The Dirac operator under collapse with bounded curvature and diameter

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Saskia Christine Roos
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1. Gutachter: Prof. Dr. Werner Ballmann
2. Gutachter: Prof. Dr. Bernd Ammann
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Summary

Let $\mathcal{M}(n,d)$ be the set of all isometry classes of closed n-dimensional Riemannian manifolds (M,g) with $|\sec(M)| \leqslant 1$ and $\operatorname{diam}(M) \leqslant d$. It is a well-known result by Gromov that any sequence in $\mathcal{M}(n,d)$ admits a subsequence converging to a compact metric space Y in the Gromov-Hausdorff topology. A convergent sequence is said to collapse if the dimension of the limit space Y is strictly less than n.

The aim of this thesis is to study the behavior of the spectrum of the Dirac operator on collapsing sequences of spin manifolds in $\mathcal{M}(n,d)$. Since a limit space Y has in general many singularities we focus on two special cases. We assume that Y is a Riemannian manifold or that the Hausdorff dimension of Y is $(n-1)$.

In Chapter 1 we state the basic definitions and properties of the Gromov-Hausdorff distance. Afterwards we summarize the known results for collapse with bounded curvature and diameter. One of the most important results is that a collapsing sequence in $\mathcal{M}(n,d)$ can be approximated by a collapsing sequence of singular Riemannian affine fiber bundles.

The main result of Chapter 2 is that the Hausdorff dimension of a limit space Y of a convergent sequence $(M_i,g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ is larger than or equal to $(n-1)$, if and only if there are positive constants C, r such that $C \leqslant \frac{\operatorname{vol}(B_r^{M_i}(x))}{\operatorname{inj}^{M_i}(x)}$ for all $x \in M_i$ and $i \in \mathbb{N}$. To show this we first prove that for a Riemannian submersion $f: M \to Y$ there is a constant C, such that $\operatorname{inj}(f^{-1}(p)) \leqslant C \operatorname{inj}^M(x)$ for all $x \in f^{-1}(p)$ if the injectivity radius of M is sufficiently small compared to the injectivity radius of Y. As a conclusion, we define the set $\mathcal{M}(n,d,C)$ that contains all isometry classes of closed Riemannian manifolds in $\mathcal{M}(n,d)$ satisfying $C \leqslant \frac{\operatorname{vol}(M)}{\operatorname{inj}(M)}$. Moreover, we show that the arising limit spaces are n-dimensional Riemannian manifolds or $(n-1)$-dimensional Riemannian orbifolds with a $C^{1,\alpha}$-metric and bounded curvature in the weak sense.

Since any collapsing sequence in $\mathcal{M}(n,d)$ with a smooth limit space can be approximated by a collapsing sequence of Riemannian affine fiber bundles $f: M \to B$, we discuss these special bundles thoroughly in Chapter 3. The results proven in that chapter are mainly a preparation for the study of Dirac eigenvalues on collapsing sequences in Chapter 4. Using O'Neill's formulas we discuss how the metric on the total space M is related to the metric on the base space B and derive various bounds that are needed in the next chapter. Then we show by various examples that, in general, a spin structure on the total space M does not induce a spin structure on the base space B. If we restrict to the case of S^1-principal bundles $f: M \to B$ then a spin structure on M induces a spin structure on B if the S^1-action lifts to the spin structure on M. As the limit of a collapsing sequence can be non orientable we also briefly discuss pin structures. Loosely speaking, pin structures are a generalization of spin structures to non orientable spaces. Afterwards we restrict our attention to spin structures on the total space M that admit affine parallel spinors, which can be interpreted as spinors that are "invariant" along the fibers. We show that the space of affine parallel spinors is isometric to the sections of a twisted spinor bundle $\mathfrak{P}$ over the base space B. Furthermore, we show that there is an elliptic first order self-adjoint differential operator on $\mathfrak{P}$ that is isospectral to the Dirac operator on M restricted to the space of affine parallel spinors.

In Chapter 4 we first consider collapsing sequences in $\mathcal{M}(n,d)$ converging to a Riemannian manifold. In that case we show that the spectrum of the Dirac operator restricted

to the space of affine parallel spinors converges to the spectrum of a twisted Dirac operator with a Hölder continuous symmetric potential. This determines the behavior of the Dirac spectrum on collapsing sequences with smooth limit space because it was shown by Lott that the remaining part of the spectrum diverges in the limit. In addition, we state conditions such that the spectrum of the Dirac operator converges to the spectrum of the Dirac operator on the limit space up to multiplicity. Afterwards we extend a result by Ammann regarding Dirac eigenvalues on collapsing S^1-principal bundles to arbitrary collapsing sequences of spin manifolds in the set $\mathcal{M}(n+1, d, C)$ introduced in Chapter 2. Similar to the results for collapsing sequences with smooth limit space we show that the spectrum of the Dirac operator restricted to the space of affine parallel spinors converges to the spectrum of a twisted Dirac operator with symmetric Hölder continuous potential. In addition, we study the structure of the Dirac spectrum on S^1-orbifold bundles and prove a lower and an upper bound for the Dirac eigenvalues.

We included a small introduction to infranilmanifolds in Appendix A. Appendix B deals with Riemannian submersions with a fixed spin structure on the total space. There we derive formulas for the spinorial connection and the Dirac operator. These formulas describe explicitly how the vertical and the horizontal components interact with each other which is helpful for the considerations in Chapter 3. Moreover, in Appendix B we also restate O'Neill's formulas for Riemannian submersions. In Appendix C, we recall that the Dirac spectrum is continuous under a C^1-variation of the metric and in Appendix D we discuss the convergence of S^1-principal bundles with connection.

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Introduction

In differential geometry many results for Riemannian manifolds are proved under assumptions on the geometry of the manifold like curvature and volume. But can we also say something about the set of all isometry classes of Riemannian manifolds satisfying different assumptions on the geometry? Let $\mathcal{M}(n,d,v)$ be the set of all isometry classes of closed n-dimensional Riemannian manifolds (M,g) with $|\sec| \leqslant 1$, $\operatorname{diam}(M) \leqslant d$ and $\operatorname{vol}(M) \geqslant v$. Cheeger showed that the set of diffeomorphism classes in $\mathcal{M}(n,d,v)$ is finite [Che70, Theorem 3.1, Theorem 4.2]. If we remove the lower volume bound then the resulting set $\mathcal{M}(n,d)$ contains infinitely many diffeomorphism classes. Nevertheless, Gromov was able to show that any sequence in $\mathcal{M}(n,d)$ contains a convergent subsequence with respect to the Gromov-Hausdorff topology [Gro81, Théorème 5.3].

In this thesis we are interested in those convergent sequences in $\mathcal{M}(n,d)$ where the volume of the manifolds goes to zero in the limit. It follows that the limit space of such sequences is a compact metric space Y of strictly lower dimension, i.e. the sequence *collapses*. Easy examples of collapsing sequences arise by scaling flat manifolds like the torus, see Examples 1.12, 1.13. To the author's knowledge, the first nontrivial example of a collapsing sequence was pointed out by Marcel Berger in about 1962. He considered the Hopf fibration $S^1 \to S^3 \to S^2$. Starting with the standard round metric on S^3, Berger rescaled the metric tangent to the fibers by $\varepsilon > 0$ while keeping the metric in the directions orthogonal to the fibers fixed. As $\varepsilon \to 0$ the sectional curvature remains bounded while the volume converges to 0. Furthermore, S^3 resembles more and more a two-sphere with constant sectional curvature equal to 4 as $\varepsilon \to 0$ (see Example 1.14 for more details).

One of the first results on collapse with bounded curvature is Gromov's characterization of almost flat manifolds [Gro78, Main Theorem 1.4]. Gromov showed that there is a positive $\varepsilon(n)$ such that any closed n-dimensional Riemannian manifold (M,g) with $\operatorname{diam}(M) \leqslant 1$ and $|\sec| \leqslant \varepsilon(n)$ is an almost flat manifold, i.e. for any $\varepsilon > 0$ there is a metric g_ε such that $\operatorname{diam}(M, g_\varepsilon) = 1$ and $|\sec_\varepsilon| \leqslant \varepsilon$. Moreover, Gromov showed that any almost flat manifold is finitely covered by a nilmanifold $\tilde{M}$. Employing additional analytic arguments, Ruh showed that M is an *infranilmanifold*, i.e. the deck transformation group of $\tilde{M} \to M$ consists of affine diffeomorphisms with respect to the canonical affine connection on $\tilde{M}$ [Ruh82].

Further important results are Fukaya's fibration theorems [Fuk87b, Fuk89]. Fukaya showed that if two manifolds $M \in \mathcal{M}(n+k,d)$ and $B \in \mathcal{M}(n,d)$ are sufficiently close with respect to the Gromov-Hausdorff distance then there is a fibration $M \to B$ such that the fibers are infranilmanifolds. In a next step, Fukaya applied his fibration theorems to the sequence of orthonormal frame bundles of a collapsing sequence in $\mathcal{M}(n,d)$ and derived a description of the boundary of $\mathcal{M}(n,d)$ [Fuk88, Theorem 0.12, Theorem 10.1].

Around the same time, Cheeger and Gromov studied collapse with bounded curvature from a different point of view [CG86, CG90]. Cheeger and Gromov define local group actions and the action of a sheaf of groups on Riemannian manifolds. Using these definitions, Cheeger and Gromov showed that on each sufficiently collapsed Riemannian manifold there is a sheaf of tori with additional regularity conditions acting on it. This structure is called an *F-structure*, where "F" stands for flat. One of the advantages of this approach is that they do not need to assume a uniform diameter bound.

These two approaches are combined in [CFG92]. Generalizing Fukaya's fibration theorem, Cheeger, Fukaya and Gromov show that the orthonormal frame bundle FM of a sufficiently collapsed Riemannian manifold M is locally the total space of a fibration with infranil fibers and affine structure group. Then Cheeger, Fukaya and Gromov generalized the theory developed by Cheeger and Gromov to show that there is a sheaf of nilpotent groups acting on FM with additional regularity conditions generalizing the notion of an F-structure. These generalized structures are called *N-structures*.

Our first main result gives a characterization for codimension one collapse, i.e. convergent sequences in $\mathcal{M}(n,d)$ with $(n-1)$-dimensional limit space. The motivation here is that, in general, the limit space of a collapsing sequence in $\mathcal{M}(n,d)$ has many singularities. However, for codimension one collapse, the limit space is always a Riemannian orbifold [Fuk90, Proposition 11.5], while the limit space has in general non orbifold singularities if its dimension is less than $(n-1)$, [NT11, Theorem 1.1].

Theorem 0.1. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n,d)$ converging to a compact metric space (Y, d_Y) in the Gromov-Hausdorff topology. Then the following are equivalent*

(1) $\dim_{\mathrm{Haus}}(Y) \geqslant (n-1)$,

(2) for all $r > 0$ there is a positive constant $C(n,r,Y)$ such that

$$C \leqslant \frac{\mathrm{vol}(B_r^{M_i}(x))}{\mathrm{inj}^{M_i}(x)}$$

holds for all $x \in M_i$ and $i \in \mathbb{N}$,

(3) for some $r > 0$ there is a positive constant $C(n,r,Y)$ such that the above inequality holds for all $x \in M_i$ and $i \in \mathbb{N}$.

The intuition behind this theorem is that for a collapsing sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ the volume of a ball represents all collapsed and non collapsed directions, while the injectivity radius represents only the fastest scale of collapse. If we have a codimension one collapse then then it happens on the scale of the injectivity radius. But if a collapsing sequence $(M_i, g_i)_{i\in\mathbb{N}}$ loses two or more dimensions in the limit then the loss of volume of the balls in M_i is larger than the injectivity radius. In particular, the sequence $\left(\frac{\mathrm{vol}(B_r^{M_i}(x))}{\mathrm{inj}^{M_i}(x)}\right)_{i\in\mathbb{N}}$ has to vanish in the limit.

It follows from this theorem that the possible limit spaces of the set $\mathcal{M}(n,d,C)$ of all isometry classes of Riemannian manifolds $(M,g) \in \mathcal{M}(n,d)$ with $C \leqslant \frac{\mathrm{vol}(M)}{\mathrm{inj}(M)}$ are n-dimensional Riemannian manifolds or $(n-1)$-dimensional Riemannian orbifolds. Moreover, the sectional curvatures of the limit spaces are uniformly bounded in the weak sense.

A further interesting question concerning collapse in $\mathcal{M}(n,d)$ is the behavior of the spectra of geometric operators. We would like to know how the limit of the spectra is related to the spectrum of the corresponding geometric operator on the limit space.

For a sequence in $\mathcal{M}(n,d)$ converging to a limit space B in the measured Gromov-Hausdorff topology, Fukaya showed that the Laplace spectrum converges to the spectrum of a self-adjoint operator over B [Fuk87a, Theorem 0.4]. If B happens to be a manifold, then the limit of the Laplace spectrum does in general not coincide with the Laplace spectrum of B, see Example 4.4. In [Lot02b, Lot02c] Lott generalized this behavior to the spectrum of the Laplace operator acting on differential forms. Then Lott combined these results with Bochner-type formulas for Dirac operators to prove similar results for Dirac-type operators on G-Clifford bundles, where $G \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$ [Lot02a]. If the limit space B is a Riemannian manifold then the results of [Lot02a] state that the Dirac spectrum converges to the spectrum of an elliptic first order differential operator $\mathcal{D}^B = \sqrt{\Delta + H}$ acting on a G-Clifford bundle over B. Here Δ is the Laplacian with respect to a limit measure and H is a symmetric potential arising as the weak-$*$-limit of curvature terms.

In this thesis we will give an explicit description of the limit operator $\mathcal{D}^B$ for sequences of spin manifolds in $\mathcal{M}(n,d)$ converging to Riemannian manifolds of lower dimensions and for collapsing sequences of spin manifolds in $\mathcal{M}(n,d,C)$.

One of the main ingredients of the proofs is that any collapsing sequence $(M_i,g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ with a smooth limit space (B,h) can be realized as a collapsing sequence $(f_i : (M_i,g_i) \to (B,h_i))_{i\in\mathbb{N}}$ of fiber bundles with infranil fibers and affine structure group [Fuk87b, Fuk89]. Using this property of collapsing sequences in $\mathcal{M}(n,d)$ we describe the behavior of the Dirac spectrum.

Theorem 0.2. *Let $(M_i,g_i)_{i\in\mathbb{N}}$ be a sequence of spin manifolds in $\mathcal{M}(n+k,d)$ converging to a closed n-dimensional Riemannian manifold (B,h). Then for all $i \in \mathbb{N}$ the space of L^2-spinors on M_i can be decomposed into*

$$L^2(\Sigma M_i) = \mathcal{S}_i \oplus \mathcal{S}_i^\perp$$

such that all eigenvalues of the Dirac operator on M_i restricted to $\mathcal{S}_i^\perp$ go to $\pm\infty$ as $i \to \infty$ and the eigenvalues of the Dirac operator on M_i restricted to $\mathcal{S}_i$ converge to the spectrum of the self-adjoint elliptic first-order differential operator

$$\mathcal{D}^B = \check{D}^{\mathcal{T}} + H$$

acting on a twisted Clifford bundle $\mathfrak{P}$ over B. Here, $\check{D}^{\mathcal{T}}$ is a Dirac operator on $\mathfrak{P}$ and H a $C^{0,\alpha}$-symmetric potential for $\alpha \in [0,1)$.

In fact, we will give a complete description of the twisted Clifford bundle $\mathfrak{P}$ and of the potential H. We show that the following three geometric objects of the fiber bundles $f_i : M_i \to B$ contribute to $\mathcal{D}^B$: The holonomy of the vertical distribution, the integrability of the horizontal distributions and the intrinsic curvature of the fibers. These three different conditions are independent from each other as can be seen in the Examples 3.6, 3.5, 3.7, 3.9.

Corollary 0.3. *Let $(f_i : (M_i, g_i) \to (B, h_i))_{i\in\mathbb{N}}$ be a collapsing sequence of fiber bundles with infranil fiber such that $(M_i, g_i)_{i\in\mathbb{N}}$ is a spin manifold in $\mathcal{M}(n+k, d)$ for all $i \in \mathbb{N}$ and B is a closed n-dimensional manifold. Further, we denote by Z_i the closed k-dimensional infranilmanifold which is diffeomorphic to the fiber of $f_i : (M_i, g_i) \to (B, h_i)$. If in the limit $i \to \infty$ the holonomy of the vertical distribution is trivial, the instrinsic curvature of the fibers is flat and the horizontal distribution is integrable then there is a subsequence such that the spectrum of the Dirac operator $D^{M_i}_{|\mathcal{S}_i}$ converges, up to multiplicity, to the spectrum of D^B if n or k is even, and to the spectrum of $D^B \oplus -D^B$ if n and k are odd.*

Next, we consider the behavior of the Dirac operator on sequences of spin manifolds in $\mathcal{M}(n+1, d, C)$. As collapsing sequences in $\mathcal{M}(n+1, d, C)$ can always be approximated by a sequence of S^1-orbifold bundles [Fuk88], [Fuk90, Proposition 11.5], we are able to give a complete description of the behavior of the Dirac spectrum. This extends results of [Amm98a], [Amm98b, Kapitel 7], [AB98, Section 4] where collapsing S^1-principal bundles under slightly different assumptions were considered. First we show that the results of Theorem 0.2 and Corollary 0.3 carry over to collapsing sequences in $\mathcal{M}(n + 1, d, C)$. Further, we prove a lowerand an upper bound on the Dirac eigenvalues on collapsing sequences in $\mathcal{M}(n + 1, d, C)$ generalizing the bounds stated in [Amm98a, Theorem 3.1, Theorem 4.1], [Amm98b, Satz 7.2.1, Satz 7.3.2].

Proposition 0.4. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence of spin manifolds in $\mathcal{M}(n+1, d, C)$ converging to an n-dimensional Riemannian orbifold (B, h). Then we can number the Dirac eigenvalues $(\lambda_{j,k}(i))_{j\in\mathbb{Z},k}$ where $k \in \mathbb{Z}$ if there is an induced spin structure on B and $k \in (\mathbb{Z} + \frac{1}{2})$ else, such that for any $\varepsilon > 0$ there is an index $I > 0$ such that for all $i \geqslant I$ there are S^1-fibrations $f_i : M_i \to B$ such that for all j, k,*

$$|\lambda_{j,k}(i)| \geqslant \sinh\left(\operatorname{arsinh}\left(\frac{|k|}{\|l_i\|_\infty} - \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A - \varepsilon\right) - \varepsilon\right).$$

Here $2\pi l_i$ is the length of the fibers and C_A is a constant depending on n, d and C. In particular, $\lim_{i\to\infty} |\lambda_{j,k}(i)| = \infty$ whenever $k \neq 0$ since $\lim_{i\to\infty} l_i = 0$.

For any $i \geqslant I$, let $\omega_i \in \Omega(M_i, \mathcal{V}_i)$ be the orthogonal projection onto $\mathcal{V}_i := \ker(\mathrm{d}f_i)$, where $f_i : M_i \to B$. If, in addition, there is a constant C such that

$$\|\mathrm{d}\omega_i\|_{C^{0,1}} \leqslant C$$

for all $i \geqslant I$, then for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ (projectable spin structures), resp. $k \in (\mathbb{Z} + \frac{1}{2})$ (non projectable spin structures),

$$\limsup_{i\in\mathbb{N}} \left(\min_{p\in B} l_i(p)|\lambda_{j,k}(i)|\right) \leqslant |k|.$$

As we will see in Example 4.4 these results cannot be extended to the Dirac operator acting on differential forms.

A brief outline of this thesis is as follows: In Chapter 1 we recall the definition and basic properties of the Gromov-Hausdorff distance and discuss the basic results for collapse with bounded curvature and diameter. The characterization of codimension one collapse, Theorem 0.1, is proven in Chapter 2. Then we start with the preparation for the study of Dirac eigenvalues in Chapter 3. Since any collapsing sequence in $\mathcal{M}(n,d)$ with smooth limit space can be approximated by a sequence of fiber bundles $f: M \to B$ with infranil fibers and affine structure group, we discuss the relation between the geometry of the total space M and the base space B on such fiber bundles in great detail. There we also show that, in general, a spin structure on M does not induce a spin structure on B. Nevertheless we show that the space of affine parallel spinors on M, i.e. spinors that are "invariant" along the fibers, is isometric to the space of sections of a twisted Clifford bundle over B. All these results are used in Chapter 4 to show the results for Dirac eigenvalues on collapsing sequences in $\mathcal{M}(n,d)$ with smooth limit spaces, proving Theorem 0.2, Corollary 0.3, and on collapsing sequences in $\mathcal{M}(n+1,d,C)$, where we prove Proposition 0.4. Moreover, we show that the limit operator $\mathcal{D}^B$ is a twisted Dirac operator with symmetric $H^{1,\infty}$-potential. Thus, we generalize our convergence results to the spectrum of Dirac operators with symmetric potentials that are uniformly bounded in the $H^{1,\infty}$-topology. In that generality we conclude that the spectra of Dirac operators with symmetric $H^{1,\infty}$-potential restricted to the space of affine parallel spinors converges again to the spectrum of a Dirac operator with a symmetric $H^{1,\infty}$-potential over the limit space.

In Appendix A we define infranilmanifolds and discuss under which assumptions there is a spin structure with affine parallel spinors. Then we consider Riemannian submersions $f: M \to B$ where M is a spin manifold in Appendix B. We derive formulas for the spinorial connection and the Dirac operator on M decomposing them into their vertical and horizontal parts. Moreover, we recall O'Neill's formulas for Riemannian submersions. Afterwards we review the continuity of Dirac spectra under a C^1-variation of Riemannian metrics following [Now13] in Appendix C. In Appendix D we discuss the convergence of S^1-principal bundles with connection. These results are used to prove the upper bound in Proposition 0.4.

We would like to remark that the results of this thesis have been published in several articles. The characterization of the codimension one collapse, which is proven in Chapter 2, is the content of [Roo18a]. In [Roo18c] the behavior of the Dirac spectrum on collapsing sequences in $\mathcal{M}(n+1,d,C)$ is discussed and its sequel [Roo18b] deals with the Dirac operators on collapsing sequences in $\mathcal{M}(n,d)$ with smooth limit spaces. The content of [Roo18c] and [Roo18b] corresponds to Chapter 3 and Chapter 4. The results of Appendix D can be found in [Roo17, Section 4.1].

Chapter 1

Convergence of Riemannian manifolds

Let $\mathcal{M}(n,d)$ be the set of isometry classes of closed n-dimensional Riemannian manifolds (M,g) such that $|\sec^M| \leqslant 1$ and $\operatorname{diam}(M) \leqslant d$. It follows from [Gro81, Théorèm 5.3] that any sequence $(M_i,g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ has a subsequence that converges with respect to the Gromov-Hausdorff distance to a compact metric space with dimension less than or equal to n. If the dimension of the limit space is strictly smaller than n then one says that the sequence $(M_i,g_i)_{i\in\mathbb{N}}$ *collapses*. The structure of collapsing sequences with bounded curvature and diameter was intensively studied by Cheeger, Fukaya and Gromov [CG86, CG90, Fuk87b, Fuk88, Fuk89, CFG92].

Before discussing collapse with bounded curvature and diameter in detail we first briefly recall the definition and properties of the Gromov-Hausdorff distance d_{GH}. Then we state the known results regarding the structure of the boundary of $\mathcal{M}(n,d)$ in Section 1.2. For later use, we carry out the following two special cases in more detail: Collapsing sequences $(M_i,g_i)_{i\in\mathbb{N}}$ converging to a smooth manifold and collapsing sequences in $\mathcal{M}(n,d)$ with $(n-1)$-dimensional limit space.

We will roughly follow the lines of [Ron07] to give a summary of the known results regarding the boundary of $\mathcal{M}(n,d)$.

1.1 The Gromov-Hausdorff distance

Let A and B be two compact subsets of a fixed metric space (Z,d_Z). The *Hausdorff distance* between A and B is defined as

$$d_{\mathrm{H}}^{Z}(A,B) := \min\{\varepsilon > 0 : B \subset T_\varepsilon(A) \text{ and } A \subset T_\varepsilon(B)\},$$

where $T_\varepsilon(A) := \{x \in Z : d_Z(x,A) < \varepsilon\}$ is an open ε-neighborhood of A.

By construction the Hausdorff distance is symmetric and satisfies the triangle inequality. Furthermore, $d_{\mathrm{H}}^{Z}(A,B) = 0$ if and only if $A = B$. Thus, the Hausdorff distance defines a metric on the space of all compact subsets of Z. Loosely speaking, the Hausdorff distance measures the "uniform closeness" of two compact subsets in a fixed metric space.

In [Gro81, Chapitre 3] Gromov studies the space of isometry classes of compact metric spaces Met_c. To define a metric on Met_c the Hausdorff distance is modified in the following way:

Definition 1.1. Let (X, d_X) and (Y, d_Y) be two compact metric spaces. A metric $\tilde{d}$ on the disjoint union $X \sqcup Y$ is called an *admissible metric* if it extends the metrics on X and Y, i.e. $\tilde{d}(x_1, x_2) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$ and $\tilde{d}(y_1, y_2) = d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$. The Gromov-Hausdorff distance between X and Y is defined as

$$d_{\mathrm{GH}}(X, Y) := \inf\{\, d_H^{\tilde{d}}(X, Y) : \tilde{d} \text{ is an admissible metric on } X \sqcup Y\}.$$

One can show that d_{GH} also satisfies the triangle inequality. But in contrast to the Hausdorff distance, two compact metric spaces (X, d_X) and (Y, d_Y) satisfy $d_{\mathrm{GH}}(X, Y) = 0$ if and only if they are isometric to each other. As d_{GH} is symmetric by construction it follows that the Gromov-Hausdorff distance defines a complete metric on the set of isometry classes of compact metric spaces Met_c.

Remark 1.2. The definition of the Gromov-Hausdorff distance given above is an equivalent formulation of the original definition [Gro81, Définition 3.4]. There the Gromov-Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as

$$d_{\mathrm{GH}} = \inf\{d_H^Z(\varphi(X), \psi(Y))\},$$

where the infinuum is taken over all metric spaces (Z, d_Z) such that there are isometric embeddings $\varphi : X \hookrightarrow Z$ and $\psi : Y \hookrightarrow Z$.

Moreover, (Met_c, d_{GH}) is a complete metric space. To get an intuition for the Gromov-Hausdorff distance we give a proof of that result (see also [Ron07, Section 2]).

Proposition 1.3. *(Met_c, d_{GH}) is a complete metric space.*

Proof. Let $(X_i, d_{X_i})_{i\in\mathbb{N}}$ be a Cauchy sequence in Met_c with respect to the Gromov-Hausdorff distance. It is clear that for any Cauchy sequence

(1) there is a uniform bound on the diameter,

(2) for any $\varepsilon > 0$ there is an $N(\varepsilon)$ such thatfor any $i \in \mathbb{N}$ there is an ε-dense subset $X_i(\varepsilon) \subset X_i$ whose cardinality is bounded by $N(\varepsilon)$.

By passing to a subsequence, if necessary, we assume that for any $i \in \mathbb{N}$ there is an admissible metric on $X_i \sqcup X_{i+1}$ such that $d_H^{i,i+1}(X_i, X_{i+1}) < 2^{-i}$. In what follows x_i will always denote an element of X_i.

In the next step we define a metric d_Y on $Y := \bigsqcup_{i\in\mathbb{N}} X_i$ by setting

$$d_Y(x_i, x_{i+j}) := \min_{x_{i+k}\in X_{i+k}} \left\{ \sum_{k=0}^{j-1} d_{i+k,i+k+1}(x_{i+k}, x_{i+k+1}) \right\}.$$

Loosely speaking, $d(x_i, x_{i+j})$ is the distance of the shortest path from x_i to x_{i+j} passing $X_{i+1}, \ldots, X_{i+j-1}$. By construction, $(X_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in (Y, d_Y) with respect to the Hausdorff-distance d_H^Y.

Now we construct a possible candidate (X, d_X) for the limit of the sequence $(X_i)_{i\in\mathbb{N}}$. We define

$$X := \{(x_i)_{i\in\mathbb{N}} \text{ Cauchy sequence in } Y \text{ with } x_i \in X_i\} \big/ \sim,$$

where $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$ if $\lim_{i\to\infty} d_Y(x_i, y_i) = 0$. The metric d_X on X is defined as

$$d_X((x_i)_i, (y_i)_i) := \lim_{i\to\infty} d_Y(x_i, y_i).$$

It follows from the properties (1) and (2) that (X, d_X) is a compact metric space. We want to show that (X, d_X) is the Gromov-Hausdorff limit of $(X_i)_{i\in\mathbb{N}}$. Therefore, we define an admissible metric $d_{Y\sqcup X}$ on $Y \sqcup X$ by

$$d_{Y\sqcup X}(y, (x_i)_i) := \lim_{i\to\infty} d_Y(y, x_i).$$

Then X is the Hausdorff limit of $(X_i)_{i\in\mathbb{N}}$ in $Y \sqcup X$. Hence, X is also the Gromov-Hausdorff limit since

$$d_{\mathrm{GH}}(X_i, X) \leqslant d_H^{Y\sqcup X}(X_i, X) \xrightarrow{i\to\infty} 0 \; . \qquad \square$$

In fact, one can show that the properties (1) and (2) in the above proof are also sufficient to choose a convergent subsequence. Considering the set of all isometry classes of closed n-dimensional Riemannian manifolds (M, g) with $\operatorname{diam}(M) \leqslant d$, Gromov uses Bishop-Gromov volume comparison to observe that a lower bound on the Ricci curvature controls the size of ε-dense subsets [Gro81, Théorème 5.3].

Theorem 1.4. *Let k, d be positive numbers. Any sequence of closed n-dimensional Riemannian manifolds $(M_i, g_i)_{i\in\mathbb{N}}$ with $\operatorname{Ric}^{M_i} \geqslant -k$ and $\operatorname{diam}(M_i) \leqslant d$ contains a d_{GH}-convergent subsequence.*

Remark 1.5. Note that the limit of a d_{GH}-convergent sequence of Riemannian manifolds does not need to be a Riemannian manifold, see for instance Example 1.15.

For later use, we also discuss how the symmetry of compact metric spaces is preserved under the Gromov-Hausdorff convergence. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of compact metric spaces converging to X in the Gromov-Hausdorff topology. Further, we assume that for each $i \in \mathbb{N}$ there is an isometric and effective action of a compact group G_i on X_i. Is there a compact group G acting as isometries on X whose action is related to the actions of G_i on X_i? If yes, does the sequence of quotients $\left(X_i/G_i\right)_{i\in\mathbb{N}}$ converges to the quotient space X/G in the Gromov-Hausdorff topology? To answer these questions we briefly discuss the notion of equivariant Gromov-Hausdorff convergence. This equivariant extension of the Gromov-Hausdorff distance was first introduced by Fukaya [Fuk86, Chapter 1] and achieved its final form with Fukaya and Yamaguchi [FY92, Section 3].

Before defining the equivariant Gromov-Hausdorff distance, we first have to introduce the following "equivalent" concept of Gromov-Hausdorff convergence.

Let (X, d_X) and (Y, d_Y) be two compact metric spaces. A map $f : X \to Y$ is called an *ε-Gromov-Hausdorff approximation* if

- Y is contained in $T_\varepsilon(f(X))$,
- $|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon$ for all $x_1, x_2 \in X$.

We define

$$\hat{d}_{\mathrm{GH}}(X,Y) := \inf \left\{ \varepsilon > 0 : \begin{array}{c} \exists\ \varepsilon\text{-GH approximations } f : X \to Y \\ \text{and } g : Y \to X \end{array} \right\}.$$

The map $\hat{d}_{\mathrm{GH}}$ on Met_c is symmetric and $\hat{d}_{\mathrm{GH}}(X,Y) = 0$ if and only if X is isometric to Y. But in general $\hat{d}_{\mathrm{GH}}$ does not satisfy the triangle inequality, since for three compact metric spaces $(X, d_X), (Y, d_Y)$ and (Z, d_Z) with an ε_1-Gromov-Hausdorff approximation $f : X \to Y$ and an ε_2-Gromov-Hausdorff approximation $g : Y \to Z$ it does not necessarily follow that Z is a subset of $T_{\varepsilon_1+\varepsilon_2}(g(f(X)))$. Nevertheless, it can be shown that

$$\frac{1}{2} d_{\mathrm{GH}} \leqslant \hat{d}_{\mathrm{GH}} \leqslant 2 d_{\mathrm{GH}}.$$

Hence, d_{GH} and $\hat{d}_{\mathrm{GH}}$ have the same Cauchy-sequences and limits.

Next we define an equivariant version of $\hat{d}_{\mathrm{GH}}$. Let (X, d_X) and (Y, d_Y) be two compact metric spaces and assume that there are compact groups G, H acting on X, respectively Y as isometries. Then a pair of maps (f, φ), where $f : X \to Y$ and $\varphi : G \to H$, is an *ε-equivariant Gromov-Hausdorff approximation* if

- f is an ε-Gromov-Hausdorff approximation,
- for any $g \in G$ and $x \in X$, $d_Y\big(\varphi(g)\big(f(x)\big), f(g(x))\big) < \varepsilon$.

Similarly to $\hat{d}_{\mathrm{GH}}$ we define

$$d_{\mathrm{eq.GH}}\left((X,G),(Y,H)\right) := \inf \left\{ \varepsilon > 0 : \begin{array}{c} \exists\ \varepsilon\text{- equivariant-GH approximations} \\ (f,\varphi) : (X,G) \to (Y,H) \\ \text{and } (g,\psi) : (Y,H) \to (X,G) \end{array} \right\}.$$

With this definition we obtain the following convergence result (see for instance [Ron07, Lemma 2.2]).

Lemma 1.6. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of compact metric spaces such that each X_i admits an isometric and effective group action by a compact group G_i. If $(X_i)_{i\in\mathbb{N}}$ converges to X in the Gromov-Hausdorff topology, then there is a compact group G of isometries on X such that $(X_i, G_i)_{i\in\mathbb{N}}$ converges to (X, G) with respect to $d_{eq.GH}$ and $\left(X_i/G_i\right)_{i\in\mathbb{N}}$ converges to X/G in the Gromov-Hausdorff topology.*

Proof. Since $(X_i)_{i\in\mathbb{N}}$ converges to X in the Gromov-Hausdorff topology there is a vanishing sequence $(\varepsilon_i)_{i\in\mathbb{N}}$, i.e. $\lim_{i\to\infty} \varepsilon_i = 0$, such that for any $i \in \mathbb{N}$ there are ε_i-Gromov-Hausdorff approximations $f_i : X_i \to X$ and $h_i : X \to X_i$. Furthermore, we choose for any i an ε_i-dense subset $X(\varepsilon_i) \subset X$ such that $X(\varepsilon_i) \subset X(\varepsilon_j)$ for all $i < j$. Let $(C(X(\varepsilon_i), X), d_{\varepsilon_i})$ be the space of all maps from $X(\varepsilon_i)$ to X endowed with the metric

$d_{\varepsilon_i}(\varphi, \psi) := \max_{x \in X(\varepsilon_i)} d_X(\varphi(x), \psi(x))$. It is easy to check that $(C(X(\varepsilon_i), X), d_{\varepsilon_i})$ is a compact metric space.

Fix an $i \in \mathbb{N}$. For any $j \geqslant i$ we define the map

$$\begin{aligned} \phi_j : G_j &\to C((X(\varepsilon_i), X)), \\ g &\mapsto f_j \circ \chi^j_g \circ h_j, \end{aligned}$$

where $\chi^j : G_j \times X_j \to X_j$ denotes the isometric group action. Since $(C(X(\varepsilon_i), X), d_{\varepsilon_i})$ is compact, there is a subsequence $(\phi_j(G_j))_{j \geqslant i}$ converging to a compact group $G'_i \subset (C(X(\varepsilon_i), X), d_{\varepsilon_i})$. By construction it follows that $g : X(\varepsilon_i) \hookrightarrow X$ is an isometric embedding for any $g \in G'_i$. Let G be the direct limit of $(G'_i)_{i \in \mathbb{N}}$. Then G is a closed group of isometric embeddings $\bigcup_{i \in \mathbb{N}} X(\varepsilon_i) \hookrightarrow X$. These embeddings extend to isometries of X. It is immediate that G is a closed subgroup of isometries of X with the claimed properties.□

In the next section we use the following precompactness result for the equivariant Gromov-Hausdorff distance due to Fukaya [Fuk88, Lemma 1.11 and Lemma 1.13].

Lemma 1.7. *Let $\mathcal{M}(n, d; G)$ be the set of all pairs (M, χ) of isometry classes of closed n-dimensional Riemannian manifolds (M, g) with $|\sec| \leqslant 1$ and $\operatorname{diam}(M) \leqslant d$ and an isometric and effective action $\chi : G \times M \to M$ of a compact group G. Then any sequence $(M_i, \chi_i)_{i \in \mathbb{N}}$ admits a subsequence that converges with respect to $d_{eq.GH}$ to a compact metric space Y with an isometric action χ of G on Y. Furthermore,*

$$\lim_{i \to \infty} d_{GH}\left({}^{M_i}\!/_{G}, {}^{Y}\!/_{G} \right) = 0.$$

1.2 Collapse with bounded curvature and diameter

In this section we discuss sequences $(M_i, g_i)_{i \in \mathbb{N}}$ in the set $\mathcal{M}(n, d)$ of isometry classes of closed n-dimensional Riemannian manifolds with $|\sec^M| \leqslant 1$ and $\operatorname{diam}(M) \leqslant d$. By Theorem 1.4 any sequence in $\mathcal{M}(n, d)$ contains a Gromov-Hausdorff converging subsequence. In the following we discuss the structure of these converging sequences and their limit spaces. By scaling, the results discussed here also hold for the set $\mathcal{M}(n, d \,|\, k)$ of isometry classes of closed n-dimensional Riemannian manifolds with $\operatorname{diam}(M) \leqslant d$ and $|\sec^M| \leqslant k$.

There are two kinds of converging sequences in $\mathcal{M}(n, d)$: the *non collapsing* and the *collapsing* sequences. These two cases are characterized by the behavior of the injectivity radius of the considered manifolds. If the injectivity radius remains bounded away from zero then the sequence is said to be non collapsing. Otherwise the sequence collapses in the limit.

The behavior of non collapsing sequences and the structure of their limit spaces is characterized in the Cheeger-Gromov compactness theorem, summarizing results from [Che67, Che70, Gro81, GW88, Pet87]. Before stating this theorem, we need to define the notion of $C^{1,\alpha}$-convergence of Riemannian manifolds.

Definition 1.8. A sequence of n-dimensional Riemannian manifolds $(M_i, g_i)_{i \in \mathbb{N}}$ converges to a Riemannian manifold (M_∞, g_∞) in the $C^{1,\alpha}$-topology if there are diffeomorphisms

$f_i : M_\infty \to M_i$ such that the pullback metrics $f_i^* g_i$ converge to g_∞ in the $C^{1,\alpha}$-sense. More precisely, there is a $C^{2,\alpha}$-atlas on M_∞ compatible with its smooth structure, such that $(f_i^* g_i)_{i \in \mathbb{N}}$ converges to g_∞ in local coordinates.

Theorem 1.9. *Any sequence in*

$$\mathcal{M}(n, d, \iota) := \{(M, g) \in \mathcal{M}(n, d) : \operatorname{inj}(M) \geqslant \iota\}$$

contains a subsequence that converges to a Riemannian manifold M_∞ with a $C^{1,\alpha}$-metric g_∞ in the $C^{1,\alpha}$-topology. In particular, $\mathcal{M}(n, d, \iota)$ contains only finitely many diffeomorphism types of Riemannian manifolds.

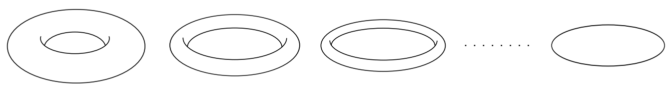
Remark 1.10. In [Che70, Corollary 2.2] Cheeger showed that for any n-dimensional Riemannian manifold (M, g) with $|\sec^M| \leqslant K$, $\operatorname{diam}(M) \leqslant d$ and $\operatorname{vol}(M) \geqslant v$ there is a positive constant $C := C(n, K, d, v)$ such that $\operatorname{inj}(M) > C$. Hence, the lower bound on the injectivity radius in the above theorem can be replaced by a lower volume bound.

Remark 1.11. The above theorem also holds if the two-sided bound on the sectional curvature is replaced by a two-sided bound on the Ricci curvature [And90, Theorem 1.1]. In that generality the lower bound on the injectivity radius cannot be replaced by a lower volume bound.

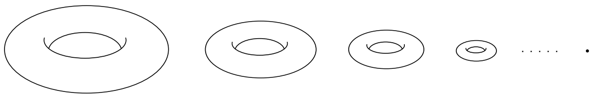
The main step of the proof of Theorem 1.9 is to construct an atlas whose charts are Riemannian normal coordinates on balls of a definite size such that the transition functions are controlled, see [Che67, Theorem 1], [Che70, Theorem 3.1, Theorem 4.2]. Cheeger concluded that the limit space has to be a Riemannian manifold of lower regularity. Regarding the regularity, Gromov proved uniform C^2-bounds on the transition functions of the above atlas [Gro81, Théorème 8.25], hence, uniform C^1-bounds on the metric. In [GW88, Pet87] the authors used harmonic coordinates to show that the regularity of the limit metric can be improved to $C^{1,\alpha}$ with $\alpha \in [0, 1)$. This regularity is optimal and cannot be improved under the assumptions of Theorem 1.9, c.f. [Pet87, Example 5.1].

The Cheeger-Gromov compactness theorem shows that the behavior of non collapsing sequences in $\mathcal{M}(n, d)$ is well understood. For the remainder of this section we focus on *collapsing* sequences, i.e. Gromov-Hausdorff-convergent sequences $(M_i, g_i)_{i \in \mathbb{N}}$ in $\mathcal{M}(n, d)$ such that $\lim_{i \to \infty} \operatorname{inj}(M_i) = 0$. Such sequences converge in the Gromov-Hausdorff topology to compact metric spaces of strictly lower dimension. One of the easiest examples of such sequences are collapsing tori.

Example 1.12. Let $\mathbb{T}^2 := S^1 \times S^1$ be the torus with the flat metric $g := g_S \oplus g_S$, where g_S is the standard metric on S^1. For any $i \in \mathbb{N}$ we set $g_i := g_S \oplus \frac{1}{i} g_S$. By construction, the sequence $(\mathbb{T}^2, g_i)_{i \in \mathbb{N}}$ is contained in $\mathcal{M}(2, 2\pi)$. As $i \to \infty$ we observe that the torus becomes thinner and thinner and collapses to a circle in the limit.



Example 1.13. Consider for each $i \in \mathbb{N}$ the metric $g_i := \frac{1}{i} g_S \oplus \frac{1}{i} g_S$ on the torus $\mathbb{T}^2$. As above, the sequence $(\mathbb{T}^2, g_i)_{i\in\mathbb{N}}$ is contained in $\mathcal{M}(2, 2\pi)$. As $i \to \infty$ the torus just gets smaller and smaller and in the limit it collapses to a point.



To the author's knowledge, the first nontrivial example of collapse with bounded curvature was pointed out by Marcel Berger in 1962. In the following we discuss this example in detail.

Example 1.14. Let $S^3 := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ be the unit 3-sphere. The circle S^1 acts on S^3 by multiplication,

$$\begin{aligned} \phi : S^1 \times S^3 &\to S^3, \\ (\theta, (z, w)) &\mapsto (\theta \cdot z, \theta \cdot w). \end{aligned} \tag{1.14.1}$$

The orbits of this S^1-action are the *Hopf circles*. The corresponding *Hopf map* is defined as

$$\begin{aligned} f : S^3 &\to S^2, \\ (z, w) &\mapsto zw^{-1}. \end{aligned} \tag{1.14.2}$$

It is easy to check that f is a submersion and that the preimage of any $p \in S^2$ is a Hopf circle. Moreover, $f : S^3 \to S^2$ is an S^1-principal bundle called the *Hopf fibration*. If g is the standard metric on S^3 and (S^2, h) is the round sphere of radius $\frac{1}{2}$, then f is a Riemannian submersion.

Now we want to construct a sequence of metrics g_i on S^3 such that $(S^3, g_i)_{i\in\mathbb{N}}$ converges to the round 2-sphere S^2 of radius $\frac{1}{2}$ in the Gromov-Hausdorff topology as $i \to \infty$. It is well-known that S^3 is diffeomorphic to SU(2). Let X, Y, Z be a basis of the Lie algebra $\mathfrak{su}(2)$ such that X is parallel to the fibers of the Hopf map (1.14.2) and such that

$$[X, Y] = 2Z, \ [Y, Z] = 2X, \ [Z, X] = 2Y.$$

Considering the dual basis X^*, Y^* and Z^* we define for each $i \in \mathbb{N}$ the metric

$$g_i = \frac{1}{i^2} X^* \otimes X^* + Y^* \otimes Y^* + Z^* \otimes Z^*.$$

For $i = 1$ the metric g_1 is the standard metric on S^3. Furthermore, the metrics g_i are left-invariant under the S^1-action (1.14.1), for any $i \in \mathbb{N}$. A straightforward calculation shows that the sectional curvatures of g_i are given by

$$\sec^i(X, Y) = \frac{1}{i^2}, \ \sec^i(X, Z) = \frac{1}{i^2}, \ \sec^i(Y, Z) = 4 - \frac{3}{i^2}.$$

Hence, $(S^3, g_i)_{i\in\mathbb{N}}$ is a collapsing sequence with bounded curvature and diameter whose Gromov-Hausdorff limit is the round 2-sphere S^2 of radius $\frac{1}{2}$.

In general, the limit of a collapsing sequence has singularities. Such examples can already be constructed by modifying the above example of the collapsing Hopf fibration.

Example 1.15. Let $\frac{p}{q}$ be a complete reduced fraction. We consider the circle action on S^3 defined via

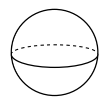
$$\begin{aligned} \phi_{pq} : S^1 \times S^3 &\to S^3, \\ (\theta, (z, w)) &\mapsto (\theta^p \cdot z, \theta^q \cdot w). \end{aligned} \tag{1.15.1}$$

Analogous to Example 1.14, we take a global frame X, Y, Z of S^3 such that X is parallel to the orbits of the S^1-action (1.15.1) and Y and Z are perpendicular to the orbits with respect to the standard metric on S^3. Let X^*, Y^* and Z^* be the dual basis. For any $i \in \mathbb{N}$ we define the metric

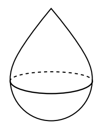
$$g_i = \frac{1}{i^2} X^* \otimes X^* + Y^* \otimes Y^* + Z^* \otimes Z^*.$$

The sequence $(S^3, g_i)_{i \in \mathbb{N}}$ collapses with bounded curvature and diameter to a compact metric space (Y, d_Y). Depending on the fraction $\frac{p}{q}$ we obtain the following different limit spaces:

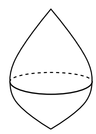
If $p = q = 1$ then the sequence $(S^3, g_i)_{i \in \mathbb{N}}$ coincides with the collapsing sequence in Example 1.14. Thus, the limit space (Y, d_Y) is the round 2-sphere of radius $\frac{1}{2}$.



If $p \neq 1$ and $q = 1$ then (Y, d_Y) is a Riemannian orbifold with one singular point at the north pole. The singularity at the north pole is locally isometric to the disk modulo the $\mathbb{Z}_p$-action of rotations around the origin. Such orbifolds are also called "teardrop" orbifolds.



If $p \neq 1$ and $q \neq 1$ then (Y, d_Y) is a Riemannian orbifold with two singular points at the poles. At the north pole (Y, d_Y) is locally isometric to the disk modulo the $\mathbb{Z}_p$-action of rotations around the origin and at the south pole (Y, d_Y) is locally isometric to the disk modulo the $\mathbb{Z}_q$-action of rotations around the origin.



We see that, if $p \neq q$ then the limit space (Y, d_Y) is not a manifold but an orbifold with singular points. In particular, we observe that (Y, d_Y) does not have to be a manifold.

Before we study collapse in $\mathcal{M}(n, d)$ in general we first discuss the special case of sequences $(M_i, g_i)_{i \in \mathbb{N}}$ in $\mathcal{M}(n, d)$ converging to a Riemannian manifold (B, h) of lower dimension. Fukaya studied such collapsing sequences in [Fuk87b, Fuk89] and summarized their behavior in his fibration theorem.

Notation 1.16. Here and subsequently $\tau(\varepsilon | x_1, \ldots, x_k)$ denotes a non negative continuous function such that for any fixed choice of $x_1, \ldots, x_k$, $\lim_{\varepsilon \to 0} \tau(\varepsilon | x_1, \ldots, x_k) = 0$. During calculations, the explicit value of τ might change. Since we are only interested in the behavior as $\varepsilon \to 0$ we omit putting indices if the explicit expression of τ changes.

Theorem 1.17. *For any integer n and positive constant μ there is a positive constant $\varepsilon(n,\mu)$ such that for any two closed Riemannian manifold (M,g) and (B,h) with*

$$\dim(B) \leqslant \dim(M) = n,$$
$$|\sec(M)| \leqslant 1,\ |\sec(B)| \leqslant 1,$$
$$\operatorname{inj}(B) \geqslant \mu,$$

that are ε-close in the Gromov-Hausdorff sense, i.e. $d_{GH}(M,B) \leqslant \varepsilon < \varepsilon(n,\mu)$, then there is a map $f : M \to B$ such that

(1) (M,B,f) is a fiber bundle,

(2) the fibers of f are diffeomorphic to a connected infranilmanifold Z,

(3) the structure group of the fibration lies in $\operatorname{Aff}(Z)$,

(4) f is an almost Riemannian submersion, i.e. if X is perpendicular to a fiber of f then

$$e^{-\tau(\varepsilon)} \leqslant \frac{|df(X)|}{|X|} \leqslant e^{\tau(\varepsilon)},$$

(5) the second fundamental form of the fibers is bounded by a positive constant $c(n)$.

Definition 1.18. Z is an *infranilmanifold* if it is diffeomorphic to the quotient ${}_{\Gamma}\backslash^{N}$, where N is a connected and simply-connected nilpotent Lie group and Γ is a cocompact discrete subgroup of $\operatorname{Aff}(N) = N_L \rtimes \operatorname{Aut}(N)$. Here N_L is the group of left-translations acting on N and $\operatorname{Aut}(N)$ is the automorphism group of N. Furthermore, $\operatorname{Aff}(Z)$ denotes those diffeomorphisms of Z that lift to diffeomorphisms in $\operatorname{Aff}(N)$.

We refer to Appendix A for more details and the basic properties of infranilmanifolds.

If B is a point, Theorem 1.17 coincides with Gromov's theorem on almost flat manifolds [Gro78, Ruh82].

Theorem 1.19. *For any $n \in \mathbb{N}$ there is an $\varepsilon(n) > 0$ such that any closed n-dimensional Riemannian manifold (M,g) with $\operatorname{diam}(M) = 1$ and $|\sec| \leqslant \varepsilon(n)$ is diffeomorphic to an infranilmanifold ${}_{\Gamma}\backslash^{N}$. Furthermore, there is a positive constant $w(n)$ such that we have $[\Gamma : \Gamma \cap N_L] \leqslant w(n)$.*

At this point we want to remark that any infranilmanifold Z admits a sequence of metrics $(g_\varepsilon)_{\varepsilon \leqslant 1}$ such that the sectional curvature of (Z, g_ε) is uniformly bounded in ε and $(Z, g_\varepsilon)_{\varepsilon \leqslant 1}$ collapses to a point as $\varepsilon \to 0$. In the following example we show how such a sequence of metrics is constructed on a nilpotent Lie group N. As the universal cover of any infranilmanifold is a connected and simply-connected nilpotent Lie group, the construction of the following example can be modified to the case of infranilmanifolds.

Example 1.20. Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}$, i.e. there is a k such that its lower central series

$$\mathfrak{n}_1 = \mathfrak{n},\ \mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}_1],\ \mathfrak{n}_3 = [\mathfrak{n}, \mathfrak{n}_2], \dots$$

terminates at $\mathfrak{n}_{k+1} = 0$. It is an easy observation that $\mathfrak{n}_{i+1} \subset \mathfrak{n}_i$ and that $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$.

For any left-invariant metric g the sectional curvature satisfies the inequality

$$|R(X,Y)Z|_g \leqslant 6\|\operatorname{ad}\|_g^2 |X|_g |Y|_g |Z|_g$$

for all vector fields X, Y, Z, where

$$\|\operatorname{ad}\|_g := \max\{|[X,Y]|_g : |X|_g = |Y|_g = 1, X, Y \in \mathfrak{n}\}.$$

To construct a collapsing sequence of metrics with uniform bounded curvature we first fix a left-invariant metric g_1 on N. Let $E_k := \{e_{\alpha_k}\}_{\alpha_k \in A_k}$ be an orthonormal basis of $\mathfrak{n}_k$. Then there is an orthonormal set of vectors $E_{k-1} := \{e_{\alpha_{k-1}}\}_{\alpha_{k-1} \in A_{k-1}}$ such that $E_k \cup E_{k-1}$ is an orthonormal basis for $\mathfrak{n}_{k-1}$. In that way we construct an orthonormal basis $\bigcup_{i=1}^k E_i$ for $\mathfrak{n}$ such that $\bigcup_{i=j}^k E_i$ is an orthonormal basis for $\mathfrak{n}_j$ for any $j \in \{1, \dots, k\}$. Since $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$ it follows that for all $e_{\alpha_i} \in E_i$ and $e_{\beta_j} \in E_j$ their Lie bracket is determined by their Lie algebra coefficients $\{\tau_{\alpha_i \beta_j}^{\gamma_l}\}_{\alpha_i, \beta_j, \gamma_l}$, defined by

$$[e_{\alpha_i}, e_{\beta_j}] = \sum_{l \geqslant i+j} \sum_{\gamma_l = 1}^{A_l} \tau_{\alpha_i \beta_j}^{\gamma_l} e_{\gamma_l}.$$

For any $\varepsilon > 0$ we define the metric g_ε via

$$g_\varepsilon(e_{\alpha_i}, e_{\alpha_i}) = \varepsilon^{2i}$$

for all $e_{\alpha_i} \in E_i$ and $1 \leqslant i \leqslant k$. It follows immediately that $\{\varepsilon^{-i} e_{\alpha_i}\}_{\substack{\alpha_i \in A_i \\ 1 \leqslant i \leqslant k}}$ is an orthonormal basis for g_ε. This kind of scaling is called an *inhomogeneous scaling.*

The sequence $(N, g_\varepsilon)_\varepsilon$ converges to a point in the Gromov-Hausdorff topology as $\varepsilon \to 0$. Furthermore, the sectional curvature of (N, g_ε) remains bounded since

$$\begin{aligned} \frac{1}{|e_{\alpha_i}|_{g_\varepsilon}} \cdot \frac{1}{|e_{\beta_j}|_{g_\varepsilon}} |\,[e_{\alpha_i}, e_{\beta_j}]\,|_{g_\varepsilon} &= \varepsilon^{-(i+j)} \left| \sum_{l \geqslant i+j} \sum_{\gamma_l = 1}^{A_l} \tau_{\alpha_i \beta_j}^{\gamma_l} e_{\gamma_l} \right|_{g_\varepsilon} \\ &\leqslant \varepsilon^{-(i+j)} \varepsilon^{i+j} \left| \sum_{l \geqslant i+j} \sum_{\gamma_l = 1}^{A_l} \tau_{\alpha_i \beta_j}^{\gamma_l} \right| \leqslant \|\operatorname{ad}\|_{g_1}, \end{aligned}$$

for all $e_{\alpha_i} \in E_i$, $e_{\beta_j} \in E_j$ and $\varepsilon \in (0,1]$. In particular, $\|\operatorname{ad}\|_{g_\varepsilon} \leqslant \|\operatorname{ad}\|_{g_1}$ for all $\varepsilon \leqslant 1$ and therefore

$$|\sec^{g_\varepsilon}| \leqslant 6\|\operatorname{ad}\|_{g_\varepsilon}^2 \leqslant 6\|\operatorname{ad}\|_{g_1}^2.$$

As we have seen in Example 1.15 the limit of a collapsing sequence can have singularities. Therefore, we can not apply Theorem 1.17 directly to an arbitrary collapsing sequence in $\mathcal{M}(n,d)$. In [Fuk88] Fukaya dealt with this problem in the following way: Instead of studying a convergent sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ he considered the associated sequence of orthogonal frame bundles FM_i. We recall that the orthogonal frame bundle FM of an n-dimensional Riemannian manifold (M,g) is defined as

$$FM := \bigsqcup_{p\in M} \{A : T_pM \to \mathbb{R}^n : \ A \text{ is an isometry}\}.$$

Clearly, FM is an $\mathrm{O}(n)$-principal bundle. Up to the choice of a biinvariant metric on $\mathrm{O}(n)$ there is a canonical metric g^F on FM such that the projection $\pi : FM \to M$ is a Riemannian submersion with totally geodesic fibers. By construction, the quotient manifold $FM/_{\mathrm{O}(n)}$ with the induced quotient metric is isometric to (M,g).

Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n,d)$ converging to a compact metric space Y in the Gromov-Hausdorff topology. It follows from Lemma 1.7 that there is a compact metric space $\tilde{Y}$ on which $\mathrm{O}(n)$ acts as isometries such that $\tilde{Y}/_{\mathrm{O}(n)}$ is isometric to Y, i.e.

$$\begin{tikzcd} (FM_i, \mathrm{O}(n)) \arrow[r, "d_{eq.\mathrm{GH}}"] \arrow[d, "\pi_i"'] & (\tilde{Y}, \mathrm{O}(n)) \arrow[d, "\pi"] \\ M_i \arrow[r, "d_{\mathrm{GH}}"] & Y \end{tikzcd}.$$

Fukaya showed that the metric space $\tilde{Y}$ is in fact a Riemannian manifold [Fuk88, Section 6 - 8] and that Theorem 1.17 can be generalized to the G-equivariant Gromov-Hausdorff topology [Fuk88, Theorem 9.1]. Therefore, the G-equivariant version of Theorem 1.17 can be applied to the sequence $(FM_i, g_i^F)_{i\in\mathbb{N}}$ of frame bundles.

Theorem 1.21. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n,d)$ converging with respect to the Gromov-Hausdorff metric to a compact metric space Y. For sufficiently large i there is a map $f_i : M_i \to Y$, and a compact metric space $\tilde{Y}$ on which $\mathrm{O}(n)$ acts isometrically, and an $\mathrm{O}(n)$-equivariant map $\tilde{f}_i : FM_i \to \tilde{Y}$ such that the diagram*

$$\begin{tikzcd} FM_i \arrow[r, "\tilde{f}_i"] \arrow[d, "\pi_i"'] & \tilde{Y} \arrow[d, "\pi"] \\ M_i \arrow[r, "f_i"] & Y \end{tikzcd}$$

commutes, and

(1) $\tilde{Y}$ is a Riemannian manifold with $C^{1,\alpha}$-metric tensors,

(2) $\tilde{f}_i$ is a fiber bundle with affine structure group and infranil fibers,

(3) $\tilde{f}_i$ is an almost Riemannian submersion, i.e. if $X \in T_xFM_i$ is perpendicular to the fibers of $\tilde{f}_i$ then

$$e^{-\tau(d_{GH}(M_i,Y)|n,d)} < \frac{|d\tilde{f}_i(X)|}{|X|} < e^{\tau(d_{GH}(M_i,Y)|n,d)},$$

(4) M_i *and* Y *are isometric to* ${}^{FM}\!/_{\mathrm{O}(n)}$ *and* ${}^{\tilde{Y}}\!/_{\mathrm{O}(n)}$ *respectively,*

(5) for each $p \in Y$ *the groups* $G_{\tilde{p}} = \{g \in O(n) | g(\tilde{p}) = \tilde{p}\}$ *for* $\tilde{p} \in \pi^{-1}(p)$ *are isomorphic to each other. We set* $G_p := G_{\tilde{p}}$ *for some fixed* $\tilde{p} \in \pi^{-1}(p)$.

Remark 1.22. Let $\mathcal{FM}(n,d)$ be the set of all isometry classes of frame bundles (FM, g^F) of Riemannian manifolds $(M,g) \in \mathcal{M}(n,d)$. There are positive constants $C_1(n)$ and $C_2(n)$ such that

$$\mathcal{FM}(n,d) \subset \mathcal{M}\left(n + \frac{(n-1)(n-2)}{2}, d + C_1(n) \,\middle|\, C_2(n)\right).$$

Recall that $\mathcal{M}(n,d\,|\,k)$ denotes the set of all isometry classes of closed n-dimensional Riemannian manifolds (M,g) with $\operatorname{diam}(M) \leqslant d$ and $|\sec^M| \leqslant k$. Let $\overline{\mathcal{FM}(n,d)}$ be the closure of $\mathcal{FM}(n,d)$. Then there is a further constant $C_3(n) > 0$ such that

$$\overline{\mathcal{FM}(n,d)} \cap \bigcup_{k=0}^{n+\frac{(n-1)(n-2)}{2}} \mathcal{M}(k,d\,|\,C_3(n))$$

is a dense subset of $\overline{\mathcal{FM}(n,d)}$ with respect to the Lipschitz distance, see [Fuk88, Theorem 6.1].

In [Fuk88, Theorem 0.5] it is shown that every limit space Y has a well-defined Hausdorff dimension $k \in \mathbb{N}$. Moreover, Y is a stratified space, i.e. $Y = S_0(Y) \supset S_1(Y) \supset \ldots \supset S_k(Y)$ such that $S_j(Y)\backslash S_{j+1}(Y)$ is a $(k-j)$-dimensional smooth Riemannian manifold. Using this structure, we can say a little bit more about the fibers of the singular fibrations $f_i : M_i \to Y$ (see also [Fuk88, Theorem 0.12]).

Corollary 1.23. *Let* $(M_i, g_i)_{i\in\mathbb{N}}$ *be a convergent sequence in* $\mathcal{M}(n,d)$ *with limit space* Y *and let* $f_i : M_i \to Y$ *be the singular fibrations from Theorem 1.21. Setting* $k := \dim_{Haus}(Y)$ *we have that*

(1) for any $j \in \{0, \ldots, k\}$, *the restriction of* f_i *to* $f_i^{-1}(S_j(Y)\backslash S_{j+1}(Y))$ *is a fiber bundle with infranil fibers,*

(2) for any $p \in Y\backslash S_1(Y)$, G_p *acts freely on the fiber* $F_i = \tilde{f}_i^{-1}(\tilde{p})$, *where* $\tilde{f}_i : FM_i \to \tilde{Y}$ *and* $\pi(\tilde{p}) = p$. *Here* G_p *is defined as in Theorem 1.21. In particular, the fiber* $f_i^{-1}(p)$ *is diffeomorphic to the quotient space* ${}^{F_i}\!/_{G_p}$,

Another approach to study the structure of collapse with bounded curvature was carried out by Cheeger and Gromov [CG86, CG90]. They generalized local group actions and introduced an action of a sheaf of groups. In particular, they considered actions of sheaves of tori with additional regularity conditions. This defines the so-called F-structure (where "F" stands for *flat*). Cheeger and Gromov proved that each sufficiently collapsed complete Riemannian manifold admits an F-structure of positive rank. An advantage of this approach is that no uniform bound on the diameter is required. However, the

Hausdorff dimension of the limit space of a collapsing sequence without a uniform diameter bound is not necessarily well-defined (see for instance [CG86, Example 0.2]).

Combining these two approaches, Cheeger, Fukaya and Gromov defined in [CFG92] the notion of nilpotent structures (N-structures) and showed that they exist on each sufficiently collapsed part of a complete Riemannian manifold (M, g) with $|\sec| \leqslant 1$. Roughly speaking, if M is sufficiently collapsed, its frame bundle FM is locally the total space of a fibration with infranil fibers and affine structure group. Thus, there is a sheaf on FM whose local sections are given by locally defined right invariant vector fields on the fiber. These local right invariant vector fields represent the locally defined left multiplications by the simply connected nilpotent Lie group covering the infranil fiber. Similar to [CG86, CG90] no upper bound on the diameter is required.

A further main result of [CFG92] is the existence of *invariant metrics* on manifolds admitting an N-structure. These metrics are invariant in the sense that the local sections of the sheaf on the frame bundle are given by locally defined Killing vector fields. To construct such a metric the authors first applied the following theorem due to Abresch [Abr88, Theorem 1.1] to obtain uniform bounds on the derivatives of the curvature (see also [Shi89, Theorem 1.2] for a proof using the Ricci flow).

Theorem 1.24. *For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there is a smoothing operator S_ε such that on any complete Riemannian manifold (M, g) with $|\sec^g| \leqslant 1$ the metric $\tilde{g} := S_\varepsilon(g)$ satisfies*

(1) $e^{-\varepsilon} g < \tilde{g} < e^{\varepsilon} g$,

(2) $|\nabla - \tilde{\nabla}| < \varepsilon$,

(3) $|\tilde{\nabla}^j \tilde{R}| < A_j(n, \varepsilon)$ *for all* $j \geqslant 0$.

In addition, Rong showed that, for sufficiently small ε, we have the following bounds on the sectional curvature of $S_\varepsilon(g)$, c.f. [Ron96, Proposition 2.5].

Proposition 1.25. *There is a constant $\delta(n) > 0$ such that for any complete Riemannian manifold (M, g) with $|\sec^g| \leqslant 1$ and any $0 \leqslant \varepsilon \leqslant \delta$, there is a positive constant $C(n)$ such that the metric $\tilde{g} := S_\varepsilon(g)$ satisfies*

$$\min \sec^{\tilde{g}} - C(n)\varepsilon \leqslant \sec^g \leqslant \max \sec^{\tilde{g}} + C(n)\varepsilon.$$

Let $A = (A_j)_{j \in \mathbb{N}}$ be a sequence of non negative numbers. Then a Riemannian manifold is called *A-regular* if $|\nabla^j R| \leqslant A_j$ for all $j \in \mathbb{N}$. Further, we denote by $C(A)$ a constant depending only on finitely many A_j.

Applying Theorem 1.24, it suffices to study sequences of A-regular manifolds. For such sequences, these so-called *invariant metrics* were constructed in [CFG92, Section 4, Section 7, Section 8]. Since we are interested in collapsing sequences in $\mathcal{M}(n, d)$ we summarize the results of [CFG92, Section 4, Section 7, Section 8] restricted to this special case.

Theorem 1.26. *Let $(M_i, g_i)_{i \in \mathbb{N}}$ be a sequence of A-regular Riemannian manifolds in $\mathcal{M}(n, d)$ converging to a lower dimensional space Y. In addition, let $(\tilde{Y}, h)$ be the* $\mathrm{O}(n)$*-equivariant Gromov-Hausdorff limit of the corresponding sequence $(FM_i, g_i^F)_{i \in \mathbb{N}}$ of frame* *bundles. For any i sufficiently large there is an* $\mathrm{O}(n)$*-invariant metric* $\tilde{g}_i^F$ *on* FM_i *which induces a Riemannian metric* $\tilde{g}_i$ *on* M_i *and an* $\mathrm{O}(n)$*-invariant metric* $\tilde{h}_i$ *on* $\tilde{Y}$ *such that for all* $j \geqslant 0$,

$$\begin{aligned} |\nabla^j(g_i^F - \tilde{g}_i^F)| &\leqslant c(n, A, j) d_{GH}(M, Y), \\ |\nabla^j(h - \tilde{h}_i)| &\leqslant c'(n, A, j) d_{GH}(M, Y), \end{aligned}$$

and such that the map $\tilde{f}_i : (FM_i, \tilde{g}_i^F) \to (\tilde{Y}, \tilde{h}_i)$ *is a Riemannian submersion. In addition, the second fundamental form of the fibers is bounded by a positive constant* $C(n)$ *and for any* $p \in \tilde{Y}$*, the induced metric on the infranil fiber* Z_p *is affine parallel, i.e. it lifts to a left-invariant metric on its universal cover.*

From Theorem 1.26, Theorem 1.24, and Proposition 1.25 the next lemma follows immediately.

Lemma 1.27. *Let* $(M_i, g_i)_{i\in\mathbb{N}}$ *be a collapsing sequence in* $\mathcal{M}(n, d)$*. Then there is an index* $I > 0$ *such that for all* $i \geqslant I$ *there is an invariant metric* $\tilde{g}_i$ *on* M_i *in the sense of the above theorem such that*

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0.$$

Furthermore, the sectional curvature and the diameter of the sequence $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$ *are uniformly bounded.*

Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{M}(n, d)$ with limit space Y such that the metrics g_i are invariant in the sense of Theorem 1.26. We observe that the corresponding sequence $(FM_i, g_i^F)_{i\in\mathbb{N}}$ can be viewed as a sequence of special fiber bundles, see Theorem 1.21 and Theorem 1.26. To simplify notation we summarize the properties of these arising fiber bundles.

Definition 1.28. A fiber bundle $f : (M, g) \to (B, h)$ is called a *Riemannian affine fiber bundle* if

- f is a Riemannian submersion,
- for each $p \in B$ the fiber $Z_p := f^{-1}(p)$ is an infranilmanifold with an induced affine parallel metric $\hat{g}_p$,
- the structure group lies in the group of affine diffeomorphisms, $\mathrm{Aff}(Z)$.

In the following corollary we summarize the results of Theorem 1.21, Theorem 1.26 and Lemma 1.27.

Corollary 1.29. *Let* $(M_i, g_i)_{i\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}(n, d)$ *converging to a compact metric space* Y *in the Gromov-Hausdorff topology. Then there is a Riemannian manifold* $\tilde{Y}$ *with* *a $C^{1,\alpha}$-metric h on which $\mathrm{O}(n)$ acts as isometries such that ${}^{\tilde{Y}}/_{\mathrm{O}(n)}$ is isometric to Y and an $I > 0$ such that for any $i \geqslant I$ the following diagram commutes.*

$$\begin{array}{ccc} FM_i & \xrightarrow{\tilde{f}_i} & \tilde{Y} \\ {\scriptstyle \pi_i}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M_i & \xrightarrow{f_i} & Y\,. \end{array} \tag{1.29.1}$$

Furthermore, the index I can be chosen such that for all $i \geqslant I$ there is an invariant metric $\tilde{g}_i$ on M_i and an $\mathrm{O}(n)$-invariant metric $\tilde{h}_i$ on $\tilde{Y}$ such that $\tilde{f}_i : (FM_i, \tilde{g}_i^F) \to (\tilde{Y}, \tilde{h}_i)$ is an $\mathrm{O}(n)$-equivariant Riemannian affine fiber bundle and

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0,$$
$$\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0.$$

Moreover, the second fundamental forms of the fibers of the Riemannian submersions $\tilde{f}_i$ are uniformly bounded in norm by a positive constant $C(n)$ and the sectional curvature of $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$ are uniformly bounded by a positive constant $K(n)$.

Remark 1.30. If a sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ converges to a Riemannian manifold (Y, h) then the above corollary simplifies to the statement that for all $i \geqslant I$ there are metrics $\tilde{g}_i$ on M_i and $\tilde{h}_i$ on Y such that $f_i : (M_i, \tilde{g}_i) \to (Y, \tilde{h}_i)$ is Riemannian affine fiber bundles with $(M_i, \tilde{g}_i) \in \mathcal{M}(n, d \,|\, K(n))$ for all $i \geqslant I$ and

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0,$$
$$\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0.$$

Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{M}(n,d)$ with limit space Y. In general, Y is not a Riemannian manifold, see for instance Example 1.15. We recall from Corollary 1.29 that Y is isometric to the quotient ${}^{\tilde{Y}}/_{\mathrm{O}(n)}$ of a Riemannian manifold $\tilde{Y}$. In particular, all singularities of Y have at least quotient singularity structure.

In Theorem 1.21 we defined for any $p \in Y$ the group G_p which is the stabilizer group of the $\mathrm{O}(n)$-action on $\tilde{Y}$. In [NT11, Theorem 2.1] it is shown that $\dim(G_p) < n - \dim(Y)$. Thus, it follows that as the difference $n - \dim_{\mathrm{Haus}}(Y)$ gets large the possible singularities that can occur get worse.

If $n - \dim(Y) = 1$ then $\dim(G_p) = 0$ for all $p \in Y$. In particular, the group G_p is finite for all $p \in Y$. This leads to the following proposition which was already proven in [Fuk90, Proposition 11.5].

Proposition 1.31. *Any $(n-1)$-dimensional metric space Y in the boundary of $\mathcal{M}(n,d)$ is a Riemannian orbifold Y with a $C^{1,\alpha}$-metric h.*

Loosely speaking, Riemannian orbifolds are locally modeled as quotients of Riemannian manifolds by a finite isometric group action. As Riemannian orbifolds are, by definition, locally finitely covered by Riemannian manifolds many geometric concepts like

Toponogov's triangle comparison and Bishop-Gromov volume comparison carry over to Riemannian orbifolds. For a thorough introduction to Riemannian orbifolds we refer the reader to [BG08, Chapter 4], [Thu80, Chapter 13].

If $n - \dim_{\mathrm{Haus}}(Y) \geqslant 2$ we do not know, a priori, if the group G_p is trivial, finite or infinite. In [NT11, Theorem 1.1], the authors characterize the set of singularities as follows.

Theorem 1.32. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n,d)$ converging to a compact metric space Y in the Gromov-Hausdorff topology. Then there is a closed set S of Hausdorff dimension*

$$\dim_{\mathrm{Haus}}(S) \leqslant \min\{n-5, \dim(Y)-3\}$$

such that $Y \backslash S$ is a Riemannian orbifold with a $C^{1,\alpha}$-metric h.

The upper bound on the Hausdorff dimension of the singular set S is sharp as the authors show in various examples [NT11, Example 1.1 - 1.4].

Chapter 2

Codimension one collapse

We say that a sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ is a *codimension one collapse* if $(M_i, g_i)_{i\in\mathbb{N}}$ converges to an $(n-1)$-dimensional compact metric space Y. In this special case we already know that Y is a Riemannian orbifold with a $C^{1,\alpha}$-metric, see Proposition 1.31. As shown in Theorem 1.32 this is a special case because if the limit space Y of a collapsing sequence in $\mathcal{M}(n,d)$ has $\dim_{\mathrm{Haus}} Y < (n-1)$ then Y has in general non orbifold singularities. Since Riemannian orbifolds are locally modeled as quotients of Riemannian manifolds by a finite isometric group action many geometric concepts carry over to Riemannian orbifolds. This motivates the main result of [Roo18a] which characterizes codimension one collapse in $\mathcal{M}(n,d)$ and proves Theorem 0.1.

Theorem 2.1. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n,d)$ converging to a compact metric space Y in the Gromov-Hausdorff topology. Then the following are equivalent*

(1) $\dim_{\mathrm{Haus}}(Y) \geqslant (n-1)$,

(2) for all $r > 0$ there is a positive constant $C(n,r,Y)$ such that

$$C \leqslant \frac{\mathrm{vol}(B_r^{M_i}(x))}{\mathrm{inj}^{M_i}(x)} \tag{2.1.1}$$

holds for all $x \in M_i$ and $i \in \mathbb{N}$,

(3) for some $r > 0$ there is a positive constant $C(n,r,Y)$ such that the inequality (2.1.1) *holds for all $x \in M_i$ and $i \in \mathbb{N}$.*

The idea behind Theorem 2.1 is the following illustrative observation. Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{M}(n,d)$. Then the r-balls around a sequence of points $x_i \in M_i$ contain all collapsing directions, while the injectivity radii at the points x_i only represent the fastest scale of collapse. If we have a codimension one collapse then the collapse happens on the scale of the injectivity radius. Hence, the volume of the balls $B_r^{M_i}(x_i)$ and the injectivity radii $\mathrm{inj}^{M_i}(x_i)$ converge to 0 at the same rate. In particular, the ratio (2.1.1) can be uniformly bounded from below. However, if the collapse has codimension larger than or equal to two then the injectivity radius "sees" only a part of the collapse. Thus, the volume of the balls $B_r^{M_i}(x_i)$ converges on a larger scale to 0 than the injectivity radii $\mathrm{inj}^{M_i}(x_i)$, i.e. the quotients (2.1.1) converge to 0 as $i \to \infty$.

Example 2.2. Consider the sequence $(\mathbb{T}^2, g_i = g_S \oplus \frac{1}{i} g_S)_{i\in\mathbb{N}}$ of flat tori from Example 1.12. This sequence collapses to S^1. As $\mathrm{inj}^{g_i}(x) \equiv \frac{\pi}{i}$, it follows that for any $r > 0$ and $x \in \mathbb{T}^2$, $\mathrm{vol}(B_r^{g_i}(x)) \approx 2\min\{r, \pi\} \cdot \frac{2\pi}{i}$ as $i \to \infty$. Therefore, we derive for $r \leqslant \pi$ that

$$\lim_{i\to\infty} \frac{\mathrm{vol}(B_r^{g_i}(x))}{\mathrm{inj}^{g_i}(x)} = \lim_{i\to\infty} \frac{4r\frac{\pi}{i}}{\frac{\pi}{i}} = 4r > 0.$$

In particular, this quotient is uniformly bounded from below, as stated in Theorem 2.1.

Example 2.3. As in Example 1.13, we consider the sequence $(\mathbb{T}^2, g_i = \frac{1}{i} g_S \oplus \frac{1}{i} g_S)_{i\in\mathbb{N}}$ of flat tori. This sequence collapses to a point. No matter how small we choose $r > 0$ there exists some $I \in \mathbb{N}$ such that $\mathrm{vol}(B_r^{g_i}(x)) = \frac{2\pi}{i} \cdot \frac{2\pi}{i}$ for all $x \in \mathbb{T}^2$ and $i > I$. As $\mathrm{inj}^{g_i}(x) = \frac{\pi}{i}$ for all $x \in \mathbb{T}^2$, we conclude that

$$\lim_{i\to\infty} \frac{\mathrm{vol}(B_r^{g_i}(x))}{\mathrm{inj}^{g_j}(x)} = \lim_{i\to\infty} \frac{4\pi}{i} = 0.$$

Thus, we cannot find a uniform positive lower bound for this quotient.

The proof of Theorem 2.1 is divided into three steps. First, we show that, in order to prove Theorem 2.1, it suffices to restrict to sequences of manifolds with invariant metrics, as introduced in Theorem 1.26. Next, we show that (1) implies (2) by constructing a lower bound as required in (2.1.1) for any given $r > 0$. As the implication from (2) to (3) is trivial it remains to show that (3) implies (1). This direction will be proved by contradiction. By Corollary 1.29 any collapsing sequence in $\mathcal{M}(n,d)$ is a sequence of collapsing singular fibrations over the limit space. Thus, we will bound the volume of $B_r^{M_i}(x)$, up to a constant, from above by the injectivity radius and the diameter of the corresponding collapsing fiber. It remains to bound the injectivity radius of the collapsing fiber from above by the injectivity radius of the manifold in x. This is done by modifying results of [Tap00] for bounded Riemannian submersions. In the end, we show that the constructed upper bound on the quotient converges to 0, giving a contradiction.

The content of this chapter corresponds to [Roo18a, Section 3 and 4].

In Section 2.1 we show that under appropriate assumptions on the geometry of a Riemannian submersion $f : M \to Y$ there is a positive constant C such that we have $\mathrm{inj}(f^{-1}(p)) \leqslant C\,\mathrm{inj}^M(x)$ for all $x \in f^{-1}(p)$, if $\mathrm{inj}^M(x)$ is sufficiently small compared to the injectivity radius of Y. This proposition is essential to prove the direction (3) to (1) in Theorem 2.1.

Using this proposition together with the structure of collapsing sequences in $\mathcal{M}(n,d)$, see Corollary 1.29, we prove Theorem 2.1 in Section 2.2 with the strategy explained above.

In conclusion, we define the space $\mathcal{M}(n,d,C)$ consisting of all isometry classes of Riemannian manifolds in $\mathcal{M}(n,d)$ with $C \leqslant \frac{\mathrm{vol}(M)}{\mathrm{inj}(M)}$. We show that there is a uniform bound on the essential supremum of the sectional curvature and a uniform lower bound on the volume for all $(n-1)$-dimensional limit spaces of $\mathcal{M}(n,d,C)$.

2.1 The injectivity radius of a fiber

Let $f : (M,g) \to (Y,h)$ be a Riemannian submersion between two Riemannian manifolds. Henceforth, we denote the fiber over $p \in Y$ by $F_p := f^{-1}(p)$ and $k := \dim(F_p)$. It is well-known that $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal distribution isomorphic to f^*TY and $\mathcal{V} = \ker(df)$ is the vertical distribution. The relations between the curvatures of (M, g),(B, h) and the fibers $(F_p, \hat{g}_p)$ are given by O'Neill's formulas, see Theorem B.6. These formulas involve the two fundamental tensors T and A which are defined as

$$\begin{aligned} T(X, Y) &:= \left(\nabla_{X^V} Y^V\right)^H + \left(\nabla_{X^V} Y^H\right)^V, \\ A(X, Y) &:= \left(\nabla_{X^H} Y^V\right)^H + \left(\nabla_{X^H} Y^H\right)^V, \end{aligned} \tag{2.3.1}$$

for all vector fields $X, Y \in \Gamma(TM)$. Here X^V, X^H denote the vertical, resp. horizontal part of the vector field X. Roughly speaking, the T-tensor is related to the second fundamental form of the fibers and the A-tensor vanishes if and only if the horizontal distribution $\mathcal{H}$ is integrable. A Riemannian submersion is called *bounded* if the fundamental tensors A and T are bounded in norm by positive constants C_A resp. C_T.

The goal of this section is to show the following proposition that is essential for the proof of Theorem 2.1 (see Notation 1.16 for the τ-notation).

Proposition 2.4. *Let $f : M^{n+k} \to Y^n$ be a bounded Riemannian submersion such that $|\sec^M| \leqslant K$ for some $K > 0$. If $\mathrm{inj}^M(x) < \min\{\frac{\pi}{\sqrt{K+3C_A^2}}, \frac{1}{4}\mathrm{inj}^Y(p)\}$ for some $x \in F_p$ then*

$$\mathrm{inj}(F_p) \leqslant \Big(1 + \tau(\mathrm{inj}^M(x)|C_A, C_T, k, K)\Big) \cdot \mathrm{inj}^M(x).$$

The explicit expression of the constant $\Big(1+\tau(\mathrm{inj}^M(x)|C_A, C_T, k, K)\Big)$ is given in the proof.

The main ingredient of the proof of Proposition 2.4 is a homotopy with fixed endpoints between a curve γ with endpoints in a fiber F_p and a curve $\tilde{\gamma}$ lying completely in the fiber F_p. Such a homotopy was constructed in the proof of Theorem 3.1 in [Tap00].

Proposition 2.5. *Let $f : M \to Y$ be a bounded Riemannian submersion with Y being compact and simply-connected. Then there exists a positive constant $C := C(Y, k, C_T, C_A)$ such that any curve $\gamma : [a, b] \to M$ with endpoints in the fiber F_p, $p = f(\gamma(a))$, is homotopic to a curve $\tilde{\gamma}$ in the fiber F_p satisfying*

$$l(\tilde{\gamma}) \leqslant C\, l(\gamma).$$

We observe that there are a few differences between the assumptions of Proposition 2.5 and the assumptions of Proposition 2.4. First, in Proposition 2.5, Tapp requires Y to be compact and simply-connected. These assumptions are needed to guarantee that for any loop $\alpha : [0, 1] \to Y$ there is a nullhomotopy $H : [0, 1] \times [0, 1] \to Y$, i.e. H satisfies $H(1, t) = \alpha(t)$, $H(0, t) = \alpha(0)$ and $H(s, 0) = H(s, 1) = \alpha(0)$ for all $s \in [0, 1]$, whose derivatives are uniformly bounded [Tap00, Lemma 7.2]. As we are only interested in the local behavior around a chosen fiber F_p of the Riemannian submersion $f : M \to Y$ it suffices to consider a small neighborhood of $p \in Y$. It follows from the assumptions of Proposition 2.4 that the considered non contractible geodesic loop γ based at $x \in F_p$ has length $l(\gamma) = 2\,\mathrm{inj}^M(x) < \frac{1}{2}\mathrm{inj}^Y(p)$. Since the ball $B_{\frac{1}{4}\mathrm{inj}^Y(p)}(p)$ is convex and contractible, the loop $f \circ \gamma$ is contractible in Y. Furthermore, by assuming a bound on the sectional curvature of Y, there is a nullhomotopy with bounded derivatives for curves with length less than $\frac{1}{2}\mathrm{inj}^Y(p)$.

Lemma 2.6. *Let Y be a Riemannian manifold with $-\lambda^2 \leqslant \sec^Y \leqslant \Lambda^2$ for some $\lambda, \Lambda > 0$. Furthermore, let $\alpha : [0,1] \to Y$ be a loop in Y based at p and $l(\alpha) < \min\{\frac{2\pi}{\Lambda}, \frac{1}{2}\operatorname{inj}^Y(p)\}$. Then there is a smooth nullhomotopy $H : [0,1] \times [0,1] \to Y$, i.e. $H(0,t) = p$ and $H(1,t) = \alpha(t)$ and $H(s,0) = H(s,1) = p$ for all $s \in [0,1]$, such that*

$$\left|\frac{\partial}{\partial t}H\right| \leqslant \frac{\Lambda}{\lambda} \cdot \frac{\sinh(\lambda\, \frac{l(\alpha)}{2})}{\sin(\Lambda\, \frac{l(\alpha)}{2})} \cdot l(\alpha),$$

$$\left|\frac{\partial}{\partial s}H\right| \leqslant \frac{\sinh(\lambda\, \frac{l(\alpha)}{2})}{\lambda\frac{l(\alpha)}{2}} \cdot \frac{l(\alpha)}{2},$$

for all $s, t \in [0,1]$.

Proof. Let α be parametrized proportional to arclength, such that $\left|\frac{\partial\alpha}{\partial t}\right| \equiv l(\alpha)$. Since α satisfies $l(\alpha) < \frac{1}{2}\operatorname{inj}^Y(p)$, it lifts to a contractible loop $\tilde{\alpha} := \exp_p^{-1} \circ \alpha$ in T_pY. Thus, the nullhomotopy $\tilde{H}(s,t) := s \cdot \tilde{\alpha}(t)$ with $s, t \in [0,1]$ is well-defined. Clearly, we have that

$$\left|\frac{\partial}{\partial s}\tilde{H}\right| = |\tilde{\alpha}| \leqslant \frac{l(\alpha)}{2}.$$

To estimate $\left|\frac{\partial}{\partial t}\tilde{H}\right|$, we first observe that it follows from the assumption $-\lambda^2 \leqslant \sec^Y \leqslant \Lambda^2$ that

$$\frac{\sin(\Lambda|v|)}{\Lambda|v|}|w| \leqslant |(\mathrm{D}_v \exp_p)(w)| \leqslant \frac{\sinh(\lambda|v|)}{\lambda|v|}|w|$$

for all $v \in T_pY$ with $|v| < \frac{\pi}{\Lambda}$ and $w \in T_vT_pY$, see e.g. [Jos05, Corollary 5.6.1]. Therefore, we obtain for $q \in B_{\frac{1}{4}\operatorname{inj}^Y(p)}(p)$ with $d(p,q) < \frac{\pi}{\Lambda}$ and $u \in T_qY$ that

$$|(\mathrm{D}_q \exp_p^{-1})u| \leqslant \frac{\Lambda|\exp_p^{-1}(q)|}{\sin(\Lambda|\exp_p^{-1}(q)|)}|u|.$$

As by assumption $d(p, \alpha(t)) \leqslant \frac{l(\alpha)}{2} < \frac{\pi}{\Lambda}$, we conclude that

$$\left|\frac{\partial}{\partial t}\tilde{H}\right| \leqslant \left|\frac{\partial}{\partial t}\tilde{\alpha}\right| \leqslant \frac{\Lambda|\tilde{\alpha}|}{\sin(\Lambda|\tilde{\alpha}|)}\left|\frac{\partial}{\partial t}\alpha\right| \leqslant \frac{\Lambda\frac{l(\alpha)}{2}}{\sin(\Lambda\, \frac{l(\alpha)}{2})} \cdot l(\alpha).$$

By construction $H := \exp_p(\tilde{H})$ is a smooth nullhomotopy of α in Y with

$$\left|\frac{\partial}{\partial s}H\right| \leqslant \frac{\sinh(\lambda|\tilde{H}(s,t)|)}{\lambda|\tilde{H}(s,t)|}\left|\frac{\partial}{\partial s}\tilde{H}\right| \leqslant \frac{\sinh(\lambda\, \frac{l(\alpha)}{2})}{\lambda\frac{l(\alpha)}{2}} \cdot \frac{l(\alpha)}{2}.$$

The corresponding bound on $\left|\frac{\partial}{\partial t}H\right|$ follows similarly. $\square$

The next corollary follows immediately by adjusting the bounds on the derivative of the exponential map.

Corollary 2.7. *Let Y be a Riemannian manifold with $-\lambda^2 \leqslant \sec^Y \leqslant -\Lambda^2$ for some $\lambda \geqslant \Lambda \geqslant 0$. Furthermore, let $\alpha : [0,1] \to Y$ be a loop in Y based at p and $l(\alpha) < \frac{1}{2}\operatorname{inj}^Y(p)$. Then there is a smooth nullhomotopy $H : [0,1] \times [0,1] \to Y$, as in Lemma 2.6, such that*

$$\left|\frac{\partial}{\partial t}H\right| \leqslant \frac{\Lambda}{\lambda} \cdot \frac{\sinh(\lambda\,\frac{l(\alpha)}{2})}{\sinh(\Lambda\,\frac{l(\alpha)}{2})} \cdot l(\alpha),$$

$$\left|\frac{\partial}{\partial s}H\right| \leqslant \frac{\sinh(\lambda\,\frac{l(\alpha)}{2})}{\lambda\frac{l(\alpha)}{2}} \cdot \frac{l(\alpha)}{2}.$$

If $\Lambda = 0$, we set $\frac{\sinh(\Lambda)}{\Lambda} = 1$.

Next, we prove Proposition 2.4. Therein we keep carefully track of the dependence of the constants on $\operatorname{inj}^M(x)$ because this is the quantity going to 0 in a collapsing sequence while the other quantities will be uniformly bounded.

Proof (Proof of Proposition 2.4). We assume without loss of generality that the curve γ is parametrized proportional to arclength such that $|\dot{\gamma}| \equiv l(\gamma)$. Since $\operatorname{inj}^M(x) < \frac{\pi}{\sqrt{K}}$ there is a non contractible geodesic loop γ based at x such that $l(\gamma) = 2\operatorname{inj}^M(x)$. As $\operatorname{inj}^M(x) < \frac{1}{4}\operatorname{inj}^Y(p)$, the composition $f \circ \gamma$ is a contractible loop in Y. By Proposition 2.5, γ is homotopic to a non contractible loop $\tilde{\gamma}$ in the fiber F_p such that $l(\tilde{\gamma}) \leqslant C \cdot l(\gamma)$ for a positive constant $C := C(Y, k, C_A, C_T)$. Thus,

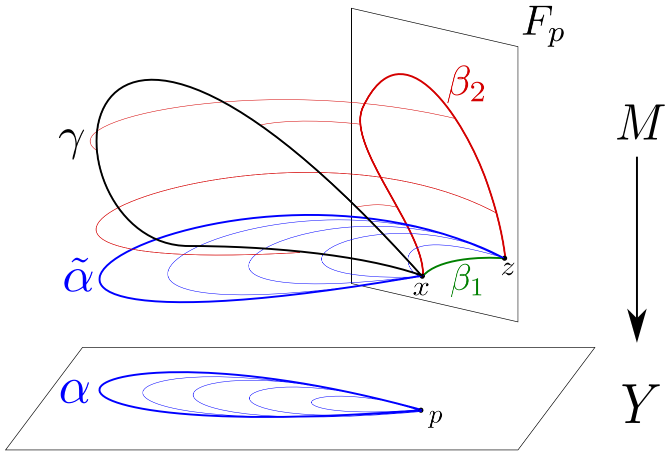
$$2\operatorname{inj}^{F_p} \leqslant l(\tilde{\gamma}) \leqslant C \cdot l(\gamma) = C \cdot 2\operatorname{inj}^M(x).$$

We claim that $C = \big(1 + \tau(l(\gamma)|C_A, C_T, k, K)\big)$. The proof consists of a careful study of the constant C, following the proof of [Tap00, Theorem 3.1]. In this proof, Tapp modifies the path γ such that it is a concatenation of paths $(\gamma_i)_i$ with endpoints in the fiber F_p such that $l(\gamma_i) < (2\operatorname{diam}(Y) + 1)$ for all i. Since in our case $l(\gamma) < (2\operatorname{diam}(Y) + 1)$ already holds by assumption we do not need this modification.

Set $\alpha(t) := f \circ \gamma$ and let $\tilde{\alpha}$ be the horizontal lift of $\alpha(t)$ with $\tilde{\alpha}(0) = \gamma(0)$. We observe that $\tilde{\alpha}(1) = h^\alpha(x) =: z$, where $h^\alpha : F_p \to F_p$ is the holonomy diffeomorphism associated to α. The vertical curve $\tilde{\gamma}$ is the concatenation of the paths β_1 and β_2 which are constructed as follows [Tap00, proof of Theorem 3.1]:

Let $H : [0,1] \times [0,1] \to Y$ be the nullhomotopy for α from Lemma 2.6. We lift H horizontally to a homotopy $\tilde{H} : [0,1] \times [0,1] \to M$ such that $\tilde{H}(1,t) = \tilde{\alpha}(t)$ and such that $\tilde{H}(s,i) = \tilde{\alpha}(i)$ for $i \in \{0,1\}$ and all $s \in [0,1]$. Then $\beta_1(t) := \tilde{H}(0,t)$ is a path in the fiber F_p connecting $x = \tilde{\alpha}(0)$ and $z = \tilde{\alpha}(1)$.

The path β_2 can be understood as a "horizontal transport" of γ to the fiber F_p. To be concrete, $\beta_2(t) := h^t(\gamma(t))$, where $h^t : F_{\alpha(t)} \to F_p$ is the holonomy diffeomorphism associated to $\alpha_{|[t,1]}$. By construction, $\beta_2 : [0,1] \to F_p$ is a path from z to x.



Tapp showed that the length of the vertical curve

$$\tilde{\gamma}(t) := \begin{cases} \beta_1(2t), & \text{if } t \leqslant \frac{1}{2} \\ \beta_2(2t-1), & \text{if } t > \frac{1}{2} \end{cases}$$

is bounded by

$$l(\tilde{\gamma}) = l(\beta_1) + l(\beta_2) \leqslant P \cdot l(\gamma) + L \cdot l(\gamma) = C \cdot l(\gamma)$$

for some explicit positive constants P and L, compare with [Tap00, p. 645]. We will study these constants P and L in detail.

First we consider the inequality $l(\beta_1) \leqslant P \cdot l(\gamma)$. The constant P is an upper bound on the derivative of the function $l \mapsto \rho_l(1)$ between $l = 0$ and $l = l(\gamma)$, where $\rho_{l(\gamma)}(t)$ is the solution to the differential equation

$$\begin{aligned} (\rho_{l(\gamma)})'(t) &= kC_AQ_sQ_tl(\gamma)(1+4^kk!) + kQ_tl(C_T + 4^kk!C_A)\rho_l(t)\ , \\ \rho_{l(\gamma)}(0) &= 0. \end{aligned} \tag{2.7.1}$$

The constants Q_t and Q_s are the bounds on the nullhomotopy H of the path $\alpha(t)$ in Y, i.e. $\left|\frac{\partial}{\partial t}H\right| \leqslant Q_tl(\gamma)$ and $\left|\frac{\partial}{\partial s}H\right| \leqslant Q_s$.

Since $-K \leqslant \sec^Y \leqslant (K + 3C_A^2)$, by O'Neill's formula B.7.3, and, by assumption, $l(\alpha) \leqslant l(\gamma) < \min\{\frac{2\pi}{\sqrt{K+3C_A^2}}, \frac{1}{2}\operatorname{inj}^Y(p)\}$, we apply Lemma 2.6 and derive that

$$\begin{aligned} Q_t &= \frac{\sqrt{K+3C_A^2}}{\sqrt{K}} \cdot \frac{\sinh(\sqrt{K}\frac{l(\gamma)}{2})}{\sin(\sqrt{K+3C_A^2}\frac{l(\gamma)}{2})}, \\ Q_s &= \frac{\sinh(\sqrt{K}\frac{l(\gamma)}{2})}{\sqrt{K}\frac{l(\gamma)}{2}} \cdot \frac{l(\gamma)}{2} =: \tilde{Q}_sl(\gamma). \end{aligned} \tag{2.7.2}$$

Note that for any loop $\bar{\alpha}$ in Y of length less than or equal to $l(\gamma)$ the corresponding nullhomotopy $\bar{H}$ of $\bar{\alpha}$ satisfies the bounds $\left|\frac{\partial}{\partial t}\bar{H}\right| \leqslant Q_tl(\bar{\alpha})$ and $\left|\frac{\partial}{\partial s}\bar{H}\right| \leqslant \tilde{Q}_sl(\bar{\alpha})$.

Thus, in our case, the differential equation (2.7.1) reads as

$$\begin{aligned} (\rho_l)'(t) &= kC_A\tilde{Q}_sQ_tl^2(1+4^kk!) + kQ_tl(C_T + 4^kk!C_A)\rho_l(t)\ , \\ \rho_l(0) &= 0, \end{aligned}$$

for $0 \leqslant l \leqslant l(\gamma)$, compare [Tap00, Lemma 3.3]. For simplicity, we set

$$\begin{aligned} G_1 &:= kC_A\tilde{Q}_sQ_t(1+4^k k!), \\ G_2 &:= kQ_t(C_T+4^k k! C_A). \end{aligned}$$

Using the variation of constants, we conclude

$$\rho_l(1) = \frac{G_1}{G_2} l\big(e^{G_2 l} - 1\big).$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}l}\rho_l(1) &= \frac{G_1}{G_2}\big(e^{G_2 l} - 1\big) + G_1 l e^{G_2 l} \\ &\leqslant \frac{G_1}{G_2}\big(e^{G_2 l(\gamma)} - 1\big) + G_1 l(\gamma) e^{G_2 l(\gamma)} =: P. \end{aligned} \tag{2.7.3}$$

It remains to check the behavior of this term as $l(\gamma)$ becomes small. Since Q_t and $\tilde{Q}_s$ are the only appearing quantities depending on $l(\gamma)$ we first conclude from (2.7.2) that

$$\begin{aligned} \lim_{l(\gamma)\to 0} Q_t &= 1, \\ \lim_{l(\gamma)\to 0} \tilde{Q}_s &= \frac{1}{2}. \end{aligned} \tag{2.7.4}$$

Hence, we extract the quantities in Q_t and $\tilde{Q}_s$ in 2.7.3,

$$\begin{aligned} \frac{G_1}{G_2} &= \frac{kC_A\tilde{Q}_sQ_t(1+4^k k!)}{kQ_t(C_T+4^k k! C_A)} = \tilde{Q}_s\frac{C_A(1+4^k k!)}{C_T+4^k k! C_A} =: \tilde{Q}_s \cdot C_1(C_A, C_T, k), \\ G_1 l(\gamma) &= kC_A\tilde{Q}_sQ_t(1+4^k k!) l(\gamma) =: \tilde{Q}_sQ_t l(\gamma)\cdot C_2(C_A, k), \\ G_2 l(\gamma) &= kQ_t(C_T+4^k k! C_A) l(\gamma) =: Q_t l(\gamma) \cdot C_3(C_A, C_T, k). \end{aligned}$$

Therefore, it follows with (2.7.4) that

$$\begin{aligned} \lim_{l(\gamma)\to 0} \frac{G_1}{G_2} &= \frac{1}{2}\cdot C_1(C_A, C_T, k), \\ \lim_{l(\gamma)\to 0} G_1 l(\gamma) &= 0 \cdot C_2(C_A, k) = 0, \\ \lim_{l(\gamma)\to 0} G_2 l(\gamma) &= 0 \cdot C_3(C_A, C_T, k) = 0. \end{aligned}$$

Summarizing these observations we conclude

$$\begin{aligned} \lim_{l(\gamma)\to 0} P &= \lim_{l(\gamma)\to 0} \frac{G_1}{G_2}\big(e^{G_2 l(\gamma)} - 1\big) + G_1 l(\gamma) e^{G_2 l(\gamma)} \\ &= \frac{1}{2} C_1(C_A, C_T, k)\big(e^0 - 1\big) + 0 \cdot e^0 = 0. \end{aligned}$$

This shows that $P = \tau(l(\gamma)|C_A, C_T, k, K)$.

Next, we consider the inequality $l(\beta_2) \leqslant L \cdot l(\gamma)$. Here, L is the maximum of the Lipschitz constants of the holonomy diffeomorphism h^α associated to paths α in Y. Since

h^α satisfies the Lipschitz constant $e^{C_T \cdot l(\alpha)}$ (c.f. [GW00, Lemma 4.2]) and $l(\alpha)$ is bounded from above by $l(\gamma)$ we conclude that

$$L = e^{C_T \cdot l(\gamma)} = 1 + \tau(l(\gamma)|C_T).$$

Summarizing all these observations we conclude

$$\begin{aligned} 2\operatorname{inj}^{F_p} \leqslant l(\tilde{\gamma}) &\leqslant C \cdot l(\gamma) \\ &= (P + L) \cdot l(\gamma) \\ &= \Big(1 + \tau(l(\gamma)|C_A, C_T, k, K)\Big) \cdot l(\gamma) \\ &= \Big(1 + \tilde{\tau}(\operatorname{inj}^M(x)|C_A, C_T, k, K)\Big) \cdot 2\operatorname{inj}^M(x). \end{aligned}$$

□

Remark 2.8. If $-K \leqslant \sec^M \leqslant -\kappa$ for some $\kappa > 0$ such that $(-\kappa + 3C_A^2) \leqslant 0$ then the assumption $\operatorname{inj}^M(x) < \frac{1}{4}\operatorname{inj}^Y(p)$ is already sufficient for Proposition 2.4 to hold, as M, as well as Y, do not have any conjugate points. In particular, the injectivity radius at some point $x \in M$ equals half of the length of the shortest non contractible geodesic loop based at x.

2.2 Characterization of codimension one collapse

In this section we prove Theorem 2.1. Further, we discuss the properties of the set $\mathcal{M}(n, d, C)$ consisting of all isometry classes of Riemannian manifolds (M, g) in $\mathcal{M}(n, d)$ with $C \leqslant \frac{\operatorname{vol}(M)}{\operatorname{inj}(M)}$.

First we note that in the case of a non collapsing sequence in $\mathcal{M}(n, d)$ the statement of Theorem 2.1 is obviously true as the limit space is a closed n-dimensional Riemannian manifold, see Theorem 1.9. Therefore, we only consider the case of collapsing sequences in $\mathcal{M}(n, d)$.

The first step of the proof of Theorem 2.1 is to reduce the statement to sequences of sufficiently collapsed manifolds with invariant metrics in the sense of Theorem 1.26. This is done in the following lemma. Then it suffices to prove Theorem 2.1 for that special case.

Lemma 2.9. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{M}(n, d)$ with limit space Y. There is a small positive δ and an index $I > 0$ such that for any $i \geqslant I$, there is an invariant metric $\tilde{g}_i$ on M_i with*

$$\begin{aligned} |g_i - \tilde{g}_i| &< (e^\delta - 1) + C(n, \delta) d_{GH}(M_i, Y), \\ |\nabla_i - \tilde{\nabla}_i| &\leqslant \delta + C_1(n, \delta) d_{GH}(M_i, Y), \\ |\tilde{\nabla}_i^j \tilde{R}_i| &\leqslant C(j, n, \delta)(1 + d_{GH}(M_i, Y)). \end{aligned}$$

In particular,

$$\begin{aligned} e^{-\tau(d_{GH}(M_i,Y)|n,\delta) - \tau(\delta|n)} \frac{\operatorname{vol}(\tilde{B}_r^{M_i}(x))}{\widetilde{\operatorname{inj}}^{M_i}(x)} &\leqslant \frac{\operatorname{vol}(B_r^{M_i})(x)}{\operatorname{inj}^{M_i}(x)} \\ &\leqslant e^{\tau(d_{GH}(M_i,Y)|n,\delta) + \tau(\delta|n)} \frac{\operatorname{vol}(\tilde{B}_r^{M_i}(x))}{\widetilde{\operatorname{inj}}^{M_i}(x)}, \end{aligned}$$

where $\tilde{B}_r^{M_i}(x)$ and $\widetilde{\operatorname{inj}}^{M_i}(x)$ are taken with respect to the metric $\tilde{g}_i$. Moreover, the Hausdorff dimension of the limit space $\tilde{Y}$ of $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$ equals the Hausdorff dimension of the limit space Y of the original sequence $(M_i, g_i)_{i\in\mathbb{N}}$.

Proof. First, we apply Abresch's smoothing theorem, Theorem 1.24 for some small $\delta > 0$ to the sequence $(M_i, g_i)_{i\in\mathbb{N}}$. We obtain the sequence $(M_i, \hat{g}_i)_{i\in\mathbb{N}}$ consisting only of A-regular manifolds, with $(A_j(n,\delta))_{j\in\mathbb{N}}$, i.e. for all $i, j \in \mathbb{N}$,

$$|\hat{\nabla}_i^j \hat{R}_i| \leqslant A_j,$$

where $\hat{\nabla}_i$, $\hat{R}_i$ is the Levi-Civita connection, respectively the curvature of the metric $\hat{g}_i$. Moreover, by choosing δ sufficiently small, Proposition 1.25 implies that

$$|\widehat{\sec}^{M_i}| \leqslant (1 + c(n)\delta).$$

It follows from the estimates for the metrics g_i and $\hat{g}_i$ in Theorem 1.24 that for all $i \geqslant I_1$,

$$e^{-\tau(\delta|n)} \frac{\operatorname{vol}(\hat{B}_r^{M_i}(x))}{\widehat{\operatorname{inj}}^{M_i}(x)} \leqslant \frac{\operatorname{vol}(B_r^{M_i})(x)}{\operatorname{inj}^{M_i}(x)} \leqslant e^{\tau(\delta|n)} \frac{\operatorname{vol}(\hat{B}_r^{M_i}(x))}{\widehat{\operatorname{inj}}^{M_i}(x)}. \tag{2.9.1}$$

Here, I_1 is chosen to be sufficiently large such that for all $i \geqslant I_1$, $\operatorname{inj}^{M_i}(x)$, resp. $\widehat{\operatorname{inj}}^{M_i}(x)$, is smaller than the conjugate radius of (M_i, g_i), resp. $(M_i, \hat{g}_i)$, which is uniformly bounded from below in terms of the upper sectional curvature bound. Thus, the bound on the conjugate radius for $(M_i, \hat{g}_i)$ only changes slightly if we choose $\delta > 0$ to be sufficiently small.

By Theorem 1.26 there is a further index I_2 such that for each element of $(M_i, \hat{g}_i)_{i\in\mathbb{N}}$ with $i \geqslant I_2$ there is an invariant metric $\tilde{g}_i$ on M_i. This leads to a new sequence $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$. The claimed bounds on $\tilde{g}_i$ follow by combining the inequalities given in Theorem 1.24 and Theorem 1.26. In particular, after a small rescaling, $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$ lies again in $\mathcal{M}(n,d)$.

Moreover, as $|\hat{g}_i - \tilde{g}_i|_{C^\infty} \leqslant \tau(d_{\mathrm{GH}}(M_i, Y)|n,\delta)$, see Theorem 1.26, it follows that

$$e^{-\tau(d_{\mathrm{GH}}(M_i,Y)|n,\delta)} \frac{\operatorname{vol}(\tilde{B}_r^{M_i}(x))}{\widetilde{\operatorname{inj}}^{M_i}(x)} \leqslant \frac{\operatorname{vol}(\hat{B}_r^{M_i})(x)}{\widehat{\operatorname{inj}}^{M_i}(x)} \leqslant e^{\tau(d_{\mathrm{GH}}(M_i,Y)|n,\delta)} \frac{\operatorname{vol}(\tilde{B}_r^{M_i}(x))}{\widetilde{\operatorname{inj}}^{M_i}(x)}, \tag{2.9.2}$$

uniformly for $i \geqslant \max\{I_1, I_2\} =: I$, as before.

Since $|\hat{g}_i - \tilde{g}_i|_{C^\infty} \leqslant \tau(d_{\mathrm{GH}}(M_i, Y)|n,\delta)$, the sequences $(M_i, \hat{g}_i)_{i\in\mathbb{N}}$ and $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$ converge to the same limit space $\tilde{Y}$. Furthermore, as $d_{\mathrm{GH}}((M_i, g_i), (M_i, \tilde{g}_i)) \leqslant \tau(\delta)$ it follows from [Fuk88, Lemma 2.3] that the Lipschitz-distance between the limit spaces Y and $\tilde{Y}$ is also bounded by $\tau(\delta)$ (the explicit value of $\tau(\delta)$ might change). In particular, Y and $\tilde{Y}$ are homeomorphic to each other. Thus, they have the same Hausdorff dimension. Together with (2.9.1) and (2.9.2) the claim follows. □

Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n,d)$ converging to a compact metric space Y of lower dimension. By the above lemma, we assume without loss of generality, that for all $i \in \mathbb{N}$ the metric g_i is an invariant metric in the sense of Theorem 1.26. Moreover, we can assume without loss of generality that $\mathrm{inj}^{M_i}(x) < \pi$ for all $x \in M_i$ and $i \geqslant I$. Consequently, we can restrict our attention to collapsing sequences $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ such that for every $i \in \mathbb{N}$ the manifold (M_i, g_i) is A-regular, $\mathrm{inj}^{M_i}(x) < \pi$ for all $x \in M_i$, and the metric g_i is invariant.

The next proposition together with Lemma 2.9 proves the implication (1) to (2) in Theorem 2.1.

Proposition 2.10. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence of A-regular manifolds in $\mathcal{M}(n,d)$ converging to a compact metric space Y in the Gromov-Hausdorff topology. Suppose that for each $i \in \mathbb{N}$ the metric g_i is invariant. If $\dim_{\mathrm{Haus}}(Y) = (n-1)$ then, for each $r > 0$, there is a positive constant $C := C(n, r, Y)$ such that*

$$C \leqslant \frac{\mathrm{vol}(B_r^{M_i}(x))}{\mathrm{inj}^{M_i}(x)} \tag{2.10.1}$$

for all $x \in M_i$ and $i \in \mathbb{N}$.

Proof. Since $\dim_{\mathrm{Haus}}(Y) = (n-1)$ it follows from Proposition 1.31 that Y is a compact Riemannian orbifold. Furthermore, Corollary 1.29 implies that we have for any $i \in \mathbb{N}$ an S^1-bundle $\tilde{f}_i : (FM_i, g_i^F) \to \tilde{Y}$. Here $\tilde{Y}$ is the Gromov-Hausdorff limit of the sequence $(FM_i, g_i^F)_{i\in\mathbb{N}}$. By Theorem 1.21, $\tilde{Y}$ is Riemannian manifold and the quotient $\tilde{Y}/_{\mathrm{O}(n)}$ is isometric to Y. Hence, $f_i : M_i \to Y$ is an S^1-orbifold bundle. As by assumption, for any $i \in \mathbb{N}$ the metric g_i is invariant there is a Riemannian orbifold metric h_i on Y such that $f_i : (M_i, g_i) \to (Y, h_i)$ is a Riemannian orbifold submersion. As (M_i, g_i) is an A-regular manifold it follows that the metric h_i on Y is $B(A)$-regular for all $i \in \mathbb{N}$. Thus, there is a subsequence such that $(h_i)_{i\in\mathbb{N}}$ converges to a smooth metric on Y in the C^∞-topology.

Now we fix some $r > 0$. As $i \to \infty$, the ball $B_r^{M_i}(x)$ resembles more and more $f_i^{-1}(B_r^Y(f_i(x)))$. Hence, there is an index I such that for any $i > I$,

$$f_i^{-1}(B^Y_{\frac{r}{2}}(p)) \subset B_r^{M_i}(x)$$

for all $p \in Y$ and $x \in F_p^i := f_i^{-1}(p)$. This is a direct consequence of Toponogov's triangle comparison.

Since the T-tensor of the Riemannian submersions $\tilde{f}_i : FM_i \to \tilde{Y}$ is uniformly bounded by a constant $C_T(n)$, see Corollary 1.29, it follows that for any $r > 0$ there is a positive constant $C_1 := C_1(r, n, C_T)$ such that, for all $i > I$,

$$\begin{aligned}\mathrm{vol}(B_r^{M_i}(x)) &\geqslant C_1 \,\mathrm{vol}(B^{Y_i}_{\frac{r}{2}}(p))\,\mathrm{vol}(F_p^i)\\ &= C_1 \,\mathrm{vol}(B^{Y_i}_{\frac{r}{2}}(p))\, 2\,\mathrm{inj}(F_p^i).\end{aligned}$$

For the last equality we used that $F_p^i \cong S^1$ for all $i \in \mathbb{N}$. In the above estimate, Y_i denotes the Riemannian orbifold (Y, h_i). Now the claim follows from

$$\begin{aligned}\frac{\mathrm{vol}(B_r^{M_i}(x))}{\mathrm{inj}^{M_i}(x)} &\geqslant 2C_1 \,\mathrm{vol}(B^{Y_i}_{\frac{r}{2}}(p))\frac{\mathrm{inj}(F_p)}{\mathrm{inj}^{M_i}(x)}\\ &\geqslant 2C_1 \inf_{i\in\mathbb{N}} \min_{p\in Y} \mathrm{vol}(B^{Y_i}_{\frac{r}{2}}(p)) > 0.\end{aligned}$$

□

To finish the proof of Theorem 2.1 it remains to show that (3) implies (1). The main idea here is to derive a contradiction by constructing an upper bound on the inequality (2.1.1) that vanishes in the limit. Together with Lemma 2.9, the next proposition completes the proof of Theorem 2.1.

Proposition 2.11. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence of A-regular manifolds in $\mathcal{M}(n,d)$ converging to a compact metric space Y in the Gromov-Hausdorff topology. Suppose that for each $i \in \mathbb{N}$, $\operatorname{inj}^{M_i}(x) < \pi$ for all $x \in M_i$ and that the metric g_i is invariant. If there exist positive constants r and C such that*

$$C \leqslant \frac{\operatorname{vol}(B_r^{M_i}(x))}{\operatorname{inj}^{M_i}(x)} \tag{2.11.1}$$

for all $x \in M_i$ and all $i \in \mathbb{N}$, then $\dim_{\mathrm{Haus}}(Y) = n-1$.

Proof. Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{M}(n,d)$ such that the Gromov-Hausdorff limit Y satisfies $n - \dim_{\mathrm{Haus}}(Y) =: k \geqslant 2$. Assume further that there are positive numbers r and C such that (2.11.1) holds for all $x \in M_i$ and $i \in \mathbb{N}$.

By Theorem 1.32 there is a closed set $S \subset Y$ with $\dim_{\mathrm{Haus}}(S) \leqslant \dim_{\mathrm{Haus}}(Y) - 3$ such that $\hat{Y} := Y\backslash S$ is a Riemannian orbifold.

Moreover, it follows from Corollary 1.29 that the second fundamental form of the Riemannian submersion $\tilde{f}_i : (FM_i, g_i^F) \to (\tilde{Y}, \tilde{h}_i)$ is uniformly bounded by a constant $\tilde{C}_T(n)$, where g_i^F is the metric induced by the metric g_i and a biinvariant metric on $\mathrm{O}(n)$. Considering the commutative diagram (1.29.1), it follows that for any $r > 0$ there is a constant $C_1(r, n, \tilde{C}_T)$ such that

$$\operatorname{vol}(B_r^{M_i}(x)) \leqslant C_1 \operatorname{vol}(B_r^{Y_i}(f_i(x))) \operatorname{vol}(F^i_{f_i(x)})$$

for any $x \in M_i$, $i \in \mathbb{N}$. Here, Y_i stands for the metric space (Y, h_i), where h_i is the quotient metric of ${(\tilde{Y}, \tilde{h}_i)}/{\mathrm{O}(n)}$.

Let $p \in \hat{Y}$ be a regular point, i.e. p has an open neighborhood that is diffeomorphic to an open manifold. Then, there is a $\kappa > 0$ such that $B_\kappa^{Y_i}(p)$ is an open Riemannian manifold for all $i \in \mathbb{N}$.

Now the maps f_i restricted to the preimage $f_i^{-1}\big(B_\kappa^{Y_i}(p)\big)$ are Riemannian submersions between manifolds for all $i \in \mathbb{N}$. Since the T-tensor of the Riemannian submersions $\tilde{f}_i$ is uniformly bounded, by Corollary 1.29 it follows that the T-tensor of f_i restricted to the preimage of $B_\kappa^{Y_i}(p)$ is also uniformly bounded by a constant C_T.

As the sequence $(M_i, g_i)_{i\in\mathbb{N}}$ only consists of A-regular manifolds, we can extract a subsequence, denoted by $(M_i, g_i)_{i\in\mathbb{N}}$, such that the Riemannian metrics $(\tilde{f}_i)_*(g_i^F)$ on $\tilde{Y}$ converge in C^∞. Thus, the metrics $(f_i)_*(g_i)$ converge in C^∞ on $B_\kappa^Y(p)$. In particular, the sectional curvature on $B_\kappa^Y(p)$ can be uniformly bounded in i. Therefore, it follows from O'Neill's formula, B.7.3, that the A-tensor is uniformly bounded in norm by a constant C_A on $B_\kappa^Y(p)$.

Since $\operatorname{inj}^{M_i}(x) < \pi$, there is a non contractible geodesic loop γ based at $x \in F_p^i$ such that $l(\gamma) = 2\operatorname{inj}^{M_i}(x)$. We observe that for all i sufficiently large, the assumptions of Proposition 2.4 are fulfilled. Hence, there is an $I \in \mathbb{N}$ such that for all $i \geqslant I$,

$$\operatorname{inj}(F_p^i) \leqslant \big(1 + \tau(\operatorname{inj}^{M_i}(x) | k, C_T, C_A)\big) \cdot \operatorname{inj}^{M_i}(x) =: C_2 \operatorname{inj}^{M_i}(x). \tag{2.11.2}$$

By O'Neill's formula, B.7.1, it follows that

$$|\sec^{F_p^i}| \leqslant |\sec^{M_i}| + 2C_T^2 =: K^2.$$

Therefore, we can apply [HK78, Corollary 2.3.2], to obtain

$$\operatorname{vol}(F_p^i) \leqslant C_3(k)\operatorname{inj}(F_p^i)\left(\frac{\sinh(\operatorname{diam}(F_p^i)K)}{K}\right)^{k-1}.$$

Together with (2.11.2) we conclude

$$\begin{aligned}
C &\leqslant \frac{\operatorname{vol}(B_r^{M_i}(x))}{\operatorname{inj}^{M_i}(x)} \\
&\leqslant \frac{C_1\operatorname{vol}(B_r^{Y_i}(p))\operatorname{vol}(F_p^i)}{\operatorname{inj}^{M_i}(x)} \\
&\leqslant \frac{C_1\operatorname{vol}(B_r^{Y_i}(p))\left(C_3(k)\operatorname{inj}(F_p^i)\left(\frac{\sinh(\operatorname{diam}(F_p^i)K)}{K}\right)^{k-1}\right)}{\operatorname{inj}^{M_i}(x)} \\
&\leqslant C_1C_2C_3\operatorname{vol}(B_r^{Y_i}(p))\left(\frac{\sinh\big(\operatorname{diam}(F_p^i)K\big)}{K}\right)^{k-1}.
\end{aligned}$$

As $(M_i, g_i)_{i\in\mathbb{N}}$ is a collapsing sequence, $\lim_{i\to\infty}\operatorname{diam}(F_p^i) = 0$. In particular, since $k \geqslant 2$ by assumption, it follows that

$$\lim_{i\to\infty}\left(\frac{\sinh(\operatorname{diam}(F_p^i)K)}{K}\right)^{k-1} = 0.$$

Hence, we obtain in the limit $i \to \infty$ that $C \leqslant 0$ which contradicts our assumption that C is a positive constant. □

As an example we consider Berger's example of the collapsing Hopf fibration (see Example 1.14) and show that the characterization of Theorem 2.1 applies.

Example 2.12. Consider the collapsing sequence $(S^3, g_i)_{i\in\mathbb{N}}$ from Example 1.14 whose Gromov-Hausdorff limit (S^2, h) is the round two-sphere of radius $\frac{1}{2}$. It is easy to check that the Hopf maps $f_i : (S^3, g_i) \to (S^2, h)$ are totally geodesic Riemannian submersions with uniformly bounded A-tensors. Let $r = \pi$, and $x \in f_i^{-1}(p) =: F_p^i$. Then

$$\operatorname{vol}(S^3, g_i) = \operatorname{vol}(B_\pi^{g_i}(x)) = \operatorname{vol}(F_p^i)\operatorname{vol}(B_{\frac{\pi}{2}}^h(p)) = \frac{2\pi}{i}\operatorname{vol}(S^2, h) = \frac{2\pi^2}{i}.$$

Therefore, we derive for $r = \pi$,

$$\lim_{i\to\infty}\frac{\operatorname{vol}(B_\pi^{g_i}(x))}{\operatorname{inj}^{g_i}(x)} = \lim_{i\to\infty}\frac{\frac{2\pi^2}{i}}{\frac{\pi}{i}} = 2\pi = 2\operatorname{vol}(S^2, h).$$

We conclude this chapter, by examining the following subset of $\mathcal{M}(n,d)$.

Definition 2.13. For given positive numbers n, d, and C, we define $\mathcal{M}(n,d,C)$ to be the set of all isometry classes of closed Riemannian manifolds (M,g) in $\mathcal{M}(n,d)$ satisfying

$$C \leqslant \frac{\mathrm{vol}(M)}{\mathrm{inj}(M)}.$$

By Theorem 2.1 and the following lemma it follows that the closure $\mathcal{CM}(n,d,C)$ of $\mathcal{M}(n,d,C)$ with respect to the Gromov-Hausdorff distance only consists of n-dimensional Riemannian manifolds and $(n-1)$-dimensional Riemannian orbifolds. For simplicity we consider each limit of a sequence in $\mathcal{M}(n,d,C)$ as an orbifold and understand a manifold as a special case.

Lemma 2.14. *Let $(M,g) \in \mathcal{M}(n,d)$ with $\mathrm{inj}(x) < \frac{\pi}{2}$ for all $x \in M$. Then there is a constant $C := C(\max_{x\in M} \mathrm{inj}(x), d)$ such that for all $x, y \in M$,*

$$C^{-1}\,\mathrm{inj}(x) \leqslant \mathrm{inj}(y) \leqslant C\,\mathrm{inj}(x).$$

Proof. The idea of this proof is to find a constant C' such that

$$|D\,\mathrm{inj}(x)| \leqslant C'\,\mathrm{inj}(x) \tag{2.14.1}$$

holds for all $x \in M$. Here we interpret the injectivity radius as a map $\mathrm{inj} : M \to \mathbb{R}$. Then it follows from this inequality that for all $x, y \in M$,

$$\mathrm{inj}(y) \leqslant \mathrm{inj}(x) \cdot e^{C' d(x,y)}, \tag{2.14.2}$$

where $d(x,y)$ denotes the geodesic distance between x and y. Since $\mathrm{diam}(M) \leqslant d$ the lemma is an immediate consequence of (2.14.2).

To construct a constant C' as in (2.14.1) we consider the map

$$\begin{aligned} F : TM &\to M \times M,\\ (x,v) &\mapsto (x, \exp_x(v)). \end{aligned}$$

Since $|\sec^M| \leqslant 1$,

$$\frac{\sin(|v|)}{|v|}|w| \leqslant |(D_v \exp_p)(w)| \leqslant \frac{\sinh(|v|)}{|v|}|w|. \tag{2.14.3}$$

By assumption $\mathrm{inj}(x) < \frac{\pi}{2}$. Thus, the injectivity radius is everywhere strictly smaller than the conjugate radius, which is bounded from below by π. Hence, for every $x \in M$ there is a geodesic loop γ with $l(\gamma) = 2\,\mathrm{inj}(M)$. In particular, for every $x \in M$ there is at least one $v \in T_xM$ with $\exp_x(v) = x$ and $|v| = 2\,\mathrm{inj}(x)$.

Thus, let $(x_0, v_0) \in TM$ be such that $\exp_{x_0}(v_0) = x_0$ and $|v_0| = 2\,\mathrm{inj}(x_0)$. Then, $F(x_0,v_0) = (x_0,x_0)$. Since $\frac{\partial F}{\partial v}$ is invertible by (2.14.3), it follows by the implicit function theorem that there is a small open neighborhood $U \subset M$ of x_0 and a map $h : U \to TM$ such that $h(x_0) = v_0 \in T_{x_0}M$ and $F(x, h(x)) = (x,x)$ for all $x \in U$. Furthermore, it follows from the implicit function theorem that we can bound the derivative of the function h as follows:

$$\begin{aligned}
|\nabla h| &\leqslant |(D_v F)^{-1}_{(x,h(x))}||D_x F_{(x,h(x))}| \\
&= |(D_{h(x)} \exp_x)^{-1}||h(x)| \\
&\leqslant \frac{|h(x)|}{\sin(|h(x)|)}|h(x)|.
\end{aligned}$$

Next, we observe that for every point $x_0 \in h$ and direction $\xi \in T_{x_0}M$, $|\xi| = 1$ there is a $v_0 \in T_{x_0}M$ such that the corresponding implicit function h satisfies

$$|\xi(\mathrm{inj})| = \frac{1}{2}\Big|\xi(|h|)\Big|.$$

Hence, we conclude that

$$\begin{aligned}
|D\,\mathrm{inj}(x)| &= \frac{1}{2}\Big|d(|h(x)|)\Big| \\
&\leqslant \frac{1}{2}|\nabla h(x)| \\
&\leqslant \frac{1}{2}\frac{|h(x)|}{\sin(|h(x)|)}|h(x)| \\
&\leqslant \left(\max_{y\in M} \frac{2\,\mathrm{inj}(y)}{\sin(2\,\mathrm{inj}(y))}\right) \cdot \mathrm{inj}(x) =: C'\,\mathrm{inj}(x).
\end{aligned}$$

In the second line we used Kato's inequality, which states that any section S of a smooth Riemannian vector bundle $E \to M$ satisfies $|d|S|| \leqslant |\nabla S|$ (see for instance [CGH00]). □

For later use we want to remark that the closure $\mathcal{CM}(n,d,C)$ of $\mathcal{M}(n,d,C)$ has a dense subspace that only consists of *smooth elements*, as defined in [Fuk88, Definition 0.4].

Definition 2.15. An element Y of the closure of $\mathcal{M}(n,d)$ is *smooth* if for any $p \in Y$ there is a neighborhood U of p and a compact Lie group G_p with a faithful representation into the orthogonal group $\mathrm{O}(n)$ such that U is isometric to the quotient ${}^{V}/_{G_p}$ for a neighborhood V of 0 in $\mathbb{R}^m$ together with a G_p-invariant smooth Riemannian metric $\bar{g}$.

This observation is necessary because we want to use the following lemma due to Fukaya (c.f. [Fuk88, Lemma 7.8]).

Lemma 2.16. *Let $(M_i, x_i)_{i\in\mathbb{N}}$ be a sequence of pointed manifolds in the d_{GH}-closure of $\mathcal{M}(n,d)$ converging to a smooth element (Y,p). Suppose that the sectional curvature of M_i at x_i are unbounded. Then the dimension of the group G_p, defined in Theorem 1.17, is positive.*

Combining this with Proposition 1.31 we conclude the following properties of the set $\mathcal{CM}(n,d,C)$.

Theorem 2.17. *Any sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d,C)$ contains a subsequence that either converges to an n-dimensional closed Riemannian manifold in the $C^{1,\alpha}$-topology or to a compact $(n-1)$-dimensional Riemannian orbifold (Y,h) with a $C^{1,\alpha}$-metric h in the Gromov-Hausdorff topology. Furthermore, there are positive constants $v := v(n,d,C)$ and $K := K(n,d,C)$ such that any element Y in $\mathcal{CM}(n,d,C)$ with $\dim(Y) = (n-1)$ satisfies $\| \sec^Y \|_{L^\infty} \leqslant K$ and $\mathrm{vol}(Y) \geqslant v$.*

Proof. Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n,d,C)$. Then there exists by Theorem 1.4 a d_{GH}-convergent subsequence converging to a compact metric space Y.

If $\dim(Y) = n$ then the injectivity radius of the manifolds M_i is uniformly bounded from below by a constant ι. Thus, this sequence lies in $\mathcal{M}(n,d,\iota)$ and the claim follows from Theorem 1.9.

If $\dim(Y) < n$, it follows from Lemma 2.14 and Theorem 2.1 that $\dim(Y) = (n-1)$. Thus, Y is a Riemannian orbifold by Proposition 1.31. In particular, it follows from Theorem 1.32 that Y has a $C^{1,\alpha}$-metric h. This proves the first part of the theorem.

For the second part, we assume that there is a sequence $(Y_i, h_i)_{i\in\mathbb{N}}$ of $(n-1)$-dimensional Riemannian orbifolds in $\mathcal{CM}(n,d,C)$ such that there is a sequence of points $p_i \in Y_i$ where the sectional curvatures are unbounded as $i \to \infty$. Without loss of generality, we assume that the metrics h_i are smooth for all $i \in \mathbb{N}$. As each element Y_i can be realized as the limit space of a codimension one collapse in $\mathcal{M}(n,d)$, there is a subsequence $(Y_i)_{i\in\mathbb{N}}$ converging to an element Y_∞ in $\mathcal{CM}(n,d,C)$ and a point p_∞ with unbounded sectional curvature. Since smooth elements, see Definition 2.15, are dense in the closure of $\mathcal{M}(n,d,C)$, we assume without loss of generality that Y_∞ is a smooth element. By a diagonal sequence argument there is a sequence $(M_j, g_j)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d,C)$ converging to Y_∞. Thus, it follows from Theorem 2.1 and Lemma 2.14 that Y_∞ is an $(n-1)$-dimensional Riemannian orbifold. As $\mathcal{CM}(n,d,C)$ is a subset of the d_{GH}-closure of $\mathcal{M}(n,d)$ we can apply Lemma 2.16. It follows that the group G_{p_∞} has positive dimension. This is a contradiction because the group G_{p_∞} has to be finite by Proposition 1.31. Consequently, there exists a constant $K := K(n,d,C)$ as claimed.

For the volume bound we assume that there exists a sequence $(Y_i, h_i)_{i\in\mathbb{N}}$ in $\mathcal{CM}(n,d,C)$ such that $\dim(Y_i) = (n-1)$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \mathrm{vol}(Y_i) = 0$. We see at once that $(Y_i)_{i\in\mathbb{N}}$ defines a collapsing sequence with limit space Y_∞. By a diagonal sequence argument we can construct a converging sequence $(M_j, g_j)_{j\in\mathbb{N}}$ in $\mathcal{M}(n,d,C)$ whose limit space Y_∞ is of dimension less than $(n-1)$. But this is a contradiction to Theorem 2.1. In particular, there is a constant $v := v(n,d,C)$ such that $\mathrm{vol}(Y) \geqslant v$ for all $(n-1)$-dimensional spaces in $\mathcal{CM}(n,d,C)$. □

Chapter 3

Riemannian affine fiber bundles

In Chapter 1.2 we have seen that for any sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n,d)$ converging to a lower dimensional Riemannian manifold (B,h) in the Gromov-Hausdorff topology there is an index I such that for any $i \geqslant I$ there is a fibration $f_i : M_i \to B$ such that the fibers are infranilmanifolds, see Theorem 1.17. Furthermore, there are metrics $\tilde{g}_i$ on M_i and $\tilde{h}_i$ on B such that $\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0$, $\lim_{i\to 0} \|\tilde{h}_i - h\|_\infty = 0$ and such that $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ is a *Riemannian affine fiber bundle*, see Corollary 1.29 and Remark 1.30. We recall from Definition 1.28 that a fibration $f : (M,g) \to (B,h)$ between two closed Riemannian manifolds is a Riemannian affine fiber bundle if

- f is a Riemannian submersion,
- for each p the fiber $Z_p := f^{-1}(p)$ is an infranilmanifold with an induced affine parallel metric $\hat{g}_p$,
- the structure group lies in $\mathrm{Aff}(Z)$.

Our goal is to study the behavior of Dirac eigenvalues on a collapsing sequence of spin manifolds in $\mathcal{M}(n,d)$ with smooth limit space. Since Dirac eigenvalues are continuous under a C^1-variation of metrics, see Appendix C, it suffices to study the behavior of Dirac eigenvalues on the total space of Riemannian affine fiber bundles. For this reason we will study Riemannian affine fiber bundles in detail in this chapter.

The content of this chapter is a mix of [Roo18c, Section 3 and 4] and [Roo18b, Section 3 and 4] and a preparation for the proofs of the main results regarding the behavior of Dirac eigenvalues on codimension one collapse [Roo18c] and on collapsing sequences in $\mathcal{M}(n,d)$ with smooth limit space [Roo18b].

Here and subsequently we fix a Riemannian affine fiber bundle $f : (M,g) \to (B,h)$ with $\dim(M) = (n+k)$ and $\dim(B) = n$. In particular, the fibers Z are closed k-dimensional infranilmanifolds. In the first section we exploit the fact that f is a Riemannian submersion and show via various examples how the geometry of the fiber bundle $f : M \to B$ influences the relation between the Levi-Civita connection on (M,g) and the Levi-Civita connection on (B,h). For the second section, we assume in addition that the total space (M,g) is a spin manifold with a fixed spin structure. First, we discuss whether the spin structure on M induces a spin structure on the fibers Z_p, $p \in B$, and on the base space B. We show that if the fibers are one-dimensional, i.e. $k = 1$ then there is an induced structure on B. But for $k \geqslant 2$ we cannot make such a statement without further assumptions. Nevertheless, as the induced metrics on the fibers are affine parallel there is an induced affine connection ∇^{aff} on the spinor bundle of M. Thus, the notion of affine parallel spinors is well-defined, see [Lot02a, Section 3]. We will show that the subspace of affine parallel spinors on M is isometric to the space of spinors of a twisted Clifford bundle over the base space B. In particular, there is an elliptic first order self-adjoint differential operator $\mathcal{D}^B$ on B that is isospectral to the Dirac operator on M restricted to the space of affine parallel spinors.

3.1 The Geometry of Riemannian affine fiber bundles

Let $f : (M, g) \to (B, h)$ be a Riemannian affine fiber bundle. Since f is a Riemannian submersion $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal distribution isomorphic to f^*TB and $\mathcal{V} = \ker(\mathrm{d}f)$ is the vertical distribution. The relations between the curvatures of (M, g), (B, h) and the fibers $(Z_p, \hat{g}_p)$, $p \in B$, are given by O'Neill's formulas, see for instance Theorem B.6. These formulas involve the two fundamental tensors T and A, see (B.4.1) for the definition. In the remainder of this chapter many calculations are carried out in a special local orthonormal frame defined as follows:

Definition 3.1. Let $f : (M, g) \to (B, h)$ be a Riemannian affine fiber bundle. A local orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$ around a point $x \in M$ is called a *split orthonormal frame* if $(\xi_1, \ldots, \xi_n)$ is the horizontal lift of a local orthonormal frame $(\check{\xi}_1, \ldots, \check{\xi}_n)$ around the point $p = f(x) \in B$ and $(\zeta_1, \ldots, \zeta_k)$ are locally defined affine parallel vector fields tangent to the fibers.

Here and subsequently we label the vertical components $a, b, c, \ldots$, and the horizontal components $\alpha, \beta, \gamma, \ldots$. The Christoffel symbols with respect to a split orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$ can be calculated simply with the Koszul formula:

$$\begin{aligned} \Gamma^c_{ab} &= \hat{\Gamma}^c_{ab}, \\ \Gamma^\alpha_{ab} = -\Gamma^b_{a\alpha} &= g(T(\zeta_a, \zeta_b), \xi_\alpha), \\ \Gamma^b_{\alpha a} &= g([\xi_\alpha, \zeta_a], \zeta_b) + g(T(\zeta_a, \xi_\alpha), \zeta_b), \\ \Gamma^a_{\alpha\beta} = -\Gamma^\beta_{\alpha a} = -\Gamma^\beta_{a\alpha} &= g(A(\xi_\alpha, \xi_\beta), \zeta_a), \\ \Gamma^\gamma_{\alpha\beta} &= \check{\Gamma}^\gamma_{\alpha\beta}. \end{aligned} \tag{3.1.1}$$

Here $\hat{\Gamma}^c_{ab}$ are the Christoffel symbols of the fiber $(Z, \hat{g})$ with respect to $(\zeta_1, \ldots, \zeta_k)$, see (A.0.1), and $\check{\Gamma}^\gamma_{\alpha\beta}$ are the Christoffel symbols of (B, h) with respect to $(\check{\xi}_1, \ldots, \check{\xi}_n)$. For later use we need to consider the following two operators characterized by their action on vector fields X, Y.

$$\nabla^Z_X Y := \left(\nabla_{X^V} Y^V\right)^V,$$
$$\nabla^{\mathcal{V}}_X Y := \left(\nabla_{X^H} Y^V\right)^V.$$

We observe that for each $p \in B$, ∇^Z restricted to a fiber Z_p is the Levi-Civita connection with respect to the induced metric $\hat{g}_p$ on Z_p. Since $\hat{g}_p$ is by assumption affine parallel, it follows that ∇^Z preserves the space of affine parallel vector fields. The difference $\mathcal{Z} := \nabla^Z - \nabla^{\mathrm{aff}}$ is a one-form with values in $\mathrm{End}(TZ)$, where we view TZ as a vector bundle over M. We observe that $\mathcal{Z} = 0$ if and only if the induced metric $\hat{g}_p$ is flat for all $p \in B$.

The operator $\nabla^{\mathcal{V}}$ can be interpreted as a connection of the vertical distribution $\mathcal{V}$ in horizontal directions. Since the metric g is affine parallel it is immediate that $\nabla^{\mathcal{V}}$ also preserves the space of affine parallel vector fields.

As the space of affine parallel vector fields on an infranilmanifold is finite dimensional, see Appendix A, there is a finite dimensional vector bundle P over B such that, for any $p \in B$, the fiber P_p is given by the space of affine parallel vector fields of the fiber $Z_p = f^{-1}(p)$. It follows that P is a well-defined vector bundle. By the discussion above, it follows that $\mathcal{Z}$ descends to a well-defined operator on P and $\nabla^{\mathcal{V}}$ induces a connection on P. In addition, there is an $\mathcal{A} \in \Omega^2(B, P)$ characterized by

$$\mathcal{A}(X, Y) = A(\tilde{X}, \tilde{Y}),$$

for any vector fields X, Y on B. Here $\tilde{X}$, $\tilde{Y}$ denote the horizontal lifts of X and Y.

It will be shown that exactly these three operators, $\nabla^{\mathcal{V}}$, $\mathcal{Z}$, and $\mathcal{A}$ contribute additionally to the limit of Dirac operators on a collapsing sequence of spin manifolds in $\mathcal{M}(n+k, d)$ with smooth n-dimensional limit space. To ensure the continuity of the corresponding spectra, we will choose subsequences such that these three operators converge in the $C^{0,\alpha}$-topology for any $\alpha \in [0, 1)$. Our strategy is to prove uniform a priori $C^1(B)$-bounds. Then it follows from the compactness of the embedding $C^1 \hookrightarrow C^{0,\alpha}$, for $\alpha \in [0, 1)$ that there is a subsequence such that these three operators $\nabla^{\mathcal{V}}$, $\mathcal{Z}$, and $\mathcal{A}$ converge in $C^{0,\alpha}$ for any $\alpha \in [0, 1)$. For a fixed Riemannian affine fiber bundle $f : M \to B$, the $C^1(B)$-bounds on $\nabla^{\mathcal{V}}$, $\mathcal{Z}$, and $\mathcal{A}$ will depend on the following three bounds:

$$\begin{aligned} \|A\|_\infty &\leqslant C_A, \\ \|T\|_\infty &\leqslant C_T, \\ \|R^M\|_\infty &\leqslant C_R. \end{aligned}$$

We show in the following lemma that such constants exist uniformly for any sequence $(M_i, g_i)_{i \in \mathbb{N}}$ in $\mathcal{M}(n+k, d)$ converging to an n-dimensional Riemannian manifold (B, h).

Lemma 3.2. *Let $(M_i, g_i)_{i \in \mathbb{N}}$ be a sequence in $\mathcal{M}(n+k, d)$ converging to an n-dimensional Riemannian manifold (B, h). Then there is an index I such that for all $i \geqslant I$ there are metrics $\tilde{g}_i$ on M_i and $\tilde{h}_i$ on B such that $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ is a Riemannian affine fiber bundle and*

$$\begin{aligned} \lim_{i \to \infty} \|\tilde{g}_i - g_i\|_{C^1} &= 0, \\ \lim_{i \to \infty} \|\tilde{h}_i - h\|_{C^1} &= 0. \end{aligned} \tag{3.2.1}$$

In particular, there is a positive constant $C_R(n)$, such that $|\sec^{\tilde{g}_i}| \leqslant C_R$ for all $i \geqslant I$. Moreover, there are positive constants $C_A(n, k, B)$, $C_T(n+k)$ such that the fundamental tensors A_i and T_i of the Riemannian submersion $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ are uniformly bounded in norm, i.e. for all $i \geqslant I$,

$$\begin{aligned} \|A_i\|_\infty &\leqslant C_A, \\ \|T_i\|_\infty &\leqslant C_T. \end{aligned}$$

Proof. Applying Corollary 1.29, see also Remark 1.30, there is an index I such that for all $i \geqslant I$ there are metrics $\tilde{g}_i$ on M_i, and $\tilde{h}_i$ on B such that $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ is a Riemannian affine fiber bundle. By Lemma 1.27, the metrics $(\tilde{g}_i)_{i \geqslant I}$ and $(\tilde{h}_i)_{i \geqslant I}$ satisfy (3.2.1). Moreover, there is a positive constant $C_R(n)$ such that $|\sec^{\tilde{g}_i}| \leqslant C_R$ for all $i \geqslant I$, see e.g. Remark 1.30.

We recall from Theorem 1.21 that for all $i \geqslant I$ there is a Riemannian manifold $(\tilde{B}, \tilde{h}_i^F)$ with an isometric $\mathrm{O}(n+k)$-action such that $\tilde{B}/\mathrm{O}(n+k)$ is isometric to $(B, \tilde{h}_i)$. Furthermore, there is a $\Lambda_1(n+k) > 0$ such that $|\sec^{\tilde{h}_i^F}| \leqslant \Lambda_1$ for all $i \geqslant I$, see for instance Remark 1.22. Since $\tilde{B}/\mathrm{O}(n+k)$ is a Riemannian manifold, there is a $\Lambda_2(n+k, B) > 0$ such that the sectional curvature of $(B, \tilde{h}_i)$ is uniformly bounded, i.e. $|\sec^{\tilde{h}_i}| \leqslant \Lambda_2$ for all $i \geqslant I$. Thus, it follows via O'Neill's formula, B.7.3, that

$$\|A_i\|_\infty^2 \leqslant \frac{n(n-1)}{6}(|\sec^{\tilde{g}_i}| + |\sec^{\tilde{h}_i}|) \leqslant \frac{n(n-1)}{6}(C_R(n) + \Lambda_2) =: C_A^2.$$

The uniform bound on the T-tensor follows directly from Corollary 1.29. □

In the remainder of this section we prove $C^1(B)$ bounds for $\nabla^{\mathcal{V}}$, $\mathcal{Z}$, and $\mathcal{A}$ on a fixed Riemannian affine fiber bundle $f : (M, g) \to (B, h)$. As before, we set $\dim(M) = (n+k)$, $\dim(B) = n$.

First we deal with $\nabla^{\mathcal{V}}$. We denote by $(\xi_1, \ldots, \xi_n)$ the horizontal lift of a local orthonormal frame $(\check{\xi}_1, \ldots, \check{\xi}_n)$ of B. Let $(e_1, \ldots, e_k)$ be a locally affine parallel frame for the vertical distribution $\mathcal{V}$ such that $[\xi_\alpha, e_a] = 0$ for all $\alpha \in \{1, \ldots, n\}$. We write $\langle ., . \rangle$ for the locally defined metric on $\mathcal{V}$ characterized by $\langle e_a, e_b \rangle = \delta_{ab}$. There is a positive definite symmetric operator $\widehat{W}$ satisfying

$$g(\widehat{W}U, V) = \langle U, V \rangle,$$

for all vertical vector fields U, V. We consider its unique positive definite square root $W := \sqrt{\widehat{W}}$. Then,

$$g(U, V) = \langle W(U), W(V) \rangle \tag{3.2.2}$$

holds for all vertical vector fields U, V. Since the induced metric on the fiber is affine parallel, it follows that W is affine parallel as well. Setting $\zeta_a := W^{-1}(e_a)$ it is immediate that $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$ is a split orthonormal frame. A short computation shows that

$$g(T(\zeta_a, \xi_\alpha), \zeta_b) = \frac{1}{2}\langle (W^{-1}\xi_\alpha(W) + \xi_\alpha(W)W^{-1})e_a, e_b \rangle. \tag{3.2.3}$$

Hence,

$$\begin{aligned}
\Gamma_{\alpha a}^b &= g([\xi_\alpha, \zeta_a], \zeta_b) + g(T(\zeta_a, \xi_\alpha), \zeta_b) \\
&= g(\xi_\alpha(W^{-1})e_a, \zeta_b) + \frac{1}{2}\langle (W^{-1}\xi_\alpha(W) + \xi_\alpha(W)W^{-1})e_a, e_b \rangle \\
&= \frac{1}{2}\langle (W^{-1}\xi_\alpha(W) - \xi_\alpha(W)W^{-1})e_a, e_b \rangle =: \langle \mathcal{W}_{\xi_\alpha} e_a, e_b \rangle.
\end{aligned}$$

By abuse of notation, we use the same letter $\mathcal{W}$ for the connection one-form of $\nabla^{\mathcal{V}}$. As discussed above $\nabla^{\mathcal{V}}$ induces a well-defined connection on the vector bundle $P \to B$. To show that for a collapsing sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n+k, d)$ as in Lemma 3.2 there is a subsequence such that $(\nabla^{\mathcal{V}_i})_{i\in\mathbb{N}}$ converges in $C^{0,\alpha}$ for any $\alpha \in [0,1)$, it suffices to bound the norm of $\mathcal{W}$ and of its derivatives $X(\mathcal{W})$ for basic vector fields X, i.e. X is horizontal and projectable.

Lemma 3.3. *Let $f : M \to B$ be a Riemannian affine fiber bundle such that*

$$\begin{aligned} \|T\|_\infty &\leqslant C_T, \\ \|A\|_\infty &\leqslant C_A, \\ \|R^M\|_\infty &\leqslant C_R. \end{aligned}$$

Then,

$$\begin{aligned} \|\mathcal{W}_X\|_\infty &\leqslant 2C_T\|X\|, \\ \|X(\mathcal{W})\|_\infty &\leqslant C(C_T, C_A, C_R)\|X\|, \end{aligned}$$

for any basic vector field X.

Proof. It suffices to do all calculations pointwise. Hence, let $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$ be a split orthonormal frame around a fixed point $x \in M$ with $\zeta_a = W^{-1}e_a$, as above. Clearly, W can be viewed as a field of symmetric positive definite matrices.

For the first inequality, we use (3.2.3) to calculate

$$\begin{aligned} |T|^2 &= \sum_{\alpha=1}^{n}\sum_{a,b=1}^{k} g(T(\zeta_a, \xi_\alpha), \zeta_b)^2 \\ &= \frac{1}{4}\sum_{\alpha=1}^{n}\sum_{a,b=1}^{k} \langle(\xi_\alpha(W)W^{-1} + W^{-1}\xi_\alpha(W))e_a, e_b\rangle^2 \\ &= \frac{1}{4}\sum_{\alpha=1}^{n} |\xi_\alpha(W)W^{-1} + W^{-1}\xi_\alpha(W)|^2 \\ &= \frac{1}{2}\sum_{\alpha=1}^{n} \operatorname{tr}\left((W^{-1}\xi_\alpha(W))^2\right) + \operatorname{tr}\left(W^{-2}\xi_\alpha(W)^2\right). \end{aligned}$$

As W^{-1} and $\xi_\alpha(W)$ are symmetric, it follows that

$$\operatorname{tr}\left(W^{-2}\xi_\alpha(W)^2\right) = |W^{-1}\xi_\alpha(W)|^2 \geqslant 0.$$

Since W^{-1} is also symmetric and positive definite it has a unique symmetric positive-definite square root C, i.e. $C^2 = W^{-1}$. Replacing W^{-1} by C^2 leads to

$$\operatorname{tr}\left((W^{-1}\xi_\alpha(W))^2\right) = \operatorname{tr}\left(C^2\xi_\alpha(W)C^2\xi_\alpha(W)\right) = |C\xi_\alpha(W)C|^2 \geqslant 0.$$

Thus,

$$\frac{1}{2}\sum_{\alpha=1}^{n} \|W^{-1}\xi_\alpha(W)\|_\infty^2 \leqslant \|T\|_\infty^2 \leqslant C_T^2. \tag{3.3.1}$$

Now the first inequality follows immediately from

$$\|\mathcal{W}_{\xi_\alpha}\|_\infty = \frac{1}{2}\|W^{-1}\xi_\alpha(W) - \xi_\alpha(W)W^{-1}\|_\infty \leqslant 2C_T.$$

For the second inequality we also fix a point $x \in M$. Suppose that $(\xi_1, \dots, \xi_n)$ is the horizontal lift of an orthonormal frame parallel in $p = f(x)$. All following calculations are done with respect to x. We compute

$$\begin{aligned}\xi_\beta(\mathcal{W}_{\xi_\alpha}) = \frac{1}{2}\Big(&W^{-1}\xi_\beta\xi_\alpha(W) - \xi_\beta\xi_\alpha(W)W^{-1} \\ &+ \xi_\alpha(W)W^{-1}\xi_\beta(W)W^{-1} - W^{-1}\xi_\beta(W)W^{-1}\xi_\alpha(W)\Big).\end{aligned}$$

By the inequality (3.3.1), it remains to bound the second derivatives.

A straightforward calculation shows that

$$\begin{aligned}g((\nabla_{\xi_\beta}T)(\zeta_a, \zeta_b), \xi_\alpha) = \frac{1}{2}\Big\langle\Big(&\xi_\beta\xi_\alpha(W)W^{-1} + W^{-1}\xi_\beta\xi_\alpha(W) \\ &+ \xi_\beta(W)W^{-1}\xi_\alpha(W)W^{-1} \\ &- W^{-1}\xi_\beta(W)W^{-1}\xi_\alpha(W) \\ &- \xi_\beta(W)W^{-1}W^{-1}\xi_\alpha(W) \\ &+ W^{-1}\xi_\alpha(W)\xi_\beta(W)W^{-1}\Big)e_a, e_b\Big\rangle\end{aligned} \tag{3.3.2}$$

for all $1 \leqslant a, b \leqslant k$. Since $\xi_\beta\xi_\alpha(W) = \xi_\alpha\xi_\beta(W)$, it follows from the inequality (3.3.1) that

$$\chi_{\alpha,\beta,i,j} := g((\nabla_{\xi_\alpha}T)(\zeta_i, \zeta_j), \xi_\beta) - g((\nabla_{\xi_\beta}T)(\zeta_i, \zeta_j), \xi_\alpha)$$

is bounded by

$$|\chi_{\alpha,\beta,i,j}| \leqslant 8C_T^2.$$

Next, we apply the identity B.8.16 to conclude that

$$\chi_{\alpha,\beta,i,j} = g((\nabla_{\zeta_i}A)(\xi_\beta, \xi_\alpha), \zeta_j) + g((\nabla_{\zeta_j}A)(\xi_\beta, \xi_\alpha), \zeta_i).$$

Inserting this equality in the identity B.6.4 we obtain

$$\begin{aligned}g(R^M(\zeta_j, \zeta_i)\xi_\beta, \xi_\alpha) &= g((\nabla_{\zeta_i}A)(\xi_\beta, \xi_\alpha), \zeta_j) - g((\nabla_{\zeta_j}A)(\xi_\beta, \xi_\alpha), \zeta_i) \\ &\quad + g(A(\xi_\beta, \zeta_j), A(\xi_\alpha, \zeta_i)) - g(A(\xi_\beta, \zeta_i), A(\xi_\alpha, \zeta_j)) \\ &\quad - g(T(\zeta_j, \xi_\beta), T(\zeta_i, \xi_\alpha)) + g(T(\zeta_i, \xi_\beta), T(\zeta_j, \xi_\alpha)) \\ &= 2g((\nabla_{\zeta_i}A)(\xi_\beta, \xi_\alpha), \zeta_j) - \chi_{\alpha,\beta,i,j} \\ &\quad + g(A(\xi_\beta, \zeta_j), A(\xi_\alpha, \zeta_i)) - g(A(\xi_\beta, \zeta_i), A(\xi_\alpha, \zeta_j)) \\ &\quad - g(T(\zeta_j, \xi_\beta), T(\zeta_i, \xi_\alpha)) + g(T(\zeta_i, \xi_\beta), T(\zeta_j, \xi_\alpha)).\end{aligned}$$

Thus,

$$|g((\nabla_{\zeta_i}A)(\xi_\beta, \xi_\alpha), \zeta_j)| \leqslant \frac{1}{2}C_R + 5C_T^2 + C_A^2 =: C_1.$$

Using the formula B.6.3 we conclude

$$\begin{aligned} g(R(\zeta_j,\xi_\beta)\xi_\alpha,\zeta_i) &= g((\nabla_{\xi_\beta}T)(\zeta_i,\zeta_j),\xi_\alpha)) - g(T(\zeta_j,\xi_\beta),T(\zeta_i,\xi_\alpha)) \\ &\quad + g((\nabla_{\zeta_i}A)(\xi_\beta,\xi_\alpha),\zeta_j) + g(A(\xi_\beta,\zeta_j),A(\xi_\alpha,\zeta_i)). \end{aligned}$$

Hence,

$$|g((\nabla_{\xi_\beta}T)(\zeta_i,\zeta_j),\xi_\alpha))| \leqslant C_R + C_T^2 + C_A^2 + C_1 =: C_2.$$

Together with the inequality (3.3.1), we conclude from (3.3.2) that

$$\|W^{-1}\xi_\beta\xi_\alpha(W) + \xi_\beta\xi_\alpha(W)W^{-1}\|_\infty \leqslant C_2 + 4C_T^2 := C_3.$$

Using the same strategy as in the proof of the inequality (3.3.1), it follows that

$$\|W^{-1}\xi_\beta\xi_\alpha(W)\|_\infty \leqslant C_3.$$

Collecting everything so far, the claim follows by linearity and from the inequality

$$\|\xi_\beta(\mathcal{W}_{\xi_\alpha})\|_\infty \leqslant C_3 + 2C_T^2. \qquad \square$$

Remark 3.4. The connection $\nabla^{\mathcal{V}}$ is gauge equivalent to the trivial connection if and only if the holonomy $\mathrm{Hol}(\mathcal{V},\nabla^{\mathcal{V}})$ is trivial, see for instance [Bau14, Section 4.3].

In the following examples we show that $\nabla^{\mathcal{V}}$ can be gauge equivalent to the trivial connection although the T-tensor is nontrivial.

Example 3.5. Let $M = B \times \mathbb{T}^k$ be the trivial $\mathbb{T}^k$-bundle over a closed n-dimensional Riemannian manifold (B,h). In this situation the vertical distribution $\mathcal{V}$ is the trivial vector bundle $B \times \mathbb{R}^k$. For any $i \in \mathbb{N}$ we endow M with the Riemannian product metric

$$g_i := h + \frac{1}{i^2}u^2\hat{g},$$

where $u : B \to \mathbb{R}$ is a fixed smooth non constant function and $\hat{g}$ is the standard flat metric on $\mathbb{T}^k$. Then $(M,g_i)_{i\in\mathbb{N}}$ is a collapsing sequence with bounded sectional curvature and diameter. Consider the Riemannian submersions $f_i : (M,g_i) \to (B,h)$. The fibers are embedded flat tori and the horizontal distribution is integrable for all $i \in \mathbb{N}$. The T-tensor is given by

$$T_i(U,V) = \frac{\mathrm{grad}(u)}{u} g_i(U,V) \neq 0$$

for any two vertical vectors U, V and any $i \in \mathbb{N}$. In particular, the T-tensor is nontrivial for all $i \in \mathbb{N}$. We claim that the induced connection $\nabla^{\mathcal{V}}$ is trivial with respect to an isometric trivialization. To see this claim, we adapt the notation of Lemma 3.3. Let $(e_1,\dots,e_n)$ be a global orthonormal vertical frame on (M,g) such that $[X,e_i] = 0$ for all basic vector fields X and all $i \in \{1,\dots,k\}$. By construction, it follows that $(e_1,\dots,e_k)$ is a global orthogonal vertical frame for all $i \in \mathbb{N}$. With respect to this orthogonal vertical frame, W_i, defined as in (3.2.2), is given by

$$W_i = \frac{u}{i} \operatorname{Id}.$$

Hence,

$$\begin{aligned}(\mathcal{W}_i)_X &= \frac{1}{2}(W_i^{-1}X(W_i) - X(W_i)W_i^{-1}) \\ &= \frac{1}{2}\left(\frac{i}{u}\frac{X(u)}{i} - \frac{X(u)}{i}\frac{i}{u}\right)\operatorname{Id} \\ &= 0.\end{aligned}$$

Therefore, $\mathcal{W}_i = 0$ for all $i \in \mathbb{N}$, although the T-tensor is nontrivial. In particular, $\nabla^{\mathcal{V}}$ is the trivial connection on the trivial vector bundle $\mathcal{V} = B \times \mathbb{R}^k$.

Next, we give an example of a collapsing sequence such that the corresponding connections $\nabla^{\mathcal{V}_i}$ do not converge to a connection that is gauge equivalent to the trivial connection.

Example 3.6. Consider the two-dimensional torus $\mathbb{T}^2$ and choose an arbitrary nontrivial element of $\operatorname{Aut}(\mathbb{T}^2) \cong \operatorname{GL}(2,\mathbb{Z})$, e.g.

$$H := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $C := [0,1] \times \mathbb{T}^2$ be the cylinder over $\mathbb{T}^2$ and set

$$M := {C}\big/{\sim},$$

where we identify $(0,x)$ with $(1,Hx)$ for all $x \in \mathbb{T}^2$. This defines a nontrivial $\mathbb{T}^2$-bundle $f : M \to S^1$. Note that $f : M \to S^1$ is not a $\mathbb{T}^2$-principal bundle. Using a partition of unity we can find a Riemannian metric $g = h + \hat{g}$ on M such that h is the standard metric on S^1 and $(\hat{g}_p)_{p \in S^1}$ is a family of flat metrics on $\mathbb{T}^2$. Then $(M, g_i)_{i\in\mathbb{N}}$ with $g_i := h \oplus \frac{1}{i^2}\hat{g}$ defines a collapsing sequence with bounded sectional curvature and diameter such that the vertical distribution $\mathcal{V}_i$ is nontrivial for all $i \in \mathbb{N}$. In particular, $\nabla^{\mathcal{V}_i}$ is never gauge equivalent to the trivial connection.

Next, we take a look at $\mathcal{Z} = \nabla^Z - \nabla^{\mathrm{aff}} \in \Omega^1(M, \operatorname{End}(TZ))$. By construction, ∇^Z restricted to a fiber Z_p is the Levi-Civita connection of $(Z_p, \hat{g}_p)$, where $\hat{g}_p$ is the induced affine parallel metric. Moreover, $\mathcal{Z} = 0$ if and only if the induced metric $\hat{g}_p$ is flat for all $p \in B$. The following example shows that for a collapsing sequence of Riemannian affine fiber bundles with non flat fibers the one-form $\mathcal{Z}$ does not have to vanish in the limit.

Example 3.7. Let $M = {}_{\Gamma}\backslash^{N}$ be the nilmanifold, where N is the 3-dimensional Heisenberg group

$$N := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \; x, y, z \in \mathbb{R} \right\}$$

and

$$\Gamma := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \; x, y, z \in \mathbb{Z} \right\}.$$

The Lie algebra $\mathfrak{n}$ of N is given by

$$\mathfrak{n} := \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : \; x, y, z \in \mathbb{R} \right\}.$$

We fix the basis

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let X^*, Y^*, Z^* be the dual basis. We observe that the Lie brackets are given by

$$[X, Y] = Z, \; [X, Z] = 0, \; [Y, Z] = 0.$$

For any $i \in \mathbb{N}$ we consider the affine parallel metric

$$g_i := \frac{1}{i^2} X^* \cdot X^* + \frac{1}{i^2} Y^* \cdot Y^* + \frac{1}{i^4} Z^* \cdot Z^*.$$

At this point we want to remark that this construction is exactly the *inhomogeneous scaling* that we introduced in the example of collapsing nilpotent Lie groups, Example 1.20. It is not hard to check that $(M, g_i)_{i \in \mathbb{N}}$ defines a collapsing sequence with bounded curvature and diameter that converges to a point as $i \to \infty$. For each $i \in \mathbb{N}$ we consider the orthonormal frame (e_1, e_2, e_3) defined by

$$e_1 = iX, \; e_2 = iY, \; e_3 = i^2 Z.$$

The Koszul formula shows that the Christoffel symbol $\Gamma_{12}^3(i) = \frac{1}{2}$ for all $i \in \mathbb{N}$. Therefore, the Levi-Civita connection ∇^{Z_i} does not converge to the affine connection ∇^{aff} as $i \to \infty$, i.e. $\mathcal{Z}_i$ does not vanish in the limit. That $\mathcal{Z}_i = \mathcal{Z}_1$ for all $i \in \mathbb{N}$ does also follow from the fact that

$$(M, g_1) \to (M, g_i),$$

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ix & i^2 z \\ 0 & 1 & iy \\ 0 & 0 & 1 \end{pmatrix},$$

is an i^4-fold isometric covering.

Since $M \to B$ is a Riemannian affine fiber bundle, the induced metric $\hat{g}_p$ on the fiber Z_p is affine parallel. Therefore, the space of affine parallel vector fields is invariant under the action of $\mathcal{Z} \in \Omega^1(M, \mathrm{End}(TZ))$, where we view TZ as a vector bundle over M. We recall the vector bundle $P \to B$, where for any $p \in B$ the fiber P_p is given by the space of affine parallel vector fields on the fiber Z_p of $f : M \to B$. As the action of $\mathcal{Z}$ preserves the space of affine parallel vector fields, there is an induced operator $\mathcal{Z}$ on the vector bundle $P \to B$. Further, we want to remark that the actions of ∇^Z and $\mathcal{Z}$ coincide on the space of affine parallel vector fields.

Lemma 3.8. *Let $f : M \to B$ be a Riemannian affine fiber bundle such that*

$$\begin{aligned}\|T\|_\infty &\leqslant C_T,\\ \|A\|_\infty &\leqslant C_A,\\ \|R^M\|_\infty &\leqslant C_R.\end{aligned}$$

Then

$$\begin{aligned}\|\mathcal{Z}\|_\infty &\leqslant C(k, C_T, C_R),\\ \|X(\mathcal{Z})\|_\infty &\leqslant C(k, C_t, C_A, C_R)\|X\|,\end{aligned}$$

for all basic vector fields X.

Proof. Let $(\zeta_1, \dots, \zeta_k)$ be a local orthonormal vertical frame such that $\zeta_1, \dots, \zeta_k$ are affine parallel. Their structural coefficients τ_{ab}^c are defined via

$$[\zeta_a, \zeta_b] = \sum_{c=1}^{k} \tau_{ab}^c \zeta_c.$$

These are the structural coefficients of the Lie algebra $\mathfrak{n}$ of the connected and simply-connected nilpotent Lie group N that covers Z. We recall from (3.1.1) and (A.0.1) that

$$\Gamma_{ab}^c = \hat{\Gamma}_{ab}^c = \frac{1}{2}\left(\tau_{ab}^c - \tau_{ac}^b - \tau_{bc}^a\right).$$

By [Lot02c, Lemma 1] we have

$$\sum_{a,b,c=1}^{k} (\tau_{ab}^c)^2 = -4\,\mathrm{scal}(Z).$$

Thus,

$$\begin{aligned}\|\mathcal{Z}\|^2 &= \sum_{a,b,c=1}^{k} (\Gamma_{ab}^c)^2\\ &= \frac{3}{4}\sum_{a,b,c=1}^{k} (\tau_{ab}^c)^2 + \frac{1}{2}\sum_{a,b,c=1}^{k} (\tau_{ac}^b\tau_{bc}^a - \tau_{ab}^c\tau_{ac}^b - \tau_{ab}^c c\tau_{bc}^a)\\ &= -3\,\mathrm{scal}(Z),\end{aligned}$$

because $\sum_{b,c=1}^{k} \tau_{ab}^{c}\tau_{ac}^{b} = 0$ since $\mathfrak{n}$ is nilpotent. The first inequality follows from O'Neill's formula, B.7.1,

$$\begin{aligned}
|\operatorname{scal}(Z)| &\leqslant \sum_{a,b=1} |\sec^{Z}(\zeta_a, \zeta_b)| \\
&= \sum_{a,b=1} |\sec^{M}(\zeta_a, \zeta_b) - |T(\zeta_a, \zeta_b)|^2 + g(T(\zeta_a, \zeta_a), T(\zeta_b, \zeta_b))| \\
&\leqslant k^2(C_R + 2C_T^2).
\end{aligned}$$

The second inequality is also proven in local coordinates. First we observe that

$$|X(\mathcal{Z})|^2 = \sum_{a,b,c=1}^{k} |X(\Gamma_{ab}^{c})|^2$$

for any basic vector field X. Since $\Gamma_{ab}^{c} = g(\nabla_{\zeta_a}\zeta_b, \zeta_c)$ it follows that

$$\begin{aligned}
|X(\Gamma_{ab}^{c})| &= |g(\nabla_X \nabla_{\zeta_a}\zeta_b, \zeta_c) - g(\nabla_{\zeta_a}\zeta_b, \nabla_X \zeta_c)| \\
&= |g(R^M(X, \zeta_a)\zeta_b + \nabla_{[X,\zeta_a]}\zeta_b + \nabla_{\zeta_a}\nabla_X \zeta_b, \zeta_c) - g(\nabla_{\zeta_a}\zeta_b, \nabla_X \zeta_c)| \\
&= |g(R^M(X, \zeta_a)\zeta_b + \nabla_{[X,\zeta_a]}\zeta_b, \zeta_c) - g(\nabla_X \zeta_b, \nabla_{\zeta_a}\zeta_c) - g(\nabla_{\zeta_a}\zeta_b, \nabla_X \zeta_c)|.
\end{aligned}$$

As the Lie bracket $[X, \zeta_a]$ is vertical, for any basic vector field X and $1 \leqslant a \leqslant k$, we use Lemma 3.3 to conclude that

$$\begin{aligned}
|g(\nabla_{[X,\zeta_a]}\zeta_b, \zeta_c)| &\leqslant |[X, \zeta_a]| \, \|\mathcal{Z}\| \\
&= |\nabla_X^{\mathcal{V}}\zeta_a + T(\zeta_a, X)| \|\mathcal{Z}\| \\
&\leqslant 3C_T \|\mathcal{Z}\| |X|,
\end{aligned}$$

and

$$\begin{aligned}
|g(\nabla_X \zeta_b, \nabla_{\zeta_a}\zeta_c)| &\leqslant |g(\nabla_X^{\mathcal{V}}\zeta_b, \nabla_{\zeta_a}^{Z}\zeta_c)| + |g(A(X, \zeta_a), T(\zeta_a, \zeta_c))| \\
&\leqslant (2C_T\|\mathcal{Z}\| + C_A C_T)|X|,
\end{aligned}$$

hold for any basic vector field X. Now we combine these inequalities to conclude

$$|X(\Gamma_{ab}^{c})| \leqslant \Big(C_R + 7C_T\|\mathcal{Z}\| + 2C_A C_T\Big)|X|. \qquad \square$$

Finally, we consider $\mathcal{A} \in \Omega^2(B, \mathcal{P})$. On a Riemannian affine fiber bundle $f : M \to B$ this two-form is characterized by the property that $(f^*\mathcal{A})(X, Y) = A(X, Y)$ for all basic vector fields X, Y. Recall that for any $p \in B$ the fiber P_p of the vector bundle $P \to B$ is the space of affine parallel vector fields of the fiber Z_p of the Riemannian affine fiber bundle $f : M \to B$. In the following example we see that this two-form $\mathcal{A}$ can be non zero while $\mathcal{Z}$ and $\nabla^{\mathcal{V}}$ are trivial.

Example 3.9. Let $f : (M, g) \to (B, h)$ be an S^1-principal bundle such that f is a Riemannian submersion with totally geodesic fibers of length 2π. Suppose further that the curvature form $\mathcal{A}$ of the S^1-principal bundle is nontrivial. We observe that for any $i \in \mathbb{N}$, the cyclic subgroup $\mathbb{Z}_i < S^1$ acts on M as isometries. Thus, the sequence $(M/\mathbb{Z}_i, g_i)_{i\in\mathbb{N}}$ converges with bounded sectional curvature and diameter to (B, h). Here g_i is the induced quotient metric. By construction, the T-tensor and the one-form $\mathcal{Z} \in \Omega(TS^1)$ vanish identically for any $i \in \mathbb{N}$. But the A-tensor of the fibration $(M/\mathbb{Z}_i, g_i) \to (B, h)$ is given by

$$A_i(X, Y) = -\frac{1}{2}\mathcal{A}(X, Y)V,$$

where X, Y are basic vector fields and V is a vertical vector field of unit length with respect to the metric g_i. In particular, $\mathcal{A}_i = \mathcal{A}$ for all $i \in \mathbb{N}$.

Lemma 3.10. *Let $f : M \to B$ be a Riemannian affine fiber bundle. Then there exists an $\mathcal{A} \in \Omega^2(B, P)$ such that $f^*\mathcal{A}(X, Y) = A(X, Y)$ for all basic vector fields X, Y. If in addition,*

$$\begin{aligned}\|A\|_\infty &\leqslant C_A,\\ \|T\|_\infty &\leqslant C_T,\\ \|R^M\|_\infty &\leqslant C_R,\end{aligned}$$

then

$$\|\mathcal{A}\|_{C^1(B)} \leqslant C(k, n, C_A, C_T, C_R).$$

Proof. As $f : M \to B$ is a Riemannian affine fiber bundle, the induced metrics on the fibers are affine parallel. Therefore, $A(X, Y)$ is an affine parallel vector field for all basic vector fields X, Y. In particular, there is an $\mathcal{A} \in \Omega^2(B, P)$ such that $f^*\mathcal{A}(X, Y) = A(X, Y)$ for all basic vector fields X, Y.

It remains to bound $\mathcal{A}$ in $C^1(B)$. We use a split orthonormal frame, see Definition 3.1. For any pair of basic vector fields X, Y we obtain

$$\begin{aligned}(f^*\mathcal{A})(X, Y) = A(X, Y) &= \sum_{a=1}^{k} g(A(X, Y), \zeta_a)\,\zeta_a\\ &=: \sum_{a=1}^{k} A^a(X, Y)\,\zeta_a\\ &=: \sum_{a=1}^{k} (f^*\mathcal{A}^a)(X, Y)\,\zeta_a\ .\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathcal{A}\|_{C^1(B)} &\leqslant \sum_{a=1}^{k} (\|\mathcal{A}^a\|_\infty + \|\nabla(\mathcal{A}^a)\|_\infty)\\ &\leqslant kC_A + \sum_{a=1}^{k} \|\nabla(\mathcal{A}^a)\|_\infty.\end{aligned}$$

We calculate the second term pointwise. Let $p \in B$ be arbitrary and $x \in f^{-1}(p)$. Further, let $(\xi_1, \ldots, \xi_k)$ be the horizontal lift of a local orthonormal frame $\left(\check{\xi}_1, \ldots, \check{\xi}_n\right)$ that is parallel in $p \in B$. Using the explicit description of the Christoffel symbols, (3.1.1), we derive

$$\begin{aligned}
|\nabla(\mathcal{A}^a)|^2 &= \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{n} |(\nabla_{\check{\xi}_\alpha} \mathcal{A}^a)(\check{\xi}_\beta, \check{\xi}_\gamma)|^2 \\
&= \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{n} \sum_{a=1}^{k} |g\big((\nabla_{\xi_\alpha} A)(\xi_\beta, \xi_\gamma), \zeta_a\big)|^2 \\
&= \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{n} \sum_{a=1}^{k} \Big| g\big(R^M(\xi_\beta, \xi_\gamma)\xi_\alpha, \zeta_a\big) - g\big(A(\xi_\beta, \xi_\gamma), T(\zeta_a, \xi_\gamma)\big) \\
&\qquad\qquad + g\big(A(\xi_\gamma, \xi_\alpha), T(\zeta_a, \xi_\beta)\big) + g\big(A(\xi_\alpha, \xi_\beta), T(\zeta_a, \xi_\gamma)\big) \Big|^2 \\
&\leqslant \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{n} (C_R + 3C_A C_T)^2.
\end{aligned}$$

Here, we used in the first line that $\left(\check{\xi}_1, \ldots, \check{\xi}_n\right)$ is parallel in p and in the third line we applied the O'Neill formula B.6.5. □

3.2 Spin structures on Riemannian affine fiber bundles

In this section we study Riemannian affine fiber bundles $f : (M, g) \to (B, h)$ where (M, g) is a spin manifold with a fixed spin structure. Let us briefly review the definition of a spin structure: For any positive integer n the group $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$. If $n \geqslant 3$, $\mathrm{Spin}(n)$ is actually the universal cover of $\mathrm{SO}(n)$. A *spin structure* on an orientable Riemannian manifold (M, g) is a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}}M$ that defines a double cover of the oriented orthonormal frame bundle $P_{\mathrm{SO}}M$ that is compatible with the group double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, i.e.

$$\begin{tikzcd}
\mathrm{Spin}(n) \times P_{\mathrm{Spin}}M \arrow[r] \arrow[d] & P_{\mathrm{Spin}}M \arrow[d] \arrow[dr] & \\
& & M . \\
\mathrm{SO}(n) \times P_{\mathrm{SO}}M \arrow[r] & P_{\mathrm{SO}}M \arrow[ur] &
\end{tikzcd}$$

Locally there is always a double cover of $P_{\mathrm{SO}}M$, but it might not extend to a globally well-defined $\mathrm{Spin}(n)$-principal bundle over M. The condition for a manifold to be spin is the vanishing of the second Stiefel-Whitney class which is defined purely topologically. The *spinor bundle* ΣM is the associated complex vector bundle of $P_{\mathrm{Spin}}M$ and the canonical complex spinor representation $\theta_n : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n)$ where Σ_n is a vector space with complex dimension $\dim_{\mathbb{C}}(\Sigma_n) = 2^{\left[\frac{n}{2}\right]}$. The standard literature for spin structures and spin manifolds is [LM89]. Further good references, we want to mention here, are [Fri97] and [BHM+15].

First we consider a fixed Riemannian affine fiber bundle $f : M \to B$ and discuss whether the spin structure on M induces a spin structure on B or on the fibers Z_p, $p \in B$. The main observation of the first subsection is that there is an induced spin structure on each fiber Z_p, $p \in B$. Moreover, we cannot determine whether there is an induced structure on B if $\dim(Z) = k \geqslant 2$. If $k = 1$ then the fibers of the Riemannian affine fiber bundle $f : M \to B$ have to be diffeomorphic to S^1. In that special case there is always an induced structure on B. For instance, if $f : M \to B$ is an S^1-principal bundle then there is an induced spin structure on B if the S^1-action lifts to $P_{\mathrm{Spin}}M$. Otherwise there is an induced spin^c structure. For the case of B being non orientable we have similar results with pin structures. Roughly speaking, pin structures can be interpreted as a generalization of spin structures to non orientable spaces. Since the limit space of a codimension one collapse in $\mathcal{M}(n, d)$ is a Riemannian orbifold, see Proposition 1.31, we extend our observation to Riemannian orbifolds in the case $k = 1$.

In the second subsection we restrict ourselves to spin structures with affine parallel spinors. For a Riemannian affine fiber bundle $f : M \to B$ the induced metrics on the fibers are affine parallel. Thus, there is an induced affine connection ∇^{aff} on the spinor bundle ΣM. Hence, we can consider the space of affine parallel spinors on M. It follows that the Dirac operator leaves the space of affine parallel spinors invariant. The main result of the second subsection is that the space of affine parallel spinors is isometric to the space of sections of a twisted Clifford bundle over the base manifold B. Furthermore, there is an elliptic first order self-adjoint differential operator $\mathcal{D}^B$ that is isospectral to the Dirac operator on M restricted to the space of affine parallel spinors.

3.2.1 Induced structures

Let $f : (M, g) \to (B, h)$ be a Riemannian affine fiber bundle with infranil fiber Z. In the remainder of this section we assume that M is a spin manifold with a fixed spin structure. Since any fiber Z_p, $p \in B$, is an embedded oriented submanifold with trivial normal bundle, it follows that there is an induced spin structure on Z_p. Moreover, each path in B connecting two points $p, q \in B$ induces an isomorphism between the induced spin structure on Z_p and the induced spin structure on Z_q. In particular, the spin structures on Z_p and Z_q are equivalent, for all $p, q \in B$. The construction of this isomorphism is analogous to the construction given in Appendix C. Nevertheless, there is in general no induced spin structure on B as can be seen in the example of the Hopf fibration $S^5 \to \mathbb{C}\mathrm{P}^2$, c.f. Example 3.16. There are even examples of Riemannian affine fiber bundles $M \to B$ where M is spin and B is non orientable.

Example 3.11. Let $M := \mathrm{U}(1) \times_{\mathbb{Z}_2} S^2$, where $\mathbb{Z}_2$ acts on $\mathrm{U}(1)$ via complex conjugation and on S^2 via the antipodal map. Then M is spin and $f : M \to \mathbb{R}\mathrm{P}^2$ is a nontrivial S^1-bundle over the non orientable manifold $\mathbb{R}\mathrm{P}^2$.

For this reason, we also consider pin$^\pm$ structures. Loosely speaking, pin$^\pm$ structures are a generalization of spin structures to a non orientable setting. In the following, we briefly sketch the definition and basic properties of pin$^\pm$ structures. For further details, we refer to [KT90] and [Gil89, Appendix A].

The double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ can be extended to a double cover of $\mathrm{O}(n)$ in two inequivalent ways, called $\rho^+ : \mathrm{Pin}^+(n) \to \mathrm{O}(n)$ and $\rho^- : \mathrm{Pin}^-(n) \to \mathrm{O}(n)$. As topological spaces $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$ are both homeomorphic to $\mathrm{Spin}(n) \sqcup \mathrm{Spin}(n)$ but the group structures of $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$ are different. To see this, we consider the subgroup $\{\mathrm{Id}, r\} \subset \mathrm{O}(n)$, where r is a reflection along a hyperplane. Then

$$\begin{aligned}(\rho^+)^{-1}(\{\mathrm{Id}, r\}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,\\ (\rho^-)^{-1}(\{\mathrm{Id}, r\}) &\cong \mathbb{Z}_4.\end{aligned}$$

The notion of $\mathrm{pin}^\pm$ structures is an extension of the definition of spin structures to the double covers $\mathrm{Pin}^\pm(n) \to \mathrm{O}(n)$.

Definition 3.12. A *$\mathrm{pin}^\pm$ structure* on an n-dimensional Riemannian manifold (M, g) is a $\mathrm{Pin}^\pm$-principal bundle $P_{\mathrm{Pin}^\pm}M$ that is a double cover of the orthonormal frame bundle $P_{\mathrm{O}}M$ compatible with the group double cover $\mathrm{Pin}^\pm(n) \to \mathrm{O}(n)$.

$$\begin{tikzcd}
\mathrm{Pin}^\pm(n) \times P_{\mathrm{Pin}^\pm}M \arrow[r] \arrow[d] & P_{\mathrm{Pin}^\pm}M \arrow[d] \arrow[dr] & \\
 & & M . \\
\mathrm{O}(n) \times P_{\mathrm{O}}M \arrow[r] & P_{\mathrm{O}}M \arrow[ur] &
\end{tikzcd}$$

Example 3.13.

$$\text{The real projective space } \mathbb{RP}^n \text{ is } \begin{cases} \mathrm{pin}^+, & \text{if } n = 4k, \\ \mathrm{pin}^-, & \text{if } n = 4k+2, \\ \mathrm{spin}, & \text{if } n = 4k+3. \end{cases}$$

Similar to spin structures, the existence of a $\mathrm{pin}^\pm$ structure is a topological property characterized by the vanishing of specific Stiefel-Whitney classes. The proof of the following theorem can be found in [KT90, Lemma 1.3].

Theorem 3.14. *A manifold M admits a pin^+ structure if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes and a pin^- structure if and only if the Stiefel-Whitney classes satisfy the equation $w_2(M) + w_1(M)^2 = 0$. The topological condition for a spin structure is $w_2(M) = w_1(M) = 0$.*

Since the first Stiefel-Whitney class of a manifold M vanishes if and only if M is orientable it is an immediate consequence of the above theorem that an orientable manifold is spin if and only if it admits a $\mathrm{pin}^\pm$ structure.

As we are interested in the question whether the spin structure on the total space M of a Riemannian affine fiber bundle $f : (M, g) \to (B, h)$ induces a spin or $\mathrm{pin}^\pm$ structure on B we also consider the interplay between spin and $\mathrm{pin}^\pm$ structures on short exact sequences of vector bundles,

$$0 \to E \to F \to G \to 0.$$

We recall that a short exact sequence of vector bundles splits, i.e. $F \cong E \oplus G$. Moreover, it is well-known that the i-th Stiefel-Whitney class w_i of the Whitney sum $E \oplus G$ is given by

$$w_i(E \oplus G) = \sum_{k=1}^{i} w_i(E) \cup w_{i-k}(G),$$

where $\cup$ is the cup product. Together with the above theorem, we conclude the following lemma, see [Gil89, Lemma A.1.5].

Lemma 3.15. *Let*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

be a short exact sequence of real vector bundles over a manifold M. For any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, we have that

(1) if V_i and V_j are spin, there is an induced spin structure on V_k,

(2) if V_i is spin and V_j is pin$^{\pm}$, there is an induced pin$^{\mp}$ structure on V_k,

(3) if V_i is pin$^{\pm}$ and V_j is pin$^{\mp}$ and V_k is orientable, then there is an induced spin structure on V_k.

Let $f : (M, g) \to (B, h)$ be a Riemannian affine fiber bundle and assume that M is spin. Then we have the following short exact sequence of vector bundles,

$$0 \to f^*TB \to TM \to \mathcal{V} \to 0.$$

Here $\mathcal{V} = \ker(\mathrm{d}f)$ is the vertical distribution. As TM is by assumption spin, it follows from Lemma 3.15 that f^*TB is spin if and only if $\mathcal{V}$ is spin and that f^*TB is pin$^{\pm}$ if and only if $\mathcal{V}$ is pin$^{\mp}$. But a spin or pin$^{\pm}$ structure on f^*TB does not induce a corresponding structure on B itself as can be seen in the following example.

Example 3.16. Consider $S^5 \to \mathbb{C}\mathrm{P}^2$. Then $f^*w_2(\mathbb{C}\mathrm{P}^2) \in H^2(S^5, \mathbb{Z}_2)$. But $H^2(S^5, \mathbb{Z}_2)$ is trivial. Hence, $f^*w_2(\mathbb{C}\mathrm{P}^2) = 0$ although $\mathbb{C}\mathrm{P}^2$ is not spin.

Nevertheless, if B is spin or pin$^{\pm}$ then the corresponding structure can be pulled back to f^*TB. Thus, if B is pin$^{\pm}$ and $\mathcal{V}$ is pin$^{\mp}$ then there is an induced spin structure on M by Lemma 3.15.

If $k = 1$ the fiber Z of a Riemannian affine fiber bundle has to be diffeomorphic to S^1. As discussed in Chapter 2, a sequence $(M_i, g_i)_{i \in \mathbb{N}}$ in $\mathcal{M}(n, d)$ that converges to a compact metric space Y with $\dim(Y) \geqslant (n-1)$ can be characterized by a uniform lower bound $C \leqslant \frac{\mathrm{vol}(M_i)}{\mathrm{inj}(M_i)}$ for all $i \in \mathbb{N}$. Furthermore, the limit has to be a Riemannian orbifold, see Proposition 1.31. We recall the set

$$\mathcal{M}(n+1, d, C) := \left\{ (M, g) \in \mathcal{M}(n+1, d) : C \leqslant \frac{\mathrm{vol}(M)}{\mathrm{inj}(M)} \right\}$$

of isometry classes of closed Riemannian manifolds from Definition 2.13. Combining Theorem 2.17 and Corollary 1.29, it follows that for any collapsing sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n+1, d, C)$ converging to (B, h) in the Gromov-Hausdorff topology, there is, for any i sufficiently large, an S^1-orbifold bundle $f_i : M_i \to B$ with structure group in $\mathrm{Aff}(S^1) \cong S^1 \rtimes \{\pm 1\}$.

For a fixed S^1-orbifold bundle $f : M \to B$ with M being spin there are exactly two cases that can occur. If B is orientable then $f : M \to B$ is an S^1-principal orbifold bundle. If B is non orientable then the vertical distribution $\mathcal{V}$ has to be isomorphic to the pullback of the determinant bundle $\det(TB)$ of B.

Similar to [Amm98a] and [Mor96] we distinguish between two types of spin structures on the total space M: The projectable and the non projectable spin structures. Projectable spin structures and projectable spinors were studied for G-principal bundles with compact Lie group G in [Mor96, Chapitre 1]. Since, in general, S^1 does not act by isometries we have replaced the spin structure by the larger so-called *topological spin structure* $P_{\widetilde{\mathrm{GL}}_+}M \to P_{\mathrm{GL}_+}M$. Here $P_{\mathrm{GL}_+}M$ is the $\mathrm{GL}_+(n)$-principal bundle consisting of all positively oriented frames and $P_{\widetilde{\mathrm{GL}}_+}M$ is a double cover of $P_{\mathrm{GL}_+}M$ that is compatible with the corresponding group double cover $\widetilde{\mathrm{GL}}_+(n) \to \mathrm{GL}_+(n)$, i.e.

$$\begin{tikzcd}
\widetilde{\mathrm{GL}}_+(n) \times P_{\widetilde{\mathrm{GL}}_+}M \arrow[r] \arrow[d] & P_{\widetilde{\mathrm{GL}}_+}M \arrow[d] \arrow[dr] & \\
 & & M \, . \\
\mathrm{GL}_+(n) \times P_{\mathrm{GL}_+}M \arrow[r] & P_{\mathrm{GL}_+}M \arrow[ur] &
\end{tikzcd}$$

Definition 3.17. Let $M \to B$ be an S^1-orbifold bundle with M being spin. Then the spin structure of M is called *projectable* if all local S^1-actions along the fibers lift to the topological spin structure.

Remark 3.18. For Riemannian affine fiber bundles $f : (M, g) \to (B, h)$ with S^1 fibers a spin structure on M is projectable if and only if there are nontrivial affine parallel spinors.

In the case of an S^1-principal bundle $f : M \to B$, where f is a Riemannian submersion, a projectable spin structure on M induces a spin structure on B, [Amm98a, Section 2]. We first show that a projectable spin structure on the total space of an S^1-principal orbifold bundle $f : M \to B$ induces a spin structure on the orbifold B. To the author's knowledge, the first definition of spin orbifolds appeared in [DLM02].

Definition 3.19. An oriented Riemannian orbifold (B, h) is spin if there exists a two-sheeted cover of the oriented orthonormal frame bundle $P_{\mathrm{SO}}B$ such that for any orbifold chart $\left(\tilde{U} \to \tilde{U}/G_U \cong U \subset B\right)$ there exists a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}}\tilde{U}$ on $\tilde{U}$ such that the spin structure $P_{\mathrm{Spin}}B_{|U} \to P_{\mathrm{SO}}B_{|U}$ is induced by $P_{\mathrm{Spin}}\tilde{U} \to P_{\mathrm{SO}}\tilde{U}$.

Hence, the spin structure on a Riemannian orbifold can be understood as a locally G_p-invariant spin structure on the locally defined smooth cover around $p \in B$. Here, G_p is the stabilizer group of the Riemannian orbifold (B, h) at p. This requires a lift of the group G_p of isometries to the spin bundle.

Definition 3.20. A singular point $p \in B$ is said to be *spin* if there exists a lift $\widetilde{G}_p$ of the group $G_p \subset \mathrm{SO}(n)$ that projects isomorphically onto G_p via the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

From now on, a spin orbifold is a Riemannian orbifold with a fixed spin structure.

Proposition 3.21. *Let $f : M \to B$ be an S^1-principal orbifold bundle. If M is a spin orbifold with a projectable spin structure then there is an induced spin structure on B. On the other hand, if B is a spin orbifold then there is an induced projectable spin structure on M.*

Proof. Since all metric spin structures are isomorphic to each other, see Appendix C, we can assume without loss of generality that $f : M \to B$ is a Riemannian submersion, i.e. S^1 acts on M as isometries. The following proof is a locally equivariant version of the construction given in [Mor96, Chapter 1].

For $p \in B$ we consider a local trivialization U around p. The local situation is described by

$$\begin{tikzcd}
S^1 \times \tilde{U} \arrow[r] \arrow[d] & (S^1 \times \tilde{U})/_{G_U} \cong f^{-1}(U) \arrow[d] \\
\tilde{U} \arrow[r] & \tilde{U}/_{G_U} .
\end{tikzcd}$$

It follows that the spin structure on $S^1 \times \tilde{U}$ is G_U invariant.

If the spin structure on M is projectable, i.e. S^1-invariant, the spin structure on $S^1 \times \tilde{U}$ is $S^1 \times G_U$ invariant. It follows that the spin structure on M induces a G_U-invariant spin structure on $\tilde{U}$ which in turn defines a spin structure on the quotient U.

On the other hand, if B is a spin orbifold it follows that the spin structure on M induced by the pullback of the spin structure on B has to be S^1-invariant, i.e. projectable. □

However, it can happen that a collapsing sequence of spin manifolds converges to a non orientable space as we have seen in Example 3.11. In that situation we have to modify the proof of the above proposition.

Proposition 3.22. *Let $f : M \to B$ be an S^1-orbifold bundle where B is a non orientable Riemannian orbifold. Then any projectable spin structure on M induces a pin$^-$ structure on B. Conversely, if B is pin$^-$ and M is orientable then there is an induced projectable spin structure on M.*

Proof. As in the proof of Proposition 3.21 we assume that f is a Riemannian submersion. In our situation it follows that $TM \cong f^*(TB \oplus \det(TB))$. If the spin structure on M is projectable then the quotient $P_{\mathrm{Spin}}M/_{S^1}$ induces a spin structure on $TB \oplus \det(TB)$. Since $\det(TB)$ is a non orientable line bundle its first Stiefel-Whitney class is nontrivial and its second Stiefel-Whitney class vanishes. Thus, $\det(TB)$ is pin$^+$ by Theorem 3.14.

Now it is a direct consequence of Lemma 3.15 that there is an induced pin$^-$ structure on B.

On the other hand, if B is pin$^-$ then there is an induced spin structure on $TB \oplus \det(TB)$ by Lemma 3.15. This spin structure pulls back to a projectable spin structure on M. $\square$

Next, we consider an S^1-principal orbifold bundle $f : M \to B$ such that the spin structure on M is non projectable. As before, we assume without loss of generality that S^1 acts as isometries. Since the spin structure of M is non projectable the S^1-action does not lift to $P_{\text{Spin}}M$. Nevertheless, the double cover of S^1 acts on $P_{\text{Spin}}M$, where we use the double cover $S^1 \to S^1, \lambda \mapsto \lambda^2$. At this point we want to remark that a non projectable spin structure on B does not imply that B is not spin. If B is spin, then there exists a group homomorphism $\psi : \pi_1(M) \to \mathbb{Z}_2$ such that the composition $\pi_1(S^1) \hookrightarrow \pi_1(M) \to \mathbb{Z}_2$ is surjective. In this case, we can twist the spin structure on M with ψ to obtain a projectable spin structure. In short, B is spin if and only if $M \to B$ has a square root as a S^1-principal bundle (cf. [Amm98b, Chapter 7.3]). Even if we can not determine whether B is spin or not, we still have an induced structure on B. In the following lemma we repeat the proof of [Amm98a, Section 4].

Lemma 3.23. *Let $f : M \to B$ be an S^1-principal orbifold bundle. If M is a spin orbifold with a non projectable spin structure then there is an induced spin^c structure on B.*

Proof. Let $P_{\text{SO}(n)}M$ be the $\text{SO}(n)$-principal bundle over M consisting of all positively oriented orthonormal frames whose first vector is vertical. Its preimage defines a $\text{Spin}(n)$-principal bundle P. Since the spin structure on M is non projectable, it follows that not the S^1-action itself but its double cover acts on P. This S^1-action together with the $\text{Spin}(n)$-action on P induces a free $\text{Spin}^c(n) := (\text{Spin}(n) \times_{\mathbb{Z}_2} S^1)$-action on P. Hence,there is an induced spin^c structure on B. $\square$

Conversely, if we have a fixed S^1-principal orbifold bundle $f : M \to B$ such that B is spin^c, then it does not follow that M is a spin manifold.

Example 3.24. Let $M := S^1 \times \mathbb{C}\text{P}^2$ be the trivial S^1-bundle over the complex projective space $\mathbb{C}\text{P}^2$. It is known that $\mathbb{C}\text{P}^2$ is spin^c but not spin. Since any spin structure on M would induce a spin structure on $\mathbb{C}\text{P}^2$, M cannot be spin.

An easy modification of the proof of Lemma 3.23 shows that if $f : M \to B$ is an S^1-orbifold bundle such that the spin structure on M is non projectable and B is non orientable then there is an induced a pin^c structure on B, where $\text{Pin}^c(n) := \text{Pin}^-(n) \times_{\mathbb{Z}_2} S^1$.

3.2.2 Spin structures with affine parallel spinors

Let $f : (M, g) \to (B, h)$ be a fixed Riemannian affine fiber bundle and let Z be the fiber. In Appendix B we discuss in detail Riemannian submersions whose total space has a fixed spin structure. In the following, we use the results derived in Appendix B.

We set $n := \dim(B)$ and $k := \dim(Z)$. Suppose that (M, g) has a fixed spin structure. Since f is a Riemannian submersion, we know from Appendix B that

$$\Sigma M \cong f^*({}^{\diamond}\boldsymbol{\Sigma} B) \otimes \Sigma \mathcal{V},$$

where

$$^{\diamond}\mathbf{\Sigma} B := \begin{cases} \Sigma B, & \text{if } n \text{ or } k \text{ is even,} \\ \Sigma^+ B \oplus \Sigma^- B, & \text{if } n \text{ and } k \text{ are odd.} \end{cases}$$

Here ΣB and $\Sigma \mathcal{V}$ are the locally defined spinor bundles of the base manifold B and the vertical distribution $\mathcal{V}$. We refer to Appendix B for details. Although they might not be defined globally their tensor product is well-defined on all of M. Furthermore, $\Sigma^+ B$ and $\Sigma^- B$ denote two copies of ΣB where Clifford multiplication γ with vector fields $X \in \Gamma(TB)$ acts on $\Sigma^+ B$ as $\gamma(X)$ and as $-\gamma(X)$ on $\Sigma^- B$. We recall the basic properties of Clifford multiplication in Appendix B. For a thorough introduction to spin geometry we recommend [LM89, BHM+15].

Since $\Sigma M \cong f^*(^{\diamond}\mathbf{\Sigma} B) \otimes \Sigma \mathcal{V}$, any spinor Φ can locally be written as a finite linear combination $\Phi = \sum_l f^* \varphi_l \otimes \nu_l$. This decomposition allows us to study the influence of the horizontal and the vertical distributions on the spinors on M. The following formulas are a special case of the formulas in Lemma B.4. They follow from straightforward calculations using the local formulas for the spinorial connection, (B.3.1) and the Dirac operator, (B.3.2).

Lemma 3.25. *Let $f : (M, g) \to (B, h)$ be a Riemannian affine fiber bundle such that M has a fixed spin structure. With respect to a split orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$, see Definition 3.1, any spinor $\Phi = f^* \varphi \otimes \nu$ satisfies the following identities.*

$$\begin{aligned}
\nabla^M_{\xi_\alpha} \Phi &= (f^* \nabla^B_{\xi_\alpha} \varphi) \otimes \nu + f^* \varphi \otimes \nabla^{\mathcal{V}}_{\xi_\alpha} \nu + \frac{1}{2} \sum_{\beta=1}^{n} \gamma(\xi_\beta) \gamma \left(A(\xi_\alpha, \xi_\beta) \right) \Phi \\
&=: \nabla^{\mathcal{T}}_{\xi_\alpha} \Phi + \frac{1}{2} \sum_{\beta=1}^{n} \gamma(\xi_\beta) \gamma \left(A(\xi_\alpha, \xi_\beta) \right) \Phi, \\
\nabla^M_{\zeta_a} \Phi &= f^* \varphi \otimes \nabla^Z_{\zeta_a} \nu + \frac{1}{2} \sum_{b=1}^{k} \gamma(\zeta_b) \gamma \left(T(\zeta_a, \zeta_b) \right) \Phi + \frac{1}{4} \sum_{\alpha=1}^{n} \gamma(\xi_\alpha) \gamma \left(A(\xi_\alpha, \zeta_i) \right) \Phi \\
&=: \nabla^Z_{\zeta_a} \Phi + \frac{1}{2} \sum_{b=1}^{k} \gamma(\zeta_b) \gamma \left(T(\zeta_a, \zeta_b) \right) \Phi + \frac{1}{4} \sum_{\alpha=1}^{n} \gamma(\xi_\alpha) \gamma \left(A(\xi_\alpha, \zeta_a) \right) \Phi, \\
D^M \Phi &= \sum_{\alpha=1}^{n} \gamma(\xi_\alpha) \nabla^T_{\xi_\alpha} \Phi + \sum_{a=1}^{k} \gamma(\zeta_a) \nabla^Z_{\zeta_a} \Phi - \frac{1}{2} \sum_{a=1}^{k} \gamma \left(T(\zeta_a, \zeta_a) \right) \Phi \\
&\quad + \frac{1}{2} \sum_{\substack{\alpha, \beta = 1 \\ \alpha < \beta}}^{n} \gamma \left(A(\xi_\alpha, \xi_\beta) \right) \gamma(\xi_\alpha) \gamma(\xi_\beta) \Phi \\
&=: D^{\mathcal{T}} \Phi + D^Z \Phi - \frac{1}{2} \sum_{a=1}^{k} \gamma \left(T(\zeta_a, \zeta_a) \right) \Phi + \frac{1}{2} \gamma(A) \Phi.
\end{aligned}$$

Here ∇^B, $\nabla^{\mathcal{V}}$ and ∇^Z are the induced connections by the respective connections on TM, defined in Section 3.1, and $D^{\mathcal{T}}, D^Z$ are the associated Dirac operators.

Since for any $p \in B$ the induced metric $\hat{g}_p$ on the fiber Z_p is affine parallel it follows from the discussion in Appendix A that the affine connection ∇^{aff} induces an affine connection, also denoted by ∇^{aff}, on the spinor bundle ΣM. Hence, the space of affine parallel spinors is well-defined,

$$\mathcal{S}^{\mathrm{aff}} := \{\Phi \in L^2(\Sigma M) : \nabla^{\mathrm{aff}}\Phi = 0\}.$$

The main goal of this section is to construct an isometry between $\mathcal{S}^{\mathrm{aff}}$ and the L^2-sections of a twisted Clifford bundle over the base manifold B. We recall from Appendix A that the space of affine parallel spinors on Z is finite dimensional. In addition, the spin structure on M induces a spin structure on each fiber Z_p, $p \in B$ and for all $p, q \in B$ the induced spin structures on Z_p and Z_q are equivalent. In particular, the dimension of the space of affine parallel spinors on a fiber Z_p is the same for all $p \in B$. Hence, there is a locally well-defined vector bundle $\mathcal{P} \to B$ such that for each p, the fiber $\mathcal{P}_p$ is given by the space of affine parallel spinors on Z_p with respect to the induced spin structure. In the next lemma, we construct an isometry in the spirit of [Amm98b, Lemma-Definition 7.2.3].

Lemma 3.26. *Let $f : M \to B$ be a Riemannian affine fiber bundle such that M is spin. Then there is an isometry*

$$Q : L^2({}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P}) \to \mathcal{S}^{\mathrm{aff}}.$$

Moreover, the connection $\nabla^{\mathcal{T}}$ induces a connection $\check{\nabla}^{\mathcal{T}}$ on ${}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P}$. Taking a split orthonormal frame $(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_k)$ around x, we have the identity

$$\nabla^M_{\xi_\alpha} Q(\check{\Phi})_x = Q(\check{\nabla}^{\mathcal{T}}\check{\Phi})_x + \frac{1}{2}\sum_{\beta=1}^{n} \gamma(\xi_\beta)\gamma(A(\xi_\alpha, \xi_\beta))Q(\check{\Phi})_x - \frac{\check{\xi}_\alpha(\mathrm{vol}(Z_{f(x)}))}{2\,\mathrm{vol}(Z_{f(x)})} Q(\check{\Phi})_x.$$

Proof. It follows from the discussion in Appendix B that $\Sigma M \cong f^*({}^{\diamond}\mathbf{\Sigma} B) \otimes \Sigma\mathcal{V}$, where the spinor bundles ${}^{\diamond}\mathbf{\Sigma} B$ and $\Sigma\mathcal{V}$ are in general only defined locally. Let $f^{-1}(U) \cong U \times Z$ be a trivializing neighborhood on which $\Sigma\mathcal{B}$ and $\Sigma\mathcal{V}$ are well-defined. Then any spinor Φ restricted to $f^{-1}(U)$ can be written as finite linear combination $\Phi = \sum_l f^*\varphi_l \otimes \nu_l$. By linearity it suffices to consider elementary tensors $f^*\varphi \otimes \nu$. We observe that

$$\nabla^{\mathrm{aff}}(f^*\varphi \otimes \nu) = f^*\varphi \otimes \nabla^{\mathrm{aff}}\nu.$$

Thus, a spinor is affine parallel if and only if ν, restricted to any fiber, is an affine parallel spinor of Z. Hence, on any trivializing neighborhood we obtain an isomorphism

$$\begin{aligned} Q^{-1} : \mathcal{S}^{\mathrm{aff}} &\to L^2({}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P}), \\ (f^*\varphi \otimes \nu)_x &\mapsto \sqrt{\mathrm{vol}(Z_{f(x)})}\; \varphi_{f(x)} \otimes \nu_{f(x)}, \end{aligned}$$

where $\nu_{f(x)}$ denotes the restriction of ν to the fiber $Z_{f(x)}$. Due to the factor $\sqrt{\mathrm{vol}(Z_{f(x)})}$ it is evident that Q^{-1} defines an isometry. Here $\mathrm{vol}(Z_{f(x)})$ is the volume of the fiber $Z_{f(x)}$ with respect to the induced affine parallel metric $\hat{g}_{f(x)}$. Since the structure group of the Riemannian affine fiber bundle $f : M \to B$ lies in $\mathrm{Aff}(Z)$ it follows that Q^{-1} extends to a well-defined global isometry. Taking its inverse gives the desired map Q.

It follows from the discussion in Chapter 3.1 that $\mathcal{S}^{\mathrm{aff}}$ is invariant under the action of $\nabla^{\mathcal{T}}_{\xi_\alpha}$ for any $\alpha \in \{1, \dots, n\}$. Hence, there is an induced connection $\check{\nabla}^{\mathcal{T}}$ on ${}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P}$. Using Lemma 3.25 the claimed identity follows from a straightforward calculation. $\square$

Corollary 3.27. *Let $f : M \to B$ be a Riemannian affine fiber bundle with a fixed spin structure on M such that $\mathcal{S}^{\mathrm{aff}}$ is nontrivial. With respect to a split orthonormal frame $(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_k)$, any spinor $\Phi \in \mathcal{S}^{\mathrm{aff}}$ satisfies*

$$
\begin{aligned}
D^M\Phi &= Q \circ \check{D}^{\mathcal{T}} \circ Q^{-1}\Phi + \frac{1}{2} \sum_{\substack{a,b,c=1\\ b<c}}^{k} \Gamma^c_{ab}\gamma(\zeta_a)\gamma(\zeta_b)\gamma(\zeta_c)\Phi + \frac{1}{2}\gamma(A)\Phi \\
&=: Q \circ \check{D}^{\mathcal{T}} \circ Q^{-1}\Phi + \frac{1}{2}\gamma(\mathcal{Z})\Phi + \frac{1}{2}\gamma(A)\Phi.
\end{aligned}
$$

Here, $\check{D}^{\mathcal{T}}$ is the Dirac operator on ${}^{\diamond}\mathbf{\Sigma}B \otimes \mathcal{P}$ associated to the connection $\check{\nabla}^{\mathcal{T}}$.

Proof. Let Φ be an affine parallel spinor. Since $f : M \to B$ is a Riemannian affine fiber bundle, $\mathcal{S}^{\mathrm{aff}}$ is invariant under the action of the Dirac operator D^M. With respect to a split orthonormal frame $(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_k)$, see Definition 3.1, we obtain

$$
\begin{aligned}
D^M\Phi &= \sum_{\alpha=1}^{n} \gamma(\xi_\alpha)\nabla_{\xi_\alpha}\Phi + \sum_{i=1}^{k} \gamma(\zeta_a)\nabla_{\zeta_a}\Phi \\
&= \sum_{\alpha=1}^{n} \gamma(\xi_\alpha)\nabla_{\xi_\alpha}\left(Q \circ Q^{-1}(\Phi)\right) + \frac{1}{4}\sum_{a=1}^{k}\gamma(\zeta_a)\left(\sum_{b,c=1}^{k}\Gamma^c_{ab}\gamma(\zeta_b)\gamma(\zeta_c)\Phi\right) \\
&\quad - \frac{1}{2}\sum_{a=1}^{k}\gamma(T(\zeta_a,\zeta_a))\Phi + \frac{1}{4}\sum_{a=1}^{k}\sum_{\alpha=1}^{n}\gamma(\zeta_a)\gamma(\xi_\alpha)\gamma(A(\xi_\alpha,\zeta_a))\Phi \\
&= Q \circ \check{D}^{\mathcal{T}} \circ Q^{-1}\Phi + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\gamma(\xi_\alpha)\gamma(\xi_\beta)\gamma(A(\xi_\alpha,\xi_\beta))Q(\check{\Phi})_x \\
&\quad - \frac{1}{2}\gamma\left(\frac{\mathrm{grad}(\mathrm{vol}(Z_{f(x)}))}{\mathrm{vol}(Z_{f(x)})}\right)\Phi + \frac{1}{2}\sum_{\substack{a,b,c=1\\ b<c}}^{k}\Gamma^c_{ab}\gamma(\zeta_a)\gamma(\zeta_b)\gamma(\zeta_c)\Phi \\
&\quad - \frac{1}{2}\sum_{a=1}^{k}\gamma(T(\zeta_a,\zeta_a))\Phi + \frac{1}{4}\sum_{a=1}^{k}\sum_{\alpha=1}^{n}\gamma(\zeta_a)\gamma(\xi_\alpha)\gamma(A(\xi_\alpha,\zeta_a))\Phi \\
&=: Q \circ \check{D}^{\mathcal{T}} \circ Q^{-1}\Phi + \frac{1}{2}\gamma(\mathcal{Z})\Phi + \frac{1}{2}\gamma(A)\Phi.
\end{aligned}
$$

Here we used Lemma 3.25, Lemma 3.26, and the formulas for the Christoffel symbols, see (3.1.1). The last line follows from the fact (see for instance [GLP99, Lemma 1.17.2]) that

$$
\sum_{a=1}^{k} T(\zeta_a, \zeta_a) = -\,\mathrm{grad}(\ln(\mathrm{vol}(Z_p))).
$$

□

If $k = 1$ the Riemannian affine fiber bundle $f : M \to B$ is an S^1-bundle with structure group in $\mathrm{Aff}(S^1) \cong S^1 \rtimes \{\pm 1\}$. From the discussion in Chapter 3.2.1 we know that if $\mathcal{S}^{\mathrm{aff}}$ is nontrivial then there is an induced spin structure on B, if B is orientable, and an induced pin$^-$ structure, if B is non orientable. If B is orientable then $\mathcal{P}$ is just the trivial complex line bundle. In particular, ${}^{\diamond}\mathbf{\Sigma}B \otimes \mathcal{P} \cong {}^{\diamond}\mathbf{\Sigma}B$. If B is non orientable we denote by ${}^{\diamond}\mathbf{\Sigma}^P B$ the corresponding pin$^-$ bundle. We consider the embedding

$$\begin{aligned} \iota : \mathrm{O}(n) &\hookrightarrow \mathrm{SO}(n+1), \\ A &\mapsto \begin{pmatrix} \det(A) & 0 \\ 0 & A \end{pmatrix}, \end{aligned}$$

and its lift $\tilde{\iota} : \mathrm{Pin}^-(n) \hookrightarrow \mathrm{Spin}(n+1)$. Let $\theta_{n+1} : \mathrm{Spin}(n+1) \to \mathrm{GL}(\Sigma_{n+1})$ be the canonical complex spin representation. Then

$$\begin{aligned} \Sigma M &= P_{\mathrm{Spin}} M \times_{\theta_{n+1}} \Sigma_{n+1} \\ &\cong (f^* P_{\mathrm{Pin}^-(n)}(B) \times_{\tilde{\iota}} \mathrm{Spin}(n+1)) \times_{\theta_{n+1}} (\Sigma_n \otimes \mathbb{C}) \\ &\cong f^{*\diamond}\mathbf{\Sigma}^P B \otimes f^*(\det(TB) \otimes \mathbb{C}) \\ &=: f^{*\diamond}\mathbf{\Sigma}^P B \otimes f^* \det(TB)^{\mathbb{C}}, \end{aligned}$$

where ${}^{\diamond}\mathbf{\Sigma}^P B$ is defined similarly to the notation for spin bundles,

$${}^{\diamond}\mathbf{\Sigma}^P B := \begin{cases} \Sigma^P B, & \text{if } n \text{ is even}, \\ \Sigma^{P+} B \oplus \Sigma^{P-} B, & \text{if } n \text{ is odd}. \end{cases}$$

Thus, we obtain the isomorphism ${}^{\diamond}\mathbf{\Sigma}B \otimes \mathcal{P} \cong {}^{\diamond}\mathbf{\Sigma}^P B \otimes \det(TB)^{\mathbb{C}}$.

Now we fix a Riemannian affine fiber bundle $f : (M, g) \to (B, h)$ such that (M, g) is an $(n + k)$-dimensional spin manifold and B is a closed n-dimensional Riemannian manifold. We conclude this section by taking a closer look at elements $W \in H^{1,\infty}(\Sigma M)$. In the following proposition we relate them with an affine parallel operator $\mathcal{W}$ such that the difference $\|W - \mathcal{W}\|_{L^\infty}$ depends on the derivatives of W and the diameter of the fibers. The following results will be used in the next chapter to study the eigenvalues of Dirac operators with symmetric potentials.

Proposition 3.28. *Let $f : (M^{n+k}, g) \to (B^n, h)$ be a Riemannian affine fiber bundle with fibers $Z_p := f^{-1}(p)$, $p \in B$, such that*

$$\begin{aligned} \|T\|_\infty &\leqslant C_T, \\ \|R^M\|_\infty &\leqslant C_R, \end{aligned}$$

and such that (M, g) is spin. For any $W \in \mathrm{Hom}(\Sigma M, \Sigma M)$ there is an affine parallel operator $\mathcal{W}$, i.e. $\nabla^{\mathrm{aff}}\mathcal{W} = 0$, such that

$$\begin{aligned} \|W - \mathcal{W}\|_{L^\infty} &\leqslant 2 \max_{p \in B}(\mathrm{diam}(Z_p)) \|\nabla^{\mathrm{aff}} W\|_{L^\infty} \\ &\leqslant 2 \max_{p \in B}(\mathrm{diam}(Z_p)) \left(\|\nabla W\|_{L^\infty} + C(k, C_R, C_T)\|W\|_{L^\infty}\right). \end{aligned}$$

Proof. First we consider a fixed closed infranilmanifold $Z = {}_{\Gamma}\backslash^{N}$. Here, N is a connected and simply-connected Lie group and Γ is a cocompact discrete subgroup of the group $\mathrm{Aff}(N) \cong N_L \rtimes \mathrm{Aut}(N)$, where N_L is the group of left-translations and $\mathrm{Aut}(N)$ is the automorphism group. In particular, $\pi : N \to Z$ is the universal cover. Since N is nilpotent, it has a biinvariant Haar measure μ. It is a general fact, that μ is unique up to multiplication with a positive constant. Hence, we assume without loss of generality that $\mu(\mathcal{F}) = 1$, where $\mathcal{F} \subset N$ is a fundamental domain of Z, i.e. $\pi : \mathcal{F} \to Z$ is an isomorphism.

Let $W : Z \to V$ be a map from Z into an arbitrary vector space V. Then its pullback $\widetilde{W} := \pi^* W : N \to V$ is invariant under the action of Γ. We define

$$\widetilde{\mathcal{W}} := \int_{\mathcal{F}} (L_g^* \widetilde{W}) \mathrm{d}\mu(g).$$

By construction $\widetilde{\mathcal{W}}$ is affine parallel, i.e. left-invariant, since

$$\begin{aligned} L_h^* \widetilde{\mathcal{W}} &= \int_{\mathcal{F}} L_h^* L_g^* \widetilde{W} \, \mathrm{d}\mu(g) \\ &= \int_{\mathcal{F}} L_{gh}^* \widetilde{W} \, \mathrm{d}\mu(g) \\ &= \int_{R_{h^{-1}}(\mathcal{F})} L_g^* \widetilde{W} \, \mathrm{d}\mu(gh^{-1}) \\ &= \int_{\mathcal{F}} L_g^* \widetilde{W} \, \mathrm{d}\mu(g) = \widetilde{\mathcal{W}}. \end{aligned}$$

Here, we used in the last line, that μ is a biinvariant Haar measure. It remains to show that $\widetilde{W}$ descends to a well-defined operator on Z. So let $\gamma \in \Gamma$. Then

$$\begin{aligned} \gamma^* \widetilde{\mathcal{W}} &= \int_{\mathcal{F}} \gamma^* L_g^* \widetilde{W} \, \mathrm{d}\mu(g) \\ &= \int_{\mathcal{F}} (L_g \circ \gamma)^* \widetilde{W} \mathrm{d}\mu(g) \\ &= \int_{\mathcal{F}} L_{\gamma^{-1}(g)}^* \gamma^* \widetilde{W} \, \mathrm{d}\mu(g) \\ &= \int_{\gamma^{-1}(\mathcal{F})} L_g^* \widetilde{W} \, \mathrm{d}\mu(\gamma^{-1}(g)) = \widetilde{\mathcal{W}} \end{aligned}$$

since $\widetilde{W}$ is Γ invariant and Γ preserves the Haar measure μ. Hence, there is an induced affine parallel operator $\mathcal{W}$ on Z such that $\pi^* \mathcal{W} = \widetilde{\mathcal{W}}$.

Now, let $f : M \to B$ be a Riemannian affine fiber bundle with fiber Z such that M is spin and let $W \in \mathrm{Hom}(\Sigma M, \Sigma M)$. We can interpret W as a map $M \ni x \mapsto W_x$, where W_x is a homomorphism of the fiber ΣM_x. Let $(U_\alpha)_\alpha$ be a bundle atlas for M. For each α and $(p, z) \in U_\alpha \times Z$ we define the operator $\mathcal{W}_\alpha$ on $f^{-1}(U_\alpha) \cong U_\alpha \times Z$ as explained above. Since the structure group of a Riemannian affine fiber bundle lies in $\mathrm{Aff}(Z)$ it follows, that the transition maps preserve the Haar measure. In particular, the transition maps commute with the averaging operator. Hence, the locally defined operators $(\mathcal{W}_\alpha)_\alpha$ glue together to a well-defined affine parallel operator $\mathcal{W} \in \mathrm{Hom}(\Sigma M, \Sigma M)$.

It remains to estimate the difference between the operators W and $\mathcal{W}$. Although we are interested in the essential supremum, it suffices to do the calculation at a fixed point $x \in M$ where W_x and its derivative are defined. As before, we lift the whole situation to

N via the universal cover $\pi : N \to Z$. In the following calculation, we set $p := f(x)$ and let $\hat{g}_p$ be the induced affine parallel metric on the fiber $Z_p = f^{-1}(p)$. Then $g_p^N := \hat{g}_p$ is a left-invariant metric on N. From the calculations in the beginning of the proof we recall the fundamental domain $\mathcal{F}$ of the quotient $Z = {}_{\Gamma}\diagdown^{N}$. Let $\tilde{x}$ be the unique point in $\mathcal{F}$ such that $\pi(\tilde{x}) = x \in f^{-1}(p)$. Further, we consider for each $g \in N$ the geodesic γ_g from $\tilde{x}$ to $L_g(\tilde{x})$ with respect to g_p^N. We assume without loss of generality that γ_g is parametrized by arc length, i.e. $|\gamma_g'| \equiv 1$. At last, we set d_p to be the distance function on (N, g_p^N). Using this notation we calculate the norm of the difference $(\mathcal{W} - W)$ with respect to the metric g,

$$\begin{aligned}
\|(\mathcal{W} - W)_x\|_\infty &= \left\| \int_{\mathcal{F}} (L_g^* \widetilde{W})_{\tilde{x}} - \widetilde{W}_{\tilde{x}} \mathrm{d}\mu(g) \right\|_\infty \\
&= \left\| \int_{\mathcal{F}} \int_0^{d_p(\tilde{x}, L_g(\tilde{x}))} L^*_{\gamma_g(t)} \left(\nabla^{\mathrm{aff}}_{\gamma_g'(t)} \widetilde{W} \right)_{\tilde{x}} \mathrm{d}t \mathrm{d}\mu(g) \right\|_\infty \\
&\leqslant 2 \operatorname{diam}(Z, \hat{g}_p) \|\nabla^{\mathrm{aff}} W_x\|_\infty \\
&\leqslant 2 \operatorname{diam}(Z, \hat{g}_p) \left(\|\nabla^Z W_x\|_\infty + \|\mathcal{Z}\|_\infty \|W_x\|_\infty \right).
\end{aligned}$$

Since $\|\mathcal{Z}\|_{L^\infty} \leqslant C(k, C_T, C_R)$ by Lemma 3.8 the claim follows. $\square$

Chapter 4

The behavior of Dirac eigenvalues

To the author's knowledge, the first result regarding the behavior of Dirac eigenvalues on collapsing sequences of spin manifolds was proved by Ammann in his PhD thesis [Amm98b, Kapitel 7], see also [Amm98a, AB98]. There he considered collapsing S^1-principal bundles over a fixed manifold. As discussed in Section 3.2.1, we distinguish between projectable and non projectable spin structures. In the case of projectable spin structures Ammann proved:

Theorem 4.1. *Let (B,h) be a closed n-dimensional Riemannian spin manifold with a fixed spin structure. Further, let $(M_i,g_i)_{i\in\mathbb{N}}$ be a sequence of $(n+1)$-dimensional Riemannian spin manifolds such that there are Riemannian submersions $f_i:(M_i,g_i)\to(B,h)$ defining an S^1-principal bundle. Let $2\pi l_i$ be the length of the fibers and $\mathrm{i}\omega_i$ be the unique imaginary connection one-form such that $\ker(\omega_i)$ is orthogonal to the fibers with respect to g_i. If for all $i\in\mathbb{N}$ the spin structure on (M_i,g_i) is projectable and induces the fixed spin structure on B and if in addition*

$$\lim_{i\to\infty}\|l_i\|_\infty = 0,$$
$$\lim_{i\to\infty}\|l_i\cdot \mathrm{d}\omega_i\|_\infty = 0,$$
$$\alpha := \limsup_{i\to\infty}\|\operatorname{grad} l_i\|_\infty < 1,$$

then the eigenvalues $(\lambda_{j,k}(i))_{\substack{j\in\mathbb{Z},\\ k\in\mathbb{Z}}}$ of the Dirac operator D^{M_i} can be numbered in such a way that:

(1) For all $\varepsilon>0$ there is an $I\in\mathbb{N}$ such that for any $i\geqslant I$ and $j\in\mathbb{Z}$, $k\in\mathbb{Z}$

$$\|l_i\|_\infty^2\lambda_{j,k}(i)^2 \geqslant |k|(|k|-\alpha)-\varepsilon.$$

In particular, $\lambda_{j,k}(i)^2\to\infty$ as $i\to\infty$ whenever $k\neq 0$.

Furthermore, if M_i and ω_i do not depend on i, then we also have for $j\in\mathbb{Z}$ and $k\in\mathbb{Z}\backslash\{0\}$ that

$$\limsup_{i\to\infty}\left(\min_{p\in B} l_i(p)\right)^2\lambda_{j,k}(i)^2 \leqslant |k|(|k|+\alpha).$$

This upper bound of $\lambda_{j,k}(i)^2$ is not uniform in j and k.

(2) Let $(\mu_j)_{j\in\mathbb{N}}$ be the eigenvalues of the Dirac operator D^B on B. If n is even then

$$\lim_{i\to\infty}\lambda_{j,0}(i)=\mu_j.$$

However, for n odd we obtain

$$\begin{aligned}\lim_{i\to\infty}\lambda_j(0)(i)&=\mu_j\\ \lim_{i\to\infty}\lambda_{-j}(0)(i)&=-\mu_j.\end{aligned}$$

In both cases the convergence of the eigenvalues $\lambda_{j,0}(i)$ is uniform in j.

In the case of non projectable spin structures the eigenvalues $(\lambda_{j,k}(i))_{\substack{j\in\mathbb{Z},\\ k\in(\mathbb{Z}+\frac{1}{2})}}$ of the Dirac operators D^{M_i} can be numbered in such a way that $k\in(\mathbb{Z}+\frac{1}{2})$. The same lower bound as in the above theorem holds also for non projectable spin structures [Amm98b, Satz 7.3.1]. Since k can never be 0 all eigenvalues diverge to $\pm\infty$ as $i\to\infty$. We explain the details concerning the numbering of the Dirac eigenvalues in Section 4.2.

The setting in Theorem 4.1 differs slightly from our setting of collapsing sequences in $\mathcal{M}(n+1,d,C)$. Let $(f_i:(M_i,g_i)\to(B,h))_{i\in\mathbb{N}}$ be a collapsing sequence of S^1-principal bundles such that f_i is a Riemannian submersion and $(M_i,g_i)\in\mathcal{M}(n+1,d,C)$, for all $i\in\mathbb{N}$. It follows from Corollary 1.29 and Theorem 2.17 that the corresponding A- and T-tensor of the Riemannian submersions f_i are uniformly bounded in norm by positive constants $C_A(n,d,C)$, and $C_T(n)$ respectively. A straightforward calculation shows that

$$\begin{aligned}\left\|\frac{\operatorname{grad} l_i}{l_i}\right\|_\infty&\leqslant C_T,\\ \|l_i\cdot \mathrm{d}\omega_i\|_\infty&\leqslant 2C_A.\end{aligned}$$

In particular, $\alpha:=\limsup_{i\to\infty}\|\operatorname{grad} l_i\|_\infty=0$ for any collapsing sequence in $\mathcal{M}(n+1,d,C)$. But we do not necessarily have that $\lim_{i\to\infty}\|l_i\mathrm{d}\omega_i\|=0$, as can be seen in Example 3.9. It will be shown in Chapter 4.2 that for such sequences the eigenvalues $\lambda_{j,0}(i)$ do not converge to the eigenvalues of the Dirac operator D^B on B but to the eigenvalues of the Dirac operator with a symmetric potential depending on the limit behavior of the A-tensors.

In [Lot02a] Lott studied the behavior of Dirac eigenvalues on arbitrary collapsing sequences in $\mathcal{M}(n,d)$. There Lott combined his results for the eigenvalues of the p-form Laplacian on collapsing sequences [Lot02c,Lot02b] with the Bochner-type formulas for the Dirac operator. Moreover, Lott's results also hold for the Dirac operator on differential forms, i.e. the operator

$$D=\mathrm{d}+\mathrm{d}^*:\Omega^*(M)\to\Omega^*(M),$$

where d^* is the adjoint of the exterior derivative d with respect to the L^2-inner product.

Theorem 4.2. *Given $n\in\mathbb{N}$ and $G\in\{\mathrm{SO}(n),\mathrm{Spin}(n)\}$, let $(M_i,g_i)_{i\in\mathbb{N}}$ be a sequence of connected closed oriented n-dimensional Riemannian manifolds with a G-structure. Let V be a G-Clifford module. Suppose that for some $d,K>0$ and for each $i\in\mathbb{N}$ we have $\operatorname{diam}(M_i)\leqslant d$ and $\|R^{M_i}\|_\infty\leqslant K$. Then there is*

(1) a subsequence of $(M_i, g_i)_{i\in\mathbb{N}}$ *which we relabel as* $(M_i)_{i\in\mathbb{N}}$*,*

(2) a smooth closed G-manifold $\check{X}$ *with a G-invariant Riemannian metric* $g^{T\check{X}}$ *which is* $C^{1,\alpha}$*-regular for all* $\alpha \in [0, 1)$*,*

(3) a positive G-invariant function $\chi \in C(\check{X})$ *with* $\int_{\check{X}} \chi \,\mathrm{dvol} = 1$*,*

(4) a G-invariant function $\mathcal{V} \in L^\infty(\check{X})\otimes\mathrm{End}(V)$ *such that if* $\Delta^{\check{X}}$ *denotes the Laplacian on* $L^2(\check{X}, \chi\,\mathrm{dvol})\otimes V$ *and* $|D^X|$ *denotes the operator* $\sqrt{\Delta^{\check{X}} + \mathcal{V}}$ *acting on the G-invariant subspace* $(L^2(\check{X}, \chi\,\mathrm{dvol}) \otimes V)^G$ *then for all* $k \in \mathbb{N}$*,*

$$\lim_{i\to\infty} \lambda_k(|D^{M_i}|) = \lambda_k(|D^X|).$$

The limit measure χ is also necessary for the analogous results regarding the behavior of the eigenvalues of the Laplacian on functions [Fuk87a] and the eigenvalues of the Laplacian on forms [Lot02c, Lot02b], see Example 4.4.

For the special case of collapsing sequences in $\mathcal{M}(n, d)$ with a smooth limit space Lott proved an accentuation of the above theorem [Lot02a, Theorem 2]. In the following we say that for a given $\varepsilon > 0$ two collections of real numbers $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ are *ε-close* if there is a bijection $\alpha : I \to J$ such that $|b_{\alpha(i)} - a_i| \leqslant \varepsilon$ holds for all $i \in I$.

Theorem 4.3. *Let B be a fixed smooth connected closed Riemannian manifold and let* $n \in \mathbb{N}$*,* $G \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$ *and V be a G-Clifford module. For any* $\varepsilon > 0$ *and* $K > 0$*, there are positive constants* $A(B, n, V, \varepsilon, K), A'(B, n, V, \varepsilon, K)$ *and* $C(B, n, V, \varepsilon, K)$ *such that the following holds. Let M be an n-dimensional connected closed oriented Riemannian manifold with a G-structure such that* $\|R^M\|_\infty \leqslant K$ *and* $d_{GH}(M, B) \leqslant A'$*. Then there are a Clifford module* E^B *on B and a certain first order differential operator* $\mathcal{D}^B$ *on* $C^\infty(B; E^B)$ *such that*

(1) $\{\mathrm{arsinh}(\frac{\lambda}{\sqrt{2K}} : \lambda \in \sigma(D^M),\ \lambda^2 \leqslant Ad_{GH}(M, B)^{-2} - C\}$ *is ε-close to a subset of* $\{\mathrm{arsinh}(\frac{\lambda}{\sqrt{2K}} : \lambda \in \sigma(\mathcal{D}^B)\}$*,*

(2) $\{\mathrm{arsinh}(\frac{\lambda}{\sqrt{2K}} : \lambda \in \sigma(\mathcal{D}^B),\ \lambda^2 \leqslant Ad_{GH}(M, B)^{-2} - C\}$ *is ε-close to a subset of* $\{\mathrm{arsinh}(\frac{\lambda}{\sqrt{2K}} : \lambda \in \sigma(D^M)\}$*.*

In the following section we discuss the special case of collapsing sequences $(M_i, g_i)_{i\in\mathbb{N}}$ of spin manifolds in $\mathcal{M}(n, d)$ converging to a Riemannian manifold (B, h). There we will give an explicit description of the limit operator $\mathcal{D}^B$ as a twisted Dirac operator with a symmetric $C^{0,\alpha}$-potential for all $\alpha \in [0, 1)$. Moreover, we show that the limit operator $\mathcal{D}^B$ satisfies the conclusions of Theorem 4.2 with $\chi \equiv 1$. This is a special behavior for the eigenvalues of the Dirac operator on spin manifolds and does not extend to the eigenvalues of the Dirac operator acting on differential forms.

Example 4.4. Consider the torus $\mathbb{T}^2 = \{(e^{is}, e^{it}) : s, t \in \mathbb{R}\}$ with the Riemannian metric

$$g_\varepsilon := \mathrm{d}s^2 + \varepsilon^2 c(s)^2 \mathrm{d}t^2,$$

for a fixed positive function $c : S^1 \to \mathbb{R}_+$. The sequence $(\mathbb{T}^2, g_\varepsilon)_{\varepsilon>0}$ is a collapsing sequence with bounded sectional curvature and diameter converging to $(S^1, \mathrm{d}s^2)$ in the Gromov-Hausdorff topology. We observe that the integrability tensor A_ε vanishes identically for all ε and the T-tensor is always characterized by $\frac{c'(s)}{c(s)}$ independent of ε. In particular, the T-tensor is nontrivial for all ε if we choose the function c to be non constant.

We endow $(\mathbb{T}^2, g_\varepsilon)$ with the spin structure induced by the pullback of a chosen spin structure on S^1. It will be shown in Theorem 4.5 that the spectrum of the Dirac operator D_ε on $(\mathbb{T}^2, g_\varepsilon)$ restricted to the S^1-invariant spinors converges to the spectrum of the Dirac operator D^{S^1} of the chosen spin structure on S^1. For our result, it does not matter, which spin structure was chosen on S^1.

Next, we take a look at the Dirac operator acting on differential forms. In that case the space of affine parallel forms is given by

$$\mathcal{S}^{\mathrm{aff}} := \{f \in C^\infty(T^2) : \frac{\partial}{\partial t} f = 0\} \cup \{\alpha \, \mathrm{d}s \in \Omega^1(T^2) : \frac{\partial}{\partial t}\alpha = 0\}.$$

The Dirac operator $D_\varepsilon = \mathrm{d} + \mathrm{d}^*$ acts on $(f + \alpha \mathrm{d}s) \in \mathcal{S}^{\mathrm{aff}}$ as

$$D_\varepsilon(f(s) + \alpha(s)\mathrm{d}s) = \frac{\partial}{\partial s} f(s)\mathrm{d}s - c(s)^{-1}\frac{\partial}{\partial s}(c(s)\alpha(s)).$$

We observe that $(D_\varepsilon)_{|\mathcal{S}^{\mathrm{aff}}}$ is independent of ε. In particular, in the limit $\varepsilon \to 0$, the sequence $(D_\varepsilon)_{|\mathcal{S}^{\mathrm{aff}}}$ induces a first order differential operator D_0 on $\Omega^*(S^1)$. For any eigenform $f(s) + \alpha(s)\mathrm{d}s \in \Omega^1(S^1)$ of D_0 with eigenvalue λ we have that

$$\begin{aligned}
\lambda f(s) &= -c(s)^{-1}\frac{\partial}{\partial s}(c(s)\alpha(s)), \\
\lambda \alpha(s)\mathrm{d}s &= \frac{\partial}{\partial s} f(s)\mathrm{d}s.
\end{aligned}$$

It follows at once that the eigenvalue problem for D_0 is equivalent to the eigenvalue problem

$$\lambda^2 f(s) = -c(s)^{-1}\frac{\partial}{\partial s}\left(c(s)\frac{\partial}{\partial s} f\right). \tag{4.4.1}$$

Furthermore, the eigenvalues of D_0 are symmetric around 0, because, if $f(s) + \alpha(s)ds$ is an eigenform of D_0 with eigenvalue λ then $f(s) - \alpha(s)ds$ is an eigenform with eigenvalue $-\lambda$.

For a generic choice of $c(s)$ the spectrum of D_0 differs from the spectrum $\sigma(D^{S^1}) = \mathbb{Z}$ of the Dirac operator on S^1.

For example, if $c(s) = e^{\cos(s)}$ then one can calculate numerically, adapting the algorithm from [Str16, page 3 - 6], that, without counting multiplicities, the first eigenvalues are approximately given by

$$\lambda_0 = 0, \; \lambda_1 \approx 0{,}990, \; \lambda_2 \approx 1{,}137.$$

In particular, the spectrum of D_0 does not coincide with the spectrum of the Dirac operator D^{S^1} as $\sigma(D^{S^1}) = \mathbb{Z}$.

We can do a similar calculation for the Laplacian acting on functions. It follows that for $f(s) \in \mathcal{S}^{\mathrm{aff}}$,

$$\Delta_\varepsilon f(s) = -c(s)^{-1}\frac{\partial}{\partial}\left(c(s)\frac{\partial}{\partial s} f\right).$$

Since this equation is independent of ε it is immediate that $\left(\Delta_{\varepsilon\,|\mathcal{S}^{\mathrm{aff}}}\right)_\varepsilon$ induces a second order differential operator Δ_0 on S^1 as $\varepsilon \to 0$. Furthermore, the eigenvalue problem for Δ_0 is similar to the eigenvalue problem (4.4.1). In particular, the spectrum of Δ_0 differs from the spectrum of $\Delta^{S^1} = -\frac{\partial^2}{\partial s}$ which is given by $\{k^2 : k \in \mathbb{Z}\}$.

At this point we want to remark that Section 4.1 is an extended version of [Roo18b, Section 5] where we added the behavior of Dirac eigenvalues with a symmetric uniformly bounded $H^{1,\infty}$-potential.

In Section 4.2 we give an explicit description of the behavior of Dirac eigenvalues for any collapsing sequence in $\mathcal{M}(n+1, d, C)$ extending the results of Theorem 4.1. The results stated in Section 4.2 also have been proved in [Roo18c, Section 5].

4.1 Dirac eigenvalues under collapse to smooth spaces

Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence of Riemannian spin manifolds in $\mathcal{M}(n+k, d)$ converging to an n-dimensional Riemannian manifold (B, h). By Corollary 1.29 there is an $I \in \mathbb{N}$ such that for any $i \geqslant I$ there are metrics $\tilde{g}_i$ on M_i and $\tilde{h}_i$ on B such that $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ is a Riemannian affine fiber bundle (see Definition 1.28). Moreover,

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0,$$
$$\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0.$$

Let ΣM_i and $\widetilde{\Sigma M}_i$ be the spinor bundles of (M_i, g_i) and $(M_i, \tilde{g}_i)$ respectively. In Appendix C we constructed an explicit isometry

$$\Theta_i : L^2(\Sigma M_i) \to L^2(\widetilde{\Sigma M}_i), \tag{4.4.1}$$

following [BG92, Mai97].

For a Riemannian affine fiber bundle $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ the space of affine parallel spinors $\mathcal{S}_i^{\mathrm{aff}} \subset L^2(\widetilde{\Sigma M}_i)$ is well-defined. The Dirac operator $\tilde{D}^{M_i}$ on $(M_i, \tilde{g}_i)$ acts diagonally with respect to the splitting

$$L^2(\widetilde{\Sigma M}_i) = \mathcal{S}_i^{\mathrm{aff}} \oplus \left(\mathcal{S}_i^{\mathrm{aff}}\right)^{\perp}.$$

In general, for the original fibration $f_i : (M_i, g_i) \to (B, h)$ the induced metrics on the fibers is not affine parallel. Thus, the affine connection ∇^{aff} does not induce a well-defined connection on the spinor bundle ΣM_i. Instead we use the isometry Θ_i to define

$$\mathcal{S}_i := \Theta_i^{-1}(\mathcal{S}_i^{\mathrm{aff}}).$$

This induces the splitting

$$L^2(\Sigma M_i) = \mathcal{S}_i \oplus \mathcal{S}_i^{\perp}.$$

But in contrast to Riemannian affine fiber bundles, the Dirac operator D^{M_i} on (M_i, g_i), in general, does not act diagonally with respect to this splitting. Nevertheless, it follows from the continuity of the spectra of Dirac operators, see Theorem C.4, that

$$d_a\left(\sigma(D^{M_i}), \sigma(\tilde{D}^{M_i})\right) \xrightarrow{i\to\infty} 0 ,$$

where d_a is the distance in the arsinh-topology, see Definition C.3. Thus, the spectra of the restrictions of D^{M_i} and $\tilde{D}^{M_i}$ to $\mathcal{S}_i$ and $\mathcal{S}_i^{\mathrm{aff}}$, respectively to their orthogonal complements, have the same limit as $i \to \infty$.

Next we recall, that for a Riemannian affine fiber bundle $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ we defined the following objects in Chapter 3.1:

- The operator $\nabla^{\mathcal{V}_i}$ that acts on vector fields X, Y as $\nabla^{\mathcal{V}_i}_X Y := \left(\nabla_{X^H} Y^V\right)^V$, where the superscripts H and V denote the horizontal, resp. the vertical part of a vector field.
- The one-form $\mathcal{Z}_i \in \Omega^1(M_i, \mathrm{End}(TZ_i))$, where we view TZ_i as a vector bundle over M_i, is defined as $\mathcal{Z}_i := \nabla^{Z_i} - \nabla^{\mathrm{aff}}$. Here $\nabla^{Z_i}_X Y := \left(\nabla_{X^V} Y^V\right)^V$ for any two vector fields X, Y and ∇^{aff} is the induced affine connection.
- The two-form $\mathcal{A}_i \in \Omega^2(B, P)$ that is characterized by $(f_i^* \mathcal{A}_i)(X, Y) = A_i(X, Y)$ for all basic vector fields X, Y. Here A_i is the A-tensor of the Riemannian submersion $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ and $P \to B$ is a vector bundle whose fiber P_p is given by the affine parallel vector fields on $Z_p := f_i^{-1}(p)$ for all $p \in B$.

Similar to (4.4.1) there is also an isometry

$$\theta_i : L^2(TM_i) \to L^2(\widetilde{TM}_i).$$

In the next theorem we interpret the operators $\nabla^{\mathcal{V}_i}$, $\mathcal{Z}_i$ and $\mathcal{A}_i$ on (M_i, g_i) as the pullbacks of the respective operators on $(M_i, \tilde{g}_i)$ via the map θ_i.

Using this terminology and the notations introduced in Chapter 3.2.2 we state in the following theorem the explicit description of the limit operator $\mathcal{D}^B$ in Theorem 4.3 for collapsing sequences of spin manifolds converging to a Riemannian manifold of lower dimension. Theorem 0.2 follows immediately from the following more specific theorem.

Theorem 4.5. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence of spin manifolds in $\mathcal{M}(n+k, d)$ converging to a smooth n-dimensional Riemannian manifold (B, h) such that the space $\mathcal{S}_i$ is nontrivial for almost all $i \in \mathbb{N}$. Then there is a subsequence $(M_i, g_i)_{i\in\mathbb{N}}$ such that the spectrum of $D^{M_i}_{|\mathcal{S}_i}$ converges uniformly with respect to the* arsinh*-topology to the spectrum of the elliptic self-adjoint first order differential operator*

$$\begin{aligned} \mathcal{D}^B : \mathrm{dom}(\mathcal{D}^B) &\to L^2({}^{\diamond}\boldsymbol{\Sigma} B \otimes \mathcal{P}), \\ \Phi &\mapsto \check{D}^{\mathcal{T}_\infty}\Phi + \frac{1}{2}\gamma(\check{\mathcal{Z}}_\infty)\Phi + \frac{1}{2}\gamma(\check{\mathcal{A}}_\infty)\Phi, \end{aligned}$$

where

$$^{\diamond}\mathbf{\Sigma}B := \begin{cases} \Sigma B, & \text{if } n \text{ or } k \text{ is even,} \\ \Sigma^{+}B \oplus \Sigma^{-}B, & \text{if } n \text{ and } k \text{ are odd.} \end{cases}$$

Further,

(1) $\mathcal{P}$ represents the affine parallel spinors of the fibers Z_i,

(2) $\check{D}^{\mathcal{T}_\infty}$ is the twisted Dirac operator on $^{\diamond}\mathbf{\Sigma}B \otimes \mathcal{P}$ with respect to the twisted connection $\check{\nabla}^{\mathcal{T}_\infty} = \nabla^h \otimes \nabla^{\mathcal{V}_\infty}$, where ∇^h is the spinorial connection on (B, h) and $\nabla^{\mathcal{V}_\infty}$ is induced by the $C^{0,\alpha}$-limit of $\nabla^{\mathcal{V}_i}$ for any $\alpha \in [0,1)$,

(3) $\check{\mathcal{Z}}_\infty$ is induced by the $C^{0,\alpha}$-limit of $(\mathcal{Z}_i)_{i\in\mathbb{N}}$ for any $\alpha \in [0,1)$,

(4) $\check{\mathcal{A}}_\infty$ is the $C^{0,\alpha}$-limit of $(\mathcal{A}_i)_{i\in\mathbb{N}}$ for any $\alpha \in [0,1)$.

In particular, $\mathcal{D}^B$ is self-adjoint with respect to the standard measure and the metric h on B is $C^{1,\alpha}$ for any $\alpha \in [0,1)$.

Proof. The proof of this theorem is divided into several steps.

Step 1: Switching to invariant metrics. It follows from Corollary 1.29 that there is an index I such that for all $i \geqslant I$ there exists a fibration $f_i : (M_i, g_i) \to (B, h)$. Moreover, there are metrics $\tilde{g}_i$ on M_i and $\tilde{h}_i$ such that $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ is a Riemannian affine fiber bundle and

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0,$$
$$\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0.$$

Applying Theorem C.4 it follows that the spectra of the Dirac operator D^{M_i} on (M_i, g_i) and $\tilde{D}^{M_i}$ on $(M_i, \tilde{g}_i)$ are close in the arsinh-topology, see Definition C.3, i.e.

$$d_a(\sigma(D^{M_i}), \sigma(\tilde{D}^{M_i})) \leqslant C\|g_i - \tilde{g}_i\|_{C^1}. \tag{4.5.1}$$

Let $\mathcal{S}_i$ and $\mathcal{S}_i^{\mathrm{aff}}$ be defined as above. It follows from the proof of Theorem 4.3 that all eigenvalues of the restriction $\tilde{D}^{M_i}_{|(\mathcal{S}_i^{\mathrm{aff}})^\perp}$, and therefore also of $D^{M_i}_{|\mathcal{S}_i^\perp}$ go to $\pm\infty$ as $i \to \infty$. Furthermore, it was shown in the proof of Theorem 4.3 that the spectrum of $\tilde{D}^{M_i}_{|\mathcal{S}_i^{\mathrm{aff}}}$ has a well-defined limit as $i \to \infty$. Since the spectra of D^{M_i} and $\tilde{D}^{M_i}$ have the same limit by (4.5.1) it suffices to show the claim for the sequence $\left(f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)\right)_{i\in\mathbb{N}}$ of Riemannian affine fiber bundles.

In this setting we can use the isometry $Q_i : L^2(^{\diamond}\mathbf{\Sigma}_i B \otimes \mathcal{P}) \to \mathcal{S}_i^{\mathrm{aff}}$, see Lemma 3.26, and apply Corollary 3.27 to write

$$\tilde{D}^{M_i} = Q_i \circ \check{D}^{\mathcal{T}_i} \circ Q_i^{-1} + \frac{1}{2}\gamma(\mathcal{Z}_i) + \frac{1}{2}\gamma(\mathcal{A}_i).$$

We refer to Chapter 3 for the notation and definition of the separate terms. From the discussion in Chapter 3.1 it follows that $\gamma(\mathcal{Z}_i)$ and $\gamma(\mathcal{A}_i)$ act diagonally with respect to the splitting $L^2(\widetilde{\Sigma M_i}) = \tilde{\mathcal{S}}_i^{\mathrm{aff}} \oplus \left(\tilde{\mathcal{S}}_i^{\mathrm{aff}}\right)^{\perp}$. Thus, there are well-defined operators $\check{\mathcal{Z}}_i$ and $\check{\mathcal{A}}_i$ such that

$$\begin{aligned}\tilde{D}^{M_i}{}_{|\tilde{\mathcal{S}}^{\mathrm{aff}}} &= Q_i \circ \left(\check{D}^{\mathcal{T}_i} + \frac{1}{2}\gamma(\check{\mathcal{Z}}_i) + \frac{1}{2}\gamma(\check{\mathcal{A}}_i)\right) \circ Q_i^{-1} \\ &=: Q_i \circ \mathfrak{D}_i \circ Q_i^{-1}.\end{aligned}$$

Since $Q_i : L^2({}^{\diamond}\mathbf{\Sigma}_i B \otimes \mathcal{P}_i) \to \tilde{\mathcal{S}}^{\mathrm{aff}}$ is an isometry, it follows that $\tilde{D}_{i\,|\tilde{\mathcal{S}}^{\mathrm{aff}}}$ is isospectral to $\mathfrak{D}_i$. Furthermore, the operator $\mathfrak{D}_i$ is densely defined on $H^{1,2}({}^{\diamond}\mathbf{\Sigma}_i B \otimes \mathcal{P}_i)$.

Now, we are going to choose a subsequence such that the spectrum $\sigma(\mathfrak{D}_i)$ converges to the spectrum of the claimed operator $\mathcal{D}^B$. The main problem here is that the operators $\mathfrak{D}_i$ are defined on different spaces. Thus, we need to find a common space on which we can study the behavior of the spectrum of the sequence $(\mathfrak{D}_i)_{i \in \mathbb{N}}$. This is done in the next three steps.

By abuse of notation, we use the same index i for any subsequence we choose. The following identifications are based on the constructions in [Lot02a, Section 4]. The idea here is similar to Fukaya's main idea in [Fuk88]. Namely, we will consider the corresponding sequence of Spin$(n+k)$-principal bundles and identify the spinors on $(M_i, \tilde{g}_i)$ with Spin$(n+k)$-invariant functions on the corresponding Spin$(n+k)$-principal bundle $P_{\mathrm{Spin}}(M_i, \tilde{g}_i)$.

Step 2: Identification of the spinors. Let $\tilde{P}_i := P_{\mathrm{Spin}}(M_i, \tilde{g}_i)$ be the Spin$(n+k)$-principal bundle of $(M_i, \tilde{g}_i)$. Further, we set $\tilde{g}_i^P$ to be a Riemannian metric on $\tilde{P}_i$ such that

$$\tilde{\pi}_i : (\tilde{P}_i, \tilde{g}_i^P) \to (M_i, \tilde{g}_i)$$

is a Riemannian submersion with totally geodesic fibers such that $\mathrm{vol}(\tilde{\pi}_i^{-1}(x)) = 1$ for all $x \in M_i$. Further, denote by

$$\begin{aligned}R : \mathrm{Spin}(n+k) \times \tilde{P}_i &\to \tilde{P}_i, \\ (A, \tilde{x}) &\mapsto R_A(\tilde{x})\end{aligned}$$

the isometric Spin$(n+k)$ action on $\tilde{P}_i$.

Next, we recall the canonical complex spinor representation

$$\theta_{n+k} : \mathrm{Spin}(n+k) \to \Sigma_{n+k},$$

where Σ_{n+k} is a complex vector space of dimension $2^{\left[\frac{n+k}{2}\right]}$. In spin geometry, it is well-known that there is a Hermitian product $\langle ., . \rangle$ on Σ_{n+k} such that Clifford multiplication is skew-symmetric with respect to $\langle ., . \rangle$. This product is unique up to a positive scalar (see for instance [BHM+15, Proposition 1.35]).

Hence, there is an isometric Spin$(n+k)$ action on the tensor product $L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}$ defined by

$$\begin{aligned}\rho : \mathrm{Spin}(n+k) \times L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k} &\to L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}, \\ (A, \sum_l s_l \otimes \varphi_l) &\mapsto \sum_l (s_l \circ R_A) \otimes \theta_{n+k}(A)(\varphi).\end{aligned}$$

Here we used that any element $S \in L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}$ can be written as a finite linear combination $\sum_l s_l \otimes \varphi_l$ of elementary tensors. It is well-known that the tensor product $L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}$ is isometric to the space of all L^2-functions $\sigma : \tilde{P}_i \to \Sigma_{n+k}$ with respect to the volume measure of $(\tilde{P}_i, \tilde{g}_i^P)$. We denote this space by $L^2(\tilde{P}_i, \Sigma_{n+k})$.

We observe that the $\mathrm{Spin}(n+k)$-invariant subspace of $L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}$ is given by

$$\begin{aligned}\left(L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}\right)^{\mathrm{Spin}(n+k)} &:= \left\{S \in L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k} : \rho(A)S = S,\ \forall A \in \mathrm{Spin}(n+k)\right\} \\ &\cong \left\{\sigma \in L^2(\tilde{P}_i, \Sigma_{n+k}) : \sigma \circ R_A = \theta_{n+k}^{-1}(A) \circ \sigma, \forall A \in \mathrm{Spin}(n+k)\right\}.\end{aligned}$$

Since the spinor bundle $\widetilde{\Sigma M}_i$ of $(M_i, \tilde{g}_i)$ is defined as $\widetilde{\Sigma M}_i = \tilde{P}_i \times_{\theta_{n+k}} \Sigma_{n+k}$ it follows at once that there is a canonical isomorphism

$$\widetilde{\Pi}_i : \left(L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}\right)^{\mathrm{Spin}(n+k)} \to L^2(\widetilde{\Sigma M}_i). \tag{4.5.2}$$

To be more concrete, for an element $\sigma \in \left(L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}\right)^{\mathrm{Spin}(n+k)}$ the induced spinor $\widetilde{\Pi}_i(\sigma)$ is given by

$$\widetilde{\Pi}_i(\sigma)_x = [\tilde{x}, \sigma(\tilde{x})], \tag{4.5.3}$$

for any $x \in M_i$ and $\tilde{x} \in \tilde{\pi}_i^{-1}$. Since σ is $\mathrm{Spin}(n+k)$-invariant the definition (4.5.3) does not depend on the choice of $\tilde{x}$. It follows from our choice of the metric $\tilde{g}_i^P$ that the map $\widetilde{\Pi}_i$ is in fact an isometry.

Step 3: Convergence of the $\mathrm{Spin}(n+k)$*-principal bundles.* After we have identified the L^2-spinors of $(M_i, \tilde{g}_i)$ with the $\mathrm{Spin}(n+k)$-invariant subset of $L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}$ we now consider the sequence $(\tilde{P}_i, \tilde{g}_i^P)_{i\in\mathbb{N}}$. By construction, the sectional curvatures and the diameter of this sequence are uniformly bounded in i. Thus, we can apply the G-equivariant version of Gromov's compactness theorem, Theorem 1.7, to the sequence $(\tilde{P}_i, \tilde{g}_i^P)_{i\in\mathbb{N}}$. It follows that there is a subsequence, which we denote again by $(\tilde{P}_i, \tilde{g}_i^P)_{i\in\mathbb{N}}$, that converges to a compact metric space $(\tilde{B}, \tilde{h}^P)$ on which $\mathrm{Spin}(n+k)$ acts as isometries. In particular, $(\tilde{B}, \tilde{h}^P)/\mathrm{Spin}(n+k)$ is isometric to the limit space (B, h) of the sequence $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$. Using the same strategy as in [Fuk88, Theorem 6.1] it follows that $(\tilde{B}, \tilde{h}^P)$ is a Riemannian manifold. Moreover, the metric $\tilde{h}^P$ is $C^{1,\alpha}$ by Theorem 1.32.

As Fukaya's fibration theorem, Theorem 1.17, also holds in a G-equivariant setting [Fuk88, Theorem 9.1], it follows that there is a further subsequence $(\tilde{P}_i, \tilde{g}_i^P)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ there is a $\mathrm{Spin}(n+k)$-equivariant fibration $\tilde{f}_i : \tilde{P}_i \to \tilde{B}$ with infranil fibers and affine structure group. Since for every $i \in \mathbb{N}$, the metric $\tilde{g}_i$ on M_i is invariant, see Corollary 1.29, it follows that $\tilde{g}_i^P$ is also an invariant metric, i.e. there is a $\mathrm{Spin}(n+k)$-invariant metric $\tilde{h}_i^P$ on $\tilde{B}$ such that

$$\tilde{f}_i : (\tilde{P}_i, \tilde{g}_i^P) \to (\tilde{B}, \tilde{h}_i^P)$$

is a $\mathrm{Spin}(n+k)$-equivariant Riemannian affine fiber bundle. Further, we want to remark that $\pi_i : (\tilde{B}, \tilde{h}_i^P) \to (B, \tilde{h}_i)$ is a Riemannian submersion with totally geodesic fibers.

Summarizing these observations, we conclude that the following diagram commutes for every $i \in \mathbb{N}$.

$$\begin{tikzcd}
(\tilde{P}_i, \tilde{g}_i^P) \arrow[r, "\tilde{f}_i"] \arrow[d, "\tilde{\pi}_i"'] & (\tilde{B}, \tilde{h}_i^P) \arrow[d, "\pi_i"] \\
(M_i, \tilde{g}_i) \arrow[r, "f_i"'] & (B, \tilde{h}_i)\ .
\end{tikzcd}$$

Step 4: The space of affine parallel spinors. First we fix an $i \in \mathbb{N}$ and recall the affine connection ∇^{aff} on $(M_i, \tilde{g}_i)$ that is induced by the affine connection on the infranil fiber Z_i of the Riemannian affine fiber bundle $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$. Since for all $p \in B$ the induced metric $\hat{g}_p$ on the fiber $Z_p = f_i^{-1}(p)$ is affine parallel, the connection ∇^{aff} induces an affine connection, also denoted by ∇^{aff} on $\tilde{P}_i$. Hence, there is a well-defined subspace $L^2(\tilde{P}_i, \tilde{g}_i^P)^{\text{aff}} \subset L^2(\tilde{P}_i, \tilde{g}_i^P)$ consisting of affine parallel functions. As we already know that the space $\left(L^2(\tilde{P}_i, \tilde{g}_i^P) \otimes \Sigma_{n+k}\right)^{\text{Spin}(n+k)}$ is isometric to $L^2(\widetilde{\Sigma M_i})$, see (4.5.2), (4.5.3), it follows that there is an induced isometry

$$\widetilde{\Pi}_i : \left(L^2(\tilde{P}_i, \tilde{g}_i^P)^{\text{aff}} \otimes \Sigma_{n+k}\right)^{\text{Spin}(n+k)} \to \mathcal{S}_i^{\text{aff}}. \tag{4.5.4}$$

Here $\mathcal{S}_i^{\text{aff}}$ denotes as usual the space of affine parallel spinors.

Next, we observe that we can view $L^2(\tilde{P}_i, \tilde{g}_i^P)^{\text{aff}}$ also as the space of functions in $L^2(\tilde{P}_i, \tilde{g}_i^P)$ that are constant along the fibers of the fibration $\tilde{f}_i : \tilde{P}_i \to \tilde{B}$. In particular, for any $s \in L^2(\tilde{P}_i, \tilde{g}_i^P)^{\text{aff}}$ there is an $\check{s} \in L^2(\tilde{B}, \tilde{h}_i^P)$ such that $\tilde{f}_i^* \check{s} = s$. Hence, there is a natural isomorphism between $L^2(\tilde{P}_i, \tilde{g}_i^P)^{\text{aff}}$ and $L^2(\tilde{B}, \tilde{h}_i^P)$. But we want to have an isometry between these spaces. Therefore, we consider the function

$$\begin{aligned} v_i : \tilde{B} &\to \mathbb{R}, \\ \tilde{p} &\mapsto \operatorname{vol}(\tilde{f}_i^{-1}(\tilde{p})). \end{aligned}$$

Then,

$$\begin{aligned} L^2(\tilde{B}, \tilde{h}_i^P) &\to L^2(\tilde{P}_i, \tilde{g}_i^P)^{\text{aff}}, \\ \check{s} &\mapsto f_i^*(v_i^{-\frac{1}{2}} \check{s}) \end{aligned}$$

is an isometry. Combined with (4.5.4) we obtain the isometry

$$\tilde{Q}_i : \left(L^2(\tilde{B}, \tilde{h}_i^P) \otimes \Sigma_{n+k}\right)^{\text{Spin}(n+k)} \to \mathcal{S}_i^{\text{aff}},$$

such that the following diagram commutes

$$\begin{tikzcd}
\left(L^2(\tilde{B}, \tilde{h}_i^P) \otimes \Sigma_{n+k}\right)^{\text{Spin}(n+k)} \arrow[d] \arrow[dr, "\tilde{Q}_i"] & \\
L^2({}^\diamond\boldsymbol{\Sigma}_i B \otimes \mathcal{P}_i) \arrow[r, "Q_i"'] & \mathcal{S}_i^{\text{aff}}\ .
\end{tikzcd}$$

Here we used that there is an isometry $\left(L^2(\tilde{B}, \tilde{h}_i^P) \otimes \Sigma_{n+k}\right)^{\mathrm{Spin}(n+k)} \to L^2({}^\diamond\mathbf{\Sigma} B \otimes \mathcal{P}_i)$ similar to (4.5.2), (4.5.3).

We recall that

$$\Sigma_{n+k} \cong {}^\diamond\Sigma_n \otimes \Sigma_k,$$

with

$${}^\diamond\Sigma_n := \begin{cases} \Sigma_n, & \text{if } n \text{ or } k \text{ is even,} \\ \Sigma_n^+ \oplus \Sigma_n^-, & \text{if } n \text{ and } k \text{ are odd,} \end{cases}$$

where Σ_n^+ and Σ_n^- are two isomorphic copies of Σ_n (compare (B.0.1), (B.1.1)). Then

$$\begin{aligned} \left(L^2(\tilde{B}, \tilde{h}_i^P) \otimes \Sigma_{n+k}\right)^{\mathrm{Spin}(n+k)} &\cong \left(L^2(\tilde{B}, \tilde{h}_i^P) \otimes \left({}^\diamond\Sigma_n \otimes \Sigma_k\right)\right)^{\mathrm{Spin}(n+k)} \\ &\cong L^2({}^\diamond\mathbf{\Sigma}_i B \otimes \mathcal{P}) \end{aligned}$$

for some fixed locally defined vector bundle $\mathcal{P}$ over B independent of i. In particular, there are isomorphisms $\mathcal{P}_i \to \mathcal{P}$ for all $i \in \mathbb{N}$. Using pullback metrics we obtain the isometry

$$Q_i : L^2({}^\diamond\mathbf{\Sigma}_i B \otimes \mathcal{P}) \to \mathcal{S}_i^{\mathrm{aff}}. \tag{4.5.5}$$

Now, ${}^\diamond\mathbf{\Sigma}_i B$ is the only object left that depends on i. To remove also this i-dependency, let us assume for the moment that B is a spin manifold. For any $i \in \mathbb{N}$ we consider the isometry

$$\hat{\beta}_h^{\tilde{h}_i} : L^2(\Sigma B) \to L^2(\Sigma_i B),$$

that was constructed in Appendix C. Here ΣB is the spinor bundle of (B, h) and $\Sigma_i B$ is the spinor bundle of $(B, \tilde{h}_i)$.

Going back to the original case, where B is not necessarily spin, we can still apply a local version of the isometry $\hat{\beta}_h^{\tilde{h}_i}$ to obtain an isometry

$$\Theta_i : L^2({}^\diamond\mathbf{\Sigma} B \otimes \mathcal{P}) \to L^2({}^\diamond\mathbf{\Sigma}_i B \otimes \mathcal{P}).$$

Step 5: The convergence of the Dirac eigenvalues. We recall from Step 1 that the Dirac operator $\tilde{D}^{M_i}$ on $(M_i, \tilde{g}_i)$ restricted to $\mathcal{S}_i^{\mathrm{aff}}$ can be written as

$$\begin{aligned} \tilde{D}^{M_i}{}_{|\tilde{\mathcal{S}}^{\mathrm{aff}}} &= Q_i \circ \left(\check{D}^{\mathcal{T}_i} + \frac{1}{2}\gamma(\check{\mathcal{Z}}_i) + \frac{1}{2}\gamma(\check{\mathcal{A}}_i)\right) \circ Q_i^{-1} \\ &=: Q_i \circ \mathfrak{D}_i \circ Q_i^{-1}, \end{aligned}$$

where Q_i is now the isometry (4.5.5). It is obvious that the calculations that we did in Step 1 just carry over.

It follows by construction and Theorem C.2 that for any $i \in \mathbb{N}$ the operator

$$\mathcal{D}_i := \Theta_i^{-1} \circ \mathfrak{D}_i \circ \Theta_i$$

is isospectral to the restriction $\tilde{D}^{M_i}_{|\mathcal{S}_i^{\mathrm{aff}}}$ and densely defined on $H^{1,2}({}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P})$. By a small abuse of notation, we continue to write

$$\mathcal{D}_i = \check{D}^{\mathcal{T}_i} + \frac{1}{2}\gamma(\check{\mathcal{Z}}_i) + \frac{1}{2}\gamma(\check{\mathcal{A}}_i).$$

First, we observe that the C^1-norms corresponding to $(B, \tilde{h}_i)$ are all equivalent to the C^1-norm on (B, h) because $\lim_{i\in\mathbb{N}} \|\tilde{h}_i - h\|_{C^1} = 0$. By Lemma 3.8 it follows that the sequence of operators $\big(\gamma(\check{\mathcal{Z}}_i)\big)_{i\in\mathbb{N}}$ is uniformly bounded in $C^1(B, h)$. Further, we conclude from Lemma 3.10 that also the sequence $\big(\gamma(\check{\mathcal{A}}_i)\big)_{i\in\mathbb{N}}$ is uniformly bounded in $C^1(B, h)$. Since $C^1 \hookrightarrow C^{0,\alpha}$ is a compact embedding for all $\alpha \in [0, 1)$ there is a subsequence such that $\big(\gamma(\check{\mathcal{Z}}_i)\big)_{i\in\mathbb{N}}$ and $\big(\gamma(\check{\mathcal{A}}_i)\big)_{i\in\mathbb{N}}$ converge to well-defined operators $\gamma(\check{\mathcal{Z}}_\infty)$ and $\gamma(\check{\mathcal{A}}_\infty)$ in $C^{0,\alpha}$ for any $\alpha \in [0, 1)$. As $\check{\nabla}^{\mathcal{T}_i}$ corresponds to the twisted connection $\nabla^{\tilde{h}_i} \otimes \check{\nabla}^{\mathcal{V}_i}$, it follows from Theorem C.2 and Lemma 3.3 that there is a further subsequence $(M_i, g_i)_{i\in\mathbb{N}}$ such that the corresponding sequence $(\mathcal{D}_i)_{i\in\mathbb{N}}$ is a sequence of operators that are densely defined on $H^{1,2}({}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P})$. Furthermore, the sequence $(\mathcal{D}_i)_{i\in\mathbb{N}}$ converges in $B(H^{1,2}({}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P}), L^2({}^{\diamond}\mathbf{\Sigma} B \otimes \mathcal{P}))$ to the claimed limit operator $\mathcal{D}^B$. Here, $B(.,.)$ is the space of bounded linear operators endowed with the operator norm.

Thus, the sequence $(\mathcal{D}_i)_{i\in\mathbb{N}}$ satisfies the assumptions of Theorem C.5. In particular, it follows that the spectra $(\sigma(\mathcal{D}_i))_{i\in\mathbb{N}}$ converge to $\sigma(\mathcal{D}^B)$ uniformly in the arsinh-topology as $i \to \infty$. □

As a conclusion we can characterize the special case where the spectrum of the limit operator $\mathcal{D}^B$ coincides with the spectrum of the Dirac operator on the manifold B up to multiplicity. We formulate the following corollary for collapsing sequences of Riemannian affine fiber bundles and use the notation of Chapter 3. Corollary 0.3 follows immediately from Corollary 4.6 because any collapsing sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n+k, d)$ with smooth limit space (B, h) can be approximated by a collapsing sequence of Riemannian affine fiber bundles $(M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ such that $\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0$ and $\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0$, see Corollary 1.29 and Remark 1.30. Further, we know that Dirac eigenvalues are continuous under a C^1-variation in the arsinh-topology, Theorem C.4.

Corollary 4.6. *Let $(f_i : (M_i, g_i) \to (B, h_i))_{i\in\mathbb{N}}$ be a collapsing sequence of Riemannian affine fiber bundles such that $(M_i, g_i)_{i\in\mathbb{N}}$ is a spin manifold in $\mathcal{M}(n + k, d)$ and B is a closed n-dimensional manifold. Further, we denote by Z_i the closed k-dimensional infranilmanifold which is diffeomorphic to the fibers of $f_i : (M_i, g_i) \to (B, h_i)$. If*

$$\limsup_{i\to\infty} \| \operatorname{Hol}(\mathcal{V}_i, \nabla^{\mathcal{V}_i}) - \operatorname{Id} \|_\infty = 0,$$

$$\limsup_{i\to\infty} \left(\sup_{p\in B} \| \operatorname{scal}(Z^i_p)\|_\infty \right) = 0,$$

$$\limsup_{i\to\infty} \|A_i\|_\infty = 0,$$

and $\mathcal{S}_i$ is nontrivial for almost all $i \in \mathbb{N}$, then there is a subsequence also denoted by $(M_i, g_i)_{i\in\mathbb{N}}$ such that the spin structure on (M_i, g_i) induces the same spin structure on B for all $i \in \mathbb{N}$ and such that the spectrum of the Dirac operator $D^{M_i}_{|\mathcal{S}_i}$ converges, up to multiplicity, to the spectrum of D^B, if n or k is even, and to the spectrum of $D^B \oplus -D^B$, if n and k are odd. Each eigenvalue is counted $\operatorname{rank}(\mathcal{P})$*-times.*

Proof. From the above theorem it follows that the limit operator $\mathcal{D}^B$ equals $D^B \otimes \mathrm{Id}$, respectively $(D^B \oplus -D^B) \otimes \mathrm{Id}$ if

(1) $\nabla^{\mathcal{V}_\infty}$ is gauge equivalent to the trivial connection,

(2) $\mathcal{Z}_\infty = 0$,

(3) $\mathcal{A}_\infty = 0$.

Regarding the first point, we recall that $\nabla^{\mathcal{V}_\infty}$ is gauge equivalent to the trivial connection if $(\mathcal{V}_i, \nabla^{\mathcal{V}_i})$ is in the limit $i \to \infty$ a trivial vector bundle with trivial holonomy, see Remark 3.4. In this case it is immediate that $\mathcal{P}$ is the trivial vector bundle. Therefore, it follows that also ${}^{\diamond}\Sigma B$ is globally well-defined. In particular, there is a well-defined induced spin structure on B. As there are only finitely many equivalence classes of spin structures on a fixed closed Riemannian manifold (see for instance [LM89, Chapter II, Theorem 1.7]), we can choose a subsequence, again denoted by $(M_i, g_i)_{i \in \mathbb{N}}$ such that the spin structure on (M_i, g_i) induces the same spin structure on B for all $i \in \mathbb{N}$. Since $\|\mathcal{Z}_i\|_\infty \leqslant 3\|\operatorname{scal}^{Z_i}\|_\infty$, see the proof of Lemma 3.8, the second condition implies that the limit $\mathcal{Z}_\infty$ vanishes identically. Finally, it is immediate that $\mathcal{A}_\infty = 0$ is equivalent to the vanishing of the A-tensor in the limit since $\|\mathcal{A}_i\|_\infty = \|A_i\|_\infty$ by definition.

For the last statement, let $l := \operatorname{rank}(\mathcal{P})$. Since $\mathcal{P}$ is the trivial vector bundle there is a global frame $(\rho_1, \ldots, \rho_l)$. Let φ be an eigenspinor of D^B, resp. $D^B \oplus -D^B$. Then for any $1 \leqslant j \leqslant l$ the spinor $\varphi \otimes \rho_j$ is an eigenspinor of $\mathcal{D}^B$ with the same eigenvalue. Hence, any eigenvalue of D^B, resp. $D^B \oplus -D^B$ is counted l-times. □

We conclude that there are three geometric obstructions for a convergence to the Dirac operator on the base space. As discussed in the examples 3.5, 3.6, 3.7, 3.9 in Chapter 3.1 these geometric obstructions are all independent of each other. In the following example we discuss a class of collapsing sequences that satisfy the assumptions of Corollary 4.6.

Example 4.7. Let G be a compact m-dimensional Lie group with Lie algebra $\mathfrak{g}$. Since G is compact we can choose a biinvariant metric g. It is well-known that any Lie group is parallelizable. Since we have chosen a biinvariant metric g it follows that $P_{\mathrm{SO}}G \cong G \times \mathrm{SO}(m)$. We see at once that (G, g) is spin. In the following we fix the spin structure

$$P_{\mathrm{Spin}}G \cong G \times \mathrm{Spin}(m). \tag{4.7.1}$$

If G is connected and simply-connected, then the spin structure (4.7.1) is, up to equivalence, the only spin structure on G.

Next we fix a maximal torus $\mathbb{T}^k$ in G. The torus $\mathbb{T}^k$ acts on G via left multiplication. Since the metric g is biinvariant, the maximal torus $\mathbb{T}^k$ acts on G as isometries. In the following we consider the homogeneous space $B := {}_{\mathbb{T}^k}\backslash^{G}$ with the induced quotient metric h. We would like to point out that the quotients ${}_{\mathbb{T}^k}\backslash^{G}$ are called *flag manifolds*. Let $\mathfrak{g} = \mathfrak{t} + \mathfrak{b}$ be the splitting of the Lie algebra of G into the Lie algebra $\mathfrak{t}$ of $\mathbb{T}^k$ and its orthogonal complement $\mathfrak{b} \cong T_{f(e)}B$ with respect to the biinvariant metric g. Here e is the neutral element in G and $f : G \to B$ is the quotient map. Since g is a biinvariant metric it follows that $\mathfrak{b}$ is an $\mathrm{Ad}(\mathbb{T}^k)$-invariant subspace, where Ad is the adjoint representation.

In particular, the map $\chi : \mathbb{T}^k \to \mathrm{SO}(\mathfrak{b}), \chi(t)(X) = \mathrm{Ad}_t(X)$ for all $t \in \mathbb{T}^k$ and $X \in \mathfrak{b}$ is well-defined and satisfies

$$G \times_\chi \mathrm{SO}(\mathfrak{b}) \cong P_{\mathrm{SO}} B.$$

Let $\rho : \mathrm{Spin}(\mathfrak{b}) \to \mathrm{SO}(\mathfrak{b})$ be the usual double cover. Then the spin structure (4.7.1) of G induces a spin structure on the homogeneous space B if and only if there is a Lie group homomorphism $\tilde{\chi} : \mathbb{T}^k \to \mathrm{Spin}(\mathfrak{b})$ lifting χ, see for instance [Bär92, Lemma 3],

$$\begin{tikzcd} & \mathrm{Spin}(\mathfrak{b}) \arrow[d, "\rho"] \\ \mathbb{T}^k \arrow[ur, "\tilde{\chi}"] \arrow[r, "\chi"'] & \mathrm{SO}(\mathfrak{b}) \end{tikzcd} .$$

This is for example the case if the maximal torus equals the center of the Lie group as, in that case, χ acts trivially.

By construction $f : (G, g) \to (B, h)$ is a $\mathbb{T}^k$-principal bundle. Furthermore, f is a Riemannian submersion with totally geodesic fibers. We can write

$$g = \breve{g} + f^*h,$$

where $\breve{g}$ vanishes on vectors orthogonal to the fibers. For any $\varepsilon > 0$ we define

$$g_\varepsilon := \varepsilon^2 \breve{g} + f^*h.$$

We observe that g_ε is left invariant for all $\varepsilon > 0$ and biinvariant if and only if $\varepsilon = 1$. As $\varepsilon \to 0$ the sequence $(G, g_\varepsilon)_\varepsilon$ converges to (B, h) in the Gromov-Hausdorff topology. For abbreviation we denote by G_ε the Riemannian manifold (G, g_ε). It follows from [CG86, Theorem 2.1] that there are constants C and d such that $|\sec(G_\varepsilon)| \leqslant C$ and $\mathrm{diam}(G_\varepsilon) \leqslant d$ for all $\varepsilon \in (0, 1)$.

Now we want to show that the assumptions of Corollary 4.6 are fulfilled. Thus, we take a closer look at the Riemannian submersions $f_\varepsilon : (G, g_\varepsilon) \to (B, h)$. Since the fibers of f_ε are totally geodesic for all $\varepsilon > 0$ it follows that the tensor T_ε vanishes identically for all $\varepsilon > 0$. Moreover, the vertical distribution $\mathcal{V}_\varepsilon$ is a trivial $\mathbb{R}^k$ vector bundle over G_ε as $\mathbb{T}^k$ acts on G_ε as isometries for all $\varepsilon > 0$. Thus, it follows form Lemma 3.3 that for all $\varepsilon > 0$, $\nabla^{\mathcal{V}_\varepsilon}$ is gauge equivalent to the trivial connection. Applying this gauge transformation if necessary, we can assume without loss of generality that $\nabla^{\mathcal{V}_\varepsilon}$ is the trivial connection on $\mathcal{V}_\varepsilon$ for all $\varepsilon > 0$. By construction, the fibers of f_ε are embedded flat tori. Thus, the induced Levi-Civita connection on the fiber is the affine connection, i.e. $\mathcal{Z}_\varepsilon = 0$ for all $\varepsilon > 0$. Next, we take a global orthonormal vertical frame $(\zeta_1, \ldots, \zeta_k)$ that trivializes the vertical distribution $\mathcal{V}_1$ of (G, g_1). Then $(\varepsilon^{-1}\zeta_1, \ldots, \varepsilon^{-1}\zeta_k)$ is an orthonormal vertical frame for the vertical distribution $\mathcal{V}_\varepsilon$ of G_ε. For any two horizontal vectors X, Y we calculate

$$\begin{aligned} A_\varepsilon(X, Y) &= \frac{1}{2} \sum_{a=1}^{k} g_\varepsilon([X, Y], \varepsilon^{-1}\zeta_a) \\ &= \frac{1}{2} \sum_{a=1}^{k} \varepsilon^{2-1} \breve{g}([X, Y], \zeta_a) \\ &= \varepsilon A_1(X, Y). \end{aligned}$$

In particular, it follows that $\lim_{\varepsilon\to 0}\|A_\varepsilon\|_{g_\varepsilon} = 0$. Hence, all assumptions of Corollary 4.6 are fulfilled. Thus, if for almost all $\varepsilon \in (0,1]$ the space of affine parallel spinors is nontrivial then there is an induced spin structure on B and the spectra of the Dirac operators restricted to the space of affine parallel spinors converges, up to multiplicity, to the spectrum of the Dirac operator D^B of $B = {}_{\mathbb{T}^k}\backslash^{G}$, if k or $\dim(B)$ is even, respectively to the spectrum of $D^B \oplus -D^B$, if $\dim(B)$ and k are odd.

It follows from Theorem 4.5 that the limit operator $\mathcal{D}^B$ is a twisted Dirac operator with a symmetric potential lying in $C^{0,\alpha} \cap H^{1,\infty}$ for any $\alpha \in [0,1)$. In the following proposition we show that the spectrum of a Dirac operator with a symmetric $H^{1,\infty}$-potential converges to the spectrum of a Dirac operator with a symmetric $H^{1,\infty}$-potential. To simplify notation we define

$$\mathcal{O}(n,d) := \{(M,g,W) : (M,g) \in \mathcal{M}(n,d) \text{ and spin}, W \in \mathrm{Hom}(\Sigma M, \Sigma M)\}.$$

Proposition 4.8. *Let $(M_i, g_i, W_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{O}(n+k,d)$ converging to a smooth n-dimensional Riemannian manifold (B,h) such that for almost all $i \in \mathbb{N}$ the space $\mathcal{S}_i$ is nontrivial. Further, we suppose that, for all $i \in \mathbb{N}$, W_i is symmetric and $\|W_i\|_{H^{1,\infty}} \leqslant \Lambda$ for some positive constant Λ. Then there is a subsequence $(M_i, g_i)_{i\in\mathbb{N}}$ such that the spectrum of $(D^{M_i} + W_i)_{|\mathcal{S}_i}$ converges uniformly in the* arsinh*-topology to the spectrum of $\mathcal{D}^B + \breve{\mathcal{W}}_\infty$. Here $\mathcal{D}^B$ is as in Theorem 4.5 and $\breve{\mathcal{W}}_\infty$ is a symmetric $H^{1,\infty}$-potential induced by the limit of the sequence $\left(W_{i\,|\mathcal{S}_i}\right)_{i\in\mathbb{N}}$.*

Proof. Similar to the proof of Theorem 4.5 it suffices to consider the associated sequence $\left(f_i : (M_i, \tilde g_i) \to (B, \tilde h_i)\right)_{i\in\mathbb{N}}$ of Riemannian affine fiber bundles. We recall that there are explicitly constructed isometries $\Theta_i : L^2(\widetilde{\Sigma M}_i) \to L^2(\Sigma M_i)$, see Appendix C. Hence, we can pull back W_i to an element $\widetilde{W}_i$ of $\mathrm{Hom}(\widetilde{\Sigma M}_i, \widetilde{\Sigma M}_i)$. For any $i \in \mathbb{N}$, let $\widetilde{\mathcal{W}}_i$ be the associated affine parallel operator for $\widetilde{W}_i$ defined in Proposition 3.28. Since $\widetilde{\mathcal{W}}_i$ is symmetric for all $i \in \mathbb{N}$ it follows from [Kat76, Chapter 5, Theorem 4.10] that

$$\begin{aligned}
\lim_{i\to\infty} \mathrm{dist}\left(\sigma(\tilde D^{M_i} + \widetilde{W}_i), \sigma(\tilde D^{M_i} + \widetilde{\mathcal{W}}_i)\right) &\leqslant \lim_{i\to\infty} \|\widetilde{W}_i - \mathcal{W}_i\|_\infty \\
&\leqslant \lim_{i\to\infty}\Big(2\max_{p\in B}(\mathrm{diam}(f_i^{-1}(p))\cdot \\
&\qquad (\|\nabla \widetilde{W}_i\|_\infty + C(k, C_R, C_T)\|\widetilde{W}_i\|_\infty\Big) \\
&\leqslant 2\lim_{i\to\infty}\left(\max_{p\in B}(\mathrm{diam}(f_i^{-1}(p))\right)(1 + C(k,C_R,C_T))\Lambda \\
&= 0.
\end{aligned}$$

Therefore it suffices to study the spectrum of $\tilde D^{M_i} + \widetilde{\mathcal{W}}_i$. First we note that this operator acts diagonally with respect to the splitting $L^2(\widetilde{\Sigma M}_i) = \mathcal{S}_i^{\mathrm{aff}} \oplus \left(\mathcal{S}_i^{\mathrm{aff}}\right)^\perp$. Thus the same proof as for Theorem 4.5 applies in this setting. Let $\breve{\mathcal{W}}_i$ be the induced element on $\mathrm{Hom}({}^\diamond\boldsymbol{\Sigma}_i B \otimes \mathcal{P}, {}^\diamond\boldsymbol{\Sigma}_i B \otimes \mathcal{P})$, where we used the same notation as in the proof of Theorem 4.5. Since $\|W_i\|_{H^{1,\infty}} \leqslant \Lambda$ by assumption it is a simple matter to check that $\|\breve{\mathcal{W}}_i\|_{H^{1,\infty}} \leqslant \Lambda$.

In particular, there is a subsequence $(\breve{\mathcal{W}}_i)_{i\in\mathbb{N}}$ converging in the L^∞-topology to an operator $\breve{\mathcal{W}}_\infty \in \mathrm{Hom}({}^\diamond\boldsymbol{\Sigma}B\otimes\mathcal{P}, {}^\diamond\boldsymbol{\Sigma}B\otimes\mathcal{P})$. The remaining steps are similar to the proof of Theorem 4.5. □

4.2 The Dirac operator and codimension one collapse

In Chapter 2.2 we introduced the set

$$\mathcal{M}(n+1,d,C) := \left\{(M,g) \in \mathcal{M}(n+1,d) : C \leqslant \frac{\mathrm{vol}(M)}{\mathrm{inj}(M)}\right\}$$

of isometry classes of closed $(n+1)$-dimensional Riemannian manifolds. In Theorem 2.17 we showed that any n-dimensional limit space (B,h) of a sequence in $\mathcal{M}(n+1,d,C)$ is a Riemannian orbifold with a $C^{1,\alpha}$-metric h. In addition, the second derivatives of h exist almost everywhere and $\|\sec^h\|_{L^\infty} \leqslant K(n,d,C)$. As discussed in Chapter 3.2.1 spin structures and Dirac operators are also defined on Riemannian orbifolds. In this section we give an explicit description of the structure and behavior of the Dirac spectrum on any collapsing sequence in $\mathcal{M}(n+1,d,C)$.

Let $(M_i,g_i)_{i\in\mathbb{N}}$ be a sequence of Riemannian spin manifolds in $\mathcal{M}(n+1,d,C)$ converging to an n-dimensional Riemannian orbifold (B,h). By Corollary 1.29 there is an index I such that for any $i \geqslant I$ there is a fibration $f_i : M_i \to B$ defining an S^1-orbifold bundle with structure group in $\mathrm{Aff}(S^1) \cong S^1 \rtimes \{\pm 1\}$. If B is orientable, then $f_i : M_i \to B$ is an S^1-principal bundle. Otherwise, we take the orientation cover $\hat{B}$ of B and consider the pullback bundle $f_i : \hat{M}_i \to \hat{B}$. As the structure group of the fibration $f_i : M_i \to B$ lies in $\mathrm{Aff}(S^1)$ this pullback bundle is an S^1-principal bundle. Hence, it often suffices to consider sequences of S^1-principal orbifold bundles. Moreover, for all $i \geqslant I$ there are metrics $\tilde{g}_i$ on M_i and $\tilde{h}_i$ on B such that the fibrations $f_i : (M_i,\tilde{g}_i) \to (B,\tilde{h}_i)$ are Riemannian submersions, see Corollary 1.29. In addition, we have $\lim_{i\to\infty}\|\tilde{g}_i - g_i\|_{C^1} = 0$ and $\lim_{i\to\infty}\|\tilde{h}_i - h\|_{C^1} = 0$. Since Dirac eigenvalues are continuous under a C^1-variation of metrics, see Theorem C.4, it suffices to consider sequences of *Riemannian S^1-principal bundles* i.e. S^1-principal bundles $f : (M,g) \to (B,h)$ such that f is a Riemannian submersion.

For the moment we fix a Riemannian S^1-principal bundle $f : (M,g) \to (B,h)$ such that (M,g) is a spin manifold with a fixed spin structure. As discussed in Chapter 3.2.1 we distinguish between two kinds of spin structures on the total space (M,g), the projectable spin structures, if the S^1-action lifts to $P_{\mathrm{Spin}}M$, and the non projectable spin structures, where a double cover of the S^1-action acts on $P_{\mathrm{Spin}}M$. If the spin structure on M is projectable then the spin structure on M induces a spin structure on B. Otherwise there is an induced spin^c structure. In the following, we discuss the structure of the spinor bundle ΣM following [Amm98a], [Amm98b, Kapitel 7].

The isometric S^1-action on (M,g) induces a Killing vector field K. Furthermore, the length of the S^1-fibers of $f : M \to B$ equals $2\pi l$, where $l := |K|$. In particular, we can view l as a function defined on the base space B. As explained above, there is always an induced isometric S^1-action on $P_{\mathrm{Spin}}M$. Thus, we can define the Lie-derivative of a spinor

φ in the direction of K as

$$\mathcal{L}_K\varphi(x) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \kappa_{-s}(\varphi(\kappa_s(x))),$$

where κ denotes the S^1-action on ΣM and on M respectively. By construction $\mathcal{L}_K$ is the differential of the S^1-action on $L^2(\Sigma M)$. It follows from representation theory for compact abelian groups that $\mathcal{L}_K$ has the eigenvalues $\mathrm{i}k$, where $k \in \mathbb{Z}$ if the spin structure on M is projectable and $k \in (\mathbb{Z} + \frac{1}{2})$ if the spin structure on M is non projectable. Thus, $L^2(\Sigma M)$ decomposes as

$$L^2(\Sigma M) = \bigoplus_k V_k,$$

where V_k is the eigenspace of $\mathcal{L}_K$ with respect to the eigenvalue $\mathrm{i}k$.

Since S^1 acts on $L^2(\Sigma M)$ as isometries, $\mathcal{L}_K$ commutes with the Dirac operator D^M. Therefore, $\mathcal{L}_K$ and D^M are simultaneously diagonalizable, i.e. for any eigenspinor φ of D^M there is a k such that $\varphi \in V_k$. For any fixed k, let $\lambda_{j,k}$ be the eigenvalues of $D^M_{|V_k}$ such that

$$\ldots \leqslant \lambda_{-1,k} \leqslant \lambda_{0,k} < 0 \leqslant \lambda_{1,k} \leqslant \lambda_{2,k} \leqslant \ldots .$$

Remark 4.9. It is easy to check that $\mathcal{L}_K$ is the same as ∇^{aff}_K in this setting. Hence, V_0 is the space of affine parallel spinors.

Let $L := M \times_{S^1} \mathbb{C}$ be the associated line bundle. For any $k \in \mathbb{Z}$, resp. $k \in (\mathbb{Z} + \frac{1}{2})$, Ammann constructed the isometry

$$Q_k : L^2({}^\diamond\boldsymbol{\Sigma} B \otimes L^{-k}) \to V_k,$$

in [Amm98b, Lemma-Definition 7.2.3]. For $k = 0$ this isometry coincides with the isometry constructed in Lemma 3.26. Using the isometry Q_k we can generalize the bounds on Dirac eigenvalues given in Theorem 4.1 to any collapsing sequence in $\mathcal{M}(n+1, d, C)$, proving Proposition 0.4.

Proposition 4.10. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence of spin manifolds in $\mathcal{M}(n+1, d, C)$ converging to an n-dimensional Riemannian orbifold (B, h). Suppose that the spin structures on M_i are either all projectable or non projectable. Then we can number the Dirac eigenvalues $(\lambda_{j,k}(i))_{j,k}$ with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ (projectable spin structures), resp. $k \in (\mathbb{Z}+\frac{1}{2})$(non projectable spin structures), such that for any $\varepsilon > 0$ there is an index $I > 0$ such that for all $i \geqslant I$ there are fibrations $f_i : M_i \to B$ with fibers diffeomorphic to S^1 such that for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ (projectable spin structures), resp. $k \in (\mathbb{Z} + \frac{1}{2})$ (non projectable spin structures),*

$$|\lambda_{j,k}(i)| \geqslant \sinh\left(\operatorname{arsinh}\left(\frac{|k|}{\|l_i\|_\infty} - \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A - \varepsilon\right) - \varepsilon\right).$$

Here $2\pi l_i$ is the length of the fibers and C_A is a constant depending on n, d and C.

In particular, $\lim_{i\to\infty} |\lambda_{j,k}(i)| = 0$ *whenever* $k \neq 0$ *since* $\lim_{i\to\infty} l_i = 0$.

For any $i \geqslant I$, let $\omega_i \in \Omega(M_i, \mathcal{V}_i)$ be the orthogonal projection onto $\mathcal{V}_i := \ker(\mathrm{d}f_i)$, where $f_i : M_i \to B$. If, in addition, there is a constant C such that

$$\|\mathrm{d}\omega_i\|_{C^{0,1}} \leqslant C$$

for all $i \geqslant I$, then for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ (projectable spin structures), resp. $k \in (\mathbb{Z} + \frac{1}{2})$ (non projectable spin structures),

$$\limsup_{i \in \mathbb{N}} \left(\min_{p \in B} l_i(p) |\lambda_{j,k}(i)| \right) \leqslant |k|.$$

Proof. By Corollary 1.29 and Proposition 1.31, there is an index I_1 such that for any $i \geqslant I_1$ there is an S^1-orbifold bundle $f_i : M_i \to B$ with affine structure group. If B is orientable then this is an S^1-principal orbifold bundle. If B is non orientable we consider the pullback bundle $\hat{f}_i : \hat{M}_i \to \hat{B}$ over the orientation covering $\hat{B}$. Since the structure group of the fibration $f_i : M_i \to B$ lies in $\mathrm{Aff}(S^1) \cong S^1 \rtimes \{\pm 1\}$ this pullback is an S^1-principal orbifold bundle. As (non) projectable spin structures pull back to (non) projectable spin structures and as the spectrum $\sigma(D^{M_i})$ is a subset of $\sigma(D^{\hat{M}_i})$ we can assume without loss of generality that the limit space B is orientable

Applying Corollary 1.29 to the sequence $(M_i, g_i)_{i \in \mathbb{N}}$ there are metrics $\tilde{g}_i$ on M_i and metrics $\tilde{h}_i$ on B such that the fibration $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ is a Riemannian S^1-principal orbifold bundle for all $i \geqslant I_1$. Moreover,

$$\begin{aligned} \lim_{i \to \infty} \|\tilde{g}_i - g_i\|_{C^1} &= 0, \\ \lim_{i \to \infty} \|\tilde{h}_i - h\|_{C^1} &= 0. \end{aligned}$$

The change of the Dirac spectra is controlled by

$$|\operatorname{arsinh}(\lambda^{\tilde{D}}_{j,k}(i)) - \operatorname{arsinh}(\lambda^{D}_{j,k}(i))| \leqslant C\|g_i - \tilde{g}_i\|_{C^1}, \tag{4.10.1}$$

for a positive constant C, see Theorem C.4. Here $\lambda_{j,k}(i)^D$ denotes an eigenvalue of D^{M_i} and $\lambda_{j,k}(i)^{\tilde{D}}$ an eigenvalue of $\tilde{D}^{M_i}$, where the eigenvalues are numbered as explained in the beginning of this section. At this point we want to remark, that the numbering of the Dirac eigenvalues was derived for Riemannian S^1-principal orbifold bundles only. Hence, this numbering is a priori only defined for the eigenvalues $(\lambda^{\tilde{D}}_{j,k}(i))_{j,k}$ of the Dirac operator $\tilde{D}^{M_i}$ on $(M_i, \tilde{g}_i)$. Nevertheless, it follows from Theorem C.4 that there is an induced numbering $(\lambda^{D}_{j,k}(i))_{j,k}$ of the eigenvalues of the original Dirac operator D^{M_i} on (M, g_i) such that the inequality (4.10.1) holds.

As shown in [Amm98b, Beweis von Satz 7.2.1], see also [Amm98a], the Dirac operator can be written as

$$\tilde{D}^{M_i} = \frac{1}{l_i}\gamma\left(\frac{K_i}{l_i}\right)\mathcal{L}_{K_i} + D^{\mathcal{H}_i} - \frac{1}{4}\gamma\left(\frac{K_i}{l_i}\right)\gamma(l_i F_i),$$

where $F_i := \mathrm{d}\tilde{\omega}_i$ is the curvature of the unique connection one-form $\mathrm{i}\tilde{\omega}_i$, whose kernel is orthogonal to the fibers, and $D^{\mathcal{H}_i}$ is described by its action on the eigenspaces $V_k(i)$ of $\mathcal{L}_{K_i}$,

$$D^{\mathcal{H}_i}{}_{|V_k(i)} := Q_{k,i} \circ D_{k,i} \circ Q_{k,i}^{-1}.$$

In the above equation, $D_{k,i}$ is the twisted Dirac operator on $\Sigma_i B \otimes L_i^{-k}$ if n is even, and it is the twisted Dirac operator on $(\Sigma_i^+ B \oplus \Sigma_i^- B) \otimes L_i^{-k}$, if n is odd. Here $\Sigma_i^+ B$ and $\Sigma_i^- B$ are two copies of the spinor bundle $\Sigma_i B$. However, Clifford multiplication by vector fields $X \in \Gamma(TB)$ acts on $\Sigma_i^+ B$ as $\gamma(X)$ and on $\Sigma^- B$ as $-\gamma(X)$. For more details we refer the reader to Appendix B.

Moreover, a straightforward calculation shows that

$$\|l_i F_i\|_\infty = 2\|A_i\|_\infty, \tag{4.10.2}$$

where A_i denotes the A-tensor of the Riemannian submersion $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$. By Theorem 2.17 there is a constant $K(n,d,C)$ such that $\|\sec^B\|_{L^\infty} \leqslant K(n,d,C)$. Now we fix a positive constant $\tilde{K}(n,d,C) > K(n,d,C)$ and a positive constant $\tilde{C}(n) > 1$. It follows from Lemma 1.27 that there is an index $I_2 \geqslant I$ such that for all $i \geqslant I_2$ the sectional curvature of $(B, \tilde{h}_i)$ is bounded by $\tilde{K}$, i.e. $\|\sec^{\tilde{h}_i}\|_\infty \leqslant \tilde{K}$ and the sectional curvature of $(M_i, \tilde{g}_i)$ is bounded by $\tilde{C}$, i.e. $\|\sec^{\tilde{g}_i}\|_\infty \leqslant \tilde{C}$.

Let $(\xi_1, \ldots, \xi_n, \zeta_1)$ be a split orthonormal frame, see Definition 3.1. In particular, $(\xi_1, \ldots, \xi_n)$ is the horizontal lift of an orthonormal frame $(\check{\xi}_1, \ldots, \check{\xi}_n)$ in $(B, \tilde{h}_i)$. Now it follows directly from O'Neill's formula (B.7.3) that

$$\begin{aligned}
|A_i|^2 &= \sum_{i<j} |A(\xi_i, \xi_j)|^2 \\
&= \frac{1}{3} \sum_{i<j} \sec^{\tilde{h}_i}(\check{\xi}_i, \check{\xi}_i) - \sec^{\tilde{g}_i}(\xi_i, \xi_j) \\
&\leqslant \frac{n(n-1)}{6} \left(\tilde{K} + \tilde{C}(n) \right) =: C_A(n,d,C)^2.
\end{aligned}$$

Applying [HM99, Lemma 3.3] and the identity (4.10.2) we obtain

$$\left\| \frac{1}{4} \gamma \left(\frac{K_i}{l_i} \right) \gamma(l_i F_i) \right\|_\infty \leqslant \frac{1}{4} \left[\frac{n}{2} \right]^{\frac{1}{2}} \|l_i F_i\|_\infty \leqslant \frac{1}{2} \left[\frac{n}{2} \right]^{\frac{1}{2}} \|A_i\|_\infty \leqslant \frac{1}{2} \left[\frac{n}{2} \right]^{\frac{1}{2}} C_A.$$

Since $\frac{1}{4}\gamma\left(\frac{K_i}{l_i}\right)\gamma(l_i F_i)$ is symmetric it follows from [Kat76, Chapter 5, Theorem 4.10] that

$$\operatorname{dist}\left(\sigma(\tilde{D}^{M_i}), \sigma\left(\frac{1}{l_i} \gamma \left(\frac{K_i}{l_i} \right) \mathcal{L}_{K_i} + D^{\mathcal{H}_i} \right) \right) \leqslant \left\| \frac{1}{4} \gamma \left(\frac{K_i}{l_i} \right) \gamma(l_i F_i) \right\|_\infty \leqslant \frac{1}{2} \left[\frac{n}{2} \right]^{\frac{1}{2}} C_A. \tag{4.10.3}$$

Let $\lambda_{j,k}^W(i)$ be an eigenvalues of $W_i := \frac{1}{l_i}\gamma\left(\frac{K_i}{l_i}\right)\mathcal{L}_{K_i} + D^{\mathcal{H}_i}$. It was shown in [Amm98a] that for any $\varepsilon > 0$ there is an $I \geqslant I_2$ such that

$$|\lambda_{j,k}^W(i)| \geqslant \frac{|k|}{\|l_i\|_\infty} - \varepsilon,$$

for all $i \geqslant I$. Applying the inequalities (4.10.3) and (4.10.1) we obtain the claimed lower bound.

It remains to prove the upper bound. As explained in the beginning of this proof it suffices to consider the S^1-principal bundles $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$. Let $\mathrm{i}\tilde{\omega}_i$ be the unique connection one-form such that $\ker(\tilde{\omega}_i)$ is orthogonal to the fibers with respect to $\tilde{g}_i$. Since

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0,$$
$$\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0,$$

and $\tilde{\omega}_i$ coincides with the orthogonal projection onto $\ker(\mathrm{d}f_i)$ it follows from the additional assumptions that the curvatures $\tilde{F}_i := \mathrm{d}\tilde{\omega}_i$ are all uniformly bounded in $C^{0,1}(B, h)$ for all $i \geqslant I$. Thus, we can apply Theorem D.8 to deduce that there is subsequence $(M_i, \tilde{g}_i)_{i\in\mathbb{N}}$ such that all M_i are diffeomorphic to a fixed manifold M and such that the sequence $(\tilde{\omega}_i)_{i\in\mathbb{N}}$ of connection one-forms converge in $C^{1,\alpha}$ for any $\alpha \in [0, 1)$. Applying these isomorphisms it suffices to consider the sequence $\left(f_i : (M, \tilde{g}_i) \to (B, \tilde{h}_i)\right)_{i\in\mathbb{N}}$ of S^1-principal bundles.

Next, we recall that for any k,

$$D^{\mathcal{H}_i}_{|V_k(i)} = Q_{k,i} \circ D_{k,i} \circ Q_{k,i}^{-1}.$$

As the manifold M does not depend on i the same holfd for the associated line bundle $L = M \times_{S^1} \mathbb{C}$. Moreover, as in Step 4 of the proof of Theorem 4.5, we obtain an isometry

$$L^2({}^{\diamond}\mathbf{\Sigma}_i B \otimes L^{-k}) \to L^2({}^{\diamond}\mathbf{\Sigma} B \otimes L^{-k}),$$

for any $i \in \mathbb{N}$. Applying these isometries and that $\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0$ and the connection one-forms $(\tilde{\omega}_i)_{i\in\mathbb{N}}$ converge in $C^{1,\alpha}$ for any $\alpha[0, 1)$ it follows from Theorem C.5 that the spectrum $(\mu_j(i))_{j\in\mathbb{Z}}$ of the twisted Dirac operators $D_{k,i}$ converges in the arsinh-topology to the spectrum $(\mu_j)_{j\in\mathbb{Z}}$ of the twisted Dirac operator $D_{k,\infty}$ on ${}^{\diamond}\mathbf{\Sigma} B \otimes L^{-k}$. As before, we consider the operator

$$W_i := \frac{1}{l_i}\gamma\left(\frac{K_i}{l_i}\right)\mathcal{L}_{K_i} + D^{\mathcal{H}_i}.$$

It is straightforward to check that

$$W_i^2 = -\frac{1}{l_i^2}(\mathcal{L}_{K_i})^2 + (D^{\mathcal{H}_i})^2 - \gamma\left(\frac{\operatorname{grad}(l_i)}{l_i^2}\right).$$

As W_i is a self-adjoint operator it is immediate that W_i^2 is a nonnegative self-adjoint operator. A straightforward calculation using the Rayleigh quotient shows that any eigenvalue $\lambda^W_{j,k}(i)$ of $W_{i\,|V_k(i)}$ satisfies

$$(\lambda^W_{j,k}(i))^2 \leqslant \frac{|k|^2}{\min_{p\in B} l_i(p)^2} + \mu_j(i)^2 + \frac{\|\operatorname{grad}(l_i)\|_\infty}{\min_{p\in B} l_i(p)^2}.$$

Multiplying this equation with $\min_{p\in B} l_i(p)^2$ leads to

$$\min_{p\in B} l_i(p)^2(\lambda^W_{j,k}(i))^2 \leqslant |k|^2 + \min_{p\in B} l_i(p)^2\mu_j(i)^2 + \min_{p\in B} l_i(p)\|\operatorname{grad}(l_i)\|_\infty$$

Next, we observe that the T-tensor of the Riemannian submersion $f_i : (M, \tilde{g}_i) \to (B, \tilde{h}_i)$ is given by

$$T_i \left(\frac{K_i}{l_i}, \frac{K_i}{l_i} \right) = \frac{\operatorname{grad} l_i}{l_i}.$$

As the tensors $(T_i)_{i\in\mathbb{N}}$ are uniformly bounded in i, see for instance Corollary 1.29,

$$\limsup_{i\to\infty} \| \operatorname{grad}(l_i)\|_\infty = 0.$$

Furthermore, the spectrum $(\mu_j(i))_{j\in\mathbb{Z}}$ of $D_{k,i}$ converges to the spectrum $(\mu_j)_{j\in\mathbb{Z}}$ of $D_{k,\infty}$ in the arsinh-topology. Combining these observations we obtain that

$$\limsup_{i\to\infty} \min_{p\in B} l_i(p)^2 (\lambda^W_{j,k}(i))^2 \leqslant |k|^2.$$

As for the lower bound, we conclude the claim by applying the inequalities (4.10.3), (4.10.1) and taking the limit $i \to \infty$. □

An immediate consequence of this proposition is that the eigenvalues of the restrictions $D^{M_i}_{|V_k(i)}$ tend to $\pm\infty$ in the limit $i \to \infty$ whenever $k \neq 0$. In the case of non projectable spin structures this means that all eigenvalues diverge. Whereas in the case of projectable spin structures all eigenvalues diverge except those corresponding to the subspace $V_0(i)$, which is the space of affine parallel spinors. This coincides with the previous results by Ammann, Theorem 4.1 and Lott, Theorem 4.2, Theorem 4.3. In the following theorem we summarize the complete behavior of Dirac eigenvalues on collapsing sequences in $\mathcal{M}(n+1, d, C)$.

Theorem 4.11. *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a collapsing sequence of Riemannian spin manifolds in $\mathcal{M}(n+1, d, C)$. Then there is a $C^{1,\alpha}$-Riemannian orbifold (B, h) such that for a subsequence, relabeled as $(M_i, g_i)_{i\in\mathbb{N}}$, there are S^1-orbifold bundles $f_i : M_i \to B$ for which one of the following two cases occur:*

<u>Case 1:</u> There is a subsequence, relabeled as $(M_i, g_i)_{i\in\mathbb{N}}$ such that the spin structures on (M_i, g_i) are non projectable. Then the eigenvalues of the Dirac operator D^{M_i} can be numbered as $(\lambda_{j,k}(i))_{\substack{j\in\mathbb{Z}\\ k\in(\mathbb{Z}+\frac{1}{2})}}$ such that for all $\varepsilon > 0$ there is an index $I > 0$ such that for all $i \geqslant I$,

$$|\lambda_{j,k}(i)| \geqslant \sinh\left(\operatorname{arsinh}\left(\frac{|k|}{\|l_i\|_\infty} - \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A - \varepsilon \right) - \varepsilon \right).$$

Here, $2\pi l_i$ is the length of the fibers and C_A is a constant depending on n, d and C. In particular, $\lim_{i\to\infty} |\lambda_{j,k}(i)| = \infty$ for all $j \in \mathbb{Z}$ and $k \in (\mathbb{Z} + \frac{1}{2})$ since $\lim_{i\to\infty} l_i = 0$.

For any $i \geqslant I$, let $\omega_i \in \Omega(M_i, \mathcal{V}_i)$ be the orthogonal projection onto $\mathcal{V}_i := \ker(\mathrm{d}f_i)$, where $f_i : M_i \to B$. If, in addition, there is a constant C such that

$$\|\mathrm{d}\omega_i\|_{C^{0,1}} \leqslant C$$

for all $i \geqslant I$, then for all $j \in \mathbb{Z}$ and $k \in (\mathbb{Z} + \frac{1}{2})$,

$$\limsup_{i\in\mathbb{N}} \left(\min_{p\in B} l_i(p) |\lambda_{j,k}(i)| \right) \leqslant |k|.$$

Case 2: *There is a subsequence, relabeled as $(M_i, g_i)_{i\in\mathbb{N}}$, such that the spin structures on (M_i, g_i) are projectable and all of them induce the same spin structure on B, if B is orientable, resp. the same pin$^-$-structure on B, if B is nonorientable. Then the eigenvalues of the Dirac operator D^{M_i} can be numbered as $(\lambda_{j,k}(i))_{\substack{j\in\mathbb{Z}\\k\in\mathbb{Z}}}$ such that for all $\varepsilon > 0$ there is an $I > 0$ such that for all $i \geqslant I$,*

$$|\lambda_{j,k}(i)| \geqslant \sinh\left(\operatorname{arsinh}\left(\frac{|k|}{\|l_i\|_\infty} - \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A - \varepsilon\right) - \varepsilon\right).$$

In particular, $\lim_{i\to\infty} |\lambda_{j,k}(i)| = \infty$ for all $j \in \mathbb{Z}$ and $k \neq 0$ since $\lim_{i\to\infty} l_i = 0$.

For any $i \geqslant I$, let $\omega_i \in \Omega(M_i, \mathcal{V}_i)$ be the orthogonal projection onto $\mathcal{V}_i := \ker(\mathrm{d}f_i)$, where $f_i : M_i \to B$. If, in addition, there is a constant C such that

$$\|\mathrm{d}\omega_i\|_{C^{0,1}} \leqslant C$$

for all $i \geqslant I$, then for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$\limsup_{i\in\mathbb{N}} \left(\min_{p\in B} l_i(p)|\lambda_{j,k}(i)|\right) \leqslant |k|.$$

For $k = 0$, the eigenvalues $\lambda_{j,0}(i)$ converge uniformly with respect to the arsinh*-topology to the eigenvalues of the operator*

$$D^B + \frac{i}{4}\omega_n^{\mathbb{C}}\gamma(\mathcal{F}), \qquad \text{if } n \text{ is even,}$$

$$\begin{pmatrix} D^B & \frac{i}{4}\gamma(\mathcal{F}) \\ \frac{i}{4}\gamma(\mathcal{F}) & -D^B \end{pmatrix}, \qquad \text{if } n \text{ is odd.}$$

If B is orientable,

- *D^B is the Dirac operator of B,*
- *$\omega_n^{\mathbb{C}}$ is the complex volume element of ΣB, i.e. $\omega_n^{\mathbb{C}} = \mathrm{i}^{\left[\frac{n+1}{2}\right]}\gamma(e_1)\cdots\gamma(e_n)$ for any orthonormal frame $(e_1, \dots, e_n)$,*
- *$\mathcal{F}$ is a $C^{0,\alpha}$-two-form for $\alpha \in [0, 1)$.*

If B is non orientable

- *D^B is the twisted Dirac operator on the twisted pin$^-$ bundle $\Sigma^P B \otimes \det(TB)^{\mathbb{C}}$, where $\det(TB)^{\mathbb{C}}$ is the complexified determinant bundle,*
- *$\omega_n^{\mathbb{C}}$ is the complex volume element of $\Sigma^P B \otimes \det(TB)^{\mathbb{C}}$,*
- *$\mathcal{F}$ is a $C^{0,\alpha}$-two-form for $\alpha \in [0, 1)$.*

Proof. The behavior of the divergent eigenvalues as well as the upper bound follow directly from Proposition 4.10. As usual, we switch to invariant metrics and work with the resulting sequence of Riemannian S^1-orbifold bundles $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$. Under the assumption that all spin structures of $(M_i, \tilde{g})_{i \in \mathbb{N}}$ are projectable, each of them induces a spin structure on B. As there are only finitely many equivalence classes of spin structures on B (see for instance [LM89, Chapter II, Theorem 1.7]), there is a subsequence relabeled as $(M_i, \tilde{g}_i)_{i \in \mathbb{N}}$, such that the spin structure on $(M_i, \tilde{g}_i)$ induces, up to equivalence, the same spin structure on B for all $i \in \mathbb{N}$. We recall that for any $i \in \mathbb{N}$ there is a unique imaginary connection one-form $\mathrm{i}\tilde{\omega}_i$ of the Riemannian S^1-orbifold bundle $f_i : (M_i, \tilde{g}_i) \to (B, \tilde{h}_i)$ such that $\ker(\omega_i)$ is orthogonal to the fibers with respect to $\tilde{g}_i$. Let $F_i := \mathrm{d}\tilde{\omega}_i$ be the curvature form of $\tilde{\omega}_i$. To show the convergence behavior of the eigenvalues $(\lambda_{j,0}(i))_{j \in \mathbb{Z}}$, it suffices to observe that $A_{i\,|\mathcal{H}_i \times \mathcal{H}_i} = \frac{1}{2} l_i F_i$ if B is orientable. If B is non orientable we define $\mathcal{F}_i := 2\mathcal{A}_i$. Then the claim follows from Theorem 4.5 and the rules for Clifford multiplication derived in Appendix B. □

Finally, we want to mention that the same statement as in Proposition 4.8 also hold for general collapsing sequences in $\mathcal{M}(n+1, d, C)$. It is easy to check that all arguments given in the proof of Proposition 4.8 also work when B is a Riemannian orbifold.

Appendix A

Infranilmanifolds

In this appendix we recall the basic properties and definitions of infranilmanifolds. For a thorough introduction to infranilmanifolds we refer to [Dek17], [CFG92, Section 3] and [Lot02c, Section 3].

Let N be a connected and simply-connected nilpotent Lie group. The Lie algebra $\mathfrak{n}$ of N is nilpotent, i.e. there is a $k \in \mathbb{N}$ such that the lower central series

$$\mathfrak{n}_1 = \mathfrak{n},\ \mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}_1],\ \mathfrak{n}_3 = [\mathfrak{n}, \mathfrak{n}_2], \ldots$$

terminates at $\mathfrak{n}_k = 0$.

On the Lie group N, there is a canonical flat connection ∇^{aff} defined by the requirement that all left-invariant vector fields are parallel. Let $\mathrm{Aff}(N)$ denote the subgroup of the diffeomorphism group $\mathrm{Diff}(N)$ that preserves ∇^{aff}. It follows that $\mathrm{Aff}(N)$ is isomorphic to the semi-product $N_L \rtimes \mathrm{Aut}(N)$. Here N_L denotes the left-action of N on itself. Note that N_L isomorphic to N via the isomorphism $N \to N_L, g \mapsto L_g$. As usual, $\mathrm{Aut}(N)$ denotes the automorphism group of N.

An *infranilmanifold* Z is a quotient ${}_{\Gamma}\backslash^{N}$ of a connected and simply-connected nilpotent Lie group N by a cocompact discrete subgroup Γ of $\mathrm{Aff}(N)$. By the generalized first Bieberbach Theorem (see for instance [Dek17, Theorem 3.4]), the subgroup $\hat{\Gamma} := \Gamma \cap N_L$ is of finite index in Γ. In fact, there is a constant $C(k)$ depending only on $k := \dim(Z)$ such that $[\Gamma : \hat{\Gamma}] < C(k)$ [Gro78, Main result]. Thus, we have the following diagram of short exact sequences.

$$\begin{tikzcd}
1 \arrow[r] & N_L \arrow[r] \arrow[d] & \mathrm{Aff}(N) \arrow[r, "p"] \arrow[d] & \mathrm{Aut}(N) \arrow[r] \arrow[d] & 1\ , \\
1 \arrow[r] & \hat{\Gamma} \arrow[r] & \Gamma \arrow[r] & p(\Gamma) \arrow[r] & 1\ .
\end{tikzcd}$$

It follows that Z is finitely covered by the nilmanifold $\hat{Z} := {}_{\hat{\Gamma}}\backslash^{N}$. The finite deck transformation group is given by $F := p(\Gamma)$. Since Γ is a subgroup of $\mathrm{Aff}(N)$ it follows that the flat connection ∇^{aff} on N descends to a well-defined flat connection on $\hat{Z}$ and on Z. Let $\mathfrak{n}$ be the Lie algebra of N. The space of affine parallel vector fields on N and $\hat{Z}$ is isomorphic to $\mathfrak{n}$. Thus, the space of affine parallel vector fields on an infranilmanifold Z is isomorphic to the subspace $\mathfrak{n}^F$ consisting of those elements that are invariant under the induced action of F on $\mathfrak{n}$. Obviously, $\mathfrak{n}^F$ is finite dimensional.

Let g be a left-invariant metric on N. For a local orthonormal frame $(e_1, \dots, e_k)$ of (N, g) the structural coefficients of the Lie algebra $\mathfrak{n}$ of N are given by

$$[e_a, e_b] = \sum_{c=1}^{k} \tau_{ab}^c e_c.$$

The Christoffel symbols of ∇^{aff} are trivial and the Christoffel symbols of the Levi-Civita connection can be calculated using the Koszul formula,

$$\Gamma_{ab}^c = \frac{1}{2}\left(\tau_{ab}^c - \tau_{ac}^b - \tau_{bc}^a\right). \tag{A.0.1}$$

It follows that ∇^{aff} is identical to the Levi-Civita connection if and only if N is abelian, i.e. N is isometric to the additive group $(\mathbb{R}^k, +)$ with the euclidean metric. We fix the following terminology for tensor fields on a Riemannian infranilmanifold (Z, g).

Notation A.1. Let (Z, g) be a Riemannian infranilmanifold. A tensor field X on Z is called *affine parallel* if it is parallel with respect to the affine connection, i.e. $\nabla^{\text{aff}} X = 0$. This is equivalent to say that X lifts to a left-invariant tensor field $\tilde{X}$ on the universal cover N. On the other hand, a *parallel* tensor field X on Z is parallel with respect to the Levi-Civita connection on (Z, g).

In the remainder of this appendix we consider a closed k-dimensional Riemannian infranilmanifolds (Z, g) with an affine parallel metric g. Let g_N be the lift of g to N. Since g is affine parallel g_N is a left-invariant metric on N. Hence, the oriented orthonormal frame bundle is trivial, i.e. $P_{\mathrm{SO}}N \cong N \times \mathrm{SO}(k)$. Thus, there is a canonical spin structure on N given by

$$N \times \mathrm{Spin}(k) \to N \times \mathrm{SO}(k). \tag{A.1.1}$$

We recall that the equivalence classes on a spin manifold M are in one-to-one correspondence with the cohomology class $\mathrm{H}^1(M, \mathbb{Z}_2)$, [LM89, Chapter II, Theorem 1.7]. Since N is connected and simply-connected the cohomology group $\mathrm{H}^1(N, \mathbb{Z}_2)$ is trivial. Thus, the spin structure defined by (A.1.1) is, up to equivalence, the only spin structure on N.

As the metric g on $Z = {}_{\Gamma}\backslash^{N}$ is affine parallel it follows that Γ is a discrete group of isometries of (N, g_N). Hence, the oriented orthonormal bundle of Z is isomorphic to

$$P_{\mathrm{SO}}Z \cong {}_{\Gamma}\backslash^{(N \times \mathrm{SO}(k))}.$$

At this point we want to remark that there are examples of infranilmanifolds that are not spin, e.g. the Kleinian Bottle. An infranilmanifold Z is spin if and only if $F \subset \mathrm{SO}(k)$ and if there exists a lift

$$\begin{tikzcd} & & \mathrm{Spin}(k) \arrow[d] \\ \Gamma \arrow[r, "\rho"] \arrow[urr, dashed, "\tilde{\rho}"] & F \arrow[r, hook] & \mathrm{SO}(k) \end{tikzcd}.$$

The different equivalence classes of spin structures of Z correspond to different lifts of the map $\Gamma \to \mathrm{SO}(k)$ to $\Gamma \to \mathrm{Spin}(k)$. Moreover, the group

$$\mathrm{Hom}(\Gamma, \mathbb{Z}_2) \cong \mathrm{H}^1(\Gamma, \mathbb{Z}_2) \cong \mathrm{H}^1(Z, \mathbb{Z}_2)$$

acts freely and transitively on the set of equivalence classes.

It follows that the $\mathrm{Spin}(k)$-principal bundle of Z is given by

$$P_{\mathrm{Spin}}(Z) \cong {}_{\Gamma}\backslash^{(N \times \mathrm{Spin}(k))}.$$

Let $\theta_k : \mathrm{Spin}(k) \to \mathrm{Aut}(\Sigma_k)$ be the canonical complex spinor representation, where Σ_k is a complex vector space with $\dim_{\mathbb{C}}(\Sigma_k) = 2^{\left[\frac{k}{2}\right]}$. If there is a given lift $\tilde{\rho} : \Gamma \to \mathrm{Spin}(k)$, then Γ acts on Σ_k via

$$\begin{aligned} \Gamma \times \Sigma_k &\to \Sigma_k, \\ (\gamma, \varphi) &\mapsto \theta_k(\tilde{\rho}(\gamma))(\varphi). \end{aligned}$$

Thus, the spinor bundle of Z is defined as

$$\Sigma Z = P_{\mathrm{Spin}} Z \times_{\theta_k} \Sigma_k \cong {}_{\Gamma}\backslash^{(N \times \Sigma_k)}.$$

Next, we recall the affine connection ∇^{aff} on Z that is induced by the canonical flat connection ∇^{aff} on N for which all left-invariant vector fields are parallel. Since the metric g on Z is affine parallel ∇^{aff} induces a connection on $P_{\mathrm{SO}}Z$ and on $P_{\mathrm{Spin}}Z$. For brevity, we continue to write ∇^{aff} for these induced connections. In this thesis we are mainly interested in the space of affine parallel spinors on an infranilmanifold (Z, g) with an affine parallel metric g and a fixed spin structure, i.e.

$$\mathcal{P} := \{\varphi \in L^2(\Sigma Z) : \nabla^{\mathrm{aff}} \varphi = 0\}.$$

First we observe that the space of affine parallel spinors is isomorphic to

$$\Sigma_k^{\Gamma} = \{\nu \in \Sigma_k : \theta_k(\tilde{\rho}(\gamma))(\nu) = \nu, \ \forall \gamma \in \Gamma\}.$$

Since $\hat{\Gamma} \subset N_L$ it is immediate that $\rho(\gamma) = \mathrm{Id}$ for all $\gamma \in \hat{\Gamma}$. Thus, $\tilde{\rho}(\hat{\Gamma})$ takes values in $\{\pm 1\}$. Here ± 1 denote the two preimages of the identity $\mathrm{Id} \in \mathrm{SO}(k)$ under the double cover $\mathrm{Spin}(k) \to \mathrm{SO}(k)$. We conclude that $\Sigma_k^{\Gamma} = \{0\}$ if there exists a $\gamma \in \hat{\Gamma}$ such that $\tilde{\rho}(\gamma) = -1$. If $\tilde{\rho}_{|\hat{\Gamma}} = 1$ then $\Sigma_k^{\Gamma} = \Sigma_k^F$, where the latter is the space of all elements in Σ_k that are fixed by the action of the finite group $F = \rho(\Gamma)$. Since $\Sigma_k^F \subset \Sigma_k$, the space of affine parallel spinors on Z is finite dimensional.

Appendix B

Spinors on Riemannian submersions

This appendix deals with Riemannian submersions $f : M \to B$ where M is a spin manifold. We discuss how the Clifford multiplication of vertical and horizontal vectors acts on the spinors of M. The main goal of this appendix is to derive formulas for the spinorial connection and the Dirac operator on the total space M expressing the influence of the "vertical" and the "horizontal" geometry. Afterwards we recall O'Neill's formulas for Riemannian submersions which are constantly used in this thesis.

We start with an elementary discussion of the canonical complex spin representation (see [LM89, Chapter I, §5] for more details). First, we recall that the group $\mathrm{Spin}(n)$ is contained in the Clifford algebra $\mathbb{C}l(n) := Cl(\mathbb{C}^n)$ of $\mathbb{C}^n$.

If n is even, then there is a unique irreducible representation $\chi_n : \mathbb{C}l(n) \to \mathrm{GL}(\Sigma_n)$. In the other case, i.e. n is odd, there are two inequivalent irreducible representations $\chi_n^{\pm} : \mathbb{C}l(n) \to \mathrm{GL}(\Sigma_n)$. Here Σ_n is a complex vector space of complex dimension $2^{\left[\frac{n}{2}\right]}$. The *canonical complex spin representation* is defined as

$$\theta_n := \begin{cases} \chi_{n\,|\,\mathrm{Spin}(n)}\,, & \text{if } n \text{ is even,} \\ \chi^+_{n\,|\,\mathrm{Spin}(n)}\,, & \text{if } n \text{ is odd.} \end{cases}$$

It is important to remark here, that the restrictions $\chi^+_{n\,|\,\mathrm{Spin}(n)}$ and $\chi^-_{n\,|\,\mathrm{Spin}(n)}$ are equivalent to each other, although the non restricted representations χ_n^+ and χ_n^- are inequivalent.

If n is even the canonical complex spin representation splits $\Sigma_n = \widehat{\Sigma}_n^+ \oplus \widehat{\Sigma}_n^-$ such that the restrictions $\theta_{n\,|\widehat{\Sigma}_n^{\pm}}$ are inequivalent irreducible representations of $\mathrm{Spin}(n)$. This splitting corresponds to the ± 1 eigenspaces of the *complex volume element*

$$\omega_n^{\mathbb{C}} := \mathrm{i}^{\left[\frac{n+1}{2}\right]}\gamma(e_1)\cdots\gamma(e_n).$$

Here $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$ and $\gamma : \mathbb{R}^n \to \mathrm{GL}(\Sigma_n)$ denotes Clifford multiplication, i.e. $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2\langle v, w\rangle$, where $\langle ., . \rangle$ is the standard scalar product on $\mathbb{R}^n$. The map

$$\begin{gathered} \Sigma_n = \widehat{\Sigma}_n^+ \oplus \widehat{\Sigma}_n^- \to \Sigma_n = \widehat{\Sigma}_n^+ \oplus \widehat{\Sigma}_n^-, \\ \psi = \psi^+ + \psi^- \mapsto \bar{\psi} = \psi^+ - \psi^- \end{gathered}$$

is called *complex conjugation.*

If n is odd the canonical complex spin representation θ_n is irreducible. The complex volume element $\omega_n^{\mathbb{C}}$ acts trivially on Σ_n. For later use, we want to define $\theta_n^- := \chi_{n\,|\,\mathrm{Spin}(n)}^-$ for which the complex volume element $\omega_n^{\mathbb{C}}$ acts as -1. With respect to this representation, the Clifford multiplication of $x \in \mathbb{R}^n$ acts as $-\gamma(x)$. Recall from the discussion above that the irreducible representations θ_n and θ_n^- are equivalent to each other.

Now we consider a Riemannian submersion $f : (M, g) \to (B, h)$ such that (M, g) is a spin manifold with a fixed spin structure. Here and subsequently, we set $\dim(M) = n + k$ and $\dim(B) = n$. In Chapter 3.2.1 we discuss the problem that, in general, we cannot determine whether (B, h) has an induced spin or pin$^\pm$ structure. Nevertheless, we know that there is an induced spin structure on each fiber Z_p, $p \in B$.

First, we discuss how the involved spinor representations interact with each other. Since $\dim_{\mathbb{C}}(\Sigma_m) = 2^{\left[\frac{m}{2}\right]}$ it follows at once that

$$\dim_{\mathbb{C}}(\Sigma_{n+k}) = \begin{cases} 2\dim_{\mathbb{C}}(\Sigma_n)\dim_{\mathbb{C}}(\Sigma_k), & \text{if } n \text{ and } k \text{ are odd,} \\ \dim_{\mathbb{C}}(\Sigma_n)\dim_{\mathbb{C}}(\Sigma_k), & \text{if } n \text{ or } k \text{ is even.} \end{cases}$$

Thus, if n or k is even, there is a vector space isomorphism

$$\Sigma_{n+k} \cong \Sigma_n \otimes \Sigma_k, \tag{B.0.1}$$

Here $\Sigma_n \otimes \Sigma_k$ is to be understood as the tensor product of two complex vector spaces. We discuss the behavior of the Clifford multiplication under this isomorphism later in this appendix.

Counting dimensions, it follows that such an isomorphism cannot exist if n and k are both odd. In that case, we proceed as follows:

Using the standard basis $(e_1, \ldots, e_{n+k})$ of $\mathbb{R}^{n+k}$ we consider the operator

$$\omega_n^{\mathbb{C}} := \mathrm{i}^{\left[\frac{n+1}{2}\right]}\gamma(e_1)\cdots\gamma(e_n).$$

A short computation shows that $(\omega_n^{\mathbb{C}})^2 = \mathrm{Id}$. Hence, the action of $\omega_n^{\mathbb{C}}$ decomposes Σ_{n+k} into the two eigenspaces Σ_{n+k}^+ and Σ_{n+k}^- with respect to the eigenvalues ± 1.

Remark B.1. If n and k are odd, the splitting $\Sigma_{n+k} = \Sigma_{n+k}^+ \oplus \Sigma_{n+k}^-$ defined above is *different* from the canonical splitting $\Sigma_{n+k} = \widehat{\Sigma}_{n+k}^+ \oplus \widehat{\Sigma}_{n+k}^-$ for even dimensions, as $\omega_n^{\mathbb{C}}$ and $\omega_{n+k}^{\mathbb{C}}$ are not simultaneously diagonalizable.

Next we observe that the operator

$$\omega_k^{\mathbb{C}} := \mathrm{i}^{\left[\frac{k+1}{2}\right]}\gamma(e_{n+1})\cdots\gamma(e_{n+k})$$

anticommutes with $\omega_n^{\mathbb{C}}$. Moreover, $(\omega_k^{\mathbb{C}})^2 = \mathrm{Id}$. Hence, the action of $\omega_k^{\mathbb{C}}$ defines an involution

$$\omega_k^{\mathbb{C}} : \Sigma_{n+k}^\pm \to \Sigma_{n+k}^\mp.$$

In the following, we identify Σ_{n+k}^- with the image $\omega_k^{\mathbb{C}}(\Sigma_{n+k}^+)$. Since

$$\dim_{\mathbb{C}}(\Sigma_{n+k}^\pm) = \frac{1}{2}\dim_{\mathbb{C}}(\Sigma_{n+k}) = \dim_{\mathbb{C}}(\Sigma_n)\dim_{\mathbb{C}}(\Sigma_k),$$

there is an vector space isomorphism

$$\Sigma^{\pm}_{n+k} \cong \Sigma^{\pm}_{n} \otimes \Sigma_k.$$

Here, the notation $\Sigma^{\pm}_n$ symbolizes that $\omega^{\mathbb{C}}_n$ acts as ± 1. Later we will see, that in fact $\Sigma^{\pm}_n$ corresponds to the two irreducible spinor representations θ_n, θ^-_n. Summarizing the above discussion, we conclude

$$\begin{aligned} \Sigma_{n+k} &= \Sigma^+_{n+k} \oplus \Sigma^-_{n+k}, && \text{if } n \text{ and } k \text{ are odd.} \\ &\cong \left(\Sigma^+_n \otimes \Sigma_k\right) \oplus \left(\Sigma^-_n \otimes \Sigma_k\right) \\ &\cong \left(\Sigma^+_n \oplus \Sigma^-_n\right) \otimes \Sigma_k. \end{aligned} \tag{B.1.1}$$

Next we want to determine how Clifford multiplication with vectors in $\mathbb{R}^{n+k}$ "separates" into Clifford multiplications on Σ_n and Σ_k. For all natural numbers n and k the Clifford algebra $\mathbb{C}l(n+k)$ is canonical isomorphic to the graded tensor product $\mathbb{C}l(n) \hat{\otimes} \mathbb{C}l(k)$, endowed with the multiplication

$$(a \hat{\otimes} \varphi) \cdot (b \hat{\otimes} \psi) = (-1)^{\deg(\varphi)\deg(b)} (a \cdot b) \hat{\otimes} (\varphi \cdot \psi),$$

see for instance [BHM+15, Proposition 1.12].

If n or k is even, the multiplication of the graded tensor product $\mathbb{C}l(n) \hat{\otimes} \mathbb{C}l(k)$ carries over to $\Sigma_n \otimes \Sigma_k$. In the remaining case, n and k odd, we recall the eigenvalue decomposition $\Sigma_{n+k} = \Sigma^+_{n+k} \oplus \Sigma^-_{n+k}$ together with the involution $\omega^{\mathbb{C}}_k : \Sigma^{\pm}_{n+k} \to \Sigma^{\mp}_{n+k}$. Since $\omega^{\mathbb{C}}_k$ anticommutes with $\omega^{\mathbb{C}}_n$ it follows that Clifford multiplication with vectors $v \in \operatorname{Span}\{e_1, \dots, e_n\}$ acts as $\gamma(v)$ on Σ^+_{n+k} and as $-\gamma(v)$ on Σ^-_{n+k}. On the other hand, Clifford multiplication with any vector in $\operatorname{Span}\{e_{n+1}, \dots, e_{n+k}\}$ interchanges the eigenspaces $\Sigma^{\pm}_{n+k}$ and commutes with $\omega^{\mathbb{C}}_k$. Using the isomorphisms (B.0.1) and (B.1.1) combined with the above discussion, we obtain the following identifications for Clifford multiplication with vectors $(x, v) \in \mathbb{R}^n \times \mathbb{R}^k$:

$$\gamma((x,v))(\psi \otimes \nu) \cong \begin{cases} (\gamma(x)\psi) \otimes \nu + \bar{\psi} \otimes (\gamma(v)\nu), & \text{if } n \text{ is even,} \\ (\gamma(x)\psi) \otimes \bar{\nu} + \psi \otimes (\gamma(v)\nu), & \text{if } k \text{ is even,} \\ \\ (\gamma(x)\psi^+ \oplus -\gamma(x)\psi^-) \otimes \nu + (\psi^- \oplus \psi^+) \otimes (\gamma(v)\nu), \\ \text{if } n \text{ and } k \text{ are odd.} \end{cases} \tag{B.1.2}$$

Here, $\bar{\psi}$ is the complex conjugation, introduced in the beginning of this appendix, with respect to the $\mathbb{Z}_2$ grading on Σ_p, whenever p is even. In the case, where n and k are even, both possibilities are isomorphic to each other.

Now we return to the case of a Riemannian submersion $f : (M, g) \to (B, h)$ where M is a spin manifold with a fixed spin structure. Applying the above discussion pointwise we conclude that

$$\Sigma M \cong \begin{cases} f^*(\Sigma B) \otimes \Sigma\mathcal{V}, & \text{if } n \text{ or } k \text{ is even,} \\ (f^*(\Sigma^+ B) \oplus f^*(\Sigma^- B)) \otimes \Sigma\mathcal{V}, & \text{if } n \text{ and } k \text{ are odd.} \end{cases} \tag{B.1.3}$$

Remark B.2. Instead with the usual complex spinor bundles, we could also work with the corresponding real spinor bundles, see [LM89, Example 3.7]. For an n-dimensional Riemannian spin manifold (M, g) the corresponding *real spinor bundle* is defined as

$$Cl_{\mathrm{Spin}}(M) := P_{\mathrm{Spin}}(M) \times_l Cl(\mathbb{R}^n),$$

where $Cl(\mathbb{R}^n)$ is the Clifford algebra of R^n which can be viewed as a module over itself by left multiplication l. Since $\mathrm{Spin}(n) \subset Cl(\mathbb{R}^n)$ it follows that $Cl_{\mathrm{Spin}}(M)$ is a $Cl(\mathbb{R}^n)$-principal bundle.

Now let $f : (M^{n+k}, g) \to (B^n, h)$ be a Riemannian submersion such that M is a spin manifold with a fixed spin structure. Similar to the complex Clifford algebras there is a canonical isomorphism $Cl(\mathbb{R}^{n+k}) \cong Cl(\mathbb{R}^n)\hat{\otimes}Cl(\mathbb{R}^k)$ for all $n, k \in \mathbb{N}$. Since $Cl_{\mathrm{Spin}}(M)$ is a $Cl(\mathbb{R}^{n+k})$-principal bundle the isomorphism $Cl(\mathbb{R}^{n+k}) \cong Cl(\mathbb{R}^n)\hat{\otimes}Cl(\mathbb{R}^k)$ just carries over, i.e.

$$Cl_{\mathrm{Spin}}(M) \cong Cl_{\mathrm{Spin}}(B)\hat{\otimes}Cl_{\mathrm{Spin}}(\mathcal{V}).$$

Therefore, we do not need the case distinction (B.1.3). Nevertheless it is more common to work with complex spinors and the spinor bundle ΣM.

As can be seen in Chapter 3.2.1, the base manifold B and the vertical distribution $\mathcal{V}$, in general, are not spin. Thus, the spinor bundles ΣB and $\Sigma\mathcal{V}$ are only defined locally but their tensor product is defined globally. The rules for Clifford multiplication (B.1.2) carry over to the spinor bundle ΣM. In the setting of Riemannian manifolds, these rules allow us to distinguish Clifford multiplication with horizontal and vertical vector fields.

Notation B.3. Let $f : M \to B$ be a Riemannian submersion such that M has a fixed spin structure. For abbreviation we write

$$\Sigma M \cong f^*({}^{\diamond}\boldsymbol{\Sigma} B) \otimes \Sigma\mathcal{V},$$

where

$${}^{\diamond}\boldsymbol{\Sigma} B := \begin{cases} \Sigma B, & \text{if } n \text{ or } k \text{ is even,} \\ \Sigma^+ B \oplus \Sigma^- B, & \text{if } n \text{ and } k \text{ are odd.} \end{cases}$$

We recall, that there is a canonical Hermitian product $\langle ., . \rangle$ on ΣM, see e.g. [BHM+15, Proposition 2.5], such that

$$\langle \gamma(X)\varphi, \psi \rangle = -\langle \varphi, \gamma(X)\psi \rangle$$

for all vector fields X and spinors φ, ψ.

Next we calculate the spinorial connection on ΣM with respect to a local orthonormal frame $(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k)$ such that$(\xi_1, \ldots, \xi_n)$ is the horizontal lift of a local orthonormal frame $(\check{\xi}_1, \ldots, \check{\xi}_n)$ in the base space B and $(\zeta_1, \ldots, \zeta_k)$ is a locally defined vertical orthonormal frame. In the following, we use the indices $a, b, c, \ldots$ for the vertical components and the indices $\alpha, \beta, \gamma, \ldots$ for the horizontal components.

For any vector field X and spinor Φ the spinorial connection ∇^M on M is locally given by

$$\nabla^M_X \Phi = X(\Phi) + \frac{1}{4}\sum_{i,j=1}^{n+k} g(\nabla_X e_i, e_j)\gamma(e_i)\gamma(e_j)\Phi, \tag{B.3.1}$$

where $(e_1, \ldots, e_{n+k})$ is a local orthonormal frame. This connection is metric with respect to the canonical Hermitian product and satisfies the Leibniz rule relative to the Clifford product, i.e.

$$\nabla_Y(\gamma(X)\varphi) = \gamma(\nabla_Y X)\varphi + \gamma(X)\nabla_Y \varphi$$

for all vector fields X, Y and spinors φ.

The action of the Dirac operator on a spinor Φ is locally define via

$$D^M \Phi = \sum_{i=1}^{n+k} \gamma(e_i)\nabla^M_{e_i}\Phi. \tag{B.3.2}$$

It follows from (B.1.3) that locally any spinor Φ on M can be written as a finite linear combination $\Phi = \sum_l f^*\varphi_l \otimes \nu_l$. The next lemma follows from straightforward calculations, where we use that the Christoffel symbols of M are given by

$$\begin{gathered}
\Gamma^c_{ab} = \hat{\Gamma}^c_{ab},\\
\Gamma^\alpha_{ab} = -\Gamma^b_{a\alpha} = g(T(\zeta_a, \zeta_b), \xi_\alpha),\\
\Gamma^b_{\alpha a} = g([\xi_\alpha, \zeta_a], \zeta_b) + g(T(\zeta_a, \xi_\alpha), \zeta_b),\\
\Gamma^a_{\alpha\beta} = -\Gamma^\beta_{\alpha a} = -\Gamma^\beta_{a\alpha} = g(A(\xi_\alpha, \xi_\beta), \zeta_a),\\
\Gamma^\gamma_{\alpha\beta} = \check{\Gamma}^\gamma_{\alpha\beta}.
\end{gathered}$$

Here $(\hat{\Gamma}^c_{ab})_{1\leqslant a,b,c\leqslant k}$ are the Christoffel symbols of the fibers with respect to $(\zeta_1, \ldots, \zeta_k)$ and $(\check{\Gamma}^\gamma_{\alpha\beta})_{1\leqslant \alpha,\beta,\gamma\leqslant n}$ are the Christoffel symbols of the base space B with respect to $(\check{\xi}_1, \ldots, \check{\xi}_n)$.

Lemma B.4. *Let $f : (M^{n+k}, g) \to (B^n, h)$ be a Riemannian submersion. Suppose that M is a spin manifold with a fixed spin structure. With respect to a local orthonormal frame*

$(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_k)$ *as above, any spinor* $\Phi = f^*\varphi \otimes \nu$ *satisfies the following identities:*

$$\begin{aligned}
\nabla^M_{\xi_\alpha}\Phi &= (f^*\nabla^B_{\xi_\alpha}\varphi)\otimes\nu + f^*\varphi\otimes\nabla^{\mathcal{V}}_{\xi_\alpha}\nu + \frac{1}{2}\sum_{\beta=1}^{n}\gamma(\xi_\beta)\gamma\left(A(\xi_\alpha,\xi_\beta)\right)\Phi \\
&=: \nabla^{\mathcal{T}}_{\xi_\alpha}\Phi + \frac{1}{2}\sum_{\beta=1}^{n}\gamma(\xi_\beta)\gamma\left(A(\xi_\alpha,\xi_\beta)\right)\Phi, \\
\nabla^M_{\zeta_a}\Phi &= f^*\varphi\otimes\nabla^Z_{\zeta_a}\nu + \frac{1}{2}\sum_{b=1}^{k}\gamma(\zeta_b)\gamma\left(T(\zeta_a,\zeta_b)\right)\Phi + \frac{1}{4}\sum_{\alpha=1}^{n}\gamma(\xi_\alpha)\gamma\left(A(\xi_\alpha,\zeta_i)\right)\Phi \\
&=: \nabla^Z_{\zeta_a}\Phi + \frac{1}{2}\sum_{b=1}^{k}\gamma(\zeta_b)\gamma\left(T(\zeta_a,\zeta_b)\right)\Phi + \frac{1}{4}\sum_{\alpha=1}^{n}\gamma(\xi_\alpha)\gamma\left(A(\xi_\alpha,\zeta_a)\right)\Phi, \\
D^M\Phi &= \sum_{\alpha=1}^{n}\gamma(\xi_\alpha)\nabla^{\mathcal{T}}_{\xi_\alpha}\Phi + \sum_{a=1}^{k}\gamma(\zeta_a)\nabla^Z_{\zeta_a}\Phi - \frac{1}{2}\sum_{a=1}^{k}\gamma\left(T(\zeta_a,\zeta_a)\right)\Phi \\
&\quad + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{n}\gamma\left(A(\xi_\alpha,\xi_\beta)\right)\gamma(\xi_\alpha)\gamma(\xi_\beta)\Phi \\
&=: D^{\mathcal{T}}\Phi + D^Z\Phi - \frac{1}{2}\sum_{a=1}^{k}\gamma\left(T(\zeta_a,\zeta_a)\right)\Phi + \frac{1}{2}\gamma(A)\Phi.
\end{aligned}$$

Here ∇^B, $\nabla^{\mathcal{V}}$ *and* ∇^Z *are the induced connections by the respective connections on* TM, *defined in Section 3.1.*

In the remainder of this appendix we recall O'Neill's formulas for Riemannian submersion [O'N66, Gra67]. We follow [Bes08, Chapter 9, Sections C and D], where the authors summarized the formulas from [O'N66, Gra67].

Let $f : (M, g) \to (B, h)$ be a Riemannian submersion, where $\dim(M) = n + k$ and $\dim(B) = n$. In the following we denote by $F_p := f^{-1}(p)$ the fiber over $p \in B$ and with $\hat{g}_p$ the induced metric on F_p. It is a general fact that the tangent bundle $TM = \mathcal{H} \oplus \mathcal{V}$ splits into the horizontal distribution $\mathcal{H} \cong f^*TB$ and the vertical distribution $\mathcal{V} := \ker(\mathrm{d}f)$. In particular, any $X \in TM$ can be written as $X = X^H + X^V$, where X^H, resp. X^V denotes the horizontal, resp. vertical component. O'Neill introduced the two fundamental tensors T and A that are defined by their actions on vector fields X, Y,

$$\begin{aligned}
T(X,Y) &:= \left(\nabla_{X^V}Y^V\right)^H + \left(\nabla_{X^V}Y^H\right)^V, \\
A(X,Y) &:= \left(\nabla_{X^H}Y^V\right)^H + \left(\nabla_{X^H}Y^H\right)^V.
\end{aligned} \tag{B.4.1}$$

Loosely speaking, the T-tensor corresponds to the second fundamental form of the fibers. In particular, T vanishes identically if and only if the Riemannian submersion $f : M \to B$ is totally geodesic. On the other hand, the A-tensor vanishes identically if and only if the horizontal distribution $\mathcal{H}$ is integrable. If both, T and A, vanish identically, then M is locally isometric to the Riemannian product $B \times F$.

Here and subsequently, R is the curvature tensor of g, and $\check{R} := f^*R^B$ is the pullback of the curvature tensor of B. Further, we denote by $\hat{R}_p$ the curvature tensor of the induced metric $\hat{g}_p$ on the fiber F_p, where $p \in B$.

Notation B.5. Following the notation of [O'N66], U, V, W, W' will always be vertical vector fields and X, Y, Z, Z' will always be horizontal vector fields. In addition, E, E_1, E_2 will denote arbitrary vector fields.

Theorem B.6. *For a Riemannian submersion $f : (M, g) \to (B, h)$, the curvature tensor R of M satisfies the following identities:*

$$\begin{aligned} g(R(U,V)W,W') = g(\hat{R}(U,V)W,W') + g(T(U,W),T(V,W')) \\ - g(T(V,W),T(U,W'), \end{aligned} \tag{B.6.1}$$

$$g(R(U,V)W,X) = -g((\nabla_V T)(U,W),X) + g((\nabla_U T)(V,W),X), \tag{B.6.2}$$

$$\begin{aligned} g(R(X,U)Y,V) = -g((\nabla_X T)(U,V),Y) + g(T(U,X),T(V,Y)) \\ - g((\nabla_U A)(X,Y),V) - g(A(X,U),A(Y,V)), \end{aligned} \tag{B.6.3}$$

$$\begin{aligned} g(R(U,V)X,Y) = -g((\nabla_U A)(X,Y),V) + g((\nabla_V A)(X,Y),U) \\ - g(A(X,U),A(Y,V)) + g(A(X,V),A(Y,U)) \\ + g(T(U,X),T(V,Y)) - g(T(V,X),T(U,Y)), \end{aligned} \tag{B.6.4}$$

$$\begin{aligned} g(R(X,Y)Z,U) = -g((\nabla_Z A)(X,Y),U) - g(A(X,Y),T(U,Z) \\ + g(A(Y,Z),T(U,X)) + g(A(Z,X),T(U,Y)), \end{aligned} \tag{B.6.5}$$

$$\begin{aligned} g(R(X,Y)Z,Z') = g(\check{R}(X,Y)Z,Z') + 2g(A(X,Y),A(Z,Z')) \\ - g(A(Y,Z),A(X,Z')) + g(A(X,Z),A(Y,Z') \end{aligned} \tag{B.6.6}$$

The corresponding formulas for the sectional curvatures follow immediately from the above theorem. In the following, sec denotes the sectional curvature on M, $\widecheck{\sec}$ the pullback of the sectional curvature on B and $\widehat{\sec}_p$ the intrinsic sectional curvature of the fiber F_p, $p \in B$.

Corollary B.7. *If $g(U,V) = 0$, $g(X,Y) = 0$ and all of them have unit length, then*

$$\sec(U,V) = \widehat{\sec}(U,V) + |T(U,V)|^2 - g(T(U,U),T(V,V)), \tag{B.7.1}$$

$$\sec(X,U) = g((\nabla_X T)(U,U),X) - |T(U,X)|^2 + |A(X,U)|^2, \tag{B.7.2}$$

$$\sec(X,Y) = \widecheck{\sec}(X,Y) - 3|A(X,Y)|^2. \tag{B.7.3}$$

Finally, we want to state various relations between the tensors A, T and their derivatives. All of these relation can be proven by straightforward calculations.

Proposition B.8. *Let $f : (M, g) \to (B, h)$ be a Riemannian submersion. Then the tensors $(\nabla_{E_1} T)(E_2, \cdot\,)$ and $(\nabla_{E_1} A)(E_2, \cdot\,)$ are alternating and*

$$g((\nabla_E T)(U,V),X) = g((\nabla_E T)(V,U),X), \tag{B.8.1}$$

$$g((\nabla_E A)(X,Y),U) = -g((\nabla_E A)(Y,X),U). \tag{B.8.2}$$

Moreover,

$$(\nabla_X T)(Y, \cdot\,) = -T(A(X,Y), \cdot\,), \tag{B.8.3}$$

$$(\nabla_U T)(X, \cdot\,) = -T(T(U,X), \cdot\,), \tag{B.8.4}$$

$$(\nabla_U A)(V, \cdot\,) = -A(T(U,V), \cdot\,), \tag{B.8.5}$$

$$(\nabla_X A)(U, \cdot\,) = -A(A(X,U), \cdot\,). \tag{B.8.6}$$

Furthermore, the derivatives of T satisfy

$$g((\nabla_X T)(U,V),W) = g(A(X,V),T(U,W)) - g(A(X,W),T(U,V)), \tag{B.8.7}$$

$$g((\nabla_X T)(U,Y),Z) = g(A(X,Y),T(U,Z)) - g(A(X,Z),T(U,Y)), \tag{B.8.8}$$

$$g((\nabla_U T)(V,W),W') = g(T(U,W),T(V,W') - g(T(U,W'),T(V,W)), \tag{B.8.9}$$

$$g((\nabla_U T)(V,X),Y) = g(T(U,X),T(V,Y)) - g(T(V,X),T(U,Y)), \tag{B.8.10}$$

and the derivatives of A satisfy

$$g((\nabla_U A)(X,V),W) = g(T(U,V),A(X,W)) - g(T(U,W),A(X,V)), \tag{B.8.11}$$

$$g((\nabla_U A)(X,Y),Z) = g(A(X,Z),T(U,Y)) - g(A(X,Y),T(U,Z)), \tag{B.8.12}$$

$$g((\nabla_X A)(Y,U),V) = g(A(X,U),A(Y,V)) - g(A(X,V),A(Y,U)), \tag{B.8.13}$$

$$g((\nabla_X A)(Y,Z),Z') = g(A(X,Z),A(Y,Z')) - g(A(X,Z'),A(Y,Z)), \tag{B.8.14}$$

and

$$\begin{aligned} &g((\nabla_X A)(Y,Z),U) + g(\nabla_Y A)(Z,X),U) + g(\nabla_Z)A(X,Y),U) \\ &\qquad = g(A(X,Y),T(U,Z)) + g(A(Y,Z),T(U,X)) + g(A(Z,X),T(U,Y)). \end{aligned} \tag{B.8.15}$$

In addition,

$$\begin{aligned} &g((\nabla_U A)(Y,Z),U) + g((\nabla_V A)(X,Y),U) \\ &\qquad = g((\nabla_Y)(U,V),X) - g((\nabla_X T)(U,V),Y). \end{aligned} \tag{B.8.16}$$

Appendix C

Continuity of Dirac spectra

In Section 3.1 we observed that any collapsing sequence $(M_i, g_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(n+k,d)$ with smooth n-dimensional limit space (B,h) can be approximated by a sequence of Riemannian affine fiber bundles $f_i : (M_i, \tilde{g}_i) \to (B, h_i)$ such that

$$\lim_{i\to\infty} \|\tilde{g}_i - g_i\|_{C^1} = 0,$$
$$\lim_{i\to\infty} \|\tilde{h}_i - h\|_{C^1} = 0.$$

Since we are interested in Dirac eigenvalues, we have to verify that the Dirac spectrum is continuous with respect to a C^1-variation of metrics. Although this behavior is well-known we want to discuss this topic in more detail following [Now13]. Furthermore, we formulate an explicit consequence of [Now13, Theorem 4.10] that is crucial for the proof of Theorem 4.5.

Let M be an n-dimensional spin manifold with a fixed topological spin structure. We recall, that a *topological spin structure* is a $\widetilde{\mathrm{GL}}_+$-principal bundle $P_{\widetilde{\mathrm{GL}}_+}(M)$ such that it is a double cover $P_{\widetilde{\mathrm{GL}}_+}(M) \to P_{\mathrm{GL}_+}(M)$ of the $\mathrm{GL}_+(n)$-principal bundle $P_{\mathrm{GL}_+}(M)$ consisting of positively oriented frames that is compatible with the group double cover $\widetilde{\mathrm{GL}}_+(n) \to \mathrm{GL}_+(n)$, i.e.

$$\begin{tikzcd}
\widetilde{\mathrm{GL}}_+(n) \times P_{\widetilde{\mathrm{GL}}_+}M \arrow[r] \arrow[d] & P_{\widetilde{\mathrm{GL}}_+}M \arrow[d] \arrow[dr] & \\
& & M\ . \\
\mathrm{GL}_+(n) \times P_{\mathrm{GL}_+}M \arrow[r] & P_{\mathrm{GL}_+}M \arrow[ur] &
\end{tikzcd}$$

Let $\mathfrak{R}(M)$ denote the space of all Riemannian metrics on M endowed with the C^1-topology. For any $g \in \mathfrak{R}(M)$, the topological spin structure induces a spin structure on (M,g) by restricting to the preimage of the oriented orthogonal frame bundle $P_{\mathrm{SO}}(M,g)$ of (M,g). Let $\Sigma_g M$ be the spinor bundle of (M,g). The relation between spinor bundles and Dirac operators with respect to different metrics was studied in [BG92, Mai97]. In the following we recall the identifications of different metric spin structures.

For any $g, h \in \mathfrak{R}(M)$, there is a unique symmetric positive definite endomorphism field H_g such that $g(H_g X, Y) = h(X,Y)$ for all vector fields X, Y. Hence, there is a unique symmetric positive definite square root $b_g^h := \sqrt{H_g}$ such that for all vector fields X, Y,

$$g(b_g^h X, b_g^h Y) = g(H_g X, Y) = h(X, Y).$$

In particular, there is an $\mathrm{SO}(n)$-equivariant isomorphism

$$\begin{aligned}(b_g^h)^n : P_{\mathrm{SO}}(M, h) &\to P_{\mathrm{SO}}(M, g),\\ (e_1, \dots, e_n) &\mapsto (b_g^h e_1, \dots, b_g^h e_n).\end{aligned}$$

Remark C.1. Reversing the roles of g and h, it follows that

$$b_h^g = (b_g^h)^{-1}.$$

However, if we consider three metrics $g, h, k \in \mathfrak{R}(M)$, then in general

$$b_k^g \neq b_k^h \circ b_h^g.$$

Since M is by assumption a closed n-dimensional Riemannian manifold with a fixed topological spin structure, it follows that b_g^h lifts to a $\mathrm{Spin}(n)$-equivariant isomorphism

$$\widetilde{(b_g^h)^n} : P_{\mathrm{Spin}}(M, h) \to P_{\mathrm{Spin}}(M, g).$$

In particular, we obtain an isometry of the corresponding spinor bundles,

$$\begin{aligned}\beta_g^h : \Sigma_h M = P_{\mathrm{Spin}}(M, h) \times_{\theta_n} \Sigma_n &\to \Sigma_h M = P_{\mathrm{Spin}}(M, g) \times_{\theta_n} \Sigma_n,\\ \varphi = [A, \xi] &\mapsto \beta_g^h \varphi := [\widetilde{(b_g^h)^n}(A), \xi].\end{aligned}$$

Here, $\theta_n : \mathrm{Spin}(n) \to \Sigma_n$ denotes the canonical complex spinor representation.

However, the induced map $\beta_g^h : L^2(\Sigma_h M) \to L^2(\Sigma_g M)$ is not an isometry as the volume forms are not the same, i.e. $\mathrm{dvol}_h \neq \mathrm{dvol}_g$. Hence, we define

$$\hat{\beta}_g^h := f_g^h \beta_g^h, \tag{C.1.1}$$

where f_g^h is a positive function such that $\mathrm{dvol}_h = (f_g^h)^2 \,\mathrm{dvol}_g$. By construction,

$$\hat{\beta}_g^h : L^2(\Sigma_h M) \to L^2(\Sigma_g M)$$

is an isometry of Hilbert spaces. This isometry allows us, to pullback the Dirac operator D_g of $\Sigma_g M$ to an elliptic first order differential operator

$$\begin{aligned}{}^hD^g &:= \hat{\beta}_h^g \circ D_g \circ \hat{\beta}_g^h\\ &= \beta_h^g D_g \beta_g^h - f_h^g\, b_g^h(\mathrm{grad}_g\, f_g^h)\end{aligned} \tag{C.1.2}$$

We summarize the properties of this construction in the following proposition, see also [Mai97, Section 2]

Proposition C.2. *Let h be a fixed Riemannian metric on a closed spin manifold M with a fixed topological spin structure. Then for every metric $g \in \mathfrak{R}(M)$ the operator ${}^hD^g$, defined in* (C.1.2)*, is isospectral to the Dirac operator D_g on $\Sigma_g M$. Furthermore, ${}^hD^g$ is closed and densely defined on $H^{1,2}(\Sigma_h M)$. Moreover, the map*

$$\begin{aligned} \mathfrak{R}(M) &\to B\big(H^{1,2}(\Sigma_h M), L^2(\Sigma_h M)\big), \\ g &\mapsto {}^hD^g, \end{aligned}$$

is continuous. Here, $B(.,.)$ denotes the space of bounded linear operators endowed with the operator norm.

Following an idea of Lott, [Lot02a], Nowaczyk studied the continuity of Dirac eigenvalues with respect to the arsinh-topology.

Definition C.3. On $\mathbb{R}^{\mathbb{Z}}$, let the metric d_a be defined by

$$d_a(u,v) := \sup_{j\in\mathbb{Z}} |\operatorname{arsinh}(u(j)) - \operatorname{arsinh}(v(j))|$$

for all $u, v \in \mathbb{R}^{\mathbb{Z}}$. The topology induced by d_a is called the arsinh-*topology.*

In this setting, the continuity result for Dirac eigenvalues [Now13, Main Theorem 2] reads as

Theorem C.4. *Let M be a spin manifold with a fixed topological spin structure. There exists a family of functions $(\lambda_j \in C^0(\mathfrak{R}(M), \mathbb{R}))_{j\in\mathbb{Z}}$ such that the sequence $(\lambda_j(g))_{j\in\mathbb{Z}}$ are the eigenvalues of D^g. In addition, the sequence $(\operatorname{arsinh}(\lambda_j))_{j\in\mathbb{Z}}$ is equicontinuous and non decreasing.*

In fact, the above theorem is a restriction of a more general continuity statement proven in [Now13, Section 4].

Theorem C.5. *Let H be an Hilbert space, The spectrum of a family of unbounded self-adjoint operators $T : E \to C(H)$ is continuous in the* arsinh-*topology if*

(1) there exists a dense subspace $Z \subset H$, such that $\operatorname{dom} T_e = Z$ for all $e \in E$,

(2) there exists a norm $|.|$ on Z such that $T_e : Z \to H$ is bounded and the graph norm of T_e is equivalent to $|.|$,

(3) E is a topological space,

(4) the map $E \to B(Z, H), e \mapsto T_e$ is continuous.

By Theorem C.2 the family of Dirac operators associated to a C^1-convergent sequence of Riemannian metrics on a fixed spin manifold M satisfies these conditions.

Appendix D

Convergence of S^1-principal bundles

The goal of this appendix is to establish a general notion of convergence for S^1-principal bundles with connection. The content of this appendix was published in [Roo17, Section 4.1].

First, we show that for a suitable bound on the curvature of the S^1-principal bundle there are only finitely many possibilities of isomorphism classes of S^1-principal bundles satisfying it. Thus, we can focus on a sequence of connection one-forms on a fixed S^1-principal bundle where we obtain a converging subsequence by applying suitable gauge transformations.

In the beginning, we recall the basic classification results for S^1-principal bundles. For more details see e.g. [Bla10, Chapter 2] and [Bry08, Chapter VI]. These results are the main ingredients to prove the desired convergence results.

We recall the following terminology: Two S^1-principal bundles P and P' together with connections $\mathrm{i}\omega$ resp. $\mathrm{i}\omega'$ are *isomorphic with connections* if there is a principal bundle isomorphism $\Phi : P \to P'$ such that $\Phi^*\omega' = \omega$.

Isomorphism classes of S^1-principal bundles as well as gauge equivalence classes are classified by the Čech-cohomology of the underlying base manifold M. Especially the classification of isomorphism classes is a well-known result which we restate here.

Theorem D.1. *Let M be a compact manifold. Then there is a bijection between the Čech-cohomology group $\check{\mathrm{H}}^2(M, \mathbb{Z})$ and the isomorphism classes of S^1-principal bundles over M.*

Let P be an S^1-principal bundle over a compact manifold M. Then P defines, up to isomorphism, a unique class in $\check{\mathrm{H}}^2(M, \mathbb{Z})$. This class is called the *first Chern class* of P.

The curvature of a connection one-form $\mathrm{i}\omega$ on P is given by a closed two-form F on M, namely

$$\mathrm{d}\omega = F.$$

The de Rham class $[\frac{1}{2\pi}F] \in \mathrm{H}^2(M, \mathbb{R})$ is the image of the first Chern class of P under the Čech-de Rham isomorphism. A short calculation shows that $[\frac{1}{2\pi}F]$ is independent of the choice of the connection one-form $\mathrm{i}\omega$ on P. Thus, it depends only on the isomorphism class of the S^1-principal bundle.

We want to show that there is a suitable bound on the curvature of S^1-principal bundles such that there are, up to isomorphism, only finitely many S^1-principal bundles satisfying it. To do so, we recall from Hodge theory that on a compact Riemannian manifold (M, g) each de Rham class $[\omega]$ admits a unique harmonic representative $\widetilde{\omega}$. Moreover, $\widetilde{\omega}$ minimizes the L^2-norm in the class $[\omega]$. In addition, the projection from closed to harmonic forms is continuous in L^2. Thus, it is a natural choice to assume an L^2-bound on the curvature for our purpose .

Lemma D.2. *Let (M, g) be a compact Riemannian manifold and K a fixed nonnegative number. Then there are only finitely many isomorphism classes of S^1-principal bundles P with connection over M whose curvature F satisfies $\|F\|_{L^2} \leqslant K$.*

Proof. By Theorem D.1 the isomorphism classes of S^1-principal bundles over M are classified by $\check{\mathrm{H}}^2(M, \mathbb{Z})$. By the universal coefficient theorem, we have $\check{\mathrm{H}}^2(M, \mathbb{Z}) \cong \mathbb{Z}^{b_2(M)} \oplus T_1$, where T_1 is the torsion of $\check{\mathrm{H}}_1(M, \mathbb{Z})$ which is finite, and $b_2(M)$ is the second Betti number of M. The kernel of the homeomorphism $h : \check{\mathrm{H}}^2(M, \mathbb{Z}) \to \mathrm{H}^2(M, \mathbb{R})$ is given by T_1. Moreover, the cohomology class $[\frac{1}{2\pi}F] \in \mathrm{H}^2(M, \mathbb{R})$ is an integral class, i.e. it lies in the image of h.

Since the harmonic representative of a de Rham class minimizes the L^2 norm it follows that the set of isomorphism classes of S^1-principal bundles whose curvature satisfies $\|F\|_{L^2} \leqslant K$ is given by $h^{-1}(C)$, where

$$C := \left\{ [\omega] \in \mathrm{H}^2(M, \mathbb{R}) : \|\widetilde{\omega}\|_{L^2} \leqslant \frac{1}{2\pi} K \right\}.$$

Here $\widetilde{\omega}$ denotes the unique harmonic representative of $[\omega]$.

As $\mathrm{H}^2(M, \mathbb{R}) \cong \mathcal{H}^2(M) \cong \mathbb{R}^{b_2(M)}$, with $\mathcal{H}^2(M)$ denoting the space of harmonic two-forms, is a finite dimensional vector space and the projection from closed to harmonic forms is continuous in L^2, it follows that C is compact. In particular, Image$(h) \cap C$ is compact, hence finite. Since the kernel of h is also finite the claim follows. □

We recall now the characterization of the gauge equivalence classes of connections on a fixed S^1-principal bundle P over M which can be found in the standard literature.

Theorem D.3. *For a fixed S^1-principal bundle P over a compact Riemannian manifold M two principal connections are gauge equivalent if and only if their difference is represented by a closed integral one-form. In particular, the space of gauge equivalence classes of connections with fixed curvature F is given by the Jacobi torus $\check{\mathrm{H}}^1(M, \mathbb{R})/\check{\mathrm{H}}^1(M, \mathbb{Z})$.*

Using this theorem we are able to prove the following convergence result. Observe that in general we will not obtain C^∞-convergence. Therefore, we establish here the following notion: A connection one-form $\mathrm{i}\omega$ is called $C^{k,\alpha}$ if its associated Christoffel symbols are $C^{k,\alpha}$. Further on, we only consider $\alpha \in [0, 1)$.

Theorem D.4. *Let $(P_i, \mathrm{i}\omega_i)_{i \in \mathbb{N}}$ be a sequence of S^1-principal bundles with connection over a fixed compact Riemannian manifold (M, g). For each i let $F_i = \mathrm{d}\omega_i$ be the corresponding curvature. If there is a nonnegative K such that $\|F_i\|_{C^{k,\alpha}} \leqslant K$ for all i, then for any $\beta < \alpha$*

there is an S^1-principal bundle P with a $C^{k+1,\beta}$-connection ω and a subsequence, again denoted by $(P_i, \mathrm{i}\omega_i)_{i\in\mathbb{N}}$ together with principal bundle isomorphisms $\Phi_i : P \to P_i$ such that $\Phi_i^*\omega_i$ converges to ω in the $C^{k+1,\beta}$-norm

Proof. As the $C^{k,\alpha}$-norm of the curvatures of the S^1-principal bundle P_i is uniformly bounded in i, it is immediate that the L^2-norm of the curvatures is also uniformly bounded. Applying Lemma D.2 we conclude that this sequence only contains finitely many isomorphism classes of S^1-principal bundles. Hence, we find a subsequence $(P_i, \mathrm{i}\omega_i)_{i\in\mathbb{N}}$ such that for each i there is an isomorphism $\Psi_i : P_i \to P$ for some fixed P.

Using that the connections on P form an affine space over $\Omega^1(M)$, we fix ω_1 as a reference connection. The difference $\Psi_i^*\omega_i - \Psi_1^*\omega_1$ is given by a unique $\eta_i \in \Omega^1(M)$. We will apply the Hodge decomposition various times and show for each part separately how we obtain a converging subsequence.

Since P is fixed $\left[\frac{1}{2\pi}\Psi_i^*F_i\right] = \left[\frac{1}{2\pi}\Psi_k^*F_k\right]$ for all i, k. Hence, for each i there is a one-form ζ_i such that

$$\Psi_i^*F_i = \Psi_1^*F_1 + \mathrm{d}\zeta_i.$$

It follows from our assumptions on the curvatures that there is a positive constant $\widetilde{K}$ such that $\|\mathrm{d}\zeta_i\|_{C^{k,\alpha}} \leqslant \widetilde{K}$ uniformly in i.

By Hodge decomposition we can choose $\zeta_i = \mathrm{d}^*\xi_i$ for some closed two-form ξ_j which is orthogonal to $\ker(\Delta)$ in L^2. Thus, $\mathrm{d}\zeta_i = \Delta\xi_i$. Applying Schauder's estimate we find a positive constant C such that for all i

$$\|\xi_i\|_{C^{k+2,\alpha}} \leqslant C\|\Delta\xi_i\|_{C^{k,\alpha}} \leqslant C\widetilde{K}.$$

As the embedding $C^{k,\alpha} \hookrightarrow C^{k,\beta}$ is compact for any $\beta < \alpha$ there is a subsequence $(\xi_i)_{i\in\mathbb{N}}$ converging in $C^{k+2,\beta}$. Thus, $(\mathrm{d}^*\xi_i)_{i\in\mathbb{N}}$ converges in $C^{k+1,\beta}$. In general, the limit is not smooth.

For each i the connections $\Psi_i^*\mathrm{i}\omega_i$ and $\Psi_1^*\mathrm{i}\omega_1 + \mathrm{i}\mathrm{d}^*\xi_i$ have the same curvature. Thus, for each i, there is a unique closed one-form η_i such that

$$\Psi_i^*\omega_i = \Psi_1^*\omega_1 + \mathrm{d}^*\xi_i + \eta_i.$$

Again we apply Hodge decomposition and obtain for each i a smooth function f_i and a harmonic one-form ν_i such that

$$\eta_i = \mathrm{d}f_i + \nu_i.$$

If $\mathrm{d}f_i \neq 0$ we apply the gauge transformation $G_i = e^{-\mathrm{i}f_i}$ and obtain

$$G_i^*\Psi_i^*\omega_i = \Psi_1^*\omega_1 + \mathrm{d}^*\xi_i + \nu_i.$$

Now, we need to find a subsequence and suitable gauge transformations such that the sequence of the remaining harmonic parts $(\nu_i)_{i\in\mathbb{N}}$ converges. To obtain these we take a closer look at the classification of connections on a fixed S^1-principal bundle. By Theorem D.3, the gauge equivalence classes of connections for a fixed curvature form are classified by the Jacobi torus $\check{\mathrm{H}}^1(M,\mathbb{R})/\check{\mathrm{H}}^1(M,\mathbb{Z})$. By Hodge theory, there is exactly one harmonic representative in each de Rham class. Since $\check{\mathrm{H}}^1(M,\mathbb{Z})$ has no torsion elements it is embedded in $\mathrm{H}^1(M,\mathbb{R})$ via the Čech-de Rham isomorphism. Hence, we obtain the quotient of harmonic forms divided by harmonic integral forms which is isomorphic to the torus $\mathbb{T}^{b_1(M)}$. As the projection from closed to harmonic forms is continuous in L^2 the Jacobi torus is compact in the L^2-topology.

The sequence $(\nu_i)_{i\in\mathbb{N}}$ induces a sequence in the Jacobi torus. Since the Jacobi torus is a compact quotient in the L^2-topology there is a subsequence of harmonic representatives $(\widetilde{\nu}_i)_{i\in\mathbb{N}}$ converging in L^2 to a smooth harmonic one-form $\widetilde{\nu}$. Note that each ν_i is equivalent to $\widetilde{\nu}_i$. By standard elliptic estimates it follows that $(\widetilde{\nu}_i)_{i\in\mathbb{N}}$ converges in C^l for any $l > 0$.

Taking the corresponding gauge transformations H_i, we obtain the sequence

$$(H_i^* G_i^* \Psi_i^* \omega_i = \Psi_1^* \omega_1 + \mathrm{d}^* \xi_1 + \widetilde{\nu}_1)_{i\in\mathbb{N}}\,,$$

which converges in $C^{k+1,\beta}$. Setting $\Phi_i := \Psi_i \circ G_i \circ H_i$ finishes the proof. □

Remark D.5. Similarly a uniform upper bound on the $H^{k,2}$-norm of the curvature leads to a $H^{l+1,2}$-converging subsequence of the underlying connection one-forms for any $l < k$.

This theorem shows that the space of S^1-principal bundles over a fixed compact Riemannian manifold (M,g) with a uniform bound on the $C^{k,\alpha}$-norm of the curvature is "precompact in the $C^{k+1,\beta}$-topology" for any $\beta < \alpha$. Now we also want to vary the base manifold (M,g). For this we use the following compactness theorem by Anderson, [And90, Theorem 1.1], see also Remark 1.11.

Theorem D.6. *For given positive numbers Λ , ι, and d the set $\mathcal{M}(n,\Lambda,\iota,d)$ of isometry classes of closed Riemannian n-manifolds (M,g) with*

$$|\operatorname{Ric}^M| \leqslant \Lambda, \quad \operatorname{inj}(M) \geqslant \iota, \quad \operatorname{diam}(M) \leqslant d,$$

is precompact in the $C^{1,\alpha}$-topology for any $\alpha \in [0,1)$. Furthermore, the subspace consisting of Einstein manifolds is compact in the C^∞-topology.

Combining this class of manifolds with the assumptions in Theorem D.4 we define the following set of isometry classes of closed Riemannian manifolds with S^1-prinicpal bundles.

Definition D.7. Let $\mathcal{M}^{S^1}(n,\Lambda,\iota,d,K)$ be the set of isometry classes of S^1-prinicpal bundles $P \xrightarrow{\pi} M$ with principal connection $\mathrm{i}\omega$ such that $(M,g) \in \mathcal{M}(n,\Lambda,\iota,d)$ and $\|F\|_{C^{0,1}(g)} \leqslant K$ where $F = \mathrm{d}\omega$ is the corresponding curvature.

Theorem D.8. *Any sequence $(M_i,g_i,P_i,\omega_i)_{i\in\mathbb{N}}$ in $\mathcal{M}^{S^1}(n,\Lambda,\iota,d,K)$ admits a subsequence, again denoted by $(M_i,g_i,P_i,\omega_i)_{i\in\mathbb{N}}$, such that for any $\alpha \in [0,1)$ there is an S^1-principal bundle P over a closed Riemannian manifold M with a $C^{1,\alpha}$-metric g and a $C^{1,\alpha}$-connection $\mathrm{i}\omega$ such that for each i there is a principal bundle isomorphism*

$$\begin{array}{ccc} P & \xrightarrow{\Phi_i} & P_i \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi_i} & M_i \end{array}$$

with $\Phi_i^*\omega_i$ and $\phi_i^* g_i$ converging to ω resp. g in $C^{1,\alpha}$.

Proof. Let $(M_i, g_i, P_i, \omega_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{M}^{S^1}(n, \Lambda, \iota, d, K)$. Since any manifold in this sequence lies in $\mathcal{M}(n, \Lambda, \iota, d)$ there exists a subsequence again denoted by $(M_i, g_i, P_i, \omega_i)_{i\in\mathbb{N}}$ and a $C^{1,\alpha}$-Riemannian manifold (M, g) such that for each i there exists a diffeomorphism $\phi_i : M \to M_i$ such that $\phi_i^* g_i$ converges to g in $C^{1,\alpha}$, see Theorem D.6.

By pulling back each element in $(M_i, g_i, P_i, \omega_i)_{i\in\mathbb{N}}$ with the diffeomorphism ϕ_i we obtain a sequence of metrics and S^1-principal bundles with connections over a fixed compact manifold M which we call $(M, g_i, P_i, \omega_i)_{i\in\mathbb{N}}$ for simplicity.

We fix the initial metric g_1 as our background metric. Applying Theorem D.4 to the sequence $(P_i, \omega_i)_{i\in\mathbb{N}}$ viewed as S^1-principal bundles over (M, g_1) we obtain a subsequence together with principal bundle isomorphism $\Psi_i : P \to P_i$ such that $\Psi_i^* \omega_i$ converges in $C^{1,\alpha}(g_1)$.

Since $(\phi_i^* g_i)_{i\in\mathbb{N}}$ converges to g in $C^{1,\alpha}$ the claim follows. $\square$

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