On the dynamics of correlations in Scaling limits of interacting particle systems

Dissertation

vorgelegt von
Raphael Elias Winter
aus
Bonn

Bonn 2018

Acknowledgments

Contents

Bibliography   **107**

Chapter 1

Introduction

A classical topic in physics is the study of gases, liquids, and plasmas. Experimentally, one typically measures macroscopic quantities such as temperature, pressure, and density. These quantities can be theoretically studied by regarding the medium as a continuum. The adoption of this perspective led to the postulation of the associated laws of evolution, such as Fourier's law for heat transfer and Fick's law for mass transfer. However, a more precise model takes into account that these media are composed of a large number of small, identical, colliding particles. This perspective allows one to understand phenomena that emerge from the randomness of particle collisions and movement. For example, Maxwell derived the velocity distribution of particles in an ideal gas at equilibrium, which is the Gaussian with prescribed temperature T and total mass m:

$$M(v) := \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} e^{-\frac{m|v|^2}{2k_B T}}. \tag{1.0.1}$$

The utility of this perspective is further illustrated by Boltzmann's formalization of entropy based on the combinatorial properties of large particle systems.

The evolution of macroscopic observables, and the underlying Hamiltonian dynamics of particles are well-studied objects in both physics and mathematics, yet many questions concerning the connection between the two remain open. The depth of this issue is exemplified by the reversibility paradox: on the macroscopic level we observe heat being transfered from hot regions of a gas to colder regions, until the temperature becomes uniform. Hence we can tell if the evolution is shown to us forwards or backwards. On the other hand, we cannot ascertain the direction of time in the billiard ball evolution of particles. Because of this paradox, scientists were initially reluctant to accept the particle interpretation as a viable explanation for irreversible macroscopic phenomena, prominently Loschmidt (cf. [37]) and Zermelo (cf. [60]).

It is believed that this issue can be resolved by assuming that the initial particle configuration is random. Even though the evolution of every sampled configuration is reversible, and (almost) returns to its initial state after a sufficiently long time, the entropy of the statistical ensemble is monotone increasing in time. An intuition for this can be gained by considering a system at statistical equilibrium. Physical observables depend on the number of particles in a given volume of the phase space, and are invariant under permutation of the particles. Therefore, a system at equilibrium will attain those states that correspond to a high number of underlying particle combinations, that is

states of high entropy. Due to the large number of particles in physical systems, the combinatorial factors are sufficiently large to assume that only the macroscopic state with maximal entropy is observed.

In physics, the evolution of a system is typically described by integro-differential equations, such as the Boltzmann equation for ideal gases or the Landau equation for plasmas. The range of validity of these equations is determined by a small number of non-dimensional parameters describing the physical system in question, such as the fraction of volume occupied by particles or the ratio of kinetic to potential energy of a typical particle. The formal derivation of the equations is based on the use of the smallness of parameters and heuristic statistical assumptions, such as propagation of chaos. For example, this justifies the assumption that the impact parameter of colliding particles in a dilute ideal gas is random, so the probabilities for the velocities after collision are given by random scattering.

Mathematically, the transition from particle dynamics to observables on a larger scale is described using so-called scaling limits (cf. [49]). Within this framework, the validity of evolution equations and physical principles such as propagation of chaos can be formulated and proved as rigorous theorems. By a scaling limit, we mean a specific sequence of Hamiltonian equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y_j(\tau) = V_j(\tau), \quad \frac{\mathrm{d}}{\mathrm{d}\tau}V_j(\tau) = -\sum_{k \in J} \theta^2 \nabla \phi((Y_j(\tau) - Y_k(\tau))), \tag{1.0.2}$$

with random and uncorrelated initial data $(Y_j, V_j)_{j \in J}$, where J is a countable or finite set indexing the particles. The potential ϕ determines the nature of particle interaction, for example hard-sphere collision or electrostatic repulsion. For the moment, we assume that ϕ has a well-defined length scale ℓ, which we set as the microscopic unit of length. Similarly θ^2 in (1.0.2) can be thought of as the strength of interaction, and we denote by Z the average number of particles in a ball of radius ℓ. The macroscopic evolution is observed in space-time units (x, t) given by

$$x = \varepsilon y, \quad X_j = \varepsilon Y_j, \quad t = \varepsilon \tau. \tag{1.0.3}$$

Here $\varepsilon > 0$ might converge to zero or be identically one, in which case the units (x, t) and (y, τ) coincide. We note that the length scale that we consider in some scaling limits is referred to as mesoscopic rather than macroscopic in the physics literature. To simplify our terminology, we will use the term macroscopic also in these cases, as is customary in mathematical works on the topic.

Scaling limits of interacting particle systems are characterized by the specific, fixed interdependence of the parameters ε, Z, and θ^2 as one of them converges to zero. The precise rescaling of the parameters is chosen according to the non-dimensional parameters of the system. We will discuss scaling limits in more detail in Section 1.1. A variant of scaling limits are the so-called Lorentz models. Here, one considers the evolution of a single tagged particle interacting with a random, but fixed, background of scatterers.

A key objective of kinetic theory is the rigorous derivation of equations describing the evolution of the one-particle distribution from scaling limits. For an extensive overview of open and solved problems in the area we refer to [49, 50, 56]. We associate to the evolving random particle

configuration $(Y_j(\tau), V_j(\tau))_{j \in J}$ the normalized one-particle measure

$$\tilde{f}_1(\tau, y, v) \, \mathrm{d}y \, \mathrm{d}v = \frac{1}{Z} \mathbb{E} \left[\sum_{j \in J} \delta(y - Y_j(\tau)) \delta(v - V_j(\tau)) \right], \tag{1.0.4}$$

where the average $\mathbb{E}[\cdot]$ is taken with respect to the initial random distribution. Our goal is to describe the macroscopic evolution of the system, rather than the evolution on the length scale y (cf. (1.0.3)). Therefore we introduce the rescaled function $f_1(t, x, v)$ defined by

$$f_1(t, x, v) = \tilde{f}_1(t/\varepsilon, x/\varepsilon, v). \tag{1.0.5}$$

The goal is to prove that for a given scaling limit we have convergence on the macroscopic scale

$$f_1(t, x, v) \, \mathrm{d}x \, \mathrm{d}v \rightharpoonup^* f_{1,\mathrm{lim}}(t, x, v) \, \mathrm{d}x \, \mathrm{d}v, \tag{1.0.6}$$

where $f_{1,\mathrm{lim}}$ is the solution to the equation describing the macroscopic evolution of the physical system.

Lanford obtained a breakthrough in this direction (cf. [32]), proving the validity of the Boltzmann equation in the so-called Boltzmann-Grad scaling limit for short times. Moreover, he proved that the absence of statistical correlations is propagated to later times in this scaling limit. This propagation of chaos principle can be mathematically described by means of the probability distribution of n-tuples of particles. Assume this distribution is given by a function $\tilde{f}_n(\tau, y_1, v_1, \ldots, y_n, v_n)$ for $n \in \mathbb{N}$. Now we choose the function $\tilde{g}_2$ such that the following identity holds:

$$\tilde{f}_2(\tau, y_1, v_1, y_2, v_2) = \tilde{f}_1(\tau, y_1, v_1) \tilde{f}_1(\tau, y_2, v_2) + \tilde{g}_2(\tau, y_1, v_1, y_2, v_2). \tag{1.0.7}$$

If $\tilde{g}_2 \to 0$ holds under a prescribed scaling limit, we say the propagation of chaos principle holds on the level of pairs of particles. By a similar construction we can define so-called truncated n-particle correlation functions $\tilde{g}_n$. In this setting, propagation of chaos can be rigorously formulated as $\tilde{g}_n \to 0$ for $n = 2, 3, \ldots$ in the given scaling limit.

On account of the large number of particles, the genesis of particle correlations in particle systems is a complicated process. If particles interact through elastic collisions that are localized both in space and time, correlations can be analyzed using so-called collision trees, in which every node represents a particle at a given position and every edge represents a correlating event. It is interesting to remark that, despite the simple dynamics, the notion of what constitutes a correlating event is subtle. For instance, let a particle A collide with a particle B, which as a result does not collide with a particle C. In this case, the particles A and C are correlated, even though they cannot be linked by a chain of collision events. An elaborate analysis of correlations using collision trees can be found for example in [19, 46].

In the following, we will consider systems in which the trajectories of particles are governed by a large number of small deflections. We will consider the so-called weak-coupling and plasma limits, which were introduced to describe the dynamics of electrons in a plasma under electrostatic repulsion. Since in such a system every particle interacts with every other at any given point in time, we cannot apply the idea of collision trees. Instead, the dynamics results in a non-local, yet

rather regular structure of particle correlations. Remarkably, a propagation of chaos principle still holds for such systems. We can still neglect the joint correlation of more than two particles, even though in general many particles will simultaneously interact. Intuitively speaking, the interaction of n particles can be understood as the result of the interactions of each possible pair among them.

Making this assumption, the particle system can be truncated to a closed system of equations involving only $\tilde{f}_1$ and $\tilde{g}_2$ (cf. (1.0.7)). Since $|\tilde{g}_2| \to 0$ in the scaling limit, we normalize it to a function g_2^* of order one. Then the system is described by an equation of the following form:

$$\partial_\tau \tilde{f}_1 = \varepsilon B_1[g_2^*(\tau)] \tag{1.0.8}$$

$$\partial_\tau g_2^* = B_2[\varepsilon, g_2^*(\tau)] + C_2[\tilde{f}_1(\tau)]. \tag{1.0.9}$$

The specific form of the operator B_2 is different in the weak-coupling limit and the plasma limit. Due to the factor ε in (1.0.8), the evolution of $\tilde{f}_1$ has a characteristic timescale $t = \varepsilon\tau$, whereas g_2^* evolves on the timescale τ. This two-timescale approach is crucial to the (formal) derivations of kinetic equations by Bogolyubov (cf. [6]). If (1.0.9) has a stable steady state g_B, we can assume that g_2^* is close to g_B for positive macroscopic times t. Hence the (formal) equation for f_1 on the timescale t can be obtained by plugging the steady state g_B into (1.0.8).

The main results in this work take the system (1.0.8)-(1.0.9) as a starting point. Assuming propagation of chaos, these equations describe the leading order behavior of the systems up to macroscopic times. The rigorous results can be found in Chapters 3 and 4. In Chapter 3, we consider the system (1.0.8)-(1.0.9) associated to the so-called weak-coupling scaling. We prove that the macroscopic one-particle function f_1 (cf. (1.0.5)) converges to a solution of the nonlinear Landau equation as $\varepsilon \to 0$. The system (1.0.8)-(1.0.9) is non-Markovian for positive $\varepsilon > 0$, whereas the limit evolution is Markovian and parabolic. We prove the existence of solutions to the non-Markovian system by showing that the system is dissipative in a time-averaged sense. This technique allows us to extend the solutions to the long timescale t and obtain the Landau equation in the limit. Chapter 3, in almost this precise form, has been published as an article (cf. [55]).

In Chapter 4, we perform a careful estimate of the steady states g_B of (1.0.9) in the plasma limit, when f_1 is time-independent. The steady state g_B of (1.0.9) in the plasma limit case was formally computed by Lenard (cf. [34]), Guernsey (cf. [23]), Oberman and Williams (cf. [42]). We give a notion of weak solution to the steady state equation of (1.0.9), and prove that the function g_B derived by these authors is indeed a weak solution in this sense. Moreover, we prove that for particle systems interacting via the Coulomb potential, the correlations g_B have the so-called Debye screening length as characteristic length. This means that the effective range of interaction between particles is given by the Debye length, as predicted in the plasma physics literature. Due to the Debye screening, the interaction of particles with large impact factor give a lower order correction to the macroscopic evolution. On the other hand, the collisions of particles with small impact factor yield a singularity in the function g_B. Due to these collisions, the evolution of Coulomb interacting systems is described by the Landau equation (cf. [34]). Furthermore, we prove that g_B is a globally stable steady state of (1.0.9) if the interaction is given by a soft potential and f_1 is time-independent.

The Introduction is structured as follows. In Section 1.1 we introduce important scaling limits and their associated limit equations. Moreover, in Section 1.2 we explain the notion of random particle configurations more precisely, as well as their connection to correlation functions. Section 1.3 is devoted to the formal derivation of limit equations along the lines proposed by Bogolyubov (cf. [5]).

In Chapter 2 we explain our rigorous results in more detail, in the context provided by the Introduction.

1.1 Scaling limits of interacting particle systems and their limit equations

In this section, we introduce important scaling limits and their associated limit equations. For more details, see [49]. Let J be a countable or finite index set and $(X_{j,0}, V_{j,0})_{j \in J}$ be a configuration of particles in the three dimensional phase space, so $(X_{j,0}, V_{j,0}) \in \mathbb{R}^3 \times \mathbb{R}^3$ for all $j \in J$. The configuration will be chosen at random according to some measure μ on the set of possible configurations. Since particles are assumed to be identical, the measure μ is symmetric under permutations of the set J. We discuss such measures in more detail in Section 1.2. Depending on the specific system, the ensemble is chosen to be canonical or grand-canonical.

For the moment, we think of a random, uncorrelated initial distribution μ such that the average number of particles in a (macroscopic) set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ with respect to μ is given by:

$$\mathbb{E}[|\{j \in J : (X_{j,0}, V_{j,0}) \in A\}|] = N \int_A f_{1,0}(x, v) \, \mathrm{d}x \, \mathrm{d}v.$$

Here $N > 0$ is positive and $f_{1,0} \geq 0$ is a prescribed one-particle correlation function. We will consider both spatially inhomogeneous systems with a finite number of particles, and spatially homogeneous systems, i.e. $f_{1,0}(x, v) = f_{1,0}(v)$, with infinitely many particles. The normalization of $f_{1,0}$ in the respective cases we choose as:

$$\int f_{1,0}(x, v) \, \mathrm{d}x \, \mathrm{d}v = 1, \quad \text{for spatially inhomogeneous systems} \tag{1.1.1}$$

$$\int f_{1,0}(v) \, \mathrm{d}v = 1, \quad \text{for spatially homogeneous systems.} \tag{1.1.2}$$

Since we assume $f_{1,0}$ to be fixed, the temperature of the ensemble $(X_j, V_j)_{j \in J}$ is of order one. Let the evolution of the particle configuration be given by the Hamiltonian system (1.0.2), and without loss of generality all particles have unit mass. Then the properties of the system are encoded in the following:

(i) ϕ the interaction potential

(ii) $f_{1,0}(x, v)$ the (normalized) one-particle correlation function at $t = 0$

(iii) ε, ratio between radius of particles and macroscopic scale (cf. (1.0.3))

(iv) Z, N typical number of particles - Z for a microscopic, N for a macroscopic unit volume

(v) θ^2 the strength of the potential (cf. (1.0.2)).

We will introduce different scaling limits. In such a limit, the quantities ε, Z and θ^2 satisfy prescribed algebraic relations, that are motivated by the physical properties of the system. If $\varepsilon \to 0$ in the

scaling limit, the macroscopic scale (1.0.3) is larger than the scale of the potential. Notice that the rescaling (1.0.3) keeps the velocity of particles and thus the temperature of the system constant. The parameter $\varepsilon > 0$ is a dimensionless quantity describing the system.

For simplicity assume for the moment that ϕ is a fixed, smooth, radially symmetric, and quickly decaying potential. Note that this does not include the physically important case of Coulomb interacting particle systems, i.e. $\phi_C(x) = \frac{c}{|x|}$ for some $c > 0$. This potential does not have a characteristic length scale, since the function remains the same in any unit of length. In this case, a characteristic length scale emerges from the dynamics of the system, as proved in Chapter 4.

Mean field limit   In the mean field limit, the dynamics of a typical particle becomes deterministic. It is characterized by a large number of particles, each exerting a small force. In the limit, the forces average by the law of large numbers to a deterministic force field.

Consider the ratio of potential energy and kinetic energy of a typical (and thus any) particle $(X_{j,0}, V_{j,0})$ with $j \in J$:

$$\mathcal{E} = \frac{\mathbb{E}[\sum_{\ell \neq j} \theta^2 \phi(Y_{\ell,0} - Y_{j,0})]}{\mathbb{E}[\frac{1}{2} V_{j,0}^2]} \sim Z\theta^2. \tag{1.1.3}$$

The mean field limit is given by;

$$\varepsilon = 1, \quad \theta^2 = 1/N, \quad Z = N \to \infty, \tag{1.1.4}$$

so the scaling keeps $\mathcal{E}$ (cf. (1.1.3)) of order one. In the rescaling (1.1.4), as $N \to \infty$, the one particle distribution function $f_1^N(t, x, v)$ converges to a solution $f(t, x, v)$ of the Vlasov equation:

$$\partial_t f + v\nabla_x f - \nabla E_{f(t)}(x)\nabla_v f = 0, \quad f(x, v, 0) = f_{1,0}(x, v)$$
$$E_f(x) = \int f(x - y, v)\phi(y) \, \mathrm{d}v \, \mathrm{d}y. \tag{1.1.5}$$

The Vlasov equation is rigorously obtained in the scaling limit (1.1.4) in [8]. The article also proves propagation of chaos for the system and characterizes the fluctuations of the particle system around the limit solution. Despite equation (1.1.5) being time-reversible, the spatially homogeneous steady state can be proved to be stable in a suitable topology. This so-called Landau damping effect is well-known on the linearized level (cf. [10, 21, 22, 29, 36]), and was proved for the nonlinear equation in [39].

Weak-coupling limit   In the weak-coupling limit, the trajectory of a typical particle is governed by a large number of small deflections resulting in a variance of order one in the velocity, hence we observe diffusion in the velocity component. Thus, the limit equation is irreversible and has Maxwellian steady states. The scaling limit is given by:

$$\varepsilon \to 0, \quad Z = 1, \quad \theta^2 = \varepsilon^{\frac{1}{2}}, \tag{1.1.6}$$

in particular the ratio $\mathcal{E}$ (cf. (1.1.3)) of potential and kinetic energy vanishes in the limit.

In the limit $\varepsilon \to 0$, the macroscopic evolution is expected to approach to a solution of the Landau equation:

$$\partial_t f + v \nabla_x f = Q_L(f, f), \tag{1.1.7}$$

where Q_L is given by the formula

$$Q_L(f, f) = \sum_{i,j=1}^{3} \partial_{v_i} \left(\int_{\mathbb{R}^3} a_{i,j}(v - v')(\partial_{v_j} - \partial_{v'_j})\big(f(t, v)f(t, v')\big) \, \mathrm{d}v' \right)$$

$$a_{i,j}(w) = \int_{\mathbb{R}^3} k_i k_j \delta(k \cdot w)|\hat{\phi}(k)|^2 \, \mathrm{d}k = \frac{\Lambda}{|w|}\left(\delta_{i,j} - \frac{w_i w_j}{|w|^2} \right) \quad \text{for some } \Lambda > 0. \tag{1.1.8}$$

More details, including a rigorous derivation of (1.1.7) under a propagation of chaos assumption, are given in Chapter 3. Results on the Landau equation can be found in [15, 16, 24]. The derivation of the Landau equation from particle systems is still open, we refer to the consistency result proved in [5], and to [4, 14] for the derivation of the linear equation from scaling limits of Lorentz systems.

Plasma limit   The plasma limit yields trajectories of particles with a large number of small deflections as in the weak-coupling limit. The scaling limit is defined as:

$$Z \to \infty, \quad Z\theta^2 = 1, \quad \varepsilon = 1/Z, \tag{1.1.9}$$

so in particular $\mathcal{E} = 1$ and the potential energy of a particle is comparable to its kinetic energy. This results in the appearance of collective effects, which give a correction to the coefficient a of the Landau equation (1.1.7). The evolution is described by the Balescu-Lenard equation given by:

$$\partial_t f + v \nabla_x f = Q_{BL}(f, f), \tag{1.1.10}$$

where Q_{BL} can be expressed in terms of the so-called dielectric function ε and reads:

$$Q_{BL}(f, f) = \sum_{i,j=1}^{3} \partial_{v_i} \left(\int_{\mathbb{R}^3} a_{i,j}(v - v', v)(\partial_{v_j} - \partial_{v'_j})\big(f(t, v)f(t, v')\big) \, \mathrm{d}v' \right)$$

$$a_{i,j}(w, v) = \int_{\mathbb{R}^3} \frac{k_i k_j}{|\varepsilon(k, k \cdot v)|^2} \delta(k \cdot w)|\hat{\phi}(k)|^2 \, \mathrm{d}k. \tag{1.1.11}$$

A precise definition of the dielectric function and a discussion of its properties is given in Chapter 4. There are few mathematical results on this limit and the Balescu-Lenard equation. Results on the linearized Balescu-Lenard equation can be found in [52], and the equation was derived from a stochastic model in [28, 30, 44]. More precisely, these papers derive the Balescu-Lenard equation from the dynamics of a point charge moving through a random medium. The medium is given by a random Gaussian field that describes the random fluctuations of particle configurations around the homogeneous density.

Boltzmann-Grad limit   The Boltzmann-Grad limit models the evolution of particles in an ideal gas. Particles typically experience one collision per unit of time. The interaction can be given by different potentials, an important case is the hard sphere potential. The limit is characterized by the following relations:

$$N \to \infty, \quad N\varepsilon^2 \sim 1, \quad \theta^2 \sim 1. \tag{1.1.12}$$

The limit evolution can be described by the Boltzmann equation. The equation is

$$\partial_t f + v\nabla_x f = Q_B(f, f), \tag{1.1.13}$$

where Q_B is the operator

$$Q_B(f, f) = \int_{\mathbb{R}^3} \int_{S^2} (f(v')f(v'_*) - f(v)f(v_*))B(v - v_*, \sigma) \, \mathrm{d}\sigma \, \mathrm{d}v_*. \tag{1.1.14}$$

The velocities v', v'_* before collision, and v, v_* after collision, as well as the collision parameter $\sigma \in S^n$ can be computed using the assumption that collisions are elastic, i.e. momentum and kinetic energy are conserved. Moreover, the cross-section B depends on the choice of interaction potential ϕ.

The transition from interacting particle systems of hard spheres to the Boltzmann equation in the Boltzmann-Grad limit has been proved for short macroscopic times by Lanford in [32]. Further results include the limit for other interaction potentials and stronger estimates on the correlation of particles, and can be found in [19, 45, 46].

The Boltzmann equation given by (1.1.13) and (1.1.14) conserves mass and energy, and yields a monotone increasing entropy. In particular, the steady states are given by the Maxwellians (cf. (1.0.1)). Results on well-posedness and convergence to equilibrium can be found for example in [17, 18, 54, 59].

1.2   Measures on the space of locally finite particle configurations and correlation functions

We now discuss probability measures on sets of particle configurations and their connection to correlation functions, loosely following the exposition in [32]. First consider the case of a canonical ensemble, as used in the derivation of the Vlasov equation in [8]. Let $f_{1,0} \in C(\mathbb{R}^3 \times \mathbb{R}^3)$ be a probability density on the phase space. Now consider the random variables

$$(X_j, V_j)_{1 \leq j \leq N}, \quad \text{i.i.d. with} \quad (X_j, V_j) \sim f_{1,0} \, \mathrm{d}x \, \mathrm{d}v \quad \text{for } j = 1, \dots, N. \tag{1.2.1}$$

The collection $(X_j, V_j)_{1 \leq j \leq N}$ represents a collection of N independent particles in the phase space, distributed according to the probability density $f_{1,0}$.

The ensemble (1.2.1) cannot be used for scaling limits modeling spatially homogeneous systems, since such systems will necessarily have an infinite number of particles. In this case we use a grand canonical ensemble of particles, for which only the average number of particles per unit of volume is given. To this end, let J be a countable index set. Then a particle system on the phase space of $\mathbb{R}^3$ is a collection $\{(X_j, V_j)\}_{j \in J}$. Now the set Ω of locally finite particle configurations is the set of

all such $\{(X_j, V_j)\}_{j \in J}$, identifying configurations that are equal up to permutation. More precisely, we identify:

$$\{(X_j, V_j)\}_{j \in J} = \{(X_{\tau(j)}, V_{\tau(j)})\}_{j \in J}, \tag{1.2.2}$$

for every bijective map $\tau : J \to J$. We endow Ω with the σ-Algebra $\mathcal{U}$, generated by the cylinder sets. Let $n \in \mathbb{N}$, $k_1, \ldots k_n \in \mathbb{N}_0$ and $A_1, \ldots, A_n \subset \mathbb{R}^3 \times \mathbb{R}^3$ be Borel sets, and define the associated cylinder set by:

$$U(A_1, k_1, \ldots, A_n, k_n) = \{\{(X_j, V_j)\}_{j \in J} : |\{j \in J : (X_j, V_j) \in A_\ell\}| = k_\ell\}. \tag{1.2.3}$$

In the following, we will often make the assumption that the probability measure μ is translation invariant in the spatial component, that is for any $a \in \mathbb{R}^3$ and $U \in \mathcal{U}$ we have:

$$\mu(U) = \mu(T_a(U)), \tag{1.2.4}$$

where T_a is the translation operator: $T_a((X_j, V_j)_j) = (X_j + a, V_j)_j$.

In this framework, a random measure ν on the space of particle configurations Ω can be defined by the measure it assigns to the cylindrical sets $U \in \mathcal{U}$. For ease of notation, define the occupation numbers of Borel sets $U_1, \ldots, U_n \subset \mathbb{R}^3 \times \mathbb{R}^3$ and a given particle configuration by:

$$n(U_1, \ldots, U_n) = |\{(j_1, \ldots, j_n) : (X_j, V_j) \in U_k, j_k \neq j_l \text{ for } k \neq l\}|. \tag{1.2.5}$$

For spatially homogeneous scaling limits, we would like to define a notion of random initial particle configurations with an average of N particles per unit of volume, which are independent and have velocities distributed according to $\sim f_{1,0}(v) \, \mathrm{d}v$ for some initial probability density $f_{1,0}$. This is achieved by the so-called Poisson measure $\nu_{N,f_{1,0}}$. The measure $\nu_{N,f_{1,0}}$ has the property that $(n(U_\ell))_{1 \leq \ell \leq k}$ is a collection of independent random variables for every finite disjoint collection of Borel sets $(U_\ell)_{1 \leq \ell \leq k}$. Further for Borel sets $A, B \subset \mathbb{R}^3$, $U = A \times B$ we set:

$$P(n(U) = k) = \frac{(N|A| \int_B f_{1,0}(v) \, \mathrm{d}v)^k}{k!} e^{-N|A| \int_B f_{1,0}(v) \, \mathrm{d}v}. \tag{1.2.6}$$

A crucial tool in understanding measures on the space of particle configurations and their time evolutions are density and correlation functions. We will introduce them quickly here, following the approach by Klimontovich in [26]. Let μ be a probability measure on $(\Omega, \mathcal{U})$. For brevity, we write ξ as generic phase space variables in the macroscopic lengthscale, and η as:

$$\xi = (x, v), \quad \eta = (y, v), \quad x = \varepsilon y. \tag{1.2.7}$$

For particle positions in the phase space we write $(X_j, V_j) = P_j$. For any $n \in \mathbb{N}$, we define an associated measure $\mu_n \in M_+((\mathbb{R}^3 \times \mathbb{R}^3)^n)$ on the space of n-tuples by:

$$\mu_n(\xi_1, \ldots, \xi_n) = \sum_{(j_1, \ldots, j_n) \in J_0^n} \delta(\xi_1 - P_{j_1}) \ldots \delta(\xi_n - P_{j_n}). \tag{1.2.8}$$

Here J_0^n is the set of mutually disjoint indices $(j_1, \ldots, j_n)$ in J^n. Now we can define the n-particle correlation function F_n by:

$$F_n(\xi_1, \ldots, \xi_n) = \mathbb{E}[\mu_n(\xi_1, \ldots, \xi_n)], \tag{1.2.9}$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure. Notice that in general, F_n is a measure rather than a function and that (1.2.9) can be also expressed in terms of the occupation numbers n (cf. (1.2.5)). Let $U_1, \ldots, U_n$ be Borel sets, then we have:

$$\int_{U_1} \cdots \int_{U_n} F_n(\, \mathrm{d}\xi_1, \ldots, \, \mathrm{d}\xi_n) = \mathbb{E}[n(U_1, \ldots, U_n)].$$

In the cases we will consider later, it is expected that F_n is indeed absolutely continuous. For the Poisson measure $\nu_{N, f_{1,0}}$, the density functions factorize into

$$F_n(\xi_1, \ldots, \xi_n) = N^n f_{1,0}(v_1) \cdots f_{1,0}(v_n) \tag{1.2.10}$$

by a straightforward computation using the independence of occupation numbers of disjoint sets and (1.2.6).

A number of natural questions arise from the notion of correlation functions. One can ask whether the relation between density functions F_n and probability measures μ on the space of particle configurations is one-to-one. It is proved in [33] that a sequence of density functions $(F_n)_{n \in \mathbb{N}}$ defines at most one measure μ, provided that the functions satisfy the bound:

$$|F_n| \le C^n n^{2n} \quad \text{for some } C > 0. \tag{1.2.11}$$

A sufficient condition for the correlation functions F_n to define a probability measure on the set of particle correlation functions is given in [35], although the condition seems difficult to verify in practice. More generally, one can ask whether, given the first k correlation functions $F_1, \ldots, F_k$, one can find correlation functions F_n, $n > k$ such that the collection $(F_k)_{k \in \mathbb{N}}$ define a probability measure on the set of particle configurations. This so-called truncated moment problem has been studied in various settings. A result for the translation invariant problem on the discrete set $\mathbb{Z}^d$ can be found in [9].

If the initial particle configuration is distributed according to the measure $\nu_{N, f_{1,0}}$, then as observed in (1.2.10), the correlation functions F_n factorize, and the numbers of particles in disjoint regions are independent. Due to the interaction via the potential ϕ in the Hamiltonian dynamics (1.0.2), this independence cannot be expected to hold for positive times. To measure this effect, we introduce truncated correlation functions $G_n(\xi_1, \ldots, \xi_n)$. The first three truncated correlation functions are defined by:

$$\begin{aligned}
G_1(\xi_1) &= F_1(\xi) \\
G_2(\xi_1, \xi_2) &= F_2(\xi_1, \xi_2) - F_1(\xi_1)F_1(\xi_2) \\
G_3(\xi_1, \xi_2, \xi_3) &= F_3(\xi_1, \xi_2, \xi_3) - F_1(\xi_1)F_1(\xi_2)F_1(\xi_3) - F_1(\xi_3)G_2(\xi_1, \xi_2) \\
&\quad - F_1(\xi_1)G_2(\xi_2, \xi_3) - F_1(\xi_2)G_2(\xi_1, \xi_3).
\end{aligned} \tag{1.2.12}$$

An inductive definition of higher order truncated correlation functions can be found in [26]. By construction, the functions G_n are invariant under permutations of the variables.

The construction (1.2.12) allows us to measure the size of correlations in the system, and to give a more precise notion of the propagation of chaos principle as the smallness of the truncated correlations G_n relative to the correlation functions F_k. Since we will typically consider particle systems with a diverging average number of particles per unit of volume ($N \to \infty$) we introduce the normalized objects:

$$\mu_n = \frac{1}{N^n} m_n, \quad f_n = \frac{1}{N^n} F_n, \quad g_n = \frac{1}{N^n} G_n. \tag{1.2.13}$$

This has the advantage that f_n are functions of order one, and propagation of chaos can be formulated as $g_n \to 0$, in various topologies.

1.3 The BBGKY hierarchy and the formal derivation of limit equations

In the previous section, we discussed measures on the space of locally finite particle configurations, and how they can be characterized using correlation functions. Now let $(Y_j, V_j)_{j \in J}$ be a random configuration of particles. Further assume that the Hamiltonian evolution (1.0.2) is well-defined; sufficient conditions for this can be found in [50]. The pathwise evolution defines an evolution in the space of locally finite particle configurations, which in turn results in an evolution of the associated correlation functions f_n. We now take a closer look at the evolution of the normalized correlation functions f_n (cf. (1.2.13)), for simplicity first in the microscopic variables, i.e. $\tilde{f}_n(\tau, \eta_1, \ldots, \eta_n)$. Write $\alpha_n = (\eta_1, \ldots, \eta_n)$ for a generic n-tuple of phase space variables, and let $\alpha'_n[k]$ be the tuple obtained by omitting the k-th particle, so $\alpha'_n[k] = (\eta_1, ..., \eta_{k-1}, \eta_{k+1}, ..., \eta_n)$. A formal computation (c.f. [26]) reveals that these functions satisfy:

$$\partial_\tau \tilde{f}_n(\tau, \alpha_n) + \sum_{k=1}^n v_k \nabla_{y_k} \tilde{f}_n(\tau, \alpha_n) - N\theta^2 \sum_{k=1}^n \int_{\mathbb{R}^6} d\eta_{n+1} \nabla_{y_k} \phi_r(y_k - y_{n+1}) \nabla_{v_k} \tilde{f}_{n+1}(\tau, \alpha_n, \eta_{n+1})$$

$$= \theta^2 \sum_{k=1}^n \sum_{\ell=1}^n \nabla_{y_k} \phi_r(y_k - y_\ell) \nabla_{v_k} \tilde{f}_n(\tau, \alpha_n). \tag{1.3.1}$$

Now if the $\tilde{f}_n$ are a solution to the BBGKY hierarchy (1.3.1), then the associated truncated correlation functions $\tilde{g}_n$ satisfy the equation:

$$\partial_\tau \tilde{g}_n(\alpha_n) + \sum_{k=1}^n v_k \nabla_{y_k} \tilde{g}_n(\alpha_n) - \theta^2 \sum_{k \neq \ell = 1}^n \nabla_{v_k} (\tilde{f}_1(\eta_k) \tilde{g}_{n-1}(\alpha'_n[k]) + \tilde{g}_n(\alpha_n)) \nabla \phi_r(y_k - y_\ell)$$

$$= N\theta^2 \sum_{k=1}^n \int \nabla \phi_r(y_k - y_{n+1}) \nabla_{v_k} (\tilde{f}_1(\eta_k) \tilde{g}_n(\alpha'_n[k], \eta_{n+1}) + \tilde{g}_{n+1}(\alpha_n, \eta_{n+1})) \ d\eta_{n+1} \tag{1.3.2}$$

for $n \geq 1$, and $\tilde{g}_0 = \tilde{f}_1$. It is readily seen by construction that:

$$\tilde{f}_n(\tau, \eta_1, \ldots, \eta_n) = \tilde{f}_n(\tau, \eta_{\sigma(1)}, \ldots, \eta_{\sigma(n)}), \quad \tilde{g}_n(\tau, \eta_1, \ldots, \eta_n) = \tilde{g}_n(\tau, \eta_{\sigma(1)}, \ldots, \eta_{\sigma(n)}) \tag{1.3.3}$$

for any element of the permutation group $\sigma \in S_n$. Furthermore, we can use (1.3.2) to formally infer a separation of orders of magnitude for the functions $\tilde{g}_n$. Consider the equation for the two-particle truncated correlations $\tilde{g}_2$

$$\partial_\tau \tilde{g}_2 + \sum_{k=1}^{2} v_k \nabla_{y_k} \tilde{g}_2 - N\theta^2 \sum_{k=1}^{n} \int \nabla\phi(y_k - y_{n+1})\nabla_{v_k}(\tilde{f}_1(\eta_k)\tilde{g}_2(\alpha_2'[k], \eta_3) + \tilde{g}_3(\alpha_2, \eta_3)) \ d\eta_3$$
$$= \theta^2 \sum_{k \neq \ell = 1}^{2} \nabla_{v_k}(\tilde{f}_1(\eta_k)\tilde{f}_1(\eta_2) + \tilde{g}_2(\alpha_n))\nabla\phi(y_k - y_\ell). \tag{1.3.4}$$

The source term on the right-hand side of (1.3.4) is formally of order θ^2, so we expect $\tilde{g}_2 \sim \theta^2$. Iterating the argument, we obtain a separation of orders of magnitude for the truncated correlation functions $\tilde{g}_n$:

$$|\tilde{f}_n| \approx 1 \ , \quad |\tilde{g}_n| \approx (\theta^2)^{n-1}. \tag{1.3.5}$$

We observe that (1.3.5) indicates the validity of the propagation of chaos principle. This motivates us to truncate the hierarchy, by making the approximation:

$$\tilde{f}_3(\tau, \eta_1, \eta_2, \eta_3) = \tilde{f}_1(\tau, \eta_1)\tilde{f}_2(\tau, \eta_1)\tilde{f}_1(\tau, \eta_3) + \sum_{k=1}^{3} \tilde{f}_1(\tau, \eta_k)\tilde{g}_2(\tau, \alpha_3'[k]). \tag{1.3.6}$$

The resulting truncated BBGKY hierarchy is a closed system of equations for $\tilde{f}_1, \tilde{g}_2$:

$$\partial_\tau \tilde{f}_1 + v_1 \nabla_{y_1} \tilde{f}_1 - N\theta^2 \nabla \tilde{f}_1(v_1)\nabla E_f(y_1) = N\theta^2 \nabla_{v_1} \cdot \left(\int \nabla\phi(y_1 - y_3)\tilde{g}_2(\eta_1, \eta_3) \ d\eta_3 \right)$$

$$\partial_\tau \tilde{g}_2 + \sum_{k=1}^{2} v_i \nabla_{y_i} \tilde{g}_2 - N\theta^2 \sum_{k \neq \ell} \int \nabla\phi(y_k - y_3)\nabla_{v_k}(\tilde{f}_1(\eta_k)\tilde{g}_2(\eta_\ell, \eta_3)) \ d\xi_3 \tag{1.3.7}$$
$$= \theta^2 \sum_{k \neq \ell} \nabla_{v_k} \left(\tilde{f}_1(\eta_1)\tilde{f}_1(\eta_2) \right) \nabla\phi(y_k - y_\ell).$$

Formal derivation of the Vlasov equation In order to derive the formal limit equation in the mean field rescaling (1.1.4), we insert the dependence $\theta^2 = 1/N$ into (1.3.7). Further, we recall that $\varepsilon = 1$ in (1.1.4), so the functions $\tilde{f}_1(\tau, y, v)$ and $f_1(t, x, v)$ coincide. Therefore, the equation for f_1 simplifies to:

$$\partial_t f_1 + v_1 \nabla_{x_1} f_1 - \nabla f_1(v_1)\nabla E_f(x_1) = \nabla_{v_1} \cdot \left(\int \nabla\phi(x_1 - x_3)g_2(\xi_1, \xi_3) \ d\xi_3 \right). \tag{1.3.8}$$

The separation of orders of magnitude in (1.3.5) implies that $g_2 \approx 1/N$ becomes negligible as $N \to \infty$, and we recover the Vlasov equation (1.1.5) from (1.3.8).

Formal derivation of the Landau equation   Consider the weak-coupling scaling limit (1.1.6), and for simplicity assume that the initial density of particles is spatially homogeneous, and the initial configuration is distributed according to a grand-canonical ensemble. A crucial observation by Bogolyubov (cf. [6]) is that the evolution of $\tilde{f}_1$ and $\tilde{g}_2$ take place on two different timescales. To see this, we remark that, up to lower order terms in the weak-coupling limit, the equations for $\tilde{f}_1$ and $\tilde{g}_2$ (1.3.7) read

$$\partial_\tau \tilde{f}_1 = \varepsilon^{\frac{1}{2}} \nabla_{v_1} \cdot \left(\int \nabla\phi(y_1 - y_3)\tilde{g}_2(\eta_1, \eta_3) \, \mathrm{d}\eta_3 \right) \tag{1.3.9}$$

$$\partial_\tau \tilde{g}_2 + \sum_{k=1}^{2} v_i \nabla_{y_i}\tilde{g}_2 = \varepsilon^{\frac{1}{2}} \sum_{k\neq\ell} \nabla_{v_k} \left(\tilde{f}_1(\eta_1)\tilde{f}_1(\eta_2) \right) \nabla\phi(y_k - y_\ell). \tag{1.3.10}$$

Formally, all terms in (1.3.10) are of order $\varepsilon^{\frac{1}{2}}$ so $\tilde{g}_2$ is of order $\varepsilon^{\frac{1}{2}}$, and its evolution takes place on the timescale τ. In turn, this indicates that $\tilde{f}_1$ evolves on the macroscopic timescale $t = \varepsilon\tau$. If the equation (1.3.10) has a stable stable steady state, we expect that the equation for $\tilde{f}_1$ on the macroscopic timescale is given by:

$$\partial_t f_1 = \nabla_v \cdot \left(\int \nabla\phi(y_1 - y_3)\tilde{g}_B(\eta_1, \eta_3) \, \mathrm{d}\eta_3 \right), \tag{1.3.11}$$

where $\tilde{g}_B = \tilde{g}_B[f_1]$ is the rescaled steady state associated to the one-particle function $\tilde{f}_1(t)$, i.e. $\tilde{g}_B$ satisfies the equation:

$$\sum_{k=1}^{2} v_i \nabla_{y_i}\tilde{g}_B = \sum_{k\neq\ell} \nabla_{v_k} \left(\tilde{f}_1(\eta_1)\tilde{f}_1(\eta_2) \right) \nabla\phi(y_k - y_\ell). \tag{1.3.12}$$

The steady state equation (1.3.12) can be solved explicitly using the method of characteristics, and inserting the steady state $\tilde{g}_B$ into (1.3.11) yields the Landau equation (1.1.7)-(1.1.8). In Chapter 3, we prove the stability of the system (1.3.9)-(1.3.10) for short macroscopic times, and the convergence to a solution of the Landau equation.

Formal derivation of the Balescu-Lenard equation   Now we consider the plasma limit (1.1.9), for spatially homogeneous particle systems. As in the formal derivation of the Landau equation, we neglect terms in (1.3.7) that are formally of lower order in the plasma limit and obtain the system:

$$\partial_\tau \tilde{f}_1 = \nabla_{v_1} \cdot \left(\int \nabla\phi(y_1 - y_3)\tilde{g}_2(\eta_1, \eta_3) \, \mathrm{d}\eta_3 \right) \tag{1.3.13}$$

$$\partial_\tau \tilde{g}_2 + \sum_{k=1}^{2} v_i \nabla_{y_i}\tilde{g}_2 - \sum_{k\neq\ell} \int \nabla\phi(y_k - y_3)\nabla_{v_k}(\tilde{f}_1(\eta_k)\tilde{g}_2(\eta_\ell, \eta_3)) \, \mathrm{d}\xi_3 \tag{1.3.14}$$
$$= \varepsilon \sum_{k\neq\ell} \nabla_{v_k} \left(\tilde{f}_1(\eta_1)\tilde{f}_1(\eta_2) \right) \nabla\phi(y_k - y_\ell).$$

Again, the terms in the equation (1.3.14) describing the evolution of correlations are formally of the same order of magnitude, indicating that $\tilde{g}_2$ is of order ε and evolves on the timescale τ. Similarly, $\tilde{f}_1$ can be expected to evolve on the macroscopic timescale $t = \varepsilon\tau$.

Hence, on the macroscopic timescale, the kinetic equation for f_1 in the limit $\varepsilon \to 0$ is formally given by

$$\partial_t f_1 = \nabla_v \cdot \left(\int \nabla\phi(y_1 - y_3)\tilde{g}_B(\eta_1, \eta_3)\, \mathrm{d}\eta_3 \right), \tag{1.3.15}$$

where $\tilde{g}_B = \tilde{g}_B[f_1]$ solves the steady state equation

$$\sum_{k=1}^{2} v_i \nabla_{y_i}\tilde{g}_B - \sum_{k\neq\ell} \int \nabla\phi(y_k - y_3)\nabla_{v_k}(f_1(\eta_k)\tilde{g}_B(\eta_\ell, \eta_3))\, \mathrm{d}\eta_3 = \sum_{k\neq\ell} \nabla_{v_k}\left(\tilde{f}_1\tilde{f}_1\right)\nabla\phi(y_k - y_\ell). \tag{1.3.16}$$

Due to the integral term on the left-hand side of (1.3.16), the equation cannot be solved using the method of characteristics. The equation was first solved Lenard (cf. [34]) using the Wiener-Hopf method; he observed that inserting the steady state $\tilde{g}_B$ into (1.3.15) yields the Balescu-Lenard equation (1.1.10)-(1.1.11). In Chapter 4 we study the properties of the solutions $\tilde{g}_B$ of (1.3.16) in detail.

Derivation of Boltzmann equation from the BBGKY hierarchy   In the Boltzmann-Grad rescaling (1.1.12), the argument that we used for the weak-coupling limit and the plasma limit can be applied to obtain the formal limit equation. This formally yields the Boltzmann equation given by (1.1.13) and (1.1.14).

Chapter 2

Summary of the results and outlook

The overarching theme of this work is the convergence of scaling limits of long range interacting particle systems to the Landau and Balescu-Lenard equations. The full derivation of both equations seems out of reach at present, but the results presented here give us some clues about the physics of these systems, and provide mathematical tools that can be used to prove the stability of the evolution on the macroscopic timescale.

Recall the truncated BBGKY hierarchy in the weak-coupling limit, i.e. (1.3.9)-(1.3.10). In Theorem 3.2.6 in Chapter 3, we prove the well-posedness of the system (1.3.9)-(1.3.10) on the macroscopic timescale $t = \varepsilon\tau$. Theorem 3.2.8 then shows that the solutions f_ε converge to a solution f to the Landau equation. The result is subject to the following restrictions: an explicit interaction potential ϕ, the interaction of particles with small relative velocity is cut out, the system is initially close to the Maxwellian equilibrium, and the result is only valid for short macroscopic times. We now discuss why these assumptions were made and discuss whether they can be removed in future works.

The explicit choice of the interaction potential ϕ was made to simplify computations. The potential appears in the equations only in integrated quantities. For the specific choice of potential made in Chapter 3, those integrals can be explicitly computed. However, this is not an intrinsic restriction of the strategy of proof, and replacing these identities by estimates for the resulting functions, one should be able to generalize the result to a broad class of potentials.

In the region of small relative velocity $|v_1 - v_2| \leq \varepsilon$, the approximation of the BBGKY hierarchy using propagation of chaos is not valid. Since this region vanishes in the limit $\varepsilon \to 0$, this does not affect the validity of the derivation of the Landau equation. The singularity $|v_1 - v_2|^{-1}$ appears also in the Landau equation, therefore the resulting regularity properties have been studied in a number of papers, for example [24]. However, the techniques developed there do not immediately carry over to the system (1.3.9)-(1.3.10), since it does not have the same smoothing properties as the parabolic Landau equation. Nevertheless, it seems possible to use these ideas to remove the cutoff in $v_1 - v_2$ in the future.

We prove our results close to the Maxwellian equilibrium of the limit equation. This is due to the presence of a boundary layer in the evolution of f_ε close to $t = 0$. Physically, this can be explained by the Bogolyubov argument. On the one hand, we argue that the truncated correlations $g_2(t)$ evolve adiabatically as a functional of the one-particle function $f_1(t)$. On the other hand, we consider

random initial data with vanishing particle correlations. Therefore, $g_2(t)$ quickly changes to the Bogolyubov correlation at $t = 0$, which is observed as a boundary layer in f_1. This suggests that the result can be extended to a broader class of initial data if we assume that the initial configuration has nonzero initial correlations that are chosen according to the initial datum $f_{1,0}$. In the more classical setting with uncorrelated initial configurations, our technique only allows us to estimate the size of the boundary layer, but gives no precise description of it. A more detailed analysis of the initial formation of correlations is an interesting project connected to the result presented here.

Lastly, the result could be improved by removing the restriction to short macroscopic times. This requires a deeper understanding of the non-linear problem for positive ε. As presented here, the method deals with the non-linearity as a perturbation, and the lack of dissipation in the non-Markovian evolution makes it difficult to obtain a global-in-time result. For the parabolic Landau equation, the existence of global solutions is proved in [24] for initial data sufficiently close to equilibrium. Potentially, a similar result holds for the non-Markovian evolution, but this requires a better understanding of the stability properties of the evolution as $\varepsilon \to 0$.

In Chapter 4, we study the truncated correlation function g_2 in the plasma limit (1.1.9). For a broad class of potentials, including the physically important Coulomb potential, we give a precise description of the Bogolyubov correlations g_B, solving (1.3.16) for f_1 fixed. The physical implications of the results are discussed in detail in Chapter 4, and on the level of the steady state g_B our results are satisfactory for the moment. We prove the linear stability of the steady state g_B under the evolution (1.3.10) for soft potentials in Theorem 4.2.22. Since the result rests on linearized Landau damping, we can only hope for an improvement of the result if we can improve the results on linearized stability for the Vlasov equation. The existing results, for instance [21, 22, 29] are obtained under the assumption of radial symmetry. Extending the theory to anisotropic data seems to be an important challenge for the future.

Considering that we prove non-linear stability of the evolution (1.0.8)-(1.0.9) in the weak-coupling case in Chapter 3 and analyze equation (1.3.14) in the plasma limit in Chapter 4, it also seems feasible to use the results of both chapters to prove non-linear stability of (1.3.13)-(1.3.14) in the future.

Chapter 3

From a non-Markovian system to the Landau equation

This chapter has been published as an article (cf. [55]).

3.1 Introduction

A central objective in kinetic theory is the derivation of effective equations for macroscopic densities of particles in a plasma or gas. Two of the main equations in this context are the Boltzmann equation and the Landau equation, and a large portion of the mathematical research in this area is devoted to the study of these equations. For an extensive overview of mathematical kinetic theory we refer to [49, 56]. For the Boltzmann equation, rigorous results have been proved, both on the level of the equation itself, and on the level of its derivation from particle systems. Results on well-posedness, entropic properties of solutions, and rate of convergence to equilibrium can be found in [17, 18, 54, 59]. For the derivation of the equation from interacting particle systems we refer to [19, 32, 45, 46], and to [7, 13, 20, 51] for the derivation of the linear equation from Lorentz models.

Many of these problems, including the derivation starting from particle systems, are still open for the Landau equation. The goal is to describe the evolution of the macroscopic velocity distribution of (initially randomly) distributed particles $(X_i, V_i)_{i \in I} \in (\mathbb{R}^3 \times \mathbb{R}^3)^I$ (where I is a countable or finite index set) evolving according to the Hamiltonian dynamics:

$$\partial_\tau X_i(\tau) = V_i(\tau)$$
$$\partial_\tau V_i(\tau) = -\theta^2 \sum_{j \neq i} \nabla \phi(X_i(\tau) - X_j(\tau)), \quad \theta > 0 \text{ scaling parameter.} \qquad (3.1.1)$$

Here $\phi = \phi(x)$ is the interaction potential, and in the rest of this chapter we use the notation $\nabla \phi = \nabla_x \phi$ and assume ϕ is radially symmetric. When the strength of the potential is small, i.e. $\theta^2 \to 0$, and for large times $t \gg 1$, the evolution of the particles is governed by many small deflections. Let $Z > 0$ be the average number of particles per unit of volume, to be made precise later. It is widely accepted that for a suitable choice of ϕ and rescaling of $\theta \to 0$ and Z, the number density

$f(t, v)$ of a spatially homogeneous system satisfies the Landau equation (cf. [49]):

$$\partial_t f(t, v) = \sum_{i,j=1}^{3} \partial_{v_i} \left(\int_{\mathbb{R}^3} a_{i,j}(v - v')(\partial_{v_j} - \partial_{v_j'})\big(f(t, v)f(t, v')\big) \, \mathrm{d}v' \right)$$
$$f(0, v) = f_0(v).$$

(3.1.2)

Here t is a macroscopic time scale that we will specify later, and the matrix valued function a is determined by the pair interaction potential ϕ:

$$a_{i,j}(w) = \frac{\pi^2}{4} \int_{\mathbb{R}^3} k_i k_j \delta(k \cdot w)|\hat{\phi}(k)|^2 \, \mathrm{d}k = \frac{\Lambda}{|w|} \left(\delta_{i,j} - \frac{w_i w_j}{|w|^2} \right) \quad \text{for some } \Lambda > 0. \qquad (3.1.3)$$

In the most physically relevant case – that of Coulomb interaction, i.e. $\phi(x) = \frac{c}{|x|}$ – considered in [31], the constant Λ is logarithmically divergent.

The equation (3.1.2)-(3.1.3) was introduced by Landau in [31] (see also [36]). However, Landau did not take as a starting point the dynamics of the particles (cf. (3.1.1)). Instead he studied the Boltzmann equation in the limit of grazing collisions, which was assumed to be a good approximation for the dynamics of the system (3.1.1). A rigorous version of Landau's argument can be found in [1].

A rather general approach to deriving kinetic equations from (3.1.1) was later developed by Bogolyubov (cf. [6]). We will briefly summarize this method here. Consider a countable system of particles $(X_i(0), V_i(0))_{i \in I} \in (\mathbb{R}^3 \times \mathbb{R}^3)^I$, distributed according to an uncorrelated, translation invariant grand canonical ensemble. Furthermore, assume the velocities V_i are of order one. We consider scaling limits of a single scaling parameter $\varepsilon \to 0$, as is customary in the modern literature on kinetic equations (cf. [2, 19, 32, 45, 49]). We set the strength of the potential θ^2 and the particle density Z as:

$$\theta^2 = \varepsilon^\beta, \quad Z = \varepsilon^{1-2\beta}. \qquad (3.1.4)$$

For reasons we will explain later, we choose $\beta \in (0, 1)$. We can then consider the n-particle correlation functions $F_n(x_1, v_1, \ldots, x_n, v_n)$. In order to work with functions of order one, we define the rescaled functions f_n by:

$$F_n(\tau, x_1, v_1, \ldots, x_n, v_n) = Z^n f_n(\tau, x_1, v_1, \ldots, x_n, v_n).$$

Then the correlation functions f_n satisfy the so-called BBGKY hierarchy (see e.g. [2]):

$$\partial_\tau f_n + \sum_{i=1}^{n} v_i \nabla_{x_i} f_n - \varepsilon^{1-\beta} \sum_{i=1}^{n} \int \nabla\phi(x_i - x_{n+1})\nabla_{v_i} f_{n+1} \, \mathrm{d}x_{n+1} \, \mathrm{d}v_{n+1}$$
$$= \varepsilon^\beta \sum_{i \neq j} \nabla\phi(x_i - x_j)\nabla_{v_i} f_n. \qquad (3.1.5)$$

Since $\beta \in (0, 1)$, we have $Z\theta^2 = \varepsilon^{1-\beta} \to 0$. The physical meaning of this will be explained below. Under this assumption, Bogolyubov's argument yields the Landau equation (3.1.2) as the limiting

equation for f_1. In the case $\beta = 1$, i.e. $Z\theta^2 = 1$, Bogolyubov's technique can also be applied, however here the limiting equation is the Balescu-Lenard equation (see [3, 2, 34]). In this case, the particles of the system must be viewed as interacting as part of an effective medium, in which the interaction of pairs of particles is modified due to collective effects. In the physics literature this is characterized by means of the so-called dielectric function, that gives a nontrivial correction to the limit kinetic equation. We will not however consider this issue in the present chapter.

Our assumption $Z\theta^2 \to 0$ has a clear interpretation in terms of dimensionless quantities. Observe that $Z\theta^2$ describes the ratio of the average potential to the average kinetic energy of a particle:

$$\frac{\langle \theta^2 \sum_{j \in I : i \neq j} \phi(X_i - X_j) \rangle}{\langle V_i^2 \rangle} \sim Z\theta^2 = \varepsilon^{1-\beta}.$$

Since $Z\theta^2 \to 0$, the kinetic energy of the particles particles is much larger than their potential energy, hence the absence of collective effects. Our objective is to study the evolution of the one particle function f_1. We will refer to the timescale on which this evolution takes place as macroscopic time. To simplify notation, we set $(x_i, v_i) = \xi_i$ and introduce the (rescaled) truncated correlation functions g_2, g_3, ... defined by:

$$g_2(\xi_1, \xi_2) = f_2(\xi_1, \xi_2) - f_1(\xi_1)f_1(\xi_2)$$
$$g_3(\xi_1, \xi_2, \xi_3) = f_3(\xi_1, \xi_2, \xi_3) - f_1(\xi_1)g_2(\xi_2, \xi_3) - f_1(\xi_2)g_2(\xi_1, \xi_3) - f_1(\xi_3)g_2(\xi_1, \xi_2)$$
$$- f_1(\xi_1)f_1(\xi_2)f_1(\xi_3)$$
$$g_4(\xi_1, \xi_2, \xi_3, \xi_4) = \dots.$$

From (3.1.5) we can derive equations for g_2, g_3 and higher order truncated correlation functions. A crucial observation is that we can expect to have a separation of orders of magnitude $f_1 \gg g_2 \gg g_3$ as $\theta^2 = \varepsilon^\beta \to 0$. To see this, we consider now the exact equations satisfied by g_2 and f_1. For ease of notation, we introduce the function σ with $\sigma(1) = 2$, $\sigma(2) = 1$, to relabel the indexes of ξ_1, ξ_2. By a straightforward algebraic computation, the BBGKY hierarchy (3.1.5) implies:

$$\partial_\tau f_1 = \varepsilon^{1-\beta} \nabla_v \cdot \left(\int \nabla\phi(x_1 - x_3)g_2(\xi_1, \xi_3) \, \mathrm{d}\xi_3 \right)$$

$$\partial_\tau g_2 + \sum_{i=1}^{2} v_i \nabla_{x_i} g_2 - \varepsilon^{1-\beta} \sum_{i=1}^{2} \int \nabla\phi(x_i - x_3)\nabla_{v_i}(f_1(\xi_i)g_2(\xi_{\sigma(i)}, \xi_3) + g_3(\xi_1, \xi_2, \xi_3)) \, \mathrm{d}\xi_3 \qquad (3.1.6)$$

$$= \varepsilon^\beta \sum_{i=1}^{2} \nabla_{v_i}(f_1(\xi_1)f_1(\xi_2) + g_2(\xi_1, \xi_2)) \nabla\phi(x_i - x_{\sigma(i)}).$$

Indeed, the sources on the right-hand side of the equation are of order $\varepsilon^\beta \ll 1$, leading us to expect

$f_1 \gg g_2$. A similar argument suggests $g_2 \gg g_3$. Therefore, we approximate (3.1.6) by:

$$\partial_\tau f_1 = \varepsilon^{1-\beta} \nabla_v \cdot \left(\int \nabla \phi(x_1 - x_3) g_2(\xi_1, \xi_3) \, \mathrm{d}\xi_3 \right) \tag{3.1.7}$$

$$\partial_\tau g_2 + \sum_{i=1}^2 v_i \nabla_{x_i} g_2 - \varepsilon^{1-\beta} \sum_{i=1}^2 \int \nabla \phi(x_i - x_3) \nabla_{v_i} f_1(\xi_i) g_2(\xi_{\sigma(i)}, \xi_3) \, \mathrm{d}\xi_3 \tag{3.1.8}$$

$$= \varepsilon^\beta \sum_{i=1}^2 \nabla_{v_i} \left(f_1(\xi_1) f_1(\xi_2) \right) \nabla \phi(x_i - x_{\sigma(i)}).$$

Since the source term on the right-hand side of (3.1.8) is of order ε^β, it is convenient to define the function $\tilde{g}_2 = \varepsilon^{-\beta} g_2$. Then we can rewrite (3.1.7)-(3.1.8) as:

$$\partial_\tau f_1 = \varepsilon \nabla_v \cdot \left(\int \nabla \phi(x_1 - x_3) \tilde{g}_2(\xi_1, \xi_3) \, \mathrm{d}\xi_3 \right) \tag{3.1.9}$$

$$\partial_\tau \tilde{g}_2 + \sum_{i=1}^2 v_i \nabla_{x_i} \tilde{g}_2 - \varepsilon^{1-\beta} \sum_{i=1}^2 \int \nabla \phi(x_i - x_3) \nabla_{v_i} f_1(\xi_i) \tilde{g}_2(\xi_{\sigma(i)}, \xi_3) \, \mathrm{d}\xi_3 \tag{3.1.10}$$

$$= \sum_{i=1}^2 \nabla_{v_i} \left(f_1(\xi_1) f_1(\xi_2) \right) \nabla \phi(x_i - x_{\sigma(i)}).$$

It is now apparent that the contribution of the integral term in (3.1.10) is negligible, and therefore that term can be dropped. Moreover, the stabilization of $\tilde{g}_2$ to a steady state takes place in times τ of order one. On the other hand, the changes in f_1 take place in times τ of order $1/\varepsilon$, suggesting we should define the macroscopic time scale as $t = \varepsilon\tau$. The separation of time scales is a key point in the argument by Bogolyubov. It implies that, on the macroscopic timescale, the truncated correlation $\tilde{g}_2(t)$ can be expected to be a functional $\tilde{g}_2(t) = A_2[\varepsilon, f_1(t)]$ of f_1. More generally, Bogolyubov argues that on the timescale t all truncated correlation functions g_k evolve in a similar adiabatic manner. This ansatz allows us to derive the limiting kinetic equation for $f_1(t)$ in a straightforward fashion. The integral term in (3.1.10) can be neglected, since it is of lower order. Therefore (3.1.9)-(3.1.10) can be approximated by ($\nabla \phi(x) = -\nabla \phi(-x)$ by radial symmetry):

$$\partial_\tau f_1 = \varepsilon \nabla_v \cdot \left(\int \nabla \phi(x_1 - x_3) \tilde{g}_2(\xi_1, \xi_3) \, \mathrm{d}\xi_3 \right)$$

$$\partial_\tau \tilde{g}_2 + \sum_{i=1}^2 v_i \nabla_{x_i} \tilde{g}_2 = \left(\nabla_{v_1} - \nabla_{v_2} \right) \left(f_1(\xi_1) f_1(\xi_2) \right) \nabla \phi(x_1 - x_2). \tag{3.1.11}$$

Now the functional $A_2[f_1]$ can be computed explicitly by solving the steady state equation for $\tilde{g}_2$ in (3.1.11). We substitute $\tilde{g}_2 = A_2[f_1]$ in the equation for f_1 and identify the Landau equation (3.1.2) as the limiting equation on the macroscopic time scale t. For the scaling limit with $\beta = 1$ in (3.1.4), the functional $A_2[f_1]$ was computed explicitly in [34], solving the steady state equation associated to (3.1.10). The resulting limit equation for $f_1(t)$ is the Balescu-Lenard equation, which will not be considered in this chapter.

It is possible to go from (3.1.11) to the Landau equation, reformulating the problem as a non-Markovian evolution. To this end, we rewrite (3.1.11) as a single equation, involving only terms depending on f_1. We can integrate the equation for $\tilde{g}_2$ along characteristics (by assumption the initial correlations vanish):

$$\tilde{g}_2(\tau, \xi_1, \xi_2) = \int_0^\tau (\nabla_{v_1} - \nabla_{v_2})(f_1(s, \xi_1)f_1(s, \xi_2))\nabla\phi(x_1 - x_2 - (\tau - s)(v_1 - v_2)) \, \mathrm{d}s.$$

We obtain a closed equation for the function f_1 by plugging this formula back into (3.1.11). The function f_1 changes on the macroscopic timescale $t = \varepsilon\tau$. In order to keep the velocities v of order one, we must change the spatial variable, using as the unit of length the mean free path, i.e. the flight length after which the velocity of a particle deviates by an amount of order one. We therefore define the macroscopic length scale $y = \varepsilon x$. Notice that due to the translation invariance of the system, $f_1(t, y, v) = f_1(t, v)$ is independent of the spatial variable. Let $f_\varepsilon(t, v)$ be the particle density function on the macroscopic timescale, then f_ε satisfies the equation

$$\partial_t f_\varepsilon = \frac{1}{\varepsilon}\nabla_v \cdot \left(\int_0^t K[f_\varepsilon(s)]\big(\frac{t - s}{\varepsilon}, v\big)\nabla f_\varepsilon(s, v) - \nabla_v \cdot K[f_\varepsilon(s)]\big(\frac{t - s}{\varepsilon}, v\big)f_\varepsilon(s, v) \, \mathrm{d}s \right) \tag{3.1.12}$$

$$f_\varepsilon(0, v) = f_0(v),$$

where K is given by the formula

$$K[f](\tau, v) := \int \int \nabla\phi(x) \otimes \nabla\phi(x - \tau(v - v'))f(v') \, \mathrm{d}v' \, \mathrm{d}x.$$

By Bogolyubov's argument, (3.1.12) should give the leading order behavior of the one particle function f_ε, which should converge to a solution f of the Landau equation (3.1.2). Note that the equation (3.1.12) yields a nonlinear non-Markovian evolution for f_ε, while f is given by a Markovian, parabolic equation. The convergence of solutions f_ε of an equation with memory effects to a kinetic equation is a characteristic feature of kinetic particle limits, as indicated in [2, 5, 49].

Notice that in the class of scaling limits (3.1.4), for $\beta = 1/2$ we obtain the classical weak coupling limit (cf. [5, 49]). In this case, the (microscopic) density Z remains of order one. Therefore, the interaction potential takes the form $\phi_\varepsilon(y) = \sqrt{\varepsilon}\phi(y/\varepsilon)$ in macroscopic variables, which has a range of order ε. The number of collisions per macroscopic unit of time is $1/\varepsilon$, and the transferred momentum produced by each collision is of order $\sqrt{\varepsilon}$. Assuming that the collisions are independent, this makes the variance of the deflections on the macroscopic time scale of order one, due to the central limit theorem. We remark that the scaling (3.1.4) is more general than the classical weak coupling, since $Z \to 0$ or $Z \to \infty$ are possible, depending on the choice of $\beta \in (0, 1)$. In these cases, the diffusion in the velocity variable also follows from an analogue of the central limit theorem. For instance if $Z \to \infty$, a particle interacts with Z particles during a macroscopic time of order ε, which yields a deflection of $\sqrt{Z\theta^4} = \sqrt{\varepsilon}$. Since the range of the potential is of order ε, these deflections become independent after macroscopic times of order ε and therefore the deflection of a particle in a macroscopic unit of time is of order one. For $Z \to 0$, the macroscopic time between collisions is $\varepsilon/Z = \varepsilon^{2\beta}$, and the deflection in each collision is ε^β. Therefore, another central limit theorem argument gives the diffusive behavior in the velocity variable.

There are multiple gaps to bridge in order to make Bogolyubov's argument rigorous. First one has to prove the well-posedness of the infinite system of ODEs (3.1.1). Sufficient conditions on the potential and initial data for this can be found, for example in [50]. Proving the separation of orders of magnitude $f_1 \gg g_2 \gg \ldots$ and the validity of the truncation of the BBGKY hierarchy is a key problem, and still open. We will see later that this assumption cannot be expected to hold in general, at least when the relative velocity of particles becomes very small.

Actually, this fact is closely related to the onset of the singularity $|v_1 - v_2|^{-1}$ in the Landau equation (cf. the term $\Lambda/|w|$ in (3.1.3)). The easiest way to understand this singularity is through a careful analysis of the mutual deflection of two particles with very close velocities, i.e. $v_1 - v_2 \approx 0$. An implicit assumption made in the derivation of the Landau equation is that the particles move along near-rectilinear trajectories. Two particles moving along near-rectilinear trajectories with velocities v_1, v_2 which come sufficiently close to interact, will interact during a collision time of order $|v_1 - v_2|^{-1}$. Hence, the resulting deflection is of order $\theta^2 |v_1 - v_2|^{-1}$. If $|v_1 - v_2| \ll \theta^2$, this quantity is not small, and this contradicts the assumption of near-rectilinear motion. Therefore the underlying assumption behind the derivation of the Landau equation breaks down for particles with very small relative velocity. Nevertheless, if the velocities satisfy the condition $1 \gg |v_1 - v_2| \gg \theta^2$, the rectilinear approximation is valid, in spite of the fact that the collision time diverges like $|v_1 - v_2|^{-1}$. This is the reason for the onset of the factor $1/|w|$ in (3.1.3).

We remark that the introduction of this singularity does not pose a serious physical difficulty concerning the validity of the Landau equation, since it is an integrable singularity. This is due to the fact that the number of pairs of particles with small relative velocities is a sufficiently small fraction of the total number of pairs of interacting particles, and therefore can be neglected. In particular, the fraction of interacting particles with $|v_1 - v_2| \ll \theta^2$ which experience relevant deflections in their collisions vanishes in the limit $\theta \to 0$.

We emphasize that the singularity $1/|w|$ appearing in the diffusion matrix in the Landau equation (cf. (3.1.3)) is a consequence of the collision dynamics of particles with small relative velocity, and therefore independent of the particular choice of the interaction potential ϕ. In particular this singularity is not specifically related to the choice of the Coulomb interaction between the particles. It is interesting to point out the difference with the Boltzmann equation, where the homogeneity of the collision kernel is closely related to the homogeneity of the interaction potential (cf. [56]).

We notice that the assumption $f_1 \gg g_2$ can be expected to fail in the region of very small relative velocities due to the same geometric considerations as above (cf. [5]). Indeed, the function $g_2(x_1, v_1, x_2, v_2)$ measures the deflections of interacting particles with velocities v_1, v_2. For small relative velocities, the truncated correlation function g_2 can be of the same order as f_1. It is worth remarking that dropping the term g_2 on the right-hand side of (3.1.5) is equivalent to approximating the trajectories of interacting particles by straight lines. As seen before, this fails in the region $|v_1 - v_2| \ll \theta^2$, which is vanishing in the limit $\theta \to 0$. Notice that this observation yields some insight into the type of functional spaces in which the approximation $f_1 \gg g_2$ can be expected to hold.

In this chapter, we prove that Bogolyubov's adiabatic approach to deriving the Landau equation (3.1.2) from the system (3.1.12) is indeed correct, when the singularity $v \approx v'$ is cut out. To be

precise, we consider the Landau-type equation

$$\partial_t f = \sum_{i,j=1}^{3} \partial_{v_i} \left(\int_{\mathbb{R}^3} a_{i,j}(v - v')(\partial_{v_j} - \partial_{v'_j})\big(f(t,v)f(t,v')\big)\eta(|v - v'|^2)\, \mathrm{d}v' \right)$$
$$f(0,v) = f_0(v),$$

(3.1.13)

where $\eta(r)$ vanishes for r small. We will derive the equation (3.1.13) from the system (3.1.12), where K is now given by:

$$K[f](t,v) := \int \int \nabla\phi(x) \otimes \nabla\phi(x - t(v - v'))f(v')\eta(|v - v'|^2)\, \mathrm{d}v'\, \mathrm{d}x.$$

(3.1.14)

The reason for introducing the artificial cutoff $\eta(r)$ in the region of small relative velocity is that the estimates in this work are presently not strong enough to deal with the case $\eta \equiv 1$. As indicated above, the effect of collisions with small relative velocities can be expected to be small, and therefore the Landau equation (cf. (3.1.2), (3.1.3)) and the modified Landau equation (cf. (3.1.13), (3.1.14)) might be expected to exhibit similar physical properties. In particular, the asymptotic behavior of the matrix $K[f](t,v)$ as $v \to \infty$ is preserved.

The main results of this chapter are the existence of strong solutions f_ε to (3.1.12) with K as in (3.1.14), and the convergence of these solutions to a strong solution f of the Landau equation (3.1.13) for macroscopic times of order one. We assume that f_0 is close to the Maxwellian steady state of the limit equation and choose a particular short range potential ϕ. In contrast to the diffusive, parabolic Landau equation, equation (3.1.12) is hyperbolic. We show that regularity and decay of the initial datum f_0 are conserved. Furthermore, the evolution given by (3.1.12) is clearly non-Markovian, since the time derivative depends on the whole history of the function f_ε until time t. In the limit $\varepsilon \to 0$, this memory effect disappears and we recover the Markovian dynamics of the Landau equation.

As mentioned above, the derivation of the Landau-type equations from particle systems is still largely open. The linear Landau equation has been derived in [4, 14] as a scaling limit of systems with a single particle traveling through a random (but fixed) configuration of scatterers.

Furthermore, it is shown in [5] that the Landau equation (3.1.2) is consistent with a scaling limit of interacting particle systems. More precisely it is shown that the time derivative of the macroscopic density of particles in the weak coupling limit at $t = 0$ is correctly predicted by the Landau equation. The technique follows a similar line of reasoning to that of Bogolyubov, truncating the BBGKY hierarchy to a system like (3.1.12), and proving convergence to the Landau equation on a timescale shorter than the macroscopic. It is worth noticing, that in [5] the convergence of solutions of the truncated hierarchies to the solution of the the Landau equation is established in the sense of weak convergence. In this chapter, the convergence of the solutions u_ε of the non-Markovian problem (3.1.12) to the solution u of the Landau-type equation (3.1.13) is proved in strong norms, up to macroscopic times of order one. Given that estimates in stronger norms, which allow for strong convergence, are technical to obtain, it is natural to ask why this is needed. The reason for this is that our technique for controlling the nonlinearity in (3.1.12) up to macroscopic times of order one is based on a linearization of the problem in strong norms, combined with estimates of quadratic or higher order terms. This is only possible in very strong norms that in particular yield estimates

for the time derivative of the solution. It is certainly possible to prove the convergence $u_\varepsilon \to u$ in the weak topology. However, since stronger estimates were needed to prove well-posedness of the non-Markovian problem up to macroscopic times, the convergence is readily established in stronger norms.

On the other hand, convergence of the solutions of the non-Markovian evolution to the solution of the Landau equation in weak topology, as used in [5], would be in some sense the natural result, considering that the solutions of the non-Markovian equation exhibit significant changes on the microscopic time scale. Indeed, one important assumption made in this work is that the initial data for (3.1.12), i.e. the initial distribution of particles, is close to a Maxwellian equilibrium. This smallness condition is needed in order to control the effect of these oscillations on the macroscopic evolution of the one particle function.

In [24], global well-posedness of the spatially inhomogeneous Landau equation was proved for initial data close to equilibrium in a periodic box. Lower bounds on the entropy dissipation in the Landau equation can be found in [11]. A concept of weak solutions for the homogeneous Landau equation (3.1.2), namely H-solutions, was introduced in [57]. This paper also gives sufficient conditions under which the Landau equation can be obtained as a grazing collision limit, taking as a starting point the Boltzmann equation. In the grazing collision limit, the collision kernel in the Boltzmann equation is concentrated on the set of collisions with small transferred momentum. The Landau equation has also been derived from the Boltzmann equation in the grazing collision limit in the spatially inhomogeneous case (cf. [1]).

Given that the paper [24] proves global well-posedness for the Landau equation near the Gaussian distribution in the spatially inhomogeneous case, it is natural to ask why such that a result cannot be obtained for the non-Markovian equation (3.1.12). To explain this we describe the analogies and differences between the approach in [24] and that of this work.

The approach of [24] is based in a linearization near the Maxwellian distribution of velocities. A dissipation formula allows one to obtain global estimates for the difference between the solutions of the inhomogeneous Landau equation and the Maxwellian, that can be used to prove global stability results. In this work, we consider the equation (3.1.12), which unlike the Landau equation is non-Markovian and, due to this, not pointwise dissipative in time. The techniques used here are more reminiscent of the theory of symmetric hyperbolic systems ([25], [38]), which usually only yields local well-posedness in time, due to the fact that quadratic or higher order terms must be estimated. We generalize these methods to the case of a non-Markovian evolution with memory effects. The key ingredient in our approach is the derivation of a coercivity estimate averaged in time (cf. Lemma 3.3.7) for the solutions obtained with $f_\varepsilon = f_0$ frozen inside the operator $K[f_\varepsilon(s)]$ on the right-hand side of (3.1.12). Our proof strategy for Lemma 3.3.7 does not rule out solutions of the linearized problem which separate exponentially from the initial distribution function. The estimates in Lemma 3.3.7 are based on a Laplace transform argument and the derivation of estimates for some elliptic equations with complex coefficients, where the Laplace transform argument z remains at a positive distance from the imaginary axis. Obtaining global-in-time estimates for solutions of (3.1.12) would require us to prove that coercivity still holds for the operator linearized around the Maxwellian, even when the complex parameter z approaches to the imaginary axis. Such an estimate might be true, but seems to require more involved arguments than the ones presented here. Notice that a coercivity estimate strong enough to provide decay of the perturbations with respect to the Maxwellian for

long times might be easier to obtain in a compact domain (for instance a torus) than in the whole space.

Due to the mathematical difficulties arising from the singularity $|v_1 - v_2|^{-1}$ for relative velocities in the Landau equation (3.1.2)-(3.1.3), a number of Landau-type equations, in which the singularity has been weakened, have been studied. As for our modification of the Landau equation (3.1.13), these equations cannot be directly derived as a scaling limit of interacting particle systems (3.1.1). These modified Landau equations are obtained by replacing the singularity $|v - v'|^{-1}$ by $|v - v'|^{\gamma+2}$. The well-posedness of these equations, as well as stability of Maxwellians and the dissipation of entropy have been studied in [15, 16, 48, 53, 58].

The chapter is structured as follows: In Section 3.2, we give a precise formulation of the main results Theorem 3.2.6 and 3.2.8, as well as the proofs of some auxiliary results. In Section 3.3 we prove the result in the linear case. Section 3.4 proves that the a priori estimates are stable under certain small perturbations, and that these smallness assumptions are conserved by the equation. In Section 3.5 we give the proofs of the two main theorems.

3.2 Main results, notation and auxiliary lemmas

3.2.1 Formulation of the main results

Our goal is to prove the existence of a strong solution to the equation

$$
\begin{aligned}
\partial_t u_\varepsilon =& \frac{1}{\varepsilon} \nabla_v \cdot \left(\int_0^t K[u_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s, v) \, \mathrm{d}s \right) \\
&- \frac{1}{\varepsilon} \nabla_v \cdot \left(\int_0^t P[u_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s, v) \, \mathrm{d}s \right) \\
u_\varepsilon(0, v) =& u_0(v),
\end{aligned}
\tag{3.2.1}
$$

where K and P denote the following operators:

$$
\begin{aligned}
K, P : W^{1,1}(\mathbb{R}^3) &\longrightarrow L^\infty(\mathbb{R}^+ \times \mathbb{R}^3) \\
K[u](t, v) &:= \int \int \nabla \phi(x) \otimes \nabla \phi(x - t(v - v')) u(v') \eta(|v - v'|^2) \, \mathrm{d}v' \, \mathrm{d}x \\
P[u](t, v) := \nabla_v \cdot K(t, v) &= \int \int \nabla \phi(x) \otimes \nabla \phi(x - t(v - v')) \nabla u(v') \eta(|v - v'|^2) \, \mathrm{d}v' \, \mathrm{d}x.
\end{aligned}
\tag{3.2.2}
$$

We will specify the potential ϕ and the cutoff function $\eta \in C^\infty(\mathbb{R})$ below. Formally, as $\varepsilon \to 0$, the functions u_ε converge to a strong solution u of:

$$
\begin{aligned}
\partial_t u &= \nabla \cdot (\mathcal{K}[u] \nabla u) - \nabla \cdot (\mathcal{P}[u] u) \\
u(0, v) &= u_0(v) \\
\mathcal{K}[u](v) &= \frac{\pi^2}{4} \int (k \otimes k) |\hat{\phi}(k)|^2 \delta(k \cdot (v - v')) \eta(|v - v'|^2) u(v') \, \mathrm{d}k \, \mathrm{d}v' \\
\mathcal{P}[u](v) &= \frac{\pi^2}{4} \int (k \otimes k) |\hat{\phi}(k)|^2 \delta(k \cdot (v - v')) \eta(|v - v'|^2) \nabla u(v') \, \mathrm{d}k \, \mathrm{d}v'.
\end{aligned}
\tag{3.2.3}
$$

We will prove this result for u_0 close to the Maxwellian distribution m, which is the steady state of the limit equation (3.2.3). Furthermore we choose the potential ϕ to have a particular form, making the computations considerably easier.

Notation 3.2.1. *Let $\eta \in C^\infty(\mathbb{R})$ be a fixed cutoff function with $0 \le \eta \le 1$, $\eta(r) = 1$ for $|r| \ge \kappa$ and $\eta(r) = 0$ for $|r| \le \frac{\kappa}{2}$ for some $\frac{1}{2} > \kappa > 0$ that we will not further specify in the following analysis. We choose the potential $\phi(x)$ to be given by*

$$\phi(x) = \sqrt{\frac{2}{\pi}} K_0(|x|), \tag{3.2.4}$$

where K_0 is the modified Bessel function of second type.

Remark 3.2.2. *The potential ϕ is monotone decreasing, decays exponentially at infinity and diverges logarithmically at the origin. Our approach also seems to work for other potentials with analogous properties, but becomes significantly less technical with this particular choice. The Fourier transform of the potential is given by:*

$$\hat{\phi}(k) = \frac{1}{(1 + |k|^2)^{\frac{3}{2}}}. \tag{3.2.5}$$

The function spaces we are going to work with in the forthcoming analysis are the following ones.

Definition 3.2.3. *Let $\lambda(v), \tilde{\lambda}(v)$ be the weight functions given by $\lambda(v) := e^{|v|}$, $\tilde{\lambda}(v) := \frac{e^{|v|}}{1+|v|}$. For $n \in \mathbb{N}$ and $\nu = \lambda, \tilde{\lambda}$, we define the weighted Sobolev space H_ν^n as the closure of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm:*

$$\|u\|_{H_\nu^n}^2 := \sum_{\alpha \in \mathbb{N}^3, |\alpha| \le n} \|\nu^{\frac{1}{2}}(\cdot) \nabla^\alpha u(\cdot)\|_{L^2}^2. \tag{3.2.6}$$

In the case $n = 0$ we also write $H_\nu^n = L_\nu^2$. For functions $f(t, v)$ with an additional time dependence, we define the spaces $V_{A,\nu}^n$ as the closure of $C_c^\infty([0, \infty) \times \mathbb{R}^3; \mathbb{R}^d)$ with respect to:

$$\|f\|_{V_{A,\nu}^n}^2 := \int_0^\infty e^{-At} \sum_{j=1}^d \|f_j(t, \cdot)\|_{H_\nu^n}^2 \, dt, \quad \text{where } A \ge 1. \tag{3.2.7}$$

Let $X_{A,\nu}^n$ be the function space given by:

$$X_{A,\nu}^n := \{(f, g) \in V_{A,\nu}^n \times V_{A,\nu}^{n-1} : f = \nabla \cdot g, \, \text{supp } f, g \subset [0, 1] \times \mathbb{R}^3\},$$
$$\text{with norm } \|(f, g)\|_{X_{A,\nu}^n} := \|f\|_{V_{A,\nu}^n} + \|g\|_{V_{A,\nu}^{n-1}}. \tag{3.2.8}$$

For $u = (f, g) \in X_{A,\nu}^n$ we write $\partial_t u = (\partial_t f, \partial_t g)$ whenever the right-hand side is well-defined.

Remark 3.2.4. *The validity of our analysis is not subject to the choice of the particular exponent in the weight function, and weights of the form $\lambda_c(v) = e^{c|v|}$ or fast power law decay would work equally well.*

The choice of the weight functions $\lambda, \tilde{\lambda}$ is motivated by the following compactness property, that we will later use to prove the existence of fixed points.

Lemma 3.2.5. *Let* $(u_i)_{i\in\mathbb{N}} = ((f_i, g_i))_{i\in\mathbb{N}} \subset X_{A,\lambda}^{n+1}$ *be a bounded sequence, such that the sequence* $(\partial_t f_i, \partial_t g_i) \in X_{A,\lambda}^{n+1}$ *is bounded as well. Then the sequence* (u_i) *is precompact in* $X_{A,\tilde{\lambda}}^n$.

Proof. For some $C > 0$ there holds $\|(f_i, g_i)\|_{X_{A,\lambda}^{n+1}} + \|(\partial_t f_i, \partial_t g_i)\|_{X_{A,\lambda}^{n+1}} \leq C$. Denote by $(\varphi_R)_{R>0} \in C_c^\infty$ a standard sequence of cutoff functions that is one on B_R and vanishes outside of B_{R+1}. We construct a convergent subsequence $u_{\ell(k)}$ inductively. The region $[0, 1] \times B_{R+1}$ is compact, so by Rellich's theorem the sequences $(f_i \varphi_1)$, $(g_i \varphi_1)$ have convergent subsequences $f_{\ell_1(i)} \varphi_1 \to F_1$, $g_{\ell_1(i)} \varphi_1 \to G_1$ in $V_{A,\lambda}^n$ and $V_{A,\lambda}^{n-1}$ respectively. Since $V_{A,\lambda}^{n,d} \hookrightarrow V_{A,\tilde{\lambda}}^{n,d}$ embed continuously (actually Lipschitz with constant $L \leq 1$), the sequences are also convergent in the latter spaces. Now we inductively extract further convergent subsequences $f_{\ell_k(i)} \varphi_k \to F_k$ and $g_{\ell_k(i)} \varphi_k \to G_k$. By construction we have $F_m = F_k$, $G_m = G_k$ on B_k for $m \geq k$. We pick a sequence $u_{\ell(k)}$ such that:

$$\|f_{\ell(k)} \varphi_k - F_k\|_{V_{A,\tilde{\lambda}}^n} + \|g_{\ell(k)} \varphi_k - G_k\|_{V_{A,\tilde{\lambda}}^{n-1}} \leq \frac{1}{k}.$$

The sequences $f_{\ell(k)}$, $g_{\ell(k)}$ are Cauchy sequences in $V_{A,\tilde{\lambda}}^n$ and $V_{A,\tilde{\lambda}}^{n-1}$ respectively. To see this, take $i, j \geq k$ and bound:

$$\|f_{\ell(i)} - f_{\ell(j)}\|_{V_{A,\tilde{\lambda}}^n} \leq \|(f_{\ell(i)} - f_{\ell(j)})\varphi_k\|_{V_{A,\tilde{\lambda}}^n} + \|(f_{\ell(i)} - f_{\ell(j)})(1 - \varphi_k)\|_{V_{A,\tilde{\lambda}}^n}$$
$$\leq \frac{2}{k} + \frac{1}{k}\|(f_{\ell(i)} - f_{\ell(j)})(1 - \varphi_k)\|_{V_{A,\lambda}^n} \longrightarrow 0,$$

where we have used that $\tilde{\lambda}(v) \leq \frac{1}{|k|}\lambda(v)$ for $|v| \geq k$. Hence $f_{\ell(k)}$ is a Cauchy sequence. The proof for $g_{\ell(k)}$ is similar. Therefore $u_{\ell(k)}$ is precompact in $X_{A,\tilde{\lambda}}^n$. $\square$

We can now formulate the precise statement for the existence of solutions u_ε of (3.2.1) and convergence to a solution of the nonlinear Landau equation (3.2.3).

Theorem 3.2.6. *Let* $m_0, \sigma > 0$ *and* $m(\sigma^2, m_0)$ *be the Maxwellian with mass* m_0 *and standard deviation* σ:

$$m(\sigma^2, m_0)(v) := m_0 \frac{e^{-\frac{1}{2}\frac{|v|^2}{\sigma^2}}}{(\sigma\sqrt{2\pi})^3}. \tag{3.2.9}$$

Let $n \geq 6$ *and* $v_0 \in H_\lambda^n$ *satisfy:*

$$0 \leq v_0(v) \leq Ce^{-\frac{1}{2}|v|}.$$

There exist $A, C(A) > 0$, $\delta_1, \varepsilon_0 \in (0, \frac{1}{2}]$ *such that for all* $\varepsilon, \delta_2 \in (0, \varepsilon_0] > 0$ *the equation*

$$\partial_t u_\varepsilon = \frac{1}{\varepsilon}\nabla \cdot \left(\int_0^t K[u_\varepsilon(s)]\left(\frac{t-s}{\varepsilon}, v\right)\nabla u_\varepsilon(s, v) \, \mathrm{d}s\right)$$
$$- \frac{1}{\varepsilon}\nabla \cdot \left(\int_0^t P[u_\varepsilon(s)]\left(\frac{t-s}{\varepsilon}, v\right) u_\varepsilon(s, v) \, \mathrm{d}s\right) \tag{3.2.10}$$
$$u_\varepsilon(0, \cdot) = u_0(\cdot) = m(v) + \delta_2 v_0(v)$$

has a strong solution $u_\varepsilon \in V_{A,\lambda}^n \cap C^1([0,\delta_1]; H_\lambda^{n-2})$ up to time δ_1 with uniform bound:

$$\|u_\varepsilon\|_{V_{A,\lambda}^n} + \|\partial_t u_\varepsilon\|_{V_{A,\lambda}^{n-2}} \leq C(A). \tag{3.2.11}$$

Remark 3.2.7. *Our result is valid for small initial perturbations $u_0 + \delta_2 v_0$ of the Maxwellian and small times $0 \leq t \leq \delta_1$. Notice that the functions u_ε are solutions to (3.2.10) up to time δ_1, but are defined also for later times. In the following, we will write $C, c > 0$ for generic large/small constants that are not dependent on other parameters.*

Theorem 3.2.8. *For $n \geq 6$ pick $A \geq 1$, $\delta_1 \in (0, \frac{1}{2}]$ and ε, δ_2 small enough such that Theorem 3.2.6 ensures the existence of solutions $u_\varepsilon \in V_{A,\lambda}^n \cap C^1([0,\delta_1]; H_\lambda^{n-2})$ of (3.2.10). Along a sequence $\varepsilon_j \to 0$ the u_{ε_j} converge $u_{\varepsilon_j} \to u$ in $V_{A,\tilde{\lambda}}^{n-3}$, $u_{\varepsilon_j} \rightharpoonup u$ in $V_{A,\lambda}^n$, $\partial_t u_{\varepsilon_j} \rightharpoonup \partial_t u$ in $V_{A,\lambda}^{n-2}$. The function $u \in V_{A,\lambda}^n \cap C^1([0,\delta_1]; H_\lambda^{n-4})$ solves the limit equation up to times $0 \leq t \leq \delta_1$:*

$$\partial_t u = \nabla \cdot (\mathcal{K}[u]\nabla u) - \nabla \cdot (\mathcal{P}[u]u)$$

$$u(0,v) = m(v) + \delta_2 v_0(v)$$

$$\mathcal{K}[u](v) = \frac{\pi^2}{4} \int (k \otimes k)|\hat{\phi}(k)|^2 \delta(k \cdot (v - v'))\eta(|v - v'|^2)u(v')\, \mathrm{d}k\, \mathrm{d}v' \tag{3.2.12}$$

$$\mathcal{P}[u](v) = \frac{\pi^2}{4} \int (k \otimes k)|\hat{\phi}(k)|^2 \delta(k \cdot (v - v'))\eta(|v - v'|^2)\nabla u(v')\, \mathrm{d}k\, \mathrm{d}v'.$$

In order to show the existence of a strong solution to (3.2.10), we will consider mollifications of the equations first, and derive a priori estimates that are independent of the mollification. We introduce the following notation.

Notation 3.2.9. *Let φ_γ be a standard mollifier on $\mathbb{R}^3$. For $0 < \gamma \leq 1$, define the regularized gradient $^\gamma\nabla$ as $^\gamma\nabla f(v) := \nabla(\varphi_\gamma * f)$. We define $^\gamma\nabla$ to be the standard gradient for $\gamma = 0$. We will use the following conventions for Laplace transform and Fourier transform:*

$$\mathcal{L}(u)(z) = \int_0^\infty u(t)e^{-zt}\, \mathrm{d}t \tag{3.2.13}$$

$$\hat{u}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} u(v)e^{-ik\cdot v}\, \mathrm{d}v. \tag{3.2.14}$$

Now we observe that if $u_\varepsilon = u_0 + f_\varepsilon$ is a solution of (3.2.10), an equivalent way of stating this is

$$\partial_t u_\varepsilon = \frac{1}{\varepsilon}{}^\gamma\nabla \cdot \left(\int_0^t K[u_0 + f_\varepsilon(s,\cdot)]\left(\frac{t-s}{\varepsilon}, v\right){}^\gamma\nabla u_\varepsilon(s,v)\, \mathrm{d}s \right)$$

$$\quad - \frac{1}{\varepsilon}{}^\gamma\nabla \cdot \left(\int_0^t P_\gamma[u_0 + f_\varepsilon(s,\cdot)]\left(\frac{t-s}{\varepsilon}, v\right) u_\varepsilon(s,v)\, \mathrm{d}s \right) \tag{3.2.15}$$

$$u_\varepsilon(0,\cdot) = u_0(\cdot), \quad P_\gamma = {}^\gamma\nabla \cdot K, \quad K \text{ as defined in (3.2.2)}$$

holds for $\gamma = 0$. We will show a priori estimates for the above equation for $0 < \gamma \leq 1$ and later recover the case $\gamma = 0$ as a limit. We start our analysis by writing K and P in a more convenient form.

Lemma 3.2.10. *The operator K defined in (3.2.2) and $P_\gamma = {}^\gamma\nabla\cdot K$ can be expressed by the formulas:*

$$K[u](t,v) = \int (k \otimes k)|\hat{\phi}(k)|^2 \cos(t(v-v')\cdot k)\eta(|v-v'|^2)u(v')\,\mathrm{d}k\,\mathrm{d}v' \tag{3.2.16}$$

$$P_\gamma[u](t,v) = \int (k \otimes k)|\hat{\phi}(k)|^2 \cos(t(v-v')\cdot k)\eta(|v-v'|^2)^\gamma\nabla u(v')\,\mathrm{d}k\,\mathrm{d}v'. \tag{3.2.17}$$

Proof. The formula for P_γ follows from the one for K, so we only prove this one. Plancherel's theorem allows to rewrite:

$$K[u](t,v) = \int \nabla\phi(x) \otimes \nabla\phi(x - t(v-v'))u(v')\eta(|v-v'|^2)\,\mathrm{d}v'\,\mathrm{d}x$$
$$= \int (k \otimes k)|\hat{\phi}(k)|^2 e^{-itk\cdot(v-v')}u(v')\eta(|v-v'|^2)\,\mathrm{d}v'\,\mathrm{d}k.$$

Since K only takes real values, we can symmetrize the exponential and obtain

$$\int (k \otimes k)|\hat{\phi}(k)|^2 e^{-itk\cdot(v-v')}u(v')\eta(|v-v'|^2)\,\mathrm{d}v'\,\mathrm{d}k$$
$$= \int (k \otimes k)|\hat{\phi}(k)|^2 \cos\big(tk\cdot(v-v')\big)u(v')\eta(|v-v'|^2)\,\mathrm{d}v'\,\mathrm{d}k,$$

proving the claim. $\qquad\square$

We will omit the index $\gamma \geq 0$ in notation, when there is no risk of confusion. Controlling the nonlinearity inside K and P strongly relies on being able to bound spatial derivatives of u_ε. Therefore we consider differentiations of the equation. Let $\alpha \in \mathbb{N}^3$ be a multi-index. With the convention $\binom{\alpha}{\beta} = \prod_{j=1}^{3}\binom{\alpha_j}{\beta_j}$, the function $D^\alpha u_\varepsilon = \frac{\partial^\alpha u_\varepsilon}{\partial v_1^{\alpha_1}\partial v_2^{\alpha_2}\partial v_3^{\alpha_3}}$ (formally) satisfies the equation:

$$\partial_t D^\alpha u_\varepsilon = \sum_{\beta_1+\beta_2=\alpha}\binom{\alpha}{\beta_1}\frac{1}{\varepsilon}\left(\nabla\cdot\left(\int_0^t D^{\beta_1}KD^{\beta_2}\nabla u_\varepsilon\,\mathrm{d}s\right) - \nabla\cdot\left(\int_0^t D^{\beta_1}PD^{\beta_2}u_\varepsilon\,\mathrm{d}s\right)\right).$$

In order to have a short notation for the terms appearing on the right-hand side of the equation above, we introduce the following notation.

Notation 3.2.11. *Let $n \in \mathbb{N}$ and α, β be multi-indices with $\beta \leq \alpha$, $|\alpha| \leq n-1$ and $\nu, u_\varepsilon \in V_{A,\tilde{\lambda}}^n$. For $\gamma \in (0,1]$ we define:*

$$\begin{aligned}\mathcal{A}_\gamma^{\alpha,\beta}[\nu](u_\varepsilon) =&\frac{1}{\varepsilon}\left(\int_0^t D^\beta K[\nu(s)]\left(\frac{t-s}{\varepsilon},v\right){}^\gamma\nabla D^{\alpha-\beta}u_\varepsilon(s,v)\,\mathrm{d}s\right)\\ &-\frac{1}{\varepsilon}\left(\int_0^t D^\beta P_\gamma[\nu(s)]\left(\frac{t-s}{\varepsilon},v\right)D^{\alpha-\beta}u_\varepsilon(s,v)\,\mathrm{d}s\right).\end{aligned} \tag{3.2.18}$$

Furthermore, for $m \in \mathbb{N}$, $u \in V_{A,\lambda}^m$, we set:

$$|u|_{F^m}(z,v) := \sum_{|\beta|\leq m}|\mathcal{L}(D^\beta u)(z,v)|. \tag{3.2.19}$$

The equation (3.2.15) has an averaged in time coercivity property, which we will prove by showing nonnegativity for certain quadratic functionals Q. This allows to show that u_ε inherits decay and regularity properties from the initial datum. We have the following basic a priori estimate for solutions u_ε of (3.2.15):

Lemma 3.2.12. *Let $n \in \mathbb{N}$, $A, \varepsilon, \gamma > 0$ and $u_\varepsilon \in C^1([0,T]; H_\lambda^n)$ be a solution to (3.2.15) for $T > 0$ arbitrary. Then for $|\alpha| \leq n$ we can bound:*

$$A \int_0^T \int \lambda(v) |D^\alpha u_\varepsilon(t,v)|^2 e^{-At} \, \mathrm{d}t \, \mathrm{d}v \leq -2 Q_{\varepsilon,A}^\alpha [u_0 + f_\varepsilon](u_\varepsilon \mathbb{1}_{[0,T]}) + \|\lambda^{\frac{1}{2}} D^\alpha u_0\|_{L^2}^2.$$

Here $Q_{\varepsilon,A}^\alpha[\nu](u)$ is given by (we drop the index γ if there is no risk of confusion):

$$Q_{\varepsilon,A}^\alpha[\nu](u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} Q_{\varepsilon,A}^{\alpha,\beta}[\nu](u) \tag{3.2.20}$$

$$Q_{\varepsilon,A}^{\alpha,\beta}[\nu](u) = \int_0^\infty \int \frac{e^{-At}}{\varepsilon}^\gamma \nabla(D^\alpha u(t)\lambda) \int_0^t D^{\alpha-\beta} K[\nu(s)](\frac{t-s}{\varepsilon})^\gamma \nabla D^\beta u(s) \, \mathrm{d}s \, \mathrm{d}v \, \mathrm{d}t \tag{3.2.21}$$

$$- \int_0^\infty \int \frac{e^{-At}}{\varepsilon}^\gamma \nabla(D^\alpha u(t)\lambda) \int_0^t D^{\alpha-\beta} P_\gamma[\nu(s)](\frac{t-s}{\varepsilon}) D^\beta u(s) \, \mathrm{d}s \, \mathrm{d}v \, \mathrm{d}t. \tag{3.2.22}$$

Proof. Follows by a simple computation:

$$A \int_0^T \int_{\mathbb{R}^3} \lambda(v)|D^\alpha u_\varepsilon(t,v)|^2 e^{-At} \, \mathrm{d}t \, \mathrm{d}v$$

$$= -\int_0^T \int_{\mathbb{R}^3} \lambda(v) D^\alpha u_\varepsilon(t,v)^2 \partial_t(e^{-At}) \, \mathrm{d}t \, \mathrm{d}v$$

$$\leq 2 \int_0^T \int_{\mathbb{R}^3} \lambda(v) D^\alpha u_\varepsilon(t,v) \partial_t D^\alpha u_\varepsilon(t,v) e^{-At} \, \mathrm{d}t \, \mathrm{d}v + \int \lambda(v)|D^\alpha u_0|^2 \, \mathrm{d}v$$

$$= -2 Q_{\varepsilon,A}^\alpha[u_0 + f_\varepsilon](u_\varepsilon \cdot \mathbb{1}_{[0,T]}) + \|\lambda^{\frac{1}{2}} D^\alpha u_0\|_{L^2}^2,$$

where in the last line the equation is used. $\qquad \square$

The following analogue of Plancherel's theorem for Laplace transforms will be useful throughout this chapter.

Lemma 3.2.13. *Let $\mu_A(\mathrm{d}t) := e^{-At} \, \mathrm{d}t$. Then for $u,v \in L^2(\mu_A)$ we have:*

$$(2\pi)^{\frac{1}{2}} \int_0^\infty e^{-At} \overline{u}(t) v(t) \mu_A(\mathrm{d}t) = \int_{\mathbb{R}} \overline{\mathcal{L}(u)}\left(\frac{A}{2} + i\omega\right) \mathcal{L}(v)\left(\frac{A}{2} + i\omega\right) \, \mathrm{d}\omega.$$

Our proof strongly relies on the geometry of both complex and real vectors. To avoid confusion we introduce the following notation.

Definition 3.2.14. *For $v, w \in \mathbb{R}^3$ we will use the notation $v \cdot w = \sum_i v_i w_i$ for the Euclidean scalar product. The inner product of complex vectors $V, W \in \mathbb{C}^3$ we denote by $\langle V, W \rangle = \sum_i \overline{V}_i W_i$. We*

will use the notation $|\cdot|$ for the vector norms induced by each of the inner products, as well as the matrix norm induced by this norm. Moreover for $0 \neq V \in \mathbb{C}^3$ and $W \in \mathbb{C}^3$ we define the orthogonal projections $P_V W$ and $P_V^\perp W$ as:

$$P_V W := \left(\frac{\langle V, W \rangle}{|V|} \right) \frac{V}{|V|}, \quad P_V^\perp W := W - P_V W. \tag{3.2.23}$$

For future reference, we compute the Laplace transform of $K[u](t, v)$ in t. With our particular choice of potential, some of the integrals are explicitly computable, as is stated in the following auxiliary Lemma.

Lemma 3.2.15. *For $\Re(z) \geq 0$, $v \in \mathbb{R}^3$ let $M_1(z, v), M_2(z, v)$ be the matrix-valued functions defined by*

$$M_1(z, v) := \frac{\pi^2}{4|v|} \frac{1}{1 + \frac{z}{|v|}} P_v^\perp, \quad M_2(z, v) := \frac{\pi^2}{4|v|} \frac{\frac{z}{|v|}}{(1 + \frac{z}{|v|})^2} P_v. \tag{3.2.24}$$

Then we have the following identity:

$$\int (k \otimes k) |\hat{\phi}(k)|^2 \frac{z}{z^2 + (k \cdot v)^2} \, \mathrm{d}k = M_1(z, v) + M_2(z, v). \tag{3.2.25}$$

Proof. We decompose $k \in \mathbb{R}^3$ into $k = uw + w^\perp$, where $w = \frac{v}{|v|}$. We insert the explicit form of the Fourier transform of ϕ (cf. (3.2.5)) to rewrite the integral as (here $a^{\otimes 2} = a \otimes a$):

$$\int (k \otimes k) |\hat{\phi}(k)|^2 \frac{z}{z^2 + (k \cdot v)^2} \, \mathrm{d}k = \int_{\mathbb{R}} \int_{\mathrm{span}(w)^\perp} \frac{(uw + w^\perp)^{\otimes 2}}{(1 + u^2 + |w^\perp|^2)^3} \, \mathrm{d}w^\perp \frac{z}{z^2 + (u|v|)^2} \, \mathrm{d}u$$

$$= \frac{1}{|v'|} \int_{\mathbb{R}} \int_{\mathrm{span}(w)^\perp} \frac{(uw + w^\perp)^{\otimes 2}}{(1 + u^2 + |w^\perp|^2)^3} \, \mathrm{d}w^\perp \frac{\frac{z}{|v|}}{(\frac{z}{|v|})^2 + u^2} \, \mathrm{d}u$$

$$= \frac{1}{|v|} \int_{\mathbb{R}} \int_{\mathrm{span}(w)^\perp} \frac{((uw)^{\otimes 2} + (w^\perp)^{\otimes 2})}{(1 + u^2 + |w^\perp|^2)^3} \, \mathrm{d}w^\perp \frac{\frac{z}{|v'|}}{(\frac{z}{|v|})^2 + u^2} \, \mathrm{d}u,$$

where we used that the mixed terms $uw \otimes w^\perp$ do not contribute to the integral due to the symmetry of the integrand. Now the inner integral is explicit:

$$\int_{\mathrm{span}(w)^\perp} \frac{((uw)^{\otimes 2} + (w^\perp)^{\otimes 2})}{(1 + u^2 + |w^\perp|^2)^3} \, \mathrm{d}w^\perp = u^2 \int_0^\infty \frac{2\pi r P_w}{(1 + u^2 + r^2)^3} \, \mathrm{d}r + \int_0^\infty \frac{\pi r^3 P_w^\perp}{(1 + u^2 + r^2)^3} \, \mathrm{d}r$$

$$= \frac{\pi u^2}{2(1 + u^2)^2} P_w + \frac{\pi}{4(1 + u^2)} P_w^\perp.$$

Inserting this back into the full integral gives two explicit integrals:

$$\frac{1}{|v|} \int_{\mathbb{R}} \int_{\text{span}(w)^\perp} \frac{((uw)^{\otimes 2} + (w^\perp)^{\otimes 2})}{(1 + u^2 + |w^\perp|^2)^3} \, \mathrm{d}w^\perp \frac{\frac{z}{|v|}}{(\frac{z}{|v|})^2 + u^2} \, \mathrm{d}u$$

$$= \frac{1}{|v|} \int_{\mathbb{R}} \left(\frac{\pi u^2}{2(1 + u^2)^2} P_w + \frac{\pi}{4(1 + u^2)} P_w^\perp \right) \frac{\frac{z}{|v|}}{(\frac{z}{|v|})^2 + u^2} \, \mathrm{d}u$$

$$= \frac{\pi^2}{4|v|} \left(\frac{\frac{z}{|v|}}{(1 + \frac{z}{|v|})^2} P_w + \frac{1}{1 + \frac{z}{|v|}} P_w^\perp \right)$$

$$= M_1(z, v) + M_2(z, v),$$

which implies the statement of the lemma. $\qquad\square$

Now the Laplace transform $\mathcal{L}(K[u])$ can be rewritten in a more explicit form.

Lemma 3.2.16. *Let $u \in H_{\tilde{\lambda}}^n$, $n \geq 2$ and $\mathcal{L}(K[u])(z, v)$ be the Laplace transform of $K[u]$, i.e.*

$$\mathcal{L}(K[u])(z, v) = \int_0^\infty K[u](t, v) e^{-zt} \, \mathrm{d}t.$$

Then $\mathcal{L}(K[u])$ is given by the formula:

$$\mathcal{L}(K[u])(z, v) = \int (M_1 + M_2)(z, v - v') u(v') \eta(|v - v'|^2) \, \mathrm{d}v'. \tag{3.2.26}$$

In particular, the matrix $\mathcal{L}(K[u])$ is symmetric. For the operator P_γ introduced in (3.2.15) we have the formula:

$$\mathcal{L}(P_\gamma[u])(z, v) = \int (M_1 + M_2)(z, v - v')^\gamma \nabla u(v') \eta(|v - v'|^2) \, \mathrm{d}v'. \tag{3.2.27}$$

Proof. Follows from $\mathcal{L}(\cos(\alpha t))(z) = \frac{z}{z^2 + \alpha^2}$, Lemma 3.2.10 and Lemma 3.2.15. $\qquad\square$

3.2.2 Strategy of the proofs of Theorems 3.2.6 and 3.2.8

We can now outline the structure of this chapter, and introduce the key steps in the proofs of the Theorems 3.2.6 and 3.2.8:

(i) In Section 3.3 we prove that the linear equation

$$\partial_t u_\varepsilon = \frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t K[u_0] \left(\frac{t - s}{\varepsilon}, v \right) \nabla u_\varepsilon(s, v) \, \mathrm{d}s \right)$$

$$- \frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t P[u_0] \left(\frac{t - s}{\varepsilon}, v \right) u_\varepsilon(s, v) \, \mathrm{d}s \right) \tag{3.2.28}$$

$$u_\varepsilon(0, v) = u_0(v),$$

has a solution $u_\varepsilon \in V_{A,\lambda}^n \cap C^1(\mathbb{R}^+; H_\lambda^{n-2})$. The proof is based on the fact that the equation is dissipative in a time averaged sense, and strongly relies on the convolution structure of the equation in Laplace variables. Symbolically the equation in Laplace variables looks similar to:

$$z\mathcal{L}(u)(z,v) = \nabla \cdot (\tilde{K}(z,v)\nabla\mathcal{L}(u)(z,v)) + u_0(v).$$

We show that for $\Re(z) > 0$, the real part of the matrix $\tilde{K}(z,v)$ is nonnegative. This is quantified in Lemma 3.3.7 in terms of the quadratic operators $Q_{\varepsilon,A}^\alpha[u_0]$ (cf. (3.2.20)).

(ii) In order to solve the nonlinear problem, we have to allow for time dependent functions inside the operator K. We therefore consider equation (3.2.15) for a fixed function f_ε and mollified derivatives $^\gamma\nabla$:

$$
\begin{aligned}
\partial_t u_\varepsilon =& \frac{1}{\varepsilon}{}^\gamma\nabla \cdot \left(\int_0^t K[u_0 + f_\varepsilon(s,\cdot)] \left(\frac{t-s}{\varepsilon}, v \right) {}^\gamma\nabla u_\varepsilon(s,v)\, \mathrm{d}s \right) \\
&- \frac{1}{\varepsilon}{}^\gamma\nabla \cdot \left(\int_0^t P_\gamma[u_0 + f_\varepsilon(s,\cdot)] \left(\frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s,v)\, \mathrm{d}s \right)
\end{aligned}
\tag{3.2.29}
$$
$$u_\varepsilon(0,\cdot) = u_0(\cdot), \quad P_\gamma = {}^\gamma\nabla \cdot K.$$

In Subsection 3.4.1 we identify a closed, nonempty, convex subset Ω of $X_{A,\tilde{\lambda}}^n$ (defined in (3.4.6)) such that the local in time solution operator Ψ_{δ_1} to (3.2.29):

$$
\begin{aligned}
\Psi_{\delta_1} : \Omega &\longrightarrow X_{A,\lambda}^n \\
(f, F) &\mapsto \left((u - u_0)\kappa_{\delta_1}, \mathcal{A}_\gamma^{0,0}[f](u)\kappa_{\delta_1} \right), \quad \text{where } u \text{ solves (3.2.29)}
\end{aligned}
\tag{3.2.30}
$$

is well-defined. Here κ_{δ_1} is a cutoff function that localizes to small times. Notice that the solution operator maps from $X_{A,\tilde{\lambda}}^n$ to $X_{A,\lambda}^n$, thus we gain decay. The proof is based on proving that replacing the constant kernel $K[u_0]$ by $K[u_0 + f]$ amounts to a small perturbation. The main assumption for this, and the defining property of the set Ω is that for some $A, R > 0$ and small $\delta > 0$, we can bound $\mathcal{L}(f)$ on the line $\Re(z) = \frac{A}{2}$ by:

$$|\mathcal{L}(f)(z,v)| \le \left(\frac{\delta}{1+|z|^2} + \frac{R\varepsilon|z|}{(1+\varepsilon|z|)(1+|z|^2)} \right) e^{-\frac{1}{2}|v|}. \tag{3.2.31}$$

Under assumption (3.2.31) we obtain an a priori estimate on the solutions and their time derivatives:

$$
\begin{aligned}
\|\Psi_{\delta_1}(f,F)\|_{X_{A,\lambda}^n} + \|\partial_t \Psi_{\delta_1}(f,F)\|_{X_{A,\lambda}^{n-2}} &\le C \\
\|\Psi_{\delta_1}(f,F)\|_{X_{A,\lambda}^{n+1}} + \|\partial_t \Psi_{\delta_1}(f,F)\|_{X_{A,\lambda}^{n+1}} &\le C(\gamma).
\end{aligned}
\tag{3.2.32}
$$

It is crucial that the first estimate is uniform in the mollifying parameter $\gamma > 0$. In Section 3.4.2 we prove that the operator Ψ_{δ_1} introduced in (3.2.30) leaves the set Ω invariant, for $\delta_1 > 0$ small, close to the Maxwellian and $\varepsilon > 0$ small.

Now, for $\gamma > 0$, we infer the existence of a fixed point of Ψ_{δ_1} from (3.2.32) and Schauder's theorem. Here we use bounded sequences in $X_{A,\lambda}^{n+1}$ with bounded time derivative are precompact in $X_{A,\tilde{\lambda}}^n$, as proved in Lemma 3.2.5. This compactness property allows to take the limit $\gamma \to 0$ and thus to prove Theorem 3.2.6. Here we make use of the uniform estimate in (3.2.32). The proof of Theorem 3.2.8 follows by passing $\varepsilon \to 0$ using Lemma 3.2.5 yet again.

A key point of the analysis is the invariance of the set Ω under Ψ_{δ_1}, which is proved in Section 3.4.2. The proof relies on recovering the decay assumption (3.2.31). We can think of functions f satisfying (3.2.31) as a sum $f = f_1 + f_2$. Here f_1 satisfies $|\mathcal{L}(f_1)(z)| \leq \frac{\delta}{1+|z|^2}$, which can be thought of as an estimate of the form $\|\partial_{tt}^2 f_1\|_{L^1} \lesssim \delta$, and f_2 satisfies $|\mathcal{L}(f_2)(z)| \leq \frac{R\varepsilon|z|}{(1+\varepsilon|z|)(1+|z|^2)}$, which can be understood as $\|\partial_t f_2\|_{L^1} \lesssim R\varepsilon$ and $\|\partial_{tt}^2 f_2\|_{L^1} \lesssim R$. This is only a heuristic consideration, since L^∞/L^1 duality does not hold for Laplace transform. A typical function of this form is $f_2^\varepsilon(t) = \varepsilon^2 \Phi(t/\varepsilon)$. The behavior of f_1 close to $t = 0$ is more complicated, since it involves a boundary layer. Indeed, there is necessarily a boundary layer in $\partial_{tt} u_\varepsilon$ in equation (3.2.29). To see this, let u be the solution of the limit (Landau-) equation (3.2.3), and u_ε the solution to (3.2.29). Then, starting away from equilibrium, we have:

$$\partial_t u_\varepsilon(0, v) = 0, \quad \partial_t u(0, v) \neq 0.$$

So in the limit $\varepsilon \to 0$, the second derivative necessarily grows infinitely large close to the origin.

The quadratic decay of the Laplace transforms can be obtained by a bootstrap argument. To fix ideas, we observe that (3.2.29) in Laplace variables is similar to:

$$z\mathcal{L}(u - u_0) = \nabla \cdot \left(\tilde{K}(\varepsilon z)(\nabla \mathcal{L}(u) + \nabla \mathcal{L}(u) * \mathcal{L}(f)) \right). \tag{3.2.33}$$

In Subsection 3.4.1 we prove that $\nabla^m \mathcal{L}(u)$ are bounded in a weighted L^2 space in time and velocities. This can be bootstrapped to pointwise estimates: First we remark that localizing supp $u \subset [0,1] \times \mathbb{R}^3$ gives an L^∞ estimate for $\nabla^m \mathcal{L}(u)$. Assuming $|\tilde{K}(z)| \leq \frac{1}{1+|z|}$, equation (3.2.33) gives an estimate like:

$$|\nabla^m \mathcal{L}(u - u_0)(z, v)| \leq \frac{C}{(1 + \varepsilon|z|)|z|} e^{-\frac{1}{2}|v|}.$$

Plugging this estimate back into (3.2.33) proves quadratic decay of the Laplace transforms:

$$|\nabla^m \mathcal{L}(u - u_0)(z, v)| \leq \frac{C}{(1 + \varepsilon|z|)|z|^2} e^{-\frac{1}{2}|v|}.$$

In order to show invariance of the set Ω we need the same estimate with a small prefactor, as in estimate (3.2.31). We split the solution into a well-behaved part and the boundary layer mentioned before. For the first part, we use smallness of the cutoff time $\delta_1 > 0$ to get a small prefactor additional to the quadratic decay. The estimate of the boundary layer, close to the Maxwellian, is obtained by isolating and estimating it explicitly. This is the content of Subsection 3.4.2, and the most delicate part of the analysis.

We remark that there are two points where our proof is non-constructive, namely the proof of existence of solutions u_ε via Schauder's fixed point theorem, and the convergence of the sequence u_ε to the solution u of the Landau equation. Therefore, an explicit rate of convergence of the sequence u_ε to u cannot directly be derived with our method.

3.2.3   A well-posedness result for the regularized problem (3.2.29)

Before we start with the analysis of the equation in more detail, we first prove that the equation (3.2.29) with frozen nonlinearity indeed has a solution. This standard Picard-iteration argument is given in the following Lemma.

Lemma 3.2.17. *Let $n \in \mathbb{N}$, $\gamma, \varepsilon > 0$ and $u_0 \in H_\lambda^n$. Further assume there is a constant $C > 0$ such that $|f_\varepsilon(t, v)| \le Ce^{-\frac{1}{2}|v|}$ and $\operatorname{supp} f_\varepsilon \subset [0, 1]$. Then there exists a (unique) global in time solution $u_\varepsilon \in C^1([0, \infty); H_\lambda^n)$ to:*

$$
\begin{aligned}
\partial_t u_\varepsilon = & \frac{1}{\varepsilon} \gamma \nabla \cdot \left(\int_0^t K[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) \gamma \nabla u_\varepsilon(s, v) \, \mathrm{d}s \right) \\
& - \frac{1}{\varepsilon} \gamma \nabla \cdot \left(\int_0^t P_\gamma[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s, v) \, \mathrm{d}s \right) \\
u_\varepsilon(0, \cdot) = & u_0(\cdot).
\end{aligned}
\tag{3.2.34}
$$

Proof. For better notation, we introduce a shorthand for the right-hand side of the equation:

$$
\begin{aligned}
\mathcal{B}(u)(t, t', v) := & \frac{1}{\varepsilon} \gamma \nabla \cdot \left(\int_{t'}^t K[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) \gamma \nabla u(s, v) \, \mathrm{d}s \right) \\
& - \frac{1}{\varepsilon} \gamma \nabla \cdot \left(\int_{t'}^t P_\gamma[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) u(s, v) \, \mathrm{d}s \right).
\end{aligned}
$$

The claim follows from a standard Picard-type argument. Let $T > 0$ to be chosen later. Consider the mapping

$$
\begin{aligned}
\mathcal{D} : C^1([0, T]; H_\lambda^n) & \to C^1([0, T]; H_\lambda^n) \\
u & \mapsto \mathcal{D}(u),
\end{aligned}
$$

where $\mathcal{D}(u)$ is given by:

$$
\mathcal{D}(u)(t, v) := u_0(v) + \int_0^t \mathcal{B}(u)(s, v) \, \mathrm{d}s.
\tag{3.2.35}
$$

The mapping is $\mathcal{D}$ contractive for small times. More precisely we have:

$$
\|\mathcal{B}(u)(t, t', \cdot)\|_{H_\lambda^n} \le C|t - t'| \sup_{t' \le s \le t} \|u(s, \cdot)\|_{L_\lambda^2}.
\tag{3.2.36}
$$

Hence, there exists a $T_1 > 0$ such that $\mathcal{D}$ is contractive and we obtain a unique solution for $T \le T_1$. Assume we already have constructed the solution u up to time mT_1 for $m \in \mathbb{N}$. Consider the mapping:

$$
\begin{aligned}
\mathcal{D}_m : C^1([mT_1, (m+1)T_1]; H_\lambda^n) & \to C^1([mT_1, (m+1)T_1]; H_\lambda^n) \\
w & \mapsto \mathcal{D}_m(w) = u(mT_1, v) + \int_{mT_1}^T \mathcal{B}(w)(s, v) \, \mathrm{d}s.
\end{aligned}
$$

By (3.2.36) this mapping is contractive and we can pick the same small time T_1 in each step of the induction. $\qquad \square$

3.3   The linear equation (3.2.28)

The linear equation (3.2.28) has an averaged-in-time coercivity property. We will prove this using geometric arguments that resemble the ones used for the Landau equation, see for instance [15]. For shortness we introduce the following notation.

Notation 3.3.1. *For $z \in \mathbb{C}$ and $v \in \mathbb{R}^3$ define:*

$$\alpha(z,v) := \frac{|\Im(z)|}{1 + |v|}, \quad \beta(z,v) := \frac{|\Re(z)|}{1 + |v|}. \tag{3.3.1}$$

Further we define the following positive functions C_1, C_2 and C_3:

$$C_1(z,v) = \frac{1}{(1 + |v|)(1 + \alpha(z,v))^2} \tag{3.3.2}$$

$$C_2(z,v) = \frac{\beta(z,v) + \alpha(z,v)^2}{(1 + |v|)(1 + \alpha(z,v))^4} \tag{3.3.3}$$

$$C_3(z,v) = \frac{\beta(z,v) + \alpha(z,v) + \alpha(z,v)^2}{(1 + |v|)(1 + \alpha(z,v))^4}. \tag{3.3.4}$$

Let $0 \neq v \in \mathbb{R}^3$, $V, W \in \mathbb{C}^3$. We define the anisotropic norm:

$$|W|_v := |P_v^{\perp} W| + \frac{|P_v W|}{1 + |v|}, \tag{3.3.5}$$

and the weight functionals $B_1(z,v)(V,W)$, $B_2(z,v)(V,W)$ given by:

$$B_1(V,W) = C_1(z,v)|V|_v|W|_v + C_2(z,v)|P_v V||P_v W| \tag{3.3.6}$$

$$B_2(V,W) = C_1(z,v)|V|_v|W|_v + C_3(z,v)|P_v V||P_v W|. \tag{3.3.7}$$

The following straightforward analysis lemma we will use to bound real and imaginary part of the matrices M_i defined in (3.2.24) from above and below.

Lemma 3.3.2. *Let $z \in \mathbb{C}$ with $0 \leq \Re(z) \leq 1$. The following bounds hold:*

$$\Re\left(\frac{z}{(1+z)^2}\right) \geq c\frac{\Re(z) + |\Im(z)|^2}{(1 + |\Im(z)|)^4} \tag{3.3.8}$$

$$\left|\Im\left(\frac{z}{(1+z)^2}\right)\right| \leq C\frac{\Re(z) + |\Im(z)| + |\Im(z)|^2}{(1 + |\Im(z)|)^3} \tag{3.3.9}$$

$$\Re\left(\frac{1}{1+z}\right) \geq c\frac{1}{(1 + |\Im(z)|)^2} \tag{3.3.10}$$

$$\left|\Im\left(\frac{1}{(1+z)}\right)\right| \leq C\frac{|\Im(z)|}{(1 + |\Im(z)|)^2}. \tag{3.3.11}$$

Proof. To prove (3.3.8)-(3.3.9), we rewrite the fraction as:

$$\frac{z}{(1+z)^2} = \frac{z + 2|z|^2 + \overline{z}|z|^2}{|1 + z|^4}.$$

Since the real part of z is bounded and nonnegative by assumption, (3.3.8) follows immediately. For the proof of (3.3.9) we include the computation:

$$|\Im(\frac{z}{(1+z)^2})| \leq C\frac{|\Im(z)| + (\Re(z)^2 + \Im(z)^2)(1 + |\Im(z)|)}{|1+z|^4}$$
$$\leq C\frac{\Re(z) + |\Im(z)| + |\Im(z)|^2}{|1+z|^3},$$

proving also the second claim. The inequalities (3.3.10) and (3.3.11) are immediate. $\qquad\square$

The following simple lemma provides an estimate for the derivatives of the matrices M_i defined in (3.2.24).

Lemma 3.3.3. *For a multi-index $\beta \in \mathbb{N}^3$, $\Re(z) \geq 0$, $i = 1, 2$ and $v \in \mathbb{R}^3$, $V, W \in \mathbb{C}^3$, we can estimate:*

$$|\langle V, D^\beta(M_i(z,v)\eta(|v|^2))W\rangle| \leq \frac{C_{|\beta|}|V||W|}{(1+|v|^{|\beta|+1})(1+\alpha(z,v))}\eta(16|v|^2). \tag{3.3.12}$$

Here η is the cutoff function introduced in Notation 3.2.1.

Proof. With Leibniz's rule, we can split the derivative into:

$$D^\beta((M_1 + M_2)(z,v)\eta(|v|^2)) = \sum_{\beta_2 \leq \beta}\binom{\beta}{\beta_2}D^{\beta-\beta_2}((M_1 + M_2)(z,v))D^{\beta_2}(\eta(|v|^2)).$$

By construction of the fixed cutoff function η we can estimate:

$$|\nabla^m\eta(r)| \leq \frac{C}{1+|r|^m}|\eta(16r)|. \tag{3.3.13}$$

We write M_1, M_2 defined in (3.2.24) as :

$$M_1(z,v) = \frac{\pi^2}{4(z+|v|)}P_v^\perp, \quad M_2(z,v) = \frac{\pi^2 z}{4(z+|v|)^2}P_v.$$

The operators P_v, $P_v^\perp$ are zero-homogeneous in v. So for every $c > 0$ we can estimate:

$$|\nabla_v^n M_i(z,v)| \leq \frac{C|M_i(z,v)|}{1+|v|^n} \leq \frac{C}{(1+|v|)^{n+1}(1+\alpha(z,v))} \quad \text{for } i = 1, 2, |v| \geq c > 0. \tag{3.3.14}$$

Combining (3.3.13) and (3.3.14) gives the claim. $\qquad\square$

The following Lemmas prove coercivity of the matrix $\mathcal{L}(K)[u](v)$, which becomes anisotropic as $|v| \to \infty$. The crucial geometric argument is contained in the following Lemma, that in our setting needs to be valid for complex vectors (since we apply it to Laplace transforms).

Lemma 3.3.4. *For $0 \neq V \in \mathbb{C}^3$ and $0 \leq r \leq 1$, let $D_V(r)$ be given by:*

$$D_V(r) = \{v' \in \mathbb{R}^3 : \frac{1}{2} \leq |v'| \leq 1, \frac{|\langle v', V \rangle|}{|v'||V|} \geq r\}.$$

There exists a constant $c > 0$ such that for all $v \in \mathbb{R}^3$, $|v| \geq 2$ the following statements hold:

$$\text{for } 0 \neq V \in \mathbb{C}^3: \qquad\qquad \text{Vol}(D_V(1/8)) \geq c, \qquad (3.3.15)$$

$$\text{for } V \in \mathbb{C}^3 \, \exists \, 0 \neq W \in \mathbb{C}^3 \, \forall \, v' \in D_W(1/8): \qquad |P_{v-v'}^\perp V| + |P_{v-(-v')}^\perp V| \geq c|V|_v, \qquad (3.3.16)$$

where the anisotropic norm $|\cdot|_v$ was introduced in (3.3.5). Furthermore for $v \in \mathbb{R}^3$, $V \in \mathbb{C}^3$, define

$$E(v, V) = \{v' \in B_1(0) \subset \mathbb{R}^3 : |\langle v' + v, V \rangle| \geq |\langle v, V \rangle|\}.$$

There exists $c > 0$ such that for all $v \in \mathbb{R}^3$, $|v| \geq 2$:

$$|P_{v-v'}V| \geq c|P_v V| \qquad\qquad \text{for } v' \in E(v, V) \qquad (3.3.17)$$

$$\text{Vol}(E(v, V)) \geq c > 0. \qquad (3.3.18)$$

Proof. The inequality (3.3.15) is clear if $0 \neq V \in \mathbb{R}^3$ is real. Moreover, there is a constant $c > 0$ such that $\text{Vol}(D_V(r)) \geq c > 0$ for $0 \leq r \leq \frac{3}{4}$ and $V \in \mathbb{R}^3$. Let now $V = V_R + iV_I \in \mathbb{C}^3$, where at least one of the vectors $V_R, V_I \in \mathbb{R}^3$ is nonzero, and let W be the longer vector of V_R, V_I. We define $\tilde{D}_V = D_W(\frac{1}{2})$. Then we have $\frac{|\langle v', V \rangle|}{|v'||V|} \geq \frac{1}{4}\frac{|W|}{|V|} \geq \frac{1}{8}$ for $v' \in \tilde{D}_V$. Since $W \in \mathbb{R}^3$ we have $\text{Vol}(D_W(\frac{1}{2})) \geq c > 0$, so in particular

$$U(v, V) := \{v' \in \mathbb{R}^3 : \frac{|\langle v', V \rangle|}{|v'||V|} \geq \frac{1}{8}\}$$

satisfies $\text{Vol}(U(v, V)) \geq c > 0$. Since $U(v, V)$ is homogeneous, the set

$$U(v, V) \cap \{v' \in \mathbb{R}^3 : \frac{1}{2} \leq |v'| \leq 1\} \subset D_V(\frac{1}{8})$$

also has volume uniformly bounded below, which implies the claim (3.3.15). For the proof of (3.3.16), let $v \in \mathbb{R}^3$, $|v| \geq 2$ and $V \in \mathbb{C}^3$ be a unit vector such that $V = V_1 + V_2$, $V_1 = P_v V$, $V_2 = P_v^\perp V$. Let us first assume that $V_2 \neq 0$. We claim that (3.3.16) holds with $W = V_2$. To this end, let $|v| \geq 2$ and $v' \in D_{V_2}(1/8)$, so in particular $|v'| \leq 1$. Then the angle ψ between v and $v - v'$ is bounded by $|\psi| \leq \frac{\pi}{6}$, hence:

$$|P_{v-v'}V_2| = |P_{v-v'}P_v^\perp V| \leq \frac{1}{2}|V_2|, \text{ therefore:}$$

$$|P_{v-v'}^\perp V| = |V_1 - P_{v-v'}V_1 + V_2 - P_{v-v'}V_2| \geq |V_1 - P_{v-v'}V_1 + V_2| - \frac{1}{2}|V_2|$$

$$\geq |P_{V_2}(V_1 - P_{v-v'}V_1 + V_2)| - \frac{1}{2}|V_2| = |V_2 - P_{V_2}P_{v-v'}V_1| - \frac{1}{2}|V_2|. \qquad (3.3.19)$$

We rewrite the first term on the right-hand side as:

$$|V_2 - P_{V_2} P_{v-v'} V_1| = ||V_2| - \langle \frac{V_2}{|V_2|}, P_{v-v'} V_1 \rangle|. \tag{3.3.20}$$

Let $\zeta(v') = \langle \frac{V_2}{|V_2|}, P_{v-v'} V_1 \rangle$. We observe that $V_2 = P_v^\perp V$ and $V_1 = pv$ for some $p \in \mathbb{C}$, so:

$$\zeta(v') = \langle \frac{V_2}{|V_2|}, \frac{v-v'}{|v-v'|} \rangle \langle \frac{v-v'}{|v-v'|}, V_1 \rangle = \frac{p}{|v-v'|} \langle \frac{V_2}{|V_2|}, -v' \rangle \langle \frac{v-v'}{|v-v'|}, v \rangle. \tag{3.3.21}$$

Since $|v'| \leq \frac{1}{2}|v|$, we have $\langle \frac{v-v'}{|v-v'|}, v \rangle \geq \frac{1}{2}|v|$. This implies the lower bound:

$$|\zeta(v')| \geq \frac{1}{4} \frac{|pv|}{1+|v|} |\langle \frac{V_2}{|V_2|}, -v' \rangle| \geq \frac{c|V_1|}{1+|v|} \quad \text{for } v' \in D_{V_2}(1/8). \tag{3.3.22}$$

Now we claim that the real part of $\zeta(v')$ is nonpositive, after possibly changing the sign of v':

$$\Re(\zeta(v')) \leq 0, \text{ or } \Re(\zeta(-v')) \leq 0. \tag{3.3.23}$$

To see this, we use (3.3.21) and $\langle \frac{v-v'}{|v-v'|}, v \rangle \geq 0$. Inserting the estimates (3.3.22), (3.3.23) and the lower bound $|z| \geq \frac{1}{\sqrt{2}} (|\Re(z)| + |\Im(z)|)$ into (3.3.20) we obtain:

$$|V_2 - P_{V_2} P_{v-v'} V_1| + |V_2 - P_{V_2} P_{v-(-v')} V_1| \geq \frac{1}{\sqrt{2}} \left(|V_2| + \frac{c|V_1|}{1+|v|} \right).$$

We plug this back into (3.3.19) and add the corresponding term for $-v'$ to prove (3.3.16) in the case $V_2 \neq 0$. In order to prove (3.3.16) for $V_2 = 0$, we remark that the estimate is homogeneous in V, so it suffices to prove it for $|V| = 1$, when it follows by continuity from the case $V_2 = 0$.

The estimate (3.3.17) follows from the observation that for $v' \in E(v, V)$ we have

$$|P_{v-v'} V| = \left| \langle \frac{v-v'}{|v-v'|}, V \rangle \right| \geq \frac{1}{2} |P_v V|.$$

Finally (3.3.18) is a consequence of $E(v, V)$ containing either v' or $-v'$ for every $v' \in B_1(0)$. $\qquad \square$

Lemma 3.3.4 proves lower bounds for the projections $|P_{v-v'} V|$ respectively $|P_{v-v'}^\perp V|$ on a set (of v') with uniformly positive Lebesgue measure. We now show that this implies a lower bound for the integrals (3.2.26), (3.2.27) representing $\mathcal{L}(K)$, $\mathcal{L}(P)$.

Lemma 3.3.5. *Let $z \in \mathbb{C}$ with $0 \leq \Re(z) \leq 1$ and β be a multi-index. Let $V, W \in \mathbb{C}^3$ be complex vectors. Further let $n \geq 1$ and $u_0 \in H_\lambda^n$ satisfy the pointwise estimates:*

$$c\mathbb{1}_{|v| \leq 4}(v) \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad \text{for } c > 0.$$

Recall B_1, B_2 as defined in (3.3.6)-(3.3.7) and C_1 defined in (3.3.2). Then there holds:

$$\int_{\mathbb{R}^3} \langle V, \Re(M_1 + M_2)(z, v - v')V \rangle u_0(v') \eta \, dv' \geq cB_1(z, v)(V, V) \tag{3.3.24}$$

$$\int_{\mathbb{R}^3} |\langle V, (M_1 + M_2)(z, v - v')W \rangle| u_0(v') \eta \, dv' \leq C(1 + \alpha(z, v))B_2(z, v)(V, W) \tag{3.3.25}$$

$$\int_{\mathbb{R}^3} |\langle V, D^\beta \left((M_1 + M_2)(z, v - v')\eta \right) W \rangle| u_0(v') \, dv' \leq C \frac{(1 + \alpha(z, v))}{(1 + |v|)^{|\beta|}} C_1(z, v)|V||W|. \tag{3.3.26}$$

Proof. First we prove (3.3.24). We remark that the integrand is nonnegative:

$$\langle V, \Re(M_1)V \rangle = \langle V, \Re\left(\frac{\pi^2}{4|v|}\frac{1}{1+\frac{z}{|v|}}\right)P_v^\perp V\rangle$$

$$= \Re\left(\frac{\pi^2}{4|v|}\frac{1}{1+\frac{z}{|v|}}\right)|P_v^\perp V|^2 \geq 0,$$

by (3.3.10). By a similar computation the same is true for M_2. We use (3.3.8) to bound the real part of M_2 (cf.(3.2.24)) below. Using nonnegativity of the integrand, the lower bound on $u_0(v')$ and $\eta(|r|) = 1$ for $|r| \geq 1$ we can estimate from below by (C_2 as in (3.3.3)):

$$\int_{\mathbb{R}^3}\langle V, \Re(M_2)(z, v-v')V\rangle u_0(v')\eta(|v-v'|^2)\,\mathrm{d}v' \geq c\int_{B_4(0)\backslash B_1(v)}C_2(z, v-v')|P_{v-v'}V|^2\,\mathrm{d}v'.$$

Now there are $c_1, c_2 > 0$ s.t. for $|v| \leq 2$ we have $|P_{v-v'}V| \geq c_1|V|_v$ for all v' in a set $G(v, V) \subset B_4(0) \backslash B_1(v)$ with $|G(v, V)| \geq c_2$. To see this we remark that the inequality is homogeneous in V, so we can restrict to $|V| = 1$ and v bounded, when the claim follows by contradiction. For $|v| \geq 2$ we use (3.3.17)-(3.3.18) to obtain a set of positive measure on which we have $|P_{v-v'}V| \geq c|P_v V|$. We find the lower bound:

$$\int_{\mathbb{R}^3}\langle V, \Re(M_2)(z, v-v')V\rangle u_0(v')\eta(|v-v'|^2)\,\mathrm{d}v' \geq cC_2(z, v)|P_v V|^2. \tag{3.3.27}$$

We apply the same strategy for the term containing M_1 (cf. (3.2.24)):

$$\int_{\mathbb{R}^3}\langle V, \Re(M_1)(z, v-v')V\rangle u_0(v')\eta(|v-v'|^2)\,\mathrm{d}v' \geq cC_1(z, v)\int_{B_4(0)\backslash B_1(v)}|P_{(v-v')}^\perp V|^2\,\mathrm{d}v'.$$

For $|v| \geq 2$ we use (3.3.15)-(3.3.16) to obtain:

$$\int_{\mathbb{R}^3}\langle V, \Re(M_1)(z, v-v')V\rangle u_0(v')\eta(|v-v'|^2)\,\mathrm{d}v' \geq cC_1(z, v)|V|_v^2, \tag{3.3.28}$$

for $|v| \leq 2$ the same follows again by rescaling $|V| = 1$ and contradiction. Combining (3.3.27) and (3.3.28) we obtain (3.3.24). We now show the upper bound (3.3.25). The estimates (3.3.8)-(3.3.9) allow to estimate the contribution of M_2 (cf. (3.2.24)) by C_3 as defined in (3.3.4):

$$\int_{\mathbb{R}^3}|\langle V, M_2(z, v-v')W\rangle|u_0(v')\eta(|v-v'|^2)\,\mathrm{d}v'$$

$$\leq C\int_{\mathbb{R}^3}|M_2(z, v-v')||\langle P_{v-v'}V, P_{v-v'}W\rangle|u_0(v')\eta(|v-v'|^2)\,\mathrm{d}v'$$

$$\leq C\int_{\mathbb{R}^3}(1+\alpha)C_3(z, v-v')(|P_v V| + \frac{|v'|}{|v|}|P_v^\perp V|)(|P_v W| + \frac{|v'|}{|v|}|P_v^\perp W|)e^{-\frac{1}{2}|v'|}\eta\,\mathrm{d}v'$$

$$\leq C(1+\alpha(z, v))C_3(z, v)\left(|P_v V||P_v W| + |V|_v|W|_v\right).$$

Since $C_3(z,v) \leq CC_1(z,v)$ for $0 \leq \Re(z) \leq 1$, this shows the contribution of M_2 can be estimated by the right-hand side of (3.3.25). For bounding the contribution of M_1 we proceed similarly, using (3.3.11):

$$\int_{\mathbb{R}^3} |\langle V, M_1(z, v-v')W\rangle| u_0(v') \eta(|v-v'|^2) \, \mathrm{d}v'$$

$$\leq C \int_{\mathbb{R}^3} (1 + \alpha(z, v-v')) C_1(z, v-v') |P_{(v-v')}^{\perp} V| |P_{(v-v')}^{\perp} W| e^{-\frac{1}{2}|v'|} \eta(|v-v'|^2) \, \mathrm{d}v'.$$

Write $V = P_v V + P_v^{\perp} V = V_1 + V_2$ and $W = W_1 + W_2$ respectively. Then we have

$$|P_{v-v'}^{\perp} V| \leq C \left(\frac{|V_1||v'|}{1+|v|} + |V_2| \right).$$

This implies that we can bound:

$$\int_{\mathbb{R}^3} |\langle V, M_1(z, v-v')W\rangle| u_0(v') \eta \, \mathrm{d}v'$$

$$\leq C \int_{\mathbb{R}^3} (1 + \alpha(z, v-v')) C_1(z, v-v') (\frac{|V_1||v'|}{1+|v|} + |V_2|)(\frac{|W_1||v'|}{1+|v|} + |W_2|) e^{-\frac{1}{2}|v'|} \eta \, \mathrm{d}v'$$

$$\leq C(1 + \alpha(z,v)) C_1(z,v) |V|_v |W|_v,$$

which concludes the proof of (3.3.25). Estimate (3.3.26) follows from a similar computation, using Lemma 3.3.3. $\qquad\square$

The following Lemma uses the symmetry of the highest order term in the functionals Q to show it can be expressed by the real part of $\mathcal{L}(K)$, $\mathcal{L}(P)$ only, which surprisingly has a sign.

Lemma 3.3.6. *Let $n \geq 1$ and $u_0 \in H_\lambda^n$ satisfy the pointwise estimates*

$$c\mathbb{1}_{|v| \leq 4}(v) \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad \text{for } c > 0. \tag{3.3.29}$$

Furthermore let $\varepsilon > 0$, $A > 0$ such that $\varepsilon A \leq 1$ and write $z = a + i\omega = \frac{A}{2} + i\omega$. Let $u \in V_{A,\lambda}^n$ for some $n \in \mathbb{N}$ and $\gamma \in (0,1]$. The term in $Q_{\varepsilon,A}^{\alpha,\alpha}$ (as defined in (3.2.20)-(3.2.22)), where $|\alpha| \leq n$, depends on the real part of $\mathcal{L}(K)$ only. Writing $V = \nabla D^\alpha \mathcal{L}(u)(z,v)$ we have:

$$
\begin{aligned}
(2\pi)^{\frac{1}{2}} Q_{\varepsilon,A}^{\alpha,\alpha}[u_0](u) &= \int_{\mathbb{R}} \int \langle {}^\gamma V(z,v)\lambda(v), \mathcal{L}(K)[u_0](\varepsilon z, v)^\gamma V(z,v)\rangle \, \mathrm{d}v \, \mathrm{d}\omega \\
&= \int_{\mathbb{R}} \int \langle {}^\gamma V(z,v)\lambda(v), \Re(\mathcal{L}(K))[u_0](\varepsilon z, v)^\gamma V(z,v)\rangle \, \mathrm{d}v \, \mathrm{d}\omega.
\end{aligned}
\tag{3.3.30}
$$

Proof. Follows from the observation that the left-hand side is real by Plancherel's Lemma and that K is a symmetric matrix. $\qquad\square$

The following lemma amounts to a coercivity result, and shows that for a function $u \in V_{A,\lambda}^n$ the functional $Q_{\varepsilon,A}^\alpha[u_0](u)$ can be controlled by the first n derivatives of u only. Here we use that to leading order, the functional is actually dissipative. The exact form of the dissipation D is of particular importance, since we use it later to show that the nonlinearity can be handled as a perturbation.

Lemma 3.3.7. *Let $n \geq 1$ and $u_0 \in H_\lambda^n$ satisfy the pointwise estimates*

$$c\mathbb{1}_{|v| \leq 4}(v) \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad \text{for } c > 0. \tag{3.3.31}$$

For $A > 0$, let $a = \frac{A}{2}$ and assume $\varepsilon \in (0, \frac{1}{a}]$, $\gamma \in (0, 1]$ arbitrary and $|\alpha| \leq n$ for an $\alpha \in \mathbb{N}^3$. Define the dissipation $D_{\varepsilon,A}^\alpha$ as ($z = a + i\omega$):

$$D_{\varepsilon,A}^\alpha(u) := \int \int B_1(\varepsilon z, v)[^\gamma \nabla D^\alpha \mathcal{L}(u)(z, v), ^\gamma \nabla D^\alpha \mathcal{L}(u)(z, v)]\lambda(v) \, dv \, d\omega. \tag{3.3.32}$$

Then the leading order quadratic form satisfies the lower bound:

$$Q_{\varepsilon,A}^{\alpha,\alpha}[u_0](u) \geq cD_{\varepsilon,A}^\alpha(u) - C\|u\|_{V_{A,\lambda}^n}^2. \tag{3.3.33}$$

We will denote by $D_{\varepsilon,A}^\alpha$ the dissipation of the equation. The lower order terms can be estimated by the dissipation:

$$\sum_{\beta < \alpha} \binom{\alpha}{\beta} |Q_{\varepsilon,A}^{\alpha,\beta}[u_0](u)| \leq \frac{c}{2} D_{\varepsilon,A}^\alpha(u) + C\|u\|_{V_{A,\lambda}^n}^2. \tag{3.3.34}$$

The constants can depend on u_0 and n, but not on $A \geq 1$, $\varepsilon > 0$.

Proof. In the proof, we drop the dependence on γ for shortness. We start with proving the lower bound (3.3.33). As a first step we rewrite $Q_{\varepsilon,A}^{\alpha,\alpha}[u_0](u)$ in terms of Laplace transforms (write $z = a + i\omega$ for shortness):

$$\begin{aligned}
Q_{\varepsilon,A}^{\alpha,\alpha}[u_0](u) =& \frac{1}{\varepsilon} \int_0^\infty e^{-At} \int \nabla(D^\alpha u(t)\lambda) \left(\int_0^t K[u_0](\frac{t-s}{\varepsilon}, v)\nabla D^\alpha u(s) \, ds \right) dv \, dt \\
& -\frac{1}{\varepsilon} \int_0^\infty e^{-At} \int \nabla(D^\alpha u(t)\lambda) \left(\int_0^t P[u_0](\frac{t-s}{\varepsilon}, v)D^\alpha u(s) \, ds \right) dv \, dt \\
=& (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \int \langle \nabla(D^\alpha \mathcal{L}(u)(z, v))\lambda), \mathcal{L}(K)[u_0](\varepsilon z, v)\nabla D^\alpha \mathcal{L}(u)(z, v) \rangle \, dv \, d\omega \\
& -(2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \int \langle \nabla(D^\alpha \mathcal{L}(u)(z, v))\lambda), \mathcal{L}(P)[u_0](\varepsilon z, v)D^\alpha \mathcal{L}(u)(z, v) \rangle \, dv \, d\omega \\
=& J_1 + J_2. \tag{3.3.35}
\end{aligned}$$

We recall the representation of $\mathcal{L}(K)$ given in Lemma 3.2.16:

$$\mathcal{L}(K[u])(z, v) = \int (M_1 + M_2)(z, v - v')u(v')\eta(|v - v'|^2) \, dv'. \tag{3.3.36}$$

We start by estimating J_1. For shortness, we write $V = \nabla D^\alpha \mathcal{L}(u)$. Then use (3.3.36), Lemma 3.3.6 and the pointwise estimates proven in Lemma 3.3.5 :

$$\begin{aligned}
J_1 =& (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \int \langle V(z, v)\lambda(v), \mathcal{L}(K)[u_0](\varepsilon z, v)V(z, v) \rangle \, dv \, d\omega \\
& + (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \int \langle D^\alpha \mathcal{L}(u)(z, v)\nabla(\lambda(v)), \mathcal{L}(K)[u_0](\varepsilon z, v)V(z, v) \rangle \, dv \, d\omega \\
\geq& cD_{\varepsilon,A}^\alpha(u) + (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} \int \langle D^\alpha \mathcal{L}(u)(z, v)\nabla(\lambda(v)), \mathcal{L}(K)[u_0](\varepsilon z, v)V(z, v) \rangle \, dv \, d\omega \\
=& cD_{\varepsilon,A}^\alpha(u) + I_3. \tag{3.3.37}
\end{aligned}$$

It remains to estimate J_2 given by (3.3.35) and I_3 given by (3.3.37). To this end, we recall the definition of $\| \cdot \|_{V_{\varepsilon,A}^n}$ in (3.2.7) and use the Plancherel identity in Lemma 3.2.13 to estimate:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} |D^\alpha \mathcal{L}(u)(z,v)|^2 \lambda(v) \, d\omega \, dv \leq C\|u\|_{V_{A,\lambda}^n}^2. \tag{3.3.38}$$

In order to estimate I_3, we observe that $\nabla \lambda = P_v \nabla \lambda$. Then we combine (3.3.36) with (3.3.25) in Lemma 3.3.5 to obtain the estimate (recall B_2, cf. (3.3.7)):

$$
\begin{aligned}
|I_3| \leq &C \int_{\mathbb{R}} \int |D^\alpha \mathcal{L}(u)| \lambda C(1 + \alpha(\varepsilon z, v)) B_2(\varepsilon z, v)[P_v \nabla \lambda(v), V] \, dv \, d\omega \\
\leq &C \int_{\mathbb{R}} \int (|D^\alpha \mathcal{L}(u)|) \, \lambda \left(\frac{|V(z,v)|}{(1 + \alpha(\varepsilon z, v))(1 + |v|^2)} + \frac{(\beta + \alpha + \alpha^2)|P_v V(z,v)|}{(1 + \alpha)^3(1 + |v|)} \right) \, dv \, d\omega.
\end{aligned}
$$

We apply Young's inequality and (3.3.38) to get the bound ($D_{\varepsilon,A}^\alpha$ defined in (3.3.32)):

$$|I_3| \leq \frac{c}{4} D_{\varepsilon,A}^\alpha + C\|D^\alpha u\|_{V_{A,\lambda}^n}^2. \tag{3.3.39}$$

It remains to estimate J_2 to finish the proof of (3.3.33). We recall that $P[u_0] = \nabla \cdot K[u_0]$. We apply (3.3.26) with $|\beta| = 1$ and recall the definition of C_1 (cf. (3.3.2)) to obtain an upper estimate for J_2:

$$|J_2| \leq C \int_{\mathbb{R}} \int \left(\lambda^{\frac{1}{2}}(v) \frac{1 + \alpha(\varepsilon z, v)}{1 + |v|} C_1(\varepsilon z, v)|V| \right) \left(\lambda^{\frac{1}{2}}(v)|D^\alpha \mathcal{L}(u)(z,v)| \right) \, dv \, d\omega.$$

Notice that (3.3.26) provides $\frac{1}{|v|}$ more decay than naively expected, which is essential here. Young's inequality in combination with (3.3.38) implies:

$$|J_2| \leq \frac{c}{4} D_{\varepsilon,A}^\alpha(u) + C\|u\|_{V_{A,\lambda}^n}^2. \tag{3.3.40}$$

Combining the estimates (3.3.35), (3.3.39) and (3.3.40) proves (3.3.33). In the case $\beta < \alpha$ we use (3.3.26) in Lemma 3.3.5 and Young's inequality to prove (3.3.34). $\qquad\square$

The linear result follows as a corollary. The statement can be generalized significantly, the assumptions in our a priori estimates are designed for the nonlinear case and therefore more restrictive than needed for the linear equation.

Theorem 3.3.8. *Let $n \geq 6$ and $u_0 \in H_\lambda^n$ satisfy the pointwise estimate*

$$c\mathbb{1}_{|v| \leq 4}(v) \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}, \quad c, C > 0. \tag{3.3.41}$$

There exists $A > 0$ s.t. for $\varepsilon > 0$ small, there is a solution $u_\varepsilon \in V_{A,\lambda}^n \cap C^1(\mathbb{R}^+; H_\lambda^{n-2})$ to:

$$
\begin{aligned}
\partial_t u_\varepsilon = &\frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t K[u_0] \left(\frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s, v) \, ds \right) \\
&- \frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t P[u_0] \left(\frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s, v) \, ds \right) \\
u_\varepsilon(0, \cdot) = &u_0(\cdot).
\end{aligned}
\tag{3.3.42}
$$

There is a function $u \in V_{A,\lambda}^n \cap C^1(\mathbb{R}^+; H_\lambda^{n-4})$ s.t. $u_{\varepsilon_j} \rightharpoonup u$ in $V_{A,\lambda}^n$ along a sequence $\varepsilon_j \to 0$. The function u solves the limit equation ($\mathcal{K}, \mathcal{P}$ defined in (3.2.3)):

$$\partial_t u = \nabla \cdot (\mathcal{K}[u_0]\nabla u) - \nabla \cdot (\mathcal{P}[u_0]u)$$
$$u(0,v) = u_0(v). \tag{3.3.43}$$

Proof. For $0 < \gamma \le 1$, the existence of solutions $u_{\varepsilon,\gamma}$ to (3.2.34) follows from Lemma 3.2.17. In order to prove well-posedness for (3.3.42), i.e. $\gamma = 0$, we derive a priori estimates that are uniform in γ. Combining Lemma 3.2.12 and Lemma 3.3.7 shows that for $A > 0$ large enough

$$\|u_{\varepsilon,\gamma}\|_{V_{A,\lambda}^n} \le C \tag{3.3.44}$$

are uniformly bounded in $0 < \gamma, \varepsilon \le \frac{1}{A}$. Now we use the Laplace representation in Lemma 3.2.16 to infer the uniform boundedness:

$$|\nabla^m \mathcal{L}(K[u_0])(z,v)| + |\nabla^m \mathcal{L}(P_\gamma[u_0])(z,v)| \le C(m) \quad \text{for } m \in \mathbb{N}. \tag{3.3.45}$$

We rewrite (3.2.34) in Laplace variables and obtain:

$$z\mathcal{L}(u_{\varepsilon,\gamma}) = {}^\gamma\nabla \cdot (\mathcal{L}(K[u_0])(\varepsilon z)^\gamma \nabla \mathcal{L}(u_{\varepsilon,\gamma}) - \mathcal{L}(P_\gamma[u_0])(\varepsilon z)\mathcal{L}(u_{\varepsilon,\gamma})) + u_0(v). \tag{3.3.46}$$

The right-hand side of (3.3.46) is bounded in $V_{A,\lambda}^{n-2}$ due to (3.3.45) and (3.3.44), so we get a bound of:

$$\|u_{\varepsilon,\gamma}\|_{V_{A,\lambda}^n} + \|\partial_t u_{\varepsilon,\gamma}\|_{V_{A,\lambda}^{n-2}} \le C. \tag{3.3.47}$$

By the Rellich type Lemma 3.2.5, and the fact that $V_{A,\lambda}^n$ is a separable Hilbert space, there is a $u_\varepsilon \in V_{A,\lambda}^n$ and a sequence $\gamma_j \to 0$ s.t. $u_{\varepsilon,\gamma_j} \rightharpoonup u_\varepsilon$ in $V_{A,\lambda}^n$ and $u_{\varepsilon,\gamma_j} \to u_\varepsilon$ in $V_{A,\tilde{\lambda}}^{n-3}$. We need to show that the weak limit u_ε indeed solves the equation (3.3.42). Both sides of (3.3.46) converge pointwise a.e. to the respective sides with $\gamma = 0$ along a subsequence of $\gamma_j \to 0$. Since the Laplace transform defines the function uniquely, u_ε is indeed a solution. Finally, the solutions u_ε are in $C^1(\mathbb{R}^+; H_\lambda^{n-2})$ since they are bounded in $V_{A,\lambda}^n$ and the equation (3.3.42) in combination with $|\nabla^m K[u_0]| + |\nabla^m P[u_0]| \le C(m)$ allows to control the time derivative in $C^0(\mathbb{R}^+; H_\lambda^{n-2})$.

The convergence of u_ε to a solution u of (3.3.43) follows similarly. We use the uniform bound (3.3.47) to find a subsequence $\varepsilon_j \to 0$ and $u \in V_{A,\lambda}^n$ such that $u_{\varepsilon_j} \rightharpoonup u$ in $V_{A,\lambda}^n$ and $u_{\varepsilon_j} \to u$ in $V_{A,\tilde{\lambda}}^{n-3}$. Now the claim follows from the observation that for $\gamma = 0$ we can take the limits on both sides of (3.3.46) and pointwise a.e. along a subsequence there holds:

$$\mathcal{L}(u_{\varepsilon_j}) \to \mathcal{L}(u), \ \mathcal{L}(K)[u_0](\varepsilon_j z,v) \to \mathcal{K}[u_0](v), \ \mathcal{L}(P)[u_0](\varepsilon_j z,v) \to \mathcal{P}[u_0](v).$$

Repeating the argument above, we find that the weak limit $u_{\varepsilon_j} \rightharpoonup u \in V_{A,\lambda}^n$ is actually $u \in V_{A,\lambda}^n \cap C^1(\mathbb{R}^+; H_\lambda^{n-4})$ and is indeed a solution of the equation (3.3.43). $\square$

3.4 A priori estimate for the nonlinear problem

3.4.1 Continuity of the fixed point mapping Ψ

In this subsection we prove that solutions of equation (3.2.15) satisfy an a priori estimate, for small perturbations f_ε. Here smallness is measured in terms of the size and decay of the Laplace transform, i.e. the smoothness of the perturbation f_ε. The necessary framework is provided by the definition below. Notice that we always assume that $f_\varepsilon = \nabla \cdot g_\varepsilon$ is a divergence, so it has zero average. This is the key point to obtain an additional decay $\frac{1}{|v|}$ in Lemma 3.4.7. Furthermore it is essential that the highest order term $Q_{\varepsilon,\delta}^{\alpha,\alpha}[f_\varepsilon](u)$ introduced in (3.2.20) is a symmetric integral, which induces a cancellation for large Laplace frequencies. In the subsequent subsection we will prove that our smallness assumption is consistent, i.e. if the condition is satisfied by f_ε, then it is also satisfied by $u_\varepsilon - u_0$ when u_ε solves (3.2.15).

Definition 3.4.1. *We define a sequence of cutoff functions $\kappa_{\delta_1} \in C_c^\infty(\mathbb{R})$ by*

$$\kappa_{\delta_1}(s) := \kappa\Big(\frac{s}{\delta_1}\Big), \tag{3.4.1}$$

where $\kappa \in C_c^\infty(\mathbb{R})$, $0 \le \kappa \le 1$, $\kappa(s) = 1$ for $|s| \le 1$ and $\kappa(s) = 0$ for $|s| \ge 2$. Let $R, \varepsilon, \delta > 0$ and $z \in \mathbb{C}$. We define $Y_{R,\varepsilon,\delta}(z)$ by

$$Y_{R,\varepsilon,\delta}(z) := \frac{\delta}{1+|z|^2} + \frac{R\varepsilon|z|}{(1+\varepsilon|z|)(1+|z|^2)}. \tag{3.4.2}$$

We will consider $u = (f,g) \in X_{A,\tilde{\lambda}}^n$ (defined in (3.2.8)), s.t. a.e. on the line $\Re(z) = \frac{A}{2} = a > 0$:

$$|\mathcal{L}(f)(z,v)| \le Y_{R,\varepsilon,\delta}(z)e^{-\frac{1}{2}|v|}, \qquad |\mathcal{L}(g)(z,v)| \le Y_{R,\varepsilon,\delta}(z)e^{-\frac{1}{2}|v|} \tag{3.4.3}$$

$$|\mathcal{L}(f)(z,v)| \le \frac{Re^{-\frac{1}{2}|v|}}{|1+\varepsilon z|(1+|z|^2)}, \qquad |\mathcal{L}(g)(z,v)| \le \frac{Re^{-\frac{1}{2}|v|}}{|1+\varepsilon z|(1+|z|^2)} \tag{3.4.4}$$

$$|\partial_t f(t,v)| \le Re^{-\frac{1}{2}|v|}. \tag{3.4.5}$$

For $R, \delta, \varepsilon > 0$, $A \ge 1$, $a = \frac{A}{2}$ and $n \in \mathbb{N}$, let $\Omega_{A,R,\delta,\varepsilon}^n \subset X_{A,\tilde{\lambda}}^n$ be the set of functions given by:

$$\Omega_{A,R,\delta,\varepsilon}^n = \{u = (f,g) \in X_{A,\tilde{\lambda}}^n : \|u\|_{X_{A,\tilde{\lambda}}^n} \le R, (3.4.5) \text{ and } (3.4.3)\text{-}(3.4.4) \text{ for } \Re(z) = a\}. \tag{3.4.6}$$

Since the estimates (3.4.3)-(3.4.4) are stable under convex combinations of functions, we have:

Lemma 3.4.2. *For all $R, \delta, \varepsilon > 0$, $A \ge 1$ and $n \in \mathbb{N}$, the set $\Omega_{A,R,\delta,\varepsilon}^n$ is a nonempty, bounded, closed and convex subset of $X_{A,\tilde{\lambda}}^n$.*

The following theorem is the main result of this subsection, giving an a priori estimate for the solution operator to (3.2.15) under the smallness assumption $(f,g) \in \Omega_{A,R,\delta,\varepsilon}^n$ for small ε, δ. We prove the error term can be controlled by the dissipation $D_{\varepsilon,A}^\alpha$ (cf. (3.3.32)) provided by the linear equation. Observe that existence of (unique) global solutions of (3.2.15) has been proved in Lemma 3.2.17. Here we will prove a priori estimates that are uniform in the mollifying parameter $\gamma > 0$ and $\varepsilon > 0$.

Theorem 3.4.3. *Let $n \in \mathbb{N}$, $n \geq 2$. Assume $u_0 \in H_\lambda^n$ satisfies:*

$$c\mathbb{1}_{|v| \leq 4}(v) \leq u_0(v) \leq Ce^{-\frac{1}{2}|v|}.$$

Then there exist $A, \delta > 0$ such that for all $R > 0$ there is an $\varepsilon_0 > 0$ with the property that the operator ψ_{δ_1} given by:

$$\Psi_{\delta_1} : \Omega_{A,R,\delta,\varepsilon}^n \longrightarrow X_{A,\lambda}^n$$
$$(f,g) \mapsto \left((u - u_0)k_{\delta_1}, \mathcal{A}_\gamma^{0,0}[f](u)k_{\delta_1}\right), \ \mathcal{A}_\gamma^{0,0} \text{ as in (3.2.11) and } u \text{ solution to:}$$

$$\partial_t u = \frac{1}{\varepsilon}\gamma\nabla \cdot \left(\int_0^t K[u_0 + f(s)]\left(\frac{t-s}{\varepsilon}, v\right)\gamma\nabla u(s,v)\,\mathrm{d}s\right) \tag{3.4.7}$$
$$- \frac{1}{\varepsilon}\gamma\nabla \cdot \left(\int_0^t P_\gamma[u_0 + f(s)]\left(\frac{t-s}{\varepsilon}, v\right)u(s,v)\,\mathrm{d}s\right)$$
$$u(0,\cdot) = u_0(\cdot),$$

is well-defined and continuous (w.r.t. the topologies of $X_{A,\tilde{\lambda}}^n$, $X_{A,\lambda}^n$) for all $\gamma, \delta_1 \in (0, \frac{1}{2}]$ and $\varepsilon \in (0, \varepsilon_0)$. Furthermore, the solutions satisfy the following estimate:

$$\|\Psi_{\delta_1}(f,g)\|_{X_{A,\lambda}^n} + \|\partial_t \Psi_{\delta_1}(f,g)\|_{X_{A,\lambda}^{n-2}} \leq C(A, \delta_1). \tag{3.4.8}$$

Notice that the operator ψ_{δ_1} maps functions in $X_{A,\tilde{\lambda}}^n$ to functions in $X_{A,\lambda}^n$, thus yields better decay. As can be seen from Lemma 3.2.12 this follows from the fast decay of the initial datum, provided we can control the quadratic terms Q. In Section 3.3 we have shown that the quadratic functionals $Q[u_0]$ defined in (3.2.20) satisfy a coercivity estimate. In this subsection we will prove smallness for the perturbation $Q[f]$, so the sum $Q[u_0 + f]$ still has a sign. To this end we first include an auxiliary Lemma to represent those functionals in Laplace variables.

Lemma 3.4.4. *The quadratic functionals $Q_{\varepsilon,A}^{\alpha,\beta}[\nu](u)$ defined in (3.2.20) can be represented by means of the Laplace transform of u as:*

$$(2\pi)^{\frac{1}{2}}Q_{\varepsilon,A}^{\alpha,\beta}[\nu](u) = \int_\mathbb{R}\int_\mathbb{R}\int \langle \nabla(D^\alpha\mathcal{L}(u)\lambda)(z), D^{\alpha-\beta}\Lambda[\nu](\varepsilon z, \omega - \theta)\mathcal{L}(\nabla D^\beta u)(p)\rangle\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}\omega$$
$$- \int_\mathbb{R}\int_\mathbb{R}\int \langle \nabla(D^\alpha\mathcal{L}(u)\lambda)(z), \nabla D^{\alpha-\beta}\Lambda[\nu](\varepsilon z, \omega - \theta)\mathcal{L}(D^\beta u)(p)\rangle\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}\omega.$$

We use the short notation $z = a + i\omega$, $p = a + i\theta$ and Λ is given by M_1, M_2 (cf. (3.2.24)) as:

$$\Lambda[\nu](z, \tau, v) = \int_{\mathbb{R}^3}\int_\mathbb{R}(M_1 + M_2)(z, v - v')e^{-i\tau s}\eta(|v - v'|^2)\nu(s, v')\,\mathrm{d}s\,\mathrm{d}v'. \tag{3.4.9}$$

Proof. Follows directly from the elementary properties of the Laplace Transform. $\square$

Exploiting the symmetry properties of the functional $Q_{\varepsilon,A}^{\alpha,\alpha}[f_\varepsilon]$ is essential to proving that this term is small compared to the dissipation $D_{\varepsilon,A}^\alpha$ (cf. (3.3.32)). For better notation we first include some definitions.

Definition 3.4.5. *For $\varepsilon > 0$, $v \in \mathbb{R}^3$, $z = a + i\omega, p = a + i\theta \in \mathbb{C}$, define the matrices L_1, L_2:*

$$L_1(\varepsilon, z, p, v) := \frac{1}{2}(M_1(\varepsilon z, v) + M_1(\varepsilon \overline{p}, v)) \tag{3.4.10}$$

$$L_2(\varepsilon, z, p, v) := \frac{1}{2}(M_2(\varepsilon z, v) + M_2(\varepsilon \overline{p}, v)) \tag{3.4.11}$$

and the associated symmetrized kernel Λ_s by:

$$\Lambda_s[\nu](\varepsilon, z, p, v) := \Lambda_1[\nu](\varepsilon, z, p, v) + \Lambda_2[\nu](\varepsilon, z, p, v) \tag{3.4.12}$$

$$\Lambda_1[\nu](\varepsilon, z, p, v) := \int_{\mathbb{R}^3} L_1(\varepsilon, z, p, v - v') \left(\int_0^\infty e^{-is(\omega - \theta)} \nu(s, v') \, \mathrm{d}s \right) \eta(|v - v'|^2) \, \mathrm{d}v'$$

$$\Lambda_2[\nu](\varepsilon, z, p, v) := \int_{\mathbb{R}^3} L_2(\varepsilon, z, p, v - v') \left(\int_0^\infty e^{-is(\omega - \theta)} \nu(s, v') \, \mathrm{d}s \right) \eta(|v - v'|^2) \, \mathrm{d}v'.$$

We split the kernel L_2 further into:

$$N_2(\varepsilon, z, p, v) = L_2(\varepsilon, z, p, v) - N_1(\varepsilon, z, p, v), \text{ where} \tag{3.4.13}$$

$$N_1(\varepsilon, z, p, v) = \frac{1}{|v|^2} \frac{\varepsilon(a + i(\theta - \omega))}{(1 + \frac{\varepsilon z}{|v|})^2 (1 + \frac{\varepsilon \overline{p}}{|v|})^2} P_v. \tag{3.4.14}$$

Lemma 3.4.6. *Let $a > 0$ and $z = a + i\omega$, $p = a + i\theta$. Further let $\varepsilon \leq \frac{1}{a}$. For $V, W \in \mathbb{C}^3$ and L_1, N_1 as in the definition above, and $|v| \geq c > 0$, we have the estimates:*

$$|\langle V, L_1(\varepsilon, z, p, v)W \rangle| \leq C \frac{|P_v^\perp V||P_v^\perp W|}{1 + |v|} \frac{1 + \varepsilon|\theta - \omega|}{(1 + \alpha(\varepsilon z, v))(1 + \alpha(\varepsilon p, v))} \tag{3.4.15}$$

$$|\langle V, N_2(\varepsilon, z, p, v)W \rangle| \leq C \frac{|V||W|}{1 + |v|^3} \frac{\varepsilon^2|p||z| + \varepsilon^2|p||z|(1 + \varepsilon|\theta - \omega|)}{(1 + \alpha(\varepsilon z, v))^2 (1 + \alpha(\varepsilon p, v))^2}. \tag{3.4.16}$$

Proof. We start by proving (3.4.15). Using $\varepsilon \leq \frac{1}{a}$, $|v| \geq c > 0$ and the definition of L_1 (cf. (3.4.10)) and M_1 (cf. (3.2.24)) we can bound:

$$\begin{aligned} |\langle V, L_1(\varepsilon, z, p, v)W \rangle| \leq & \frac{C|P_v^\perp V||P_v^\perp W|}{|v|} \left| \frac{1}{1 + \varepsilon z/|v|} + \frac{1}{1 + \varepsilon \overline{p}/|v|} \right| \\ \leq & \frac{C|P_v^\perp V||P_v^\perp W|}{|v|} \left| \frac{1 + \varepsilon|\theta - \omega|}{(1 + \alpha(\varepsilon z, v))(1 + \alpha(\varepsilon p, v))} \right|. \end{aligned}$$

The decomposition of L_2 (defined in (3.4.11)) follows from the identity:

$$\frac{b}{(1 + b)^2} + \frac{\overline{c}}{(1 + \overline{c})^2} = \frac{b + \overline{c}}{(1 + b)^2 (1 + \overline{c})^2} + \left(\frac{4b\overline{c}}{(1 + b)^2 (1 + \overline{c})^2} + \frac{b\overline{c}(b + \overline{c})}{(1 + b)^2 (1 + \overline{c})^2} \right). \tag{3.4.17}$$

We insert $b = \frac{\varepsilon z}{|v|}$, $c = \frac{\varepsilon p}{|v|}$ and multiply (3.4.17) with $\frac{\pi^2 P_v}{4|v|}$. Then the first term on the right gives N_1, so the second gives N_2 as defined in (3.4.13). The latter is bounded by:

$$\begin{aligned} |N_2| \leq & \frac{\pi^2 P_v}{4|v|} \left(\frac{4b\overline{c}}{(1 + b)^2 (1 + \overline{c})^2} + \frac{b\overline{c}(b + \overline{c})}{(1 + b)^2 (1 + \overline{c})^2} \right) \\ \leq & \frac{C}{|v|^3} \left| \frac{\varepsilon^2|p||z| + \varepsilon^3|p||z|(a + |\theta - \omega|)}{(1 + \alpha(\varepsilon z, v))^2 (1 + \alpha(\varepsilon p, v))^2} \right| \leq \frac{C}{|v|^3} \frac{\varepsilon^2|p||z| + \varepsilon^2|p||z|(1 + \varepsilon|\theta - \omega|)}{(1 + \alpha(\varepsilon z, v))^2 (1 + \alpha(\varepsilon p, v))^2}. \end{aligned}$$

This proves estimate (3.4.16). $\qquad \square$

Our goal is to prove estimates for the functional $Q^{\alpha,\alpha}_{\varepsilon,A}[f_\varepsilon]$. This will be done estimating Λ_s as defined in (3.4.12), which is given by L_1, L_2 (cf. (3.4.10), (3.4.11)). We have decomposed $L_1 + L_2 = L_1 + N_1 + N_2$, and Lemma 3.4.6 gives estimates for L_1 and N_2. It remains to prove an estimate for N_1. Here we rely on the additional decay provided by the divergence property $f = \nabla \cdot g$ of functions in Ω. Under the divergence assumption we get the following Lemma.

Lemma 3.4.7. *Let N_1 be given by (3.4.14). Let $h = \nabla \cdot G$, where $G \in H^1_{\check\lambda}$, $|G(v)| \leq R_1 e^{-\frac{1}{2}|v|}$. For $a > 0$, $\varepsilon \in (0, \frac{1}{a}]$, $z = a + i\omega, p = a + i\theta \in \mathbb{C}$ we have:*

$$\left| \int \langle V, N_1(\varepsilon, z, p, v - v')W \rangle h(v') \eta(|v - v'|^2) \, \mathrm{d}v' \right| \leq \frac{C R_1 |V||W|(1 + \varepsilon|\omega - \theta|)}{(1 + |v|^3)(1 + \alpha(\varepsilon z, v))^2 (1 + \alpha(\varepsilon p, v))^2}. \quad (3.4.18)$$

Proof. We simply use that $h = \nabla \cdot G$ is a divergence and write:

$$\int_{\mathbb{R}^3} N_1(\varepsilon, z, p, v - v') \eta h(v') \, \mathrm{d}v' = - \int_{\mathbb{R}^3} \nabla_{v'} \left(N_1(\varepsilon, z, p, v - v') \eta(|v - v'|^2) \right) G(v') \, \mathrm{d}v'. \quad (3.4.19)$$

Explicitly computing the derivative of N_1 as defined in (3.4.14) gives:

$$\left| \nabla_v \left(N_1(\varepsilon, z, p, v) \eta(|v|^2) \right) \right| \leq C \frac{1 + \varepsilon|\theta - \omega|}{(1 + |v|^3)(1 + \alpha(\varepsilon z, v))^2 (1 + \alpha(\varepsilon p, v))^2}.$$

Now plugging the assumption $|G(v)| \leq R_1 e^{-\frac{1}{2}|v|}$ into (3.4.19) gives the claim. $\qquad\square$

Lemma 3.4.8. *For $A > 0, n \in \mathbb{N}$, $n \geq 2$, $R, \delta, \varepsilon > 0$ and all $(f, g) \in \Omega^n_{A, R, \delta, \varepsilon}$ we have:*

$$\begin{aligned}
\left| \int_0^\infty e^{-is\tau} f(s, v) \, \mathrm{d}s \right| &\leq C(A) \min\{Y_{R,\varepsilon,\delta}(\tau), \frac{R}{(1 + \varepsilon|\tau|)(1 + |\tau|^2)}\} e^{-\frac{1}{2}|v|} \\
\left| \int_0^\infty e^{-is\tau} g(s, v) \, \mathrm{d}s \right| &\leq C(A) \min\{Y_{R,\varepsilon,\delta}(\tau), \frac{R}{(1 + \varepsilon|\tau|)(1 + |\tau|^2)}\} e^{-\frac{1}{2}|v|},
\end{aligned} \quad (3.4.20)$$

for $\tau \in \mathbb{R}$. Here $Y_{R,\varepsilon,\delta}(\tau)$ is the function defined in (3.4.2).

Proof. By definition of $\Omega^n_{A, R, \delta, \varepsilon}$ (see (3.4.6)) for $\Re(z) = a$ there holds:

$$\begin{aligned}
|\mathcal{L}(f)(z, v)| &\leq \min\{Y_{R,\varepsilon,\delta}(z), \frac{R}{(1 + \varepsilon|z|)(1 + |z|^2)}\} e^{-\frac{1}{2}|v|} \\
|\mathcal{L}(g)(z, v)| &\leq \min\{Y_{R,\varepsilon,\delta}(z), \frac{R}{(1 + \varepsilon|z|)(1 + |z|^2)}\} e^{-\frac{1}{2}|v|}.
\end{aligned} \quad (3.4.21)$$

Notice that the estimate is the same for f and g. We rewrite the left-hand side of (3.4.20) as:

$$\begin{aligned}
\left| \int_0^\infty e^{-is\tau} f(s, v) \, \mathrm{d}s \right| &= \left| \int_0^\infty e^{-is\tau} e^{-as} f(s, v) \kappa_2(s) e^{as} \, \mathrm{d}s \right| \\
&= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \mathcal{L}(f)(a + i(\tau - \omega)) F(\tau - \omega) \, \mathrm{d}\omega \right|,
\end{aligned}$$

where $F(\omega) = \int_{\mathbb{R}} e^{-is\omega} \kappa_2(|s|) e^{as} \, \mathrm{d}s$. The function F is the Fourier transformation of a fixed Schwartz function, hence decays faster than any polynomial. For the rational function $Y_{R,\varepsilon,\delta}$ defined in (3.4.2), a straightforward computation shows $|Y_{R,\varepsilon,\delta} * F| \leq C|Y_{R,\varepsilon,\delta}|$ with $C > 0$ independent of $\varepsilon \in (0, \frac{1}{a}]$ and $R, \delta > 0$. $\qquad\square$

Lemma 3.4.9. *Let $A \geq 1$, $n \geq 2$, α a multi-index with $|\alpha| \leq n$ and $c > 0$ arbitrary be given. There exists $\delta_0(c, A, n) > 0$ such that for all $\delta \in (0, \delta_0]$ and $R > 0$, we can estimate:*

$$|Q_{\varepsilon,A}^\alpha[f](u)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |Q_{\varepsilon,A}^{\alpha,\beta}[f](u)| \leq c D_{\varepsilon,A}^\alpha(u) + \|u\|_{V_{A,\lambda}^n}^2, \tag{3.4.22}$$

for all $(f, g) \in \Omega_{A,R,\delta,\varepsilon}^n$, when $0 < \varepsilon \leq \varepsilon_0(\delta, R, A, c, n)$ is small.

Proof. Fix $A \geq 1$ and $n \in \mathbb{N}$, $n \geq 2$ and $c > 0$ as in the assumption. We first estimate the highest order term $\beta = \alpha$ in the quadratic form Q. We start our estimate from the representation in Lemma 3.4.4 (we write $\nabla = {}^\gamma\nabla$ for shortness):

$$(2\pi)^{\frac{1}{2}} Q_{\varepsilon,A}^{\alpha,\alpha}[\nu](u) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int \langle \lambda \mathcal{L}(\nabla D^\alpha u)(z), \Lambda(\varepsilon z, \omega - \theta) \mathcal{L}(\nabla D^\alpha u)(p) \rangle \, \mathrm{d}v \, \mathrm{d}\theta \, \mathrm{d}\omega \tag{3.4.23}$$

$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int \langle \mathcal{L}(D^\alpha u)(z) \nabla(\lambda), \Lambda(\varepsilon z, \omega - \theta) \mathcal{L}(\nabla D^\alpha u)(p) \rangle \, \mathrm{d}v \, \mathrm{d}\theta \, \mathrm{d}\omega \tag{3.4.24}$$

$$- \int_{\mathbb{R}} \int_{\mathbb{R}} \int \langle \nabla(\mathcal{L}(D^\alpha u)\lambda)(z), \nabla\Lambda(\varepsilon z, \omega - \theta) \mathcal{L}(D^\alpha u)(p) \rangle \, \mathrm{d}v \, \mathrm{d}\theta \, \mathrm{d}\omega \tag{3.4.25}$$

$$= J_1 + I_3 + J_2.$$

We start with estimating the critical term J_1. We can symmetrize in p, z, and replace Λ by Λ_s as introduced in Definition 3.4.5. The symmetrization gives (for shortness write $V = \mathcal{L}(\nabla D^\alpha u)$):

$$J_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \int \lambda \langle V(z, v), \Lambda(\varepsilon z, \omega - \theta, v) V(p, v) \rangle \, \mathrm{d}v \, \mathrm{d}\theta \, \mathrm{d}\omega$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int \lambda \langle V(z, v), (\Lambda_1 + \Lambda_2) V(p, v) \rangle \, \mathrm{d}v \, \mathrm{d}\theta \, \mathrm{d}\omega = I_1 + I_2. \tag{3.4.26}$$

We estimate I_1 using the estimate on L_1 in (3.4.15) and use Lemma 3.4.8 to bound $\mathcal{L}(f)$:

$$|I_1| \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \int \int \lambda \frac{|P_{(v-v')}^\perp V(z, v)||P_{(v-v')}^\perp V(p, v)|}{|v - v'|} \frac{(1 + \varepsilon|\theta - \omega|)|\mathcal{L}(f)(i(\theta - \omega), v')|\eta}{(1 + \alpha(\varepsilon z, v - v'))(1 + \alpha(\varepsilon p, v - v'))}$$

$$\leq C(A) \int_{\mathbb{R}} \int_{\mathbb{R}} \int \frac{\lambda|V(z, v)|_v |V(p, v)|_v}{(1 + |v|)(1 + \alpha(\varepsilon z, v))(1 + \alpha(\varepsilon p, v))} \frac{R\varepsilon|\theta - \omega|}{(1 + \varepsilon|\theta - \omega|)(1 + |\theta - \omega|)^2} \tag{3.4.27}$$

$$+ C(A) \int_{\mathbb{R}} \int_{\mathbb{R}} \int \frac{\lambda|V(z)|_v |V(p)|_v}{(1 + |v|)(1 + \alpha(\varepsilon p, v))(1 + \alpha(\varepsilon z, v))} Y_{R,\varepsilon,\delta}(\theta - \omega).$$

Observe the following straightforward integral estimates hold:

$$\int_{\mathbb{R}} \frac{R\varepsilon|\tau|}{(1 + \varepsilon|\tau|)(1 + |\tau|)^2} \, \mathrm{d}\tau \leq C R\varepsilon^{\frac{1}{2}}, \quad \int_{\mathbb{R}} Y_{R,\varepsilon,\delta}(\tau) \, \mathrm{d}\tau \leq C(\delta + \varepsilon^{\frac{1}{2}} R). \tag{3.4.28}$$

We apply Young's inequality to (3.4.27) and use (3.4.28) to obtain a total bound of:

$$|I_1| \leq \int_{\mathbb{R}} \int \frac{C(A)\lambda(v)|V(z, v)|_v^2}{(1 + |v|)(1 + \alpha(z, v))^2} \, \mathrm{d}\omega \left(\int_{\mathbb{R}} \frac{R\varepsilon|\tau|}{(1 + \varepsilon|\tau|)(1 + |\tau|)^2} + Y_{R,\varepsilon,\delta}(\tau) \right) \, \mathrm{d}v \leq \frac{c}{6} D_{\varepsilon,A}^\alpha(u),$$

for $0 < \delta < \delta_0(n, A)$, $0 < \varepsilon \leq \varepsilon_0(\delta, R, A, c, n)$ small and $D_{\varepsilon,A}^\alpha(u)$ as defined in (3.3.32). The term I_2 (cf. (3.4.26)) can be controlled similarly. We split I_2 further into:

$$|I_2| \leq \left| \int_\mathbb{R} \int_\mathbb{R} \int \langle \lambda V(z,v)(z,v) N_1(\varepsilon, z, p, v) V(p,v) \rangle \mathcal{L}(f)(i(\theta - \omega), v') \eta \, dv' \, dv \, d\theta \, d\omega \right|$$

$$+ \left| \int_\mathbb{R} \int_\mathbb{R} \int \langle \lambda V(z,v)(z,v), N_2(\varepsilon, z, p, v) V(p,v) \rangle \mathcal{L}(f)(i(\theta - \omega), v') \eta \, dv' \, dv \, d\theta \, d\omega \right|$$

$$= I_{2,1} + I_{2,2}.$$

The integral $I_{2,2}$ can be bounded using (3.4.16) and (3.4.28) (adapting $0 < \delta_0, \varepsilon_0$ if needed):

$$|I_{2,2}| \leq \int_\mathbb{R} \int_\mathbb{R} \int \int \int \lambda \frac{|V(z)||V(p)|}{|v - v'|^3} \frac{\varepsilon^2|p||z| + \varepsilon^2|p||z|(1 + \varepsilon|\theta - \omega|)}{(1 + \alpha(\varepsilon z, v - v'))^2(1 + \alpha(\varepsilon p, v - v'))^2} |\mathcal{L}(f)| \eta \, dv \, dv' \, d\theta \, d\omega$$

$$\leq C(A)\left(\delta + R\varepsilon^{\frac{1}{2}}\right) \int_\mathbb{R} \int \lambda \frac{|V(z,v)|^2}{1 + |v|} \frac{\beta(\varepsilon z, |v|) + \alpha^2(\varepsilon z, |v|)}{(1 + \alpha(\varepsilon z, v - v'))^4} \, dv \, d\omega \leq \frac{c}{4} D_{\varepsilon,A}^\alpha(u),$$

where we use that $C_2 \leq CC_1$. It remains to control $I_{2,1}$, which we estimate by means of (3.4.18). We obtain:

$$|I_{2,1}| \leq C(A)\left(\delta + R\varepsilon^{\frac{1}{2}}\right) \int_\mathbb{R} \int_\mathbb{R} \int \frac{\lambda(v)|V(z,v)|^2}{(1 + |v|^3)(1 + \alpha(\varepsilon z, v))^2} \, d\omega \, dv \leq \frac{c}{4} D_{\varepsilon,A}^\alpha(u).$$

Therefore $|J_1| \leq \frac{c}{2} D_{\varepsilon,A}^\alpha(u)$. The remaining terms can be estimated by:

$$|J_2| + |I_3| + \sum_{\beta < \alpha} \binom{\alpha}{\beta} |Q_{\varepsilon,A}^{\alpha,\beta}[f](u)| \leq \frac{c}{2} D_{\varepsilon,A}^\alpha(u) + \|u\|_{V_{A,\lambda}^n}^2. \tag{3.4.29}$$

The estimate for $Q_{\varepsilon,A}^{\alpha,\beta}$, $\beta < \alpha$ can be seen as follows: Let $V, W \in \mathbb{C}^3$ be arbitrary. By the definition (3.4.9) of Λ and the estimate for $\mathcal{L}(f)$ in Lemma 3.4.8 we can bound $\langle V, \Lambda^\beta W \rangle$ by:

$$|\langle V, D^\beta \Lambda[f](z, \tau, v) W \rangle| = \left| \int_{\mathbb{R}^3} \int_0^\infty \langle V, D^\beta \left((M_1 + M_2)(z, v - v')\eta \right) W \rangle e^{-i\tau s} f(s, v') \, ds \, dv' \right|$$

$$\leq C(A) \int_{\mathbb{R}^3} \left| \langle V, D^\beta \left((M_1 + M_2)(z, v - v')\eta \right) W \rangle \right| Y_{R,\varepsilon,\delta}(\tau) e^{-\frac{1}{2}|v'|} \, dv'.$$

We use Lemma 3.3.3 to estimate the velocity integral by:

$$|\langle V, D^\beta \Lambda[f](z, \tau, v) W \rangle| \leq \frac{C(A)(1 + \alpha(\varepsilon z, v))}{(1 + |v|)^{|\beta|}} C_1(\varepsilon z, v)|V||W| Y_{R,\varepsilon,\delta}(\tau). \tag{3.4.30}$$

Now we can argue as in the proof of Lemma 3.3.7. The term J_2 is estimated as the corresponding term in the proof of Lemma 3.3.7, using (3.4.30). For estimating I_3 (given by (3.4.24)), some care is needed. We rewrite I_3, integrating by parts (we use the shorthand $W(z, v) = \mathcal{L}(D^\alpha u)(z, v)$):

$$I_3 = -\int_\mathbb{R} \int_\mathbb{R} \int \langle W(z,v) \nabla^2(\lambda), \Lambda(\varepsilon z, \omega - \theta, v) W(p, v) \rangle \, dv \, d\theta \, d\omega$$

$$- \int_\mathbb{R} \int_\mathbb{R} \int \langle W(z,v) \nabla(\lambda), \nabla \cdot \Lambda(\varepsilon z, \omega - \theta, v) W(p, v) \rangle \, dv \, d\theta \, d\omega$$

$$- \int_\mathbb{R} \int_\mathbb{R} \int \langle \nabla W(z,v) \otimes \nabla(\lambda), \Lambda(\varepsilon z, \omega - \theta, v) W(p, v) \rangle \, dv \, d\theta \, d\omega.$$

Here we use the notation $\langle A, B \rangle = \sum_{i,j} \overline{A}_{i,j} B_{i,j}$ for matrices A, B. The first two lines are bounded by $\frac{1}{2}\|u\|^2_{V^n_{A,\lambda}}$ using (3.4.30) and the Plancherel Lemma 3.2.13. The third line can be estimated like the corresponding I_3 in Lemma 3.3.7. The lower order terms $\beta < \alpha$ are estimated in the same way using Lemma 3.3.5, so we indeed obtain (3.4.29). Combining all the estimates, we obtain the upper estimate $|Q^\alpha_{\varepsilon,A}[f](u)| \leq cD^\alpha_{\varepsilon,A}(u) + \|u\|^2_{V^n_{\varepsilon,A}}$ as claimed. $\qquad\square$

We obtain the main result of this subsection, Theorem 3.4.3, as a Corollary.

Proof of Theorem 3.4.3. We have proved the existence of solutions u to (3.4.7) in Lemma 3.2.17. We need to show continuity of the mapping Ψ_{δ_1} and the a priori estimate (3.4.8). First we use Lemma 3.2.12 to bound the norm of the solution by:

$$A\|u\|^2_{V^n_{A,\lambda}} \leq C\|u_0\|^2_{H^n_\lambda} - 2\sum_{|\alpha| \leq n} Q^\alpha_{\varepsilon,A}[u_0](u) + Q^\alpha_{\varepsilon,A}[f](u). \tag{3.4.31}$$

Applying Lemma 3.3.7 to $Q^\alpha_{\varepsilon,A}[u_0](u)$ and Lemma 3.4.9 to $Q^\alpha_{\varepsilon,A}[f](u)$ we find that for $A > 0$ and $\delta > 0$ sufficiently small, $R > 0$ and $\varepsilon > 0$ small enough we have:

$$Q^\alpha_{\varepsilon,A}[u_0](u) + Q^\alpha_{\varepsilon,A}[f](u) \geq \frac{c}{2}D^\alpha_{\varepsilon,A} - C\|u\|_{V^n_{A,\lambda}}.$$

Plugging this back into (3.4.31) we find $A, \delta > 0$ such that for all $R > 0$ and $\varepsilon > 0$ small we have, independently of $0 < \gamma \leq 1$:

$$\|u\|_{V^n_{A,\lambda}} \leq \|u_0\|_{H^n_\lambda}. \tag{3.4.32}$$

Now define $U := \int_0^t A^{0,0}_\gamma[u_0 + f](u)$ ($\mathcal{A}$ as in Notation 3.2.11). Then by equation (3.4.7) we have $(u - u_0) = \nabla \cdot U$. Using Lemma 3.2.15 we write:

$$\mathcal{L}(\partial_t U)(z, v) = \int (M_1 + M_2)(\varepsilon z, v - v')\eta \mathcal{L}\left((u_0(v') + f(\cdot, v'))^\gamma \nabla u(\cdot, v)\right)(z)\, \mathrm{d}v'$$
$$- \int {}^\gamma\nabla \cdot (M_1 + M_2)(\varepsilon z, v - v')\eta \mathcal{L}\left((u_0(v') + f(\cdot, v'))u(\cdot, v)\right)(z)\, \mathrm{d}v'. \tag{3.4.33}$$

Now M_1, M_2 as well as their derivatives are bounded. Further Lemma 3.4.8 and (3.4.28) imply:

$$\|\mathcal{L}(f)(z, v)\|_{L^1_{\Re(z)=0}} \leq C(A)(\delta + R\varepsilon^{\frac{1}{2}})e^{-\frac{1}{2}|v|}. \tag{3.4.34}$$

Hence for $\delta > 0$ and $\varepsilon(A, R) > 0$ sufficiently small, combining (3.4.33), (3.4.34), and the Plancherel Lemma 3.2.13 gives the desired estimate for U in (3.4.8). Plugging this back into (3.4.7) gives (3.4.8):

$$\|((u - u_0)\kappa_{\delta_1}, U\kappa_{\delta_1})\|_{X^n_{A,\lambda}} + \|\partial_t((u - u_0)\kappa_{\delta_1}, U\kappa_{\delta_1})\|_{X^{n-2}_{A,\lambda}} \leq C.$$

It remains to show continuity of the operator Ψ_{δ_1} for positive γ, ε. Let $(f_i, g_i) \in \Omega^n_{A,R,\delta,\varepsilon}$, $i = 1, 2$ and u_1, u_2 the corresponding solutions to (3.4.7). For shortness write

$$K_i = \frac{1}{\varepsilon}K[u_0 + f_i(s)]\left(\frac{t-s}{\varepsilon}, v\right), \quad P_i = \frac{1}{\varepsilon}P[u_0 + f_i(s)]\left(\frac{t-s}{\varepsilon}, v\right).$$

Then the difference $u_1 - u_2$ satisfies $(u_1(0) - u_2(0)) = 0$ and:

$$\partial_t(u_1 - u_2) = {}^\gamma\nabla \cdot \left(\int_0^t K_1{}^\gamma\nabla u_1(s, v) - P_1 u_1 - K_2{}^\gamma\nabla u_2(s, v) + P_2 u_2 \, ds \right). \tag{3.4.35}$$

For $m \in \mathbb{N}$ arbitrary, $\|K[f]\|_{L^2([0,1];C^m(\mathbb{R}^+;\mathbb{R}^3))} + \|P[f]\|_{L^2([0,1];C^m(\mathbb{R}^+;\mathbb{R}^3))} \leq C\|(f,g)\|_{X_{A,\lambda}^n}$ are continuous. Recalling that ${}^\gamma\nabla$ are mollifying operators, the continuity of Ψ_{δ_1} now follows from (3.4.35) by Gronwall's Lemma. $\qquad\square$

3.4.2 Invariance of the set Ω under the mapping Ψ

Recovering the quadratic decay in Laplace variables

In the last subsection we have shown that for $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ as defined in (3.4.6), the equation

$$\begin{aligned}
\partial_t u_\varepsilon = &\frac{1}{\varepsilon}{}^\gamma\nabla \cdot \left(\int_0^t K[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) {}^\gamma\nabla u_\varepsilon(s, v) \, ds \right) \\
&- \frac{1}{\varepsilon}{}^\gamma\nabla \cdot \left(\int_0^t P[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s, v) \, ds \right) \\
u_\varepsilon(0, \cdot) = &u_0(\cdot),
\end{aligned} \tag{3.4.36}$$

has solutions in $X_{A,\lambda}^n$. The goal of this section is to show that the associated solution operator Ψ_{δ_1} defined in (3.4.7) leaves the set $\Omega_{A,R,\delta,\varepsilon}^n$ (cf. (3.4.6)) invariant. More precisely, we will prove the following theorem.

Theorem 3.4.10. *Let $n \geq 6$ and assume $v_0 \in H_\lambda^n$ satisfies the bounds:*

$$0 \leq v_0(v) \leq Ce^{-\frac{1}{2}|v|}.$$

Let $A, \delta > 0$ be as in Theorem 3.4.3 and Ψ_{δ_1} the solution operator to (3.4.36):

$$\begin{aligned}
\Psi_{\delta_1} : &\Omega_{A,R,\delta,\varepsilon}^n \longrightarrow X_{A,\lambda}^n \\
&(f_\varepsilon, g_\varepsilon) \mapsto \left((u_\varepsilon - u_0)k_{\delta_1}, \mathcal{A}_\gamma^{0,0}[u_0 + f](u_\varepsilon)k_{\delta_1} \right), u_\varepsilon \text{ solves (3.4.36) with } u_0 = m + \delta_2 v_0.
\end{aligned}$$

There exist $\delta_1, \varepsilon_0, R > 0$ such that for $\delta_2, \varepsilon \in (0, \varepsilon_0]$ and for all $\gamma \in (0, 1]$, the set $\Omega_{A,R,\delta,\varepsilon}^n$ is invariant under the mapping Ψ_{δ_1}.

As a first step, we will prove estimate (3.4.4). Differentiating equation (3.4.36) yields, where $\mathcal{A}_\gamma^{\alpha,\beta}$ is defined in (3.2.18):

$$\partial_t D^\alpha u_\varepsilon = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \nabla \cdot \left(\mathcal{A}_\gamma^{\alpha,\beta}[u_0 + f_\varepsilon](u_\varepsilon) \right). \tag{3.4.37}$$

Therefore in order to characterize the properties of $D^\alpha u_\varepsilon$ in Laplace variables, we first need to understand the right-hand side of the above equation in this framework.

Lemma 3.4.11. *Let $n \geq 0$ and $(f, g) \in \Omega_{A,R,\delta,\varepsilon}^n$. Further let $u_0 \in C(\mathbb{R}^3)$ satisfy*

$$0 \leq u_0(v) \leq C e^{-\frac{1}{2}|v|}.$$

Let $a = \frac{A}{2} \geq \frac{1}{2}$, $\gamma \in (0, 1]$ and $\beta \leq \alpha$ be multi-indexes with $|\alpha| = m < n$. Then for almost every $z \in \mathbb{C}$ with $\Re(z) = a$ we can estimate ($\mathcal{A}_\gamma^{\alpha,\beta}$ and $|\cdot|_{F^m}$ as in Notation 3.2.11):

$$|\mathcal{L}(\mathcal{A}_\gamma^{\alpha,\beta}[u_0](u))(z, v)| \leq \frac{C(A)|u|_{F^{m+1}}}{|1 + \varepsilon z|}, \quad |\mathcal{L}(\mathcal{A}_\gamma^{\alpha,\beta}[f](u))(z, v)| \leq C(A)\frac{Y_{\varepsilon,\delta} *_a |u|_{F^{m+1}}}{|1 + \varepsilon z|}.$$

Here the convolution $*_a$ is to be understood as ($z = a + i\omega$):

$$(f *_a g)(a + i\omega) = \int_\mathbb{R} f(i\theta)g(a + i(\omega - \theta)) \, \mathrm{d}\theta. \tag{3.4.38}$$

Proof. Is a direct consequence of elementary properties of the Laplace transform, Lemma 3.4.8 and the defining formula (3.2.18) of $\mathcal{A}_\gamma^{\alpha,\beta}$. $\qquad\square$

Lemma 3.4.12. *Let $u \in V_{A,\lambda}^n$ for $n \geq 2$. For $a \in (0, 1]$ and $\delta_1 \in (0, 1]$ we have:*

$$\|\mathcal{L}(u\kappa_{\delta_1})(\cdot, v)\|_{L^\infty_{\Re(z)=a}} \leq C(a, \delta_1)\|\mathcal{L}(u)(\cdot, v)\|_{L^2_{\Re(z)=a}} \tag{3.4.39}$$

$$\|\mathcal{L}(u\kappa_{\delta_1})(\cdot, v)\|_{L^2_{\Re(z)=a}} \leq C(a, \delta_1)\|\mathcal{L}(u)(\cdot, v)\|_{L^2_{\Re(z)=a}}. \tag{3.4.40}$$

Proof. We start by proving (3.4.39). Consider the two-sided Laplace transform $\tilde{\mathcal{L}}$:

$$\tilde{\mathcal{L}}(f)(z) = \int_{-\infty}^\infty e^{-zt}f(t) \, \mathrm{d}t.$$

Extending $u(t) = 0$ for negative t, we find that for $\Re(z) = a \geq \frac{1}{2}$:

$$\tilde{\mathcal{L}}(u\kappa_{\delta_1}) = \tilde{\mathcal{L}}(\kappa_{\delta_1}) *_a \tilde{\mathcal{L}}(u).$$

Since $\tilde{\mathcal{L}}(\kappa_{\delta_1})$ is a Schwartz function, the claim follows from Young's inequality and the assumption $n \geq 2$ (so both sides of (3.4.39), (3.4.40) are continuous). The proof of (3.4.40) follows similarly. $\quad\square$

Now that we can characterize the properties of the operators $\mathcal{A}_\gamma^{\alpha,\beta}$ in Laplace variables, we are able to prove bounds for the Laplace transforms of the solution u_ε.

Lemma 3.4.13. *Let $n \geq 2$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 3.4.3. For $R > 0$, $\gamma \in (0, 1]$, $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ let $u_\varepsilon \in V_{A,\lambda}^n$ be the solution to (3.4.36), and let $|\alpha| = m \leq n - 2$. Recall the family of cutoff functions κ_δ defined in (3.4.1). For $\delta_3, \varepsilon \in (0, 1]$, we have:*

$$|\mathcal{L}(\kappa_{\delta_3}D^\alpha(u_\varepsilon - u_0))| \leq \frac{C(A, \delta_3)}{|1 + \varepsilon z|}\frac{1}{|z|}(|u_\varepsilon\kappa_{2\delta_3}|_{F^{m+2}} + Y_{\varepsilon,\delta} *_a |u_\varepsilon\kappa_{2\delta_3}|_{F^{m+2}}) \tag{3.4.41}$$

$$|\mathcal{L}\left(\kappa_{\delta_3}\mathcal{A}_\gamma^{0,0}[u_0 + f_\varepsilon](u_\varepsilon)\right)| \leq \frac{C(A, \delta_3)}{|1 + \varepsilon z|}\frac{1}{|z|}(|u_\varepsilon\kappa_{2\delta_3}|_{F^{m+2}} + Y_{\varepsilon,\delta} *_a |u_\varepsilon\kappa_{2\delta_3}|_{F^{m+2}}), \tag{3.4.42}$$

a.e. on the line $\Re(z) = a$. Again we use the shorthand $*_a$ as introduced in (3.4.38).

Proof. Integrating the equation (3.4.36) we find:

$$(u_\varepsilon - u_0)(T) = \int_0^T \frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t K[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s,v) \, \mathrm{d}s \right)$$
$$- \frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t P[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s,v) \, \mathrm{d}s \right) \, \mathrm{d}t.$$

Since $\kappa_{2\delta_3} = 1$ on the support of κ_{δ_3}, the Volterra structure of the equation allows to rewrite:

$$\kappa_{\delta_3}(u_\varepsilon - u) = \kappa_{\delta_3} \int_0^T \frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t K[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) \nabla(\kappa_{2\delta_3} u_\varepsilon)(s,v) \, \mathrm{d}s \right) \qquad (3.4.43)$$
$$- \frac{1}{\varepsilon} \nabla \cdot \left(\int_0^t P[u_0 + f_\varepsilon(s)] \left(\frac{t-s}{\varepsilon}, v \right) (\kappa_{2\delta_3} u_\varepsilon)(s,v) \, \mathrm{d}s \right) \, \mathrm{d}t.$$

Hence in Laplace variables we have:

$$z \mathcal{L}(D^\alpha(u_\varepsilon - u_0)\kappa_{\delta_3}) = \mathcal{L}\left(\kappa_{\delta_3} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \nabla \cdot \left(\mathcal{A}_\gamma^{\alpha,\beta}[u_0 + f_\varepsilon](u_\varepsilon \kappa_{2\delta_3}) \right) \right).$$

Estimate (3.4.41) now follows from Lemma 3.4.11 and Lemma 3.4.12. Estimate (3.4.42) is proved in the same way. $\qquad \square$

Lemma 3.4.14 (L^∞ estimate in Laplace variables). *Let $n \geq 2$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 3.4.3. For $R > 0$, $\gamma, \delta_2 \in (0,1]$, $(f_\varepsilon, g_\varepsilon) \in \Omega^n_{A,R,\delta,\varepsilon}$ let $u_\varepsilon \in V^n_{A,\lambda}$ be the solution to (3.4.36) with $u_0 = m(v) + \delta_2 v_0(v)$, where $v_0 \in H^n_\lambda$ satisfies:*

$$0 \leq v_0(v) \leq C e^{-\frac{1}{2}|v|}.$$

Then for $m \in \mathbb{N}$, $m \leq n-2$, $\varepsilon > 0$ small enough and $\delta_1 \in (0,1]$, there holds:

$$\|\mathcal{L}(\nabla^m u_\varepsilon \kappa_{\delta_1})\|_{L^\infty_{\Re(z)=a}} \leq C(A, \delta_1) e^{-\frac{1}{2}|v|}.$$

Proof. We solve equation (3.4.36) with $(f_\varepsilon, g_\varepsilon) \in \Omega^n_{A,R,\delta,\varepsilon}$. Now Theorem 3.4.9 shows that there are $A, \delta, C(A) > 0$ such that for all $R > 0$ a solution u_ε to (3.4.36) satisfies:

$$\|u_\varepsilon\|_{V^n_{A,\lambda}} \leq C(A),$$

provided $\varepsilon > 0$ is small enough. By Plancherel Lemma 3.2.13 this implies in particular

$$\|\mathcal{L}(D^\alpha u_\varepsilon)\|_{L^2_v L^2_{\Re(z)=a}} \leq C(A) \quad \text{for } |\alpha| \leq n.$$

With Sobolev inequality we can infer the existence of a constant $C(A) > 0$ such that for every multi-index α with $|\alpha| \leq n-2$ we have:

$$\|\mathcal{L}(D^\alpha u_\varepsilon(\cdot, v))\|_{L^2_{\Re(z)=a}} \leq C(A) e^{-\frac{1}{2}|v|}.$$

Now with Lemma 3.4.12 we can estimate:

$$\|\mathcal{L}(\nabla^m u_\varepsilon \kappa_{\delta_1})\|_{L^\infty_{\Re(z)=a}} \leq C(A, \delta_1) e^{-\frac{1}{2}|v|},$$

as claimed. $\qquad \square$

We can plug the L^∞ estimate for the Laplace transform back into (3.4.13) and bootstrap it to a pointwise estimate.

Lemma 3.4.15 (Linear decay in Laplace variables). *Let $n \geq 4$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 3.4.3. For $R > 0$, $\gamma, \delta_2 \in (0,1]$, $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ let $u_\varepsilon \in V_{A,\lambda}^n$ be the solution to (3.4.36) with $u_0 = m(v) + \delta_2 v_0(v)$, where $v_0 \in H_\lambda^n$ satisfies:*

$$0 \leq v_0(v) \leq C e^{-\frac{1}{2}|v|}.$$

Then for $m \in \mathbb{N}$, $m \leq n-4$, $\varepsilon > 0$ small enough and $\delta_1 \in (0,1]$ there holds:

$$|\mathcal{L}(\nabla^m((u - u_\varepsilon)\kappa_{\delta_1})(z,v))| \leq \frac{C(A,\delta_1)e^{-\frac{1}{2}|v|}}{1 + |z|}$$

$$|\mathcal{L}(\nabla^m(\mathcal{A}_\gamma^{0,0}[u_0 + f_\varepsilon](u_\varepsilon)\kappa_{\delta_1})(z,v)| \leq \frac{C(A,\delta_1)e^{-\frac{1}{2}|v|}}{1 + |z|}.$$

Proof. Follows by combining Lemma 3.4.13 with Lemma 3.4.14. $\square$

Bootstrapping the estimate in Lemma 3.4.13 gives an additional quadratic decay, which is the content of the following Lemma.

Lemma 3.4.16 (Quadratic decay of Laplace Transforms). *Let $n \geq 4$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 3.4.3. For $R > 0$, $\gamma, \delta_2 \in (0,1]$, $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ let $u_\varepsilon \in V_{A,\lambda}^n$ be the solution to (3.4.36) with $u_0 = m(v) + \delta_2 v_0(v)$, where $v_0 \in H_\lambda^n$ satisfies:*

$$0 \leq v_0(v) \leq C e^{-\frac{1}{2}|v|}.$$

Then for $m \in \mathbb{N}$, $m \leq n-4$, $\varepsilon > 0$ small enough and $\delta_1 \in (0,1]$ there holds:

$$|\mathcal{L}(\nabla^m(u_\varepsilon - u_0)\kappa_{\delta_1})(z,v)| \leq \frac{C(A,\delta_1)e^{-\frac{1}{2}|v|}}{|1 + \varepsilon z|(1 + |z|^2)}$$

$$|\mathcal{L}(\nabla^m(\mathcal{A}_\gamma^{0,0}[u_0 + f_\varepsilon](u_\varepsilon)\kappa_{\delta_1})(z,v)| \leq \frac{C(A,\delta_1)e^{-\frac{1}{2}|v|}}{|1 + \varepsilon z|(1 + |z|^2)}.$$

Proof. Follows by iterating Lemma 3.4.13 further with the estimate Lemma 3.4.15. For completeness we remark that the linear decay of $|u_\varepsilon \kappa_{\delta_1}|_{F^{m+2}}$ is stable under convolution with $Y_{\varepsilon,\delta}$. To see this we estimate the convolution explicitly ($z = a + i\omega$, $y = a + i\theta$, $a \geq \frac{1}{2}$):

$$Y_{\varepsilon,\delta} *_a |u_\varepsilon \kappa_{\delta_1}|_{F^{m+2}} \leq \int_\mathbb{R} \frac{C(A,\delta_1)}{1 + |z - y|} \left(\frac{\delta}{1 + |z|^2} + \frac{R\varepsilon|z|}{(1 + \varepsilon|z|)(1 + |z|^2)} \right) d\omega e^{-\frac{1}{2}|v|}$$

$$\leq \frac{C(A,\delta_1)}{1 + |z|} + \int_\mathbb{R} \frac{C(A,\delta_1)}{1 + |z - y|} \frac{R\varepsilon|z|}{(1 + \varepsilon|z|)(1 + |z|^2)} d\omega e^{-\frac{1}{2}|v|}.$$

It remains to show that the last integral decays linearly with a prefactor independent of $R > 0$. This can be seen by splitting the integral into the regions

$$D_d(x) := \{y : \Re(y) = a, |y| \geq 2|x| \text{ or } |y| \leq \frac{1}{2}|x|\}$$

$$D_c(x) := \{y : \Re(y) = a, \frac{1}{2}|x| \leq |y| \leq 2|x|\},$$

when the integral can be estimated as ($C(A, \delta_1)$ might change from line to line):

$$\int_{\mathbb{R}} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{R\varepsilon|y|}{(1 + \varepsilon|y|)(1 + |y|^2)} \, \mathrm{d}\theta$$

$$= \int_{D_d(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{R\varepsilon|y|}{(1 + \varepsilon|y|)(1 + |y|^2)} \, \mathrm{d}\theta + \int_{D_c(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{R\varepsilon|y|}{(1 + \varepsilon|y|)(1 + |y|^2)} \, \mathrm{d}\theta$$

$$\leq \frac{C(A, \delta_1)}{1 + |z|} \int_{D_d(x)} \frac{R\varepsilon}{(1 + \varepsilon|y|)(1 + |y|)} \, \mathrm{d}\theta + \int_{D_c(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{R\varepsilon}{(1 + \varepsilon|y|)(1 + |y|)} \, \mathrm{d}\theta$$

$$\leq \frac{C(A, \delta_1)}{1 + |z|} + \int_{D_c(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{R\varepsilon}{(1 + \varepsilon|y|)(1 + |y|)} \, \mathrm{d}\theta$$

with $C(A, \delta_1)$ is independent of $R > 0$, provided $\varepsilon(R) > 0$ is small enough. We can bound the second integral by:

$$\int_{D_c(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \frac{R\varepsilon}{(1 + \varepsilon|y|)(1 + |y|)} \, \mathrm{d}\theta \leq \frac{R\varepsilon}{(1 + \varepsilon|z|)(1 + |z|)} \int_{D_c(x)} \frac{C(A, \delta_1)}{1 + |z - y|} \, \mathrm{d}\theta$$

$$\leq \frac{C(A, \delta_1) R\varepsilon \log(1 + |z|)}{(1 + \varepsilon|z|)(1 + |z|)} \leq \frac{1}{(1 + |z|)},$$

for $\varepsilon > 0$ small enough. $\square$

As a corollary we obtain the uniform boundedness of the sequence u_ε.

Lemma 3.4.17 (Uniform boundedness). *Let $n \geq 4$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 3.4.3. For $R > 0$, $\gamma, \delta_2 \in (0, 1]$, $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ let $u_\varepsilon \in V_{A,\lambda}^n$ be the solution to (3.4.36) with $u_0 = m(v) + \delta_2 v_0(v)$, where $v_0 \in H_\lambda^n$ satisfies:*

$$0 \leq v_0(v) \leq C e^{-\frac{1}{2}|v|}.$$

Then for $m \in \mathbb{N}$, $m \leq n - 4$, $\varepsilon > 0$ small enough there holds:

$$|\nabla^m (u_\varepsilon - u_0)(t, v)| \leq C(A) e^{-\frac{1}{2}|v|}, \quad for \ 0 \leq t \leq 1. \tag{3.4.44}$$

Boundary Layer Estimate

To obtain smallness for the Laplace transforms, we separate the contributions of M_1 and M_2 to u_ε.

Lemma 3.4.18 (Decomposition). *Let $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ and $u_\varepsilon \in V_{A,\lambda}^n$ a solution to (3.4.36). Then $u_\varepsilon - u_0 = p_\varepsilon + q_\varepsilon$. Here $p_\varepsilon = \nabla \cdot P_\varepsilon$ is a divergence and P_ε is given by:*

$$\partial_t P_\varepsilon = \left(\int_0^t \int \frac{\pi^2}{4} \frac{e^{-\frac{s|v'|}{\varepsilon}} P_{v'}^\perp}{\varepsilon} (u_0 + f_\varepsilon)(t - s, v - v') \eta(|v'|^2) \nabla u_\varepsilon(t - s, v) \, \mathrm{d}v' \, \mathrm{d}s \right)$$

$$- \left(\int_0^t \int \frac{\pi^2}{4} \frac{e^{-\frac{s|v'|}{\varepsilon}} P_{v'}^\perp}{\varepsilon} \nabla(u_0 + f_\varepsilon)(t - s, v - v') \eta(|v'|^2) u_\varepsilon(t - s, v) \, \mathrm{d}v' \, \mathrm{d}s \right)$$

$$P_\varepsilon(0) = 0.$$

$$\tag{3.4.45}$$

Similarly, $q_\varepsilon = \nabla \cdot Q_\varepsilon$, where Q_ε is given by:

$$z\mathcal{L}(Q_\varepsilon) = \left(\int M_2(\varepsilon z, v')\eta(|v'|^2)\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v')\nabla u_\varepsilon(s, v)\right) \, \mathrm{d}v' \right)$$
$$- \left(\int \nabla \cdot M_2(\varepsilon z, v')\eta(|v'|^2)\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v'))u_\varepsilon(s, v)\right) \, \mathrm{d}v' \right). \tag{3.4.46}$$

Proof. We take the Laplace transform of equation (3.4.36) and use Lemma 3.2.15 to obtain:

$$z\mathcal{L}(u_\varepsilon)(z, v) - u_0(v)$$
$$= \nabla \cdot \left(\int_{\mathbb{R}^3} (M_1 + M_2)(\varepsilon z, v')\eta(|v'|^2)\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v')\nabla u_\varepsilon(s, v)\right)(z) \, \mathrm{d}v' \right)$$
$$- \nabla \cdot \left(\int_{\mathbb{R}^3} \nabla \cdot (M_1 + M_2)(\varepsilon z, v')\eta(|v'|^2)\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v')u_\varepsilon(s, v)\right)(z) \, \mathrm{d}v' \right).$$

Now introduce the functions $p_\varepsilon, q_\varepsilon$ given by the splitting:

$$z\mathcal{L}(q_\varepsilon)(z, v) = \nabla \cdot \left(\int_{\mathbb{R}^3} M_2(\varepsilon z, v')\eta(|v'|^2)\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v')\nabla u_\varepsilon(s, v)\right)(z) \, \mathrm{d}v' \right) \tag{3.4.47}$$
$$- \nabla \cdot \left(\int_{\mathbb{R}^3} \nabla \cdot M_2(\varepsilon z, v')\eta(|v'|^2)\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v')u_\varepsilon(s, v)\right)(z) \, \mathrm{d}v' \right)$$
$$z\mathcal{L}(p_\varepsilon)(z, v) = \nabla \cdot \left(\int_{\mathbb{R}^3} M_1(\varepsilon z, v')\eta\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v')\nabla u_\varepsilon(s, v)\right)(z) \, \mathrm{d}v' \right) \tag{3.4.48}$$
$$- \nabla \cdot \left(\int_{\mathbb{R}^3} \nabla \cdot M_1(\varepsilon z, v')\eta\mathcal{L}\left((u_0 + f_\varepsilon)(s, v - v')u_\varepsilon(s, v)\right)(z) \, \mathrm{d}v' \right).$$

Therefore $q_\varepsilon = \nabla \cdot Q_\varepsilon$, with Q_ε as in (3.4.46). To show $p_\varepsilon = \nabla \cdot P_\varepsilon$ we transform the equation for p_ε back to the variables (t, v). To do so we remark that M_1 is the Laplace transform of:

$$\frac{\pi^2}{4}\mathcal{L}\left(\frac{e^{-\frac{t|v|}{\varepsilon}}}{\varepsilon}\right)(z)P_v^\perp = M_1(\varepsilon z, v).$$

Therefore $p_\varepsilon = \nabla \cdot Q_\varepsilon$ and $u_\varepsilon - u_0 = q_\varepsilon + p_\varepsilon$ as claimed. □

Splitting the function u_ε into $u_\varepsilon = p_\varepsilon + q_\varepsilon$ allows to estimate the contributions of M_1 and M_2 (as in (3.2.15)) separately. The function q_ε can be estimated in a straightforward fashion.

Lemma 3.4.19 (Estimate for q_ε). *Let $n \geq 4$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 3.4.3. For $R > 0$, $\gamma, \delta_2 \in (0, 1]$, $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ let $u_\varepsilon \in V_{A,\lambda}^n$ be the solution to (3.4.36) with $u_0 = m(v) + \delta_2 v_0(v)$, where $v_0 \in H_\lambda^n$ satisfies:*

$$0 \leq v_0(v) \leq Ce^{-\frac{1}{2}|v|}.$$

Let $\nabla \cdot Q_\varepsilon = q_\varepsilon \in V_{A,\lambda}^n$ be given by (3.4.46). Then for $m \in \mathbb{N}$, $m \leq n - 4$, $\varepsilon > 0$ small enough there holds:

$$|\mathcal{L}(\nabla^m q_\varepsilon \kappa_{\delta_1})(z,v)| \leq \frac{C(A,\delta_1)\varepsilon|z|}{(1+\varepsilon|z|)^2(1+|z|^2)} e^{-\frac{1}{2}|v|} \tag{3.4.49}$$

$$|\mathcal{L}(\nabla^m Q_\varepsilon \kappa_{\delta_1})(z,v)| \leq \frac{C(A,\delta_1)\varepsilon|z|}{(1+\varepsilon|z|)^2(1+|z|^2)} e^{-\frac{1}{2}|v|}. \tag{3.4.50}$$

In particular, for $0 \leq t \leq 1$, $m \leq n - 4$ we have

$$|\partial_t \nabla^m q_\varepsilon| \leq C(A)e^{-\frac{1}{2}|v|} \qquad\qquad |\partial_t \nabla^m Q_\varepsilon| \leq C(A)e^{-\frac{1}{2}|v|}. \tag{3.4.51}$$

Lemma 3.4.20 (L^∞ estimate for time derivative). *Let $n \geq 4$ and $A = 2a \geq \frac{1}{2}$, $\delta > 0$ be as in Theorem 3.4.3. For $R > 0$, $\gamma, \delta_2 \in (0,1]$, $(f_\varepsilon, g_\varepsilon) \in \Omega_{A,R,\delta,\varepsilon}^n$ let $u_\varepsilon \in V_{A,\lambda}^n$ be the solution to (3.4.36) with $u_0 = m(v) + \delta_2 v_0(v)$, where $v_0 \in H_\lambda^n$ satisfies:*

$$0 \leq v_0(v) \leq Ce^{-\frac{1}{2}|v|}.$$

Then for $m \in \mathbb{N}$, $m \leq n - 4$, $\varepsilon > 0$ small enough there holds:

$$|\partial_t \nabla^m u_\varepsilon(t,v)| \leq C(A)e^{-\frac{1}{2}|v|} \quad \text{for } 0 \leq t \leq 1. \tag{3.4.52}$$

Proof. We use the decomposition $u_\varepsilon = p_\varepsilon + q_\varepsilon$ introduced in Lemma 3.4.18. By the previous Lemma 3.4.19 we know

$$|\partial_t \nabla^m q_\varepsilon(t,v)| \leq C(A)e^{-\frac{1}{2}|v|} \quad \text{for } 0 \leq t \leq 1.$$

It remains to estimate p_ε. The sequence $e^{-s/\varepsilon}/\varepsilon$ is bounded in L^1. Therefore the claim follows by inserting the estimate (3.4.44) into the definition (3.4.45) of $p_\varepsilon = \nabla \cdot P_\varepsilon$. $\qquad\square$

Notation 3.4.21. *Let b be the function given by:*

$$b(t,r) := \frac{e^{-tr}}{r^2} + \frac{t}{r} - \frac{1}{r^2}.$$

For $u_0 \in H_\lambda^n$, define the boundary layer $B(t,v;u_0) = \nabla \cdot B_F(t,v;u_0)$ by:

$$B_F(t,v;u_0) := \int \frac{\pi^2}{4} \frac{b(t,\frac{|v'|}{\varepsilon})P_{v'}^\perp}{\varepsilon} \eta \left(u_0(v-v')\nabla u_0(v) - \nabla u_0(v-v')\, u_0(v) \right) \, \mathrm{d}v'. \tag{3.4.53}$$

Lemma 3.4.22 (Boundary Layer property). *The function $B = \nabla \cdot B_F$, as defined in (3.4.53) satisfies:*

$$\partial_{tt} B(t,v) = \nabla \cdot \left(\int \frac{\pi^2}{4} \frac{e^{-\frac{t|v'|}{\varepsilon}} P_{v'}^\perp}{\varepsilon} \eta(|v'|^2) \left(u_0(v-v')\nabla u_0(v) - \nabla u_0(v-v')u_0(v) \right) \, \mathrm{d}v' \right),$$

$$B(0,v) = 0 \quad \partial_t B(0,v) = 0.$$

Proof. Differentiating b gives:

$$\partial_t b(t, r) = \frac{1 - e^{-rt}}{r}, \quad \partial_{tt} b(t, r) = e^{-rt}.$$

Therefore the second time derivative of B is:

$$\partial_{tt} B(t, v) = \nabla \cdot \left(\int \frac{\pi^2}{4} \frac{e^{-\frac{|v'|}{\varepsilon}t} P_{v'}^{\perp}}{\varepsilon} \eta \left(u_0(v - v') \nabla u_0(v) - \nabla u_0(v - v') u_0(v) \right) \, \mathrm{d}v' \right).$$

The initial data $B(0, v) = 0$, $\partial_t B(0, v) = 0$ follow by simply putting $t = 0$. $\qquad \square$

Lemma 3.4.23 (Remainder estimate). *Let $n \geq 4$ and p_ε solve (3.4.45) and $\|u_\varepsilon\|_{V_{A,\lambda}^n} \leq C$. There exists a $C_0 > 0$ such that for all $m \leq n - 2$ there exists ε small enough such that:*

$$|\partial_{tt}(p_\varepsilon - B)(t, v)| \leq C_0 e^{-\frac{1}{2}|v|}, \quad \text{for } t \in [0, 1].$$

Proof. Take the time derivative of (3.4.45). We can split using Lemma 3.4.22:

$$\partial_{tt} p_\varepsilon = \nabla \cdot \left(\int_0^t \int \frac{\pi^2}{4} \frac{e^{-\frac{s|v'|}{\varepsilon}} P_{v'}^{\perp}}{\varepsilon} \partial_t((u_0 + f_\varepsilon)(t - s, v - v')\eta \nabla u_\varepsilon(t - s, v)) \, \mathrm{d}v' \, \mathrm{d}s \right)$$

$$- \nabla \cdot \left(\int_0^t \int \frac{\pi^2}{4} \frac{e^{-\frac{s|v'|}{\varepsilon}} P_{v'}^{\perp}}{\varepsilon} \partial_t(\nabla(u_0 + f_\varepsilon)(t - s, v - v'))\eta u_\varepsilon(t - s, v) \, \mathrm{d}v' \, \mathrm{d}s \right)$$

$$+ \nabla \cdot \left(\int \frac{\pi^2}{4} \frac{e^{-\frac{t|v'|}{\varepsilon}} P_{v'}^{\perp}}{\varepsilon} \eta(|v'|^2) \left(u_0(v - v') \nabla u_0(v) - \nabla u_0(v - v') u_0(v) \right) \, \mathrm{d}v' \right)$$

$$= R_1 + R_2 + \partial_{tt} B.$$

Since $|\partial_t f_\varepsilon| \leq C e^{-\frac{1}{2}|v|}$ by assumption, we obtain:

$$|\partial_{tt}(p_\varepsilon - B)(t, v)| = |R_1(t, v) + R_2(t, v)| \leq C_0 e^{-\frac{1}{2}|v|}, \quad \text{for } t \in [0, 1],$$

as claimed. $\qquad \square$

Lemma 3.4.24 (Smallness of $\mathcal{L}(p_\varepsilon - B)$). *Let p_ε solve (3.4.45) and $\|u_\varepsilon\|_{V_{A,\lambda}^n} \leq C$ for some $A = 2a > 0$. We have $p_\varepsilon - B = \nabla \cdot (P_\varepsilon - B_F)$, and there is a $C_0 > 0$ such that for all $m \leq n - 2$, $\delta_1 > 0$ and $\varepsilon > 0$ small enough:*

$$|\mathcal{L}((p_\varepsilon - B)\kappa_{\delta_1})(z, v)| + |\mathcal{L}((P_\varepsilon - B_F)\kappa_{\delta_1})(z, v)| \leq \frac{\delta_1 C_0 e^{-\frac{1}{2}|v|}}{1 + |z|^2}.$$

Proof. By definition of B the difference $p_\varepsilon - B$ vanishes initially, as well as the time derivative:

$$(p_\varepsilon - B)(0, v) = \partial_t(p_\varepsilon - B)(0, v) = 0.$$

Combined with the lemma above this shows:

$$|\partial_{tt}((p_\varepsilon - B)\kappa_{\delta_1})| \le C_0 e^{-\frac{1}{2}|v|}(1 + \frac{t}{\delta_1} + \frac{t^2}{\delta_1^2})\kappa_{\delta_1}, \quad \text{for } t \in [0, 1].$$

After integrating by parts twice this allows to bound the Laplace transform by:

$$
\begin{aligned}
|\mathcal{L}((p_\varepsilon - B)\kappa_{\delta_1})(z, v)| &\le \frac{C_0 e^{-\frac{1}{2}|v|}}{|z|^2} \int_0^\infty C e^{-\frac{1}{2}|v|}(1 + \frac{t}{\delta_1} + \frac{t^2}{\delta_1^2})\kappa_{\delta_1} \ \mathrm{d}t \\
&\le \frac{C_0 e^{-\frac{1}{2}|v|}}{|z|^2}\delta_1.
\end{aligned}
$$

The estimate for $P_\varepsilon - B_F$ is proved similarly. $\qquad\square$

Lemma 3.4.25 (Stationarity of m)**.** *Let $\sigma^2, m_0 > 0$, $m(\sigma^2, M_0)(v)$ be the Maxwellian defined in (3.2.9). Then for all $t \ge 0$, $v \in \mathbb{R}^3$ we have:*

$$B(t, v; m) = 0. \tag{3.4.54}$$

Proof. The argument is identical to the one proving that m is a stationary point of the Landau equation: First we observe that

$$\nabla m(v) = -\frac{v}{\sigma^2}m(v).$$

This however implies that:

$$P_{v'}^\perp \left(m(v - v')\nabla m(v) - \nabla m(v - v')m(v) \right) = -P_{v'}^\perp \frac{v'}{\sigma^2}m(v - v')m(v) = 0.$$

Inserting this into the definition of $B(t, v; m)$ in (3.4.53) gives the claim. $\qquad\square$

We use the stationarity of the Maxwellian m to obtain smallness of the boundary layer, provided the evolution starts sufficiently close to m.

Lemma 3.4.26 (Boundary layer estimate)**.** *Let $u_0 = m(v) + \delta_2 v_0$, for v_0 some fixed smooth function satisfying*

$$0 \le v_0(v) \le C e^{-\frac{1}{2}|v|}, \quad |\nabla^i v_0| \le C e^{-\frac{1}{2}|v|} \quad \text{for } i = 0, 1, 2.$$

Let B be the associated Boundary Layer defined by (3.4.53). Then the Laplace transforms of B and B_F satisfy:

$$|\mathcal{L}(B\kappa_{\delta_1})(z, v)| + |\mathcal{L}(B_F\kappa_{\delta_1})(z, v)| \le C(\delta_1)\frac{\delta_2 e^{-\frac{1}{2}|v|}}{1 + |z|^2}. \tag{3.4.55}$$

Proof. Using Lemma 3.4.25 we can simplify B to:

$$
\begin{aligned}
B(t,v) =& \nabla \cdot \left(\int \frac{b(t,\frac{|v'|}{\varepsilon})P_{v'}^{\perp}}{\varepsilon}(m+\delta_2 v_0)(v-v')\eta\nabla(m+\delta_2 v_0)(v)\,\mathrm{d}v' \right) \\
&- \nabla \cdot \left(\int \frac{b(t,\frac{|v'|}{\varepsilon})P_{v'}^{\perp}}{\varepsilon}\nabla(m+\delta_2 v_0)(v-v')\eta(m+\delta_2 v_0)(v)\,\mathrm{d}v' \right) \\
=& \nabla \cdot \left(\int \frac{b(t,\frac{|v'|}{\varepsilon})P_{v'}^{\perp}}{\varepsilon}\eta \left[\delta_2 v_0(v')\nabla m(v) + (\delta_2 v_0 + m)(v-v')\delta_2\nabla v_0(v) \right]\,\mathrm{d}v' \right) \\
&- \nabla \cdot \left(\int \frac{b(t,\frac{|v'|}{\varepsilon})P_{v'}^{\perp}}{\varepsilon}\eta \left[\delta_2 \nabla v_0(v')m(v) + \nabla(\delta_2 v_0 + m)(v-v')\delta_2 v_0(v) \right]\,\mathrm{d}v' \right).
\end{aligned}
$$

The Laplace transform of b can be computed explicitly:

$$
\mathcal{L}(b(\cdot,r))(z) = \frac{1}{rz^2} - \frac{1}{r(z+r)z}.
$$

Inserting this above we obtain the estimate:

$$
|\mathcal{L}(B\kappa_{\delta_1})(z,v)| + |\mathcal{L}(B_F\kappa_{\delta_1})(z,v)| \le C(\delta_1)\frac{\delta_2 e^{-\frac{1}{2}|v|}}{1+|z|^2},
$$

which is the claim of the Lemma. $\qquad\qquad\square$

We are in the position to now prove Theorem 3.4.10.

Proof of Theorem 3.4.10. Let $A, \delta > 0$ as in Theorem 3.4.3. Then the theorem ensures that for $R > 0$, $\delta_2 \in (0,1]$ arbitrary, and $(f,g) \in \Omega^n_{A,R,\delta_\varepsilon}$ the solution u_ε to (3.4.36) with $u_\varepsilon - u_0 = \nabla \cdot U_\varepsilon$ can be bounded by:

$$
\|u_\varepsilon \kappa_{\delta_1}\|_{V^n_{A,\lambda}} + \|U_\varepsilon \kappa_{\delta_1}\|_{V^{n-1}_{A,\lambda}} \le C. \tag{3.4.56}
$$

We use that $\psi_{\delta_1}(f,g) = (\kappa_{\delta_1}(u_\varepsilon - u_0), \kappa_{\delta_1}U_\varepsilon)$ and decompose u_ε into three pieces:

$$
\begin{aligned}
(u_\varepsilon - u_0)\kappa_{\delta_1} &= (p_\varepsilon - B)\kappa_{\delta_1} + B\kappa_{\delta_1} + q_\varepsilon\kappa_{\delta_1} \\
U_\varepsilon \kappa_{\delta_1} &= (P_\varepsilon - B_F)\kappa_{\delta_1} + B_F\kappa_{\delta_1} + Q_\varepsilon\kappa_{\delta_1}.
\end{aligned}
\tag{3.4.57}
$$

Using estimate (3.4.56) and Lemmas 3.4.19, 3.4.24, 3.4.26 we can find $\delta_1, \varepsilon_0 > 0$ small enough and $R > 0$ large enough, such that for $\delta_2, \varepsilon \in (0, \varepsilon_0]$ the Laplace transforms of the summands in (3.4.57) can be estimated by:

$$
|\mathcal{L}(u_\varepsilon \kappa_{\delta_1})| + |\mathcal{L}(U_\varepsilon \kappa_{\delta_1})| \le \frac{\delta e^{-\frac{1}{2}|v|}}{1+|z|^2} + \frac{R\varepsilon|z|e^{-\frac{1}{2}|v|}}{(1+\varepsilon|z|)^2(1+|z|)^2}.
$$

So we recover (3.4.3), one of the defining estimates of $\Omega^n_{A,R,\delta,\varepsilon}$. The upper bound (3.4.4) is the content of Lemma 3.4.16. The remaining estimate (3.4.5) is proved in Lemma 3.4.20. $\qquad\square$

3.5 Existence of solutions and Markovian Limit

3.5.1 Existence of a solution to the non-Markovian equation

With the a priori estimates proved in the last section, we can now prove Theorem 3.2.6.

Proof of Theorem 3.2.6. Without loss of generality, let m be the standard Gaussian, i.e. $\sigma = m_0 = 1$. First let $\gamma > 0$. We invoke Theorems 3.4.3 and 3.4.10 to find $A, \delta, R, \delta_1 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon, \delta_2 \in (0, \varepsilon_0]$ the mapping $\Psi_{\delta_1} : \Omega^n_{A,R,\delta,\varepsilon} \to \Omega^n_{A,R,\delta,\varepsilon}$ is continuous with respect to the topologies of $X^n_{A,\tilde{\lambda}}$, $X^n_{A,\lambda}$, hence also as a map from $X^n_{A,\tilde{\lambda}}$ to itself. By Lemma 3.4.2 we know that $\Omega^n_{A,R,\delta,\varepsilon}$ is a closed, convex, bounded and nonempty subset of $X^n_{A,\tilde{\lambda}}$. Therefore, existence of a fixed point of Ψ_{δ_1} follows from Schauder's theorem, provided we can show that the mapping is compact. To see this, we use that Theorem 3.4.22 gives the estimate:

$$\|\Psi_{\delta_1}(f,g)\|_{X^n_{A,\lambda}} + \|\partial_t \Psi_{\delta_1}(f,g)\|_{X^{n-2}_{A,\lambda}} \leq C(A). \tag{3.5.1}$$

Since $^\gamma\nabla$ is smoothing, the defining equation (3.4.36) of Ψ_{δ_1} implies:

$$\|\Psi_{\delta_1}(f,g)\|_{X^{n+1}_{A,\lambda}} + \|\partial_t \Psi_{\delta_1}(f,g)\|_{X^{n+1}_{A,\lambda}} \leq C(A,\gamma).$$

This implies compactness of the mapping Ψ_{δ_1} by the Rellich type Lemma 3.2.5. Hence for $\gamma \in (0, 1]$, we have proved the existence of solutions $u_{\varepsilon,\gamma}$ to:

$$\begin{aligned}
\partial_t u_{\varepsilon,\gamma} = &\frac{1}{\varepsilon} {}^\gamma\nabla \cdot \left(\int_0^t K[u_{\varepsilon,\gamma}] \left(\frac{t-s}{\varepsilon}, v \right) {}^\gamma\nabla u_{\varepsilon,\gamma}(s,v) \, \mathrm{d}s \right) \\
&- \frac{1}{\varepsilon} {}^\gamma\nabla \cdot \left(\int_0^t P[u_{\varepsilon,\gamma}] \left(\frac{t-s}{\varepsilon}, v \right) u_{\varepsilon,\gamma}(s,v) \, \mathrm{d}s \right) \\
u_{\varepsilon,\gamma}(0,\cdot) = &\, u_0(\cdot),
\end{aligned} \tag{3.5.2}$$

for times $0 \leq t \leq \delta_1$. It remains to pass $\gamma \to 0$ to obtain a solution of the non-mollified equation. The uniform estimate (3.5.1) shows that for $\varepsilon > 0$ there is a sequence $\gamma_j \to 0$ such that $u_{\varepsilon,\gamma_j} \to u_\varepsilon$ in $V^{n-3}_{A,\tilde{\lambda}}$, $u_{\varepsilon,\gamma_j} \rightharpoonup u_\varepsilon$ in $V^n_{A,\lambda}$ and $\partial_t u_{\varepsilon,\gamma_j} \rightharpoonup \partial_t u_\varepsilon$ in $V^{n-2}_{A,\lambda}$. Hence both sides of (3.5.2) converge weakly in $V^{n-2}_{A,\lambda}$, and it suffices to identify the limit of the right-hand side. Indeed, from the convergence in $V^{n-3}_{A,\tilde{\lambda}}$ we conclude that pointwise a.e. along a subsequence:

$$\begin{aligned}
&{}^{\gamma_j}\nabla \cdot \left(\int_0^t K[u_{\varepsilon,\gamma_j}] \left(\frac{t-s}{\varepsilon}, v \right) {}^{\gamma_j}\nabla u_{\varepsilon,\gamma_j}(s,v) - P_{\gamma_j}[u_{\varepsilon,\gamma_j}] \left(\frac{t-s}{\varepsilon}, v \right) u_{\varepsilon,\gamma_j}(s,v) \, \mathrm{d}s \right) \\
&\to \nabla \cdot \left(\int_0^t K[u_\varepsilon] \left(\frac{t-s}{\varepsilon}, v \right) \nabla u_\varepsilon(s,v) - P[u_\varepsilon] \left(\frac{t-s}{\varepsilon}, v \right) u_\varepsilon(s,v) \, \mathrm{d}s \right).
\end{aligned} \tag{3.5.3}$$

Estimate (3.2.11) follows from (3.5.1), and inserting the estimate back into equation (3.2.10) proves that $u_\varepsilon \in C^1([0,\delta_1]; H^{n-2}_\lambda)$. $\qquad \square$

3.5.2   Non-Markovian to Markovian limit

In this section we prove the transition from non-Markovian to Markovian dynamics on the macroscopic timescale. As $\varepsilon \to 0$, the solutions u_ε to the non-Markovian equations (3.2.10) converge to solutions of the Landau equation.

Proof of Theorem 3.2.8.   For the solutions u_ε of (3.2.10) constructed in Theorem 3.2.6 we have the a priori bound:

$$\|((u_\varepsilon - u_0)\kappa_{\delta_1}, U_\varepsilon \kappa_{\delta_1})\|_{X_{A,\lambda}^n} + \|\partial_t((u_\varepsilon - u_0)\kappa_{\delta_1}, U_\varepsilon \kappa_{\delta_1})\|_{X_{A,\lambda}^{n-2}} \leq C(A).$$

Using the compactness Lemma 3.2.5 and the fact that $V_{A,\lambda}^n$ is a separable Hilbert space, we can find $u \in V_{A,\lambda}^n$, s.t. along a sequence $\varepsilon_j \to 0$ we have $u_{\varepsilon_j} \to u$ in $V_{A,\tilde{\lambda}}^{n-3}$, $u_{\varepsilon_j} \rightharpoonup u \in V_{A,\lambda}^n$ and $\partial_t u_{\varepsilon_j} \rightharpoonup \partial_t u \in V_{A,\lambda}^{n-2}$. We need to show that u solves the equation (3.2.12). Since both sides of the equation are well-defined and have a well-defined Laplace transform, it is sufficient to show that u solves the equation in Laplace variables. To this end, we take the Laplace transform of (3.2.10):

$$\mathcal{L}(\partial_t u_{\varepsilon_j})(z, v) = \nabla \cdot \left(\int_{\mathbb{R}^3} (M_1 + M_2)(\varepsilon_j z, v') \mathcal{L}(u_{\varepsilon_j}(s, v - v') \nabla u_{\varepsilon_j}(s, v))(z) \eta \, \mathrm{d}v' \right)$$
$$- \nabla \cdot \left(\int_{\mathbb{R}^3} \nabla (M_1 + M_2)(\varepsilon_j z, v') \mathcal{L}(u_{\varepsilon_j}(s, v - v') u_{\varepsilon_j}(s, v))(z) \eta \, \mathrm{d}v' \right). \tag{3.5.4}$$

The left-hand side converges pointwise to $\mathcal{L}(\partial_t u) = z\mathcal{L}(u) + u_0$, up to choosing a further subsequence. The right-hand side of (3.5.4) converges pointwise along a subsequence to:

$$\nabla \cdot \left(\int_{\mathbb{R}^3} \frac{\pi^2}{4|v'|} \frac{1}{1 + \frac{z}{|v'|}} P_{v'}^\perp \mathcal{L}(u(s, v - v') \nabla u(s, v))(z) \eta \, \mathrm{d}v' \right)$$
$$- \nabla \cdot \left(\int_{\mathbb{R}^3} \frac{\pi^2}{4|v'|} \frac{1}{1 + \frac{z}{|v|}} P_{v'}^\perp \mathcal{L}(\nabla u(s, v - v') u(s, v))(z) \eta \, \mathrm{d}v' \right)$$
$$= \mathcal{L}(\nabla \cdot (\mathcal{K}[u]\nabla u) - \nabla \cdot (\mathcal{P}[u]u)).$$

Therefore $u \in V_{A,\lambda}^n \cap C^1([0, \delta_1]; H_\lambda^{n-4})$ solves equation (3.2.12) as claimed. □

Chapter 4

The two-particle correlation dynamics in the plasma limit

4.1 Introduction

4.1.1 Kinetic limits of particle systems with long-range interactions

A classical problem studied in statistical physics is the dynamics of systems of many identical particles which interact by means of long range potentials. In particular, this problem has received a big deal of attention in the community working on plasma physics in the case in which particles interact via the Coulomb potential.

Early contributions to this topic were made by Bogolyubov [6], and have been extended by the works of Balescu [2, 3], as well as Guernsey [23] and Lenard [34]. These authors obtained a kinetic equation which describes the behavior of the velocity distribution of a spatially homogeneous many particle system with long range interaction (in particular Coulomb forces). The equation that they obtained, which is usually called the Balescu-Lenard equation, reads as:

$$\partial_t f(t, v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v - v', v)(\nabla_v - \nabla_{v'})(f(t, v)f(t, v')) \, dv' \right) \tag{4.1.1}$$

$$a_{i,j}(w, v) = \int_{\mathbb{R}^3} k_i k_j \delta(k \cdot w) \frac{|\hat{\phi}(k)|^2}{|\varepsilon(k, k \cdot v)|^2} \, dk. \tag{4.1.2}$$

Here ϕ is the interaction potential and ε is the so-called dielectric function, which we introduce in Definition 4.2.6. We remark that the integral defining a is logarithmically divergent for large values of k in the case of Coulomb interaction. We will discuss this in detail in Subsection 4.1.3.

The equation (4.1.1) shares many properties with classical kinetic equations like for instance the Boltzmann equation. In particular, the steady states of (4.1.1) are the Maxwellian distributions:

$$M(v) := \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} e^{-\frac{m|v|^2}{2k_B T}}. \tag{4.1.3}$$

Moreover, the entropy $H[f(t, \cdot)] = - \int f(t, v) \log(f(t, v)) \, dv$ of a solution f of (4.1.1) is (formally) increasing in time, as remarked in [34].

The Balescu-Lenard equation (4.1.1), was found independently by Guernsey [23] and Lenard (cf. [34]), following the approach by Bogolyubov, and along a different line by Balescu (cf. [3]). There are also stochastic derivations of the Balescu-Lenard equation using different arguments, which are discussed in Subsection 4.1.2.

We recall here, in a more modern language, the main ideas in the original derivation of the equation (4.1.1) proposed by Bogolyubov. An overview over particle models and scaling limits in kinetic theory can be gained from [49, 50, 56].

Consider a system of particles $\{(\tilde{X}_j, \tilde{V}_j)\}_{j \in J}$ with unitary mass, where J is a countable index set and $\tilde{X}_j, \tilde{V}_j \in \mathbb{R}^3$ denote the position and velocity of particles. Let the evolution of the system be given by:

$$\partial_\tau \tilde{X}_i(\tau) = \tilde{V}_i(\tau), \quad \partial_\tau V_i = -\tilde{\theta}^2 \sum_{j \neq i} \nabla \phi(\tilde{X}_i - \tilde{X}_j). \tag{4.1.4}$$

The parameter $\tilde{\theta} > 0$ can be thought of as the charge of a particle. We will assume that the initial configuration of particles is random and distributed according to a grand canonical ensemble measure that is translation invariant in space and has $\tilde{N}$ particles per spatial unit of volume on average. The average kinetic energy of a particle, that we also call the temperature of the system, we will denote by T. By rescaling velocities and time we can assume without loss of generality that $T = 1$. We consider scaling limits of (4.1.4) and try to characterize the statistical behavior of (4.1.4) depending on the rescaling of the quantities $\tilde{\theta}, \tilde{N}$, as well as the interaction potential ϕ. We will consider two classes of potentials, namely the Coulomb potential $\phi(x) = \frac{c}{|x|}$ for some $c > 0$, and so-called soft potentials, that are radially symmetric functions in the Schwartz class.

There is a significant difference between Coulomb potentials and soft potentials. In the first case, the potential does not have a characteristic length scale, while soft potentials do. Upon changing units we assume this length to be one. This endows the system with an intrinsic unit of length in the soft potential case. In the case of Coulomb interaction, this intrinsic length emerges from the dynamics of the system. To this end, we observe that there are two independent quantities with the unit of a length that can be obtained from the quantities $\tilde{\theta}^2, \tilde{N}$ and T describing the system. One of them is the typical distance of particles $d = \tilde{N}^{-\frac{1}{3}}$. The second is the so-called Debye screening length:

$$L_D = \sqrt{\frac{T}{\tilde{N}\tilde{\theta}^2}}, \tag{4.1.5}$$

which is well-known in plasma physics. The Debye length will play a crucial role in many results of this chapter. It measures the characteristic (effective) range of interaction between the particles of the system, assuming that the velocity distribution of particles $f_1(v)$ satisfies a suitable stability condition (cf. Assumption 4.2.13). Under this assumption, L_D is the effective radius of a single particle, that is the characteristic distance to which the influence of a single particle can be felt in a system evolving according to (4.1.4), when ϕ is the Coulomb potential. We can assume $L_D = 1$ using the change of variables:

$$L_D X = \tilde{X}, \quad L_D \tau = \tilde{\tau}, \quad L_D \theta^2 = \tilde{\theta}^2, \quad N = L_D^3 \tilde{N}. \tag{4.1.6}$$

After changing units, the average number of particles per unit volume N and the rescaled strength θ^2 of the potential satisfy the relation:

$$N\theta^2 = 1, \tag{4.1.7}$$

and the particle system $\{(X_j, V_j)\}_{j \in J}$ satisfies (4.1.4) with θ replaced by $\tilde{\theta}$. Hence, for systems evolving according to (4.1.4) with ϕ the Coulomb potential, we can assume without loss of generality that (4.1.7) holds.

We consider different regimes of scaling limits and assume the quantities θ, N to be powers of a scaling parameter $\tilde{\sigma} \to 0$ given by constants $\tilde{\alpha}, \tilde{\beta} > 0$:

$$\theta^2 = \tilde{\sigma}^{\tilde{\beta}}, \quad N = \tilde{\sigma}^{-\tilde{\alpha}}. \tag{4.1.8}$$

We have seen that in the Coulomb case, up to a change of length scale we can always assume (4.1.7), i.e. $\alpha = \beta$. By relabeling $\sigma = \tilde{\sigma}^{\tilde{\beta}}$, the scaling limits for Coulomb interacting particles can be reduced to

$$\theta^2 = \sigma, \quad N = \sigma^{-1}, \quad \sigma \to 0. \qquad \text{(Coulomb-scaling)}$$

For soft potentials, not all choices of $\tilde{\alpha}, \tilde{\beta} > 0$ yield kinetic limits, i.e. limits for which the mean free path of particles is larger than the average particle distance. Let ϕ be a soft potential with characteristic length $\ell = 1$. Then per unit of time, a typical particle will interact with N many particles and each interaction yields a deflection of order θ^2 with zero average. If the forces of all particles within the range of the potential are independent, the variance of the sum of the deflections is:

$$\text{Var}(V(\tau)) \sim \tilde{\sigma}^{2\tilde{\beta} - \tilde{\alpha}} \tau. \tag{4.1.9}$$

If $2\tilde{\beta} - \tilde{\alpha} > 0$, the variance will become of order one on a macroscopic time scale $t = \tilde{\sigma}^{2\tilde{\beta} - \tilde{\alpha}} \tau$. On the other hand, if $2\tilde{\beta} - \tilde{\alpha} < 0$, the mean free path is much shorter than the range of the potential, and the limit is non-kinetic. Therefore, we make the assumption $0 < \lambda := 2\tilde{\beta} - \tilde{\alpha}$. Upon relabeling $\sigma = \tilde{\sigma}^{\lambda}$, $\beta = \tilde{\beta}/\lambda$, the different kinetic limits can be characterized by a single parameter $\beta > 0$ and are given by:

$$\theta^2 = \sigma^{\beta}, \quad N = \sigma^{1-2\beta}, \quad \sigma \to 0. \qquad \text{(soft potential scaling)}$$

This has the advantage that the macroscopic timescale is given by $t = \sigma\tau$, so σ is precisely the ratio between macroscopic and microscopic scale.

We will investigate the evolution of the distribution of particles, assuming that the particles are initially independently distributed according to a spatially homogeneous density $f_0(x, v) = f_0(v)$ in a grand canonical ensemble with an average of N particles per unit of volume. The presentation will be similar to the one in [55]. Denote phase space variables by $\xi = (x, v)$, let $F_n(\tau, \xi_1, \ldots, \xi_n)$ be the n-particle correlation function of the system, and $f_n = F_n/N^n$ be the rescaled correlation

function. Formally, these functions satisfy the BBGKY hierarchy (cf. [2]). In the scaling limits (soft potential scaling), the hierarchy reads as:

$$\partial_\tau f_n + \sum_{i=1}^{n} v_i \nabla_{x_i} f_n - \sigma^{1-\beta} \sum_{i=1}^{n} \int \nabla\phi(x_i - x_{n+1}) \nabla_{v_i} f_{n+1} \, \mathrm{d}\xi_{n+1}$$
$$= \sigma^\beta \sum_{i \neq j} \nabla\phi(x_i - x_j) \nabla_{v_i} f_n. \tag{4.1.10}$$

We recall that in the Coulomb case, the scaling limits can be reduced to the case (Coulomb-scaling), that is $\beta = 1$. Since we assume that particles are initially independently distributed, the correlation functions at the initial time $\tau = 0$ factorize: $f_n(0, \xi_1, \ldots, \xi_n) = f_1(0, \xi_1) \cdots f_1(0, \xi_n)$. The evolution given by (4.1.4) will create correlations between particles. In order to be able to study this, we introduce the (rescaled) truncated correlation functions g_n:

$$g_2(\xi_1, \xi_2) = f_2(\xi_1, \xi_2) - f_1(\xi_1)f_1(\xi_2),$$
$$g_3(\xi_1, \xi_2, \xi_3) = f_3(\xi_1, \xi_2, \xi_3) - (f_1 f_1 f_1)(\xi_1, \xi_2, \xi_3)$$
$$- f_1(\xi_1)g_2(\xi_2, \xi_3) - f_1(\xi_2)g_2(\xi_1, \xi_3) - f_1(\xi_3)g_2(\xi_1, \xi_2), \tag{4.1.11}$$
$$\ldots.$$

Rewriting the equations BBGKY hierarchy (4.1.10) in terms of the functions g_n we find that a consistent assumption on the orders of magnitudes is:

$$g_n \approx \sigma^{(n-1)\beta}. \tag{4.1.12}$$

Hence we expect that, to leading order, the equations for f_1, g_2 (cf. (4.1.10)) can be approximated by:

$$\partial_\tau f_1 = \sigma^{1-\beta} \nabla_v \cdot \left(\int \nabla\phi(x_1 - x_3) g_2(\xi_1, \xi_3) \, \mathrm{d}\xi_3 \right)$$

$$\partial_\tau g_2 + \sum_{k=1}^{2} v_k \nabla_{x_k} g_2 - \sigma^{1-\beta} \sum_{k=1}^{2} \int \nabla\phi(x_k - x_3) \nabla_{v_k} (f_1(\xi_k) g_2(\xi_{\zeta(k)}, \xi_3)) \, \mathrm{d}\xi_3 \tag{4.1.13}$$
$$= \sigma^\beta \sum_{k=1}^{2} \nabla_{v_k} (f_1(\xi_1) f_1(\xi_2)) \nabla\phi(x_k - x_{\zeta(k)}).$$

Here the function $\zeta(1) = 2, \zeta(2) = 1$ exchanges the variables. Since the source term for g_2 in (4.1.13) is of order σ^β, the function $\tilde{g}_2 = \sigma^{-\beta} g_2$ can be expected to be of order one. With this definition, (4.1.13) is equivalent to:

$$\partial_\tau f_1 = \sigma \nabla_v \cdot \left(\int \nabla\phi(x_1 - x_3) \tilde{g}_2(\xi_1, \xi_3) \, \mathrm{d}\xi_3 \right) \tag{4.1.14}$$

$$\partial_\tau \tilde{g}_2(t) + \sum_{i=1}^{2} v_i \nabla_{x_i} \tilde{g}_2(t) - \sigma^{1-\beta} \sum_{i=1}^{2} \int \nabla\phi(x_i - x_3) \nabla_{v_i} f_1(t, \xi_i) \tilde{g}_2(, \xi_{\zeta(i)}, \xi_3) \, \mathrm{d}\xi_3 \tag{4.1.15}$$
$$= \sum_{i=1}^{2} \nabla_{v_i} (f_1(t, \xi_1) f_1(t, \xi_2)) \nabla\phi(x_i - x_{\zeta(i)}).$$

Since $\tilde{g}_2$ is of order one, it is now apparent that f_1 can be expected to evolve on the longer timescale $t = \sigma\tau$, as predicted earlier by means of a central limit type argument. For $\beta \in (0, 1]$, the characteristic timescale of evolution for $\tilde{g}$ can be expected to be τ, since the function is of order one and all terms in the equation (4.1.15) are at most of order one. For $\beta > 1$, the integral term in (4.1.15) is formally the dominant, and in this case the available information is more fragmentary. In particular, the limit behavior and the kinetic timescale depends strongly on the analyticity properties of the potential ϕ.

Assuming $\tilde{g}_2$ evolves on the scale τ and has a globally stable equilibrium, one can directly predict the limiting kinetic equation for f_1 on the timescale t from (4.1.14)-(4.1.15). Since $\tilde{g}_2$ evolves on a faster timescale, we expect it to be close to an equilibrium truncated correlation g_B, which is given as a functional of $f_1(t)$. Bringing this steady state to the equation (4.1.14), we obtain the formal kinetic equation for f_1.

Following this approach, f_1 can be expected to be (almost) frozen, which motivates to consider the equation (4.1.15) for fixed f_1, when for $\beta = 1$ the equation becomes:

$$\partial_\tau \tilde{g}_2 + \sum_{i=1}^{2} v_i \nabla_{x_i} \tilde{g}_2 - \sum_{i=1}^{2} \int \nabla\phi(x_i - x_3)\nabla_{v_i} f_1(\xi_i)\tilde{g}_2(\xi_{\zeta(i)}, \xi_3) \, \mathrm{d}\xi_3$$
$$= (\nabla_{v_1} - \nabla_{v_2})(f_1(\xi_1)f_1(\xi_2))\nabla\phi(x_1 - x_2). \tag{4.1.16}$$

In the paper [6], Bogolyubov assumes that there is a steady state g_B of (4.1.16), satisfying the boundary condition:

$$g_B(x - \tau v_1, v_1, x_2 - \tau v_2, v_2) \to 0 \quad \text{as } \tau \to \infty . \tag{4.1.17}$$

This condition can be interpreted as particles being uncorrelated before they come close enough to interact. We will call the steady state equation

$$\sum_{i=1}^{2} v_i \nabla_{x_i} g_B - \sum_{i=1}^{2} \nabla_{v_i} f_1 \int \nabla\phi(x_i - x_3) g_B(\xi_{\zeta(i)}, \xi_3) \, \mathrm{d}\xi_3 = (\nabla_1 - \nabla_2)(f_1 f_1)\nabla\phi(x_1 - x_2). \tag{4.1.18}$$

the Bogolyubov equation and the solution g_B the (truncated) Bogolyubov correlation.

In the case $\beta < 1$, the integral term in (4.1.15) is negligible, so the steady state equation reads:

$$\sum_{i=1}^{2} v_i \nabla_{x_i} g_B = (\nabla_1 - \nabla_2)(f_1 f_1)\nabla\phi(x_1 - x_2). \tag{4.1.19}$$

The equation (4.1.19) can be solved explicitly using the method of characteristics. In this case the resulting kinetic equation for f_1 is formally the Landau equation. The case $\beta < 1$, including nonlinear terms, is considered in [5, 55]. Global well-posedness and stability for the Landau equation has been proved in [24]. For $\beta \geq 1$, the integral term is not negligible, and the steady state equation (4.1.18) is more involved. These regimes are of particular importance since for the Coulomb potential $\phi(x) = c/|x|$ the system can be always reduced to the case $\beta = 1$.

The approach by Bogolyubov, in the case $\beta = 1$, was completed by Lenard in [34], who derived the Balescu-Lenard equation. The Lenard approach, which is based on a Wiener-Hopf argument,

yields an explicit formula for the right-hand side of (4.1.14), when g_B is a steady state of (4.1.15) with f_1 fixed. A Fourier representation of the full steady state g_B was found later by Oberman and Williams [42] using a similar approach. There are few rigorous results on the Balescu-Lenard equation (4.1.1). The linearized equation has been studied in [52]. We observe that the Balescu-Lenard equation takes into account the collective effects in the medium, which cannot be observed in the limits with $\beta < 1$ which yield the Landau equation.

The problems considered in this chapter are the following. First we study the well-posedness of (4.1.18). Secondly, we study the stability properties of the steady state g_B under the evolution given by (4.1.16). Thirdly, we study the decay properties of the steady states g_B. We have already discussed the relevance of the first two problems to the Bogolyubov approach. Concerning the third issue, the steady state g_B encodes the information on the range of interaction of particles within the system. To understand this, consider two particles at phase space positions $\xi_j = (x_j, v_j)$, $j = 1, 2$. Let $b(\xi_1, \xi_2)$ be the impact parameter, and $d(\xi_1, \xi_2)$ be the distance of the first particle to the collision point. More precisely, the impact parameter b is defined as the vector from x_2 to x_1 at their time of closest approach along the free trajectories, so b and d, (and the negative part d_-) are given by:

$$b(\xi_1, \xi_2) = P^\perp_{v_1-v_2}(x_1 - x_2), \quad d(\xi_1, \xi_2) = (x_1 - x_2) \cdot \frac{v_1 - v_2}{|v_1 - v_2|}, \quad d_- = \max\{0, -d\}. \qquad (4.1.20)$$

Since we expect the system to have the Debye length L_D (cf. (4.1.5)) as characteristic length, the correlation of particles that remain at a distance much larger than the Debye length, i.e. $|b| \gg L_D$, should be negligible. Moreover, we expect negligible correlations for particles that (so far) have remained at a distance larger than the Debye length, that is $d_- \gg L_D$. In other words, studying the decay properties of the function g_B, we can show the onset of Debye screening in the system. In equation (4.1.18), we have taken the Debye length L_D as unit of length, and rescaled g_B to order one. In this chapter, we prove that for Coulomb interacting systems, the equilibrium correlations g_B satisfy the following estimate, for every compact set $K \subset \mathbb{R}^3$ and $\delta > 0$

$$|g_B(\xi_1, \xi_2)| \leq \frac{C(\delta, K)}{|v_1 - v_2|} \frac{1}{(|b| + d_-)(1 + |b| + d_-)^{\gamma-\delta}}, \quad v_1, v_2 \in K. \qquad (4.1.21)$$

Here $\gamma = 0$ if $f_1(v)$ decays exponentially, and $\gamma = 1$ if f_1 behaves like a Maxwellian for large velocities. We observe that the result only shows the onset of a characteristic length scale, when the one-particle function f_1 behaves like a Maxwellian for large velocities, but not for exponentially decaying functions, indicating that the Debye screening can only be expected for functions f_1 with Maxwellian decay.

We further note that (4.1.21) indicates that the correlations become singular for particles that have approached closely. This is crucial for identifying the kinetic equation for Coulombian particle systems and is discussed in Subsection 4.1.3.

In the case of soft potential interaction, we prove that the equilibrium correlations g_B satisfy the estimate (4.1.21) with $\gamma = 2$, even if the potential decays exponentially. In this case, we do not observe a singularity at small distances.

A fact that will play a crucial role in the proof of (4.1.21) is the existence of zeros of the function $\Re(\varepsilon(k, u))$ for $k \to 0$ (ε as in (4.1.1)), for which $\Im(\varepsilon(k, u))$ is exponentially small. These zeros are well-known in the physics literature, and related to the so-called Langmuir waves (cf. [36]). These

are plasma density waves with very large wavelength which damp out only very slowly. This issue is responsible for the slow Landau damping of Maxwellian plasmas observed in [21], and also crucial to the analysis of the linearized Balescu-Lenard equation in [52]. Moreover, this fact accounts for the dependence of the screening properties (cf. (4.1.21)) on the behavior of the one-particle function for large velocities.

We study the linearized evolution of the truncated correlation function g_2 (4.1.16) with fixed one-particle function. Similar to the Vlasov equation, the equation can be solved in Fourier-Laplace variables (cf. [27]). We introduce in Definition 4.2.10 the representation of the solution in terms of Vlasov propagators, and in Section 4.4 we show linear stability of the Bogolyubov steady states g_B

$$g(t, \cdot) \longrightarrow g_B(\cdot) \quad \text{in } D'(\mathbb{R}^9) \text{ as } t \to \infty, \tag{4.1.22}$$

as well as stability of the fluxes on the right-hand side of (4.1.14), for soft potentials ϕ. The result (4.1.22) can be understood as a linear Landau damping result for two particles.

We remark that the reduction of the evolution problem to Vlasov equations stresses the importance of a good understanding of the Vlasov-Poisson equation, in particular the stability of steady states. It was proved in [21, 22] that the linear Vlasov-Poisson equation admits for stable steady states, however at the moment the theory only covers the radially symmetric case. In a one-dimensional periodic domain, the spectral theory of the linearized Vlasov equation is studied in [10]. It is interesting to remark that in [21, 22] it has been shown that the rate of convergence to equilibrium is only logarithmic when f_1 is a Maxwellian. Due to the shortcomings of the current stability theory of the (linear) Vlasov-Poisson equation, the rigorous stability results for the truncated correlations g_2 in this work are derived for soft potentials.

We then summarize the main implications of the results for the study of scaling limits of Coulomb particle systems. Most importantly, for Coulomb interacting particles, using as unit of length the Debye length, the only kinetic limit is given by the scaling (Coulomb-scaling), and the Debye screening becomes visible in the length scale of the two-particle correlation function. It is worth mentioning that the different decay exponents γ in the result suggests that the screening properties depend on the behavior of the one-particle function f_1 for large velocities.

Further, the argument identifies two regions in which the assumption $f_1 \gg g_2$ breaks down, namely for particles ξ_1, ξ_2 with very small relative velocity $v_1 - v_2 \approx 0$, and very fast particles. The critical region of particles with very small relative velocity is a result of the fact that the collision time diverges, when particles only very slowly separate (see [55]). The Debye screening, and the lack of screening for very fast particles can be observed on the level of the linearized Vlasov equation. We will take a closer look at this in Subsection 4.1.2.

A mathematical description of scaling limits of Coulomb particle systems requires to understand the following aspects: Firstly, the emergence of the Debye length L_D from the particle system (4.1.4). Secondly, one needs to estimate the deflections due to the interaction of particles with an impact parameter much larger than the Debye length. Due to the screening, the influence of a single charge decays much faster than the Coulomb potential itself. Thirdly, one needs to understand the deflections produced by particles that approach closer than the Debye length. The influence of these deflections turns out to be dominant by a logarithmic factor and yields the Landau equation in the kinetic limit. This is discussed in Subsection 4.1.3.

4.1.2 Debye Screening in the Vlasov equation

In this subsection, we discuss the onset of a screening length in the linearized Vlasov equation. To this end, we will take a closer look at the steady states of the Vlasov-Poisson equation in the presence of a point charge. The Debye screening can be observed in the decay of the equilibrium spatial profile, which has a characteristic length scale that is given by the Debye length L_D (cf. (4.1.5)), in spite of the fact that the Coulomb potential does not have a length scale. The screening effect is related to the classical subjects in the Vlasov theory such as Landau damping and Langmuir waves (cf. [21, 22, 29, 36, 39, 43]).

We prove in this chapter, that the evolution problem (4.1.16) can be reduced to the Vlasov system. We remark that one can formally derive the Balescu-Lenard equation from a stochastic model. The method consists in describing the evolution of the probability density of a tagged particle which interacts with a random medium. The random medium is assumed to evolve according to the Vlasov equation, linearized around the velocity distribution of the tagged particle. The approach of a Vlasov medium is well-studied in the formal theory in plasma physics [44, 47]. Rigorous results on a related model can be found in [28, 30].

Let (X, V) be the phase space coordinates of the tagged particle traveling through a continuous background, with which it interacts via the Coulomb potential. Here $f_0(v)$ is a fixed velocity distribution, and $h(t, x, v)$ the correction that is induced by the particle. Taking as unit of length the Debye length L_D (cf. (4.1.5)) as before, let the system be given by:

$$\partial_\tau h + v \nabla_x h - \nabla_x(\phi * \varrho)\nabla_v f_0 = \sigma \nabla_v f_0 \nabla \phi(x - X(\tau)), \qquad h(0, x, v) = 0 \qquad (4.1.23)$$

$$\varrho(x) = \int h(x, v)\, \mathrm{d}v \qquad (4.1.24)$$

$$\partial_\tau X = V, \quad \partial_\tau V = -\sigma \nabla_x(\phi * \varrho)(X(\tau)), \quad (X(0), V(0)) = (X_0, V_0). \qquad (4.1.25)$$

In the derivations of the Balescu-Lenard equation in [28, 30, 44], the initial datum $h(0, \cdot)$ in (4.1.23) is random. Then the dynamics describing the evolution of (X, V) becomes a stochastic differential equation. Notice that the evolution of random measures under the Vlasov equation has already been considered in Braun and Hepp (cf. [8]). In the system (4.1.23)-(4.1.25), (X, V) can be interpreted as a particle traveling through a random background of particles, and $h(x, v)$, $\varrho(x)$ as the correction of the homogeneous density (or "cloud") induced by the particle. It is worth noting that the well-posedness of the problem of a moving point charge interacting with a fully nonlinear Vlasov-Poisson system has been studied in [12].

For simplicity, assume $f_0(v)$ in (4.1.23) is radially symmetric. In the derivation of the Landau equation and the Balescu-Lenard equation, we make the assumption that the trajectories of particles are approximately rectilinear on the microscopic timescale. This suggests to approximate $X(\tau)$ in (4.1.23) by

$$X(\tau) \approx X_0 - \tau V_0. \qquad (4.1.26)$$

For the special case $V_0 = 0$, it was observed in [36] that the Debye screening can be derived from the equation (4.1.23). The spatial density of the steady state of (4.1.23) with a point charge at rest can be computed explicitly (without loss of generality $X_0 = 0$):

$$\varrho_{eq}(x) = \frac{\sigma}{4\pi|x|}e^{-|x|}. \qquad (4.1.27)$$

Remarkably, even though the potential $\phi(x) = 1/|x|$ does not have a length scale, the spatial profile of ϱ_{eq} decays exponentially with characteristic scale given by the Debye length L_D.

Now consider the case of $V_0 \neq 0$. Making the assumption of rectilinear motion (4.1.26), we can again solve (4.1.23) explicitly. For $\tau \to \infty$, the solution converges to traveling wave with velocity V_0. The spatial profile of the traveling wave can be represented in Fourier variables. Let f_0 be a given one-particle function, then the formula reads:

$$\hat{\varrho}_{trav}(k) = \frac{\sigma \int \frac{k \nabla f_0(v)}{k(v - V_0) - i0} \, dv}{|k|^2 D(k, k \cdot V_0)}, \tag{4.1.28}$$

where $D(k, u)$ is given by:

$$D(k, u) := 1 - \frac{1}{|k|^2} \int_{\mathbb{R}^3} \frac{k \cdot \nabla f_0(v)}{k \cdot v - u + i0} \, dv. \tag{4.1.29}$$

We remark that (4.1.29) suggests that for $|V_0| \to \infty$, the spatial profile $\varrho_{trav}(x)$ can have large oscillations with long wavelength $\lambda = 1/|k| \to \infty$. To see this, we decompose $D = D_R + iD_I$ into its real and imaginary part. For $|k| \to 0$ and u of order one, we have the asymptotic formula

$$D_R(k, u) \sim 1 - 1/|u|^2, \quad D_I(k, u) = 1/|k|^2 \int_{k \cdot v = u} k/|k| \nabla f_0(v) \, dv. \tag{4.1.30}$$

Hence, the real part of D in (4.1.29) has a zero for $|k| \to 0$, $u \sim 1$, and the imaginary part depends on the tail behavior of the one-particle function f_0. This suggests that the traveling wave ϱ_{trav} (cf. (4.1.28)) surrounding the particle (X, V) can lead to large deflections in other particles for $|V_0| \gg 1$, depending on the decay of $f_0(v)$ for large velocities. In the presence of very fast particles, the rectilinear approximation (4.1.26) does not hold. However, this should not affect the validity of the final kinetic equation in the limit $\sigma \to 0$, since the number of particles with velocity $|V_0| \gg 1$ becomes negligible.

This observation explains why the exponent in the estimate (4.1.21) depends on the decay properties of the one-particle functions, and the estimate is only valid for velocities varying on a compact set.

The zero of the real part D_R (cf. (4.1.30)) is also related to other important phenomena in plasma physics, such as the so-called Langmuir waves. The length of the Langmuir waves is much larger than the Debye length and the oscillation frequency has been normalized to $\Omega_{\text{Langmuir}} = 1$ in our setting. The amplitudes of these waves decrease exponentially at a rate proportional to D_I (cf. (4.1.30)), so the rate strongly depends on the background distribution of particles. For a Maxwellian distribution of particles $f_0 = M$, the imaginary part is exponentially small, which results in a very slow Landau damping as observed in [21, 22].

4.1.3 On the range of validity of the Balescu-Lenard equation for Coulomb potentials

The goal of this subsection is to determine the correct kinetic equation for scaling limits of particle systems interacting with the Coulomb potential, or the Coulomb potential smoothed out at

the origin. It was already remarked by Lenard in [34], that the integral (4.1.1) is not well-defined for $\phi(x) = 1/|x|$, since the integral

$$a_{i,j}(w,v) = \int_{\mathbb{R}^3} k_i k_j \delta(k \cdot w) \frac{|\hat{\phi}(k)|^2}{|\varepsilon(k, k \cdot v)|^2}\, \mathrm{d}k \tag{4.1.31}$$

is logarithmically divergent for large k. This corresponds to the divergence (4.1.21) for small values of the spatial variable x, so main contribution comes from the singularity of the Coulomb potential at the origin.

In the scaling limit (Coulomb-scaling), particle interaction is given by the potential $\sigma\phi(x) = \sigma/|x|$. Therefore, an interaction of particles with impact parameter $|b| \leq \sigma$ will result in a deflection of order one. This yields a Boltzmann collision term in the limit equation, as observed in [41]. We now analyze the influence of interactions with impact parameter $|b| \geq \sigma$. This corresponds to a truncation $\tilde{a}_{i,j}$ of the integral (4.1.31) to $|k| \leq \sigma^{-1}$. As Lenard observed in [34], the function $\varepsilon(k, k \cdot v) \to 1$ becomes constant for $k \to \infty$. Therefore, the truncated coefficient $\tilde{a}$ satisfies:

$$\tilde{a}_{i,j}(w,v) = \lim_{\sigma \to 0} |\log(\sigma)| \int_{B_{\sigma^{-1}}} \frac{k_i k_j \delta(k \cdot w)|\hat{\phi}(k)|^2}{|\varepsilon(k, k \cdot v)|^2}\, \mathrm{d}k \sim \delta_{i,j} - \frac{w_i w_j}{|w|^2}. \tag{4.1.32}$$

Hence, we obtain the Landau kernel in this limit. Now we discuss how this observation connects to the scaling limit (Coulomb-scaling). Due to (4.1.32), the kinetic timescale is not given by $t = \sigma\tau$, but slightly shorter by a logarithmic correction. Therefore, the mathematically rigorous kinetic equation associated to the scaling limit (Coulomb-scaling) is expected to be the Landau equation, and the main contribution is due to the interaction of particles with very small impact factor. However a more accurate description of physical systems might be obtained by keeping the terms of the order $|\log(1/\sigma)|^{-1}$ in the equation, since in physical systems, $|\log(1/\sigma)|$ cannot be expected to be very large (cf. the discussion in §41 of [36]). Therefore, the physical equation describing plasmas can be expected to involve a Balescu-Lenard term, the Landau collision operator and a Boltzmann collision operator. The precise numerical factors would depend on the physical system in question. The Balescu-Lenard equation is the correct limit equation for systems with soft potential interaction in the scaling limits (soft potential scaling) with $\beta = 1$.

Consider particle systems interacting via the Coulomb potential and take as unit of length the Debye length L_D (4.1.5). As a simplified problem, one can study a smooth variant of the Coulomb potential, that is $\phi_{C,r} \in C^\infty$ radially symmetric and $\phi_{C,r}(x) = 1/|x|$ for $|x| \geq 1$. Then the kinetic equation associated to the scaling limit (Coulomb-scaling) can be expected to be the Balescu-Lenard equation. Notice that the equation includes the screening effect, that is expected since $\phi_{C,r}(x)$ coincides with the Coulomb potential for large $|x|$.

A characterization of the limit equations for scaling limits of Lorentz models (i.e. a tagged particle in a random, but fixed, background of scatterers) can be found in [41].

4.2 Preliminary and main results

4.2.1 Definitions and assumptions

For future reference we fix the notation for some classical integral transforms.

Notation 4.2.1. *We will use the following conventions for the Laplace transform $\mathcal{L}(f)$, the Fourier transform $\hat{f}$ and the Fourier-Laplace transform $\tilde{f}$:*

$$\mathcal{L}(f)(z) = \int_0^\infty e^{-zt} f(t) \, \mathrm{d}t \tag{4.2.1}$$

$$\mathcal{F}(f)(k) = \hat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot k} \, \mathrm{d}x \tag{4.2.2}$$

$$\tilde{f}(z,k) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \int_0^\infty f(t,x) e^{-zt} e^{-ix \cdot k} \, \mathrm{d}t \, \mathrm{d}x. \tag{4.2.3}$$

Definition 4.2.2. *We define operators P^+, P^- and P on $L^2(\mathbb{R})$, that on Schwartz functions $f \in \mathcal{S}(\mathbb{R})$ are given by:*

$$P^\pm[f](x) := \lim_{\delta \to 0^+} \int_{\mathbb{R}} \frac{f(x')}{x' - x \mp i\delta} \, \mathrm{d}x', \quad P[f](x) := \mathrm{PV} \int_{\mathbb{R}} \frac{f(x')}{x' - x} \, \mathrm{d}x' \tag{4.2.4}$$

where the principal value integral PV *is defined as:* $\mathrm{PV} \int \, \mathrm{d}x' = \lim_{\delta \to 0^+} \int \mathbb{1}(|x - x'| \geq \delta) \, \mathrm{d}x'$.

Notation 4.2.3 (Relative velocity and impact parameter)**.** *For vectors $k, v_1, v_2 \in \mathbb{R}^3$, $v_1 \neq v_2$, $k \neq 0$, we will use the following shorthand notation:*

$$\omega = \frac{k}{|k|}, \quad v_r = v_1 - v_2, \quad v_r = \frac{v_r}{|v_r|}. \tag{4.2.5}$$

The impact parameter $b \in \mathbb{R}^3$ and the distance to the collision point $d \in \mathbb{R}$ of particles (x_1, v_1), (x_2, v_2) with relative position $x = x_1 - x_2$ and relative velocity $v_r = v_1 - v_2$ is defined as:

$$d(x, v_r) = \frac{x \cdot v_r}{|v_r|}, \quad b(x, v_r) = x - P_{v_r}(x) = x - \frac{v_r(x \cdot v_r)}{|v_r|^2}. \tag{4.2.6}$$

Due to the translation invariance of the system, the truncated correlation function $g_2(x, v, x', v')$ is a function of $x - x', v, v'$ only. By a slight abuse of notation, we identify g_2 with the function:

$$g_2(x - x', v, v') = g_2(x, v, x', v'). \tag{4.2.7}$$

Also the function should be invariant under exchanging the two particles, so we impose the symmetry:

$$g_2(x, v, v') = g_2(-x, v', v). \tag{4.2.8}$$

This symmetry we include in the space of functions in which we solve the Bogolyubov equation.

Definition 4.2.4. *Define the functionals $|h|[g]$, $h[g]$ given by the following formulas:*

$$|h|[g] = \int |g(x, v_1, v_2)| \, \mathrm{d}v_2, \quad h[g] = \int g(x, v_1, v_2) \, \mathrm{d}v_2. \tag{4.2.9}$$

Let W be the function space given by:

$$W = \{g \in L^1_{loc}(\mathbb{R}^9) : (4.2.8) \text{ holds}, |h|[g] \in L^1_{loc}, \sup_{|v| \leq R} \|h[g](\cdot, v)\|_{L^2} \leq C(R) \text{ for } R > 0\}. \tag{4.2.10}$$

We now give a definition of a solution to the Bogolyubov equation. We recall the space $L^1 + L^2$ of functions ζ that can be decomposed as $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in L^1$, $\zeta_2 \in L^2$.

Definition 4.2.5 (Bogolyubov correlation)**.** *Let $\nabla\phi \in L^1 + L^2$, and $f \in W^{1,1}(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ be a probability density. We say $g_B \in W$ is a solution to the Bogolyubov equation if for all $\psi \in C_c^\infty(\mathbb{R}^9)$*

$$
\begin{aligned}
&- \int (v_1 - v_2) g_B \partial_x \psi - \int \nabla f(v_1) \nabla\phi(x+y) h[g_B](y, v_2)\psi(x, v_1, v_2) \\
&- \int \nabla f(v_2)\nabla\phi(-x+y)h[g_B](y,v_1)\psi(x,v_1,v_2) = \int (\nabla_{v_1} - \nabla_{v_2})[f \otimes f]\nabla\phi(x)\psi,
\end{aligned}
\tag{4.2.11}
$$

and it satisfies the Bogolyubov boundary condition

$$
g_B(x - \tau(v_1 - v_2), v_1, v_2) \to 0, \quad \text{as } \tau \to \infty, \text{ a.e.}
\tag{4.2.12}
$$

Definition 4.2.6 (Radon transform and dielectric function)**.** *Let $f \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We define the Radon transform $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ associated to f by ($\omega = \omega(k)$ as in (4.2.5)):*

$$
F(k, u) := \int_{\{v : \omega \cdot v = u\}} f(v) \, \mathrm{d}v.
\tag{4.2.13}
$$

Further we define the dielectric function $\varepsilon : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ associated to $f \in W^{1,1}(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ and a potential ϕ by:

$$
\varepsilon(k, -|k|u) := 1 - \hat{\phi}(k) P^-[\partial_u F(k, \cdot)](u).
\tag{4.2.14}
$$

Here the operator P^- defined in (4.2.4) is applied in the second variable of $\partial_u F$. As a shorthand we also introduce the functions α, α^- given by:

$$
\alpha(\chi, u) := P[\partial_u F(\chi, \cdot)](u), \quad \alpha^-(\chi, u) := P^-[\partial_u F(\chi, \cdot)](u).
\tag{4.2.15}
$$

Remark 4.2.7. *Note that the dielectric function ε coincides with the function D introduced in (4.1.28), which quantifies the correction to the homogeneous density induced by a single point charge.*

The following definitions will be useful in studying the linear evolution problem (4.1.16) for g. When f is time independent, the equation (4.1.16) for g can be solved explicitly. To this end we introduce some notation.

Notation 4.2.8. *We introduce the function:*

$$
Q(k, v) = k\nabla f(v)\hat{\phi}(k).
\tag{4.2.16}
$$

Furthermore, for a function $h(x, v)$ and a potential ϕ we set E_h to be the self-consistent potential associated to h:

$$
E[h](x) = E_h(x) = \int \int \phi(x - y) h(y, v) \, \mathrm{d}v \, \mathrm{d}y.
\tag{4.2.17}
$$

Definition 4.2.9 (Vlasov and transport propagator). *Let ϕ be a radially symmetric Schwartz potential. Let $\mathcal{V}$ be the linear Vlasov propagator associated to f, so let $\mathcal{V}(t)[h_0] = h(t)$ be the solution to:*

$$\partial_t h + v\nabla_x h - \nabla E_h \nabla f = 0, \quad h(0,\cdot) = h_0(\cdot), \tag{4.2.18}$$

with E_h as in (4.2.17). In Fourier-Laplace variables (cf. (4.2.3)) the solution is given by:

$$\tilde{h}(z,k,v) = \frac{\hat{h}_0(k,v)}{z+ikv} + \frac{iQ(k,v)\tilde{\varrho}(z,k)}{z+ikv}, \quad \tilde{\varrho}(z,k) = \frac{\int \frac{\hat{h}_0(k,v')}{z+ikv'}\,\mathrm{d}v'}{\varepsilon(k,-iz)}, \tag{4.2.19}$$

with Q as introduced in (4.2.16). Further let T be the free transport propagator so

$$T(t)[g](\xi_1,\xi_2) := g(x - v_1 t, v_1, x_2 - v_2 t, v_2). \tag{4.2.20}$$

Definition 4.2.10. *Let $\tilde{g}_0(\xi_1,\xi_2) = g_0(x_1 - x_2, v_1, v_2)$, $g_0 \in \mathcal{S}((\mathbb{R}^3)^3)$ be symmetric in exchanging the variables ξ_1, ξ_2, and set $S(\xi_1,\xi_2) = \delta(\xi_1 - \xi_2)f(v_1)$. We define the Bogolyubov propagator $\mathcal{G}$ by:*

$$\mathcal{G}(t)[\tilde{g}_0] := \mathcal{V}_{\xi_1}(t)\mathcal{V}_{\xi_2}(t)[S + \tilde{g}_0] - T(t)[S], \tag{4.2.21}$$

where $\mathcal{V}_{\xi_1}$ is the Vlasov propagator acting the set of variables $(x_1, v_1) = \xi_1$, and $\mathcal{V}_{\xi_2}$ the propagator acting on $(x_2, v_2) = \xi_2$.

We will analyze the equilibrium two-particle correlations for so-called soft potentials and the Coulomb potential. Notice that we restrict our attention to radially symmetric potentials.

Assumption 4.2.11 (Potentials). *Let $\phi_C \in C(\mathbb{R}^3\backslash\{0\})$ be the Coulomb potential, so $\phi_C(x) = \frac{c}{|x|}$ for some $c > 0$. Assume without loss of generality that $c = \sqrt{\frac{\pi}{2}}$, when $\hat{\phi}(k) = \frac{1}{|k|^2}$. We say $\phi_S = \phi_S(|x|)$ is a soft potential if $\phi_S \in \mathcal{S}(\mathbb{R}^3)$.*

On the one-particle distribution function f we make the following regularity assumptions.

Assumption 4.2.12 (Regularity and Decay). *Let $f \in C^8(\mathbb{R}^3)$ be nonnegative and*

$$|\nabla^m f(v)| \leq Ce^{-|v|}, \quad \text{for } m = 0, 1, \ldots, 8. \tag{4.2.22}$$

Further let f be normalized to:

$$\int f(v)\,\mathrm{d}v = 1. \tag{4.2.23}$$

Our proof of existence of Bogolyubov correlations requires the plasma to be stable. This can be mathematically formulated in terms of the dielectric function ε (cf. (4.2.14)) associated to f.

Assumption 4.2.13 (Plasma stability). *We say f is stable if for all $k \in \mathbb{R}^3$, $\chi \in S^2$, $u \in \mathbb{R}$ we have:*

$$|k|^2 \neq P^-[\partial_u F(\chi,\cdot)](u), \quad \text{in particular } |\varepsilon(k,u)| \neq 0, \quad \varepsilon \text{ as in (4.2.14)}. \tag{4.2.24}$$

Remark 4.2.14. *The physical relevance of this condition is discussed in [36]. A necessary and sufficient condition for stability (cf. (4.2.24)) was given by Penrose in [43]. For example the condition (4.2.24) is satisfied by functions f, for which $F(u)$ has precisely one maximum and no other critical points.*

In order to prove (exponential) linear stability of the equilibrium correlations and their fluxes we make a stronger analytic stability assumption on the plasma, which requires that we can extend the dielectric function to a strip in the complex plane.

Assumption 4.2.15 (Strong plasma stability). *Let $f > 0$ be a Schwartz probability density on $\mathbb{R}^3$. Let F be the Radon transform defined in (4.2.13) and $\phi = \phi_S$ a soft potential. Assume that there exists $c > 0$ such that for all $\chi \in S^2$, $F(\chi, iz)$ has a holomorphic extension to the strip $H_c := \{z \in \mathbb{C} : |\Re(z)| \leq c\}$ and on H_c satisfies the estimate*

$$|F(\chi, iz)| \leq \frac{C}{1 + \Im(z)^2}. \tag{4.2.25}$$

We will assume that the associated extension of the dielectric function $z \mapsto \varepsilon(k, -i|k|z)$ to the shifted right half-plane $H^-_c := \{z \in \mathbb{C} : \Re(z) \geq -c\}$ is bounded below uniformly:

$$|\varepsilon(k, -i|k|z)| \geq c_0 > 0, \quad \text{for } 0 \neq k \in \mathbb{R}^3, \, z \in H^-_c. \tag{4.2.26}$$

We now introduce some technical assumptions, that we later use to quantify the rate of decay of the equilibrium correlations. We distinguish functions f that behave like an exponential as $|v| \to \infty$, specified in Assumption 4.2.17, and functions that behave like Gaussians, as specified in Assumption 4.2.18.

Notation 4.2.16. *We recall the function α introduced in (4.2.15). For $k \in \mathbb{R}^3$, $\chi \in S^2$, let $u_0^+(k, \chi) > 0$, $u_0^+(k, \chi) < 0$ be the solutions to:*

$$|k|^2 - \alpha(\chi, u_0^\pm) = 0, \tag{4.2.27}$$

whenever (4.2.27) has a unique solution with the prescribed sign. Further write $I(k, \chi)$ for the set

$$I(k, \chi) = (u_0^-(k, \chi) - 1, u_0^-(k, \chi) + 1) \cup (u_0^+(k, \chi) - 1, u_0^+(k, \chi) + 1). \tag{4.2.28}$$

Let $L^\pm(k, \chi)$, $\Psi^\pm(k, \chi, y)$ be given by:

$$L^\pm(k, \chi) = \frac{\partial_u F(\chi, u_0(k, \chi))}{\partial_u \alpha(\chi, u_0(k, \chi))}, \qquad\qquad \text{for } k \in \mathbb{R}^3, \, \chi \in S^2, \tag{4.2.29}$$

$$\Psi^\pm(k, \chi, y) = u_0(k, \chi) + y \frac{\partial_u F(\chi, u_0(k, \chi))}{\partial_u \alpha(\chi, u_0(k, \chi))}, \qquad \text{for } k \in \mathbb{R}^3, \, \chi \in S^2, \, y \in \mathbb{R}. \tag{4.2.30}$$

Assumption 4.2.17 (Asymptotically exponential behavior). *Let f satisfy the Assumptions 4.2.12-4.2.13. Let $L^\pm = L^\pm(k, \chi)$ and $\Psi^\pm$ be as in Notation 4.2.16. We say f behaves asymptotically like*

an exponential if it satisfies the following for some $r, c, C > 0$:

$$|\nabla_{k,\chi,y}^6 \left(\frac{|k|^3}{\partial_u \alpha(\chi, \Psi^\pm)} \right)| \le C, \quad \text{for } |k| \le r, \, \chi \in S^2, \, |y| \le L^{\pm -1}, \tag{4.2.31}$$

$$|\nabla_{k,\chi,y}^6 \left(\frac{|k|^2 - \alpha(\chi, \Psi^\pm)}{y \partial_u F(\chi, \Psi^\pm)} \right)| \le C, \quad \text{for } |k| \le r, \, \chi \in S^2, \, |y| \le L^{\pm -1}, \tag{4.2.32}$$

$$| \left(\frac{|k|^2 - \alpha(\chi, \Psi^\pm)}{y \partial_u F(\chi, \Psi^\pm)} \right) | \ge c, \quad \text{for } |k| \le r, \, \chi \in S^2, \, |y| \le L^{\pm -1}, \tag{4.2.33}$$

$$|\nabla_{k,\chi,y}^6 \left(\frac{F(\chi, \Psi^\pm)}{\partial_u F(\chi, \Psi^\pm)} \right)| \le C, \quad \text{for } |k| \le r, \, \chi \in S^2, \, |y| \le L^{\pm -1}. \tag{4.2.34}$$

Assumption 4.2.18 (Asymptotically Maxwellian behavior)**.** *Let f satisfy the Assumptions 4.2.12-4.2.13. Let $L^\pm = L^\pm(k, \chi)$ and $\Psi^\pm$ be as in Notation 4.2.16. We say f behaves asymptotically like a Gaussian if it satisfies (4.2.31)-(4.2.33) and the following for some $r, C > 0$:*

$$|\nabla_{k,\chi,y}^6 \left(\frac{F(\chi, \Psi^\pm)}{|k|\partial_u F(\chi, \Psi^\pm)} \right)| \le C, \quad \text{for } |k| \le r, \, \chi \in S^2, \, |y| \le L^{\pm -1}. \tag{4.2.35}$$

Remark 4.2.19. *For example, the Assumptions 4.2.17 and 4.2.18 are satisfied by probability densities of the form:*

$$f(v) \sim \left(1 + \frac{\Phi(v)}{(2 + |v|^2)^\alpha} \right) e^{-(1+|v|^2)^{\frac{\gamma}{2}}}. \tag{4.2.36}$$

Here $\gamma = 1$ if f satisfies Assumption 4.2.17, $\gamma = 2$ if f satisfies Assumption 4.2.18, $\alpha > 0$ and $\Phi \in C_b^\infty$ is smooth with bounded derivatives and $|\Phi| \le 1$. Note that this includes anisotropic velocity distributions.

4.2.2 Results of the chapter

The first result of the chapter is the well-posedness of the steady state equation (4.1.18). We prove that the solutions formally obtained by Oberman and Williams [42] by means of the method introduced by Lenard in [34] are indeed well-defined solutions to the equation in the sense of Definition 4.2.5.

Theorem 4.2.20 (Bogolyubov correlations)**.** *Let f satisfy the Assumptions 4.2.12 and 4.2.13 and ϕ be either the Coulomb potential or a soft potential. In the Coulomb case, assume further that f satisfies Assumption 4.2.17 or 4.2.18. Then there exists a weak solution g_B to the Bogolyubov equation in the sense of Definition 4.2.5.*

The proof of this theorem is the content of Subsection 4.2.4.

After making precise the well-posedness of the equation, we study screening properties of the Bogolyubov correlations. The following theorem describes the decay of the solutions of the Bogolyubov equation (4.1.18). Note that the equation is written taking as unit of length the characteristic length ℓ of the potential in the case $\phi = \phi_S$ soft or the Debye length L_D (4.1.5) for the Coulomb potential. Therefore, the following estimate proves that the characteristic range of interaction is given by ℓ or L_D respectively. Furthermore, we find that the decay rate of the Bogolyubov correlations differs from the decay rate of the potential.

Theorem 4.2.21 (Screening estimate for the Bogolyubov correlations). *Let f be a function that satisfies the Assumptions 4.2.12-4.2.13 and ϕ be either Coulomb potential or a soft potential. We recall the definition of the impact parameter b and the distance to collision d, as well as d_- (cf. (4.2.6)). Then for $x \in \mathbb{R}^3$, and $v_1, v_2 \in K$ varying on a compact set $K \subset \mathbb{R}^3$ the following estimate holds:*

$$|g_B(x, v_1, v_2)| \leq \frac{C(K, \delta)}{|v_r|} \frac{1}{|b| + d_-} \frac{1}{(1 + |b| + d_-)^{\gamma - \delta}}, \quad \text{for } \delta > 0. \qquad (4.2.37)$$

If $\phi = \phi_C$, we can choose $\gamma = 1$ for f behaving like a Maxwellian in the sense of Assumption 4.2.18, and $\gamma = 0$ for f satisfying Assumption 4.2.17. For $\phi = \phi_S$ the statement holds for $\gamma = 1$ and $C(K, \delta)$ can be chosen independently of K.

More precise estimates can be found in the Theorems 4.3.1 and 4.3.6.

The derivation of the Balescu-Lenard equation proposed by Bogolyubov postulates that steady states do not only exist, but are also stable in microscopic times. More precisely, Bogolyubov's argument requires that the fluxes in f_1 induced by the function g_2 (cf. (4.1.13)) converge to the fluxes associated to the equilibrium correlations $g_B[f_1]$. In the case of soft potential interaction, we prove the stability of the equilibrium correlations if f_1 in (4.1.13) is assumed to be time-independent.

Theorem 4.2.22. *Let ϕ be a soft potential and f satisfy the strong stability Assumption 4.2.15. Further let $\tilde{g}_0(\xi_1, \xi_2) = g_0(x_1 - x_2, v_1, v_2)$, $g_0 \in \mathcal{S}((\mathbb{R}^3)^3)$ be translation invariant and symmetric:*

$$\tilde{g}_0(\xi_1, \xi_2) = \tilde{g}_0(\xi_2, \xi_1) \qquad \qquad \text{for all } \xi_1, \xi_2 \in \mathbb{R}^3 \times \mathbb{R}^3, \qquad (4.2.38)$$

$$\tilde{g}_0(x_1, v_1, x_2, v_2) = \tilde{g}_0(x_1 + a, v_1, x_2 + a, v_2) \qquad \text{for all } x_1, x_2, a, v_1, v_2 \in \mathbb{R}^3. \qquad (4.2.39)$$

Consider the function $\tilde{g}(t) := (\mathcal{G}(t)\tilde{g}_0)$ given by (4.2.21), which (using (4.2.39)) we identify with

$$g(t, x_1 - x_2, v_1, v_2) = \tilde{g}(t, x_1, v_1, x_2, v_2). \qquad (4.2.40)$$

Then we have $g, \partial_t g \in C(\mathbb{R}^+, \mathcal{S}(\mathbb{R}^9))$ and g solves the Bogolyubov equation (4.1.16) with initial datum g_0. The steady state g_B given in Theorem 4.2.20 is linearly stable, more precisely:

$$g(t) \longrightarrow g_B \quad \text{in } D'(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3) \text{ as } t \to \infty. \qquad (4.2.41)$$

Furthermore, the associated fluxes in the space of velocities are stable, i.e. for all $v \in \mathbb{R}^3$ we have:

$$\nabla_v \cdot \left(\int \nabla\phi(x) g(t, x, v, v')\, \mathrm{d}v'\, \mathrm{d}x \right) \longrightarrow \nabla_v \cdot \left(\int \nabla\phi(x) g_B(x, v, v')\, \mathrm{d}v'\, \mathrm{d}x \right) \quad \text{as } t \to \infty. \quad (4.2.42)$$

This theorem is proved in Section 4.4.

4.2.3   Auxiliary results

The following lemmas provide a version of the well-known Plemelj-Sokhotski formula, which allows us to write the original function f in terms of $P^+[f]$ and $P^-[f]$ as introduced in Definition 4.2.2. In a more general setting, such formulas are discussed in [40].

Lemma 4.2.23. *The operators $P^\pm$ and P are bounded from L^2 to L^2. Let $f \in L^2(\mathbb{R}; \mathbb{R})$, then we have $P^+[f] = \overline{P^-[f]}$. Furthermore for $f \in L^2(\mathbb{R}; \mathbb{C})$ there holds:*

$$f = \frac{1}{2\pi i}(P^+[f] - P^-[f]). \tag{4.2.43}$$

Proof. By a classical result, $P^\pm$ are Fourier multiplication operators with symbols $\pm 2\pi i \mathbb{1}_{\xi > 0}$. The same holds for P with multiplier $i\pi \operatorname{sign} \xi$. Combining this with Plancherel's theorem, we find that the operators are bounded on L^2 and satisfy the identity (4.2.43). For real-valued functions f, the identity $P^+[f] = \overline{P^-[f]}$ holds, since these operators are obtained in a limit $\delta \to 0$ (cf. (4.2.4)) and the identity holds for all $\delta > 0$. $\qquad\square$

Lemma 4.2.24. *Let $f \in L^2(\mathbb{R})$, and q^+ be analytic on the upper half plane, q^- analytic on the lower half plane and decaying: $|q^\pm(z)| \to 0$, $|z| \to \infty$. Assume that $\lim_{\delta \to 0^+} q^\pm(\cdot \pm i\delta)$ exists in $L^2(\mathbb{R})$ and:*

$$\lim_{\delta \to 0^+} \frac{1}{2\pi i}\left(q^+(\cdot + i\delta) - q^-(\cdot - i\delta)\right) = f. \tag{4.2.44}$$

Then we have: $P^\pm[f] = q^\pm$.

Proof. We consider the differences $\zeta^\pm := q^\pm - P^\pm[f]$. The functions are analytic in the upper, respectively the lower half-plane and decay as $|z| \to \infty$, $|\Im(z)| \geq 1$. We claim the function ζ, given by ζ^+ on the upper half-plane and ζ^- on the lower half-plane, is an entire function. To see this, fix $z_0 \in \mathbb{C}$ arbitrary and consider $Z(z) := \int_{\gamma[z_0, z]} \zeta(z')d\gamma(z')$, where $\gamma[z_0, z]$ is an arbitrary curve connecting z_0 and z. Then Z is an analytic function above and below and is continuous at the real line by (4.2.43) and (4.2.44), hence an entire function. Using $Z' = \zeta$, we infer that ζ is an entire function as well. Outside the strip with $|\Im(z)| \leq 1$, ζ is bounded and decays for $|z| \to \infty$. On the strip, we use the L^2 convergence of $P^\pm[f]$ and $q^\pm$ together with the mean value property of $\Re(\zeta), \Im(\zeta)$ to obtain:

$$|\zeta(z)| \leq C \int_{B_1(z)} |\zeta(z')| \, \mathrm{d}z' \leq C \left(\|f\|_{L^2} + \sup_{|r| < 2} \|q^\pm(\cdot \pm ir)\|_{L^2(\mathbb{R})}\right) \leq C.$$

So ζ is a bounded entire function, hence constant. By $\lim_{R \to \infty} \zeta(iR) = 0$ we get $\zeta \equiv 0$ as claimed. $\quad\square$

We make Assumption 4.2.13 to ensure that the dielectric function ε does not vanish. In many arguments later we will make use of quantitative lower bounds on $|\varepsilon|$, one of which is provided by the following lemma.

Lemma 4.2.25 (Estimate on the degeneracy of ε). *Let f satisfy the Assumptions 4.2.12-4.2.13. If $\phi = \phi_S$ is a soft potential, there exists $c_1 > 0$ such that for all $k \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ we have:*

$$|\varepsilon(k, -k \cdot v)| \geq c_1 > 0. \tag{4.2.45}$$

If $\phi = \phi_C$ is the Coulomb potential, for any $K \subset \mathbb{R}^3$ compact and $\delta > 0$ we have:

$$|\varepsilon(k, -k \cdot v)| \geq c_1(K) > 0, \qquad \text{for all } 0 \neq k \in \mathbb{R}^3, \, v \in K \tag{4.2.46}$$

$$|\varepsilon(k, -k \cdot v)| \geq c_2(\delta) > 0, \qquad \text{for all } |k| \geq \delta, \, v \in \mathbb{R}^3. \tag{4.2.47}$$

Proof. Let $\phi = \phi_C$ be the Coulomb potential. Then we have:

$$|\varepsilon(k, -k \cdot v)| = \left| 1 - \frac{1}{|k|^2} P^-[\partial_u F(\omega, \cdot)](\omega \cdot v) \right|. \tag{4.2.48}$$

Since $|P^-[\partial_u F(\omega, \cdot)]|$ is bounded, $|\varepsilon(k - k \cdot v)|$ attains its minimum on $(k, v) \in \left(\mathbb{R}^3 \setminus B_\delta(0)\right) \times \mathbb{R}^3$ for any $\delta > 0$. This minimum is nonzero by (4.2.24), so (4.2.47) holds.

On the other hand, since $P^-[\partial_u F] \neq 0$ (cf. (4.2.24)), the mapping $v \mapsto \inf_{k \in \mathbb{R}^3} |\varepsilon(k, -k \cdot v)|$ is continuous, so (4.2.46) holds on compact sets K.

The estimate (4.2.45) for soft potentials is immediate. $\qquad\square$

Remark 4.2.26. *In the Coulomb case, the estimates (4.2.46)-(4.2.47) cannot be improved, since it is known (cf. [43]) that:*

$$\inf_{k \in \mathbb{R}^3, v \in \mathbb{R}^3} |\varepsilon(k, -k \cdot v)| = 0.$$

Lemma 4.2.27 (Asymptotics of $\alpha(\chi, u)$). *Let f satisfy the Assumptions 4.2.12-4.2.13. We recall the function α introduced in (4.2.15). There exist constants $C, R > 0$ such that for $|u| \geq R$:*

$$\left| \partial_u^j \alpha(\chi, u) - \frac{(-1)^j (j+1)!}{u^{j+2}} \right| \leq \frac{C}{u^{j+3}} \quad \text{for } j \in \mathbb{N}_0, \ j \leq 6, \tag{4.2.49}$$

$$\left| \partial_\chi^\ell \partial_u^j \alpha(\chi, u) \right| \leq \frac{C}{u^{j+3}} \quad \text{for } j \in \mathbb{N}_0, \ \ell \in \mathbb{N}, \ j + \ell \leq 6 . \tag{4.2.50}$$

Proof. The derivative ∂_u^j can be taken inside the operator P:

$$\partial_u^j \alpha(\chi, u) = P[\partial_u^{j+1} F(\chi, \cdot)](u). \tag{4.2.51}$$

Using that P is a Fourier multiplication operator with multiplier $i\pi \operatorname{sign}(\xi)$ we write:

$$\widehat{\partial_u^j \alpha(\chi, \cdot)}(\xi) = i\pi \operatorname{sign}(\xi) \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi).$$

Now we perform the Fourier inversion integral and integrate by parts:

$$\partial_u^j \alpha(\chi, u) = -\int_{-\infty}^0 (\pi/2)^{\frac{1}{2}} i e^{i\xi \cdot u} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi) \, d\xi + \int_0^\infty (\pi/2)^{\frac{1}{2}} i e^{i\xi \cdot u} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi) \, d\xi$$

$$= (\pi/2)^{\frac{1}{2}} \int_{-\infty}^0 \frac{e^{i\xi \cdot u}}{u} \partial_\xi \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi) \, d\xi + (\pi/2)^{\frac{1}{2}} \frac{1}{u} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(0)$$

$$-(\pi/2)^{\frac{1}{2}} \int_0^\infty \frac{e^{i\xi \cdot u}}{u} \partial_\xi \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi) \, d\xi + (\pi/2)^{\frac{1}{2}} \frac{1}{u} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(0).$$

Since $\partial_u^{j+1} F$ is a derivative, we have $\mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi) = 0$. Iterating the argument we find:

$$\partial_u^j \alpha(\chi, u) = -\frac{(2\pi)^{\frac{1}{2}} i}{(-iu)^{j+2}} \partial_\xi^{j+1} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(0) - (2\pi)^{\frac{1}{2}} - \frac{(2\pi)^{\frac{1}{2}} i}{(-iu)^{j+3}} \partial_\xi^{j+2} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(0) \tag{4.2.52}$$

$$+ \int_0^\infty \frac{e^{i\xi \cdot u} i}{(-iu)^{j+3}} \partial_\xi^{j+3} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi) \, d\xi - \int_{-\infty}^0 \frac{e^{i\xi \cdot u} i}{(-iu)^{j+3}} \partial_\xi^{j+3} \mathcal{F}(\partial_u^{j+1} F(\chi, \cdot))(\xi) \, d\xi.$$

The leading order term is explicit by (4.2.23):

$$\partial_\xi^{j+1}\mathcal{F}(\partial_u^{j+1}F(\chi,\cdot))(0) = \frac{i^{j+1}(j+1)!}{(2\pi)^{\frac{1}{2}}}. \tag{4.2.53}$$

Combining (4.2.52), (4.2.53) gives (4.2.49). The derivative of (4.2.53) in χ vanishes, so we obtain (4.2.50). $\qquad\square$

The implicit function theorem gives the following Lemma on the function u_0 defined in Notation 4.2.16.

Lemma 4.2.28. *Let f satisfy the Assumptions 4.2.12-4.2.13. Using (4.2.49), for $|k| \leq r$, $r > 0$ small enough there are unique $u_0^\pm(k,\chi)$ such that (4.2.27) holds, and we have the estimates:*

$$|\partial^j u_0^\pm(k,\chi)| \leq \frac{C}{|k|^{j+1}} \quad \text{for } j \in \mathbb{N}_0,\ j \leq 6, \tag{4.2.54}$$

$$|\partial_\chi^\ell \partial^j u_0^\pm(k,\chi)| \leq \frac{C}{|k|^j} \quad \text{for } j \in \mathbb{N}_0,\ \ell \in \mathbb{N},\ j + \ell \leq 6 . \tag{4.2.55}$$

We can represent the solution to the Bogolyubov equation (4.1.18) explicitly in Fourier variables. The decay properties of the solution are encoded in the singularity of their Fourier transform at the origin, which motivates to make the following definition.

Definition 4.2.29. *Let $0 < \kappa \leq 1$ and $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. Define the functional $[f]_\kappa$ by:*

$$[f]_\kappa(x) := \sup_{\substack{0 < |h| \leq 1 \\ x+h \neq 0}} \frac{|f(x+h) - f(x)|}{|h|^\kappa}.$$

The following lemma gives sharp decay estimates for functions that have an isolated singularity in Fourier variables.

Lemma 4.2.30. *Let $l \in \mathbb{N}$, $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be ℓ times continuously differentiable with $|\nabla^j f| \in L^1$ for $0 \leq j \leq \ell$. Further let $0 < \kappa \leq 1$ and $[\nabla^\ell f]_\kappa \in L^1$. Then the Fourier transform $\hat{f}$ decays like:*

$$|\hat{f}(x)| \leq \frac{C}{1 + |x|^{\ell+\kappa}}. \tag{4.2.56}$$

Proof. Since $f \in L^1$ we know $\hat{f} \in L^\infty$ with $\|\hat{f}\|_{L^\infty} \leq C\|f\|_{L^1}$. For the additional decay we inspect the transformation formula directly. We distinguish the cases ℓ even and ℓ odd. For $\ell = 2m$ even, we use

$$e^{-i\pi kx} = \frac{1}{(\pi|x|)^{2m}}\Delta^m(e^{-i\pi xk}). \tag{4.2.57}$$

Further we use that f is in $f \in W^{l,1}(\mathbb{R}^n)$ to compute

$$\hat{f}(\pi x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int f(k)e^{-i\pi xk}\,\mathrm{d}k = \frac{1}{(\pi|k|)^{2m}}\frac{1}{(2\pi)^{\frac{n}{2}}}\int \Delta^m f(k)e^{-i\pi xk}\,\mathrm{d}k. \tag{4.2.58}$$

Now $g := \Delta^m f$ satisfies $|g| + [g]_\kappa \in L^1$. Therefore we can estimate

$$\hat{g}(\pi x) = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int g(k) e^{-i\pi(k - \frac{x}{|x|^2})x} \, \mathrm{d}k = \frac{1}{2(2\pi)^{\frac{n}{2}}} \int \left(g(k) - g(k + \frac{x}{|x|^2}) \right) e^{-\pi k x} \, \mathrm{d}k.$$

Taking absolute values and using $[g]_\kappa \in L^1$ gives

$$|\hat{g}(\pi x)| \leq \frac{1}{2(2\pi)^{\frac{n}{2}}} \int [g]_\kappa(k)/|x|^\kappa \, \mathrm{d}k \leq \frac{C}{|x|^\kappa}.$$

Inserting this into (4.2.58) gives $|\hat{f}(x)| \leq \frac{C}{1+|x|^{l+\kappa}}$ as claimed. For $\ell = 2m + 1$ odd we repeat the computation, except that we now use $e^{-i\pi kx} = \frac{ix}{(\pi|x|)^{2m}} \cdot \nabla \Delta^m(e^{-i\pi xk})$ instead of (4.2.57). $\qquad\square$

As a corollary we obtain bounds for the (inverse) Fourier transform of functions that depend on the modulus $\omega = \frac{k}{|k|}$.

Lemma 4.2.31. *Let $\ell \in \mathbb{N}$, $\Phi(k, \chi) \in C_c^{n+\ell}(B_1(0) \times S^{n-1})$. Then the Fourier transform of the mapping $T(k) = |k|^\ell \Phi(k, \frac{k}{|k|})$ on $\mathbb{R}^n$ decays like:*

$$|\hat{T}(x)| \leq \frac{C(\delta)}{1 + |x|^{n+\ell-\delta}}, \quad \textit{for } \delta > 0 \textit{ arbitrary.}$$

Proof. Follows by applying Lemma 4.2.30 to T. Differentiating the function we obtain the estimates:

$$[\nabla^{n+\ell-1} T]_{1-\delta}(k) \leq \frac{C(\delta)|k|^\ell \|\Phi\|_{C^{n+\ell}}}{|k|^{\ell+n-\delta}}, \quad |\nabla^j T(k)| \leq \frac{C|k|^\ell \|\Phi\|_{C^{n+\ell}}}{|k|^j} \quad 0 \leq j \leq n + \ell - 1.$$

Since T is compactly supported in the unit ball, we can apply Lemma 4.2.30 and obtain the claim. $\quad\square$

4.2.4   The Oberman-Williams-Lenard solution

The Fourier representation formula for the Bogolyubov correlations, more precisely a Fourier representation $\hat{g}_B$ of the solution to (4.1.18) has been obtained by Oberman and Williams in [42], following the complex-variable approach by Lenard in [34]. We will briefly restate their result in the mathematically rigorous framework of this work. We will define a function g_B via its Fourier transform $\hat{g}_B$. In order to complete the proof that g_B is a solution of the Bogolyubov equation in the sense of Definition 4.2.5, we need to show that g_B is in W and satisfies the Bogolyubov condition (4.2.12). This is the content of Section 4.3, in particular of the Theorems 4.3.1, 4.3.6.

Notation 4.2.32. *We introduce functions $A^\pm, B^\pm$, derived from ε and F (cf. (4.2.6),(4.2.13)):*

$$A^\pm(k, u) := (1 - B^\pm) P^\pm [\frac{F(k, \cdot)}{|\varepsilon(k, -|k|\cdot)|^2}](u) \tag{4.2.59}$$

$$B^\pm(k, u) := \hat{\phi}(k) P^\pm [\partial_u F(k, \cdot)](u). \tag{4.2.60}$$

Definition 4.2.33. *For $v_1, v_2 \in \mathbb{R}^3$, consider the Schwartz distribution $\hat{g}_B(\cdot, v_1, v_2) \in \mathcal{S}'(\mathbb{R}^3)$ given by the following linear functional $(\varphi, \hat{g}_B(v_1, v_2))_{\mathcal{S}, \mathcal{S}'}$ on $\mathcal{S}(\mathbb{R}^3)$ (ω as defined in (4.2.5)):*

$$(\varphi, \hat{g}_B(v_1, v_2)) = \int \frac{\varphi(k)\hat{\phi}(k)\omega\left((\nabla_{v_1} - \nabla_{v_2})(ff) + \nabla f(v_1)\overline{\hat{h}}_B(k, v_2) - \nabla f(v_2)\hat{h}_B(k, v_1)\right)}{\omega(v_1 - v_2) - i0} \, \mathrm{d}k. \tag{4.2.61}$$

Here $-i0$ represents taking the limit $\delta \to 0^+$ with $-i\delta$ in (4.2.61), and $\hat{h}_B$ is given by the formula:

$$\hat{h}_B(k, v) := f(v)\frac{(1 - \varepsilon(k, -kv))}{\varepsilon(k, -kv)} - \hat{\phi}(k)\frac{A^-(k, \omega v)}{\varepsilon(k, -kv)}(\omega \nabla f(v)). \tag{4.2.62}$$

Then we will call $g_B(\cdot, v_1, v_2) \in \mathcal{S}'(\mathbb{R}^3) = \mathcal{F}^{-1}(\hat{g}_B(\cdot, v_1, v_2))$ the Bogolyubov correlation associated to f.

The strategy for solving (4.1.18) is solving integrated versions of the equation first. To fix ideas, let g be a solution and consider the functions $h(x, v)$, $H(k, u)$ defined by

$$h(x, v_1) = \int_{\mathbb{R}^3} g(x, v_1, v_2) \, \mathrm{d}v_2$$

$$\hat{H}(k, u) = \int_{\mathbb{R}^3} \hat{h}(k, v)\delta(u - \frac{kv}{|k|}) \, \mathrm{d}v.$$

The key observation is that g, h and H solve the equations (as before: $\zeta(1) = 2$, $\zeta(2) = 1$)

$$(v_1 - v_2)\partial_x g = \sum_{j=1}^{2} \nabla f(v_j) \int \nabla\phi((-1)^{j+1}x + y)h(y, v_{\zeta(j)}) \, \mathrm{d}y + (\nabla_{v_1} - \nabla_{v_2}f)(ff)\nabla\phi(x) \tag{4.2.63}$$

$$\hat{h}(k, v) = \int_{\mathbb{R}^3} \frac{-\omega\hat{\phi}(k)((\nabla_{v_1} - \nabla_{v_2}f)(ff) + \nabla f(v_1)\overline{\hat{h}}(k, v_2) - \nabla f(v_2)\hat{h}(k, v_1))}{\omega(v_1 - v_2) - i0} \, \mathrm{d}v_2 \tag{4.2.64}$$

$$\hat{H}(k, u) = -\hat{\phi}(k)\left(\partial_u F P^-[F] - P^-[\partial_u F]F + \partial_u F P^-[\overline{\hat{H}}] - P^-[\partial_u F]\hat{H}\right). \tag{4.2.65}$$

Note that the equation for H is closed. This suggests to solve the equations (4.2.63)-(4.2.65) in reverse order: Once we have found the solution $\hat{H}$ to (4.2.65), we can use (4.2.64) to compute $\hat{h}$ and then compute $\hat{g}$ using (4.2.63). Following this reasoning, we show the existence of a solution to (4.2.65) in the first step of our rigorous analysis.

Lemma 4.2.34. *Let f satisfy the Assumptions 4.2.12-4.2.13. We recall the definitions of F in (4.2.13) and $A^\pm$ in (4.2.59). The function $\hat{H}_B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ given by*

$$\hat{H}_B(k, u) := \frac{1}{2\pi i}(A^+ - A^-) - F(k, u) \tag{4.2.66}$$

is measurable in $\mathbb{R}^3 \times \mathbb{R}$ and satisfies $\hat{H}_B(k, \cdot) \in L^2$ a.e. in $k \in \mathbb{R}^3$. Further, for a.e. $k \in \mathbb{R}^3$ it solves the equation:

$$\hat{H}_B(k, u) = -\hat{\phi}(k)\left(\partial_u F P^-[F] - P^-[\partial_u F]F + \partial_u F P^-[\overline{\hat{H}_B}] - P^-[\partial_u F]\hat{H}_B\right). \tag{4.2.67}$$

Proof. As a pointwise a.e. limit of measurable functions, $\hat{H}_B$ is measurable again. By Lemma 4.2.23 we know that $A^+ = \overline{A^-}$, so $\hat{H}_B$ is real-valued. By (4.2.47) $|\varepsilon|$ is bounded below, so $\frac{F}{|\varepsilon|}$ is L^2. We can rewrite A^- using ε (as in cf. (4.2.14)):

$$A^-(k,\cdot) = \varepsilon(k,-|k|\cdot)\int_{\mathbb{R}} \frac{F(\omega,u')}{|\varepsilon(k,-|k|u')|^2(u'-\cdot+i0)}\,\mathrm{d}u', \tag{4.2.68}$$

and find this function is in L^2, since $P^\pm$ are bounded on L^2. It remains to show that $\hat{H}_B$ satisfies the equation. Since $\hat{H}_B$ is real-valued, equation (4.2.67) is equivalent to

$$\hat{H}_B + F = F - \hat{\phi}(k)\left(\partial_u F P^-[F + \hat{H}_B] - (F + \hat{H}_B)P^-[\partial_u F]\right).$$

Using that $|1 - \hat{\phi}(k)P^+[\partial_u F]| = |\varepsilon|$ is non-zero, Lemma 4.2.23 shows the equation is equivalent to:

$$\frac{P^+[\hat{H}_B + F]}{1 - \hat{\phi}(k)P^+[\partial_u F]} - \frac{P^-[\hat{H}_B + F]}{1 - \hat{\phi}(k)P^-[\partial_u F]} = \frac{2\pi i F(u)}{(1 - \hat{\phi}(k)P^+[\partial_u F])(1 - \hat{\phi}(k)P^-[\partial_u F])}. \tag{4.2.69}$$

So it remains to check (4.2.69) is satisfied for $\hat{H}_B$ as defined in (4.2.66) above. The equation is satisfied, if we can show that

$$P^\pm[\hat{H}_B] = A^\pm - P^\pm[F]. \tag{4.2.70}$$

By the definition (4.2.66) of $\hat{H}_B$, this is the case if for $A^\pm$ as in (4.2.59) we have:

$$A^\pm = P^\pm[\frac{1}{2\pi i}(A^+ - A^-)]. \tag{4.2.71}$$

This however follows from the uniqueness proved in Lemma 4.2.24. $\square$

Lemma 4.2.35. *Let f satisfy the Assumptions 4.2.12-4.2.13 and consider the function $\hat{h}_B$ defined by the Fourier representation (4.2.62). Then $\hat{h}_B$ is a measurable function in $\mathbb{R}^3 \times \mathbb{R}^3$ and for $k \neq 0$ it satisfies:*

$$|\hat{h}_B(k,v)| \leq C(k)e^{-|v|} \tag{4.2.72}$$

Furthermore, for $k \neq 0$ the function $\hat{h}_B(k,\cdot)$, $k \neq 0$ solves the equation:

$$\hat{h}_B(k,v) = \int_{\mathbb{R}^3} \frac{\omega\hat{\phi}(k)((\nabla_{v_1} - \nabla_{v_2}f)(ff) + \nabla f(v_1)\overline{\hat{h}}_B(k,v_2) - \nabla f(v_2)\hat{h}_B(k,v_1))}{\omega(v_1 - v_2) - i0}\,\mathrm{d}v_2. \tag{4.2.73}$$

Proof. Measurability and decay of $\hat{h}_B$ follow from the regularity and decay properties of f. It remains to show $\hat{h}_B(k,\cdot)$ solves (4.2.73). To this end, we first show $H_*(k,\cdot) := \int_{\mathbb{R}^3} \hat{h}_B(k,v)\delta(\cdot - \omega v)\,\mathrm{d}v$ coincides with the function $\hat{H}_B(k,\cdot)$ (cf. (4.2.66)). This can be seen by integrating (4.2.62):

$$H_*(k,u) = F(k,u)\frac{1 - \varepsilon(k,-|k|u)}{\varepsilon(k,-|k|u)} - \frac{A^-(k,u)}{\varepsilon(k,-|k|u)}\frac{1}{2\pi i}(B^+ - B^-).$$

Since $\varepsilon(k, -|k|) = 1 - B^-(k, u)$, the claim $\hat{H}_B = H_*$ is equivalent to verifying

$$\hat{H} = \frac{1}{2\pi i}(P^+[\hat{H}] - P^-[\hat{H}]) = \frac{FB^-}{1 - B^-} - \frac{A^-}{1 - B^-}\frac{1}{2\pi i}(B^+ - B^-). \qquad (4.2.74)$$

We add F on both sides and use (4.2.70) to see this is equivalent to

$$\frac{1}{2\pi i}(A^+ - A^-) = \frac{FB^-}{1 - B^-} - \frac{A^-}{1 - B^-}\frac{1}{2\pi i}(B^+ - B^-) + F.$$

Rearranging terms, the claim can be rewritten as:

$$\frac{1}{2\pi i}(A^+(1 - B^-) - A^-(1 + B^+)) = F,$$

which is equivalent to (4.2.69). Hence we have verified (4.2.74) and proven $H_* = \hat{H}_B$. Using this we can prove $\hat{h}_B$ as defined above solves (4.2.73). To this end, we integrate in v_2 and bring the last summand in (4.2.73) to the left-hand side, when the equation reads:

$$\varepsilon(k, -kv)\hat{h}_B(k, v_1) = \int_{\mathbb{R}^3} \frac{\hat{\phi}(k)\omega}{\omega \cdot (v_1 - v_2) - i0}\left((\nabla_{v_1} - \nabla_{v_2}f)(ff)(v_1, v_2) + \nabla f(v_1)\overline{\hat{h}_B}(k, v_2)\right)\,\mathrm{d}v_2$$
$$= -\hat{\phi}(k)\left(\omega\nabla f(v_1)P^-[F + \hat{H}_B] - P^-[F]f(v)\right).$$

Replacing $P^-[F + \hat{H}_B] = A^-$ by means of (4.2.70), we have shown the claim to be equivalent to (4.2.62), the definition of $\hat{h}_B$. $\qquad \square$

Now it is straightforward to check that g_B defined in Definition 4.2.33 is a weak solution of the Bogolyubov equation, assuming that g_B has marginal $\int \hat{g}_B(x, v_1, v_2) = h_B(x, v_1)$ and satisfies the Bogolyubov boundary condition (4.2.12). These conditions will be proved in the Theorems 4.3.1, 4.3.6, whose proof does not depend on the results in this section.

Theorem 4.2.36. *Let f satisfy the Assumptions 4.2.12 and 4.2.13 and ϕ be either the Coulomb potential or a soft potential. In the Coulomb case, assume further that f satisfies Assumption 4.2.17 or 4.2.18. If g_B defined by (4.2.33) satisfies $\int \hat{g}_B(x, v_1, v_2) = h_B(x, v_1)$, and the Bogolyubov boundary condition (4.2.12), then g_B is a weak solution to the Bogolyubov equation.*

Proof. Since $g \in W$ by assumption, the equation (4.2.11) holds weakly if the Fourier-transformed equation

$$(v_1 - v_2)ik\hat{g}_B - ik\hat{\phi}\nabla f(v_1)\hat{h}_B(k, v_2) + i\hat{\phi}\nabla f(v_1)\overline{\hat{h}}_B(k, v_2) = ik(\nabla_{v_1} - \nabla_{v_2})(ff)\hat{\phi}, \qquad (4.2.75)$$

holds in the sense of distributions. This is true by the definition of g_B (cf. (4.2.33)). $\qquad \square$

4.3 Characteristic length scale of the equilibrium correlations

In this section, we estimate the Bogolyubov correlations g_B, and give sufficient conditions for the onset of a characteristic length scale. In the Coulomb case, we observe the onset of a characteristic length scale for one-particle functions f that behave like Maxwellians for large velocities, and the characteristic length is given by the Debye length L_D (cf. (4.1.5)). In the soft potential case, the Bogolyubov correlations always have a characteristic length scale, which coincides with the length scale of the potential. For both types of potentials, we derive the rate of decay. This will provide the assumptions on h_B, g_B made in Theorem 4.2.36, and hence complete the proof of Theorem 4.2.20.

To this end, for $v_1, v_2 \in \mathbb{R}^3$ we define $\hat{\Gamma}(\cdot, v_1, v_2) \in \mathcal{S}'(\mathbb{R}^3)$ by:

$$\hat{\Gamma}(k, v_1, v_2) := \hat{\phi}(k)k \left((\nabla_{v_1} - \nabla_{v_2}f)(ff) + \nabla f(v_1)\overline{\hat{h}}_B(k, v_2) - \nabla f(v_2)\hat{h}_B(k, v_1) \right). \tag{4.3.1}$$

This allows us to get a representation of g_B (cf. (4.2.61)) of the form:

$$\hat{g}_B(k, v_1, v_2) = \frac{1}{k(v_1 - v_2) - i0}\hat{\Gamma}(k, v_1, v_2). \tag{4.3.2}$$

Using the notation introduced in (4.2.5), this yields the identity:

$$g_B(x, v_1, v_2) = \frac{2\pi i}{|v_r|}\Gamma(x, v_1, v_2) *_x \left(\mathbb{1}_{(0,\infty)}(x \cdot v_r) \cdot \mathcal{H}^1 \llcorner \mathrm{span}\{v_r\} \right). \tag{4.3.3}$$

Here we have used the one-dimensional Fourier transform $\mathcal{F}^{-1}(\frac{1}{\cdot - i0}) = (2\pi)^{\frac{1}{2}}i\mathbb{1}_{(0,\infty)}(\cdot)$, and the notation $\mathcal{H}^1 \llcorner Y$ for the one-dimensional Hausdorff-measure supported on a line Y. The properties of the equilibrium correlations g_B can be analyzed by first characterizing the properties of Γ, and then using the convolution representation (4.3.3).

4.3.1 Coulomb interaction

In this paragraph, we analyze the onset of a characteristic length in the Bogolyubov correlations g_B (cf. (4.2.61)) in the case of Coulomb interacting particles. Taking the Debye length L_D (cf. (4.1.5)) as unit of length, the Bogolyubov equation has the form (4.1.18) with $\phi = \phi_C$. The result we will prove in this paragraph is the following.

Theorem 4.3.1 (Screening in the Coulomb case)**.** *Let g_B be defined by (4.2.61), where f satisfies the Assumptions 4.2.12-4.2.13 and $\phi = \phi_C$ is the Coulomb potential (cf. Definition 4.2.11). Further let f satisfy Assumption 4.2.18 (Maxwellian behavior for $|v| \to \infty$) or Assumption 4.2.17 (Exponential behavior for $|v| \to \infty$). Then the marginal of g_B coincides with h_B:*

$$\int g_B(x, v_1, v_2)\,\mathrm{d}v_2 = h_B(x, v_1). \tag{4.3.4}$$

We recall the definition of v_r in (4.2.5), and b, d, d_- in (4.1.20). Let $K \subset \mathbb{R}^3$ be compact and $\delta \in (0, 1)$. Under Assumption 4.2.18, g_B, h_B satisfy the following estimates for $x \in \mathbb{R}^3$, $v_1, v_2 \in K$:

$$|g_B(x, v_1, v_2)| \leq \frac{C(K, \delta)}{|v_r|}\frac{1}{(|b| + d_-)(1 + |b| + d_-)^{1-\delta}}, \tag{4.3.5}$$

$$|h_B(x, v_1)| \leq \frac{C(K, \delta)}{|x|(1 + |x|^{3-\delta})}. \tag{4.3.6}$$

Under Assumption 4.2.17, g_B, h_B satisfy the following estimates for $x \in \mathbb{R}^3$, $v_1, v_2 \in K$:

$$|g_B(x, v_1, v_2)| \le \frac{C(K, \delta)}{|v_r|} \frac{(1 + |b| + d_-)^\delta}{(|b| + d_-)}, \tag{4.3.7}$$

$$|h_B(x, v)| \le \frac{C(K, \delta)}{|x|(1 + |x|^{2-\delta})}. \tag{4.3.8}$$

Note that the result (4.3.5) shows the onset of a characteristic length in the correlations g_B if f satisfies Assumption 4.2.18, but the estimate (4.3.7) indicates this is not in general true for functions satisfying Assumption 4.2.17. Furthermore, the estimates (4.3.5) and (4.3.7) prove that g_B satisfies the Bogolyubov boundary condition (4.2.12).

For estimating the decay of the function g_B, we use Lemma 4.2.31, i.e. we expand the Fourier transform of h_B near $k = 0$ into

$$\hat{h}_B(k, v) = |k|^r T(k, \omega, v), \tag{4.3.9}$$

where T is some smooth function. Note that the representation formula for $\hat{h}_B$ (4.2.62) suggests that (4.3.9) holds with $r = -2$, in which case Lemma 4.2.31 gives an estimate of $|h(x, v)| \le C/|x|$ for $|x| \to \infty$. In other words, naively one might expect the decay of the correlations to be the same as the decay of the Coulomb potential. However, since $\hat{\phi}(k)$ appears also in the dielectric constant ε in the denominator, we obtain $r > -2$ in (4.3.9). Computing the precise value of r is subtle, since the denominator $|\varepsilon(k, -|k|u')|^2$ in $P^-[A]$ (appearing in (4.2.62)) becomes singular for $|u'| \sim 1/|k|$, $k \to 0$ as observed in Remark 4.2.26. The following lemma allows to separate the critical region from the remainder.

Lemma 4.3.2. *Assume that f satisfies the Assumptions 4.2.12-4.2.13 and $\phi = \phi_C$ is the Coulomb potential. There exists $r_0 > 0$ and $T(k, \chi, v) \in C^6(B_{r_0}(0) \times S^2 \times \mathbb{R}^3)$ such that for $|k| \in (0, r_0)$, $\chi \in S^2$, $v \in \mathbb{R}^3$:*

$$\int_{\mathbb{R}} \frac{\hat{\phi}(k)(\omega \cdot \nabla f(v)) F(\omega, u')}{|1 - \hat{\phi}(k)\alpha^-(\omega, u')|^2 (\omega \cdot v - u' + i0)} \, du' = D(k, \omega, v) + |k|^2 T(k, \omega, v). \tag{4.3.10}$$

Here D is given by the formula ($u_0^\pm, I$ as in Notation 4.2.16)

$$D(k, \chi, v) = \int_{I(k, \chi)} \frac{\hat{\phi}(k)(\chi \cdot \nabla f(v)) F(\chi, u')}{|1 - \hat{\phi}(k)\alpha^-(\chi, u')|^2 (\chi \cdot v - u')} \, du'. \tag{4.3.11}$$

Moreover, T satisfies the estimate:

$$\|T(\cdot, \cdot, v)\|_{C^6(B_{r_0}(0) \times S^2)} \le C. \tag{4.3.12}$$

Proof. We decompose α^- (cf. (4.2.15)) into its real and imaginary part:

$$\alpha^-(\chi, u) = \alpha(\chi, u) - i\pi \partial_u F(\chi, u). \tag{4.3.13}$$

By Lemma 4.2.28, for $|k| \in (0, r_0)$ small enough and $\chi \in S^2$ there exist $u_0^{\pm}(k, \chi)$ such that (4.2.27) holds. By the estimate (4.2.49), after possibly choosing a smaller $r_0 > 0$, the following holds for $|k| \in (0, r_0)$ and $u \neq I(k, \chi)$:

$$\frac{1}{||k|^2 + \alpha^-(\chi, u)|} \leq C(1 + |u|^3). \tag{4.3.14}$$

Now the claim follows by decomposing:

$$\int_{\mathbb{R}} \frac{\hat{\phi}(k)(\omega \cdot \nabla f(v))F(\omega, u')}{|1 - \hat{\phi}(k)\alpha^-(\omega, -|k|u')|^2(\omega \cdot v - u' + i0)} \, \mathrm{d}u'$$

$$= |k|^2 \int_{\mathbb{R} \backslash I(k)} \frac{(\omega \cdot \nabla f(v))F(\omega, u')}{||k|^2 + \alpha^-(\omega, u')|^2(\omega v - u' + i0)} \, \mathrm{d}u' + \int_{I(k)} \frac{\hat{\phi}(k)(\omega \cdot \nabla f(v))F(\omega, u')}{|1 - \hat{\phi}(k)\alpha^-(\omega, u')|^2(\omega v - u')} \, \mathrm{d}u',$$

since by (4.3.14) the function T given by:

$$T(k, \chi, v) := \int_{\mathbb{R} \backslash I(k)} \frac{(\chi \cdot \nabla f(v))F(\chi, u')}{||k|^2 + \alpha^-(\chi, u')|^2(\chi \cdot v - u' + i0)} \, \mathrm{d}u' \tag{4.3.15}$$

satisfies the estimate (4.3.12). $\square$

Now we have decomposed the integral (4.3.10) into a well-behaved part T, and the singular integral D. The behavior of D for large v depends on the behavior of f as $v \to \infty$. If f behaves like a Maxwellian, we have $D(k, v) \approx |k|$ for small k. If f behaves like an exponential, the function is of order one close to the origin.

Lemma 4.3.3 (Expansion of D at $k = 0$)**.** *Let f satisfy the Assumptions 4.2.12-4.2.13. Rewrite the function D defined by (4.3.11) in the following form:*

$$D(k, \chi, v) = \gamma_h(k, \chi, v) \qquad \text{if } f \text{ satisfies Assumption 4.2.17,} \tag{4.3.16}$$
$$D(k, \chi, v) = |k|\gamma_h(k, \chi, v) \qquad \text{if } f \text{ satisfies Assumption 4.2.18.} \tag{4.3.17}$$

We can choose $\gamma_h \in C(B_{r_0}(0) \times S^2 \times \mathbb{R}^3)$ (r_0 as in Lemma 4.3.2) such that for any $K \subset \mathbb{R}^3$ compact

$$|\nabla_{k,\chi}^j \gamma_h(k, \chi, v)| \leq C(K), \quad \text{for } j = 0, 1 \ldots, 6, \ k \in B_{r_0}(0), \ \chi \in S^2 \text{ and } v \in K. \tag{4.3.18}$$

Similarly, for $\chi \in S^2$, $k \in B_{r_0}(0)$, $v_1, v_2 \in \mathbb{R}^3$ write :

$$\chi(\nabla f(v_2)D(k, \chi, v_1) - \nabla f(v_1)D(k, \chi, v_2)) = |k|\gamma_g(k, \chi, v_1, v_2) \quad \text{under Assumption 4.2.17,} \tag{4.3.19}$$
$$\chi(\nabla f(v_2)D(k, \chi, v_1) - \nabla f(v_1)D(k, \chi, v_2)) = |k|^2\gamma_g(k, \chi, v_1, v_2) \quad \text{under Assumption 4.2.18.} \tag{4.3.20}$$

In both cases, we can choose $\gamma_g \in C(B_{r_0}(0) \times S^2 \times \mathbb{R}^3 \times \mathbb{R}^3)$ such that for all $K \subset \mathbb{R}^3$ compact:

$$|\nabla_{k,\chi}^j \gamma_g(k, \chi, v_1, v_2)| \leq C(K), \quad \text{for } 0 \leq j \leq 6, \ k \in B_{r_0}(0), \ \chi \in S^2 \text{ and } v_1, v_2 \in K. \tag{4.3.21}$$

Proof. After changing variables with $\Psi(k, \chi, \cdot)$, D reads:

$$D(k, \chi, v) = \int_{-1/L}^{1/L} \frac{|k|^2 \chi \nabla f(v) F(\chi, \Psi(y)) L(k, \chi)}{||k|^2 - \alpha(\Psi(y))|^2 + |\partial_u F(\chi, \Psi(y))|^2} \frac{1}{\chi \cdot v - \Psi(y)} \, \mathrm{d}y \tag{4.3.22}$$

$$= \int_{-1/L}^{1/L} \frac{|k|^3}{\partial_u \alpha(\chi, \Psi)} \frac{\chi \nabla f(v)(F/\partial_u F)(\chi, \Psi)}{\left|\frac{|k|^2 - \alpha(\Psi)}{y \partial_u F(\chi, \Psi)}\right|^2 y^2 + 1} \frac{|k|^{-1}}{\chi \cdot v - \Psi(y)} \, \mathrm{d}y. \tag{4.3.23}$$

If f satisfies Assumption 4.2.17, then for $|k| \leq \lambda$ small enough, the functions $F/\partial_u F$, $\frac{|k|^3}{\partial_u \alpha(\chi, \Psi)}$ and $\frac{|k|^2 - \alpha(\Psi)}{y \partial_u F(\chi, \Psi)}$ are bounded, as well as their derivatives in k, χ. Furthermore, $\left|\frac{|k|^2 - \alpha(\Psi)}{y \partial_u F(\chi, \Psi)}\right| \geq c > 0$ is bounded below. Additionally, we use $\psi(k, \chi, y) \in I(k, \chi)$ and $|\chi \cdot v| \leq C(K)$ to infer that the function

$$z(k, \chi, v, y) = \frac{|k|^{-1}}{\chi \cdot v - \Psi(y)} \tag{4.3.24}$$

is bounded as well as its derivatives in k, χ. Hence, under Assumption 4.2.17 the expansion (4.3.16) with the estimate (4.3.18) follow by differentiating through the integral. Similarly, we prove (4.3.17) with the estimate (4.3.18) under Assumption 4.2.18.

The expansions (4.3.19)-(4.3.20) with the estimate (4.3.21) are proved analogously, using the fact that

$$z_{sym}(k, \chi, v_1, v_2, y) = \left(\frac{|k|^{-2}}{\chi \cdot v_1 - \Psi(y)} - \frac{|k|^{-2}}{\chi \cdot v_2 - \Psi(y)}\right) = \frac{|k|^{-2}\chi(v_2 - v_1)}{(\chi \cdot v_1 - \Psi(y))(\chi \cdot v_1 - \Psi(y))}, \tag{4.3.25}$$

is a bounded function, as well as its derivatives in k, χ. $\qquad\square$

We now prove an integral estimate for $\hat{h}_B(k, v)$ (cf. (4.2.62)).

Lemma 4.3.4. *Let f satisfy the Assumptions 4.2.12-4.2.13, and Assumption 4.2.17 or 4.2.18. Further let $\phi = \phi_C$ be the Coulomb potential and h_B be given by (4.2.62). Then there exists $C > 0$ such that*

$$\int_{B_2} \left|\int_{\mathbb{R}^3} \hat{h}_B(k, v) \, \mathrm{d}v\right| \, \mathrm{d}k \leq C. \tag{4.3.26}$$

Proof. We start by performing the integration in the direction orthogonal to ω using Fubini's Theorem:

$$\int_{B_2} \left|\int_{\mathbb{R}^3} \hat{h}_B(k, v) \, \mathrm{d}v\right| \, \mathrm{d}k = \int_{B_2} \left|\int_{\mathbb{R}} \int_{\mathbb{R}^3} \hat{h}_B(k, v)\delta(u - \omega v) \, \mathrm{d}v \, \mathrm{d}u\right| \, \mathrm{d}k$$

$$\leq \int_{B_2} \left|\int_{\mathbb{R}} F(\omega, u)\frac{1 - \varepsilon(k, -|k|u)}{\varepsilon(k, -|k|u)} - \hat{\phi}(k)\frac{A^-(k, u)}{\varepsilon(k, -|k|u)}\partial_u F(k, u) \, \mathrm{d}u\right| \, \mathrm{d}k$$

$$\leq C + \int_{B_2} \left|\int_{\mathbb{R}} \frac{F(\omega, u)}{\varepsilon(k, -|k|u)} \, \mathrm{d}u\right| \, \mathrm{d}k + \int_{B_2} \left|\int_{\mathbb{R}} \hat{\phi}(k)\frac{A^-(k, u)}{\varepsilon(k, -|k|u)}\partial_u F(k, u) \, \mathrm{d}u\right| \, \mathrm{d}k. \tag{4.3.27}$$

Now the estimates follow similar to the proof of the last Lemma. We observe that for $|k| \geq \lambda > 0$ bounded away from the origin, the integrand in the first integral in (4.3.27) is bounded. Further, for $\lambda > 0$ small enough we know that $|F(u)/\varepsilon(k, -|k|u)| \leq |F(u)/\partial_u F|$ is bounded for $|u - u_0^\pm(k, \omega)| \leq 1$. Finally, on the region $|k| \leq \lambda$, $|u - u_0| \geq 1$, the integral is bounded since $|\varepsilon(k, -|k|u)|^{-1} \leq C(1 + |u|^3)$.

In order to bound the second integral in (4.3.27), we recall the definition of A^- (4.2.59) to rewrite:

$$\int_{B_2} \left| \int_{\mathbb{R}} \hat{\phi}(k) \frac{A^-(k, u)}{\varepsilon(k, -|k|u)} \partial_u F(k, u) \, \mathrm{d}u \right| \, \mathrm{d}k = \int_{B_2} \left| \int_{\mathbb{R}} \hat{\phi}(k) P^- \left[\frac{F(k, \cdot)}{|\varepsilon(k, -|k|\cdot)|^2} \right](u) \partial_u F(k, u) \, \mathrm{d}u \right| \, \mathrm{d}k.$$

Now the claim follows if we can show that $\left| \int P^- \left[\frac{F}{|\varepsilon|^2} \right](u) \partial_u F(k, u) \, \mathrm{d}u \right| \leq C$ is uniformly bounded, for $|k|$ sufficiently small. For $I(k, \omega)$ as introduced in (4.2.28) we can estimate

$$\left| \int P^- \left[\frac{F}{|\varepsilon|^2} \right](u) \partial_u F(k, u) \, \mathrm{d}u \right| \leq C + \left| \int_{I(k)} \int_{I(k)} \frac{F(k, u') \partial_u F(u)}{|\varepsilon(k, -|k|u')|^2 (u - u' - i0)} \, \mathrm{d}u' \, \mathrm{d}u \right|. \tag{4.3.28}$$

Now since f satisfies Assumption 4.2.17 or 4.2.18, the function $\frac{F(k, u') \partial_u F(u)}{|\varepsilon(k, -|k|u')|^2}$ and its derivative in u' is bounded for $u, u' \in I(k)$ and $|k|$ sufficiently small. Therefore, the integral (4.3.28) is uniformly bounded and the claim follows. $\square$

From the expansion of D near $k = 0$ in Lemma 4.3.3, we can now obtain an expansion of $\hat{h}_B$ and $\hat{g}_B$ near $k = 0$.

Lemma 4.3.5 (Expansion of $\hat{h}_B$ for $|k| \to 0$ and $|k| \to \infty$)**.** *Assume that f satisfies the Assumptions 4.2.12-4.2.13 and $\phi = \phi_C$ is the Coulomb potential. Let $\hat{h}_B$ be given by (4.2.62) and $K \subset \mathbb{R}^3$ compact. Then there exists a function $\hat{h}_{B,0}(k, \chi, v) \in C^6(B_1(0) \times S^2 \times \mathbb{R}^3)$ such that:*

$$\|\hat{h}_{B,0}(\cdot, \cdot, v)\|_{C^6(B_1(0) \times S^2)} \leq C(K), \qquad\qquad\qquad for\ v \in K \quad (4.3.29)$$

$$\hat{h}_B(k, v) = -f(v) + |k| \hat{h}_{B,0}(k, k/|k|, v), \qquad under\ Assumption\ 4.2.18 \quad (4.3.30)$$

$$\hat{h}_B(k, v) = -f(v) + \hat{h}_{B,0}(k, k/|k|, v), \qquad under\ Assumption\ 4.2.17. \quad (4.3.31)$$

Furthermore for $|k| \geq 1$ and $\ell \in 1, \cdots, 6$ we have:

$$|\nabla_k^\ell \hat{h}_B(k, v)| \leq \frac{C}{1 + |k|^{\ell+2}} e^{-|v|}. \tag{4.3.32}$$

Proof. On the region $|k| \in (r_0, 1)$, the function $\hat{h}_B(k, v)$ is smooth by (4.2.47). For $|k| \in (0, r_0)$ small, we use $\hat{\phi}(k) = \frac{1}{|k|^2}$ and the decomposition (4.3.10):

$$\hat{h}_B(k, v) = -f(v) + |k|^2 \left(\frac{f(v)}{|k|^2 - \alpha^-(\omega \cdot v) + i\partial_u F(\omega, \omega \cdot v)} - T(k, \omega, v) \right) + D(k, \omega, v). \tag{4.3.33}$$

The first two summands can be written in the forms (4.3.30), (4.3.31) respectively, as can be inferred from from Lemma 4.3.2 and (4.2.24). For the last summand, the claim follows from Lemma 4.3.3. It remains to prove the estimate (4.3.32). This however follows from the lower bound (4.2.47) on $|\varepsilon|$ for $|k| \geq 1$. $\square$

Proof of Theorem 4.3.1. Let $\eta \in C_c^\infty$ be a cutoff function with $\eta(k) = 1$ for $|k| \leq 1/2$ and $\eta(k) = 0$ for $|k| \geq 1$. We recall the functions Γ (cf. (4.3.1)) and h_B (cf. (4.2.62)), and separate the contributions of large and small Fourier modes:

$$\hat{\Gamma}(k, v_1, v_2) = \eta(k)\hat{\Gamma} + (1 - \eta)(k)\hat{\Gamma} =: \hat{\Gamma}_1 + \hat{\Gamma}_2 \tag{4.3.34}$$

$$\hat{h}_B(k, v) = \eta(k)\hat{h}_B + (1 - \eta)(k)\hat{h}_B =: \hat{h}_{B,1} + \hat{h}_{B,2}. \tag{4.3.35}$$

The function $h_{B,1}$ satisfies the estimates (4.3.6),(4.3.8), which can be seen by applying Lemma 4.2.30 to the expansions (4.3.30),(4.3.31). The function $h_{B,2}$ satisfies the estimates (4.3.6),(4.3.8) by (4.3.32).

In order to estimate Γ_1, we again apply Lemma 4.2.30. To this end, we insert the expansion of $\hat{h}_B$ into the definition of Γ (cf. (4.3.1)) to find:

$$\hat{\Gamma}(k, v_1, v_2) = k/|k|^2(\nabla f(v_1)h_{B,0}(k, v_2) - \nabla f(v_2)h_{B,0}(k, v_1)), \quad \text{for } |k| \leq 1.$$

Hence for any $\delta > 0$ and $R > 0$, Lemma 4.2.30 shows that Γ_1 decays like

$$|\Gamma_1(x, v_1, v_2)| \leq \frac{C(K, \delta)}{1 + |x|^{m-\delta}}, \quad \text{for } x \in \mathbb{R}^3, |v_1|, |v_2| \leq R, \tag{4.3.36}$$

where $m = 3$ if f_1 satisfies Assumption 4.2.18, and $m = 2$ under Assumption 4.2.17. On the other hand, the estimate (4.3.32) shows that

$$\left|\nabla_k^j\left(\hat{\Gamma}(k, v_1, v_2) - k/|k|^2(\nabla_{v_1} - \nabla_{v_2})(ff)(v_1, v_2)\right)\right| \leq \frac{C(K)}{1 + |k|^{2+j}}, \quad \text{for } j = 0, \ldots, 6, |k| \geq 1.$$

Therefore, Γ_2 satisfies the estimate:

$$|\Gamma_2(x, v_1, v_2)| \leq \frac{Ce^{-(|v_1|+|v_2|)}}{|x|(1 + |x|)^4}. \tag{4.3.37}$$

Now inserting the estimates (4.3.36) and (4.3.37) into the representation (4.3.3) shows the estimates (4.3.5) and (4.3.7).

It remains to show that g_B is in the space W introduced in (4.2.10)). We remark that by construction $\hat{g}_B(k, v_1, v_2) = \hat{g}_B(-k, v_2, v_1)$, so g_B satisfies the symmetry property (4.2.8).

To show that $|h|[g_B] \in L_{loc}^1$ we use the decomposition (4.3.34):

$$g_B = \frac{2\pi i}{|v_r|}(\Gamma_1 + \Gamma_2)(x, v_1, v_2) *_x \left(\mathbb{1}_{(0,\infty)}(x \cdot v_r) \cdot \mathcal{H}^1 \llcorner \text{span}\{v_r\}\right) =: g_{B,1} + g_{B,2}. \tag{4.3.38}$$

From the estimate (4.3.37) we deduced that $g_{B,2}$ satisfies $|h|[g_{B,2}] \in L_{loc}^1$.

We now estimate $|h|[g_{B,1}]$. To this end, we decompose the function further into:

$$\hat{g}_{B,1}(k, v_1, v_2) = \mathbb{1}_{|\omega(v_1-v_2)|>1}\hat{g}_{B,1} + \mathbb{1}_{|\omega(v_1-v_2)|\leq 1}\hat{g}_{B,1} =: \hat{g}_{B,a} + \hat{g}_{B,b}. \tag{4.3.39}$$

Inserting the definition of g_B (4.2.61), and using $|v_1| \leq R$ we can estimate $g_{B,a}$ by:

$$\int_{\mathbb{R}^3} |g_{B,a}(x,v_1)| \, \mathrm{d}v_2 \leq C \left(1 + \int \int_{B_2} |\nabla f(v_2)| |\hat{h}_B(k,v_1)| \, \mathrm{d}k \, \mathrm{d}v_2 \right) + \int_{B_2} \left| \int_{\mathbb{R}^3} \hat{h}_B(k,v) \, \mathrm{d}v \right| \mathrm{d}k$$

$$\leq C(R) + C \int_{B_2} \left| \int_{\mathbb{R}^3} e^{ikx} \hat{h}_B(k,v) \, \mathrm{d}v \right| \mathrm{d}k,$$

which is bounded by (4.3.26). Hence $|h|[g_{B,a}] \in L^1_{loc}$.

In order to estimate $g_{B,b}$ given by (4.3.39), we use the fact that $|\omega(v_1 - v_2)| \leq 1$ and $|v_1| \leq R$ implies $|\omega v_2| \leq R + 1$. Hence $|\varepsilon(k, -kv_2)| \geq c > 0$ is bounded below uniformly on the support of $\hat{g}_{B,b}$, and $|h|[g_{B,b}] \in L^1_{loc}$ follows. Hence also $|h|[g_B] \in L^1_{loc}$ as claimed.

It then immediately follows that h_B is indeed the marginal of g_B (cf. (4.2.61)), since:

$$\int \hat{g}_B(k,v_1,v_2) \, \mathrm{d}v_2 = \int \frac{\hat{\phi}(k)\omega \left((\nabla_{v_1} - \nabla_{v_2})(ff) + \nabla f(v_1)\overline{\hat{h}}_B(k,v_2) - \nabla f(v_2)\hat{h}_B(k,v_1) \right)}{\omega(v_1 - v_2) - i0} \, \mathrm{d}v_2,$$

and $\hat{h}_B$ satisfies the equation (4.2.73). The estimates (4.3.6)-(4.3.8) imply $\sup_{|v| \leq R} \|h[g_B](\cdot,v)\|_{L^2} \leq C(R)$ as claimed. $\qquad\square$

4.3.2 Soft potential interaction

Theorem 4.3.6 (Decay estimate for soft potentials). *We recall g_B as introduced in Definition 4.2.33, and assume f satisfies the Assumptions 4.2.12-4.2.13 and $\phi = \phi_S$ is a soft potential (cf. Definition 4.2.11). Further we use the shorthand notation v_r, v_r in (4.2.5), and b, d, d_- introduced in (4.1.20). Write $v_r = v_1 - v_2$, $v_r = v_r/|v_r|$ and let $\delta \in (0,1)$. For almost every $(x, v_1) \in \mathbb{R}^3 \times \mathbb{R}^3$, there holds $g_B(z, v_1, \cdot) \in L^1(\mathbb{R}^3)$, and the marginal of g coincides with h_B:*

$$\int g_B(x, v_1, v_2) \, \mathrm{d}v_2 = h_B(x, v_1). \tag{4.3.40}$$

Furthermore, for $n \in \mathbb{N}$ the function g_B satisfies the estimate:

$$|g_B(x, v_1, v_2)| \leq \frac{C(\delta)}{|v_r|} \frac{1}{(1 + |b| + d_-)^{2-\delta}} e^{-(|v_1| + |v_2|)}, \tag{4.3.41}$$

$$|h_B(x, v_1)| \leq \frac{C(\delta)}{1 + |x|^{3-\delta}} e^{-(|v_1| + |v_2|)}. \tag{4.3.42}$$

Proof. The identity (4.3.40) follows analogously to the Coulomb case. For proving the estimates (4.3.41), (4.3.42), we recall the definition of h in Fourier variables:

$$\hat{h}_B(k,v) := f(v)\frac{(1 - \varepsilon(k, -kv))}{\varepsilon(k, -kv)} - \hat{\phi}(k)\frac{A^-(k, kv)}{\varepsilon(k, -kv)}(\omega \nabla f(v)). \tag{4.3.43}$$

Since ε is non-degenerate by Assumption, the functions $(1 - \varepsilon)/\varepsilon$ and A^-/ε are bounded, as well as their first three derivatives in k. Using the exponential decay of $f(v)$ and $\nabla f(v)$, the decay estimate (4.3.42) follows from Lemma 4.2.31. A similar argument proves (4.3.41). $\qquad\square$

We observe that the result shows that the rate of decay is independent of the rate of the decay of the soft potential. Further, we do not observe a singularity for small impact parameters b.

4.4 Stability of the linearized evolution of the truncated two-particle correlation function

4.4.1 The linearized evolution semigroup

The goal of this subsection is to prove that the Bogolyubov propagator $\mathcal{G}$ introduced in Definition 4.2.10 provides a strong solution to the linear Bogolyubov evolution equation (4.1.16). We start by proving the well-posedness of the propagator. Since the definition involves the action of the Vlasov semigroup both on smooth initial data and on Dirac masses, we first derive properties for both cases. We recall that for translation invariant functions, we can reduce the number of variables using (4.2.7).

Since we prove the well-posedness of the linear evolution problem in the Schwartz space, we recall the seminorms generating this space.

Definition 4.4.1. *For $k, l \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $\| \cdot \|_{C^{k,l}(\mathbb{R}^n)}$ be the seminorm defined by:*

$$\|f\|_{C^{k,l}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} (1 + |x|)^l (|f(x)| + |\nabla^k f(x)|). \tag{4.4.1}$$

Remark 4.4.2. *The collection of norms $\| \cdot \|_{C^{k,l}(\mathbb{R}^n)}$ with $k, l \in \mathbb{N}_0$ generates the Schwartz space, which can be equipped with the associated Frechèt-metric.*

Lemma 4.4.3 (Solution of the Vlasov equation for Dirac masses)**.** *Let $\phi = \phi_S$ be a soft potential, let $f \in \mathcal{S}(\mathbb{R}^3)$ satisfy Assumption 4.2.15 and let $x_0, v_0 \in \mathbb{R}^3$. We set $h_0(x, v) = \delta(x - x_0)\delta(v - v_0)f(v)$. Consider the function $h(t) = \mathcal{V}(t)[h_0]$ defined by the Fourier-Laplace representation (4.2.19). Then there exists a function $Y \in C(\mathbb{R}^+, \mathcal{S}((\mathbb{R}^3)^3))$ such that $\partial_t Y(t, x) \in C(\mathbb{R}^+, \mathcal{S}((\mathbb{R}^3)^3))$ and:*

$$h(t, x, v) = Y(t, x - x_0, v, v_0) + \delta(x - x_0 - tv)\delta(v - v_0)f(v). \tag{4.4.2}$$

Furthermore, h is a weak solution to the Vlasov equation (4.2.18), and Y solves:

$$\partial_t Y + v\nabla_x Y - \nabla E_h \nabla f = 0, \quad Y(0, \cdot) = 0. \tag{4.4.3}$$

Proof. We start by proving that h can be decomposed as claimed in (4.4.2). W.l.og. let $x_0 = 0$. By the Fourier-Laplace representation of h in (4.2.19) we have:

$$\hat{h}(t, x, v) = \frac{1}{2\pi i}\int_{L_1} \tilde{h}(z, k, v)e^{zt}\,\mathrm{d}z = \frac{1}{2\pi i}\left(\int_{L_1} \frac{\hat{h}_0(k, v)}{z + ikv}e^{zt}\,\mathrm{d}z + \int_{L_1} \frac{iQ(k, v)\tilde{\varrho}(z, k)}{z + ikv}e^{zt}\,\mathrm{d}z\right) \tag{4.4.4}$$

where $L_\gamma := \{z \in \mathbb{C} : \Re(z) = \gamma\}$ is the line with real part γ, oriented upwards. The line integral is evaluated in the improper sense

$$\int_{L_\gamma} f(z)\,\mathrm{d}z = \lim_{T \to \infty}\int_{L_\gamma} f(z)\mathbb{1}(|z| \leq T)\,\mathrm{d}z. \tag{4.4.5}$$

The first line integral in (4.4.4) is explicit and yields:

$$\frac{1}{2\pi i}\int_{L_1} \frac{\hat{h}_0(k, v)}{z + ikv}\,\mathrm{d}z = e^{-ikvt}\hat{h}_0(k, v),$$

so we obtain the second term in (4.4.2). It remains to show that the second line integral in (4.4.4) gives a function Y with the desired properties. Using the formula (4.2.19), the term can be rewritten as:

$$\hat{Y}(t,k,v,v_0) = \frac{f(v_0)}{(2\pi)^{\frac{3}{2}}} \frac{1}{2\pi i} \int_{L_1} \frac{iQ(k,v)e^{zt}}{\varepsilon(k,-iz)(z+ikv)(z+ikv_0)}\,\mathrm{d}z. \tag{4.4.6}$$

Now $\varepsilon(k,-iz)$ is smooth and bounded below by Assumption 4.2.15. The line integral is absolutely convergent and differentiating through it shows that for all $\ell_1, \ell_2, \ell_3 \in \mathbb{N}_0$, $T > 0$, there exists a $C > 0$ such that:

$$\left\|\nabla_{v_0}^{\ell_1} \nabla_v^{\ell_2} \nabla_k^{\ell_3} \frac{1}{2\pi i} \int_{L_1} \frac{e^{zt}}{\varepsilon(k,-iz)(z+ikv)(z+ikv_0)}\,\mathrm{d}z\right\|_{C([0,T]\times\mathbb{R}^9)} \leq C. \tag{4.4.7}$$

Using that Q and f in (4.4.6) are Schwartz functions, we obtain $Y \in C(\mathbb{R}^+, \mathcal{S}(\mathbb{R}^9))$. Next we observe that $\int h(t,x,v)\,\mathrm{d}v = \varrho(t,x)$. To see this, we use $\int \tilde{h}(z,k,v)\,\mathrm{d}v = \tilde{\varrho}(z,k)$. The integration in v commutes with the Laplace inversion (4.4.4), so ϱ is the spatial density of h. Hence the Fourier-Laplace definition (4.2.19) of h gives a weak solution of the Vlasov equation. Combining this with the decomposition (4.4.2) we find that Y is a weak solution to (4.4.3). Using equation (4.4.3) we find $\partial_t Y \in C(\mathbb{R}^+, \mathcal{S}(\mathbb{R}^9))$ as claimed. $\qquad\square$

Lemma 4.4.4 (Vlasov equation with Schwartz initial data). *Let $\phi = \phi_S$ be a soft potential, let $f \in \mathcal{S}(\mathbb{R}^3)$ satisfy Assumption 4.2.15. Further assume $h_0 \in \mathcal{S}((\mathbb{R}^3)^2)$. Let $h(t) = \mathcal{V}(t)[h_0]$ be defined by formula (4.2.19). There exists an $m \in \mathbb{N}_0$ such that for any $k, l \in \mathbb{N}_0$, there is a $C > 0$ such that:*

$$\|h\|_{C^1([0,T];C^{k,l})} \leq C\|h_0\|_{C^{k+m,l+m}}. \tag{4.4.8}$$

Further, the function is a strong solution to the Vlasov equation (4.2.18).

Proof. For proving the estimate (4.4.8), we use the definition of $\mathcal{V}(t)[h_0]$ in Fourier-Laplace variables (cf. (4.2.19)) to obtain the representation:

$$\hat{h}(t,x,v) = \frac{1}{2\pi i}\left(\int_{L_1} \frac{\hat{h}_0(k,v)}{z+ikv}e^{zt}\,\mathrm{d}z + \int_{L_1} \frac{iQ(k,v)\tilde{\varrho}(k,z)}{z+ikv}e^{zt}\,\mathrm{d}z\right), \tag{4.4.9}$$

$$\tilde{\varrho}(k,z) := \frac{\int \frac{\hat{h}_0(k,v')}{z+ikv'}\,\mathrm{d}v'}{\varepsilon(k,-iz)}. \tag{4.4.10}$$

Since $\varepsilon(k,-iz)$ is uniformly bounded below on the line L_1, the claim follows by differentiating through the integrals in (4.4.9). $\qquad\square$

We recall the Bogolyubov propagator $\mathcal{G}$ introduced in (4.2.21). The previous two lemmas allow us to prove that the Bogolyubov propagator is well-defined. In order to show that the function $g(t) := \mathcal{G}(t)[g_0]$ indeed solves the Bogolyubov equation, we show commutativity for Vlasov operators acting on different sets of variables. To this end we introduce the following shorthand notation.

Notation 4.4.5. *Let S be the Schwartz distribution given by:*

$$S(\xi_1,\xi_2) = \delta(\xi_1 - \xi_2)f(v_1). \tag{4.4.11}$$

Lemma 4.4.6. *Let $g_0(\xi_1, \xi_2) = \bar{g}_0(x_1 - x_2, v_1, v_2) + S(\xi_1, \xi_2)$, where $\bar{g}_0 \in \mathcal{S}$ and S as introduced in (4.4.11). Then the compositions of operators $\mathcal{V}_{\xi_1} \mathcal{V}_{\xi_2}[g_0]$, $\mathcal{V}_{\xi_2} \mathcal{V}_{\xi_1}[g_0]$ as introduced in Definition 4.2.10 are well-defined and the following commutation relation between $\mathcal{V}_{\xi_1}$ and $\mathcal{V}_{\xi_2}$ holds:*

$$\mathcal{V}_{\xi_1}(t')\mathcal{V}_{\xi_2}(t)[g_0] = \mathcal{V}_{\xi_2}(t)\mathcal{V}_{\xi_1}(t')[g_0]. \tag{4.4.12}$$

Proof. By Lemma 4.4.3, $\mathcal{V}_{\xi_2}(t)[g_0]$ is the sum of a Schwartz function and a Dirac mass, so the composition with $\mathcal{V}_{\xi_1}(t')$ is well defined. The commutativity relation (4.4.12) follows from the explicit Fourier-Laplace representation (4.2.19). $\qquad\square$

Now can now prove that $\mathcal{G}(t)$ gives the solution of the Bogolyubov equation (4.1.16). For convenience we introduce the following notation.

Notation 4.4.7. *We write $E_j[g]$, $j = 1, 2$ for the following expressions:*

$$E_2[g](x, v_2) = \int \phi(x + y)g(y, v_1, v_2)\,\mathrm{d}v_1, \quad E_1[g](x, v_1) = \int \phi(-x + y)g(y, v_1, v_2)\,\mathrm{d}v_2. \tag{4.4.13}$$

Theorem 4.4.8 (Solution of the linearized evolution equation). *Let g_0, f be as in Theorem 4.2.22. The function g given by $g(t) = \mathcal{G}(t)[g_0]$ satisfies $g \in C(\mathbb{R}^+, \mathcal{S}((\mathbb{R}^3)^3))$, $\partial_t g \in C(\mathbb{R}^+, \mathcal{S}((\mathbb{R}^3)^3))$ and solves the Bogolyubov equation (4.1.16).*

Proof. First we observe that using the notation (4.4.13), the Bogolyubov equation (4.1.16) reads:

$$\begin{aligned}\partial_\tau g + (v_1 - v_2)\nabla_x g &- \nabla f(v_1)\nabla_x E_2[g](x, v_2) - \nabla f(v_2)\nabla_x E_1[g](x, v_1) \\ &= (\nabla_{v_1} - \nabla_{v_2})(f(v_1)f(v_2))\nabla\phi(x).\end{aligned} \tag{4.4.14}$$

We decompose $g(t) = \mathcal{G}(t)[g_0]$ into two parts:

$$g(t) = \mathcal{V}_{\xi_1}\mathcal{V}_{\xi_2}[g_0] + (\mathcal{V}_{\xi_1}\mathcal{V}_{\xi_2}[S] - T(t)S) = G_1 + G_2. \tag{4.4.15}$$

We take the time derivative of both expressions. For the first term, the existence of the time derivative follows from Lemma 4.4.4, and using Lemma 4.4.6 we find:

$$\partial_t G_1 = -\sum_{i \neq j} v_i \nabla_{x_i} G_1 + \nabla f(v_j)\nabla_{x_i} E_i[G_1]. \tag{4.4.16}$$

To prove differentiability in time for G_2 we observe that

$$G_2(t) = \mathcal{V}_{\xi_1}(t)[\mathcal{V}_{\xi_2}(t)[S] - T(t)S] + (\mathcal{V}_{\xi_1}(t)[S] - T(t)S) \tag{4.4.17}$$

satisfies $G_2, \partial_t G_2 \in C(\mathbb{R}^+, \mathcal{S}((\mathbb{R})^9))$ by Lemma 4.4.3 and Lemma 4.4.4. Differentiating G_2 yields:

$$\partial_t G_2(t) = -\sum_{i \neq j} v_i \nabla_{x_i} G_2 + \nabla f(v_j)\nabla_{x_i} E_i[\mathcal{V}_{\xi_1}\mathcal{V}_{\xi_2}[S]]. \tag{4.4.18}$$

Now the claim follows from $\sum_{i \neq j=1}^{2} \nabla f(v_j)E_i[T(t)[S]] = (\nabla_{v_1} - \nabla_{v_2})(f(v_1)f(v_2))\nabla\phi(x)$. $\qquad\square$

4.4.2 Distributional stability of the Bogolyubov correlations

In Theorem 4.4.8 we have proved that the Bogolyubov propagator $\mathcal{G}(t)$ gives a solution to the Bogolyubov equation. In this subsection we prove the result (4.2.41) claimed in Theorem 4.2.22, that is the distributional stability of the Bogolyubov correlations. We split the problem into analyzing the solution Λ of (4.4.14) with non-zero initial datum g_0, but without the right-hand side in (4.4.14), and the solution Ψ of (4.4.14) with zero initial datum. The following lemma gives this decomposition in Fourier-Laplace variables.

Lemma 4.4.9. *Let $g_0 \in \mathcal{S}((\mathbb{R}^3)^3)$ be a function such that $g_0(x_1 - x_2, v_1, v_2)$ is symmetric in exchanging $\xi_1 = (x_1, v_1)$, $\xi_2 = (x_2, v_2)$. We make the decomposition*

$$g(t, \xi_1, \xi_2) = \mathcal{G}(t)[g_0] = \Psi(t, t, \xi_1, \xi_2) + \Lambda(t, t, \xi_1, \xi_2), \qquad (4.4.19)$$

where $\Psi(t, t', \xi_1, \xi_2) := \mathcal{V}_{\xi_1}(t)\mathcal{V}_{\xi_2}(t')[S] - T(t)[S]$, $\Lambda(t, t') = \mathcal{V}_{\xi_1}(t)\mathcal{V}_{\xi_2}(t')[g_0]$. Then the Fourier-Laplace representation of Ψ, written in the form (4.2.7), satisfies:

$$\Psi(z, z', k, v_1, v_2) := \Psi_1(z, z', k, v_1, v_2) + \Psi_2(z, z', k, v_1, v_2) + \Psi_2(z', z, -k, v_2, v_1)$$

$$\Psi_1(z, z', k, v_1, v_2) := -\frac{Q(k, v_1)Q(-k, v_2) \int \frac{\delta(v_1' - v_2')f(v_1')}{(z + ikv_1')(z' - ikv_2')} \, dv_1' \, dv_2'}{\varepsilon(k, -iz)\varepsilon(-k, -iz')(z + ikv_1)(z' - ikv_2)} \qquad (4.4.20)$$

$$\Psi_2(z, z', k, v_1, v_2) := \frac{f(v_1)}{(z + ikv_1)} \frac{iQ(-k, v_2)}{\varepsilon(-k, -iz')(z_2 - ikv_1)(z' - ikv_2)}$$

and the Fourier-Laplace representation of Λ is given by:

$$\Lambda(z, z', k, v_1, v_2) = \Lambda_1(z, z', k, v_1, v_2) + \Lambda_2(z, z', k, v_1, v_2) + \Lambda_2(z', z, -k, v_2, v_1)$$

$$\Lambda_1(z, z', k, v_1, v_2) := \frac{g_0(k, v_1, -k, v_2)}{(z + ikv_1)(z_2 - ikv_2)} - \frac{Q(k, v_1)Q(-k, v_2) \int \frac{\hat{g}_0(k, v_1', -k, v_2')}{(z + ikv_1')(z' - ikv_2')} \, dv_1' \, dv_2'}{\varepsilon(k, -iz_1)\varepsilon(-k, -iz_2)(z_1 + ikv_1)(z_2 - ikv_2)} \qquad (4.4.21)$$

$$\Lambda_2(z, z', k, v_1, v_2) := \frac{iQ(-k, v_2) \int \frac{\hat{g}_0(k, v_1, -k, v')}{z' + ikv'} \, dv'}{\varepsilon(-k, -iz_2)(z_1 + ikv_1)(z_2 - ikv_2)}.$$

Proof. Follows directly from the Fourier-Laplace representation of $\mathcal{V}$ in (4.2.19) and the definition of the Bogolyubov propagator in Definition 4.2.10. $\qquad \square$

We will start by proving two Lemmas that we will use throughout this whole section.

Lemma 4.4.10. *Let $H_\gamma = \{z \in \mathbb{C} : |\Re(z)| \leq \gamma\}$ and $f(k, z) \in L^1_{loc}(\mathbb{R}^3, \mathbb{C})$, such that there exist $R, c > 0$ with $\|f(k, i\cdot)\|_{L^\infty(H_{c|k|})} \leq R$ for all $k \in \mathbb{R}^3$. Define the function*

$$I(t, k, v, v') := \int_{i\mathbb{R} - |c|k} \frac{e^{zt}f(k, iz)}{(z + ikv)(z + ikv')} \, dz.$$

Then for all $M, N \in \mathbb{N}_0$, there exists $C > 0$ such that

$$|\nabla_v^M \nabla_{v'}^N I(t, k, v, v')| \leq \frac{Ce^{-c|k|t}}{|k|}. \qquad (4.4.22)$$

Moreover, let I be a function satisfying (4.4.22) *and* $\kappa \in \mathcal{S}(\mathbb{R}^3)$ *be a Schwartz function. Then for* $p(k,v) := \mathrm{PV} \int \frac{\kappa(v')I(t,k,v,v')}{k(v-v')} \, \mathrm{d}v'$ *we have*

$$\|p(k,\cdot)\|_{C_b^1(\mathbb{R}^3)} \leq \frac{Ce^{-c|k|t}}{|k|}. \tag{4.4.23}$$

Proof. We start by proving (4.4.22). To this end, let $M, N \in \mathbb{N}_0$ be arbitrary. Since f is bounded on $H_{c|k|}$, we can differentiate through the integral:

$$
\begin{aligned}
|\nabla_v^M \nabla_{v'}^N I(t,k,v,v')| &\leq e^{-c|k|t} \int_{i\mathbb{R}-c|k|} \frac{|k|^{N+M}|f(k,iz)|}{|z+ikv|^{M+1}|z+ikv'|^{N+1}} \, \mathrm{d}z \\
&\leq Ce^{-c|k|t} \int_{\mathbb{R}} \frac{|k|^{N+M}}{(|k|+|r-kv|)^{M+1}(|k|+|r-kv'|)^{N+1}} \, \mathrm{d}r \\
&\leq Ce^{-c|k|t} \int_{\mathbb{R}} \frac{|k|^{N+M+1}}{(|k|+|r|k|-kv|)^{M+1}(|k|+|r|k|-kv'|)^{N+1}} \\
&\leq \frac{Ce^{-c|k|t}}{|k|} \sup_{a,b\in\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1+|t-a|)^{M+1}(1+|t-b|)^{N+1}} \, \mathrm{d}t \leq \frac{Ce^{-c|k|t}}{|k|}.
\end{aligned}
$$

To prove (4.4.23) we remark that $P(t,k,v,u) := \int I(t,k,v,v')\kappa(v')\delta(kv'-u) \, \mathrm{d}v'$ satisfies

$$|\nabla_v^M \nabla_u^N P(t,k,v,u)| \leq \frac{Ce^{-c|k|t}}{|k|(1+|u|)^2}.$$

On the other hand $p(k,v) = \mathrm{PV} \int \frac{P(t,k,v,u')}{kv-u'} \, \mathrm{d}u'$ and the principal value integral can be bounded by

$$\left| \mathrm{PV} \int \frac{P(u')}{u-u'} \, \mathrm{d}u' \right| \leq C \left(\|P\|_{C^1} + \|P\|_{L^1} \right).$$

$\square$

Lemma 4.4.11. *Let* $f \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ *be a Schwartz function.*

(i) *For* $t \to \infty$, *the following convergence holds in the sense of Schwartz distributions:*

$$\mathrm{PV} \frac{e^{-ik(v_1-v_2)t}}{k(v_1-v_2)} \longrightarrow -i\pi\delta(k(v_1-v_2)) \in \mathcal{S}'(\mathbb{R}^9). \tag{4.4.24}$$

(ii) *For* $M \in \mathbb{N}_0$ *arbitrary, the following convergence holds in* $C_b^M(\mathbb{R}^3)$ *as* $t \to \infty$:

$$\mathrm{PV} \int f(k,v_2) \frac{e^{-ik(v_1-v_2)t}}{k(v_1-v_2)} \, \mathrm{d}k \, \mathrm{d}v_2 \to -i\pi \int_{\mathbb{R}^3\times\mathbb{R}^3} \delta(k(v_1-v_2))f(k,v_2) \, \mathrm{d}v_2 \, \mathrm{d}k. \tag{4.4.25}$$

Proof. We start by proving the convergence (4.4.24). Let $w(k, v_1, v_2)$ be a Schwartz function and $W(k, u) := \int_{\mathbb{R}^6} \delta(k(v_1 - v_2) - u) w(k, v_1, v_2) \, dv_1 \, dv_2$. Let $\hat{W}$ be the Fourier transform in u, then:

$$\text{PV} \int \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{e^{-ik(v_1 - v_2)t}}{k(v_1 - v_2)} w(k, v_1, v_2) \, dv_1 \, dv_2 \, dk = \text{PV} \int \int_{\mathbb{R}} \frac{e^{-iut}}{u} W(k, u) \, du \, dk$$

$$= \int -i\sqrt{\frac{\pi}{2}} \, \text{sign}(\xi + t) \hat{W}(k, \xi) \, d\xi \, dk \to -i\pi \int W(k, 0) \, dk, \quad \text{as } t \to \infty.$$

For proving (4.4.25), we observe that $f \in \mathcal{S}$ implies that $F(k, u) := \int \delta(kv + u) f(k, v) \, dv$ is also Schwartz. Furthermore, we have

$$\text{PV} \int f(k, v_2) \frac{e^{-ik(v_1 - v_2)t}}{k(v_1 - v_2)} \, dk \, dv_2$$

$$= \int \text{PV} \int_{\mathbb{R}} \frac{F(k, u) e^{-i(kv_1 + u)t}}{kv_1 + u} \, du \, dk = \int \text{PV} \int_{\mathbb{R}} \frac{F(k, u - kv_1) e^{-iut}}{u} \, du \, dk$$

$$\to \int F(k, k \cdot v_1) \, dk, \quad \text{as } t \to \infty.$$

Differentiating through the integral, we obtain the convergence for arbitrary derivatives in v_1. $\qquad \square$

Lemma 4.4.12. *The solution $g(t) = \mathcal{G}(t)[N_0]$ to (4.1.16) with zero initial datum $N_0 :\equiv 0$ converges to the Lenard solution in the sense of distributions, so*

$$\mathcal{G}(t)[N_0] \longrightarrow g_B \quad \text{in } \mathcal{S}'(\mathbb{R}^9) \text{ as } t \to \infty.$$

Proof. By Lemma 4.4.9 we have $g(t, \cdot) = \mathcal{G}(t)[N_0](\cdot) = \Psi(t, t, \cdot)$. We use the Fourier-Laplace representation $\Psi(z_1, z_2, k, v_1, v_2) = \Psi_1(z_1, z_2, k, v_1, v_2) + \Psi_2(z_1, z_2, k, v_1, v_2) + \Psi_2(z_2, z_1, -k, v_2, v_1)$ in (4.4.20). We will show the distributional convergence term by term, starting with Ψ_1.

Lemma 4.4.13. *The following convergence holds in the sense of distributions:*

$$\Psi_1(t, t, k, v_1, v_2) \longrightarrow \frac{Q(k, v_1) Q(-k, v_2)}{k(v_1 - v_2) - i0} \int \frac{\frac{f(v')}{|\varepsilon(k, -kv')|^2}}{k(v_1 - v') - i0} \, dv' \tag{4.4.26}$$

$$+ \frac{Q(k, v_1) Q(-k, v_2)}{k(v_1 - v_2) - i0} \int \frac{\frac{f(v')}{|\varepsilon(k, -kv')|^2}}{k(v_2 - v') - i0} \, dv', \quad \text{as } t \to \infty. \tag{4.4.27}$$

Proof. First we perform the integration in v_2'

$$\Psi_1(z_1, z_2, k, v_1, v_2) = -\frac{Q(k, v_1) Q(-k, v_2) \int \frac{f(v')}{(z_1 + ikv')(z_2 - ikv')} \, dv'}{\varepsilon(k, -iz_1) \varepsilon(k_2, -iz_2)(z_1 + ikv_1)(z_2 - ikv_2)}$$

$$= -\int \frac{Q(k, v_1) Q(-k, v_2) \frac{f(v')}{(z_1 + ikv')(z_2 - ikv')}}{\varepsilon(k, -iz_1) \varepsilon(k_2, -iz_2)(z_1 + ikv_1)(z_2 - ikv_2)} \, dv'.$$

Now for k fixed, we can perform the Laplace inversion integral both in z_1 and z_2. For $\Re(z_i) > 0$ the integrand has no singularities, so we can carry out the Laplace inversion on the contour with

$\Re(z_i) = 1$. By Assumption (4.2.26), $|\varepsilon(k, -iz)|$ is bounded below for $\Re(z) = -ic|k|$ and some $c > 0$. The estimate (4.2.25) allows to use Cauchy's residual theorem to move the contour to the left of the imaginary line:

$$
\frac{1}{2\pi i} \int_{i\mathbb{R}+c} \frac{Q(k,v)e^{zt}}{\varepsilon(k,-iz)(z+ikv)(z+ikv')} \, \mathrm{d}z
$$

$$
= \frac{1}{2\pi i} \int_{i\mathbb{R}-c|k|} \frac{Q(k,v)e^{zt}}{\varepsilon(k,-iz)(z+ikv)(z+ikv')} \, \mathrm{d}z + \mathrm{PV}\, \frac{Q(k,v)e^{-ikvt}}{\varepsilon(k,-kv)ik(v'-v)} + \mathrm{PV}\, \frac{Q(k,v)e^{-ikv't}}{\varepsilon(k,-kv')ik(v-v')}
$$

$$
= Q(k,v)\left(\frac{1}{2\pi i} \int_{i\mathbb{R}-c|k|} \frac{e^{zt}}{\varepsilon(k,-iz)(z+ikv)(z+ikv')} \, \mathrm{d}z + \mathrm{PV}\, \frac{\frac{e^{-ikvt}}{\varepsilon(k,-kv)} - \frac{e^{-ikv't}}{\varepsilon(k,-kv')}}{ik(v'-v)} \right)
$$

$$
=: Q(k,v)(I(t,k,v,v') + R(t,k,v,v')).
$$

Writing Ψ_1 in terms of the functions I and R we obtain

$$
\Psi_1(t_1,t_2,k,v_1,v_2) = -\int f(v')Q(k,v_1)Q(-k,v_2)(I+R)(t,v_1,v')(I+R)(t,v_2,v') \, \mathrm{d}v'.
$$

We expand the product $(I+R)(I+R)$ inside the integral. We claim all terms containing an integral term I tend to zero in the limit $t \to \infty$ by Lemma 4.4.10. For the terms containing products of the form IR this follows from (4.4.22), for the products of the form II this can be inferred from (4.4.23) and the fact that the singularity in k in estimate (4.4.23) is integrable. It remains to study the limiting behavior of the residual part:

$$
\Psi_1(t,k,v_1,v_2) + \int f(v')R(t,v_1,v')R(t,v_2,v') \, \mathrm{d}v' \to 0 \quad \text{in } D'(\mathbb{R}^9).
$$

In order to find the distributional limit of Ψ_1 we have to determine the limit of

$$
\Psi_\infty(t,k,v_1,v_2) := -\int f(v')R(t,v_1,v')R(t,v_2,v') \, \mathrm{d}v'
$$

$$
= -Q(k,v_1)Q(-k,v_2)\, \mathrm{PV} \int f(v') \frac{\frac{e^{-ikv_1t}}{\varepsilon(k,-kv_1)} - \frac{e^{-ikv't}}{\varepsilon(k,-kv')}}{k(v'-v_1)} \frac{\frac{e^{ikv_2t}}{\varepsilon(-k,kv_2)} - \frac{e^{ikv't}}{\varepsilon(-k,kv')}}{k(v'-v_2)} \, \mathrm{d}v'.
$$

The denominator we split as

$$
\frac{1}{k(v'-v_1)k(v'-v_2)} = \frac{1}{k(v_1-v_2)}\left(\frac{1}{k(v'-v_1)} - \frac{1}{k(v'-v_2)} \right). \tag{4.4.28}
$$

Using this we can split $\Psi_\infty = \sum_{j=1}^{2} \sum_{l=1}^{4} \Psi_\infty^{j,l}$, where $\Psi_\infty^{j,l}$ are given by (here $\zeta(1) = 2$, $\zeta(2) = 1$):

$$\Psi_\infty^{j,1}(t,k,v_1,v_2) := (-1)^j Q(k,v_1) Q(-k,v_2) \int f(v') \frac{\frac{e^{-ik(v_1-v_2)t}}{\varepsilon(k,-kv_1)\varepsilon(-k,kv_2)}}{k(v'-v_j)k(v_1-v_2)} \, \mathrm{d}v'$$

$$\Psi_\infty^{j,2}(t,k,v_1,v_2) := (-1)^j Q(k,v_1) Q(-k,v_2) \int f(v') \frac{-\frac{e^{(-1)^j ik(v_j-v')t}}{\varepsilon(k,-kv_1)\varepsilon(-k,kv')}}{k(v'-v_j)k(v_1-v_2)} \, \mathrm{d}v'$$

$$\Psi_\infty^{j,3}(t,k,v_1,v_2) := (-1)^j Q(k,v_1) Q(-k,v_2) \int f(v') \frac{\frac{1}{\varepsilon(k_1,-k_1v')\varepsilon(-k_1,k_1v')}}{k(v'-v_j)k(v_1-v_2)} \, \mathrm{d}v'$$

$$\Psi_\infty^{j,4}(t,k,v_1,v_2) := (-1)^j Q(k,v_1) Q(-k,v_2) \int f(v') \frac{-\frac{e^{(-1)^j ik(v'-v_{\zeta(j)})t}}{\varepsilon(k,-kv')\varepsilon(-k,kv_{\zeta(j)})}}{k(v'-v_j)k(v_1-v_2)} \, \mathrm{d}v'.$$

We compute the limits of these terms separately. Applying the Lemmas 4.4.10 and 4.4.11 yields for $t \to \infty$:

$$\Psi_\infty^{j,1}(t,v_1,v_2) \to (-1)^{j+1} \frac{i\pi\delta(k(v_1-v_2))}{\varepsilon(k,-kv_1)\varepsilon(-k,kv_2)} Q(k,v_1) Q(-k,v_2) \, \mathrm{PV} \int \frac{f(v')}{k(v'-v_1)} \, \mathrm{d}v'$$

$$\Psi_\infty^{j,2}(t,v_1,v_2) \to \frac{i\pi}{k(v_1-v_2)} Q(k,v_1) Q(-k,v_2) \int f(v') \frac{\delta(k(v'-v_j))}{|\varepsilon(k,-kv')|^2} \, \mathrm{d}v'$$

$$\Psi_\infty^{j,3}(t,v_1,v_2) \to (-1)^j \frac{Q(k,v_1) Q(-k,v_2)}{k(v_1-v_j)} \int \frac{f(v')}{|\varepsilon(k,-kv')|^2 k(v'-v_1)} \, \mathrm{d}v'$$

$$\Psi_\infty^{j,4}(t,v_1,v_2) \to 0 \quad \text{for } v_1 \neq v_2.$$

The terms $\Psi_\infty^{1,1}$ and $\Psi_\infty^{2,1}$ cancel. The remaining terms can be rearranged to:

$$\Psi_1(t,v_1,v_2) \to \frac{Q(v_1) Q(-k,v_2)}{k(v_1-v_2) - i0} \int \frac{f(v')}{|\varepsilon(k,-kv')|^2 k(v'-v_1) - i0} \, \mathrm{d}v'$$
$$+ \frac{Q(k,v_1) Q(-k,v_2)}{k(v_1-v_2) - i0} \int \frac{f(v')}{|\varepsilon(k,-kv')|^2 k(v'-v_2) - i0} \, \mathrm{d}v', \quad \text{as } t \to \infty,$$

using Plemelj's formula. $\qquad\square$

Lemma 4.4.14. *For Ψ_2 we have the following convergence in the sense of distributions:*

$$\Psi_2(t,t,k,v_1,v_2) \to -\frac{f(v_1) Q(-k,v_2)}{\varepsilon(-k,-kv_1)k(v_1-v_2) - i0}, \quad \text{as } t \to \infty.$$

Proof. We argue similarly to the case of Ψ_1. We start from the definition of Ψ_2

$$\Psi_2(z_1,z_2,k,v_1,v_2) = \frac{f(v_1)}{(z_1+ikv_1)} \frac{iQ(-k,v_2)}{\varepsilon(-k,-iz_2)(z_2-ikv_1)(z_2-ikv_2)}$$

and invert the Laplace transforms to obtain:

$$\Psi_2(t_1, t_2, v_1, v_2) = R(t_1, t_2, v_1, v_2) + I(t_1, t_2, v_1, v_2)$$

$$R(t_1, t_2, v_1, v_2) := e^{-ikv_1 t_1} f(v_1) Q(-k, v_2) \frac{\frac{e^{ikv_2 t_2}}{\varepsilon(-k, -kv_2)} - \frac{e^{ikv_1 t_2}}{\varepsilon(-k, -kv_1)}}{-k(v_1 - v_2)}$$

$$I(t_1, t_2, v_1, v_2) = f(v_1) e^{-ikv_1 t_1} \frac{1}{2\pi i} \int_{i\mathbb{R} - c|k|} \frac{i e^{z_2 t_2} Q(-k, v_2)}{\varepsilon(-k, -iz_2)(z_2 - ikv_1)(z_2 - ikv_2)} \, dz_2.$$

We have $I(t, t, \cdot) \to 0$ for $t \to \infty$, arguing as in the previous lemma. Hence we are left with the residual term R, which by Lemma 4.4.11 converges to

$$R(t, t, v_1, v_2) = e^{-ikv_1 t} f(v_1) Q(-k, v_2) \frac{\frac{e^{ikv_2 t}}{\varepsilon(-k, -kv_2)} - \frac{e^{ikv_1 t}}{\varepsilon(-k, -kv_1)}}{-k(v_1 - v_2)}$$

$$\to \delta(v_1 - v_2) \frac{i\pi f(v_1) Q(-k, v_2)}{\varepsilon(-k, -kv_2)} - \frac{f(v_1) Q(-k, v_2)}{\varepsilon(-k, -kv_1) k(v_1 - v_2)},$$

as $t \to \infty$. Using Plemelj's formula this proves the claim of the lemma. $\qquad\square$

Combining the two previous lemmas, we obtain the following convergence in the sense of distributions:

$$g(t, v_1, v_2) \to \frac{Q(k, v_1) Q(-k, v_2)}{k(v_1 - v_2) - i0} \int \frac{f(v')}{|\varepsilon(k, -kv')|^2 k(v' - v_1) - i0} \, dv'$$

$$+ \frac{Q(k, v_1) Q(-k, v_2)}{k(v_1 - v_2) - i0} \int \frac{f(v')}{|\varepsilon(k, -kv')|^2 k(v' - v_2) - i0} \, dv'$$

$$- \frac{f(v_1) Q(-k, v_2)}{\varepsilon(-k, -kv_1) k(v_1 - v_2) - i0} + \frac{f(v_2) Q(-k, v_2)}{\varepsilon(-k, -kv_2) k(v_1 - v_2) + i0},$$

which by a rearrangement of terms coincides with g_B (cf. (4.2.33)). This finishes the proof of Lemma 4.4.12. $\qquad\square$

We now prove that the memory of the initial datum is erased by the evolution.

Lemma 4.4.15. *Let $g_0 \in \mathcal{S}((\mathbb{R}^3)^3)$ be a function such that $g_0(x_1 - x_2, v_1, v_2)$ is symmetric in exchanging ξ_1, ξ_2. Then the following holds:*

$$\Lambda(t, t, x, v_1, v_2) = \mathcal{V}_{\xi_1}(t) \mathcal{V}_{\xi_2}(t)[g_0](x, v_1, v_2) \longrightarrow 0 \quad \text{in } S'(\mathbb{R}^9) \text{ as } t \to \infty.$$

Proof. We start with the Fourier Laplace representation in (4.4.21):

$$\Lambda(z_1, z_2, k, v_1, v_2) = \Lambda_1(z_1, z_2, k, v_1, v_2) + \Lambda_2(z_1, z_2, k, v_1, v_2) + \Lambda_2(z_2, z_1, -k, v_2, v_1)$$

The first term in Λ_1 is simply given by the action of the transport operator

$$T(t) g_0(x, v_1, v_2) = g_0(x - t(v_1 - v_2), v_1, v_2).$$

Since $g_0 \in \mathcal{S}(\mathbb{R}^9)$, this term converges to zero in distribution. In the second term we perform the Laplace inversion, to split into a residual part and a contour integral left of the imaginary line:

$$\int_{\gamma_c} \int_{\gamma_c} \frac{e^{z_1 t} e^{z_2 t} Q(k, v_1) Q(-k, v_2) \int \int \frac{\frac{1}{2} g_0(k, v_1', -k, v_2')}{(z_1 + ikv_1')(z_2 - ikv_2')} \, dv_1' \, dv_2'}{\varepsilon(k, -iz_1) \varepsilon(-k, -iz_2)(z_1 + ikv_1)(z_2 - ikv_2)}$$

$$= Q(k, v_1) Q(-k, v_2) \int \int \frac{1}{2} g_0(k, v_1', -k, v_2')(I + R)(t, k, v_1, v_1')(I + R)(t, -k, v_2, v_2') \, dv_1' \, dv_2'$$

$$I(t, k, v, v') := \int_{\gamma_{-c|k|}} \frac{e^{zt}}{\varepsilon(k, -iz_1)(z_1 + ikv_1)(z + ikv')} \, dz$$

$$R(t, k, v, v') := \frac{e^{-ikvt}}{\varepsilon(k, -kv)} + \frac{e^{-ikv't}}{\varepsilon(k, -kv')}.$$

Arguing as in the proof of Lemma (4.4.13), all terms containing an I converge to zero in distribution after expanding the product $(I + R)(I + R)$. The residual part R converges to zero since $e^{i(v-w)t} \to 0$ in $\mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}^3)$. The convergence $\Lambda_2 \to 0$ follows by an analogous computation. $\qquad \square$

4.4.3   Stability of the velocity fluxes

In this Subsection we prove the convergence result (4.2.42) in Theorem 4.2.22. Consider the marginal $j(t, x, v_1) := \int g(t, x, v_1, v_2) \, dv_2$ of $g(t, \cdot)$. From (4.2.19) we obtain the representation formula

$$j(t, x, v_1) = \psi(t, t, x_1 - x_2, v_1) + \lambda(t, t, x_1 - x_2, v_1)$$

$$\psi(t, t, k, v_1) = \psi_1(t, t, k, v_1) + \psi_2(t, t, k, v_1) - f(v_1)$$

$$\psi_1(z_1, z_2, k, v_1) := \frac{iQ(k, v_1) \int \int \frac{\delta(v_1' - v_2') f(v_1')}{(z_1 + ikv_1')(z_2 - ikv_2')} \, dv_1' \, dv_2'}{(z_1 + ikv_1) \varepsilon(k, -iz_1) \varepsilon(-k, -iz_2)}$$

$$\psi_2(z_1, z_2, k, v_1) := \frac{\int \frac{\delta(v_1 - v_2') f(v_1)}{z_2 - kv_2'} \, dv_2'}{(z_1 + ikv_1) \varepsilon(-k, -iz_2)} \tag{4.4.29}$$

$$\lambda(z_1, z_2, k, v_1) = \lambda_1(z_1, z_2, k, v_1) + \lambda_2(z_1, z_2, k, v_1)$$

$$\lambda_1(z_1, z_2, k, v_1) := \int \frac{g_0(k, v_1, v_2)}{(z_1 + ikv_1)(z_2 - ikv_2)} \, dv_2 + \frac{\frac{1}{2} \int \int \frac{iQ(k, v_1) g_0(k, v_1', v_2')}{(z_1 + ikv_1')(z_2 - ikv_2')} \, dv_1' \, dv_2'}{\varepsilon(k, -iz_1) \varepsilon(-k, -iz_2)(z_1 + ikv_1)}$$

$$\lambda_2(z_1, z_2, k, v_1) := \int \frac{\frac{1}{2} \int \frac{\hat{g}_0(k, v_1, v')}{z_2 + ikv'} \, dv'}{\varepsilon(-k, -iz_2)(z_1 + ikv_1)} \, dv_2.$$

Further, we define the flux operator J given by

$$J[\psi](v_1) := \nabla \cdot \left(\int -ik\hat{\phi}(k) \psi(k, v_1) \, dk \right). \tag{4.4.30}$$

Lemma 4.4.16. *The flux $J[\Psi]$ (cf. (4.4.30)) converges to*

$$J[\psi](t, v_1) \longrightarrow \nabla_{v_1}\left(\int \psi_\infty(k, v_1) \, dk\right) \quad \text{for all } v_1 \in \mathbb{R}^3 \text{ as } t \to \infty$$

$$\psi_\infty(k, v_1) := \int (\nabla_{v_1} - \nabla_{v'}f)(ff)(v_1, v')\frac{\delta(k(v_1 - v'))(k \otimes k)|\hat\phi(k)|^2}{|\varepsilon(k, -kv_1)|^2} \, dv'.$$

which is the velocity flux on the right-hand side of the Balescu-Lenard equation (4.1.1).

Proof. We show the convergence term by term, considering $J[\Psi_1]$, $J[\Psi_2]$ separately. Observe that $J[f(v_1)] = 0$, since the function is independent of the space variable. Let us first take a look at ψ_2. The integration in v_2' can be carried out, and in the usual fashion we split the Laplace inversion in a contour integral left of the imaginary line and a residual:

$$\psi_2(t, t, k, v_1) = \frac{f(v_1)}{\varepsilon(-k, kv_1)} + I(t, k, v_1), \quad I(t, k, v_1) := e^{-itkv_1}\int_{i\mathbb{R}-c|k|}\frac{e^{z_2 t}}{\varepsilon(-k, -iz_2)(z_2 - ikv_1)} \, dz_2.$$

The contour integral vanishes in the limit $t \to \infty$, i.e. $J[I](t, v_1) \to 0$. Therefore the contribution of $J[\psi_2]$ is

$$J[\psi_2] \to -\nabla_{v_1}\left(\int ik\hat\phi(k)\frac{f(v_1)}{\varepsilon(-k, kv_1)} \, dk\right) = -\nabla_{v_1}\left(\int ik\hat\phi(k)\frac{f(v_1)\varepsilon(k, -kv_1)}{|\varepsilon(k, -kv_1)|^2} \, dk\right)$$
$$= -\nabla_{v_1}\left(\int (k \otimes k)|\hat\phi(k)|^2\frac{\delta(k(v_1 - v_1'))f(v_1)\nabla f(v_1')}{|\varepsilon(k, -kv_1)|^2} \, dk\right). \tag{4.4.31}$$

It remains to find the limit of $J[]\psi_1(t)]$. Again we can perform the integration in v_2', obtaining

$$\psi_1(z_1, z_2, k, v_1) = \frac{iQ(k, v_1)\int\int\frac{\delta(v_1' - v_2')f(v_1')}{(z_1 + ikv_1')(z_2 - ikv_2')} \, dv_1' \, dv_2'}{(z_1 + ikv_1)\varepsilon(k, -iz_1)\varepsilon(-k, -iz_2)} = \frac{iQ(k, v_1)\int\frac{f(v_1')}{(z_1 + ikv_1')(z_2 - ikv_1')} \, dv_1'}{(z_1 + ikv_1)\varepsilon(k, -iz_1)\varepsilon(-k, -iz_2)}.$$

As in the previous lemmas, the Laplace inversion integral can be proved to be exponentially decaying in time up to a residual, which is given by

$$\lim_{t\to\infty} J[\psi_1] = \lim_{t\to\infty}\nabla_{v_1}\cdot\left(\int k\hat\phi(k)Q(k, v_1)\int f(v_1')R(t, k, v_1, v_1') \, dv_1' \, dk\right)$$

$$R(t, k, v, v') = \frac{e^{itkv'}}{\varepsilon(-k, kv)}\left(\frac{e^{-itkv}}{\varepsilon(k, -kv)ik(v' - v)} - \frac{e^{-itkv'}}{\varepsilon(k, -kv')ik(v + v')}\right).$$

Applying Lemma 4.4.11, we identify the limit as:

$$\lim_{t\to\infty} J[\psi_1](t, v_1) = \nabla_{v_1}\cdot\left(\int k \otimes k|\hat\phi(k)|^2\nabla f(v)\int\frac{\delta(k(v_1 - v_1'))f(v_1')}{|\varepsilon(k, -kv_1')|^2} \, dv_1' \, dk\right). \tag{4.4.32}$$

Summing (4.4.31) and (4.4.32), we obtain as a limit of $J[\psi]$

$$\lim_{t\to\infty} J[\psi] = \nabla_{v_1}\cdot\left(\int (\nabla_{v_1} - \nabla_{v'}f)(ff)(v_1, v')\delta(k(v_1 - v'))\frac{k \otimes k|\hat\phi(k)|^2)}{|\varepsilon(k, -kv_1)|^2} \, dk \, dv'\right)$$

as claimed. $\qquad\square$

By a similar computation we obtain the following lemma.

Lemma 4.4.17. *Let J be the operator introduced in (4.4.30). For all $v_1 \in \mathbb{R}^3$ there holds:*

$$J[\lambda](t, v_1) \longrightarrow 0 \quad as \ t \to \infty.$$

Combining Lemma 4.4.17 with Lemma 4.4.16 shows the convergence of the velocity fluxes claimed in (4.2.42). This concludes the proof of Theorem 4.2.22.

Bibliography

[1] R. Alexandre and C. Villani. "On the Landau approximation in plasma physics". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 21.1 (2004), pp. 61–95.

[2] R. Balescu. *Equilibrium and nonequilibrium statistical mechanics*. Interscience Publishers John Wiley & Sons, Ltd., London-New York-Sydney, 1975.

[3] R. Balescu. *Statistical mechanics of charged particles*. Monographs in Statistical Physics and Thermodynamics, Vol. 4. Interscience Publishers John Wiley & Sons, Ltd., London-New York-Sydney, 1963.

[4] G. Basile, A. Nota, and M. Pulvirenti. "A diffusion limit for a test particle in a random distribution of scatterers". In: *J. Stat. Phys.* 155.6 (2014), pp. 1087–1111.

[5] A. Bobylev, M. Pulvirenti, and C. Saffirio. "From particle systems to the Landau equation: a consistency result". In: *Comm. Math. Phys.* 319.3 (2013), pp. 683–702.

[6] N. Bogoliubov. "Problems of a dynamical theory in statistical physics". In: *Studies in Statistical Mechanics, Vol. I*. North-Holland, Amsterdam; Interscience, New York, 1962, pp. 1–118.

[7] C. Boldrighini, L. A. Bunimovich, and Y. Sinai. "On the Boltzmann equation for the Lorentz gas". In: *J. Statist. Phys.* 32.3 (1983), pp. 477–501.

[8] W. Braun and K. Hepp. "The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles". In: *Comm. Math. Phys.* 56.2 (1977), pp. 101–113.

[9] E. Caglioti, M. Infusino, and T. Kuna. "Translation invariant realizability problem on the d-dimensional lattice: an explicit construction". In: *Electron. Commun. Probab.* 21 (2016), Paper No. 45, 9.

[10] P. Degond. "Spectral theory of the linearized Vlasov-Poisson equation". In: *Trans. Amer. Math. Soc.* 294.2 (1986), pp. 435–453.

[11] L. Desvillettes. "Entropy dissipation estimates for the Landau equation in the Coulomb case and applications". In: *J. Funct. Anal.* 269.5 (2015), pp. 1359–1403.

[12] L. Desvillettes, E. Miot, and C. Saffirio. "Polynomial propagation of moments and global existence for a Vlasov-Poisson system with a point charge". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32.2 (2015), pp. 373–400.

[13] L. Desvillettes and M. Pulvirenti. "The linear Boltzmann equation for long-range forces: a derivation from particle systems". In: *Math. Models Methods Appl. Sci.* 9.8 (1999), pp. 1123–1145.

[14] L. Desvillettes and V. Ricci. "A rigorous derivation of a linear kinetic equation of Fokker-Planck type in the limit of grazing collisions". In: *J. Statist. Phys.* 104.5-6 (2001), pp. 1173–1189.

[15] L. Desvillettes and C. Villani. "On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness". In: *Comm. Partial Differential Equations* 25.1-2 (2000), pp. 179–259.

[16] L. Desvillettes and C. Villani. "On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications". In: *Comm. Partial Differential Equations* 25.1-2 (2000), pp. 261–298.

[17] L. Desvillettes and C. Villani. "On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation". In: *Invent. Math.* 159.2 (2005), pp. 245–316.

[18] R. DiPerna and P.-L. Lions. "On the Cauchy problem for Boltzmann equations: global existence and weak stability". In: *Ann. of Math. (2)* 130.2 (1989), pp. 321–366.

[19] I. Gallagher, L. Saint-Raymond, and B. Texier. *From Newton to Boltzmann: hard spheres and short-range potentials.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[20] G. Gallavotti. "Grad-Boltzmann limit and Lorentz's Gas". In: *Statistical Mechanics. A short treatise* (1999).

[21] R. Glassey and J. Schaeffer. "On time decay rates in Landau damping". In: *Comm. Partial Differential Equations* 20.3-4 (1995), pp. 647–676.

[22] R. Glassey and J. Schaeffer. "Time decay for solutions to the linearized Vlasov equation". In: *Transport Theory Statist. Phys.* 23.4 (1994), pp. 411–453.

[23] R. Guernsey. "Kinetic equation for a completely ionized gas". In: *Phys. Fluids* 5 (1962), pp. 322–328.

[24] Y. Guo. "The Landau equation in a periodic box". In: *Comm. Math. Phys.* 231.3 (2002), pp. 391–434.

[25] F. John. *Partial differential equations.* fourth. Vol. 1. Applied Mathematical Sciences. Springer-Verlag, New York, 1991.

[26] Y. Klimontovich. *Kinetic theory of nonideal gases and nonideal plasmas.* Vol. 105. International Series in Natural Philosophy. Pergamon Press, Oxford-Elmsford, N.Y., 1982.

[27] J. Krommes. "Two new proofs of the test particle superposition principle of plasma kinetic theory". In: *Phys. Fluids* 19.5 (1976), pp. 649–655.

[28] C. Lancellotti. "On the fluctuations about the Vlasov limit for N-particle systems with mean-field interactions". In: *J. Stat. Phys.* 136.4 (2009), pp. 643–665.

[29] C. Lancellotti. "On the Glassey-Schaeffer estimates for linear Landau damping". In: *J. Comput. Theor. Transp.* 44.4-5 (2015), pp. 198–214.

[30] C. Lancellotti. "Time-asymptotic evolution of spatially uniform Gaussian Vlasov fluctuation fields". In: *J. Stat. Phys.* 163.4 (2016), pp. 868–886.

[31]   L. Landau. "Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung". In: *Phys. Zs. Sow. Union* 10.154 (1936).

[32]   O. Lanford. "Time evolution of large classical systems". In: *Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974)* (1975), 1–111. Lecture Notes in Phys., Vol. 38.

[33]   A. Lenard. "Correlation functions and the uniqueness of the state in classical statistical mechanics". In: *Comm. Math. Phys.* 30 (1973), pp. 35–44.

[34]   A. Lenard. "On Bogoliubov's kinetic equation for a spatially homogeneous plasma". In: *Ann. Physics* 10 (1960), pp. 390–400.

[35]   A. Lenard. "States of classical statistical mechanical systems of infinitely many particles. II. Characterization of correlation measures". In: *Arch. Rational Mech. Anal.* 59.3 (1975), pp. 241–256.

[36]   E. Lifshitz and L. Pitaevskii. *Course of Theoretical Physics*. Pergamon Press, Oxford, 1981.

[37]   J. Loschmidt. "Über den Zustand des Wärmegleichgewichtes eines Systems von Körpern mit Rücksicht auf die Schwerkraft, 1.-4. Teil." Deutsch. In: (1876).

[38]   A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables.* Vol. 53. Applied Mathematical Sciences. Springer-Verlag, New York, 1984.

[39]   C. Mouhot and C. Villani. "On Landau damping". In: *Acta Math.* 207.1 (2011), pp. 29–201.

[40]   N. Muskhelishvili. *Singular integral equations.* Dover Publications, Inc., New York, 1992.

[41]   A. Nota, S. Simonella, and J. Velázquez. "On the theory of Lorentz gases with long range interactions". In: *Rev. Math. Phys., to appear* (2018).

[42]   C. Oberman and E. Williams. "Theory of fluctuations in plasma". In: 1983, p. 111.

[43]   O. Penrose. "Electrostatic instabilities of a uniform non-Maxwellian plasma". In: *Phys. Fluids* 3.2 (1960), pp. 258–265.

[44]   J. Piasecki and G. Szamel. "Stochastic dynamics of a test particle in fluids with weak long-range forces". In: *Physica A Statistical Mechanics and its Applications* 143 (1987), pp. 114–122.

[45]   M. Pulvirenti, C. Saffirio, and S. Simonella. "On the validity of the Boltzmann equation for short range potentials". In: *Rev. Math. Phys.* 26.2 (2014), p. 64.

[46]   M. Pulvirenti and S. Simonella. "The Boltzmann-Grad limit of a hard sphere system: analysis of the correlation error". In: *Invent. Math.* 207.3 (2017), pp. 1135–1237.

[47]   N. Rostoker. "Superposition of Dressed Test Particles". In: *The Physics of Fluids* 7.4 (1964), pp. 479–490.

[48]   L. Silvestre. "Upper bounds for parabolic equations and the Landau equation". In: *J. Differential Equations* 262.3 (2017), pp. 3034–3055.

[49]   H. Spohn. "Kinetic equations from Hamiltonian dynamics: Markovian limits". In: *Rev. Modern Phys.* 52.3 (1980), pp. 569–615.

[50] H. Spohn. *Large Scale Dynamics of Interacting Particles*. en. Springer Science & Business Media, 2012.

[51] H. Spohn. "The Lorentz process converges to a random flight process". In: *Comm. Math. Phys.* 60.3 (1978), pp. 277–290.

[52] R. Strain. "On the linearized Balescu-Lenard equation". In: *Comm. Partial Differential Equations* 32.10-12 (2007), pp. 1551–1586.

[53] R. Strain and Y. Guo. "Exponential decay for soft potentials near Maxwellian". In: *Arch. Ration. Mech. Anal.* 187.2 (2008), pp. 287–339.

[54] G. Toscani and C. Villani. "Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation". In: *Comm. Math. Phys.* 203.3 (1999), pp. 667–706.

[55] J. Velázquez and R. Winter. "From a non-Markovian system to the Landau equation". In: *Comm. Math. Phys., to appear* (2018).

[56] C. Villani. *A review of mathematical topics in collisional kinetic theory*. Vol. 1. Handbook of mathematical fluid dynamics. Amsterdam: North-Holland, 2002.

[57] C. Villani. "On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations". In: *Arch. Rational Mech. Anal.* 143.3 (1998), pp. 273–307.

[58] C. Villani. "On the spatially homogeneous Landau equation for Maxwellian molecules". In: *Math. Models Methods Appl. Sci.* 8.6 (1998), pp. 957–983.

[59] B. Wennberg. "Stability and exponential convergence in L^p for the spatially homogeneous Boltzmann equation". In: *Nonlinear Anal.* 20.8 (1993), pp. 935–964.

[60] E. Zermelo. "Ueber einen Satz der Dynamik und die mechanische Wärmetheorie". en. In: *Ann. Phys.* 293.3 (1896), pp. 485–494.