i, j \in \mathbb{T}_N^d$ and all $w, w' \in \mathcal{W}$ by Assumption V.1.4). Hence b^N is locally bounded.

Step 3: Assumption V.3.1 b.):

Let $\mathbb{L}^{\mathrm{LMF}}_{\mu^N,\cdot,\cdot}$ be the generator of the local mean field model, defined as (V.1.6), with μ^N the empirical measure corresponding to $\underline{\theta}^N \in \mathbb{R}^{N^d}$ and $\underline{w}^N \in \mathcal{W}$. Then for N large enough

$$\begin{aligned}
&\int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \mathbb{L}^{\mathrm{LMF}}_{\mu^N,x,w} \varphi(\theta) + \frac{1}{2} \left|\partial_\theta \varphi(\theta)\right|^2 \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \\
&= 2 + 2 \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \left(-\overline{\Psi}'(\theta)\theta - w\theta^2 + \theta^2\right) \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) + 2B^N_{\underline{w}^N}\left(\underline{\theta}^N\right),
\end{aligned} \tag{V.3.125}$$

where

$$\begin{aligned}
&B^N_{\underline{w}^N}\left(\underline{\theta}^N\right) := \frac{1}{N^{2d}} \sum_{i,j \in \mathbb{T}_N^d} J\left(\frac{i-j}{N}, w^{i,N}, w^{j,N}\right) \theta^{i,N} \theta^{j,N} \\
&\leq \left(\sum_{i \in \mathbb{T}_N^d} \sup_{w,w' \in \mathcal{W}} \left|\frac{1}{N^d} J\left(\frac{i}{N}, w, w'\right) - \int_{\Delta_{i,N}} J(x, w, w')\,\mathrm{d}x\right| + \left\|\overline{J}\right\|_{L^1}\right) \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} \left(\theta^{j,N}\right)^2 \qquad \text{(V.3.126)} \\
&\leq \left(\delta + \left\|\overline{J}\right\|_{L^1}\right) \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} \left(\theta^{j,N}\right)^2,
\end{aligned}$$

with $\delta > 0$ if $N > \overline{N}_\delta$ by Assumption V.1.4. With this upper bound on B^N, Ψ being a polynomial of even degree with positive coefficient of this degree (Assumption V.1.7) and $\mathcal{W}$ being compact, we conclude that

$$\begin{aligned}
\text{(V.3.125)} &\leq C + 2 \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \left(|w| + 1 + \left\|\overline{J}\right\|_{L^1} + \delta\right) \theta^2 \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta) \\
&\leq \lambda \int_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \varphi(\theta)\, \mu^N(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta).
\end{aligned} \tag{V.3.127}$$

Here the constant λ only depends on Ψ and J for N large enough but not on μ^N. Hence the Assumption V.3.1 b.) is satisfied.

Step 4: Assumption V.3.1 c.):

Fix an arbitrary $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty} \cap \mathscr{C}^L$. We know by Step 1, that $(x, w, t) \mapsto \beta(x, w, \mu_t)$ is continuous. Moreover, the set $\mathbb{T}^d \times \mathcal{W} \times \{\mu_t\}_{t \in [0,T]}$ is compact in $\mathbb{T}^d \times \mathcal{W} \times \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$, by Prokhorov's theorem. Hence β is bounded on this set by a constant C_β. Then for all $(t, x, w, \theta) \in [0, T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$

$$\begin{aligned}
\mathbb{L}^{\mathrm{LMF}}_{\mu_t,x,w} \varphi(\theta) + \frac{1}{2} \left|\partial_\theta \varphi(\theta)\right|^2 &= -2\partial_\theta \overline{\Psi}(\theta)\theta - 2w\theta^2 + 2\theta\beta(x, \mu_t) + 2 + 2\theta^2 \\
&\leq -2\partial_\theta \overline{\Psi}(\theta)\theta + 2|w|\theta^2 + 2|\theta| C_\beta + 2 + 2\theta^2 \leq \lambda\left(\mu_{[0,T]}\right) \varphi(\theta),
\end{aligned} \tag{V.3.128}$$

because $\overline{\Psi}$ is a polynomial of even degree (Assumption V.1.7) and $\mathcal{W}$ is compact.

Step 5: Assumption V.3.1 d.):

Fix an $R > 0$ and a $\overline{\mu}_{[0,T]} \in \mathscr{C}_{\varphi,R} \cap \mathscr{C}^L$. Take an arbitrary sequence $\left\{\mu^{(n)}_{[0,T]}\right\}$ from one of the sets given in Assumptions V.3.1 d.), such that $\mu^{(n)}_{[0,T]} \to \overline{\mu}_{[0,T]}$. We show in the subsequent steps that

$$\int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left|\beta\left(x,w,\mu_t^{(n)}\right)-\beta\left(x,w,\overline{\mu}_t\right)\right|^2 \mu_t^{(n)}\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta\right)\mathrm{d}t \to 0. \tag{V.3.129}$$

Step 5.1: Case: All $\mu_{[0,T]}^{(n)} \in \mathscr{C}^L$:

Assume at first that $\mu_{[0,T]}^{(n)} \in \mathscr{C}_{\varphi,R}\cap\mathscr{C}^L$ for all $n\in\mathbb{N}$. For each $t\in[0,T]$, $\mu_t^{(n)}\to\overline{\mu}_t$ in $\mathbb{M}_{\varphi,R}$ by the uniform topology on $\mathscr{C}$. Therefore, the set $U_t := \left\{\mu_t^{(n)}\right\}_n \cup \{\overline{\mu}_t\}$ is compact. $\mathbb{T}^d\times\mathcal{W}\times U_t \ni (x,w,\mu)\mapsto |\beta(x,w,\mu)-\beta(x,w,\overline{\mu}_t)|$ is uniformly continuous (we show the continuity in Step 1). Hence for each $t\in[0,T]$, the absolute value in (V.3.129) converges uniformly in $(x,w)\in\mathbb{T}^d\times\mathcal{W}$ to zero, when n tends to infinity. Moreover, this absolute value is uniformly bounded, because for all $(x,w)\in\mathbb{T}^d\times\mathcal{W}$

$$\left|\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} J\left(x-x',w,w'\right)\theta'\mu_t^{(n)}\left(\mathrm{d}x',\mathrm{d}w',\mathrm{d}\theta'\right)\right|^2 \le \left\|\overline{J}\right\|_{L^2}^2 \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \left|\theta'\right|^2 \mu_t^{(n)}\left(\mathrm{d}x',\mathrm{d}w',\mathrm{d}\theta'\right). \tag{V.3.130}$$

The right hand side is bounded by $\left\|\overline{J}\right\|_{L^2}^2 R$, because all $\mu_{[0,T]}^{(n)}$ are in $\mathscr{C}_{\varphi,R}$. This implies the convergence (V.3.129) for sequences in $\mathscr{C}^L$.

Step 5.2: Case: All $\mu_{[0,T]}^{(n)}$ are empirical processes:

Fix a sequence of empirical processes $\left\{\mu_{[0,T]}^{(n)}\right\}_n \subset \mathscr{C}_{\varphi,R}$, such that $\mu_{[0,T]}^{(n)}\to\overline{\mu}_{[0,T]}$. Fix $N_n\in\mathbb{N}$, $\theta_{[0,T]}^{i,N_n}\in\mathsf{C}([0,T])$, $w^{i,N_n}\in\mathcal{W}$ such that $\mu_{[0,T]}^{(n)} = \frac{1}{N_n^d}\sum_{i\in\mathbb{T}_{N_n}^d}\delta_{\left(\frac{i}{N_n},w^{i,N_n},\theta_t^{i,N_n}\right)}$. Note that we do not get for this sequence the continuity of β at $t\in[0,T]$ from Step 1.

For each $t\in\mathbb{T}_N^d$ and $n\in\mathbb{N}$, the inner integral in (V.3.129) is given by

$$\frac{1}{N_n^d}\sum_{j\in\mathbb{T}_{N_n}^d}\left|\beta\left(\frac{j}{N_n},w^{j,N_n},\mu_t^{(n)}\right)-\beta\left(\frac{j}{N_n},w^{j,N_n},\overline{\mu}_t\right)\right|^2. \tag{V.3.131}$$

We show in the following that this sum converges for each $t\in[0,T]$ pointwise to zero (Step 5.2.1). Moreover, we show that this sum is uniformly bounded (Step 5.2.2). From these two results we conclude (V.3.129) by the dominated convergence theorem.

Step 5.2.1: (V.3.131) **vanishes pointwise:** To show that (V.3.131) vanishes, we divide the absolute value as in (V.3.123) into five summands. Fix an arbitrary small $\epsilon>0$. By fixing $k\in\mathbb{N}$ and $M>0$ large enough, the Ⓑ, Ⓒ and Ⓓ of these summands are smaller than ϵ for all $(x,w)\in\mathbb{T}^d\times\mathcal{W}$ for fixed k and all $n\in\mathbb{N}$ large enough, by the same arguments that we use in Step 1.2. Hence to bound (V.3.131) we only need to bound the following two summands

$$\begin{aligned}
Ⓐ &:= \frac{1}{N_n^d}\sum_{j\in\mathbb{T}_{N_n}^d}\left|\frac{1}{N_n^d}\sum_{i\in\mathbb{T}_{N_n}^d}\theta_t^{i,N_n}\left(J-J_\ell\right)\left(\frac{j-i}{N_n},w^{j,N_n},w^{i,N_n}\right)\right|,\\
Ⓔ &:= \frac{1}{N_n^d}\sum_{j\in\mathbb{T}_{N_n}^d}\left|\int J_\ell\left(\frac{j}{N_n}-x',w^j,w^i\right)\chi_M\left(\theta'\right)\left(\mu_t^{(n)}-\overline{\mu}_t\right)\left(\mathrm{d}x',\mathrm{d}\theta'\right)\right|.
\end{aligned} \tag{V.3.132}$$

We prove now that Ⓐ and Ⓔ are smaller than ϵ when n is large enough (Step 5.2.1.3 and Step 5.2.1.2). Both proofs require that N_n converges to infinity. We show in Step 5.2.1.1, that this is a consequence of the convergence of $\mu_t^{(n)}$ to a measures in $\mathbb{M}_1^L\left(\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\right)$.

Step 5.2.1.1: The sequence $N_n \to \infty$: Assume that this were not the case, i.e. that there is a subsequence $\{N_{n_\ell}\}_{\ell=1}^{\infty}$ such that $N_{n_\ell} \le \overline{N} < \infty$. This is a contradiction to the convergence of $\mu_t^{(n)}$ to $\overline{\mu}_t$. Indeed, choose $f \in \mathsf{C}_b\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ such that $f(x,w,\theta) = f(x) \ge 0$ for all $(x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, $\int_{\mathbb{T}^d} f(x)\,\mathrm{d}x > 0$ and $f\left(\frac{k}{N}\right) = 0$ for all $N \le \overline{N}$, $k \in \mathbb{T}^d_N$. Then $\int f(x)\,\mu_t^{(n_\ell)} = 0$ for all $\ell \in \mathbb{N}$, but $\int f(x)\,\overline{\mu}_t > 0$. A contradiction.

Step 5.2.1.2: Ⓔ: The function J_ℓ is uniformly continuous on $\mathbb{T}^d \times \mathcal{W}$. By the compactness of $\mathbb{T}^d \times \mathcal{W}$, there are finitely many $\{x_a\}_{a\in A} \subset \mathbb{T}^d$ and finitely many $\{w_{a'}\}_{a'\in A'} \subset \mathcal{W}$, such that

$$\text{Ⓔ} \le 2\epsilon M + \max_{a\in A, a'\in A'} \left| \int J_\ell\left(x_a - x', w_{a'}, w'\right) \chi_M\left(\theta'\right) \left(\mu_t^{(n)} - \overline{\mu}_t\right)\left(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta'\right) \right|. \qquad \text{(V.3.133)}$$

The maximum is only over a finite number of values, hence the convergence of $\mu_t^{(n)}$ to $\overline{\mu}_t$ implies that for n large enough, the maximum is bounded by ϵ.

Step 5.2.1.3: Ⓐ: We bound Ⓐ by ϵ through a similar estimate as in Step 1.1. In particular we use the following estimate instead of (V.3.121). For all $j \in \mathbb{T}^d_{N_n}$, Ⓐ is less or equal to

$$\begin{aligned} &\left| \sum_{i\in\mathbb{T}^d_{N_n}} \theta_t^{i,N_n} \int_{\Delta_{i,N_n}} \Big(J - J_\ell\Big)\left(\frac{j}{N_n} - x', w^{j,N_n}, w^{i,N_n}\right) \mathrm{d}x' \right| \\ &+ \left| \sum_{i\in\mathbb{T}^d_{N_n}} \theta_t^{i,N_n} \left(\frac{1}{N_n^d} J_\ell\left(\frac{j-i}{N_n}, w^{j,N_n}, w^{i,N_n}\right) - \int_{\Delta_{i,N_n}} J_\ell\left(\frac{j}{N_n} - x', w^{j,N_n}, w^{i,N_n}\right) \mathrm{d}x' \right) \right| \qquad \text{(V.3.134)} \\ &+ \left| \sum_{i\in\mathbb{T}^d_{N_n}} \theta_t^{i,N_n} \left(\frac{1}{N_n^d} J\left(\frac{j-i}{N_n}, w^{j,N_n}, w^{i,N_n}\right) - \int_{\Delta_{i,N_n}} J\left(\frac{j}{N_n} - x', w^{j,N_n}, w^{i,N_n}\right) \mathrm{d}x' \right) \right|. \end{aligned}$$

We denote the three summands by Ⓐ1, Ⓐ2 and Ⓐ3 and we bound them separately. By applying twice the Hölder inequality

$$\text{Ⓐ1} \le \left(\frac{1}{N_n^d} \sum_{i\in\mathbb{T}^d_{N_n}} \left|\theta_t^{i,N_n}\right|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} \left(\sup_{w,w'\in\mathcal{W}} \left|(J - J_\ell)(x', w, w')\right| \right)^2 \mathrm{d}x' \right)^{\frac{1}{2}} \le R\epsilon, \qquad \text{(V.3.135)}$$

for k large enough.

$$\text{Ⓐ2} \le \frac{1}{N_n^d} \sum_{i\in\mathbb{T}^d_{N_n}} \left|\theta_t^{i,N_n}\right| \sup_{|y-y'|\le\frac{1}{N_n}} \sup_{w,w'\in\mathcal{W}} \left|J_\ell\left(y', w, w'\right) - J_\ell\left(y, w, w'\right)\right| \le R\epsilon, \qquad \text{(V.3.136)}$$

for each k, when n (and hence N_n) is large enough. Last but not least, by a change of variables

$$\left(\text{Ⓐ3}\right)^2 \le \frac{1}{N_n^d} \sum_{i\in\mathbb{T}^d_{N_n}} \left|\theta_t^{i,N_n}\right|^2 \sum_{i\in\mathbb{T}^d_{N_n}} \sup_{w,w'\in\mathcal{W}} \left| \int_{\Delta_{i,N_n}} J\left(\frac{i}{N_n}, w, w'\right) - J\left(x', w, w'\right) \mathrm{d}x' \right|^2, \qquad \text{(V.3.137)}$$

which is also bounded by $R\epsilon$, when n is large enough by Assumption V.1.4.

Step 5.2.2: (V.3.131) **is uniformly (in $t \in [0,T]$) bounded:**

We show that each summand of (V.3.131) is bounded uniformly in $t \in [0,T]$, $j \in \mathbb{T}^d_{N_n}$, $n \in \mathbb{N}$.

By applying the Hölder inequality we get

$$\begin{aligned}\left|\beta\left(\frac{j}{N_n}, w^{j,N_n}, \mu_t^{(n)}\right)\right|^2 \leq \frac{1}{N_n^d} \sum_{i\in\mathbb{T}^d_{N_n}} \left|\theta_t^{i,N_n}\right|^2 \\ \left(\sum_{i\in\mathbb{T}^d_N} \sup_{w,w'\in\mathcal{W}} \left|\frac{1}{N_n^d} J\left(\frac{i}{N_n}, w, w'\right) - \int_{\Delta_{i,N_n}} J\left(x,w,w'\right) \mathrm{d}x\right|^2 + \left\|\overline{J}\right\|_{L^2}\right).\end{aligned} \tag{V.3.138}$$

This is bounded by $R\left(\left\|\overline{J}\right\|_{L^2} + \delta\right)$ for a $\delta > 0$, when N_n is large enough, by Assumption V.1.4. Moreover, we get a uniform upper bound on $\left|\beta\left(\frac{j}{N_n}, w^{j,N_n}, \underline{\mu}_t\right)\right|$ as in (V.3.130).

We have hence proven Assumption V.3.1 d.).

Summarized, the specific model considered in Section V.1.1, satisfies the Assumption V.3.1. □

Remark V.3.35. *When considering only continuous J, the proofs are much simpler. However, also interaction weights that are not continuous are of particular interest (for some examples see Example V.1.5).*

V.4 Representations of the rate function for the LDP of the empirical process

In this section, we state three other representations of the rate function $S_{\nu,\zeta}$, besides the two given in Theorem V.3.5. To state these representations we need the following notation.

Notation V.4.1. *For $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty}$ and $(t,x,w,\theta) \in [0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$, set*

$$b^{I,\mu_{[0,T]}}(t,x,w,\theta) := b\left(x,w,\theta,\mu_t\right). \tag{V.4.1}$$

With $b^{I,\mu_{[0,T]}}$ as drift coefficient, define the generator $\mathbb{L}^{I,\mu_{[0,T]}}_{t,x,w}$ as in (V.3.11)*. For this system, the Assumption V.3.7 are satisfied if the assumptions of Theorem V.3.5 hold (as shown in Section V.3.2.1.1). In particular the corresponding martingale problem has for each $(x,w,\theta) \in \mathbb{T}^d \times \mathcal{W} \times \mathbb{R}$ a unique solution, which we denote by $P^{I,\mu_{[0,T]}}_{x,w,\theta}$. Then we define $P^{I,\mu_{[0,T]}}_{x,w} \in \mathbb{M}_1(\mathsf{C}([0,T]))$, $P^{I,N,\mu_{[0,T]}}_{\underline{w}^N} \in \mathbb{M}_1\left(\mathsf{C}([0,T])^{N^d}\right)$ and $P^{I,N,\mu_{[0,T]}} \in \mathbb{M}_1\left(\mathcal{W}^{N^d} \times \mathsf{C}([0,T])^{N^d}\right)$ as in Notation V.3.8.*

Moreover, we denote by $U^{\mu_{[0,T]}}_{s,t}$ the operator $U_{s,t}$ defined in (V.3.17) *with P^I replaced by $P^{I,\mu_{[0,T]}}$.*

Theorem V.4.2. *Let the assumptions of Theorem V.3.5 hold. $S_{\nu,\zeta}$ has the following representations for $\mu_{[0,T]} \in \mathscr{C}$, with $S_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$.*

(i)

$$S_{\nu,\zeta}\left(\mu_{[0,T]}\right) = \inf_{\substack{Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\\ \Pi(Q)_{[0,T]}=\mu_{[0,T]}}} \mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes P^{I,\mu_{[0,T]}}_{x,w}\right) \tag{V.4.2}$$

(ii) *$S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ is equal to*

$$\begin{aligned}\sup_{\substack{r\in\mathbb{N},\\ 0\leq t_1<\ldots<t_r\leq T}} \Bigg[\sup_f \left\{\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\mu_{t_1} - \int_{\mathbb{T}^d} \log\left(\int_{\mathcal{W}\times\mathbb{R}} U^{\mu_{[0,T]}}_{0,t_1} e^f(x,w,\theta)\,\nu_x(\mathrm{d}\theta)\,\zeta_x(\mathrm{d}w)\right)\mathrm{d}x\right\} \\ + \sum_{i=2}^r \sup_f \left\{\int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} f\mu_{t_i} - \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \log U^{\mu_{[0,T]}}_{t_{i-1},t_i} e^f(x,w,\theta)\,\mu_{t_{i-1}}\right\}\Bigg],\end{aligned} \tag{V.4.3}$$

where the μ_{t_i} *integrate with respect to the variables* $\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta$ *and the functions* f *in the suprema are in the set* $\mathsf{C}_c^\infty\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$.

(iii) *There is a function* $h^{\mu_{[0,T]}} \in \widehat{L}^2_{\mu_{[0,T]}}(0,T)$ *(this space is defined in the Step 3 of the proof of Lemma V.3.26), such that* $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$ *is equal to*

$$\frac{1}{2}\int_0^T \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \frac{\sigma^2}{2}\left(h^{\mu_{[0,T]}}(t,x,w,\theta)\right)^2 \mu_t(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta)\,\mathrm{d}t + \mathsf{H}\left(\mu_0|\mathrm{d}x\otimes\zeta_x\otimes\nu_x\right). \tag{V.4.4}$$

Moreover, $\mu_{[0,T]}$ *satisfies in a weak sense (i.e. when integrated against an arbitrary function in* $\mathsf{C}_c^{1,0,2}\big([0,T]\times\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}\big)$*) the PDE*

$$\partial_t\mu_t = \left(\mathbb{L}_{\mu_t,\cdot,\cdot}\right)^*\mu_t + \sigma^2\partial_\theta\left(\mu_t h^{\mu_{[0,T]}}(t)\right). \tag{V.4.5}$$

Proof. When $S_{\nu,\zeta}\left(\mu_{[0,T]}\right) < \infty$, then $\mu_{[0,T]} \in \mathscr{C}_{\varphi,\infty}\cap\mathscr{C}^L$. Therefore, we know by Section V.3.2.1.1, that the measure $P^{I,\mu_{[0,T]}}_{x,w}$ is well defined. Moreover, all the results of Section V.3.1 hold for the independent spin system with the drift coefficient b^I of Notation V.4.1.

The representations (i) and (ii) follow directly from Lemma V.3.11 and Lemma V.3.12.

The representation (iii), follows from Lemma V.3.26 and the proof of this lemma, in particular (V.3.73) in Step 3 of this proof. That $\mu_{[0,T]}$ is a weak solution of the PDE (V.4.5), follows from (V.3.72) and (V.3.68). □

V.5 The LDP of the empirical measure

In this section we show the large deviation principle for the empirical measures L^N under the assumptions of Section V.3 and the following exponential tightness assumption.

Assumption V.5.1. *The family* $\left\{L^N, P^N\right\}$ *is exponential tight, i.e. for all* $s > 0$*, there is a compact set* $\mathcal{K}_s \subset \mathbb{M}_1\big(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\big)$*, such that*

$$\limsup_{N\to\infty} N^{-d}\log P^N\left[L^N \notin \mathcal{K}_s\right] \le -s. \tag{V.5.1}$$

To state the large deviation principle result, we need the following definitions and notations.

Definition V.5.2. *We say* $Q \in \mathcal{M}_{\varphi,R}$ *if and only if* $\Pi(Q)_{[0,T]} \in \mathbb{M}_{\varphi,R}$*, for* $R \in (0,\infty]$.

For fixed $x \in [0,T]$ and $Q \in \mathcal{M}_{\varphi,R}\cap\mathbb{M}_1^L\big(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\big)$, we define $b^{I,\Pi(Q)}$, $\mathbb{L}^{I,\Pi(Q)}_{t,x,w}$ and the measures $P^{I,\Pi(Q)}_{x,w} \in \mathbb{M}_1(\mathsf{C}([0,T]))$ and $P^{I,N,\Pi(Q)} \in \mathbb{M}_1\Big(\mathcal{W}^{N^d}\times\mathsf{C}([0,T])^{N^d}\Big)$ as in Notation V.4.1.

Theorem V.5.3. *If the assumptions of Theorem V.3.5 and the Assumption V.5.1 hold, then the family of empirical measures* $\left\{L^N, P^N\right\}$ *satisfies on* $\mathbb{M}_1\big(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\big)$ *the large deviation principle with good rate function*

$$I(Q) := \begin{cases} \mathsf{H}\left(Q\middle|P^{I,\Pi(Q)}\right) & \text{if } Q \in \mathbb{M}_1^L\big(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\big)\cap\mathcal{M}_{\varphi,\infty},\\ \infty & \text{otherwise.}\end{cases} \tag{V.5.2}$$

where $P^{I,\Pi(Q)} := \mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes P^{I,\Pi(Q)}_{x,w} \in \mathbb{M}_1\big(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\big)$.

To prove this theorem, we use the same approach as for the proof of the large deviation principle for the empirical process $\mu^N_{[0,T]}$, that we give in Section V.3. To summarise it, we show at first the large deviation principle for spins that evolve according to an independent system of SDEs. Form this we derive a local LDP for the interacting system. This requires exponential bounds on the probability that the empirical measures leave the set $\mathcal{M}_{\varphi,R}$. Finally, we infer from the local LDP, the desired LDP of the interacting system. In this last step, we have to show that I is a good rate function. This prove is different from the corresponding one in Section V.3. Moreover, we require the exponential tightness in the last step. We need to assume it in this section, because we are not able to prove it in general as in Section V.3.

In Section V.5.2 we show that the concrete example (0.9.3) of a local mean field model satisfies the exponential tightness. Moreover, we show a second representation of the rate function for this model.

V.5.1 Proof of the LDP (Theorem V.5.3)

To prove the Theorem V.5.3, we show at first that the measure in the relative entropy in (V.5.2) is actually a probability measure.

Lemma V.5.4. *For each $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$, the measure $P^{I,\Pi(Q)}$ is well defined.*

Proof of Lemma V.5.4. Fix a $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. The function $b^{I,\Pi(Q)}$ is continuous. Indeed, $t \mapsto \Pi(Q)_t$ is continuous (Lemma V.2.27), $\Pi(Q)_t \in \mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)$ for all $t \in [0,T]$ and b is continuous on $\mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \big(\mathbb{M}_{\varphi,R} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\big)\big)$ (Assumption V.3.1 a.a)). Therefore, we can apply [SV79] Theorem 11.1.4 to get the continuity of $(x,w,\theta) \mapsto P^{I,\Pi(Q)}_{x,w,\theta}$ (see also Lemma V.3.14). By this continuity, the Assumption V.1.1 and the Assumption V.1.3, we conclude (as in Lemma V.2.11) that the measure $P^{I,\Pi(Q)}$ is well defined. □

V.5.1.1 The independent system

Fix a $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. We get, as in the proof of Lemma V.3.11, the following large deviation principle for the independent system (by Lemma V.2.7 and Lemma V.2.8 with $r = 1$, $Y = \mathsf{C}([0,T])$).

Lemma V.5.5. *The family $\big\{L^N, P^{I,N,\Pi(Q)}\big\}$ satisfies on $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ the large deviation principle with rate function*

$$I^Q(\Gamma) = \mathsf{H}\left(\Gamma \middle| P^{I,\Pi(Q)}\right), \tag{V.5.3}$$

for $\Gamma \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ and infinity otherwise.

V.5.1.2 The interacting system

As in Section V.3.2, we show at first the following local version of the LDP.

Lemma V.5.6. *Under the same assumptions as in Theorem V.3.5, the following statements are true, for each $\overline{Q} \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$.*

(i) *For all open neighbourhoods $V \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ of $\overline{Q}$*

$$\liminf_{N\to\infty} N^{-d} \log P^N\left[L^N \in V\right] \geq -I\left(\overline{Q}\right). \tag{V.5.4}$$

(ii) *For each $\gamma > 0$, there is an open neighbourhood $V \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ of $\overline{Q}$ such that*

$$\limsup_{N\to\infty} N^{-d} \log P^N \left[L^N \in V\right] \leq \begin{cases} -I\left(\overline{Q}\right) + \gamma & \text{if } I\left(\overline{Q}\right) < \infty, \\ -\gamma & \text{otherwise.} \end{cases} \tag{V.5.5}$$

The Lemma V.5.6 can be proven as the Theorem V.3.27. This proof requires Lemma V.5.5 and the following exponential bound instead of Lemma V.3.32..

Lemma V.5.7. *For all $s > 0$, there is a $R = R_s > 0$, such that for all $N \in \mathbb{N}$*

$$\sup_{\underline{w}^N \in \mathcal{W}^{N^d}} P^N_{\underline{w}^N} \left[L^N \notin \mathcal{M}_{\varphi,R}\right] \leq e^{-N^d s}. \tag{V.5.6}$$

The Lemma V.5.7 follows directly from Lemma V.3.32, because $L^N \in \mathcal{M}_{\varphi,R}$ if and only if $\Pi\left(L^N\right)_{[0,T]} \in \mathscr{C}_{\varphi,R}$, i.e.

$$P^N_{\underline{w}^N} \left[L^N \in \mathcal{M}_{\varphi,R}\right] = P^N_{\underline{w}^N} \left[\mu^N_{[0,T]} \in \mathscr{C}_{\varphi,R}\right]. \tag{V.5.7}$$

Then Lemma V.5.6 and Assumption V.5.1 imply the lower and upper large deviation bound with the good rate function I. Indeed, we show in the next lemma that I is a good rate function. This finishes the proof of the Theorem V.5.3.

Lemma V.5.8. *The function $Q \mapsto I(Q)$ is a good rate function.*

Proof. We show at first that the level set $\mathcal{L}^{\leq s}(I)$ is relatively compact and then that it is closed.
Step 1: $\mathcal{L}^{\leq s}(I)$ is relatively compact:

By Assumption V.5.1 and Lemma V.5.7 we know that there is a compact set $\mathcal{K}_{s+\epsilon} \subset \mathcal{M}_{\varphi,R}$, for $R > 0$ large enough, such that (V.5.1) holds. We claim that $\mathcal{L}^{\leq s}(I) \subset \mathcal{K}_{s+\epsilon}$. Assume that there is a $Q \in \mathcal{L}^{\leq s}(I)$ that is not in $\mathcal{K}_{s+\epsilon}$. Then we know by (V.5.1) and Theorem V.5.6 (i) (because $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big) \setminus \mathcal{K}_{s+\epsilon}$ is an open neighbourhood of Q), that $s + \epsilon \leq I(Q)$, a contradiction.

Step 2: $\mathcal{L}^{\leq s}(I)$ is closed:

By the definition of I and the previous step, $\mathcal{L}^{\leq s}(I) \subset \mathcal{K}_{s+\epsilon} \cap \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. Fix an arbitrary converging sequence $\left\{Q^{(n)}\right\}_n \subset \mathcal{L}^{\leq s}(I)$. The limit point Q^* of this sequence is in $\mathcal{K}_{s+\epsilon} \cap \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$. We prove now that $Q \in \mathcal{L}^{\leq s}(I)$.

This follows if we knew that for all $(x, w) \in \mathbb{T}^d \times \mathcal{W}$, $P^{I,\Pi(Q^{(n)})}_{x,w} \to P^{I,\Pi(Q^*)}_{x,w}$. Indeed, this implies that also $\mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P^{I,\Pi(Q^{(n)})}_{x,w} \to \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P^{I,\Pi(Q^*)}_{x,w}$. Then we conclude the lower semi-continuity of I, from the lower semi-continuity of the relative entropy in both variables.

The convergence of $P^{I,\Pi(Q^{(n)})}_{x,w}$ follows from [SV79] Theorem 11.1.4. This theorem is applicable if for each $M \in \mathbb{R}$

$$\lim_{n\to\infty} \int_0^T \sup_{|\theta| \leq M} \left| b\left(x, w, \theta, \Pi\left(Q^{(n)}\right)_t\right) - b\left(x, w, \theta, \Pi\left(Q^*\right)_t\right)\right| \mathrm{d}t = 0. \tag{V.5.8}$$

This convergence follows if

$$\sup_{t\in[0,T]} \sup_{|\theta| \leq M} \left| b\left(x, w, \theta, \Pi\left(Q^{(n)}\right)_t\right) - b\left(x, w, \theta, \Pi\left(Q^*\right)_t\right)\right| \to 0. \tag{V.5.9}$$

Let us show that (V.5.9) holds. The function

$$(t, x, w, \theta, Q) \mapsto b\left(x, w, \theta, \Pi(Q)_t\right) \tag{V.5.10}$$

is continuous on $[0,T] \times \mathbb{T}^d \times \mathcal{W} \times \mathbb{R} \times \big(\mathcal{M}_{\varphi,R} \cap \mathbb{M}^L_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)\big)$, as composition of continuous functions (Assumption V.3.1 a.a)). Moreover, only the compact sets $[0,T]$, $|\theta| \leq M$ and $Q^{(n)}, Q^* \in \mathcal{K}_{s+\epsilon}$ are considered in (V.5.9). From the uniform convergence of b on this set, we conclude (V.5.9). Hence $\mathcal{L}^{\leq s}(I)$ is closed. □

V.5.2 The concrete example (0.9.3) of a local mean field model

In this section we consider the large deviation principle for the family $\left\{L^N, P^N\right\}$ of the local mean field model, defined by $\sigma = 1$ and the drift coefficient (0.9.3) with Assumption V.1.1, Assumption V.1.2, Assumption V.1.3, Assumption V.1.4 and Assumption V.1.7.

Choose $\varphi = \theta^2 + 1$. We know by Section V.3.3, that the Assumption V.3.1 is satisfied. Hence by Theorem V.5.3, the empirical measure L^N of the local mean field model, satisfies the large deviation principle, provided that the the exponential tightness Assumptions V.5.1 holds. We claim the exponential tightness in the next lemma, which we prove in Section V.5.2.3.

Lemma V.5.9. *For the concrete example* (0.9.3) *of a local mean field model, the family* $\left\{L^N, P^N\right\}$ *is exponentially tight, i.e. the Assumptions V.5.1 is satisfied.*

The measure $P^{I,\Pi(Q)}_{x,w}$ is for each $(x, w) \in \mathbb{T}^d \times \mathcal{W}$ the law of the following one dimensional SDE

$$\begin{aligned} \mathrm{d}\widehat{\theta}^x_t &= \left(-\Psi\left(\widehat{\theta}^x_t, w\right) + \beta\left(x, w, \Pi\left(Q\right)_t\right)\right)\mathrm{d}t + \mathrm{d}B_t, \\ \widehat{\theta}^x_0 &\sim \nu_x, \end{aligned} \tag{V.5.11}$$

where the function $\beta : \mathbb{T}^d \times \mathcal{W} \times \left\{\mu \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right) : \int_{\mathbb{T}^d\times\mathcal{W}\times\mathbb{R}} \theta^2 \mu\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta\right) < \infty\right\} \to \mathbb{R}$ is defined in (V.3.119). We interpret $\beta\left(x, w, \mu\right)$ as the effective field corresponding to the measure $\mu \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}\right)$ at the spatial position $x \in \mathbb{T}^d$ with fixed environment $w \in \mathcal{W}$. If we inserted μ^N_t in β, the drift coefficient of the SDE (V.5.11) would equal the drift coefficient of the SDE (0.9.3). Hence the rate function I (defined in (V.5.2)) measures the deviation of Q from the measure of the solution to the SDE with effective field Q.

Now we show that this rate function has another representation, in which the influence of the entropy and of the interaction becomes obvious. We need the following notation.

Notation V.5.10.
- *We denote by* W^0_x *the law of a Brownian motion with initial distribution* ν_x.
- *We use the symbol* $W^{-\Psi}_{x,w}$ *for the law of the solution of the SDE with drift coefficient* $-\partial_\theta\Psi\left(., w\right)$ *and with initial distribution* ν_x*, for* $w \in \mathcal{W}$.
- *With these measures we define the products of these measures* $W^{N,0}_{\underline{w}^N}, W^{N,\Psi}_{\underline{w}^N} \in \mathbb{M}_1\left(\mathsf{C}([0,T])^{N^d}\right)$ *and the product with* ζ^N *by* $W^{N,0}, W^{N,\Psi} \in \mathbb{M}_1\left(\mathcal{W}^{N^d} \times \mathsf{C}([0,T])^{N^d}\right)$ *similar as in Notation V.1.9.*

Remark V.5.11. *All these measures exist, because the corresponding martingale problems are well posed (by the Assumption V.1.7). Note that the* N^d *diffusion processes described by* $W^{N,0}_{\underline{w}^N}$ *and* $W^{N,-\Psi}_{\underline{w}^N}$ *do not interact.*

Theorem V.5.12. *If Assumption V.1.1, Assumption V.1.2, Assumption V.1.3, Assumption V.1.4 and Assumption V.1.7 hold, then the family* $\left\{L^N, P^N\right\}$ *satisfies on* $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ *the large deviation principle with good rate function*

$$\overline{I}\left(Q\right) = \mathsf{H}\left(Q\middle|\mathrm{d}x \otimes \zeta_x\left(\mathrm{d}w\right) \otimes W^{-\Psi}_{x,w}\right) - F\left(Q\right), \tag{V.5.12}$$

if $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}^L_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ *and* $\overline{I}\left(Q\right) := \infty$ *otherwise. Here*

$$\begin{aligned} F\left(Q\right) :=& \frac{1}{2}\int\int J\left(x - x', w, w'\right)\left[\theta_T\theta'_T - \theta_0\theta'_0\right] Q\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) Q\left(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta'_{[0,T]}\right) \\ &- \frac{1}{2}\int\int\int J\left(x'' - x, w'', w\right) J\left(x'' - x', w'', w'\right) Q\left(\mathrm{d}x'', \mathrm{d}w'', \mathrm{d}\eta_{[0,T]}\right) \\ &\qquad \int_0^T \theta'_t\theta_t \mathrm{d}t Q\left(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta'_{[0,T]}\right) Q\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right), \end{aligned} \tag{V.5.13}$$

where the integrals $\int$ are over the space $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$.

By the uniqueness of the rate function of a large deviation principle, $\overline{I} = I$ (for I defined in Theorem V.5.3).

V.5.2.1 Proof of Theorem V.5.12

Proof. We know already by Theorem V.5.3 and Lemma V.5.9, that L^N satisfies the large deviation principle with rate function I defined in (V.5.2). Hence we only have to show that I equals $\overline{I}$.

Note that $I(Q) = \infty$ and $\overline{I}(Q) = \infty$, if $Q \notin \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ or if not $Q << \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P_{x,w}^{I,\Pi(Q)}$. Indeed, if $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$, then $Q << \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W_{x,w}^{-\Psi}$ if and only if $Q << \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes P_{x,w}^{I,\Pi(Q)}$. This is the case because $(t,x,w) \mapsto \beta(x,w,\Pi(Q)_t)$ is uniformly bounded (see (V.3.130)). Moreover, $F(Q)$ is bounded for such a Q (because $\overline{J} \in L^2\left(\mathbb{T}^d\right)$ by Assumption V.1.4).

Hence Theorem V.5.12 follows if for all $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ with $Q << \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W_{x,w}^{-\Psi}$, $I(Q)$ equals $\overline{I}(Q)$. For such a Q, F has the following different representation.

Lemma V.5.13. *For $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ with $Q << \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W_{x,w}^{-\Psi}$,*

$$F(Q) = \int_{\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])} \log \frac{\mathrm{d}P_{x,w}^{I,\Pi(Q)}}{\mathrm{d}W_{x,w}^{-\Psi}}\left(\theta_{[0,T]}\right) Q\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right). \tag{V.5.14}$$

From this lemma, we immediately infer the equality of I and $\overline{I}$ and hence Theorem V.5.12. □

Proof of Lemma V.5.13. Fix a $Q \in \mathcal{M}_{\varphi,\infty} \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ with $Q << \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W_{x,w}^{-\Psi}$. By the well-posedness of the martingale problems, the measures $P_{x,w}^{I,\Pi(Q)}$ and $W_{x,w}^{-\Psi}$ are equivalent. By the Girsanov theorem, their Radon-Nikodym derivative can be written as

$$\begin{aligned} &\log \frac{\mathrm{d}P_{x,w}^{I,\Pi(Q)}}{\mathrm{d}W_{x,w}^{-\Psi}}\left(\theta_{[0,T]}\right) \\ &= -\frac{1}{2}\int_0^T \left(\beta\left(x,w,\Pi(Q)_t\right)\right)^2 \mathrm{d}t + \int_0^T \beta\left(x,w,\Pi(Q)_t\right) \mathrm{d}\theta_t =: \textcircled{1} + \textcircled{2}. \end{aligned} \tag{V.5.15}$$

Integrating ① w.r.t Q we get the first term in F. We show now that ② leads to the second term of F. Therefore, remark at first that

$$\begin{aligned} &\int \textcircled{2}\, Q = \iint J(x-x',w,w') \int_0^T \theta_t' \mathrm{d}\theta_t \quad Q\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta_{[0,T]}\right) Q\left(\mathrm{d}x',\mathrm{d}w',\mathrm{d}\theta_{[0,T]}'\right) \\ &= \frac{1}{2}\iint J(x-x',w,w')\left(\int_0^T \theta_t' \mathrm{d}\theta_t + \int_0^T \theta_t \mathrm{d}\theta_t'\right) Q\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta_{[0,T]}\right) Q\left(\mathrm{d}x',\mathrm{d}w',\mathrm{d}\theta_{[0,T]}'\right), \end{aligned} \tag{V.5.16}$$

where we use that J is an even function. Integrals without integration bounds integrate over the space $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$. The stochastic integrals are well defined because $\theta_{[0,T]}$ is a $Q_{x,w}$-semimartingale, because $Q_{x,w} << W_{x,w}^{-\Psi} << W_x^0$ for almost all $(x,w) \in \mathbb{T}^d \times \mathcal{W}$. By integration by parts formula for the Itô integral, (V.5.16) equals the first summand on the right hand side of (V.5.13). □

V.5.2.2 Preliminaries

In this section we state and prove some results, that we need in the proof of Lemma V.5.9.

Lemma V.5.14. *The map $\mathbb{T}^d \times \mathcal{W} \ni (x,w) \mapsto W_{x,w}^{-\Psi} \in \mathbb{M}_1(\mathsf{C}([0,T]))$ is Feller continuous.*

Proof. By similar estimates as in Lemma V.3.14 and Lemma V.2.9, we get the Feller continuity. This requires the Assumption V.1.7 and Assumption V.1.1. □

Lemma V.5.15. *For all $\kappa < c_\Psi$, there is a constant $C_\kappa > 0$ such that*

(i) $$\sup_{x\in\mathbb{T}^d}\sup_{w\in\mathcal{W}} E_{W^{-\Psi}_{x,w}}\left[e^{\kappa\left[(\theta_T)^2+(\theta_0)^2\right]}\right] < C_\kappa \quad \text{and}$$

(ii) $$\sup_{x\in\mathbb{T}^d}\sup_{w\in\mathcal{W}} E_{W^{-\Psi}_{x,w}}\left[e^{\kappa\int_0^T(\theta_t)^2\mathrm{d}t}\right] < C_\kappa.$$

Proof. ***(i)*** Fix arbitrary $(x,w)\in\mathbb{T}^d\times\mathcal{W}$.

By the Girsanov theorem and Itô's lemma we have

$$\frac{\mathrm{d}W^{-\Psi}_{x,w}}{\mathrm{d}W^0_x} = e^{\Psi(\theta_0,w)-\Psi(\theta_T,w)+\frac{1}{2}\int_0^T\partial^2_{\theta^2}\Psi(\theta_t,w)\mathrm{d}t-\frac{1}{2}\int_0^T(\partial_\theta\Psi(\theta_t,w))^2\mathrm{d}t}. \tag{V.5.17}$$

By $\overline{\Psi}$ being a polynomial of even degree (Assumption V.1.7) and by $\mathcal{W}$ being compact, the following upper bound on the Radon-Nikodym derivative holds

$$\frac{\mathrm{d}W^{-\Psi}_{x,w}}{\mathrm{d}W^0_x} \le e^{\Psi(\theta_0,w)-\Psi(\theta_T,w)+TC}. \tag{V.5.18}$$

Therefore,

$$\begin{aligned} E_{W^{-\Psi}_{x,w}}\left[e^{\kappa\left[(\theta_T)^2+(\theta_0)^2\right]}\right] &\le e^{TC}E_{W^0_x}\left[e^{\kappa(\theta_T)^2-\Psi(\theta_T,w)}e^{\kappa(\theta_0)^2+\Psi(\theta_0,w)}\right] \\ &\le e^{TC}e^{C}\int_{\mathbb{R}} e^{\kappa(\theta)^2+\overline{\Psi}(\theta)+w_1\theta}\nu_x(\mathrm{d}\theta), \end{aligned} \tag{V.5.19}$$

where we use Assumption V.1.7, $\mathcal{W}$ being compact and $\kappa < c_\Psi$ in the second inequality. The right hand side of (V.5.19) is bounded by a constant uniformly in $(x,w)\in\mathbb{T}^d\times\mathcal{W}$, by Assumption V.1.2, Assumption V.1.7 and by $\mathcal{W}$ being compact.

(ii) By Assumption V.1.7, the Radon-Nikodym derivative in (V.5.17) is also be bounded by

$$\frac{\mathrm{d}W^{-\Psi}_{x,w}}{\mathrm{d}W^0_x} \le e^{\Psi(\theta_0,w)+C-\int_0^T c(\theta_t)^2\mathrm{d}t}, \tag{V.5.20}$$

for constants $c\in(0,c_\Psi)$ and $C=C(c)>0$. Using this bound, we get

$$E_{W^{-\Psi}_{x,w}}\left[e^{\kappa\int_0^T(\theta_t)^2\mathrm{d}t}\right] \le e^C E_{W^0_x}\left[e^{\int_0^T(\kappa-c)(\theta_t)^2\mathrm{d}t}e^{\Psi(\theta_0,w)}\right] \le e^C\int_{\mathbb{R}} e^{\overline{\Psi}(\theta)+w_1\theta}\nu_x(\mathrm{d}\theta). \tag{V.5.21}$$

The right hand side is bounded by a constant uniformly in $(x,w)\in\mathbb{T}^d\times\mathcal{W}$ by Assumption V.1.2, Assumption V.1.7 and by $\mathcal{W}$ being compact. □

Now we derive the Radon-Nikodym derivative of $P^N_{\underline{w}^N}$ and $W^{N,-\Psi}_{\underline{w}^N}$ by using the Girsanov theorem.

Lemma V.5.16. *For $\underline{w}^N\in\mathcal{W}^{N^d}$ and $\underline{\theta}^N_{[0,T]}\in\mathsf{C}([0,T])^{N^d}$,*

$$\frac{\mathrm{d}P^N_{\underline{w}^N}}{\mathrm{d}W^{N,-\Psi}_{\underline{w}^N}}\left(\underline{\theta}^N_{[0,T]}\right) = e^{N^dF\left(L^N\right)-\frac{1}{2}\frac{T}{N}\sum_{i\in\mathbb{T}^d_N}J\left(0,w^{i,N},w^{i,N}\right)}, \tag{V.5.22}$$

where $L^N = L^N\left(\underline{w}^N,\underline{\theta}_{[0,T]}\right)\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$.

Proof of Lemma V.5.16. Fix $\underline{w}^N \in \mathcal{W}^{N^d}$. To shorten the notation, we define for $\underline{\theta}^N \in \mathbb{R}^{N^d}$

$$B^N_{\underline{w}^N}\left(\underline{\theta}^N\right) := \frac{1}{2N^d}\sum_{i,j\in\mathbb{T}^d_N} J\left(\frac{i-j}{N}, w^{i,N}, w^{j,N}\right)\theta^{i,N}\theta^{j,N}. \tag{V.5.23}$$

By the Girsanov theorem and J being even,

$$\begin{aligned}&\log\left(\frac{\mathrm{d}P^N_{\underline{w}^N}}{\mathrm{d}W^{N,-\Psi}_{\underline{w}^N}}\left(\underline{\theta}^N_{[0,T]}\right)\right)\\ &= -\frac{1}{2}\int_0^T \sum_{i\in\mathbb{T}^d_N}\left(\partial_{\theta^{i,N}_t} B^N_{\underline{w}^N}\left(\underline{\theta}^N_t\right)\right)^2 \mathrm{d}t + \sum_{i\in\mathbb{T}^d_N}\int_0^T \partial_{\theta^{i,N}_t} B^N_{\underline{w}^N}\left(\underline{\theta}^N_t\right)\mathrm{d}\theta^{i,N}_t =: \textcircled{1} + \textcircled{2}.\end{aligned} \tag{V.5.24}$$

The first summand of (V.5.24) equals the first summand of $N^d F\left(L^N\right)$, because for each $t \in [0,T]$

$$\begin{aligned}\textcircled{1} &= \frac{1}{N^d}\sum_{j,k\in\mathbb{T}^d_N}\left(\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N} J\left(\frac{i-j}{N}, w^{i,N}, w^{j,N}\right) J\left(\frac{i-k}{N}, w^{i,N}, w^{k,N}\right)\right)\theta^{j,N}_t\theta^{k,N}_t\\ &= N^d \int\!\!\int \theta'_t\theta_t \int J\left(x''-x, w'', w\right) J\left(x''-x', w'', w'\right) L^N\left(\mathrm{d}x'', \mathrm{d}w'', \mathrm{d}\eta_{[0,T]}\right)\\ &\qquad\qquad L^N\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) L^N\left(\mathrm{d}x', \mathrm{d}w', \mathrm{d}\theta'_{[0,T]}\right),\end{aligned} \tag{V.5.25}$$

where the integrals in the last line are over the sets $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])$.

For ② we apply Itô's lemma. Under $W^{N,-\Psi}_{\underline{w}^N}$, the $\theta^{k,N}_{[0,T]}$ is a Itô process with drift coefficient $-\partial_\theta\Psi\left(.,w^{k,N}\right)$, for each $k \in \mathbb{T}^d_N$. Hence

$$\begin{aligned}\textcircled{2} &= B^N_{\underline{w}^N}\left(\underline{\theta}^N_T\right) - B^N_{\underline{w}^N}\left(\underline{\theta}^N_0\right) - \frac{1}{2}\sum_{i\in\mathbb{T}^d_N}\int_0^T \partial^2_{(\theta^{i,N})^2} B^N_{\underline{w}^N}\left(\underline{\theta}^N_t\right)\mathrm{d}t\\ &= B^N_{\underline{w}^N}\left(\underline{\theta}^N_T\right) - B^N_{\underline{w}^N}\left(\underline{\theta}^N_0\right) - \frac{1}{2}\sum_{i\in\mathbb{T}^d_N} T\frac{J\left(0, w^{i,N}, w^{i,N}\right)}{N^d}.\end{aligned} \tag{V.5.26}$$

Using (V.5.23), we conclude that ② is equal to the second summand of $F\left(L^N\right)$. □

V.5.2.3 Proof of the exponential tightness (Lemma V.5.9)

Proof of Lemma V.5.9. To show that the family $\left\{L^N, P^N\right\}$ is exponential tight, we first construct compact sets $K^\ell \subset \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ for which the family $\left\{L^N, W^{N,-\Psi}\right\}$ is exponential tight. Then we show that this leads to the exponential tightness of $\left\{L^N, P^N\right\}$.

Step 1: Exponential tightness of $\left\{L^N, W^{N,-\Psi}\right\}$:

To show the exponential tightness of $\left\{L^N, W^{N,-\Psi}\right\}$ we generalise Lemma 6.2.6 in [DZ98]. In contrast to [DZ98], the measures $\mathcal{A} := \left\{\zeta_x(\mathrm{d}w) \otimes W^{-\Psi}_{x,w}\right\}_{x\in\mathbb{T}^d} \subset \mathbb{M}_1(\mathcal{W} \times \mathsf{C}([0,T]))$ are not identically distributed, due to the dependency of the initial distribution and of the random environment on $x \in \mathbb{T}^d$.

Therefore, we show at first that $\mathcal{A}$ is a tight set of measures. Take an arbitrary sequence in $\mathcal{A}$. Then there is a sequence $\{x_n\}_n \subset \mathbb{T}^d$ such that the sequence is given by $\left\{\zeta_{x_n}(\mathrm{d}w) \otimes W^{-\Psi}_{x_n,w}\right\}_n$. This implies that there is a converging subsequence $x_{n_k} \to x^* \in \mathbb{T}^d$ (due to the compactness of $\mathbb{T}^d$). By the continuity of $x \mapsto \zeta_x(\mathrm{d}w) \otimes W^{-\Psi}_{x,w}$ (this could be shown as the continuity of (V.2.13) by

Assumption V.1.3 and Lemma V.5.14), we get a converging subsequence. Therefore, $\mathcal{A}$ is sequentially compact. Moreover, $\mathcal{W} \times \mathsf{C}([0,T])$ is a separable metric space. Then the Prokhorov's theorem implies that $\mathcal{A}$ is tight.

The tightness of the set $\mathcal{A}$ implies that there is a compact set $\Gamma_a \subset \mathcal{W} \times \mathsf{C}([0,T])$ such that for all $x \in \mathbb{T}^d$

$$\int_{\mathcal{W}} \int_{\mathsf{C}([0,T])} \mathbb{1}_{\left\{\left(w,\theta_{[0,T]}\right) \notin \Gamma_a\right\}} W^{-\Psi}_{x_n,w} \left(\mathrm{d}\theta_{[0,T]}\right) \zeta_{x_n}(\mathrm{d}w) \leq e^{-2a^2} \left(e^a - 1\right). \tag{V.5.27}$$

Now define $K^a := \left\{Q \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right) : Q\left(\mathbb{T}^d \times \Gamma_a\right) \geq 1 - \frac{1}{a}\right\}$. The sets K^a are closed by the Portmanteau Lemma. Moreover, for each $A \in \mathbb{N}$ the sets

$$K_A := \bigcap_{a=A}^{\infty} K^a \tag{V.5.28}$$

are compact by Prokhorov's theorem and the definition of K^a. Then we get

$$\begin{aligned}
&W^{N,-\Psi}\left[L^N \notin K^a\right] = W^{N,-\Psi}\left[L^N\left(\mathbb{T}^d \times \Gamma_a\right) > \frac{1}{a}\right] \\
&\leq e^{-2N^d a} E_{W^{N,-\Psi}}\left[e^{2a^2 \sum_{i \in \mathbb{T}^d_N} \mathbf{1}_{\left(w,\theta^{i,N}_{[0,T]}\right) \notin \Gamma_a}}\right] \\
&\leq e^{-2N^d a} \prod_{i \in \mathbb{T}^d_N} \left(1 + e^{2a^2} \int_{\mathcal{W}} \int_{\mathsf{C}([0,T])} \mathbb{1}_{\left\{\left(w,\theta_{[0,T]}\right) \notin \Gamma_a\right\}} W^{-\Psi}_{x_n,w}\left(\mathrm{d}\theta_{[0,T]}\right) \zeta_{x_n}(\mathrm{d}w)\right) \\
&\leq e^{-2N^d a} \left(1 + e^a - 1\right)^{N^d} \leq e^{-N^d a}.
\end{aligned} \tag{V.5.29}$$

This implies that

$$W^{N,-\Psi}\left[L^N \notin K_A\right] \leq 2e^{-N^d A}, \tag{V.5.30}$$

hence the claimed exponential tightness.

Step 2: Transferring the exponential tightness:
We show now that $\left\{L^N, P^N\right\}$ are also exponential tight with respect to the same sets Γ_a. We get by the Radon-Nikodym derivative derived in Lemma V.5.16 and by the Hölder inequality $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$

$$\begin{aligned}
E_{P^N}\left[\mathbb{1}_{L^N \notin K_A}\right] &= E_{W^{N,-\Psi}}\left[e^{N^d F\left(L^N\right)} \mathbb{1}_{L^N \notin K_A}\right] e^{-\frac{1}{2} T \overline{J}(0)} \\
&\leq e^{-\frac{1}{2} T \overline{J}(0)} E_{W^{N,-\Psi}}\left[e^{N^d p F\left(L^N\right)}\right]^{\frac{1}{p}} W^{N,-\Psi}\left[L^N \notin K_A\right]^{\frac{1}{p'}}.
\end{aligned} \tag{V.5.31}$$

Note that $F\left(L^N\right)$ is bounded from above by

$$F\left(L^N\right) \leq \frac{1}{2} \frac{1}{N^d} \left(\left\|\overline{J}\right\|_{L^1} + \delta\right) \sum_{i \in \mathbb{T}^d_N} \left(\theta^{i,N}_T\right)^2 + \left(\theta^{i,N}_0\right)^2, \tag{V.5.32}$$

for each $\delta > 0$ when $N > N_\delta$ (see (V.3.126)), because the second summand of F in (V.5.13) is not positive. When $p > 1$ is not too large and N large enough, $p\left(\left\|\overline{J}\right\|_{L^1} + \delta\right) < c_\Psi$ (by Assumption V.1.7). Therefore, we get by Lemma V.5.15 (i),

$$E_{W^{N,-\Psi}}\left[e^{N^d p F\left(L^N\right)}\right] \leq \prod_{i \in \mathbb{T}^d_N} \sup_{w \in \mathcal{W}} E_{W^{-\Psi}_{\frac{i}{N},w}}\left[e^{p\left(\left\|\overline{J}\right\|_{L^1} + \delta\right)\left((\theta_T)^2 + (\theta_0)^2\right)}\right] \leq C^{N^d}. \tag{V.5.33}$$

Combining (V.5.31), (V.5.30) and (V.5.33), we conclude

$$E_{P^N}\left[\mathbb{1}_{L^N\notin K_A}\right]\le C^{N^d}e^{-N^d\frac{1}{p'}A}, \tag{V.5.34}$$

when N and A are large enough. □

V.6 Comparison of the LDPs of the empirical measure and of the empirical process

In this section we state at first (Section V.6.1) a one-to-one relation between the minimizer of the rate functions I (of $\{L^N, P^N\}$ derived in Theorem V.5.3) and $S_{\nu,\zeta}$ (of $\left\{\mu^N_{[0,T]}, P^N\right\}$ derived in Theorem V.3.5). Then we explain how one can easily infer from the large deviation principle for the empirical measure $\{L^N\}$, the large deviation principle for the empirical process $\left\{\mu^N_{[0,T]}\right\}$ in $\mathscr{C}$. This follows by a simple application of the contraction principle (see Theorem V.6.2). However, the derived rate function does not have the expression $S_{\nu,\zeta}$ defined in (V.3.10). We show in Section V.6.3 that the derived rate function is at least an upper bound on $S_{\nu,\zeta}$.

V.6.1 Relation between the minimiser of the rate function

We know by Theorem V.4.2 (i) and (V.5.2) the following relation between $S_{\nu,\zeta}$ and I

$$\begin{aligned} S_{\nu,\zeta}\left(\mu_{[0,T]}\right) &= \inf_{\substack{Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\\ \Pi(Q)_{[0,T]}=\mu_{[0,T]}}} \mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes P^{I,\mu_{[0,T]}}_{x,w}\right)\\ &= \inf_{\substack{Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\\ \Pi(Q)_{[0,T]}=\mu_{[0,T]}}} I(Q). \end{aligned} \tag{V.6.1}$$

We show in the next theorem a one-to-one relation between the minimizer of I and $S_{\nu,\zeta}$. Note that in general there can be two $Q, Q'\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$ with the same projection $\Pi(Q)=\Pi(Q')$ and with $I(Q)=I(Q')$. However, when $S_{\nu,\zeta}(\Pi(Q))=0$, then this is not the case.

Theorem V.6.1. (i) *If $I(Q)=0$, then $S_{\nu,\zeta}\left(\Pi(Q)_{[0,T]}\right)=0$.*

(ii) *If $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)=0$, then there is exactly one $Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$ with $\Pi(Q)_{[0,T]}=\mu_{[0,T]}$ and $I(Q)=0$. This Q equals $\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes P^{I,\mu_{[0,T]}}_{x,w}$.*

Proof. By (V.6.1), (i) is obviously satisfied.

Now we show the opposite direction (ii). Fix a $\mu_{[0,T]}\in\mathscr{C}$ with $S_{\nu,\zeta}\left(\mu_{[0,T]}\right)=0$. Then $Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$, with $\Pi(Q)_{[0,T]}=\mu_{[0,T]}$ and $I(Q)$ implies that $Q=Q^*:=\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes P^{I,\mu_{[0,T]}}_{x,w}$. This implies that there is at most one minimizer with $\Pi(Q)_{[0,T]}=\mu_{[0,T]}$ and $I(Q)$.

Now we show that there exist an arbitrary $\overline{Q}\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$, with $\Pi\left(\overline{Q}\right)_{[0,T]}=\mu_{[0,T]}$ such that $I\left(\overline{Q}\right)=0$. This implies in particular that $\Pi(Q^*)_{[0,T]}=\mu_{[0,T]}$. By Section V.3.2.1.1 the results of Section V.3.1 hold for the SDE with fixed interaction $\mu_{[0,T]}$. Then we get by the beginning of Step 1 of the proof of Lemma V.3.15, that there is a $Q^*\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$, with $\Pi(Q^*)_{[0,T]}=\mu_{[0,T]}$ and with $I(Q^*)=0$. □

V.6.2 From the LDP of the empirical measure to the LDP of the empirical process

We infer the large deviation principle for the empirical process $\left\{\mu^N_{[0,T]}, P^N\right\}$ from the large deviation principle for the empirical measure $\left\{L^N, P^N\right\}$, in the following theorem. This is a simple application of the contraction principle. This theorem requires only the large deviation principle for $\left\{L^N\right\}$ (in contrast to the relation (V.6.1) between the rate function). However, the rate function for the empirical processes is only described via a minimizing problem (see Section V.6.3 for a further discussion).

Theorem V.6.2. *If the assumptions of Theorem V.5.12 hold, then the family of the empirical processes $\left\{\mu^N_{[0,T]}, P^N\right\}$ satisfies on $\mathscr{C}$ the large deviation principle with rate function*

$$j\left(\mu_{[0,T]}\right) := \inf_{Q\in\mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])):\Pi(Q)_{[0,T]}=\mu_{[0,T]}} I(Q). \tag{V.6.2}$$

Proof. The family $\left\{L^N, P^N\right\}$ satisfies by Theorem V.5.12 on $\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$ the large deviation principle with rate function I. Moreover, the map $\Pi : \mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\to\mathscr{C}$ is continuous (Lemma V.2.28). Then the contraction principle implies the LDP of $\left\{\mu^N_{[0,T]}, P^N\right\}$ with the rate function j. □

V.6.3 An upper bound on the rate function $S_{\nu,\zeta}$

By Theorem V.6.2, j is the rate function of the large deviation principle for $\left\{\mu^N_{[0,T]}, P^N\right\}$. Moreover, by Theorem V.3.5 and the uniqueness of rate functions, j has to be equal to $S_{\nu,\zeta}$ and $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}$. We show now that j is equal to $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}$ at least when j is finite, without using the Theorem V.3.5 (we need only Lemma V.3.11 and Lemma V.3.15). However, j is not everywhere finite (see also Remark V.3.16 for the concept of admissible flows). Therefore, this is only an upper bound on $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}$. Nevertheless, the upper bound on $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}$, implies at least a large deviation upper bound with $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}$ as rate function. For the large deviation lower bound (and another proof of the upper bound) we refer to Section V.3.

Lemma V.6.3. *Let the assumptions of Theorem V.5.12 hold.*
If $j\left(\mu_{[0,T]}\right) < \infty$ for a $\mu_{[0,T]}\in\mathscr{C}$, then $j\left(\mu_{[0,T]}\right) = S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}\left(\mu_{[0,T]}\right)$.
In particular this implies $j\left(\mu_{[0,T]}\right) \geq S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}\left(\mu_{[0,T]}\right) \geq S_{\nu,\zeta}\left(\mu_{[0,T]}\right)$.

Remark V.6.4. *In [DPdH96] a proof of the equality between the counterparts of j and $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}$ is given. However, in that proof the authors accidentally use a circular reasoning (in the equality (2.24) in [DPdH96]). We are also not able to prove the missing lower bound on $S^{\mathbb{T}^d\times\mathcal{W}}_{\nu,\zeta}$, without using Theorem V.3.5,.*

Proof of Lemma V.6.3. Fix a $\mu_{[0,T]}\in\mathscr{C}$ with $j\left(\mu_{[0,T]}\right) < \infty$. Then there is a $R > 0$, such that $\mu_{[0,T]}\in\mathscr{C}_{\varphi,R}$, because there has to be a $Q\in\mathcal{M}_{\varphi,\infty}$ with $I(Q) < \infty$ and $\Pi(Q)_{[0,T]} = \mu_{[0,T]}$. By the same argument $\mu_{[0,T]}\in\mathscr{C}^L$.

Define $b^{I,\mu_{[0,T]}}(t,x,w,\theta) := b(x,w,\theta,\mu_t)$ as in Notation V.4.1. With this $b^{I,\mu_{[0,T]}}$, we can define a system of independent SDEs as in (V.3.12). This system satisfies the Assumption V.3.7 as shown in Section V.3.2.1.1. Then the Lemma V.3.11 is applicable and we denote the rate function (V.3.15) by $S^{I,1,\mu_{[0,T]}}_{\nu,\zeta}$, i.e.

$$j\left(\mu_{[0,T]}\right) = \inf_{Q\in\mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])):\Pi(Q)_{[0,T]}=\mu_{[0,T]}} I(Q) = S^{I,1,\mu_{[0,T]}}_{\nu,\zeta}\left(\mu_{[0,T]}\right). \tag{V.6.3}$$

From this equality and Lemma V.3.15 (which is applicable for the same reasons), we conclude the Lemma V.6.3. □

V.7 LDP of the empirical measure for (0.9.3), via a generalisation of Varadhan's lemma

We show in Section V.5.2 that the family of empirical measure $\{L^N\}$ of the local mean field model (0.9.3), satisfies the large deviation principle and we derive two representations of the rate function (Theorem V.5.3 and Theorem V.5.12). In the proof of Theorem V.5.3 we use the same approach as in the proof of the large deviation principle for the empirical process $\left\{\mu^N_{[0,T]}\right\}$ in Section V.3. In particular we investigate the SDE with a fixed effective field and derive a LDP for this system. Then we infer from this LDP a LDP of the the SDE with interaction. From Theorem V.5.3 we infer Theorem V.5.12, by showing an equality of the two formulas of the rate function.

In this section we prove Theorem V.5.12 by another approach. We look at first at the SDEs with drift coefficient $-\Psi'$, i.e. at spins that are distributed according to $W^{N,-\Psi}$ (defined in Notation V.5.10). Then we apply the generalised Varadhan's Lemma (Theorem C.1.1). By the Laplace principle we infer finally the claimed large deviation principle with the rate function (V.5.12).

The Theorem V.5.3 can then be derived from Theorem V.5.12. Indeed, one only has to show the equality of the two representations of the rate function, which follows by the proof of Theorem V.5.12 given in in Section V.5.2.1.

Notation V.7.1. *We fix $\varphi(\theta) = 1 + \theta^2$. To simplify the notation we use $\mathcal{M}_R$ and $\mathcal{M}_\infty$ instead of $\mathcal{M}_{\varphi,R}$, $\mathcal{M}_{\varphi,\infty}$ in this section, i.e.*

$$\mathcal{M}_R := \left\{ Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big) : \sup_{t\in[0,T]} \int (\theta_t)^2 \, Q(\mathrm{d}\theta_{[0,T]}) \leq R - 1 \right\}, \tag{V.7.1}$$

and

$$\mathcal{M}_\infty := \bigcup_R \mathcal{M}_R \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big), \tag{V.7.2}$$

equipped with the subspace topology induced by $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$.

Proof (second) of Theorem V.5.12.

We know from Lemma V.2.8 that $\{L^N, W^{N,-\Psi}\}$ satisfies the large deviation principle with rate function $\mathsf{H}\left(Q \middle| \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W^{-\Psi}_{x,w}\right)$ if $Q \in \mathbb{M}_1^L\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$ and infinity otherwise. To infer the LDP of $\{L^N, P^N\}$ from the LDP of $\{L^N, W^{N,-\Psi}\}$, we need at first the following result, which states the validity of the Laplace principle.

Lemma V.7.2. *For any $G \in \mathsf{C}_b\big(\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)\big)$ bounded continuous functional,*

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N^d} \log E_{P^N}\left[e^{N^d G(L^N)}\right] \\ &= \sup_{\substack{Q\in\mathcal{M}_\infty \\ Q\in\mathbb{M}_1^L(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T]))}} \left\{(G+F)(Q) - \mathsf{H}\left(Q \middle| \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W^{-\Psi}_{x,w}\right)\right\} \\ &= \sup_{Q\in\mathbb{M}_1(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T]))} \left\{G(Q) - \overline{I}(Q)\right\} < \infty, \end{aligned} \tag{V.7.3}$$

with F and $\overline{I}$ defined in Theorem V.5.12.

In the proof of this lemma, we apply at first the Girsanov theorem to replace the integral with respect to P^N, by an integral with respect to $W^{N,-\Psi}$. Thus we get in the exponent $G-F$, by Lemma V.5.16. However, the function F is neither bounded nor continuous, due to the unbounded terms in the integrals in F. Therefore, we can not apply the original Varadhan's lemma, but we have to use a generalised version of it (see Appendix C).

Moreover, we need that $\overline{I}$ is a good rate function.

Lemma V.7.3. *The rate function $\overline{I}$ is good, i.e. the level sets $\mathcal{L}^{\leq c}\left(\overline{I}\right) := \left\{Q : \overline{I}\,(Q) \leq c\right\}$ are compact for each $c \geq 0$.*

The validity of the Laplace principle for all $G \in \mathsf{C}_b\big(\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)\big)$ (Lemma V.7.2) and $\overline{I}$ being a good rate function (Lemma V.7.3), implies the claimed large deviation principle for $\left\{L^N\right\}$ under $\left\{P^N\right\}$ (see [DE97] Theorem 1.2.3). □

V.7.1 Proof of Lemma V.7.2

In this section we prove Lemma V.7.2. We explain at first the strategy of this proof. To prove the first equality in Lemma V.7.2, we need to show that for any $G \in \mathsf{C}_b\big(\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)\big)$,

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{N^d} \log E_{W^{N,-\Psi}} \left[e^{N^d (G+F)\left(L^N\right)}\right] \\ &= \sup_{Q \in \mathcal{M}_\infty} \left\{(G+F)\,(Q) - \mathsf{H}\left(Q \middle| \mathrm{d}x \otimes \zeta_x\,(\mathrm{d}w) \otimes W^{-\Psi}_{x,w}\right)\right\},\end{aligned} \tag{V.7.4}$$

by Lemma V.5.16. The second equality in Lemma V.7.2 follows from the definition of $\overline{I}$.

The equation (V.7.4) would follow directly from Varadhan's Lemma (see Theorem 4.3.1 in [DZ98]), if F were continuous. But this is not the case, because the functions in the integrals in F are not bounded. Therefore, we can not use the usual Varadhan's lemma, but we need a generalisation (Theorem C.1.1). We prove at the end of this section that the conditions of this generalisation are satisfied. This requires some results, that we state now. Also larger sets than $\mathcal{M}_R$ are required in that proof (we refer to Section V.7.4 for a discussion why we need larger sets). For each $R \in \mathbb{R}_+$ define

$$\begin{aligned}\mathcal{N}_R := \Bigg\{ & Q \in \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big) : \int\limits_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} \int_0^T (\theta_t)^2 \,\mathrm{d}t\, Q\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) \leq R \text{ and} \\ & \int\limits_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} (\theta_T)^2\, Q\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) \leq R \text{ and} \int\limits_{\mathbb{T}^d \times \mathcal{W} \times \mathbb{R}} (\theta_0)^2\, Q\left(\mathrm{d}x, \mathrm{d}w, \mathrm{d}\theta_{[0,T]}\right) \leq R \Bigg\},\end{aligned} \tag{V.7.5}$$

and denote the subspace of $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$, of the union of these sets, by

$$\mathcal{N}_\infty := \bigcup_R \mathcal{N}_R \subset \mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)\,, \tag{V.7.6}$$

equipped with the topology induced by $\mathbb{M}_1\big(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\big)$.

In the first lemma, we show that the probability of being outside of $\mathcal{N}_R$ under $W^{N,-\Psi}$, decays exponentially fast.

Lemma V.7.4. *For all $\kappa < c_\Psi$ (defined in Assumption V.1.7), there is a constant $C > 0$, such that for all N and R large enough*

$$W^{N,-\Psi}\left[L^N \notin \mathcal{N}_R\right] \leq e^{-N^d \kappa R} C^{N^d}. \tag{V.7.7}$$

Then we show that the probability of being outside of $\mathcal{N}_R$ also decays (at least asymptotically) exponential fast under P^N.

Lemma V.7.5.
$$\begin{aligned}&\limsup_{R\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\log P^N\left[L^N\notin\mathcal{N}_R\right]\\&=\limsup_{R\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\log E_{W^{N,-\Psi}}\left[e^{N^dF(L^N)}\mathbb{1}_{L^N\notin\mathcal{N}_R}\right]=-\infty.\end{aligned}\tag{V.7.8}$$

Moreover, we show that the sets $\mathcal{N}_R$ are closed and that the restriction of F to particular sequences in these sets is continuous.

Lemma V.7.6. *The sets $\mathcal{N}_R$ are closed.*

Lemma V.7.7. *For each $R>0$, for each $Q\in\mathcal{N}_R$ with $\mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)<\infty$ and for each sequence of empirical measures $\{L_{N_n}\}\subset\mathcal{N}_R$ with $L_{N_n}\to Q$, the sequence $F(L_{N_n})\to F(Q)$.*

Last but not least we need that the relative entropy is infinite when $Q\notin\mathcal{M}_\infty$.

Lemma V.7.8. $\mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)=\infty$ *if $Q\notin\mathcal{M}_\infty$. Therefore, this also holds for $Q\notin\mathcal{N}_\infty$.*

We state the proofs of these lemmas in Section V.7.2. In the rest of this section, we infer Lemma V.7.2 from these lemmas.

Proof of Lemma V.7.2. To prove Lemma V.7.2, we show that (V.7.4) holds by applying the generalised Varadhan's Lemma (Theorem C.1.1). In Step 1 we show that the conditions of Theorem C.1.1 hold. Then we derive in Step 2, that the supremum on the right hand side of (V.7.4) is finite.

Step 1: Applying the Theorem C.1.1: To apply Theorem C.1.1 we show that the model we consider here is within the class defined in Section C.4.2. We take as increasing sets the $\mathcal{N}_R$.
Step 1.1: (C.4.2.ii): See Lemma V.7.6.
Step 1.2: (C.4.2.iii): See Lemma V.7.8, because $\mathcal{M}_\infty\subset\mathcal{N}_\infty$.
Step 1.3: (C.4.2.iv): See Lemma V.7.7.
Step 1.4: (C.4.2.v): For an empirical process $L^N\in\mathcal{N}_R$, we get by (V.5.32), that

$$F\left(L^N\right)\le\left(\left\|\overline{J}\right\|_{L^1}+\delta\right)\frac{1}{2}\frac{1}{N^d}\sum_{i\in\mathbb{T}^d_N}\left(\theta^{i,N}_T\right)^2+\left(\theta^{i,N}_0\right)^2\le R\left(\left\|\overline{J}\right\|_{L^1}+\delta\right).\tag{V.7.9}$$

for all $\delta>0$, when $N>N_\delta>0$ (by Assumption V.1.4).

If $Q\in\mathcal{N}_R$ such that $\mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)<\infty$, then $Q\in\mathbb{M}^L_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$. Then

$$F(Q)\le\left\|\overline{J}\right\|_{L^1}\frac{1}{2}\int_{\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])}\theta_T^2+\theta_0^2\,Q\left(\mathrm{d}x,\mathrm{d}w,\mathrm{d}\theta_{[0,T]}\right)\le R\left\|\overline{J}\right\|_{L^1}.\tag{V.7.10}$$

Hence we set $\alpha(R)=R\left(\left\|\overline{J}\right\|_{L^1}+\delta\right)$, for δ small enough.
Step 1.5: (C.4.2.vi): This follows from Lemma V.7.4 with $\beta(R)=c_\Psi R-C$ for a constant $C>0$.
Step 1.6: (C.4.2.vii): $\alpha(R)-\beta(R)\to-\infty$ by Assumption V.1.7.
Step 1.7: (C.4.2.viii): See Lemma V.7.5.
Step 1.8: (C.4.2.ix): The sufficient moment condition is satisfied, because G is bounded and because there is a $C>0$ and a $\gamma>1$ not too large, such that

$$E_{W^{N,-\Psi}}\left[e^{\gamma N^dF(L^N)}\right]\le C^{N^d},\tag{V.7.11}$$

for all $N\in\mathbb{N}$, by (V.7.19) (in the proof of Lemma V.7.5).

Hence the model we consider here is within the class defined in Section C.4.2 and Theorem C.1.1 is applicable, such that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^d} \log E_{W^{N,-\Psi}} \left[e^{N^d (G+F)\left(L^N\right)} \right] \\
&= \sup_{\substack{Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right):\\ \mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)<\infty}} \left\{ (G+F)(Q) - \mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)\right\} \\
&= \sup_{Q\in\mathcal{M}_\infty\cap\mathbb{M}_1^L(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T]))} \left\{ (G+F)(Q) - \mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)\right\}.
\end{aligned} \tag{V.7.12}$$

For the last equality we use that F is finite for each $\mu \in \mathcal{M}_\infty \cap \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ (see (V.7.10)) and Lemma V.7.8.

Step 2: The suprema in (V.7.12) **are finite:** For a lower bound on the right hand side of (V.7.12), take $Q = \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W^{-\Psi}_{x,w} \in \mathcal{N}_R$. Moreover, the left hand side of (V.7.12) is bounded from above, because

$$E_{W^{N,-\Psi}} \left[e^{N^d(G+F)\left(L^N\right)} \right] \le e^{N^d|G|_\infty} \left(E_{W^{N,-\Psi}} \left[e^{N^d \gamma F\left(L^N\right)} \right] \right)^{\frac{1}{\gamma}} \le e^{N^d|G|_\infty} C^{\frac{N^d}{\gamma}}, \tag{V.7.13}$$

for each $N \in \mathbb{N}$, by (V.7.11). Hence also the suprema in (V.7.12) are finite. □

V.7.2 Proofs of the lemmas of Section V.7.1

In this section we prove the lemmas that we state in Section V.7.1.

Proof of Lemma V.7.4. At first we split the set $\mathcal{N}_R$ in its three conditions. Then we show separately for each of the three terms an exponential small upper bound.

$$\begin{aligned}
W^{N,-\Psi}\left[L^N \notin \mathcal{N}_R\right] \le\ & W^{N,-\Psi}\left[\frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \left(\theta_T^{i,N}\right)^2 > R \right] \\
&+ W^{N,-\Psi}\left[\frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \left(\theta_0^{i,N}\right)^2 > R \right] \\
&+ W^{N,-\Psi}\left[\frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \int_0^T \left(\theta_t^{i,N}\right)^2 \mathrm{d}t > R \right] =: ① + ② + ③.
\end{aligned} \tag{V.7.14}$$

Fix a $\kappa < c_\Psi$. With the exponential Chebychev inequality, we get

$$① \le e^{-N^d R\kappa} \prod_{i\in\mathbb{T}_N^d} \sup_{w\in\mathcal{W}} E_{W^{-\Psi}_{\frac{i}{N},w}} \left[e^{\kappa(\theta_T)^2} \right] \le e^{-N^d R\kappa} C^{N^d}, \tag{V.7.15}$$

where we use Lemma V.5.15 (i) to get the last inequality. By the Chebychev inequality we get also for ②

$$② = \nu^N \left[\frac{1}{N^d} \sum_{i\in\mathbb{T}_N^d} \left(\theta_0^{i,N}\right)^2 > R \right] \le e^{-N^d R\kappa} \prod_{i\in\mathbb{T}_N^d} \int_{\mathbb{R}} e^{\kappa\theta^2} \nu_{\frac{i}{N}}(\mathrm{d}\theta) \le e^{-N^d R\kappa} C^{N^d}, \tag{V.7.16}$$

where we use Assumption V.1.2 and Assumption V.1.7 in the last inequality. For ③ we get from the exponential Chebyshev inequality and Lemma V.5.15 (ii)

$$③ \leq e^{-N^d R\kappa} \prod_{i\in\mathbb{T}_N^d} \sup_{w\in\mathcal{W}} E_{W^{-\Psi}_{\frac{i}{N},w}} \left[e^{\kappa \int_0^T (\theta_t)^2 \mathrm{d}t} \right] \leq e^{-N^d R\kappa} C^{N^d}. \tag{V.7.17}$$

□

Proof of Lemma V.7.5. With $\frac{1}{p} + \frac{1}{p'} = 1$, we get by the Hölder inequality

$$E_{W^{N,-\Psi}} \left[e^{N^d F(L^N)} \mathbb{1}_{L^N \notin \mathcal{N}_R} \right] \leq E_{W^{N,-\Psi}} \left[e^{N^d p F(L^N)} \right]^{\frac{1}{p}} W^{N,-\Psi} \left[L^N \notin \mathcal{N}_R \right]^{\frac{1}{p'}}. \tag{V.7.18}$$

For $\delta > 0$ and $p > 1$ small enough, such that $p\left(\left\|\overline{J}\right\|_{L^1} + \delta\right) < c_\Psi$, we can bound F form above as in (V.7.9) when $N > N_\delta$. Then

$$E_{W^{N,-\Psi}} \left[e^{N^d p F(L^N)} \right] \leq \prod_{i\in\mathbb{T}_N^d} \sup_{w\in\mathbb{W}} E_{W^{-\Psi}_{\frac{i}{N},w}} \left[e^{p\left(\left\|\overline{J}\right\|_{L^1}+\delta\right)\left((\theta_T)^2+(\theta_0)^2\right)} \right] \leq C^{N^d}, \tag{V.7.19}$$

where we use Lemma V.5.15 (i) in the last inequality. With Lemma V.7.4, we conclude

$$E_{W^{N,-\Psi}} \left[e^{N^d F(L^N)} \mathbb{1}_{L^N \notin \mathcal{N}_R} \right] \leq C^{N^d} e^{-N^d \frac{1}{p'} \kappa R}, \tag{V.7.20}$$

for $\kappa < c_\Psi$, when N and R are large enough. This proves Lemma V.7.5. □

Proof of Lemma V.7.6. Fix a sequence $Q^{(n)} \in \mathcal{N}_R$ that converges in $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$ to a $Q \in \mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$. We want to show that $Q \in \mathcal{N}_R$. To this end, define the cutoff functions for $M \in \mathbb{R}_+$

$$\chi_M(\theta) := (\theta \wedge M) \vee -M. \tag{V.7.21}$$

The function $\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T]) \ni \left(x, w, \theta_{[0,T]}\right) \mapsto \chi_M\left(\theta_T^2\right) \in \mathbb{R}$ is a continuous, bounded function. By the weak convergence of $Q^{(n)}$, $\int \chi_M\left(\theta_T^2\right) Q^{(n)} \to \int \chi_M\left(\theta_T^2\right) Q$ for each $M \in \mathbb{R}_+$. Hence

$$\int \chi_M\left(\theta_T^2\right) Q \leq R. \tag{V.7.22}$$

By the monotone convergence theorem this implies

$$\int \theta_T^2 Q \leq R. \tag{V.7.23}$$

Similar calculations with $\chi_M\left(\theta_0^2\right)$ and $\chi_M\left(\int_0^T \theta_t^2 \mathrm{d}t\right)$ imply also the boundedness by R of the other two conditions in $\mathcal{N}_R$. Hence $\mathcal{N}_R$ is closed. □

Proof of Lemma V.7.7. Fix an $R > 0$, a measure $Q \in \mathcal{N}_R$, with $\mathsf{H}\left(Q \middle| \mathrm{d}x \otimes \zeta_x(\mathrm{d}w) \otimes W_{x,w}^{-\Psi}\right) < \infty$. Hence in particular that $Q \in \mathbb{M}_1^L\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$. Moreover, fix a weakly converging sequence of empirical measures $L^{N_n} \to Q$ in $\mathcal{N}_R$. We show now that $F\left(L^{N_n}\right) \to F(Q)$.

Because the integrands in F are neither continuous nor bounded, we approximate J by continuous J_ℓ and use cutoff functions for the spins $\chi_M(\theta, \theta') = \chi_M(\theta)\chi_M(\theta')$ as in (V.3.123). The five arising summands (similar to (V.3.123)), can all be bounded by ϵ, by the same approach that we use in Step 5.2.1 in Section V.3.3. This implies the convergence of $F\left(L^{N_n}\right) \to F(Q)$.

This approach requires that $L^{N_n}, Q \in \mathcal{N}_R$, the Assumption V.1.4, that $L^{N_n} \otimes L^{N_n}$ is tight and that on compact subsets of $\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)^2$ the paths of the spins are equibounded.

We need the last two properties to bound Ⓒ and Ⓓ of (V.3.123). Indeed, when considering the second summand of F we get for we get for Ⓒ

$$\begin{aligned}
&\left|\int J_\ell\left(x-x',w,w'\right)\left(\theta'_T\theta_T-\chi_M\left(\theta'_T\right)\chi_M\left(\theta_T\right)\right)\left(L^{N_n}\otimes L^{N_n}\right)\right|\\
&\le L^{N_n}\otimes L^{N_n}\left[\left(\theta_{[0,T]},\theta'_{[0,T]}\right):\left|\theta_{[0,T]}\right|_\infty>M\text{ or }\left|\theta'_{[0,T]}\right|_\infty>M\right]\left|J_\ell\right|_\infty\int\left(\theta_T\right)^2L^{N_n} \qquad \text{(V.7.24)}\\
&\le L^{N_n}\otimes L^{N_n}\left[\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)^2\setminus K_\epsilon\right]\left|J_\ell\right|_\infty R\le C\epsilon^{\frac{1}{2}},
\end{aligned}$$

for a suitably chosen compact set K_ϵ and $M>0$, such that $\theta_{[0,T]}\in K_\epsilon$ implies that $\left|\theta_{[0,T]}\right|_\infty\le M$. The first summand of F can be bounded analogue.

Note that we get the tightness of $L^{N_n}\otimes L^{N_n}$ by Prokhorov's theorem and because the convergence of $L^{N_n}\to Q$ implies the convergence of $L^{N_n}\otimes L^{N_n}\to Q\otimes Q$. Moreover, on compact subsets of $\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)^2$, the paths of the spins are equibounded by Lemma V.2.20 (i). □

Proof of Lemma V.7.8. To prove this lemma, we use that the probability of L^N being outside of $\mathcal{M}_R$ under P^N decays exponentially fast, i.e. there is a $\lambda\in\mathbb{R}_+$ and a $C>0$ such that for all $N\in\mathbb{N}$ and R large enough

$$W^{N,-\Psi}\left[L^N\notin\mathcal{M}_R\right]\le e^{-N^d\lambda R}C^{N^d}. \qquad \text{(V.7.25)}$$

We get this exponential bound from Lemma V.3.32, because $L^N\in\mathcal{M}_R$ if and only if $\mu^N_{[0,T]}\in\mathscr{C}_{\varphi,R}$ with $\varphi=\theta^2$. The necessary assumptions for this lemma (i.e. Assumption V.3.1 a.a)-c.)) are satisfied for the drift coefficient $-\Psi'$ (by the Assumption V.1.7 and the same arguments as in Section V.3.3).

If $Q\in\mathbb{M}_1\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)\setminus\mathcal{M}_\infty$, then $Q\notin\mathcal{M}_R$ for all $R>0$. Then for R large enough

$$\begin{aligned}
-\mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)&\le-\inf_{Q'\notin\mathcal{M}_R}\mathsf{H}\left(Q'\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)\\
&\le\liminf_{N\to\infty}\frac{1}{N^d}\log W^{N,-\Psi}\left[L^N\notin\mathcal{M}_R\right]\le-\lambda R+\log C,
\end{aligned} \qquad \text{(V.7.26)}$$

by (V.7.25), by the large deviation principle for $\left\{L^N,W^{N,-\Psi}\right\}$ and by $\mathcal{M}_R$ being closed (by a similar proof as for Lemma V.7.6). □

Remark V.7.9. *We could also prove* (V.7.25) *without using the Lemma V.3.32. One would have to show at first the exponential decay of the probability of being outside of $\mathcal{M}_R$ under $W^{N,-\Psi}$. This can be done by the direct approach to transfer the problem to the measure $W^{N,0}$ (by the Girsanov theorem) and then to use the Doob submartingale inequality.*

V.7.3 $\overline{I}$ is a good rate function (Lemma V.7.3)

Proof of Lemma V.7.3. We have to prove that $\mathcal{L}^{\le c}\left(\overline{I}\right)$ is compact. Therefore, we show at first that $\mathcal{L}^{\le c}\left(\overline{I}\right)$ is a subset of $\mathcal{N}_R$ for R large enough, then that it is closed and finally that it is compact.

Step 1: $\mathcal{L}^{\le c}\left(\overline{I}\right)$ is a subset of $\mathcal{N}_R$ for R large enough:

Fix a $Q\in\mathcal{L}^{\le c}\left(\overline{I}\right)$. Then $\overline{I}(Q)\le c$ implies that $Q\in\mathcal{N}_\infty$ and $Q\in\mathbb{M}_1^L\left(\mathbb{T}^d\times\mathcal{W}\times\mathsf{C}([0,T])\right)$. Choose $R>0$ such that $Q\in\mathcal{N}_{R+\frac{1}{R}}\setminus\mathcal{N}_R$. Then

$$\overline{I}(Q)=\mathsf{H}\left(Q\middle|\mathrm{d}x\otimes\zeta_x(\mathrm{d}w)\otimes W^{-\Psi}_{x,w}\right)-F(Q)\ge\kappa R+C_R-\left\|\overline{J}\right\|_{L^1}\left(R+\frac{1}{R}\right), \qquad \text{(V.7.27)}$$

with $\kappa \in \mathbb{R}$ such that $\left\|\overline{J}\right\|_{L^1} < \kappa < c_\Psi$ (possible due to Assumption V.1.7). In this equality we use a similar calculation as in (V.7.26) to bound the relative entropy (with $\mathcal{N}_R$ instead of $\mathcal{M}_R$ and with Lemma V.7.4). The upper bound on F holds by Assumption V.1.4 and by Q having the Lebesgue measure as projection to $\mathbb{T}^d$.

The right hand side of (V.7.27) tends to infinity when R increases (by Assumption V.1.7). Hence there is a R large enough such that $Q \in \mathcal{N}_R$ if $Q \in \mathcal{L}^{\leq c}\left(\overline{I}\right)$.

Step 2: $\mathcal{L}^{\leq c}\left(\overline{I}\right)$ is a closed:

Take a sequence $\left\{Q^{(n)}\right\}_n \subset \mathcal{L}^{\leq c}\left(\overline{I}\right) \subset \mathcal{N}_R$, such that $Q^{(n)} \to Q$ in $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$. Then $Q \in \mathcal{N}_R$ because this set is closed (Lemma V.7.6). By F being continuous on $\mathcal{N}_R$ and the relative entropy being lower semi-continuous,

$$\overline{I}\left(Q\right) \leq \liminf_{n\to\infty} \overline{I}\left(Q^{(n)}\right) \leq c. \tag{V.7.28}$$

Step 3: $\mathcal{L}^{\leq c}\left(\overline{I}\right)$ is compact:

We use now the exponential tightness of $\left\{L^N, W^{N,-\Psi}\right\}$ derived in Section V.5.2.3. The corresponding compact sets K_A are defined in (V.5.28). We claim that there is a $A > 0$ such that $\mathcal{L}^{\leq c}\left(\overline{I}\right) \subset K_A$. Take a $Q \in \mathcal{L}^{\leq c}\left(\overline{I}\right) \subset \mathcal{N}_R$ with $Q \notin K^A$ for a $A > 0$. Then

$$\overline{I}\left(Q\right) = \mathsf{H}\left(Q\middle|\mathrm{d}x \otimes \zeta_x\left(\mathrm{d}w\right) \otimes W^{-\Psi}_{x,w}\right) - F\left(Q\right) \geq A - \left\|\overline{J}\right\|_{L^1} R, \tag{V.7.29}$$

where we bound F as in (V.7.27) and the lower bound on the relative entropy follows by the same calculation as in (V.7.26) with K_A instead of $\mathcal{M}_R$ and with (V.5.30). This implies that there has to be a $A > 0$, such that $\mathcal{L}^{\leq c}\left(\overline{I}\right) \subset K^A$.

Therefore, we conclude that the sets $\mathcal{L}^{\leq c}\left(\overline{I}\right)$ are compact as closed subsets of a compact set. □

V.7.4 Discussion of the different subsets that we use in the proofs

In the previous sections we use the three different subsets of $\mathbb{M}_1\left(\mathbb{T}^d \times \mathcal{W} \times \mathsf{C}([0,T])\right)$, $\mathcal{M}_R$ (defined in (V.7.1)), $\mathcal{N}_R$ (defined in (V.7.5)) and K^L (defined in (V.5.28)). Let us briefly discuss why we do not restrict the attention to one of these sets.

On the one hand, we need a compact set when proving that I is a good rate function (see Lemma V.7.3). On the other hand, we need a closed set on which the function F is continuous and bounded in Lemma V.7.2. Each of the sets K_L and $\mathcal{N}_R$ only satisfies one of these properties. The set $\mathcal{N}_R$ is not compact, because the definition does not include equicontinuity. Whereas the set K_L is only abstractly defined, what seems to make it impossible to show that F is continuous on this set.

Moreover, we need for the proof that one representation of the rate function implies the other (see the proof of Theorem V.5.12 in Section V.5.2.1), that the function β (defined in (V.3.119)) is uniformly bounded in time for $t \in [0,T]$, such that the martingale problem for the SDE (V.5.11) well posed. This is the case for all measure in $\mathcal{M}_\infty$, but not for all measures in $\mathcal{N}_\infty$.

Then the natural question arises, why not using $\mathcal{M}_R$ in the proof in Section V.7 of the Theorem V.5.12. In Step 1 of the proof of Lemma V.7.2, we apply the extended Varadhan's Lemma (Theorem C.1.1). To apply this lemma, we need that the probability $W^{N,-\Psi}\left[L^N \notin \mathcal{M}_R\right]$ decays like $e^{\kappa R}$ with $\kappa > \left\|\overline{J}\right\|_{L^1}$. However, in general we can show an exponential decay but not with a $\kappa > \left\|\overline{J}\right\|_{L^1}$ (e.g. by the approach in (V.7.25) or the one sketched in Remark V.7.9),

This is different for the sets $\mathcal{N}_R$, because in the proof of the exponential decay for these sets (Lemma V.7.4), we can benefit from the integrals with respect to time arising by the Girsanov theorem. These integrals can however not approximate the supremum in $\mathcal{M}_R$.

Remark V.7.10. *The rate function is infinite outside of the set $\mathcal{M}_\infty$. Moreover, $P^N\left[L^N \in \mathcal{M}_\infty\right]$ equals one (this follows e.g. by similar arguments we use to show* (V.7.25)*). Therefore, $\left\{L^N, P^N\right\}$ satisfies the large deviation principle on $\mathcal{M}_\infty$ (by Lemma 4.1.5 (b) in [DZ98]).*

We could consider the subspace $\mathcal{M}_\infty$ *from the very first, i.e. we consider on the subspace* $\mathcal{M}_\infty$ *the LDP for* $\{L^N, W^{N,-\Psi}\}$ *(this LDP exists due to Lemma 4.1.5 (b) in [DZ98], by* (V.7.25) *and by Lemma V.7.8). Then by the same arguments and steps as in the proof in Section V.7 of Theorem V.5.12, we would get a LDP of* $\{L^N, P^N\}$ *on* $\mathcal{M}_\infty$*, with the rate function* I*.*

However, it simplifies the proof of Section V.7 only marginally and the result for the whole space $\mathbb{M}_1(\mathbb{T}^d \times \mathcal{W} \times \mathbb{R})$ *is stronger.*

Appendix

Appendix A

Properties of the space $\mathbb{M}\left(\mathbb{T}^d\right)$

Notation A.1. *With* $\mathbb{M}(\mathbb{T}^d)$ *we denote the space of regular, finite (in the total variation norm* $\|\mu\|_{TV}$*), real valued, signed Borel measures on* $\mathbb{T}^d$*.*

We equip $\mathbb{M}(\mathbb{T}^d)$ *with the* topology of weak-*-convergence*, i.e. the topology generated by the sets* $\left\{\mu \in \mathbb{M}(\mathbb{T}^d) : \left|\int f(x)\,\mu(\mathrm{d}x) - a\right| \le \delta\right\}$ *for all* $\delta > 0$*,* $a \in R$ *and* $f \in \mathsf{C}_b(\mathbb{T}^d)$*.*

In the following sections we state properties of the space $\mathbb{M}(\mathbb{T}^d)$ (in Section A.1), of subsets of $\mathbb{M}(\mathbb{T}^d)$ (in Section A.2). Moreover we show in Chaper A.3 a result concerning the convergence of products of measures of this space.

A.1 General properties

(i) $\mathbb{M}(\mathbb{T}^d)$ (with the total variation norm) is isometrically isomorphic to the continuous (topological) dual space of $\mathsf{C}(\mathbb{T}^d)$ (by the Riesz-Representation theorem, also Riesz-Markov-Kakutani representation theorem, see [DS88] Theorem IV.6.3 and also Example 1.10.6 in [Meg98] (remember that $\mathbb{T}^d$ is a compact Hausdorff space)).

(ii) The continuous (topological) dual space to $\mathbb{M}(\mathbb{T}^d)$ with the weak-*-topology, can be identified by $\mathsf{C}(\mathbb{T}^d)$ (by Theorem V.3.9 in [DS88], with $\{\mu \to \langle \mu, \phi \rangle\}_{\phi \in \mathsf{C}(\mathbb{T}^d)}$ the total (separating) subset of the linear functions on $\mathbb{M}(\mathbb{T}^d)$, see also [DZ98] Section 6.2 page 261).

(iii) $\mathbb{M}(\mathbb{T}^d)$ with the weak-*-topology is a Hausdorff, topological vector space. Therefore, it is regular (by Theorem 2.2.14 [Meg98]) and hence T_0 - $T_{3\frac{1}{2}}$, i.e. in particular completely regular.

(iv) $\mathbb{M}(\mathbb{T}^d)$ with the weak-*-topology is a locally convex space (by Lemma V.3.3 in [DS88], see also [DZ98] Theorem B.8).

Now we state three properties that are *not* satisfied by $\mathbb{M}(\mathbb{T}^d)$:

(v) The weak-*-topology on $\mathbb{M}(\mathbb{T}^d)$ is not metrisable, because the weak-*-topology on the dual of an infinite dimensional Banach space (in this case $\mathsf{C}(\mathbb{T}^d)$) is never metrisable (Proposition 2.6.12 in [Meg98]).

Hence $\mathbb{M}(\mathbb{T}^d)$ is as a vector space also not first countable.

(vi) The weak-*-topology on $\mathbb{M}(\mathbb{T}^d)$ is not complete (by Proposition 2.6.13 in [Meg98]).

(vii) $\mathbb{M}(\mathbb{T}^d)$ with the weak-*-topology is not a sequential space (by Theorem 2.5 in [HS96], there exists a countable subset of $\mathbb{M}(\mathbb{T}^d)$ that is weak-*- sequentially closed and weak-*-dense, hence not weak-*-closed).

A.2 Properties of subsets

(viii) The weak-*-topology is on a bounded (in $\|.\|_{TV}$) subsets of $\mathbb{M}(\mathbb{T}^d)$ metrisable (by [Meg98] Corollary 2.6.20).

For example, for each $R > 0$, the weak-*-topology is metrisable on the set

$$K_R := \left\{\mu \in \mathbb{M}(\mathbb{T}^d) : \|\mu\|_{TV} \leq R\right\}. \tag{A.1}$$

(ix) The sets K_R are weak-*-compact. This follows from the Banach-Alaoglu theorem ([Meg98] Theorem 2.6.18), because each closed ball (in the $\|.\|_{TV}$) is weak-*-compact.

(x) More general, each bounded (in $\|.\|_{TV}$) and weak-*-closed subset of $\mathbb{M}(\mathbb{T}^d)$ is weak-*-compact ([Meg98] Corollary 2.6.19).

A.3 Convergence of product measures

Lemma A.2. *If $\mu^{(n)} \to \mu$ weak-* on $\mathbb{M}(\mathbb{T}^d)$, then also $\mu^{(n)} \otimes \mu^{(n)} \to \mu \otimes \mu$ weak-* on $\mathbb{M}(\mathbb{T}^d \times \mathbb{T}^d)$.*

Proof. Fix $f_1, f_2 \in \mathsf{C}(\mathbb{T}^d)$. Then

$$\int\int f_1(x) f_2(y) \mu^{(n)}(\mathrm{d}x) \mu^{(n)}(\mathrm{d}y) \to \int\int f_1(x) f_2(y) \mu(\mathrm{d}x) \mu(\mathrm{d}y). \tag{A.2}$$

This implies that also the integral of finite sums of products $f_1 f_2$ converge. By the Stone-Weierstrass theorem, these sums are dense in $\mathsf{C}\left(\left(\mathbb{T}^d\right)^2\right)$. Hence for a given $f \in \mathsf{C}\left(\left(\mathbb{T}^d\right)^2\right)$ and a given $\epsilon > 0$, there is a $f^\epsilon = \sum_{i=1}^{N_\epsilon} f_{1,i}^\epsilon(x) f_{2,i}^\epsilon(y)$ such that $|f - f^\epsilon|_\infty < \epsilon$. This implies for n large enough

$$\begin{aligned} &\left|\int\int f(x,y)\left(\mu^{(n)}(\mathrm{d}x)\mu^{(n)}(\mathrm{d}y) - \mu(\mathrm{d}x)\mu(\mathrm{d}y)\right)\right| \\ &\leq |f - f^\epsilon|_\infty \left(\left\|\mu^{(n)}\right\|_{TV}^2 + \|\mu\|_{TV}^2\right) + \left|\int\int f^\epsilon(x,y)\left(\mu^{(n)}(\mathrm{d}x)\mu^{(n)}(\mathrm{d}y) - \mu(\mathrm{d}x)\mu(\mathrm{d}y)\right)\right|. \end{aligned} \tag{A.3}$$

The right hand side is bounded by ϵ for n large enough. Indeed, the total variation norms in the first summand are uniformly bounded, because $\left\{\mu^{(n)}\right\}$ is a weak-*-convergent sequence (see [Bog07] Proposition 8.1.7) □

Appendix B

Girsanov formula for locally bounded drift and diffusion coefficients

In this dissertation, we use several times the Radon-Nikodym derivative of the laws of two SDEs. We know a priori that the martingale problems for both SDEs are well posed. If the drift and diffusion coefficients of both SDEs are bounded, then we know the Radon-Nikodym derivative by the Girsanov formula (see for example [SV79] Theorem 6.4.2). However, in the SDEs we consider, the coefficients (or at least the diffusion coefficient) are only locally bounded. We give therefore a proof of the Girsanov formula for locally bounded coefficients that is based on a spatial localisation argument given in [SV79] (see [SV79] Theorem 10.1.1 and Exercise 10.3.2 in [SV79]). Depending on the setting it is easier to verify the conditions of the following theorem than other conditions that imply the Girsanov formula. These are for example the Novikov condition (see [IW89] Theorem 5.3 in Chapter 3, [IW89] Chapter 4.4.1 and [Nov73]) or more general conditions (e.g. the Kazamaki condition, see "Notes and References" in Chapter 6 in [LS01]).

To keep the result of this chapter very general, we compare the following two m-dimensional SDEs

$$\mathrm{d}X_t = b\left(t, X_t\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}W_t^X, \tag{B.1}$$

$$\mathrm{d}Y_t = c\left(t, Y_t\right)\mathrm{d}t + \sigma\left(t, Y_t\right)\mathrm{d}W_t^Y. \tag{B.2}$$

Theorem B.1. *Let $b, c : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0,\infty) \times \mathbb{R}^m \to S^m$ (the space $m \times m$ matrices) be measurable and locally bounded, such that the martingale problems for* (B.1) *and* (B.2) *starting from (t,x) are well posed with solution $\{P_{t,x}\}$ and $\{Q_{t,x}\}$, respectively. Moreover, assume that σ^{-1} is locally bounded and for all $T > 0$, $x \in \mathbb{R}^m$*

$$\begin{aligned} &\inf_{0\le t\le T}\ \inf_{z\in\mathbb{R}^m,|z|=1} z\ \left(\sigma\sigma^T\right)(t,x)\ z > 0 \qquad \text{and} \\ &\lim_{y\to x}\sup_{0\le t\le T}\left|\left(\sigma\sigma^T\right)(t,x) - \left(\sigma\sigma^T\right)(t,y)\right| = 0. \end{aligned} \tag{B.3}$$

Then for all $s \in [0,\infty)$ and $x \in \mathbb{R}^m$, $\frac{\mathrm{d}Q_{s,x}}{\mathrm{d}P_{s,x}} = M^s = M$, where

$$M_t = \exp\left\{\int_s^t \left(\sigma^{-1}\left(c-b\right)\right)\left(u, X_u\right)\mathrm{d}W_u^X - \frac{1}{2}\int_s^t \left|\left(\sigma^{-1}\left(c-b\right)\right)\left(u, X_u\right)\right|^2 \mathrm{d}u\right\}. \tag{B.4}$$

Proof. We know by [IW89] Theorem 5.2 in Chapter 3, that M_t is a supermartingale, and that it is a martingale if and only if $\mathbb{E}_{P_{s,x}}\left[M_t\right] = 1$. That this expectation is less or equal to one follows from the supermartingale property and $t = 0$. To show that $\mathbb{E}_{P_{s,x}}\left[M_t\right] \ge 1$, we localize the problem spatially by the local boundedness of the coefficients.

Define for each $n \in \mathbb{N}$, the set $G_n := [0,n) \times \{x \in \mathbb{R}^m : |x| \leq n\}$ and the functions

$$b_n := \Psi_n b \quad , \quad c_n := \Psi_n c \quad \text{and} \quad \sigma_n := \Psi_n \sigma + (1-\Psi_n) I, \tag{B.5}$$

where $\Psi_n \in \mathsf{C}^\infty([0,\infty) \times \mathbb{R}^m, [0,1])$, such that $\Psi_n \equiv 1$ on G_n and $\Psi_n \equiv 0$ on G_{n+1}. Due to this definitions, the martingale problems for (b_n, σ_n) and (c_n, σ_n) are well defined (by Theorem 7.2.1 in [SV79]). We denote the families of solution by $\left\{P^n_{s,x}\right\}$ and $\left\{Q^n_{s,x}\right\}$. By Theorem 6.4.2 in [SV79], $\frac{\mathrm{d}Q^n_{s,x}}{\mathrm{d}P^n_{s,x}} = M^n$, with

$$M^n_t = \exp\left\{\int_s^t \left(\sigma_n^{-1}(c_n - b_n)\right)(u, X_u)\,\mathrm{d}W^X_u - \frac{1}{2}\int_s^t \left|\left(\sigma_n^{-1}(c_n - b_n)\right)(u, X_u)\right|^2 \mathrm{d}u\right\}. \tag{B.6}$$

Denote by $\tau_n(X) := \inf\{t \geq s : (t, X_t) \notin G_n\}$ the first hitting time of the set G_n. We know by Corollary 10.1.2 in [SV79], that $Q_{s,x}[\tau_n \leq T] \to 0$. Hence for each $n \in \mathbb{N}$, $t, T \in [0,\infty)$, $T > t$,

$$\mathbb{E}_{P_{s,x}}[M_t] \geq \mathbb{E}_{P_{s,x}}[M_t \mathbb{1}_{\tau_n \geq T}] = \mathbb{E}_{P^n_{s,x}}[M^n_t \mathbb{1}_{\tau_n \geq T}] = Q_{s,x}[\tau_n \geq T] \to 1\,. \tag{B.7}$$

This implies that M_t is a martingale, hence it is the Radon-Nikodym derivative of $Q_{s,x}$ and $P_{s,x}$. □

If the diffusion coefficient is constant, then the conditions of Theorem B.1 on the diffusion coefficient are satisfied. Hence the following corollary holds:

Corollary B.2. *If the diffusion coefficient is constant $\sigma \in \mathbb{R}$, $\sigma \neq 0$, then* (B.3) *is satisfied. Hence, if b, c are locally bounded and the martingale problems for* (B.1) *and* (B.2) *are well posed with families of solutions $\{P_{s,x}\}$ and $\{Q_{s,x}\}$, then $\frac{\mathrm{d}Q_{s,x}}{\mathrm{d}P_{s,x}} = M$, with*

$$M_t = \exp\left\{\sigma^{-1}\int_s^t (c-b)(u, X_u)\,\mathrm{d}W^X_u - \frac{1}{2}|\sigma|^{-2}\int_s^t |(c-b)(u, X_u)|^2\,\mathrm{d}u\right\}. \tag{B.8}$$

Appendix C

A generalisation of Varadhan's lemma to nowhere continuous functions

The results and proofs of this chapter can also be found in [Mül16]. Varadhan proved in [Var66] in Chapter 3 a generalisation of the Laplace method, that is referred to as Varadhan's lemma. The lemma is a consequence of the large deviation principle. It gives a precise description of the logarithmic asymptotic (for $N \to \infty$) of expectations like

$$\mathbb{E}\left[e^{N\phi(\xi_N)}\right]. \tag{C.0.1}$$

For example Varadhan's lemma can be used to transfer the LDP from $\mathbb{P}$ to $e^{\phi}\mathbb{P}$, by the relation between the Laplace principle and the large deviation principle (see [DE97] Chapter 1). It requires usually that the function ϕ is continuous and satisfies a tail condition or that it is even bounded ([Var66] Chapter 3, [DZ98] Theorem 4.3.1, [DE97] Theorem 1.2.1, [dH00] Theorem III.13).

In the following we generalise this to functions ϕ that are not continuous. We only require that ϕ can be approximated (in an appropriate sense) by two sequences of measurable functions. Moreover, we need beside the tail condition, further conditions concerning the difference of ϕ and the approximating functions. For continuous ϕ, the sequences can be chosen to equal ϕ everywhere and our conditions shrink to the usual tail condition.

In [JMTW15], the upper bound of Varadhan's lemma is extended to functions ϕ that are not upper semi-continuous. However, the authors require that either the rate function is continuous or that the level sets $\phi^{-1}([a,\infty])$ are closed for all a large enough. In the models that we have in mind none of these two conditions is satisfied.

We state the main result in Section C.1. Then we discuss in Section C.3 some less general conditions, that might be simpler to prove, under which the main results still holds. We state our proof of the extended Varadhan's lemma in Section C.2. Finally, we give some examples, that indicate why this extension is useful.

C.1 The generalised Varadhan's lemma

Let X be a regular topological space and $\{\xi_N\}$ be a family of random variables with values in X. We denote by $\{\mathbb{P}^N\}$ the probability measures associated with $\{\xi_N\}$.

In the following theorem we state a Varadhan type lemma for a non-continuous and unbounded function ϕ, that satisfies the usual tail condition. Moreover, we require the existence of two sequences of functions $\underline{\phi}_R$ and $\overline{\phi}_R$, that approximate (in a appropriate lower/upper semicontinous way) ϕ. These sequences have to satisfy two conditions. We show in Section C.3 how the conditions of this Theorem can be simplified.

Set

$$S^* := \sup_{x \in X : I(x) < \infty} \{(\phi - I)(x)\} \in \mathbb{R} \cup \{-\infty, \infty\}. \tag{C.1.1}$$

Theorem C.1.1. *Assume that $(\xi_N, \mathbb{P}^N)$ satisfies a LDP with speed a_N on X with good rate function $I : X \to [0, \infty]$. Let ϕ be a measurable function $X \to \mathbb{R} \cup \{-\infty, \infty\}$. Assume that the following conditions are satisfied.*

a.) *There is a family of measurable functions $\left\{\underline{\phi}_R\right\}_{R \in \mathbb{R}_+}$ from $X \to \mathbb{R} \cup \{-\infty, \infty\}$, such that*

$$\forall (x \in X \text{ with } I(x) < \infty) \quad \forall \delta > 0 \quad \exists R^*_{x,\delta} > 0 \quad \text{such that} \quad \forall R > R^*_{x,\delta}$$

exists an open neighbourhood $U_{x,\delta,R} \subset X$ of x, such that

$$\inf_{y \in U_{x,\delta,R} \cap \mathrm{supp}\{\xi_N\}} \underline{\phi}_R(y) \geq \phi(x) - \delta. \tag{C.1.2}$$

b.) *There is a family of measurable functions $\{\overline{\phi}_R\}_{R \in \mathbb{R}_+}$ from $X \to \mathbb{R} \cup \{-\infty, \infty\}$, such that*

$$\exists R^* > 0 \quad \text{such that} \quad \forall R > R^* \quad \forall (x \in X \text{ with } I(x) < \infty) \quad \forall \delta > 0$$

exists an open neighbourhood $U_{x,\delta,R} \subset X$ of x, such that

$$\sup_{y \in U_{x,\delta,R} \cap \mathrm{supp}\{\xi_N\}} \overline{\phi}_R(y) \leq \max\{S^*, \phi(x)\} + \delta. \tag{C.1.3}$$

c.) *For all $\epsilon > 0$*

$$\begin{aligned} &\lim_{R \to \infty} \liminf_{N \to \infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\}}\right] \\ &= \lim_{R \to \infty} \liminf_{N \to \infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)}\right]. \end{aligned} \tag{C.1.4}$$

d.) *For all $\epsilon > 0$*

$$\limsup_{R \to \infty} \limsup_{N \to \infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)} \mathbb{1}_{\{\phi(\xi_N) > \overline{\phi}_R(\xi_N) + \epsilon\}}\right] \leq S^*. \tag{C.1.5}$$

e.) *The following tail condition holds*

$$\lim_{M \to \infty} \limsup_{N \to \infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)} \mathbb{1}_{\phi(\xi_N) \geq M}\right] \leq S^*. \tag{C.1.6}$$

Then

$$\lim_{N \to \infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \phi(\xi_N)}\right] = S^*. \tag{C.1.7}$$

Remark C.1.2. *The condition $I(x) < \infty$ in the supremum in the definition* (C.1.1) *of S^* can be dropped, if $\phi(x) < \infty$ for all $x \in X$.*

Remark C.1.3. *Theorem C.1.1 implies the usual Varadhan's lemma. If* $\phi : X \to \mathbb{R} \cup \{-\infty, \infty\}$ *is continuous, then set* $\underline{\phi}_R = \overline{\phi}_R = \phi$ *and all the conditions except the tail condition e.) are immediately satisfied.*

Remark C.1.4. *We need the conditions c.), d.), to reduce the proofs of the Varadhan lower and upper bound for* ϕ *to the analysis of Varadhan lower and upper bounds on* $\underline{\phi}_R$ *and* $\overline{\phi}_R$ *respectively. The Varadhan lower and upper bounds for these sequences can finally be shown by a similar proof as for the original Varadhan's lemma (e.g. [DZ98] Theorem 4.3.1), due to the conditions a.) and b.).*

Remark C.1.5. *The conditions a.) and b.) on* $\underline{\phi}_R$ *and* $\overline{\phi}_R$ *differ. For the latter the lower bound* R^* *on* R *is uniformly in* δ *and* x*. Whereas for* $\underline{\phi}_R$ *it does not have to be uniformly in these variables.*

Remark C.1.6. *As in [Var66] Chapter 3, we could also treat a different function* ϕ^N *for each* $N \in \mathbb{N}$*. Then we would get*

$$\lim_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \phi^N(\xi_N)}\right] = S^*, \tag{C.1.8}$$

if the conditions c.), d.) and e.) hold with ϕ *replaced by* ϕ^N*.*

Remark C.1.7. *To apply Bryc's Lemma (inverse Varadhan Lemma), one can use* $\phi = F + G$ *with* $G \in \mathsf{C}_b(X)$ *and* $F : X \to \mathbb{R}$ *such that the* $F - I$ *is the new rate function. By the continuity and boundedness of* G*, it is enough to find pointwise (to* F*) converging sequences* $\{\underline{F}_R\}$ *and* $\{\overline{F}_R\}$ *and to look at* $\underline{\phi}_R = \underline{F}_R + G$ *and* $\overline{\phi}_R = \overline{F}_R + G$*.*

We split the proof in showing that the right hand side of (C.1.7) is a lower bound (Lemma C.1.8) and an upper bound (Lemma C.1.9) for the left hand side.

Lemma C.1.8. *Let* ϕ *and* $\left\{\underline{\phi}_R\right\}$ *be defined as in Theorem C.1.1. and the large deviation lower bound for* $\left(\xi_N, \mathbb{P}^N\right)$ *with speed* a_N *holds with rate function* I*. When a.) and c.) hold, then*

$$\liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \phi(\xi_N)}\right] \geq S^*. \tag{C.1.9}$$

Lemma C.1.9. *Let* ϕ *and* $\left\{\overline{\phi}_R\right\}$ *be defined as in Theorem C.1.1. and the large deviation upper bound for* $\left(\xi_N, \mathbb{P}^N\right)$ *with speed* a_N *holds with rate function* I*. When b.), d.) and e.) hold, then*

$$\limsup_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \phi(\xi_N)}\right] \leq S^*. \tag{C.1.10}$$

C.2 Proof of the generalised Varadhan's lemma

C.2.1 Proof of the lower bound (Lemma C.1.8)

Proof. This proof is organised as follows. At first we show that the function ϕ in the exponent on the left hand side of (C.1.9) can be replaced by $\underline{\phi}_R$, with an arbitrary small error ϵ for R large enough (see (C.2.1) and (C.2.2)). This requires the condition c.). Then we use a similar idea as in the proof of the lower bound of the usual Varadhan Lemma in [DZ98] (proof of Lemma 4.3.4). Here we use condition a.) (compare this to the application of the lower semi continuity condition in the proof in [DZ98]). This leads to the claimed lower bound for each $x \in X$ (see (C.2.4)).

Fix a $\epsilon > 0$ and a R, then

$$\begin{aligned}
&\liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)}\right] \\
&\geq \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right] \\
&\geq -\epsilon + \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right].
\end{aligned} \tag{C.2.1}$$

By condition c.), we get for R large enough that the right hand side of (C.2.1) is greater or equal to

$$\geq -2\epsilon + \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \underline{\phi}_R(\xi_N)}\right]. \tag{C.2.2}$$

We fix an arbitrary $x \in X$ with $I(x) < \infty$ and an arbitrary $\delta > 0$. By the condition a.) there is a open neighbourhood $U_{x,\delta,R}$ of x such that

$$\inf_{y\in U_{x,\delta,R}\cap \operatorname{supp}\{\xi_N\}} \underline{\phi}_R(y) \geq \phi(x) - \delta. \tag{C.2.3}$$

Using this the large deviation lower bound of ξ_N, the right hand side of (C.2.1) is greater or equal to

$$\begin{aligned}
&\geq -2\epsilon + \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\{\xi_N \in U_{x,\delta,R}\}}\right] \\
&\geq -2\epsilon + \inf_{y\in U_{x,\delta,R}\cap \operatorname{supp}\{\xi_N\}} \underline{\phi}_R(y) + \liminf_{N\to\infty} a_N^{-1} \log \mathbb{P}_N\left[\xi_N \in U_{x,\delta,R}\right] \\
&\geq -2\epsilon + \phi(x) - \delta - I(x).
\end{aligned} \tag{C.2.4}$$

Now let δ and ϵ tend to zero. Hence we get for all $x \in X$ (with $I(x) < \infty$),

$$\liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)}\right] \geq \phi(x) - I(x). \tag{C.2.5}$$

This implies the Varadhan lower bound (C.1.9). □

C.2.2 Proof of the upper bound (Lemma C.1.9)

Proof. To prove the Varadhan upper bound we replace the function ϕ in the exponent of the left hand side of (C.1.10) by the function $\overline{\phi}_R$. Therefore, we split the expectation on the left hand side of (C.1.10) into the events when ϕ is greater or lower than a $M \in \mathbb{R}$. Moreover, we split it again in the events that ϕ exceeds $\overline{\phi}_R$ more or less than ϵ. Then we interchange the max and lim sup to investigate the three situations separately (see (C.2.6)). Only the situation when ϕ exceeds $\overline{\phi}_R$ less than ϵ needs further investigation due to the conditions d.) and e.) (see also (C.2.10)). In that situation we replace ϕ in the exponent by $\overline{\phi}_R$ with error ϵ. Then for $\overline{\phi}_R$ in the exponent we use parts of the usual proof of Varadhan's upper bound of [DZ98] Lemma 4.3.6. This leads to (C.2.9). Here we use condition b.) (compare this to the application of the upper semi continuity condition in the proof in [DZ98]). Finally in (C.2.10), we combine these calculations and conclude the claimed upper bound.

Fix an $M \in \mathbb{R}$, an $\epsilon > 0$ and an $R > 0$.

$$\begin{aligned}
&\limsup_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)}\right] \\
&\leq \left\{\limsup_{N\to\infty} a_N^{-1} \log\left(\mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\left(\overline{\phi}_R(\xi_N)\wedge M\right)}\right] e^{a_N\epsilon}\right)\right\} \\
&\qquad \vee \left\{\limsup_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)} \mathbb{1}_{\phi(\xi_N)\geq M}\right]\right\} \\
&\qquad \vee \left\{\limsup_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \phi(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \overline{\phi}_R(\xi_N)+\epsilon\right\}}\right]\right\}.
\end{aligned} \tag{C.2.6}$$

We show that the first lim sup of the right hand side leads to the claimed upper bound. Therefore, we use parts of the proof of [DZ98] Lemma 4.3.6 for the upper bound of Varadhan's Lemma. Fix

$\alpha, \delta \in \mathbb{R}_+$. The function I is lower semi continuous and $\overline{\phi}_R$ satisfies b.). Hence (for R large enough) for each $x \in X$ there is a open neighbourhood $A_x = A_{x,\delta,R} \subset X$ of x such that

$$\sup_{y \in A_x \cap \operatorname{supp}\{\xi_N\}} \overline{\phi}_R(y) \leq \max\{S^*, \phi(x)\} + \delta \qquad \text{and} \qquad \inf_{y \in \overline{A_x}} I(y) \geq I(x) - \delta. \quad \text{(C.2.7)}$$

Using these A_x we find a finite cover $U_{i=1}^{N(\alpha)} A_{x_i}$ of the level sets $I^{-1}([0,\alpha])$ with $x_i \in I^{-1}([0,\alpha])$ by the compactness of $I^{-1}([0,\alpha])$. Then

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\left(\underline{\phi}_R(\xi_N)\wedge M\right)}\right] \\ &\leq \sum_{i=1}^{N(\alpha)} e^{a_N(\max\{S^*,\phi(x_i)\}+\delta)}\mathbb{P}^N\left[\xi_N \in \overline{A_{x_i}}\right] + e^{a_N M}\mathbb{P}^N\left[\left(\bigcup_{i=1}^{N(\alpha)} A_{x_i}\right)^c\right]. \end{aligned} \quad \text{(C.2.8)}$$

By the large deviation upper bound of ξ_N with rate function I we get

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{a_N} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\left(\overline{\phi}_R(\xi_N)\wedge M\right)}\right] \\ &\leq \max\left\{\left\{\max_{i=1}^{N(\alpha)}\left\{\max\{S^*,\phi(x_i)\} - \inf_{x\in\overline{A_{x_i}}} I(x)\right\} + \delta\right\}, \{M-\alpha\}\right\} \\ &\leq \max\left\{S^* + \delta, \left\{\max_{i=1}^{N(\alpha)}\{(\phi - I)(x_i)\} + 2\delta\right\}, \{M-\alpha\}\right\} \\ &\leq \max\left\{\left\{\sup_{x\in X:I(x)<\infty}\{(\phi - I)(x)\} + 2\delta\right\}, \{M-\alpha\}\right\}. \end{aligned} \quad \text{(C.2.9)}$$

Combining (C.2.9) into (C.2.6), we get for $R > R(\alpha,\gamma,\delta)$

$$\begin{aligned} &\limsup_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\phi(\xi_N)}\right] \\ &\leq \left\{\epsilon + \sup_{x\in X, I(x)<\infty}\{\phi(x) - I(x)\} + 2\delta\right\} \vee \{\epsilon + M - \alpha\} \\ &\quad \vee \left\{\limsup_{N\to\infty} a_N^{-1}\log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\phi(\xi_N)}\mathbb{1}_{\phi(\xi_N)\geq M}\right]\right\} \\ &\quad \vee \left\{\limsup_{N\to\infty} a_N^{-1}\log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\phi(\xi_N)}\mathbb{1}_{\{\phi(\xi_N)>\overline{\phi}_R(\xi_N)+\epsilon\}}\right]\right\}. \end{aligned} \quad \text{(C.2.10)}$$

Take now at first the limit $R \to \infty$. Then the last lim sup vanishes due to condition d.). Because α, δ are arbitrary take then the limits $\alpha \to \infty$, $\delta \to 0$. Afterwards let M tend to infinity and apply condition e.). Finally, ϵ is also arbitrary small. Hence we have proven Lemma C.1.9. □

C.3 Simplifications of the conditions in the generalised Varadhans's lemma

C.3.1 The condition a.) on $\underline{\phi}_R$

Lemma C.3.1. *The condition a.) on $\underline{\phi}_R$ is satisfied if*

- *each $\underline{\phi}_R : X \to \mathbb{R}$ is lower semi continuous and*
- *$\left\{\underline{\phi}_R\right\}$ converge pointwise to ϕ on $\{x \in X : I(x) < \infty\}$ for $R \to \infty$.*

Proof. By the lower semi continuity there is for each $x \in X$ with $I(x) < \infty$, each $\delta > 0$ and each $R > 0$, a neighbourhood $U_{x,\delta,R} \subset X$ of x such that

$$\inf_{y \in U_{x,\delta,R}} \underline{\phi}_R(y) \geq \underline{\phi}_R(x) - \frac{\delta}{2}. \tag{C.3.1}$$

Moreover, by the pointwise convergence there is a $R^*_{x,\delta}$ such that for all $R > R^*_{x,\delta}$

$$\underline{\phi}_R(x) > \phi(x) - \frac{\delta}{2}. \tag{C.3.2}$$

Therefore, the claimed condition a.) is proven. □

C.3.2 The condition b.) on $\overline{\phi}_R$

Lemma C.3.2. *The condition b.) on $\overline{\phi}_R$ is satisfied if*

- *each $\overline{\phi}_R : X \to \mathbb{R}$ is upper semi continuous and*
- *for each $\delta > 0$ there is a $R^*_\delta \in \mathbb{R}_+$ such that for all $R \geq R^*_\delta$, $\overline{\phi}_R(x) \leq \max\{S^*, \phi(x)\} + \delta$ for all $x \in \{x \in X : I(x) < \infty\}$.*

Proof. By the upper semi continuity we have that for each $x \in \{x \in X : I(x) < \infty\}$, each δ and each $R \in \mathbb{R}_+$ there is a open set $A_{x,\delta,R} \subset X$ such that

$$\sup_{y \in A_{x,\delta,R}} \overline{\phi}_R(y) \leq \overline{\phi}_R(x) + \delta. \tag{C.3.3}$$

If we have $R > R^*_\delta$, then we know by the second property that the right hand side is lower or equal to

$$\leq \max\{S^*, \phi(x)\} + 2\delta. \tag{C.3.4}$$

□

The class of functions that satisfies b.) is in general larger than the class defined in Lemma C.3.2. For an abstract example, see Section C.4.2.

C.3.3 The condition c.)

Lemma C.3.3. *The condition c.) holds if*

$$\forall \epsilon > 0 \quad \exists R_\epsilon > 0 \quad \text{such that} \quad \forall R > R_\epsilon \quad \exists N_{\epsilon,R} \in \mathbb{N} \quad \text{such that} \quad \forall N > N_{\epsilon,R}$$

$$\mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right] \geq \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) \leq \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right]. \tag{C.3.5}$$

Proof. For $a, b \geq 0$,

$$\log(a+b) \leq \max\{\log(2a), \log(2b)\} = \max\{\log(a), \log(b)\} + \log(2) \leq \log(a+b) + \log(2). \tag{C.3.6}$$

This implies by the assumption of this lemma that

$$\begin{aligned} &\lim_{R\to\infty} \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)}\right] \\ &= \lim_{R\to\infty} \liminf_{N\to\infty} a_N^{-1} \log \max\left\{\mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right]\right. \\ &\qquad\qquad \left., \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) \leq \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right]\right\} \\ &= \lim_{R\to\infty} \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right]. \end{aligned} \tag{C.3.7}$$

□

Lemma C.3.4. *The condition c.) holds if*

(i) $\forall C > 0 \quad \forall \epsilon > 0 \quad \exists R_{C,\epsilon} > 0$ such that $\forall R > R_{C,\epsilon} \quad \exists N_{C,\epsilon,R} \in \mathbb{N}$ such that $\forall N > N_{C,\epsilon,R}$

$$\mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) \leq \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right] \leq e^{-a_N C} \qquad \text{or} \tag{C.3.8}$$

(ii) $\forall \epsilon > 0 \quad \exists \beta_\epsilon \in (0,1]\, \exists R_\epsilon > 0$ such that $\forall R > R_\epsilon \quad \forall N \in \mathbb{N}$

$$\mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right] \geq \beta \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)}\right]. \tag{C.3.9}$$

Proof. We only have to show that the left hand side of (C.1.4) is greater or equal to the right hand side.

(i) By the condition of this lemma we get as in (C.3.7) that for each $C > 0$

$$\begin{aligned}
&\lim_{R\to\infty} \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)}\right] \\
&\leq \lim_{R\to\infty} \liminf_{N\to\infty} \max\left\{a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right], -C\right\} \\
&= \max\left\{\lim_{R\to\infty} \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right], -C\right\}.
\end{aligned} \tag{C.3.10}$$

Now we let C tend to infinity, what proves the claim.

(ii) In this case the equality (C.1.4) follows by inserting (C.3.9) and using that $\frac{\log(\beta)}{a_N} \to 0$. □

The following lemma is a corollary of Lemma C.3.4 (i).

Lemma C.3.5. *If* $\forall C > 0 \quad \forall \epsilon > 0 \quad \exists R_{C,\epsilon} > 0$ such that $\forall R > R_{C,\epsilon} \quad \exists N_{C,\epsilon,R} \in \mathbb{N}$ such that

$$e^{a_N \sup_{x\in \operatorname{supp}\{\xi_N\}} \underline{\phi}_R(x)} \mathbb{P}^N\left[\phi(\xi_N) \leq \underline{\phi}_R(\xi_N) - \epsilon\right] \leq e^{-a_N C}, \tag{C.3.11}$$

for all $N > N_{C,\epsilon,R}$*, then condition Lemma C.3.4 (i) and hence Condition c.) hold.*

Remark C.3.6. *The supremum in* (C.3.11) *could be restricted to* $\left\{x : \phi(x) \leq \underline{\phi}_R(x) - \epsilon\right\}$.

In the following lemma we show that a simpler condition (than the condition Lemma C.3.4 (i)) implies the condition c.). The proof of this lemma requires parts of the proof of the Varadhan lower bound (of Section C.2.1). Of course not those parts that require the condition c.).

Lemma C.3.7. *Let the condition a.) hold. If*

$$\forall \epsilon > 0 \quad \exists \gamma_\epsilon > 0 \quad \exists R_\epsilon^* > 0 \quad \text{such that} \quad \forall R > R_\epsilon \quad \exists N_R \in \mathbb{N} \quad \text{such that} \quad \forall N > N_R^*$$

$$\mathbb{E}_{\mathbb{P}^N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) \leq \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right] \leq e^{a_n(S^* - \gamma_\epsilon)}, \tag{C.3.12}$$

then the condition c.) holds.

Proof. Let us fix a $\epsilon > 0$ small enough. Then

$$\begin{aligned}
&\liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \underline{\phi}_R(\xi_N)}\right] \\
&\leq \max\left\{\liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right], S^* - \gamma_\epsilon\right\}.
\end{aligned} \tag{C.3.13}$$

The left hand side is larger or equal to S^* as shown in the proof of Lemma C.1.8 ((C.2.2) to (C.2.4) requires only condition a.)). This implies that

$$\liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \underline{\phi}_R(\xi_N)}\right] = \liminf_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \underline{\phi}_R(\xi_N)} \mathbb{1}_{\left\{\phi(\xi_N) > \underline{\phi}_R(\xi_N) - \epsilon\right\}}\right], \tag{C.3.14}$$

for all $R > R_\epsilon^*$, i.e. the condition c.) holds. □

C.3.4 The condition e.)

Lemma C.3.8. *The tail condition e.) is satisfied if for some $\gamma > 1$*

$$\limsup_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{\gamma a_N \phi(\xi_N)}\right] < \infty. \tag{C.3.15}$$

Proof. As shown in Lemma 4.3.8 in [DZ98] the moment condition implies the tail condition. The continuity of ϕ is not required in that proof. □

Lemma C.3.9. *The condition d.) and the following asymptotic tail condition*

$$\lim_{M\to\infty} \lim_{R\to\infty} \limsup_{N\to\infty} a_N^{-1} \log \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N \overline{\phi}_R(\xi_N)} \mathbb{1}_{\overline{\phi}_R(\xi_N)\geq M}\right] \leq S^*, \tag{C.3.16}$$

imply the tail condition e.).

Proof. For all $R > 0$ we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\phi(\xi_N)}\mathbb{1}_{\phi(\xi_N)\geq M}\right] \leq e^{a_N\epsilon}\mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\overline{\phi}_R(\xi_N)}\mathbb{1}_{\overline{\phi}_R(\xi_N)\geq M-\epsilon}\right] \\ + \mathbb{E}_{\mathbb{P}_N}\left[e^{a_N\phi(\xi_N)}\mathbb{1}_{\left\{\phi(\xi_N)>\overline{\phi}_R(\xi_N)+\epsilon\right\}}\right].\end{aligned} \tag{C.3.17}$$

By taking first the maximum of the summands of the right hand side and then the limit $R \to \infty$, the contribution of the second summand vanishes due to condition d.). Finally, we take the limit $M \to \infty$ and use (C.3.16) to conclude the claimed condition e.). □

C.4 Example

If ϕ is continuous, then all conditions of Theorem C.1.1 simplify to the usual conditions of Varadhan's lemma (see Remark C.1.3).

We state now at first a simple example of sums Bernoulli random variables and show that the generalised Varadhan's lemma (Theorem C.1.1) hold for functions with one jump point. Finally, we show that an abstract setting implies the conditions of Theorem C.1.1. In this setting the function ϕ might be nowhere continuous.

C.4.1 A simple example

Let θ^i be i.i.d. random variables with distribution $Bern(p)$ for $p > \frac{1}{2}$, i.e. $\theta^i = 1$ with probability p and $\theta^i = -1$ with probability $1-p$. Define the random variables $\xi_N = \frac{1}{N}\sum_{i=1}^N \theta^i$. These random variables satisfy the large deviation principle with rate function

$$\Lambda^*(x) = \frac{1}{2}\left[(x+1)\log\left(\frac{x+1}{p}\right) + (1-x)\log\left(\frac{1-x}{1-p}\right)\right] - \log(2), \tag{C.4.1}$$

for $x \in [-1, 1]$ and infinity otherwise.

For a $\alpha \in (0, \Lambda^*(0))$, set

$$\phi(x) = \begin{cases} 0 & \text{if } x > 0, \\ \alpha & \text{else.} \end{cases} \tag{C.4.2}$$

We show now that the conditions of Theorem C.1.1 hold. Set $\overline{\phi}_R = \phi$. Then condition d.) holds. Moreover, condition b.) is satisfied, because ϕ is upper semi continuous. By the boundedness of ϕ also the tail condition e.) holds. Define

$$\underline{\phi}_R(x) = \begin{cases} 0 & \text{if } x \geq \frac{1}{R}, \\ \alpha & \text{else.} \end{cases} \tag{C.4.3}$$

This function is lower semi-continuous and converges pointwise to ϕ. This implies by Lemma C.3.1 the condition a.).

We only have to show that condition c.) holds. We do this now with help of the Lemma C.3.3. Then

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}^N}\left[e^{N\underline{\phi}_R(\xi_N)}\mathbb{1}_{\left\{\phi(\xi_N)<\overline{\phi}_R(\xi_N)-\epsilon\right\}}\right]=e^{N\alpha}\mathbb{P}^N\left[\xi_N\in\left(0,\frac{1}{R}\right)\right]\\&\leq e^{N\alpha}\mathbb{P}^N\left[\xi_N<\frac{1}{R}\right]\leq e^{N\alpha}e^{-N\inf_{x\leq\frac{1}{R}}(\Lambda^*(x)+o_N(1))}\leq e^{N\alpha}e^{-N\left(\Lambda^*\left(\frac{1}{R}\right)+o_N(1)\right)},\end{aligned}\tag{C.4.4}$$

for R large enough, because $p>\frac{1}{2}$. Moreover

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^N}\left[e^{a_N\underline{\phi}_R(\xi_N)}\mathbb{1}_{\left\{\phi(\xi_N)>\underline{\phi}_R(\xi_N)-\epsilon\right\}}\right]&\geq\mathbb{P}^N\left[\xi_N\geq\frac{1}{R}\right]\\&\geq e^{-N\left(\inf_{x\geq\frac{1}{R}-\epsilon}\Lambda^*(x)+o_N(1)\right)}=e^{-No_N(1)},\end{aligned}\tag{C.4.5}$$

for R large enough, because $p>\frac{1}{2}$. Hence the condition of Lemma C.3.3 is satisfied because $\alpha<\Lambda^*(0)$.

C.4.2 Class of examples

A class of examples is given by the following abstract setting. We show in Chapter V.7, that the empirical measures L^N corresponding to the spin system (0.1.1) satisfies these conditions.

(C.4.2.i) I be the good rate function as stated in Theorem C.1.1 and ϕ be a measurable function $X\to\mathbb{R}\cup\{-\infty,\infty\}$.

(C.4.2.ii) For each $R\in\mathbb{N}$, let $M_R\subset X$ be a closed subset, such that $M_R\subset M_{R+1}$.
We set $\Xi:=\bigcup M_R$.

(C.4.2.iii) $I(x)=\infty$ if $x\notin\Xi$.

(C.4.2.iv) For each $R\in\mathbb{R}$, $x\in M_R$ with $I(x)<\infty$ and each sequence $\left\{x^{(n)}\right\}\subset M_R\cap\operatorname{supp}\{\xi_N\}$ with $x^{(n)}\to x$, the following convergence holds: $\phi\left(x^{(n)}\right)\to\phi(x)$.
This holds in particular when the restriction of ϕ to M_R is continuous for each R.

(C.4.2.v) There is an $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ and a $N^*\in\mathbb{N}$ such that, for all $N>N^*$, $\phi(x)\leq\alpha(R)$, when $x=\xi_N\in M_R$ or when $x\in M_R$ and $I(x)<\infty$.

(C.4.2.vi) There is a function $\beta:\mathbb{R}_+\to\mathbb{R}$, such that $\mathbb{P}^N[\xi_N\notin M_R]\leq e^{-a_N\beta(R)}$.

(C.4.2.vii) $\lim_{R\to\infty}\alpha(R)-\beta(R)=-\infty$.

(C.4.2.viii)

$$\limsup_{R\to\infty}\limsup_{N\to\infty}\frac{1}{a_N}\log\mathbb{E}_{\mathbb{P}^N}\left[e^{a_N\phi(\xi_N)}\mathbb{1}\{\xi_N\notin M_R\}\right]=-\infty.\tag{C.4.6}$$

(C.4.2.ix) ϕ satisfies the tail condition e.) or it satisfies (C.3.15).

Remark C.4.1. *Note that the most important condition is (C.4.2.iv). In this abstract setting, the function ϕ might be nowhere continuous, as long as this condition holds. The other conditions are only necessary to reduce the analysis to sequences the fit in the setting of (C.4.2.iv).*

We define

$$\underline{\phi_R}(x) = \begin{cases} \phi(x) & \text{if } x \in M_R, \\ \alpha(R) & \text{otherwise} \end{cases} \qquad \overline{\phi_R}(x) = \begin{cases} \phi(x) & \text{if } x \in M_R, \\ S^* & \text{otherwise.} \end{cases} \tag{C.4.7}$$

Lemma C.4.2. *These conditions allow the application of Theorem C.1.1.*

Proof. **Step 1: Condition a.):**

The $\underline{\phi_R}$ is measurable, because $\underline{\phi_R}$ restricted to M_R is measurable (by (C.4.2.i)), the M_R are closed and $\underline{\phi_R}$ is constant outside of M_R.

To show the other part of this condition, fix an $R > 0$, a $\delta > 0$ and an $x \in M_R$, with $I(x) < \infty$. On M_R, $\underline{\phi_R} = \phi$. Hence by (C.4.2.iv), there is a open set $\widehat{U}_{x,\delta,R} \subset M_R$ such that

$$\inf_{y \in \widehat{U}_{x,\delta,R} \cap \operatorname{supp}\{\xi_N\}} \underline{\phi_R}(y) \geq \phi(x) - \delta. \tag{C.4.8}$$

Denote by $U_{x,\delta,R}$ the open subset of X, such that $\widehat{U}_{x,\delta,R} = U_{x,\delta,R} \cap M_R$. For each $y \in U_{x,\delta,R} \backslash \widehat{U}_{x,\delta,R}$, with $y \in \operatorname{supp}\{\xi_N\}$, $\underline{\phi_R}(y) = \alpha(R) \geq \phi(x)$ by (C.4.2.v). Hence $\underline{\phi_R}$ satisfies condition a.) for $x \in M_R$.

For $x \notin M_R$, there is a open neighbourhood $U_{x,\delta,R}$ of x with $U_{x,\delta,R} \cap M_R = \emptyset$ (because M_R is closed). On $U_{x,\delta,R}$, $\underline{\phi_R}$ is constant and equals $\alpha(R)$. This implies condition a.).

Step 2: Condition b.):

The function $\overline{\phi_R}$ is measurable by the same arguments as $\underline{\phi_R}$.

Fix an arbitrary $R > 0$, a $\delta > 0$ and a $x \in M_R$ with $I(x) < \infty$. By (C.4.2.iv), there is an open neighbourhood $\widehat{U}_{x,\delta,R} \subset M_R$ such that

$$\sup_{y \in \widehat{U}_{x,\delta,R} \cap \operatorname{supp}\{\xi_N\}} \overline{\phi}_R(y) \leq \phi(x) + \delta. \tag{C.4.9}$$

Denote by $U_{x,\delta,R}$ the open subset of X, such that $\widehat{U}_{x,\delta,R} = U_{x,\delta,R} \cap M_R$. Then

$$\sup_{y \in U_{x,\delta,R} \cap \operatorname{supp}\{\xi_N\}} \overline{\phi}_R(y) \leq \max\left\{ \sup_{y \in \widehat{U}_{x,\delta,R} \cap \operatorname{supp}\{\xi_N\}} \overline{\phi}_R(y), S^* \right\} \leq \max\{\phi(x) + \delta, S^*\}. \tag{C.4.10}$$

The case when $x \notin M_R$, can be treat as in condition a.). This implies condition b.).

Step 3: Condition c.) holds:

To show this condition we show the sufficient condition of Lemma C.3.5. By the definition of $\underline{\phi_R}$ and (C.4.2.v), we know that $\phi(\xi_N) < \underline{\phi}_R(\xi_N) - \epsilon$ implies that $\xi_N \notin M_R$. Hence

$$\begin{aligned} & e^{a_N \sup_{x \in \operatorname{supp}\{\xi_N\}} \underline{\phi_R}(x)} \mathbb{P}^N \left[\phi(\xi_N) < \underline{\phi}_R(\xi_N) - \epsilon \right] \\ & \leq e^{a_N \alpha(R)} \mathbb{P}^N \left[\xi_N \notin M_R \right] \leq e^{a_N (\alpha(R) - \beta(R))}, \end{aligned} \tag{C.4.11}$$

by (C.4.2.v), (C.4.2.vi). Finally, (C.4.2.vii) implies the condition of Lemma C.3.5.

Step 4: Condition d.) holds:

This condition is satisfied by (C.4.2.viii) and because $\phi(\xi_N) > \overline{\phi}_R(\xi_N) + \epsilon$ implies that $\xi_N \notin M_R$.

Step 5: Condition e.) holds:

We assume this in (C.4.2.ix). □

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