Pathwise methods in regularisation by noise

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A Laura

(ma anche a tutti i gatti del mondo)

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Introduction

In a broad sense, *regularisation by noise* refers to the frequently observed phenomenon where the qualitative properties of a system evolving in time significantly change (and possibly improve) upon the insertion of an external noisy perturbations.

Phisycally, the presence of such perturbations is quite natural: every physical model arises from a mathematical abstraction procedure, where the less relevant factors in the dynamics (which are usually impossible to keep track of) are not taken into account; reinserting them in the system, by means of a statistical description, should provide a more faithful description of the real process one is trying to study.

Mathematically, the simplest yet most striking instance of this phenomenon is given by finite dimensional ODEs of the form

$$\dot{x}_t = b(t, x_t). \tag{1}$$

It is well-known that, if b is not Lipschitz, existence and uniqueness of solutions to (1) might not hold; a classical example, displaying the so called *Peano paintbrush phenomenon*, is given by $b(x) = 2\,\mathrm{sgn}(x)|x|^{1/2}$, which admits infinitely many solutions starting from $x_0 = 0$ of the form

$$x_t = \pm(t - t_0)^2\,\mathbb{1}_{t \geqslant t_0} \tag{2}$$

for any $t_0 \geqslant 0$. On the other hand, if we pass to consider the SDE

$$\mathrm{d}X_t^\varepsilon = b(t, X_t^\varepsilon)\mathrm{d}t + \varepsilon\,\mathrm{d}W_t \tag{3}$$

where $\varepsilon > 0$ and W is sampled as a Brownian motion, then existence and uniqueness of solutions holds for (3), for any continuous drift b (thus including the above choice).

Examples like the one presented above are not the only type of regularisation by noise; among others, it is worth mentioning a qualitative change of the long time behaviour of the system (e.g. by stabilization by noise [12, 11] or synchronization by noise [121]), the prevention of finite time blow-up [217], or the role played by the presence of random initial data (especially in dispersive equations, see [264] and the references therein). However, “restoration of wellposedness by noise”, in the style of a comparison between (1) and (3), is the most common acception of regularisation by noise and the one we will stick with in the rest of this thesis.

Historically, first regularisation by noise results are usually attributed to Zvonkin [285] and Veretennikov [269] (although existence and uniqueness in law for (3) was already well known, by a simple application of Girsanov transform). It is quite hard to pinpoint when the modern terminology was born, with the earliest work I could find with the exact expression *regularisation by noise* in the title being [226]. Nowadays the literature on the topic is huge, to the point where it recently gained its own spot in the Mathematics Subject Classification System MSC2020, as subject 60H50.

It is almost impossible to give a complete account on the existing theory (for some presentations, we refer to the monograph [115], the review paper [155] and Section 1.6 from [218]), but let us at least try to provide some main highlights on the existing techniques, their advantages and shortcomings; from now on for simplicity we will take $\varepsilon = 1$ in (3).

In the case of multidimensional SDEs driven by Brownian motion, a landmark result was obtained by Krylov and Röckner in [191], who established strong well-posedness of (3) for b merely satisfying some local integrability conditions, e.g. $b \in L_t^q L_x^p$ with $2/q + d/p < 1$. The next pioneering contribution was given by Flandoli, Gubinelli and Priola in [122], where they realized that in many situations the SDE also admits a sufficiently regular flow of diffeomorphisms, which in turn allows to solve the *stochastic transport equation* associated to (3), given by

$$\mathrm{d}u + b \cdot \nabla u \mathrm{d}t + \circ \mathrm{d}W \cdot \nabla u = 0. \tag{4}$$

These papers led to a by now standard technique, referred to as either *Zvonkin transform* or *Itô–Tanaka trick*; by stochastic calculus and parabolic PDEs tools, it allows to construct a transformation of the phase space Φ such that the new variable $Y_t := \Phi_t(X_t)$ solves another SDE with more regular coefficients; see among others the works of Fedrizzi and Flandoli [105, 106] and Zhang [279, 280, 282] for the case of additive and multiplicative Brownian noise and Priola [242] for Lévy type of noise. In all the afomentioned papers, strong existence and pathwise uniqueness is established; moreover, similar results can be stablished for infinite dimensional SDEs, see [82, 83].

Other techniques, based on martingale problems, allow to even treat even distributional drifts, see among others the works [25, 123, 190, 63, 283]; the price one has to pay in this case is that only weaker statements (i.e. weak existence and uniqueness in law) can be obtained (with the notable exception of $d = 1$, where one recovers strong statements, see [25] and [15]).

Among works which do not fit exactly the dychotomy presented above, let us mention: compactness arguments, first developed in [220] and recently revisited in [246] to construct strong solutions in the critical case $b \in L_t^q L_x^p$ with $2/q + d/p = 1$; the Eulerian approach from [29], which first establishes existence, uniqueness and regularity for (4), and subsequently construct a *generalised Lagrangian flow* for the SDE in the style of [97]; finally, a class of *Wiener uniqueness* results (at the level of transport equations (4)) developed in [216, 108, 223] (see also the discussion from [116]).

The case of stochastically perturbed PDEs is even more varied, for two main reasons: on one hand, like in the analytical setting, each PDE requires the development of ad hoc techniques; on the other, there is no canonical way to insert a noise in the equation, and the correct (physically meaningful) choice should again depend on the specific structural properties of the system in consideration. In any case, let us mention the works [53] and [14] for stochastic heat equations with additive noise; [16], [107] and [125] for transport type equations with multiplicative gradient noise (like the one appearing in (4)); [88], [89] for Schrödinger equations with white noise dispersion; finally, [113] and [92] for results concerning interacting particle systems with environmental background noise.

Given the above (very incomplete!) class of results, one might wonder whether there is still something interesting left to say concerning regularisation by noise. Luckily for this thesis, the answer is positive! There are two main reasons for this:

i. The use of heavily probabilistic tools, although partially necessary, introduces into the picture considerations (e.g. the exact notion of existence and/or uniqueness we are working with, which varies from result to result) which are not quite germane to the original problem. Observe that, due to the additive nature of the noise, equations (1) and (3) can be perfectly meaningful for any choice of a continuous path W, without any need for Itô calculus to be involved.

ii. Similarly, although these techniques work very well for a large class of Markovian noises, they break down for other choices. In particular, Brownian noise is very peculiar, for its properties of martingality and absence of memory, while in real-life examples it is often reasonable to expect the noise to be depend on the past history of the system (and possibly exhibit so called long-range dependence).

These problems are intimately related to the *pathwise approach* to regularisation by noise, which justifies the title of this thesis; in a nutshell, it can be summarised as splitting the problem in two distinct steps:

1. Find *analytic properties* of a continuous path w such that the perturbed ODE (written directly in integral form)
$$x_t = x_0 + \int_0^t b(s, x_s)\, \mathrm{d}s + w_t \tag{5}$$
admits exactly one solution.

2. Show that such properties are satisfied by typical realizations of the noise $W(\omega)$ one is interested in (which is the only step where probabilistic considerations enter the picture).

In the case of Brownian W, the question whether such a problem can be fullfilled (although not stated exactly in this modern reformulation) is due to Krylov, see also Problem 7.1.7 from [41]; a successful answer was provided in a landmark paper by Davie [86], establishing the first *path-by-path uniqueness* result in the literature (the terminology is due to Flandoli [115]). Since then, similar results have been established for several classes of SDEs, see e.g. [254, 255, 243, 244].

Compared to the aforementioned fully probabilistic techniques, the pathwise program encoded by Ponts 1. and 2. presents substantial additional difficulties, as it tries to exploit as least information on the noise w itself as possible. At the same time, whenever it works, it has a huge payoff: it gives us a better understanding of the mechanisms underlying the regularising effect and it provides a stronger notion of uniqueness, involving all possible solutions to the pathwise equation (which can be treated as a *random ODE* rather than an SDE), without any adaptability or probabilistic requirement; once Point 1. is available, the verification at Point 2. can be accomplished for a larger class of noises, which can be naturally non-martingale or non-Markovian.

Even further, one can start asking questions of a different flavour, like:

Do generic continuous perturbations w regularise the ODE (5)?

This last formulation is now completely detached from any specific probabilistic setting, although it perfectly fits the regularisation by noise framework. One of the main goals of this thesis will be to provide the following positive, quantitative answer to the previous question. For simplicity, we formulate it here only in the case of time-independent drifts b.

Theorem 1. (cf. Theorem 3.59 from Chapter 3) *Let $b \in C^\alpha_x$, $\alpha \in (-\infty, 1)$, $\delta \in (0, 1)$ and consider the perturbed ODE (5). The following hold:*

i. If $\delta < (2 - 2\alpha)^{-1}$, then for almost every $w \in C^\delta_t$ the ODE has a meaningful analytical interpretation; moreover, if $x_0 \in \mathbb{R}^d$ is fixed, then for almost every $w \in C^\delta_t$ there exists a unique solution to (5).

ii. If $\delta < (4 - 2\alpha)^{-1}$, then for almost every $w \in C^\delta_t$ the ODE is wellposed for all $x_0 \in \mathbb{R}^d$ and solutions form a flow of diffeomorphisms.

iii. More generally, if $\delta < (2n + 2 - 2\alpha)^{-1}$ for some $n \geqslant 1$, then for almost every $w \in C^\delta_t$ the associated flow is n-times differentiable.

iv. Finally, for almost every continuous w, the ODE (5) admits a smooth flow.

In the statement above, C^α_x denote Besov–Hölder spaces (also usually denoted by $B^\alpha_{\infty,\infty}$, see Appendix A.2), while C^δ_t denotes the space of δ-Hölder continuous paths $w: [0, T] \to \mathbb{R}^d$, namely such that $|w_t - w_s| \leqslant C\, |t - s|^\delta$ for some constant $C > 0$. Finally, the terminology "for almost every $w \in C^\delta_t$" must be understood in the measure-theoretic sense of prevalence, which is recalled in Appendix A.3.

Let us make some comments on Theorem 1:

- The statement covers the regime of *distributional drifts*, i.e. the case of $b\in C^\alpha_x$ with $\alpha<0$; here it is then unclear how to interpret the equation, as one cannot pointwise evaluate $b(x_s)$, thus the integral appearing in (5) is undefined in the Lebesgue sense; nontheless, for suitable values of $\delta>0$, Theorem 1 guarantees that the ODE is well-defined and might even have a regular flow of solutions.

- There is a nontrivial interplay between the *irregularity* of the perturbation w, as measured by the smallness the parameter δ, and its regularising effect on the ODE, as measured either by the allowed range of values for the parameter α associated to the regularity of the drift b, or by the number of derivatives n that the flow of solutions admits. In particular, Theorem 1 is a mathematical formalization of the principle *"the rougher the noise, the better the regularisation"*.

- In the limit case where δ gets to 0, there are no conditions left on α nor n and we assist to the *infinitely regularising effect* of continuous additive perturbations. Moreover, in all the statements there are no conditions on the dimension $d\in\mathbb{N}$ of the state space $\mathbb{R}^d$, which can be arbitrarily large (but finite).

The strategy we will employ in order to prove Theorem 1 builds on the philosophy initiated in [57], which started the study of analytic properties of paths ensuring the regularisation of ODEs. In particular, the authors therein identify the space-time regularity of the *averaged field*

$$T^w b(t,y) := \int_0^t b(r, y+w_r)\,\mathrm{d}r, \quad T^w_{s,t} b(y) := T^w b(t,y) - T^w b(s,y) \tag{6}$$

as a key property ensuring both the meaningfulness and the possibly the wellposedness of the perturbed ODE (5). The reasoning behind this fact roughly goes through the following lines:

1. In order to see a regularising effect, we need the path w to be particularly "active", so that whenever the solution x gets close to a critical point of the drift b, it cannot spend too much time there (this is usually believed to be the source of non-uniqueness, think of the Peano phenomenon presented at the beginning). If this "activity" is understood as w presenting very fast oscillations (think of Brownian trajectories), then it is reasonable to expect a lot of cancellations whenever integrating a function f along the curve w. In particular, there is some hope that the field $T^w b$ is actually spatially more regular than the original b.

2. Given the structure of the equation, we expect any solution x to be of the form $w+\theta$, where θ at least formally corresponds to the integral $\int_s^t b(r,x_r)\mathrm{d}r$ term; for measurable and bounded b this is indeed the case, implying that θ is Lipschitz, in particular more regular than w. It is reasonable to expect this last property to be preserved even in the distributional case and thus to *impose* the solution ansatz $x=w+\theta$ for $\theta\in C^\gamma_t$ for some value $\gamma>\delta$.

3. Intuitively, any x admitting such a decomposition "locally looks like w" up to higher order corrections, which suggest that for $|t-s|\ll 1$ it should be possible to approximate θ by
$$\theta_t-\theta_s = \int_s^t b(r,\theta_r+w_r)\mathrm{d}r \approx \int_s^t b(r,\theta_s+w_r)\mathrm{d}r = T^w b(t,\theta_s) - T^w b(s,\theta_s) = T^w_{s,t} b(\theta_s).$$

The key intuition from [57] is that, for sufficiently regular $T^w b$, the last step can be made fully rigorous by means of so called *nonlinear Young integrals*.

The correct solution ansatz for distributional drifts b is $x=w+\theta$ with $\theta\in C^{1/2}_t$, which allows to show that, if $T^w b\in C^\gamma_t C^1_x$ (see the definitions given in Chapters 1 and 3) for some $\gamma>1/2$, then $\int_s^t b(r,x_r)\mathrm{d}r = \int_s^t b(r,\theta_r+w_r)\mathrm{d}r$ is well-defined (but it is not an integral in the Lebesgue sense anymore!) and characterized as the unique limit of suitable Riemann-Stjeltes sums:

$$\int_0^t b(r,\theta_r+w_r)\mathrm{d}r = \int_0^t T^w b(\mathrm{d}r,\theta_r) = \lim_{n\to\infty}\sum_i T^w_{t_i^n,t_{i+1}^n} b(\theta_{t_i^n}) = \lim_{n\to\infty}\sum_i \int_{t_i^n}^{t_{i+1}^n} b(r,\theta_{t_i^n}+w_r)\mathrm{d}r$$

where the limit is taken along any sequence of partitions $\Pi^n = \{0 = t_0^n < t_1^n < \ldots < t_n^n = t\}$ with mesh $|\Pi^n| := \sup_i |t_{i+1}^n - t_i^n|$ converging to 0 as $n \to \infty$. Moreover, the resulting "integral" inherits the time regularity of $T^w b \in C_t^\gamma C_x^1$; in other terms, for any x as above, $t \mapsto \int_0^t b(r, x_r) \in C_t^\gamma \hookrightarrow C_t^{1/2}$.

As a consequence, we can set up and solve a fixed point map on the path space $\mathcal{D}^w := w + C_t^{1/2} = \{w + \theta : \theta \in C_t^{1/2}\}$, given by

$$F(x)_t = x_0 + \int_0^t b(r, x_r) \mathrm{d}r + w_t =: x_0 + \int_0^t T^w b(\mathrm{d}r, \theta_r) + w_t.$$

In order to show contractivity of F on $\mathcal{D}^w$ (endowed with a suitable metric), it is enough to know that $T^w b$ is regular enough (e.g. $T^w b \in C_t^\gamma C_x^2$ for some $\gamma > 1/2$); in this case, there exists a unique fixed point $x \in \mathcal{D}^w$ for F, which we *define* to be the solution to the perturbed ODE (5) (this definition is consistent with the classical one whenever b is regular). Equivalently, passing to the variable $\theta = x - w$, this amounts to establishing existence and uniqueness of solutions to

$$\theta_t = \theta_0 + \int_0^t T^w b(\mathrm{d}r, \theta_r) \tag{7}$$

which is now a *nonlinear Young differential equation*.

One can draw a nice analogy between the arguments developed in [57] (and their revisitation in [145]) and the general philosophy of rough paths (see the monographs [132, 134]). The main idea in rough path theory is that, in order to analytically define and solve an equation of the form

$$\mathrm{d}X_t = \sigma(X_t) \mathrm{d}Y_t,$$

driven by an irregular signal Y (typically $Y \in C_t^\delta$ with $\delta \leqslant 1/2$), the information contained in Y alone is not enough. Instead, one needs to *enhance* the imput data Y into $(Y, \mathbb{Y})$ by additionally considering a sufficient amount of iterated integrals of Y against itself (or in general its signature); once this is done, the resulting solution map $Y \mapsto X$ will admit a so called *Itô-Lyons decomposition*, namely it will consist of the composition of a measurable lift of Y into $(Y, \mathbb{Y})$ and a continuous (actually, differentiable) map sending $(Y, \mathbb{Y})$ into the corresponding solution X.

The situation here is similar: in order to give meaning to the perturbed ODE (5), we *postulate* the existence and regularity of $T^w b$, which plays the role of the enhancement $(Y, \mathbb{Y})$; once this is done and the correct solution ansatz $x \in \mathcal{D}^w$ is identified, we can solve the ODE in a purely analytical manner and construct a continuous (in suitable topologies) solution map $(b, T^w b) \mapsto x$, which yields an analogue of the Itô-Lyons decomposition. Heuristically, the information given by b is not enough to solve the equation, but we need to take into account the "activity" of w as encoded by the regularity of $T^w b$. Moreover, similarly to the rough paths setting, whenever successful, this strategy immediately yields the construction of an associated flow of diffeomorphisms to the equation (something usually more difficult to achieve in the SDE setting, due to the adaptability requirements and thus the impossiblity to apply time reversal arguments in a naive way).

Among the interesting processes W to which these results apply (by being able to verify Point 2. of the above programme, i.e. the $\mathbb{P}$-a.s. regularity of $T^{W(\omega)} b$), but are outside the scope of classical SDE techniques (due to W not being Markovian nor a semimartingale), the most prominent example is given by *fractional Brownian motion* (fBm) of Hurst parameter $H \neq 1/2$ (whose basic properties are recalled in Appendix A.1). It is a fundamental class of Gaussian processes, first introduced by Kolmogorov [189] in the study of turbulence and later rediscovered by Mandelbrot and Van Ness [211]; we refer to [90] and the references therein for an overview on its modelling applications. FBm can be regarded as a generalization of standard Brownian motion (corresponding to $H = 1/2$) and it shares many similar properties, including self-similarity and stationary increments. However for $H \neq 1/2$ its increments are not independent and can be either positively correlated (for $H > 1/2$, corresponding to long range dependence of increments) or negatively correlated ($H < 1/2$, short range dependence). More generally, this thesis unveils a large class of Gaussian processes with regularising trajectories, whose fundamental feature is to satisfy a suitable form of *local nondeterminism* (cf. Section 5.1.3).

In this introductory discussion, we have indulged a lot on explaining (some of) the ideas coming from [57], but they constitute only a part of the results contained here (mostly those from Chapter 3). The reason is that they laid the ground for several subsequent developments (see e.g. [65, 66]), going far beyond the initial applications from [57], revealing a general strategy: in order develop pathwise regularisation by noise results for a given system, one should find a correct reformulation of the initial problem (like the ansatz $x \in w + C_t^\gamma$) which allows an application of the *nonlinear Young formalism*. Indeed, the starting point of our analysis will be exactly to analyse in detail the latter class of equations, in an abstract setting, so to specialize them afterwards to different classes of problems.

Structure of the thesis

I've tried as much as possible to present the topics included in this thesis in their most natural logical order, with each chapter being a natural prosecution of the ones that preceeded it. Nontheless, each chapter has its own brief introduction, explaining the main motivations as well the notations and conventions adopted therein, and a closing section of bibliographical remarks, explaining more in detail the main sources of all the results presented and how they fit in the existing literature. For this reason, here I will only give a very short overview of the contents of the chapters.

Chapter 1 is based on the paper [141] and contains a preliminary study of abstract nonlinear Young differential equations (YDEs) in Banach spaces, i.e. equations of the form

$$y_t = y_0 + \int_0^t A(\mathrm{d}s, y_s).$$

Results concerning existence, uniqueness, differentiability of the associated flow, as well as convergence of numerical schemes and topological properties of the set of solution are discussed.

Chapter 2 is again based on [141] and can be seen as a natural continuation of Chapter 1, passing to the more complicated case of evolutionary nonlinear Young partial differential equations, especially of transport and parabolic type.

Chapter 3 applies the abstract theory from the previous chapters to study the regularising effect of continuous additive perturbations w on ODEs; the material is taken mostly from the paper [145], made in collaboration with Massimiliano Gubinelli, and partially from the preprint [146], joint with Fabian A. Harang and Avi Mayorcas.

Specifically, in order to study the perturbed ODE (5), we apply the change of variables $\theta = x - w$ and pass to consider the nonlinear YDE (7), which is a special case of the ones from Chapter 1 for the choice $A = T^w b$.

The first part of the chapter deals with $w = W(\omega)$ sampled as an fBm of parameter $H \in (0,1)$, presenting several techniques to estimate the $\mathbb{P}$-a.s. regularity of the averaged field $T^{W(\omega)} b$; this information is properly combined with an application of *Girsanov transform* to recover path-by-path uniqueness results à la Davie. It is then shown how, under suitable requirements on $T^W b$ (thus on b and the value H), it is even possible to construct a flow of diffeomorphisms for the ODE and solve the associated perturbed transport equation, by directly invoking the results from Chapters 1 and 2.

The last part of the chapter switches to the perspective of establishing results for *generic* perturbations w, in the sense of the prevalence. Although this notion of genericity is purely analytical, it does allow for probabilistic tools in the proofs, and indeed our results (especially the aforementioned Theorem 1) build on the theory designed for fBm in the first part.

Chapter 4 is based on the preprint [139] in collaboration with Fabian A. Harang. It extends the results from Chapter 3 by considering the regularising effect of addivite perturbations w on multiplicative SDEs of the (integral) form

$$x_t = x_0 + \int_0^t b^1(s, x_s)\mathrm{d}s + \int_0^t b^2(s, x_s)\mathrm{d}\beta_s + w_t;$$

here β is another fBm of parameter $\delta \in (1/2,1)$, so that in principle the term $\int_0^t b^2(s,x_s)\mathrm{d}\beta_s$ would be defined pathwise in the Young sense. Compared to the analysis developed in Chapter 3, the main difficulty lies in rigorously defining and studying the *multiplicative averaged field*

$$\Gamma^w b(t,y) = \int_0^t b^2(s,y+w_s)\mathrm{d}\beta_s;$$

once this is done, the general theory from Chapter 1 can be again implemented successfully.

Chapter 5 takes a slightly different perspective from the previous chapters and is based on the preprint [143], jointly with Massimiliano Gubinelli. So far, all our considerations were done at the level of ODEs (or SDEs) with fixed prescribed drift b, which results in statements of the form "if $T^w b$ is regular enough, then the perturbed ODE is wellposed". One might look instead for an intrinsic property of the path w, ensuring the regularity of $T^w b$ for *all* drifts b in a suitable class; this idea was again first developed in [57], where the concept of ρ-irregularity was first introduced. We say that a path $w: [0,T] \to \mathbb{R}^d$ is (γ,ρ)-irregular if there exists a constant $C>0$ such that

$$\left| \int_s^t e^{i\xi\cdot w_r}\,\mathrm{d}r \right| \leqslant C|\xi|^{-\rho}|t-s|^{\gamma} \quad \forall \xi \in \mathbb{R}^d, s \leqslant t. \tag{8}$$

The results presented in this chapter can be considered as a spiritual continuation of [57], carrying a detailed study of properties of ρ-irregular paths and their relation to instances of regularisation by noise for ODEs and PDEs. In particular, several useful criteria for stochastic processes to be ρ-irregular are presented and then applied to a large class of Gaussian processes, satisfying a suitable *local nondeterminism* condition. Moreover the relation between ρ-irregularity and analytical properties, like α-Hölder roughness or the dimension of the image sets $w([s,t])$, are presented. The final part of the chapter is the devoted to a detailed study of the *perturbation problem*, whose simplest instance can be phrased as follows:

If w is ρ-irregular, under which conditions on φ the same holds for $\tilde{w} = w + \varphi$?

Chapter 6 follows the preprint [144], jointly with Massimiliano Gubinelli, and focuses on the inviscid mixing and enhanced dissipation properties of shear flows $u: \mathbb{T} \to \mathbb{R}$; here $\mathbb{T}$ denotes the 1-dimensional torus and the class of PDEs in study is given by

$$\partial_t f(t,x,y) + u(y)\partial_x f(t,x,y) = \nu\Delta f(t,x,y); \tag{9}$$

here $\nu \geqslant 0$, $(t,x,y) \in [0,+\infty) \times \mathbb{T}^2$ and f has a prescribed initial condition $f_0 = 0$. The goal is to understand how the long-time behaviour of solutions to (9) are affected by the presence of u, in terms of the polynomial (resp. exponential) decay of $\|f\|_{H^{-s}}$ (resp. $\|f\|_{L^2}$), as $t \to \infty$, when $\nu = 0$ (resp. $\nu > 0$).

Although at fist glance this topic might not seem connected to regularisation by noise, it turns out it can be tackled by means of the same philosophy underlying our pathwise approach. Specifically, we will show that a sufficient condition for w to be mixing (resp. diffusion enhancing) is for it to be ρ-irregular (respectively satisfy *Wei's irregularity condition*) and that such notions of irregularity are safisfied by generic Hölder paths. In this way, we obtain a variant of the principle from Theorem 1 that can be summarised as "*the rougher the shear flow, the faster the mixing*".

Finally, Appendix A contains several useful results that have been used throughout the thesis, like fundamental properties of stochastic processes and function spaces, chaining lemmas, a recap of stochastic integration in Banach spaces and more.

What is not included here

There are three topics, which will not appear in this thesis for different reasons, but are still worth mentioning and discussing here shortly.

The first one can be regarded as one of the long-standing objectives of the overall programme of regularisation by noise and is usually called the *zero noise limit*. To explain what we mean, let us go back to the Peano phenomenon represented by the family of solutions (2). If we assume reality to be deterministic, we cannot expect any behaviour of this kind to be allowed in physical systems; thus what should we do when such pathologies arise in the model in consideration? One possibility is to regard some of these solutions as a mathematical artefact, which simply do not exist in reality; we are then led to the problem of identifying *physically meaningful* solutions by devising *selection principles* for them. Given that noise is present in any physical system, the only solutions to the mathematical model one would expect to observe in reality are those which are stable under arbitrarily small perturbations. In other terms, the physical solutions to (1) should be (a subset of) those recovered as the limit of X^ε solving (3) as $\varepsilon \to 0^+$.

When W is sampled as a Brownian motion, $d=1$ and the drift b has one singular point, the problem was solved in the 80's by Bafico and Baldi [18] using martingale techniques and explicit formulas solutions to elliptic PDEs; unfortunately, the theory hasn't made significant progresses since then. Still in the one dimensional setting, the presence of singular large deviations has been observed in [159, 171], while the result from [18] has been revisited using alternative techniques in [91, 262]; the novelty of these works is the use of more intrinsic tools applied directly at the level of the SDE, given respectively by the identification of the relevant dynamical time scales and the use of local times. In general dimension d, it is worth mentioning the structural properties of zero noise limits given by [51, 44], which are however not enough to fully characterize them; in particular, [44] establishes an important link between the Feller transition kernel obtained in this way and the viscosity solutions of the associated Kolmogorov equation (for which we refer to the monograph [126]). Recent further works on the multidimensional case include [93, 237, 238]. Finally, let us mention [17], regarding the study of the behaviour of zero noise limits not at the level of the SDE, but instead looking at the associated transport equation.

In the aforementioned works, Markov and martingale properties of the solutions enter crucially in the analysis; the zero noise limit is examined at the level of the law (or more generally transition kernel) of the solutions, i.e. by studying the limit of $\mathcal{L}(X^\varepsilon)$ as $\varepsilon \to 0^+$.

The pathwise setting poses even more challenges: for a given continuous regularising path w, suppose we considered $x^\varepsilon(w)$ solution to (5) with w replaced by εw. Is it be possible to characterize the limit points of $x^\varepsilon(w)$ as $\varepsilon \to 0^+$? If so, do they depend on the choice of the perturbation w, or are they universal objects? In particular, if we take $w = W^H(\omega)$ to be typical realisations of fBm, does the limit depend on the value of $H \in (0,1)$? We are still completely lacking the right tools to address these difficult questions; for this reason, they will be never further mentioned in the rest of the thesis.

The second topic that will not be covered is related to the paper [147], made in collaboration with Fabian A. Harang and Avi Mayorcas. Therein we studied the wellposedness properties of *distribution dependent SDEs* (henceforth DDSDEs) of the form

$$\mathrm{d}X_t = B_t(X_t, \mu_t)\mathrm{d}t + \mathrm{d}W_t, \quad \mu_t = \mathcal{L}(X_t), \tag{10}$$

which are expected to be the *mean-field limit* as $N \to \infty$ of the interacting particle systems

$$\mathrm{d}X_t^{i,N} = B_t(X_t^{i,N}, \mu_t^N)\mathrm{d}t + \mathrm{d}W_t^i, \quad \mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}, \quad \forall i = 1, \ldots, N \tag{11}$$

where $\{W^i\}_{i\in\mathbb{N}}$ are i.i.d. random variables sampled as W and μ_t^N represents the *empirical measure* of the system at time t. The literature on DDSDEs (also referred to as *McKean-Vlasov* SDEs) and interacting particle systems, especially for W sampled as a Brownian motion, is enormous and we will not attempt to cover it here; we refer instead to the introduction from [147].

The reason why [147] is not presented here (apart from reasons of length), is because the philosophy adopted therein is exactly the opposite of the one presented here: our goal was to extract as much information as possible on the solution theory for (10) and (11) without imposing any assumption on W; in particular, the noise in consideration can be very degenerate (one can even take $W \equiv 0$) and thus cannot provide any regularising effect.

Our main techniques, based on the "pathwise McKean Vlasov theory" first developed by Tanaka [259] and recently nicely revisited in [69], allow to automatically deduce mean-field convergence of system (11) to (10) as soon as the DDSDE is wellposed and satisfies suitable stability estimates. One of the main *raisons d'etre* for [147] is the attempt to establish a family of "baseline results" for the DDSDE, true for *any* choice of the process W, to be compared with what can be obtained in the (more interesting) case where W is truly nondegenerate and possibly strongly regularising. We started examining the latter case in the work [146], whose results are also partially contained here (cf. Theorem 3.27 from Chapter 3), by establishing the wellposedness of the DDSDE (10) for suitable distributional drifts B and W sampled as fBm.

Last but not least, during my PhD I also focused on another line of research, jointly with Franco Flandoli and Dejun Luo, which resulted in the papers [140, 119, 117, 118, 120, 148].

Roughly speaking, the main focus of this series of works is the effect of suitable *multiplicative transport noise* on evolutionary SPDEs of the form

$$\mathrm{d}u_t + \circ\mathrm{d}W_t \cdot \nabla u_t = [F(u_t) + \kappa\Delta u_t]\mathrm{d}t. \tag{12}$$

Here F is an abstract nonlinearity (possibly nonlocal and/or also depending on higher derivatives of u_t), $\kappa \geqslant 0$, while W is a Gaussian field which is Brownian in time and coloured in space, i.e. it can be written as $W(t,x) = \sum_k \sigma_k(x)\beta^k(t)$, where $\{\beta^k\}_k$ is a family of independent standard Brownian motions and σ_k are suitable vector field; the symbol $\circ\mathrm{d}W$ denotes Stratonovich integration, which is physically justified by the Wong–Zakai principle. Although for the sake of this brief discussion there is no need to specify all the details, in most works we considered the torus $\mathbb{T}^d \simeq [0,2\pi]^d$ with periodic boundary conditions.

Starting with [140] it was observed that, for any fixed $\nu > 0$, one can construct a sequence of divergence free noises W^n, undergoing a suitable *scaling limit*, such that the associated solutions u^n to (12) converge weakly to the solution u to the deterministic PDE with enhanced viscosity

$$\partial_t u = F(u_t) + (\kappa + \nu)\Delta u_t. \tag{13}$$

The work [140] only covers the linear case $F(u_t) = b_t \cdot \nabla u_t$, but since then the theory has then been considerably expanded. For instance [119] treats the case of 2D Euler equation in vorticity form, corresponding to

$$F(u_t) = \nabla\Delta^{-1}u_t = K * u_t, \quad \kappa = 0$$

where K denotes the Biot–Savart kernel; in this case, the result can be stated as a scaling limit of stochastic Euler equations to deterministic Navier–Stokes (thus inverting the classical paradigm where solutions to Euler are recovered from the vanishing viscosity limit of Navier–Stokes).

The theory is still thriving, as we are starting to understand how to make the arguments concerning the scaling limit more quantitative [118]; we also recently established the presence of underlying large deviations and Gaussian fluctuations in [148]. The scaling limit of (12) to (13) has important applications in regularisation by noise phenomena, specifically in showing that *noise prevents blow-up* of solutions, see [124, 117].

However, the arguments in the aforementioned papers heavily rely on the availability of stochastic calculus (the key point is the correct computation of the *Itô-Stratonovich corrector* arising in (12)) and would immediately break down if the noise W were not Brownian in time; moreover the SPDE (12) does not have in general a pathwise interpretation. Although extremely interesting, this line of research does not fit the general philosophy presented in this thesis and is thus omitted.

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Chapter 1
Nonlinear Young Differential Equations

The aim of this chapter is to present a systematic treatment and a well-developed solution theory for so called *nonlinear Young differential equations* (henceforth nonlinear YDEs), namely equations of the form

$$x_t = x_0 + \int_0^t A(\mathrm{d}s, x_s). \tag{1.1}$$

Nonlinear YDEs can be regarded as the basic building block for many results presented in this thesis, as their importance lies in their *versatility*:differential systems which a priori do not present the structure (1.1), may be *recast* as nonlinear YDEs, after a suitable change of variables. This approach is particularly convenient for two main reasons: i) it allows to give meaning to such systems also in situations where classical theory breaks down; ii) after the change of variables has been applied, wellposedness results for the original problem follow almost automatically by an application of the abstract theory of nonlinear YDEs presented here. As we shall see, in the stochastic setting this allows for an entirely *pathwise treatment*, making it possible in particular to establish *genericity* results.

Let us shortly describe the setting for nonlinear Young equations. Given a Banach space V and a time interval $[0, T]$, the unknown $x: [0, T] \to V$ appearing in (1.1) is an α-Hölder continuous path, while the vector field $A: [0, T] \times V \to V$ is given and enjoys suitable space-time Hölder regularity. If A is sufficiently smooth in time, then $A(\mathrm{d}s, x_s)$ can be interpreted as $\partial_t A(s, x_s)\mathrm{d}s$, so that (1.1) can be regarded as an ODE in integral form; here however we are interested in the case $\partial_t A$ does not exist, so that (1.1) does not admit a classical interpretation.

In the case $A(t, z) = f(z) y_t$, where y is an U-valued α-Hölder continuous path and f maps V into the space of linear maps from U to V, equation (1.1) can be rewritten as

$$x_t = x_0 + \int_0^t f(x_s)\, \mathrm{d}y_s \tag{1.2}$$

which can be regarded as a rough differential equation driven by a signal y.

In the regime $\alpha > 1/2$, for sufficiently regular f, equation (1.2) can be rigorously interpreted by means of Young integrals, introduced in [277]; wellposedness of Young differential equations (YDEs) was first studied in [209]. After that, several alternative approaches to (1.2) have been developed, either by means of fractional calculus [278] or numerical schemes [87]; see the review [199] for a self-contained exposition of the main results for YDEs. YDEs have found several applications in the study of SDEs driven by fractional Brownian motion (fBm) of parameter $H > 1/2$, see for instance [224].

Although equation (1.1) may be seen as a natural generalization of (1.2), its development is much more recent. Nonlinear Young integrals like the one appearing in (1.1) were first defined in [57] in applications to additively perturbed ODEs, and subsequently rediscovered in [176], where they were employed to give a pathwise interpretation to Feynman-Kac formulas and SPDEs with random coefficients.

In this thesis we will consider exclusively the case of time regularity $\alpha > 1/2$, also known as the Young regime or level-1 rough path. However it is now well understood, since the pioneering work of Lyons [210], that it is possible to give meaning to equation (1.2) even in the case $\alpha \leqslant 1/2$ by means of the theory of rough paths, see the monographs [134], [132] for a detailed account on the topic. An analogue extesion of (1.1) to the case of *nonlinear rough paths* has been recently achieved in [70], [230]; so far however it hasn't found the same variety of applications, discussed above, as the nonlinear Young case. For more bibliographic references and further extensions of nonlinear Young integrals, we refer the reader to Section 1.5.

Motivated by the above discussion, we collect here several results for abstract nonlinear YDEs, providing general criteria for existence, uniqueness and stability of solutions to (1.1), as well as convergence of numerical schemes and differentiability of the flow. The content of this chapter is taken from the paper [141], which in turn is deeply inspired by the review [199]; all the theory is developed in (possibly infinite dimensional) Banach spaces and relies systematically on the use of the sewing lemma, a by now standard feature of the rough path framework.

The content given here will then set the stage for applications (especially to regularisation by noise phenomena) which will be presented in Chapters 3, 4 and Section 5.2.2 from Chapter 5.

Structure of the chapter. Section 1.1 is entirely devoted to the definition of the nonlinear Young integral and its basic properties; once they are established, we will pass to a detailed study of nonlinear YDEs in Section 1.2 and of the associated flow in Section 1.3. Finally, Section 1.4 contains several less canonical results, yet extremely relevant for our analysis, ranging from measurable selections of solutions to conditional uniqueness statements.

Notations. Here are the most relevant notations and conventions adopted through this chapter:

- We write $a \lesssim b$ if $a \leqslant Cb$ for a suitable constant, $a \lesssim_\lambda b$ to stress the dependence $C = C(\lambda)$; $a \sim b$ stands for $a \lesssim b$ and $b \lesssim a$.
- We will always work on a finite time interval $[0, T]$; the Banach spaces V, W appearing might be infinite dimensional, but will be always assumed separable for simplicity.
- Given a Banach space $(E, \|\cdot\|_E)$, we set $C_t^0 E = C([0, T]; E)$ endowed with supremum norm

$$\|f\|_\infty = \sup_{t \in [0, T]} \|f_t\|_E \quad \forall f \in C_t^0 E$$

  where $f_t := f(t)$ and we adopt the incremental notation $f_{s,t} := f_t - f_s$. Similarly, for any $\alpha \in (0, 1)$ we set $C_t^\alpha E = C^\alpha([0, T]; E)$, the space of α-Hölder continuous functions, with

$$[\![f]\!]_\alpha := \sup_{0 \leqslant s < t \leqslant T} \frac{\|f_{s,t}\|_E}{|t - s|^\alpha}, \qquad \|f\|_\alpha := \|f\|_\infty + [\![f]\!]_\alpha.$$

  The above notation will be applied to several choice of E, such as $C_t^\alpha V$, $C_t^\alpha \mathbb{R}^d$, but also $C_t^\alpha C_{V,W}^{\beta,\lambda}$ or $C_t^\alpha C_{V,W,\mathrm{loc}}^\beta$, for which we refer to Definitions 1.2 and 1.4.
- We denote by $\mathcal{L}(V, W)$ the set of all linear bounded operators from V to W, $\mathcal{L}(V) = \mathcal{L}(V, V)$.
- Whenever we will refer to differentiability, this must be understood in the sense of Frechét, unless specified otherwise; given a map $F: V \to W$ we denote by DF its Frechét differential. We will use indifferently $DF(x, y) = DF(x)(y)$ for the differential at point x evaluated along the direction y.
- As a rule of thumb, whenever $\mathcal{J}(\Gamma)$ appears, it denotes the sewing of $\Gamma: \Delta_2 \to E$; we refer to Section 1.1 for more details on the sewing map. Similarly, in proofs based on a Banach fixed point argument, $\mathcal{I}$ will denote the map whose constractivity must be established.
- As a rule of thumb, we will use C_i, $i \in \mathbb{N}$ for the constants appearing in the main statements and κ_i for those only appearing inside the proofs; the numbering restarts at each statement and is only meant to distinguish the dependence of the constants from relevant parameters.

1.1 The nonlinear Young integral

1.1.1 Preliminaries

This subsections contains an exposition of the sewing lemma and the definition of the joint space-time Hölder continuous drifts A we will work with; the reader already acquainted with these concepts may skip it.

Given a finite interval $[0,T]$, consider the n-simplex $\Delta_n := \{(t_1,\ldots,t_n) : 0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant T\}$. Let V be a Banach space, for any $\Gamma : \Delta_2 \to V$ we define $\delta\Gamma : \Delta_3 \to V$ by

$$\delta\Gamma_{s,u,t} := \Gamma_{s,t} - \Gamma_{s,u} - \Gamma_{u,t}.$$

We say that $\Gamma \in C_2^{\alpha,\beta}([0,T];V) = C_2^{\alpha,\beta} V$ if $\Gamma_{t,t} = 0$ for all $t \in [0,T]$ and $\|\Gamma\|_{\alpha,\beta} < \infty$, where

$$\|\Gamma\|_\alpha := \sup_{s<t} \frac{\|\Gamma_{s,t}\|_V}{|t-s|^\alpha}, \quad \|\delta\Gamma\|_\beta := \sup_{s<u<t} \frac{\|\delta\Gamma_{s,u,t}\|_V}{|t-s|^\beta}, \quad \|\Gamma\|_{\alpha,\beta} := \|\Gamma\|_\alpha + \|\delta\Gamma\|_\beta.$$

For a map $f : [0,T] \to V$, we still denote by $f_{s,t}$ the increment $f_t - f_s$. The next result, usually refereed to as "sewing lemma", gives a quantitative information on how well the 2-parameter map Γ can be locally approximated by a difference $f_{s,t}$. It is a fundamental tool in modern rough path theory, dating back to [162], [112], allowing to define abstract Riemann–Stjeltes type of integrals.

Lemma 1.1. (Lemma 4.2 from [132]) *Let α, β be such that $0 < \alpha < 1 < \beta$. For any $\Gamma \in C_2^{\alpha,\beta} V$ there exists a unique map $\mathcal{J}(\Gamma) \in C_t^\alpha V$ such that $\mathcal{J}(\Gamma)_0 = 0$ and*

$$\|\mathcal{J}(\Gamma)_{s,t} - \Gamma_{s,t}\|_V \leqslant C_1 \|\delta\Gamma\|_\beta \, |t-s|^\beta \tag{1.3}$$

where the constant C_1 can be taken as $C_1 = (1 - 2^{\beta-1})^{-1}$. Thus the sewing map $\mathcal{J} : C_2^{\alpha,\beta} V \to C_t^\alpha V$ is linear and bounded and there exists $C_2 = C_2(\alpha,\beta,T)$ such that

$$\|\mathcal{J}(\Gamma)\|_\alpha \leqslant C_2 \, \|\Gamma\|_{\alpha,\beta}. \tag{1.4}$$

For a given Γ, $\mathcal{J}(\Gamma)$ is characterized as the unique limit of Riemann-Stjeltes sums: for any $t > 0$

$$\mathcal{J}(\Gamma)_t = \lim_{|\Pi| \to 0} \sum_i \Gamma_{t_i,t_{i+1}}.$$

The notation above means that for any sequence of partitions $\Pi_n = \{0 = t_0 < t_1 < \ldots < t_{k_n} = t\}$ with mesh $|\Pi_n| = \sup_{i=1,\ldots,k_n} |t_i - t_{i-1}| \to 0$ as $n \to \infty$, it holds

$$\mathcal{J}(\Gamma)_t = \lim_{n\to\infty} \sum_{i=0}^{k_n-1} \Gamma_{t_i,t_{i+1}}.$$

Next we need to introduce suitable classes of Hölder continuous maps on Banach spaces.

Definition 1.2. *Let V, W Banach spaces, $f \in C(V;W)$, $\beta \in (0,1)$. We say that f is locally β-Hölder continuous and write $f \in C^\beta_{V,W,\mathrm{loc}}$ if for any $R > 0$ the following quantities are finite:*

$$\llbracket f \rrbracket_{\beta,R} := \sup_{\substack{x \neq y \in V \\ \|x\|_V, \|y\|_V \leqslant R}} \frac{\|f(x) - f(y)\|_W}{\|x-y\|_V^\beta}, \quad \|f\|_{\beta,R} := \llbracket f \rrbracket_{\beta,R} + \sup_{\substack{x \in V \\ \|x\|_V \leqslant R}} \|f(x)\|_V.$$

For $\lambda \in (0,1]$, we define the space $C^{\beta,\lambda}_{V,W}$ as the collection of all $f \in C(V;W)$ such that

$$\llbracket f \rrbracket_{\beta,\lambda} := \sup_{R \geqslant 1} R^{-\lambda} \llbracket f \rrbracket_{\beta,R}, \quad \|f\|_{\beta,\lambda} := \llbracket f \rrbracket_{\beta,\lambda} + \|f(0)\|_V < \infty.$$

Finally, the classical Hölder space $C^\beta_{V,W}$ is defined as the collection of all $f \in C(V;W)$ such that

$$\llbracket f \rrbracket_\beta := \sup_{x \neq y \in V} \frac{\|f(x) - f(y)\|_W}{\|x-y\|_V^\beta}, \quad \|f\|_\beta = \llbracket f \rrbracket_\beta + \sup_{x \in V} \|f(x)\|_V < \infty.$$

Remark 1.3. We ask the reader to keep in mind that although linked, $\llbracket f \rrbracket_{\beta,R}$ and $\llbracket f \rrbracket_{\beta,\lambda}$ denote two different quantities. $C^\beta_{V,W,\mathrm{loc}}$ is a Fréchet space with the topology induced by the seminorms $\{\|f\|_{\beta,R}\}_{R \geqslant 0}$, while $C^{\beta,\lambda}_{V,W}$ and $C^\beta_{V,W}$ are Banach spaces. Observe that if $f \in C^{\beta,\lambda}_{V,W}$, we have an upper bound on its growth at infinity, since for any $x \in V$ with $\|x\|_V \geqslant 1$ it holds

$$\|f(x)\|_V \leqslant \|f(x) - f(0)\|_V + \|f(0)\|_V \leqslant \|x\|_V^\beta \, \llbracket f \rrbracket_{\beta,\|x\|_V} + \|f(0)\|_V \leqslant \|f\|_{\beta,\lambda} (1 + \|x\|_V^{\beta+\lambda}).$$

In particular, if $\beta+\lambda\leqslant 1$, then f has at most linear growth.

We can now introduce fields $A\colon[0,T]\times V\to W$ satisfying a joint space-time Hölder continuity. We adopt the incremental notation $A_{s,t}(x):=A(t,x)-A(s,x)$, as well as $A_t(x)=A(t,x)$; from now on, whenever A appears, it is implicitly assumed that $A(0,x)=0$ for all $x\in V$.

Definition 1.4. *Given A as above, $\alpha,\beta\in(0,1)$, we say that $A\in C^\alpha_t C^\beta_{V,W,\mathrm{loc}}$ if for any $R\geqslant 0$ it holds*

$$[\![A]\!]_{\alpha,\beta,R}:=\sup_{0\leqslant s<t\leqslant T}\frac{[\![A_{s,t}]\!]_{\beta,R}}{|t-s|^\alpha},\quad \|A\|_{\alpha,\beta,R}:=\sup_{0\leqslant s<t\leqslant T}\frac{\|A_{s,t}\|_{\beta,R}}{|t-s|^\alpha}<\infty.$$

We say that $A\in C^\alpha_t C^{\beta,\lambda}_{V,W}$ if

$$[\![A]\!]_{\alpha,\beta,\lambda}:=\sup_{0\leqslant s<t\leqslant T}\frac{[\![A_{s,t}]\!]_{\beta,\lambda}}{|t-s|^\alpha},\quad \|A\|_{\alpha,\beta,\lambda}:=\sup_{0\leqslant s<t\leqslant T}\frac{\|A_{s,t}\|_{\beta,\lambda}}{|t-s|^\alpha}<\infty;$$

analogous definitions hold for $C^\alpha_t C^\beta_{V,W}$, $[\![\cdot]\!]_{\alpha,\beta}$, $\|\cdot\|_{\alpha,\beta}$.

The definition can be extended to the cases $\alpha=0$ or $\beta=0$ by interpreting the norm in the supremum sense; for instance $A\in C^0_t C^\beta_{V,W}$ if $\|A\|_{0,\beta}=\sup_{t\in[0,T]}\|A_t\|_\beta<\infty$.

Given a smooth map $F\colon V\to W$, we regard its Frechét differential D^kF of order k as a map from V to $\mathcal{L}^k(V,W)$, the set of bounded k-linear forms from V^k to W.

Definition 1.5. *We say that $A\in C^\alpha_t C^{n+\beta}_{V,W}$ if $A\in C^\alpha_t C^\beta_{V,W}$ and it is k-times Frechét differentiable in x, with $D^kA\in C^\alpha_t C^\beta_{V,\mathcal{L}^k(V,W)}$ for all $k\leqslant n$. $C^\alpha_t C^{n+\beta}_{V,W}$ is a Banach space with norm*

$$\|A\|_{\alpha,n+\beta}=\sum_{k=0}^n\|D^kA\|_{\alpha,\beta}.$$

Analogue definitions hold for $C^\alpha_t C^{n+\beta}_{V,W,\mathrm{loc}}$ and $C^\alpha_t C^{n+\beta,\lambda}_{V,W}$.

1.1.2 Construction and first properties

We are now ready to construct nonlinear Young integrals, following the line of proof from [176], [170]; other constructions are possible, see the discussion in Appendix A.2 from [141].

Theorem 1.6. *Let $\alpha,\beta,\gamma\in(0,1)$ such that $\alpha+\beta\gamma>1$, $A\in C^\alpha_t C^\beta_{V,W,\mathrm{loc}}$ and $x\in C^\gamma_t V$. Then for any $[s,t]\subset[0,T]$ and for any sequence of partitions of $[s,t]$ with infinitesimal mesh, the following limit exists and is independent of the chosen sequence of partitions:*

$$\int_s^t A(\mathrm{d}u,x_u):=\lim_{|\Pi|\to 0}\sum_i A_{t_i,t_{t+1}}(x_{t_i}).$$

The limit will be referred as a nonlinear Young integral. Furthermore:

1. *For all $(s,r,t)\in\Delta_3$ it holds $\int_s^r A(\mathrm{d}u,x_u)+\int_r^t A(\mathrm{d}u,x_u)=\int_s^t A(\mathrm{d}u,x_u)$.*
2. *If $\partial_t A$ exists continuous, then $\int_s^t A(\mathrm{d}u,x_u)=\int_s^t\partial_t A(u,x_u)\,\mathrm{d}u$.*
3. *There exists a constant $C_1=C_1(\alpha,\beta,\gamma)$ such that*

$$\left\|\int_s^t A(\mathrm{d}u,x_u)-A_{s,t}(x_s)\right\|_W\leqslant C_1|t-s|^{\alpha+\beta\gamma}\,[\![A]\!]_{\alpha,\beta,\|x\|_\infty}[\![x]\!]_\gamma^\beta. \tag{1.5}$$

4. *The map $(A,x)\mapsto\int_0^\cdot A(\mathrm{d}u,x_u)$ is continuous as a function from $C^\alpha_t C^\beta_{V,W,\mathrm{loc}}\times C^\gamma_t V\to C^\alpha_t W$. More precisely, it is a linear map in A and there exists $C_2=C_2(\alpha,\beta,\gamma,T)$ such that*

$$\left\|\int_0^\cdot A^1(\mathrm{d}u,x_u)-\int_0^\cdot A^2(\mathrm{d}u,x_u)\right\|_\alpha\leqslant C_2\|A^1-A^2\|_{\alpha,\beta,\|x\|_\infty}(1+[\![x]\!]_\gamma); \tag{1.6}$$

it is locally δ-Hölder continuous in x for any $\delta \in (0,1)$ such that $\delta < (\alpha + \beta\gamma - 1)/\gamma$ and there exists $C_3 = C_3(\alpha, \beta, \gamma, \delta, T)$ such that, for any $R \geqslant \|x\|_\infty \vee \|y\|_\infty$, it holds

$$\left\| \int_0^\cdot A(\mathrm{d}u, x_u) - \int_0^\cdot A(\mathrm{d}u, y_u) \right\|_\alpha \leqslant C_3 \|A\|_{\alpha,\beta,R} (1 + \|x\|_\gamma + \|y\|_\gamma) [\![x - y]\!]_\gamma^\delta. \tag{1.7}$$

Proof. In order to show convergence of the Riemann sums, it is enough to apply the sewing lemma to the choice $\Gamma_{s,t} := A_{s,t}(x_s) = A(t, x_s) - A(s, x_s)$. Indeed we have

$$\|\Gamma\|_\alpha = \sup_{s<t} \frac{\|A_{s,t}(x_s)\|_W}{|t-s|^\alpha} \leqslant \sup_{s<t} \frac{\|A_{s,t}\|_{0,\|x\|_\infty}}{|t-s|^\alpha} \leqslant \|A\|_{\alpha,0,\|x\|_\infty}$$

and

$$\begin{aligned} \|\delta\Gamma_{s,u,t}\|_W &= \|A_{u,t}(x_s) - A_{u,t}(x_u)\|_W \leqslant [\![A_{u,t}]\!]_{\beta,\|x\|_\infty} \|x_{u,s}\|_V^\beta \\ &\leqslant |t-u|^\alpha |u-s|^{\beta\gamma} [\![A]\!]_{\alpha,\beta,\|x\|_\infty} [\![x]\!]_\gamma^\beta \end{aligned}$$

which implies $\|\delta\Gamma\|_{\alpha+\beta\gamma} \leqslant [\![A]\!]_{\alpha,\beta,\|x\|_\infty} [\![x]\!]_\gamma^\beta$. In particular $\Gamma \in C_2^{\alpha,\alpha+\beta\gamma} W$ with $\alpha + \beta\gamma > 1$, therefore by the sewing lemma we can set

$$\int_0^t A(\mathrm{d}s, x_s) := \mathcal{J}(\Gamma)_t = \lim_{|\Pi| \to 0} \sum_i \Gamma_{t_i, t_{t+1}}.$$

Property 1. then follows from $\mathcal{J}(\Gamma)_{s,t} = \mathcal{J}(\Gamma)_{s,r} + \mathcal{J}(\Gamma)_{r,t}$ and Property 3. from the above estimates on $\|\delta\Gamma\|_{\alpha+\beta\gamma}$. Similarly estimate (1.6) is obtained by the previous estimates applied to $A = A^1 - A^2$. Property 2. follows from the fact that if $\partial_t A$ exists continuous, then necessarily

$$\lim_{|\Pi| \to 0} \sum_i A_{t_i, t_{t+1}}(x_{t_i}) = \int_0^t \partial_t A(u, x_u) \, \mathrm{d}u.$$

It remains to show estimate (1.7). To this end, for fixed $x, y \in C_t^\gamma V$ and R as above, we need to provide estimates for $\|\delta\tilde{\Gamma}\|_{1+\varepsilon}$ for $\tilde{\Gamma}_{s,t} := A_{s,t}(x_s) - A_{s,t}(y_s)$ and suitable $\varepsilon > 0$. It holds

$$\begin{aligned} |\delta\tilde{\Gamma}_{s,u,t}| &\leqslant |A_{u,t}(x_u) - A_{u,t}(x_s)| + |A_{u,t}(y_u) - A_{u,t}(y_s)| \leqslant \|A\|_{\alpha,\beta,R} ([\![x]\!]_\gamma^\beta + [\![y]\!]_\gamma^\beta) |t-s|^{\alpha+\beta\gamma}, \\ |\delta\tilde{\Gamma}_{s,u,t}| &\leqslant |A_{u,t}(x_u) - A_{u,t}(y_u)| + |A_{u,t}(x_s) - A_{u,t}(y_s)| \lesssim \|A\|_{\alpha,\beta,R} \|x-y\|_0^\beta |t-s|^\alpha, \end{aligned}$$

which interpolated together give

$$\|\delta\Gamma\|_{(1-\theta)(\alpha+\beta\gamma)+\theta\alpha} \lesssim \|A\|_{\alpha,\beta,R} (1 + [\![x]\!]_\gamma + [\![y]\!]_\gamma) \|x-y\|_0^{\beta\theta}$$

for any $\theta \in (0,1)$ such that $(1-\theta)(\alpha+\beta\gamma) + \theta\alpha = 1 + \varepsilon > 1$, namely such that

$$\beta\theta < \frac{\alpha + \beta\gamma - 1}{\gamma}.$$

The sewing lemma then implies that

$$\begin{aligned} \left\| \int_s^t A(\mathrm{d}r, x_r) - \int_s^t A(\mathrm{d}r, y_r) \right\|_W &\lesssim_\theta \left\| \int_s^t A(\mathrm{d}r, x_r) - \int_s^t A(\mathrm{d}r, y_r) - \tilde{\Gamma}_{s,t} \right\|_W + \|\tilde{\Gamma}_{s,t}\|_W \\ &\lesssim \|\delta\tilde{\Gamma}\|_{1+\varepsilon} |t-s|^{1+\varepsilon} + \|A\|_{\alpha,\beta,R} |t-s|^\alpha \|x-y\|_0^\beta \\ &\lesssim_{\theta,T} |t-s|^\alpha \|A\|_{\alpha,\beta,R} (1 + \|x\|_\gamma + \|y\|_\gamma) \|x-y\|_0^{\beta\theta}. \end{aligned}$$

Dividing by $|t-s|^\alpha$ and taking the supremum we obtain (1.7). □

Remark 1.7. Several other variants of the nonlinear Young integrals can be constructed, for instance integrals of the form

$$\int_0^\cdot y_s \, A(\mathrm{d}s, x_s)$$

for $y \in C_t^\delta \mathbb{R}$ such that $\alpha + \delta > 1$ and A, x as above. This can be either interpreted as a more classical Young integral of the form $\int_0^\cdot y_t \, \mathrm{d}(\int_0^t A(\mathrm{d}s, x_s)) = \mathcal{J}(\Gamma)$ for $\Gamma_{s,t} = y_s \int_s^t A(\mathrm{d}r, x_r)$, or as the sewing of $\tilde{\Gamma}_{s,t} = y_s \, A_{s,t}(x_s)$, the two definitions being equivalent; see Remark 2.8 from [141].

Nonlinear Young integrals are a generalisation of classical ones, as the next example shows.

Example 1.8. Let $f \in C^\beta(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and $y\in C^\alpha_t \mathbb{R}^m$, then $A(t,x) := f(x) y_t$ is an element of $C^\alpha_t C^\beta_{\mathbb{R}^d}$, since

$$|A_{s,t}(x) - A_{s,t}(y)| = |[f(x) - f(y)] y_{s,t}| \leqslant |f(x) - f(y)|\, |y_{s,t}| \leqslant [\![f]\!]_\beta [\![y]\!]_\alpha |t-s|^\alpha |x-y|^\beta.$$

In particular, for any $x\in C^\gamma_t \mathbb{R}^d$ with $\alpha+\beta\gamma>1$, we can consider $\int_0^\cdot A(\mathrm{d}s, x_s)$; this corresponds to the classical Young integral $\int_0^\cdot f(x_s)\mathrm{d}y_s$, since both are defined as sewings of

$$\Gamma_{s,t} = A_{s,t}(x_s) = f(x_s) y_t - f(x_s) y_s = f(x_s) y_{s,t};$$

the same reasoning holds for infinite sums of Young integrals of the form $\sum_n \int_0^\cdot f^n(x_s)\mathrm{d}y^n_s$.

On the other hand, if f is time-dependent and sufficiently regular, although it is possible to define $\int_0^\cdot f(s,x_s)\mathrm{d}y_s$, this does not necessarily coincide with the nonlinear Young integral associated to $A(t,x) = f(t,x) y_t$; for the exact relations between them, see Remark 2.10 from [141].

1.1.3 Nonlinear Young calculus

Theorem 1.6 establishes continuity of the map $(A,x)\mapsto \int_0^\cdot A(\mathrm{d}s, x_s)$; if A is sufficiently regular, then we can even establish its differentiability.

Proposition 1.9. *Let $\alpha,\beta,\gamma\in(0,1)$ such that $\alpha+\beta\gamma>1$, $A\in C^\alpha_t C^{1+\beta}_{V,W,\mathrm{loc}}$. Then the nonlinear Young integral, seen as a map $F: C^\gamma_t V\to C^\alpha_t W$, $F(x) = \int_0^\cdot A(\mathrm{d}s,x_s)$, is Frechét differentiable with*

$$DF(x)\colon y\mapsto \int_0^\cdot DA(\mathrm{d}s, x_s) y_s. \tag{1.8}$$

Proof. For notational simplicity we will assume $A\in C^\alpha_t C^{1+\beta}_{V,W}$. It is enough to show that, for any $x,y\in C^\gamma_t V$, the Gateaux derivative of F at x in the direction y is given by the expression above, i.e.

$$\lim_{\varepsilon\to 0} \frac{F(x+\varepsilon y) - F(x)}{\varepsilon} = \int_0^\cdot DA(\mathrm{d}s, x_s) y_s \tag{1.9}$$

where the limit is in the $C^\alpha_t W$-topology. Indeed, once this is shown, it follows easily from reasoning as in Theorem 1.6 that the map $(x,y)\mapsto \int DA(\mathrm{d}s,x_s)y_s$ is jointly uniformly continuous in bounded balls and linear in the second variable; Frechét differentiability then follows from existence and continuity of the Gateaux differential.

In order to show (1.9), setting for any $\varepsilon>0$

$$\Gamma^\varepsilon_{s,t} := \frac{A_{s,t}(x_s+\varepsilon y_s) - A_{s,t}(x_s)}{\varepsilon} - DA_{s,t}(x_s) y_s,$$

it suffices to show that $\mathcal{J}(\Gamma^\varepsilon)\to 0$ in $C^\alpha_t W$. In particular, by Lemma A.33 in Appendix A.6, we only need to check that $\|\Gamma^\varepsilon\|_\alpha\to 0$ as $\varepsilon\to 0$, while $\|\delta\Gamma^\varepsilon\|_{\alpha+\beta\gamma}$ stays uniformly bounded. It holds

$$\begin{aligned}\|\Gamma^\varepsilon_{s,t}\|_W &= \left\| \int_0^1 [DA_{s,t}(x_s+\lambda\varepsilon y_s) - DA_{s,t}(x_s)] y_s \,\mathrm{d}\lambda \right\|_W\\ &\leqslant \varepsilon^\beta \|DA_{s,t}\|_\beta \,\|y_s\|_V^{\beta+1} \leqslant \varepsilon^\beta |t-s|^\alpha \,\|A\|_{\alpha,1+\beta}\,\|y\|_\delta^{\beta+1}\end{aligned}$$

which implies that $\|\Gamma^\varepsilon\|_\alpha \lesssim \varepsilon^\beta\to 0$; similar calculations show that

$$\begin{aligned}\|\Gamma^\varepsilon_{s,u,t}\|_W &= \left\| \int_0^1 [DA_{u,t}(x_s+\lambda\varepsilon y_s) - DA_{u,t}(x_s)] y_s\,\mathrm{d}\lambda - \int_0^1 [DA_{u,t}(x_u+\lambda\varepsilon y_u) - DA_{u,t}(x_u)] y_u\,\mathrm{d}\lambda \right\|_W\\ &= \left\| -\int_0^1 [DA_{u,t}(x_s+\lambda\varepsilon y_s) - DA_{u,t}(x_s)] y_{s,u}\,\mathrm{d}\lambda \right.\\ &\quad \left. + \int_0^1 [DA_{u,t}(x_s+\lambda\varepsilon y_s) - DA_{u,t}(x_s) - DA_{u,t}(x_u+\lambda\varepsilon y_u) + DA_{u,t}(x_u)] y_u\,\mathrm{d}\lambda \right\|_W\\ &\lesssim |t-s|^{\alpha+\gamma} \|DA\|_{\alpha,\beta}\,\|y\|_\gamma^{1+\beta} + |t-s|^{\alpha+\beta\gamma}\,\|DA\|_{\alpha,\beta}\|y\|_\gamma([\![x]\!]^\beta_\gamma + [\![y]\!]^\beta_\gamma)\end{aligned}$$

which implies that $\|\delta\Gamma\|_{\alpha+\beta\gamma}\lesssim 1$ uniformly in $\varepsilon>0$. The conclusion the follows. □

Proposition 1.9 provides an alternative proof of Lemma 4.5 from [145].

Corollary 1.10. *Let $\alpha,\beta,\gamma\in(0,1)$ such that $\alpha+\beta\gamma>1$, $A\in C_t^\alpha C_{V,W,\mathrm{loc}}^{1+\beta}$, $x^1,x^2\in C_t^\gamma V$. Then*

$$\int_0^\cdot A(\mathrm{d}s,x_s^1)-\int_0^\cdot A(\mathrm{d}s,x_s^2)=\int_0^\cdot v_{\mathrm{d}s}(x_s^1-x_s^2) \tag{1.10}$$

with v given by

$$v_t:=\int_0^t\int_0^1 DA(\mathrm{d}s,x_s^2+\lambda(x_s^1-x_s^2))\,\mathrm{d}\lambda; \tag{1.11}$$

formula (1.11) meaningfully defines an element of $C_t^\alpha\mathcal{L}(V,W)$, which satisfies

$$[\![v]\!]_\alpha\leqslant C\|DA\|_{\alpha,\beta,R}(1+[\![x^1]\!]_\gamma+[\![x^2]\!]_\gamma) \tag{1.12}$$

where $R\geqslant\|x\|_\infty\vee\|y\|_\infty$ and $C=C(\alpha,\beta,\gamma,T)$.

Proof. It follows from the hypothesis on A that the map

$$y\in V\mapsto\int_0^1\left[\int_0^t DA(\mathrm{d}s,x_s^2+\lambda(x_s^1-x_s^2))y\right]\mathrm{d}\lambda\in W \tag{1.13}$$

is well defined (the outer integral being meaningful in the Bochner sense) and linear in y; moreover estimate (1.5) combined with the trivial inequality $1+[\![x^2+\lambda(x_s^1-x_s^2)]\!]_\gamma^\beta\lesssim 1+[\![x^1]\!]_\gamma+[\![x^2]\!]_\gamma$, valid for any $\lambda,\beta\in[0,1]$, yields

$$\left\|\int_0^1\left[\int_0^t DA(\mathrm{d}s,x^2+\lambda(x_s^1-x_s^2))y\right]\mathrm{d}\lambda\right\|_W\lesssim\|DA\|_{\alpha,\beta,R}\,(1+[\![x^1]\!]_\gamma+[\![x^2]\!]_\gamma)\,\|y\|_V.$$

In particular, if we define v_t as the linear map appearing (1.13), it is easy to check that similar estimates yield $v\in C_t^\alpha\mathcal{L}(V,W)$. The fact that this definition coincide with the one from (1.11), i.e. that we can exchange integration in $\mathrm{d}\lambda$ and in "$\mathrm{d}s$", follows from the Fubini theorem for the sewing map, see Lemma A.32 in Appendix A.6. Inequality (1.12) then follows from estimates analogous to the ones obtained above. Identity (1.10) is an application of the more abstract classical identity

$$F(x^1)-F(x^2)=\left[\int_0^1 DF(x^2+\lambda(x^1-x^2))\mathrm{d}\lambda\right](x^1-x^2)$$

applied to $F(x)=\int_0^\cdot A(\mathrm{d}s,x_s)$, for which the exact expression for DF is given by Proposition 1.9. □

The following Itô-type formula is taken from [176], Theorem 3.4.

Proposition 1.11. *Let $F\in C_t^\alpha C_{V,W,\mathrm{loc}}^\beta$ and $x\in C_t^\gamma V$ with $\alpha+\beta\gamma>1$, then it holds*

$$F(t,x_t)-F(0,x_0)=\int_0^t F(\mathrm{d}s,x_s)+\int_0^t F(s,\mathrm{d}x_s); \tag{1.14}$$

if in addition $F\in C_t^0 C_{V,W,\mathrm{loc}}^{1+\beta'}$ with $\beta'\in(0,1)$ s.t. $\gamma(1+\beta')>1$, then

$$F(t,x_t)-F(0,x_0)=\int_0^t F(\mathrm{d}s,x_s)+\int_0^t DF(s,x_s)(\mathrm{d}x_s). \tag{1.15}$$

In particular, if $x=\int_0^\cdot A(\mathrm{d}s,y_s)$ for some $A\in C_t^\gamma C_V^\delta$, $y\in C_t^\eta V$ with $\gamma+\eta\delta>1$, then (1.15) becomes

$$F(t,x_t)-F(0,x_0)=\int_0^t F(\mathrm{d}s,x_s)+\int_0^t DF(s,x_s)(A(\mathrm{d}s,y_s)). \tag{1.16}$$

Proof. Let $0=t_0<t_1<\cdots<t_n=t$, then it holds

$$\begin{aligned}F(t,x_t)-F(0,x_0)=&\sum_i\,[F_{t_{i+1}}(x_{t_{i+1}})-F_{t_i}(x_{t_i})]\\=&\sum_i\,F_{t_i,t_{i+1}}(x_{t_i})+\sum_i\,[F_{t_i}(x_{t_{i+1}})-F_{t_i}(x_{t_i})]+\sum_i\,R_{t_i,t_{i+1}}=:I_1^n+I_2^n+I_3^n\end{aligned}$$

where $R_{t_i,t_{i+1}}=F_{t_i,t_{i+1}}(x_{t_i+1})-F_{t_i,t_{i+1}}(x_{t_i})$ satisfies

$$\|R_{t_i,t_{i+1}}\|\leqslant\|F\|_{\alpha,\beta,\|x\|_\infty}[\![x]\!]_\gamma^\beta|t_{i+1}-t_i|^{\alpha+\beta\gamma},$$

while I_1^n and I_2^n are Riemann-Stjeltes sums associated to $\Gamma^1_{s,t} = F_{s,t}(x_s)$ and $\Gamma^2_{s,t} = F_s(x_t) - F_s(x_s)$. Taking a sequence of partitions Π_n with $|\Pi_n| \to 0$, by the above estimate we have $I_3^n \to 0$; on the other hand, by the sewing lemma we obtain

$$F(t, x_t) - F(0, x_0) = \mathcal{J}(\Gamma^1)_t + \mathcal{J}(\Gamma^2)_t,$$

which is exactly (1.14). If $F \in C^0_t C^{1+\beta'}_{V,W,\mathrm{loc}}$, then setting $\Gamma^3_{s,t} := DF(s, x_s)(x_{s,t})$, it holds

$$\begin{aligned}\|\Gamma^2_{s,t} - \Gamma^3_{s,t}\|_V &= \|F(s, x_t) - F(s, x_s) - DF(s, x_s)(x_{s,t})\|_V\\ &= \left\| \int_0^1 [DF(s, x_s + \lambda x_{s,t}) - DF(s, x_s)](x_{s,t}) \mathrm{d}\lambda \right\|_V\\ &\lesssim \|DF(s, \cdot)\|_{\beta', \|x\|_\infty} \|x_{s,t}\|^{1+\beta'} \lesssim \|F\|_{0,1+\beta',\|x\|_\infty} [\![x]\!]_\gamma^{\beta'} |t-s|^{\gamma(1+\beta')}\end{aligned}$$

which under the assumption $\gamma(1 + \beta') > 1$ implies by the sewing lemma that $\mathcal{J}(\Gamma^2) = \mathcal{J}(\Gamma^3)$ and thus (1.15). The proof of (1.16) is analogous, only this time consider $\Gamma^4_{s,t} := DF(s, x_s)(A_{s,t}(y_s))$, then it's easy to check that $\|\Gamma^3_{s,t} - \Gamma^4_{s,t}\|_V \lesssim |t-s|^{\gamma+\eta\delta}$ which implies that $\mathcal{J}(\Gamma^3) = \mathcal{J}(\Gamma^4)$. □

The identities from Proposition 1.11 admit further variants, see Remark 2.14 from [141].

1.2 Existence, uniqueness, numerical schemes for YDEs

This section is devoted to the study of nonlinear Young differential equations (YDE for short), defined below; it provides sufficient conditions for existence and uniqueness of solutions, as well as convergence of numerical schemes. Before proceeding further, let us point out that by Example 1.8 any Young differential equation

$$x_t = x_0 + \int f(x_s) \mathrm{d}y_s$$

can be reinterpreted as a nonlinear YDE associated to $A := f \otimes y$. Nonlinear YDEs therefore are a natural extension of the standard ones; most results regarding their existence and uniqueness which will be presented are perfect analogues (in terms of regularity requirements) to the well known classical ones (which can be found for instance in [199] or Section 8 of [132]).

Definition 1.12. *Let $A \in C^\alpha_t C^\beta_{V,\mathrm{loc}}$, $x_s \in V$. We say that x is a solution to the YDE associated to (x_s, A) on an interval $[s,t] \subset [0,T]$ if $x \in C^\gamma([s,t]; V)$ for some γ such that $\alpha + \beta\gamma > 1$ and it satisfies*

$$x_r = x_s + \int_s^r A(\mathrm{d}u, x_u) \quad \forall r \in [s,t]. \tag{1.17}$$

Throughout this section, for $x: [0,T] \to V$ and $I \subset [0,T]$, we set

$$[\![x]\!]_{\gamma;I} := \sup_{\substack{s,t \in I\\ s \neq t}} \frac{\|x_{s,t}\|_V}{|t-s|^\gamma}$$

as well as $[\![x]\!]_{\gamma;s,t}$ in the case $I = [s,t]$; similarly for $\|x\|_{\infty;I}$ and $\|x\|_{\gamma;I}$. For any $\Delta > 0$ we also define

$$[\![x]\!]_{\gamma,\Delta,V} = [\![x]\!]_{\gamma,\Delta} := \sup_{\substack{s,t \in [0,T]\\ |s-t| \in (0,\Delta]}} \frac{\|x_{s,t}\|_V}{|t-s|^\gamma}.$$

1.2.1 Existence

We provide here sufficient conditions for the existence of either local or global solutions to the YDE, under suitable compactness assumptions on A. The proof is based on an Euler scheme in the style of those from [87], [199]; its rate of convergence will be studied later on. Other proofs, based on compactness techniques or Leray–Schauder fixed point theorem, are possible, see [57], [176].

Theorem 1.13. *Let $A \in C_t^\alpha C_{V,W}^\beta$ where W is compactly embedded in V and $\alpha(1+\beta) > 1$. Then for any $s > 0$ and $x_s \in V$ there exists a solution to the YDE*

$$x_t = x_s + \int_s^t A(\mathrm{d}r, x_r) \quad \forall t \in [s, T]. \tag{1.18}$$

Proof. Up to rescaling and shifting, we can assume for simplicity $T = 1$ and $s = 0$.

Fix $n \in \mathbb{N}$, set $t_k^n = k/n$ for $k \in \{0, \ldots, n\}$ and define recursively $(x_k^n)_{k=1}^n$ by $x_0^n = x_0$ and

$$x_{k+1}^n = x_k^n + A_{t_k^n, t_{k+1}^n}(x_k^n).$$

We can embed $(x_k^n)_{k=1}^n$ into $C_t^0 V$ by setting

$$x_t^n := x_0 + \sum_{0 \leqslant k \leqslant \lfloor nt \rfloor} A_{t_k^n, t \wedge t_k^{n+1}}(x_k^n);$$

note that by construction $x^n - x_0$ is a path in $C_t^\alpha W$. Using the identity

$$A_{s,t}(x_s^n) = \int_s^t A(\mathrm{d}r, x_r^n) + \int_s^t [A(\mathrm{d}r, x_s^n) - A(\mathrm{d}r, x_r^n)]$$

we deduce that x^n satisfies a YDE of the form

$$x_t^n = x_0 + \int_0^t A(\mathrm{d}s, x_s^n) + \psi_t^n \tag{1.19}$$

where

$$\psi_t^n = \sum_{0 \leqslant k \leqslant n} \psi_t^{n,k} = \sum_{0 \leqslant k \leqslant n} \int_{t_k^n}^{(t \wedge t_{k+1}^n) \vee t_k^n} [A(\mathrm{d}r, x_{t_k^n}^n) - A(\mathrm{d}r, x_r^n)].$$

By the properties of Young integrals, ψ^n satisfies

$$\|\psi_{t_k^n, t_{k+1}^n}^n\|_W = \left\| \int_{t_k^n}^{t_{k+1}^n} [A(\mathrm{d}r, x_{t_k^n}^n) - A(\mathrm{d}r, x_r^n)] \right\|_W \lesssim n^{-\alpha(1+\beta)} \|A\|_{\alpha,\beta} \llbracket x^n \rrbracket_{\alpha, 1/n, V}^\beta. \tag{1.20}$$

We first want to obtain a bound for $\llbracket \psi^n \rrbracket_{\gamma, \Delta, W}$; we can assume wlog $\Delta > 1/n$, since we want to take $n \to \infty$. Estimates depend on whether s and t lie on the same interval $[t_k^n, t_{k+1}^n]$ or not; assume first this is the case, then

$$\begin{aligned}
\|\psi_{s,t}^n\|_W = & \left\| \int_s^t [A(\mathrm{d}r, x_{t_k^n}^n) - A(\mathrm{d}r, x_r^n)] \right\|_W \\
\lesssim & \|A_{s,t}(x_{t_k^n}^n) - A_{s,t}(x_s^n)\|_W + |t - s|^{\alpha(1+\beta)} \|A\|_{\alpha,\beta} \llbracket x^n \rrbracket_{\alpha, \Delta, V}^\beta \\
\lesssim & n^{-\alpha\beta} |t - s|^\alpha \|A\|_{\alpha,\beta} \llbracket x^n \rrbracket_{\alpha, \Delta, V}^\beta.
\end{aligned}$$

Next, given $s < t$ such that $|t - s| < \Delta$ which are not in the same interval, there are around $n|t - s|$ intervals separating them, i.e. there exist $l < m$ such that $m - l \sim n|t - s|$ and $s \leqslant t_l^n < \cdots < t_m^n \leqslant t$. Therefore in this case we have

$$\begin{aligned}
\|\psi_{s,t}^n\|_W \leqslant & \|\psi_{s, t_l^n}^n\|_W + \sum_{k=l}^{m-1} \|\psi_{t_k^n, t_{k+1}^n}^n\|_W + \|\psi_{t_m^n, t}^n\|_W \\
\lesssim & \|A\|_{\alpha,\beta} \llbracket x^n \rrbracket_{\alpha, \Delta, V}^\beta [|t - s|^\alpha n^{-\alpha\beta} + (m - l) n^{-\alpha(1+\beta)}] \\
\lesssim & \|A\|_{\alpha,\beta} \llbracket x^n \rrbracket_{\alpha, \Delta, V}^\beta [|t - s|^\alpha n^{-\alpha\beta} + |t - s| \, n^{1-\alpha(1+\beta)}] \\
\lesssim & \|A\|_{\alpha,\beta} \llbracket x^n \rrbracket_{\alpha, \Delta, V}^\beta |t - s|^\alpha n^{1-\alpha(1+\beta)}
\end{aligned}$$

where in the second line we used both (1.20) and the previous bound for $\psi_{s, t_l^n}^n$ and $\psi_{t_m^n, t}^n$, while in the last one the fact that $-\alpha\beta \leqslant 1 - \alpha(1+\beta)$. Overall we conclude that

$$\llbracket \psi^n \rrbracket_{\alpha, \Delta, W} \leqslant \kappa_1 \, n^{1-\alpha(1+\beta)} \|A\|_{\alpha,\beta} \llbracket x^n \rrbracket_{\alpha, \Delta, V}^\beta \tag{1.21}$$

for a suitable constant $\kappa_1 = \kappa_1(\alpha, \beta)$ independent of Δ and n.

Our next goal is a uniform bound for $[\![x^n]\!]_{\alpha,\Delta,W}$. Since x^n solves (1.19), it holds

$$\begin{aligned}\|x^n_{s,t}\|_W &\lesssim \|A_{s,t}(x^n_s)\|_W + |t-s|^{\alpha(1+\beta)}\|A\|_{\alpha,\beta}\,[\![x^n]\!]^\beta_{\alpha,\Delta,W} + \|\psi^n_{s,t}\|_W\\ &\lesssim |t-s|^\alpha\|A\|_{\alpha,\beta} + |t-s|^\alpha\,\Delta^{\alpha\beta}\|A\|_{\alpha,\beta}[\![x^n]\!]^\beta_{\alpha,\Delta,W} + |t-s|^\alpha[\![\psi^n]\!]_{\alpha,\Delta,W}\\ &\lesssim |t-s|^\alpha\|A\|_{\alpha,\beta} + |t-s|^\alpha\|A\|_{\alpha,\beta}[\![x^n]\!]^\beta_{\alpha,\Delta,W}(\Delta^{\alpha\beta} + n^{1-\alpha(1+\beta)}).\end{aligned}$$

Let us divide both sides by $|t-s|$, take the supremum over all $|t-s|<\Delta$ and choose Δ such that $\Delta^{\alpha\beta}\,\|A\|_{\alpha,\beta}\leqslant 1/4$; then for all n large enough, so that $n^{1-\alpha(1+\beta)}\|A\|_{\alpha,\beta}\leqslant 1/4$, it holds

$$[\![x^n]\!]_{\alpha,\Delta,W} \lesssim \|A\|_{\alpha,\beta} + \frac{1}{2}[\![x^n]\!]^\beta_{\alpha,\Delta,W} \lesssim \|A\|_{\alpha,\beta} + \frac{1}{2} + \frac{1}{2}[\![x^n]\!]_{\alpha,\Delta,W};$$

we used the trivial bound $a^\beta\leqslant 1+a$, which holds for all $\beta\in[0,1]$ and $a\geqslant 0$. Overall this implies the uniform bound $[\![x^n]\!]_{\alpha,\Delta,W}\lesssim 1+\|A\|_{\alpha,\beta}$ for all n big enough.

The subspace $\{y\in C^\alpha([0,1];W): y_0=0\}$ is a Banach space endowed with the seminorm $[\![y]\!]_{\alpha,\Delta,W}$, which in this case is equivalent to the norm $\|y\|_{\alpha,W}$; $\{x_n-x_0\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence in this space. By Ascoli–Arzelà, since W compactly embeds in V, we can extract a subsequence (not relabelled for simplicity) such that $x_n-x_0\to x-x_0$ in $C^{\alpha-\varepsilon}_t V$ for any $\varepsilon>0$, for some $x\in C^\alpha_t V$ such that $x(0)=x_0$. Observe that ψ^n satisfy (1.21) and $[\![x^n]\!]^\beta_{\alpha,\Delta,V}$ are uniformly bounded, therefore $\psi^n\to 0$ in $C^\alpha_t W$ as $n\to\infty$; choosing ε small enough s.t. $\alpha+\beta(\alpha-\varepsilon)>1$, by continuity of the nonlinear Young integral it holds

$$\int_0^\cdot A(\mathrm{d}s, x^n_s)\to\int_0^\cdot A(\mathrm{d}s,x_s)\quad\text{in } C^\alpha_t W.$$

Passing to the limit in (1.19) we obtain the conclusion. □

Remark 1.14. If V is finite dimensional, the compactness condition is trivially satisfied by taking $V=W$. The proof also works for non uniform partitions Π_n of $[0,T]$, under the condition that their mesh $|\Pi_n|\to 0$ and that there exists $c>0$ such that $|t^n_{i+1}-t^n_i|\geqslant c|\Pi_n|$ for all $n\in\mathbb{N}$, $i\in\{0,\dots,N_n\}$.

Remark 1.15. The proof provides several estimates, some of which are true even without the compactness assumption. For instance, by $[\![x^n]\!]_{\alpha,\Delta}\lesssim 1+\|A\|_{\alpha,\beta}$ and Exercise 4.24 from [132], choosing Δ satisfying $\Delta^{\alpha\beta}\|A\|_{\alpha,\beta}\sim 1$, we deduce that there exists $C_1=C_1(\alpha,\beta,T)$ such that

$$[\![x^n]\!]_\alpha\leqslant C_1\left(1+\|A\|^{1+\frac{1-\alpha}{\alpha\beta}}_{\alpha,\beta}\right)\quad\forall n\in\mathbb{N}.$$

Estimate (1.21) is true for any choice of $\Delta>0$, in particular for $\Delta=T$, which gives a global bound; combining it with the above one, we deduce that

$$[\![\psi^n]\!]_\alpha\leqslant C_2\,n^{1-\alpha(1+\beta)}\left(1+\|A\|^{\frac{1+\alpha\beta}{\alpha}}_{\alpha,\beta}\right)\quad\forall n\in\mathbb{N}$$

for some $C_2=C_2(\alpha,\beta,T)$. Also observe that from the assumptions on α and β it always holds

$$1+\frac{1-\alpha}{\alpha\beta}\leqslant 2,\qquad \frac{1+\alpha\beta}{\alpha}\leqslant 3.$$

Under the compactness assumption, since $x^n\to x$ in $C^0_t V$, the solution x obtained also satisfies

$$[\![x]\!]_\alpha\leqslant\liminf_{n\to\infty}[\![x^n]\!]_\alpha\leqslant C_1\left(1+\|A\|^{1+\frac{1-\alpha}{\alpha\beta}}_{\alpha,\beta}\right)\leqslant 2C_1(1+\|A\|^2_{\alpha,\beta}).\tag{1.22}$$

Finally observe that by going through the same proof of (1.21), for any $T>0$ and α,β,γ such that $\alpha+\beta\gamma>1$, there exists $C_3=C_3(\alpha,\beta,\gamma,T)$ such that

$$[\![\psi^n]\!]_{\alpha,\Delta,V}\leqslant C_3\,n^{1-\alpha-\beta\gamma}\|A\|_{\alpha,\beta}\,[\![x^n]\!]^\beta_{\gamma,\Delta,V}\quad\forall n\in\mathbb{N}.\tag{1.23}$$

This estimate is rather useful when A enjoys different space-time regularity at different scales, like in Section 1.4.3.

Corollary 1.16. *Let $A \in C_t^\alpha C_{V,W,\text{loc}}^\beta$ where W is compactly embedded in V and $\alpha(1+\beta) > 1$. Then for any $s \in [0,T)$ and any $x_s \in V$, there exists $\tau^* \in (s,T]$ and a solution to the YDE (1.18) defined on $[s,T^*)$, with the property that either $T^* = T$ or*

$$\lim_{t \uparrow T^*} \|x_t\|_V = +\infty.$$

Proof. The full proof is based on classical localization arguments and iterations techniques and can be found in [141]. Here we only present a simple a priori estimate on the maximal time T^* of existence; as before it is enough to treat the case $s = 0, T = 1$.

Fix $R > 0$ and consider a localization of A, namely a drift $A^R \in C_t^\alpha C_{V,W}^\beta$ s.t. $A^R(t,x) = A(t,x)$ for any (t,x) with $\|x\|_V \leqslant 2R$ and $A^R(t,x) \equiv 0$ for $\|x\|_V \geqslant 3R$; let $C_R := C(1 + \|A\|_{\alpha,\beta,3R}^2)$, where C is the constant appearing in (1.22). Then for any $x_0 \in V$ with $\|x_0\| \leqslant R$, by Theorem 1.13 there exists a solution x to the YDE associated to (x_0, A^R) on the interval $[0,1]$; setting $\tau := \inf\{t \in [0,1]: \|x_t\|_V \geqslant 2R\}$, by (1.22) it holds $[\![x]\!]_{\alpha;[0,\tau]} \leqslant C_R$, and so

$$2R = \|x_\tau\|_V \leqslant \|x_0\|_V + \tau^\alpha [\![x]\!]_{\alpha;[0,\tau]} \leqslant R + \tau^\alpha C_R$$

which implies

$$\tau \geqslant \left(\frac{C_R}{R}\right)^{-\alpha}. \tag{1.24}$$

In particular, since $A = A^R$ on $[0,T] \times B_{2R}$, we conclude that $x.$ is also a solution to the YDE associated to (x_0, A) on the interval $[0,\tau]$ and that $T^* \geqslant \tau$. □

1.2.2 A priori estimates

A classical way to pass from local to global solutions is to establish suitable a priori estimates, which are also of fundamental importance for compactness arguments. Throughout this section, we assume that a solution x to the YDE is already given and focus exclusively on obtainig bounds on it; for simplicity we work on $[0,T]$, but all the statements immediately generalise to $[s,T]$.

Proposition 1.17. *Let $\alpha > 1/2$, $\beta \in (0,1)$ such that $\alpha(1+\beta) > 1$, $A \in C_t^\alpha C_V^\beta$, $x_0 \in V$ and $x \in C_t^\alpha V$ be a solution to the associated YDE. Then there exists $C = C(\alpha,\beta,T)$ such that*

$$[\![x]\!]_\alpha \leqslant C(1 + \|A\|_{\alpha,\beta}^2), \qquad \|x\|_\alpha \leqslant C\,(1 + \|x_0\|_V + \|A\|_{\alpha,\beta}^2). \tag{1.25}$$

We omit the proof of Proposition 1.17, for which we refer to [141], as it is very similar (actually simpler) to the one of Proposition 1.18 below; observe that estimate (1.25) is in perfect agreement with (1.22).

The assumption of a global bound on A of the form $A \in C_t^\alpha C_V^\beta$ is sometimes too strong for practical applications. It can be relaxed to suitable growth conditions, as the next result shows; it is based on Theorem 3.1 from [176], see also Theorem 2.9 from [57].

Proposition 1.18. *Let $A \in C_t^\alpha C_V^{\beta,\lambda}$ with $\alpha(1+\beta) > 1$, $\beta + \lambda \leqslant 1$. Then there exists a constant $C = C(\alpha,\beta,T)$ such that any solution x on $[0,T]$ to the YDE associated to (x_0, A) satisfies*

$$\|x\|_\alpha \leqslant C \exp\!\left(\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1 + \|x_0\|_V). \tag{1.26}$$

Proof. Fix an interval $[s,t] \subset [0,T]$, set $R = \|x\|_{\infty;s,t}$. Since x is a solution, for any $[u,r] \subset [s,t]$ it holds

$$\begin{aligned}
\|x_{u,r}\|_V &\lesssim \|A_{u,r}(x_u)\|_V + |r-u|^{\alpha(1+\beta)} [\![A]\!]_{\alpha,\beta,R} [\![x]\!]_{\alpha;s,t}^\beta \\
&\lesssim \|A_{u,r}(x_u) - A_{u,r}(x_s)\|_V + |r-u|^\alpha \|A\|_{\alpha,\beta,\lambda} (1 + \|x_s\|_V) \\
&\quad + |r-u|^\alpha |t-s|^{\alpha\beta} \|A\|_{\alpha,\beta,\lambda}\,(1 + \|x\|_{\infty;s,t}^\lambda) [\![x]\!]_{\alpha;s,t}^\beta \\
&\lesssim |r-u|^\alpha \|A\|_{\alpha,\beta,\lambda} [1 + \|x_s\|_V + |t-s|^{\alpha\beta} (1 + \|x\|_{\infty;s,t}^\lambda) [\![x]\!]_{\alpha;s,t}^\beta]
\end{aligned}$$

which implies, dividing by $|r-u|^\alpha$ and taking the supremum, that

$$[\![x]\!]_{\alpha;s,t} \lesssim \|A\|_{\alpha,\beta,\lambda} (1 + \|x_s\|_V) + |t-s|^{\alpha\beta} \|A\|_{\alpha,\beta,\lambda}\,(1 + \|x\|_{\infty;s,t}^\lambda) [\![x]\!]_{\alpha;s,t}^\beta.$$

By Young's inequality, for any $a, b \geqslant 0$ it holds $a^{\lambda} b^{\beta} \leqslant a^{\beta+\lambda} + b^{\beta+\lambda}$; moreover $\beta + \lambda \leqslant 1$ so that $a^{\beta+\lambda} \leqslant 1 + a$. Therefore we obtain

$$\begin{aligned}
[\![x]\!]_{\alpha;s,t} &\lesssim \|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V) + |t-s|^{\alpha\beta}\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_{\infty;s,t}+[\![x]\!]_{\alpha;s,t})\\
&\lesssim \|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V) + \|A\|_{\alpha,\beta,\lambda}|t-s|^{\alpha\beta}[\![x]\!]_{\alpha;s,t}
\end{aligned}$$

where in the second passage we used the estimate $\|x\|_{\infty;s,t} \lesssim_T \|x_s\|_V + [\![x]\!]_{\alpha;s,t}$. Overall we deduce the existence of a constant $\kappa_1 = \kappa_1(\alpha, \beta, T)$ such that

$$[\![x]\!]_{\alpha;s,t} \leqslant \frac{\kappa_1}{2}\|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V) + \frac{\kappa_1}{2}\|A\|_{\alpha,\beta,\lambda}|t-s|^{\alpha\beta}[\![x]\!]_{\alpha;s,t}.$$

Choosing $[s,t]$ such that $|t-s| = \Delta$ satisfies $\kappa_1\|A\|_{\alpha,\beta,\lambda}\Delta^{\alpha\beta} \leqslant 1$, we obtain

$$[\![x]\!]_{\alpha;s,t} \leqslant \kappa_1\|A\|_{\alpha,\beta,\lambda}(1+\|x_s\|_V). \tag{1.27}$$

If T satisfies $\kappa_1\|A\|_{\alpha,\beta,\lambda}T^{\alpha\beta} \leqslant 1$, then we can take $s=0, t=T, \Delta=T$, which gives a global estimate and thus the conclusion. If this is not the case, we can choose $\Delta < T$ s.t. $\kappa_1\|A\|_{\alpha,\beta,\lambda}\Delta^{\alpha\beta} = 1$ and then (1.27) implies that

$$[\![x]\!]_{\alpha,\Delta} \leqslant \kappa_1\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_\infty) \tag{1.28}$$

as well as

$$[\![x]\!]_{\alpha} \lesssim \Delta^{\alpha-1}[\![x]\!]_{\alpha,\Delta} \lesssim \|A\|_{\alpha,\beta,\lambda}^{\frac{1-\alpha}{\alpha\beta}}\|A\|_{\alpha,\beta,\lambda}(1+\|x\|_\infty).$$

Therefore

$$[\![x]\!]_{\alpha} \leqslant \kappa_2\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}(1+\|x\|_\infty)$$

where again $\kappa_2 = \kappa_2(\alpha, \beta, T)$. In particular, in order to obtain the final estimate, we only need to focus on $\|x\|_\infty$. Let us consider, for Δ as above, the intervals $I_n := [(n-1)\Delta, n\Delta]$ and set $J_n := 1 + \|x\|_{\infty;I_n}$, with the convention $J_0 = 1 + \|x_0\|_V$. Then estimates analogue to (1.27) yield

$$\begin{aligned}
J_n &\leqslant 1 + \|x_{(n-1)\Delta}\|_V + \Delta^\alpha[\![x]\!]_{\alpha;I_n}\\
&\leqslant (1+\kappa_1\Delta^\alpha\|A\|_{\alpha,\beta,\lambda})(1+\|x_{(n-1)\Delta}\|_V)\\
&\leqslant (1+\kappa_1\Delta^\alpha\|A\|_{\alpha,\beta,\lambda})\, J_{n-1}
\end{aligned}$$

which iteratively implies

$$J_n \leqslant [1+\kappa_1\Delta^\alpha\|A\|_{\alpha,\beta,\lambda}]^n J_0 \leqslant \exp(\kappa_1\, n\,\Delta^\alpha\|A\|_{\alpha,\beta,\lambda})\,(1+\|x_0\|_V),$$

where we used the basic inequality $1 + x \leqslant e^x$. Since $[0,T]$ is covered by $N \sim T\Delta^{-1}$ intervals and we chose $\Delta^{-1} \sim \|A\|^{1/\alpha\beta}$, up to relabelling κ_1 into a new constant κ_3 we obtain

$$1 + \|x\|_\infty = \sup_{n\leqslant N} J_n \leqslant \exp\left(\kappa_3\,\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1+\|x_0\|_V).$$

Finally, combining this with the estimate for $[\![x]\!]_\alpha$ above we obtain

$$\begin{aligned}
[\![x]\!]_{\alpha} &\leqslant \kappa_2\,\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\exp\left(\kappa_3\,\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1+\|x_0\|_V)\\
&\leqslant \kappa_4\exp\left(\kappa_4\|A\|_{\alpha,\beta,\lambda}^{1+\frac{1-\alpha}{\alpha\beta}}\right)(1+\|x_0\|_V)
\end{aligned}$$

where we used the inequality $x e^{\lambda x} \leqslant \lambda^{-1} e^{2\lambda x}$. The conclusion follows. □

Remark 1.19. Since $\alpha(1+\beta) > 1$, it holds $1 + \|A\|_{\alpha,\beta,\lambda}^{1+(1-\alpha)/(\alpha\beta)} \lesssim 1 + \|A\|_{\alpha,\beta,\lambda}^2$ and so

$$\|x\|_\alpha \leqslant C\exp(C\|A\|_{\alpha,\beta,\lambda}^2)(1+\|x_0\|_V) \tag{1.29}$$

up to relabelling constant $C = C(\alpha, \beta, T)$. As in several other estimates appearing later, the dependence of C on T can be established by a rescaling argument, reducing the equation to an equivalent one defined on $[0,1]$.

In classical ODEs, a key role in establishing a priori estimates (as well as uniqueness) is played by Gronwall's lemma; the following result can be regarded as a suitable replacement in the Young setting. One of the main cases of applicability is for linear equations, i.e. $A \in C_t^\alpha \mathcal{L}(V)$.

In the next statement (and the rest of the chapter more in general), we will say that $f \in \mathrm{Lip}_V$ if it is a globally Lipschitz map, with bounded seminorm

$$[\![f]\!]_{\mathrm{Lip}} := [\![f]\!]_1 = \sup_{x \neq y \in V} \frac{\|f(x) - f(y)\|_V}{\|x - y\|_V};$$

similarly we will write $f \in C_t^\alpha \mathrm{Lip}_V$ to denote time-dependent field $f\colon [0,T] \times V \to V$ such that

$$[\![f]\!]_{\alpha,1} := \sup_{s \neq t \in [0,T]} \frac{\|f_t - f_s\|_1}{|t - s|^\alpha} < \infty.$$

Theorem 1.20. *Let $\alpha > 1/2$, $A \in C_t^\alpha \mathrm{Lip}_V$ such that $A(t,0) = 0$ for all $t \in [0,T]$ and $h \in C_t^\alpha V$. Then there exists a constant $C = C(\alpha)$ such that any solution x to the YDE*

$$x_t = x_0 + \int_0^t A(\mathrm{d}s, x_s) + h_t \tag{1.30}$$

satisfies the a priori bounds

$$[\![x]\!]_\alpha \leqslant C([\![A]\!]_{\alpha,1} \|x\|_\infty + [\![h]\!]_\alpha); \tag{1.31}$$

$$\|x\|_\infty \leqslant C \exp\big(CT\, [\![A]\!]_{\alpha,1}^{1/\alpha}\big)\, (\|x_0 + h_0\|_V + T^\alpha [\![h]\!]_\alpha); \tag{1.32}$$

$$\|x\|_\alpha \leqslant C \exp(CT(1 + [\![A]\!]_{\alpha,1}^2))[\|x_0 + h_0\|_V + (1 + T^\alpha)[\![h]\!]_\alpha]. \tag{1.33}$$

Proof. We can assume without loss of generality that $T = 1$, as the general case follows by rescaling. It is also clear that, up to changing constant C, inequality (1.33) follows from combining together (1.31) and (1.32) and using the fact that $[\![A]\!]_{\alpha,1}^{1/\alpha} \lesssim 1 + [\![A]\!]_{\alpha,1}^2$ since $\alpha > 1/2$. Up to renaming x_0, we can also assume $h_0 = 0$. The proof is similar to that of Proposition 1.18, but we provide it for the sake of completeness.

Let $\Delta > 0$ to be chosen later, $s < t$ such that $|t - s| \leqslant \Delta$, then by (1.30) it holds

$$\begin{aligned}
\|x_{s,t}\|_V &\leqslant \left\| \int_s^t A(\mathrm{d}u, x_u) \right\|_V + \|h_{s,t}\|_V \\
&\leqslant \|A_{s,t}\,(x_s)\|_V + \kappa_1 |t - s|^{2\alpha} [\![A]\!]_{\alpha,1}\, [\![x]\!]_{\alpha,\Delta} + |t - s|^\alpha [\![h]\!]_\alpha \\
&\leqslant |t - s|^\alpha ([\![A]\!]_{\alpha,1} \|x\|_\infty + [\![h]\!]_\alpha + \kappa_1 \Delta^\alpha [\![A]\!]_{\alpha,1} [\![x]\!]_{\alpha,\Delta})
\end{aligned}$$

and so dividing both sides by $|t - s|^\alpha$, taking the supremum over s, t and choosing Δ such that $\kappa_1\, \Delta^\alpha [\![A]\!]_{\alpha,1} \leqslant 1/2$ we obtain

$$[\![x]\!]_{\alpha,\Delta} \leqslant 2\, ([\![A]\!]_{\alpha,1} \|x\|_\infty + [\![h]\!]_\alpha). \tag{1.34}$$

As usual, if $\kappa_1\, [\![A]\!]_{\alpha,1} \leqslant 1/2$, then the conclusion follows from (1.34) with the choice $\Delta = 1$ and the trivial estimate $\|x\|_\infty \leqslant \|x_0\|_V + [\![x]\!]_\alpha$. Suppose instead the opposite, choose $\Delta < 1$ such that $\kappa_1\, \Delta^\alpha [\![A]\!]_{\alpha,1} = 1/2$; define $I_n = [(n-1)\Delta, n\Delta]$, $J_n = \|x\|_{\infty; I_n}$, then estimates similar to the ones done above show that

$$\begin{aligned}
J_{n+1} &\leqslant \|x_{n\Delta}\|_V + \Delta^\alpha\, [\![x]\!]_{\alpha; I_n} \\
&\leqslant \|x_{n\Delta}\|_V\, (1 + 2\Delta^\alpha [\![A]\!]_{\alpha,1}) + 2 [\![h]\!]_\alpha \\
&\lesssim J_n + [\![h]\!]_\alpha
\end{aligned}$$

which implies recursively that for a suitable constant κ_2 it holds $J_n \lesssim e^{\kappa_2 n} (\|x_0\|_V + [\![h]\!]_\alpha)$. Since $n \sim \Delta^{-1} \sim [\![A]\!]_{\alpha,1}^{1/\alpha}$ we deduce that

$$\|x\|_\infty = \sup_n J_n \lesssim \exp\big(\kappa_3\, [\![A]\!]_{\alpha,1}^{1/\alpha}\big) (\|x_0\|_V + [\![h]\!]_\alpha)$$

which gives (1.32); combined with $\Delta^{-\alpha} \sim [\![A]\!]_{\alpha,1}$, estimate (1.34) and the basic inequality

$$[\![x]\!]_\alpha \lesssim \Delta^{-\alpha} \|x\|_\infty + [\![x]\!]_{\alpha,\Delta}$$

it also yields estimate (1.31). $\square$

Another way to establish that solutions don't blow-up in finite time is to the show that the YDE admits (coercive) invariants. The next lemma gives simple conditions to establish their existence.

Lemma 1.21. *Let $A \in C_t^\alpha C_V^\beta$ with $\alpha(1+\beta)>1$, $x \in C_t^\alpha V$ be a solution to the YDE associated to (x_0, A) and assume $F \in C^2(V;\mathbb{R})$ is such that*

$$DF(z)(A_{s,t}(z)) = 0 \quad \forall z \in V, 0 \leqslant s \leqslant t \leqslant T.$$

Then F is constant along x, i.e. $F(x_t) = F(x_0)$ for all $t \in [0,T]$.

Proof. It follows immediately from the Itô-type formula (1.16), since it holds

$$F(x_t) - F(x_0) = \int_0^t DF(x_s)(A(\mathrm{d}s, x_s)) = \mathcal{J}(\Gamma)$$

for the choice $\Gamma_{s,t} = DF(x_s)(A_{s,t}(x_s)) \equiv 0$ by hypothesis. □

Remark 1.22. If V is an Hilbert space with $\|z\|_V^2 = \langle z, z\rangle_V$, then $\|\cdot\|_V$ is constant along solutions of the YDE under the condition $\langle z, A_{s,t}(z)\rangle_V = 0$ for all $z \in V$ and $s \leqslant t$. In this case, blow up cannot occurr, thus under the hypothesis of Corollary 1.16, global existence of solutions holds. Similarly, if in addition $A \in C_t^\alpha C_{V,\mathrm{loc}}^{1+\beta}$, then by Corollary 1.24 below, global existence and uniqueness holds.

1.2.3 Uniqueness

We now turn to sufficient conditions for uniqueness of solutions; some of the results below also establish existence under different sets of assumptions than those from Section 1.2.1.

Theorem 1.23. *Let $A \in C_t^\alpha C_V^{1+\beta}$, $\alpha(1+\beta)>1$. Then for any $x_0 \in V$ there exists a unique global solution to the YDE associated to (x_0, A).*

Proof. The proof is based on an application of Banach fixed point theorem. Let M, τ be positive parameters to be fixed later and set

$$E := \{x \in C^\alpha([0,\tau]; V) : x(0) = x_0, \llbracket x \rrbracket_\alpha \leqslant M\},$$

which is complete metric space with the metric $d(x,y) = \llbracket x-y \rrbracket_\alpha$; define the map $\mathcal{I}$ by

$$x \mapsto \mathcal{I}(x)_\cdot = x_0 + \int_0^\cdot A(\mathrm{d}s, x_s).$$

We want to show that $\mathcal{I}$ is a contraction from E to itself, for suitable choice of M and τ. It holds

$$\begin{aligned}
\|\mathcal{I}(x)_{s,t}\|_V &\leqslant \|A_{s,t}(x_s)\|_V + \kappa_1 \llbracket A \rrbracket_{\alpha,1} \llbracket x \rrbracket_\alpha \, |t-s|^{2\alpha}\\
&\leqslant \|A_{s,t}(x_s) - A_{s,t}(x_0)\|_V + \|A_{s,t}(x_0)\|_V + \kappa_1 \llbracket A \rrbracket_{\alpha,1} \llbracket x \rrbracket_\alpha \, |t-s|^{2\alpha}\\
&\leqslant \|A\|_{\alpha,1} \llbracket x \rrbracket_\alpha \, s^\alpha |t-s|^\alpha + \|A\|_{\alpha,1} |t-s|^\alpha + \kappa_1 \llbracket A \rrbracket_{\alpha,1} \llbracket x \rrbracket_\alpha \, |t-s|^{2\alpha}\\
&\leqslant \tau^\alpha (1+\kappa_1) \|A\|_{\alpha,1} \llbracket x \rrbracket_\alpha \, |t-s|^\alpha + \|A\|_{\alpha,1} |t-s|^\alpha.
\end{aligned}$$

Choosing τ and M such that

$$\tau^\alpha (1+\kappa_1) \|A\|_{\alpha,1} \leqslant \frac{1}{2}, \quad M \geqslant 2 \|A\|_{\alpha,1},$$

for any $x \in V$ it holds

$$\llbracket \mathcal{I}(x) \rrbracket_\alpha \leqslant \tau^\alpha \, \|A\|_{\alpha,1} (1+\kappa_1) \llbracket x \rrbracket_\alpha + \|A\|_{\alpha,1} \leqslant M/2 + M/2 \leqslant M$$

which shows that $\mathcal{I}$ maps E into itself.

By the hypothesis and Corollary 1.10, for any $x, y \in V$ it holds

$$\begin{aligned}
\|\mathcal{I}(x)_{s,t} - \mathcal{I}(y)_{s,t}\|_V &= \left\| \int_s^t v_{\mathrm{d}u}(x_u - y_u) \right\|_V\\
&\leqslant \|v_{s,t}(x_s - y_s)\|_V + \kappa_1 \llbracket v \rrbracket_\alpha \llbracket x-y \rrbracket_\alpha |t-s|^{2\alpha}\\
&\leqslant \llbracket v \rrbracket_\alpha \llbracket x-y \rrbracket_\alpha (s^\alpha + \kappa_1 |t-s|^\alpha) |t-s|^\alpha\\
&\leqslant \kappa_2 \|A\|_{\alpha,1+\beta} (1 + \llbracket x \rrbracket_\alpha + \llbracket y \rrbracket_\alpha) \llbracket x-y \rrbracket_\alpha \tau^\alpha |t-s|^\alpha,
\end{aligned}$$

which implies

$$[\![\mathcal{I}(x)-\mathcal{I}(y)]\!]_\alpha \leqslant \kappa_2 \|A\|_{\alpha,1+\beta}(1+2M)\tau^\alpha [\![x-y]\!]_\alpha < [\![x-y]\!]_\alpha$$

as soon as we choose τ such that $\kappa_2\|A\|_{\alpha,1+\beta}(1+2M)\tau^\alpha<1$. Therefore in this case $\mathcal{I}$ is a contraction from E to itself; for any $x_0\in V$ there exists a unique solution $x\in C^\alpha([0,\tau];V)$ starting from x_0. The same procedure allows to show existence and uniqueness of solutions $x\in C^\alpha([s,s+\tau]\cap[0,T];V)$ for any $s\in[0,T]$ and any $x_s\in V$, where τ does not depend on (s,x_s); by iteration, global existence and uniqueness follows. □

By applying classical localization arguments, we immediately deduce the following local well-posedness result (see [141] for a more detailed proof).

Corollary 1.24. *Let $A\in C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$, $\alpha(1+\beta)>1$. Then for any $x_0\in V$ there exists a unique maximal solution x to the YDE associated to (x_0,A), defined on $[0,T^*)\subset[0,T]$, such that either $T^*=T$ or*

$$\lim_{t\to T^*}\|x_t\|_V=+\infty.$$

In particular if $A\in C^\alpha_t C^{\beta,\lambda}_V\cap C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$ with $\alpha(1+\beta)>1$, $\beta+\lambda\leqslant 1$, then global existence and uniqueness holds.

Once existence of solutions is established, their uniqueness can be alternatively shown by obtaining more general stability estimates, i.e. comparing solutions to YDEs driven by different drifts; such results were obtained in [57] and later revisited in [145].

Theorem 1.25. *Let $R,M>0$ fixed. For $i=1,2$, let $x^i_0\in V$ such that $\|x^i_0\|_V\leqslant R$, $A^i\in C^\alpha_t C^{\beta,\lambda}_V$ with $\alpha(1+\beta)>1$, $\beta+\lambda\leqslant 1$ and $\|A^i\|_{\alpha,\beta,\lambda}\leqslant M$, as well as $A^1\in C^\alpha_t C^{1+\beta,\lambda}_V$ with $\|A^1\|_{\alpha,1+\beta,\lambda}\leqslant M$; let x^i be two given solutions associated respectively to (x^i_0,A^i). Then it holds*

$$[\![x^1-x^2]\!]_\alpha \leqslant C(\|x^1_0-x^2_0\|_V+\|A^1-A^2\|_{\alpha,\beta,\lambda})$$

for a constant $C=C(\alpha,\beta,T,R,M)$ increasing in the last two variables.

Proof. Let x^i be the two given solutions and set $e_t:=x^1_t-x^2_t$, then e satisfies

$$\begin{aligned}
e_t &= e_0+\int_0^t A^1(\mathrm{d}s,x^1_s)-\int_0^t A^2(\mathrm{d}s,x^2_s)\\
&= e_0+\int_0^t A^1(\mathrm{d}s,x^1_s)-\int_0^t A^1(\mathrm{d}s,x^2_s)+\int_0^t (A^1-A^2)(\mathrm{d}s,x^2_s)\\
&= e_0+\int_0^t v_{\mathrm{d}s}(e_s)+\psi_t
\end{aligned}$$

for the choice

$$v_t := \int_0^t\int_0^1 DA^1(\mathrm{d}s,x^2_s+\lambda(x^1_s-x^2_s))\mathrm{d}\lambda, \qquad \psi_t:=\int_0^t (A^1-A^2)(\mathrm{d}s,x^2_s)$$

where we applied Corollary 1.10. By the same result, combined with estimate (1.29), it holds

$$\begin{aligned}
[\![v]\!]_{\alpha,1} &\leqslant \kappa_1\|DA^1\|_{\alpha,\beta,\lambda}(1+\|x^1\|_\alpha+\|x^2\|_\alpha)\\
&\leqslant \kappa_2\exp(\kappa_2(\|A^1\|^2_{\alpha,1+\beta,\lambda}+\|A^2\|^2_{\alpha,\beta,\lambda}))(1+R)\\
&\leqslant \kappa_2\exp(2\kappa_2 M^2)(1+R);
\end{aligned}$$

similarly, by Point 4. of Theorem 1.6,

$$\begin{aligned}
[\![\psi]\!]_\alpha &\leqslant \kappa_3\,\|A^1-A^2\|_{\alpha,\beta,\lambda}(1+\|x^2\|^\lambda_\infty)(1+[\![x^2]\!]_\alpha)\\
&\leqslant \kappa_4\,\|A^1-A^2\|_{\alpha,\beta,\lambda}\exp(\kappa_4(1+M^2))(1+R).
\end{aligned}$$

Applying Theorem 1.20 to e, we have

$$[\![x^1-x^2]\!]_\alpha \leqslant \kappa_5\, e^{\kappa_5[\![v]\!]^2_{\alpha,1}}(\|x^1_0-x^2_0\|_V+[\![\psi]\!]_\alpha)$$

which combined with the previous estimates implies the conclusion. □

Remark 1.26. If $A\in C^\alpha_t C^{1+\beta}_V$ and we consider solutions x^i associated to (x^i_0,A), going through the same proof but applying instead estimate (1.25), we obtain

$$[\![v]\!]_{\alpha,1}\lesssim \|DA\|_{\alpha,\beta}(1+\|x^1\|_\alpha+\|x^2\|_\alpha)\lesssim 1+\|A\|^3_{\alpha,1+\beta};$$

together with (1.33), this implies the existence of a constant $C=C(\alpha,\beta,T)$ such that

$$[\![x^1-x^2]\!]_\alpha\leqslant C\exp(C\|A\|^6_{\alpha,1+\beta})\|x^1_0-x^2_0\|_V. \tag{1.35}$$

Namely, the solution map $F^A: x_0\mapsto x$ associated to A, seen as a map from V to $C^\alpha_t V$, is globally Lipschitz. Similar estimates show that, if $\{A_n\}_n$ is a sequence such that $A_n\to A$ in $C^\alpha_t C^{1+\beta}_V$, then $F^{A_n}\to F^A$ uniformly on bounded sets.

As a corollary, we obtain convergence of the Euler scheme introduced in Section 1.2.1, with rate $2\alpha-1$. For simplicity we state the result in the case $A\in C^\alpha_t C^{1+\beta}_V$, but the same results follow for $A\in C^\alpha_t C^{1+\beta,\lambda}_V$ by the usual localization procedure.

Corollary 1.27. *Given $A\in C^\alpha_t C^{1+\beta}_V$ with $\alpha(1+\beta)>1$ and $x_0\in V$, denote by x^n the element of $C^\alpha_t V$ constructed by the n-step Euler approximation from Theorem 1.13, and by x the unique solution associated to (x_0,A). Then there exists a constant $C=C(\alpha,\beta,T)$ such that*

$$\|x-x^n\|_\alpha\leqslant C\exp(C\,\|A\|^6_{\alpha,1+\beta})\,n^{1-2\alpha}\quad \text{as } n\to\infty.$$

Proof. Recall that by the proof of Theorem 1.13, x^n satisfies the YDE

$$x^n_t=x_0+\int_0^t A(\mathrm{d}s,x^n_s)+\psi^n_t,$$

where by Remark 1.15, for the choice $\beta=1$, it holds

$$[\![\psi^n]\!]_\alpha\lesssim\left(1+\|A\|^{1+1/\alpha}_{\alpha,1}\right)n^{1-2\alpha}.$$

Define $e^n:=x-x^n$, then by Corollary 1.10 it satisfies

$$e^n_t=\int_0^t A(\mathrm{d}s,x^n_s)-A(\mathrm{d}s,x_s)+\psi^n_t=\int_0^t v^n_{\mathrm{d}s}(e^n_s)+\psi^n_t$$

where again by Remark 1.15 it holds

$$[\![v^n]\!]_{\alpha,1}\lesssim\|A\|_{\alpha,1+\beta}\,(1+[\![x]\!]_\alpha+[\![x^n]\!]_\alpha)\lesssim 1+\|A\|^3_{\alpha,1+\beta}.$$

Applying Theorem 1.20, we deduce the existence of $\kappa_1=\kappa_1(\alpha,\beta,T)$ such that

$$\|e^n\|_\alpha\leqslant\kappa_1\exp(\kappa_1\|A\|^6_{\alpha,1+\beta})\,[\![\psi^n]\!]_\alpha,$$

which combined with the estimate for $[\![\psi^n]\!]_\alpha$ yields the conclusion. □

1.3 Flow

Having established sufficient conditions for the wellposedness of solutions associated to (x_0,A), the next natural step is the study of their dependence on the data of the problem. This section is devoted to the study of the flow, seen as the ensemble of all possible solutions, and its Frechét differentiability w.r.t. (x_0,A).

In order to avoid technicalities we will only consider $A\in C^\alpha_t C^{1+\beta,\lambda}_V$ or even $A\in C^\alpha_t C^{1+\beta}_V$, but everything extends easily by localisation arguments to $A\in C^\alpha_t C^{\beta,\lambda}_V\cap C^\alpha_t C^{1+\beta}_{V,\mathrm{loc}}$.

1.3.1 Flow of diffeomorphisms

We start by giving a proper definition of a flow for the YDE associated to A; recall here that Δ_n denotes the n-simplex on $[0,T]$.

Definition 1.28. *Given $A \in C_t^\alpha C_V^{\beta,\lambda}$ with $\alpha(1+\beta)>1$, $\beta+\lambda \leqslant 1$, we say that $\Phi: \Delta_2 \times V \to V$ is a flow of homeomorphisms for the YDE associated to A if the following hold:*

i. *$\Phi(t,t,x)=x$ for all $t\in[0,T]$ and $x\in V$;*

ii. *$\Phi(s,\cdot,x)\in C^\alpha([s,T];V)$ for all $s\in[0,T]$ and $x\in V$;*

iii. *for all $(s,t,x)\in\Delta_2\times\mathbb{R}^d$ it holds*

$$\Phi(s,t,x)=x+\int_s^t A(\mathrm{d}r,\Phi(s,r,x));$$

iv. *Φ satisfies the semigroup property, namely*

$$\Phi(u,t,\Phi(s,u,x))=\Phi(s,t,x) \quad \text{for all } (s,u,t)\in\Delta_3 \text{ and } x\in V;$$

v. *for any $(s,t)\in\Delta_2$, the map $\Phi(s,t,\cdot)$ is an homeomorphism of V, i.e. it is continuous with continuous inverse.*

From now on, whenever talking about a flow Φ, we will use the notation $\Phi_{s\to t}(x)=\Phi(s,t,x)$; we will denote by $\Phi_{s\leftarrow t}(\cdot)$ the inverse of $\Phi_{s\to t}(\cdot)$ as a map from V to itself.

Definition 1.29. *Given A as in Definition 1.28 and $\gamma\in(0,1)$, we say that the YDE admits a locally γ-Hölder continuous flow Φ (Φ is C^γ_{loc} for short) if for any $(s,t)\in\Delta_2$ the maps $\Phi_{s\to t}$, $\Phi_{s\leftarrow t}$ belong to $C^\gamma_{\mathrm{loc}}(V;V)$; we say that Φ is a flow of diffeomorphisms if $\Phi_{s\to t},\Phi_{s\leftarrow t}\in C^1_{\mathrm{loc}}(V;V)$. Similar definitions hold for a locally Lipschitz flow, or a $C^{n+\gamma}_{\mathrm{loc}}$-flow with $\gamma\in[0,1)$ and $n\in\mathbb{N}$.*

When $V=\mathbb{R}^d$, we say that Φ is a Lagrangian flow if there exists a constant C such that

$$C^{-1}\lambda_d(E)\leqslant\lambda_d(\Phi_{s\leftarrow t}(E))\leqslant C\lambda_d(E) \qquad \forall E\in\mathcal{B}(\mathbb{R}^d), \forall(s,t)\in\Delta_2,$$

where λ_d denotes the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{B}(\mathbb{R}^d)$ the collection of Borel sets.

It follows from Remark 1.26 that, if $A\in C_t^\alpha C_V^{1+\beta}$ with $\alpha(1+\beta)>1$, then the solution map $(x_0,t)\mapsto x_t$ is Lipschitz in space, uniformly in time. However we cannot yet talk about a flow, as we haven't shown the invertibility of the solution map, nor the flow property; this is accomplished by the following lemma.

Lemma 1.30. *Let $A\in C_t^\alpha C_V^\beta$ and $x\in C_t^\alpha V$ such that $\alpha(1+\beta)>1$, x be a solution of the YDE associated to (x_0,A). Then setting $\tilde{A}(t,z):=A(T-t,z)$ and $\tilde{x}_t:=x_{T-t}$, $\tilde{x}$ is a solution to the time-reversed YDE*

$$\tilde{x}_t=\tilde{x}_0+\int_0^t\tilde{A}(\mathrm{d}s,\tilde{x}_s).$$

Similarly, setting $\tilde{x}_t=x_{t+s}$, $\tilde{A}(t,x)=A(t+s,x)$ for $t\in[s,T]$, then $\tilde{x}$ is a solution to the time-shifted YDE

$$\tilde{x}_{\tilde{t}}=\tilde{x}_0+\int_0^{\tilde{t}}\tilde{A}(\mathrm{d}r,\tilde{x}_r) \quad \forall\tilde{t}\in[0,T-s].$$

The proof is elementary but a bit tedious, thus omitted; see Lemmas 4.26-4.27 from [145].

As a consequence of Lemma 1.30, Theorem 1.25 and Remark 1.26, we immediately deduce sufficient conditions for the existence of a Lipschitz flow.

Corollary 1.31. *Let $A\in C_t^\alpha C_V^{1+\beta}$ with $\alpha(1+\beta)>1$, then the associated YDE admits a locally Lipschitz flow Φ^A. Moreover there exists $C=C(\alpha,\beta,T,\|A\|_{\alpha,1+\beta})$ such that*

$$\|\Phi^A_{s\to\cdot}(x)-\Phi^A_{s\to\cdot}(y)\|_{\alpha;s,T}\leqslant C\|x-y\|_V, \quad [\![\Phi^A_{s\to\cdot}(x)]\!]_{\alpha;s,T}\leqslant C \quad \forall s\in[0,T], x,y\in V \tag{1.36}$$

together with a similar inequality for $\Phi_{\cdot\leftarrow t}(\cdot)$. Analogous estimates also hold for $A\in C_t^\alpha C_V^{1+\beta,\lambda}$ with $\alpha(1+\beta)>1$, $\beta+\lambda\leqslant1$, in which case we need to restrict to $\|x\|_V,\|y\|_V\leqslant R$ and take $C=C_R$.

By Theorem 1.25, we can also infer continuity of the flow with respect to the driver A.

Corollary 1.32. *Consider the map Φ given by $A \mapsto \Phi^A$ where $A \in C_t^\alpha C_V^{1+\beta,\lambda}$, Φ^A is the flow given by Corollary 1.31. Then Φ is continuous from $C_t^\alpha C_V^{1+\beta,\lambda}$ to $C([0,T] \times V; V)$, the latter being endowed with the topology of uniform convergence on bounded sets.*

Actually, under the same hypothesis as in Corollary 1.31 it is possible to prove that the YDE admits a flow of diffeomorphisms, which satisfies a variational equation. In the rest of this section, for simplicity we will always assume $A \in C_t^\alpha C_V^{1+\beta}$ with global bounds.

Theorem 1.33. *Let $A \in C_t^\alpha C_V^{1+\beta}$ with $\alpha(1+\beta) > 1$, then the YDE associated to A admits a flow of diffeomorphisms. For any $x \in V$, $D_x \Phi_{s\to t}(x) = J^x_{s\to t}$, where $J^x_{s\to\cdot} \in C_t^\alpha \mathcal{L}(V;V)$ is the unique solution to the variational equation*

$$J^x_{s\to t} = I + \int_s^t DA(\mathrm{d}r, \Phi_{s\to r}(x)) \circ J^x_{s\to r} \qquad \forall t \in [s,T] \tag{1.37}$$

where $\circ$ denotes the composition of linear operators.

We postpone the proof of this result to Section 1.3.2, as the variation equation will follow from a more general result on the differentiability of the Itô map. Following [176], we give an alternative proof in the case of finite dimensional V, where more precise information on Φ is known.

Theorem 1.34. *Let A satisfy the hypothesis of Theorem 1.33, $V = \mathbb{R}^d$ for some $d \in \mathbb{N}$; then the associated YDE admits a flow of diffeomorphisms and the following hold:*

i. For any $x \in \mathbb{R}^d$ and $s \in [0,T]$, $D_x \Phi_{s\to\cdot}(x)$ corresponds to $J^x_{s\to\cdot} \in C^\alpha([s,T]; \mathbb{R}^{d\times d})$ satisfying

$$J^x_{s\to t} = I + \int_s^t DA(\mathrm{d}r, \Phi_{s\to r}(x)) J^x_{s\to r}. \tag{1.38}$$

ii. The Jacobian $\jmath_{s\to t}(x) := \det(D_x \Phi_{s\to t}(x))$ satisfies the identity

$$\jmath_{s\to t}(x) = \exp\left(\int_s^t \operatorname{div} A(\mathrm{d}r, \Phi_{s\to r}(x)) \right) \tag{1.39}$$

and there exists a constant $C = C(\alpha, \beta, T, \|A\|_{\alpha,1+\beta}) > 0$ such that

$$C^{-1} \leqslant \jmath_{s\to t}(x) \leqslant C \quad \forall (s,t,x) \in \Delta_2 \times \mathbb{R}^d.$$

In particular, Φ is a Lagrangian flow of diffeomorphisms.

Proof. For simplicity we will prove all the statements for $s=0$, the general case being similar. By Corollary 1.31, the existence of a locally Lipschitz flow Φ is known; to show differentiability, it is enough to establish existence and continuity of the Gateaux derivatives.

Fix $x, v \in \mathbb{R}^d$ and consider for any $\varepsilon > 0$ the map $\eta_t^\varepsilon := \varepsilon^{-1}(\Phi_{0\to\cdot}(x+\varepsilon v) - \Phi_{0\to\cdot}(x))$; by estimate (1.36), the family $\{\eta^\varepsilon\}_{\varepsilon>0}$ is bounded in $C_t^\alpha \mathbb{R}^d$. Thus by Ascoli-Arzelà we can extract a subsequence $\varepsilon_n \to 0$ such that $\eta^{\varepsilon_n} \to \eta$ in $C_t^{\alpha-\delta}\mathbb{R}^d$ for some $\eta \in C_t^\alpha \mathbb{R}^d$ and any $\delta > 0$. Choose $\delta > 0$ small enough such that $(\alpha - \delta)(1+\beta) > 1$; by Proposition 1.9, the map $F(y) = \int_0^\cdot A(\mathrm{d}s, y_s)$ is differentiable from $C_t^{\alpha-\delta}\mathbb{R}^d$ to itself, with DF given by (1.8). Using this fact and the chain rule, we deduce that

$$\begin{aligned}
\eta_\cdot &= \lim_{\varepsilon_n \to 0} \frac{\Phi_{0\to\cdot}(x+\varepsilon_n v) - \Phi_{0\to\cdot}(x)}{\varepsilon_n} \\
&= v + \lim_{\varepsilon_n \to 0} \frac{F(\Phi_{0\to\cdot}(x+\varepsilon_n v)) - F(\Phi_{0\to\cdot}(x))}{\varepsilon_n} \\
&= v + DF(\Phi_{0\to\cdot}(x))(\eta_\cdot);
\end{aligned}$$

namely, η satisfies the YDE

$$\eta_t = v + \int_0^t D_x A(\mathrm{d}r, \Phi_{0\to r}(x)) \eta_r \tag{1.40}$$

whose meaning was defined in Remark 1.7. Equation (1.40) is an affine YDE, which admits a unique solution by Corollary 1.24; moreover it's easy to check that the unique solution must have the form $\eta_t = J^x_{0\to t} v$, where $J^x_{0\to\cdot} \in C^\alpha_t \mathbb{R}^{d\times d}$ is the unique solution to the affine $\mathbb{R}^{d\times d}$-valued YDE

$$J^x_{0\to t} = I + \int_0^t D_x A(\mathrm{d}r, \Phi_{0\to r}(x)) J^x_{0\to r},$$

whose global existence and uniqueness follows from Corollary 1.24 and Theorem 1.20. As the reasoning holds for any subsequence ε_n we can extract and any $v \in \mathbb{R}^d$, we conclude that $\Phi_{0\to t}(\cdot)$ is Gateaux differentiable with $D\Phi_{0\to t}(x) = J^x_{0\to t}$ satisfying (1.38). A similar argument shows that $J^x_{0\to t}$ depends continuously on x, from which Frechét differentiability follows.

Part *ii.* can be established for instance by means of an approximation procedure; indeed by Lemma A.35 in Appendix A.6, given $A \in C^\alpha_t C^{1+\beta}_{\mathbb{R}^d}$, we can find $A^n \in C^1_t C^{1+\beta}_{\mathbb{R}^d}$ such that $A^n \to A$ in $C^{\alpha-}_t C^{1+\beta-}_{\mathbb{R}^d}$ and by Theorem 1.25, the solutions $y^n_\cdot = \Phi^n_{0\to\cdot}(x)$ associated to (x, A^n) converge to $\Phi_{0\to\cdot}(x)$ associated to (x, A). Moreover for A^n the YDE is meaningful as the more classical ODE associated to $\partial_t A^n$, so we can apply to it all the classical results from ODE theory; the Jacobian associated to A^n is given by

$$\det(D_x \Phi^n_{0\to t}(x)) = \exp\left(\int_0^t \operatorname{div} \partial_t A^n(r, \Phi^n_{0\to r}(x)) \mathrm{d}r\right) = \exp\left(\int_0^t \operatorname{div} A^n(\mathrm{d}r, \Phi^n_{0\to r}(x))\right).$$

Passing to the limit as $n \to \infty$, by the continuity of nonlinear Young integrals, we obtain (1.39). Moreover by equation (1.36) we have the estimate

$$\sup_{t\in[0,T]} \left| \int_0^t \operatorname{div} A(\mathrm{d}r, \Phi_{0\to r}(x)) \right| \lesssim \|\operatorname{div} A\|_{\alpha,\beta} (1 + [\![\Phi_{0\to\cdot}(x)]\!]_\alpha) \lesssim \|A\|_{\alpha,1+\beta},$$

which gives Lagrangianity. □

It's possible to show that the flow inherits regularity from the drift, namely that to a spatially more regular A corresponds a more regular Φ.

Theorem 1.35. *Let $n \in \mathbb{N}$, $\alpha, \beta \in (0,1)$ be such that $\alpha(1+\beta) > 1$ and assume $A \in C^\alpha_t C^{n+\beta}_V$. Then the flow Φ associated to A is locally C^n-regular.*

We omit the proof, which follows similar lines to those of Theorems 1.33 and 1.34 and is mostly technical; we refer the interested reader to [145, 170] and the discussion at the end of Section 3 from [199]. Given Theorem 1.35, we may strenghten Corollary 1.32 as follows. We recall that convergence in $C^0_t C^n_{V,\mathrm{loc}}$ stands for uniform convergence in $[0,T] \times B_R$ of all derivatives up to order n, for all $R > 0$.

Corollary 1.36. *For any $n \geqslant 1$, the map $A \mapsto \Phi^A$ is continuous from $C^\alpha_t C^{n+\beta}_V$ to $C^0_t C^n_{V,\mathrm{loc}}$.*

We omit the proof, as it follows from the very same steps from that of Theorem 1.35. Like in Theorems 1.33-1.34, also in the case of Theorem 1.35 and Corollary 1.36 the assumptions on A can be weakened to local regularity and growth conditions: it is enough to require $A \in C^\alpha_t C^{\beta,\lambda}_V$ together with $A \in C^\alpha_t C^{n+\beta}_{V,\mathrm{loc}}$ (and one still retains continuity of the map $A \mapsto \Phi^A$ in suitable topologies).

1.3.2 Differentiability of the Itô map

Denote by $\Phi^A_{s\to\cdot}(x)$ the solution to the YDE associated to (x, A); the aim of this section is to study the dependence of the flow Φ^A as a function of $A \in C^\alpha_t C^{1+\beta}_V$, namely to identify $D_A \Phi^A_{s\to\cdot}(x)$.

For simplicity we will restrict to the case $s = 0$; we will actually fix $A \in C^\alpha_t C^{1+\beta}_V$, consider $\Phi^{A+\varepsilon B}$ with B varying and set $X^x_t := \Phi^A_{0\to t}(x)$.

Theorem 1.37. *Let $\alpha(1+\beta) > 1$, $x_0 \in V$ and consider the Itô map $\Phi^\cdot_{0\to\cdot}(x): C^\alpha_t C^{1+\beta}_V \to C^\alpha_t V$, $A \mapsto \Phi^A_{0\to\cdot}(x)$. Then $\Phi^\cdot_{0\to\cdot}(x)$ is Frechét differentiable and for any $B \in C^\alpha_t C^{1+\beta}_V$ the Gateaux derivative*

$$D_A \Phi^A_{0\to\cdot}(x)(B) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} (\Phi^{A+\varepsilon B}_{0\to\cdot}(x) - \Phi^A_{0\to\cdot}(x)) \in C^\alpha_t V$$

satisfies the affine YDE

$$Y_t^x = \int_0^t DA(\mathrm{d}s, X_s^x)(Y_s^x) + \int_0^t B(\mathrm{d}s, X_s^x) \quad \forall t \in [0,T] \tag{1.41}$$

and is given explicitly by

$$D_A \Phi_{0\to t}^A(x)(B) = J_{0\to t}^x \int_0^t (J_{0\to s}^x)^{-1} B(\mathrm{d}s, X_s^x) \quad \forall t \in [0,T] \tag{1.42}$$

where $J_{0\to\cdot}^x$ is the unique solution to (1.37) and $(J_{0\to s}^x)^{-1}$ denotes its inverse as an element of $\mathcal{L}(V)$.

The proof requires the following preliminary lemma.

Lemma 1.38. *For any $L \in C_t^\alpha \mathcal{L}(V)$, there exists a unique solution $M \in C_t^\alpha \mathcal{L}(V)$ to the YDE*

$$M_t = \mathrm{Id}_V + \int_0^t L_{\mathrm{d}s} \circ M_s \qquad \forall t \in [0,T]; \tag{1.43}$$

moreover M_t is invertible for any $t \in [0,T]$ and $N_\cdot := (M_\cdot)^{-1} \in C_t^\alpha \mathcal{L}(V)$ is the unique solution to

$$N_t = \mathrm{Id}_V - \int_0^t N_s \circ L_{\mathrm{d}s} \qquad \forall t \in [0,T]. \tag{1.44}$$

Finally, for any $y_0 \in V$ and any $\psi \in C_t^\alpha V$, the unique solution to the affine YDE

$$y_t = y_0 + \int_0^t L_{\mathrm{d}s} y_s + \psi_t \tag{1.45}$$

is given by

$$y_t = M_t\, y_0 + M_t \int_0^t N_s\, \mathrm{d}\psi_s. \tag{1.46}$$

Proof. Setting $A(t,M) := L_t \circ M$, it holds $A \in C_t^\alpha C_{\mathcal{L}(V),\mathrm{loc}}^2$ and so existence and uniqueness of a global solution to (1.43) follows from Corollary 1.24 and Theorem 1.20; similarly for (1.44) with $\tilde{A}(t,N) = N \circ L_t$. Let $M_\cdot, N_\cdot \in C_t^\alpha \mathcal{L}(V)$ be solutions respectively to (1.43), (1.44); we claim that they are inverse of each other. Indeed, by the product rule for Young integrals, it holds

$$\mathrm{d}(N_t \circ M_t) = (\mathrm{d}N_t) \circ M_t + N_t \circ (\mathrm{d}M_t) = -N_t \circ L_{\mathrm{d}t} \circ M_t + N_t \circ L_{\mathrm{d}t} \circ M_t = 0$$

which implies $N_t \circ M_t = N_0 \circ M_0 = \mathrm{Id}_V$ and thus $N_t = (M_t)^{-1}$. Let $y_\cdot \in C_t^\alpha V$ be the unique solution to (1.45), whose global existence and uniqueness follows as above, and set $z_t := N_t\, y_t$; then again by Young product rule, it holds $\mathrm{d}z_t = N_t\, \mathrm{d}\psi_t$ and thus

$$N_t\, y_t = z_t = z_0 + \int_0^t \mathrm{d}z_s = y_0 + \int_0^t N_s \mathrm{d}\psi_s$$

which gives (1.46) by applying M_t on both sides. □

Proof. (of Theorem 1.37) Given $A, B \in C_t^\alpha C_V^{1+\beta}$, it is enough to show that the limit

$$\lim_{\varepsilon \to 0} \frac{\Phi_{0\to\cdot}^{A+\varepsilon B}(x) - \Phi_{0\to\cdot}^A(x)}{\varepsilon} \text{ exists in } C_t^\alpha V$$

and that it is a solution to (1.41). Once this is established, we can apply Lemma 1.38 for the choice $L_t = \int_0^t D_x A(\mathrm{d}s, X_s^x)$, $y_0 = 0$ and $\psi_t = \int_0^t B(\mathrm{d}s, X_s^x)$ to deduce that the limit is given by formula (1.42), which is meaningful since $J_{0\to\cdot}^x$ is defined as the solution to (1.43) and is therefore invertible (again by Lemma 1.38). The explicit formula (1.42) for the Gateaux derivatives readily implies existence and continuity of the Gateux differential $D_A \Phi_{0\to\cdot}^A(x)$ and thus also Frechét differentiability.

In order to prove the claim, let $Y^x \in C_t^\alpha V$ be the solution to (1.41), which exists and is unique by Lemma 1.38; then we need to show that

$$\lim_{\varepsilon \to 0} \left\| \frac{\Phi_{0\to\cdot}^{A+\varepsilon B}(x) - X_\cdot^x}{\varepsilon} - Y_\cdot^x \right\|_\alpha = 0.$$

Set $X_\cdot^{\varepsilon,x} := \Phi_{0\to\cdot}^{A+\varepsilon B}(x)$; recall that by Theorem 1.25, we have

$$\|X^{\varepsilon,x} - X^x\|_\alpha \lesssim \varepsilon \|B\|_{\alpha,\beta}. \tag{1.47}$$

Setting $e^\varepsilon := \varepsilon^{-1}[X^{\varepsilon,x} - X^x] - Y^x$, it holds

$$\begin{aligned}
e_t^\varepsilon &= \frac{1}{\varepsilon}\left[\int_0^t (A+\varepsilon B)(\mathrm{d}s, X_s^{\varepsilon,x}) - A(\mathrm{d}s, X_s^x)\right] - \int_0^t DA(\mathrm{d}s, X_s^x)(Y_s^x) - \int_0^t B(\mathrm{d}s, X_s^x)\\
&= \int_0^t \left[\frac{A(\mathrm{d}s, X_s^{\varepsilon,x}) - A(\mathrm{d}s, X_s^x)}{\varepsilon} - DA(\mathrm{d}s, X_s^x)(Y_s)\right] + \int_0^t [B(\mathrm{d}s, X_s^{\varepsilon,x}) - B(\mathrm{d}s, X_s^x)]\\
&= \int_0^t DA(\mathrm{d}s, X_s^x)(e_s^\varepsilon) + \psi_t^\varepsilon
\end{aligned}$$

where ψ^ε is given by

$$\begin{aligned}
\psi_t^\varepsilon &= \int_0^t \frac{A(\mathrm{d}s, X_s^{\varepsilon,x}) - A(\mathrm{d}s, X_s^x) - DA(\mathrm{d}s, X_s^x)(X_s^{\varepsilon,x} - X_s^x)}{\varepsilon} + \int_0^t B(\mathrm{d}s, X_s^{\varepsilon,x}) - B(\mathrm{d}s, X_s^x)\\
&=: \psi_t^{\varepsilon,1} + \psi_t^{\varepsilon,2}.
\end{aligned}$$

In order to conclude, it is enough to show that $\|\psi^\varepsilon\|_\alpha \to 0$ as $\varepsilon \to 0$, since then we can apply the usual a priori estimates from Theorem 1.20 to e^ε, which solves an affine YDE starting at 0. We already know that $X^{\varepsilon,x} \to X^x$ as $\varepsilon \to 0$, which combined with the continuity of nonlinear Young integrals implies that $\psi_t^{\varepsilon,2} \to 0$ as $\varepsilon \to 0$. Observe that $\psi^{\varepsilon,1} = \mathcal{J}(\Gamma^\varepsilon)$ for

$$\Gamma_{s,t}^\varepsilon = \varepsilon^{-1}[A_{s,t}(X_s^{\varepsilon,x}) - A_{s,t}(X_s^x) - DA_{s,t}(X_s^x)(X_s^{\varepsilon,x} - X_s^x)]$$

which by virtue of (1.47) satisfies

$$\|\Gamma_{s,t}^\varepsilon\|_V \lesssim \varepsilon^{-1} \|A_{s,t}\|_{C_V^{1+\beta}} \|X_s^{\varepsilon,x} - X_s^x\|_V^{1+\beta} \lesssim \varepsilon^\beta |t-s|^\alpha \|A\|_{\alpha,1+\beta}$$

which implies that $\|\Gamma^\varepsilon\|_\alpha \to 0$ as $\varepsilon \to 0$. On the other hand we have

$$\begin{aligned}
\|\delta\Gamma_{s,u,t}^\varepsilon\|_V \;=\;& \varepsilon^{-1}\left\| \int_0^1 [DA_{u,t}(X_s^x + \lambda(X_s^{\varepsilon,x} - X_s^x)) - DA_{u,t}(X_s^x)](X_s^{\varepsilon,x} - X_s^x)\mathrm{d}\lambda \right.\\
& \left. - \int_0^1 [DA_{u,t}(X_u^x + \lambda(X_u^{\varepsilon,x} - X_u^x)) - DA_{u,t}(X_u^x)](X_u^{\varepsilon,x} - X_u^x)\mathrm{d}\lambda \right\|_V\\
\leqslant\;& \varepsilon^{-1}\left\| \int_0^1 [DA_{u,t}(X_s^x + \lambda(X_s^{\varepsilon,x} - X_s^x)) - DA_{u,t}(X_s^x)](X_{s,u}^{\varepsilon,x} - X_{s,u}^x)\mathrm{d}\lambda \right\|_V\\
& + \varepsilon^{-1}\left\| \int_0^1 [DA_{u,t}(X_u^x + \lambda(X_u^{\varepsilon,x} - X_u^x)) - DA_{u,t}(X_s^x + \lambda(X_s^{\varepsilon,x} - X_s^x))](X_u^{\varepsilon,x} - X_u^x)\mathrm{d}\lambda \right\|_V\\
& + \varepsilon^{-1}\left\| \int_0^1 [DA_{u,t}(X_u^x) - DA_{u,t}(X_s^x)](X_u^{\varepsilon,x} - X_u^x)\mathrm{d}\lambda \right\|_V\\
\lesssim\;& \varepsilon^{-1}|t-s|^{\alpha(1+\beta)} \|A\|_{\alpha,1+\beta} [\![X^{\varepsilon,x} - X^x]\!]_\alpha (1 + [\![X^{\varepsilon,x} - X^x]\!]_\alpha + [\![X^x]\!]_\alpha)\\
\lesssim\;& |t-s|^{\alpha(1+\beta)} \|A\|_{\alpha,1+\beta} (1 + [\![X^x]\!]_\alpha)
\end{aligned}$$

which implies that $\|\delta\Gamma^\varepsilon\|_{\alpha(1+\beta)}$ are uniformly bounded in ε. We can finally apply Lemma A.33 from Appendix A.6 to conclude. □

Remark 1.39. Although $A \mapsto \Phi^A$ is defined only on $C_t^\alpha C_V^{1+\beta}$, observe that $(A,B) \mapsto D_A \Phi_{0\to\cdot}^A(x)(B)$ as given by formula (1.42) is well defined and continuous for any $(A,B) \in C_t^\alpha C_V^{1+\beta} \times C_t^\alpha C_V^\beta$.

We can use Theorem 1.37 to complete the proof of Theorem 1.33.

Proof. (of Theorem 1.33) The existence of a Lipschitz flow Φ is granted by Corollary 1.31, so it suffices to show its differentiability and the variational equation; for simplicity we take $s=0$. Existence of a unique solution $J^x_{0\to\cdot}\in C^\alpha_t\mathcal{L}(V)$ to (1.37) follows from Lemma 1.38 applied to

$$L_t=\int_0^t DA(\mathrm{d}r,\Phi_{0\to r}(x));$$

by linearity, it's also easy to check that for any $h\in V$, $Y^h_t:=J^x_{0\to t}(h)$ is the unique solution to

$$Y^h_t=h+\int_0^t DA(\mathrm{d}r,\Phi_{0\to r}(x))(Y^h_r). \tag{1.48}$$

Therefore in order to conclude it suffices to show that the directional derivatives

$$D_x\Phi^A_{0\to\cdot}(x)(h)=\lim_{\varepsilon\to 0}\frac{\Phi^A_{0\to\cdot}(x+\varepsilon h)-\Phi^A_{0\to\cdot}(x)}{\varepsilon}$$

exist in $C^\alpha_t V$ and are solutions to (1.48), as this implies that $D_x\Phi^A_{0\to\cdot}(x)=J^x_{0\to\cdot}$. Now fix $x,h\in V$ and let $y^\varepsilon=\Phi^A_{0\to\cdot}(x+\varepsilon h)$, then $z^\varepsilon:=y^\varepsilon-\varepsilon h$ solves

$$z^\varepsilon_t=x+\int_0^t A^\varepsilon(\mathrm{d}s,z^\varepsilon_s)$$

with $A^\varepsilon(t,v)=A(t,v+\varepsilon h)$, i.e. $z^\varepsilon_\cdot=\Phi^{A^\varepsilon}_{0\to\cdot}(x)$. It's easy to see that, if the first limit below exists, then

$$\lim_{\varepsilon\to 0}\frac{z^\varepsilon-z^0}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{y^\varepsilon-y^0}{\varepsilon}-h,\quad \lim_{\varepsilon\to 0}\frac{A^\varepsilon-A}{\varepsilon}=B\quad\text{for}\quad B(t,x):=DA(t,x)(h).$$

By the Frechét differentiability of $A\mapsto\Phi^A_{0\to\cdot}(x)$ and the chain rule, it holds

$$\lim_{\varepsilon\to 0}\frac{z^\varepsilon-z^0}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\Phi^{A^\varepsilon}_{0\to\cdot}(x)-\Phi^A_{0\to\cdot}(x)}{\varepsilon}=D_A\Phi^A_{0\to\cdot}(x)(B)$$

which is characterized as the unique solution Z^h to

$$Z^h_t=\int_0^t DA(\mathrm{d}r,\Phi^A_{0\to r}(x))(Z^h_r)+\int_0^t DA(\mathrm{d}r,\Phi^A_{0\to r}(x))(h).$$

This implies by linearity that

$$Y^h:=Z^h_t+h=\lim_{\varepsilon\to 0}\frac{y^\varepsilon-y}{\varepsilon}=D_x\Phi^A_{0\to\cdot}(x)(h)$$

solves exactly (1.48). The conclusion follows. □

Example 1.40. Here are some examples of applications of Theorem 1.37.

i. Consider the simple case of an additive perturbation, i.e. for fixed (x_0,A) we want to understand how the solution x of

$$x_t=x_0+\int_0^t A(\mathrm{d}s,x_s)+\psi_t$$

depends on ψ, where $\psi\in C^\alpha_t V$ with $\psi_0=0$. Identifying ψ with the spatially constant drift $B_\psi(t,z):=\psi_t$ for all $z\in V$, it holds $x_\cdot=\Phi^{A+B_\psi}_{0\to\cdot}(x_0)=:F(\psi)$, which implies that F is Frechét differentiable in 0 with

$$DF(0)(\psi)_\cdot=J^x_{0\to\cdot}\int_0^\cdot (J^x_{0\to s})^{-1}\mathrm{d}\psi_s.$$

ii. Consider the classical Young case, namely $V=\mathbb{R}^d$, with

$$A(t,z)=A_\omega(t,z)=\sigma(z)\omega_t=\sum_{i=1}^m\sigma_i(z)\omega^i_t,\quad (t,z)\in[0,T]\times\mathbb{R}^d$$

for regular vector fields $\sigma_i: \mathbb{R}^d \to \mathbb{R}^d$ and $\omega \in C_t^\alpha \mathbb{R}^m$, $\alpha > 1/2$; let σ_i be fixed and consider the dependence on the drivers ω, namely the map $\omega \mapsto \Phi_{0\to\cdot}^{A_\omega}(x)$.

For fixed $\omega \in C_t^\alpha \mathbb{R}^m$ and $x \in \mathbb{R}^d$, set $X_t^x(\omega) := \Phi_{0\to t}^{A_\omega}(x)$, $J_{0\to t}^x(\omega) := D_x \Phi_{0\to t}^{A_\omega}(x)$; then the map $\omega \mapsto X_t^x(\omega)$ is Frechét differentiable at ω with directional derivatives

$$D_\omega X_t^x(\psi) = J_{0\to t}^x \int_0^t \sum_{i=1}^m (J_{0\to r}^x)^{-1} \sigma_i(X_r^x) \mathrm{d}\psi_r^i. \tag{1.49}$$

The above formula uniquely extends by continuity to the case $\psi \in W_t^{1,1}$, in which case we can write it in compact form as

$$D_\omega X_t^x(\psi) = \int_0^T K(t,r) \dot{\psi}_r \,\mathrm{d}r, \quad K(t,r) = \mathbb{1}_{[0,t]}(r)\, J_{0\to t}^x (J_{0\to r}^x)^{-1} \sigma(X_r^x). \tag{1.50}$$

Formulas (1.49) and (1.50) are well known by Malliavin calculus, mostly in the case ω is sampled as an fBm of parameter $H > 1/2$, see Section 11.3 from [132]; formula (1.42) can be regarded as a generalisation of them.

1.4 Further results

The theory presented so far, although satisfactory, requires quite a lot of spatial regularity on A: if we allow for α arbitrarily close to $1/2$, the conditions of Theorem 1.33 roughly become $A \in C_t^\alpha C_V^2$. At the same time, we have already seen that, under the weaker requirement $A \in C_t^\alpha C_V^1$, the equation is meaningful and existence of solutions holds (at least for $V = \mathbb{R}^d$, cf. Theorem 1.13). It is then natural to wonder whether we can obtain a better understanding of the structure of solutions, or develop different criteria (possibly depending on terms different from A like $\partial_t A$, or the solutions themselves) to ensure uniqueness in this weaker regularity regime.

1.4.1 Topological properties of the set of solutions

We restrict in this subsection to the case $V = \mathbb{R}^d$; we will adopt the shorter notations $C_t^\alpha = C_t^\alpha \mathbb{R}^d$ and $C_t^\alpha C_x^{\beta,\lambda} = C_t^\alpha C_{\mathbb{R}^d}^{\beta,\lambda}$.

Inspired by a series of results by Stampacchia, Vidossich, Browder, Gupta and others (see [270] and the references therein), we want to study the topological structure of the set

$$C(x_0, A) = \left\{ x \in C_t^\alpha \text{ such that } x_t = x_0 + \int_0^t A(\mathrm{d}s, x_s) \text{ for all } t \in [0,T] \right\}$$

where $A \in C_t^\alpha C_x^{\beta,\lambda}$ with $\alpha(1+\beta) > 1$ and $\beta + \lambda \leqslant 1$; namely, $C(x_0, A)$ is the set of solutions to the Cauchy problem associated to (x_0, A). Recall that by Corollary 1.16 and Proposition 1.18, existence of global solutions is granted, but uniqueness is not unless $A \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$; therefore $C(x_0, A)$ may not consist of a singleton. The following result is an extension of Proposition 43 from [139], where the structure of the set $C(x_0, A)$ was already partially addressed.

Theorem 1.41. *Let $A \in C_t^\alpha C_x^{\beta,\lambda}$ with α, β, λ as above, $x_0 \in \mathbb{R}^d$; then the set $C(x_0, A)$ is nonempty, compact and simply connected. Moreover, for any fixed $y \in \mathbb{R}^d$, the map*

$$\mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda} \ni (x_0, A) \mapsto d(y, C(x_0, A)) \in \mathbb{R}$$

is lower semincontinuous.

Here we recall that for $y \in C_t^\alpha$, $K \subset C_t^\alpha$, the distance of an element from a set is defined by

$$d(y, K) = \inf_{z \in K} \|y - z\|_\alpha.$$

A main tool in the proof of Theorem 1.41 is the use of the Browder–Gupta theorem from [49]; we recite here a slight modification due to Gorniewicz. Recall that a map f is proper if it is continuous and the preimage $f^{-1}(K)$ is compact whenever K is so.

Theorem 1.42. **(Theorem 69.1, Chapter VI from [158])** *Let X be a metric space, $(E, \|\cdot\|)$ a Banach space and $f: X \to E$ a proper map. Assume further that for each $\varepsilon > 0$ a proper map f_ε: $X \to E$ is given and the following two conditions are satisfied:*

i. $\|f_\varepsilon(x) - f(x)\| \leqslant \varepsilon$ for all $x \in X$;

ii. for any $\varepsilon > 0$ and $u \in E$ such that $\|u\| \leqslant \varepsilon$, the equation $f_\varepsilon(x) = u$ has exactly one solution.

Then the set $S = f^{-1}(0)$ is R^δ in the sense of Aronszajn.

Let us recall that an R^δ-set is the intersection of a decreasing sequence of compact absolute retracts, see Definition 2.11 from [158]; here we will only need the basic fact that R^δ-sets are always simply connected. In order to prove Theorem 1.41, we need the a preliminary lemma.

Lemma 1.43. *For A as above and for any $y \in C_t^\alpha$, there exists at least one solution $x \in C_t^\alpha$ to*

$$x_t = y_t + \int_0^t A(\mathrm{d}s, x_s) \qquad \forall t \in [0, T]; \tag{1.51}$$

moreover, there exists $C = C(\alpha, \beta, T)$ such that any solution satisfies the a priori estimate

$$\|x\|_\alpha \leqslant C \exp(C\|A\|_{\alpha,\beta,\lambda}^2 + \|y\|_\alpha^2)(1 + |y_0|). \tag{1.52}$$

If in addition $A \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$, then the solution is unique.

Proof. Set $\tilde{A}(t, x) = A(t, x) + y_t$, then x is a solution to (1.51) if and only if it solves

$$x_t = y_0 + \int_0^t \tilde{A}(\mathrm{d}s, x_s)$$

where $\tilde{A} \in C_t^\alpha C_x^{\beta,\lambda}$ with $\|\tilde{A}\|_{\alpha,\beta,\lambda} \leqslant \|A\|_{\alpha,\beta,\lambda} + \|y\|_\alpha$. Existence and the estimate (1.52) then follow from Corollary 1.16 and Proposition 1.18; $A \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$ implies $\tilde{A} \in C_t^\alpha C_{\mathrm{loc}}^{1+\beta}$ and so uniqueness follows from Corollary 1.24. □

Proof. **(of Theorem 1.41)** We divide the proof in several steps.

Step 1: $C(x_0, A)$ nonempty, compact. Nonemptiness follows immediately from Lemma 1.43 applied to $y \equiv x_0$; let x^n be a sequence of elements of $C(x_0, A)$, then by (1.52) they are uniformly bounded in C_t^α and so by Ascoli–Arzelà we can extract a (not relabelled) subsequence $x^n \to x$ in $C_t^{\alpha-\varepsilon}$ for all $\varepsilon > 0$, for some $x \in C_t^\alpha$. Choosing $\varepsilon > 0$ sufficiently small such that $\alpha + \beta(\alpha - \varepsilon) > 1$, by Theorem 1.6 the map $z_\cdot \mapsto \int_0^\cdot A(\mathrm{d}s, z_s)$ is continuous from $C_t^{\alpha-\varepsilon}$ to C_t^α, therefore

$$x_\cdot^n = x_0 + \int_0^\cdot A(\mathrm{d}s, x_s^n) \to x_0 + \int_0^\cdot A(\mathrm{d}s, x_s) = x_\cdot \text{ in } C_t^\alpha,$$

which shows compactness.

Step 2: $C(x_0, A)$ connected. Given $A \in C_t^\alpha C_x^{\beta,\lambda}$, consider a sequence $A^\varepsilon \in C_t^\alpha C_x^{1+\beta,\lambda}$ such that

$$\|A^\varepsilon\|_{\alpha,\beta,\lambda} \leqslant 2\|A\|_{\alpha,\beta,\lambda}, \quad A^\varepsilon \to A \text{ in } C_t^\alpha C_{\mathrm{loc}}^\beta \quad \text{as } \varepsilon \to 0;$$

this is always possible, for instance by taking $A^\varepsilon = \rho^\varepsilon * A$, $\{\rho^\varepsilon\}_{\varepsilon > 0}$ being a family of standard spatial mollifiers. For $x_0 \in \mathbb{R}^d$ fixed, take $R > 0$ big enough such that

$$C \exp(C\|A^\varepsilon\|_{\alpha,\beta,\lambda}^2 + \|x_0 + y\|_\alpha^2)(1 + |y_0 + x_0|) \leqslant R \quad \forall \varepsilon \in (0, 1), y \in C_t^\alpha \text{ s.t. } \|y\|_\alpha \leqslant 1,$$

where C is the constant appearing in (1.52); this is always possible due to the uniform bound on $\|A^\varepsilon\|_{\alpha,\beta,\lambda}$. Define the metric space E to be

$$E = \{z \in C_t^\alpha : \|z\|_\alpha \leqslant R\}, \quad d_E(z^1, z^2) = \|z^1 - z^2\|_\alpha;$$

also define the maps $f, f_\varepsilon: E \to C_t^\alpha$ by

$$f(z) = z_\cdot - x_0 - \int_0^\cdot A(\mathrm{d}s, z_s), \quad f_\varepsilon(z) = z_\cdot - x_0 - \int_0^\cdot A^\varepsilon(\mathrm{d}s, z_s).$$

By Theorem 1.6, they are continuous from E to C_t^α; reasoning exactly as in Step 1, it is easy to check that they are proper. Observe that an element $x\in E$ satisfies $f(x)=y$ if and only if it satisfies

$$x\in C_t^\alpha, \quad x_t = x_0 + y_t + \int_0^t A(\mathrm{d}s, x_s) \quad \forall t\in[0,T], \quad \|x\|_\alpha \leqslant R,$$

similarly for f_ε; moreover the bound $\|x\|_\alpha \leqslant R$ is trivially satisfied for all y such that $\|y\|_\alpha \leqslant 1$, by our choice of R and Lemma 1.43. It follows that, for any such y, $f_\varepsilon(x)=y$ has exactly one solution $x\in E$. In order to apply Theorem 1.42 and get the conclusion, it remains to show that $f_\varepsilon \to f$ uniformly in E; but by Theorem 1.6 it holds

$$\begin{aligned}\|f(z)-f_\varepsilon(z)\|_\alpha &= \left\| \int_0^\cdot A(\mathrm{d}s, z_s) - \int_0^\cdot A^\varepsilon(\mathrm{d}s, z_s)\right\|_\alpha\\ &\lesssim \|A-A^\varepsilon\|_{\alpha,\beta,R}\,(1+\|z\|_\alpha)\\ &\lesssim \|A-A^\varepsilon\|_{\alpha,\beta,R}(1+R) \to 0 \text{ as } \varepsilon\to 0\end{aligned}$$

and the can conclude that $f^{-1}(0)=C(x_0,A)$ is simply connected in E, thus also in C_t^α.

Step 3: lower semicontinuity. Consider now a sequence $(x_0^n, A^n)\to(x_0,A)$ in $\mathbb{R}^d\times C_t^\alpha C_x^{\beta,\lambda}$, we need to show that for any fixed $y\in C_t^\alpha$ it holds

$$d(y, C(x_0,A)) \leqslant \liminf_{n\to\infty} d(y, C(x_0^n, A^n)).$$

Since by Step 1 the set $C(x_0^n,A^n)$ is compact, it is always possible to find $x^n\in C(x_0^n,A^n)$ such that

$$\|y-x_0^n\| = (y, C(x_0^n, A^n));$$

we can assume wlog that $\lim d(y, C(x_0^n,A^n))$ exists, since otherwise we can extract a subsequence realizing the liminf. Since (x_0^n,A^n) is convergent, it is also bounded in $\mathbb{R}^d\times C_t^\alpha C_x^{\beta,\lambda}$, which implies by estimate (1.52) that the sequence $\{x^n\}_n$ is bounded in C_t^α. It is not difficult to see, invoking Ascoli–Arzelà and going through the same reasoning as in Step 1, that we can extract a (not relabelled) subsequence such that $x^n\to x$ in C_t^α where $x\in C(x_0,A)$. As a consequence

$$d(y, C(x_0,A)) \leqslant \|y-x\|_\alpha = \lim_{n\to\infty} \|y-x^n\|_\alpha = \liminf_{n\to\infty} d(y, C(x_0^n, A^n))$$

which gives the conclusion. □

Remark 1.44. For simplicity we have only treated the case $V=\mathbb{R}^d$, but it's clear that Theorem 1.41 admits several variants; for instance one can consider a general Banach space V and $A\in C_t^\alpha C_{V,W}^{\beta,\lambda}$ with W compactly embedded in V, which implies global existence by Corollary 1.16 and the usual a priori estimates. Other extensions of interest can be those to mixed or fractional equation, which have been trated in Section 3.5 from [141].

Theorem 1.41 has relevant consequence when considering $C(x_0,A)$ as a multivalued map; here we only recall some key concepts and refer the reader to [55] for a complete overview.

Given a complete metric space (E,d), the space

$$\mathcal{K}(E) = \{K\subset E : K \text{ is compact}\}$$

is itself a complete metric space with the Hausdorff metric

$$d_H(K_1,K_2) = \max\left\{ \sup_{a\in K_1} d(a,K_2), \sup_{b\in K_2} d(b,K_1)\right\};$$

moreover

$$d_H(K_1,K_2) = \sup_{a\in E} |d(a,K_1)-d(a,K_2)| = \max_{a\in K_1\cup K_2} |d(a,K_1)-d(a,K_2)|.$$

If we endow the space $(\mathcal{K}(E), d_H)$ with its Borel σ-algebra, then it's possible to show that a map $F\colon(\Omega,\mathcal{A})\to(\mathcal{K}(E),d_H)$ is measurable if and only if, for all $a\in E$, the map

$$\Omega\ni\omega\mapsto d(a,F(\omega))\in\mathbb{R}$$

is measurable.

Corollary 1.45. *The map from* $\mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda}$ *to* $\mathcal{K}(C_t^\alpha)$ *given by* $(x_0, A) \mapsto C(x_0, A)$ *is a measurable multifunction.*

Proof. It follows immediately from Theorem 1.41 and the fact that lower semicontinuous maps are measurable. □

Since composition of measurable functions is still measurable, we readily obtain the following:

Corollary 1.46. *Let* $(\Omega, \mathcal{F})$ *be a measurable space on which an* $\mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda}$*-valued random variable* (ξ_0, Ξ) *is defined; then* $\omega \mapsto C(\xi_0(\omega), \Xi(\omega))$ *defines a* $\mathcal{K}(C_t^\alpha)$*-valued random variable (this is usually referred to as a random set in* C_t^α*).*

We can also obtain the existence of measurable selections for $C(x_0, A)$. To this end, let us recall the following classical result.

Theorem 1.47. (Theorem 12.1.10 from [258]) *Let* (G, σ) *be a measurable space,* (E, d) *a separable metric space and* $y \mapsto K_y$ *be a measurable map from* G *to* $\mathcal{K}(E)$*. Then there exists a measurable selection of* K*, namely a measurable map* $k: G \to E$ *such that* $k(y) \in K_y \, \forall y \in G$*.*

Corollary 1.48. *There exists a measurable map* $X: \mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda} \to C_t^\alpha$ *such that* $X(x_0, A)$ *is a solution to the YDE associated to* (x_0, A) *for all* $(x_0, A) \in \mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda}$*. In particular, in the setting of Corollary 1.46,* $\omega \mapsto X(\xi_0(\omega), \Xi(\omega))$ *defines a selection for the random set* $\omega \mapsto C(\xi_0(\omega), \Xi(\omega))$*.*

Proof. The statement immediately follows from Theorem 1.47 for $G = \mathbb{R}^d \times C_t^\alpha C_x^{\beta,\lambda}$.[1.1] □

Let us finally point out a trivial fact: if A is regular, then necessarily $C(x_0, A) = \{\Phi^A_{0\to\cdot}(x_0)\}$, where Φ^A is the flow associated to A, and $X(x_0, A) = \Phi^A_{0\to\cdot}(x_0)$.

1.4.2 Conditional uniqueness

This section provides several criteria for uniqueness of the YDE, under additional assumptions on the associated solutions. Typically such properties can't be established directly, at least not under mild regularity assumptions on A; yet the criteria are rather useful in application to SDEs, where the analytic theory can be combined with more probabilistic techniques (see the forthcoming Chapter 3 and the use of Girsanov transform).

We start with the following result, which states that the existence of a sufficiently regular flow necessarily implies uniqueness of solutions. It is inspired by the analogue results for ODEs in the style of van Kampen and Shaposhnikov, see [265], [254].

Theorem 1.49. *Suppose* $A \in C_t^\alpha C_V^{\beta,\lambda}$ *with* $\alpha(1+\beta) > 1$, $\beta + \lambda \leqslant 1$ *and that the associated YDE admits a spatially locally* γ*-Hölder continuous flow. If*

$$\alpha\gamma(1+\beta) > 1,$$

then for any $x_0 \in V$ *there exists a unique solution to the YDE in the class* $x \in C_t^\alpha V$*.*

Proof. Let $x_0 \in V$ and x be a given solution to the YDE starting at x_0. By the a priori estimate (1.26), we can always find $R = R(x_0)$ big enough such that

$$\sup_{s \in [0,T]} \{\|x\|_\alpha + \|\Phi(s, \cdot, x_s)\|_{\alpha; s, T}\} \leqslant R;$$

therefore in the following computations, up to a localisation argument, we can assume without loss of generality that $A \in C_t^\alpha C_V^\beta$ and that Φ is globally γ-Hölder.

1.1. Technically speaking, C_t^α is not separable, so we couldn't apply Theorem 1.47 directly. However, we may replace C_t^α with $\bar{C}_t^{\alpha-\varepsilon}$, where $\bar{C}_t^\delta$ denotes the closure of C_c^∞ under the C^δ-norm; $\mathcal{K}(C_t^\alpha)$ naturally embeds into $\mathcal{K}(\bar{C}_t^{\alpha-\varepsilon})$ and we may take $E = \bar{C}_t^{\alpha-\varepsilon}$. As usual such a loss of regularity is harmless, since we may choose $\varepsilon > 0$ small enough so that $(\alpha-\varepsilon)(1+\beta) > 1$, so that the nonlinear YDE is still meaningful.

It suffices to show that $f_t := \Phi(t,T,x_t) - \Phi(0,T,x_0)$ satisfies $\| f_{s,t}\|_V \lesssim |t-s|^{1+\varepsilon}$ for some $\varepsilon > 0$; if that is the case, then necessarily $f \equiv 0$, namely $\Phi(t,T,x_t) = \Phi(0,T,x_0)$ for all $t \in [0,T]$; inverting the flow we find $x_t = \Phi(0,t,x_0)$, implying that $\Phi(0,\cdot,x_0)$ is the unique solution starting from x_0.

By the flow property

$$\begin{aligned}\| f_{s,t}\|_V &= \|\Phi(t,T,x_t) - \Phi(s,T,x_s)\|_V \\ &= \|\Phi(t,T,x_t) - \Phi(t,T,\Phi(s,t,x_s))\|_V \\ &\lesssim \|x_t - \Phi(s,t,x_s)\|_V^\gamma.\end{aligned}$$

Since both x and $\Phi(s,\cdot,x_s)$ are solutions to the YDE starting from x_s, it holds

$$\begin{aligned}\|x_t - \Phi(s,t,x_s)\|_V &= \left\| \int_s^t A(\mathrm{d}r, x_r) - \int_s^t A(\mathrm{d}r, \Phi(s,r,x_s)) \right\|_V \\ &\lesssim \|A_{s,t}(x_s) - A_{s,t}(\Phi(s,s,x_s))\|_V + |t-s|^{\alpha(1+\beta)} \|A\|_{\alpha,\beta} (1 + [\![x]\!]_\alpha + [\![\Phi(s,\cdot,x_s)]\!]_\alpha) \\ &\lesssim |t-s|^{\alpha(1+\beta)},\end{aligned}$$

where we used the basic property $\Phi(s,s,x) = x$; overall we obtain $\| f_{s,t}\|_V \lesssim |t-s|^{\gamma\alpha(1+\beta)}$, which implies the conclusion. □

Remark 1.50. The assumptions of Theorem 1.49 can be weakened in several ways. For instance, the existence of a γ-Hölder regular semiflow is enough to establish that $\Phi(t,T,x_t) = \Phi(0,T,x_0)$, even when Φ is not invertible. Uniqueness only requires $\Phi(t,T,\cdot)$ to be invertible for $t \in D$, D dense subset of $[0,T]$; indeed this implies $x_t = \Phi(0,t,x_0)$ on D and then by continuity the equality can be extended to the whole $[0,T]$. Similarly, it is enough to require

$$\sup_{t \in D} \|\Phi(t,T,\cdot)\|_{\gamma,R} < \infty \quad \text{for all } R \geqslant 0$$

for D dense subset of $[0,T]$ as before.

The next conditional uniqueness statements are slightly more subtle and require more setup. We start by introducing the concept of averaged translation, originally due to [57], cf. Definition 2.13; we provide a different construction of it based on the sewing lemma (although with the same underlying idea).

Definition 1.51. *Let $A \in C^\alpha_t C^\beta_V$, $y \in C^\gamma_t V$ with $\alpha + \beta\gamma > 1$. The averaged translation $\tau_x A$ is defined as*

$$\tau_y A(t,x) = \int_0^t A(\mathrm{d}s, z + y_s) \quad \forall t \in [0,T], z \in V.$$

Lemma 1.52. *Let $A \in C^\alpha_t C^{n+\beta}_V$, $y \in C^\gamma_t V$ with $\alpha + \beta\gamma > 1$, $\eta \in (0,1)$ satisfying $\eta < n + \beta$, $\alpha + \eta\gamma > 1$. The operator τ_y is continuous from $C^\alpha_t C^{n+\beta}_V$ to $C^\alpha_t C^{n+\beta-\eta}_V$ and there exists $C = C(\alpha,\beta,\gamma,\eta,T)$ s.t.*

$$\|\tau_y A\|_{\alpha,n+\beta-\eta} \leqslant C \|A\|_{\alpha,n+\beta} (1 + [\![y]\!]_\gamma). \tag{1.53}$$

Proof. Observe that $\tau_y A$ corresponds to the sewing of $\Gamma: \Delta_2 \to C^{n+\beta}_V$ given by

$$\Gamma_{s,t} := A_{s,t}(\,\cdot\, + y_s).$$

It holds $\|\Gamma_{s,t}\|_{n+\beta} \leqslant |t-s|^\alpha \, \|A\|_{\alpha,n+\beta}$; moreover by Lemma A.34 in Appendix A.6 it holds

$$\begin{aligned}\|\delta\Gamma_{s,u,t}\|_{n+\beta-\eta} &= \|A_{u,t}(\,\cdot\, + y_s) - A_{u,t}(\,\cdot\, + y_u)\|_{n+\beta-\eta} \\ &\lesssim \| y_s - y_u\|_V^\eta \, \|A_{u,t}\|_{n+\beta} \\ &\lesssim |t-s|^{\alpha+\gamma\eta} \, [\![y]\!]_\gamma \, \|A\|_{\alpha,n+\beta}.\end{aligned}$$

Since $\alpha + \gamma\eta > 1$, by the sewing lemma we deduce that $\mathcal{J}(\Gamma) = \tau_y A \in C^\alpha_t C^{n+\beta-\eta}_V$, together with estimate (1.53). □

Young integrals themselves can indeed be regarded as averaged translations evaluated at $z = 0$. Iterating translations is a consistent procedure, as the following lemma shows.

Lemma 1.53. *Assume that $\alpha+\beta\gamma>1$ and $A\in C_t^\alpha C_V^\beta$, $x\in C_t^\gamma V$ and $\tau_x A\in C_t^\alpha C_V^\beta$. Then for any $y\in C_t^\gamma V$ it holds*

$$\int_0^t (\tau_x A)(\mathrm{d}s, y_s) = \int_0^t A(\mathrm{d}s, x_s+y_s) \quad \forall t\in[0,T].$$

Proof. The statement follows immediately from the observation that for any $s\leqslant t$ it holds

$$\begin{aligned}\left\| \int_s^t (\tau_x A)(\mathrm{d}r, y_r) - \int_s^t A(\mathrm{d}r, x_r+y_r)\right\| &\lesssim \|(\tau_x A)_{s,t}(y_s) - A_{s,t}(x_s+y_s)\| + |t-s|^{\alpha+\beta\gamma}\\ &\lesssim \|(A_{s,t}(\cdot+x_s))(y_s) - A_{s,t}(x_s+y_s)\| + |t-s|^{\alpha+\beta\gamma}\\ &\lesssim |t-s|^{\alpha+\beta\gamma}\end{aligned}$$

so that the two integrals must coincide. □

The main reason for introducing averaged translations is the following key result, which is a generalization of Theorem 4.8 from [145].

Theorem 1.54. (Conditional Comparison Principle) *Let $A^1, A^2\in C_t^\alpha C_V^\beta$ with $\alpha(1+\beta)>1$ for some $\alpha,\beta\in(0,1)$ and let $x^i\in C_t^\alpha V$ be given solutions respectively to the YDE associated to (x_0^i, A^i). Suppose in addition that x^1 is such that $\tau_{x^1}A^1\in C_t^\alpha \mathrm{Lip}_V$. Then there exists $C=C(\alpha,\beta,T)$, increasing in the last variable, such that*

$$\|x^1-x^2\|_\alpha \leqslant C\exp\big(C\|\tau_{x^1}A^1\|_{\alpha,1}^{1/\alpha}\big)(1+\|A^2\|_{\alpha,\beta}^2)\,(\|x_0^1-x_0^2\|+\|A^1-A^2\|_{\alpha,\beta}). \tag{1.54}$$

In particular, uniqueness holds in the class $C_t^\alpha V$ to the YDE associated to (x_0^1, A^1).

Proof. The final uniqueness claim immediately follows from inequality (1.54), since in that case we can consider $A^1=A^2$, $x_0^1=x_0^2$. Now let x^i be two solutions as above, then their difference $e=x^1-x^2$ satisfies

$$\begin{aligned} e_t &= e_0 + \int_0^t A^1(\mathrm{d}s, x_s^1) - \int_0^t A^2(\mathrm{d}s, x_s^2)\\ &= e_0 + \int_0^t A^1(\mathrm{d}s, x_s^1) - \int_0^t A^1(\mathrm{d}s, e_s+x_s^1) + \int_0^t (A^2-A^1)(\mathrm{d}s, x_s^2)\\ &= e_0 - \int_0^t \tau_{x^1}A^1(\mathrm{d}s, e_s) + \int_0^t \tau_{x^1}A^1(\mathrm{d}s, 0) + \int_0^t (A^2-A^1)(\mathrm{d}s, x_s^2)\\ &= e_0 + \int_0^t B(\mathrm{d}s, e_s) + \psi_t\end{aligned}$$

where in the third line we applied Lemma 1.53 and we take

$$B(t,z) = -\tau_{x_1}A^1(t,z) + \tau_{x_1}A^1(t,0), \qquad \psi_\cdot = \int_0^\cdot (A^2-A^1)(\mathrm{d}s, x_s^2).$$

By the hypothesis, $B\in C_t^\gamma \mathrm{Lip}_V$ with $B(t,0)=0$ for all $t\in[0,T]$, while $\psi\in C_t^\alpha V$. Therefore from Theorem 1.20 applied to v we deduce the existence of a constant $\kappa_1=\kappa_1(\alpha,T)$ such that

$$\|x^1-x^2\|_\alpha \leqslant \kappa_1 \exp\big(\kappa_1 [\![\tau_{x^1}A^1]\!]_{1,\alpha}^{1/\alpha}\big)(\|x_0^1-x_0^2\|_V + [\![\psi]\!]_\alpha).$$

On the other hand, estimates (1.6) and (1.25) imply that

$$[\![\psi]\!]_\alpha \leqslant \kappa_2\,\|A^1-A^2\|_{\alpha,\beta}\,(1+\|A^2\|_{\alpha,\beta}^2)$$

for some $\kappa_2=\kappa_2(\alpha,\beta,T)$. Combining the above estimates, the conclusion follows. □

Remarkably, the hypothesis $\tau_x A\in C_t^\alpha \mathrm{Lip}_V$ allows not only to show that x is the unique solution starting at x_0, but also that any other solution will not get too close to it.

Lemma 1.55. *Let $A\in C_t^\alpha C_V^\beta$ with $\alpha(1+\beta)>1$, $x,y\in C_t^\alpha V$ solutions respectively to the YDEs associated to (x_0,A), (y_0,A) and assume that $\tau_x A\in C_t^\alpha \mathrm{Lip}_V$. Then there exists $C=C(\alpha,T)$ s.t.*

$$C^{-1}\exp\big(-C\|\tau_x A\|_{\alpha,1}^{1/\alpha}\big) \leqslant \inf_{t\in[0,T]} \frac{\|x_t-y_t\|_V}{\|x_0-y_0\|_V} \leqslant \sup_{t\in[0,T]} \frac{\|x_t-y_t\|_V}{\|x_0-y_0\|_V} \leqslant C\exp\big(C\|\tau_x A\|_{\alpha,1}^{1/\alpha}\big) \tag{1.55}$$

Proof. The special case of an Hilbert space V was considered in [141], Lemma 5.7; here we give an alternative, simpler proof, which covers general Banach V.

The first inequality is an immediate consequence of Theorem 1.54, so we only need to prove the second one. For any fixed $\tau \in [0, T]$, by applying the time reversal $\tilde{x}_t = x_{\tau - t}$, $\tilde{y}_t = y_{\tau - t}$ (cf. Lemma 1.30) and the first inequality, we then deduce that

$$\begin{aligned}\|x_0 - y_0\|_V = \|\tilde{x}_\tau - \tilde{y}_\tau\|_V &\leqslant C_{\alpha,\tau} \exp\big(C_{\alpha,\tau} \|\tau_x A\|_{\alpha,1;[0,\tau]}^{1/\alpha}\big) \|\tilde{x}_0 - \tilde{y}_0\|_V\\ &\leqslant C_{\alpha,T} \exp\big(C_{\alpha,T} \|\tau_x A\|_{\alpha,1}^{1/\alpha}\big) \|x_\tau - y_\tau\|_V,\end{aligned}$$

where we used the fact that $\tau \mapsto C_{\alpha,\tau}$ can be chosen in an increasing way. Taking the infimum over $\tau \in [0, T]$ readily yields the conclusion. □

Under the assumption of regularity of $\tau_x A$, convergence of the Euler scheme to the unique solution can be established, with the same rate $2\alpha - 1$ as in the more regular case of $A \in C_t^\alpha C_V^{1+\beta}$. The following result is a direct analogue of Corollary 1.27.

Corollary 1.56. *Let $A \in C_t^\alpha \mathrm{Lip}_V$ with $\alpha > 1/2$, $x_0 \in V$ and suppose there exists a solution x associated to (x_0, A) such that $\tau_x A \in C_t^\alpha \mathrm{Lip}_V$ (which is therefore the unique solution); denote by x^n the element of $C_t^\alpha V$ constructed by the n-step Euler approximation from Theorem 1.13. Then there exists $C = C(\alpha, T)$ such that*

$$\|x - x^n\|_\alpha \leqslant C \exp\big(C \|\tau_x A\|_{\alpha,1}^{1/\alpha}\big) (1 + \|A\|_{\alpha,1}^3)\, n^{1-2\alpha} \quad \text{as } n \to \infty.$$

Proof. As in the proof of Corollary 1.27, recall that x^n satisfies the YDE

$$x^n = x_0 + \int_0^t A(\mathrm{d}s, x_s^n) + \psi_t^n, \qquad [\![\psi^n]\!]_\alpha \lesssim_{\alpha,T} (1 + \|A\|_{\alpha,1}^3)\, n^{1-2\alpha}.$$

Therefore $v^n = x^n - x$ satisfies

$$v_t^n = \int_0^t B(\mathrm{d}s, v_s^n) + \psi_t^n, \quad B(t,z) = \tau_x A(t,z) - \tau_x A(t,0), \quad [\![B]\!]_{\alpha,1} = [\![\tau_x A]\!]_{\alpha,1}.$$

Applying Theorem 1.20 we obtain that, for suitable $\kappa = \kappa(\alpha, T)$ it holds

$$\|x - x^n\|_\alpha \leqslant \kappa \exp\big(\kappa \|\tau_x A\|_{\alpha,1}^{1/\alpha}\big)\, [\![\psi^n]\!]_\alpha$$

which combined with the above inequality for $[\![\psi^n]\!]_\alpha$ gives the conclusion. □

1.4.3 The case of continuous $\partial_t A$

In this section we study how the wellposedness theory changes when, in addition to the regularity condition $A \in C_t^\alpha C_t^\beta$, we impose $\partial_t A$: $[0,T] \times V \to V$ to exist continuous and uniformly bounded (we assume boundedness for simplicity, but it could be replaced by a growth condition).

The key point is that, by Point 2. from Theorem 1.6, any solution to the YDE is also a solution to the classical ODE associated to $\partial_t A$; as such, it is Lipschitz continuous with constant $\|\partial_t A\|_\infty$. We can exploit this additional time regularity, combined with nonlinear Young theory, to obtain well-posedness under weaker conditions than those from Theorem 1.23, which are at the same time not covered by the case of Lipschitz $\partial_t A$.

While the existence of $\partial_t A$ is not a very meaningful requirement for classical YDEs, i.e. for $A(t,x) = f(x) y_t$, as it would imply that $y \in C_t^1$, there are other situations in which it becomes a natural assumption. One example is for perturbed ODEs $\dot{x} = b(x) + \dot{w}$, in which the associated A is the averaged field

$$A(t,x) = \int_0^t b_s(x + w_s)\, \mathrm{d}s$$

for which $\partial_t A$ exists continuous as soon as b is continuous field; this case will be studied more in detail in Chapter 3.

Theorem 1.57. *Let A be such that $A\in C^\alpha_t C^{1+\beta}_V$ and $\partial_t A\in C^0([0,T]\times V;V)$ with $\alpha+\beta>1$. Then for any $x_0\in V$ there exists a unique global solution to the YDE associated to (x_0,A).*

Proof. Similarly to Theorem 1.23, the proof is by Banach fixed point theorem. For suitable values of $M,\tau>0$ to be fixed later, consider the space $E:=\{x\in \mathrm{Lip}([0,\tau];V): x(0)=x_0, [\![x]\!]_{\mathrm{Lip}}\leqslant M\}$; it is a complete metric space with the metric $d(x,y)=[\![x-y]\!]_\gamma$ (the condition $[\![x]\!]_{\mathrm{Lip}}\leqslant M$ is essential for this to be true). Define the map $\mathcal{I}$ by

$$\mathcal{I}(x)_t=x_0+\int_0^t \partial_t A(s,x_s)\mathrm{d}s=x_0+\int_0^t A(\mathrm{d}s,x_s);$$

observe that, under the condition $\|\partial_t A\|_\infty\leqslant M$, $\mathcal{I}$ maps E into itself. By the hypothesis and Corollary 1.10, for any $x,y\in E$ it holds

$$\begin{aligned}\|\mathcal{I}(x)_{s,t}-\mathcal{I}(y)_{s,t}\|_V=&\left\|\int_s^t v_{\mathrm{d}u}(x_u-y_u)\right\|_V\\ \leqslant&\|v_{s,t}(x_s-y_s)\|_V+\kappa_1\,[\![v]\!]_\alpha[\![x-y]\!]_{\mathrm{Lip}}|t-s|^{2\alpha}\\ \leqslant&[\![v]\!]_\alpha[\![x-y]\!]_\alpha(s^\alpha+\kappa_1|t-s|^\alpha)|t-s|^\alpha\\ \leqslant&\kappa_2\tau^\alpha\|A\|_{\alpha,1+\beta}\,(1+[\![x]\!]_{\mathrm{Lip}}+[\![y]\!]_{\mathrm{Lip}})[\![x-y]\!]_\alpha|t-s|^\alpha\end{aligned}$$

which implies

$$[\![\mathcal{I}(x)-\mathcal{I}(y)]\!]_\alpha\leqslant\kappa_2\,\tau^\alpha\|A\|_{\alpha,1+\beta}(1+2M)[\![x-y]\!]_\alpha<[\![x-y]\!]_\alpha$$

as soon as we choose τ small enough such that $\kappa_2\,\tau^\alpha\|A\|_{\alpha,1+\beta}(1+2M)<1$. Therefore $\mathcal{I}$ is a contraction on E and for any $x_0\in V$ there exists a unique associated solution $x\in C^\gamma([0,\tau];V)$. Global existence and uniqueness then follows from the usual iterative argument. □

We can also establish an analogue of Theorem 1.25 in this setting.

Theorem 1.58. *Let $M>0$ fixed. For $i=1,2$, let $A^i\in C^\alpha_t C^\beta_V$ such that $\partial_t A^i\in C^0([0,T]\times V;V)$, $\alpha+\beta>1$ and $\|A^i\|_{\alpha,\beta}+\|\partial_t A^i\|_\infty\leqslant M$, as well as $A^1\in C^\alpha_t C^{1+\beta}_V$ with $\|A^1\|_{\alpha,1+\beta}\leqslant M$, and $x^i_0\in V$; let x^i be two given solutions associated respectively to (x^i_0,A^i). Then it holds*

$$[\![x^1-x^2]\!]_\alpha\leqslant C(\|x^1_0-x^2_0\|_V+\|A^1-A^2\|_{\alpha,\beta})$$

for a constant $C=C(\alpha,\beta,T,M)$ increasing in the last variable. A more explicit formula for C is given by (1.56).

Proof. The proof is analogous to that of Theorem 1.25, so we will mostly sketch it; it is based on an application of Corollary 1.10 and Theorem 1.20.

Given two solutions as above, their difference $e=x^1-x^2$ satisfies the affine YDE

$$e_t=e_0+\int_0^t v_{\mathrm{d}s}\,e_s+\psi_t$$

with

$$v_t=\int_0^t\int_0^1 DA^1(\mathrm{d}s,x^2_s+\lambda e_s)\mathrm{d}\lambda,\quad \psi_t=\int_0^t(A^1-A^2)(\mathrm{d}s,x^2_s).$$

We have the estimates

$$\begin{aligned}\|v\|_{\alpha,1}&\lesssim_{\alpha,\beta,T}\ \|A^1\|_{\alpha,1+\beta}(1+[\![x^1]\!]_{\mathrm{Lip}}+[\![x^2]\!]_{\mathrm{Lip}})\lesssim\|A^1\|_{\alpha,1+\beta}(1+\|\partial_t A^1\|_\infty+\|\partial_t A^2\|_\infty),\\ \|\psi_t\|_\alpha&\lesssim_{\alpha,\beta,T}\ \|A^1-A^2\|_{\alpha,\beta}(1+[\![x^2]\!]_{\mathrm{Lip}})\lesssim\|A^1-A^2\|_{\alpha,\beta}(1+\|\partial_t A^2\|_\infty);\end{aligned}$$

combined with Theorem 1.20, they yield

$$\begin{aligned}\|e\|_\alpha\leqslant&\kappa_1\,e^{\kappa_1(1+\|A^1\|^2_{\alpha,1+\beta})(1+\|\partial_t A^1\|^2_\infty+\|\partial_t A^2\|^2_\infty)}(\|e_0\|_V+\|A^1-A^2\|_{\alpha,\beta}(1+\|\partial_t A^2\|_\infty))\\ \leqslant&\kappa_2\,e^{\kappa_2(1+\|A^1\|^2_{\alpha,1+\beta})(1+\|\partial_t A^1\|^2_\infty+\|\partial_t A^2\|^2_\infty)}(\|e_0\|_V+\|A^1-A^2\|_{\alpha,\beta})\end{aligned}$$

for some $\kappa_2=\kappa_2(\alpha,\beta,T)$. In particular, C can be taken of the form

$$C(\alpha,\beta,T,M)=\kappa_3(\alpha,\beta,T)\exp(\kappa_3(\alpha,\beta,T)(1+M^4)).\tag{1.56}$$ □

Corollary 1.59. *Given A as in Theorem 1.57, denote by x^n the element of $C_t^\alpha V$ constructed by the n-step Euler approximation from Theorem 1.13 and by x the solution associated to (x_0, A). Then there exists a constant $C = C(\alpha, \beta, T, \|A\|_{\alpha,1+\beta}, \|\partial_t A\|_\infty)$ such that*

$$\|x - x^n\|_\alpha \leqslant C n^{-\alpha} \quad \text{as } n \to \infty.$$

A more explicit formula for C is given by (1.57).

Proof. By Theorem 1.13, x^n satisfies the YDE

$$x^n = x_0 + \int_0^t A(\mathrm{d}s, x_s^n) + \psi_t^n = x_0 + \int_0^t A^n(\mathrm{d}s, x_s^n)$$

where $A^n(t,z) := A(t,z) + \psi_t^n$ and that by estimate (1.23), for the choice $\Delta = T$, $\beta = \gamma = 1$, we have

$$[\![\psi^n]\!]_\alpha \lesssim_{\alpha,T} \|A\|_{\alpha,1} [\![x^n]\!]_{\mathrm{Lip}}\, n^{-\alpha} \lesssim \|A\|_{\alpha,1} \|\partial_t A\|_\infty\, n^{-\alpha}.$$

Defining $e^n := x - x^n$, by the basic estimates $\|A - A^n\|_{\alpha,\beta} \lesssim_T [\![\psi^n]\!]_\alpha$ and $\|\partial_t A^n\|_\infty \lesssim \|\partial_t A\|_\infty$, going through the same proof as in Theorem 1.58 we deduce that

$$\|e^n\|_\alpha \leqslant \kappa_1 \exp(\kappa_2 (1 + \|A\|_{\alpha,1}^2)(1 + \|\partial_t A\|_\infty^2))\, \|A - A^n\|_{\alpha,\beta}$$

and so finally that, for a suitable constant $\kappa_2 = \kappa_2(\alpha, T)$, it holds

$$\|e^n\|_\alpha \leqslant \kappa_2 \exp(\kappa_2 (1 + \|A\|_{\alpha,1}^2)(1 + \|\partial_t A\|_\infty^2))\, n^{-\alpha}. \tag{1.57}$$ □

Finally, we can provide an analogue of Corollary 1.56 to this setting.

Corollary 1.60. *Let A be such that $A \in C_t^\alpha C_V^\beta$ and $\partial_t A \in C^0([0,T] \times V; V)$ with $\alpha(1+\beta) > 1$, $x_0 \in V$ and suppose there exists a solution x associated to (x_0, A) such that $\tau_x A \in C_t^\alpha \mathrm{Lip}_V$ (which is therefore the unique solution); denote by x^n the element of $C_t^\alpha V$ constructed by the n-step Euler approximation from Theorem 1.13. Then there exists $C = C(\alpha, T)$ such that*

$$\|x - x^n\|_\alpha \leqslant C \exp\big(C \|\tau_x A\|_{\alpha,1}^{1/\alpha}\big) \|A\|_{\alpha,1} \|\partial_t A\|_\infty\, n^{-\alpha} \quad \text{as } n \to \infty.$$

Proof. As usual, x^n satisfies the YDE

$$x^n = x_0 + \int_0^t A(\mathrm{d}s, x_s^n) + \psi_t^n, \qquad [\![\psi^n]\!]_\alpha \lesssim_{\alpha,T} \|A\|_{\alpha,1} \|\partial_t A\|_\infty\, n^{-\alpha};$$

From here, the proof is mostly identical to that of Corollary 1.56. □

Remark 1.61. The results from Section 1.3 can be generalized to this setting as well. For instance one can show that, under the assumptions of Theorem 1.58, there exist an associated flow of diffeomorphisms Φ^A and that the variational equation 1.37 still holds. Moreover, if $A \in C_t^\alpha C_V^{n+\beta}$ with $\partial_t A \in C^0([0,T] \times V; V)$ and $\alpha + \beta > 1$, then Φ^A is locally C^n (which is the analogue of Theorem 1.35); again we refer to Theorem 4.33 from [145] for a proof. Finally, everything can be further extended by only requiring local regularity: e.g. in order to have a locally C^n flow, it suffices to know that $\partial_t A \in C^0([0,T] \times V; V)$ and $A \in C_t^\alpha C_{V,\mathrm{loc}}^{n+\beta}$; in analogy with Corollary 1.36, one then has a continuous driver-to-flow map $A \mapsto \Phi^A$ from $C_t^1 C_V^0 \cap C_t^\alpha C_{V,\mathrm{loc}}^{n+\beta}$ to $C_t^0 C_{V,\mathrm{loc}}^n$.

1.5 Bibliographical comments

As already mentioned, all the material presented here is taken from [141]. There are a few modifications: I[1.2] have decided to omit some (not particularly interesting) proofs (e.g. Proposition 1.17, Theorem 1.35) and slightly extended Section 1.4.1, with the new Corollary 1.48.

1.2. I have decided to drop the *plurale maiestatis* in these bibliographical, more colloquial, sections. This is in order to emphasize that several of the statements appearing are not entirely of mathematical nature, but rather reflect my personal opinion; the reader is of course free to disagree with them.

I have also omitted Section 3.5 from [141], where two further variants of the ODEs considered here were treated, namely *mixed equations*

$$\mathrm{d}x_t = F(t, x_t)\mathrm{d}t + A(\mathrm{d}t, x_t)$$

and *fractional Young equations*

$$(D^{\delta}_{0+}x)_t = A(\mathrm{d}t, x_t)$$

where D^{δ}_{0+} denotes the Riemann–Liouville fractional derivative of order $\delta \in (0, 1]$.

Young integrals first appeared in [277] and standard YDEs (i.e. for $A(t, x) = f(x) y_t$) were first studied by Lyons in [209]. Sharp results, including explicit counterexample to existence or uniqueness, were given by Davie in [87], who also established convergence of numerical schemes and provided growth criteria for absence of blow-up like those from Proposition 1.18.

In this sense, all the statements presented here are natural extensions of the standard case. There is one (extremely sharp!) result which I didn't cover, again due to [87] and later revisited in [134]: for $A = f \otimes y$, uniqueness of solutions holds when $f \in C^{1/\alpha}$ and $y \in C^{\alpha}$ (which would correspond to the borderline case $A \in C^{\alpha}_t C^{1+\beta}_V$ for $\alpha(1+\beta) = 1$). In this case however, the associated flow needs not be Lipschitz nor differentiable (one should consider the analogy with standard ODE theory, when passing from Lipschitz drifts to Log-Lipschitz ones).

In another direction, here I focused for simplicity only on time regularity measured in Hölder scales; however Young integration theory has a natural formulation in the setting of paths of bounded p-variation with the use of controls (again see the monography [134] and the review [199]). Passing from Hölder continuity to p-variation can be often established rather simply by a time reparametrization argument; but there are applications where the p-variation perspective can be fundamental, see e.g. the discussion on "complementary Young regularity", Section 11.1 from [132]. An extension of nonlinear Young integration with p-variation has been recently given in [10].

In connection to the above and Malliavin calculus, one could in fact in Example 1.40 consider general ψ of bounded $1/2$-variation, instead of just $\psi \in W^{1,1}$. This fact is particularly important in the case of SDEs driven by fractional Brownian motion, as it is known that the associated Cameron-Martin space is made of elements of bounded $1/2$-variation for any value of the Hurst parameter $H \in (0, 1)$, see [137].

In relation to the a priori estimates like Proposition 1.18 and Theorem 1.20, let me mention the existence of general Gronwall rough lemmas, see Lemma 3.2 from [175] and Lemma 2.12 from [94]; in the case of rough differential equations, finding sharp growth conditions to avoid finite time blow-up of solutions is a topic of ongoing research, see Section 10.7 from [134], [87], [200] and [22] for some results.

Like standard Young equations can be regarded as a particular case of rough differential equations, nonlinear YDEs are a subclass of the rough flows developed by Bailleul in [20, 21]; alternatively, they can also be solved by means of the nonlinear sewing lemma developed in [45, 47, 46]. Still, there are many good reasons to develop a detailed theory for nonlinear YDEs, as will become more clear by the applications presented in the next chapters; my impression is that, deriving the same statements from these more general theories, often results in the need for stronger assumptions (e.g. more regular coefficients). In a different direction, let me also mention that nonlinear Young integrals are a subcase of the nonlinear paraproducts developed in [138].

Chapter 2
Nonlinear Young PDEs

This chapter serves as a natural follow-up to Chapter 1. While therein we only focused at the "particle" level, namely the trajectories x_t satisfying suitable differential equations indexed on $[0,T]$, here we pass to deal with maps indexed on $[0,T]\times\mathbb{R}^d$ which satisfy suitable PDE-type equations. The common theme however is that the objects in consideration will not be differentiable in time, even in a weak sense, making it necessary to employ Young integration techniques. We will only consider two prototypical type of equations, given respectively by: i) linear first order PDEs, treated in Section 2.1; ii) parabolic nonlinear equations, which are still however of semilinear type (thus allowing for a mild formulation of the problem), cf. Section 2.2.

All the material given here is taken from Sections 6 and 7 from [141], with [145] also being a precursor regarding Section 2.1. We will adopt the same notations and conventions as in Chapter 1.

2.1 Young transport equations

This section is devoted to the study of nonlinear Young transport equations of the form

$$u_{\mathrm{d}t}+A_{\mathrm{d}t}\cdot\nabla u_t+c_{\mathrm{d}t}\,u_t=0, \tag{2.1}$$

which we will henceforth refer to as the YTE associated to (A,c). We realize the terminology can be slightly misleading: equation (2.1) is linear in the unknown u and the term "nonlinear" rather comes from the underlying nonlinear Young integral theory needed to solve it.

We restrict here to the case $V=\mathbb{R}^d$; similarly to Section 1.4.1, we adopt the simpler notations $C_t^\alpha C_x^\beta$ in place of $C_t^\alpha C_{\mathbb{R}^d}^\beta$. Like in Section 1.3, for simplicity we assume some global bounds on the driver A like $A\in C_t^\alpha C_x^{1+\beta}$, but standard (a bit tedious) localisation arguments allow to relax them to growth conditions and local regularity requirements.

Classical results on weak solutions to (2.1) in the case $A_{\mathrm{d}t}=b_t\,\mathrm{d}t$, $c_{\mathrm{d}t}=\tilde{c}_t\,\mathrm{d}t$ can be found in [97], [4]. Our approach here mostly follows the one given in [145], although slightly less based on the method of characteristics, and more on a duality approach. We refer to Section 2.3 for more references in the literature.

Before explaining the meaning of (2.1), we need some preparations. Given any compact $K\subset\mathbb{R}^d$, we denote by $C_K^\beta=C_K^\beta(\mathbb{R}^d)$ the Banach space of $f\in C^\beta(\mathbb{R}^d)$ with $\operatorname{supp} f\subset K$ and by $C_c^\beta=C_c^\beta(\mathbb{R}^d)$ the set of all compactly supported β-Hölder continuous functions. C_c^β is a direct limit of Banach spaces and thus it is locally convex; we denote its topological dual by $(C_c^\beta)^*$. Given $\gamma,\beta\in(0,1)$, we say that $f\in C_t^\alpha C_c^\beta$ if there exists a compact K such that $f\in C_t^\alpha C_K^\beta$; similarly, a distribution $u\in C_t^\gamma(C_c^\beta)^*$ if $u\in C_t^\gamma(C_K^\beta)^*$ for all compact $K\subset\mathbb{R}^d$. We will use the bracket $\langle\cdot,\cdot\rangle$ to denote both the classical L^2-pairing and the one between C_c^β and its dual. Finally, $\mathcal{M}_{\mathrm{loc}}$ denotes the space of Radon measures on $\mathbb{R}^d$, $\mathcal{M}_K$ the space of finite signed measure supported on K; observe that the above notation is consistent with $\mathcal{M}_{\mathrm{loc}}=(C_c^0)^*$.

We are now ready to give a notion of solution to the YTE.

Definition 2.1. *Let $\alpha,\beta\in(0,1)$ such that $\alpha(1+\beta)>1$. We say that $u\in L_t^\infty\mathcal{M}_{\mathrm{loc}}\cap C_t^{\alpha\beta}(C_c^\beta)^*$ is a weak solution to the YTE associated to $A\in C_t^\alpha C_x^\beta$, $c\in C_t^\alpha C_x^\beta$ with $\operatorname{div} A\in C_t^\alpha C_x^\beta$ if*

$$\langle u_t,\varphi\rangle-\langle u_0,\varphi\rangle=\int_0^t\langle A_{\mathrm{d}s}\cdot\nabla\varphi+(\operatorname{div}A_{\mathrm{d}s}-c_{\mathrm{d}s})\varphi,u_s\rangle\quad\forall\varphi\in C_c^\infty, t\in[0,T] \tag{2.2}$$

Observe that under the above assumptions, for any $\varphi \in C_c^\infty$, $A \cdot \nabla \varphi$ and $(\operatorname{div} A - c)\varphi$ belong to $C_t^\alpha C_c^\beta$; since $u \in C_t^{\alpha\beta}(C_c^\beta)^\ast$ with $\alpha(1+\beta) > 1$, the integral appearing in (2.2) is meaningful as a functional Young integral.[2.1]

Remark 2.2. For practical purposes, it is useful to consider the following equivalent characterization of solutions: under the above regularity assumptions, u is a solution if and only if for any compact $K \subset \mathbb{R}^d$ and $\varphi \in C_K^\infty$ it holds

$$|\langle u_{s,t}, \varphi\rangle - \langle A_{s,t} \cdot \nabla \varphi + (\operatorname{div} A_{s,t} - c_{s,t})\varphi, u_s\rangle| \lesssim_K \|\varphi\|_{C_K^{1+\beta}} |t-s|^{\alpha(1+\beta)} [\![u]\!]_{C_t^{\alpha\beta}(C_K^\beta)^\ast} \times \times (\|A\|_{\alpha,\beta} + \|\operatorname{div} A - c\|_{\alpha,\beta}). \tag{2.3}$$

Indeed, (2.2) implies (2.3) by definition; conversely, if (2.3) holds, then necessarily $t \mapsto \langle u_t, \varphi\rangle$ must coincide with the sewing of $\Gamma_{s,t} = \langle A_{s,t} \cdot \nabla\varphi + (\operatorname{div} A_{s,t} - c_{s,t})\varphi, u_s\rangle$, recovering (2.2).

Clearly, in the l.h.s. of (2.3) one can replace u_s with u_t to get a similar estimate.

Remark 2.3. The presence of c in (2.1) also allows to consider nonlinear Young continuity equations (YCE for short) of the form

$$v_{\mathrm{d}t} + \nabla \cdot (A_{\mathrm{d}t}\, v_t) + c_{\mathrm{d}t}\, v_t = 0;$$

weak solutions to the above equation must be understood as weak solutions to the YTE associated to $(A, \tilde{c})$ with $\tilde{c} = c + \nabla \cdot A$.

Let us quickly recall some results from Section 1.3: given $A \in C_t^\alpha C_x^{1+\beta}$, $\alpha(1+\beta) > 1$, the YDE admits a flow of diffeomorphisms $\Phi_{s\to t}(x)$ and there exists $C = C(\alpha, \beta, T, \|A\|_{\alpha,1+\beta})$ such that

$$\begin{aligned} \|\Phi_{s\to\cdot}(x) - \Phi_{s\to\cdot}(y)\|_{\alpha;s,T} &\leqslant C|x-y| \\ |\Phi_{s\to t}(x) - x| &\leqslant C|t-s|^\alpha \\ [\![\Phi_{s\to\cdot}(x)]\!]_{\alpha;s,T} + |D_x \Phi_{s\to t}(x)| &\leqslant C \end{aligned}$$

for all $x, y \in \mathbb{R}^d$, $(s,t) \in \Delta_2$, together with similar estimates for $\Phi_{\cdot\leftarrow t}$. Moreover

$$\det D_x \Phi_{s\to t}(x) = \exp\left(\int_s^t \operatorname{div} A(\mathrm{d}r, \Phi_{s\to r}(x)) \right)$$

and similarly

$$\det D_x\, \Phi_{s\leftarrow t}(x) = (\det D_x \Phi_{s\to t}(\Phi_{s\leftarrow t}(x)))^{-1} = \exp\left(-\int_s^t \operatorname{div} A(\mathrm{d}r, \Phi_{r\leftarrow t}(x)) \right).$$

Proposition 2.4. *Let $A \in C_t^\alpha C_x^{1+\beta}$, $c \in C_t^\alpha C_x^\beta$. Then for any $\mu_0 \in \mathcal{M}_{\mathrm{loc}}$, a solution to the YTE is given by the formula*

$$\langle u_t, \varphi\rangle = \int \varphi(\Phi_{0\to t}(x)) \exp\left(\int_0^t (\operatorname{div} A - c)(\mathrm{d}s, \Phi_{0\to s}(x)) \right) \mu_0(\mathrm{d}x) \quad \forall \varphi \in C_c^\infty. \tag{2.4}$$

If $\mu_0(\mathrm{d}x) = u_0(x)\mathrm{d}x$ for $u_0 \in L_{\mathrm{loc}}^p$, then u_t corresponds to the measurable function

$$u(t,x) = u_0(\Phi_{0\leftarrow t}(x)) \exp\left(-\int_0^t c(\mathrm{d}s, \Phi_{s\leftarrow t}(x)) \right) \tag{2.5}$$

which belongs to $L_t^\infty L_{\mathrm{loc}}^p$ and satisfies

$$\int_K |u(t,x)|^p\, \mathrm{d}x = \int_{\Phi_{0\leftarrow t}(K)} |u_0(x)|^p \exp\left(\int_0^t (\operatorname{div} A - c)(\mathrm{d}s, \Phi_{0\to s}(x)) \right).$$

If in addition $c \in C_t^\alpha C_x^{1+\beta}$, then for any $u_0 \in C_{\mathrm{loc}}^1$ it holds $u \in C_t^\alpha C_{\mathrm{loc}}^0 \cap C_t^0 C_{\mathrm{loc}}^1$.

Proof. Since $|\Phi_{0\to t}(x) - x| \lesssim T^\alpha$, it is always possible to find $R \geqslant 0$ big enough such that $\operatorname{supp} \varphi(\Phi_{0\to t}(\cdot)) \subset \operatorname{supp} \varphi + B_R$ for all $t \in [0,T]$; by estimates (1.6) and (1.25), it holds

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left| \int_0^t (\operatorname{div} A - c)(\mathrm{d}s, \Phi_{0\to s}(x)) \right| \lesssim \|\operatorname{div} A - c\|_{\alpha,\beta} \sup_{x \in \mathbb{R}^d} (1 + [\![\Phi_{0\to\cdot}(x)]\!]_\alpha) < \infty.$$

2.1. What we mean here is that, given $f \in C_t^\gamma E$ and $g \in C_t^\eta E^\ast$ with $\gamma + \eta > 1$, one can define the Young-type integral $\int_0^t \langle g_s, \mathrm{d} f_s\rangle$, for instance as the sewing of $\Gamma_{s,t} := \langle g_s, f_{s,t}\rangle$.

It is therefore clear that u_t defined as in (2.4) belongs to $L_t^{\infty}(C_c^0)^*$. Similarly, combining the estimates

$$\begin{aligned}|\varphi(\Phi_{0\to t}(x))-\varphi(\Phi_{0\to s}(x))| &\leqslant |t-s|^{\alpha\beta}[\![\varphi]\!]_\beta[\![\Phi_{0\to\cdot}(x)]\!]_\alpha^\beta \lesssim |t-s|^{\alpha\beta}[\![\varphi]\!]_\beta\\ \left|\int_s^t(\operatorname{div}A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\right| &\lesssim |t-s|^\alpha\,\|\operatorname{div}A-c\|_{\alpha,\beta}(1+[\![\Phi_{0\to\cdot}(x)]\!]_\alpha)\lesssim|t-s|^\alpha,\end{aligned}$$

it is easy to check that $u\in C_t^{\alpha\beta}(C_c^\beta)^*$.

Let us show that it is a solution to the YTE in the sense of Definition 2.1. Given $\varphi\in C_K^\infty$ and $x\in\mathbb{R}^d$, define

$$z_t(x):=\varphi(\Phi_{0\to t}(x))\exp\biggl(\int_0^t(\operatorname{div}A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\biggr).$$

By the Itô-type formula for Young integrals (cf. Proposition 1.11), z satisfies

$$\begin{aligned}z_{s,t}(x) &= \int_s^t\varphi(\Phi_{0\to r}(x))\exp\biggl(\int_0^r(\operatorname{div}A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\biggr)(\operatorname{div}A-c)(\mathrm{d}r,\Phi_{0\to r}(x))\\ &\quad+\int_s^t\exp\biggl(\int_0^r(\operatorname{div}A-c)(\mathrm{d}s,\Phi_{0\to s}(x))\biggr)\nabla\varphi(\Phi_{0\to r}(x))\cdot A(\mathrm{d}r,\Phi_{0\to r}(x)).\end{aligned}$$

By the properties of Young integrals and the previous estimates, which are uniform in x, it holds

$$\begin{aligned}z_{s,t}(x) &\sim \exp\biggl(\int_0^s(\operatorname{div}A-c)(\mathrm{d}r,\Phi_{0\to r}(x))\biggr)\times\\ &\quad\times[\varphi(\Phi_{0\to s}(x))(\operatorname{div}A-c)_{s,t}(\Phi_{0\to s}(x))+\nabla\varphi(\Phi_{0\to s}(x))\cdot A_{s,t}(\Phi_{0\to s}(x))]\end{aligned}$$

in the sense that the two quantities differ by $O(|t-s|^{\alpha(1+\beta)})$, uniformly in $x\in\mathbb{R}^d$. Therefore

$$\begin{aligned}\langle u_{s,t},\varphi\rangle &= \int_{K+B_R}z_{s,t}(x)\mu_0(\mathrm{d}x)\\ &\sim\int_{K+B_R}[A_{s,t}\cdot\nabla\varphi+(\operatorname{div}A-c)_{s,t}\varphi](\Phi_{0\to t}(x))\exp\biggl(\int_0^s(\operatorname{div}A-c)(\mathrm{d}r,\Phi_{0\to r}(x))\biggr)\mu_0(\mathrm{d}x)\\ &\sim\langle u_s,A_{s,t}\cdot\nabla\varphi+(\operatorname{div}A-c)_{s,t}\varphi\rangle\end{aligned}$$

where the two quantities differ by $O(\|\varphi\|_{C_K^{1+\beta}}|t-s|^{\alpha(1+\beta)})$. By Remark 2.2 we deduce that u is indeed a solution.

The statements for $u_0\in L_{\mathrm{loc}}^p$ are an easy application of formula (1.39); it remains to prove the claims for $u_0\in C_{\mathrm{loc}}^1$, under the additional assumption $c\in C_t^\alpha C_x^{1+\beta}$. First of all observe that, for any $(s,t)\in\Delta_2$, by the aforementioned estimates for $\Phi_{\cdot\leftarrow s}$, it holds

$$\|\Phi_{\cdot\leftarrow t}(x)-\Phi_{\cdot\leftarrow s}(x)\|_\alpha=\|\Phi_{\cdot\leftarrow s}(\Phi_{s\leftarrow t}(x))-\Phi_{\cdot\leftarrow s}(x)\|_\alpha\lesssim|\Phi_{s\leftarrow t}(x)-x|\lesssim|t-s|^\alpha; \tag{2.6}$$

as a consequence, the map $(t,x)\mapsto u_0(\Phi_{0\leftarrow t}(x))$ belongs to $C_t^\alpha C_{\mathrm{loc}}^0$. Consider now the map

$$g(t,x):=\int_0^t c(\mathrm{d}r,\Phi_{r\leftarrow t}(x)).$$

It holds

$$\int_0^t c(\mathrm{d}r,\Phi_{r\leftarrow t}(x))-\int_0^s c(\mathrm{d}r,\Phi_{r\leftarrow s}(x))=\int_s^t c(\mathrm{d}r,\Phi_{r\leftarrow t}(x))+\int_0^s[c(\mathrm{d}r,\Phi_{r\leftarrow t}(x))-c(\mathrm{d}r,\Phi_{r\leftarrow s}(x))];$$

by Corollary 1.10 and estimate (2.6), we have

$$\begin{aligned}\left\|\int_0^\cdot[c(\mathrm{d}r,\Phi_{r\leftarrow t}(x))-c(\mathrm{d}r,\Phi_{r\leftarrow s}(x))]\right\|_\alpha &\lesssim \|c\|_{\alpha,1+\beta}(1+[\![\Phi_{\cdot\leftarrow t}(x)]\!]_\alpha+[\![\Phi_{\cdot\leftarrow s}(x)]\!]_\alpha)\times\\ &\quad\times\|\Phi_{\cdot\leftarrow t}(x)-\Phi_{\cdot\leftarrow s}(x)\|_\alpha\\ &\lesssim |t-s|^\alpha.\end{aligned}$$

As a consequence, $g\in C_t^\alpha C_{\mathrm{loc}}^0$ and so does u. The verification that $u\in C_t^0C_{\mathrm{loc}}^1$ is similar and thus omitted. □

Remark 2.5. Analogous computations show that a solution to the YTE with terminal condition $u(T,\cdot)=\mu_T(\cdot)$ is given by

$$\langle u_t,\varphi\rangle = \int \varphi(\Phi_{t\leftarrow T}(x))\exp\bigg(\int_t^T (c-\operatorname{div} A)(\mathrm{d}s,\Phi_{s\leftarrow T}(x))\bigg)\,\mu_T(\mathrm{d}x) \quad \forall \varphi\in C_c^\infty.$$

In the case $\mu_T(\mathrm{d}x)=u_T(x)\mathrm{d}x$ with $u_T\in L^p_{\mathrm{loc}}$, it corresponds to

$$u_t(x)=u_T(\Phi_{t\to T}(x))\exp\bigg(\int_t^T c(\mathrm{d}s,\Phi_{t\to s}(x))\bigg).$$

This solution satisfies the same space-time regularity as in Proposition 2.4. Moreover by the properties of the flow, if μ_0 (resp. μ_T) has compact support, then it's possible to find $K\subset\mathbb{R}^d$ compact such that $\operatorname{supp} u_t\subset K$ uniformly in $t\in[0,T]$. In particular, if $c\in C_t^\alpha C_x^{1+\beta}$ and $u_0\in C_c^1$ (resp. $u_T\in C_c^1$), then the associated solution belongs to $C_t^\alpha C_c^0\cap C_t^0C_c^1$.

The next result is at the heart of the duality approach and our main tool to establish uniqueness.

Proposition 2.6. *Let $u\in C_t^\alpha C_c^0\cap C_t^0 C_c^1$ be a solution of the YTE*

$$u_{\mathrm{d}t}+A_{\mathrm{d}t}\cdot\nabla u_t+c_{\mathrm{d}t}\,u_t=0 \tag{2.7}$$

and let $v\in L_t^\infty(C_c^0)^*\cap C_t^{\alpha\beta}(C_c^\beta)^*$ be a solution to the YCE

$$v_{\mathrm{d}t}+\nabla\cdot(A_{\mathrm{d}t}\,v_t)-c_{\mathrm{d}t}\,v_t=0. \tag{2.8}$$

Then it holds $\langle v_t,u_t\rangle=\langle v_s,u_s\rangle$ for all $(s,t)\in\Delta_2$. A similar statement holds for $u\in C_t^\alpha C^0_{\mathrm{loc}}\cap C_t^0 C^1_{\mathrm{loc}}$ and v as above and compactly supported uniformly in time.

The proof requires some preparations. Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of standard spatial mollifiers (say ρ_1 supported on B_1 for simplicity) and define R^ε, for sufficiently regular g and h, as the following bilinear operator:

$$R^\varepsilon(g,h)=(g\cdot\nabla h)^\varepsilon-g\cdot\nabla h^\varepsilon=\rho^\varepsilon*(g\cdot\nabla h)-g\cdot\nabla(\rho^\varepsilon*h); \tag{2.9}$$

the following commutator lemma is a slight variation on Lemma 5.11 from [145], which in turn is inspired by the general technique first introduced in the pioneering work [97].

Lemma 2.7. *The operator $R^\varepsilon\colon C^{1+\beta}_{\mathrm{loc}}\times C^1_{\mathrm{loc}}\to C^\beta_{\mathrm{loc}}$ defined by (2.9) satisfies the following.*

i. There exists a constant C independent of ε and R such that

$$\|R^\varepsilon(g,h)\|_{\beta,R}\leqslant C\|g\|_{1+\beta,R+1}\|h\|_{\beta,R+1}.$$

ii. For any fixed $g\in C^{1+\beta}_{\mathrm{loc}}, h\in C^\beta_{\mathrm{loc}}$ it holds $R^\varepsilon(g,h)\to 0$ in $C^{\beta'}_{\mathrm{loc}}$ as $\varepsilon\to 0$, for any $\beta'<\beta$.

Proof. It holds

$$\begin{aligned}R^\varepsilon(g,h)(x)&=\int_{B_1} h(x-\varepsilon z)\,\frac{g(x-\varepsilon z)-g(x)}{\varepsilon}\cdot\nabla\rho(z)\,\mathrm{d}z-(h\operatorname{div} g)^\varepsilon(x)\\&=:\tilde{R}^\varepsilon(g,h)(x)-(h\operatorname{div} g)^\varepsilon(x).\end{aligned}$$

Thus claim *i.* follows from $\|(h\operatorname{div} g)^\varepsilon\|_{\beta,R}\leqslant\|h\|_{1,R+1}\|g\|_{1+\beta,R+1}$ and

$$\begin{aligned}|\tilde{R}^\varepsilon(g,h)(x)-\tilde{R}^\varepsilon(g,h)(y)| &\leqslant \left|\int_{B_1}[h(x-\varepsilon z)-h(y-\varepsilon z)]\frac{g(x-\varepsilon z)-g(x)}{\varepsilon}\cdot\nabla\rho(z)\mathrm{d}z\right|\\&\quad+\left|\int_{B_1}h(x-\varepsilon z)\left[\frac{g(x-\varepsilon z)-g(x)}{\varepsilon}-\frac{g(y-\varepsilon z)-g(y)}{\varepsilon}\right]\cdot\nabla\rho(z)\mathrm{d}z\right|\\&\leqslant |x-y|^\beta\|h\|_{\beta,R+1}\|g\|_{1,R+1}\|\nabla\rho\|_{L^1}\\&\quad+\|h\|_{0,R+1}\int_{B_1}\left|\int_0^1[\nabla g(x-\varepsilon\theta z)-\nabla g(y-\varepsilon\theta z)]\right||z||\nabla\rho(z)|\,\mathrm{d}z\\&\lesssim |x-y|^\beta\|h\|_{\beta,R+1}\|g\|_{1+\beta,R+1}\end{aligned}$$

where the estimate is uniform in $x, y \in B_R$ and in $\varepsilon > 0$. Claim *ii.* follows from the above uniform estimate, the fact that $R^{\varepsilon}(g, h) \to 0$ in C^0_{loc} (cf. Lemma 5.11 from [145]) and a standard interpolation argument. □

Remark 2.8. It is difficult to stress enough the tremendous importance that commutator estimates have in modern PDE theory. In order to explain why Lemma 2.7 is so important and nontrivial, let us point out the following fact: as $\varepsilon \to 0$, we know that $(g \cdot \nabla h)^{\varepsilon}$ (resp. $g \cdot \nabla h^{\varepsilon}$) converges to $g \cdot \nabla h$, which is only known to belong to C^0_{loc} since $h \in C^1_{\text{loc}}$, therefore we cannot expect the family $\{(g \cdot \nabla h)^{\varepsilon}\}_{\varepsilon > 0}$ to be bounded in C^{β}_{loc}. The commutator lemma is telling us that, if don't deal with the terms $(g \cdot \nabla h)^{\varepsilon}$ and $g \cdot \nabla h^{\varepsilon}$ separately, but rather take their difference, we can transfer the higher regularity of g (in some kind of integration by parts fashion) to boundedness of the family $\{R^{\varepsilon}(g, h)\}$ in the better space C^{β}_{loc}.

Proof. (of Proposition 2.6) We only treat the case $u \in C^{\alpha}_t C^0_c \cap C^0_t C^1_c$, $v \in L^{\infty}_t (C^0_c)^* \cap C^{\alpha\beta}_t (C^{\beta}_c)^*$, the other one being similar. Applying a mollifier ρ^{ε} on both sides of (2.7), it holds

$$u^{\varepsilon}_{\mathrm{d}t} + A_{\mathrm{d}t} \cdot \nabla u^{\varepsilon}_t + (c_{\mathrm{d}t}\, u_t)^{\varepsilon} + R^{\varepsilon}(A_{\mathrm{d}t}, u_t) = 0$$

where we used the definition of R^{ε}; by Remark 2.2, the above expression can be equivalently interpreted as

$$\| u^{\varepsilon}_{s,t} + A_{s,t} \cdot \nabla u^{\varepsilon}_s + (c_{s,t}\, u_s)^{\varepsilon} + R^{\varepsilon}(A_{s,t}, u_s) \|_{C^0} \lesssim_{\varepsilon} |t - s|^{\alpha(1+\beta)} \quad \text{uniformly in } (s,t) \in \Delta_2.$$

Since v is a weak solution to (2.8), it holds

$$\begin{aligned}
\langle u^{\varepsilon}_t, v_t \rangle - \langle u^{\varepsilon}_s, v_s \rangle &= \langle u^{\varepsilon}_{s,t}, v_s \rangle + \langle u^{\varepsilon}_t, v_{s,t} \rangle \\
&\sim_{\varepsilon} - \langle A_{s,t} \cdot \nabla u^{\varepsilon}_t + (c_{s,t}\, u_t)^{\varepsilon} + R^{\varepsilon}(A_{s,t}, u_t), v_s \rangle + \langle A_{s,t} \cdot \nabla u^{\varepsilon}_t + c_{s,t}\, u^{\varepsilon}_t, v_s \rangle \\
&\sim \langle c_{s,t}\, u^{\varepsilon}_t - (c_{s,t} u_t)^{\varepsilon} - R^{\varepsilon}(A_{s,t}, u_t), v_s \rangle
\end{aligned}$$

where by $a \sim_{\varepsilon} b$ we mean that $|a - b| \lesssim_{\varepsilon} |t - s|^{\alpha(1+\beta)}$. As a consequence, defining $f^{\varepsilon}_t := \langle u^{\varepsilon}_t, v_t \rangle$, we deduce by the sewing lemma that $f^{\varepsilon}_t - f^{\varepsilon}_0 = \mathcal{J}(\Gamma^{\varepsilon}_{s,t})$ for the choice

$$\Gamma^{\varepsilon}_{s,t} := \langle c_{s,t}\, u^{\varepsilon}_t - (c_{s,t}\, u_t)^{\varepsilon} - R^{\varepsilon}(A_{s,t}, u_t), v_s \rangle.$$

Our aim is to show that $\mathcal{J}(\Gamma^{\varepsilon}_{s,t}) \to 0$ as $\varepsilon \to 0$; to this end, we start estimating $\|\Gamma^{\varepsilon}\|_{\alpha, \alpha(1+\beta)}$.

It holds

$$\begin{aligned}
\delta \Gamma^{\varepsilon}_{s,r,t} \;=\; & \langle c_{s,r}\, u^{\varepsilon}_{r,t}, v_s \rangle - \langle c_{r,t}\, u^{\varepsilon}_t, v_{s,r} \rangle \\
& + \langle c_{r,t}\, u_{s,r}, v^{\varepsilon}_t \rangle - \langle c_{s,r}\, u_s, v^{\varepsilon}_{r,t} \rangle \\
& + \langle R^{\varepsilon}(A_{r,t}, u_t), v_{s,r} \rangle - \langle R^{\varepsilon}(A_{s,r}, u_{r,t}), v_s \rangle.
\end{aligned}$$

Therefore, up to choosing a suitable compact $K \subset \mathbb{R}^d$, we have the estimates

$$\begin{aligned}
|\Gamma^{\varepsilon}_{s,t}| &\leqslant (\|c_{s,t}\, u^{\varepsilon}_t\|_{C^0_K} + \|(c_{s,t}\, u^{\varepsilon}_t)\|_{C^0_K} + \|R^{\varepsilon}(A_{s,t}, u_t)\|_{C^0_K}) \|v_s\|_{(C^0_K)^*} \\
&\lesssim |t - s|^{\alpha} (\|c\|_{\alpha,\beta} + \|A\|_{\alpha,1}) \|u\|_{C^0_t C^0_c} \|v_s\|_{(C^0_K)^*}
\end{aligned}$$

as well as

$$\begin{aligned}
|\delta \Gamma^{\varepsilon}_{s,r,t}| \;\leqslant\; & \|c_{s,r}\, u^{\varepsilon}_{r,t}\|_{C^0_K} \|v_s\|_{(C^0_K)^*} + \|c_{r,t}\, u^{\varepsilon}_t\|_{C^{\beta}_K} \|v_{s,r}\|_{(C^{\beta}_K)^*} \\
& + \|c_{r,t}\, u_{s,r}\|_{C^0_K} \|v^{\varepsilon}_t\|_{(C^0_K)^*} + \|c_{s,r}\, u_s\|_{C^{\beta}_K} \|v^{\varepsilon}_{r,t}\|_{(C^{\beta}_K)^*} \\
& + \|R^{\varepsilon}\| \, \|A_{r,t}\|_{1+\beta} \|u_t\|_{C^1_K} \|v_{s,r}\|_{(C^{\beta}_K)^*} + \|R^{\varepsilon}\| \, \|A_{s,r}\|_{1+\beta} \|u_{r,t}\|_{C^0_K} \|v_s\|_{(C^0_K)^*} \\
\lesssim\; & |t - s|^{\alpha(1+\beta)} (\|c\|_{\alpha,\beta} + \|R^{\varepsilon}\| \|A\|_{\alpha,1+\beta}) \times \\
& \times (\|u\|_{C^0_t C^1_K} \|v\|_{C^{\alpha\beta}_t (C^{\beta}_K)^*} + \|u\|_{C^{\alpha}_t C^0_K} \|v\|_{L^{\infty}_t (C^0_K)^*}).
\end{aligned}$$

Overall we deduce that $\|\Gamma^{\varepsilon}\|_{\alpha}$ and $\|\delta \Gamma^{\varepsilon}\|_{\alpha(1+\beta)}$ are bounded uniformly in $\varepsilon > 0$; moreover by properties of convolutions and Lemma 2.7, it holds $\Gamma^{\varepsilon}_{s,t} \to 0$ as $\varepsilon \to 0$ for any $(s,t) \in \Delta_2$ fixed. By Lemma 1.1 it holds

$$|f^{\varepsilon}_{s,t} - \Gamma^{\varepsilon}_{s,t}| \lesssim |t - s|^{\alpha(1+\beta)}$$

uniformly in $\varepsilon > 0$, so passing to the limit as $\varepsilon \to 0$ we deduce that

$$|\langle u_t, v_t\rangle - \langle u_s, v_s\rangle| \lesssim |t-s|^{\alpha(1+\beta)} \quad \forall (s,t) \in \Delta_2.$$

This immediately implies that $t \mapsto \langle u_t, v_t\rangle$ is constant and thus the conclusion. □

We are now ready to establish uniqueness of solutions to the YTE and YCE under suitable regularity conditions on (A, c).

Theorem 2.9. *Let $A \in C_t^\alpha C_x^{1+\beta}$, $c \in C_t^\alpha C_x^{1+\beta}$ with $\alpha(1+\beta) > 1$. Then for any $u_0 \in C_{\mathrm{loc}}^1$ there exists a unique solution to the YTE (2.7) with initial condition u_0 in the class $C_t^\alpha C_{\mathrm{loc}}^0 \cap C_t^0 C_{\mathrm{loc}}^1$, which is given by formula (2.5); similarly, for any $\mu_0 \in \mathcal{M}_{\mathrm{loc}}$ there exists a unique solution to the YCE (2.8) with initial condition μ_0 in the class $L_t^\infty (C_c^0)^* \cap C_t^{\alpha\beta} (C_c^\beta)^*$, which is given by formula (2.4).*

Proof. Existence follows from Proposition 2.4, so we only need to establish uniqueness. By linearity of YTE, it suffices to show that the only solution u to (2.7) in the class $C_t^\alpha C_{\mathrm{loc}}^0 \cap C_t^0 C_{\mathrm{loc}}^1$ with $u_0 \equiv 0$ is given by $u \equiv 0$. Let u be such a solution and fix $\tau \in [0, T]$; since $(\operatorname{div} A - c) \in C_t^\alpha C_x^\beta$, by Proposition 2.4 and Remark 2.5, for any compactly supported $\mu \in \mathcal{M}$ there exists a solution $v \in L_t^\infty \mathcal{M}_K \cap C_t^{\alpha\beta} (C_c^\beta)^*$ to (2.8) with terminal condition $v_\tau = \mu$, up to taking a suitable compact set K. By Proposition 2.6 it follows that

$$\langle u_\tau, \mu\rangle = \langle u_\tau, v_\tau\rangle = \langle u_0, v_0\rangle = 0;$$

as the reasoning holds for any compactly supported $\mu \in \mathcal{M}$, $u_\tau \equiv 0$ and thus $u \equiv 0$.

Uniqueness of solutions to YCE (2.8) in the class $L_t^\infty (C_c^0)^* \cap C_t^{\alpha\beta} (C_c^\beta)^*$ follows similarly. □

2.2 Parabolic Young PDEs

We present in this section a generalization to the nonlinear Young setting of some of the results contained in [161]. Specifically, we are interested in studying a parabolic nonlinear evolutionary problem of the form

$$\mathrm{d}x_t = -A x_t \,\mathrm{d}t + B(\mathrm{d}t, x_t) \tag{2.10}$$

where $-A$ is the generator of an analytical semigroup.

In order not to create confusion, in this section the nonlinear Young term will be always denoted by B. As we will use a one-parameter family of spaces $\{V_\alpha\}_{\alpha \in \mathbb{R}}$, the regularity of B will be denoted by $B \in C_t^\gamma C_{W,U}^\beta$, with W and U being taken from that family; whenever it doesn't create confusion, we will still denote the associated norm by $\|B\|_{\gamma,\beta}$.

Let us first recall the functional setting from [161], Section 2.1. It is based on the theory of analytical semigroups and infinitesimal generators, see [234] for a general reference, but the reader not acquainted with the topic may think for simplicity of $A = I - \Delta$, $V = L^2(\mathbb{R}^d)$ and $V_\alpha = H^{2\alpha}(\mathbb{R}^d)$ fractional Sobolev spaces.

Let $(V, \|\cdot\|_V)$ be a separable Banach space, $(A, \mathrm{Dom}(A))$ be an unbounded linear operator on V, $\mathrm{rg}(A)$ be its range; suppose its resolvent set is contained in $\Sigma = \{z \in \mathbb{C} : |\arg(z)| > \pi/2 - \delta\} \cup U$ for some $\delta > 0$ and some neighbourhood U of 0. Further assume that there exist positive constants C, η such that its resolvent R_α satisfies

$$\|R_\alpha\|_{\mathcal{L}(V)} \leqslant C(\eta + |\alpha|)^{-1} \quad \forall \alpha \in \Sigma.$$

Under these assumptions, $-A$ is the infinitesimal generator of an analytical semigroup $(S(t))_{t \geqslant 0}$ and there exist positive constants M, λ such that

$$\|S(t)\|_{\mathcal{L}(V)} \leqslant M e^{-\lambda t} \quad \forall t \geqslant 0.$$

Moreover, $-A$ is one-to-one from $\mathrm{Dom}(A)$ to V and the fractional powers $(A^\alpha, \mathrm{Dom}(A^\alpha))$ of A can be defined for any $\alpha \in \mathbb{R}$; if $\alpha < 0$, then $\mathrm{Dom}(A^\alpha) = V$ and A^α is a bounded operator, while for $\alpha \geqslant 0$ $(A^\alpha, \mathrm{Dom}(A^\alpha))$ is a closed operator with $\mathrm{Dom}(A^\alpha) = \mathrm{rg}(A^{-\alpha})$ and $A^\alpha = (A^{-\alpha})^{-1}$.

For $\alpha \geqslant 0$, let V_α be the space $\mathrm{Dom}(A^\alpha)$ with norm $\|x\|_{V_\alpha} = \|A^\alpha x\|_V$; for $\alpha = 0$ it holds $A^0 = \mathrm{Id}$ and $V_0 = V$. For $\alpha < 0$, let V_α be the completion of V w.r.t. the norm $\|x\|_{V_\alpha} = \|A^\alpha x\|_V$, which is thus a bigger space than V. The one-parameter family of spaces $\{V_\alpha\}_{\alpha \in \mathbb{R}}$ defined in this way is such that V_δ embeds continuously in V_α whenever $\delta \geqslant \alpha$ and $A^\alpha A^\delta = A^{\alpha+\delta}$ on the common domain of definition; moreover $A^{-\delta}$ maps V_α onto $V_{\alpha+\delta}$ for all $\alpha \in \mathbb{R}$ and $\delta \geqslant 0$.

The operator $S(t)$ can be extended to V_α for all $\alpha < 0$ and $t > 0$ and maps V_α to V_δ for all $\alpha \in \mathbb{R}$, $\delta \geqslant 0$, $t > 0$; finally, it satisfies the following properties:

$$\|A^\alpha S(t)\|_{\mathcal{L}(V)} \leqslant M_\alpha t^{-\alpha} e^{-\lambda t} \text{ for all } \alpha \geqslant 0,\ t > 0; \tag{2.11}$$

$$\|S(t)x - x\|_V \leqslant C_\alpha t^\alpha \|A^\alpha x\|_V \text{ for all } x \in V_\alpha,\ \alpha \in (0,1]. \tag{2.12}$$

Remark 2.10. It follows from the statements above and the semigroup property of $S(t)$ that for any $\alpha \in \mathbb{R}$, $\delta \in (0,1]$, $\rho \geqslant 0$ and any $s \leqslant t$ it holds

$$\begin{aligned}\|S(t)x - S(s)x\|_{V_\alpha} &= \|[S(t-s) - I]\, S(s)x\|_{V_\alpha}\\ &\lesssim_{\alpha,\delta} |t-s|^\delta \|S(s)x\|_{V_{\alpha+\delta}} \lesssim_{\alpha,\delta,\rho} |t-s|^\delta |s|^{-\rho} \|x\|_{V_{\alpha+\delta-\rho}}.\end{aligned}$$

In particular, for $\rho = 0$ we find $\|S(t) - S(s)\|_{\mathcal{L}(V_{\alpha+\delta}, V_\alpha)} \lesssim |t-s|^\delta$, equivalently $S(\cdot) \in C_t^\delta \mathcal{L}(V_{\alpha+\delta}, V_\alpha)$. It also follows that for any given $x_0 \in V_{\alpha+\delta}$, the map $t \mapsto S(t)x_0$ belongs to $C_t^\delta V_\alpha$ with

$$[\![S(\cdot)x_0]\!]_{\delta,V_\alpha} \lesssim_{\alpha,\delta} \|x_0\|_{V_{\alpha+\delta}}. \tag{2.13}$$

Before dealing with (2.10), it makes sense to consider the linear equation

$$\mathrm{d}x_t = -Ax_t\,\mathrm{d}t + \mathrm{d}y_t,$$

whose solution is formally given by the mild formulation

$$x_t = S(t)x_0 + \int_0^t S(t-s)\mathrm{d}y_s,$$

which is actually rigorous for $y \in C_t^1 V_{-\alpha}$. The next result, which is taken from [161], states that the solution map can be extended by continuity to suitable non differentiable functions $y \in C_t^0 V$.

Theorem 2.11. *Let $\alpha \in \mathbb{R}$ and consider the map Ξ defined for any $y \in C_t^1 V_{-\alpha}$ by*

$$\Xi(y)_t = \int_0^t S(t-s)\dot{y}_s\,\mathrm{d}s.$$

Then for any $\gamma > \alpha$, Ξ extends uniquely to a map $\Xi \in \mathcal{L}(C_t^\gamma V_{-\alpha}; C_t^\kappa V_\delta)$ for all $\delta \in (0, \gamma - \alpha)$ and all $\kappa \in (0, \gamma - \alpha - \delta)$. Moreover there exists a constant $C = C(\alpha, \kappa, \delta, \gamma)$ such that

$$[\![\Xi(y)]\!]_{\kappa,V_\delta} \leqslant C\,[\![y]\!]_{\gamma,V_{-\alpha}}, \qquad \sup_{t\in[0,T]} \|\Xi(y)_t\|_{V_\delta} \leqslant C T^{\gamma-\delta-\alpha} [\![y]\!]_{\gamma,V_{-\alpha}}. \tag{2.14}$$

Proof. (Sketch) For the full proof, we refer to Theorem 1 from [161]. Here we will only show one of the main steps, namely the estimate for $\sup_{t\in[0,T]} \|\Xi(y)_t\|_{V_\delta}$; it is quite a nice proof, as it presents the general "sewing procedure" which is also at the heart of Lemma 1.1. By time rescaling, we can assume wlog $T = 1$ and simplify the problem by only estimating $\|\Xi(y)_1\|_{V_\delta}$; we only need to consider $y \in C_t^1 V_{-\alpha}$, as the general case follows by a standard density argument.

For fixed $n \in \mathbb{N}$, consider the dyadic points $t_k^n = k 2^{-n}$ and define $\Xi^n(y)_1 = \Xi^n$ by

$$\Xi^n = \sum_{k=0}^{2^n-1} S(1 - t_k^n)\, y_{t_k^n, t_{k+1}^n}.$$

It is cleat that Ξ^n is a Riemann-Stjeltes sum converging to $\Xi^\infty := \Xi(y)_1 = \int_0^1 S(1-s)\dot{y}_s$ as $n \to \infty$. In order to obtain an estimate on Ξ^∞ in V_δ, by telescopic sums it suffices to control

$$\|\Xi^\infty - \Xi^0\|_{V_\delta} \leqslant \sum_{n=0}^{\infty} \|\Xi^{n+1} - \Xi^n\|_{V_\delta}.$$

For any $n \in \mathbb{N}$, it holds

$$\begin{aligned}
\Xi^{n+1} - \Xi^n &= \sum_{k=0}^{2^n-1} \left[S(1-t^{n+1}_{2k}) y_{t^{n+1}_{2k}, t^{n+1}_{2k+2}} - S(1-t^{n+1}_{2k}) y_{t^{n+1}_{2k,2k+1}} - S(1-t^{n+1}_{2k+1}) y_{t^{n+1}_{2k+1,2k+2}}\right] \\
&= \sum_{k=0}^{2^n-1} [S(1-t^{n+1}_{2k}) - S(1-t^{n+1}_{2k+1})]\, y_{t^{n+1}_{2k+1,2k+2}}.
\end{aligned}$$

In view of Remark 2.10, choosing $\beta > 1 - \alpha - \delta$, we can then estimate the term as follows:

$$\begin{aligned}
\|\Xi^{n+1} - \Xi^n\|_{V_\delta} &\leqslant \sum_k (1 - t^{n+1}_{2k+1})^{-(\delta+\alpha+\beta)} |t^{n+1}_{2k} - t^{n+1}_{2k+1}|^\beta \, \|y_{t^{n+1}_{2k+1,2k+2}}\|_{V_{-\alpha}} \\
&\lesssim \sum_k \left(1 - \frac{k}{2^n}\right)^{-(\delta+\alpha+\beta)} 2^{-n(\beta+\gamma)} [\![y]\!]_{\gamma, V_{-\alpha}} \\
&\lesssim 2^{-n(\gamma-\alpha-\delta)}\, [\![y]\!]_{\gamma, V_{-\alpha}} \sum_{k=0}^{\infty} |k|^{-(\delta+\alpha+\beta)} \lesssim 2^{-n(\gamma-\alpha-\delta)}\, [\![y]\!]_{\gamma, V_{-\alpha}}.
\end{aligned}$$

Since $\gamma - \alpha - \delta > 0$ by assumption, we can conclude that

$$\begin{aligned}
\|\Xi^\infty\|_{V_\delta} &\leqslant \|\Xi^0\|_{V_\delta} + \sum_{n=0}^{\infty} \|\Xi^{n+1} - \Xi^n\|_{V_\delta} \\
&\lesssim \|S(1) y_{0,1}\|_{V_\delta} + [\![y]\!]_{\gamma, V_{-\alpha}} \sum_n 2^{-n(\gamma-\alpha-\delta)} \lesssim [\![y]\!]_{\gamma, V_{-\alpha}}
\end{aligned}$$

which yields the conclusion. □

Definition 2.12. *Given A as above and $B \in C^\gamma_t C^\beta_{V_\delta, V_\rho}$, $\rho \leqslant \delta$, we say that $x \in C^\kappa_t V_\delta$ is a mild solution to equation (2.10) with initial data $x_0 \in V_\delta$ if $\gamma + \beta\kappa > 1$, so that $\int_0^\cdot B(\mathrm{d}s, x_s)$ is well defined as a nonlinear Young integral, and moreover x satisfies*

$$x_t = S(t)x_0 + \int_0^t S(t-s) B(\mathrm{d}s, x_s) = S(t)x_0 + \Xi\left(\int_0^\cdot B(\mathrm{d}s, x_s)\right)_t \quad \forall t \in [0,T] \tag{2.15}$$

where Ξ is the map defined by Theorem 2.11 and the equality holds in V_α for suitable α.

We are now ready to prove the main result of this section.

Theorem 2.13. *Assume A as above, $B \in C^\gamma_t C^{1+\beta}_{V_\delta, V_\rho}$ with $\rho > \delta - 1$ and suppose there exists $\kappa \in (0,1)$ such that*

$$\begin{cases} \gamma + \beta\kappa > 1 \\ \kappa < \gamma + \rho - \delta \end{cases}. \tag{2.16}$$

Then for any $x_0 \in V_{\delta+\kappa}$ there exists a unique solution with initial data x_0 to (2.10), in the sense of Definition 2.12, in the class $C^\kappa_t V_\delta \cap C^0_t V_{\delta+\kappa}$.

Moreover, the solution depends in a Lipschitz way on (x_0, B), in the following sense: for any $R > 0$ exists a constant $C = C(\beta, \gamma, \delta, \rho, \kappa, T, R)$ such that for any (x_0^i, B^i), $i = 1, 2$, satisfying $\|x_0^i\|_{V_{\delta+\kappa}} \vee \|B^i\|_{\gamma, 1+\beta} \leqslant R$, denoting by x^i the associated solutions, it holds

$$[\![x^1 - x^2]\!]_{\kappa, V_\rho} \leqslant C(\|x_0^1 - x_0^2\|_{V_{\delta+\kappa}} + \|B^1 - B^2\|_{\gamma, 1+\beta}).$$

Remark 2.14. If $B \in C^\gamma_t C^2_{V_\delta, V_\rho}$, then it is possible to find κ satisfying (2.16) if and only if

$$2\gamma + \rho - \delta > 1;$$

in the case $\rho = \delta$ we recover the usual condition $\gamma > 1/2$. Instead the regime $\rho < \delta - 1$ would enforce the requirement $\gamma > 1$, making the statement vacuous.

Proof. The basic idea is to apply a Banach fixed point argument to the map

$$x \mapsto \mathcal{I}(x)_t := S(t)x_0 + \Xi\left(\int_0^{\cdot} B(\mathrm{d}s, x_s)\right)_t \tag{2.17}$$

defined on a suitable domain.

By Remark 2.10, if $x_0 \in V_{\delta+\kappa}$, then $S(\cdot)x_0 \in C_t^{\kappa} V_\delta$; moreover $B \in C_t^{\gamma} C^1_{V_\delta, V_\rho}$, so under the condition $\gamma + \kappa > 1$ the nonlinear Young integral in (2.17) is well defined for $x \in C_t^{\kappa} V_\delta$, $y_t = \int_0^t B(\mathrm{d}s, x_s) \in C_t^{\gamma} V_\rho$ and then $\Xi(y) \in C_t^{\kappa} V_\delta$ under the condition $\kappa < \gamma + \rho - \delta$. So under our assumptions $\mathcal{I}$ maps $C_t^{\kappa} V_\delta$ into itself; our first aim is to find a closed bounded subset which is invariant under $\mathcal{I}$.

For suitable τ, M to be chosen later, consider the set

$$E := \left\{ x \in C^{\kappa}([0,\tau]; V_\delta) : x(0) = x_0, \llbracket x \rrbracket_{\kappa, V_\delta} \leqslant M, \sup_{t \in [0,\tau]} \|x_t\|_{V_{\delta+\kappa}} \leqslant M \right\};$$

E is a complete metric space endowed with the distance $d_E(x_1, x_2) = \llbracket x_1 - x_2 \rrbracket_{\kappa, V_\delta}$. It holds

$$\llbracket \mathcal{I}(x) \rrbracket_{\kappa, V_\delta} \leqslant \llbracket S(\cdot)x_0 \rrbracket_{\kappa, V_\delta} + \left\llbracket \Xi\left(\int_0^{\cdot} B(\mathrm{d}s, x_s)\right) \right\rrbracket_{\kappa, V_\delta} \lesssim \|x_0\|_{V_{\delta+\rho}} + \left\llbracket \int_0^{\cdot} B(\mathrm{d}s, x_s) \right\rrbracket_{\gamma, V_\rho};$$

for the nonlinear Young integral we have the estimate

$$\begin{aligned} \left\| \int_s^t B(\mathrm{d}r, x_r) \right\|_{V_\rho} &\lesssim \|B_{s,t}(x_s)\|_{V_\rho} + |t-s|^{\gamma+\kappa} \llbracket B \rrbracket_{\gamma,1} \llbracket x \rrbracket_{\kappa, V_\delta} \\ &\lesssim \|B_{s,t}(x_s) - B_{s,t}(x_0)\|_{V_\rho} + |t-s|^{\gamma} \|B\|_{\gamma,0} + |t-s|^{\gamma} \tau^{\kappa} \llbracket B \rrbracket_{\gamma,1} \llbracket x \rrbracket_{\kappa} \\ &\lesssim |t-s|^{\gamma} \|B\|_{\gamma,1} (1 + \tau^{\kappa} \llbracket x \rrbracket_{\kappa, V_\delta}) \end{aligned}$$

and so

$$\left\llbracket \int_0^{\cdot} B(\mathrm{d}r, x_r) \right\rrbracket_{\gamma, V_\rho} \lesssim \|B\|_{\gamma,1} (1 + \tau^{\kappa} \llbracket x \rrbracket_{\kappa, V_\delta}).$$

Overall, we can find a constant κ_1 such that

$$\llbracket \mathcal{I}(x) \rrbracket_{\kappa, V_\delta} \leqslant \kappa_1 \|x_0\|_{V_{\delta+\kappa}} + \kappa_1 \|B\|_{\gamma,1} (1 + \tau^{\kappa} \llbracket x \rrbracket_{\kappa, V_\delta}).$$

Similar computations, together with estimate (2.14), show the existence of κ_2 such that

$$\sup_{t \in [0,\tau]} \|\mathcal{I}(x)_t\|_{V_{\delta+\kappa}} \leqslant \kappa_2 \|x_0\|_{V_{\delta+\kappa}} + \kappa_2 \|B\|_{\gamma,1} \tau^{\gamma-\delta+\rho} (1 + \tau^{\kappa} \llbracket x \rrbracket_{\kappa, V_\delta}).$$

Therefore taking $\tau \leqslant 1$, $\kappa_3 = \kappa_1 \vee \kappa_2$, in order for $\mathcal{I}$ to map E into itself it suffices

$$\kappa_3 \|x_0\|_{V_{\delta+\kappa}} + \kappa_3 \|B\|_{\gamma,1} (1 + \tau^{\kappa} M) \leqslant M,$$

which is always possible, for instance by requiring

$$2\kappa_3 \|B\|_{\gamma,1} \tau^{\kappa} \leqslant 1, \quad 2\kappa_3 \|x_0\|_{V_{\delta+\kappa}} + 2\kappa_3 \|B\|_{\gamma,1} \leqslant M.$$

Observe that τ can be chosen independently of $\|x_0\|_{V_{\delta+\kappa}}$; moreover for the same choice of τ, analogous computations show that any solution x to (2.10) defined on $[0, \tilde{\tau}]$ with $\tilde{\tau} \leqslant \tau$ satisfies the a priori estimate

$$\llbracket x \rrbracket_{\kappa, V_\delta; 0, \tilde{\tau}} + \sup_{t \in [0, \tilde{\tau}]} \|x_t\|_{V_{\delta+\kappa}} \leqslant \kappa_4 (\|x_0\|_{V_{\delta+\kappa}} + \|B\|_{\gamma,1}) \tag{2.18}$$

for another constant κ_4, independent of x_0.

We now want to find $\tilde{\tau} \in [0, \tau]$ such that $\mathcal{I}$ is a contraction on $\tilde{E}$, $\tilde{E}$ being defined as E in terms of $\tilde{\tau}, M$. Given $x^1, x^2 \in \tilde{E}$, it holds

$$\begin{aligned} d_E(\mathcal{I}(x^1), \mathcal{I}(x^2)) &= \left\llbracket \Xi\left(\int_0^{\cdot} B(\mathrm{d}s, x_s^1) - \int_0^{\cdot} B(\mathrm{d}s, x_s^2)\right) \right\rrbracket_{\kappa, V_\delta} \\ &\lesssim \left\llbracket \left(\int_0^{\cdot} B(\mathrm{d}s, x_s^1) - \int_0^{\cdot} B(\mathrm{d}s, x_s^2)\right) \right\rrbracket_{\gamma, V_\rho} \end{aligned}$$

and under the assumptions we can apply Corollary 1.10, so we have

$$\begin{aligned}\left\|\int_s^t B(\mathrm{d}r, x_r^1) - \int_s^t B(\mathrm{d}r, x_r^2)\right\|_{V_\rho} &= \left\|\int_s^t v_{\mathrm{d}r}(x_r^1 - x_r^2)\right\|_{V_\rho}\\ &\lesssim |t-s|^\gamma [\![v]\!]_{\gamma,\mathcal{L}} \|x_s^1 - x_s^2\|_{V_\rho} + |t-s|^{\gamma+\kappa} [\![v]\!]_{\gamma,\mathcal{L}} [\![x^1 - x^2]\!]_{\kappa,V_\rho}\\ &\lesssim |t-s|^\gamma \|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^\kappa [\![x^1 - x^2]\!]_{\kappa,V_\rho}.\end{aligned}$$

This implies

$$\left[\!\!\left[\int_0^\cdot B(\mathrm{d}r, x_r^1) - B(\mathrm{d}r, x_r^2)\right]\!\!\right]_{\gamma,V_\rho} \lesssim \|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^\kappa [\![x^1 - x^2]\!]_{\kappa,V_\rho}$$

and so overall, for a suitable constant κ_5,

$$d_E(\mathcal{I}(x^1), \mathcal{I}(x^2)) \leqslant \kappa_5 \|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^\kappa\, d_E(x^1, x^2).$$

Choosing $\tilde{\tau}$ small enough such that $\kappa_5\|B\|_{\gamma,1+\beta}(1+M)\tilde{\tau}^\kappa < 1$, we deduce that there exists a unique solution to (2.10) defined on $[0,\tilde{\tau}]$. Since we have the uniform estimate (2.18), we can iterate the contraction argument to construct a unique solution on $[0,\tau]$; but since the choice of τ does not depend on x_0 and $x_\tau \in V_{\delta+\kappa}$, we can iterate further to cover the whole interval $[0,T]$ with subintervals of size τ.

To check the Lipschitz dependence on (x_0, B), one can reason using Theorem 1.25 as usual, but let us give an alternative proof; we only check Lipschitz dependence on B, as the proof for x_0 is similar.

Given B^i, $i=1,2$ as above, denote by $\mathcal{I}_{B^i}$ the map associated to B^i defined as in (2.17); we can choose $\tilde{\tau}$ and M such that they are both strict contractions of constant $\kappa_6 < 1$ on E defined as before. Observe that for any $z \in E$ it holds

$$\begin{aligned}d_E(\mathcal{I}_{B^1}(z), \mathcal{I}_{B^2}(z)) &= \left[\!\!\left[\Xi\left(\int_0^\cdot B^1(\mathrm{d}s, z_s) - \int_0^\cdot B^2(\mathrm{d}s, z_s)\right)\right]\!\!\right]_{\kappa,V_\delta}\\ &\lesssim \left[\!\!\left[\int_0^\cdot B^1(\mathrm{d}s, z_s) - \int_0^\cdot B^2(\mathrm{d}s, z_s)\right]\!\!\right]_{\gamma,V_\rho}\\ &\lesssim (1+M)\|B^1 - B^2\|_{\gamma,\beta}.\end{aligned}$$

Denote by x^i the unique solutions on E associated to B^i, then by the above computation we get

$$\begin{aligned}[\![x^1 - x^2]\!]_{\kappa,V_\delta} &= d_E(\mathcal{I}_{B^1}(x^1), \mathcal{I}_{B^2}(x^2))\\ &\leqslant d_E(\mathcal{I}_{B^1}(x^1), \mathcal{I}_{B^1}(x^2)) + d_E(\mathcal{I}_{B^1}(x^2), \mathcal{I}_{B^2}(x^2))\\ &\leqslant \kappa_6\, [\![x^1 - x^2]\!]_{\kappa,V_\delta} + \kappa_7(1+M)\|B^1 - B^2\|_{\gamma,\beta}\end{aligned}$$

which implies that

$$[\![x^1 - x^2]\!]_{\kappa,V_\delta} \leqslant \frac{\kappa_7}{1-\kappa_6}(1+M)\|B^1 - B^2\|_{\gamma,\beta}$$

which shows Lipschitz dependence on B^i on the interval $[0,\tilde{\tau}]$. As before, a combination of a priori estimates and iterative arguments allows to extend the local estimate to a global one. □

By the usual localization and blow-up alternative arguments, we obtain the following result.

Corollary 2.15. *Assume A as in Theorem 2.13, $B \in C_t^\gamma C^{1+\beta}_{V_\delta,V_\rho,\mathrm{loc}}$ with $\rho > \delta - 1$ and suppose there exists $\kappa \in (0,1)$ satisfying (2.16). Then for any $x_0 \in V_{\delta+\kappa}$ there exists a unique maximal solution x starting from x_0, defined on an interval $[0,T^*) \subset [0,T]$, such that either $T^* = T$ or*

$$\lim_{t \uparrow T^*} \|x_t\|_{V_{\delta+\kappa}} = +\infty.$$

Remark 2.16. For simplicity we have only treated here uniqueness results, but if the embedding $V_\delta \hookrightarrow V_\alpha$ for $\delta > \alpha$ is compact (as is often the case, at least on bounded domains) one can use compactness arguments to deduce existence of solutions under weaker regularity conditions on B, in analogy with Theorem 1.13. Once can also consider equations of the form

$$\mathrm{d}x_t = -Ax_t\,\mathrm{d}t + F(x_t)\mathrm{d}t + B(\mathrm{d}t, x_t),$$

in which case uniqueness can be achieved under the same conditions on B as above and a Lipschitz condition on F, see also Remark 1 from [161].

2.3 Bibliographical comments

As already mentioned, all the material presented here is taken from [141], with [145] as a precursor for Section 2.1.

In the style of Section 1.4.3, one could have treated the special case of continuous $\partial_t A$, obtaining an exact analogue of Theorem 2.9 (now under the assumption that $\partial_t A, \partial_t c$ are continuous and bounded and $A, c \in C_t^\alpha C_x^{1+\beta}$ with $\alpha + \beta > 1$). The proof in this case is much simpler, as it doesn't require the use of commutator lemmas, see Section 5.1 from [145].

Transport equations of the form $\partial_t u + b \cdot \nabla u = 0$ can be solved classically by the method of characteristics, whenever the coefficients are sufficiently smooth (e.g. b Lipschitz). There are now much more advanced theories, dealing with only weakly differentiable coefficients, giving rise to the concepts of renormalized solutions and generalized Lagrangian flows, see [97, 4].

Quite surprising, I haven't found in the literature any systematic study of transport equations in the standard Young setting (i.e. with $b \cdot \nabla u$ replaced by $b \cdot \nabla u \mathrm{d} y_t$ for some $y \in C^\alpha$), even in the regularity regime where the existence of a flow associated to the YDE is granted. There are some recent works, all coming from the nonlinear Young setting, see [56], [176] and Chapter 9 from [218]. However [56] and [218] only deal with the (much simpler) case of continuous $\partial_t A$, while the validity of the results from [176] (which claim an analogue of Theorem 2.9, see Section 3.4 therein) is quite debatable, see Remark 5.10 from [145]. In particular, the authors in [176] try to go through the standard proof based on characteristic functions, but the validity of the chain rule breaks down due to insufficient space-time regularity of the inverse flow $t \mapsto \Phi_{0\leftarrow t}(\cdot)$ (such regularity is restored under the stronger, non-optimal assumption $A \in C_t^\alpha C_x^{2+\beta}$). It is interesting to observe how in the Young setting, even when the flow is known to exist (which immediately suggest the validity of the classical solution formula $u_t = u_0 \circ \Phi_{0\leftarrow t}$), a complete proof still requires a weak definition of solution (cf. Definition 2.1) and the use of advanced tools like commutators (Lemma 2.7).

I have the impression that the case of Young transport equation has been overlooked since many authors have immediately passed to the study of the more advanced rough path case, see for instance [23, 96, 35], as well as [94] and the references therein for more general equations. Nontheless, I find it extremely interesting to understand whether many concepts from the theory of generalized Lagrangian flows, carry to the Young regime, in particular if one can only impose weak differentiability assumptions on A; for instance one could investigate the validity of Ambrosio's superposition principle ([4, 5]), the uniqueness of renormalized solutions [97], or try to develop a quantitative approach in the style of [81]. The only formidable (but incomplete) attempt I know in this direction is [23]; I believe that studying the simpler Young setting first would provide a better insight to the problem.

Finally, let me mention Chapter 4 from [27] for the study of a nonlinear, Burgers type equation, in the nonlinear Young framework.

Section 2.2 is an extension to the nonlinear Young setting of the work [161], which was first able to solve classical parabolic Young equations. The general strategy in the proof of Theorem 2.11 has then lead to the development of so called "mild sewing lemma", which has allowed to tackle the rough path case as well, see [164] and subsequently [153] and Exercises 4.16-4.17 from [132]. Recently, this mild version has been incorporated in the stochastic sewing, see [202].

In a similar direction, "Volterra sewing lemmas" have been developed in [169]. The recent work [58] incorporates such sewing in the study of the parabolic problem

$$\partial_t u = \Delta u + b(u) + g(u)\xi + \dot{\omega}_t;$$

after a suitable change of variable, this equation is "translated" to the nonlinear Young setting, where the authors develop a solution theory comparable to the one given here, cf. Section 4 from [58].

Chapter 3

Regularising ODEs by additive perturbations

We are now ready to harvest the fruits of our preparations from Chapters 1-2 to present a well developed theory for perturbed ODEs of the form

$$x_t = x_0 + \int_0^t b_s(x_s)\mathrm{d}s + w_t \tag{3.1}$$

where w is a continuous path in $\mathbb{R}^d$ and $b: [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a time-dependent drift; to convey the main ideas, in this introduction we will restrict to autonomous drifts $b_s(x_s) = b(x_s)$, although the general case will be treated throughout the chapter.

Let us shortly recall the main well-known results for (3.1) when $w \equiv 0$, which transfer easily to the case of any given continuous w (for the proofs, we refer to [115] and the references therein):

- If b is globally Lipschitz, then existence and uniqueness of solutions to (3.1) holds for any $x_0 \in \mathbb{R}^d$; they form a Lipschitz flow of homeomorphisms. If b is more regular, say C^n_{loc}, the same regularity transfers to the flow.

- If b is merely continuous and enjoys some linear growth conditions, existence of solutions holds by Peano's theorem; however, for any $\alpha \in [0,1)$, one can find examples of α-Hölder continuous drifts and $x_0 \in \mathbb{R}^d$ such that there are infinitely many solutions to (3.1).

- There are examples of measurable and bounded b for which there exist no solutions to (3.1).

- Finally, if b is not even a measurable function but only a distribution, it's not even clear how to give meaning to (3.1), since we cannot pointwise evaluate $b(x_s)$ and so we cannot define the integral in the Lebesgue sense.

The question we want to address is whether a well-chosen additive perturbation w can cure the above pathologies; in particular, the endgoal is to identify some intrinsic properties of w (hopefully, but not necessarily, of analytic type) which imply such a regularising effect. Further, we would like these properties to be satisfied by a large class of paths, either in the sense of sampling w from suitable stochastic processes, or by showing that they hold for *generic* functions. This program is carried out here and in the upcoming Chapter 5 (based respectively on [145] and [143]), which constitute the main body of this thesis.

The fundamental intuitions in order to tackle (3.1), which we are now going to illustrate, are due to [57]; the material presented here can be regarded as a follow-up and refinement of that work. Firstly, in order to expect something different for (3.1) compared to its counterpart with $w \equiv 0$, we need the path to be sufficiently "active", in some sense to be formalized rigorously; then, given the structure of the pertubed ODE, we expect any solution to be of the form $x = \theta + w$, where formally θ should solve

$$\theta_t = x_0 + \int_0^t b(\theta_s + w_s)\mathrm{d}s. \tag{3.2}$$

In particular, if there is any way to make sense of the integral appearing in (3.2), then θ should be fairly regular in time and so x should be "as active as w". Put in other terms, w and θ should represent respectively the *fast and slow components* of the system; any oscillation of w should happen at a time scale where θ is almost constant and the solution x should behave like w. If we accept this philosophy to be true for the moment, then it should be possible to approximate the integral in (3.2) by finite sums of the form

$$\int_0^t b(\theta_r + w_r)\mathrm{d}r \approx \sum_i \int_{t_i}^{t_{i+1}} b(\theta_{t_i} + w_r)\mathrm{d}r$$

over finer and finer partitions of $[0,T]$. Defining the *averaged field* $T^w b$ as

$$T_t^w b(z) = T^w b(t,z) := \int_0^t b(z + w_r)\mathrm{d}r, \quad T_{s,t}^w b(z) = T^w b(t,z) - T^w b(s,z) := \int_s^t b(z + w_r)\mathrm{d}r,$$

we can rewrite the previous sums as

$$\sum_i \int_{t_i}^{t_{i+1}} b(\theta_{t_i} + w_s)\mathrm{d}s = \sum_i T_{t_i,t_{i+1}}^w b(\theta_{t_i}).$$

But now the last term is an old friend of ours, namely the Riemann-Lebesgue sums used to approximate nonlinear Young integrals $\int_0^t A(\mathrm{d}s, \theta_s)$ from Chapter 1 for $A = T^w b$!

Although the above reasoning is very heuristical, it can be at least shown that such an identification is in fact rigorous whenever b is at least continuous, see Lemma 3.31; yet, we do not know how to treat the case of distributional b. But we can now apply a standard principle from rough path theory, which amounts to *postulating* the existence and space-time regularity of $T^w b$! If for instance we assume $T^w b \in C_t^\gamma C_x^1$ for some $\gamma > 1/2$, then we can rigorously construct, for any $\theta \in C_t^\gamma$, the nonlinear Young integral

$$\int_0^t T^w b(\mathrm{d}s, \theta_s)$$

which now *defines* the object $\int_0^t b(\theta_s + w_s)\mathrm{d}s$ even when the intepretation in the Lebesgue sense breaks down. It then makes sense to look for solutions x into the restricted subset of continuous paths of the form $\mathcal{D}^w = w + C_t^\gamma = \{w + \theta : \theta \in C_t^\gamma\}$, which rigorously and quantitatively encodes the idea that θ should be regular (or slowly varying) compared to w.

All in all, by applying the abstract theory of nonlinear Young integration from Chapter 1, the problem of existence and uniqueness of solutions to (3.1) entirely reduces to that of establishing regularity of $T^w b$, without any reference to the ODE problem; for instance, it would be enough to verify that $T^w b \in C_t^\gamma C_x^2$. This now provides a rigorous mathematical way of measuring how "active" or "oscillatory" the path w is (but it is not the only possible one! See Chapter 5 for a deeper discussion of this point). Furthermore, the nonlinear Young interpretation allows to construct an associated flow of solutions (which gets more and more regular as $T^w b$ does so) and treat the transport equation as well.

In order to find interesting examples of paths regularising the equation, we will first sample w as a typical realization of fractional Brownian motion (fBm) W of Hurst parameter $H \in (0,1)$; we refer the reader to Appendix A.1 for its definitions and main properties.

This choice is relevant for the following main reasons:

- When $H \neq 1/2$, W is not a semimartingale nor a Markov process; in particular, all standards tools employed to analyse the associated SDE (Itô calculus, martingale problem, Zvonkin transform, Dirichlet forms, etc.) completely break down. Regularisation by noise results for SDEs driven by fBm are in general difficult to obtain and of interest on their own.
- Working with a specified stochastic process provides us with a specific structure, which we can exploited in order to get as sharp results as possible. This is embodied by Theorem 3.30, whose derivation relies heavily on the so called *Girsanov transform*.

- By varying $H \in (0,1)$, we get a better picture of how the regularity of the path W interplays with its regularising effect on the ODE. In particular, we will be able to rigorously prove the (both natural and counterintuitive) principle that "*the rougher the noise, the better the regularisation*", cf. Theorem 3.54.

Another extremely nice feature of working with fBm is that it will naturally enlarge our perspective by allowing to derive *genericity results* in the sense of *prevalence*, as will be shown in Section 3.3. For more details on its exact mathematical definition and main properties, we refer the reader to Appendix A.3. Here let us only mention that it is a natural generalization of the "for Lebesgue a.e. $x \in \mathbb{R}^d$" property to the infinite dimensional setting (where the Lebesgue measure is ill-defined) and that it allows for probabilistic tools in the proof (which is why we can obtain it starting from our analysis of fBm trajectories). In particular, all our main results will transfer to statements involving "almost every Hölder continuous function", see Theorem 3.59.

Structure of the chapter. We start by introducing rigorously averaged fields $T^w b$ in Section 3.1, first by defining them analytically for any given path w and then passing to study their space-time regularity when w is a typical realization of fBm. We then apply these results, together with probabilistic arguments based on Girsanov transform, to prove strong existence and path-by-path uniqueness for SDEs in Section 3.2; results concerning the regularity of the flow of solutions and wellposedness of the associated transport equation are given as well. In Section 3.3, we extend all our considerations to the case of generic (in the sense of prevalence) continuous additive perturbations. Finally, in Section 3.4 we provide some additional bibliographic references and outline some open problems for future research.

Notations and conventions. We will keep adopting all the same notations from Chapter 1, with the only major difference that we will use γ as a parameter denoting time regularity in nonlinear Young considerations, as α will be used instead to measure the regularity of the drift b appearing in the SDE. This means for instance that we will use the notation $C_t^\gamma C_x^{\beta,\lambda} = C_t^\gamma C_{\mathbb{R}^d,\mathbb{R}^d}^{\beta,\lambda}$ to denote the weighted Hölder spaces already given in Definition 1.4.

We will use the notation $B_p^s = B_{p,p}^s = B_{p,p}^s(\mathbb{R}^d;\mathbb{R}^m)$ for Besov spaces (see Appendix A.2 for their definition and main properties), $L_x^p = L^p(\mathbb{R}^d;\mathbb{R}^m)$ for standard Lebesgue spaces, $H_x^s = W_x^{s,2} = B_2^s$ for fractional Sobolev spaces, $\mathcal{S} = \mathcal{S}(\mathbb{R}^d;\mathbb{R}^m)$ for the space of Schwarz functions and $\mathcal{S}'$ for its dual. Finally, $\mathcal{M}_x = \mathcal{M}(\mathbb{R}^d;\mathbb{R}^m)$ denotes the space of (vector-valued) finite Radon measures, endowed with the total variation norm.

We will always work on a finite fixed time interval $[0,T]$. Given a Banach space E, we will denote by $L_t^q E = L^q(0,T;E)$ the space of all strongly measurable functions $f\colon [0,T] \to E$ (up to equivalence class) such that

$$\|f\|_{L^q E}^q := \int_0^T \|f_t\|_E^q \,\mathrm{d}t < \infty;$$

when $q = \infty$, the above norm is replaced by the essential supremum as usual. We might use the longer expression $\|f\|_{L^q(0,\tau;E)}$ to stress that the integral is taken over $[0,\tau] \subset [0,T]$. We denote by $\mathcal{P}(E)$ the space of (tight) probability measures on E.

We will keep using as in Chapter 1 the notation $C_t^\gamma E = C^\gamma([0,T];E)$ for the space of E-valued, γ-Hölder continuous functions; similarly for $C_t^0 E$, endowed with the supremum norm. When $E = \mathbb{R}^d$, we will drop it for simplicity and just write $w \in L_t^q$ or $w \in C_t^\gamma$; in the latter case, we might sometime use the notation $[\![w]\!]_{\gamma;[0,\tau]}$ for some $\tau < T$ to denote the Hölder seminorm restricted to $t \in [0,\tau]$.

The above conventions are adopted in particular for the choices of E outlined above, thus giving rise to spaces like $L_t^q B_p^\alpha$, $C_t^0 \mathcal{M}_x$, $C_t^\gamma C_x^{\beta,\lambda}$ and so on, with norms denoted by $\|\cdot\|_{L^q B_p^\alpha}$, $\|\cdot\|_{C^0 \mathcal{M}}$, etc.

With a slight abuse of notation, we will also use expression like $L_t^q F$, where F is a Frechét space with distance induced by a family of seminorms $\|\cdot\|_{n,F}$, to mean that $\int_0^T \|f_t\|_{n,F}^q < \infty$ for all n. This applies in particular to the choice $F = \mathcal{S}$, so that we might write $f \in L_t^\infty \mathcal{S}$ to mean that $f \in L_t^\infty H_x^s$ for all $s \in \mathbb{R}$; similarly for the locally Hölder spaces $C_{\mathrm{loc}}^\beta = C_{\mathrm{loc}}^\beta(\mathbb{R}^n;\mathbb{R}^m)$ as in Definition 1.4.

Given a space-time dependent drift $b: [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$, we will use the shortcut notation $b(t,x) = b_t(x)$. The standard heat kernel will be denoted by P_t, so that $P_t f = p_t * f$, where $*$ stands for convolution and $p_t(x) := (2\pi t)^{-d/2} e^{-|x|^2/2t}$. We will use the notation $B_R = B_R(0)$ for the closed ball around 0 of radius R in $\mathbb{R}^d$.

Even when not specified, whenever talking about a random variable X, we will assume it to be defined on an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$; we will use $L^p_\omega E = L^p(\Omega; E)$ as a shortcut for $L^p(\Omega, \mathcal{F}, \mathbb{P}; E)$ and directly drop E whenever it coincides with $\mathbb{R}^m$ (in which case we might directly write L^p_ω). Typical elements of Ω are denoted by ω, so that $X(\omega)$ stands for the evaluation of the r.v. X, and $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$. Whenever a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ appears, it is implicitly assumed to satisfy the usual assumption (completeness, right-continuity); we will use the shortcut notation $\mathbb{E}_t$ to denote the conditional expectation $\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$. If $\mathcal{F}_t$ is not specified, it is taken as the natural filtration of the stochastic process W in consideration.

3.1 Averaged fields: analytic definition, stochastic estimates

3.1.1 Analytic definition

We provide here the definition of the averaging operator T^w for measurable paths $w: [0,T] \to \mathbb{R}^d$, together with some basic properties which will be fundamental for later sections. Our definition is rather abstract and works for a general class of Banach spaces E, but keep in mind that for our purposes E will always be a Besov space $B^s_{p,q}$ with $p \in [2,\infty)$. Also, we will consider for simplicity the scalar-valued case, i.e. $E \subseteq \mathcal{S}'(\mathbb{R}^d) = \mathcal{S}'$; everything immediately generalises to the vector-valued case $\mathcal{S}'(\mathbb{R}^d; \mathbb{R}^m)$ by reasoning componentwise.

Let us assume that E is a separable Banach space that continuously embeds into $\mathcal{S}'$ (so that there is also a dual embedding $\mathcal{S} \hookrightarrow E^*$) such that translation $\tau^v: f \mapsto \tau^v f = f(\cdot + v)$ act continuously on it and leave the norm invariant: $\|\tau^v f\|_E = \|f\|_E$ for all $v \in \mathbb{R}^d$ and $f \in E$. Assume moreover that the map $v \mapsto \tau^v$ is continuous, in the sense that if $v_n \to v$, then $\tau^{v_n} f \to \tau^v f$ for all $f \in E$.

Definition 3.1. *Let $w: [0,T] \to \mathbb{R}^d$ be a measurable function, E as above. Then we define the averaging operator T^w as the continuous linear map from $L^1_t E$ to $C^0_t E$ given by*

$$T^w_t b = (T^w b)_t = T^w b(t) := \int_0^t \tau^{w_s} b_s \,\mathrm{d}s \quad \forall t \in [0,T].$$

We will refer to $T^w b$ as an averaged field to stress that b is fixed, while w might be varying.

The definition is meaningful, since by the continuity properties of $v \mapsto \tau^v$, the map $s \mapsto \tau^{w_s} b_s$ is still measurable and, by the invariance of $\|\cdot\|_E$ under translations, $\|b_\cdot\|_{L^1 E} = \|\tau^{w_\cdot} b_\cdot\|_{L^1 E}$. Continuity of $T^w b$ and the bound $\|T^w b\|_{C^0 E} \leqslant \|b\|_{L^1 E}$ follow from standard properties of Bochner integral, as well as the linearity of the map $b \mapsto T^w b$. Similarly, it is easy to see that, in the case b enjoys higher integrability, T^w can also be defined as a linear bounded operator from $L^q_t E$ to $C^{1/q'}_t E$, where q' is the conjugate exponent of q. Furthermore, if w and $\tilde{w}$ are such that $w_t = \tilde{w}_t$ for Lebesgue-a.e. $t \in [0,T]$, then $T^w b$ and $T^{\tilde{w}} b$ coincide for all b, so that T^w can be defined for w in an equivalence class.

Lemma 3.2. *Let $w^n \to w$ in L^1_t and $b \in L^q_t E$, then $T^{w^n} b \to T^w b$ in $C^{1/q'}_t E$.*

Proof. Up to an extracting subsequence argument, we can further assume wlog that $w^n_t \to w_t$ for Lebesgue-a.e. t. Since $\tau^{w^n_t} b_t \to \tau^{w_t} b_t$ for a.e. t and $\|\tau^{w^n_t} b_t - \tau^{w_t} b_t\|^q \lesssim \|b_t\|^q \in L^1_t$, by dominated convergence it holds

$$\|T^{w^n} b - T^w b\|_{C^{1-1/q} E} \lesssim \int_0^T \|\tau^{w^n_t} b_t - \tau^{w_t} b_t\|^q_E \,\mathrm{d}t \to 0 \quad \text{as } n \to \infty,$$

which gives the conclusion. □

The advantage of the above definition of T^w is that it is intrinsic and does not depend on any approximation procedure by mollifiers. However, a possibly more intuitive description of $T^w b$ can be given by duality. Recall that in the sense of distributions $(\tau^v)^* = \tau^{-v}$, so that for any $\varphi \in \mathcal{S} \hookrightarrow E^*$ it holds

$$\langle T_t^w b, \varphi \rangle = \int_0^t \langle b_s, \varphi(\cdot - w_s) \rangle \mathrm{d}s$$

where the pairing is integrable since $|\langle b_s, \varphi(\cdot - w_s)\rangle| \lesssim_\varphi \|b_s\|_E$. The above relation holds for all $\varphi \in \mathcal{S}$ and therefore uniquely identifies $T_t^w b$ as an element of $\mathcal{S}'$, for all $t \in [0, T]$. The advantage now is that the map $(t, x) \mapsto \varphi(x - w_t)$ can be regarded as an element of $L_t^\infty \mathcal{S}$,[3.1] to which standard operations on $\mathcal{S}$ such as differentiation and convolution can be applied.

Lemma 3.3. *Let w and b be as above. Then:*

i. Averaging and spatial differentiation commute, i.e. for all $i = 1, \ldots, d$, $\partial_i T^w b = T^w \partial_i b$.

ii. Averaging and spatial convolution commute, i.e. for any $K \in C_c^\infty$ it holds

$$K * (T^w b) = T^w (K * b) = (T^w K) * b.$$

Proof. The statements follow easily from the duality formulation. For any $\varphi \in \mathcal{S}$, $t \in [0, T]$ it holds

$$\begin{aligned}\langle \partial_i T_t^w b, \varphi \rangle &= -\langle T_t^w b, \partial_i \varphi \rangle = -\int_0^t \langle b_r, \partial_i \varphi(\cdot - w_r) \rangle \mathrm{d}r \\ &= \int_0^t \langle \partial_i b_r, \varphi(\cdot - w_r) \rangle \mathrm{d}r = \langle T_t^w \partial_i b, \varphi \rangle.\end{aligned}$$

If $K \in C_c^\infty$, then denoting by $\tilde{K}$ its reflection, by duality it holds

$$\begin{aligned}\langle K * T_t^w b, \varphi \rangle &= \langle T_t^w b, \tilde{K} * \varphi \rangle = \int_0^t \langle b_r, \tau^{-w_r}(\tilde{K} * \varphi) \rangle \mathrm{d}r = \int_0^t \langle b_r, \tilde{K} * (\tau^{-w_r} \varphi) \rangle \mathrm{d}r \\ &= \int_0^t \langle K * b_r, \tau^{-w_r} \varphi \rangle \mathrm{d}r = \langle T_t^w (K * b), \varphi \rangle.\end{aligned}$$

A similar computation shows the other part of the identity. □

Remark 3.4. Let us point out that if $w \in L_t^\infty$, then the averaging operator has finite speed of propagation and so behaves well under localisation. Indeed, if $b \in L_t^1 E$ is such that $\operatorname{supp} b_t \subset B_R$ for all $t \in [0, T]$, then $\operatorname{supp} T_t^w b \subset B_{R + \|w\|_\infty}$ for all $t \in [0, T]$; similarly, if b and $\tilde{b}$ are such that their restrictions to B_R coincide for all t, then $T^w b$ and $T^w \tilde{b}$ coincide on $B_{R - \|w\|_\infty}$.

In view of the applications in Sections 3.2-3.3, our main goal is to establish conditions under which $T^w b \in C_t^\gamma F$, where $\gamma > 1/2$ and F is another Banach space enjoying better regularity properties than E; typically $F = C^{\beta, \lambda}$ for suitable values of β, λ, so that the theory outlined in Chapter 1 can be applied.

For this reason, we are going to assume from now on that $b \in L_t^q E$ for some $q > 2$. The idea behind this restriction is that sometimes averaging allows to trade off time regularity for space regularity (think of the analogy with parabolic regularity theory) and therefore in order to have $T^w b \in C_t^\gamma F$, knowing a priori only that $T^w b \in C_t^{1/q'} E$, we need to require at least[3.2]

$$1 - \frac{1}{q} > \gamma > \frac{1}{2} \quad \Rightarrow \quad q > 2.$$

3.1. Technically, this is only true if we impose $w \in L_t^\infty$, which we will do in general in this chapter. In the case of less abstract spaces like $E = B_{p,q}^s$, we can take φ to belong to the dual E^*, which can be identified with $B_{p',q'}^{-s}$, which is now a normed space with a translation invariant norm; in this case, the claim that $(t, x) \mapsto \varphi(x - w_t)$ can be regarded as an element of $L_t^\infty B_{p',q'}^{-s}$ is true for any choice of w, regardless of its integrability.

3.2. The situation could be slightly different if we allowed for the use of controls; indeed, the most general version of Young's integration theory (and associated solvability of YDEs) only requires to work with paths of bounded p-variation for $p < 2$. Setting $p = 1/\gamma$, this amounts to an estimate of the form $|x_{s,t}| \leqslant \omega(s,t)^\gamma$, where ω is a so called *control*, see [134] for more details. The case we consider corresponds to $\omega(s,t) = |t - s|$, $\gamma > 1/2$.

Remark 3.5. Despite our use of the terminology "regularisation by averaging", what we mean is really that we *fix* a drift b and we want to establish that, for almost every path w (either in a probabilistic or prevalent sense), the averaged field $T^w b$ has nice regularity properties. This is *different* from trying to establish that the averaging operator T^w as a linear operator from $L^q_t E$ to $C^\gamma_t F$ is bounded, which is false due to the time dependence of the drifts we consider. Indeed, given any $b \in E$, defining $\tilde{b}_t = \tau^{-w_t} b$, by definition of averaging we obtain $T^w_{s,t}\tilde{b} = (t-s)b$, which shows that $T^w \tilde{b}$ cannot have better spatial regularity than $\tilde{b}$. The situation is more interesting if one defines T^w for time independent drifts only; this situation will be analysed in Chapter 5.

In order to show prevalence of regularisation by averaging, we first need to show that such a property indeed defines Borel sets in suitable spaces of paths. To this end, we require F to be another Banach spaces which embeds into $\mathcal{S}'$ which enjoys the following *Fatou type property*: if $\{x_n\}_n$ is a bounded sequence in F and x_n converge to x in $\mathcal{S}'$, then $x \in F$ and $\|x\|_F \leqslant \liminf \|x_n\|_F$.

In the next lemma we allow any $\bar{\gamma} \in (0,1)$, but our primary focus will be $\bar{\gamma} = 1/2$.

Lemma 3.6. *Let F be as above, $b \in L^q_t E$ for some $q > 2$. Then for any $\bar{\gamma} \in (0,1)$ the set*

$$\mathcal{A}^{\bar{\gamma}} = \{w\colon [0,T] \to \mathbb{R}^d \text{ such that } T^w b \in C^\gamma_t F \text{ for some } \gamma > \bar{\gamma}\}$$

is Borel measurable w.r.t. the following topologies: L^p_t with $p \in [1,\infty]$, C^α_t with $\alpha \geqslant 0$.

Proof. We can write $\mathcal{A}^{\bar{\gamma}}$ as a countable union of sets as follows:

$$\mathcal{A}^{\bar{\gamma}} = \bigcup_{m,n\in\mathbb{N}} \mathcal{A}^{\bar{\gamma}}_{m,n} := \bigcup_{m,n\in\mathbb{N}} \{w\colon [0,T] \to \mathbb{R}^d \text{ such that } \|T^w b\|_{C^{\bar{\gamma}+1/m}F} \leqslant n\};$$

in order to show the statement, it suffices to show that for every m,n the set $\mathcal{A}^{\bar{\gamma}}_{m,n}$ is closed in the above topologies. We only need to deal with the L^1-topology, since it is weaker than any of the others considered. Let w^k be a sequence in $\mathcal{A}^{\bar{\gamma}}_{m,n}$ such that $w^k \to w$ in L^1_t, then by Lemma 3.2 we know that $T^{w^k} b \to T^w b$ in $C^0_t E$ and so that for any $s < t$, $T^{w^k}_{s,t} b \to T^w_{s,t} b$ in E and in $\mathcal{S}'$. On the other hand, by definition of $\mathcal{A}^{\bar{\gamma}}_{m,n}$ it holds

$$\sup_k \|T^{w^k}_{s,t} b\|_F \leqslant n\, |t-s|^{\bar{\gamma}+1/m}$$

which implies by the Fatou property of F that $T^w_{s,t} b \in F$ and that $\|T^w_{s,t} b\|_F \leqslant n\, |t-s|^{\bar{\gamma}+1/m}$ as well. As the reasoning holds for any $s < t$, it follows that $T^w b \in \mathcal{A}^{\bar{\gamma}}_{m,n}$. □

Remark 3.7. Any weakly-$*$ compact Banach space F which embeds in $\mathcal{S}'$ satisfies the Fatou property. In the following we will always work with L^p_x-based function spaces with $p \in [2,\infty]$, so that this property holds automatically. Let us also point out that the proof actually works more generally for conditions of the form $T^w b \in C^\phi_t F$, where ϕ is a prescribed modulus of continuity.

We conclude this section with some lemmas on approximation by mollifications which will be very useful in Sections 3.2-3.3.

Lemma 3.8. *Let $\varphi \in C^{\beta,\lambda}_x$ for some $\beta, \lambda > 0$ and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a family of standard spatial mollifiers; set $\varphi^\varepsilon := \rho^\varepsilon * \varphi$. Then $\varphi^\varepsilon \to \varphi$ in $C^{\beta-\delta,\lambda}_x$ for any $\delta > 0$ and it holds*

$$\|\varphi^\varepsilon\|_{\beta,\lambda} \lesssim \|\varphi\|_{\beta,\lambda}, \quad \|\varphi^\varepsilon - \varphi\|_{\beta-\delta,\lambda} \lesssim_\beta \varepsilon^\delta \quad \forall \varepsilon \in (0,1].$$

Proof. We can consider $\beta \in (0,1)$, as the general case follows by considering $D^k\varphi$ in place of φ; we will only prove the second inequality in (3.8), the first one being similar. By standard properties of mollifiers, for any $\varepsilon \in (0,1]$ and any $x, y \in B_R$ we have the estimates

$$\begin{aligned} |\varphi^\varepsilon(x) - \varphi^\varepsilon(y)| &\leqslant |x-y|^\beta \llbracket \varphi \rrbracket_{\beta,R+1} \lesssim \llbracket \varphi \rrbracket_{\beta,\lambda} |x-y|^\beta R^\lambda \\ |\varphi^\varepsilon(x) - \varphi(x)| &\leqslant \varepsilon^\beta \llbracket \varphi \rrbracket_{\beta,R+1} \lesssim \varepsilon^\beta \llbracket \varphi \rrbracket_{\beta,\lambda} R^\lambda; \end{aligned}$$

interpolated together, for any $\theta \in (0,1)$ they provide

$$|\varphi^\varepsilon(x) - \varphi^\varepsilon(y) - \varphi(x) + \varphi(y)| \lesssim_\beta \varepsilon^{(1-\theta)\beta} |x-y|^{\beta\theta} [\![\varphi]\!]_{\beta,\lambda} R^\lambda \quad \forall x, y \in B_R.$$

Taking the supremum over $x, y \in B_R$ and choosing $\theta = 1 - \delta / \beta$ yields $[\![\varphi^\varepsilon - \varphi]\!]_{\beta-\delta,\lambda} \lesssim \varepsilon^\delta$; together with a similar estimate for $|\varphi^\varepsilon(0) - \varphi(0)|$, we deduce (3.8). □

Lemma 3.9. *Let $b \in L^q_t E$ such that $T^w b \in C^\gamma_t C^{\beta,\lambda}_x$ for some $\gamma \in (0,1]$, $\beta, \lambda > 0$ and set $b^\varepsilon := \rho^\varepsilon * b$ for some spatial mollifiers $\{\rho^\varepsilon\}_{\varepsilon > 0}$. Then for any $\delta > 0$, $T^w b^\varepsilon \to T^w b$ in $C^\gamma_t C^{\beta-\delta,\lambda}_x$ as $\varepsilon \to 0$; a similar statement holds with C^β in place of $C^{\beta,\lambda}$.*

Proof. The lemma is a slight improvement on Lemma 3.9 from [145] and Lemma 11 from [139]; for simplicity we only give the proof for $C^\gamma_t C^{\beta,\lambda}_x$. In view of Lemmas 3.3 and 3.8, for any $s < t$ it holds $\rho^\varepsilon * T^w_{s,t} b = T^w_{s,t} b^\varepsilon$ and we have the estimate

$$\|T^w_{s,t} b^\varepsilon - T^w_{s,t} b\|_{\beta-\delta,\lambda} \lesssim \varepsilon^\delta |t-s|^\gamma \|T^w b\|_{C^\gamma C^{\beta,\lambda}}$$

which readily implies $[\![T^w b^\varepsilon - T^w b]\!]_{C^\gamma C^{\beta-\delta,\lambda}} \lesssim \varepsilon^\delta$ and thus the conclusion (recall that $T^w b(0,\cdot) \equiv 0$ by definition and that we work on a finite interval $[0,T]$). □

3.1.2 Itô–Tanaka formula for averaged fields

We are going to prove an Itô–Tanaka type formula for averaged functionals, in the same spirit of the one considered in [80]. We first need to recall the Clark–Ocone formula, see [225]. Given a two-sided standard Brownian motion B on a space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t = \sigma(B_s, s \leqslant t)$, and a Malliavin differentiable random variable A with Malliavin derivative $D.A$, the Clark–Ocone formula states that

$$A = \mathbb{E}[A] + \int_{-\infty}^{+\infty} \mathbb{E}_r[D_r A] \mathrm{d}B_r \tag{3.3}$$

where we recall the shortcut notation $\mathbb{E}_r[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_r]$. From (3.3) it follows immediately that, for any $s \in \mathbb{R}$, we have the more general identity

$$A = \mathbb{E}_s A + \int_s^{+\infty} \mathbb{E}_r[D_r A] \mathrm{d}B_r.$$

We do not provide here the general definition of Malliavin derivative of a Brownian variable, which can be found in [225]. Let us only mention that in the specific case of our interest, for $A := f(X)$, where f is a smooth function and $X = \int_{-\infty}^{+\infty} K_s \mathrm{d}B_s$, the Malliavin derivative is given by

$$D_t A = \nabla f(X) \cdot K_t\,. \tag{3.4}$$

Lemma 3.10. *Let $b: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a smooth, compactly supported function, then for any fixed $0 \leqslant s \leqslant t \leqslant T$, $H \in (0,1)$ and $x \in \mathbb{R}^d$, there exist $c_H, \tilde{c}_H > 0$ s.t. the following identity holds $\mathbb{P}$-a.s.:*

$$\begin{aligned} \int_s^t b_r(x + W_r) \mathrm{d}r \;\; &= \;\; \int_s^t P_{\tilde{c}_H |r-s|^{2H}} b_r(x + \mathbb{E}_s W_r) \mathrm{d}r \\ &\quad + c_H \int_s^t \int_u^t P_{\tilde{c}_H |r-u|^{2H}} \nabla b_r(x + \mathbb{E}_s W_r) |r-u|^{H-1/2} \mathrm{d}r \cdot \mathrm{d}B_u. \end{aligned} \tag{3.5}$$

Proof. For $H = 1/2$ the above formula is well known and coincides with a standard application of the Itô–Tanaka trick, combined with the mild formulation of solutions to the heat equation, see for instance the discussion in [80]; so we can assume $H \neq 1/2$. Let us fix $x \in \mathbb{R}^d$. Since b is smooth, for fixed r we can apply Clark–Ocone formula to $b_r(x + W^H_r)$ to obtain

$$\begin{aligned} b_r(x + W_r) =& \mathbb{E}_s [b_r(x + W_r)] + \int_s^r \mathbb{E}_u [\nabla b_r(x + W_r)] c_H (r-u)^{H-1/2} \cdot \mathrm{d}B_u \\ =& P_{\tilde{c}_H |r-s|^{2H}} b_r(x + \mathbb{E}_s W_r) + c_H \int_s^r P_{\tilde{c}_H |r-u|^{2H}} \nabla b_r(x + \mathbb{E}_u W_r) |r-u|^{H-1/2} \cdot \mathrm{d}B_u; \end{aligned}$$

in the above, we used both the representation of W in terms of a stochastic integral (cf. (A.1)), as well as the decomposition $W_r = (W_r - \mathbb{E}_u W_r) + \mathbb{E}_u W_r$ with the first term independent of $\mathcal{F}_u$ (see Appendix A.1). Integrating over $[s,t]$ and applying stochastic Fubini's theorem (which is allowed, since we are assuming b smooth and compactly supported) we obtain

$$\begin{aligned}\int_s^t b_r(x+W_r)\mathrm{d}t &= \int_s^t P_{\tilde{c}_H|r-s|^{2H}} b_r(x+\mathbb{E}_s W_r)\mathrm{d}r \\ &\quad + c_H \int_s^t \int_s^r P_{\tilde{c}_H|r-u|^{2H}} \nabla b_r(x+\mathbb{E}_u W_r)|r-u|^{H-1/2}\cdot \mathrm{d}B_u\,\mathrm{d}r \\ &= \int_s^t P_{\tilde{c}_H|r-s|^{2H}} b_r(x+\mathbb{E}_s W_r)\mathrm{d}r \\ &\quad + c_H \int_s^t \int_u^t P_{\tilde{c}_H|r-u|^{2H}} \nabla b_r(x+\mathbb{E}_u W_r)|r-u|^{H-1/2}\mathrm{d}r\cdot \mathrm{d}B_u\end{aligned}$$

which gives the conclusion. □

The previous result can be strengthened by considering for instance $b \in C^1_x$ instead of smooth, or showing that we can find a set of full probability on which the identity holds for all $0 \leqslant s \leqslant t \leqslant T$; we don't do it here since it is not needed for our purposes. Instead, we need to strengthen the result to the following *functional equality*.

Theorem 3.11. *Let $b: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a smooth, compactly supported function, $H \in (0,1)$. Then for any fixed $0 \leqslant s \leqslant t \leqslant T$, $\mathbb{P}$-a.s. it holds*

$$T^W_{s,t} b = \int_s^t P_{\tilde{c}_H|r-s|^{2H}} b_r(\cdot + \mathbb{E}_s W_r)\mathrm{d}r + c_H \int_s^t \int_u^t P_{\tilde{c}_H|r-u|^{2H}} \nabla b_r(\cdot + \mathbb{E}_u W_r)|r-u|^{H-1/2}\mathrm{d}r\cdot \mathrm{d}B_u$$

where the first integral must be interpreted in the Bochner sense, while the second one as a functional stochastic integral.

We omit the proof, as it is quite technical and requires some knowledge of stochastic integration in UMD spaces; up to technical details, it is mostly a rewriting of the statement already contained in Lemma 3.10, without further insights. The interested reader can find the details in Appendix A.3 from [145].

3.1.3 Stochastic estimates in martingale type 2 Banach spaces

We provide here the regularity estimates for $T^W b$ when W is sampled as a fBm of parameter H; besides being results of independent interest (for fixed x, $T^W b(x)$ is usually referred to as an *additive function* of the process W), they will pave the way for our development of a solution theory for (possibly singular) SDEs associated to W, as well as prevalence statements.

The main ingredients in the proofs are the use of the functional Itô–Tanaka formula from Theorem 3.11, Burkholder's inequality (Theorem A.24 from Appendix A.4), heat kernel and interpolation estimates (Lemma A.13 and Proposition A.9 from Appendix A.2).

We will restrict to the case of $b \in L^q_t B^s_p$, but the strategy is fairly general and works for other classes of spaces (e.g. Lebesgue and Bessel spaces), up to the requirement that the aforementioned tools are still available. However, in order to apply Burkholder's inequality, we will restrict to scales of L^p-based spaces with $p \in [2,\infty)$; see Appendix A.3 from [145] for a deeper discussion on this point. We will see later in Section 3.1.4 how to treat other values of p.

Theorem 3.12. *Let W be a fBm of parameter H and let $b \in L^q_t B^s_p$ for some $p, q \in [2,\infty)$. Then for any $\rho > 0$ satisfying*

$$\rho < \frac{1}{H}\left(\frac{1}{2} - \frac{1}{q}\right), \tag{3.6}$$

there exists $\gamma > 1/2$ such that $T^W b \in C^\gamma_t B^{s+\rho}_p$ with probability 1; moreover, there exist positive constants λ, K such that

$$\mathbb{E}\left[\exp\left(\lambda \frac{\|T^W b\|^2_{C^\gamma B^{s+\rho}_p}}{\|b\|^2_{L^q B^s_p}}\right)\right] \leqslant K \quad \forall b \in L^q_t B^s_p \setminus \{0\}. \tag{3.7}$$

Proof. To simplify the notation, we give the proof in the case $s=0$ (s being the regularity parameter in B_p^s), the other ones being identical; from now on, we will freely use s as a time parameter instead.

First assume b to be a smooth function; by Theorem 3.11, $\int_s^t b_r(\cdot + W_r)\mathrm{d}r = I^1_{s,t} + I^2_{s,t}$ for

$$I^1_{s,t} = \int_s^t P_{\tilde{c}_H|r-s|^{2H}} b_r(\cdot + \mathbb{E}_s W_r)\mathrm{d}r, \quad I^2_{s,t} = c_H \int_s^t \int_u^t P_{\tilde{c}_H|r-u|^{2H}} \nabla b_r(\cdot + \mathbb{E}_u W_r)|r-u|^{H-1/2}\mathrm{d}r \cdot \mathrm{d}B_u.$$

From now on for simplicity we will drop the constants c_H, $\tilde{c}_H$, as they don't play any significant role in the calculations. For the first term, we have the deterministic estimate

$$\begin{aligned}
\|I^1_{s,t}\|_{B_p^\rho} = \left\| \int_s^t P_{|r-s|^{2H}} b_r(\cdot + \mathbb{E}_s W_r)\mathrm{d}r \right\|_{B_p^\rho} &\leqslant \int_s^t \|P_{|r-s|^{2H}} b_r\|_{B_p^\rho} \mathrm{d}r \\
&\lesssim \int_s^t |r-s|^{-\rho H} \|b_r\|_{B_p^0} \mathrm{d}r \\
&\leqslant \|b\|_{L^q B_p^0} \left| \int_s^t |r-s|^{-\rho H q'} \mathrm{d}r \right|^{1/q'} \\
&\lesssim \|b\|_{L^q B_p^0} |t-s|^{1-1/q-\rho H},
\end{aligned}$$

where we used heat kernel estimates for Besov spaces (Lemma A.13) and the fact that the B_p^ρ-norm of b_r is not affected by a translation by $\mathbb{E}_s W_r$. By condition (3.6), $\rho H q' < 1$ and $1 - 1/q - \rho H > 1/2$; we deduce that there exists $\gamma > 1/2$ such that, uniformly in $\omega \in \Omega$,

$$\|I^1\|_{C^\gamma B_p^\rho} \lesssim \|b\|_{L^q B_p^0}. \tag{3.8}$$

For the second term, we apply Burkholder's inequality (A.16) (which is allowed since B_p^ρ with $p \geqslant 2$ is a martingale type 2 space, see Appendix A.4); we obtain

$$\mathbb{E}\big[\|I^2_{s,t}\|_{B_p^\rho}^{2k}\big] \leqslant (Ck)^k\, \mathbb{E}\left[\left(\int_s^t \left\| \int_u^t P_{|r-u|^{2H}} \nabla b_r(\cdot + \mathbb{E}_u W_r)|r-u|^{H-1/2}\mathrm{d}r \right\|_{B_p^\rho}^2 \mathrm{d}u \right)^k\right]. \tag{3.9}$$

We can control the inner integral by deterministic estimates, similar to the ones above:

$$\begin{aligned}
\left\| \int_u^t P_{|r-u|^{2H}} \nabla b_r(\cdot + W^2_{u,r})|r-u|^{H-1/2}\mathrm{d}r \right\|_{B_p^\rho} &\leqslant \int_u^t \|P_{|r-u|^{2H}} \nabla b_r\|_{B_p^\rho} |r-u|^{H-1/2}\mathrm{d}r \\
&\lesssim \int_u^t |r-u|^{-H(\rho+1)+H-1/2} \|b_r\|_{B_p^0} \mathrm{d}r \\
&\leqslant \|b\|_{L^q B_p^0} \left(\int_u^t |r-u|^{-(H\rho+1/2)q'} \mathrm{d}r \right)^{1/q'} \\
&\lesssim \|b\|_{L^q B_p^0} |t-u|^{1/2-1/q-H\rho},
\end{aligned}$$

where again we used the fact that $(H\rho + 1/2)q' < 1$, thanks to (3.6). Set $\varepsilon := 1 - 2/q - 2H\rho$; inserting the estimate inside (3.9) we find that, up to relabelling C, it holds

$$\mathbb{E}\big[\|I^2_{s,t}\|_{B_p^\rho}^{2k}\big] \leqslant (Ck)^k \|b\|_{L^q B_p^0}^{2k} |t-s|^{k(1+\varepsilon)}.$$

But then we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\lambda \frac{\|I^2_{s,t}\|_{B_p^\rho}^2}{|t-s|^{1+\varepsilon} \|b\|_{L^q B_p^0}^2}\right)\right] &= \sum_k \frac{\lambda^k}{k!} \mathbb{E}\left[\frac{\|I^2_{s,t}\|_{B_p^\rho}^{2k}}{|t-s|^{k(1+\varepsilon)} \|b\|_{L^q B_p^0}^{2k}}\right] \\
&\leqslant \sum_k \frac{(\lambda C)^k k^k}{k!} \lesssim \sum_k (\lambda C e)^k < \infty
\end{aligned}$$

as soon as $\lambda < (Ce)^{-1}$. It follows from Lemma A.27 that, for any $\varepsilon' < \varepsilon$, $I^2 \in C_t^{1/2+\varepsilon'} B_p^\alpha$ and that there exists another $\lambda > 0$ (not relabelled for simplicity) such that

$$\mathbb{E}\left[\exp\left(\lambda \frac{\|I^2\|_{C^{1/2+\varepsilon'} B_p^\rho}^2}{\|b\|_{L^q B_p^0}^2}\right)\right] \leqslant K \tag{3.10}$$

for a constant K independent of b. Together with (3.8), this proves the claim for smooth b.

Now let b be a generic element of $L_t^q B_p^0$; we can then find a sequence b_n of smooth functions such that $\|b - b_n\|_{L^q B_p^0} \to 0$ as $n \to \infty$. We know that in this case $\|T^W(b_n - b)\|_{C^0 B_p^0} \to 0$, uniformly on $\omega \in \Omega$. On the other hand, it follows from (3.7), applied to $b_n - b_m$, that for any k it holds

$$\mathbb{E}\big[\|T^W(b_n - b_m)\|_{C^\gamma B_p^\rho}^{2k}\big] \lesssim_k \|b_n - b_m\|_{L^q B_p^0}^{2k};$$

hence the sequence $T^W b_n$ is Cauchy in $L^{2k}(\Omega,; C_t^\gamma B_p^\rho)$ and it admits a limit, which must then coincide with $T^W b \in L^{2k}(\Omega; C_t^\gamma B_p^\rho)$. By Fatou's lemma, it holds

$$\mathbb{E}\left[\exp\left(\lambda \frac{\|T^W b\|_{C^\gamma B_p^\rho}^2}{\|b\|_{L^q B_p^0}^2}\right)\right] \leqslant \liminf_{n \to \infty} \mathbb{E}\left[\exp\left(\lambda \frac{\|T^W b_n\|_{C^\gamma B_p^\rho}^2}{\|b_n\|_{L^q B_p^0}^2}\right)\right] \leqslant K$$

which gives the conclusion. □

Remark 3.13. Theorem 3.12 formally does not cover $q = \infty$, since $L_t^\infty B_p^s$ is not separable. However, we can readily embed $L_t^\infty B_p^s \hookrightarrow L_t^q B_p^s$ with q arbitrarily large (recall that we are on a finite $[0, T]$) and apply the result therein to conclude that, if $b \in L_t^\infty B_p^s$, then $\mathbb{P}$-a.s. $T^W b \in C_t^{1/2+} B_p^{s+\rho}$ for any $\rho < 1/(2H)$.

Remark 3.14. Theorem 3.12 immediately implies that, under assumption (3.6), the *random* averaging operator $T^W: b \mapsto T^W b$ is a linear bounded map from $L_t^q B_p^s$ into $L^k(\Omega; C_t^\gamma B_p^{s+\rho})$, for any $k \in \mathbb{N}$. Observe the difference with Remark 3.5.

We can actually further improve the regularity result of Theorem 3.12, by allowing an arbitrarily large constant λ appearing in (3.7).

Corollary 3.15. *Let $b \in L_t^q B_p^s$ with $p \in [2, \infty)$, $\rho > 0$ and assume (3.6) holds. Then there exists $\gamma > 1/2$ and a function $K(\lambda)$ independent of b such that*

$$\mathbb{E}\left[\exp\left(\lambda \frac{\|T^W b\|_{C^\gamma B_p^{s+\rho}}^2}{\|b\|_{L^q B_p^s}^2}\right)\right] \leqslant K(\lambda) < \infty \quad \forall \lambda \in \mathbb{R}.$$

Proof. As before, we can assume wlog $s = 0$. If ρ satisfies (3.6), then there exists $\varepsilon > 0$ such that also $\rho + \varepsilon$ satisfies (3.6); it then follows from Lemma A.9 that

$$\|T^W b\|_{C^\gamma B_p^\rho} \lesssim \|T^W b\|_{C^\gamma B_p^0}^{1-\theta} \|T^W b\|_{C^\gamma B_p^{\rho+\varepsilon}}^\theta \leqslant \|b\|_{L^q B_p^0}^{1-\theta} \|T^W b\|_{C^\gamma B_p^{\rho+\varepsilon}}^\theta$$

where $\theta = \varepsilon/(s + \varepsilon)$ and we used the fact that $q > 2$ due to condition (3.6). It follows that

$$\frac{\|T^W b\|_{C^\gamma B_p^\rho}^{2/\theta}}{\|b\|_{L^q B_p^0}^{2/\theta}} \lesssim \frac{\|T^W b\|_{C^\gamma B_p^{\rho+\varepsilon}}^2}{\|b\|_{L^q B_p^0}^2}$$

where $1/\theta = (s + \varepsilon)/\varepsilon =: \beta$. Applying Theorem 3.12 to $\rho + \varepsilon$, we obtain that there exist $\bar{\lambda}$, $\bar{K}$ independent of b such that

$$\mathbb{E}\left[\exp\left(\bar{\lambda} \frac{\|T^W b\|_{C^\gamma B_p^\rho}^{2\beta}}{\|b\|_{L^q B_p^0}^{2\beta}}\right)\right] \leqslant \mathbb{E}\left[\exp\left(C_\varepsilon \bar{\lambda} \frac{\|T^W b\|_{C^\gamma B_p^{\rho+\varepsilon}}^2}{\|b\|_{L^q B_p^0}^2}\right)\right] \leqslant \bar{K}.$$

Since $\beta > 1$, the conclusion follows with $K(\lambda)$ given by the (β-dependent) optimal deterministic constant such that $\exp(\lambda x^2) \leqslant K(\lambda) \exp(\bar{\lambda} x^{2\beta}) / \bar{K}$ for all $x \geqslant 0$. □

In the limiting case in which (3.6) becomes an equality, slightly more careful estimates still allow to obtain a regularity result in space, at the cost of lower time regularity.

Theorem 3.16. *Let $b \in L_t^q B_p^s$ with $p \in [2, \infty)$, $q \in (2, \infty)$ and let $\rho > 0$ satisfy*

$$\rho = \frac{1}{2H}\left(1 - \frac{1}{q}\right). \tag{3.11}$$

Then $\mathbb{P}$-a.s. $T^W b \in C_t^0 B_p^{s+\rho}$ and there exist positive constant λ, K such that

$$\mathbb{E}\left[\exp\left(\lambda \frac{\|T^W b\|_{C^0 B_p^{s+\rho}}^2}{\|b\|_{L^q B_p^s}^2}\right)\right] < K \quad \forall b \in L_t^q B_p^s \setminus \{0\}.$$

Proof. As before, we can assume $s = 0$, b smooth; again we decompose $T^W b = I^1 + I^2$. Going through the same calculations for I^1, we obtain

$$\|I_{s,t}^1\|_{B_p^\rho} \lesssim \|b\|_{L^q B_p^0} |t - s|^{1 - 1/q - \rho H} = \|b\|_{L^q B_p^0} |t - s|^{1/2}$$

where the estimate is uniform in $\omega \in \Omega$; it follows immediately that

$$\mathbb{E}\left[\exp\left(\lambda \|I^1\|_{C^0 B_p^\rho}^2\right)\right] < \infty$$

and so we only need to focus on I^2. By Burkholder's inequality, we have

$$\mathbb{E}\left[\|I^2\|_{C^0 B_p^\rho}^{2k}\right] \leqslant (Ck)^k \mathbb{E}\left[\left(\int_0^T \left\| \int_u^T P_{|r-u|^{2H}} \nabla b_r(\cdot + \mathbb{E}_u W_r) |r - u|^{H - 1/2} \mathrm{d}r \right\|_{B_p^\rho}^2 \mathrm{d}s\right)^k\right]$$

and as before we want to estimate the integral inside in a deterministic manner. Going through similar calculations we obtain

$$\int_0^T \left\| \int_u^T P_{|r-u|^{2H}} \nabla b(\cdot + \mathbb{E}_u W_r) |r - u|^{H - 1/2} \mathrm{d}r \right\|_{B_p^\rho}^2 \mathrm{d}u \lesssim \int_0^T \left(\int_u^T |r - u|^{-H\alpha - 1/2} \|b_r\|_{B_p^0} \mathrm{d}r\right)^2 \mathrm{d}u.$$

Due to the assumption on the coefficients, we can now apply the Hardy-Littlewood-Sobolev inequality to obtain

$$\left(\int_0^T \left(\int_u^T |r - u|^{-H\alpha - 1/2} \|b_r\|_{B_p^0} \mathrm{d}r\right)^2 \mathrm{d}u\right)^{1/2} \lesssim \|b\|_{L_t^q B_p^0}$$

so that

$$\mathbb{E}\left[\|I^2\|_{C^0 B_p^\rho}^{2k}\right] \leqslant (C' k)^k \, \|b\|_{L_t^q B_p^\rho}^{2k}.$$

The conclusion follows by expanding the exponential and choosing λ small as before. □

We end this section with several remarks discussing various technical point and extensions, and which can be skipped on a first reading.

Remark 3.17. Heuristically, condition (3.6) can be seen as a time-space weighted regularity condition, where time counts as $1/H$ times space (which is in agreement with parabolic regularity in the case $H = 1/2$ of Brownian motion). Indeed, we know that the averaging operator T^w maps $L_t^q B_p^s$ into $W_t^{1,q} B_p^s$; if we assume that regularity can be distributed between time and space, it should also map $L_t^q B_p^s$ into $W_t^{\theta,q} B_p^{s + (1-\theta)/H}$ for any $\theta \in (0, 1)$. In order to achieve $1/2 + \varepsilon$ regularity in time it is then required $\theta - 1/q > 1/2$, which implies that the regularity gain in space is at most

$$\frac{1 - \theta}{H} < \frac{1}{H}\left(\frac{1}{2} - \frac{1}{q}\right)$$

which matches exactly condition (3.6) for ρ.

Remark 3.18. The restriction to work with $B_{p,q}^s$ with $p = q$, is not particularly relevant since by Besov embedding if $b \in L_t^q B_{p,\tilde{p}}^s$, then it also belongs to $L_t^q B_p^s$ if $\tilde{p} < p$ and to $L_t^q B_p^{s-\varepsilon}$ for any $\varepsilon > 0$ if $\tilde{p} \geqslant p$, so that we can first embed it and then apply the estimate there. Also the restriction $p \neq \infty$ can be overcome, as will be shown in the upcoming Section 3.1.4. In the special case where b is known to be compactly supported (e.g. $b = \delta_0$), one can more directly embed B_∞^s into $B_p^{s-\varepsilon}$ for any $\varepsilon > 0$ and $p < \infty$.

Remark 3.19. The restriction to work with L^p-based spaces with $p \geqslant 2$ is more restrictive and it would be of fundamental importance to weaken it, especially reaching the case $p=1$; this was already pointed out in Conjecture 1.2 from [57]. The reason is that, by the properties of averaging, we know that for any $K \in C_c^\infty$ and time independent b it holds $K * T^w b = T^w(K * b) = (T^w K) * b$; if we were able to show that $T^w K \in C_t^\gamma B_1^\rho$ with an estimate that only depends on the L^1-norm of K, then we could automatically deduce regularity estimates of the form $K * T^w b \in C_t^\gamma B_p^{s+\rho}$ with $b \in B_p^s$ for any $p \in [1,\infty]$. We could then consider a family of mollifiers obtained by rescaling K (which all have the same L^1-norm, so the same estimate in $C_t^\gamma B_1^\rho$) to get estimates for the map $b \mapsto T^w b$ in any L^p based space with $p \in [1,\infty]$ (as above, only time independent b considered). Unfortunately, it is now understood that Conjecture 1.2 from [57] is at least partially false, see the upcoming discussion at the beginning of Chapter 5 and Remark 3.7 from [170]; nontheless, a complete picture is still missing. A partial contribution to the case $p \in (1,2]$ will be presented in the upcoming Section 3.1.4.

Remark 3.20. A closer look at the proofs shows that both the Itô–Tanaka formula from Theorem 3.11 and the regularity estimates from Theorems 3.12 can be generalised to Gaussian processes X different from fBm, but still satisfying a strong form of *local nondetermism*. For instance one can take X of the form

$$X_t = \int_0^t K(t,s) \mathrm{d}B_s,$$

for some deterministic matrix-valued function K, such that for some $H \in (0,1)$ it holds

$$\operatorname{Var}(X_t|\mathcal{F}_s) = \operatorname{Var}(X_t - \mathbb{E}_s X_t) \gtrsim |t-s|^{2H} \quad \forall s < t \tag{3.12}$$

where $\mathcal{F}_t = \sigma(B_s : s \leqslant t)$. Condition (3.12) is a type of strong local nondeterminism (SLND); these type of processes satisfy many interesting properties, which will be discussed in detail in Chapter 5.

Remark 3.21. It follows immediately from the above results and from Besov embeddings that if $b \in L_t^q B_p^s$ for some $q > 2$, $p \in [2,\infty)$, then for any β such that

$$\beta < s + \frac{1}{H}\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{d}{p} \tag{3.13}$$

there exists $\gamma > 1/2$ such that $T^W b \in C_t^\gamma C_x^\beta$ with full probability. For instance in the case $s=0$, i.e. $b \in L_t^\alpha B_p^0$, in order to require $T^W b \in C_t^\gamma C_x^0$ it is enough

$$\frac{1}{q} + H\frac{d}{p} < \frac{1}{2},$$

while in order to require $T^W b \in C_t^\gamma C_x^1$ it suffices

$$\frac{1}{q} + H\frac{d}{p} < \frac{1}{2} - H.$$

If $b \in L_t^q B_\infty^s$ with spatially compact support, uniform in time, then $T^W b \in C_t^\gamma C_x^n$ if

$$H < \frac{1}{n-s}\left(\frac{1}{2} - \frac{1}{q}\right).$$

3.1.4 Stochastic estimates in other Banach spaces

As already mentioned in Remark 3.19, the techniques employed in Section 3.1.3 come with the fundamental restriction of using L^p-based spaces with $p \in [2,\infty)$. Here we present some alternative strategies, which cover the cases $p \in (1,2)$ and $p = \infty$, but come at a certain price.

We start by dealing with $p \in (1,2)$; to this end, we first present a lemma of independent interest, which is a generalization of Lemma 45 from [143] and Theorem 4.3 from [57]. We recall to the reader that the notion of martingale $\mathfrak{p}$ space and its properties may be found in Appendix A.4.

Lemma 3.22. *Let E be a Banach space of martingale type $\mathfrak{p} \in (1,2]$, $(X_t)_{t\in[0,T]}$ be an E-valued stochastic process. Assume that there exist deterministic constants C_1, C_2 such that, for any $s<t$, $\mathbb{P}$-a.s. it holds*

$$\|X_{s,t}\|_E \leqslant C_1|t-s|, \quad \|\mathbb{E}_s X_{s,t}]\|_E \leqslant C_2. \tag{3.14}$$

Then there exist constants $\mu, K > 0$, only depending on E, such that

$$\mathbb{E}\left[\exp\left(\mu\frac{\|X_{s,t}\|_E^{\mathfrak{p}}}{C_1 C_2^{\mathfrak{p}-1}|t-s|}\right)\right] \leqslant K \quad \forall s < t. \tag{3.15}$$

Proof. By linearity, we may assume $C_1 = 1$. Suppose first that $|t-s| \leqslant C_2$, then we have the trivial estimate

$$\|X_{s,t}\|_E^{\mathfrak{p}} \leqslant |t-s|^{\mathfrak{p}} \leqslant |t-s| C_2^{\mathfrak{p}-1},$$

which immediately implies (3.15) in this case.

Suppose now that $C_2 < |t-s|$. Let $n \in \mathbb{N}$ to be fixed later; define $t_k = s + k(t-s)/n$ for $k \in \{0, \ldots, n\}$ and

$$Z_k = \mathbb{E}_{t_{k+1}} X_{s,t} - \mathbb{E}_{t_k} X_{s,t}.$$

Setting $S_k = \sum_{j=0}^{k-1} Z_j$, $\{S_k\}_k$ is an E-valued martingale and it holds $X_{s,t} = S_n + \mathbb{E}_s X_{s,t}$. We have a trivial bound on $\mathbb{E}_s X_{s,t}$ by the assumption and the condition $C_2 < |t-s|$; thus it suffices to estimate $\|S_n\|_E$, which we plan to do by means of the Azuma–Hoeffding inequality, see Theorem A.25 in Appendix A.4.

It holds $Z_k = \mathbb{E}_{t_{k+1}} X_{t_{k+1},t} - \mathbb{E}_{t_k} X_{t_k,t} + X_{t_k,t_{k+1}}$ and so by assumption 3.14 we have the estimate

$$\|Z_k\|_E \leqslant 2C_2 + |t-s|/n \quad \mathbb{P}\text{-a.s.}$$

Applying Theorem A.25, there exist constants $\mu, K > 0$ only depending on E such that

$$\mathbb{E}[\exp(\mu|S_n|^{\mathfrak{p}}/C_n)] \leqslant K$$

where the constant C_n is given by

$$C_n = \sum_{k=0}^{n-1} (2C_f + |t-s|/n)^{\mathfrak{p}} \lesssim n C_2^{\mathfrak{p}} + n^{1-\mathfrak{p}}|t-s|^{\mathfrak{p}}.$$

Choosing n s.t. $n \sim |t-s|/C_2$, we obtain $C_n \sim |t-s| C_2^{\mathfrak{p}-1}$, and so for μ sufficiently small it holds

$$\mathbb{E}[\exp(\mu|S_n|^{\mathfrak{p}}/C_2^{\mathfrak{p}-1}|t-s|)] \leqslant K;$$

the estimate is uniform over $s < t$, which yields the conclusion. □

We can use Lemma 3.22 to get averaging estimates in L^p-based spaces with $p \in (1,2)$.

Corollary 3.23. *Let f be a smooth function with $\hat{f}$ supported on $\lambda\mathcal{A}$, where $\mathcal{A} \subset \mathbb{R}^d$ is a fixed annulus, let $p \in (1,2]$ and p' denote its conjugate exponent. Then there exist constants μ, K, depending on $p, H, \mathcal{A}$, such that*

$$\mathbb{E}\left[\exp\left(\mu\left(\frac{\lambda^{\frac{1}{Hp'}}\|T_{s,t}^W f\|_{L^p}}{|t-s|^{\frac{1}{p}}\|f\|_{L^p}}\right)^p\right)\right] \leqslant K. \tag{3.16}$$

Proof. We can assume $\|f\|_{L^p} = 1$. Setting $X_{s,t} = \int_s^t f(\cdot + W_r)\mathrm{d}r$, it holds $\|X_{s,t}\|_{L^p} \leqslant |t-s|$; moreover an application of Lemma A.4 from Appendix A.2 gives

$$\|\mathbb{E}_s X_{s,t}]\|_{L^p} = \left\|\int_s^t P_{c_H|r-s|^{2H}} f(\cdot + \mathbb{E}_s W_r)\,\mathrm{d}r\right\|_{L^p} \lesssim \int_s^t e^{-\lambda^2 c_H^2 |r-s|^{2H}} \mathrm{d}r \lesssim \lambda^{-1/H}.$$

Since L^p spaces with $p \in (1,2]$ have martingale type p, we can apply Lemma 3.22 to conclude. □

Proposition 3.24. *Let $s \in \mathbb{R}$, $p \in (1,2]$, p' its conjugate exponent. Then there exist constants μ, K, depending on p, H, such that*

$$\mathbb{E}\left[\exp\left(\mu\left|\frac{\|T_{s,t}^W f\|_{B_p^{s+1/(p'H)}}}{|t-s|^{1/p}\|f\|_{B_p^s}}\right|^p\right)\right] \leqslant K \quad \forall f \in B_p^s \setminus \{0\} \tag{3.17}$$

As a consequence, for any $q\in[1,\infty]$ and any $\alpha<1/p$, $\gamma<1/(p'H)$, there exist $\tilde\mu,\tilde K$ depending on the previous parameters such that

$$\mathbb{E}\left[\exp\left(\mu\frac{\|T^W f\|^p_{C^\alpha B^{s+\gamma}_{p,q}}}{\|f\|^p_{B^s_{p,q}}}\right)\right]\leqslant K \quad \forall f\in B^s_{p,q}\setminus\{0\}. \tag{3.18}$$

Proof. In the following we will apply the convention that $0/0=0$. Let $f\in B^s_p$ with $\|f\|_{B^s_p}=1$, then $T^W_{s,t}f=\sum_n X^n_{s,t}$ for $X^n_{s,t}=\int_s^t\Delta_n f(\cdot+W_r)\mathrm{d}r$; since $\Delta_n f$ is a smooth function supported on $2^n\mathcal{A}$ for a given annulus $\mathcal{A}$, we can apply Corollary 3.23 to it. Together with Jensen's inequality, this gives

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\mu|t-s|^{-1}\|T^W_{s,t}f\|^p_{B^{s+p'/H}_p}\Big)\Big]&=\mathbb{E}\left[\exp\left(\mu|t-s|^{-1}\sum_n\|\Delta_n f\|^p_{L^p}\,2^{n\frac{p-1}{H}}\frac{\|X^n_{s,t}\|^p_{L^p}}{\|\Delta_n f\|^p_{L^p}}\right)\right]\\&\leqslant\sum_n\|\Delta_n f\|^p_{L^p}\,\mathbb{E}\left[\exp\left(\mu 2^{n\frac{p-1}{H}}\frac{\|X^n_{s,t}\|^p_{L^p}}{|t-s|\|\Delta_n f\|^p_{L^p}}\right)\right]\leqslant K\\&\leqslant K\sum_n\|\Delta_n f\|^p_{L^p}=K\end{aligned}$$

which proves (3.17). Next, observe that given any $q\in[1,\infty]$, we can always embed $B^s_{p,q}$ into $B^{s-\varepsilon}_p$ and apply (3.17) therein, which provides

$$\mathbb{E}\left[\exp\left(\mu\left|\frac{\|T^W_{s,t}f\|_{B^{s+1/(p'H)-\varepsilon}_p}}{|t-s|^{1/p}\,\|f\|_{B^s_{p,q}}}\right|^p\right)\right]\leqslant K\quad\forall f\in B^s_{p,q}\setminus\{0\};$$

estimate (3.18) then follows from an application of Lemma A.27. □

Remark 3.25. Let us comment on both the advantages and disadvantages of Proposition 3.24, compared to results from the previous section like Theorems 3.12 and 3.16. On one hand, it becomes clear that the expected regularity gain, for $b\in L^\infty_t B^s_p$, is morally $T^W b\in C^\gamma_t B^{s+\rho}_p$ with

$$\rho=\frac{1}{(p'\wedge 2)H}, \tag{3.19}$$

where the expression is now expected to be true for all $p\in(1,\infty)$. Also observe that, as $p\downarrow 1$, $\rho\downarrow 0$, confirming that working with L^1-based spaces is particularly difficult and might not even possible for general $H\in(0,1)$. Proposition 3.24 reaches the critical regularity $\rho=1/(p'H)$, while not giving up on the time regularity $\alpha\sim 1/p$ (compare with Theorem 3.16!). However the proof is not very robust: already treating $f\in L^\infty_t B^s_p$ provides some technical challenges. Moreover, while Theorems 3.12 and 3.16 generalize easily to $B^s_{p,\tilde p}$ with $p\neq\tilde p$ (heat kernel estimates are still available), the proof of Proposition 3.24 crucially requires $p=\tilde p$ as the explicit summability of the LP blocks is needed when invoking Jensen's inequality (at least for proving (3.17)). On the upside, the proof generalizes to the case $p\in[2,\infty)$ as well, up to requiring this time $f\in B^s_{p,2}$ instead of B^s_p.

We now move to the case of Besov-Hölder spaces B^s_∞; here we can obtain very similar results to those of Section 3.1.3, up to the price of dealing with $T^W b$ belonging to suitable weighted spaces $C^{\beta,\lambda}_x$, like the ones already considered in Chapter 1.

Proposition 3.26. *Let $b\in L^q_t B^s_\infty$ with $s<0$, $q\in(2,\infty]$, W be a fBm of parameter $H\in(0,1)$; suppose (s,q) satisfy*

$$\gamma:=1-\frac{1}{q}+sH>\frac{1}{2}. \tag{3.20}$$

Then for any $\tilde\gamma<\gamma$ there exists an increasing function K (depending on d,T and the above parameters) such that

$$\mathbb{E}\left[\exp\left(\frac{\eta}{\|b\|^2_{L^q B^s_\infty}}\left[\!\!\left[\int_0^\cdot b_r(x+W_r)\mathrm{d}r\right]\!\!\right]^2_{\tilde\gamma}\right)\right]\leq K(\eta)\quad\forall\eta>0,\,x\in\mathbb{R}^d,\,b\in L^q_t B^s_\infty\setminus\{0\}. \tag{3.21}$$

Proof. Arguing by density, we may assume b to be smooth and compactly supported; up to reasoning componentwise, scaling and shifting, wlog $x=0$, $b\in C_c^\infty([0,T]\times\mathbb{R}^d)$ and $\|b\|_{L^q B^s_\infty}=1$.

Since B^s_∞ is not a martingale type 2 space, we cannot apply Theorem 3.11; yet, for fixed $x=0$, Lemma 3.10 is still available, so we can decompose $\int_\tau^t b_r(W_r)\mathrm{d}r = I^1_{\tau,t}+I^2_{\tau,t} = I^1_{\tau,t}+\int_\tau^t J_{u,t}\cdot \mathrm{d}B_u$ for

$$I^1_{\tau,t} := \int_\tau^t P_{\tilde c_H |r-\tau|^{2H}} b_r(\mathbb{E}_\tau W^r)\mathrm{d}r, \quad J_{u,t} := c_H \int_u^t P_{\tilde c_H |r-u|^{2H}} \nabla b_r(\mathbb{E}_\tau W^r) |r-u|^{H-1/2}\mathrm{d}r.$$

As before, in the following we will drop the constants $c_H,\tilde c_H$. The terms $I_{\tau,t}$ and $J_{u,t}$ can still be estimated by means of heat kernel estimates in B^s_∞ (Lemma A.13), similarly to the proof of Theorem 3.12. For instance, one obtains

$$|J_{u,t}| \lesssim \int_u^t \|P_{|r-u|^{2H}}\nabla b_r\|_{L^\infty_x} |r-u|^{H-1/2}\mathrm{d}r \lesssim |t-u|^{\gamma-1/2}.$$

Applying Burkholder-Davis-Gundy inequality with optimal asymptotic constant, combining the estimate for $I^2_{\tau,t}$ with a similar one for $I^1_{\tau,t}$, we deduce the existence of $\bar\eta, C>0$ such that

$$\mathbb{E}\left[\exp\left(\eta \left|\frac{\int_\tau^t b_r(W_r)\mathrm{d}r}{|t-s|^\gamma}\right|^2\right)\right] \leqslant C \quad \forall \eta\leqslant\bar\eta,\ s<t.$$

Together with Lemma A.27, this implies that for any $\tilde\gamma<\gamma$ there exist (relabelled) $\bar\eta, C>0$ s.t.

$$\mathbb{E}\left[\exp\left(\eta \left[\!\!\left[\int_0^\cdot b_r(W_r)\mathrm{d}r\right]\!\!\right]_{\tilde\gamma}^2\right)\right] \leqslant C \quad \forall \eta\leqslant\bar\eta. \tag{3.22}$$

In order to conclude, it remains to show that we can improve inequality (3.22) by allowing any value $\eta>0$; to do so, we will resort to an interpolation trick, similar in style to Corollary 3.15 (although slightly more involved).

First, observe that if s,q,H satisfy (3.20) and we fix $\tilde\gamma<\gamma$, then we can find $\varepsilon>0$ sufficiently small so that $\gamma^\varepsilon := 1-1/q-(s-\varepsilon)H>1/2$ and $\tilde\gamma<\gamma^\varepsilon$; then by estimate (3.22) (for $s-\varepsilon$ in place of s) and linearity, there exist $\bar\eta>0$ and $C>0$ such that

$$\mathbb{E}\left[\exp\left(\frac{\bar\eta}{\|\tilde b\|^2_{L^q B^{s-\varepsilon}_\infty}} \left[\!\!\left[\int_0^\cdot \tilde b_r(W_r)\mathrm{d}r\right]\!\!\right]_{\tilde\gamma}^2\right)\right] \leqslant C \quad \forall \tilde b\in L^q_t B^{s-\varepsilon}_\infty\setminus\{0\}. \tag{3.23}$$

We may assume $\|b\|_{L^q B^s_\infty}=1$; fix $\varepsilon>0$ as above. For $N\in\mathbb{N}$ to be chosen later, we decompose b as

$$b_t = b^1_t + b^2_t, \quad b^1_t = \sum_{j\leqslant N} \Delta_j b_t, \quad b^2_t = \sum_{j>N}\Delta_j b_t,$$

where Δ_j denote Littlewood-Paley blocks; there exists $C>0$ such that

$$\|b^1\|_{L^q L^\infty} \leqslant C\, 2^{-Ns}, \quad \|b^2\|_{L^q B^{s-\varepsilon}_\infty} \leqslant C\, 2^{-N\varepsilon}.$$

Now for a given $\eta>0$, choose $N=N(\eta)\in\mathbb{N}$ such that

$$\eta\leqslant C^{-2}\, 2^{2N\varepsilon-1}\bar\eta$$

and decompose b as above; we may assume that $b^{2,N}\neq 0$, otherwise the stated estimate is trivial. Clearly, under (3.20) it holds $\tilde\gamma\leqslant 1-1/q$; therefore setting $\beta=1-1/q-\tilde\gamma$, we have the $\mathbb{P}$-a.s.

$$\left[\!\!\left[\int_0^\cdot b^1_r(W_r)\mathrm{d}r\right]\!\!\right]_{\tilde\gamma} \leqslant (1+T)^\beta\, \|b^1\|_{L^q C^0} \leqslant C(1+T)^\beta 2^{-Ns} =: C_{N(\eta)}.$$

Combining this with (3.23) applied to $\tilde b = b^2$, we get

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\eta \left[\!\!\left[\int_0^\cdot b_r(W_r)\mathrm{d}r\right]\!\!\right]_{\tilde\gamma}^2\right)\right] &\leqslant \mathbb{E}\left[\exp\left(2\eta \left[\!\!\left[\int_0^\cdot b^1_r(W_r)\mathrm{d}r\right]\!\!\right]_{\tilde\gamma}^2 + 2\eta \left[\!\!\left[\int_0^\cdot b^2_r(W_r)\mathrm{d}r\right]\!\!\right]_{\tilde\gamma}^2\right)\right]\\
&\leqslant \exp\left(2\eta C^2_{N(\eta)}\right) \mathbb{E}\left[\exp\left(\frac{\bar\eta}{\|b^2\|^2_{L^q B^{s-\varepsilon}_\infty}} \left[\!\!\left[\int_0^\cdot b^2_r(W_r)\,\mathrm{d}r\right]\!\!\right]_{\tilde\gamma}^2\right)\right]\\
&\lesssim \exp\left(2\eta C^2_{N(\eta)}\right)
\end{aligned}$$

where the estimate now holds for all $\eta > 0$. □

Corollary 3.27. *Let $b \in L_t^q B_\infty^s$ with $s < 0$, $q \in (2, \infty]$, W be a fBm of parameter $H \in (0,1)$ and let $\rho \in (0,1]$; suppose (s, ρ, q) satisfy*

$$\gamma := 1 - \frac{1}{q} + (s - \rho) H > \frac{1}{2}. \tag{3.24}$$

Then for any $\tilde{\gamma} < \gamma$ there exists an increasing function K (depending on d, T and the above parameters) such that

$$\mathbb{E}\left[\exp\left(\eta \left| \frac{[\![\int_0^\cdot b_r(x + W_r) \mathrm{d}r - \int_0^\cdot b_r(y + W_r) \mathrm{d}r]\!]_{\tilde{\gamma}}}{\|b\|_{L^q B_\infty^s} |x - y|^\rho} \right|^2\right)\right] \leqslant K(\eta) \quad \forall \eta > 0, \tag{3.25}$$

uniformly over $x \neq y$ and $b \in L_t^q B_\infty^s$, $b \neq 0$; moreover, for any $\lambda, \varepsilon_i > 0$, $\mathbb{P}$-a.s. $T^W b \in C_t^{\gamma - \varepsilon_1} C_x^{\rho - \varepsilon_2, \lambda}$.

Suppose now that $b \in L_t^q B_\infty^\alpha$ for $\alpha \in (-\infty, 1)$, $q \in (2, \infty]$ satisfying

$$\alpha - \frac{1}{Hq} > 1 - \frac{1}{2H}, \tag{3.26}$$

then the following hold:

1. *There exists $\tilde{\gamma} > 1/2$ such that, for any $\lambda > 0$, $\mathbb{P}$-a.s. $T^W b \in C_t^{\tilde{\gamma}} C_x^{1,\lambda}$.*
2. *There exists $\tilde{\gamma} > 1/2$ such that for any $b^1, b^2 \in L_t^q B_\infty^\alpha$ and any $n \in \mathbb{N}$*

$$\mathbb{E}\left[\left[\!\!\left[\int_0^\cdot b^1(r, W_r) \mathrm{d}r - \int_0^\cdot b^2(r, W_r) \mathrm{d}r\right]\!\!\right]_{\tilde{\gamma}; [0,\tau]}^n\right]^{1/n} \lesssim_n \left(\int_0^\tau \|b_r^1 - b_r^2\|_{B_\infty^{\alpha - 1}}^q \mathrm{d}r\right)^{1/q} \quad \forall \tau \in [0, T].$$

3. *If $H < 1/2$, $\alpha < 0$, there exists $\tilde{\gamma} > H + 1/2$ and an increasing function K such that*

$$\mathbb{E}\left[\exp\left(\frac{\eta}{\|b\|_{L^q B_\infty^\alpha}^2} \left[\!\!\left[\int_0^\cdot b_r(W_r) \mathrm{d}r\right]\!\!\right]_{\tilde{\gamma}}^2\right)\right] \leqslant K(\eta) \quad \forall \eta > 0,\ b \in L_t^q B_\infty^\alpha, b \neq 0.$$

Proof. Given b as above, $x \neq y$ fixed, define $\tilde{b}_t(\cdot) = |x - y|^{-\rho} [b_t(x + \cdot) - b_t(y + \cdot)]$; by the behaviour of Besov spaces under translations (Lemma A.7) it holds

$$\|\tilde{b}\|_{L^q B_\infty^{s - \rho}} \lesssim \|b\|_{L^q B_\infty^s}.$$

Inequality (3.25) then follows from (3.21) applied to $\tilde{b}$, since by assumption (3.24) $\tilde{s} = s - \rho$ satisfies condition (3.20); $T^W b$ belonging to $C_t^{\gamma - \varepsilon_1} C_x^{\rho - \varepsilon_2, \lambda}$ is a consequence of Garsia-Rodemich-Rumsay lemma, as stated in the version of Corollary A.30.

We now assume (3.26) holds and prove Points 1-3.

If $b \in L_t^q B_\infty^\alpha$, then $D_x b \in L_t^q B_\infty^{\alpha - 1}$ with $s = \alpha - 1$ satisfying (3.20), so we can find $\rho > 0$ small enough such that (s, ρ) satisfy (3.24) as well. It follows that $\mathbb{P}$-a.s. $T^W D_x b = D_x T^W b \in C_t^{\gamma - \varepsilon} C_x^{0, \lambda}$, namely $T^W b \in C_t^{\gamma - \varepsilon} C_x^{1, \lambda}$, for any $\varepsilon > 0$, showing Point 1.

For $\tau = T$, the statement in part 2 is again a consequence of (3.21) (for $x = 0$ and $s = \alpha - 1$) and linearity of $b \mapsto T^W b$. For general $\tau \in [0, T]$, define $\tilde{b}_t^i = b_t^i \mathbb{1}_{[0,\tau]}(t)$ and observe that

$$\left[\!\!\left[\int_0^\cdot (b^1 - b^2)(r, W_r) \mathrm{d}r\right]\!\!\right]_{\gamma; [0,\tau]} = \left[\!\!\left[\int_0^\cdot (\tilde{b}^1 - \tilde{b}^2)(r, W_r) \mathrm{d}r\right]\!\!\right]_{\gamma; [0,T]};$$

the estimate for general τ thus follows applying the one for $\tau = T$ to $\tilde{b}^i$.

Finally, in order to prove 3 it is enough to show that

$$\gamma = 1 - \frac{1}{q} + \alpha H > H + 1/2,$$

as in that case we can find $\tilde{\gamma} \in (H + 1/2, \gamma)$ such that (3.21) holds. But the above condition on γ is exactly (3.26). □

Let us finally write more explicitly a specific consequence of Corollary 3.27, which will be of fundamental importance in order to apply the theory from Chapter 1.

Corollary 3.28. *Let* $b\in L^q_t B^\alpha_\infty$ *for parameters* (q,α) *satisfying (3.26); then we can find parameters* $\gamma,\beta,\lambda\in(0,1)$ *such that* $\gamma>1/2$*,* $\gamma(1+\beta)>1$*,* $\beta+\lambda\leqslant 1$ *and* $\mathbb{P}$*-a.s.* $T^W b\in C^\gamma_t C^{\beta,\lambda}_x\cap C^\gamma_t C^{1,\lambda}_x$*.*

3.2 Main results for SDEs perturbed by fBm

In the following, whenever referring to a *distributional field*, we will mean an object of the form $b\in L^q_t C^\alpha_x$ for some $q\in[2,\infty]$ and $\alpha\in\mathbb{R}$ (so that on one hand $T^W b$ is analytically well-defined by Section 3.1.1, while on the other the results from Sections 3.1.3-3.1.4 are available, at least under suitable conditions).

We are now ready to present the main class of drifts we will work with in this section.

Assumption 3.29. *Given* $H\in(0,1)$*, the distributional drift* b *satisfies the following:*

- *If* $H>1/2$*, then* $b\in C^{\alpha H}_t C^0_x\cap C^0_t C^\alpha_x$ *for some* $\alpha>1-\frac{1}{2H}>0$*; this is equivalent to requiring the existence of a constant* $C>0$ *such that*

$$|b_t(x)|\leqslant C,\quad |b_t(x)-b_s(y)|\leqslant C(|t-s|^{\alpha H}+|x-y|^\alpha)\quad \forall s,t,x,y.$$

- *If* $H\leqslant 1/2$*, then* $b\in L^q_t B^\alpha_\infty$ *for some* $(\alpha,q)\in\mathbb{R}\times[2,\infty]$ *satisfying condition (3.26), namely*

$$\alpha-\frac{1}{Hq}>1-\frac{1}{2H}.$$

In both cases, we will use the notation $\|b\|_E$ *for* $E=C^{\alpha H}_t C^0_x\cap C^0_t C^\alpha_x$ *when* $H>1/2$*, respectively* $E=L^q_t B^\alpha_\infty$ *when* $H\leqslant 1/2$*.*

The main result of this section, which is a combination of the ones obtained in [145] and [146], can then be summarized as follows.

Theorem 3.30. *Let* W *be an fBm of parameter* $H\in(0,1)$ *and let* b *satisfy Assumption 3.29. Then for any* $x_0\in\mathbb{R}^d$ *strong existence, pathwise uniqueness, path-by-path uniqueness and uniqueness in law hold for the SDE*

$$X_t=x_0+\int_0^t b_r(X_r)\,\mathrm{d}r+W_t\quad \forall t\in[0,T].$$

Moreover we have strong stability estimates for SDEs driven by different drifts, in the following sense. Given $x^i_0\in\mathbb{R}^d$ *and* b^i *satisfying Assumption 3.29,* $i=1,2$*, denote by* X^i *the solutions associated to* (x^i_0,b^i) *and driven by the same noise W; let* $M>0$ *be a constant such that* $\|b^i\|_E\leqslant M$ *for* $i=1,2$*. Finally, let* $(\alpha,\tilde q)$ *be another pair satisfying (3.26), with the same* α *as in Assumption 3.29 and* $\tilde q\leqslant q$ *(*$\tilde q\in[2,\infty]$ *if* $H>1/2$*).*

Then there exists $\gamma>1/2$ *such that, for any* $p\in[1,\infty)$*, there exists a constant* $C>0$ *(depending on* $\gamma,p,M,T,d,\tilde q$ *and the parameters appearing in Assumption 3.29) such that*

$$\mathbb{E}[\|X^1-X^2\|^p_{\gamma;[0,\tau]}]^{\frac{1}{p}}\leqslant C\,(|x^1_0-x^2_0|+\|b^1-b^2\|_{L^{\tilde q}(0,\tau;B^{\alpha-1}_\infty)})\quad \forall\tau\in[0,T].\tag{3.27}$$

The concepts of strong existence, pathwise uniqueness, path-by-path uniqueness and uniqueness in law will be introduced in the upcoming Section 3.2.1, and are especially needed in the regime $H<1/2$ where $\alpha<0$ is allowed (namely, b can be a true distribution and not a function, think of $b=\delta_0$) so that the interpretation of the SDE is a priori unclear.

However, let us immediately spend a few words on the stability estimate (3.27), which was first obtained in [146]. A first important consequence is that, given any sequence of smooth drifts b^n such that $\|b^n-b\|_{L^{\tilde q}B^{\alpha-1}_\infty}$, the associated solutions X^n will converge to X; this allows to compare our notion of solution to others that might be given, especially in the case of a truly distributional drift b, where there is no canonical definition. In particular, if we were given a completely different concept of solution, which still shared the fundamental property of being the unique limit of smooth approximations (a basic requirement that any reasonable notion of solution should satisfy), then it will automatically coincide with ours.

Additionally, observe that in equation (3.27), the difference of the two drifts is measured in the weaker norm $L_t^{\tilde{q}} B_\infty^{\alpha-1}$, while in order to achieve wellposedness the drifts b^i are required to belong to at least $L_t^{\tilde{q}} B_\infty^{\alpha}$ (or even more when $H > 1/2$). This idea the "stability should come at one regularity less than uniqueness" is ubiquitous in analysis, think of the basic results from ODE and SDE theory (where tipycally $b \in W^{1,\infty}$ is needed for uniquess but stability comes in C^0) but also the more advanced versions from DiPerna–Lions theory like in [81] (where uniqueness requires $b \in W^{1,p}$ but stability comes in L^p). Finally, even when $H > 1/2$, since we are always considering non-Lipschitz drifts (otherwise the wellposedness result is trivial), it holds $\alpha - 1 < 0$; in other terms, even if b^i in this case must be continuous, convergence only needs to hold in a negative Besov norm.

The rest of the section is devoted to proving Theorem 3.30, which requires some preparations.

3.2.1 Solution concepts

We start by introducing the exact notions of solutions we will work with, so to make the statement of Theorem 3.30 fully rigorous, and explain the relations between them. We begin with a simple lemma, establishing that the interpretation of the perturbed ODE as a nonlinear Young equation is in fact a natural generalization of the classical setting.

Lemma 3.31. *Let $b: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $w: [0, T] \to \mathbb{R}^d$ be continuous functions and assume that $T^w b \in C_t^\gamma C_{\mathrm{loc}}^\beta$ for parameters $\gamma \in (1/2, 1]$, $\beta \in (0, 1]$ satisfying $\gamma(1 + \beta) > 1$. Then x is a solution to the ODE*[3.3]

$$x_t = x_0 + \int_0^t b_s(x_s) \mathrm{d}s + w_t \quad \forall t \in [0, T]$$

if and only if $x = \theta + w$, where $\theta \in C_t^\gamma$ solves the nonlinear YDE

$$\theta_t = \theta_0 + \int_0^t T^w b(\mathrm{d}s, \theta_s) \quad \forall t \in [0, T].$$

Proof. The proof is a simple application of Theorem 1.6. Assume first that x is a solution to the ODE, then θ as defined above solves $\theta_t = \theta_0 + \int_0^t b_s(\theta_s + w_s) \mathrm{d}s$. By definition of $T^w b$ and continuity of b, $\partial_t T_t^w b(z) = b_t(z + w_t)$, so that $\theta_t = \theta_0 + \int_0^t \partial_t T_s^w b(\theta_s) \mathrm{d}s = \theta_0 + \int_0^t T^w b(\mathrm{d}s, \theta_s)$. It is also clear that $\theta \in C_t^\gamma$ (in fact θ is even Lipschitz continuous).

Conversely, assume that $\theta \in C_t^\gamma$ solves the YDE , then $\int_0^t T^w b(\mathrm{d}s, \theta_s)$ corresponds to the limit of the Riemann-type sums

$$\int_0^t T^w b(\mathrm{d}s, \theta_s) = \lim_{|\pi| \to 0} \sum_i T_{t_i, t_{i+1}}^w b(\theta_{t_i}) = \lim_{|\pi| \to 0} \sum_i \int_{t_i}^{t_{i+1}} b_s(\theta_{t_i} + w_s) \mathrm{d}s.$$

Using the fact that the b and θ are uniformly continuous on compact sets, it is easy to check that the last expression must coincide with $\int_0^t b_s(\theta_s + w_s) \mathrm{d}s = \int_0^t b_s(x_s) \mathrm{d}s$, which readily implies that x solves the ODE. □

Lemma 3.31 motivates the following definition, which is taken from [139]. It covers a general setting, which does not require W to be sampled as an fBm and also allows for the presence of random initial data.

Definition 3.32. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, (ξ, W) an $\mathbb{R}^d \times C_t^0$-valued random variable defined on it and let b be a distributional field. We say that another C_t^0-valued random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a **pathwise solution** to the SDE*

$$X_t = \xi + \int_0^t b_s(X_s) \mathrm{d}s + W_t \quad \forall t \in [0, T] \tag{3.28}$$

3.3. In the following we are always implicitly assuming $w_0 = 0$ (which comes wlog since the term w_0 can be reabsorbed in x_0 if needed). Otherwise, in the ODE (3.28) the term w_t should be replaced by $w_t - w_0$, since it is formally obtained by integrating $\mathrm{d}w_s$ over $[0, t]$. A similar observation applies the process W and the SDE (3.28) in Definition 3.32, as well as this thesis in general.

associated to (b,ξ,W) *if there exists* $\Omega'\subset\Omega$ *with* $\mathbb{P}(\Omega')=1$ *and deterministic* γ,β,λ *satisfying*

$$\gamma\in\left(\frac{1}{2},1\right),\quad \beta,\lambda\in[0,1],\quad \gamma(1+\beta)>1,\quad \beta+\lambda\leqslant 1 \tag{3.29}$$

such that for all $\omega\in\Omega'$ *the following hold:*

1. $T^{W(\omega)}b\in C^\gamma_t C^{\beta,\lambda}_x$;
2. $\theta(\omega):=X(\omega)-W(\omega)\in C^\gamma_t$;
3. $\theta(\omega)$ *satisfies the pathwise meaningful nonlinear Young equation*

$$\theta_t(\omega)=\xi(\omega)+\int_0^t T^{W(\omega)}b(\mathrm{d}s,\theta_s(\omega))\quad\forall t\in[0,T]. \tag{3.30}$$

Remark 3.33. Let us comment on Definition 3.32, which is nonclassical and inspired by the work [114] (which in turn builds on [4]). Under the above assumptions, $\omega\mapsto T^{W(\omega)}b$ is a well-defined $C^\gamma_t C^{\beta,\lambda}_x$-valued random variable (adapted to W); we can therefore invoke Corollary 1.46 to define the random compact set in C^γ_t given by $\omega\mapsto C(\xi(\omega),W(\omega)):=\tilde{C}(\xi(\omega),T^{W(\omega)}b)$, where the latter denotes the set of solutions to the YDE associated to $(\xi(\omega),T^{W(\omega)}b)$. Condition 3. in Definition 3.32 can then be rephrased as

$$\theta(\omega)\in C(\xi(\omega),W(\omega))\quad\text{for }\mathbb{P}\text{-a.e. }\omega.$$

Alternatively, we can disintegrate the measure $\mathcal{L}_{\mathbb{P}}(X,\xi,W)$ w.r.t. $\nu=\mathcal{L}(\xi,W)$ (see e.g. Theorem 5.3.1 from [5][3.4]), i.e. write it has

$$\mathcal{L}_{\mathbb{P}}(X,\xi,W)(\mathrm{d}x,\mathrm{d}y)=k_y(\mathrm{d}x)\,\nu(\mathrm{d}y)$$

where $\{k_y\}_{y\in\mathbb{R}^d\times C^0_t}$ is a probability kernel. In this sense, the definition of pathwise solution can be purely rephrased at the level of the laws by imposing the following equivalent of Condition 3.:

$$\operatorname{supp}(k_y)\subset C(y)\quad\text{for }\nu\text{-a.e. }y;$$

here we used with a slight abuse $C(y)=C(y_1,y_2)$ whenever $y=(y_1,y_2)\in\mathbb{R}^d\times C^0_t$.

In a sense, X is a *random solution to a random YDE*, rather than a solution to an SDE; in other terms, differently from classical SDEs driven by Brownian motion, all integrals appearing are pathwise analytically defined, which is why we chose the terminology of pathwise solution.

Finally, let us mentioned that, depending on the context, other notions of solutions for SDEs with distributional drifts are possible, see e.g. [15] and [123].

Definition 3.34. *Let* b *be a distributional field,* $\nu\in\mathcal{P}(\mathbb{R}^d\times C^0_t)$. *A tuple* $(\Omega,\mathcal{F},\mathbb{P};X,\xi,W)$ *given by a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *and a* $C^0_t\times\mathbb{R}^d\times C^0_t$*-valued random variable is a* ***weak solution*** *to the SDE (3.28), associated to the imput data* (b,ν), *if* $\mathcal{L}_{\mathbb{P}}(\xi,W)=\nu$ *and* X *is a pathwise solution associated to* (b,ξ,W) *in the sense of Definition 3.32. We say that* X *is a* ***strong solution*** *if it is adapted to the filtration* $\mathcal{F}_t=\sigma\{\xi,W_s\,|\,s\leqslant t\}$.

Definition 3.35. *Consider the SDE (3.28) associated to imput data* (b,ν) *as in Definition 3.34. Then we say that:*

i. ***uniqueness in law*** *holds if any pair of weak solutions* $(\Omega^i,\mathcal{F}^i,\mathbb{P}^i;X^i,\xi^i,W^i)$, $i=1,2$, *satisfies* $\mathcal{L}_{\mathbb{P}^1}(X^1)=\mathcal{L}_{\mathbb{P}^2}(X^2)$.

ii. ***pathwise uniqueness*** *holds if any two given solutions* (X^i,ξ,W) *defined on the same probability space, w.r.t. the same* (b,ξ,W), *satisfies* $X^1=X^2$ $\mathbb{P}$*-a.s.*

iii. ***strong existence*** *holds if there exists a strong solution* $(\Omega,\mathcal{F},\mathbb{P};X,\xi,W)$*; similarly for* ***weak existence***.

3.4. Technically speaking, in order to apply the disintegration theorem, we need to be on a Polish space, so Hölder spaces are ill-suited. The issue can be resolved easily like in Section 1.4.2: for any $\varepsilon>0$, C^α_t embeds into $\mathcal{C}^{\alpha-\varepsilon}_t$, the latter being the separable space obtained as the closure of C^∞_t w.r.t. the $C^{\alpha-\varepsilon}_t$-norm; similarly, $C^\gamma_t C^{\beta,\lambda}_x$ embeds into $\mathcal{C}^{\gamma-\varepsilon}_t\mathcal{C}^{\beta-\varepsilon,\lambda}_x$ (as partially already shown by Lemma 3.9). We can choose ε small enouch so that $(\gamma-\varepsilon)(1+\beta-\varepsilon)>1$, so that the YDE is still perfectly meaningful.

As standard in the probabilistic literature, if strong existence holds, then $(\Omega,\mathcal{F},\mathbb{P})$ can be chosen to be the canonical space, namely with $\Omega=\mathbb{R}^d\times C_t^0$, $\mathbb{P}=\nu$ and $\mathcal{F}$ the completion of $\mathcal{B}(\mathbb{R}^d\times C_t^0)$ under ν. Consequently, one can construct a strong solution on any other probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ where a random variable $(\tilde{\xi},\tilde{W})$ with law ν is defined.

Although we have given Definitions 3.32-3.35 in full generality, from now we will focus for simplicity only one the case of deterministic initial data $\xi=x_0\in\mathbb{R}^d$ (the other standard alternative is to take ξ independent of W, which can be reduced to the deterministic setting by first conditioning on ξ). We will need another (very strong!) notion of uniqueness, declined in two variants; the first one comes from [86], for the terminology involving the second one we follow [139] (although the notion already appeared before in the literature, at least outside of the nonlinear Young context).

Definition 3.36. *Let b be a distributional field, W a C_t^0-random variable defined on $(\Omega,\mathcal{F},\mathbb{P})$ such that $\mathbb{P}$-a.s. $T^W b\in C_t^\gamma C_x^{\beta,\lambda}$ for some (γ,β,λ) satisfying (3.29); we can therefore define the random set $C(x_0,W)$ as in Remark 3.33. We say that:*

i. ***path-by-path uniqueness*** *holds for the SDE (3.28) if*

$$\mathbb{P}(\omega\in\Omega\colon C(x_0,W(\omega)) \text{ is a singleton})=1\quad\forall x_0\in\mathbb{R}^d;$$

ii. ***path-by-path wellposedness*** *holds for the SDE (3.28) if*

$$\mathbb{P}(\omega\in\Omega\colon C(x_0,W(\omega)) \text{ is a singleton }\ \forall x_0\in\mathbb{R}^d)=1.$$

Remark 3.37. It is clear that path-by-path wellposedness implies path-by-path uniqueness; the opposite needs not to be true, since we are not allowed to exchange expectation and the uncountable quantifier $\forall x_0\in\mathbb{R}^d$.

Observe that, since $C(x_0,W(\omega))$ only depends on $W(\omega)$, both conditions depend exclusively on the law of W and not the specific probability space in consideration. The condition "$C(x_0,W(\omega))$ is a singleton" encodes the idea that, for the fixed realization $W(\omega)$, existence and uniqueness of solutions to the YDE holds (recall that under our assumption, by Theorem 1.41, $C(x_0,W(\omega))$ is always nonempty, so uniqueness is truly the important part here).

It follows that path-by-path implies pathwise uniqueness: if $C(x_0,W(\omega))=\{F(W(\omega))\}$ for $\mathbb{P}$-a.e. ω, where $F(W(\omega))\in C_t^\gamma$, and X^i are two solutions associated to (x_0,W,b), then $\mathbb{P}$-a.s. it must hold $X^1(\omega)=F(W(\omega))=X^2(\omega)$ Moreover, path-by-path implies uniqueness in law[3.5]: by the disintegration argument from Remark 3.33, we have $\mathcal{L}_{\mathbb{P}}(X,x_0,W)(\mathrm{d}x,\mathrm{d}y)=k_y(\mathrm{d}x)\,\nu(\mathrm{d}y)$ where now for ν-a.e. y it holds $\operatorname{supp}k_y\subset C(y)=\{F(y)\}$, which implies $k_y=\delta_{F(y)}$ for ν-a.e. As a consequence, $\mathcal{L}_{\mathbb{P}}(X,x_0,W)$ (and thus $\mathcal{L}_{\mathbb{P}}(X)$) is entirely determined by ν.

Remark 3.38. As we will shortly, in our setting path-by-path uniqueness will in fact imply strong existence (and thus effectively give us everything we can desire from the SDE). We have however cheated a little bit: in line of principle, the full condition $T^W b\in C_t^\gamma C_x^{\beta,\lambda}$ with parameters satisfying (3.29) is not needed to give pathwise meaning to the equation, as it would suffice to ask for $T^W b\in C_t^\gamma C_{\mathrm{loc}}^\beta$ and $\theta\in C_t^\gamma$ with $\gamma(1+\beta)>1$, or even more generally $T^W b\in C_t^\gamma C_{\mathrm{loc}}^\beta$, $\theta\in C_t^\delta$ with $\gamma+\beta\delta>1$. A good reason for asking the stronger $T^W b\in C_t^\gamma C_x^{\beta,\lambda}$ (besides it being always satisfied in our cases of interest) is that otherwise we cannot define associated the random compact set $C(x_0,W)$ (solutions might blow-up in finite time, we cannot guarantee their existence up to T) and thus compromise part of the arguments presented so far. Several facts can still be recovered, up to the price of considering the lifetime of the solution as a stopping time τ (so that the pathwise relation (3.30) is satisfied for $t\in[0,\tau)$ and potentially $\mathbb{P}(\tau<T)>0$); however this requires much more technical work and makes it difficult to develop arguments at the level of the laws of solutions.

Proposition 3.39. *Let b be a distributional field, W a C_t^0-random variable defined on $(\Omega,\mathcal{F},\mathbb{P})$ such that $\mathbb{P}$-a.s. $T^W b\in C_t^\gamma C_x^{\beta,\lambda}$ for some (γ,β,λ) satisfying (3.29) and assume that path-by-path uniqueness holds, in the sense of Definition 3.36. Then for any $x_0\in\mathbb{R}^d$ there exists a (unique) strong solution for the associated SDE, in the sense of Definition 3.35.*

3.5. In fact, one could readapt Yamada-Watanabe's argument (Proposition 1 from [276]), based on disintegration of measures, to show that pathwise uniqueness implies uniqueness in law also in the nonlinear Young framework, but for the sake of simplicity we will not do it here.

Proof. Fix any $\omega\in\Omega$ s.t. $T^{W(\omega)}b\in C_t^\gamma C_x^{\beta,\lambda}$ and $C(x_0,W(\omega))$ is a singleton, $C(x_0,W(\omega))=\{\theta(\omega)\}$. Since b is distributional, setting $b^\varepsilon=\rho^\varepsilon * b$, it holds $b^\varepsilon\in L_t^2C_x^1$ and therefore there exist a unique solution $X^\varepsilon(\omega)$ to

$$X_t^\varepsilon(\omega)=x_0+\int_0^t b^\varepsilon(s,X_s^\varepsilon)\mathrm{d}s+W_t(\omega)\quad \forall t\in[0,T].$$

Moreover, by Lemma 3.31, $X_t^\varepsilon(\omega)=\theta_t^\varepsilon(\omega)+W_t(\omega)$, where $\theta_t^\varepsilon(\omega)$ solves the YDE associated to $(x_0,T^{W(\omega)}b^\varepsilon)$. By Lemma 3.9, $T^{W(\omega)}b^\varepsilon\to T^{W(\omega)}b$ in $C_t^\gamma C_x^{\beta-\delta,\lambda}$ as $\varepsilon\to 0$; in particular, since $\{T^{W(\omega)}b^\varepsilon\}_{\varepsilon>0}$ is also bounded therein, by Proposition 1.18, we have a uniform bound on $[\![\theta^\varepsilon(\omega)]\!]_{C^\gamma}$. We can therefore extract a (not relabelled) subsequence such that $\theta^\varepsilon(\omega)\to\tilde\theta(\omega)$ in $C_t^{\gamma-\delta}$ for some $\tilde\theta(\omega)\in C_t^\gamma$; combined with $T^{W(\omega)}b^\varepsilon\to T^{W(\omega)}b$, up to choosing δ small enough, we conclude that $\tilde\theta(\omega)\in C(x_0,W(\omega))$. But since $C(x_0,W(\omega))=\{\theta(\omega)\}$, it must hold $\tilde\theta(\omega)=\theta(\omega)$; as the reasoning holds for any possible subsequence we can extract, we conclude that the whole sequence $\{\theta^\varepsilon(\omega)\}_{\varepsilon>0}$ converges to $\theta(\omega)$ and so that $X^\varepsilon(\omega)\to X(\omega):=\theta(\omega)+W(\omega)$ in C_t^0.

As the argument works on a set of ω of full probability, we conclude that $X^\varepsilon\to X$ $\mathbb{P}$-a.s.; on the other hand, by classical SDE results, the solutions X^ε are adapted to the filtration $\mathcal{F}_t^W$ generated by W, and so must be their $\mathbb{P}$-a.s. limit X. In other terms, X is a strong solution to the SDE. □

Without the assumption of path-by-path uniqueness, we have a much weaker (but still of interest) result, concerning existence of pathwise solutions.

Lemma 3.40. *Let b be a distributional field, W a C_t^0-random variable defined on $(\Omega,\mathcal{F},\mathbb{P})$ such that $\mathbb{P}$-a.s. $T^Wb\in C_t^\gamma C_x^{\beta,\lambda}$ for some (γ,β,λ) satisfying (3.29). Then for any $x_0\in\mathbb{R}^d$, weak existence holds for the associated SDE, in the sense of Definition 3.35.*

Proof. The proof is actually an immediate consequence of Corollary 1.48. In fact, given the map F as defined therein, it is enough to define $X(\omega):=F(x_0,T^{W(\omega)}b)+W(\omega)$; it is a well-defined random variable since it is given by the composition of measurable maps and a pathwise solution to the SDE by construction. □

Remark 3.41. As the solution X appearing in the proof of Lemma 3.40 can be expressed as $X(\omega)=G(W(\omega))$ for some measurable map G, one might be tempted to conclude that it is a strong solution. This needs not to be the case. Indeed, setting as before $\mathcal{F}_t^W=\sigma(W_r:r\leqslant t)$, it is true that X as a path is $\mathcal{F}_T^W$-adapted, but this is not enough to guarantee that X_t is $\mathcal{F}_t^W$-adapted for all $t\in[0,T]$.

3.2.2 Girsanov transform and path-by-path uniqueness

Thanks to Remark 3.37 and Proposition 3.39, in order to establish strong existence, path-by-path uniqueness, pathwise uniqueness and uniqueness in law for the SDE (thus completing the proof of the first part of Theorem 3.30), we are reduced to the verification of two facts:

1. under Assumption 3.29, $\mathbb{P}$-a.s. $T^Wb\in C_t^\gamma C_x^{\beta,\lambda}$ for some parameters satisfying (3.29);
2. path-by-path uniqueness holds for the SDE.

Point 1. will easily follows from the results from Sections 3.1.3-3.1.4 (in the case $H>1/2$, under Assumption 3.29, b is actually continuous and bounded, so we don't even need Point 1.).

Thus we are left with verifying path-by-path uniqueness; this is accomplished, as in [86] and [57], by the use of Girsanov's theorem (which in the setting of fBm is recalled in Appendix A.1, see Theorem A.1). The importance of Girsanov transform comes from the fact that, whenever Novikov's condition can be checked successfully, not only it immediately yields existence and uniqueness in law, but it also provides a huge deal of information on the behaviour of the solution X. In particular, it provides another measure $\mathbb{Q}$, equivalent to $\mathbb{P}$[3.6], under which X is distributed as an fBm; this implies that X *has all the same pathwise properties as* W! We will see shortly how we can exploit this information crucially.

3.6. Actually, the upcoming Proposition 3.44 does not even need Novikov to be verified, and in fact we will exhibit a measure $\mathbb{Q}\gg\mathbb{P}$ such that X under $\mathbb{Q}$ is distributed as a fBm (the notation is a bit misleading since in the proof of Lemma 3.44 the roles of $\mathbb{P}$ and $\mathbb{Q}$ will be inverted). However, in our cases of interest (i.e. b satisfying Assumption 3.29) Novikov condition will also always be satisfied, see the proof of the upcoming Proposition 3.49.

Before proceeding further, let us present two short lemmas (still formulated at the pathwise level) which will be needed in the following.

Lemma 3.42. *Let b be a distributional fields and let w,θ be continuous paths such that $T^w b$, $T^{w+\theta} b \in C_t^\gamma C_{\mathrm{loc}}^\beta$, where $\gamma(1+\beta)>1$ and $\theta \in C_t^\gamma$. Then for any $\tilde\theta \in C_t^\gamma$ it holds*

$$\int_0^t T^{w+\theta} b(\mathrm{d}s, \tilde\theta_s) = \int_0^t T^w b(\mathrm{d}s, \theta_s + \tilde\theta_s) \quad \forall t \in [0,T].$$

Proof. Although the result follows easily from an application of Lemma 1.53, let us give a self-contained proof. Since $\theta, \tilde\theta$ are continuous, we can assume wlog that $T^w b, T^{w+\theta} b \in C_t^\gamma C_x^\beta$.

Assume first that b is continuous, then again by Theorem 1.6 it holds

$$\int_0^t T^{w+\theta} b(\mathrm{d}s, \tilde\theta_s) = \int_0^t b_s(w_s + \theta_s + \tilde\theta_s)\mathrm{d}s = \int_0^t T^w b(\mathrm{d}s, \theta_s + \tilde\theta_s).$$

For general b, we can consider the mollifications $b^\varepsilon = \rho^\varepsilon * b$; by Lemma 3.9, $T^w b^\varepsilon \to T^w b$ and $T^{w+\theta} b^\varepsilon \to T^{w+\theta} b$ in $C_t^\gamma C_x^{\beta-\delta}$ for any $\delta>0$; moreover by the previous step, identity (3.42) is true for b replaced by b^ε. Choosing δ small enough so that $\gamma(1+\beta-\delta)>1$ and taking the limit as $\varepsilon\to 0$ we can conclude that (3.42) holds due to continuity of Young integrals (Point 4. of Theorem 1.6). $\square$

Lemma 3.43. *Let b be a distributional field, w a continuous path such that $T^w b \in C_t^\gamma C_{\mathrm{loc}}^1$ for some $\gamma > 1/2$; assume there exists a solution $\theta \in C_t^\gamma$ to the nonlinear YDE*

$$\theta_t = \theta_0 + \int_0^t T^w b(\mathrm{d}s, \theta_s) \quad \forall t \in [0,T] \tag{3.31}$$

such that $T^{w+\theta} b \in C_t^\gamma C_{\mathrm{loc}}^1$ as well. Then θ is the unique element of C_t^γ solving (3.31).

Proof. The proof follows from the general abstract framework of Theorem 1.54, but again we give a self-contained argument. Assume we are given another solution $\tilde\theta \in C_t^\gamma$ to (3.31), defined on $[0,T]$; since $\{\theta_t: t\in[0,T]\} \cup \{\tilde\theta_t: t \in [0,T]\}$ is compact in $\mathbb{R}^d$, we can localize everything and assume wlog that $T^w b, T^{w+\theta} b \in C_t^\gamma C_x^1$. Setting $e := \tilde\theta - \theta \in C_t^\gamma$, by Lemma 3.42 it holds

$$\begin{aligned} e_t &= \int_0^t T^w b(\mathrm{d}s, \theta_s + e_s) - \int_0^t T^w b(\mathrm{d}s, \theta_s) \\ &= \int_0^t T^{w+\theta} b(\mathrm{d}s, e_s) - \int_0^t T^{w+\theta} b(\mathrm{d}s, 0) = \int_0^t A(\mathrm{d}s, \theta_s) \end{aligned}$$

for the choice $A_{s,t}(x) := T^{w+\theta}_{s,t} b(x) - T^{w+\theta}_{s,t} b(0)$. Observe that under our assumptions $A \in C_t^\gamma C_x^1$ and $A(t,0)=0$ for all $t \in [0,T]$, therefore by Theorem 1.20 we deduce that $e \equiv 0$, namely $\theta \equiv \tilde\theta$. $\square$

We are now ready to explain how Girsanov transform enters our framework with a general abstract result.

Proposition 3.44. *Let W be an $\mathcal{F}_t$-fBm of parameter H on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and let b be a distributional drift. Suppose that:*

1. *There exist parameters γ,β,λ satisfying (3.29) s.t. $\mathbb{P}$-a.s. $T^W b \in C_t^\gamma C_x^{\beta,\lambda} \cap C_t^\gamma C_{\mathrm{loc}}^1$;*
2. *for any $x_0 \in \mathbb{R}^d$, Girsanov's theorem is applicable to the process $W-h$, where*

$$h_t := \int_0^t b_s(x_0 + W_s)\mathrm{d}s = T_t^W b(x_0).$$

Then path-by-path uniqueness holds for the SDE (3.28), in the sense of Definition 3.36.

Proof. As already mentioned in Remark 3.37, the notion of path-by-path uniqueness does not depend on the specific $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ in consideration, so we can take it to be the canonical space $\Omega = C_t^0$ endowed with the fBm law $\mathbb{P} = \mu^H$. Let us define the set

$$A = \{\omega \in \Omega : T^{W(\omega)} b \in C_t^\gamma C_x^{\beta,\lambda} \cap C_t^\gamma C_{\mathrm{loc}}^1\}.$$

By the usual decomposition $X = \theta + W$ for pathwise solutions, we can rephase the content of Lemma 3.43 in the following way: it holds

$$A \cap \{\omega \in \Omega \colon C(x_0, W(\omega)) \text{ is a singleton}\} \supseteq \Gamma_{x_0}$$

for the choice

$$\Gamma_{x_0} \colon = A \cap \{\omega \in \Omega \colon \exists \theta \in C(x_0, W(\omega)) \text{ such that } T^{W(\omega)+\theta} b \in C_t^\gamma C_{\mathrm{loc}}^1\}.$$

In light of the above, in order to conclude, it suffices to show that $\mu^H(\Gamma_{x_0}) = 1$ for all $x_0 \in \mathbb{R}^d$.

By hypothesis, $\mu^H(A) = 1$, so we only need to focus on the other set defining Γ_{x_0}. By definition of h, the process $X = x_0 + W$ satisfies

$$X_t = x_0 + \int_0^t T^W b(\mathrm{d}s, x_0) + [W_t - h_t] =: x_0 + \int_0^t T^X b(\mathrm{d}s, 0) + \tilde{W}_t; \tag{3.32}$$

by hypothesis, Girsanov's theorem is applicable, so we can construct a new probability measure $\mathbb{Q}$, absolutely continuous w.r.t. to μ^H, such that $\tilde{W}$ is an $\mathcal{F}_t$-fBm under $\mathbb{Q}$. Observe that by the assumptions, since $T^W b \in C_t^\gamma C_x^{\beta,\lambda}$, the process $h = T^W b(x_0) \in C_t^\gamma$ $\mathbb{P}$-a.s. and thus by absolute continuity also $\mathbb{Q}$-a.s.; similarly, $T^X b = \tau^{x_0} T^W b \in C_t^\gamma C_{\mathrm{loc}}^1$ $\mathbb{P}$-a.s. and $\mathbb{Q}$-a.s. Applying Lemma 3.42, we can then deduce from (3.32) that $\mathbb{Q}$-a.s. $X(\omega) = h(\omega) + \tilde{W}(\omega)$ with $h(\omega) \in C_t^\gamma$ and

$$X_t(\omega) = x_0 + \int_0^t T^{\tilde{W}(\omega)} b(\mathrm{d}s, h_s(\omega)) + \tilde{W}_t(\omega);$$

namely, $(\Omega, \mathcal{F}, \mathbb{Q}; X, x_0, \tilde{W})$ is a weak solution to the SDE associated to b, in the sense of Definition 3.34. Since $\tilde{W}$ has law μ^H under $\mathbb{Q}$, we can finally obtain

$$\begin{aligned}
\mu^H(\Gamma_{x_0}) &= \mathbb{Q}\big(A \cap \big\{\omega \in \Omega \colon \exists \theta \in C(x_0, \tilde{W}(\omega)) \text{ such that } T^{\tilde{W}(\omega)+\theta} b \in C_t^\gamma C_{\mathrm{loc}}^1\big\}\big)\\
&\geqslant \mathbb{Q}\big(A \cap \big\{\omega \in \Omega \colon h(\omega) \in C(x_0, \tilde{W}(\omega)) \text{ and } T^{\tilde{W}(\omega)+h(\omega)} b \in C_t^\gamma C_{\mathrm{loc}}^1\big\}\big)\\
&\geqslant \mathbb{Q}\big(\omega \in \Omega \colon h(\omega) \in C(x_0, \tilde{W}(\omega)) \text{ and } T^{\tilde{W}(\omega)+h(\omega)} b \in C_t^\gamma C_{\mathrm{loc}}^1\big) = 1,
\end{aligned}$$

which gives the conclusion. □

With Proposition 3.44 at hand, we are only left with establishing sufficient conditions in order for Girsanov's theorem (Theorem A.1 from Appendix A.1) to hold, which is the main goal of the rest of this section. In particular, one needs to understand under which assumptions condition (A.10) holds, which requires a good control of the quantity $\|K_H^{-1} h\|_{L^2}$ in terms of h; we recall that the operator K_H^{-1} is defined by formula (A.7), in terms of fractional derivatives D^α as defined in (A.4). The following basic fact will be quite useful.

Lemma 3.45. *Let $f \in C_t^\beta$ such that $f_0 = 0$, then $D^\alpha f$ is well defined in C_t^0 for any $\alpha < \beta$; moreover $D^\alpha f \in C^\gamma$ for any $\gamma < \beta - \alpha$ and we have the estimate*

$$\|D^\alpha f\|_{C^\gamma} \lesssim_{\gamma,\alpha} \|f\|_{C^\beta}. \tag{3.33}$$

Proof. The statement follows from an application of Theorem 2.8 from [236]: on a finite interval $[0, T]$, the space $\mathbb{H}^{\beta,0}$ considered therein corresponds to the functions $f \in C_t^\beta$ such that $f_0 = 0$. □

Lemma 3.46. *Let $\alpha \in (0, 1/2)$ and $h \in C_t^\beta$ for some $\beta > \alpha$, $h_0 = 0$. Then $s^\alpha D^\alpha s^{-\alpha} h \in L_t^2$ and there exists a constant $C = C(\alpha, \beta, T)$ such that*

$$\|s^\alpha D^\alpha s^{-\alpha} h\|_{L^2} \leqslant C \|h\|_{C^\beta}. \tag{3.34}$$

In particular, for any $H \in (0, 1)$, if $h \in C_t^\beta$ for some $\beta > H + 1/2$, $h_0 = 0$, then $K_H^{-1} h \in L_t^2$ and there exists a constant $C = C(H, \beta, T)$ such that

$$\|K_H^{-1} h\|_{L^2} \leqslant C \|h\|_{C^\beta}. \tag{3.35}$$

Proof. We have

$$(s^\alpha D^\alpha s^{-\alpha} h)(t) = \Gamma(1-\alpha)^{-1}\left[h_t + \alpha t^\alpha \int_0^t \frac{t^{-\alpha} h_t - s^{-\alpha} h_s}{(t-s)^{\alpha+1}} \mathrm{d}s\right].$$

Since $h \in C_t^\beta$, it clearly also belongs to L_t^2, so we only need to control the term

$$\begin{aligned}
t^\alpha \left|\int_0^t \frac{t^{-\alpha} h_t - s^{-\alpha} h_s}{(t-s)^{\alpha+1}} \mathrm{d}s\right| &\leqslant t^\alpha \int_0^t \frac{t^{-\alpha}|h_t - h_s| + (s^{-\alpha} - t^{-\alpha})|h_s|}{(t-s)^{\alpha+1}} \mathrm{d}s\\
&\leqslant \|h\|_{C^\beta} t^\alpha \int_0^t \frac{t^{-\alpha}(t-s)^\beta + (s^{-\alpha} - t^{-\alpha})}{(t-s)^{\alpha+1}} \mathrm{d}s\\
&= \|h\|_{C^\beta} t^{-\alpha}\left[t^\beta \int_0^1 \frac{1}{(1-u)^{1+\alpha-\beta}} \mathrm{d}u + \int_0^1 u^{-\alpha} \frac{(1-u^\alpha)}{(1-u)^{1+\alpha}} \mathrm{d}u\right]\\
&\lesssim_T \|h\|_{C^\beta} t^{-\alpha}.
\end{aligned}$$

Since $\alpha \in (0, 1/2)$, $t^{-\alpha} \in L_t^2$ and so we deduce that the overall expression belongs to L_t^2, as well as estimate (3.34). Regarding the second statement, the case $H = 1/2$ is straightforward since $K_H^{-1} h = h'$. In the case $H > 1/2$, combining formula (A.7) for K_H^{-1} with estimates (3.33) and (3.34) for the choice $\alpha = H - 1/2$, choosing $\varepsilon > 0$ sufficiently small we have

$$\|K_H^{-1} h\|_{L^2} \lesssim \|h'\|_{C^{H-1/2+\varepsilon}} \lesssim \|h'\|_{C^\beta};$$

the case $H < 1/2$ is analogous. □

Remark 3.47. We have given an explicit proof of Lemma 3.46, but a similar (stronger) type of result can be achieved by a more abstract argument. Indeed, it follows from the proof of Theorem 5.4 from [236] that $\|s^\alpha D^\alpha(s^{-\alpha} h)\|_{L^2} \sim \|D^\alpha h\|_{L^2}$ and similarly $\|K_H^{-1} h\|_{L^2} \sim \|D^{H+1/2} h\|_{L^2}$; we have already seen that if $h \in C_t^\beta$ with $\beta > \alpha$ and $h_0 = 0$, then $D^\alpha h$ is a continuous function, so its L^2-norm is trivially finite. The inclusion $C_t^\beta \subset I^\alpha(L_t^2)$ is strict and therefore the hypothesis of Lemma 3.46 are non optimal, but they are rather useful when dealing with functions h which are not of bounded variation.

We can now state a general result on the applicability of Girsanov transform together with a good control on the density defining $\mathbb{Q}$.

Theorem 3.48. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space, W be an $\mathcal{F}_t$-fBm of parameter $H \in (0,1)$ and h be an $\mathcal{F}_t$-adapted process with trajectories in C_t^β, $\beta > H + 1/2$, s.t. $h_0 = 0$ and*

$$\mathbb{E}[\exp(\eta \|h\|_{C^\beta}^2)] < \infty \quad \forall \eta \in \mathbb{R}. \tag{3.36}$$

Then Girsanov transform for $\tilde{W} = h + W$ is applicable, i.e. $\tilde{W}$ is an $\mathcal{F}_t$-fBm of parameter H under the probability measure $\mathbb{Q}$ given by (A.9). Moreover the measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent and it holds

$$\mathbb{E}_{\mathbb{P}}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^n + \left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right)^n\right] < \infty \quad \forall n \in \mathbb{N}.$$

Proof. By hypothesis (3.36) and Lemma 3.46, it follows immediately that

$$\mathbb{E}[\exp(\eta \|K^{-1} h\|_{L^2}^2)] < \infty \quad \forall \eta \in \mathbb{R};$$

therefore Novikov's criterion is satisfied and Theorem A.1 is applicable. The proof of the second part of the statement follows from classical arguments, but we include it for the sake of completeness. Let us prove integrability of the moments: for any $\alpha \geqslant 1$, it holds

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^\alpha\right] &= \mathbb{E}_{\mathbb{P}}\left[\exp\left(\alpha \int_0^T (K_H^{-1} h) \cdot \mathrm{d}B - \alpha^2 \|K_H^{-1} h\|_{L^2}^2 + \left(\alpha^2 - \frac{\alpha}{2}\right) \|K_H^{-1} h\|_{L^2}^2\right)\right]\\
&\leqslant \mathbb{E}_{\mathbb{P}}\left[\exp\left(2\alpha \int_0^T (K_H^{-1} h) \cdot \mathrm{d}B - 2\alpha^2 \|K_H^{-1} h\|_{L^2}^2\right)\right]^{1/2} \mathbb{E}_{\mathbb{P}}[\exp((2\alpha^2 - \alpha)\|K_H^{-1} h\|_{L^2}^2)]^{1/2}\\
&= \mathbb{E}_{\mathbb{P}}[\exp((2\alpha^2 - \alpha)\|K_H^{-1} h\|_{L^2}^2)]^{1/2} < \infty,
\end{aligned}$$

where in the second line we used the fact that the integrand in the first term is again a probability density by Novikov's criterion, this time applied to the process $\tilde{h} = 2\alpha h$. Now in order to show that the measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, we need to show that the inverse density $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$ is integrable w.r.t. $\mathbb{Q}$. Again by Girsanov, since we have $W = \tilde{W} - h$, the inverse density is given by

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = \exp\left(\int_0^T (K_H^{-1} h)_s \cdot \mathrm{d}\tilde{B}_s - \frac{1}{2} \int_0^T |(K_H^{-1} h)_s|^2 \,\mathrm{d}s \right)$$

where $\tilde{B}$ now denotes the standard Bm associated to $\tilde{W}$, i.e. such that $\tilde{W}_t = \int_0^t K_H(t,s)\,\mathrm{d}\tilde{B}_s$. Since we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\frac{1}{2} \int_0^T |(K_H^{-1} h)_s|^2 \,\mathrm{d}s \right) \right] &= \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2} \int_0^T |(K_H^{-1} h)_s|^2 \,\mathrm{d}s \right) \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right] \\ &\leqslant \mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^T |(K_H^{-1} h)_s|^2 \,\mathrm{d}s \right) \right]^{1/2} \mathbb{E}_{\mathbb{P}}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right)^2 \right]^{1/2} < \infty\end{aligned}$$

we can conclude, again by applying Novikov, that $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$ is integrable w.r.t. $\mathbb{Q}$. Reasoning as before it can be shown that $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$ admits moments of any order w.r.t. $\mathbb{Q}$, which gives the conclusion. □

We are now finally ready to verify that Girsanov's theorem holds for the class of SDEs we are interested in.

Proposition 3.49. *Let $H \in (0,1)$ and b be a distributional drift satisfying Assumption 3.29. Then for any $x_0 \in \mathbb{R}^d$, Girsanov is applicable to the process $W - h$, for h as defined in Proposition 3.44. In particular, h satisfies condition (3.36) and all the conclusion of Theorem 3.48 hold.*

Proof. It suffices to check that the conditions of Theorem 3.48 are satisfied by $h_\cdot = \int_0^\cdot b_s(x_0 + W_s)\mathrm{d}s$; up to shifting b, we can assume without loss of generality $x_0 = 0$. We treat separately the cases $H > 1/2$ and $H < 1/2$ (the case $H = 1/2$ being classical).

Let $H > 1/2$; the process h belongs to $C_t^{H+1/2+\varepsilon}$ if and only if the map $t \mapsto b_t(W_t) \in C_t^{H-1/2+\varepsilon}$. Recall that for any $\varepsilon > 0$, $W \in C_t^{H-\varepsilon}$; then by Assumption 3.29 it holds

$$|b_t(W_t) - b_s(W_s)| \leqslant \|b\|_E (|t-s|^{\alpha H} + |W_t - W_s|^{\alpha}) \leqslant \|b\|_E (|t-s|^{\alpha H} + [\![W]\!]_{H-\varepsilon}^{\alpha} |t-s|^{\alpha(H-\varepsilon)}).$$

Since $\alpha > 1 - \frac{1}{2H}$, we can find $\varepsilon > 0$ small enough and a constant $C = C(T, \|b\|_E, \alpha, \varepsilon)$ such that

$$[\![b_\cdot(W_\cdot)]\!]_{C^{H+1/2+\varepsilon}} \leqslant C(1 + [\![W]\!]_{C^{H-\varepsilon}}^{\alpha}).$$

As the exponent $\alpha < 1$, by Fernique's Theorem [111] we deduce that

$$\mathbb{E}[\exp(\eta \|h\|_{C^{H+1/2-\varepsilon}}^2)] \lesssim \mathbb{E}[\exp(\eta C\, [\![W]\!]_{C^{H-\varepsilon}}^{2\alpha})] < \infty \quad \forall \eta \in \mathbb{R},$$

which by Theorem 3.48 implies the conclusion in this case.

The case $H < 1/2$ is actually even simpler, since we already have Proposition 3.26 at hand. Indeed, observe that here Assumption 3.29 on b is exactly equivalent to requiring

$$\gamma := 1 - \frac{1}{q} + \alpha H > H + \frac{1}{2};$$

therefore, up to choosing $\varepsilon > 0$ such that $\gamma - \varepsilon > H + 1/2$ as well, we can conclude that

$$\mathbb{E}[\exp(\eta\, [\![h]\!]_{C^{\gamma-\varepsilon}}^2)] < \infty \quad \forall \eta \in \mathbb{R}$$

which allows again to apply Theorem 3.48. □

We can summarize everything we have achieved so far in the next statement, which proves the first part of Theorem 3.30.

Theorem 3.50. *Let W be an fBm of parameter $H\in(0,1)$ and let b satisfy Assumption 3.29. Then for any $x_0\in\mathbb{R}^d$ strong existence, pathwise uniqueness, path-by-path uniqueness and uniqueness in law hold for the SDE (3.28). Moreover, given any sequence of smooth drifts b^n converging to b in E, the associated solutions $X^n\to X$ in C_t^0 $\mathbb{P}$-a.s.*

Proof. By Corollary 3.28, if the drift b satisfies Assumption 3.29, then $\mathbb{P}$-a.s. $T^W b\in C_t^\gamma C_x^{\beta,\lambda}\cap C_t^\gamma C^1_{\mathrm{loc}}$ for parameters satisfying (3.29); on the other hand, by Proposition 3.49, Girsanov is applicable, so by Proposition 3.44 path-by-path uniqueness holds. Pathwise uniqueness and uniqueness in law then follow from Remark 3.37, while strong existence from Proposition 3.39.

The last claim can be obtained arguing as in Proposition 3.39. Indeed, by the linearity of the operator $b\mapsto T^W b$ and the estimates coming from Corollaries 3.27-3.28, it is easy to check that if $\|b^n-b\|_E\to 0$, then $T^W b^n\to T^W b$ in $C_t^\gamma C_x^{\beta,\lambda}\cap C_t^\gamma C^1_{\mathrm{loc}}$. Then one can fix any $\omega\in\Omega$ s.t. $C(x_0,T^{W(\omega)}b)$ is a singleton (hence coinciding with $\{\theta(\omega)\}$, where $X=\theta+W$) and, arguing by compactness as usual starting from the Young a priori estimates (Proposition 1.18), deduce that the sequence $\{\theta^n(\omega)\}_{n\in\mathbb{N}}$ must converge in C_t^γ to $\theta(\omega)$, which implies $X^n(\omega)\to X(\omega)$ for every ω belonging to a set of full probability. □

Assumption 3.29 is fairly general and can be applied to several classes of drifts b, especially when we combine it with functional embeddings.

Corollary 3.51. *Let $H<1/2$ and $b\in L_t^q B^\alpha_{p,\infty}$ with $q\in[2,\infty]$, $p\in[1,\infty]$ and $\alpha\in\mathbb{R}$ such that*

$$\frac{1}{q}+H\left(\frac{d}{p}-\alpha\right)<\frac{1}{2}-H. \tag{3.37}$$

Then path-by-path uniqueness for the SDE (3.28), in the sense of Definition 3.36. Special cases of the above result include $b\in L_t^q L_x^p$ with

$$\frac{1}{q}+H\frac{d}{p}<\frac{1}{2}-H \tag{3.38}$$

and $b\in L_t^q\mathcal{M}_x$ (here $\mathcal{M}_x$ denotes the space of finite signed Radon measure) with

$$\frac{1}{q}+Hd<\frac{1}{2}-H. \tag{3.39}$$

Proof. Let $\tilde\alpha:=\alpha-d/p$, then by Besov embeddings $L_t^q B^\alpha_{p,\infty}\hookrightarrow L_t^q B^{\tilde\alpha}_\infty$ and condition (3.37) becomes

$$\frac{1}{q}-\tilde\alpha H<\frac{1}{2}-H\quad\Longleftrightarrow\quad\tilde\alpha-\frac{1}{Hq}>1-\frac{1}{2H};$$

therefore Assumption 3.29 is satisfied and Theorem 3.50 applies. The cases (3.38) and (3.39) are similar and based on the embeddings $L_x^p\hookrightarrow B^0_{p,\infty}$ and $\mathcal{M}_x\hookrightarrow B^0_{1,\infty}$. □

Let us conclude this subsection by shortly comparing Theorem 3.50 and Corollary 3.51 to other results existing in the literature.

In the regime $H>1/2$, Assumption 3.29 coincides with the one from [226], although therein pathwise uniqueness is shown only in the case $d=1$, while here we obtain path-by-path uniqueness in any dimension. In the case $H=1/2$, we can allow $b\in L_t^\infty C_x^\alpha$ for any $\alpha>0$; the result is comparable to the one from [86], where sharper estimates allow to reach $b\in L^\infty_{t,x}$, see also [254, 255] for further extensions. Finally, in the regime $H<1/2$ we can allow b to be truly distributional; in this case, we recover the results from [57] and further extend them to time-dependent drifts, which might be only integrable in time (in particular, our work [145] is to the best of our knowledge the first one to consider the case $b\in L_t^q B_p^\alpha$ with $\alpha<0$ and $q<\infty$). In the case $b\in L_t^q L_x^p$ with (q,p) satisfying (3.38), it was already shown in [194] that strnog existence and pathwise uniqueness holds[3.7]; here we strengthen the result to path-by-path uniqueness.

3.7. Actually, although not stated, in the work [194] the restriction to $q,p\in[2,\infty]$ must be imposed. Indeed the proof of Theorem 6.2 therein builds on the availability of Girsanov transform as checked in Theorem 6.1, whose proof is based on the fact that $|b|^2\in L_t^{q/2}L_x^{p/2}$ and Lemma 6.4, which requires $q/2,p/2\in[1,\infty]$. Instead in our result such a restriction is never needed (although $q>2$ arises naturally from (3.38)).

3.2.3 Stability estimates

We are now ready to complete the proof of our main result.

Proof. (of Theorem 3.30) It remains to prove the stability estimate (3.27); the argument is taken from the proof of Theorem 3.13 from [146]. First observe that, thanks to Theorem 3.50, we can assume without loss of generality b^1 and b^2 to be smooth functions, as the general case will then follow from a standard approximation procedure. For simplicity, we will only prove (3.27) in the case $\tau = T$, the general case being almost identical.

Recall that, by Proposition 3.49, to any solution X^i we can associate a measure $\mathbb{Q}^i$, equivalent to $\mathbb{P}$, such that X^i is distributed as $x_0^i + W$ under $\mathbb{Q}^i$; moreover, as the proof of Proposition 3.49 shows, all the estimates involving $\mathbb{E}_{\mathbb{P}}[(\mathrm{d}\mathbb{Q}^i / \mathrm{d}\mathbb{P})^n]$ with $n \in \mathbb{Z}$ only depend on $\|b^i\|_E$ and are therefore uniform over $\|b^i\|_E \leqslant M$ as in our assumptions. This also implies that both solutions admit a decomposition $X^i = x_0^i + \theta^i + W$, where there exists $\varepsilon > 0$ such that

$$\mathbb{E}[\exp(\eta \|\theta^i\|_{C^{H+1/2+\varepsilon}}^2)] \leqslant K(\eta, M) < \infty \quad \forall \eta \in \mathbb{R}. \tag{3.40}$$

Next, define $Y = X^1 - X^2$, $y_0 = x_0^1 - x_0^2$; for $\lambda \in [0,1]$, set $x_0^\lambda = \lambda x_0^1 + (1-\lambda) x_0^2$, $\theta^\lambda = \lambda \theta^1 + (1-\lambda)\theta^2$. Since b^i are smooth, by the mean value theorem it holds

$$\begin{aligned}
Y_t &= y_0 + \int_0^t [b_s^1(X_s^1) - b_s^1(X_s^2)]\mathrm{d}s + \int_0^t (b^1 - b^2)_s(X_s^2)\mathrm{d}s\\
&= y_0 + \int_0^t \left[\int_0^1 D b_s^1(\lambda X_s^1 + (1-\lambda) X_s^2)\mathrm{d}\lambda\right] \cdot Y_s\,\mathrm{d}s + \int_0^t (b^1 - b^2)_s(X_s^2)\mathrm{d}s\\
&= y_0 + \int_0^t \left[\int_0^1 D b_s^1(x_0^\lambda + \theta_s^\lambda + W_s)\mathrm{d}\lambda\right] \cdot Y_s\,\mathrm{d}s + \int_0^t (b^1 - b^2)_s(X_s^2)\mathrm{d}s.
\end{aligned}$$

Setting

$$A_t := \int_0^t \int_0^1 D b_s^1(x_0^\lambda + \theta_s^\lambda + W_s)\mathrm{d}\lambda\mathrm{d}s, \quad \psi_t := \int_0^t (b^1 - b^2)_s(X_s^2)\mathrm{d}s,$$

we see that the above can be written compactly as an affine equation for Y of the form

$$Y_t = y_0 + \int_0^t \dot{A}_s Y_s\,\mathrm{d}s + \psi_t = y_0 + \int_0^t A_{\mathrm{d}s} Y_s + \psi_t$$

where we changed notation in the last passage to stress that the integral can be interpreted in the Young sense, i.e. $\int_0^t A_{\mathrm{d}s} Y_s$ corresponds to the sewing of $\Gamma_{s,t} := A_{s,t} Y_s$.

We claim that there exists $\gamma > 1/2$ such that

$$\mathbb{E}[\exp(\eta\, [\![A]\!]_{C^\gamma}^2)] \leqslant K(\eta, M) \quad \forall \eta \in \mathbb{R}, \quad \mathbb{E}[\|\psi\|_{C^\gamma}^p]^{\frac{1}{p}} \lesssim \|b^1 - b^2\|_{L^{\tilde{q}} B_\infty^{\alpha-1}}. \tag{3.41}$$

Let us first show how we can obtain the conclusion once we have accomplished (3.41). Since $\gamma > 1/2$, intepreting the affine equation for Y as a Young differential equation, we can apply Theorem 1.20 to deduce the existence of some $C > 0$ (also depending on T) such that

$$\|Y\|_{C^\gamma} \lesssim_T e^{C [\![A]\!]_{C^\gamma}^{1/\gamma}} (|y_0| + \|\psi\|_{C^\gamma}) \lesssim_T e^{C [\![A]\!]_{C^\gamma}^2} (|y_0| + \|\psi\|_{C^\gamma}) \quad \mathbb{P}\text{-a.s.}$$

Next, taking the L_Ω^p norm on both sides and applying Minkowsky's inequality, we find

$$\begin{aligned}
\mathbb{E}[\|Y\|_{C^\gamma}^p]^{\frac{1}{p}} &\lesssim \mathbb{E}[e^{Cp [\![A]\!]_{C^\gamma}^2}]^{\frac{1}{p}} |y_0| + \mathbb{E}\big[e^{Cp [\![A]\!]_{C^\gamma}^2} \|\psi\|_{C^\gamma}^p\big]^{\frac{1}{p}}\\
&\lesssim \mathbb{E}[e^{Cp [\![A]\!]_{C^\gamma}^2}]^{\frac{1}{p}} |y_0| + \mathbb{E}[e^{C2p [\![A]\!]_{C^\gamma}^2}]^{\frac{1}{2p}}\, \mathbb{E}[\|\psi\|_{C^\gamma}^{2p}]^{\frac{1}{2p}}\\
&\lesssim |y_0| + \|b^1 - b^2\|_{L^{\tilde{q}} B_\infty^{\alpha-1}}
\end{aligned}$$

where we used Cauchy's inequality and (3.41) several times; given the definition of Y and y_0, the final estimate is exactly (3.27) for $\tau = T$.

It remains to prove the claim (3.41). The second inequality is realtively simple: since by asumption $(\tilde{q},\alpha)$ satisfy (3.26) and X^2 is distributed as an fBm under $\mathbb{Q}^2$, it holds

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\|\psi\|_{C^\gamma}^p]^{\frac{1}{p}} =& \mathbb{E}_{\mathbb{Q}^2}\left[\left\|\int_0^\cdot (b^1-b^2)_s(X_s^2)\mathrm{d}s\right\|_{C^\gamma}^p \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^2}\right]^{\frac{1}{p}}\\
\leqslant& \mathbb{E}_{\mathbb{Q}^2}\left[\left\|\int_0^\cdot (b^1-b^2)_s(X_s^2)\mathrm{d}s\right\|_{C^\gamma}^{2p}\right]^{\frac{1}{2p}} \mathbb{E}_{\mathbb{Q}^2}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^2}\right)^2\right]^{\frac{1}{2}}\\
=& \mathbb{E}_{\mathbb{P}}\left[\left\|\int_0^\cdot (b^1-b^2)_s(x_0^2+W_s)\mathrm{d}s\right\|_{C^\gamma}^{2p}\right]^{\frac{1}{2p}} \mathbb{E}_{\mathbb{Q}^2}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^2}\right)^2\right]^{\frac{1}{2}}\\
\lesssim_M& \|b^1-b^2\|_{L^{\tilde{q}}B_\infty^{\alpha-1}},
\end{aligned}$$

where in the last passage we used the properties of $\mathbb{Q}^2$ and Point 2. of Corollary 3.27.

We pass to proving the first inequality in claim (3.41). Observe that, for any $\lambda\in[0,1]$, by Jensen's inequality the process θ^λ satisfies

$$\mathbb{E}[\exp(\eta\,\|\theta^\lambda\|_{C^{H+1/2+\varepsilon}}^2)] \leqslant \lambda\mathbb{E}[\exp(\eta\,\|\theta^1\|_{C^{H+1/2+\varepsilon}}^2)] + (1-\lambda)\mathbb{E}[\exp(\eta\,\|\theta^2\|_{C^{H+1/2+\varepsilon}}^2)] \leqslant K(\eta,M).$$

Therefore by Proposition 3.49, for any $\lambda\in[0,1]$ there exists $\mathbb{Q}^\lambda$, equivalent to $\mathbb{P}$, such that $W+\theta^\lambda$ is distributed as W under $\mathbb{Q}^\lambda$; moreover we have controls on $\mathbb{E}[(\mathrm{d}\mathbb{Q}^\lambda/\mathrm{d}\mathbb{P})^n]$, for $n\in\mathbb{Z}$, which are independent of $\lambda\in[0,1]$. As a consequence, applying again Jensen's inequality (and Fubini's theorem), we have

$$\begin{aligned}
\mathbb{E}[\exp(\eta[\![A]\!]_{C^\gamma}^2)] =& \mathbb{E}\left[\exp\left(\eta\left[\!\!\left[\int_0^1\int_0^\cdot Db_s^1(x_0^\lambda+\theta_s^\lambda+W_s)\mathrm{d}s\,\mathrm{d}\lambda\right]\!\!\right]_{C^\gamma}^2\right)\right]\\
\leqslant& \int_0^1 \mathbb{E}\left[\exp\left(\eta\left[\!\!\left[\int_0^\cdot Db_s^1(x_0^\lambda+\theta_s^\lambda+W_s)\mathrm{d}s\right]\!\!\right]_{C^\gamma}^2\right)\right]\mathrm{d}\lambda =: \int_0^1 I_\lambda \mathrm{d}\lambda
\end{aligned}$$

and now it suffices to control the quantity I_λ uniformly in $\lambda\in[0,1]$. But by the previous considerations, we have

$$\begin{aligned}
I_\lambda =& \mathbb{E}_{\mathbb{Q}^\lambda}\left[\exp\left(\eta\left[\!\!\left[\int_0^\cdot Db_s^1(x_0^\lambda+\theta_s^\lambda+W_s)\mathrm{d}s\right]\!\!\right]_{C^\gamma}^2\right)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^\lambda}\right]\\
\leqslant& \mathbb{E}_{\mathbb{P}}\left[\exp\left(2\eta\left[\!\!\left[\int_0^\cdot Db_s^1(x_0^\lambda+W_s)\mathrm{d}s\right]\!\!\right]_{C^\gamma}^2\right)\right]^{\frac{1}{2}} \mathbb{E}_{\mathbb{Q}^\lambda}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^\lambda}\right)^2\right]^{\frac{1}{2}} \leqslant K(\eta,M)
\end{aligned}$$

where in the final passage we applied Proposition 3.26 (for the choice $s=\alpha-1$). This proves the claim (3.41) and concludes the proof. □

3.2.4 Existence of the flow and application to transport equations

Theorem 3.30 provides very sharp conditions for the strong wellposedness of the SDE, among the best known in the literature for $H\neq 1/2$ (see Section 3.4 for more details on other works and what are expected to be the optimal results). On the other hand, we have seen that its proof requires a sophisticated interplay of analytical (nonlinear Young integrals) and stochastic (Girsanov transform) tools.

Here we take a step back and apply more directly the nonlinear Young theory presented in Chapter 1, whenever $T^w b$ is sufficiently regular to allow so, thus keeping the analysis entirely at the pathwise level. While this comes at the cost of requiring more generous assumptions on the drift b (which is still non-Lipschitz and possibly distributional), it presents several advantages: it immediately yields an associated *random flow of diffeomorphisms* for the SDE (something which is typically quite difficult and technical to obtain working at the stochastic level) and allows to solve the associated transport and continuity equations as well. As we will see in the Section 3.3, this approach also works well in the context of *generic additive perturbations*.

We start again by considering again the ODE driven by a drift b perturbed by a continuous path w (which for the moment is not required to be a realization of fBm). In light of Lemma 3.31, we will say that x solves the Cauchy problem

$$x_t = x_0 + \int_0^t b_s(x_s)\mathrm{d}s + w_t \quad \forall t \in [0,T] \tag{3.42}$$

if $T^w b \in C_t^\gamma C_{\mathrm{loc}}^\beta$ for some $\gamma, \beta \in (0,1)$ with $\gamma(1+\beta) > 1$, $x = \theta + w$ for some $\theta \in C_t^\gamma$ and

$$\theta_t = x_0 + \int_0^t T^w b(\mathrm{d}s, \theta_s) \quad \forall t \in [0,T]$$

where the integral is meaningful in the nonlinear Young sense. The next statement collects some of the main results from Chapter 1 specialized to equation (3.42).

Theorem 3.52. *Suppose that either one between the following holds:*

a) there exist γ, β, λ satisfying (3.29) such that $T^w b \in C_t^\gamma C_x^{\beta,\lambda} \cap C_t^\gamma C_{\mathrm{loc}}^2$, or

b) $b \in L_{t,x}^\infty$ and there exists $\gamma > 1/2$ such that $T^w b \in C_t^\gamma C_x^{3/2}$.

Then for any $x_0 \in \mathbb{R}^d$ there exists a unique global solution to (3.42); moreover the perturbed ODE admits a flow of diffeomorphisms. If additionally $T^w b \in C_t^\gamma C_x^{n+1}$in point a), respectively $T^w b \in C_t^\gamma C_x^{n+1/2}$ in point b), then the flow is locally C^n.

Proof. The statement is useful rewriting of the results from Sections 1.3 and 1.4.3, applied to the choice $A = T^w b$, keeping in mind as stressed several times throughout them that the global regularity condition on A can be weakened by standard localization arguments, once growth conditions are available. The claims under assumption *a)* follow from Theorems 1.34 and 1.35; instead under *b)* they come from Remark 1.61. Here we are cheating a little bit, since Section 1.4.3 would require $\partial_t A(t,x) = \partial_t T_t^w b(x) = b_t(x + w_t)$ to be continuous, while here we only have $\partial_t A \in L_{t,x}^\infty$, but we invite the reader to observe that that this doesn't create any real issue. In fact, the hypothesis $\partial_t A \in C_{t,x}^0$ is only needed in order to guarantee that any solution θ to the nonlinear YDE is Lipschitz continuous, something which is still true for $b \in L_{t,x}^\infty$. Alternatively, one could first replace b by suitable mollifications b^ε, construct the flow therein, and then pass to the limit as $\varepsilon \to 0$, which is allowed since the estimate on $\|b^\varepsilon\|_{C^0} = \|b^\varepsilon\|_{L^\infty} \leqslant \|b\|_{L^\infty}$ is uniform. □

Remark 3.53. It also follows from the results of Chapter 1 that, under the assumptions of Theorem 3.52, the map that sends $T^w b$ to the associated flow is continuous in suitable topologies. For instance, in the setting *a)*, it will be continuous from $C_t^\gamma C_x^{\beta,\lambda} \cap C_t^\gamma C_{\mathrm{loc}}^{n+1}$ to $C_t^0 C_{\mathrm{loc}}^n$ for any $n \geqslant 1$.

At this point, we can go back to the study of the SDE (3.28) and understand whether $T^{W(\omega)} b$ $\mathbb{P}$-a.s. satisfies the assumptions of Theorem 3.52; since its conclusion at fixed ω involve the set of all possible initial data, and not just a fixed $x_0 \in \mathbb{R}^d$ in consideration, we obtain a path-by-path wellposedness result, in the sense of Definition 3.36.

Theorem 3.54. *Let W be a fBm of Hurst parameter $H \in (0,1)$ and let $b \in L_t^\infty B_\infty^\alpha$ with*

$$\alpha > 2 - \frac{1}{2H} \quad \text{or} \quad \alpha > \max\left\{\frac{3}{2} - \frac{1}{2H}, 0\right\}. \tag{3.43}$$

Then path-by-path wellpossedness holds for the SDE (3.28), which moreover admits an associated random flow of diffeomorphisms. If moreover

$$\alpha > n + 1 - \frac{1}{2H} \quad \text{or} \quad \alpha > \max\left\{n + \frac{1}{2} - \frac{1}{2H}, 0\right\} \tag{3.44}$$

for some $n \geqslant 1$, then the flow is locally C^n.

Proof. If $\alpha > 2 - 1/(2H)$, we can apply Corollary 3.28 to both b and Db to deduce that $\mathbb{P}$-a.s. $T^W b \in C_t^\gamma C_x^{\beta,\lambda} \cap C_t^\gamma C_x^{2,\lambda}$; we can then invoke the pathwise statement from Theorem 3.52 to any $\omega \in \Omega$ such that $T^{W(\omega)} b$ satisfies the above regularity to get the conclusion. More generally, if $\alpha > n + 1 - 1/(2H)$, the same argument shows that $\mathbb{P}$-a.s $T^W b \in C_t^\gamma C_x^{\beta,\lambda} \cap C_t^\gamma C_x^{n,\lambda}$.

The case $\alpha > 0$, $\alpha > 3/2 - 1/(2H)$ proceeds among similar lines, only this time we cannot immediately invoke Corollary 3.28 to get the conclusion. However, since $Db \in L_t^\infty B_\infty^{\alpha-1}$, by the assumptions and Corollary 3.27 we can find $\rho > 1/2$ such that

$$\gamma = 1 + (\alpha - 1 - \rho)H > 1/2,$$

namely such that condition (3.24) is satisfied for $s = \alpha - 1$, $\rho > 1/2$ and $q = \infty$. It then follows that (up to relabelling $\gamma > 1/2$)

$$\sup_{x \neq y \in \mathbb{R}^d} \mathbb{E}\left[\exp\left(\eta \frac{\|DT^W b(x) - DT^W b(y)\|_\gamma^2}{|x-y|^\rho}\right)\right] \leqslant K(\eta),$$

which combined with standard Garsia-Rodemich-Rumsay estimates yields $DT^W b \in C_t^\gamma C_{\mathrm{loc}}^{\rho-\varepsilon}$ $\mathbb{P}$-a.s.; overall $T^W b \in C_t^\gamma C_{\mathrm{loc}}^{1+\rho-\varepsilon}$ and we can then conclude by Theorem 3.52. The case of higher regularity of the flow can be treated similarly. □

One can draw a nice analogy between the pathwise techniques developed here, which ultimately lead to Theorem 3.54, and the general philosophy of rough paths theory (see e.g. the monographs [132, 134]). The idea in rough path theory is that, in order to analytically define and study an equation of the form

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}Y_t,$$

driven by an irregular map Y (typically $Y \in C_t^\gamma$ with $\gamma \leqslant 1/2$), the information contained in Y alone is not enough. Instead, one needs to *enhance* the imput data Y into $(Y, \mathbb{Y})$ by additionally considering a sufficient amount of iterated integrals of Y against itself (or in general its signature); if Y is a stochastic process, the resulting solution map will admit a so called *Itô-Lyons decomposition*, namely it will consist of the composition of a measurable lift of Y into $(Y, \mathbb{Y})$ and a continuous (actually, differentiable) map sending $(Y, \mathbb{Y})$ into the corresponding solution X.

The situation here is analogous: if b is a truly distributional drift, it is a priori unclear how to give meaning to the perturbed ODE (3.42); in order to do so, one needs to enhance the data of the problem by not only considering b and w singularly but also taking into account the regularity of $T^w b$. In the stochastic setting, this leads to a decomposition solution map decomposing into the measurable map $W \mapsto T^W b$ and the continuous maps $T^W b \mapsto \Phi$, where Φ is the flow of the associated YDE.

There is however one major difference in the two settings: in rough path theory, there are multiple (infinite) choices for the enhancement $(Y, \mathbb{Y})$, which lead to different solutions; instead here, as we have seen in Section 3.1.1, $T^W b$ is always uniquely defined in the sense of distributions, and the question is really just whether it is regular enough to carry out the analytical part of the argument.

Remark 3.55. Let us shortly compare Theorems 3.30 and 3.54. The proof of Theorem 3.30 relies crucially on Proposition 3.44, and thus the availability of Girsanov transform, in order to achieve path-by-path uniqueness. In the regime $H > 1/2$, this comes at the price of requiring the drift b to be Hölder continuous in the variable t, as prescribed by Assumption 3.29. However, Theorem 3.54 covers nontrivial situations where Girsanov is not applicable, e.g. when $H > 1/2$, $b \in L_t^\infty C_x^\alpha$ for $\alpha \in (0,1)$ such that

$$\alpha > \frac{3}{2} - \frac{1}{2H};$$

observe that the above condition allows for nontrivial values $\alpha < 1$ for every $H \in (1/2, 1)$.

Remark 3.56. A statement in the style of Theorem 3.54 could also be given for $b \in L_t^q B_\infty^\alpha$; going through the same proof, applying Corollary 3.28, in order to have a locally C^n flow it would be enough to impose the condition

$$\alpha - \frac{1}{Hq} > n + 1 - \frac{1}{2H}.$$

Observe however how it gets more and more restrictive as $q \downarrow 2$ and/or $n \uparrow +\infty$.

Next, we pass to the study of transport and continuity equations associated to the perturbed ODE; again, we start by considering the case of a given fixed continuous path w, where they are formally given respectively by

$$\partial_t u + b \cdot \nabla u + \frac{\mathrm{d}w}{\mathrm{d}t} \cdot \nabla u = 0, \quad \partial_t \nu + \nabla \cdot (b\nu) + \frac{\mathrm{d}w}{\mathrm{d}t} \cdot \nabla \nu = 0. \tag{3.45}$$

As the path w is not differentiable in general, there is no classical way to give meaning to the above equations; however, we can reabsorb the term $\mathrm{d}w/\mathrm{d}t \cdot \nabla$ by a simple Galilean transformation (which is the equivalent at the Eulerian level of the change of variables $\theta = x - w$ at the Lagrangian level). Indeed, if we set $\tilde{u}(t,x) = u(t, x + w_t)$, $\tilde{\nu}(t,x) = \nu(t, x + w_t)$ and $\tilde{b}(t,x) = b(t, x + w_t)$, under the assumption that w were differentiable and b smooth, equations (3.45) would be equivalent to

$$\partial_t \tilde{u} + \tilde{b} \cdot \nabla \tilde{u} = 0, \quad \partial_t \tilde{\nu} + \nabla \cdot (\tilde{b} \tilde{\nu}) = 0. \tag{3.46}$$

Now, if $\tilde{b}$ is a measurable function, equation (3.46) is perfectly meaningful (possibly in a weak sense) without any regularity assumption on w, so we can *define* u (resp. ν) to be a solution to (3.45) if and only if $\tilde{u}$ (resp. $\tilde{\nu}$) solves (3.46)[3.8]. If $\tilde{b}$ is not a function, observing that $\int_0^t \tilde{b}_s(x) \mathrm{d}s = T_t^w b(x)$, we can rewrite (3.46) as

$$\tilde{u}_{\mathrm{d}t} + T_{\mathrm{d}t}^w b \cdot \nabla \tilde{u}_t = 0, \quad \tilde{\nu}_{\mathrm{d}t} + \nabla \cdot (T_{\mathrm{d}t}^w b \tilde{\nu}_t) = 0 \tag{3.47}$$

which can be finally given meaning as a Young transport (resp. continuity) equation like the ones studied in Section 2.1, at least under the assumption that $A = T^w b$ is regular enough.

In conclusion, from now on, whenever referring to the transport and continuity equations appearing in (3.45), we will systematically interpret them as (3.47). We can then restate the results from Section 2.1 in our setting as follows.

Theorem 3.57. *Let $T^w b$, b satisfy one between assumptions a) and b) from Theorem 3.52. Then:*

i. For any $u_0 \in C^1_{\mathrm{loc}}$, there exists a unique solution to the transport equation in (3.45), in the class $u \in C_t^\gamma C^0_{\mathrm{loc}} \cap C_t^0 C^1_{\mathrm{loc}}$, which is given by $u_t(x) = u_0(\Phi_{0 \leftarrow t}(x))$, where Φ is the flow associated to the perturbed ODE.

ii. For any $\nu_0 \in \mathcal{M}_x$, there exists a unique solution to the transport equation in (3.45), in the class $\nu \in L_t^\infty \mathcal{M}_x \cap C_t^\gamma (C_c^1)^*$, which is given by duality by $\langle \varphi, \nu_t \rangle = \langle \varphi(\Phi_{0 \to t}(\cdot)), \nu_0 \rangle$, where Φ is the flow associated to the perturbed ODE.

Proof. The statement is mostly an application of Theorem 2.9 to our setting. Although therein global bounds $T^w b \in C_t^\gamma C_x^{1+\beta}$, the usual localization arguments allow to weaken them to local bounds combined with growth conditions (in particular, $\beta = 1$ would correspond to $T^w b$ satisfying condition *a)* in Theorem 3.52). The case of $b, T^w b$ satisfying assumption *b)* instead does not follow from Theorem 2.9 but, as explained at the beginning of Section 2.3, can be found in Section 5.1 from [145]. □

Next, we can specialize Theorem 3.57 to the case where w is sampled as an fBm.

Theorem 3.58. *Let W be a fBm of Hurst parameter $H \in (0,1)$ and b satisfy the assumptions of Theorem 3.54. Then existence and uniqueness holds (in the suitable classes of solutions) for the stochastic transport and continuity equations*

$$\partial_t u + b \cdot \nabla u + \frac{\mathrm{d}W}{\mathrm{d}t} \cdot \nabla u = 0, \quad \partial_t \nu + \nabla \cdot (b\nu) + \frac{\mathrm{d}W}{\mathrm{d}t} \cdot \nabla \nu = 0$$

which are $\mathbb{P}$-a.s. pathwise meaningful in the nonlinear Young sense.

Proof. Arguing as in the proof of Theorem 3.54, the assumptions therein guarantee that $\mathbb{P}$-a.s. b and $T^W b$ satisfy either condition *a)* or *b)* from Theorem 3.52; the conclusion then follows from Theorem 3.57. □

3.8. There is a nontrivial aspect concerning this type of arguments, that will also appear in Section 5.2.2. The idea that we can freely pass from (3.45) to (3.46) and viceversa is based on the assumption that the chain rule holds; in the case of irregular w, this is usually achieved by assuming that we are working with a *geometric rough path*. For instance, if w were sampled as a Brownian motion, to be rigorous we should be working with $\mathrm{d}u + b \cdot \nabla u + \circ \mathrm{d}W \cdot \nabla u = 0$, where $\circ \mathrm{d}W$ denotes the Stratonovich differential.

3.3 Prevalence statements

The advantage of the approach presented in Section 3.2.4 is that it solely relies on the pathwise properties of the process W as encoded by the regulartiy of the averaged field $T^{W(\omega)}b$. As such, it allows to extend our considerations outside of the probabilistic framework, by addressing the more sophisticated question:

"Do *generic additive perturbations* w regularise the ODE?"

Before moving further, we need to explain what we mean here by generic, as there are several options in the literature, including topological ones like those based on Baire sets. Here instead we will adopt the measure theoretic concept of *prevalence*, which is explained in detail in Appendix A.3. Without going into too much detail here, let us only mention the following facts: i) while being a purely analytical concept, prevalence allows for the use of probabilistic tools in the proof; ii) in $\mathbb{R}^d$, a set is prevalent if and only if it is of full Lebesgue measure. In light of the second property, from now on whenever we say that a property holds for almost every (a.e.) $\varphi \in E$, E being a suitable Banach space, it will be interpreted in the prevalence sense.

For simplicity, in the following we will often restrict to the case of time-independent drifts b[3.9]; the main result of this section goes as follows (we will use the variable φ for generic continuous paths, instead of w which will still sometimes be used for typical fBm realizations).

Theorem 3.59. *Let $b \in B^{\alpha}_{\infty}$, $\alpha \in (-\infty, 1)$, $\delta \in (0,1)$ and consider the perturbed ODE*

$$x_t = x_0 + \int_0^t b(x_s) \mathrm{d}s + \varphi_t - \varphi_0. \tag{3.48}$$

Then the following hold:

i. If $\delta < (2 - 2\alpha)^{-1}$, then for a.e. $\varphi \in C^{\delta}_t$ it holds $T^{\varphi} b \in C^{\gamma}_t C^1_{\mathrm{loc}}$ and ODE has a meaningful interpretation; moreover for any initial $x_0 \in \mathbb{R}^d$ there exists a solution to the ODE.

ii. If $\delta < (2 - 2\alpha)^{-1}$ and we fix $x_0 \in \mathbb{R}^d$, then for a.e. $\varphi \in C^{\delta}_t$ there exists a unique solution to the ODE with initial condition x_0.

iii. If $\delta < (4 - 2\alpha)^{-1}$ then for a.e. $\varphi \in C^{\delta}_t$ the ODE is wellposed and it admits a locally C^1 flow.

iv. If $\delta < (2n + 2 - 2\alpha)^{-1}$ for some $n \geqslant 1$, then for a.e. $\varphi \in C^{\delta}_t$ the flow is locally C^n.

v. Finally, for a.e. $\varphi \in C^0_t$ the ODE admits a smooth flow.

Let us shortly comment on Theorem 3.59. First of all, we see that, as δ gets smaller (in other terms, generic elements $\varphi \in C^{\delta}_t$ become more irregular), the solution properties of the SDE improve. This is a rigorous formalization in the case of ODEs of the general principle "*the rougher the noise, the better the regularisation*", which is expected to hold also in many other contexts. Secondly, we see that in the limit case of generic $\varphi \in C^0_t$, there are no conditions on α, n anymore; this means that we can take $\alpha \in \mathbb{R}$ arbitrarily low and we will still obtain a smooth flow for the equation, displaying the *infinite regularising effect* of continuous functions.

Finally, on a more technical side, as the concept of prevalence is of measure-theoretic type, the *order of quantifiers* matters. This is why, in Point *ii.* above, we need to first fix $x_0 \in \mathbb{R}^d$ and only then we can give a statement for a.e. $\varphi \in C^{\delta}_t$; we not allowed to exchange quantifiers, namely to provide a statement of the form "for a.e. $\varphi \in C^{\delta}_t$ there exists a unique solution to the ODE with initial condition x_0 for any $x_0 \in \mathbb{R}^d$", unless we impose more regularity on b (which yields the stronger conclusion of Point *iii.*). The same argument applies for the quantifier associated to $b \in B^{\alpha}_{\infty}$, which is why we fix b at the very beginning of Theorem 3.59; understanding whether generic φ regularise the ODEs associated to *all* $b \in E$, for a suitable class E, is in general a more difficult question, which will be addressed in Chapter 5.

3.9. As will become evident from the proofs, many statement can be generalized to the time dependent framework; however, like in Theorem 3.30, one then needs to take into account the time regularity of the drift b, which makes the statements more technical and less uniform in δ (with a fundamental distinction between $\delta > 1/2$ and $\delta < 1/2$). In the regime $\delta < 1/2$, one could also consider $b \in L^q_t B^{\alpha}_{\infty}$ with $q > 2$, under the condition

$$\delta < \left(\frac{1}{2} - \frac{1}{q}\right)(n + 1 - \alpha)^{-1}$$

which leads to an associated C^n_{loc} flow.

The rest of this section is devoted to the proof of Theorem 3.59; we start by an intermediate result of interest on its own, concerning the regularity of $T^\varphi b$ for generic φ.

Theorem 3.60. *Let $b \in L_t^q B_p^s$ for some $q \in (2, \infty]$, $s \in \mathbb{R}$, $p \in [2, \infty)$. Let $\delta \in [0, 1)$ and $\beta \in \mathbb{R}$ satisfy*

$$\beta < \frac{1}{\delta}\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{d}{p}, \tag{3.49}$$

where d is the space dimension and we adopt the convention that (3.49) is satisfied for any β if $\delta = 0$. Then there exists $\gamma > 1/2$ such that, for almost every $\varphi \in C_t^\delta$, $T^\varphi b \in C_t^\gamma C_x^{s+\beta}$.

Similarly, if $b \in L_t^q B_\infty^s$ and $\delta \in [0, 1)$, $\beta \in \mathbb{R}$ satisfy $\beta < 1/(2\delta)$

$$\beta < \frac{1}{\delta}\left(\frac{1}{2} - \frac{1}{q}\right), \tag{3.50}$$

then there exists $\gamma > 1/2$ such that, for almost every $\varphi \in C_t^\delta$, $T^\varphi b \in C_t^\gamma C_x^{s+\beta,\lambda}$ for all $\lambda > 0$.

Proof. Since this is the first example of a several prevalence results we will present throughout this thesis, let us explain in detail the strategy. To show that a certain property $\mathcal{P}$ is satisfied by almost every $\varphi \in E$, E being a suitable Banach space, it suffices to show that:

i. the set $\mathcal{A} := \{w \in E : w \text{ satisfies } \mathcal{P}\}$ is Borel measurable in E;

ii. there exists a tight probability μ measure on E such that $\mu(\varphi + \mathcal{A}) = 1$ for all $\varphi \in E$.

This basic technique then allows for several variations, based on the properties of prevalence (for which we refer again to Appendix A.3): for instance, instead of showing that $\mathcal{A}$ is Borel, it suffices to check that it contains a prevalent Borel set; or we could write the set as a countable union $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ and show separately that each $\mathcal{A}_n$ is prevalent by using a different measure μ^n.

Back to the actual proof of Theorem 3.60. Arguing as in Lemma 3.6, for any $\gamma > 1/2$, the sets

$$\mathcal{A} = \{w \in C_t^\delta : T^w b \in C_t^\gamma C_x^{s+\beta}\}, \quad \tilde{\mathcal{A}} = \{w \in C_t^\delta : T^w b \in C_t^\gamma C_x^{s+\beta,\lambda} \text{ for all } \lambda > 0\}$$

are Borel in C_t^δ (this is because $F = C_x^{s+\beta}$ and $F = C_x^{s+\beta,\lambda}$ both satisfy the Fatou property, as can be checked easily by means of Remark 3.7). In order to prove the first statement, for any given δ and β as above, it remains to find $\gamma > 1/2$ and a tight probability distribution μ on C_t^δ such that

$$\mu(\varphi + \mathcal{A}) = \mu(w \in C_t^\delta : T^{\varphi + w} b \in C_t^\gamma C_x^{s+\beta}) = 1 \quad \forall \varphi \in C_t^\delta. \tag{3.51}$$

Thanks to the translation invariance of the B_p^s-norm, we can reduce the problem to an easier one. Indeed, setting $\tilde{b}_t := \tau^{\varphi_t} b_t$ for all $t \in [0, T]$, it holds $\tilde{b} \in L_t^q B_p^s$ and $T^{\varphi + w} b = T^w \tilde{b}$. In particular, in order to show that (3.51) holds for fixed $b \in L_t^q E$ and for all $\varphi \in C_t^\delta$, it actually suffices to find $\gamma > 1/2$ and a tight measure $\mu = \mu_{\beta,\delta,\gamma}$ on C_t^δ such that

$$\mu(w \in C_t^\delta : T^w \tilde{b} \in C_t^\gamma C_x^{s+\beta}) = 1 \text{ for all } \tilde{b} \in L_t^q B_p^s; \tag{3.52}$$

observe that the dependence on the path φ has completely disappeared in (3.52). Using Theorem 3.12 combined with Remark 3.21, we can choose $\mu = \mu^{\delta+\varepsilon}$ to be the law of a fBm of parameter $\delta + \varepsilon$, up to choosing ε small enough so that

$$\beta < \frac{1}{\delta + \varepsilon}\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{d}{p},$$

which is always possible thanks to the hypothesis (3.49); this yields the validity of (3.52) and thus the conclusion for $b \in B_p^s$ and $\delta \in (0, 1)$.

The case $b \in L_t^q B_\infty^s$ follows from an almost identical argument; again we take μ to be the law of a suitable fBm of parameter $\delta + \varepsilon$ with $\varepsilon > 0$ sufficiently small, ensuring that condition (3.24) from Corollary 3.27 is satisfied by (s, ρ, q) with $\rho = s + \beta + \varepsilon$; the application of Corollary 3.27 then yields the conclusion. □

Next, we need to identify an analytical property $\mathcal{P}$ ensuring that, for fixed x_0, there exists a unique solution to the ODE (3.48). In fact, this is already given us by Lemma 3.43: it suffices to show that the ODE admits a solution x satisfying $T^x b \in C_t^\gamma C_{\mathrm{loc}}^1$, which implies it being the only possible solution.

Proposition 3.61. *Let $b \in L_t^\infty B_\infty^\alpha$, $\alpha \in (-\infty, 1)$, $\delta \in [0,1)$ with $\delta < (2-2\alpha)^{-1}$ and fix $x_0 \in \mathbb{R}^d$. Then for a.e. $\varphi \in C_t^\delta$, the ODE (3.48) with initial datum x_0 has a unique solution.*

Proof. By Theorem 3.60, under our assumptions, we can find γ, β, λ satisfying the usual conditions such that the set

$$\tilde{\mathcal{A}} = \{w \in C_t^\delta : T^w b \in C_t^\gamma C_x^{\beta,\lambda}\}$$

is prevalent in C_t^δ; therefore the ODE has a rigorous interpretation as a nonlinear YDE and we only need to focus on establishing uniqueness. By the aforementioned Lemma 3.43, it suffices to show the existence of another parameter $\tilde{\lambda} > 0$ such that the set

$$\mathcal{A} := \tilde{\mathcal{A}} \cap \left\{ w \in C_t^\delta : \text{there exists a solution } x \in w + C^\gamma \text{ such that } T^x b \in C_t^\gamma C_x^{1,\tilde{\lambda}} \right\}.$$

is prevalent. We start by showing that $\mathcal{A}$ is Borel measurable; to this end, we write it as

$$\mathcal{A} = \bigcup_{N \geqslant 1} \mathcal{A}_N := \bigcup_{N \geqslant 1} \{w \in C_t^\delta : \|T^w b\|_{C^\gamma C^{\beta,\lambda}} \leqslant N, \exists \text{ a solution } x \text{ such that } \|T^x b\|_{C^\gamma C^{1,\tilde{\lambda}}} \leqslant N\}.$$

It then suffices to show that each $\mathcal{A}_N$ is closed in C_t^0 (thus also in C_t^δ).

Let w^n be a sequence of elements of $\mathcal{A}_N$ such that $w^n \to w$, then by the proof of Lemma 3.6 we know that $T^w b \in C_t^\gamma C_x^{\beta,\lambda}$ with the bound $\|T^w b\|_{C^\gamma C^{\beta,\lambda}} \leqslant N$; since $b \in B_\infty^\alpha$, by Lemma 3.2 $T^{w^n} b \to T^w b$ in $C_t^\gamma B_\infty^\alpha$, which we can interpolated with the above bound to deduce that $T^{w^n} b \to T^w b$ in $C_t^\gamma C_x^{\beta-\varepsilon,\lambda}$ for any $\varepsilon > 0$. For each n, denote by $x^n = \theta^n + w^n$ the unique solution associated to (b^n, x_0) such that $\|T^{x^n} b\|_{C^\gamma C^{1,\tilde{\lambda}}} \leqslant N$; by Proposition 1.18, since $\|T^{w^n} b\|_{C^\gamma C^{\beta,\lambda}} \leqslant N$, we deduce that $\sup_n \|\theta^n\|_{C^\gamma} < \infty$. By Ascoli-Arzelà, we can then find $\theta \in C_t^\gamma$ and extract a subsequence such that $\theta^n \to \theta$ in $C_t^{\gamma-\varepsilon}$ for any $\varepsilon > 0$; it follows that $x^n = \theta^n + w^n \to \theta + w =: x$ in C_t^0 and that

$$\|T^x b\|_{C^\gamma C^{1,\tilde{\lambda}}} \leqslant \liminf_{n \to \infty} \|T^{x^n} b\|_{C^\gamma C^{1,\tilde{\lambda}}} \leqslant N.$$

Moreover, using the fact that $T^{w^n} b \to T^w b$ in $C_t^\gamma C_x^{\beta-\varepsilon,\lambda}$ and $\theta^n \to \theta$ in $C_t^{\gamma-\varepsilon}$, choosing $\varepsilon > 0$ small enough so that $\gamma + (\beta - \varepsilon)(\gamma - \varepsilon) > 1$, the continuity of the nonlinear Young integral ensures that x is a solution to the ODE associated to (x_0, b, w). Overall, this shows that $w \in \mathcal{A}_N$ and so that $\mathcal{A}_N$ is closed and $\mathcal{A}$ is Borel measurable.

It remains to find a tight measure μ on C_t^δ such that $\mu(\varphi + \mathcal{A}) = 1$ for all $\varphi \in C_t^\delta$. As before, we will take $\mu = \mu^H$ the law of fBm for suitable parameter $H = H(\alpha, \delta) > \delta$; indeed, it will be enough to choose $H = \delta + \varepsilon$ for $\varepsilon > 0$ small enough so that $\delta + \varepsilon < (2-2\alpha)^{-1}$.

Let us present the argument in the case $H < 1/2$ first. Observe that $x \in (w + \varphi) + C_t^\gamma$ is a solution to the SDE pertubed by $w + \varphi$ if and only if $\tilde{x} := x - \varphi \in w + C_t^\gamma$ is a solution to the SDE perturbed by w and associated to a new time dependent drift $\tilde{b}$ given by $\tilde{b}_t(\cdot) = b(\cdot + \varphi_t)$ and a new initial datum; indeed, by definition of solution, $\theta = x - (w + \varphi) = \tilde{x} - w$ must solve

$$\theta_t = x_0 - \varphi_0 + \int_0^t T^{w+\varphi} b(\mathrm{d}s, \theta_s) = \tilde{x}_0 + \int_0^t T^w \tilde{b}(\mathrm{d}s, \theta_s).$$

By the translation invariance of the B_∞^α-norm, it holds $\tilde{b} \in C_x^\alpha$, $\|\tilde{b}\|_{L^\infty B_\infty^\alpha} = \|b\|_{B_\infty^\alpha}$. By construction, our choice of H implies $\alpha > 1 - (2H)^{-1}$, so that Proposition 3.49 applies; going through the proof of Proposition 3.44 we can then find $\gamma > 1/2$ and $\tilde{\lambda} > 0$ (independent of φ, b) such that

$$\begin{aligned}
1 &= \mu^H\Big(w \in C^\delta : \exists \theta \in C(\tilde{x}_0, T^w \tilde{b}) \text{ such that } T^{w+\theta} b \in C_t^\gamma C_x^{1,\tilde{\lambda}} \Big)\\
&= \mu^H\Big(w \in C_t^\delta : T^w \tilde{b} \in C_t^\gamma C_x^{\beta,\lambda},\ \exists \tilde{x} \in w + C_t^\gamma \text{ solution associated to } \tilde{x}_0, \tilde{b} \text{ s.t. } T^{\tilde{x}} \tilde{b} \in C_t^\gamma C_x^{1,\tilde{\lambda}} \Big)\\
&= \mu^H\Big(w \in C_t^\delta : T^{w+\varphi} b \in C_t^\gamma C_x^{\beta,\lambda},\ \exists x \in (w+\varphi) + C_t^\gamma \text{ solution to ass. to } x_0, b \text{ s.t. } T^x b \in C_t^\gamma C_x^{1,\tilde{\lambda}} \Big)
\end{aligned}$$

which gives the conclusion in this case.

The case $H > 1/2$ is essentially identical, up to a small technical detail: since the new drift $\tilde{b}$ is time dependent, it's not a priori clear if Proposition 3.49 applies, unless we verify the Assumption 3.29 holds. But using the fact that here $\alpha > 0$ and $\varphi \in C^\delta_t$, it's easy to check that $\tilde{b}$ as defined above belongs to $C^{\alpha\delta}_t C^0_x \cap C^0_t C^\alpha_x$ and so conditions $\alpha > 1 - 1/(2H)$, $\alpha\delta > H + 1/2$ are fullfilled as soon as we take $H = \delta + \varepsilon$ with $\varepsilon > 0$ small enough. □

We are finally ready to give complete the proof of our main prevalence statement.

Proof. (of Theorem 3.59) The rigorous interpretation of the ODE mentioned in Point *i.* of the statement follows from Theorem 3.60, as already explained at the beginning of the proof of Proposition 3.61; the exstence of a solution for any $x_0 \in \mathbb{R}^d$ then follows from the general theory of nonlinear Young equations, e.g. Theorem 1.41. Proposition 3.61 implies the validity of Point *ii.*

Points *iii.*-*iv.* follow from a combination of Theorems 3.60 and 3.52: if $\delta < (2n + 2 - 2\alpha)^{-1}$, then for a.e. $\varphi \in C^\delta_t$ it holds $T^\varphi b \in C^\gamma_t C^n_{\mathrm{loc}}$ (in addition to the usual bounds in $C^\gamma_t C^{\beta,\lambda}_x$). Finally, Point *v.* follows again from Theorem 3.52, only observing that for a.e. $\varphi \in C^0_t$ it holds $T^\varphi b \in C^\gamma_t C^n_{\mathrm{loc}}$ for all $n \in \mathbb{N}$ (which is a consequence of countable intersection of prevalent sets being prevalent). □

3.4 Bibliographical comments and future directions

As already mentioned, the material presented in this chapter is taken from the work [145] and its subsequent development [146]. However, compared to those works, the presentation has undergone major modifications, for a number of reasons. On one hand, most of the material concerning nonlinear YDEs and Young transport equations has already been exposed more abstractly in Chapters 1-2, so that here it could be applied directly to the choice $A = T^w b$. On the other, I have stressed here the more probabilistic side of the results, leading to the comparison of various notions of solutions from Section 3.2.1 (which contains some nice and elementary unpublished results like Proposition 3.39) and stability estimates like those in Theorem 3.30 (which are applied in [146] to prove wellposedness of distribution dependent SDEs; unfortunately, I didn't have enough time to expose those results here, although I find them extremely interesting). Instead, in [145] the focus is more on analytical prevalence statements like presented in Section 3.3.

On the technical side, I have implemented some improvements over the results from [145]. Therein, regularity estimates for $T^W b$ were only obtained for martingale type 2 spaces like in Section 3.1.3, which led to the fundamental restriction of dealing with $b \in L^q_t B^\alpha_\infty$ with compact support (so that $B^\alpha_\infty \hookrightarrow B^\alpha_p$ and results in the latter space could be invoked). Here this assumption is no longer needed thanks to the results of Section 3.1.4, up to the price of working with the more technical spaces $C^\gamma_t C^{\beta,\lambda}_x$ instead of having global bounds. The results for martingale type $\mathfrak{p}$ spaces with $\mathfrak{p} < 2$ (Proposition 3.24) presented here are new, although close in spirit to those obtained in [195], where the author also observes a regularity improvement of order at most $((p' \wedge 2)H)^{-1}$; the basic Lemma 3.22 is a far reaching generalization of Lemma 45 from [143].

Let me now pass to contextualize the results here with the existing literature (which is advancing very fast as I'm writing). First results on regularisation by noise for SDEs driven by additive fBm were obtained by Nualart and Oukine in [226, 227], by means of Girsanov transform and comparison principles for one-dimensional SDEs. The next major breakthrough was given by Catellier and Gubinelli [57], who first developed the nonlinear Young integral and obtained stochastic estimates for $T^W b$ for a large class of b; the work [57] was also the first one able to provide path-by-path uniqueness results for SDEs driven by fBm and to show wellposedness for distributional drifts. The concept of path-by-path uniqueness, which stems from the work of Davie [86] (although the terminology is due to Flandoli [115]), has been established successfully for several classes of SDEs, although mostly in the Markovian case (i.e. Brownian or Lévy noise), see e.g. [254, 255, 243, 244, 29]. The paper [57] is, to the best of my knowledge, also the first one highlighting the fact that fBm trajectories with small $H \in (0, 1)$ have a higher regularising effect.

An alternative approach to the study of SDEs with additive fBm has been developed by the school of Frank Proske and collaborators, based on Malliavin calculus and a form of "local time variational calculus", see [6, 24, 9, 8, 7]. Among them, [24] is first one to study the higher regularity of the flow associated to the SDE, at least in a weakly diffentiable sense, [6] is the first one to provide examples of "infinitely regularising processes" (a feature present in the current thesis in the behaviour of a.e. $\varphi \in C_t^0$ in Theorem 3.59) and [9] presents some applications to transport equations. However, I have the impression that this technique usually yields weaker results than the one based on nonlinear Young integrals. In most of the aforentioned works, the drift b is taken in the class $L^{1,\infty}_{\infty,\infty}$ (a particular subcase of $L^\infty_{t,x}$) and in order to obtain a locally $W^{k,p}$ flow, it is usually required $H < \frac{1}{2(d-1+2k)}$, cf. Theorem 5.2 from [24]; instead using $L^\infty_{t,x} \hookrightarrow L^\infty_t B^0_\infty$, in our setting we find a locally C^k flow under the less restrictive $H < \frac{1}{2(k+1)}$. Similarly, Theorem 3.6 from [9] shows solvability of the transport equation under $H < \frac{1}{2(d+3)}$, while here we only need $H < 1/4$, regardless of the dimension d of the space (actually, since $b \in L^\infty_{t,x}$ gives better a priori estimates on the time regularity of the flow, even $H < 1/3$ would suffice).

Finally, let me mention the work [170], which appeared around the same time I was writing the manuscript for [145] with Max. [170] combines the approaches from [57] and [6] in a very elegant and self-contained way; along the way, it highlights the importance of local nondeterminism properties (cf. Remark 3.20) and yields new examples of infinitely regularising processes; it however suffers the same "curse of dimensionality" presented above (and that will also appear in the upcoming Chapter 5 in the context of ρ-irregularity): given a singular drift $b \in B^\alpha_\infty$, in order to solve the equation one must require $H < C(d, \alpha)$ for some function C such that $C(d, \alpha) \to 0$ as $d \to \infty$, a feature not present in [57] nor here.

Natural questions in view of the above are the following: a) are the results from [57] (possibly up to the extension to time-dependent drifts presented here) close to optimal? b) can they can be improved further by means of other approaches?

Question a) can be partially addressed by means of a scaling argument. Recall that if W is a fBm of parameter H, then it is H-self-similar, in the sense that $W^\lambda_t := \lambda^{-H} W_{\lambda t}$ is still distributed as W. Applying the same transformation to the SDE (let $x_0 = 0$ for simplicity)

$$X_t = \int_0^t b_s(X_s) \mathrm{d}s + W_t$$

one obtaines that $X^\lambda_t := \lambda^{-H} X_{\lambda t}$ solves

$$X^\lambda_t = \int_0^t b^\lambda_s(X^\lambda_s) \mathrm{d}s + W^\lambda_t, \quad b^\lambda_t(x) := \lambda^{1-H} b_{\lambda t}(\lambda^H x).$$

One can now analyse how suitable norms $\|\cdot\|_E$ behaves under the transformation $b \mapsto b^\lambda$, which should allow to identify the *(sub)critical* spaces for the SDE. Intuitively, if $\|b^\lambda\|_E \to 0$ as $\lambda \to 0$, the noise component of the equation should be stronger than the nonlinearity at very small times, yielding uniqueness; conversely, if the nonlinear part is dominant for $t \ll 1$, no local wellposedness theory is expected to hold for the SDE, unless it was true even without the noise (so that, in this regime, the noise cannot bring any regularising effect either way). Applying the above to the choice $E = L^q_t B^\alpha_\infty$ yields

$$\|b^\lambda\|_{L^q B^\alpha_\infty} = \lambda^{1-H-\frac{1}{q}+\alpha H} \|b\|_{L^q B^\alpha_\infty}$$

and thus suggests the subcritical regime to be

$$\alpha - \frac{1}{Hq} > 1 - \frac{1}{H}. \tag{3.53}$$

In particular, the results presented here are expected to be non-optimal; at the same time, for $q=\infty$, equation (3.53) yields $\alpha>1-1/H$, which is open even in the case of Brownian noise (where it would become $\alpha>-1$). On the other hand, using the embedding $L_x^p \hookrightarrow B_\infty^{-d/p}$, in the case $b\in L_t^q L_x^p$ and $H=1/2$, (3.53) reveals itself to be the classical Krylov–Röckner condition

$$\frac{2}{q}+\frac{d}{p}<1$$

which was first treated in [191]. Therefore, condition (3.53) sets the benchmark for any subsequent development, which leads me to question b).

In my opinion, the most promising tool to further advance in the above problem is the stochastic sewing lemma (SSL) first developed by Lê in [194] (which by now admits several variants, see [154, 14, 195, 133]), which bypasses the purely pathwise approach presented here. For instance, it has been successfully applied in the study of convergence of numerical schemes for singular SDEs in [52, 196] and can be used to provide alternative estimates for averaged fields, see [194, 195]. The case of "threshold" regularity $b\in B^\alpha_\infty$ with $\alpha=1-1/(2H)$ has been treated succefully in [14, 10][3.10], at least in the regime $H<1/2$. Regarding the more established setting $\alpha>1-1/(2H)$, it is worth mentioning the results obtained by Máté Gerensér in [154]; applying a *shifted SSL*, he extends the results to the regime $H\in(1,\infty)\setminus\mathbb{N}$[3.11] without requiring any use of Girsanov theorem (which start struggling as H increases, as already testified by Remark 3.55).

As a side problem, observe that there is a relevant gap between Theorem 3.30 (yielding existence and uniquess of solutions) and Theorem 3.54 (where higher regularity is needed to induce a C^1_{loc} stochastic flow). I'm currently working with Máté in [142] in order to improve the existing results and we expect it to be possible to establish strong existence, path-by-path uniqueness and existence of a C^1_{loc} stochastic flow under the condition $b\in L_t^2 B_\infty^\alpha$ for

$$\alpha>1-\frac{1}{2H},$$

for all $H\in(0,\infty)\setminus\mathbb{N}$. Observe that the above coincides with (3.53) for the choice $q=2$, and is thus probably close to optimal; on the other hand, we currently don't know how to extend our techniques in order to allow smaller values of α when q gets larger. Thus we are not even close to obtaining any answer concernig the regime $\alpha>1-1/H$.

Finally, another natural question is whether one can apply similar techniques to establish regularisation by noise for SPDEs or nonlinear SDEs with non-Lipschitz drifts. Examples of the latter based on nonlinear Young integration, include [168] and Chapter 4 from [27]. Regarding SPDEs (excluding the case $H=1/2$ of space-time white noise, for which the literature is enormous), early results are again due to Nualart and collaborators, see [103, 228]; more recent developments include [43, 42] (where averaging estimates w.r.t. fractional Brownian sheet are also established). Compared to the SDE case, this setting is however much less understood and there is no unifying framework on how to treat these problems (assuming it's even possible to find one).

3.10. More precisely, the work [14] treats a singular SPDE driven by space-time white noise, whose regularity theory is expected to be comparable to that of an SDE driven by fBm of parameter $H=1/4$. The tools developed therein, including *stochastic sewing with random controls* and *Davie-type SSL*, have then been rigorously applied to the SDE case in [10].

3.11. For $H>1$, fBm of parameter H is defined iteratively by integrating an fBm of parameter $H-1$, see also the upcoming Section 5.1.3.

Chapter 4
Regularisation of multiplicative SDEs

Having established in Chapter 3 the strong regularising effect that additive perturbations w can have on ODEs, one can pass to other classes of equations to see if these techniques are robust enough to cover them as well. In particular, we deal here with multidimensional stochastic differential equations of the form

$$\mathrm{d}x_t = b^1_t(x_t)\mathrm{d}t + b^2_t(x_t)\mathrm{d}\beta_t + \mathrm{d}w_t, \qquad x_0 \in \mathbb{R}^d, \tag{4.1}$$

where β is a fractional Brownian motion with Hurst parameter $\delta > 1/2$[4.1] and w is a deterministic continuous path. As before, we are interested in understanding how the additive perturbation affects the SDE, by first identifying *analytic* conditions on w which ensure wellposedness for (4.1) even when it fails for $w \equiv 0$, and then verifying that such properties are satisfied by interesting examples of stochastic processes.

Like in Chapter 3, let us first provide an account of the main known results for (4.1) with $w \equiv 0$. Since $\delta > 1/2$, the SDE is pathwise meaningful either in the sense of Young integrals or fractional calculus; for b^1 and b^2 sufficiently smooth, existence of a unique solution is classical, see e.g. [224, 132], as well as Appendix D from [38] for a general survey. Sharp conditions for wellposedness, in the form of Osgood-type regularity for b^1 and b^2, are given in [273], generalizing to the case $\delta > 1/2$ the results from [275, 260] for $\delta = 1/2$; this includes the case of b^1 and b^2 spatially Lipschitz.

For $d = 1$ and $b^2 \equiv 1$, we have already seen that [226] first established pathwise uniqueness for b^1 satisfying suitable Hölder regularity; this result can be extended to a broader class of non-degenerate diffusion coefficients b^2 by means of a Doss–Sussman transformation, in the style of [13].

Recently, [172] investigated the case with $b^1 \equiv 0$ and b^2 non-degenerate of bounded variation; however, the conditions included therein for wellposedness are fairly specific and require verification for each choice of b^2.

None of the results mentioned above includes the case of general Hölder continuous diffusion b^2 and smooth drift b^1. This is not due to technical limitations of the proofs; in fact, uniqueness in general does not hold. To see this, let $d = 1$, y be a solution to the ODE $\dot{y}_t = f(y_t)$ with $y_0 = 0$, and define the process $x_t := y(\beta_t)$. Under the assumption that f is α-Hölder with $\delta(1+\alpha) > 1$, Young chain rule shows that x satisfies the SDE

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}\beta_t, \quad x_0 = 0.$$

As a consequence, to any solution of the ODE we can associate a solution of the SDE; if uniqueness fails for the first, it will also fail for latter. For instance we can take

$$f(z) = \frac{1}{1-\alpha}\,|z|^\alpha, \quad y^1_t = 0, \quad y^2_t = t^{\frac{1}{1-\alpha}},$$

which implies that $x^1_t = 0$ and $x^2_t = (\beta_t)^{1/(1-\alpha)}$ are two different solutions starting from 0 to the same SDE; the above procedure actually allows to construct infinitely many of them.

Therefore the wellposedness theory for SDEs driven by fBm with $\delta > 1/2$ cannot be better than the one for classical ODEs. At the same time, it is interesting to understand how the presence of w affects these equations and whether it can cure such pathologies. The results that will be presented here are all based on the paper [139]; to the best of our knowledge, it is the first work to consider the presence of the additional term b^2 in the regularisation by noise context.

4.1. Throughout this section we will use the noncanonical parameter δ to denote the Hurst parameter of β; the reason, as will become clear in a second, is that we will use H to measure instead the Hölder regularity of the perturbation w (which itself can be sampled as another independent fBm).

There are several reasons why it is interesting to consider (4.1), including the following:

i. For b^2 sufficiently regular, the term $\int b^2(x)\mathrm{d}\beta$ is analytically well defined, so it looks like a "mild perturbation" of (4.1) in the case $b^2 \equiv 0$.

ii. Since we are also allowed to vary the regularity of β, we can now study the nontrivial interplay between the parameter δ, the (possibly distributional) regularity of b^2 and the Hölder regularity of w, which will be measured by the parameter $H \in (0,1)$.

iii. Since β is not Markovian nor a semimartingale, many classical probabilistic tools (martingale problems, generators, Itô formula) are again not available, which creates new challenges and requires to adopt different strategies.

iv. As will be discussed in more detail in Section 4.3 at the end of the chapter, the case $\delta = 1/2$ (i.e. β sampled as Bm, which might be regarded as the most natural class to consider after the ODE case), seems to be significantly harder and therein our techniques break down.

From now on, in order not to hinder the main ides with technical details, we will focus for simplicity on the additively perturbed SDE (in integral form)

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}\beta_s + w_t, \tag{4.2}$$

namely with $b^1 \equiv 0$ and $b^2 \equiv b$ not depending on time, but being possibly distributional. Indeed equation (4.2) presents the same main difficulties as (4.1); once they are properly understood, generalising the results to (4.1) is almost straightforward, as will be shown in Section 4.2.3.

Our main strategy is a variation of the one from Chapter 3, based on a change of variable which allows for the use of the nonlinear Young formalism. Given a solution x to (4.2), $\theta := x - w$ formally solves

$$\theta_t = \theta_0 + \int_0^t b(\theta_s + w_s)\mathrm{d}\beta_s. \tag{4.3}$$

If both b and w are sufficiently regular, then (4.3) can be reinterpreted as the nonlinear YDE

$$\theta_t = \theta_0 + \int_0^t \Gamma^w b(\mathrm{d}s, \theta_s), \tag{4.4}$$

where we denote by $\Gamma^w b$ the *multiplicative averaged field*, formally defined as

$$\Gamma^w b(t,y) = \int_0^t b(y + w_r)\mathrm{d}\beta_r, \qquad t \in [0,T],\ y \in \mathbb{R}^d; \tag{4.5}$$

it plays in this context the same role as the *classical averaged field* $T^w b$ from Chapter 3.

We can then *define* x to be a solution to (4.2) by imposing the ansatz $x = w + \theta$, with θ solution to (4.4); in this way we can give meaning to (4.2) for less regular choices of b and w, assuming we are able to define $\Gamma^w b$ and establish its regularity. Existence and uniqueness of x then reduces to that of θ, which in turn follows from the abstract theory of nonlinear YDEs (see Chapter 1) applied to the random field $\Gamma^w b$.

There are however some major problems in achieving the program outlined above, compared to the case of perturbed ODEs treated in Chapter 3. Recall that by Section 3.1.1, the classical averaged field $T^w b$ is always analytically well defined as a distribution, so the only problem is to establish its regularity, which can be accomplished probabilistically like in Sections 3.1.3-3.1.4. In contrast, in order to define the integral appearing in (4.5) as a Young integral, we need at least to require w to be H-Hölder continuous with $\delta + H > 1$; without this assumption, it is unclear how to interpret neither (4.2) nor (4.5), even when b is a smooth function! At the same time, it is clear from our earlier results (cf. Theorems 3.54-3.59) that a strong regularisation effect is expected to hold for especially rough w, i.e. for very small values of H, thus making the requirement $\delta + H > 1$ too restrictive.

In order to overcome this difficulty, we will invoke recently developed stochastic estimates by Hairer and Li [166], regarding Wiener integrals of the form $\int f_s \,\mathrm{d}\beta_s$, where β is an fBm with $\delta > 1/2$ and $f\colon [0,T] \to \mathbb{R}$ is a possibly distributional process, see the upcoming Proposition 4.7. Remarkably, this not only allows to define $\Gamma^w b$ as a random field, but also relates its space-time Hölder regularity to that of $T^w b$, with no restrictions on the value $H \in (0,1)$ (cf. Theorem 4.12). With this tool at hand, we can then apply the already existing results for $T^w b$ in order to define $\Gamma^w b$ and solve the associated equation (4.4).

Our approach presents several nice features: it identifies sufficient analytic conditions for w to regularise the SDE, in the form of regularity requirements for $T^w b$; it provides a pathwise solution concept for (4.2) in terms of equation (4.4), which should be regarded as a random nonlinear YDE rather than an SDE; no adaptedness requirements are needed to guarantee uniqueness; finally, the existence of an associated Lipschitz flow is a direct consequence of the nonlinear YDE theory.

As an illustrative example of the kind of results we will obtain, let us provide the following statement, which can be regarded as the main result of this chapter.

Theorem 4.1. *Let $b \in B^\alpha_\infty$ with $\alpha \in \mathbb{R}$, β and w be sampled as independent fBms of parameters respectively δ and H with $\delta > 1/2$; further assume that*

$$\alpha > 2 - \frac{1}{H}\left(\delta - \frac{1}{2}\right). \tag{4.6}$$

Then strong existence and path-by-path wellposedness holds for the SDE (4.2), which moreover admits a C^1_{loc} flow of solutions.

The exact meaning of the statement will be clear from Section 4.2.2, which contains the rigorous definition of solutions to the SDE, as well as the concepts of strong existence and path-by-path wellposedness.

Remark 4.2. Let us make some observations on condition (4.6):

i. For any fixed $\delta > 1/2$ and $\alpha \in \mathbb{R}$ arbitrarily low, we can find $H > 0$ small enough such that condition (4.6) is satisfied; in particular, given an arbitrarily irregulr drift b, we can find an additive perturbation w restoring wellposedness of the SDE (4.2) and yielding a flow of solutions (in fact, the flow can be made arbitrarily regular, see Theorem 4.34).

ii. With a slight abuse, we can consider the fBm of parameter $\delta = 1$ to be given by $\beta_t = N t$, where N is a standard Gaussian variable in $\mathbb{R}^m$ (this is the only possible 1-self-similar centered Gaussian process); observe that in the limit $\delta \uparrow 1$ condition (4.6) becomes $\alpha > 2 - \frac{1}{2H}$, namely we recover the first condition in equation (3.43) from Theorem 3.54.

iii. On the other hand, if we set $\delta = 1/2$, condition (4.6) degenerates to $\alpha > 2$, with H not playing any role; thus our techniques do not extend to the case of β sampled as a Bm.

iv. In order to treat non-Lipschitz drifts b, i.e. allow $\alpha < 1$ in condition (4.6), we must impose $\delta > H + 1/2$ (which always enforces $H < 1/2 < \delta$ since $\delta < 1$); in order to handle distributional b, i.e. allow $\alpha < 0$, we must impose $\delta > 2H + 1/2$ (which always enforces $H < 1/4$).

Structure of the chapter. Section 4.1 is devoted to the rigorous definition of $\Gamma^w b$ and the study of its regularity, first analytically and then stochastically. We can then apply the results to establish wellposedness of the SDE (4.2) in Section 4.2, which contains the proof of Theorem 4.1, as well as a discussion on further generalizations allowing to cover (4.1) as well. Finally, Section 4.3 points out some current open problems and future directions.

Notations and conventions. Let us make here slightly more explicit the setting of equation (4.2), which will hold throughout the entire chapter. We will always work on a finite time interval $[0,T]$ and β will always be sampled as an $\mathbb{R}^m$-valued fBm of parameter $\delta\in(1/2,1)$. Instead the perturbation w will be always considered to be a deterministic, $\mathbb{R}^d$-valued continuous path (although, up to conditioning, it can be sampled as another process independent of β, like in the setting of Theorem 4.1). The distributional field b will be matrix-valued, $b\in\mathcal{D}'(\mathbb{R}^d;\mathbb{R}^{d\times m})$ (say regular for the moment), so that everyhing is consistent. The Hölder regularity of w will be measured by means of the parameter $H\in(0,1)$ and we will always tacitly assume $H\leqslant\delta$, which is natural in view of our purposes: in order to regularise the SDE, we expect w to be rougher than β.

Like in Chapter 3, we will always assume for simplicity $w_0=0$, so that whenever we apply the change of variables $\theta=x-w$ we don't need to to modify the initial condition $x_0\in\mathbb{R}^d$.

Since β now denotes an fBm, we will not be allowed to use it in the notations for function spaces; except for this detail, we will adopt mostly all the same notations from Chapters 1 and 3 concerning function spaces $L^q_t B^\alpha_\infty$, $C^\gamma_t C^{\eta,\lambda}_x$, etc. To this we add the notation C^n_x for the space of continuous and bounded functions with continuous and bounded derivatives up to order n, the notation $\mathcal{D}=C^\infty_c$ for the space of test functions and $\mathcal{D}'$ for its dual.

When dealing with continuous paths, to the established notations $w\in C^0_t$, $w\in C^\alpha_t$ we will add also $w\in C^{\alpha-}_t:=\cap_{\varepsilon>0}C^{\alpha-\varepsilon}_t$ to denote the space of all functions with are "almost" α-Hölder continuous (it has a natural Frechét topology induced by the family of seminorms $\|\cdot\|_{C^{\alpha-\varepsilon}}$). Let us also recall that we will frequently use the incremental notation $x_{s,t}=x_t-x_s$.

We will use both $B(0,R)$, $B_R(0)$ and B_R to denote the ball of radius R in $\mathbb{R}^n$ for suitable n.

As before, even when not stated explicitly, whenever we work with a stochastic process (e.g. β) we assume the existence of an underlying complete probability space $(\Omega,\mathcal{F},\mathbb{P})$; if an unspecified filtration $\mathcal{F}_t$ appears, it can be taken as the natural one generated by the process. The notations for $\mathbb{E}$, $L^p_\omega E=L^p(\Omega;E)$ and $L^p_\omega=L^p_\omega\mathbb{R}^n$ are the same as in Chapter 3.

4.1 Multiplicative averaged fields

The main purpose of this section is to give meaning to what we will always refer in the following to as a *multiplicative averaged field*, formally given by formula

$$\Gamma^w_{s,t}b(x)=\int_s^t b(x+w_r)\,\mathrm{d}h_r \tag{4.7}$$

for suitable continuous pahts w and h. We will proceed as follows: we will first define $\Gamma^w b$ *analytically* as a distribution, for any pair of regular enough paths w and h (not necessarily sampled as stochastic processes), in Section 4.1.1; then we will show in Section 4.1.2 how, in the case where $h=\beta(\omega)$ is sampled as an fBm of parameter $\delta>1/2$, we can weaken the regularity on w by defining $\Gamma^w b$ *probabilistically*; finally in Section 4.1.3 we develop some interpolation estimates that will be needed in order to relate the regularity of $\Gamma^w b$ to that of $T^w b$.

For notational simplicity, throughout this section (and the chapter in general), we will often write regularity assumptions of the form $w\in C^H_t$, $h\in C^\delta_t$; however it is easy to check that, since the parameters H and δ always appear in strict inequality, everything generalizes to the case $w\in C^{H-}_t$, $h\in C^{\delta-}_t$, so that it can be applied to the case where w and h are sampled as suitable fBms.

4.1.1 Analytic definition

The purpose of this section is to analytically define the multiplicative averaging operator Γ^w as a map from $\mathcal{D}'$ to itself; to this end, we need to impose some regularity on w and h, namely require $H+\delta>1$ (recall that we are also always imposing $H\leqslant\delta$). We will shortly see in Section 4.1.2 that, for β sampled as a fBm, we can drop the condition $H+\delta>1$, by defining $\Gamma^w b$ as a random field.

Recall that for any $v\in\mathbb{R}^d$, τ^v denotes the translation operator by v, i.e. $\tau^v b(\cdot)=b(\cdot+v)$.

Lemma 4.3. *Let $\alpha\in\mathbb{R}$, $w\in C^H_t$, $h\in C^\delta_t$ and $\eta\in(0,1]$ such that*

$$\delta+\eta H>1.$$

Then for any $b \in B_\infty^{\alpha+\eta}$ *there exists a unique element of* $C_t^\delta B_\infty^\alpha$*, which we denote by* $\Gamma^w b$ *and which we will refer to as a* ***multiplicative averaged field****, such that for any* $s \leqslant t$ *in* $[0,T]$ *it holds*

$$\|\Gamma^w_{s,t} b - b(\cdot + w_s)\, h_{s,t}\|_{B_\infty^\alpha} \lesssim |t-s|^{\delta+\eta H}.$$

Moreover there exists a constant $C = C(\delta + \eta H, T)$ *such that for any* $b \in B_\infty^{\alpha+\eta}$ *it holds*

$$\|\Gamma^w b\|_{C^\delta B_\infty^\alpha} \leqslant C \|b\|_{B_\infty^{\alpha+\eta}} [\![h]\!]_{C^\delta} (1 + [\![w]\!]_{C^H}). \tag{4.8}$$

In particular, the map $\Gamma^w : b \mapsto \Gamma^w b$ *is an element of* $\mathcal{L}(B_\infty^{\alpha+\eta}; C_t^\delta B_\infty^\alpha)$*. If* $\alpha > 0$*, then* $\Gamma^w b$ *defined as above coincides with the pointwise map defined by the Young integral*

$$(\Gamma^w_{s,t} b)(x) = \int_s^t b(x + w_r) \mathrm{d} h_r. \tag{4.9}$$

Proof. All the statements easily follow from an application of sewing techniques (cf. Lemma 1.1). For any $s \leqslant t$, set $\Xi_{s,t} := (\tau^{w_s} b) h_{s,t} \in B_\infty^\alpha$; it holds $\delta \Xi_{s,u,t} = (\tau^{w_s} b - \tau^{w_u} b) h_{u,t}$ with the estimates

$$\begin{aligned}\|\delta \Xi_{s,u,t}\|_{B_\infty^\alpha} &= \|\tau^{w_s} b - \tau^{w_u} b\|_{B_\infty^\alpha} |h_{u,t}| \lesssim \|b\|_{B_\infty^{\alpha+\eta}} |w_{s,u}|^\eta |h_{u,t}| \\ &\leqslant \|b\|_{B_\infty^{\alpha+\eta}} [\![w]\!]_{C^H}^\eta [\![h]\!]_{C^\delta} |t-s|^{\delta+\eta H},\end{aligned}$$

where we used the basic estimate $\|\tau^y b - \tau^z b\|_{B_\infty^\alpha} \lesssim |y-z|^\eta \|b\|_{B_\infty^{\alpha+\eta}}$ given by Lemma A.7.

The sewing lemma thus implies the existence and uniqueness of $\Gamma^w b$, as well as the bound

$$\|\Gamma^w_{s,t} b - b(\cdot + w_s) h_{s,t}\|_{B_\infty^\alpha} \lesssim \|b\|_{B_\infty^{\alpha+\eta}} [\![w]\!]_{C^H}^\eta [\![h]\!]_{C^\delta} |t-s|^{\delta+\eta H}.$$

We then have

$$\begin{aligned}\|\Gamma^w_{s,t} b\|_{B_\infty^\alpha} &\lesssim \|\tau^{w_s} b\|_{B_\infty^\alpha} |h_{s,t}| + \|b\|_{B_\infty^{\alpha+\eta}} [\![w]\!]_{C^H}^\eta [\![h]\!]_{C^\delta} |t-s|^{\delta+\eta H} \\ &\lesssim |t-s|^\delta \|b\|_{B_\infty^{\alpha+\eta}} [\![h]\!]_{C^\delta} (1 + [\![w]\!]_{C^H}),\end{aligned}$$

which implies the bound (4.8). The last claim follows from the fact that the Young integral in (4.9) corresponds to the sewing of $\langle \Xi_{s,t}, \delta_x \rangle$ and thus must coincide with $\langle \Gamma^w_{s,t} b, \delta_x \rangle$. □

The operator Γ^w behaves similarly to the classical averaging operator T^w; we summarize some of its properties in the following two lemmas, which are exact analogues of Lemmas 3.3 and 3.9 from Chapter 3.

Lemma 4.4. *Let* $\Gamma^w b$ *be given as in Lemma 4.3. Then the following properties hold:*

i *Averaging and space differentiation (in the distributional sense) commute:*

$$\partial_i \Gamma^w b = \Gamma^w \partial_i b \quad \forall\, b \in B_\infty^\alpha,\ i = 1, \ldots, d.$$

ii *Averaging and spatial convolution commute: for any* $K \in C_c^\infty$ *it holds*

$$K * (\Gamma^w b) = \Gamma^w (K * b) \quad \forall\, b \in B_\infty^\alpha.$$

iii *If* b *is compactly supported, then so is* $\Gamma^w b$*, with*

$$\operatorname{supp} \Gamma^w_{s,t} b \subseteq \operatorname{supp} b + B(0, \|w\|_\infty) \quad \forall s \leqslant t \leqslant T$$

Similarly, if b^1 *and* b^2 *coincide on* $B(0,R)$*, then* $\Gamma^w b^1$ *and* $\Gamma^w b^2$ *coincide on* $B(0, R - \|w\|_\infty)$*.*

iv *The operator* Γ^w *can be extended to an operator from* $\mathcal{D}'$ *to itself by the duality formula*

$$\langle \Gamma^w_{s,t} \psi, \varphi \rangle := \langle \psi, \Gamma^{-w}_{s,t} \varphi \rangle \quad \forall \psi \in \mathcal{D}',\ \varphi \in C_c^\infty.$$

Proof. The proof is analogue to that of Lemma 4.3. Indeed, by setting $\Xi[b]_{s,t} := (\tau^{w_s} b) h_{s,t}$, it is immediate to check that

$$\partial_{x_i} \Xi[b] = \Xi[\partial_{x_i} b], \quad K * \Xi_{s,t}[b] = \Xi_{s,t}[K * b]$$

and so the same relations must hold between the respective sewings, proving points *i.* and *ii.*.

The first part of point *iii.* follows from the fact that, for any $s<t$, $\Xi_{s,t}[b]$ is supported on $\operatorname{supp} b + B(0,w_s) \subset \operatorname{supp} b + B(0,\|w\|_\infty)$ and the second part by applying a similar reasoning to $b^1 - b^2$. Finally, it follows from Lemma 4.3 and point *iii.* that $\Gamma^w_{s,t}$ continuously maps C^∞_c into itself; therefore also the dual definition from $\mathcal{D}'$ to itself is meaningful. Whenever ψ and φ are both smooth, we have the relation

$$\langle (\tau^{w_s}\psi) h_{s,t}, \varphi\rangle = \langle \psi, (\tau^{-w_s}\varphi) h_{s,t}\rangle$$

which implies the same relation for the respective sewings, i.e. $\langle \Gamma^w_{s,t}\psi, \varphi\rangle = \langle \psi, \Gamma^{-w}_{s,t}\varphi\rangle$. □

Lemma 4.5. *Let $b\in\mathcal{D}'$ be such that $\Gamma^w b\in C^\gamma_t C^{\eta,\lambda}_x$ for some $\gamma,\lambda\in(0,1)$ and $\eta\in(0,\infty)$. Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers and set $b^\varepsilon = \rho^\varepsilon * b$. Then for any $\varepsilon>0$ it holds $\Gamma^w b^\varepsilon \in C^\gamma_t C^{\eta,\lambda}_x$ with*

$$\|\Gamma^w b^\varepsilon\|_{\gamma,\eta,\lambda} \lesssim \|\Gamma^w b\|_{\gamma,\eta,\lambda}; \tag{4.10}$$

moreover $\Gamma^w b^\varepsilon \to \Gamma^w b$ as $\varepsilon\to 0$ in $C^\gamma_t C^{\eta',\lambda}_x$ for any $\eta'<\eta$.

Proof. It is enough to prove the claim for $\eta\in(0,1)$, as the other cases follow by repeating the same argument for $D^k\Gamma^w b = \Gamma^w D^k b$. The bound (4.10) follows from point *iii.* of Lemma 4.4, since we have

$$\|\Gamma^w b^\varepsilon\|_{\gamma,\eta,R} = \|\rho^\varepsilon * \Gamma^w b\|_{\gamma,\eta,R} \lesssim \|\Gamma^w b\|_{\gamma,\eta,R+\varepsilon} \lesssim R^\lambda \|\Gamma^w b\|_{\gamma,\alpha,\lambda}$$

where we used the fact that ρ^ε is supported in B_ε and $(R+\varepsilon)^\lambda \sim R^\lambda$ since $R\geqslant 1$ and $\varepsilon\in(0,1)$. By properties of convolutions, for any $s<t$ and any $x\in B_R(0)$ it holds

$$|\Gamma^w_{s,t} b^\varepsilon(x) - \Gamma^w_{s,t} b(x)| \lesssim \varepsilon^\eta \|\Gamma^w_{s,t} b\|_{\eta,R+\varepsilon} \lesssim \varepsilon^\eta R^\lambda |t-s|^\gamma \|\Gamma^w b\|_{\gamma,\eta,\lambda};$$

from here, the argument is based on interpolation estimates like in Lemma 3.9. □

Remark 4.6. For simplicity, we focused on the case $b\in B^\alpha_\infty$, as it is the natural class of spaces where to formulate the main results of this chapter. But it is clear that many statements from this specific section, like Lemma 4.3, extend immediately to general $b\in B^\alpha_{p,q}$, as the proofs essentially rely only on the property $\|\tau^y b - \tau^z b\|_{B^\alpha_{p,q}} \lesssim |y-z|^\eta \|b\|_{B^{\alpha+\eta}_{p,q}}$ which is granted by Lemma A.7.

4.1.2 Stochastic estimates

We will now assume that $h=\beta(\omega)$ is sampled as a fractional Brownian motion with $\delta>1/2$, with trajectories in $C^{\delta-}_t$ (recall that all the results from Section 4.1.1 still apply under this regularity condition). In this case, thanks to the more specific probabilistic structure, we can extend the definition of $\Gamma^w b$ to allow less restrictive assumptions of b and w; moreover, we will show that $\Gamma^w b$ inherits the spatial regularity of $T^w b$.

Our starting point is an estimate for Wiener-type integrals due to Hairer and Li [166], which we recall first. In the next statement, for $\kappa\in[0,1]$, $f\in C^{-\kappa}_t$ must be interpreted as $t\mapsto \int_0^t f_r\,\mathrm{d}r \in C^{1-\kappa}_t$; the quantity $\|f\|_{C^{-\kappa}_{[s,t]}}$ then correspond to the $C^{1-\kappa}_t$-norm of $u\mapsto\int_s^u f_r\,\mathrm{d}r$, for $u\in[s,t]$.

Proposition 4.7. (Lemma 3.4 from [166]) *Let β be a fBm with Hurst parameter $\delta>1/2$ and fix $0\leqslant\kappa<\delta-1/2$ as well as $s>0$. Let $r\mapsto f_r$ be smooth and such that f_r with $r\geqslant s$ measurable w.r.t. $\mathcal{F}_s$. Then, for $t\geqslant s$ with $|t-s|\leqslant 1$ and $2\leqslant p<q$, one has the bound*

$$\left\| \int_s^t f_r\,\mathrm{d}\beta_r \right\|_{L^p_\omega} \lesssim \left\| \|f\|_{C^{-\kappa}_{[s,t]}} \right\|_{L^q_\omega} |t-s|^{\delta-\kappa}$$

where $\|f\|_{C^{-\kappa}_{[s,t]}}$ denotes the negative Hölder norm on $[s,t]$.

By linearity and density, this immediately allows to extend the notion of integral against β to any integrand in $L^q_\omega C^{-\kappa}_t$ (satisfying the above measurabily assumption), for any $0\leqslant\kappa<\delta-1/2$; such an integral may no longer agree with the Young one.

In fact, we will not need Proposition 4.7 in its fulls generality, rather only the simpler case where we allow f to be a deterministic (possibly distributional) function.

Proposition 4.8. *Let β be a fBm with Hurst parameter $\delta > 1/2$, $f\colon [0,T] \to \mathbb{R}$ be a smooth deterministic function; for any $s<t$, $\int_s^t f_r \,\mathrm{d}\beta_r$ is a well defined centered Gaussian variable. For any $\gamma > 3/2 - \delta$ there exists a constant $C = C(\gamma,\delta,T)$ such that its variance can be controlled by*

$$\left\| \int_s^t f_r \,\mathrm{d}\beta_r \right\|_{L^2_\omega} \leqslant C \left\| \int_0^\cdot f_r \,\mathrm{d}r \right\|_{C^\gamma} |t-s|^{\gamma+\delta-1} \quad \forall [s,t] \subset [0,T].$$

By density and linearity, this immediately allows to extend the definition of $\int_s^t f_r \,\mathrm{d}\beta_r$ as a centered Gaussian variable satisfying (4.8) to any distribution f such that $\int_0^\cdot f_r \,\mathrm{d}r \in C^\gamma_t$ for some $\gamma > 3/2 - \delta$.

Proof. For smooth f, the fact that $\int_s^t f_r \,\mathrm{d}\beta_r$ is well defined, Gaussian and centered is classical: for instance we can define it by means of Young integrals (so that it is the limit of Gaussian sums of the form $\sum f_{t_i} \beta_{t_i,t_{i+1}}$) or by integration by parts as $\int_s^t f_r \,\mathrm{d}\beta_r = (f\beta)_{s,t} - \int_s^t \beta_r f'_r \mathrm{d}r$. Since f is deterministic, estimate (4.8) then follows from Proposition 4.7 for the choice $p=2$, $q=\infty$ and $\gamma = 1-\kappa$: indeed $\kappa < \delta - 1/2$ is then equivalent to $\gamma > 3/2 - \delta$. The general case follows by standard density arguments. □

Remark 4.9. From an analytic point of view, Propositions 4.7-4.8 are quite magical: if both f and $\mathrm{d}\beta$ are negative distributions, there shouldn't be any way to define their product $f\,\mathrm{d}\beta$ as a distribution (which is somewhat equivalent to defining the process $\int_0^\cdot f \mathrm{d}\beta$). There is however a simple heuristical argument explaining why this works and leads to the above prescriptions on the parameter κ, δ.

Recall that, for $\delta > 1/2$, fBm can be loosely written as $\mathrm{d}\beta = K_{\delta-1/2}\,\mathrm{d}W$, where W is a standard Bm and K_δ is a fractional operator of integral type; in terms of regularity counting, both $K_{\delta-1/2}$ and its dual act as the fractional integral $I^{\delta-1/2}$, in the sense that they (almost) map $W^{s,p}$ into $W^{s+\delta-1/2,p}$. Assuming this to be true, we have

$$\int_s^t f_r \,\mathrm{d}\beta_r = \int_s^t f_r\,(K_{\delta-1/2}\mathrm{d}W)_r = \int_s^t (K^*_{\delta-1/2} f)_r \,\mathrm{d}W_r$$

where the latter integral would be well defined as soon as $K^*_{\delta-1/2} f \in L^2_t$, which roughly amounts to $f \in W^{1/2-\delta,2}_t$; if we already know that $f \in C^{-\kappa}_t$, it then suffices $-\kappa > 1/2 - \delta$, which yields $K^*_{\delta-1/2} f \in C^{\delta-1/2-\kappa}_t$. Assuming further that $K^*_{\delta-1/2}$ has the right "recentering" property, namely that $(K^*_{\delta-1/2} f)_s = 0$, by Itô isometry we find

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t f_r \,\mathrm{d}\beta_r\right|^2\right]^{\frac{1}{2}} &= \mathbb{E}\left[\int_s^t |(K^*_{\delta-1/2} f)_r|^2 \mathrm{d}r\right]^{\frac{1}{2}} \\ &\lesssim \|K^*_{\delta-1/2} f\|_{C^{\delta-1/2-\kappa}}\, \mathbb{E}\left[\int_s^t |r|^{2(\delta-k)-1}\,\mathrm{d}r\right]^{\frac{1}{2}} \lesssim \|f\|_{C^{-\kappa}} |t-s|^{\delta-\kappa}\end{aligned}$$

which is exactly estimate (4.8). In fact, after writing the work [139], we realized that in the special case of deterministic f, a sharper version of Proposition 4.8 was already available in the literature, see Theorem 3.3 from [185].

Proposition 4.8 immediately extends to the case where $f \in \mathcal{D}'(\mathbb{R};\mathbb{R}^{d\times m})$ and β is a $\mathbb{R}^m$-valued fBm by reasoning componentwise. Keeping in mind that we are interested in estimating $\Gamma^w b$, setting $f_r = b(x+w_r)$, we see that $\Gamma^w_{s,t} b(x) := \int_s^t f_r \,\mathrm{d}\beta_r$ is a well-defined Gaussian random variable as soon as

$$\int_0^\cdot b(x+w_r)\mathrm{d}r = T^w b(\cdot,x) \in C^\gamma_t \text{ for some } \gamma > \frac{3}{2} - \delta.$$

The above argument however only concerns the variable $\Gamma^w_{s,t} b(x)$ at fixed s,t,x; defining the whole random field $(t,x) \mapsto \Gamma^w b(t,x)$ and establishing its space-time Hölder regularity requires a bit more work. We start by deriving a priori estimates in the case where $\Gamma^w b$ is alearedy rigorously well-defined.

Lemma 4.10. *Let $b\in C^2_x$, β be an fBm of parameter $\delta>1/2$ and $w\in C^H_t$ a deterministic path such that $\delta+H>1$. Define the multiplicative averaged field $\Gamma^w b$ pathwise as in Lemma 4.3; namely, for any $\omega\in\Omega$ such that $\beta(\omega)\in C^{\delta-}_t$, set*

$$\Gamma^w_{s,t}b(x)(\omega):=\int_s^t b(x+w_r)\mathrm{d}\beta_r(\omega). \tag{4.11}$$

Then for any $\gamma>3/2-\delta$ there exist constants μ,K, only depending on δ,γ,T, such that for any $\eta,\lambda\in(0,1)$ it holds:

$$\mathbb{E}\left[\exp\left(\mu\frac{\|\Gamma^w_{s,t}b(x)\|^2}{\|T^w b\|^2_{C^\gamma C^{0,\lambda}}|t-s|^{2(\gamma+\delta-1)}R^{2\lambda}}\right)\right]\leqslant K\quad\forall x\in B_R,\forall R\geqslant 1, \tag{4.12}$$

$$\mathbb{E}\left[\exp\left(\mu\frac{\|\Gamma^w_{s,t}b(x)-\Gamma^w_{s,t}b(y)\|^2}{\|T^w b\|^2_{C^\gamma C^{\eta,\lambda}}|x-y|^{2\eta}|t-s|^{2(\gamma+\delta-1)}R^{2\lambda}}\right)\right]\leqslant K\quad\forall x,y\in B_R,\forall R\geqslant 1, \tag{4.13}$$

$$\mathbb{E}\left[\exp\left(\mu\frac{\|\nabla\Gamma^w_{s,t}b(x)\|^2}{\|T^w b\|^2_{C^\gamma C^{1+\eta,\lambda}}|x-y|^{2\eta}|t-s|^{2(\gamma+\delta-1)}R^{2\lambda}}\right)\right]\leqslant K\quad\forall x,y\in B_R,\forall R\geqslant 1. \tag{4.14}$$

Proof. The results are a direct application of Proposition 4.8. It follows from Lemma 4.3, for the choice $\alpha=\eta=1$, that $\Gamma^w b\in C^{\delta-}_t C^1_x$, as well as $T^w b\in C^1_t C^2_x$, so that $\Gamma^w_{s,t}b(x)$ and $\nabla\Gamma^w_{s,t}b(x)$ are classically well defined. Since they are also centered Gaussian variables, in order to obtain estimates of the form (4.12)-(4.14) it is enough to control their variance; it holds

$$\begin{aligned}\|\Gamma^w_{s,t}b(x)\|_{L^2_\omega}&=\left\|\int_s^t b(x+w_r)\mathrm{d}\beta_r\right\|_{L^2_\omega}\\&\lesssim\left\|\int_0^\cdot b(x+w_r)\mathrm{d}r\right\|_{C^\gamma}|t-s|^{\gamma+\delta-1}\lesssim\|T^w b\|_{C^\gamma C^{0,\lambda}}|t-s|^{\gamma+\delta-1}R^\lambda\end{aligned}$$

whenever $x\in B_R$; similarly for $x,y\in B_R$ we have

$$\begin{aligned}\|\Gamma^w_{s,t}b(x)-\Gamma^w_{s,t}b(y)\|_{L^2_\omega}&\lesssim\|T^w b(\cdot,x)-T^w b(\cdot,y)\|_{C^\gamma}|t-s|^{\gamma+\delta-1}\\&\lesssim\|T^w b\|_{C^\gamma C^{\eta,\lambda}}|x-y|^\eta|t-s|^{\gamma+\delta-1}R^\lambda.\end{aligned}$$

Finally (4.14) follows from $\nabla\Gamma^w b=\Gamma^w\nabla b$ and an application of 4.13 with b replaced by ∇b (we are also using the fact that $T^w\nabla b=\nabla T^w b$). □

We can now combine the estimates from Lemma 4.10 with suitable versions of the Garsia-Rodemich-Rumsey lemma like those in Appendix A.5; we then obtain the following result.

Corollary 4.11. *Let b^1, b^2, w^1, w^2, β be as in Lemma 4.10, $\gamma>3/2-\delta$ and $\eta,\lambda\in(0,1)$ be fixed parameters. Then for any choice of $(p,\gamma',\eta',\lambda')$ such that*

$$p\in[1,\infty],\quad\gamma'<\gamma+\delta-1,\quad\eta'<\eta,\quad\lambda<\lambda'$$

there exists a constant C (depending on T and the above parameters) such that

$$\mathbb{E}[\|\Gamma^{w^1}b^1-\Gamma^{w^2}b^2\|^p_{C^{\gamma'}C^{\eta',\lambda'}}]\leqslant C\,\|T^{w^1}b^1-T^{w^2}b^2\|^p_{C^\gamma C^{\eta,\lambda}}. \tag{4.15}$$

Proof. As the multiplicative averaging acts linearly, it suffices to show the statement for a single $T^w b$. Combining estimates (4.12)-(4.13) of Lemma 4.10 with Corollary A.30 and Remark A.31 from Appendix A.5, for any $\varepsilon>0$ we fine the existence of constant $\mu',K'>0$ (depending on ε and the previous parameters, but not on the specific $T^w b$ in consideration) such that

$$\mathbb{E}\left[\exp\left(\mu'\frac{\|\Gamma^w b\|^2_{C^{\gamma+\delta-1-\varepsilon}C^{\eta-\varepsilon,\lambda+\varepsilon}}}{\|T^w b\|^2_{C^\gamma C^{\eta,\lambda}}}\right)\right]\leqslant K'.$$

Moments estimates of the form (4.15) for any $p\in[1,\infty)$ then immediately follow by choosing $\varepsilon>0$ small enough in function of γ',η',λ'. $\square$

We are now finally ready to extend the definition of $\Gamma^w b$ to suitable distributional fields. The next statement represents the main result of this section; for completeness, it includes some of the properties already established so far.

Theorem 4.12. *Let β be a fBm of Hurst parameter $\delta>1/2$. Then for any deterministic $b\in C^2_x$ and $w\in C^H_t$ with $H+\delta>1$, it's possible to define the averaged field $\Gamma^w b$ in (4.7) pathwise as a Young integral with $h=\beta(\omega)$; $\Gamma^w b$ can be regarded as a random field from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}^d$.*

The definition extends continuously in a unique way to any pair (b,w) with $b\in\mathcal{D}'$, $w\in C^0_t$ such that $T^w b\in C^\gamma_t C^{\eta,\lambda}_x$ for some $\gamma>3/2-\delta$, $\eta,\lambda\in(0,1)$. In that case

$$\Gamma^w b\in L^p(\Omega; C^{\gamma'}_t C^{\eta',\lambda'}_x)\quad \forall\, p<\infty,\ \gamma'<\gamma+\delta-1,\ \eta'<\eta,\ \lambda'>\lambda$$

and there exists $C>0$ (depending on all the above parameters) such that for any (b^i,w^i) satisfying $T^{w^i}b^i\in C^\gamma_t C^{\eta,\lambda}_x$ it holds

$$\mathbb{E}[\|\Gamma^{w^1}b^1-\Gamma^{w^2}b^2\|^p_{C^{\gamma'}C^{\eta',\lambda'}}]\leqslant C\|T^{w^1}b^1-T^{w^2}b^2\|^p_{C^\gamma C^{\eta,\lambda}}. \tag{4.16}$$

More generally, estimate (4.16) holds replacing η', η with $n+\eta'$, $n+\eta$ respectively, for any $n\in\mathbb{N}$; namely, $\Gamma^w b$ inherits higher space regularity from $T^w b$.

Proof. The proof is divided in two natural steps: we will first show that, thanks to Corollary 4.11, we can extend the definition of $\Gamma^w b$ to the case of regular b and continuous (but not necessarily Hölder regular) w; then we will show that, under the assumption that $T^w b$ is sufficiently regular, the definition further extends to the case of distributional b.

Step 1. Let $b\in C^2_x$, w^n be a sequence in C^1_t such that $w^n\to w$ in C^0_t. Our aim is to show that the sequence $\Gamma^{w^n}b$ is Cauchy in a suitable weighted Hölder space and thus admits a unique limit, which we define to be $\Gamma^w b$. In particular, while we cannot define anymore the field $\Gamma^w b$ analytically as done in Section 4.1.1, it is still well defined as a random variable.

Since $b\in C^2_x$, for any $n,m\in\mathbb{N}$ we have the estimates

$$\left|\int_s^t b(x+w^n_r)\mathrm{d}r-\int_s^t b(x+w^m_r)\mathrm{d}r\right|\leqslant\|b\|_{C^1}\int_s^t|w^n_r-w^m_r|\mathrm{d}r\leqslant\|b\|_{C^1}\|w^n-w^m\|_{C^0}|t-s|$$

and similarly, for fixed n and any $x,y\in\mathbb{R}^d$,

$$\left|\int_s^t b(x+w^n_r)\,\mathrm{d}r-\int_s^t b(y+w^n_r)\,\mathrm{d}r\right|\leqslant\|b\|_{C^1}|x-y||t-s|.$$

One can then apply triangular inequality and interpolate the two inequalities above to deduce that, for any $\eta\in(0,1)$, it holds

$$|T^{w^n}_{s,t}b(x)-T^{w^m}_{s,t}b(y)|\lesssim\|b\|_{C^1}|x-y|^\eta\|w^n-w^m\|^{1-\eta}_\infty|t-s|.$$

Since $w^n\to w$ in C^0_t, the sequence $\{w^n\}_n$ is Cauchy in C^0_t; by the above estimate, $\{T^{w^n}b\}_n$ is also Cauchy in $C^1_t C^\eta_x$, for any $\eta<1$. Combined with (4.15), this implies that for any $\gamma'<\delta$, $\eta'<\eta$, $\lambda'>0$ and $p\in[2,\infty)$ it holds

$$\mathbb{E}[\|\Gamma^{w^n}b-\Gamma^{w^m}b\|^p_{C^{\gamma'}C^{\eta',\lambda'}}]\lesssim\|T^{w^n}b-T^{w^m}b\|^p_{C^1C^\eta}\lesssim\|b\|_{C^1}\|w^n-w^m\|^{1-\eta}_\infty.$$

Therefore the sequence $\{\Gamma^{w^n}b\}_n$ is Cauchy in $L^p(\Omega;C^{\gamma'}_t C^{\eta',\lambda'}_x)$ and it admits a unique limit, which we define to be $\Gamma^w b$. It follows from the arguments above that this is a good definition, as it does not depend on the chosen sequence $\{w_n\}_n$ such that $w_n\to w$. By construction, inequality (4.16) extends to any pairs (w^i,b^i) with $w^i\in C^0_t$ and $b^i\in C^2_x$.

More generally, by iterating the reasoning to $D^k b$ for $k\leqslant n$, the above procedure shows that if $b\in C^{n+1}_x$ and w is a continuous path, then $\Gamma^w b$ belongs to $C^{\gamma'}_t C^{n+\eta',\lambda'}_x$.

Step 2. We now want to pass to the case in which b is distributional, w is continuous and $T^w b\in C^\gamma_t C^{\eta,\lambda}_x$ (resp. $C^\gamma_t C^{n+\eta,\lambda}_x$) for some $\gamma>3/2-\delta$.

By Lemma 3.9, we can choose a family of mollifiers $\{\rho_\varepsilon\}_{\varepsilon>0}$, a parameter $\vartheta>0$ arbitrarily small and a sequence $\varepsilon_n\to 0$ such that, setting $b_n=b^{\varepsilon_n}=\rho_{\varepsilon_n}*b$, it holds that $T^w b_n\to T^w b$ in $C^\gamma_t C^{\eta-\vartheta}_x$. In particular, $\{T^w b_n\}_n$ is a Cauchy sequence in $C^\gamma_t C^{\eta-\vartheta,\lambda}_x$ and choosing ϑ such that $\gamma-\vartheta>3/2-\delta$, by the previous step $\Gamma^w b_n$ are well defined random fields; moreover for any $\gamma'<\gamma+\delta-\vartheta-1$, $\eta'<\eta-\vartheta$, $\lambda'>\lambda$ and $p\in[2,\infty)$ it follows from Corollary 4.11 that they satisfy

$$[\|\Gamma^w b_n-\Gamma^w b_m\|^p_{C^{\gamma'}C^{\eta',\lambda'}}]\lesssim\|T^w b_n-T^w b_m\|^p_{C^\gamma C^{\eta-\vartheta,\lambda}}. \tag{4.17}$$

This implies that $\{\Gamma^w b_n\}_n$ is a Cauchy sequence in $L^p(\Omega;C^{\gamma'}_t C^{1+\eta',\lambda'}_x)$ and thus admits a unique limit, which we define to be $\Gamma^w b$. It follows from Lemma 3.9 that $\Gamma^w b$ does not depend on the chosen family of mollifiers. More generally, given any sequence of smooth functions b_n s.t. $T^w b_n\to T^w b$ in $C^\gamma_t C^{\eta-\vartheta,\lambda}_x$, the above estimates show that the associated multiplicative averaged fields $\Gamma^w b_n$ must converge to $\Gamma^w b$. Moreover, for any pair of random fields $\Gamma^{w_1}b_1$, $\Gamma^{w_2}b_2$ defined in this way, for w^i continuous paths and b_i possibly distributional fields, we have the inequality

$$\mathbb{E}[\,\|\Gamma^{w_1}b_1-\Gamma^{w_2}b_2\|^p_{C^{\gamma'}C^{\eta',\lambda'}}]\lesssim\|T^{w_1}b_1-T^{w_2}b_2\|^p_{C^\gamma C^{\eta,\lambda}}$$

which can be rephrased as the fact that the multiplicative averaging, seen as a map $T^w b\mapsto\Gamma^w b$ from $C^\gamma_t C^{\eta,\lambda}_x$ to $L^p(\Omega;C^{\gamma'}_t C^{\eta',\lambda'}_x)$, is linear and continuous.

The general case of $T^w b\in C^\gamma_t C^{n+\eta,\lambda}_x$ follows as before by iterating the reasoning to the derivatives $D^k T^w b=T^w D^k b$. □

Remark 4.13. If $w\in C^H_t$ with $\delta+H>1$, the procedure from Theorem 4.12 is consistent with the one from Section 4.1.1, namely the random field $\Gamma^w b$ is a regular representative of the random distribution defined pathwise by means of Lemma 4.3.

Remark 4.14. Several properties satisfied by the analytical definition of $\Gamma^w b$ from Lemma 4.4 extend by the approximation procedure to the more general definition of Theorem 4.12, once they are interpreted as equalities between random variables. For instance it is still true that, for $K\in C^\infty_c$, $K*\Gamma^w b=\Gamma^w(K*b)$; similarly, if both $T^w b$ and $T^w\nabla b$ are regular enough, then $\Gamma^w\nabla b=\nabla\Gamma^w b$.

Remark 4.15. The proof of Theorem 4.12 also contains the following fact: if $T^w b\in C^\gamma_t C^{n+\eta,\lambda}_x$, then it's possible to find a sequence (b^n,w^n) with $b^n\in C^\infty_x$, $w^n\in C^1_t$ such that $b^n\to b$ in the sense of distributions, $w^n\to w$ in C^0_t and $\Gamma^{w^n}b^n\to\Gamma^w b$ in $L^p(\Omega;C^{\gamma'}_t C^{n+\eta',\lambda'}_t)$ for any $\gamma'<\gamma+H-1$, $\eta'<\eta$ and $\lambda'>\lambda$.

4.1.3 Regularity in terms of classical averaged field

With Theorem 4.12 at hand, we are able to relate the regularity of $\Gamma^w b$ to that of $T^w b$, although in a slightly abstract fashion. The next step is to derive explicit conditions on $T^w b$, ensuring that $\Gamma^w b$ is regular enough to apply the "nonlinear Young machinery"; we start with a simple observation.

Remark 4.16. By Theorem 4.12, if $T^w b\in C^\gamma_t C^{1+\eta,\lambda}_x$ for any $\eta\in(0,1]$ and any $\lambda>0$ and it holds

$$\gamma>\frac{3}{2}-\delta,$$

then it's possible to find parameters γ', η', λ' so that $\mathbb{P}$-a.s. $\Gamma^w b\in C^{\gamma'}_t C^{1+\eta',\lambda'}_x$ and

$$\gamma'>\frac{1}{2},\quad \gamma'(1+\eta')>1,\quad \eta'+\lambda'\leqslant 1,$$

namely the usual conditions under which we know how to solve the associated nonlinear Young equation (e.g. by Corollary 1.24).

Next, we need some conditions relating the regularity of $T^w b$ in a suitable space-time Hölder scale to another. This is accomplished by means of the following more general interpolation estimate in weighted Hölder spaces, which is of interest on its own.

Lemma 4.17. *Let $f \in C_x^{\eta,\lambda} \cap B_\infty^\alpha$ with $\alpha \in \mathbb{R}$, $\beta, \lambda > 0$ and $\eta > \alpha$. For any $\theta \in [0,1]$, set*

$$\eta_\theta := (1-\theta)\alpha + \theta\eta, \quad \lambda_\theta := \theta\lambda.$$

Then for any $\theta \geqslant -\alpha/(\eta-\alpha)$ it holds

$$\|f\|_{C^{\eta_\theta,\lambda_\theta}} \lesssim \|f\|_{B_\infty^\alpha}^{1-\theta} \|f\|_{C^{\eta,\lambda}}^\theta.$$

Proof. Although weighted Hölder (and more generally weighted Besov) spaces have already appeared in the literature (see e.g. [221]), we haven't found results for our specific setting, so we give a self-contained proof.

For simplicity, we will only consider the case $\eta \in (0,1]$, the general one being similar. Also, we will only consider $\bar{\theta} = -\alpha/(\eta-\alpha)$, namely we will provide a control on $\|f\|_{C^{0,\lambda_{\bar{\theta}}}}$; the general inequality then follows by interpolating between $\|f\|_{C^{0,\lambda_{\bar{\theta}}}}$ and $\|f\|_{C^{\eta,\lambda}}$, which can be done in an elementary way.

Let us start by providing a bound in $\|\Delta_n f\|_{C^{0,\lambda_{\bar{\theta}}}}$ for $n \in \mathbb{N}$. Recall that $\Delta_n f = K_n * f$ where $K_n(x) = 2^{nd} K(2^n x)$ for a function $K \in \mathcal{S}$ satisfying $\int_{\mathbb{R}^d} K(x) \mathrm{d}x = 0$; therefore

$$\begin{aligned}
|\Delta_n f(x)| &= \left| \int_{\mathbb{R}^d} K(y)[f(x+2^{-n}y) - f(x)]\,\mathrm{d}y \right| \\
&\lesssim \int_{\mathbb{R}^d} |K(y)|\, \|f\|_{C^{\eta,\lambda}} 2^{-n\eta} |y|^\eta (|x|^\lambda + 2^{-n\lambda}|y|^\lambda)\,\mathrm{d}y \\
&\lesssim \|f\|_{C^{\eta,\lambda}} 2^{-n\eta} (1+|x|^\lambda).
\end{aligned}$$

As the bound holds for any $x \in \mathbb{R}^d$, we conclude that $\|\Delta_n f\|_{C^{0,\lambda}} \lesssim \|f\|_{C^{\eta,\lambda}} 2^{-n\eta}$; recall that from the definition of $f \in B_\infty^\alpha$ we also have $\|\Delta_n f\|_{C^0} \leqslant \|f\|_{B_\infty^\alpha} 2^{-n\alpha}$.

Now consider $x \in B_R(0)$; we can estimate $|f(x)|$ by means of the series of LP blocks as follows:

$$\begin{aligned}
|f(x)| &\leqslant \sum_{n<N} |\Delta_n f(x)| + \sum_{n \geqslant N} |\Delta_n f(x)| \\
&\leqslant \sum_{n<N} \|\Delta_n f\|_{C^0} + \sum_{n \geqslant N} \|\Delta_n f\|_{C^{0,\lambda}} R^\lambda \\
&\lesssim \|f\|_{B_\infty^\alpha} \sum_{n<N} 2^{-n\alpha} + \|f\|_{C^{\eta,\lambda}} R^\lambda \sum_{n \geqslant N} 2^{-n\eta} \\
&\lesssim \|f\|_{B_\infty^\alpha} 2^{-N\alpha} + \|f\|_{C^{\eta,\lambda}} R^\lambda 2^{-N\eta}.
\end{aligned}$$

Choosing N such that $2^{N(\eta-\alpha)} \sim \|f\|_{C^{\eta,\lambda}} R^\lambda / \|f\|_{B_\infty^\alpha}$ then yields

$$\sup_{x \in B_R(0)} |f(x)| \lesssim \|f\|_{B_\infty^\alpha}^{1-\bar{\theta}} \|f\|_{C^{\eta,\lambda}}^{\bar{\theta}} R^{\lambda_{\bar{\theta}}} \quad \forall R \geqslant 1$$

and thus the conclusion. □

Remark 4.18. We imposed the restriction $\theta \geqslant -\alpha/(\eta-\alpha)$ because technically we haven't defined the spaces $C_x^{\eta,\lambda}$ for $\eta < 0$. This could be accomplished by working with Littlewood-Paley blocks, imposing that $f \in C_x^{\eta,\lambda}$ if $\|f\|_{C^{\eta,\lambda}} := \sup_n \{2^{n\eta} \|\Delta_n f\|_{C^{0,\lambda}}\} < \infty$; in that case, one could extend the interpolation inequality from Lemma 4.17 for any $\theta \in [0,1]$. We refrain from going into further detail on the topic since it is outside the scope of our applications.

Corollary 4.19. *Let $b \in B_\infty^\alpha$ be such that $T^w b \in C_t^\gamma C_x^{\eta,\lambda}$ with $\alpha < \eta$ and set η_θ, λ_θ as in Lemma 4.17; also define $\gamma_\theta := 1 - \theta + \theta\gamma$. Then for any $\theta \in [0,1]$ with $\theta \geqslant -\alpha/(\eta-\alpha)$ it holds*

$$\|T^w b\|_{C^{\gamma_\theta} C^{\eta_\theta,\lambda_\theta}} \lesssim \|b\|_{B_\infty^\alpha}^{1-\theta} \|T^w b\|_{C^\gamma C^{\eta,\lambda}}^\theta.$$

Proof. Recall that if $b \in B_\infty^\alpha$, then $T^w b \in C_t^1 B_\infty^\alpha$ with $\|T_{s,t}^w b\|_{B_\infty^\alpha} \leqslant |t-s| \|b\|_{B_\infty^\alpha}$; combined with Lemma 4.17, this implies

$$\|T_{s,t}^w b\|_{C^{\eta_\theta,\lambda_\theta}} \lesssim \|T_{s,t}^w b\|_{B_\infty^\alpha}^{1-\theta} \|T_{s,t}^w b\|_{C^{\eta,\lambda}}^\theta \lesssim \|b\|_{B_\infty^\alpha}^{1-\theta} \|T^w b\|_{C^\gamma C^{\eta,\lambda}}^\theta |t-s|^{1-\theta+\theta\lambda}$$

which readily implies the conclusion. □

Remark 4.20. We conclude this section with a short observation for future use, concerning the regularity of the process $t \mapsto \Gamma^w b(t, x_0)$ at a fixed $x_0 \in \mathbb{R}^d$. If $T^w b \in C_t^\gamma C_{\mathrm{loc}}^0$ for some $\gamma > 3/2 - \delta$, then for any $\gamma' < \gamma + \delta - 1$ and any $x_0 \in \mathbb{R}^d$ there exists $\mu = \mu(x_0, \gamma', \gamma, \delta)$ such that

$$\mathbb{E}[\exp(\mu \|\Gamma^w b(\cdot, x_0)\|_{C^{\gamma'}}^2)] < \infty;$$

indeed this follows immediately from estimate (4.12) and Corollary A.27.

4.2 Applications to perturbed multiplicative SDEs

Having successfully defined $\Gamma^w b$, we are now ready to study the associated SDE.

We will first show in Section 4.2.1 how the nonlinear Young interpretation of the equations is consistent with the standard one, whenever w, h and b are regular enough, but $\Gamma^w b$ already provides useful information in order to establish uniqueness of solutions. Then in Section 4.2.2 we pass to the more interesting case where the equation doesn't have a meaningful standard interpretation anymore (due to lack of regularity of either b or w); this part contains the main results of the chapter and shows the efficiency of the nonlinear Young machinery in solving the SDE and quantifying the regularising effect of the perturbation w. Finally, we discuss in Section 4.2.3 some further generalizations of the main results.

4.2.1 Classical Young equations as averaged equations

The content of this section, similarly to that of Section 4.1.1, is entirely analytic and holds also when β is replaced by any deterministic path $h \in C_t^\delta$ (as before, all the statements generalize immediately to the $h \in C_t^{\delta-}$ as well). As before, we will always tacitly assume $H \leqslant \delta$.

We start by showing that the nonlinear YDE formulation of the problem is a natural generalisation of the original one, provided b and w are regular enough.

Lemma 4.21. *Let $b \in C_x^2$, $w \in C_t^H$ and $h \in C_t^\delta$ with $\delta > 1/2$, $\delta + H > 1$. Then for any $x_0 \in \mathbb{R}^d$ there exists a unique solution $x \in C_t^H$ to the perturbed Young differential equation*

$$x_t = x_0 + \int_0^t b(x_s) \mathrm{d}h_s + w_t \quad \forall t \in [0, T]; \tag{4.18}$$

in particular, $x = \theta + w$, where $\theta \in C_t^\delta$ is the unique solution to the nonlinear YDE

$$\theta_t = \theta_0 + \int_0^t \Gamma^w b(\mathrm{d}s, \theta_s). \tag{4.19}$$

For any $\alpha \in (0,1)$ satisfying $\delta + \alpha H > 1$ there exists a constant $C = C(\alpha, \delta, H, T)$ such that θ satisfies the a priori estimate

$$[\![\theta]\!]_{C^\delta} \leqslant C(1 + \|b\|_{C^\alpha}^2 [\![h]\!]_{C^\delta}^2)(1 + [\![w]\!]_{C^H}). \tag{4.20}$$

Proof. It is easy to check that $x \in C_t^H$ solves (4.18) if and only if $\theta = x - w \in C_t^H$ satisfies

$$\theta_t = \theta_0 + \int_0^t b(\theta_s + w_s) \mathrm{d}h_s = \theta_0 + \int_0^t \tilde{b}(s, \theta_s) \mathrm{d}h_s \qquad \forall t \in [0, T]$$

where $\tilde{b}(t, z) := b(z + w_t)$; by properties of Young integrals, any such θ must also belong to C_t^δ. The drift $\tilde{b}$ satisfies

$$|\tilde{b}(t, z_1) - \tilde{b}(s, z_2)| + |\nabla \tilde{b}(t, z_1) - \nabla \tilde{b}(s, z_2)| \lesssim \|b\|_{C^2} |z_1 - z_2| + \|b\|_{C^2} [\![w]\!]_{C^H} |t - s|^H$$

which by classical results implies existence and uniqueness of solutions to the YDE associated to $\tilde{b}$ in the class C_t^δ, see for instance Theorem 2.1 from [224] or Section 3 from [72].

In order to show that θ solves (4.19), it is enough to prove that

$$\int_0^\cdot b(w_s + \theta_s) \mathrm{d}h_s = \int_0^\cdot \Gamma^w b(\mathrm{d}s, \theta_s).$$

Since $b \in C_x^2$ and $\delta + H > 1$, by Lemma 4.3 we have $\Gamma^w b \in C_t^\delta C_x^1$ and the nonlinear Young integral $\int_0^\cdot \Gamma^w b(\mathrm{d}s, \theta_s)$ is well defined (because $\theta \in C_t^\delta$ and $\delta > 1/2$). By the respective definition of the two integrals, it holds

$$\left| \int_s^t b(w_r + \theta_r) \mathrm{d}h_r - \int_s^t \Gamma^w b(\mathrm{d}r, \theta_s) \right| = \left| \int_s^t b(w_r + \theta_r) \mathrm{d}h_r \pm b(w_s + \theta_s) h_{s,t} \pm \Gamma_{s,t}^w b(\theta_s) - \int_s^t \Gamma^w b(\mathrm{d}r, \theta_s) \right|$$
$$\lesssim |t-s|^{H+\delta} + \left| b(w_s + \theta_s) h_{s,t} - \int_s^t b(\theta_s + w_r) \mathrm{d}h_r \right| \lesssim |t-s|^{H+\delta}$$

which implies that they must coincide.

We now move on to prove (4.20). For any $0 < \Delta < T$, denote by $[\![\theta]\!]_{\delta,\Delta}$ (resp. $[\![\theta]\!]_{H,\Delta}$) the quantity

$$[\![\theta]\!]_{\delta,\Delta} = \sup_{|t-s| \leqslant \Delta} \frac{|\theta_{s,t}|}{|t-s|^\delta}.$$

By properties of Young integrals, for any $s < t$ such that $|t-s| < \Delta$ it holds

$$\begin{aligned}
|\theta_{s,t}| &= \left| \int_s^t b(w_r + \theta_r) \mathrm{d}h_r \right| \\
&\lesssim |b(w_s + \theta_s) h_{s,t}| + |t-s|^{\delta + \alpha H} [\![b]\!]_{C^\alpha} [\![h]\!]_{C^\delta} [\![\theta + w]\!]_{H,\Delta}^\alpha \\
&\lesssim |t-s|^\delta \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} + |t-s|^\delta \Delta^{\alpha H} [\![b]\!]_{C^\alpha} [\![h]\!]_{C^\delta} (1 + [\![w]\!]_{C^H} + [\![\theta]\!]_{\delta,\Delta}) \\
&\lesssim |t-s|^\delta \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} (1 + \Delta^{\alpha H} + \Delta^{\alpha H} [\![w]\!]_{C^H}) + |t-s|^\delta \Delta^{\alpha H} \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} [\![\theta]\!]_{\delta,\Delta}.
\end{aligned}$$

Dividing by $|t-s|^\delta$, taking the supremum over $|t-s| \leqslant \Delta$, we find $\kappa = \kappa(\alpha, \delta, H, T)$ such that

$$[\![\theta]\!]_{\delta,\Delta} \leqslant \kappa \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} (1 + \Delta^{\alpha H} + \Delta^{\alpha H} [\![w]\!]_{C^H}) + \kappa \Delta^{\alpha H} \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} [\![\theta]\!]_{\delta,\Delta};$$

choosing Δ such that $1/4 \leqslant \kappa \Delta^{\alpha H} \|b\|_{C^\alpha} [\![h]\!]_{C^H} \leqslant 1/2$, we obtain

$$[\![\theta]\!]_{H,\Delta} \lesssim 1 + \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} + [\![w]\!]_{C^H}.$$

Applying Exercise 4.24 from [132], we deduce that

$$[\![\theta]\!]_{C^H} \lesssim \Delta^{H-1} (1 + \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} + [\![w]\!]_{C^H}) \lesssim (\|b\|_{C^\alpha} [\![h]\!]_{C^H})^{\frac{1-\delta}{\alpha H}} (1 + \|b\|_{C^\alpha} [\![h]\!]_{C^\delta} + [\![w]\!]_{C^H})$$

and the conclusion follows from the fact that by hypothesis $(1-\delta)/(\alpha H) < 1$. □

In the case b and w are regular enough for the classical SDE (4.18) to be meaningful, the non-linear Young formalism still gives non trivial criteria in order to establish uniqueness of solutions, as the next proposition shows.

Proposition 4.22. *Let $b \in C_x^\alpha$ for some $\alpha \in (0,1)$ such that $\delta + \alpha H > 1$. Then for any $x_0 \in \mathbb{R}^d$ there exists at least one solution $x \in C_t^H$, $x \in w + C_t^\delta$ to the YDE (4.18). If $\Gamma^w b \in C_t^\gamma C_{\mathrm{loc}}^{1+\eta}$ for some γ, $\eta \in (0,1)$ satisfying*

$$\gamma + \eta \delta > 1,$$

then such solution x is unique in the class $w + C_t^\delta$.

Proof. The proof follows several arguments that we frequently employed in Chapter 1 (it shares a strong similarity to Section 1.4.3 in the role played by the higher regularity $\theta \in C_t^\delta$), so we will mostly sketch it.

Step 1: Existence. Let b^ε be a sequence of mollifications of b and denote by x^ε the unique solution of the YDE (4.18) associated to b^ε with initial data x_0. Then $x^\varepsilon = \theta^\varepsilon + w$ satisfy the a priori bound (4.20), uniformly in $\varepsilon > 0$ and so by Ascoli–Arzelà we can extract a subsequence θ^{ε_n} such that $\theta^{\varepsilon_n} \to \theta$ in $C_t^{\delta'}$ for any $\delta' < \delta$. Combining this fact with $b^{\varepsilon_n} \to b$ in $C_x^{\alpha'}$ for any $\alpha' < \alpha$, it is easy to check by the continuity properties of Young integrals that $x := \theta + w$ must be a solution to the YDE associated to b, with initial data x_0.

Step 2: Averaging formulation. Reasoning as in the proof of Proposition 4.21, it can be shown that θ is also a solution of (4.19).

Step 3: Uniqueness. Given any two solutions x^1, x^2 for the same x_0, $x^i=\theta^i+w$ with $\theta^i\in C^\delta_t$, we claim that their difference $e=x^1-x^2=\theta^1-\theta^2$ satisfies a linear YDE of the form

$$\mathrm{d}e_t=\mathrm{d}v_t\,e_t,\quad v_t:=\int_0^t\int_0^1\nabla\Gamma^w b(\mathrm{d}s,\lambda\theta^1_s+(1-\lambda)\theta^2_s)\,\mathrm{d}\lambda,$$

which follows from an application of Corollary 1.10. Finally, since e satisfies a linear YDE with initial data $v_0=0$, it must hold $v\equiv 0$, which yields uniqueness. □

Proposition 4.22 covers the case where the SDE (4.18) is already meaningful in the Young sense and both formulations (as an SDE for x and a nonlinear YDE for θ) provide some useful information on the solutions. We now pass instead to the case where $w\in C^H_t$ with $H+\delta<1$, so that the Young formulation completely breaks down (even for smooth b!) and the only meaningful interpretation is given by (4.19), provided $\Gamma^w b$ is smooth enough; in the nonlinear YDE framework, this amounts as usual to require $\Gamma^w b\in C^\gamma_t C^{\eta,\lambda}_x$ for parameters satisfying

$$\gamma,\eta,\lambda\in(0,1],\quad \gamma>\frac{1}{2},\quad \gamma(1+\eta)>1,\quad \eta+\lambda\leqslant 1. \tag{4.21}$$

Lemma 4.23. *Consider a sequence b^n of regular functions (say in C^2_x), $x^n_0\in\mathbb{R}^d$ and $w^n\in C^H_t$ with $\delta+H>1$; denote by x^n the unique solution starting from x^n_0 to the classical YDE*

$$\mathrm{d}x^n=b^n(x^n)\,\mathrm{d}h+\mathrm{d}w^n.$$

Suppose that

$$x^n_0\to x_0\ \text{in}\ \mathbb{R}^d,\quad w^n\to w\ \text{in}\ C^0_t,\quad \Gamma^{w_n}b_n\to A\ \ \text{in}\ C^\gamma_t C^{1+\eta,\lambda}_x$$

where γ,η,λ are parameters satisfying (4.21). Then x^n converge uniformly to $w+\theta$, where θ is the unique solution starting from $\theta_0:=x_0-w_0$ to the nonlinear YDE associated to A.

Proof. We know from Lemma 4.21 that in the smooth case, $\theta^n:=x^n-w^n$ is a solution to the nonlinear YDE associated to $(\Gamma^{w^n}b^n,x^n_0-w^n_0)$, where the multiplicative averaging operator $\Gamma^{w^n}b^n$ is classically defined pointwise and by hypothesis $(\Gamma^{w^n}b^n,x^n_0-w^n_0)\to(A,\theta_0)$ in $C^\gamma_t C^{1+\eta,\lambda}_x\times\mathbb{R}^d$. It then follows from Theorem 1.25 that $\theta^n\to\theta$ in C^γ_t; since $w^n\to w$, it follows that $x^n=w^n+\theta^n\to w+\theta$. □

We stated Lemma 4.23 in a general fashion, so that it can be applied in many situations, but it is clear that our main aim is to combine it with the stochastic construction of $\Gamma^w b$ from Theorem 4.12, which truly relies on $h=\beta(\omega)$ being a typical realization of fBm. Nontheless, in the regime $H+\delta>1$, if the regularity of $\Gamma^w b$ is known, the approximating sequence can be constructed explicitly and we obtain the following result, which holds for any given continuous path $h\in C^\delta_t$, not necessarily sampled as a stochastic process.

Proposition 4.24. *Let $w\in C^H_t$, $h\in C^\delta_t$ with $H+\delta>1$ and let $b\in\mathcal{D}'$ be such that $\Gamma^w b\in C^\gamma_t C^{1+\eta,\lambda}_x$ for some γ,η,λ satisfying (4.21). Then for any $\theta_0\in\mathbb{R}^d$ there exists a unique solution $\theta\in C^\gamma_t$ to the nonlinear YDE*

$$\theta_t=\theta_0+\int_0^t\Gamma^w b(\mathrm{d}s,\theta_s). \tag{4.22}$$

Moreover, denoting by b^ε a sequence of mollifications of b and by x^ε the solutions associated to

$$x^\varepsilon_t=\theta_0+\int_0^t b^\varepsilon(x^\varepsilon_s)\mathrm{d}h_s+w_t,$$

then setting $\theta^\varepsilon=x^\varepsilon-w$, it holds $\theta^\varepsilon\to\theta$ in C^γ_t as $\varepsilon\to 0$.

Proof. The first claim follows from Corollary 1.24. By Lemma 4.5, $\Gamma^w b^\varepsilon$ are uniformly bounded in $C^\gamma_t C^{1+\eta,\lambda}_x$ and they are converging to $\Gamma^w b$ in $C^\gamma_t C^{1+\eta',\lambda}_x$ for any $\eta'<\eta$; we can choose it so that $\gamma(1+\eta')>1$. The conclusion then follows from Theorem 1.25. □

4.2.2 Main results

In this section we are alwats going to assume that we are dealing with a distributional drift b and that $w \in C_t^H$ with $H + \delta < 1$, so that we cannot analytically define the SDE nor $\Gamma^w b$; nontheless, the results from Section 4.2.1 suggest the following consistent concept of solution, which is in line with Definition 3.32 from the previous chapter.

Definition 4.25. *Let β be a fBm of parameter $\delta > 1/2$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $w \in C_t^0$ a deterministic path and b a distributional field. We say that a process x is a **pathwise solution** starting at $x_0 \in \mathbb{R}^d$ to the SDE*

$$\mathrm{d}x_t = b(x_t)\mathrm{d}\beta_t + \mathrm{d}w_t$$

if there exist parameters γ, η, λ satisfying (4.21) and a set $\Omega' \subset \Omega$ of full probability such that, for all $\omega \in \Omega'$, the following hold:

i. *$\Gamma^w b(\omega)$ is well defined in the sense of Theorem 4.12 and $\Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta,\lambda}$.*

ii. *$x(\omega)_0 = x_0$ and $x(\omega) \in w + C_t^\gamma$.*

iii. *$\theta(\omega) := x(\omega) - w$ satisfies the nonlinear YDE*

$$\theta_t(\omega) = x_0 + \int_0^t \Gamma^w b(\omega)(\mathrm{d}s, \theta_s(\omega)). \tag{4.23}$$

Remark 4.26. Recall that, to any $x_0 \in \mathbb{R}^d$ and $A \in C_t^\gamma C_x^{\eta,\lambda}$, we can associate the set $C(x_0, A)$ as defined in Section 1.4.1. Then conditions *i.* and *iii.* from Definition 4.25 may be rephrased as

$$\mathbb{P}(\omega \in \Omega \colon \Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta,\lambda}, \theta(\omega) \in C(x_0, \Gamma^w b(\omega))) = 1.$$

Similarly to Remark 3.33, one can then apply the disintegration theorem to deduce that the conditional law of θ given the knowledge of β must be supported on $C(x_0, \Gamma^w b(\omega))$.

Next, we can formulate an analogue of (the second part of) Definition 3.36.

Definition 4.27. *Let β, w, b and the parameters γ, η, λ be as in Definition 4.25. We say that **path-by-path wellposedness** holds for the SDE if*

$$\mathbb{P}(\omega \in \Omega \colon \Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta,\lambda}, C(x_0, \Gamma^w b(\omega)) \textit{ is a singleton for all } x_0 \in \mathbb{R}^d) = 1. \tag{4.24}$$

Remark 4.28. Many of the definitions given in Section 3.2.1 from Chapter 3 (strong and weak existence, pathwise uniqueness, uniqueness in law) carry almost identically to this setting, so we refrain from writing them explicitly. Also the relations between them (like Remark 3.37 and Proposition 3.39) transfer to this setting similarly. For instance, it is still true that the property of path-by-path wellposedness is exclusively a requirement on w and the law of β, but not of the specific probability space in consideration (which might be taken as the canonical one).

As a consequence of the nonlinear Young theory presented in Chapter 1 (for instance by Corollary 1.24), we immediately deduce the following.

Lemma 4.29. *Let β, w, b and the parameters γ, η, λ be as in Definition 4.25 and suppose that*

$$\mathbb{P}(\omega \in \Omega \colon \Gamma^w b(\omega) \in C_t^\gamma C_x^{1+\eta,\lambda}) = 1;$$

then path-by-path wellposedness holds for the SDE.

Remark 4.30. For future reference, let us collect here some known facts from abstract nonlinear Young theory and how they adapt to our setting.

If $A \in C_t^\gamma C_x^{1+\eta,\lambda}$ for parameters satisfying (4.21), then by Theorem 1.34 the associated YDE is wellposed for any $x_0 \in \mathbb{R}^d$ and there exists an associated flow of diffeomorphism; let us denote it by $\mathcal{I}(A)$. Theorem 1.35 then ensures that, if $A \in C_t^\gamma C_x^{n+\eta,\lambda}$, then $\mathcal{I}(A) \in C_t^\gamma C_{\mathrm{loc}}^n$, while Corollary 1.36 ensures the continuity of the map $A \mapsto \mathcal{I}(A)$ from $C_t^\gamma C_x^{n+\eta,\lambda}$ to $C_t^0 C_{\mathrm{loc}}^n$ for any $n \geqslant 1$. In particular, given a *random driver* A, we can associate to it a *random flow* $\mathcal{I}(A)$ defined on the same probability space; moreover by construction the values $\mathcal{I}(A)_{0\to t}(x)$ only depend on the history of the driver A up to time t, or more precisely

$$\sigma(\mathcal{I}(A)_{s\to u}(x) \colon s \leqslant u \leqslant t, x \in \mathbb{R}) \subseteq \sigma(A_{s,u}(x) \colon s \leqslant u \leqslant t, x \in \mathbb{R})$$

where σ denotes the filtration generated by the family of random variables. In our setting, we will take $A = \Gamma^w b$, so that the associated random flow is adapted to the filtration generated by $\Gamma^w b$; by Theorem 4.12, the latter process is adapted to the filtration generated by β. Overall we deduce that, whenever $\Gamma^w b$ satisfies the assumptions of Lemma 4.29, the SDE (4.25) admits a C^1_{loc} random flow of strong solutions (which are also unique).

We are now finally ready to provide explicit conditions on the deterministic averaged field $T^w b$ to ensure wellposedness of the SDE (4.25) and provide the statements and proofs of our main results. As a consequence, we also prove Theorem 4.1 from the introduction of this chapter.

Theorem 4.31. *Let $\delta > 1/2$, $b \in \mathcal{D}'$ and $w \in C^0_t$ a deterministic path such that*

$$T^w b \in C^\gamma_t C^{1+\eta,\lambda}_x \text{ for some } \gamma \in \left(\frac{3}{2} - \delta, 1\right) \text{ and any } \eta \in (0,1] \text{ and any } \lambda > 0; \tag{4.25}$$

then path-by-path wellposedness holds for the SDE (4.25), in the sense of Definition 4.27.

In particular, for any $x_0 \in \mathbb{R}^d$, any two pathwise solutions defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P}; \beta)$ starting from x_0 are indistinguishable. Moreover, solutions are adapted to the filtration generated by β and they form a random C^1_{loc} flow; specifically, the unique solution starting at x_0 is given by

$$x_t(\omega) = w_t + \mathcal{I}(\Gamma^w b(\omega))(t, x_0) \tag{4.26}$$

where $\mathcal{I}(\Gamma^w b)$ is the random C^1_{loc} flow defined in Remark 4.30.

Proof. It follows from Theorem 4.12 and Remark 4.16 that, under the regularity assumption (4.25), the multiplicative averaged field $\Gamma^w b$ is a welldefined random field and we can find γ', η', λ' satisfying (4.21) such that $\mathbb{P}$-a.s. $\Gamma^w b \in C^{\gamma'}_t C^{1+\eta',\lambda'}_x$.

Path-by-path wellposedness then readily follows from Lemma 4.29. Indistinguishability of solutions is now a consequence a standard procedure: given any two pathwise solutions $x^i = \theta^i + w$ starting at x_0, by definition it must hold $\theta^i \in C(x_0, \Gamma^w b)$ $\mathbb{P}$-a.s.; we can then find a set $\Omega' \subseteq \Omega$ of sull probability such that, for all $\omega \in \Omega'$, $\theta^i(\omega) \in C(x_0, \Gamma^w b(\omega))$ and $\Gamma^w b(\omega) \in C^{\gamma'}_t C^{1+\eta',\lambda'}_x$, so that $C(x_0, \Gamma^w b(\omega))$ is a singleton. This implies $\theta^1(\omega) = \theta^2(\omega)$ and thus $x^1(\omega) = x^2(\omega)$ for all $\omega \in \Omega'$.

Formula (4.26) follows from the definition $x_t = w_t + \theta_t$, the fact that θ solves the YDE associated to $(x_0, \Gamma^w b)$ and the definition of $\mathcal{I}(\Gamma^w b(\omega))$ as the flow of solutions to $\Gamma^w b(\omega)$. Adaptability of solutions to the filtration generated by β follows from equation (4.26) and the final part of Remark 4.30. □

The next statement recollects many of the properties we have established for solutions throughout our construction: whenever b and w are regular enough, then our concept of pathwise solution is consistent with the classical one from Young integration; when they are not regular anymore, we can still find "*recovery sequences*" $b^n \to b$ and $w^n \to w$ such that the associated classical solutions converge to ours (thus making it a meaningful notion of solution).

Proposition 4.32. *Let δ, b, w, β as in Theorem 4.31. Then:*

i *If $b \in C^2_x$ and $w \in C^H_t$ with $\delta + H > 1$, then any pathwise solution to the Young SDE*

$$x_t(\omega) = x_0 + \int_0^t b(x_s(\omega)) \mathrm{d}\beta_s(\omega) + w_t,$$

is also a solution in the sense of Definition 4.25.

ii *If condition (4.25) holds, then it's possible to find sequences $(b^n, w^n) \in C^2_x \times C^1_t$ such that $b^n \to b$ in $\mathcal{D}'$, $w^n \to w$ in C^0_t and the associated pathwise solutions x^n converge in probability to the unique pathwise solution x given by Theorem 4.31.*

iii *More generally, if condition (4.25) holds, for any sequence $(b^n, w^n) \in C^2_x \times C^1_t$ satisfying $b^n \to b$ in $\mathcal{D}'$ and $w^n \to w$ in C^0_t and such that additionally*

$$T^{w^n} b^n \text{ is Cauchy in } C^\gamma_t C^{\eta,\lambda}_x \text{ for some } \gamma \in \left(\frac{3}{2} - \delta, 1\right) \text{ and any } \eta \in (0,1] \text{ and any } \lambda > 0,$$

the associated pathwise solutions x^n converge in probability to x.

Proof. Part *i.* is just a consequence of Lemma 4.21.

Under condition (4.25), by Remark 4.15, we can find a sequence (b^n, w^n) with the above properties such that $\Gamma^{w^n} b^n(\omega) \to \Gamma^w b(\omega)$ in $C_t^{\gamma'} C_x^{1+\eta',\lambda'}$ for $\mathbb{P}$-a.e. ω, where we can choose the parameters γ', η', λ' so that they satisfy condition (4.21). Therefore Point *ii.* follows from an application of Lemma 4.23.

Suppose now (b^n, w^n) is a sequence in $C_x^2 \times C_t^1$ satisfying the assumptions of Point *iii.*; by properties of classical averaged fields (cf. Section 3.1.1), $T^{w^n} b^n \to T^w b$ in the sense of distributions, which implies by the assumption that $T^w b \in C_t^\gamma C_x^{\eta,\lambda}$ and $T^{w^n} b^n \to T^w b$ in $C_t^\gamma C_x^{\eta,\lambda}$. But then by Theorem 4.12 and Remark 4.16, we can find γ', η', λ' satisfying (4.21) such that $\Gamma^{w^n} b^n \to \Gamma^w b$ in $L^p(\Omega; C_t^{\gamma'} C_x^{\eta',\lambda'})$. The conclusion then follows again from an application of Lemma 4.23. □

We can now specialize the assumptions of Theorem 4.31 by only requiring $T^w b$ to enjoy some space-time regularity with fixed time parameter $\gamma = 1/2$; this is clearly convenient in view of combining this theory with the results from Sections 3.1.3-3.1.4 and Theorem 3.60.

Proposition 4.33. *Let $b \in B_\infty^\alpha$, $\alpha \in \mathbb{R}$, w be such that $T^w b \in C_t^{1/2} C_x^{\alpha+\nu,\lambda}$ for all $\lambda > 0$ and all $\nu > 0$ satisfying*

$$\alpha + \nu(2\delta - 1) > 2; \tag{4.27}$$

Then the hypothesis of Theorem 4.31 are met. If in addition $T^w b \in C_t^{1/2} C_x^{\alpha+\nu,\lambda}$ for all $\lambda > 0$ and all $\nu > 0$ satisfying

$$\alpha + \nu(2\delta - 1) > n + 1, \tag{4.28}$$

then the random flow associated to the SDE is C^n_{loc}.

Proof. To show the first statement, we need to verify that under condition (4.27), $T^w b \in C_t^\gamma C_x^{\eta,\lambda}$ for some $\gamma > 3/2 - \delta$ and all $\eta \in (0, 2]$; by the assumption and Corollary 4.19, we deduce that $T^w b$ belongs to $C_t^{\gamma_\theta} C_x^{\eta_\theta,\lambda}$ for all $\lambda > 0$, where

$$\gamma_\theta = 1 - \frac{\theta}{2}, \quad \eta_\theta = \alpha + \theta\nu.$$

It remains to show that, for any choice of $\eta \in (0, 2]$, we can find $\theta \in (0, 1)$ and ν satisfying (4.27) such that $\gamma_\theta > 3/2 - \delta$ and $\eta_\theta = \eta$; it is enough to impose the set of conditions

$$\begin{cases} 1 - \frac{\theta}{2} > \frac{3}{2} - \delta \\ \alpha + \theta\nu \geqslant 2 \end{cases}. \tag{4.29}$$

Short algebraic manipulations show the equivalence between system (4.29) and the hypothesis (4.27). Similar computations show that, under (4.28), $T^w b \in C_t^\gamma C_x^{n+1}$, which implies that we can find γ', η', λ' satisfying (4.21) such that $\Gamma^w b \in C_t^{\gamma'} C_x^{n+\eta',\lambda'}$; the regularity of the flow then follows from Remark 4.30. □

The proofs of Theorem 4.31 and Proposition 4.33 only rely on the analytical regularity of $T^w b$, where w is a deterministic continuous path. There is plenty of choice for w, as the next statement shows; it include as a special case Theorem 4.1 from the introduction of this chapter.

Theorem 4.34. *Let w be sampled as an fBm (independent of β) of parameter $H \in (0, 1)$, $b \in B_\infty^\alpha$ for some $\alpha \in \mathbb{R}$ such that*

$$\alpha > 2 - \frac{1}{H}\left(\delta - \frac{1}{2}\right). \tag{4.30}$$

Then almost every realisation of w satisfies condition (4.27). If in addition

$$\alpha > n + 1 - \frac{1}{H}\left(\delta - \frac{1}{2}\right), \tag{4.31}$$

then almost every realisation satisfies condition (4.28).

Moreover, under (4.30) (resp. (4.31)), almost every $w\in C_t^H$ satisfies (4.27) (resp. (4.28)), genericity being understood in the sense of prevalence. Finally, almost every $w\in C_t^0$ satisfies condition (4.28) for any choice of $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}$.

Proof. The case of w sampled as an fBm follows from the results from Sections 3.1.3-3.1.4, e.g. Corollary 3.27. Indeed, for $b\in B_\infty^\alpha$, by condition (3.24) almost every realisation of w satisfies

$$T^w b\in C_t^{1/2}C_x^{\rho,\lambda} \text{ for all } \lambda>0 \text{ and } \rho<\alpha+\frac{1}{2H},$$

or equivalently $T^w b\in C_t^{1/2}C_x^{\alpha+\nu,\lambda}$ for all $\lambda>0$ and $\nu<1/(2H)$.

Under condition (4.30), it's possible to find $\varepsilon>0$ small enough such that $\nu=1/(2H)-\varepsilon$ satisfies (4.27); similarly under condition (4.31), we can choose $\nu=1/(2H)-\varepsilon$ so that (4.28) holds. The conclusion follows from an application of Proposition 4.33.

The statement for generic $w\in C_t^\delta$ follows from the exact same reasoning, only applying Theorem 3.60. The last statement follows from the fact that for almost every $w\in C_t^0$, $T^w b\in C_t^{1/2}C_x^{\alpha+\nu,\lambda}$ for any fixed $\lambda>0$ and $\nu>0$, combined with the property that countable intersection of prevalent sets is still prevalent.[4.2] □

Remark 4.35. The result shows that, for fixed $\delta>1/2$, the introduction of a suitable continuous perturbation w allows to give meaning and solve the SDE with arbitrarily irregular distributional drift b; moreover the associated flow of solutions can become arbitrarily regular in space. As before, there is a nontrivial interplay between the irregularity of w (measured by how small H gets) and its regularizing effect needed to define $\Gamma^w b$ (which can be written in function of $1/H$). Regarding the case of continuous stochastic processes having an infnitely regularising effect on the equations, more examples will be given in Section 5.3.2, like an infinite series of fBms (Example 5.50) or the β-log Brownian motion (the process X^β from Proposition 5.51).

4.2.3 Further generalizations

We describe here how our techniques can be readapted to extend the results to other settings; most of the content given here is taken from Section 5 of [139]. In order to avoid unnecessary repetitions, the arguments will be mostly sketched, highlighting the main ideas but without delving too much into technical details.

Time inhomogeneous diffusion coefficients. So far we assumed the diffusion coefficient b to be homogeneous, i.e. $b_t(x)=b(x)$; however, our method can be easily extended to the general case of time-dependent drifts. We will outline here sufficient conditions for wellposedness of the SDE in this case.

The first basic step amounts to defining the multiplicative averaged field $\Gamma^w b$; it is easy to check that that if $(t,x)\mapsto b_t(x)$ is smooth in both variables and $w\in C_t^H$ with $\delta+H>1$, the analytical definition of $\Gamma^w b$ from Lemma 4.3 still holds. In fact, if $b\in C_t^\rho B_\infty^{\alpha+\eta}$ with $\alpha\in\mathbb{R}$ and $\rho,\eta\in(0,1]$, under the assumptions $\delta+\eta H>1$, $\delta+\rho>1$, there exists a unique distribution $\Gamma^w b\in C_t^\delta B_\infty^\alpha$ such that

$$\|\Gamma_{s,t}^w b-b_s(\cdot+w_s)\beta_{s,t}\|_{B_\infty^\alpha}\lesssim|t-s|^{\delta+\eta H\wedge\rho}. \tag{4.32}$$

To see this, one can apply the sewing lemma to $\Xi_{s,t}=b_s(\cdot+w_s)\beta_{s,t}$, which satifies

$$\begin{aligned}\|\delta\Xi_{s,u,t}\|_{B_\infty^\alpha}\lesssim&[\|b_s(\cdot+w_u)-b_u(\cdot+w_u)\|_{B_\infty^\alpha}+\|b_s(\cdot+w_u)-b_s(\cdot+w_s)\|_{B_\infty^\alpha}]|\beta_{u,t}|\\ \lesssim&\|b\|_{C^\rho B_\infty^{\alpha+\eta}}[\![\beta]\!]_{C^\delta}(1+[\![w]\!]_{C^H})|t-s|^{\delta+\eta H\wedge\rho}\end{aligned}$$

which readily implies (4.32); there in this case $\Gamma^w b$ is analytically well-defined.

4.2. Alternatively, one could use the property that almost every $w\in C_t^0$ is *infinitely regularising* in an even stronger sense, defined in terms of ρ-irregularity, as will be presented in the upcoming Chapter 5.

To extend the definition to less regular w, we can again exploit Proposition 4.8 and Lemma 4.10 to get an equivalent of Theorem 4.12; indeed the fundamental assumption therein is the regularity $T^w b \in C_t^\gamma C_x^{1+\eta,\lambda}$, regardless of whether b is time-dependent or not.

Having defined $\Gamma^w b$ and quantified its regularity in function of $T^w b$, one can then go through the same abstract procedure for existence and uniqueness of nonlinear Young equations by setting $A_{s,t}(x) = \Gamma^w_{s,t} b(x)$ and invoking the results from Chapter 1. Overall, one can obtain path-by-path wellponedness results for drifts $b \in L_t^q B_\infty^\alpha$ for suitable values of q and α; for instance, if w is sampled as an fBm of parameter H independent of β, then an analogue of condition (4.30) from Theorem 4.34, given the regularity results for $T^w b$ from Corollary 3.27, is given by

$$\alpha > 2 - \frac{1}{H}\left(1 - \frac{2}{q}\right)\left(\delta - \frac{1}{2}\right). \tag{4.33}$$

In particular, if $b \in L_t^q B_\infty^\alpha$ with $q \in (2, \infty)$ and $\alpha \in \mathbb{R}$ satisfying (4.33), then path-by-path wellposedness holds for the associated SDE, which admits a random flow of diffeomorphisms.

Including a non-Lipschitz drift term. Up until now we have only considered equation (4.25), which amounts to (4.1) in the case when $b^1 \equiv 0$ and $b^2 = b$. However, our results immediately extend to equations with both non trivial drift and diffusion, namely of the form

$$x_t = x_0 + \int_0^t b_s^1(x_s) \mathrm{d}s + \int_0^t b_s^2(x_s) \mathrm{d}\beta_s + w_t, \qquad x_0 \in \mathbb{R}^d.$$

The extension to time-dependent diffusions has been explained in the paragraph above, so we only need to focus on how to handle the additional presence of b^1. By the usual change of variables $\theta = x - w$, we see that θ formally solves the equation

$$\theta_t = x_0 + \int_0^t b_s^1(\theta_s + w_s) \mathrm{d}s + \int_0^t b_s^2(\theta_s + w_s) \mathrm{d}\beta_s.$$

Setting

$$A_{s,t}(x) := T^w_{s,t} b^1(x) + \Gamma^w_{s,t} b^2(x),$$

we can interpret the equation in the Young integral sense as

$$\theta_t = x_0 + \int_0^t A(\mathrm{d}s, \theta_s).$$

Under the condition that A is sufficiently regular, existence and uniqueness for the YDE holds (cf. Corollary 1.24) and moreover there is an associated flow of solutions $\mathcal{I}(A)$, which depends continuously on A and inherits its spatial regularity (cf. Remark 4.30).

It is therefore enough to require $T^w b^1$ and $\Gamma^w b^2$ to belong to $C_t^\gamma C_x^{1+\beta,\lambda}$ for suitable γ, β, λ; then the results from Section 4.2.2 can be extended directly. As an example, one can take $b^1 \in L_t^{q_1} B_\infty^{\alpha_1}$ for (q_1, α_1) satisfying

$$\alpha_1 > 2 - \frac{1}{2H}\left(1 - \frac{2}{q_1}\right)$$

and $b^2 \in L_t^{q_2} B_\infty^{\alpha_2}$ for (q_2, α_2) satisfying (4.33). One might even consider more complicated situations, like the presence of multiple independent fBms β^i with different parameters $\delta^i > 1/2$ and associated to different diffusion terms b^i, but let's leave it like that for simplicity.

Random initial conditions. So far we have only considered deterministic initial data $x_0 \in \mathbb{R}^d$. However, especially in view of applications to optimal transport and fluid dynamics equations, it is often interesting to allow random initial data for the SDE. This extension can be easily implemented in our framework, as we are now going to explain; for simplicity we restrict to equation (4.25), namely with $b^1 \equiv 0$ and b^2 time-independent, but everything can be extended easily to the setting of the previous paragraph. We can readapt Definition 4.25 to accommodate the presence of random initial data.

Definition 4.36. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which an fBm β of Hurst parameter $\delta > 1/2$, as well as an independent $\mathbb{R}^d$-valued random variable ξ, are defined; consider also a continuous deterministic path w and a distributional field b. We say that a process x is a* ***pathwise solution*** *to the SDE*

$$\mathrm{d}x_t = b(x_t)\mathrm{d}\beta_t + \mathrm{d}w_t, \quad x_0 = \xi$$

if there exist parameters γ, η, λ satisfying (4.21) such that $\Gamma^w b$ is well defined in the sense of Theorem 4.12 and, setting $\theta = x - w$, it holds

$$\mathbb{P}(\omega \in \Omega : \Gamma^w b(\omega) \in C_t^\gamma C_x^{\eta,\lambda}, \theta(\omega) = C_t^\gamma, \theta(\omega) \in C(\xi(\omega), \Gamma^w b(\omega))) = 1.$$

Here we have adopted directly the equivalent formulation of Definition 4.25 coming from Remark 4.26, where we recall that $C(x_0, A)$ is the set defined in Section 1.4.1. As a consequence of the nonlinear Young theory (cf. Remark 4.30), we deduce the following result.

Corollary 4.37. *Let β, b, w, ξ be as in Definition 4.36 and such that the assumptions of Lemma 4.29 are satisfied. Then any pathwise solution x to the SDE with initial condition ξ, $x = \theta + w$, satisfies*

$$\mathbb{P}(\omega \in \Omega : \theta(\omega)_t = \mathcal{I}(\Gamma^w b(\omega))(t, \xi(\omega)) \text{ for all } t \in [0, T]) = 1$$

where $\mathcal{I}$ is the map defined in Remark 4.30, i.e. $\mathcal{I}(\Gamma^w b(\omega))$ is the flow associated to $\Gamma^w b(\omega)$. In particular all the conclusions follow if the assumptions of Theorem 4.31 are satisfied; if w is sampled as an independent fBm of parameter H, it suffices to enforce condition (4.30) from Theorem 4.34.

Associated transport equation. Allowing for random initial conditions can be alternatively be interpreted as looking at the continuity equation associated to the equation (and in particular to how the initial law $\mathcal{L}(\xi)$ evolves at positive times). It then shouldn't come as a surprise, given the abstract results from Section 2.1 and their application to perturbed ODEs presented in Section 3.2.4, that we can handle the associated transport equation in this setting as well.

For the SDE (4.25), it is formally given by

$$\mathrm{d}u_t + b \cdot \nabla u_t \, \mathrm{d}\beta_t + \mathrm{d}w_t \cdot \nabla u_t = 0;$$

by the usual Galilean transformation $\tilde{u}_t(x) = u_t(x + w_t)$ the equation then becomes

$$\mathrm{d}\tilde{u}_t + \tilde{b}_t \cdot \nabla \tilde{u}_t \, \mathrm{d}\beta_t = 0$$

which can be interpreted in the nonlinear Young framework as

$$\tilde{u}_{\mathrm{d}t} + \Gamma^w_{\mathrm{d}t} b \cdot \nabla \tilde{u}_t = 0.$$

Given this identification, assuming $\Gamma^w b$ is sufficiently regular (namely satisfying the assumptions of Lemma 4.29), existence and uniqueness of solutions to the transport equation (e.g. for $u_0 \in C^1_{\mathrm{loc}}$) then follows from the abstract results of Section 2.1.

4.3 Open problems and future directions

As I already mentioned in the introduction, to the best of my knowledge, the paper [139] with Fabian is the first one to study regularisation by noise phenomena in the setting of SDE (4.1); so there aren't really many bibliographic references to talk about, apart from the already mentioned ones for $b^2 \equiv 0$. The only other paper I'm aware of is [257], which consider the one-dimensional case and piecewise Lipschitz drifts with jump discontinuities, buinding on previous works [192, 165, 219] treating mixed SDEs (i.e. allowing for both Young integral terms $\int b_s^2(x_s)\mathrm{d}\beta_s$ and Itô terms $\int b_s^3(x_s)\mathrm{d}W_s$ with W standard Bm).

The material presented here is all taken from [139], up to some small modifications. Like in Chapter 3, working with the weighted Hölder spaces $C_t^\gamma C_x^{\eta,\lambda}$, although a bit technical, requires less restrictive assumptions on b (in [139] we had to additionally impose compactness of the support of b, which is no longer needed here); I have also omitted some additionally results concerning the more regular case $b \in B^\alpha_\infty$ with $\delta + \alpha H > 1$, which mostly result in Theorem 4 from [139].

For the sake of discussion, in the following I will restrict to the case of the SDE with only diffusive part, namely (4.1) with $b^1 \equiv 0$ and $b^2 = b$.

The first major question one could ask concerning Theorem 4.1 is whether condition (4.6) is optimal for strong existence and uniqueness to hold. Unfortunately, this is probably not the case, and it should be most likely possible to weaken it to $b \in B^\alpha_\infty$ with

$$\alpha > 1 - \frac{1}{H}\left(\delta - \frac{1}{2}\right). \tag{4.34}$$

There are several facts hinting to this:

i. In line with Remark 4.2, as $\delta \uparrow 1$, condition (4.34) becomes $\alpha > 1 - 1/(2H)$ and thus is in agreement with Theorem 3.30.

ii. Going through similar regularity counting arguments to those of Theorems 4.31-4.34, it can be shown that in this case $\mathbb{P}$-a.s. $\Gamma^w b \in C^\gamma_t C^{\eta,\lambda}_x$ for some $\gamma > 1/2$ and all $\eta \in (0,1)$ and all $\lambda > 0$; as a consequence, a priori estimates on the solution θ to the YDE are available, which yield the existence of weak solutions by tightness arguments. Alternatively, one can also establish the existence of pathwise solution by a readaptation of Lemma 3.40.

iii. Moreover, again under condition (4.34), it can be checked (cf. Corollary 3.27) that $\mathbb{P}$-a.s. $T^w b \in C^{\alpha H + 1 - \varepsilon}_t C^{0,\lambda}_x$ for all $\varepsilon, \lambda > 0$; by Remark 4.20, this implies that $\Gamma^w b \in C^{\gamma'}_t C^{0,\lambda}_x$ for any $\lambda > 0$ and any $\gamma' < \alpha H + \delta$. Observe that, under (4.34), one can choose $\gamma' > H + 1/2$; since $\theta_{s,t} \sim \Gamma^w_{s,t} b(\theta_s) + o(|t-s|)$, this implies that any solution θ will belong to $C^{\gamma'}_t$ as well. But then the solution x is of the form $x = \theta + w$, with w sampled as an fBm of parameter H and θ belonging $\mathbb{P}$-a.s. to the Cameron-Martin space $\mathcal{H}^{H+1/2}$. By Girsanov's theorem, we then expect the existence of another equivalent measure $\mathbb{Q}$ such that x is distributed as an fBm of parameter H under $\mathbb{Q}$; one then expects at least uniqueness in law under (4.34).

The reason why the above argument is not formalized into a rigorous proof is because I have realized that the solution concept we adopted in Definition 4.25 (which still has a lot of nice consequences, like an easy construction of the stochastic flow) is too "rigid" to allow for the use of Girsanov, since it really relies on treating w as a deterministic perturbation in order to exploit the stochastic estimates from Proposition 4.8 (so probabilistic tools at the level of w are hard to implement here). One could try to invert the reasoning at Point iii. above, by first looking at

$$y_t = x_0 + w_t = x_0 \pm \int_0^t b(x_0 + w_s) \mathrm{d}\beta_s + w_t = x_0 + \int_0^t b(y_s) \mathrm{d}\beta_s + w_t - h_t$$

and then trying to show that under a new chang of measure $w - h = \tilde{w}$ is an fBm of parameter H, thus making y a weak solution to the SDE. There are two difficulties in doing so:

a) It is hard to verify Novikov's condition, due to the presence of both Gaussians w and β; think of $b(x) = x$, where this would amount to estimating Gaussian tails for $\int_0^t w_s \,\mathrm{d}\beta_s$, which morally behaves as the product of two independent real Gaussians.

b) Even if Novikov were successful, I still wouldn't be able to show that the pair $(y, \tilde{w})$ is a weak solution to the SDE in the sense of Definition 4.25, which is why I'm calling it too rigid. Compared to Chapter 3, what is lacking is an equivalent of Lemma 3.42, i.e. something ensuring that $\int_0^t \Gamma^w b(\mathrm{d}s, \theta_s + \tilde{\theta}_s) = \int_0^t \Gamma^{w+\theta} b(\mathrm{d}s, \tilde{\theta}_s)$. The reason for this is that, since the process θ is constructed starting from β, one cannot regard $w + \theta$ as β-independent, which interferes with the construction of $\Gamma^{w+\theta} b$ from Theorem 4.12.

Given such restrictions, it is natural to wonder whether the pathwise approach here is the most suitable one, or other more probabilistic strategies have a higher chance of success.[4.3]

4.3. In fact, Khoa Lê showed me privately some computations, based on the use of the stochastic sewing lemma [194], which seem to confirm that strong existence and uniqueness holds for $b \in B^\alpha_\infty$ under (4.34).

If one tried to run a similar scaling argument to that of Section 3.4, the situation is even worse: it would predict wellposedness in the subcritical regime for $b \in L_t^q B_\infty^\alpha$ with

$$\alpha > 1 - \frac{1}{H}\left(\delta - \frac{1}{q}\right) \tag{4.35}$$

which would become a natural generalisation of (3.53) allowing for $\delta \in (1/2, 1]$. The validity of the scaling argument is debatable, as can be very sensitive to the choice of the Banach space in consideration (e.g. replacing B_∞^α with L_x^p for $\alpha = -d/p$ might give more reasonable expectations).

Even ignoring (4.35), condition (4.34) reveals quite an interesting structure, compared to similar observations provided by Remark 4.2. Indeed, if (4.34) were true, than *any* choice of H would yield a regularising effect on the equation, in the sense of allowing non-Lipschitz drifts $b \in B_\infty^\alpha$ for suitable $\alpha < 1$. The condition $H \leqslant \delta$, although natural, doesn't seem to play any relevant role; this might be explained as the fact that, while b is allowed to be very degenerate, the presence of a purely additive term w (thus strongly nondegenerate) still improves the solution theory, even when w is not the roughest term in the decomposition $x = \theta + w$. In order to reach distributional drifts $\alpha < 0$, condition (4.34) still enforces $\delta > H + 1/2$, which suggests a strong link to the considerations based on Girsanov's theorem. Finally, in the case $H = 1/2$ (i.e. w sampled as Bm), the condition becomes $\alpha > 2(1 - \delta)$, which is still better (to the best of my knowledge) than known results on wellposedness of the associated Kolmogorov equation (which becomes a Young-type parabolic equation like those treated in Section 2.2), formally given by

$$\mathrm{d}u = \Delta u \mathrm{d}t + b \cdot \nabla u \mathrm{d}\beta.$$

All the above considerations reveal that there is still quite a lot to be understood about equation (4.1), even though we start having regularisation by noise results for it.

Among possible extensions which go beyond the setting of (4.1), one possibility is to replace β by a *deterministic* δ-Hölder continuous path, with no further information on it. In this case, one cannot invoke Proposition 4.7 anymore and instead should exploit heavily the regularising effect of w sampled as an fBm of parameter H. Although there are no published results on the matter, stochastic sewing techniques seem to be more effective in tackling this problem (see again Footnote 4.3).

A second interesting problem of course concerns decreasing the value of δ to cover the range $\delta \in (0, 1/2]$. The problem is harder already at the analytical level: in order to solve

$$x_t = x_0 + \int_0^t b(x_s) \mathrm{d}\beta_s + w_t$$

one would need to define first $\int_0^t b(x_s) \mathrm{d}\beta_s$. This can be accomplished in the setting of rough path theory, but only under the fundamental assumption that (β, w) as an $\mathbb{R}^{m+d}$-valued path admits a rough lift (in particular, we not only need $\int_0^t \beta_s \mathrm{d}\beta_s$ to be well-defined, but also $\int_0^t w_s \mathrm{d}\beta_s$). Besides that, applying the change of variables $\theta = x - w$, the problem can be reduced as usual to the study of the regularity of the averaged field $\Gamma^w b \in C_t^\gamma C_{\mathrm{loc}}^\eta$; it is natural to expect the value γ to be always less or equal to δ (i.e. the regularity of β), which means that even having defined $\Gamma^w b$, one couldn't apply the nonlinear Young machinery afterwards. There is some small hope given by analogous theories of *nonlinear rough paths*, see [230, 70], but overall it seems to me that the problem is beyond reach.

The case $\delta = 1/2$ (namely β sampled as Bm) is quite special, due to the Itô isometry (assuming we interpret the integral appearing in the SDE in the Itô sense, which I will do in the following, although it's not the only option). Indeed, considering w as a fixed deterministic path, defining the Gaussian random variable

$$\Gamma_{s,t}^w b(y) = \int_s^t b(y + w_s) \mathrm{d}\beta_s$$

is equivalent to showing that

$$\mathbb{E}[|\Gamma^w_{s,t} b(y)|^2] = \int_s^t |b|^2(y + w_s)\mathrm{d}s = T^w_{s,t}|b|^2(y) < \infty.$$

This shows a nontrivial relation between $\Gamma^w b$ and the classical averaging T^w of another object, namely $|b|^2$. The problem is, there is no need for $|b|^2$ to be a well-defined distribution! We are then forced to impose this assumption, which means (due to it positivity) that $|b|^2$ should be at least a Radon measure (and the most natural requirement then becomes $b \in L^2_x$). Unfortunately, trying to iterate the argument at the level of spatial derivative $\partial_i \Gamma^w_{s,t} b = \Gamma^w_{s,t} \partial_i b$ becomes now quite painful, as it then enforces $\partial_i b \in L^2_x$ as well, so that even in order to have something like $\Gamma^w b \in C^\gamma_t C^1_{\mathrm{loc}}$ we must already impose at least $b \in W^{1,2}_x$, regardless of the regularising effect of w. This regularity condition starts being quite expensive and makes the whole approach not very effective, compared to other techniques (e.g. the DiPerna-Lions approach, readapted to the case of SDEs e.g. in [281, 59]).

Nontheless, given that one can define and estimate both integrals of the form $\int_0^t b(\theta_s + w_s)\, \mathrm{d}\beta_s$ and the multiplicative averaged field $\Gamma^w b$, there is some hope to derive a priori estimates for the SDE and establish weak existence of solutions, which is in fact what is accomplished in the upcoming work [28].

Chapter 5
ρ-irregularity

We have seen in Chapter 3 that, given a path w and a drift b, we can analyse the regularity of the averaged field $T^w b$, which in turn allows to solve perturbed (possibly singular) ODEs. In particular, for fixed b and W fBm of parameter H, we expect a spatial regularizing effect of order at most $1/(2H)$; in practical terms, this means that to any drift, say $b \in L_t^\infty B_p^\alpha$ for simplicity, we can associated a set Γ_b of full probability such that

$$T^{W(\omega)} b \in C_t^{1/2} B_p^{\alpha + \frac{1}{2H} -} \quad \forall \omega \in \Gamma_b. \tag{5.1}$$

One can ask the following more ambitious question: can we find a *universally regularising* path w? Namely, can we find w such that

$$T^w b \in C_t^{1/2} B_p^{\alpha + \frac{1}{2H} -} \quad \text{for all drifts } b \text{ in a given class } E?$$

To some extent, this amounts to inverting the (uncountable) quantifiers involving statement (5.1) for $T^{W(\omega)} b$, i.e. finding a b-independent set $\Gamma \subset \Omega$, hopefully with $\mathbb{P}(\Gamma) = 1$, where (5.1) holds.

Remark 3.5 already informs us that this problem has no solution if the class E is allowed to contain time-dependent drifts; still, there is some hope in the autonomous case, e.g. for $E = B_p^\alpha$. This question was raised in [57] and shown to be closely related to Conjecture 1.2, left therein open. It turns out that, even in this formulation, the statement is somewhat too strong to hold true; anticipating some of the concepts that will appear throughout the chapter[5.1], let us briefly explain why. The argument is a courtesy of N. Perkowski (see also Remark 3.7 from [170]).

Suppose that there exist parameter $\alpha, \gamma > 0$, and a continuous path w, such that T^w is a bounded operator from $B_\infty^{-\alpha}$ to B_∞^γ (let us ignore the presence of possible growth conditions for simplicity); choose p large enough so that $B_p^{-\alpha/2} \hookrightarrow B_\infty^{-\alpha/2 - d/p} \hookrightarrow B_\infty^{-\alpha}$. Then it holds

$$\left| \int_0^T b(w_s) \mathrm{d}s \right| = |T^w b(0)| \leqslant \|T^w b\|_{B_\infty^\gamma} \leqslant \|b\|_{B_p^{-\alpha/2}}; \tag{5.2}$$

on the other hand, by the *occupation time formula*, for b smooth it holds

$$\int_0^T b(w_s) \mathrm{d}s = \int_{\mathbb{R}^d} b(x) \mu^w(\mathrm{d}x) = \langle b, \mu^w \rangle$$

where μ^w is the *occupation measure* of w on $[0, T]$. Therefore equation (5.2) is telling us that the linear operator $b \mapsto \langle b, \mu^w \rangle$, which is well defined for smooth b, extends uniquely to a bounded operator on $B_p^{-\alpha/2}$; by duality this implies $\mu^w(\mathrm{d}x) = \ell^w(x)\mathrm{d}x$ for some $\ell^w \in B_{p'}^{\alpha/2}$. By standard facts from geometric measure theory, we can deduce that the support of μ^w, given by $w([0, T])$, must be of Hausdorff dimension d; in turn, this implies that w cannot belong to C_t^δ for any $\delta > 1/d$.

The above argument technically doesn't disprove the existence of "universal regularizers" (which, as we will see, still exist); but it informs us that, in order to cover the class $E = B_\infty^\alpha$, they must have very limited Hölder regularity. In particular it tells us that, if W is a fBm of parameter $H > 1/d$, although the results from Chapter 3 still hold, we cannot "invert the quantifiers" and obtain a negligible set Γ outside of which T^W improves the regularity of Besov-Hölder functions by a factor of almost $1/(2H)$.

5.1. The reader who is not acquainted with concepts like occupation measures or Hausdorff dimension might skip the next paragraph and come back here after reading the rest of the chapter; Section 5.1 would mostly suffice.

A partial solution to this conundrum (which, given the above observations, seems to be the best we can hope for) is again presented in [57], where the authors choose to consider the class E given by Fourier–Lebesgue spaces $\mathcal{F}L^{\alpha,p}$ (see Appendix A.2), which in particular includes the fractional Sobolev spaces H^{α}.[5.2] By introducing the concept of (γ,ρ)-irregularity (see the upcoming Definition 5.1), they establish the following fact: if w is (γ,ρ)-irregular, then its averaged field T^w satisfies

$$\|T^w_{s,t} b\|_{\mathcal{F}L^{\alpha+\rho,p}} \leqslant C(w)|t-s|^{\gamma}\|b\|_{\mathcal{F}L^{\alpha,p}} \quad \forall b \in \mathcal{F}L^{\alpha,p},$$

with an estimate uniform in $\alpha \in \mathbb{R}$, $p \in [1,\infty]$. Further, it is proved [57] that, if w is a typical trajectory of fBm, then for any $\rho < 1/(2H)$ there exists $\gamma > 1/2$ such that it is (γ,ρ)-irregular. We see therefore that we recover analogous results to those from Section 3.1.3 (i.e. a spatial regularity improvement of almost $1/(2H)$, at the price of a time regularity close to $1/2$); we pay the price of dealing with the less standard class $E = \mathcal{F}L^{\alpha,p}$, but we gain the set Γ being independent of b.

In this chapter, we explore in detail the analytical concept of (γ,ρ)-irregularity; it tries to capture quantitatively the idea, due to the erratic behaviour of w, we should see cancellations whenever considering averages along the path of the form $\int_s^t f(w_r)\mathrm{d}r$; this is accomplished by looking at the decay of oscillatory integrals of the form $\int_s^t e^{i\xi\cdot w_r}\mathrm{d}r$. Almost all the material presented here is taken from [143].

It turns out that the notion of (γ,ρ)-irregularity is in fact quite rich and finds applications well behind the scope it was originally introduced for in [57]. On one hand, useful criteria can be developed to check that typical trajectories of many stochastic processes are (γ,ρ)-irregular; on the other, this purely analytic notion can be combined with notions coming from different areas (gemetric measure theory, rough paths) to obtain new insight on the properties of w. It also poses some new, challenging problems (see in particular Sections 5.4.3-5.4.5 and the open Conjectures 5.72, 5.75) and allows to establish pathwise regularisation by noise results, not only for ODEs, but PDEs too (Section 5.2.2). Last but not least, we can show that almost every Hölder continuous function (in the sense of prevalence) is (γ,ρ) irregular (for suitable γ,ρ), so that many of the aforementioned results can be stated without any reference to an underlying probability space. We will see in Chapter 6 another application of ρ-irregularity, in relation to the mixing properties of shear flows.

Structure of the chapter. We introduce in Section 5.1 all our main actors: the concepts of ρ-irregularity and occupation measure, their relation to averaging operators and the class of strongly locally nondeterministic Gaussian fields. Then we present in Section 5.2 the main results of this chapter, together with their applications to several regularisation by noise phenomena in ODEs and PDEs.

Sections 5.3 and 5.4 constitue the main body of the paper, presenting the proofs. The first mostly deals with sufficient conditions for stochastic processes to be ρ-irregular; instead the latter examines (mostly) analytical properties of ρ-irregular paths, with particular emphasis on the so called "perturbation problem". Finally, we present further bibliographical remarks in Section 5.5.

5.2. One could draw here a nice analogy with the solution theory for PDEs like wave equations. Indeed, if one considers the linear wave PDE on $\mathbb{R}^d$

$$\partial_t^2 u = \Delta u, \quad u|_{t=0}=0, \quad \partial_t u|_{t=0}=f,$$

then the solution at time $t>0$ is given by the linear operator

$$T_t f = \frac{\sin(t|\nabla|)}{|\nabla|} f,$$

in the sense that S_t acts in Fourier space by multiplying $\hat{f}$ by $\sin(t|\xi|)/|\xi|$. It is then clear that T_t maps H^s into H^{s+1} for any $s \in \mathbb{R}$; but if one tries to have a similar result in L^p-based results, this is not true in general, with explicit counterexamples given in [235]. What is more, when $d=3$, T_t has the alternative harmonic mean representation $T_t f = \mu_t * f$, where μ_t is the unit measure on the sphere $S_t = \{x \in \mathbb{R}^d : |x| = t\}$, which is singular w.r.t. the Lebesgue measure; in particular, a similar argument to the one above readily implies that T_t cannot map B^{α}_{∞} into $B^{\alpha+\varepsilon}_{\infty}$ for any $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Roughly speaking T_t here behaves exactly like T^w_t would, in the case where the path w is 1-irregular but also δ-Hölder continuous with $\delta > 1/d$ (e.g. for $d=3$, typical realization of fBm with $H \in (1/3, 1/2)$ would work).

Notations and conventions. We will adopt mostly the same notations as in Chapter 3, e.g. for $a \lesssim b$, the function spaces $C_t^\gamma B_p^s$, the heat semigroup $P_t = e^{t\Delta}$, etc. We will frequently work with the Fourier–Lebesgue spaces $\mathcal{F}L^{s,p}$, for which we refer to Appendix A.2. For $\alpha \in (0,\infty) \setminus \mathbb{N}$, we will use both notations B_∞^α and C_x^α to denote Besov–Hölder spaces. Like in Chapter 4, we will sometimes use $C_t^{\gamma-}$ to denote $\cap_{\varepsilon>0} C_t^{\gamma-\varepsilon}$; similarly for $C_t^{\gamma-} C_x^{\alpha-}$, etc.

As usual, C_c^∞ denotes the set of smooth, compactly supported functions; instead we will use C_b^∞ to denote the class of infinitely differentiable functions with all bounded derivatives and by C_{loc}^∞ that of smooth functions, whose derivatives might however explode at infinity.

As before, statements of the form "for almost every (a.e.) φ" must be understood in the prevalence sense, see Appendix A.3. Given a matrix A, A^* denotes its transpose, A^{-1} its inverse.

Whenever we work with a stochastic process X, even if not specified, we implicitly assume the existence of an underlying filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with $\mathcal{F}, \mathcal{F}_t$ satisfying the standard assumptions. We adopt the short notation $\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_s]$. Whenever it doesn't creat confusion, we will use L_ω^p for $L^p(\Omega, \mathbb{P}; \mathbb{R}^m)$ for suitable $m \in \mathbb{N}$.

5.1 Preliminaries

5.1.1 Definition and first properties

The concept of ρ-irregularity was introduced in [57] as an analytic property of continuous functions, which allows to quantitatively measure their oscillatory behaviour, as well as their smoothing effect on perturbations of ODEs.

Definition 5.1. *Let $\gamma, \rho > 0$. A measurable path $w: [0,T] \to \mathbb{R}^d$ is* ***(γ, ρ)-irregular*** *if there exists a constant C such that*

$$\left| \int_s^t e^{i\xi \cdot w_r} \mathrm{d}r \right| \leqslant C |\xi|^{-\rho} |t-s|^\gamma \quad \forall \xi \in \mathbb{R}^d_{\neq 0},\, s,t \in [0,T]. \tag{5.3}$$

We denote by $\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}$ the optimal constant C; with the notation $\Phi_t^w(\xi) = \int_0^t e^{i\xi \cdot w_r} \mathrm{d}r$, it holds

$$\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} := \sup_{\xi \in \mathbb{R}^d, s \neq t} \frac{|\Phi_{s,t}^w(\xi)| |\xi|^\rho}{|t-s|^\gamma}.$$

We say that w is ρ-irregular if there exists $\gamma > 1/2$ such that w is (γ, ρ)-irregular.

We have the trivial bound $|\Phi_{s,t}^w(\xi)| \leqslant |t-s|$, so that (5.3) is always satisfies for $|\xi|$ small; the relevant information in the above definition is given by the uniform bound as $|\xi| \to \infty$. Therefore we can replace $|\xi|$ with any other function with same asymptotics (in the original definition from [57], $1 + |\xi|$ appeared, here instead we adopt $|\xi|$ for its better scaling properties).

Let us collect some elementary facts on ρ-irregular functions. In the next statement, $\mathrm{SO}(d)$ denotes the special orthonormal group on $\mathbb{R}^d$.

Lemma 5.2. *Let $w: [0,T] \to \mathbb{R}^d$ be a (γ, ρ)-irregular continuous path. Then the following hold:*

i. Symmetry invariance: $-w$ is (γ, ρ)-irregular with $\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} = \|\Phi^{-w}\|_{\mathcal{W}^{\gamma,\rho}}$.

ii. Translation invariance: for any $r \leqslant T$, $w_\cdot - w_r$ is (γ, ρ)-irregular, $\|\Phi^{w_\cdot - w_r}\|_{\mathcal{W}^{\gamma,\rho}} = \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}$.

iii. Scaling invariance: for any $\lambda \in (0,1)$, $w^\lambda(t) := \lambda^{-(1-\gamma)/\rho} w(\lambda t)$ is (γ, ρ)-irregular.

iv. Rotation invariance: for any $O \in \mathrm{SO}(d)$, Ow is (γ, ρ)-irregular with $\|\Phi^{Ow}\|_{\mathcal{W}^{\gamma,\rho}} = \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}$.

v. More generally, if $A \in \mathbb{R}^{d \times d}$ is invertible, then Aw is (γ, ρ)-irregular with

$$\|\Phi^{Aw}\|_{\mathcal{W}_T^{\gamma,\rho}} \leqslant \|(A^*)^{-1}\|^\rho \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}.$$

Proof. All the statements follow from elementary calculations; let us prove only *iii.* and *v.* . Fix $\lambda \in (0,1)$, then

$$\Phi^{w^\lambda}_{s,t}(\xi) = \int_s^t e^{i\xi\cdot\lambda^{-(1-\gamma)/\rho} w_{\lambda r}}\mathrm{d}r = \lambda^{-1}\int_{\lambda s}^{\lambda t} e^{i\lambda^{-(1-\gamma)/\rho}\xi\cdot w_r}\mathrm{d}r = \lambda^{-1}\Phi_{\lambda s,\lambda t}\Big(\lambda^{-\frac{1-\gamma}{\rho}}\xi\Big),$$

so that

$$\frac{\big|\Phi^{w^\lambda}_{s,t}(\xi)\big||\xi|^\rho}{|t-s|^\gamma} = \frac{|\Phi_{\lambda s,\lambda t}(\lambda^{-(1-\gamma)/\rho}\xi)||\xi|^\rho}{\lambda|t-s|^\gamma} = \frac{|\Phi_{\lambda s,\lambda t}(\lambda^{-(1-\gamma)/\rho}\xi)||\lambda^{-(1-\gamma)/\rho}\xi|^\rho}{|\lambda t-\lambda s|^\gamma} \leqslant \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}.$$

Regarding Point *v.*, similarly we have

$$\frac{|\Phi^{Aw}_{s,t}(\xi)||\xi|^\rho}{|t-s|^\gamma} = \left|\int_s^t e^{i(A^*\xi)\cdot w_r}\mathrm{d}r\right|\frac{|\xi|^\rho}{|t-s|^\gamma} \leqslant \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\left(\frac{|A^*\xi|}{|\xi|}\right)^{-\rho} \leqslant \|(A^*)^{-1}\|^\rho\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}. \qquad \square$$

Remark 5.3. There is a striking analogy between Points *i.–iii.* and properties of stochastic processes like symmetry, stationarity and self-similarity; however the latter properties are of statistical nature, namely they preserve the *law* of the process, while ρ-irregularity is an analytical property which holds for deterministic trajectories. By property *iii.*, we deduce the existence of a critical scaling parameter associated to the pair (γ,ρ), given by

$$\delta^*_{\gamma,\rho} = \frac{1-\gamma}{\rho}; \tag{5.4}$$

we will see in Theorem 5.31 how $\delta^*_{\gamma,\rho}$ relates to regularity of (γ,ρ)-irregular paths.

Remark 5.4. Clearly, (γ,ρ)-irregularity for w is equivalent to the following: there exists a constant C such that, for any $v\in\mathbb{S}^{d-1}$, $v\cdot w$ is (γ,ρ)-irregular with $\|\Phi^{v\cdot w}\|_{\mathcal{W}^{\gamma,\rho}}\leqslant C$. The latter is not equivalent to checking (γ,ρ)-irregularity of the coordinates $w^{(i)}$ (i.e. to $v=e_i$): for instance if w is a 1-dimensional (γ,ρ)-irregular function and we define $\tilde{w}_t := (w_t, -w_t)$, then the single coordinates of $\tilde{w}$ are (γ,ρ)-irregular but $\tilde{w}$ is not, since $(1,1)\cdot\tilde{w}\equiv 0$.

Lemma 5.5. *Let w be (γ,ρ)-irregular, then for any $\theta\in[0,1]$ it is also $(\gamma^\theta,\rho^\theta)$-irregular for the choice $\gamma^\theta = 1-\theta+\theta\gamma$, $\rho^\theta=\theta\rho$ and it holds $\|\Phi^w\|_{\mathcal{W}^{\gamma^\theta,\rho^\theta}}\leqslant\|\Phi^w\|^\theta_{\mathcal{W}^{\gamma,\rho}}$.*

Proof. The conclusion follows immediately by interpolating the two inequalities

$$|\Phi^w_{s,t}(\xi)|\leqslant|t-s|, \quad |\Phi^w_{s,t}(\xi)|\leqslant\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}|t-s|^\gamma|\xi|^{-\rho}. \qquad \square$$

The above lemma shows that we can always trade space regularity for time regularity, i.e. we can decrease the parameter ρ in order to increase γ. Observe that

$$\delta^*_{\gamma,\rho} = \frac{1-\gamma}{\rho} = \frac{1-\gamma^\theta}{\rho^\theta} = \delta^*_{\gamma^\theta,\rho^\theta};$$

namely, the critical scaling parameter δ^* is left unchanged by this procedure.

In dimension $d\geqslant 2$, it is in general difficult to construct examples of (γ,ρ)-irregular paths. This fact is one of the main motivations of our interest in establishing the prevalence of this property. The situation is different in the case $d=1$, in which there are simple conditions to establish ρ-irregularity (at least for some values of ρ).

Proposition 5.6. (Proposition 1.4 from [66]) *Let $w\in C^1_t$ satisfy $\inf_t |w'_t|\geqslant\delta>0$ and $w''\in L^1_t$; then w is $(\gamma, 1-\gamma)$-irregular for any $\gamma\in(0,1)$.*

In higher dimension, we still know that ρ-irregular functions exist:

Theorem 5.7. (Theorem 1.4 from [57]) *Let $H\in(0,1)$ and denote by μ^H the law of fBm; then for any $\rho<(2H)^{-1}$ there exists $\gamma>1/2$ such that*

$$\mu^H(w\in C^0_t : w \text{ is } (\gamma,\rho)\text{-irregular}) = 1.$$

Remark 5.8. By no coincidence, the borderline parameter $\rho=1/(2H)$ is the same appearing in Section 3.1.3 in terms of the space regularity improvement of $T^W b$ compared to b (for instance when $b\in L_t^\infty B_p^s$ for some $p\in[2,\infty)$).

Combined with Lemma 5.5, Theorem 5.7 implies the existence of continuous (γ,ρ)-irregular functions for any choice of $\gamma\in(0,1)$ and $\rho<\infty$.

Let us also introduce the concept of *exponential irregularity*, which first appeared in [143].

Definition 5.9. *A measurable path $w\colon[0,T]\to\mathbb{R}^d$ is **exponentially irregular** if there exist positive constants c_1, c_2 and $\gamma\in(0,1)$ such that*

$$|\Phi^w_{s,t}(\xi)|\leqslant c_1\, e^{-c_2|\xi|}|t-s|^\gamma \quad \forall \xi\in\mathbb{R}^d_{\neq 0},\, s,t\in[0,T]. \tag{5.5}$$

We now prove that the ρ-irregularity and exponential irregularity properties define Borel sets, which is the first step in order to establish their prevalence.

Lemma 5.10. *For any $\rho>0$, the set*

$$A=\{w\colon[0,T]\to\mathbb{R}^d|\, w \text{ is } \rho\text{-irregular}\}$$

is Borel measurable w.r.t to the topology induced by any of the following norms: $\|\cdot\|_{L^p}$, $p\in[1,\infty]$, $\|\cdot\|_{C^0}$, $\|\cdot\|_{C^\alpha}$, $\alpha\in(0,1)$.

Proof. The proof is similar to that of Lemma 3.6. We can write the set A as follows:

$$A=\bigcup_{n,m\in\mathbb{N}} A_{n,m}=\bigcup_{n,m\in\mathbb{N}}\left\{w\colon[0,T]\to\mathbb{R}^d \,\middle|\, \sup_{\xi\in\mathbb{R}^d,s\neq t}\frac{|\Phi^w_{s,t}(\xi)||\xi|^\rho}{|t-s|^{1/2+1/m}}\leqslant n\right\}.$$

It will be then sufficient to show that for every m,n the set $A_{m,n}$ is closed in the aforementioned topologies. We will actually show that it is closed under convergence in measure, which is weaker than any of the norms considered and therefore yields the conclusion.

Let w_k be a sequence of elements of $A_{n,m}$ such that $w_k\to w$ in measure; by dominated convergence, for any fixed $s<t$ and $\xi\in\mathbb{R}^d$ it holds $\Phi^{w_k}_{s,t}(\xi)\to\Phi^w_{s,t}(\xi)$. But then

$$\frac{|\Phi^w_{s,t}(\xi)||\xi|^\rho}{|t-s|^{1/2+1/m}}=\lim_{k\to\infty}\frac{|\Phi^{w_k}_{s,t}(\xi)||\xi|^\rho}{|t-s|^{1/2+1/m}}\leqslant n$$

and since the reasoning holds for any fixed $s<t$ and ξ we can conclude that $f\in A_{n,m}$ as well. □

Remark 5.11. More generally, given a modulus of continuity ϕ and a function $F\colon\mathbb{R}^d\to\mathbb{R}^+$, the same proof shows that the set

$$B=\left\{w\colon[0,T]\to\mathbb{R}^d \,\middle|\, \sup_{\xi\in\mathbb{R}^d,s\neq t}\frac{|\Phi^w_{s,t}(\xi)|F(\xi)}{\phi(|t-s|)}<\infty\right\}$$

is Borel measurable in any of the above topologies. The fact that exponential irregularity defines Borel sets is established similarly.

We conclude this section with a brief detour on Carathéodory functions and their connection with the exponential irregularity property. Here λ_d denotes the Lebesgue measure on $\mathbb{R}^d$, while $\lambda_{[0,T]}$ denotes the Lebesgue measure on $[0,T]$.

Definition 5.12. *A measurable path $w\colon[0,T]\to\mathbb{R}^d$ is a **Carathéodory function** if, for any set $D\subset\mathbb{R}^d$ such that $\lambda_d(D)>0$ and any $s<t$, it holds $\lambda_{[0,T]}(w^{-1}(D)\cap[s,t])>0$.*

Observe that if w is Carathéodory, then it is unbounded on every interval, thus discontinuous.

Lemma 5.13. *Let w be an exponentially irregular measurable path, then w is Carathéodory.*

Proof. The statement follows immediately from the considerations given at the beginning of Section 6 from [36], see also Sections 11 and 28 from [152]. Let us briefly sketch the proof.

Denote by $\mu^w_{s,t}$ the occupation measure associated to w, see Definition 5.14 below; by the exponential irregularity of w, we can find $c>0$ such that for any $s<t$ it holds

$$\int_{\mathbb{R}^d} e^{c|\xi|}|\hat{\mu}^w_{s,t}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^d} e^{c|\xi|}|\Phi^w_{s,t}(\xi)|^2\,\mathrm{d}\xi < \infty.$$

It then follows from the Paley–Wiener theorem that $\mu^w_{s,t}$ is analytic and therefore it cannot vanish on any set $D\subset\mathbb{R}^d$ such that $\lambda_d(D)>0$; in particular, it must hold

$$\lambda_{[0,T]}(w^{-1}(D)\cap[s,t]) = \int_D \mu^w_{s,t}(y)\,\mathrm{d}y > 0. \qquad \square$$

5.1.2 Link with occupation measures and averaging operators

So far we have discussed several properties of ρ-irregularity, but we haven't rigorously established its relation with regularisation by noise phenomena. It turns out that ρ-irregularity is closely tied to the *occupation measure* of the path w.

In the following $\mathcal{M}_x = \mathcal{M}(\mathbb{R}^d)$ denotes the set of all finite Radon measures on $\mathbb{R}^d$, endowed with the total variation norm $\|\cdot\|_{\mathrm{TV}}$; $\mathcal{M}_+$ is the closed subset of non-negative measures.

Definition 5.14. *Given a measurable path $w\colon[0,T]\to\mathbb{R}^d$, we define its* ***occupation measure*** *as the family $(\mu^w_{s,t})_{0\leqslant s\leqslant t\leqslant T}\subset\mathcal{M}_+$ given by $\mu^w_{s,t} = w_*(\lambda_{[s,t)})$, namely*

$$\int_{\mathbb{R}^d} f(y)\,\mu^w_{s,t}(\mathrm{d}y) = \int_{[s,t)} f(w_r)\mathrm{d}r \quad \forall f\in C^0.$$

Observe that by definition $\mu^w_{s,t} = \mu^w_{0,t} - \mu^w_{0,s}$; for this reason, we will identity the family $(\mu^w_{s,t})_{s\leqslant t}$ with the map $\mu^w\in C^0_t\mathcal{M}_+$ given by $t\mapsto\mu^w_t = \mu^w_{0,t}$, so that $\mu^w_{s,t}$ represents an increment of $\mu^w_{0,\cdot}$.

Note that $\mu^w\in\mathrm{Lip}([0,T];\mathcal{M}_+)$ with $\|\mu^w_{s,t}\|_{\mathrm{TV}} = |t-s|$ and Gateaux derivative $\dot{\mu}^w_t = \delta_{w_t}$.
The Fourier transform of $\mu^w_{s,t}$ is given by

$$\widehat{\mu^w_{s,t}}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot y}\,\mu^w_{s,t}(\mathrm{d}y) = \int_s^t e^{-i\xi\cdot w_r}\,\mathrm{d}r = \overline{\Phi^w_{s,t}(\xi)}$$

which shows that w is (γ,ρ)-irregular if and only if the map $t\mapsto\mu^w_t$ belongs to $C^\gamma_t\mathcal{F}L^{\rho,\infty}$; here the Fourier–Lebesgue space $\mathcal{F}L^{\rho,\infty}$ is given by

$$\mathcal{F}L^{\rho,\infty} = \{f\in\mathcal{S}' : \langle\xi\rangle^\rho|\hat{f}(\xi)|\in L^\infty\},$$

see Appendix A.2 for more details. In particular we have

$$\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} \sim \|\mu^w\|_{C^\gamma\mathcal{F}L^{\rho,\infty}}.$$

Remark 5.15. We will mostly work with given measurable paths w, but both definitions of Φ^w and μ^w are not affected by changing w on a Lebesgue negligible subset of $[0,T]$; therefore they also makes sense when dealing with equivalence classes like $w\in L^p_t$ for $p\in[1,\infty]$. Similarly, it makes sense for w in an equivalence class to say that it is (γ,ρ)-irregular (resp. exponentially irregular).

Occupation measures are also closely related to averaging operators.

Definition 5.16. *Let $w\in L^\infty_t$; we define the averaging operator associated to w as the family of linear operators $\{T^w_{s,t}, 0\leqslant s\leqslant t\leqslant T\}$ acting on $\mathcal{S}'$ given by*

$$T^w_{s,t}b = \int_s^t b(\cdot + w_r)\,\mathrm{d}r.$$

Equivalently, $T^w_{s,t}$ can be defined by duality as follows: for any $\varphi\in\mathcal{S}$ and any $b\in\mathcal{S}'$ it holds

$$\langle T^w_{s,t}b,\varphi\rangle = \langle b, \int_s^t \varphi(\cdot - w_r)\mathrm{d}r\rangle. \tag{5.6}$$

As before, $T^w_{s,t} = T^w_{0,t} - T^w_{0,s}$ and therefore we identify $(T^w_{s,t})_{s \leqslant t}$ with the map $t \mapsto T^w_t = T^w_{0,t}$.

Remark 5.17. Differently from Definitions 5.1 and 5.14, in Definition 5.16 we required $w \in L^\infty_t$. This is because otherwise it would be a priori unclear, for a given Schwartz function $\varphi \in \mathcal{S}$, $\int_s^t \varphi(\cdot - w_r) \mathrm{d}r$ is also Schwartz, and so if the above is a good definition. However, by looking at the Fourier transform $\widehat{\varphi(\cdot - w_t)} = e^{-i\xi \cdot w_t} \hat{\varphi}$, one can check that $w \in L^\infty_t$ can be relaxed to requiring

$$\int_0^T |w_t|^n \, \mathrm{d}t < \infty \quad \text{for all } n \in \mathbb{N},$$

namely $w \in L^p_t$ for all $p < \infty$.

Averaging operators can be defined for time-dependent distributions, as done in Section 3.1.1. However, in the time-dependent case we lose the following fundamental property, which relates the averaging operator to the occupation measure.

Lemma 5.18. *Let $w \in L^\infty_t$, μ^w and T^w as above. Then for any $b \in \mathcal{S}'$, $T^w_{s,t} b = \tilde{\mu}^w_{s,t} * b$, where $\tilde{\mu}$ denotes the reflection of μ, namely $\tilde{\mu}^w_{s,t}(A) = \mu_{s,t}(-A)$.*

Proof. Observe that by definition of occupation measure, for any $s \leqslant t$ and any $R \geqslant \|w\|_{L^\infty}$, it holds supp $\mu^w_{s,t} \subset B_R$. Since $\mu^w_{s,t}$ is a measure with compact support, the convolution $b * \mu^w_{s,t}$ is well defined whenever $b \in \mathcal{S}'$; the same goes for $\tilde{\mu}^w_{s,t}$. For any $\varphi \in \mathcal{S}$ and $x \in \mathbb{R}^d$ it holds

$$\int_s^t \varphi(x - w_r) \mathrm{d}r = \int \varphi(x - y) \mu^w_{s,t}(\mathrm{d}y) = (\varphi * \mu^w_{s,t})(x).$$

The conclusion follows by the duality formula (5.6) and the identity $\langle b, \varphi * \mu^w_{s,t} \rangle = \langle \tilde{\mu}^w_{s,t} * b, \varphi \rangle$. $\square$

As a consequence, in order to quantify the regularising properties of T^w, it suffices to estimate the regularity of μ^w in suitable function spaces. This is exactly where the notion of ρ-irregularity comes into play.

Lemma 5.19. *Let $w \in L^\infty_t$ be (γ, ρ)-irregular. Then for any $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$, the averaging operator T^w belongs to $C^\gamma_t \mathcal{L}(\mathcal{F}L^{\alpha,p}, \mathcal{F}L^{\alpha+\rho,p})$ and for any $b \in \mathcal{F}L^{\alpha,p}$ it holds*

$$\|T^w_{s,t} b\|_{\mathcal{F}L^{\alpha+\rho,p}} \lesssim |t - s|^\gamma \|b\|_{\mathcal{F}L^{\alpha,p}} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}. \tag{5.7}$$

Proof. The statement follows from the considerations given in the introduction of [57]. Alternatively, using Lemma A.16 from Appendix A.2, it holds

$$\|T^w_{s,t} b\|_{\mathcal{F}L^{\alpha+\rho,p}} = \|\tilde{\mu}^w_{s,t} * b\|_{\mathcal{F}L^{\alpha+\rho,p}} \leqslant \|\tilde{\mu}^w_{s,t}\|_{\mathcal{F}L^{\rho,\infty}} \|b\|_{\mathcal{F}L^{\alpha,p}} \lesssim |t - s|^\gamma \|b\|_{\mathcal{F}L^{\alpha,p}} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}. \qquad \square$$

Unfortunately, Fourier–Lebesgue spaces are not always very useful in applications (with the exception of the scale $p = 2$, in which case $\mathcal{F}L^{\alpha,2} = H^\alpha$). We can however use Fourier–Lebesgue embeddings to deduce regularity for $\mu^w_{s,t}$ in other scales of spaces, which in turn imples different estimates for $T^w_{s,t}$. To this end, following [152], we introduce the concept of *occupation density*; in the probabilistic literature it is usually referred to as *local time* and we will indifferently use both terminologies.

Definition 5.20. *We say that a measurable $w: [0, T] \to \mathbb{R}^d$ admits an **occupation density** if, for any $s < t$, $\mu^w_{s,t}$ is absolutely continuous w.r.t. $\lambda_{[s,t)}$, in which case we denote by $\ell^w_{s,t}$ its density, so that $\mu^w_{s,t}(\mathrm{d}x) = \ell^w_{s,t}(x) \mathrm{d}x$. As usual, it holds $\ell^w_{s,t} = \ell^w_{0,t} - \ell^w_{0,s}$ and we set $\ell^w_{0,t} = \ell^w_t$; sometimes we will also use the notation $\ell^w_t(x) = \ell^w(t, x)$.*

The regularity of ℓ^w is again a property defining Borel sets.

Lemma 5.21. *For any $\alpha, \beta > 0$ and $\delta \geqslant 0$, the following set is Borel in the C^δ-topology:*

$$A = \left\{ \varphi \in C^\delta_t : \varphi \text{ admits a local time } \ell^\varphi \in C^{\tilde{\alpha}}_t C^{\tilde{\beta}}_x \text{ for all } \tilde{\alpha} < \alpha, \tilde{\beta} < \beta \right\}.$$

Proof. The proof is almost identical to that of Lemma 5.10; it holds

$$A = \bigcup_{n,m} A_{n,m} = \bigcup_{n,m} \left\{ \varphi \in C_t^\delta : \|\ell^\varphi\|_{C_t^{\alpha-1/n} C_x^{\beta-1/n}} \leqslant m \right\}$$

and usual arguments allow to show that $A_{n,m}$ is closed in C_t^0 (thus C_t^δ) for any $n, m \in \mathbb{N}$. □

With Definition 5.20 and Fourier–Lebesgue embeddings at hand, we can relate ρ-irregularity of w to the regularity of ℓ^w and to the action of T^w on Besov scales B_p^α (or more generally $B_{p,q}^\alpha$). Similar statements can be given for Sobolev spaces $W^{k,p}$ or Bessel spaces $L^{\alpha,p} = (1-\Delta)^{-\alpha/2} L^p$.

Lemma 5.22. *Let $w: [0,T] \to \mathbb{R}^d$ be a (γ, ρ)-irregular measurable path. Then:*

i. If $\rho > d/2$, then w admits an occupation density $\ell^w \in C_t^\gamma L_x^2 \cap \mathrm{Lip}_t L_x^1$.

ii. If $\rho > d$, then ℓ^w is jointly continuous in (t,x) and $\ell^w \in C_t^\gamma C_x^0$.

iii. If $\rho > d/2 + s$ for some $s > 0$, then $\ell^w \in C_t^\gamma H_x^s$; in particular, if $w \in L_t^\infty$, then ℓ is compactly supported on $[0,T] \times \mathbb{R}^d$ and therefore $\ell^w \in C_t^\gamma W_x^{s,1}$.

iv. Thuse, if $\rho > d/2 + s$ for some $s > 0$ and $w \in L_t^\infty$, then for any $\alpha \in \mathbb{R}$, $p \in [1,\infty]$ it holds

$$T^w \in C_t^\gamma \mathcal{L}(B_p^\alpha, B_p^{\alpha+s}).$$

In particular, for any $b \in B_p^\alpha$ it holds

$$\|T_{u,t}^w b\|_{B_p^{\alpha+s}} \lesssim |t-u|^\gamma \|b\|_{B_p^\alpha} \|w\|_{L^\infty}^{d/2} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} \quad \textit{uniformly in } u < t.$$

Proof. By the Fourier–Lebesgue embedding $\mathcal{F}L^{\rho,\infty} \hookrightarrow \mathcal{F}L^{0,2} = L_x^2$, which holds for $\rho > d/2$ (see Lemma A.15 in Appendix A.2), we deduce that $\mu^w \in C_t^\gamma L_x^2$, i.e. for any $s < t$ the measure $\mu_{s,t}^w$ can be identified with a function in L_x^2, which is exactly $\ell_{s,t}^w$. Moreover $\mu_{s,t}^w$ is a positive measure with total variation $\|\mu_{s,t}^w\|_{\mathcal{M}} = t - s$, which implies that $\ell_{s,t}^w \in L_x^1$ with $\|\ell_{s,t}^w\|_{L^1} = t - s$ and thus *i.*

Point *ii.* and the first part of *iii.* follow similarly by using the embeddings $\mathcal{F}L^{\rho,\infty} \hookrightarrow \mathcal{F}L^{0,1} \hookrightarrow C^0$, valid for $\rho > d$, and $\mathcal{F}L^{\rho,\infty} \hookrightarrow \mathcal{F}L^{\rho-d/2-\varepsilon,2} \hookrightarrow H_x^s$, valid for $\rho - d/2 - \varepsilon > s$.

The second half of *iii.* follows from the fact that if $w \in L_t^\infty$, then $\mu_{s,t}^w$ is supported on $B_{\|w\|_{L^\infty}}$ and so we have the estimate $\|\ell_{s,t}^w\|_{W^{s,1}} \lesssim \|w\|_{L^\infty}^{d/2} \|\ell_{s,t}^w\|_{H^s}$.

Finally, statement *iv.* can be deduced from the previous estimate and the Young-type inequality

$$\|f * g\|_{B_p^{\alpha+s}} \lesssim \|f\|_{B_p^\alpha} \|g\|_{W^{s,1}}.$$

□

As already mentioned (cf. Remark 3.5), the operator T^w in general cannot regularise time-dependent fields $b = b(t,x)$, at least not uniformly over all $b \in C_t^0 E$ for suitable Banach spaces E. Intuitively, the reason is that the oscillations in time of b could compensate those of w, and thus limit the induced cancellations. However, if b is behaves sufficiently well as a function of t, it is still possible to obtain a regularising result.

Lemma 5.23. *Let $w \in L_t^\infty$ be (γ,ρ)-irregular, $b \in C_t^\beta \mathcal{F}L^{\alpha,p}$ with $\beta > 1 - \gamma$. Then $T^w b \in C_t^\gamma \mathcal{F}L^{\alpha+\rho,p}$ and there exists a constant $C = C(\gamma+\beta, T) > 0$ such that, for all $0 \leqslant s \leqslant t \leqslant T$, it holfd*

$$\|T_{s,t}^w b\|_{\mathcal{F}L^{\alpha+\rho,p}} \leqslant C \|b\|_{C^\beta \mathcal{F}L^{\alpha,p}} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} |t-s|^\gamma. \tag{5.8}$$

Namely, the linear map $T^w: C_t^\beta \mathcal{F}L^{\alpha,p} \to C_t^\gamma \mathcal{F}L^{\alpha,p+\rho}$ is bounded with constant $C\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}$.

Proof. Let us first assume that b is smooth; in this case, for any $[s,t] \subset [0,T]$ and any sequence Π of partitions of $[s,t]$ with infinitesimal mesh, it holds

$$\begin{aligned} T_{s,t}^w b(x) &= \int_s^t b_r(x + w_r) \mathrm{d}r = \lim_{|\Pi| \to 0} \sum_i \int_{t_i}^{t_{i+1}} b_{t_i}(x + w_r) \mathrm{d}r \\ &= \lim_{|\Pi| \to 0} \sum_i T_{t_i,t_{i+1}}^w [b(t_i, \cdot)](x) = \lim_{|\Pi| \to 0} \sum_i (b_{t_i} * \tilde{\mu}_{t_i,t_{i+1}}^w)(x). \end{aligned}$$

Namely, the function $T^w_{s,t}b$ is the sewing (in the sense of Lemma 1.1) in $\mathcal{F}L^{\alpha,p+\rho}$ of $\Gamma_{s,t} := b_s * \tilde{\mu}^w_{s,t}$. By the assumptions, it holds

$$\|\Gamma_{s,t}\|_{\mathcal{F}L^{\alpha,p+\rho}} \leqslant \|b_s\|_{\mathcal{F}L^{\alpha,p}} \|\tilde{\mu}^w_{s,t}\|_{\mathcal{F}L^{\rho,\infty}} \leqslant \|b\|_{C^\beta \mathcal{F}L^{\alpha,p}} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} |t-s|^\gamma,$$
$$\|\delta\Gamma_{s,u,t}\|_{\mathcal{F}L^{\alpha,p+\rho}} = \|b_{s,u} * \tilde{\mu}^w_{u,t}\|_{\mathcal{F}L^{\alpha,p+\rho}} \leqslant \|b\|_{C^\beta \mathcal{F}L^{\alpha,p}} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} |t-s|^{\gamma+\beta},$$

where by assumption $\gamma + \beta > 1$; applying Lemma 1.1 we deduce estimate (5.8).

The case of general b follows from a standard approximation procedure: given $b \in C^\beta_t \mathcal{F}L^{\alpha,p}$, we can find a smooth sequence b^n such that $\|b^n\|_{C^\beta \mathcal{F}L^{\alpha,p}} \leqslant \|b\|_{C^\beta \mathcal{F}L^{\alpha,p}}$ and $b^n \to b$ in $C^{\beta-}_t \mathcal{F}L^{\alpha-,p}$. By properties of averaging, for any $s < t$, $T^w_{s,t} b^n$ converges to $T^w_{s,t} b$ weakly-$*$ in $\mathcal{F}L^{\alpha,p}$; the conclusion then follows from taking the liminf as $n \to \infty$ on both sides of (5.8) applied to b^n and using the Fatou property of weak-$*$ convergence. □

Remark 5.24. It is clear that the proof can be readapted in a more general setting: given E, F, G function spaces such that $*: (f, g) \mapsto f * g$ is a bilinear bounded map from $E \times F$ into G, if $\mu^w \in C^\gamma_t F$ and $\gamma + \beta > 1$, then $T^w : C^\beta_t E \to C^\gamma_t G$ is a linear bounded map. This can be applied in combination with Lemma 5.22 to obtain regularising effects of T^w when E and G are taken in suitable Besov scales.

5.1.3 Notions of local nondeterminism of stochastic processes

There is now a huge literature on local-nondeterminism and several alternative definitions, which are not in general equivalent, see [272] for a survey; here we identify two types of LND which are closely tied to ρ-irregularity and exponential irregularity of sample paths of Gaussian processes. They will play a major role in the proofs in Section 5.3.

Definition 5.25. *Let $(X_t)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued separable Gaussian process adapted to a given filtration $\mathcal{F}_t$. We say that X is* ***strongly locally nondeterministic*** *with parameter $\beta > 0$, X is β-SLND for short, if there exists $\delta > 0$ such that*

$$\mathrm{Var}(X_t | \mathcal{F}_s) \gtrsim |t-s|^{2\beta} I_d \quad \textit{uniformly in } s,t \textit{ such that } 0 < t-s < \delta. \tag{5.9}$$

Properly speaking, in the terminology of [272], Definition 5.25 is that of a *one-sided* strong local nondeterminism, but we have preferred to adopt the terminology β-SLND for simplicity.

As already pointed out in Appendix A.1, if W^H is a fractional Brownian motion of parameter $H \in (0,1)$, then it is H-SLND.

Let us extend the definition of fBm recursively as follows: given W^H, fBm of Hurst parameter $H \in (n, n+1)$, we define the process $W^{H+1}_t := \int_0^t W^H_s \mathrm{d}s$ to be the fBm of Hurst parameter $H+1$; in this way, we can cover $H \in (0,+\infty) \setminus \mathbb{N}$. It is clear that W^H is a Gaussian centered process, with trajectories in C^{H-}_t. It might be slightly less obvious that W^H is also H-SLND, for any such H.

To see this, recall from Appendix A.1 that, for $H \in (0,1)$, we have the representation

$$W^H_t = c_H \int_{-\infty}^t \big[(t-r)_+^{H-1/2} - (-r)_+^{H-1/2}\big] \mathrm{d}B_r$$

where B is a 2-sided standard Bm. Taking $\mathcal{F}_t = \sigma(B_r : r \leqslant t)$, it holds

$$\begin{aligned} W^{H+1}_t - \mathbb{E}_s W^{H+1}_t &= \int_0^t (W^H_r - \mathbb{E}_s W^H_r) \mathrm{d}r = \int_s^t (W^H_r - \mathbb{E}_s W^H_r) \mathrm{d}r \\ &= c_H \int_s^t \int_s^r (r-u)^{H-1/2} \, \mathrm{d}B_u \, \mathrm{d}r = \tilde{c}_H \int_s^t (t-u)^{(H+1)-1/2} \, \mathrm{d}B_u, \end{aligned}$$

where in the last passage we applied stochastic Fubini theorem. As a consequence,

$$\begin{aligned} \mathrm{Var}(W^{H+1}_t | \mathcal{F}_s) &= \mathrm{Var}(W^{H+1}_t - \mathbb{E}_s W^{H+1}_t) \\ &\sim_H I_d \int_s^t (t-u)^{2(H+1)-1} \, \mathrm{d}u \sim |t-u|^{2(H+1)} \, I_d \end{aligned}$$

which shows the $(H+1)$-SLND property for W^{H+1}. The general case W^{H+n}, with $H \in (0,1)$ and $n \in \mathbb{N}$, can be handled similarly.

The argument above shows a few nice properties: i) it is possible to have β-SLND Gaussian processes with arbitrarily smooth trajectories; ii) such processes may also have derivatives which are still $\tilde{\beta}$-SLND, for other values $\tilde{\beta} > 0$.

The second notion of local nondeterminism we will need is the following one.

Definition 5.26. *Let $\{X_t\}_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued separable Gaussian process adapted to a given filtration $\mathcal{F}_t$. We say that X is **exponentially locally nondeterministic** with parameter $\beta > 0$, X is β-eSLND for short, if there exists $\delta > 0$ s.t.*

$$\mathrm{Var}(X_t|\mathcal{F}_s) \gtrsim |\log(t-s)|^{-\beta} I_d \quad \textit{uniformly in } s,t \textit{ such that } 0 < t-s < \delta. \tag{5.10}$$

It is desirable to have explicit examples of processes satisfying Definition 5.26. One way to construct them is to consider, for given $\beta > 0$, the $\mathbb{R}^d$-valued Gaussian process X^β defined by

$$X_t^\beta = \int_0^t (t-s)^{-1/2} |\log(t-s)|^{-\beta/2-1/2} \mathrm{d}B_s \quad \forall t \in [0, 1/2]. \tag{5.11}$$

The proof that this process is β-eSLND is not entirely trivial and will be given later, see Proposition 5.51 in Section 5.3.2.

Remark 5.27. Definitions 5.25 and 5.26 only involve the conditional variance of the process X and are thus independent of its mean. This implies that they are all properties invariant under deterministic perturbations; namely, if X is a β-(e)SLND process and f is a given measurable function, then $X + f$ is still β-(e)SLND. This can be interpreted as the chaoticity represented by local nondeterminism being too strong to be disrupted by deterministic additive perturbations; this fundamental feature will allow us to prove prevalence of ρ-irregularity and exponential irregularity.

5.2 Main results and applications

Compared to other chapters, here we prefer to adopt a slightly different structure. Rather than delving immediately in the proofs of single statements, we will present in Section 5.2.1 below the main results of this chapter and we will apply them in Section 5.2.2 to present several instances of regularisation by noise in ODEs and PDEs. The proofs will be instead postponed to the later Sections 5.3-5.4, which constitute the main body of the current chapter.

5.2.1 Statements

The start by presenting the main probabilistic result of this chapter.

Theorem 5.28. *The following hold:*

i. Let X be a continuous β-SLND Gaussian process in $\mathbb{R}^d$; then for any $\rho < (2\beta)^{-1}$ there exists $\gamma = \gamma(\rho, \beta) > 1/2$ such that X is (γ, ρ)-irregular with probability 1.

ii. If additionally $\beta < 1/d$, then $\ell^X \in C_t^{1-\beta d-} C_x^0 \cap C_t^0 C_x^{1\wedge[1/(2\beta)-d/2]-}$ with probability 1.

iii. Let X be a β-eSLND Gaussian with measurable, L^2-integrable trajectories and $\beta \in (0,1]$; then X is exponentially irregular with probability 1.

The proof will be presented in Section 5.3; we can immediately combine Theorem 5.28 with the basic considerations from Section 5.1.3 (especially Remark 5.27) to obtain the following prevalence result (for the notion of C^∞_{loc} slowly increasing function, we refer to Definition A.17 in Appendix A.2).

Theorem 5.29. *It holds that:*

i. For any $\delta \in (0,\infty)$, almost every $\varphi \in C_t^\delta$ is ρ-irregular for any $\rho < (2\delta)^{-1}$. If $\delta \geqslant 1$, then in addition for any $k < \delta$, $D^{(k)}\varphi$ is ρ-irregular for any $\rho < (2(\delta-k))^{-1}$.

ii. *For $\delta \in (0, 1/d)$, almost every $\varphi \in C_t^\delta$ admits an occupation density ℓ^φ and moreover*

$$\ell^\varphi \in C_t^{1-\delta d-} C_x^0 \cap C_t^0 C_x^{1 \wedge \left(\frac{1}{2\delta} - \frac{d}{2}\right)-}.$$

iii. *Almost every $\varphi \in C_t^0$ is ρ-irregular for any $\rho < \infty$. In particular, its occupation measures $(\mu_{s,t}^\varphi) \subset C_c^\infty$ and its averaging operator T^φ maps $\mathcal{S}'$ into the space of C_{loc}^∞, slowly increasing functions; moreover, T^φ maps $\cup_{s \in \mathbb{R}} B_{p,q}^s$ into C_b^∞.*

iv. *For any $p \in [1, \infty)$, almost every $\varphi \in L_t^p$ is exponentially irregular; in particular it is Carathéodory and its occupation measures $(\mu_{s,t}^\varphi)$ are analytic.*

We avoid providing very similar statements, but like Point *ii.* above, Point *i.* in combinations with Lemmata 5.19, 5.22 and 5.23 provides several other prevalence statements in C_t^δ regarding the regularity of ℓ^φ and the regularising effect of T^φ acting on suitable function spaces.

Proof. We have already seen in Lemmas 5.10 and 5.21 that all the above properties (ρ-irregularity, exponential irregularity, Hölder continuity of local times) define Borel measurable sets in the function spaces appearing in points *i.-iv.*, so we only need to provide suitable measures "witnessing" their prevalence.

Let us start from the case $\delta \in (0, 1)$. Let μ^H to be the law on C_t^δ of a fBm W^H of parameter $H = \delta + \varepsilon$ for some $\varepsilon > 0$, which is tight on C_t^δ; let $\varphi \in C_t^\delta$ be fixed. The process W^H is H-SLND and by Remark 5.27 so is $\varphi + W^H$; then by Point *i.* of Theorem 5.28 it holds

$$\mu^H\left(\varphi + w \text{ is } \rho\text{-irregular for any } \rho < \frac{1}{2H}\right) = \mathbb{P}\left(\varphi + W^H \text{ is } \rho\text{-irregular for any } \rho < \frac{1}{2H}\right) = 1.$$

This implies that almost every $\varphi \in C_t^\delta$ is ρ-irregular for any $\rho < 1/(2\delta + 2\varepsilon)$; taking a sequence $\varepsilon_n \downarrow 0$, using the fact that countable intersection of prevalent sets is still prevalent, we obtain the conclusion in this case.

Consider now the case $\delta \in [n, n+1)$, $n \geqslant 1$; set $\delta = n + \theta$, $\theta \in [0, 1)$. Denote by μ^{n+H} the law of the process W^{n+H} obtained by integrating n times an fBm, which is discussed in Section 5.1.3; choose $H > \theta$, so that μ^{n+H} is tight in C_t^δ. Now fix $\varphi \in C_t^\delta$; the process Y is $(n+H)$-SLND with $D^{(k)} Y$ being $(n + H - k)$-SLND for any $k \in \{1, \ldots, n\}$ therefore $Y + \varphi$ and $D^{(k)}(Y + \varphi) = D^{(k)} Y + D^{(k)} \varphi$ have the same properties by Remark 5.27. Applying Point *i.* of Theorem 5.28 and arguing as in the previous point, taking a sequence $H_n \downarrow \delta$, the proof of claim *i.* is complete.

The proof of Point *ii.* is essentially the same, again using μ^H with $H = \delta + \varepsilon$ as a witnessing measure, only this time relying on a combination of Point *ii.* of Theorem 5.28.

The first part of Claim *iii.* is identical, relying this time on the fact that we can take any $H > 0$ and we obtain ρ-irregularity for any $\rho < 1/(2H)$, together with the property that countable intersection of prevalent sets is prevalent. The second part of claim *iii.* concerning C_{loc}^∞ follows from the fact that $C_c^\infty \hookrightarrow \mathcal{S}$ and an application of Proposition A.18; the part concerning C_b^∞ instead follows from Lemma 5.22.

The second part of Point *iv.* follows from Lemma 5.13, once we have shown the first part. For any $\beta \in (0, 1)$, denote by μ^β the law of (a measurable version of) the process X^β defined in (5.11), which is β-eSLND by Proposition 5.51. Let $\varphi \in L_t^p$; we can require φ to be an actual measurable function in its equivalence class, since the property of exponential irregularity does not depend on the chosen representative. By Remark 5.27, the process $\varphi + X^\beta$ is also β-eSLND and so by Theorem 5.28

$$\mu^\beta(\varphi + w \text{ is exponentially irregular}) = \mathbb{P}(\varphi + X^\beta \text{ is exponentially irregular}) = 1,$$

which implies the conclusion. □

Theorem 5.29, combined with geometric measure theory considerations, implies the next result. Here $\dim_F$ and $\dim_H$ denote respectively the Fourier and Hausdorff dimensions; their definitions, as well as that of Salem sets, will be recalled in Section 5.4.1, together with the proof of Theorem 5.30.

Theorem 5.30. *Let $\delta \in [0, 1)$. The following hold:*

i. *If $\delta \geqslant 1/d$, then almost every $\varphi \in C_t^\delta$ has the property that*

$$\dim_F(\varphi([s, t])) = \dim_H(\varphi([s, t])) = \frac{1}{\delta} \quad \forall [s, t] \subset [0, T].$$

ii. If $\delta < 1/d$, then almost every $\varphi \in C_t^\delta$ has the property that

$$\dim_F(\varphi([s,t])) = \dim_H(\varphi([s,t])) = d \quad \forall [s,t] \subset [0,T].$$

Moreover, for all $[s,t] \subset \mathbb{R}^d$, $\varphi([s,t])$ contains an open set.

In particular, for all $\delta \in [0,1)$, the image of almost every function $\varphi \in C_t^\delta$ is a Salem set.

The study of analytic properties of (γ, ρ)-irregular paths allows to show that, as the name suggests, they have a highly oscillatory behaviour; this can be related to other notions of roughness already existing in the literature.

Theorem 5.31. *Let w be (γ, ρ)-irregular, $\delta^*_{\gamma,\rho}$ defined as in (5.4). Then for any $\delta > \delta^*_{\gamma,\rho}$, w is nowhere δ-Hölder continuous and has infinite modulus of δ-Hölder roughness; for any $p < (\delta^*_{\gamma,\rho})^{-1}$ and any interval $[s,t] \subset [0,T]$, w has infinite p-variation on $[s,t]$.*

The proof is given in Section 5.4.2, where the concept of modulus of δ-Hölder roughness is also recalled. Quite nicely, Theorem 5.31 provides an alternative proof of Hunt's original results from [178].

Corollary 5.32. *Let $\delta \in [0,1)$, then almost every $\varphi \in C_t^\delta$ is nowhere $(\delta + \varepsilon)$-Hölder for any $\varepsilon > 0$.*

Proof. By Theorem 5.29, almost every $\varphi \in C_t^\delta$ has the following property: for any $\rho < 1/(2\delta)$, there exists $\gamma > 1/2$ such that φ is (γ, ρ)-irregular. It holds

$$\delta^*_{\gamma,\rho} = \frac{1-\gamma}{\rho} > \frac{1}{2\rho}$$

which implies by Theorem 5.31 that any such function is nowhere $\tilde{\delta}$-Hölder for any $\tilde{\delta} > 1/(2\rho)$. Taking $\rho = 1/(2\delta + 2\varepsilon)$ the conclusion follows. □

5.2.2 Applications to regularisation by noise

In this section we show how our main prevalence statement, Theorem 5.29, can be combined with already existing results to obtain results on the regularising effect of almost every $w \in C_t^\delta$ on ODEs and PDEs.

As mentioned before, (γ, ρ)-irregularity is closely related to the regularising properties of the averaging operator T^w; in turn, as already explained in Chapter 3, regularity of the averaged field $T^w b$ is all that is needed in order to develop a good solution theory for the perturbed ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x_t = b(x_t) + \frac{\mathrm{d}}{\mathrm{d}t} w_t, \tag{5.12}$$

see Theorem 3.52. We can combine it with Lemma 5.19 and Theorem 5.29 as follows.

Theorem 5.33. *Let $\delta \in [0,1)$, then a.e. $w \in C_t^\delta$ is such that, for any $\alpha \in \mathbb{R}$, the following hold:*

i. If $\alpha > \max\{3/2 - (2\delta)^{-1}, 0\}$ or $\alpha > 2 - (2\delta)^{-1}$, then for any $b \in \mathcal{F}L^{\alpha,1}$ the perturbed ODE (5.12) is well posed and admits a locally Lipschitz flow.

ii. If $\alpha > \max\{n + 1/2 - (2\delta)^{-1}, 0\}$ or $\alpha > n + 1 - (2\delta)^{-1}$, then for any $b \in \mathcal{F}L^{\alpha,1}$ the flow is locally C^n.

In particular, for almost every $\varphi \in C_t^0$, for any $\alpha \in \mathbb{R}$ and any $b \in \mathcal{F}L^{\alpha,1}$, the ODE (5.12) is wellposed and admits a C^∞ flow.

The last part of the statement is again an instance of the ∞-regularising effect of generic continuous functions on (5.12). Applying Lemma 5.22, analogous statements can be given replacing Fourier–Lebesgue $\mathcal{F}L^{\alpha,1}$ with other scales like $B^s_{p,q}$ spaces. In addition, we can also consider time-dependent fields b, for instance such that $b \in C_t^{1/2} \mathcal{F}L^{\alpha,1}$, thanks to Lemma 5.23.

The theory developed for solving (5.12) can be also successfully applied to the study first order linear PDEs of the form

$$\partial_t u + b \cdot \nabla u + cu + \frac{\mathrm{d}w}{\mathrm{d}t} \cdot \nabla u = 0, \tag{5.13}$$

once they are interpreted in a suitable nonlinear Young sense (cf. Sections 3.2.4 and 3.2.4). The abstract results from Section 2.1 (equivalently, its adaptation to this setting given by Theorem 3.57) can then be combined with Lemma 5.19 and Theorem 5.29 to obtain the following.

Theorem 5.34. *Let $\delta \in [0,1)$, then a.e. $w \in C_t^\delta$ is such that, for any $\alpha \in \mathbb{R}$, the following hold:*

i. If $\alpha > \max\{3/2 - (2\delta)^{-1}, 0\}$ or $\alpha > 2 - (2\delta)^{-1}$, then for any $b \in \mathcal{F}L^{\alpha,1}$ the transport PDE

$$\partial_t u + b \cdot \nabla u + \frac{\mathrm{d}w}{\mathrm{d}t} \cdot \nabla u = 0$$

has a unique solution $u \in C_t^0 C_{\mathrm{loc}}^1 \cap C_t^{1/2} C_{\mathrm{loc}}^0$ for any $u_0 \in C_x^1$.

ii. If $\alpha > \max\{3/2 - (2\delta)^{-1}, 0\}$ or $\alpha > 2 - (2\delta)^{-1}$, then for any $b \in \mathcal{F}L^{\alpha,1}$ the continuity equation

$$\partial_t u + \nabla \cdot (bu) + \frac{\mathrm{d}w}{\mathrm{d}t} \cdot \nabla u = 0$$

has a unique weak solution $u \in C_t^{1/2}(C_x^1)^* \cap L_t^\infty \mathcal{M}_x$ for any $u_0 \in \mathcal{M}_x$.

In the above cases, w enters the equation as a perturbation that can be reabsorbed by shifting the phase space (i.e. applying a Galilean transformation, as explained in Section 3.2.4 when passing from (3.45) to (3.46)), which is why the operator T^w appears. In the next examples instead w has the role of *modulating* a given group of transformations.

In the papers [66] and [65], the authors study the regularising properties of (γ, ρ)-irregular paths on nonlinear dispersive PDEs of the general form

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t = A \varphi_t \frac{\mathrm{d}w_t}{\mathrm{d}t} + \mathcal{N}(\varphi_t), \tag{5.14}$$

where $w \in C_t^0$, $\varphi: D \to \mathbb{R}$ (or $\mathbb{C}$), A is a linear unbounded operator and $\mathcal{N}$ is a nonlinearity (typically of polynomial type). Their cases of interest are:

1. (NLS) Nonlinear cubic Schrödinger, $D = \mathbb{T}^d$ or $\mathbb{R}^d$, $d = 1, 2$, $A = i\Delta$, $\mathcal{N}(\varphi) = i|\varphi|^2 \varphi$;
2. General NLS on $D = \mathbb{R}$ with $A = i\partial_x^2$, $\mathcal{N}(\varphi) = i|\varphi|^\mu \varphi$, $\mu \in (1, 4]$;
3. (dNLS) Nonlinear derivative cubic Schrödinger on $\mathbb{T}$, $A = i\partial_x^2$, $\mathcal{N}(\varphi) = i\partial(|\varphi|^2 - \|\varphi\|_{L^2}^2)\varphi$;
4. (KdV) Korteweg–de Vries, $D = \mathbb{T}$ or $\mathbb{R}$, $A = \partial_x^3$, $\mathcal{N}(\varphi) = \partial_x(\varphi^2)$;
5. (mKdV) Modified Korteweg–de Vries, $D = \mathbb{T}$, $A = \partial_x^3$, $\mathcal{N}(\varphi) = \partial_x(\varphi^3 - 3\varphi \|\varphi\|_{L^2}^2)$.

In all the cases above, although the original system (i.e. with $w_t = t$) would be of integrable nature, the presence of w doesn't allow to exploit this feature; moreover the group $\{e^{tA}\}_{t \in \mathbb{R}}$ associated to A acts isometrically on all H^α-spaces, thus doesn't provide a priori any regularisation.

In order to give meaning to (5.14), the authors adopt the mild formulation (which would be justified for $w \in C^1$ by the chain rule, but a posteriori is meaningful for any continuous w[5.3]):

$$\varphi_t = U_t^w \varphi_0 + U_t^w \int_0^t (U_s^w)^{-1} \mathcal{N}(\varphi_s) \mathrm{d}s, \quad U_t^w := e^{w_t A}.$$

Applying the change of variables $\psi_t = (U_t^w)^{-1} \varphi_t$, the equation becomes

$$\psi_t = \varphi_0 + \int_0^t (U_s^w)^{-1} \mathcal{N}(U_s^w \psi_s) \mathrm{d}s. \tag{5.15}$$

5.3. There is a non trivial aspect underlying this apparently simple maneuver. In particular, if w can be enhanced to a rough path, the idea that the chain rule should transfer to w by density of smooth functions actually means that we are by assumption working with a *geometric* rough path. In the stochastic setting, if w were sampled as a Brownian motion, then multiplication by $\mathrm{d}w/\mathrm{d}t$ should be interpreted in the Stratonovich sense.

The major obstacle, in order to solve equation (5.15) in spaces H^α of low regularity, is that it is unclear how to define the nonlinearity $\mathcal{N}$ in a pointwise manner. The fundamental intuition from [66, 65] is that we actually don't need it; instead, it suffices to know that that the "*time-averaged modulated nonlinearity*" is meaningful, namely the family of operators $\{X_{s,t}\}_{s<t}$ formally given by

$$X_{s,t}(\phi) := \int_s^t (U_r^w)^{-1} \mathcal{N}(U_r^w \phi) \mathrm{d}s, \quad s<t.$$

Using the ρ-irregularity of w, it is possible to show that the maps $X_{s,t}$ are continuous from H^α to itself (actually C^∞, since they are the monoid associated to an n-linear bounded operator). Then φ is defined to be a solution to (5.14) if and only if the associated ψ solves (5.15), which is interpreted as a nonlinear Young equation in a Hilbert space. We refrain from giving further details on the topic and only point out that our Theorem 5.29, combined with their results (Theorem 1.8 from [66], Theorems 1.6 and 1.7 from [65]), gives the following statements.

Theorem 5.35. *Let $\delta \in [0,1)$. Then for almost every $w \in C_t^\delta$, the w-modulated cubic NLS on $\mathbb{T}$ and $\mathbb{R}$ has a unique global solution in H^α for any $\alpha \geqslant 0$; moreover the equation admits a locally Lipschitz continuous flow.*

Theorem 5.36. *Let $\delta \in [0, 2/3)$. Then:*

i. For almost every $w \in C_t^\delta$, the w-modulated KdV on $\mathbb{T}$ has a unique local solution in H^α for any $\varphi_0 \in H^\alpha$ with $\alpha > -(2\delta)^{-1}$, which is global if $\alpha > -\min\{3/2, (4\delta)^{-1}\}$.

ii. For almost every $w \in C_t^\delta$, the w-modulated KdV on $\mathbb{R}$ has a unique local solution in H^α for any $\varphi_0 \in H^\alpha$ with $\alpha > -\min\{3/4, (2\delta)^{-1}\}$,which is global if $\alpha > -\min\{3/4, (4\delta)^{-1}\}$.

Moreover for any $\delta \in [0,1)$, for almost every $w \in C_t^\delta$, the w-modulated mKdV on $\mathbb{T}$ has a unique local solution in H^α for any $\varphi_0 \in H^\alpha$ with $\alpha \geqslant 1/2$.

Analogue statements can be obtained combining Theorem 5.29 with other results from the aforementioned papers, for instance Theorems 1.9 and 1.10 from [66].

In the setting of standard dispersive equations, a key role in establishing uniqueness of solutions is played by Strichartz estimates. In the paper [101], for a given path $w \in C_t^0 \mathbb{R}$, the authors study under which conditions the operator A given by

$$f \mapsto (Af)_t := \int_0^t |w_t - w_s|^{-\alpha} f_s \,\mathrm{d}s$$

is bounded from L_t^p to L_t^q for suitable values of (p,q); the idea is to apply this kind of modulated Hardy–Littlewood–Sobolev inequality to obtain Strichartz estimates for the modulated semigroup

$$P_{s,t}\psi(x) = e^{i\Delta(w_t - w_s)}\psi(x) = \frac{1}{(4\pi(w_t - w_s))^{d/2}} \int_{\mathbb{R}^d} \exp\left(i \frac{|x-y|^2}{4(w_t - w_s)}\right) \varphi(y) \mathrm{d}y.$$

The authors only consider w sampled as a stochastic process, specifically a fBm of parameter $H \in (0,1)$; however, up to a closer inspection of the proof, Theorem 1.1 from [101] can be restated in an analytic fashion as follows.

Theorem 5.37. *Suppose that $w \in C_t^0$ admits an occupation density $\ell^w \in C_t^\beta C_x^0$ for some $\beta \in (0,1)$. Then for any $p, q \in (1, \infty)$ and $\alpha \in (0,1)$ satisfying*

$$2 - \alpha = \frac{1}{p} + \frac{1}{q}$$

there exists a constant $C>0$ such that for all $f \in L_t^p$ and $g \in L_t^q$ it holds

$$\left| \int_0^T \int_0^T f_t \, |w_t - w_s|^{-\alpha} \, g_s \, \mathrm{d}s \mathrm{d}t \right| \leqslant C T^{\alpha\beta} \, \|f\|_{L^p} \|g\|_{L^q}.$$

For any $\delta \in [0,1)$, almost every $\varphi \in C_t^\delta$ satisfies the above assumption for any $\beta < 1 - \delta$.

Proof. The proof of Theorem 1.1 in [101] is entirely analytical, as it follows closely the proof of the standard Hardy–Littlewood–Sobolev inequality from Lieb–Loss [203], but it requires a key property satisfied by fBm paths, given in Lemma 2.1 therein: defined

$$M(r,T) := \sup_{t\in[0,T]} \int_0^T \mathbb{1}_{|w_t-w_s|<r}\, \mathrm{d}s,$$

then there must exist a constant c such that

$$M(r,T) \leqslant 2crT^{\beta} \quad \text{for all } r>0. \tag{5.16}$$

It is not difficult to see that requirement (5.16) is equivalent to the request that $\ell_T^w(x) \leqslant cT^{\beta}$ for all $x\in\mathbb{R}$; indeed, assume first that $\ell_T^w(x)\leqslant cT^{\beta}$ holds, then

$$M(r,T) = \sup_{t\in[0,T]} \int_0^T \mathbb{1}_{w_s\in B(w_t,r)}\, \mathrm{d}s = \sup_{t\in[0,T]} \int_{B(w_t,r)} \ell_T^w(x)\mathrm{d}x \lesssim 2rcT^{\beta}.$$

On the other side, if w admits a continuous density ℓ^w and (5.16) holds, then

$$\ell_T^w(w_t) = \lim_{r\to 0} \frac{1}{2r} \int_{B(w_t,r)} \ell_T^w(x)\mathrm{d}x \leqslant \lim_{r\to 0} \frac{1}{2r} M(r,T) \leqslant cT^{\beta},$$

and since we know that ℓ_T^w is supported on $w([0,T])$, the above estimate extends to all $x\in\mathbb{R}^d$.

It is now clear that requirement (5.16) can be expressed in entirely analytical terms, and so does the proof of Theorem 1.1 from [101]; the authors are only using the additional fact that almost every fBm trajectory satisfies (5.16). But then to get the last statement, it now suffices to apply Point ii. of Theorem 5.29 (for $d=1$) instead. □

Similarly, the proofs of Strichartz estimates and wellposedness for w-modulated NLS (Proposition 1.1 and Theorem 1.2 respectively) from [101] are entirely deterministic and only rely on the validity of the above modulated Hardy–Littlewood–Sobolev inequality; they can therefore be fully generalised to prevalence results, similarly to Theorems 5.35 and 5.36 above.

In [64], the authors provide regularity estimates for solutions to scalar conservation laws modulated by a path w of the form[5.4]

$$\partial_t u + \sum_{i=1}^d \partial_{x_i} A^i(u)\cdot \frac{\mathrm{d}w_t^i}{\mathrm{d}t} = 0 \qquad \text{on } \mathbb{T}^d, \quad u|_{t=0} = u_0 \in L_x^{\infty}. \tag{5.17}$$

They use the concept of (γ,ρ)-irregularity to show regularisation by noise phenomena whenever w is sampled as an fBm, but their results are of analytic (or path-by-path) type; before stating their result, let us point out a simplification: given a (γ,ρ)-irregular $w\in C_t^{\delta}$, the authors impose, on a suitable parameter $\lambda>0$, depending on another parameter $\nu\geqslant 1$, the condition

$$\lambda < \frac{\rho(\delta+1)-(1-\gamma)}{(\nu\rho\vee 1)(\delta+1)+(1-\gamma)} \wedge \frac{\rho+2(\nu\rho\vee 1)}{(\nu\rho\vee 1)(2\delta+1)+(1-\gamma)} =: c_1\wedge c_2.$$

Thanks to Theorem 5.31, we can actually simplify the above expression; we claim that $c_1\leqslant c_2$. Indeed, since $\delta\leqslant \delta_{\gamma,\rho}^*$, we have

$$\rho(\delta+1)-(1-\gamma) \leqslant \rho + \rho\delta_{\gamma,\rho}^* - (1-\gamma) \leqslant \rho;$$

to check that $c_1\leqslant c_2$, it then suffices to verify that

$$\frac{\rho}{(\nu\rho\vee 1)(\delta+1)+1-\gamma} \leqslant \frac{\rho+2(\nu\rho\vee 1)}{(\nu\rho\vee 1)(2\delta+1)+(1-\gamma)}$$

and after a few algebraic manipulations we see that this is equivalent to

$$\rho\delta \leqslant 2[(\nu\rho\vee 1)(\delta+1)+1-\gamma];$$

5.4. The authors in [64] use the notation $\circ\mathrm{d}w_t^i/\mathrm{d}t$ to stress that they are using a "Stratonovich type" noise, see the previous footnote.

this is now clearly always true since $\nu \geqslant 1$, so that $\rho\delta \leqslant (\nu\rho \vee 1)\delta \leqslant 2(\nu\rho \vee 1)(\delta + 1)$.

The main result of [64] can then be restated as follows:

Theorem 5.38. (Theorem 2.4 from [64]) *Let $w \in C_t^\delta$ be (γ, ρ)-irregular and let u be a quasi-solution to (5.17). Assume $A = (A^1, \ldots, A^d) \in C^2(\mathbb{R}; \mathbb{R}^d)$ satisfies the following non-degeneracy condition: there exist $\nu \geqslant 1$ and $c > 0$ such that, for $A' = a = (a^1, \ldots, a^d) \in C^1(\mathbb{R}; \mathbb{R}^d)$, it holds*

$$\inf_{v \in \mathbb{R}^d} \max_{i = 1, \ldots, d} |v_i(a^i(x) - a^i(y))| \geqslant c|x - y|^\nu \quad \text{for all } x, y \in \mathbb{R}.$$

Then there exists a constant $C = C(\|\Phi^w\|_{\mathcal{W}^{\gamma, \rho}})$ such that for all $T > 0$ and all

$$\lambda < \frac{\rho(\delta + 1) - (1 - \gamma)}{(\nu\rho \vee 1)(\delta + 1) + (1 - \gamma)}$$

it holds

$$\int_0^T \|u_t\|_{W^{\lambda, 1}} \mathrm{d}t \leqslant C(\|u_0\|_{L^1} + \|u\|_{L^1_{t, x}} + \|w\|_{C^\delta} \|a'(v)\, m\|_{\mathrm{TV}}). \tag{5.18}$$

If u is an entropy solution then in addition

$$\|u_t\|_{W^{\lambda, 1}} < \infty \quad \text{for all } t > 0.$$

We avoid here providing all the details on the above result, for which we refer the reader to [64]. Let us only mention that the definition of quasi-solution to (5.17) requires the existence of a finite Radon measure m associated to u, which is the one appearing in estimate (5.18); u is an entropy solution if m is non-negative. A few algebraic manipulations, combined with Theorem 5.29, imply the following result.

Corollary 5.39. *Let $\delta \in (0, 1)$. For a.e. $\varphi \in C_t^\delta$, the statement of Theorem 5.38 holds for any*

$$\lambda < \frac{1}{(\nu \vee 2\delta)(\delta + 1) + \delta}.$$

5.3 Criteria for stochastic processes

This section is devoted to the study of probabilistic properties ensuring that a stochastic process has (γ, ρ)-irregular sample paths, or even continuous occupation measure. It includes the proof of our main result (Theorem 5.28), which is the cornerstone for our prevalence statements (Theorem 5.29), but also develops several criteria of independent interest. In particular, we establish ρ-irregularity of processes like fBm, α-stable process, Ornstein–Uhlenbeck as well as $X_t = \int_0^t B_s \,\mathrm{d}s$; many of these process have already appeared in regularisation by noise phenomena, see [194, 15, 61, 60] among others.

5.3.1 General criteria

We provide here useful general criteria to establish ρ-irregularity for a given stochastic process, which will then be applied to several examples in the upcoming Sections 5.3.2 and 5.3.3.

We adopt the following convention: although we always write statements to hold for any $\xi \in \mathbb{R}^d$, they must be interpreted as "for all ξ large enough", i.e. $|\xi| \geqslant C$ for some universal deterministic constant $C > 0$, so that for instance expressions like $\log |\xi|$ are meaningful. We have seen that in the case of ρ-irregularity this is not an issue, since the only relevant information given by $\|\Phi^w\|_{\mathcal{W}^{\gamma, \rho}}$ is for large values of $|\xi|$. Similarly, for a modulus of continuity φ defined only on a neighbourhood of 0, t and s are tacitly assumed to be sufficiently close whenever $\varphi(|t - s|)$ appears.

The next statement given in a general form, but keep in mind that our primary focus is the case $F(\xi) = |\xi|^\alpha$ for suitable values of α.

Theorem 5.40. *Let $(X_t)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued stochastic process with $\mathbb{P}$-a.s. measurable trajectories, $F\colon\mathbb{R}^d\to[0,\infty)$ be a continuous function such that:*

i. there exists $c>0$ such that $F(x)\sim F(y)$ whenever $|x-y|\leqslant c$;

ii. F has at most polynomial growth, i.e. $F(\xi)\lesssim|\xi|^\alpha$ as $\xi\to\infty$ for some $\alpha<\infty$.

Also assume that there exist positive constants μ,θ,δ,K such that the following hold:

1. Integrability condition:

$$\mathbb{E}\left[\exp\left(\mu\int_0^T|X_t|^\theta\,\mathrm{d}t\right)\right]<\infty; \tag{5.19}$$

2. Continuity condition

$$\mathbb{E}\left[\exp\left(\mu\frac{|\Phi^X_{s,t}(\xi)|^2\,|F(\xi)|^2}{|t-s|}\right)\right]\leqslant K\quad\forall\xi\in\mathbb{R}^d, 0<t-s<\delta. \tag{5.20}$$

Then, for the choice $\phi(x)=\sqrt{x|\log x|}$, defining the random variable

$$Y:=\sup_{s\neq t,\xi\in\mathbb{R}^d}\frac{|\Phi^X_{s,t}(\xi)|\,F(\xi)}{\phi(|t-s|)\sqrt{\log|\xi|}},$$

there exists $\lambda>0$ such that $\mathbb{E}[\exp(\lambda Y^2)]<\infty$.

Proof. Let us first show that, starting from (5.20), we can find another constant $\tilde{\mu}$ such that the same bound holds over all $s<t$, without the restriction $|t-s|<\delta$. Let $[s,t]$ be such that $|t-s|>\delta$; we can split the interval $[s,t]$ in at most $n=\lfloor T/\delta\rfloor+1$ intervals of the form $[t_i,t_{i+1}]$, of size at most δ; we have the estimate

$$\begin{aligned}\mathbb{E}\left[\exp\left(\tilde{\mu}\frac{|\Phi^X_{s,t}(\xi)|^2\,|F(\xi)|^2}{|t-s|}\right)\right]&\leqslant\mathbb{E}\left[\exp\left(\frac{n\tilde{\mu}}{\delta}\sum_i|\Phi^X_{t_i,t_{i+1}}(\xi)|^2|F(\xi)|^2\right)\right]\\&=\mathbb{E}\left[\prod_i\exp\left(\frac{n\tilde{\mu}}{\delta}|\Phi^X_{t_i,t_{i+1}}(\xi)|^2|F(\xi)|^2\right)\right]\\&\leqslant\prod_i\mathbb{E}\left[\exp\left(n^2\tilde{\mu}\frac{|\Phi^X_{t_i,t_{i+1}}(\xi)|^2|F(\xi)|^2}{|t_{i+1}-t_i|}\right)\right]^{1/n}.\end{aligned}$$

Choosing $\tilde{\mu}$ such that $(\lfloor T/\delta\rfloor+1)^2\tilde{\mu}\leqslant\mu$ we obtain

$$\sup_{\substack{|t-s|>\delta\\\xi\in\mathbb{R}^d}}\mathbb{E}\left[\exp\left(\tilde{\mu}\frac{|\Phi^X_{s,t}(\xi)|^2\,|F(\xi)|^2}{|t-s|}\right)\right]\leqslant K.$$

From now on, with a slight abuse, we will denote by μ the new constant $\tilde{\mu}$ under which we have a bound of the form (5.20) over all possible $t\neq s$.

Let us define, for any $s\neq t$ and suitable $\lambda>0$, the following quantity:

$$J_{s,t}(\lambda):=\sum_{n\in\mathbb{N}}2^{-n}\sum_{\xi\in2^{-n}\mathbb{Z}^d}2^{-n(d+1)}(1+|\xi|)^{-(d+1)}\exp\left(\lambda\frac{|\Phi^X_{s,t}(\xi)|^2|F(\xi)|^2}{|t-s|}\right).$$

It follows from (5.20) that, for all $\lambda\leqslant\mu$, $\mathbb{E}[J_{s,t}(\lambda)]\leqslant K$ uniformly in s,t; moreover by Jensen's inequality, it's easy to see that $J_{s,t}(\lambda)^\beta\lesssim J_{s,t}(\beta\lambda)$ for any $\beta\geqslant1$. Let us also define

$$Y_{s,t}:=\frac{1}{|t-s|^{1/2}}\sup_{\xi\in\mathbb{R}^d}\frac{|\Phi^X_{s,t}(\xi)|F(\xi)}{\sqrt{\log|\xi|}}.$$

To conclude, it suffices to show the existence of $\lambda\in(0,\mu)$ such that $\mathbb{E}[\exp(\lambda Y_{s,t}^2)]\leqslant K$ uniformly in $s<t$, as we can then apply Corollary A.27 from Appendix A.5 to get the desired bound for Y.

Fix $\xi \in \mathbb{R}^d$. For any $n \in \mathbb{N}$, we can find $\tilde{\xi} \in 2^{-n}\mathbb{Z}^d$ such that $|\xi - \tilde{\xi}| \lesssim 2^{-n}$; for such $\tilde{\xi}$ it holds

$$\frac{|\Phi^X_{s,t}(\tilde{\xi})| F(\tilde{\xi})}{|t-s|^{1/2}} \lesssim \lambda^{-1/2} \sqrt{\log J_{s,t}(\lambda) + n + \log|\tilde{\xi}|}.$$

By the elementary estimate $|e^{i\xi\cdot x} - e^{i\tilde{\xi}\cdot x}| \leqslant 2^\theta |\xi - \tilde{\xi}|^\theta |x|^\theta$, valid for $\theta \in (0,1)$, we also have

$$|\Phi^X_{s,t}(\tilde{\xi}) - \Phi^X_{s,t}(\xi)| \leqslant \int_s^t |e^{i\xi\cdot X_r} - e^{i\tilde{\xi}\cdot X_r}| \mathrm{d}r \lesssim |\xi - \tilde{\xi}|^\theta \int_s^t |X_r|^\theta \mathrm{d}r \leqslant |\xi - \tilde{\xi}|^\theta \|X\|^\theta_{L^\theta},$$

which interpolated together with $|\Phi^X_{s,t}(\xi)| \leqslant |t-s|$ gives

$$|\Phi^X_{s,t}(\tilde{\xi}) - \Phi^X_{s,t}(\xi)| \lesssim |t-s|^{1/2} |\xi - \tilde{\xi}|^{\theta/2} \|X\|^{\theta/2}_{L^\theta}.$$

Gathering everything together, using the fact that for n big enough it holds $|\xi| \sim |\tilde{\xi}|$, as well as $F(\xi) \sim F(\tilde{\xi})$ (thanks to assumption *i.*), we obtain

$$\begin{aligned}
|\Phi^X_{s,t}(\xi)| F(\xi) &\lesssim |\Phi^X_{s,t}(\xi) - \Phi^X_{s,t}(\tilde{\xi})| F(\xi) + |\Phi^X_{s,t}(\tilde{\xi})| F(\tilde{\xi}) \\
&\lesssim |t-s|^{1/2} |\xi - \tilde{\xi}|^{\theta/2} \|X\|^{\theta/2}_{L^\theta} F(\xi) \\
&\quad + |t-s|^{1/2} \lambda^{-1/2} \sqrt{\log J_{s,t}(\lambda) + n + \log|\tilde{\xi}|} \\
&\lesssim |t-s|^{1/2} \|X\|^{\theta/2}_{L^\theta} 2^{-n\theta/2} F(\xi) \\
&\quad + |t-s|^{1/2} \lambda^{-1/2} \sqrt{\log J_{s,t}(\lambda) + n + \log|\xi| + c}
\end{aligned}$$

Choosing $n \sim \log|\xi|$, which is by assumption *ii.* is enough to guarantee $F(\xi)\, 2^{-n\theta/2} \lesssim 1$, we get

$$|\Phi^X_{s,t}(\xi)| F(\xi) \lesssim |t-s|^{1/2} \big[\|X\|^{\theta/2}_{L^\theta} + \lambda^{-1/2} \sqrt{\log J_{s,t}(\lambda) + \log|\xi|} \,\big].$$

Dividing by $\sqrt{\log|\xi|}\, |t-s|^{1/2}$ and taking the supremum we get

$$Y_{s,t} \lesssim \|X\|^{\theta/2}_{L^\theta} + \lambda^{-1/2} + \lambda^{-1/2} \sqrt{\log J_{s,t}(\lambda)},$$

and so there exists a constant C such that

$$\begin{aligned}
\exp(\lambda Y^2_{s,t}) &\lesssim \exp(\lambda C \|X\|^\theta_{L^\theta})\, J_{s,t}(\lambda)^C \\
&\lesssim \exp(2\lambda C \|X\|^\theta_{L^\theta}) + J_{s,t}(\lambda)^{2C} \\
&\lesssim \exp(2\lambda C \|X\|^\theta_{L^\theta}) + J_{s,t}(2\lambda C).
\end{aligned}$$

Choosing λ such that $2\lambda C \leqslant \mu$, we therefore obtain a uniform bound for $\mathbb{E}[\exp(\lambda Y^2_{s,t})]$, which by the above reasoning implies the conclusion. □

Remark 5.41. Going through the same proof, one can obtain a similar statement for F such that:

i. there exist constants $c_1, c_2, c_3 > 0$ such that $F(x) \leqslant c_1 F(c_2\, y)$ whenever $|x - y| \leqslant c_3$;

ii. F has exponential-type growth, i.e. $\log F(\xi) \leqslant c_4 |\xi|^\alpha$ as $\xi \to \infty$ for some $\alpha < \infty$, $c_4 > 0$.

Then under conditions (5.19) and (5.20), it is possible to find $c > 0$ such that, defining

$$Y := \sup_{s \neq t, \xi \in \mathbb{R}^d} \frac{|\Phi^X_{s,t}(\xi)|\, F(c\xi)}{\phi(|t-s|)\, |\xi|^\alpha}, \tag{5.21}$$

the same conclusion as in Theorem 5.40 holds. The choice $F(\xi) = \exp(\lambda|\xi|^\alpha)$ satisfies the above requirements and in this case we can get rid of $|\xi|^\alpha$ in the denominator of (5.21) by changing c.

We immediately deduce the following result.

Corollary 5.42. *Let X be a process satisfying the assumptions of Theorem 5.40 for $F(\xi) = |\xi|^\alpha$. Then for any $\rho < \alpha$, there exists $\gamma = \gamma(\rho) > 1/2$ such that X is (γ, ρ)-irregular. Moreover*

$$\mathbb{E}[\exp(\lambda \|\Phi^X\|^2_{\mathcal{W}^{\gamma,\rho}})] < \infty \quad \forall \lambda \in \mathbb{R}. \tag{5.22}$$

Proof. Let Y be as in Theorem 5.40; interpolating the trivial estimate $|\Phi^X_{s,t}(\xi)| \leqslant |t-s|$ with

$$|\Phi^X_{s,t}(\xi)| \leqslant Y |\xi|^{-\alpha} (\log|\xi|)^{1/2} |t-s|^{1/2} |\log|t-s||^{1/2}$$

we obtain that, for any fixed $\varepsilon > 0$, it holds

$$\begin{aligned}|\Phi^X_{s,t}(\xi)| &\leqslant Y^{1-2\varepsilon} |\xi|^{-\alpha(1-2\varepsilon)} (\log|\xi|)^{1/2-\varepsilon} |t-s|^{1/2+\varepsilon} |\log|t-s||^{1/2-\varepsilon} \\ &\lesssim_{\varepsilon,T} Y^{1-2\varepsilon} |\xi|^{-\alpha(1-3\varepsilon)} |t-s|^{1/2+\varepsilon/2}.\end{aligned}$$

Setting $\rho = \alpha(1-3\varepsilon) < \alpha$, $\gamma = 1/2 + \varepsilon/2 > 1/2$, we obtain

$$\|\Phi^X\|_{\mathcal{W}^{\gamma,\rho}} \lesssim Y^{1-2\varepsilon}$$

which also implies that, for a suitable $C = C(\varepsilon)$, taking $\mu > 0$ small enough, it holds

$$\mathbb{E}[\exp(\mu \|\Phi^X\|^{\beta}_{\mathcal{W}^{\gamma,\rho}})] \leqslant \mathbb{E}[\exp(\mu C Y^2)] < \infty \tag{5.23}$$

where $\beta := 2/(1-2\varepsilon) > 2$; therefore from (5.23) we immediately deduce (5.22). The reasoning holds for any $\varepsilon > 0$, so we can invert the relations between ρ, ε and γ to deduce that for any given $\rho < \alpha$ we can take $\gamma(\rho) = 1/2 + (1-\rho/\alpha)/6$. □

Theorem 5.40 and Corollary 5.42 are well suited for establishing ρ-irregularity in several situations, as we will show in the next section. However, conditions (5.19) and (5.20) are in general difficult to check, due to their exponential nature; we present now a weaker version of Theorem 5.40, which relaxes condition (5.19).

Corollary 5.43. *Let X_t be an $\mathbb{R}^d$-valued stochastic process with $\mathbb{P}$-a.s. measurable trajectories; assume that it satisfies the continuity condition (5.20) from Theorem 5.40 for $F(\xi) = |\xi|^\alpha$ and that there exists $\theta > 0$ such that*

$$\mathbb{P}\left(\int_0^T |X_t|^\theta \,\mathrm{d}t < \infty\right) = 1.$$

Then for any $\rho < \alpha$ there exists $\gamma = \gamma(\rho) > 1/2$ such that $\mathbb{P}$-a.s. $X_\cdot$ is (γ, ρ)-irregular.

Proof. Let $f \in C^0_t$ be a deterministic continuous function which is $\tilde{\rho}$-irregular for sufficiently large $\tilde{\rho} < \infty$; existence of such functions is granted by Theorem 5.7.

For any $N \in \mathbb{N}$, set $A = \{\omega \in \Omega : \|X_\cdot(\omega)\|_{L^\theta} \leqslant N\}$ and define $X^N_\cdot := \mathbb{1}_A X_\cdot + \mathbb{1}_{A^c} f_\cdot$. Then it is easy to check that by construction

$$\mathbb{E}[\exp(\lambda \|X^N_\cdot\|_{L^\theta})] < \infty \quad \forall \lambda \in \mathbb{R}.$$

Letting μ, δ, K denote the constants under which X satisfies condition (5.20), we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\mu \frac{|\Phi^{X^N}_{s,t}(\xi)|^2 |\xi|^{2\alpha}}{|t-s|}\right)\right] &\leqslant \mathbb{E}\left[\exp\left(\mu \frac{|\Phi^{X^N}_{s,t}(\xi)|^2 |\xi|^{2\alpha}}{|t-s|}\right)\right] + \exp\left(\mu \frac{|\Phi^f_{s,t}(\xi)|^2 |\xi|^{2\alpha}}{|t-s|}\right) \\ &\leqslant K + \exp(\mu \|\Phi^f\|^2_{\mathcal{W}^{\gamma,\tilde{\rho}}}) < \infty\end{aligned}$$

uniformly over $\xi \in \mathbb{R}^d$, $|t-s| < \delta$, which implies that X^N also satisfies condition (5.20). Therefore, for fixed N and $\rho < \alpha$, X^N is $\mathbb{P}$-a.s. ρ-irregular; but then

$$\mathbb{P}(X \text{ is } \rho\text{-irregular}) \geqslant \mathbb{P}(X = X^N, X^N \text{ is } \rho\text{-irregular}) = 1 - \mathbb{P}(\|X\|_{L^\theta} > N) \to 1 \quad \text{as } N \to \infty. \qquad \square$$

We conclude this section by providing easy-to-check sufficient conditions for (5.20) to hold. The result is an immediate consequence of Lemma 3.22 from Section 3.1.4, thus proof is omitted; but it is quite handy to state it explicitly, given how often it will be applied in the next sections.

Lemma 5.44. *Let X_t be an $\mathbb{R}^d$-valued stochastic process with measurable trajectories; assume there exists a deterministic function $F: \mathbb{R}^d \to [0, +\infty)$ such that, for all $t-s < \delta$, $\mathbb{P}$-a.s. it holds*

$$\left|\mathbb{E}_s\left[\int_s^t e^{i\xi\cdot X_u}\,\mathrm{d}u\right]\right| \leqslant F(\xi)^{-2} \quad \forall \xi \in \mathbb{R}^d. \tag{5.24}$$

Then there exist universal constants $\mu, K > 0$ such that

$$\mathbb{E}\left[\exp\left(\mu\frac{|\Phi_{s,t}^X(\xi)|^2\,|F(\xi)|^2}{|t-s|}\right)\right] \leqslant K \quad \forall \xi \in \mathbb{R}^d, t-s<\delta; \tag{5.25}$$

namely, condition (5.20) from Theorem 5.40 is satisfied.

5.3.2 ρ-irregularity for Gaussian processes and examples

We apply here the results of the previous section to prove part of Theorem 5.28. More quantitative results are given in the next two statements; we start with the case of β-SLND Gaussian processes.

Theorem 5.45. *Let X_t be a $\mathbb{R}^d$-valued separable Gaussian process with measurable paths; suppose that X is β-SLND and that additionally*

$$\int_0^T \mathbb{E}[|X_t|^2]\,\mathrm{d}t < \infty. \tag{5.26}$$

Then for any $\rho < (2\beta)^{-1}$, there exists $\gamma = \gamma(\rho) > 1/2$ such that X is $\mathbb{P}$-a.s. (γ, ρ) irregular with

$$\mathbb{E}[\exp(\lambda\|\Phi^X\|_{\mathcal{W}^{\gamma,\rho}}^2)] < \infty \quad \forall \lambda \in \mathbb{R}.$$

Proof. It suffices to verify that the conditions from Theorem 5.40 are satisfied for $F(\xi) = |\xi|^{1/2\beta}$, as the conclusion then follows from Corollary 5.42.

By assumption (5.26), $X.$ is an $L^2(0,T;\mathbb{R}^d)$-valued Gaussian process; by Fernique's theorem (see [111]), we can deduce that there exists $\mu > 0$ such that

$$\mathbb{E}[\exp(\mu\|X\|_{L^2}^2)] < \infty,$$

namely condition (5.19) is satisfied. It remains to check condition (5.20), which we plan to do with the help of Lemma 5.44. Let $\mathcal{F}_t$ be the natural filtration associated to X; for any $\xi \in \mathbb{R}^d$ and any $t-s<\delta$, δ being the parameter for which the β-SLND condition is satisfied, it holds

$$\begin{aligned}\left|\mathbb{E}_s\left[\int_s^t e^{i\xi\cdot X_u}\,\mathrm{d}u\right]\right| &= \left|\int_s^t \mathbb{E}_s[e^{i\xi\cdot(X_u - \mathbb{E}_s X_u + \mathbb{E}_s X_u)}]\mathrm{d}u\right| \\ &= \left|\int_s^t e^{i\xi\cdot\mathbb{E}_s X_u}\,\mathbb{E}[e^{i\xi\cdot(X_u-\mathbb{E}_s X_u)}]\mathrm{d}u\right| \\ &\leqslant \int_s^t \exp\left(-\frac{1}{2}\xi\cdot(\mathrm{Var}(X_u|\mathcal{F}_s)\xi)\right)\mathrm{d}u \\ &\leqslant \int_s^t \exp(-c|\xi|^2|u-s|^{2\beta})\mathrm{d}u \\ &\leqslant \int_0^\infty \exp(-c|\xi|^2 r^{2\beta})\mathrm{d}r \sim |\xi|^{-1/\beta}.\end{aligned}$$

Therefore assumption (5.24) from Lemma 5.44 is satisfied for $F(\xi) \sim |\xi|^{1/(2\beta)}$, which implies the verification of condition (5.20) from Theorem 5.40 and the conclusion. □

The next statement concerns the case of β-eSLND processes.

Proposition 5.46. *Let X_t be a $\mathbb{R}^d$-valued separable Gaussian process with measurable paths, which is β-eSLND and satisfies condition (5.26). Then there exist constants $c, \lambda > 0$ such that, defining*

$$Y := \sup_{s\neq t, \xi\in\mathbb{R}^d} \frac{|\Phi_{s,t}^X(\xi)|\exp(c_1|\xi|^{2/(1+\beta)})}{\phi(|t-s|)}$$

with $\phi(x) = \sqrt{x|\log x|}$, it holds

$$\mathbb{E}[\exp(\lambda Y^2)] < \infty.$$

In particular, if $\beta \leqslant 1$, then X is exponentially irregular.

Proof. As in the proof of Theorem 5.45, condition (5.26) implies condition (5.19) by Fernique's Theorem. The rest of the proof is similar, again relying on Lemma 5.44, only this time we want to apply Remark 5.41 for the choice $F(\xi)=\exp(|\xi|^{2/(1+\beta)})$.

For any $t-s<\delta$, where δ is the parameter in the β-eSLND condition, as before it holds

$$\begin{aligned}\left|\mathbb{E}_s\left[\int_s^t e^{i\xi\cdot X_u}\,\mathrm{d}u\right]\right| &= \int_s^t \exp\left(-\frac{1}{2}\xi\cdot(\mathrm{Var}(X_u|\mathcal{F}_s)\xi)\right)\mathrm{d}u\\ &\leqslant \int_0^1 \exp(-c|\xi|^2|\log r|^{-\beta})\,\mathrm{d}r\\ &\sim \int_0^{+\infty}\exp\left(-c|\xi|^2x-x^{-\frac{1}{\beta}}\right)x^{-\frac{\beta+1}{\beta}}\,\mathrm{d}x\\ &\lesssim \int_0^\infty \exp\left(-\tilde{c}\left(|\xi|^2x+x^{-\frac{1}{\beta}}\right)\right)\mathrm{d}x\end{aligned}$$

where in the third line we used the change of variables $x=|\log r|^{-\beta}$. By the general inequality $a+b\gtrsim a^\theta b^{1-\theta}$, valid for all $a,b>0$ and $\theta\in(0,1)$, it holds

$$|\xi|^2x+x^{-\frac{1}{\beta}}\gtrsim|\xi|^{2(1-\theta)}x^{1-\theta\frac{\beta+1}{\beta}}=|\xi|^{2/(\beta+1)}$$

for the choice $\theta=\beta/(\beta+1)$; therefore there exists a constant c_1 such that

$$\left|\mathbb{E}_s\left[\int_s^t e^{i\xi\cdot X_u}\mathrm{d}u\right]\right|\lesssim\exp(-2c_1|\xi|^{2/(\beta+1)})\int_0^\infty\exp\left(-\frac{\tilde{c}}{2}x^{-\frac{1}{\beta}}\right)\mathrm{d}x\lesssim\exp(-2c_1|\xi|^{2/(\beta+1)}).$$

By Lemma 5.44, we deduce that Condition (5.20) is satified for $F(\xi)\sim\exp(c_1|\xi|^{2/(\beta+1)})$; the conclusion then follows from Remark 5.41. In particular if $\beta\in(0,1]$, then $2/(\beta+1)\geqslant 1$, which implies that X is exponentially irregular. □

We have already seen in Section 5.1.3 that the (generalized) fBm of parameter $H\in(0,\infty)$ enjoys the β-SLND condition for $\beta=1/(2H)$.

The rest of this section is devoted to providing further examples of Gaussian processes satisfying the assumptions of Theorem 5.45; in the following, $(\mathcal{F}_t)_t$ denotes either the filtration generated by B or the filtration generated by a given process X, which will be clear from the context.

Example 5.47. Let B be a standard Brownian motion in $\mathbb{R}^d$ and let $A\in\mathbb{R}^{d\times d}$, $x_0\in\mathbb{R}^d$, $\sigma>0$ and a given deterministic function $f\colon[0,T]\to\mathbb{R}^d$; consider a generalised Ornstein–Uhlenbeck process in $\mathbb{R}^d$, solution to the SDE

$$\mathrm{d}X_t=(-AX_t+f_t)\mathrm{d}t+\sigma\mathrm{d}B_t,\quad X_0=x_0.$$

The explicit expression for X is given by

$$\begin{aligned}X_t&=e^{-tA}x_0+\int_0^t e^{-(t-s)A}f_s\,\mathrm{d}s+\sigma\int_0^t e^{-(t-s)A}\,\mathrm{d}B_s\\ &=e^{-(t-s)A}X_s+\int_s^t e^{-(t-r)A}f_r\,\mathrm{d}r+\sigma\int_s^t e^{-(t-r)A}\,\mathrm{d}B_r;\end{aligned}$$

in particular, it holds

$$X_t-\mathbb{E}_sX_t=\sigma\int_s^t e^{-(t-r)A}\,\mathrm{d}B_r.$$

It follows that for any $s<t$ such that $|s-t|<\delta$ and any $v\in\mathbb{R}^d$ it holds

$$\mathrm{Var}(X_t\cdot v|\mathcal{F}_s)=\mathrm{Var}((X_t-\mathbb{E}_sX_t)\cdot v)=\sigma^2\int_s^t|e^{-(t-r)A^*}v|^2\,\mathrm{d}r=\sigma^2\int_0^{t-s}|e^{-rA^*}v|^2\,\mathrm{d}r;$$

choosing δ small enough, so that $\|I_d-e^{-rA^*}\|\leqslant 1/2$ for all $r<\delta$, we deduce that

$$\mathrm{Var}(X_t\cdot v|\mathcal{F}_s)\gtrsim\sigma^2|v|^2\,|t-s|,$$

namely $\mathrm{Var}(X_t|\mathcal{F}_s)\gtrsim|t-s|\,I_d$. We conclude that X is ρ-irregular for any $\rho<1$.

Example 5.48. There is a general class of Gaussian processes for which a β-SLND condition holds, given by so called *moving averages of white noise*, as already observed by Berman (see Section 3 from [37]). Specifically, let $K\colon\Delta_2\to\mathbb{R}^{d\times d}$ be a function such that $(KK^*)(t,r)\gtrsim|t-r|^{2\beta-1}I_d$, for some $\beta>0$ and all (r,t) with $t-r<\delta$; given B standard Bm in $\mathbb{R}^d$, define the process

$$X_t:=\int_0^t K(t,r)\,\mathrm{d}B_r.$$

Then for any $s<t$ it holds $X_t-\mathbb{E}_sX_t=\int_s^t K(t,r)\mathrm{d}B_r$, which implies that

$$\mathrm{Var}(X_t|\mathcal{F}_s)=\int_s^t (KK^*)(t,r)\mathrm{d}r\gtrsim|t-s|^{2\beta}I_d$$

for all $s<t$ such that $|t-s|<\delta$. We deduce that X has ρ-irregular trajectories for any $\rho<(2\beta)^{-1}$. The standard example for this type of processes is for the choice $K(t,r)=k(t-r)\,I_d$, where k: $\mathbb{R}_+\to\mathbb{R}$ is such that $|k(t)|\gtrsim|t|^{\beta-1/2}$. Taking $k(t)=|t|^{H-1/2}$ we obtain the d-dim. Lévy fBm (sometimes also referred to as type-II fBm), for which again we have $\rho<(2H)^{-1}$.

Example 5.49. The class of moving averages is closed under integration. Given K and X as above, defining $Y:=\int_0^{\cdot}X_s\,\mathrm{d}s$, by stochastic Fubini it holds

$$Y_t=\int_0^t\int_0^s K(s,r)\mathrm{d}B_r\,\mathrm{d}s=\int_0^t\int_r^t K(s,r)\mathrm{d}s\,\mathrm{d}B_r=\int_0^t\tilde{K}(t,r)\mathrm{d}B_r$$

where $\tilde{K}(t,r):=\int_r^t K(s,r)\mathrm{d}s$. In the special case where $K(t,r)=k(t,r)\,I_d$ with $k\colon\Delta_2\to[0,+\infty)$ satisfying $k(t,r)\gtrsim|t-r|^{\beta-1/2}$ whenever $0<t-r<\delta$, we find

$$\int_s^t(\tilde{K}\tilde{K}^*)(t,r)\mathrm{d}r=I_d\int_s^t\bigg(\int_r^t k(s,u)\mathrm{d}u\bigg)^2\mathrm{d}r\gtrsim|t-r|^{2(\beta+1)}\,I_d$$

which shows that Y is $(\beta+1)$-SLND. Choosing $K(t,r)=|t-r|^{H-1/2}\,I_d$, i.e. X being a Lévy fBm of parameter H, we deduce Y is ρ-irregular for any $\rho<(2+2H)^{-1}$. The argument can be iterated, producing "Lévy fBm of parameter $n+H$", which will be ρ-irregular for any $\rho<(2n+2H)^{-1}$; see the similarity with the fBm of parameter $n+H$ as defined in Section 5.1.3.

Example 5.50. The following example is taken from [6] and it provides an explicit Gaussian process with continuous trajectories which are $\mathbb{P}$-a.s. ρ-irregular for all $\rho<\infty$. Functions with such properties can be also constructed by prevalence techniques, using the fact that countable intersection of prevalent sets is prevalent.

Let H_n be a sequence in $(0,1)$ such that $H_n\downarrow 0$ and $\{W^{H_n}\}_n$ be a sequence of independent fBms in $\mathbb{R}^d$ with parameters H_n, defined on an interval $[0,T]$; also consider a sequence λ_n of strictly positive numbers such that

$$\sum_n\lambda_n\,\mathbb{E}[\|W^{H_n}\|_{C^0}]<\infty$$

(for instance one can take $\lambda_n=(1+\mathbb{E}[\|W^{H_n}\|_{C^0}])^{-1}n^{-2}$). Then it holds

$$\mathbb{E}\bigg[\sum_n\lambda_n\,\|W^{H_n}\|_{C^0}\bigg]<\infty$$

which implies that $\mathbb{P}$-a.s. the series $\sum_n\lambda_n W^{H_n}$ is absolutely convergent, thus uniformly convergent to an element of C_t^0; we denote such limit by Y, which is therefore a Gaussian variable on C_t^0. By construction

$$\mathrm{Var}(Y_t|\mathcal{F}_s)\geqslant\lambda_n\mathrm{Var}(W_t^{H_n}|\mathcal{F}_s)=\lambda_n\,c_{H_n}|t-s|^{2H_n},$$

which implies that $\mathbb{P}$-a.s. Y is ρ-irregular for any $\rho<(2H_n)^{-1}$. As the reasoning holds for all n and $H_n\downarrow 0$ we conclude that Y is $\mathbb{P}$-a.s. ρ-irregular for all $\rho<\infty$.

Finally, we present the proof of the statement already claimed at the end of Section 5.1.3, concerning the β-eSLND property of suitable Gaussian processes.

Proposition 5.51. *Let $\beta > 0$ and consider the $\mathbb{R}^d$-valued Gaussian process X^β defined by*

$$X_t^\beta = \int_0^t (t-s)^{-1/2} |\log(t-s)|^{-\beta/2-1/2} \mathrm{d}B_s \quad \forall t \in [0,1/2] \tag{5.27}$$

where B is a standard Bm in $\mathbb{R}^d$. Then X^β admits a modification which is β-eSLND and satisfies the hypothesis of Proposition 5.46; moreover, X^β has trajectories in L_t^p for any $p < \infty$.

Proof. The process X is separable, as it is constructed from Bm, which is a separable process. Moreover, it is easy to check X is stochastically continuous, therefore by Proposition 3.2 from [85] it admits a measurable modification; from now on we will work with this modification. It holds

$$\mathrm{Var}(X_t) = I_d \int_0^t (t-s)^{-1} |\log(t-s)|^{-\beta-1} \mathrm{d}s = c_\beta \, |\log t|^{-\beta} \lesssim 1 \quad \forall t \in [0,1/2];$$

by properties of Gaussian variables, we can then find $\lambda > 0$ small enough such that

$$\mathbb{E}\left[\int_0^{1/2} \exp(\lambda |X_t|^2) \, \mathrm{d}t\right] = \int_0^{1/2} \mathbb{E}[\exp(\lambda |X_t|^2)] \, \mathrm{d}t < \infty,$$

which implies that $X \in L_t^p$ for all $p \in [1,\infty)$. Finally, since X is the moving average of white noise associated to $K(t,r) = k(t-r) I_d$, $k(t-r) = |t-r|^{-1/2} \, |\log|t-r||^{-\beta/2-1/2}$, for any $s < t$ it holds

$$\mathrm{Var}(X_t | \mathcal{F}_s) = I_d \int_s^t k^2(t-r) \, \mathrm{d}r = I_d \int_0^{t-s} u^{-1} |\log u|^{-\beta-1} \, \mathrm{d}r \sim |\log(t-s)|^{-\beta} I_d$$

which proves the β-eSLND property. □

Remark 5.52. We have constructed the process X on the interval $[0,1/2]$ for simplicity, but up to rescaling, a process with the same properties can be constructed on any finite interval $[0,T]$. Recall that for $\beta \leqslant 1$, X is exponentially irregular, thus Carathéodory and unbounded; therefore the L^p (actually, exponential) integrability obtained is optimal in this case. On the other hand, in the regime $\beta > 1$ it can be shown, using the results from [110], [212], that the resulting process has continuous trajectories. The results presented here are taken from [143]; while we were working on the manuscript, the work-[170] came out, in which the same process is independently introduced and called p-log Brownian motion (see Section 4 therein); our condition $\beta > 0$ corresponds to $p > 1$ therein. However the authors in [170] only provide estimates for μ^X in the regime $p > 1/2$, which corresponds to $\beta > 1$, and do not prove the exponential decay of $\Phi^X(\xi)$; our results in that regard are much sharper.

Further examples of ρ-irregular functions can be produced by combining a given β-SLND Gaussian process X with suitable deterministic functions.

Proposition 5.53. *The following hold:*

a) *Given a measurable, deterministic $f : [0,T] \to \mathbb{R}^d$ and a β-SLND Gaussian process X, set $Y := f + X$; then Y is also β-SLND. If moreover X satisfies condition (5.26) and $f \in L_t^\theta$ for some $\theta \in (0,\infty)$, then Y is ρ-irregular for any $\rho < 1/(2\beta)$. A similar statement holds for f as above and X β-eSLND.*

b) *Given a measurable, deterministic $f : [0,T] \to \mathbb{R}$ satisfying $c^{-1} \leqslant |f_t| \leqslant c$ for some $c > 0$ and a process X satisfying the assumptions of Theorem 5.45, $Y := fX$ also satisfies those assumptions and is therefore ρ-irregular for any $\rho < 1/(2\beta)$.*

c) *Let X is a β-SLND Gaussian process with $\beta \in (0,1]$; suppose X has trajectories in C_t^α. Let $A \in C_t^\gamma \mathbb{R}^{d\times d}$ be a deterministic function, with $\alpha + \gamma > 1$, satisfying*

$$A_t A_t^* \geqslant c I_d \quad \forall t \in [0,T].$$

Then the process Y defined by the Young integral $Y_\cdot = \int_0^\cdot A_s \, \mathrm{d}X_s$ is also Gaussian, β-SLND, with trajectories in C_t^α; it is ρ-irregular for any $\rho < 1/(2\beta)$.

Proof. Part *a)* follows from Remark 5.27 and the fact that if $f \in L_t^\theta$ and X satisfies the integrability assumptions, then so does $f + X$. Regarding *b)*, it is clear that the process Y defined in this way is still Gaussian satisfying (5.26). The process Y is β-SLND since

$$\mathrm{Var}(Y_t|\mathcal{F}_s) = |f_t|^2 \,\mathrm{Var}(X_t|\mathcal{F}_s) \sim \mathrm{Var}(X_t|\mathcal{F}_s) \gtrsim |t-s|^{2\beta} I_d.$$

It remains to prove *c)*. By properties of Young integrals, $\varphi \mapsto \int_0^\cdot A\,\mathrm{d}\varphi$ is a bounded linear map from C_t^α to itself, therefore Y is a Gaussian process on C_t^α since X is so; by Fernique's theorem, it holds $\mathbb{E}[\exp(\lambda \|Y\|_{C^\alpha})] < \infty$ for all $\lambda \in \mathbb{R}$. By definition of Young integral, we have

$$Y_t = Y_s + A_s\, X_{s,t} + R_{s,t}$$

where $|R_{s,t}| \lesssim |t-s|^{\beta+\gamma} \|A\|_{C^\gamma} \|X\|_{C^\alpha}$. This implies that Y satisfies

$$Y_t - \mathbb{E}_s Y_t = A_s\,(X_t - \mathbb{E}_s X_t) + (R_{s,t} - \mathbb{E}_s R_{s,t}),$$

where $X_t - \mathbb{E}_s X_t$ and $R_{s,t} - \mathbb{E}_s R_{s,t}$ are both Gaussian variables independent of $\mathcal{F}_s$. Moreover

$$\mathbb{E}[|R_{s,t} - \mathbb{E}_s R_{s,t}|^2] \leqslant \mathbb{E}[|R_{s,t}|^2] \lesssim |t-s|^{2(\alpha+\gamma)}\, \|A\|_{C^\gamma}^2\, \mathbb{E}[\|X\|_{C^\alpha}^2] \lesssim |t-s|^{2(\alpha+\gamma)}$$

where $\alpha + \gamma > 1 \geqslant \beta$ and so the variance above is of order $o(|t-s|^{2\beta})$ as $|s-t| \sim 0$. The decay of $\mathrm{Var}(Y_t|\mathcal{F}_s)$ for $s \sim t$ is thus governed by

$$\mathrm{Var}(A_s\,(X_t - \mathbb{E}_s X_t)) = A_s \mathrm{Var}(X_t|\mathcal{F}_s) A_s^* \gtrsim |t-s|^{2\beta}\, A_s A_s^* \gtrsim |t-s|^{2\beta}\, I_d$$

whenever $|t-s|$ is small enough. This implies that Y is β-SLND and ρ-irregular for any $\rho < 1/\beta$. □

Remark 5.54. The examples from Proposition 5.53 can be further combined together, for instance by considering

$$Y_t := \int_0^t A_s\,\mathrm{d}X_s + f_t X_t + g_t$$

for A, f, g satisfying the previous assumptions. One can moreover replace such deterministic objects by stochastic processes Z^i, independent of X, satisfying suitable regularity and integrability assumptions; this can be readily seen by first conditioning on Z^i and applying the deterministic result. This allows to construct processes with ρ-irregular trajectories which are not Gaussian nor Markovian.

5.3.3 ρ-irregularity for α-stable processes

Section 5.3.2 deals exclusively with Gaussian processes, but the criteria developed in Section 5.3.1 apply in more general situations, including Markov processes. Here we treat the case of suitable α-stable processes.

Let us first recall some important facts on stable processes (see [250] for a detailed overview). A process X with values in $\mathbb{R}^d$ is a symmetric α-stable process with spherical measure μ (up to a renormalising constant) if it is a Lévy process such that, for any $s < t$,

$$\mathbb{E}[\exp(i\langle \xi, X_t - X_s\rangle)] = \exp\left(-(t-s)\int_{\mathbb{S}^{d-1}} |\langle \xi, z\rangle|^\alpha \mu(\mathrm{d}z)\right).$$

From now on, we will say that X satisfies the *non-degeneracy condition* if there exists $c > 0$ such that

$$G(\xi) := \int_{\mathbb{S}^{d-1}} |\langle \xi, z\rangle|^\alpha \mu(\mathrm{d}z) \geqslant c|\xi|^\alpha. \tag{5.28}$$

Similar conditions have already appeared in the literature on regularisation by noise, see e.g. [61].

Proposition 5.55. *Let X be a symmetric α-stable process satisfying the non-degeneracy condition (5.28). Then X is $\mathbb{P}$-a.s. ρ-irregular for any $\rho < \alpha/2$.*

Proof. Let $\mathcal{F}_t$ be the natural filtration associated to X; for any $\xi \in \mathbb{R}^d$ and $s < t$, by the independence of increments and the non degeneracy condition, it holds

$$\left|\mathbb{E}_s\left[\int_s^t e^{i\langle \xi, X_r\rangle}\mathrm{d}r\right]\right| = \left|e^{i\langle \xi, X_s\rangle}\int_s^t e^{-(r-s)G(\xi)}\mathrm{d}r\right| = \int_0^{t-s} e^{-rG(\xi)}\mathrm{d}r \leqslant \int_0^{\infty} e^{-cr|\xi|^\alpha}\mathrm{d}r \sim |\xi|^{-\alpha}.$$

Applying Lemma 5.44 to the choice $F(\xi) = |\xi|^{\alpha/2}$, there exists $\mu > 0$ s.t.

$$\sup_{s \neq t, \xi \in \mathbb{R}^d} \mathbb{E}\left[\exp\left(\mu \frac{|\xi|^\alpha\, |\Phi_{s,t}^X(\xi)|^2}{|t-s|}\right)\right] < \infty.$$

We would like to conclude that the process X is ρ-irregular for any $\rho < \alpha/2$, but the integrability condition from Theorem 5.40 is not satisfied. However, the process X belongs $\mathbb{P}$-a.s. to L_t^θ for any $\theta < \alpha$ (see Example 25.10 from [250]) and therefore we can apply Corollary 5.43 to obtain the conclusion. □

Remark 5.56. The non-degeneracy condition (5.28) is for instance satisfied in the cases of μ_1 being the uniform measure on $\mathbb{S}^{d-1}$ and $\mu_2 = \sum_{i=1}^d \delta_{e_i}$. The associated processes have respectively generators $\mathcal{L}_1 = (-\Delta)^{\alpha/2}$ and $\mathcal{L}_2 = \sum_i (-\partial_{x_i}^2)^{\alpha/2}$.

The previous examples generalises to anisotropic Markov processes, which show different irregularity behaviour in different directions. Property (5.29) below could be regarded as a notion of "α-irregularity", where now α is a vector in $\mathbb{R}^d$.

Corollary 5.57. *Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in (0,2)^d$ and let X be $\mathbb{R}^d$-valued process whose components $X^{(i)}$ are independent symmetric α_i-stable processes, so that*

$$\mathbb{E}[\exp(i\langle \xi, X_t - X_s\rangle)] = \exp\left(-|t-s|\sum_i |\xi_i|^{\alpha_i}\right).$$

Define $|||\xi|||_\alpha^2 = \sum_i |\xi_i|^{\alpha_i}$. Then setting $\phi(x) = \sqrt{x|\log x|}$, $\mathbb{P}$-a.s. it holds

$$\sup_{s \neq t, \xi \in \mathbb{R}^d} \frac{|||\xi|||_\alpha |\Phi_{s,t}^X(\xi)|}{\sqrt{\log|\xi|}\,\phi(|t-s|)} < \infty. \tag{5.29}$$

Proof. Going through the same calculations as in Proposition 5.55, we deduce the existence of $\mu > 0$ such that

$$\sup_{s \neq t, \xi \in \mathbb{R}^d} \mathbb{E}\left[\exp\left(\mu \frac{|||\xi|||_\alpha^2\, |\Phi_{s,t}^X(\xi)|^2}{|t-s|}\right)\right] < \infty.$$

The conclusion then follows from Theorem 5.40 applied to the choice $F(\xi) = |||\xi|||_\alpha$ (together with a reasoning analogous to that of Corollary 5.43, in order to relax the integrability condition). □

5.3.4 Hölder continuity of local time for Gaussian processes

We prove here that, under suitable assumptions, a Gaussian process X is not only ρ-irregular, but it also possesses an Hölder continuous local time. To this end, we introduce another notion of local nondeterminism (which is actually Berman's original one from [37]).

Definition 5.58. *Let $(X_t)_{t \in [0,T]}$ be an $\mathbb{R}^d$-valued separable Gaussian process. We say that X is locally nondeterministic with parameter $\beta > 0$, X is β-LND for short, if for every integer $n \geqslant 2$ there exist positive constants c_n and δ_n such that*

$$\mathrm{Var}\left(\sum_{k=1}^n v_k \cdot (X_{t_{k+1}} - X_{t_k})\right) \geqslant c_n \sum_{k=1}^n \mathrm{Var}(v_k \cdot (X_{t_{k+1}} - X_{t_k})) = c_n \sum_{k=1}^n |v_k|^2\, |t_{k+1} - t_k|^{2\beta} \tag{5.30}$$

for all ordered points $t_1 < t_2 < \ldots < t_n$ with $t_n - t_1 < \delta_n$ and $v_k \in \mathbb{R}^m$.

Let us immediately explain how this new notion relates to Definition 5.25.

Lemma 5.59. *Let X be a β-SLND Gaussian process, then X is β-LND. The converse does not hold, i.e. there exist processes which are β-LND but not β-SLND.*

Proof. The result is classical, Remark 2.3 from [272] and the references therein. □

We are now ready to prove the following result (which, by Lemma 5.59, immediately implies Point *ii.* of Theorem 5.28)

Theorem 5.60. *Let X be a $\mathbb{R}^d$-valued continuous Gaussian process which is β-LND with parameter $\beta \in (0, 1/d)$. Then $\mathbb{P}$-a.s. $\ell^X \in C_t^{\alpha_1} C_x^{\alpha_2}$ for any $\alpha_i \in [0, 1)$ satisfying*

$$\alpha_1 + 2\beta\alpha_2 < 1 - \beta d. \tag{5.31}$$

In particular,

$$\ell^X \in C_t^{1-\beta d-} C_x^0 \cap C_t^0 C_x^{1 \wedge \left(\frac{1}{2\beta} - \frac{d}{2}\right)-} \quad \mathbb{P}\text{-a.s.}$$

Proof. Let us make a preliminary observation: even if X is not centered, we may always write it as $X = \psi + W$, where $\psi_t = \mathbb{E}[X_t]$ is continuous and deterministic and W is a centered Gaussian process. Clearly, W is β-LND if and only if X is so (with same constants c_n, δ_n); from now on we will work with this decomposition of X.

Our proof follows quite closely the one given in Sections 25-26 from [152], so we will mostly sketch it. For simplicity, we will already assume that ℓ^X is a well defined function, which we can evaluate pointwise and express in terms of its Fourier transform[5.5]. In order to conclude, since ℓ^X is $\mathbb{P}$-a.s. compactly supported (by continuity of X), it is enough to show that, for any α_i as above and any $k \in 2\mathbb{N}$, it holds

$$\mathbb{E}[(\ell_{s,t}(x) - \ell_{s,t}(y))^k] \leqslant C_k |x-y|^{\alpha_1 k} |s-t|^{\alpha_2 k} \quad \forall h \in \mathbb{R}^n, 0 \leqslant s \leqslant t \leqslant T.$$

Actually, it is enough to show the above for $|s-t| < \delta$ with $\delta = \delta(k)$ small enough; indeed, we will take $\delta = \delta_{2k}$ as in the LND property associated to W. By arbitrariness of $k \in \mathbb{N}$, the conclusion then follows from an application of a multiparameter Garsia-Rademich-Rumsay lemma (see e.g. [177]).

In order to obtain such estimates, we first rewrite the above quantity in terms of the Fourier transform of ℓ^X: we have

$$\ell_{s,t}^X(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\ell}_{s,t}^X(\xi)\, e^{i\xi\cdot x} \mathrm{d}\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{[s,t]} e^{-i\xi\cdot(\psi_r + W_r)} \mathrm{d}r\, e^{i\xi\cdot x}\, \mathrm{d}\xi.$$

Therefore (dropping the term $(2\pi)^{-d}$ for simplicity) it holds

$$\begin{aligned}
\mathbb{E}[(\ell_{s,t}^X(x) - \ell_{s,t}^X(y))^k] &\sim \mathbb{E}\left[\left(\int_{\mathbb{R}^d}\left(\int_{[s,t]} e^{-i\xi\cdot(\psi_r+W_r)}\,\mathrm{d}r\right)(e^{i\xi\cdot x} - e^{i\xi\cdot y})\mathrm{d}\xi\right)^k\right]\\
&= \mathbb{E}\left[\int_{\mathbb{R}^{dk}\times[s,t]^k} \exp\left(-i\sum_j \xi_j\cdot W_{t_j}\right)\prod_j e^{-i\xi_j\cdot\psi_{t_j}}(e^{i\xi_j\cdot x} - e^{i\xi_j\cdot y})\,\mathrm{d}\xi_j\,\mathrm{d}t_j\right]\\
&= \int_{\mathbb{R}^{dk}\times[s,t]^k} \exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_j \xi_j\cdot W_{t_j}\right)\right)\prod_j e^{-i\xi_j\cdot\psi_{t_j}}(e^{i\xi_j\cdot x} - e^{i\xi_j\cdot y})\mathrm{d}\xi_j\,\mathrm{d}t_j\\
&\lesssim |x-y|^{\alpha k}\int_{\mathbb{R}^{dk}\times[s,t]^k} \exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_j \xi_j\cdot W_{t_j}\right)\right)\prod_j |\xi_j|^\alpha\,\mathrm{d}\xi_j\,\mathrm{d}t_j
\end{aligned}$$

where in the last passage we used the fact that, since $k \in 2\mathbb{N}$, the above quantity coincide with its absolute value, together with the basic inequalities

$$\left|\prod_j (e^{i\xi_j\cdot x} - e^{i\xi_j\cdot y})\right| = 1, \quad \left|\prod_j (e^{i\xi_j\cdot x} - e^{i\xi_j\cdot y})\right| \lesssim |x-y|^{\alpha k}\prod_j |\xi_j|^\alpha \quad \text{for } \alpha \in [0,1].$$

Observe that in the last quantity the deterministic function ψ has disappeared and it does not play any role in the subsequent estimates.

5.5. Rigorously, we should first mollify ℓ^X, obtain stochastic estimates which do not depend on the mollification and then pass to the limit; we omit this mostly tedious passage.

It remains to obtain, exploiting the LND property, estimates for the quantity

$$I_k := \int_{\mathbb{R}^{dk} \times [s,t]^k} \prod_j |\xi_j|^\alpha \exp\left(-\frac{1}{2} \mathrm{Var}\left(\sum_j \xi_j \cdot X_{t_j} \right) \right) \mathrm{d}\xi_j \, \mathrm{d}t_j.$$

Up to a multiplicative combinatorial factor, it holds

$$I_k \sim_k \int_{\mathbb{R}^{dk} \times \Delta_k} \exp\left(-\frac{1}{2} \mathrm{Var}\left(\sum_j \xi_j \cdot X_{t_j} \right) \right) \prod_j |\xi_j|^\alpha \mathrm{d}t_j \, \mathrm{d}\xi_j$$

where $\Delta_k = \{s < r_1 < \ldots < r_k < t\}$ is the k-th simplex. By the change of variables $\xi_j = v_j - v_{j+1}$ (and $\xi_k = v_k$, $v_{k+1} = 0$) and the basic inequality $|a-b|^\alpha \leqslant |a|^\alpha + |b|^\alpha$, we obtain

$$\begin{aligned}
I_k &\sim \int_{\mathbb{R}^{dk} \times \Delta_k} \exp\left(-\frac{1}{2} \mathrm{Var}\left(\sum_j v_j \cdot X_{t_{j-1},t_j} \right) \right) \prod_j |v_j - v_{j+1}|^\alpha \, \mathrm{d}t_j \, \mathrm{d}v_j \\
&\leqslant \sum_m c_m \int_{\mathbb{R}^{dk} \times \Delta_k} \prod_j |v_j|^{\alpha \gamma_{m,j}} \exp\left(-\frac{1}{2} \mathrm{Var}\left(\sum_j v_j \cdot X_{t_{j-1},t_j} \right) \right) \mathrm{d}t_j \, \mathrm{d}v_j \\
&\leqslant \sum_m c_m \int_{\mathbb{R}^{dk} \times \Delta_k} \prod_j [|v_j|^{\alpha \gamma_{m,j}} \exp(-c|v_j|^2 |t_j - t_{j-1}|^{2\beta})] \mathrm{d}t_j \, \mathrm{d}v_j \\
&\leqslant \sum_m c_m \int_{\mathbb{R}^{dk} \times [s,t]^k} \prod_j [|v_j|^{\alpha \gamma_{m,j}} \exp(-c|v_j|^2 |\tilde{t}_j|^{2\beta})] \mathrm{d}\tilde{t}_j \, \mathrm{d}v_j \\
&= \sum_m c_m \prod_j \int_{\mathbb{R}^d \times [s,t]} |v|^{\alpha \gamma_{m,j}} \exp(-c|v|^2 |t|^{2\beta}) \, \mathrm{d}t \mathrm{d}v,
\end{aligned}$$

where the sum is over $m = m(k)$ is finite, the coefficients c_m and $\gamma_{m,j}$ are of combinatorial tipe and $\gamma_{m,j} \in \{0,1,2\}$. In the third line we used the change of variables $\tilde{t}_j = t_j - t_{j-1}$. Finally observe that by scaling

$$\int_{\mathbb{R}^d \times [s,t]} |v|^{\alpha \gamma_{m,j}} \exp(-c|v|^2 |t|^{2\beta}) \, \mathrm{d}t \mathrm{d}v \sim c_H \int_0^{|t-s|} u^{-\beta d - \gamma_{m,j} \alpha \beta} \mathrm{d}u \lesssim c_H \, |t-s|^{1 - \beta d - 2\alpha\beta}$$

since $\gamma_{m,j} \in \{0, 1, 2\}$ and we are imposing the condition $1 - \beta d - 2\alpha\beta > 0$. Combining all the estimates we obtain

$$\mathbb{E}[(\ell_{s,t}^X(x) - \ell_{s,t}^X(y))^k] \lesssim_{k,H} |x-y|^{\alpha k} \, |t-s|^{(1 - \beta d - 2\alpha\beta)k}$$

which implies $\ell^X \in C_t^{\alpha_1} C_x^{\alpha_2}$ for any

$$\alpha_1 < 1 - \beta d - 2\alpha\beta, \quad \alpha_2 < \alpha < \frac{1}{2\beta} - \frac{d}{2} \wedge 1. \qquad \square$$

Remark 5.61. The literature on local nondeterminism is nowadays huge, we refer the interested reader again to [272] and the references therein. Here for simplicity we only dealt with Gaussian processes, but the notion(s) can be generalized to include Lévy processes as well. It is also easy to check, going through the proof of Theorem 5.60, that a similar statement can be shown for γ-stable processes satisfying condition (5.28); the only difference is that the parameter $1/\beta$ appearing in condition (5.31) is replaced by γ (in a perfect parallelism with Theorem 5.45 and Proposition 5.55). Finally, in Theorem 5.60 the constraint $\alpha_2 < 1$ appears, but it could be overcome by also considering the regularity of derivatives of higher derivatives of ℓ^X (again expressed in terms of their Fourier transform); this way, one could achieve the full regularity $\ell^X \in C_t^0 C_x^{1/(2\beta) - d/2 -}$.

5.4 Analytic properties of ρ-irregularity

This section is devoted to the study of deterministic (γ, ρ)-irregular paths. It includes the proof of Theorems 5.30 and 5.31, which are presented respectively in Sections 5.4.1 and 5.4.2. In Section 5.4.3 and 5.4.4 we also discuss what we call the perturbation problem.

5.4.1 Fourier dimension and Salem sets

We highlight here the connection between ρ-irregularity and Fourier dimension and provide the proof of Theorem 5.30. This connection was already noticed in [65]; we start by recalling some facts of geometric measure theory, which can be found in [215].

Definition 5.62. *Given $E \subset \mathbb{R}^d$ Borel, denote by $\mathcal{M}_+(E)$ the set of positive measures supported on E. The Fourier and Hausdorff dimension of E correspond respectively to*

$$\dim_F(E) = \sup \{\alpha \in [0,d] : \exists \mu \in \mathcal{M}_+(E), \, \hat{\mu} \in \mathcal{F}L^{\alpha/2,\infty}\},$$

$$\dim_H(E) = \sup \{\alpha \in [0,d] : \exists \mu \in \mathcal{M}_+(E), \, I^\alpha(\mu) < \infty\},$$

where

$$I^\alpha(\mu) := \int_{\mathbb{R}^{2d}} \frac{\mu(\mathrm{d}x)\mu(\mathrm{d}y)}{|x-y|^\alpha} = c_{\alpha,d} \int |\xi|^{\alpha-d} |\hat{\mu}(\xi)|^2 \mathrm{d}\xi = c_{\alpha,d} \|\mu\|^2_{\mathcal{F}L^{\alpha/2-d/2,2}}.$$

It is clear from the definition and the embedding $\mathcal{F}L^{s,\infty} \hookrightarrow \mathcal{F}L^{s-d/2-\varepsilon,2}$ that

$$0 \leqslant \dim_F(E) \leqslant \dim_H(E) \leqslant d; \tag{5.32}$$

moreover, there are examples in which all inequalities in (5.32) are strict. This motivates the following definition.

Definition 5.63. *A Borel set $E \subset \mathbb{R}^d$ is a Salem set if* $\dim_F(E) = \dim_H(E)$.

If w is (γ, ρ)-irregular, it is clear that for any $[s,t] \subset [0,T]$ it holds

$$\mu^w_{s,t} \in \mathcal{F}L^{\rho,\infty}, \quad I^\alpha(\mu^w_{s,t}) \lesssim_\alpha |t-s|^{2\gamma} \|\Phi^w\|^2_{\mathcal{W}^{\gamma,\rho}} \quad \forall \alpha < 2\rho.$$

In particular, since $\mu^w_{s,t}$ is a measure supported on $w([s,t])$, it holds

$$\min(d, 2\rho) \leqslant \dim_F(w([s,t])) \quad \forall [s,t] \subset [0,T].$$

On the other hand, recall that if $f \in C^\delta_t$, then for any $[s,t] \subset [0,T]$ it holds $\dim_H(f([s,t])) \leqslant \delta^{-1}$.

We are now ready to give the proof of Theorem 5.30.

Proof. (of Theorem 5.30) Suppose first $\delta \geqslant 1/d$. By Theorem 5.29, a.e. $\varphi \in C^\delta_t$ is ρ-irregular for any $\rho < (2\delta)^{-1}$. It then follows from (5.32) and the above considerations that

$$2\rho = \min(d, 2\rho) \leqslant \dim_F(\varphi([s,t])) \leqslant \dim_H(\varphi([s,t])) \leqslant \delta^{-1};$$

since the inequality holds for all $\rho < (2\delta)^{-1}$, the conclusion follows. The case $\delta < 1/d$ is even more direct, since in this case we can find $\rho < (2\delta)^{-1}$ such that $2\rho > d$ and therefore we obtain

$$d = \min(d, 2\rho) \leqslant \dim_F(\varphi([s,t])) \leqslant \dim_H(\varphi([s,t])) \leqslant d.$$

Finally, if $\delta < 1/d$, then by Point ii. of Theorem 5.29, almost every $\varphi \in C^\delta_t$ admits a jointly continuous occupation density $\ell^\varphi_{s,t}$. Therefore, there exists $x \in \varphi([s,t])$ such that $\ell^\varphi_{s,t}(x) > \varepsilon > 0$ and by continuity the same must hold on an open ball $B(x,r)$ for some $r > 0$; this implies that $B(x,r) \subset \operatorname{supp} \ell^\varphi_{s,t} = \varphi([s,t])$. □

It is possible to show that ρ-irregular paths cannot be δ-Hölder for δ too large reasoning by dimensionality, since otherwise it wouldn't be true that $\dim_F(w([s,t])) \geqslant \min(d, 2\rho)$; in the next section we are going to provide a much sharper result.

5.4.2 ρ-irregular paths are rough

The results of this section are inspired by the similar discussion carried out in Sections 9-11 of [152], in which it is shown that functions with sufficiently regular occupation densities must exhibit a quite erratic behaviour. Let us point out however that here we only assume the function w to be (γ, ρ)-irregular, which in general does not imply the existence of an occupation density. Theorem 5.31 follows from the results of this section and implies that the prevalence results from Theorem 5.29 are sharp, see also the discussion in Remark 5.70 below.

The next statement shows that (γ, ρ)-irregularity is indeed a notion of irregularity, in a sense that can be explicitly quantified. We recall to the reader the existence of a critical parameter $\delta^*_{\gamma,\rho}$ associated to (γ, ρ), as discussed in Remark 5.3; it is given by

$$\delta^*_{\gamma,\rho} = \frac{1-\gamma}{\rho}.$$

In the next statement, λ_1 denotes the Lebesgue measure on the real line.

Theorem 5.64. *Let w be a (γ, ρ)-irregular function. Then for any $\delta > \delta^*_{\gamma,\rho}$, w is nowhere δ-Hölder continuous. In particular, for any fixed $M > 0$ and any $s \in [0, T]$, the set of points t around s which satisfy an approximate δ-Hölder condition with constant M is a zero density set:*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\lambda_1(B(s,\varepsilon) \cap [0,T])} \lambda_1(t \in B(s,\varepsilon) \cap [0,T] : |w_{s,t}| \leqslant M |t-s|^{\delta}) = 0.$$

Proof. First consider the case of $\rho < d$. Let $\psi \in C^\infty_c(\mathbb{R}^d)$ be a radially symmetric, decreasing function such that $\psi \equiv 1$ on B_1. Fix $M > 0$ and let us consider first $s \in (0, T)$, so that $\lambda_1(B(s, \varepsilon) \cap [0, T]) = 2\varepsilon$ whenever ε is small enough; up to translation, we can assume $w_s = 0$. We have

$$\begin{aligned}
\frac{1}{2\varepsilon}\lambda_1(t \in (s-\varepsilon, s+\varepsilon) : |w_{s,t}| \leqslant M|t-s|^{\delta}) &\leqslant \frac{1}{2\varepsilon}\lambda_1(t \in (s-\varepsilon, s+\varepsilon) : |w_t| \leqslant M\varepsilon^{\delta}) \\
&\leqslant \frac{1}{2\varepsilon}\lambda_1\Big(t \in (s-\varepsilon, s+\varepsilon) : \psi\Big(\frac{w_t}{M\varepsilon^{\delta}}\Big) \geqslant 1\Big) \\
&\leqslant \frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} \psi\Big(\frac{w_t}{M\varepsilon^{\delta}}\Big) \mathrm{d}t \\
&= \frac{1}{2\varepsilon}\int_{\mathbb{R}^d} \psi\Big(\frac{y}{M\varepsilon^{\delta}}\Big)\, \mu^w_{s-\varepsilon,s+\varepsilon}(\mathrm{d}y) \\
&= \frac{M^d}{2}\varepsilon^{d\delta-1}\int_{\mathbb{R}^d} \hat{\psi}(M\varepsilon^{\delta}\xi)\overline{\hat{\mu}^w_{s-\varepsilon,s+\varepsilon}(\xi)}\, \mathrm{d}\xi \\
&\lesssim_M \varepsilon^{\gamma+\delta d-1}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\int_{\mathbb{R}^d} |\hat{\psi}(M\varepsilon^{\delta}\xi)|\, (1+|\xi|)^{-\rho}\mathrm{d}\xi.
\end{aligned}$$

By Hölder's inequality, for any $q > d/\rho > 1$, setting $1/q' = 1 - 1/q$ it holds

$$\int_{\mathbb{R}^d} |\hat{\psi}(M\varepsilon^{\delta}\xi)|\, (1+|\xi|)^{-\rho}\mathrm{d}\xi \lesssim_{M,q} \varepsilon^{-\frac{\delta d}{q'}}\, \|\hat{f}\|_{L^p}.$$

Therefore we obtain

$$\frac{1}{2\varepsilon}\lambda_1(t \in (s-\varepsilon, s+\varepsilon) : |w_{s,t}| \leqslant M|t-s|^{\delta}) \lesssim_{M,q} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\varepsilon^{\gamma+\delta d/q-1},$$

where the last quantity is infinitesimal as $\varepsilon \to 0$ for any q such that $d/q < \rho$ and $\gamma + \delta d/q > 1$. In particular, if δ satisfies $\delta > \delta^*_{\gamma,\rho}$, then it's always possible to find such q, which gives the conclusion for $s \in [0, T]$. The reasoning at the endpoints $\{0, T\}$ is analogous: for instance in the case $s = 0$, similar calculations yield

$$\frac{1}{\varepsilon}\lambda_1(t \in (0, \varepsilon) : |w_t - w_0| \leqslant M|t|^{\delta}) \lesssim \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\varepsilon^{\gamma+d\delta/q-1}.$$

This concludes the proof in the case $\rho < d$; for $\rho \geqslant d$, we can just invoke Lemma 5.5 and find a new pair $(\tilde{\gamma}, \tilde{\rho})$ with $\tilde{\rho} < d$ such that w is $(\tilde{\gamma}, \tilde{\rho})$-irregular and $\delta^*_{\gamma,\rho} = \delta^*_{\tilde{\gamma},\tilde{\rho}}$. ☐

Remark 5.65. It is clear from the proof that the statement can be localised as follows. For a fixed $s \in [0, T]$, if the map $t \mapsto \mu^w_t$ satisfies an approximate γ-Holder condition in $\mathcal{F}L^{\rho,\infty}$ around s, namely there exist constants $C, r > 0$ such that

$$\|\mu^w_{s,t}\|_{\mathcal{F}^{\rho,\infty}} \leqslant C|t-s|^{\gamma} \quad \text{for all } t \in (s-r, s+r),$$

then w cannot satisfy an approximate δ-Holder condition around s for any $\delta > \delta^*_{\gamma,\rho}$.

Let us recall that for a given path $w \in C^0_t$, its p-variation on an interval $[s, t] \subset [0, T]$ is defined as

$$\|w\|_{p-\mathrm{var};[s,t]} = \Big(\sup_{\Pi} \sum_i |w_{t_i,t_{i+1}}|^p\Big)^{1/p}$$

where the supremum is taken over all possible finite partitions $\Pi = \{s = t_0 < \ldots < t_n = t\}$ of $[s, t]$.

Corollary 5.66. *Let w be (γ, ρ)-irregular. Then for any $p < (\delta^*_{\gamma,\rho})^{-1}$ and any $[s, t] \subset [0, T]$, $\|w\|_{p-\mathrm{var};[s,t]} = +\infty$.*

Proof. Since the ρ-irregularity property is scaling invariant and the p-variation is invariant under reparametrization, it suffices to show that if w is (γ, ρ)-irregular, then $\|w\|_{p-\mathrm{var},[0,1]} = \infty$ for any p as above. Going through analogous computations to those of Theorem 5.64, it can be shown that for any $\delta > \delta^*_{\gamma,\rho}$ it holds

$$\lim_{\varepsilon \to 0^+} \sup_{s \in [0,1-\varepsilon]} \frac{1}{\varepsilon} \lambda_1(t \in (s, s+\varepsilon) : |w_{s,t}| \leqslant \varepsilon^\delta) = 0.$$

In particular, for all $\varepsilon > 0$ small enough it holds

$$\sup_{s \in [0,1-\varepsilon]} \lambda_1(t \in (s, s+\varepsilon) : |w_{s,t}| > \varepsilon^\delta) > \frac{\varepsilon}{2} > 0;$$

thus for any $s \in [0, 1-\varepsilon]$, there exists $t \in (s, s+\varepsilon)$ such that $|w_{s,t}| > \varepsilon^\delta$. Taking $n \sim 1/\varepsilon$, we can construct a partition $\{0 = t_0 < \ldots < t_{2n} = 1\}$ such that $t_{2k} = k\varepsilon$ and $t_{2k+1} \in (t_{2k}, t_{2k+2})$ has the above property. We obtain

$$\|w\|^p_{p-\mathrm{var}} \geqslant \sum_{k=0}^{2n-1} |w_{t_k, t_{k+1}}|^p \gtrsim \varepsilon^{p\delta - 1};$$

since ε can be taken arbitrarily small, if $p < 1/\delta$ then $\|w\|_{p-\mathrm{var}} = \infty$. As the reasoning holds for any $\delta > \delta^*_{\gamma,\rho}$, the conclusion follows. □

Theorem 5.64 suggests that the behaviour of w is quite wild. This intuition can be captured by the following notions of irregularity, introduced in [136] and nicely presented in [132].

Definition 5.67. *We say that a path $w \in C^\delta_t$ is rough at time s, $s \in (0, T)$, if*

$$\limsup_{t \to s} \frac{|v \cdot w_{s,t}|}{|t-s|^{2\delta}} = +\infty \quad \forall v \in \mathbb{S}^{d-1}.$$

A path w is truly rough if it is rough on some dense set of $[0, T]$.

Definition 5.68. *A path $w \in C^\delta_t$ is θ-Hölder rough for $\theta \in (0, 1)$ on scale ε_0 if there exists a constant $L := L_\theta(w) := L(\theta, \varepsilon_0, T; w) > 0$ such that, for every $v \in \mathbb{S}^{d-1}$, $s \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$, there exists $t \in [0, T]$ satisfying*

$$|t - s| < \varepsilon \quad \text{and} \quad |v \cdot (w_{s,t})| \geqslant L_\theta(w) \varepsilon^\theta. \tag{5.33}$$

The largest such value of L is called the modulus of θ-Hölder roughness of w.

Corollary 5.69. *Let w be a (γ, ρ)-irregular path; then for any $\theta > \delta^*_{\gamma,\rho}$, w is θ-Hölder rough with infinite modulus of θ-Hölder roughness.*

Proof. For simplicity we show all the properties for s away from the endpoints $\{0, T\}$, but it is easy readapt the reasoning in the other case. Recall that, if w is (γ, ρ)-irregular and $v \in \mathbb{S}^{d-1}$, then $v \cdot w$ is (γ, ρ)-irregular and $\|\Phi^{v \cdot w}\|_{\mathcal{W}^{\gamma,\rho}} \leqslant \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}$. The calculations in the proof of Theorem 5.64 show that, for any $\delta > \delta^*_{\gamma,\rho}$ and any $M > 0$,

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \lambda_1\{t \in (s-\varepsilon, s+\varepsilon) : |v \cdot w_{s,t}| \geqslant M\varepsilon^\delta\} = 1,$$

where the rate of convergence only depends on M and $\|\Phi^{v \cdot w}\|_{\mathcal{W}^{\gamma,\rho}} \leqslant \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}$; therefore it is uniform in s and v. For fixed $M > 0$, we can find $\varepsilon_0 = \varepsilon_0(M, \delta)$ sufficiently small such that

$$\frac{1}{2\varepsilon} \lambda_1\{t \in (s-\varepsilon, s+\varepsilon) : |v \cdot w_{s,t}| \geqslant M\varepsilon^\delta\} \geqslant \frac{1}{2},$$

where the estimate holds for all $\varepsilon \in (0, \varepsilon_0]$, uniformly in s and v. Since the set has non-zero Lebesgue measure, it's always possible to find $t \in (s-\varepsilon, s+\varepsilon)$ such that $|v \cdot w_{s,t}| \geqslant M\varepsilon^\delta$, which shows that the definition of θ-Hölder roughness is satisfied with $\theta = \delta$ and $L \geqslant M$. By the arbitrariness of M, the conclusion follows. □

Remark 5.70. We conclude this section with a discussion on the optimality of Theorem 5.29, based on the results of the last two sections.

1. For $\delta\in(0,1)$, optimality follows from the reasoning in the proof of Corollary 5.32.
2. By applying Lemma 5.5, in the case $\delta=0$ the result can be strengthened to the fact that almost every $\varphi\in C_t^0$ is (γ,ρ)-irregular for any $\gamma<1$ and any $\rho<\infty$. Time regularity cannot be improved to ℓ^w being differentiable in time, since we know that (in the weak sense)
$$\frac{\mathrm{d}}{\mathrm{d}s}\ell(s,\cdot)|_{s=t}=\delta_{w_t}\quad\forall t\in[0,T].$$
Moreover $\{\ell^w_{s,t}\}\subset C_c^\infty$ cannot be improved to $\ell^w_{s,t}$ being analytic, since this would imply that $w([s,t])$ is an unbounded set.
3. One might wonder if, since by Lemma 5.5 we can always raise the value of γ by lowering the one of ρ, we can also do the opposite; in particular if, without imposing the restriction $\gamma>1/2$, we can find functions $\varphi\in C^\delta$ which are (γ,ρ)-irregular for a pair (γ,ρ) satisfying $\delta\leqslant\delta^*_{\gamma,\rho}$ but also $\rho>(2\delta)^{-1}$. In the case $\delta>1/d$, this possibility is ruled out by reasoning with Fourier dimensions, since it must hold
$$2\rho=\min(d,2\rho)\leqslant\dim_F(w([s,t]))\leqslant\dim_H(w([s,t]))\leqslant\delta^{-1}$$
independently of the value of γ.
4. If $\delta\leqslant 1/d$, the problem posed above is currently open. The only information we are able to provide in this case is that for $d=1$, by Proposition 5.6, there exist indeed C^1 functions which are $(\gamma,1-\gamma)$-irregular for any $\gamma\in(0,1)$.

Although Point 4. is open in terms of generic $\varphi\in C^\delta$, a simple computation shows that fBm paths do not have this property (the proof can also be readapted to consider other Gaussian processes).

Lemma 5.71. *Let W be a fBm of parameter $H\in(0,1)$; for any $s<t$ and any $\rho>(2H)^{-1}$, it holds*
$$\mathbb{E}[\|\mu^W_{s,t}\|^2_{\mathcal{F}L^{\rho,\infty}}]=\infty.$$

Proof. Up to rescaling, we can assume $s=0$, $t=1$. Since $\mathcal{F}L^{\rho,\infty}\hookrightarrow H^{\rho-d/2-}$, in order to conclude it suffices to show that
$$\mathbb{E}[\|\mu^W_1\|^2_{H^{1/(2H)-d/2}}]=\infty.$$
This quantity can now be computed explicitly:
$$\begin{aligned}\mathbb{E}[\|\mu^W_1\|^2_{H^{1/(2H)-d/2}}] &= \mathbb{E}\bigg[\int_{\mathbb{R}^d}|\xi|^{\frac{1}{H}-d}\bigg|\int_0^1 e^{i\xi\cdot W_s}\mathrm{d}s\bigg|^2\mathrm{d}\xi\bigg]\\ &= \int_{\mathbb{R}^d}\int_{[0,1]^2}|\xi|^{\frac{1}{H}-d}\,\mathbb{E}[e^{i\xi\cdot(W_{s,t})}]\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^d}\int_{[0,1]^2}|\xi|^{\frac{1}{H}-d}\exp\bigg(-\frac{|\xi|^2|t-s|^{2H}}{2}\bigg)\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\xi\\ &= \bigg(\int_{\mathbb{R}^d}|\tilde{\xi}|^{\frac{1}{H}-d}e^{-|\tilde{\xi}|^2/2}\mathrm{d}\tilde{\xi}\bigg)\bigg(\int_{[0,1]^2}|t-s|^{-1}\mathrm{d}t\,\mathrm{d}s\bigg)=\infty\end{aligned}$$
where in the last passage we use the change of variables $\tilde{\xi}=\xi\,|t-s|^H$. □

The above result leads us to the following conjecture.

Conjecture 5.72. *There exists no function $\varphi\in C_t^\delta$ which is (γ,ρ)-irregular with $\rho>1/(2\delta)\vee 1$.*

5.4.3 The general perturbation problem

The perturbation problem was first introduced in Section 12 of [152], in the context of paths which admit an occupation density. In the case of ρ-irregularity, it can be reformulated as:

Problem: If w is ρ-irregular and φ is a sufficiently regular function, is $w+\varphi$ still ρ-irregular?

We address here the more general question:

Problem: Which classes of transformations preserve the property of (γ,ρ)-irregularity?

It follows from the results of the previous section that good candidates are transformations which preserve the very oscillatory behaviour of w, namely at least the property that

$$\limsup_{t\to s}\frac{|v\cdot w_{s,t}|}{|t-s|^{\delta}}=+\infty \quad \text{for all } s\in(0,T),\, v\in\mathbb{S}^{d-1},\, \delta>\delta^*_{\gamma,\rho}. \tag{5.34}$$

Interestingly, it turns out that several transformations $F\colon C^0_t\to C^0_t$ have the property that, if w is (γ,ρ)-irregular, then $F(w)$ is $(\tilde{\gamma},\tilde{\rho})$-irregular for a new pair of parameters $(\tilde{\gamma},\tilde{\rho})$ such that $\delta^*_{\gamma,\rho}=\delta^*_{\tilde{\gamma},\tilde{\rho}}$; in particular, property (5.34) is preserved. However, we can't show that $(\gamma,\rho)=(\tilde{\gamma},\tilde{\rho})$, which remains a major open problem. A notable exception is given by the additive perturbations $F(w)=w+\varphi$ with $\varphi\in C^\infty_t$, whose treatment is postponed to the next subsection.

We start by showing that (γ,ρ)-irregularity is invariant under regular time reparametrization.

Lemma 5.73. *Let w be (γ,ρ)-irregular, $g\in C^\beta_t$ with $\beta+\gamma>1$. Then*

$$\left|\int_s^t e^{i\xi\cdot w_r}g_r\mathrm{d}r\right|\lesssim\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\|g\|_{C^\beta}|t-s|^{\gamma}|\xi|^{-\rho}\quad \textit{uniformly in } \xi\in\mathbb{R}^d.$$

In particular, for β as above, let $\tau\colon[0,T]\to[\tau(0),\tau(T)]$ be a $C^{1+\beta}_t$-diffeomorphism, i.e. $\tau\in C^{1+\beta}_t$ is invertible on its image with inverse of class $C^{1+\beta}_t$. Then $\tilde{w}_r:=w_{\tau^{-1}(r)}$ is also ρ-irregular and

$$\|\Phi^{\tilde{w}}\|_{\mathcal{W}^{\gamma,\rho}}\lesssim\|\tau^{-1}\|_{C^{1+\beta}}\|\tau\|_{C^{1+\beta}}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}. \tag{5.35}$$

Proof. Let w, g be as above. Then by properties of Young integral it holds

$$\begin{aligned}\left|\int_s^t e^{i\xi\cdot w_r}g_r\mathrm{d}r\right| &= \left|\int_s^t g_r\,\mathrm{d}\left(\int_s^r e^{i\xi\cdot w_u}\,\mathrm{d}u\right)\right|\\ &\lesssim |g_s|\left|\int_s^t e^{i\xi\cdot w_r}\mathrm{d}r\right|+|t-s|^{\beta+\gamma}[\![g]\!]_{C^\beta}\left[\!\!\left[\int_s^\cdot e^{i\xi\cdot w_r}\mathrm{d}r\right]\!\!\right]_{C^\gamma}\\ &\lesssim |t-s|^{\gamma}|\xi|^{-\rho}\|g\|_{C^\beta}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}},\end{aligned}$$

which gives the first claim. Applying the change of variables $\tilde{r}=\tau^{-1}(r)$, we then have

$$\begin{aligned}\left|\int_s^t e^{i\xi\cdot\tilde{w}_r}\mathrm{d}r\right| = \left|\int_{\tau^{-1}(s)}^{\tau^{-1}(t)} e^{i\xi\cdot w_r}\tau'_r\,\mathrm{d}r\right| &\lesssim |\tau^{-1}(t)-\tau^{-1}(s)|^{\gamma}|\xi|^{-\rho}\|\tau'\|_{C^\beta}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\\ &\lesssim |t-s|^{\gamma}|\xi|^{-\rho}\|\tau^{-1}\|_{C^{1+\beta}}\|\tau\|_{C^{1+\beta}}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\end{aligned}$$

which implies (5.35). □

Remark 5.74. Lemma 5.73 can be used to further enlarge the class of stochastic processes X which are ρ-irregular: given any such X and any random $C^{3/2}_t$-diffeomorphism, $Y_t:=X_{\tau^{-1}(t)}$ is still a ρ-irregular process.

Let us make some considerations based on the result above. Recall that if $f\in C^\delta_t$ for $\delta\in(0,1)$ and τ is sufficiently regular (i.e. bi-Lipschitz), then $f\circ\tau^{-1}$ is still C^δ_t, but this is not true for a general homeomorphism τ. On the other hand, if $f\in C^0_t$ has finite $1/\delta$-variation, then there exist a homeomorphism τ and $g\in C^\delta_t$ such that $f\circ\tau^{-1}=g$ (see for instance Proposition 5.15 from [134]). Moreover the $1/\delta$-variation is a quantity invariant under time reparametrization. Lemma 5.73 suggests that a similar situation here: (γ,ρ)-irregularity is preserved only if the reparametrization is smooth enough, but there might exist another underlying property which is invariant under a larger class of homeomorphism τ. We formulate this as a conjecture.

Conjecture 5.75. *For any pair (γ,ρ), there exists a property $\mathcal{P}$ such that:*

1. *For any $f\in C_t^0$ with property $\mathcal{P}$, there exists a homeomorphism τ such that $g=f\circ\tau^{-1}$ is (γ,ρ)-irregular.*
2. *The property $\mathcal{P}$ is invariant under time reparametrization.*

In the rest of the section, we will address the perturbation problem only for transformations $z=F(w)$ with a very specific structure, which makes z "locally look like w". The treatment is a bit abstract, but simple examples will be given in Remark 5.78.

Definition 5.76. *Let w be (γ,ρ)-irregular. We say that z is controlled by w with Gubinelli derivative z' if there exist $z'\in C^0([0,T];\mathbb{R}^{d\times d})$ and $R\in C_2^\beta([0,T];\mathbb{R}^d)$ with $\beta>\delta^*_{\gamma,\rho}$ such that*

$$z_{s,t}=z'_s\, w_{s,t}+R_{s,t}\quad \textit{for all } s<t.$$

Here $R\in C_2^\beta([0,T];\mathbb{R}^d)$ means that $R\colon\Delta_2\to\mathbb{R}^d$ (Δ_2 being the 2-simplex) and it satisfies

$$\|R\|_\beta:=\sup_{s<t}\frac{|R_{s,t}|}{|t-s|^\beta}<\infty.$$

The definition of controlled paths is usually given in the rough paths framework, see for instance [162] and [132]; here instead we do not impose $w,z\in C_t^\alpha$ with $R\in C_2^{2\alpha}$ and do not require w to admit a rough lift (interestingly, all of that structure is not needed for Definition 5.76 to be meaningful).

It follows from property (5.34) that for a given z, if such a pair (z',R) exists, then it is necessarily unique. Indeed, let $(\tilde z',\tilde R)$ be another such pair and set $A=z'-\tilde z'$, $B=R-\tilde R$. Choosing $\delta\in(0,1)$ such that $\delta^*_{\gamma,\rho}<\delta<\beta$, for any $s\in(0,T)$ and any $v\in\mathbb{S}^{d-1}$ it holds

$$\limsup_{t\to s}\frac{|(A_s^* v)\cdot w_{s,t}|}{|t-s|^\delta}=\limsup_{t\to s}\frac{|B_{s,t}|}{|t-s|^\delta}\leqslant\|B\|_\beta\limsup_{t\to s}|t-s|^{\beta-\delta}=0$$

which implies by (5.34) that $A_s^* v=0$ for all $v\in\mathbb{S}^{d-1}$ and $s\in(0,T)$, thus $A\equiv 0$ and so $B\equiv 0$ as well.

From now on, we will additionally assume in addition that there exists $c>0$ such that

$$z'_s\,(z'_s)^*\geqslant c^2 I_d\quad\forall s\in[0,T]. \tag{5.36}$$

In particular, the above non-degeneracy condition implies that z satisfies property (5.34) as well.

Proposition 5.77. *Let w be (γ,ρ)-irregular, z controlled by w with z' satisfying (5.36). Then there exists $\tilde\gamma>1/2$ such z is $(\tilde\gamma,\tilde\rho)$-irregular and $\tilde\rho$ is given by*

$$\tilde\rho=\frac{\beta}{1-\gamma+\beta}\,\rho-\frac{1-\gamma}{1-\gamma+\beta}>0;$$

moreover $\delta^*_{\gamma,\rho}=\delta^*_{\tilde\gamma,\tilde\rho}$ and we have the estimate

$$\|\Phi^z\|_{\mathcal{W}^{\tilde\gamma,\tilde\rho}}\lesssim(\|R\|_\beta+c^{-\rho})(1+\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}).$$

Proof. For any $s<t$, it holds

$$\begin{aligned}\left|\int_s^t e^{i\xi\cdot z_r}\,\mathrm{d}r\right|&=\left|\int_s^t e^{i\xi\cdot z_{s,r}}\,\mathrm{d}r\right|\\&\leqslant\left|\int_s^t [e^{i\xi\cdot z_{s,r}}-e^{i\xi\cdot z'_s w_{s,r}}]\mathrm{d}r\right|+\left|\int_s^t e^{i\xi\cdot z'_s w_{s,r}}\,\mathrm{d}r\right|\\&\lesssim\int_s^t|\xi||R_{s,r}|\,\mathrm{d}r+c^{-\rho}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}|\xi|^{-\rho}|t-s|^\gamma\\&\lesssim\|R\|_\beta|\xi||t-s|^{1+\beta}+c^{-\rho}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}|\xi|^{-\rho}|t-s|^\gamma.\end{aligned}$$

First assume that $|t-s|^{1-\gamma+\beta}|\xi|^{1+\rho}\leqslant 1$, so that $|\xi||t-s|^{1+\beta}\leqslant|\xi|^{-\rho}|t-s|^\gamma$, then in this case we trivially get

$$\left|\int_s^t e^{i\xi\cdot z_r}\,\mathrm{d}r\right|\lesssim(\|R\|_\beta+c^{-\rho}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}})|\xi|^{-\rho}|t-s|^\gamma. \tag{5.37}$$

Assume now that $|t-s|^{1-\gamma+\beta}|\xi|^{1+\rho}>1$; choose $N\in\mathbb{N}$ such that $N^{1-\gamma+\beta}\sim|t-s|^{1-\gamma+\beta}|\xi|^{1+\rho}$ and split the interval $[s,t]$ in N subinterval of size $|t-s|/N$. Applying the previous estimate to each of them and summing over we obtain

$$\begin{aligned}\left|\int_s^t e^{i\xi\cdot z_r}\,\mathrm{d}r\right| &\lesssim \|R\|_\beta N^{-\beta}|\xi||t-s|^{1+\beta}+c^{-\rho}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}N^{1-\gamma}|t-s|^\gamma|\xi|^{-\rho}\\ &\sim(\|R\|_\beta+c^{-\rho}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}})|t-s|\,|\xi|^{-\tilde{\rho}}\end{aligned}$$

where

$$\tilde{\rho}=\frac{\beta}{1-\gamma+\beta}\,\rho-\frac{1-\gamma}{1-\gamma+\beta}=\theta\rho+\theta-1$$

for suitable choice of $\theta\in(0,1)$. Since $\tilde{\rho}<\rho$, by Lemma 5.5 we can always find $\tilde{\gamma}\in(\gamma,1)$ such that w is $(\tilde{\gamma},\tilde{\rho})$-irregular and $\delta^*_{\tilde{\gamma},\tilde{\rho}}=\delta^*_{\gamma,\rho}$ and $\|\Phi^w\|_{\mathcal{W}^{\tilde{\gamma},\tilde{\rho}}}\lesssim\|\Phi^w\|^\theta_{\mathcal{W}^{\gamma,\rho}}\lesssim 1+\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}$; equation (5.37) applied with $\|\Phi^w\|_{\mathcal{W}^{\tilde{\gamma},\tilde{\rho}}}$, together with the above estimate, then implies

$$\left|\int_s^t e^{i\xi\cdot z_r}\,\mathrm{d}r\right|\lesssim(\|R\|_\beta+c^{-\rho})(1+\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}})|\xi|^{-\tilde{\rho}}|t-s|^{\tilde{\gamma}}$$

which gives the conclusion. □

Remark 5.78. If for instance w is ρ-irregular and $\beta\geqslant 1$, then we obtain that z is $\tilde{\rho}$-irregular with

$$\tilde{\rho}\geqslant\frac{2}{3}\rho-\frac{1}{3}.$$

Examples of z satisfying the above assumptions are the following:

- Take $z_t=\varphi_t w_t$ with $\varphi\in C^\beta_t\mathbb{R}$ satisfying $\varphi_t\geqslant c>0$, then $z'_s=\varphi_s I_d$, $R_{s,t}=w_t\varphi_{s,t}\in C^\beta_2$.
- Suppose $w\in C^\delta_t$ with $\delta\in(0,1)$ and take $z_t=\int_0^t A_s\,\mathrm{d}w_s$, where $A\in C^\alpha([0,T];\mathbb{R}^{d\times d})$ satisfies (5.36), $\alpha+\delta>1$ and the integral is defined in the Young sense. Then $z'_t=A_t$ and $\beta=\alpha+\delta$.
- Finally, if $z=w+\varphi$ with $\varphi\in C^\beta_t$, then $y'\equiv I_d$ and $R_{s,t}=\varphi_{s,t}\in C^\beta_2$; this case is however quite special and better estimates are available, see Section 5.4.4 below.

Let us highlight the difference between the purely analytical result of Proposition 5.77 compared to the probabilistic result of Proposition 5.53, in which instead we have examples of Gaussian processes which are ρ-irregular with parameter ρ invariant under any of the deterministic transformations from the list above.

There is another notable class of transformations which preserve some properties of the occupation measure μ^w. In this case however, it is rather complicated to consider the (γ,ρ)-irregularity property, and it is instead more natural to reason with occupation densities. Suppose that w admits an occupation density $\ell^w_{s,t}$ (which we know to be true for almost every $w\in C^\delta_t$ when $\delta<1/d$) and let $F:\mathbb{R}^d\to\mathbb{R}^d$ be a global diffeomorphism; define $z_t=F(w_t)$. Then z still admits an occupation density $\ell^z_{s,t}$; indeed for all smooth $\psi:\mathbb{R}^d\to\mathbb{R}$ it holds

$$\int_s^t\psi(z_r)\mathrm{d}r=\int_s^t\psi(F(w_r))\mathrm{d}r=\int_{\mathbb{R}^d}\psi(F(x))\ell^w_{s,t}(x)\mathrm{d}x=\int_{\mathbb{R}^d}\psi(x)\,|\det(DF^{-1}(x))|\,\ell^w_{s,t}(F^{-1}(x))\mathrm{d}x,$$

which shows that

$$\mu^z_{s,t}(\mathrm{d}x)=|\det(DF^{-1}(x))|\,\ell^w_{s,t}(F^{-1}(x))\mathrm{d}x=\ell^z_{s,t}(x)\,\mathrm{d}x.$$

In particular, $\ell^z_{s,t}$ inherits the regularity of $\ell^w_{s,t}$ and F; for instance, if $\ell^w\in C^\gamma_t C^\beta_x$ with $\beta\in(0,1)$, then

$$\begin{aligned}\|\ell^z_{s,t}\|_{C^\beta}&\lesssim\|\det(DF^{-1})\|_{C^\beta}\,\|\ell^w_{s,t}\circ F^{-1}\|_{C^\beta}\\ &\lesssim\|\det(DF^{-1})\|_{C^\beta}\,\|F^{-1}\|_{C^1}\|\ell^w_{s,t}\|_{C^\beta}\\ &\lesssim\|F^{-1}\|_{C^{1+\beta}}\,\|\ell^w\|_{C^\gamma C^\beta}|t-s|^\gamma,\end{aligned}$$

Similar estimates hold if $\beta > 1$, or $\ell^w \in C_t^\gamma L_x^p$, etc.

5.4.4 The additive perturbation problem

In this section, we will consider for simplicity only the case $w \in C_t^\delta$ with $\delta \in (0,1)$. In view of Theorem 5.29, we will always assume $\rho > 1/2$ (equivalently $(2\rho)^{-1} < 1$). As the name of the section suggests, we want to analyse the (γ,ρ) irregularity of $w+\varphi$, for suitable (regular) perturbations φ.

The next (partial) result is a slight improvement of Theorem 1.6 from [57].

Lemma 5.79. *Let w be (γ,ρ)-irregular and $\varphi \in C_t^\beta$ with $\beta \in (0,1]$, $\beta > \delta^*_{\gamma,\rho}$. Then, for any choice of $\delta \leqslant \beta$ satisfying $1-\gamma < \delta < \beta\rho$, $w+\varphi$ is $(\tilde{\gamma},\tilde{\rho})$-irregular for the choice*

$$\tilde{\gamma} = \gamma\left(1 - \frac{\delta}{\beta\rho}\right) + \frac{\delta}{\beta\rho}, \quad \tilde{\rho} = \rho - \frac{\delta}{\beta},$$

and it holds

$$\|\Phi^{w+\varphi}\|_{\mathcal{W}^{\tilde{\gamma},\tilde{\rho}}} \lesssim (1 + \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}})\big(1 + \|\varphi\|_{C^\beta}^{\delta/\beta}\big).$$

If w is ρ-irregular and $\beta > \max\{1/2, (2\rho)^{-1}\}$, then $w+\varphi$ is $\left(\frac{1}{2} + \frac{1}{4\beta\rho}, \rho - \frac{1}{2\beta}\right)$-irregular with

$$\|\Phi^{w+\varphi}\|_{\mathcal{W}^{\tilde{\gamma},\tilde{\rho}}} \lesssim (1 + \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}})\big(1 + \|\varphi\|_{C^\beta}^{1/2\beta}\big).$$

Proof. Since $\varphi \in C_t^\beta$, so does $e^{i\xi\cdot\varphi}$, for all $\xi \in \mathbb{R}^d$. For any $\delta > 1-\gamma$ we can then apply the estimates from Young integration as follows:

$$\begin{aligned}
\left|\int_s^t e^{i\xi\cdot(w_r+\varphi_r)}\mathrm{d}r\right| &= \left|\int_s^t e^{i\xi\cdot\varphi_r}\,\mathrm{d}\left(\int_s^r e^{i\xi\cdot w_u}\mathrm{d}u\right)\right| \\
&\lesssim_{\delta+\gamma} |e^{i\xi\cdot\varphi_s}|\left|\int_s^t e^{i\xi\cdot w_r}\mathrm{d}r\right| + |t-s|^{\gamma+\delta}[\![e^{i\xi\cdot\varphi}]\!]_{C^\delta}\left[\!\!\left[\int_s^\cdot e^{i\xi\cdot w_r}\mathrm{d}r\right]\!\!\right]_{C^\gamma} \\
&\leqslant \left|\int_s^t e^{i\xi\cdot w_r}\mathrm{d}r\right| + |t-s|^{\gamma+\delta}|\xi|^{-\rho}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}[\![e^{i\xi\cdot\varphi}]\!]_{C^\delta}.
\end{aligned}$$

By interpolation and the hypothesis $\delta \leqslant \beta$, we have

$$|e^{i\xi\cdot\varphi_t} - e^{i\xi\cdot\varphi_s}| \leqslant 2, \;\; |e^{i\xi\cdot\varphi_t} - e^{i\xi\cdot\varphi_s}| \leqslant |\xi||t-s|^\beta[\![\varphi]\!]_{C^\beta} \;\Rightarrow\; |e^{i\xi\cdot\varphi_t} - e^{i\xi\cdot\varphi_s}| \lesssim \|\varphi\|_{C^\beta}^{\delta/\beta}|\xi|^{\delta/\beta}|t-s|^\delta;$$

similarly, for any $\theta \in (0,1)$, it holds

$$\left|\int_s^t e^{i\xi\cdot w_r}\mathrm{d}r\right| \lesssim \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}^\theta |t-s|^{\gamma\theta+1-\theta}|\xi|^{-\theta\rho}.$$

Putting everything together, we obtain

$$\left|\int_s^t e^{i\xi\cdot(w_r+\varphi_r)}\mathrm{d}r\right| \lesssim \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}^\theta|t-s|^{\gamma\theta+1-\theta}|\xi|^{-\theta\rho} + |t-s|^{\gamma+\delta}|\xi|^{-\rho+\delta/\beta}\|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}}\|\varphi\|_{C^\beta}^{\delta/\beta}.$$

Choosing $\theta \in (0,1)$ such that $\theta\rho = \rho - \delta/\beta$, namely $\theta = 1 - \delta/(\beta\rho)$ we obtain the first statement. The second one simply follows from the assumption $\gamma > 1/2$, taking $\delta = 1/2$. □

Lemma 5.79 implies that, even if we consider a perturbation $\varphi \in C_t^1$, we should expect a loss in spatial regularity of order $1/2$, which is only partially recovered by an improvement in time regularity of order $1/(4\rho)$. The new parameters $(\tilde{\gamma},\tilde{\rho})$ given by (5.79) satisfy $\delta^*_{\gamma,\rho} = \delta^*_{\tilde{\gamma},\tilde{\rho}}$, which implies that $w+\varphi$ still satisfies property (5.34), as can be checked directly using the fact that $\varphi \in C_t^\beta$ for some $\beta > \delta^*_{\gamma,\rho}$. This hints that the above result, while not being fully satisfactory, might be optimal, even if we cannot exclude the existence of other pairs (γ',ρ') with $\rho' > \tilde{\rho}$ such that $w+\varphi$ is (γ',ρ')-irregular.

The proof above cannot provide better results in the case $\varphi \in C_t^\beta$ with $\beta > 1$. Even if it were false in general that $w+\varphi$ is (γ,ρ)-irregular whenever w is so and $\varphi \in C_t^\beta$ with $\beta > \delta^*_{\gamma,\rho}$, we would at least expect the claim to be true whenever φ is C_t^∞; this is a problem left open in [57] and [66].

Following [143], we are able to give it a positive answer, up to strengthening the notion of ρ-irregularity. Before giving the rigorous statement, let us give an intuition by considering the following case. Suppose that $\varphi \in C_t^{1+\beta}$ for some $\beta \in [0,1]$ and that w satisfies the following property: for any $a \in \mathbb{R}^d$, $t \mapsto w_t + at$ is ρ-irregular, uniformly in a, in the sense that $\sup_a \|\Phi^{w+at}\|_{\mathcal{W}^{\gamma,\rho}} < \infty$. Then we could improve the estimates in the proof of Lemma 5.79 as follows:

$$\begin{aligned}
\left|\int_s^t e^{i\xi\cdot(w_r+\varphi_r)}\mathrm{d}r\right| &= \left|e^{-i\xi\cdot(\varphi_s+s\varphi_s')}\int_s^t e^{i\xi\cdot(w_r+\varphi_r)}\right|\\
&= \left|\int_s^t e^{i\xi\cdot(\varphi_{s,r}-\varphi_s'(r-s))}\,\mathrm{d}\left(\int_s^r e^{i\xi\cdot(w_u+\varphi_s' u)}\mathrm{d}u\right)\right|\\
&\lesssim \|\Phi^{w+\varphi_s' t}\|_{\mathcal{W}^{\gamma,\rho}}|t-s|^\gamma|\xi|^{-\rho} + \|\Phi^{w+\varphi_s' t}\|_{\mathcal{W}^{\gamma,\rho}}|t-s|\,\|e^{i\xi\cdot(\varphi_{s,\cdot}-\varphi_s'(\cdot-s))}\|_{C^{1/2}}\\
&\lesssim_w |t-s|^\gamma|\xi|^{-\rho} + |t-s|\,|\xi|^{-\rho}\|e^{i\xi\cdot(\varphi_{s,\cdot}-\varphi_s'(\cdot-s))}\|_{C^{1/2}}
\end{aligned}$$

where the last norm is taken over the interval $[s,t]$. As before, we can estimate it using simple interpolation arguments, only this time we have

$$\begin{aligned}
|\varphi_{s,u} - \varphi_s'(u-s) - \varphi_{s,v} + \varphi_s'(v-s)| = |\varphi_{u,v} - \varphi_s'(v-u)| &= \left|\int_v^u \varphi_{r,s}'\mathrm{d}r\right|\\
&\leqslant \int_v^u [\![\varphi']\!]_{C^\beta}|r-s|^\beta\mathrm{d}r \lesssim_\varphi |u-v|^{1/2}|t-s|^{1/2+\beta}
\end{aligned}$$

where we used the fact that $[v,u] \subset [s,t]$. Therefore we obtain

$$\left|\int_s^t e^{i\xi\cdot(w_r+\varphi_r)}\mathrm{d}r\right| \lesssim |t-s|^\gamma|\xi|^{-\rho} + |t-s|^{3/2+\beta}|\xi|^{1-\rho};$$

we can now reason as in the proof of Lemma 5.77, i.e. split the interval $[s,t]$ into N subintervals of size $|t-s|/N$, apply the estimate on them, sum over N and choose $N \sim |\xi|^{1/(1+\beta)}$ to obtain

$$\left|\int_s^t e^{i\xi\cdot(w_r+\varphi_r)}\mathrm{d}r\right| \lesssim |t-s|^\gamma|\xi|^{-\rho+1/2(1+\beta)}.$$

This shows that $w+\varphi$ is $(\rho-(2+2\beta)^{-1})$-irregular. In particular, even if we are not able to recover ρ-irregularity, the loss of regularity for $\varphi \in C^{1+\beta}$ is now expected to be $(2+2\beta)^{-1}$, which suggests that more generally for $\varphi \in C^\beta$, $w+\varphi$ should be $(\rho-(2\beta)^{-1})$-irregular, for any $\beta \in [1/2,+\infty)$.

This motivates the following definition; here $F(\xi) = |\xi|^\rho/\sqrt{\log|\xi|}$, $\phi(x) = \sqrt{x\,|\log x|}$.

Definition 5.80. *We say that $w \in C_t^0$ is strongly ρ-irregular if the following holds: for any $n \in \mathbb{N}$, given $\eta \in \mathbb{R}^n$ and denoting by $g_r^\eta := \sum_{k=1}^n \eta_k r^k$, then*

$$\sup_{\xi\in\mathbb{R}^d,\eta\in\mathbb{R}^n,s\neq t}\frac{|\int_s^t e^{i\xi\cdot w_r+ig_r^\eta}\mathrm{d}r|F(\xi)}{\sqrt{\log(1+|\eta|)}\,\phi(|t-s|)} < \infty. \tag{5.38}$$

Definition 5.80 formalises the idea that the irregularity of w should be only mildly affected by polynomial perturbations of any degree; this allows to proceed as above, by locally expanding a more general additive perturbation φ in its Taylor series, centered around s.

Theorem 5.81. *Let w be strongly ρ-irregular. Then for any $\varphi \in C_t^\beta$, $\beta \in [1/2,\infty)$ and for any $\tilde{\rho} < \rho$, $w+\varphi$ is $(\tilde{\rho}-1/2\beta)$-irregular. In particular, if $\varphi \in C_t^\infty$, then $w+\varphi$ is $\tilde{\rho}$-irregular for any $\tilde{\rho} < \rho$.*

We omit the proof, which can be found in [143], Theorem 80; once the key idea presented above is understood, passing from $\varphi \in C_t^{1+\beta}$ to $\varphi \in C_t^{n+\beta}$ is mostly a technical matter. Of course, in order for Theorem 5.81 to be useful, we need Definition 5.80 to be non-vacuous; this is the aim of the next result.

Theorem 5.82. *For any $\delta \in (0,1)$, almost every $\varphi \in C_t^\delta$ is strongly ρ-irregular for any $\rho < (2\delta)^{-1}$. Almost every $\varphi \in C_t^0$ is strongly ρ-irregular for any $\rho < \infty$.*

Again we omit the proof and refer to Theorem 81 from [143]. Let us however briefly explain the strategy of proof:

i. It is enough to show that, if X is a β-SLND continuous Gaussian process, then it is $\mathbb{P}$-a.s. strongly ρ-irregular for any $\rho < 1/(2\beta)$. Indeed once this is established, obtaining a prevalence statement can be accomplished as usual, leveraging on Remark 5.27; the fact that the strong ρ-irregularity property defines Borel sets can be shown as in Lemma 5.10.

ii. To fix the ideas, from now on let X be a fBm of parameter $H \in (0,1)$. Then again by Remark 5.27, and the results from Section 5.3.2, for any g^η as in Definition 5.80 it holds

$$\mathbb{E}\left[\exp\left(\mu\frac{|\xi|^{\frac{1}{H}}}{|t-s|}\left|\int_s^t e^{i\xi\cdot X_r + i g_r^\eta}\mathrm{d}r\right|^2\right)\right] \leqslant K$$

for some universal constants $\mu, K > 0$.

iii. From here, one needs to run a Kolmogorov type of argument carefully tracking how many "dyadic" g^η (in the sense of having coefficients $\eta_k \in 2^{-N}\mathbb{Z}^n$) are needed; this gives rise to the logarithmic corrections appearing in Definition 5.80.

We haven't dedicated in this thesis much space to the concept of strong ρ-irregularity, because it is somehow very unsatisfactory. It is very technical, not particularly elegant, and although it works it doesn't appear to give any useful insight on how to overcome the main difficulties in manipulating ρ-irregular functions.

In comparison to the analytic results obtained here, the stochastic ones from Proposition 5.53 require much less effort, while having stronger conclusion; still, they are subject to the (major!) constraint of the perturbation φ being deterministic. In the next section we will see how to relax this condition.

5.4.5 The stochastic perturbation problem

We have already seen in Section 5.3.2 (e.g. Proposition 5.53) that ρ-irregular Gaussian processes W are often insensitive to deterministic perturbations φ, in the sense that $W + \varphi$ is ρ-irregular for the same parameters as ρ; this is in stark contrast with the pathwise results from Section 5.4.4 (e.g. Proposition 5.79), where we expect a "regularity loss" resulting in a new parameter $\tilde{\rho} < \rho$.

However, dealing only with deterministic (or more generally, independent from W) perturbations is by far too restrictive. An alternative approach to the problem relies on the use of Girsanov transform: if we know that $\mathrm{Law}(W+\theta)$ is equivalent to $\mathrm{Law}(W)$ on C_t^0, then it shares all its almost sure pathwise properties, including ρ-irregularity. For instance, this reasoning ensures that many of the solutions to SDEs constructed in Chapter 3 are ρ-irregular.

In fact, it is not even necessary to have equivalence of laws (namely, we don't need to verify Novikov): it suffices to know that $\mathrm{Law}(W+\theta) \ll \mathrm{Law}(W)$, which can be accomplished under the mere requirement that $\mathbb{P}(\theta \in \mathrm{CM}^W) = 1$, see Proposition 1 from [198] e Theorem 7.4 from [206]. Here CM^W denotes the Cameron-Martin space associated to W; if it is a fBm of parameter $H \in (0,1)$, then $\mathrm{CM}^W = \mathcal{H}^{H+1/2}$ and it is well-known that $C_t^{H+1/2+2\varepsilon} \hookrightarrow W_t^{H+1/2+\varepsilon,2} \hookrightarrow \mathcal{H}^{H+1/2}$ (see [137]), providing more practical conditions to verify this requirement.

Also this approach presents some drawbacks; if the process W is β-SLND, but is not a fBm, verifying the validity of Girsanov may be much harder (although still possible, see e.g. the recent work [229]).

We present here a different approach, which combines the "sewing analysis" from Section 5.4.4 to the stochastic properties of W, yielding a more satisfactory result, see Theorem 5.83 below. The content of this section is mostly yet unpublished, although strongly based on [154] and the ongoing work [142].

Let us first recall a basic property of conditional expectations: for any $\mathcal{F}_s$-measurable Y, it holds

$$\|X - \mathbb{E}_s X\|_{L^p} \leqslant \|X - Y\|_{L^p} + \|\mathbb{E}_s[Y - X]\|_{L^p} \leqslant 2\|X - Y\|_{L^p}. \tag{5.39}$$

We are now ready to present the main result of this section.

Theorem 5.83. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and W be an adapted, Gaussian, $\mathbb{R}^d$-valued β-SLND process w.r.t. $\mathcal{F}_t$*[5.6]*; assume further that $\mathbb{E}[|W_{s,t}|] \lesssim |t-s|^{\tilde{\beta}}$ for another parameter $\tilde{\beta} > 0$. Let θ be another adapted $\mathbb{R}^d$-valued process; suppose that there exists $\varepsilon > 0$ such that, for all $p \in [1, \infty)$, it holds*

$$\|\theta_t - \mathbb{E}_s \theta_t\|_{L^p_\omega} \lesssim_p |t-s|^{\beta + \frac{1}{2} + \varepsilon} \quad \forall s < t \tag{5.40}$$

and that $\mathbb{E}[\|\theta\|^{2d}_{L^\infty_t}] < \infty$. Then $\mathbb{P}$-a.s. the process $X := W + \theta$ is ρ-irregular for any $\rho < 1/(2\beta)$.

Remark 5.84. Condition (5.40) may be seen as a refinement of the one guaranteeing validity of Girsanov (at least when the process W is a fBm); indeed, $\beta < 1/2$ and $\mathbb{E}[\|\theta\|^p_{C^{\beta+1/2+\varepsilon}}] < \infty$, then by (5.39) it holds

$$\|\theta_t - \mathbb{E}_s \theta_t\|_{L^p_\omega} \leqslant 2 \|\theta_t - \theta_s\|_{L^p_\omega} \leqslant 2 \mathbb{E}\Big[\|\theta\|^p_{C^{\beta+1/2+\varepsilon}_t}\Big]^{1/p} |t-s|^{\beta+1/2+\varepsilon}.$$

We may somehow interpret (5.40) as a "regularity condition" for stochastic processes, where standard increments are replaced by $\theta_t - \mathbb{E}_s \theta_t$.

In this sense, in analogy with Girsanov, the process X will have the same properties as W (specifically, it will be ρ-irregular for the same values ρ) if it is of the form $W + \theta$, with θ possessing approximately $1/2$-regularity more than W. In another light, condition (5.40) ensures that θ_t can be sufficiently well predicted by all the history up to time s, opposite to the local nondeterminism of W, which implies its chaoticity.

Compare these results to the ones given in Section 5.4.4: although θ is (by far!) not smooth, there is no loss in the parameter ρ associated to X and we don't need to invoke a complicated condition like strong ρ-irregularity.

In order to give the proof of Theorem 5.83, we need a few preparations. We start with an analytical result showing that, under suitable assumptions, we can obtain $\mathcal{F}L^{\rho,\infty}$ bounds starting from $\mathcal{F}L^{\rho,p}$ with $p < \infty$.

Lemma 5.85. *Suppose $\psi \in \mathcal{F}L^{\rho,p}$ has support contained in B_R, $\rho \geqslant 0$. Then $\psi \in \mathcal{F}L^{\rho,\infty}$ and there exists a constant $C(\rho, p) > 0$ such that*

$$\|\psi\|_{\mathcal{F}L^{\rho,\infty}} \leqslant C R^{d/p} \|\psi\|_{\mathcal{F}L^{\rho,p}}. \tag{5.41}$$

Proof. Up to a rescaling argument, it suffices to prove the statement in the case $R = 1$. Let $g \in C^\infty_c$ such that $g \equiv 1$ on $B(0,1)$, then by assumption $\psi = \psi g$ and so $\hat{\psi} = \hat{\psi} * \hat{g}$. But then

$$\begin{aligned}(1+|\xi|)^\rho |\hat{\psi}(\xi)| &\leqslant (1+|\xi|)^\rho \int |\hat{\psi}(\xi-\eta)||\hat{g}(\eta)| \mathrm{d}\eta \\ &\lesssim_\rho \int (1+|\xi-\eta|)^\rho |\hat{f}(\xi-\eta)| (1+|\eta|)^\rho |\hat{g}(\eta)| \mathrm{d}\eta \\ &\lesssim \|f\|_{\mathcal{F}L^{\rho,p}} \|g\|_{\mathcal{F}L^{\rho,p'}} \sim_p \|f\|_{\mathcal{F}L^{\rho,p}}\end{aligned}$$

which gives the conclusion. □

We now provide an alternative general criterion to establish ρ-irregularity of a given process.

Lemma 5.86. *Let $(X_t)_{t \in [0,T]}$ be a $\mathbb{R}^d$-valued stochastic process with bounded trajectories. Suppose that it satisfies $\mathbb{E}[\|X\|^{2d}_{L^\infty}] < \infty$ and that there exist $\gamma \in (0,1)$, $\rho > 0$ such that*

$$\mathbb{E}\bigg[\bigg|\int_s^t e^{i\xi \cdot X_r} \mathrm{d}r\bigg|^p\bigg]^{\frac{1}{p}} \lesssim_p |t-s|^\gamma |\xi|^{-\rho} \quad \forall \xi \in \mathbb{R}^d, \ s < t$$

for all $p \in [1, \infty)$. Then X is $\mathbb{P}$-a.s. $(\tilde{\gamma}, \tilde{\rho})$-irregular for any $\tilde{\gamma} < \gamma$, $\tilde{\rho} < \rho$.

5.6. Here for simplicity, when referring to β-SLND, we will enforce $\mathrm{Var}(W_t|\mathcal{F}_s) \leqslant c\,|t-s|^{2\beta}$ uniformly over all $s < t$ (namely with $\delta = T$ in (5.9)). The general case doesn't pose any additional problems and can be treated similarly.

Proof. For simplicity, in the following computations, whenever the $\mathcal{F}L^{\rho,p}$-norm appears, we will restrict the integral on $|\xi|>1$. This comes with no harm, since we are always dealing with finite measures μ, whose Fourier transform is bounded; by the trivial estimate $|\hat{\mu}^X_{s,t}(\xi)|\leqslant|t-s|$, the presence of the additional weight $\langle\xi\rangle^\rho$ doesn't play any role for $|\xi|\leqslant 1$. By definition of $\mu^X_{s,t}$ and the assumptions, for any fixed $\varepsilon>0$ and $s<t$ it holds

$$\begin{aligned}\mathbb{E}[\|\mu^X_{s,t}\|^p_{\mathcal{F}L^{\rho-\varepsilon,p}}] &\sim \int_{|\xi|>1}\langle\xi\rangle^{(\rho-\varepsilon)p}\,\mathbb{E}\left[\left|\int_s^t e^{i\xi\cdot X_r}\mathrm{d}r\right|^p\right]\mathrm{d}\xi\\ &\lesssim_p \int_{\mathbb{R}^d}\langle\xi\rangle^{(\rho-\varepsilon)p}\,|t-s|^{\gamma p}\,|\xi|^{-\rho p}\,\mathrm{d}\xi\\ &\leqslant |t-s|^{\gamma p}\int_{|\xi|>1}|\xi|^{-\varepsilon p}\,\mathrm{d}\xi \lesssim_{\varepsilon,p}|t-s|^{\gamma p}\end{aligned}$$

up to choosing p large enough so that $p\varepsilon>d$. By Lemma 5.85, it then holds

$$\begin{aligned}\mathbb{E}\big[\,\|\mu^X_{s,t}\|^{p/2}_{\mathcal{F}L^{\rho-\varepsilon,\infty}}\big] &\lesssim \mathbb{E}\big[\,\|X\|^d_{L^\infty}\,\|\mu^X_{s,t}\|^{p/2}_{\mathcal{F}L^{\rho-\varepsilon,p}}\big]\\ &\lesssim \mathbb{E}[\|X\|^{2d}_{L^\infty}]^{1/2}\,\mathbb{E}[\|\mu^X_{s,t}\|^p_{\mathcal{F}L^{\rho-\varepsilon,p}}]^{1/2}\lesssim|t-s|^{\gamma p/2}.\end{aligned}$$

As the estimate holds for any p, an application of Kolmogorov's continuity criterion readily yields $\mu^X\in C^{\gamma-\varepsilon}_t\,\mathcal{F}L^{\rho-\varepsilon,\infty}$ $\mathbb{P}$-a.s.; since $\varepsilon>0$ was arbitrary, we obtain the conclusion. □

The final and most important tool we will need is a (heavily simplified version of) the shifted stochastic sewing lemma, which first appeared in [154]. We adopt the same notation as in Lemma 1.1, in particular $\delta A_{s,u,t}=A_{s,t}-A_{s,u}-A_{u,t}$.

Lemma 5.87. *Let $(A_{s,t})_{(s,t)\in[0,T]^2}$ be a family of random variables in L^p_ω, $p\in[1,\infty)$, such that $A_{s,t}$ is $\mathcal{F}_t$-measurable. Suppose that there exists a process $(\mathcal{A}_t)_{t\in[0,T]}$ which is the limit of the Riemann-Stjeltes sums associated to A, in the sense that*

$$\mathcal{A}_t-\mathcal{A}_s=\lim_{|\Pi|\to 0}\sum_i A_{t_i,t_{i+1}}$$

where the limit is in probability in Ω and holds along all possible sequences of deterministic partitions $\{\Pi_n\}_n$ with mesh $|\Pi_n|\to 0$. Additionally assume that there exist $\varepsilon, C_i>0$ such that

i. $\|A_{s,t}\|_{L^p_\omega}\leqslant C_1\,|t-s|^{1/2+\varepsilon}$ for all $s<t$;

ii. $\|\mathbb{E}_{s-(t-s)}\delta A_{s,u,t}\|_{L^p_\omega}\leqslant C_2\,|t-s|^{1+\varepsilon}$ for all $s<t$ with $s>t-s$, where $u=(t+s)/2$.

Then there exists a constant K, only depending on ε and p, such that

$$\|\mathcal{A}_t-\mathcal{A}_s\|_{L^p}\leqslant K(C_1|t-s|^{1/2+\varepsilon}+C_2\,|t-s|^{1+\varepsilon})\quad\forall s<t.\tag{5.42}$$

Proof. The statement is a simplified version of Lemma 2.2 from [154]. In particular, here we are taking for simplicity $M=1,\varepsilon_1=\varepsilon_2$ and we are assuming the process $\mathcal{A}$ to already exist (usually this is instead one of the conclusions of sewing type lemmas). Moreover we only consider u to be the midpoint of (s,t), which is enough to work with dyadic partitions; indeed we are only claiming the a posteriori estimate (5.42), knowing that $\mathcal{A}$ already exists. Equation (5.42) corresponds to eq. (2.15) from [154] and can be proved as therein. □

Proof. (of Theorem 5.83) Fix $\xi\in\mathbb{R}^d$; consider the family of random variables $A_{s,t}$ given by

$$A_{s,t}:=\mathbb{E}_{s-(t-s)}\int_s^t e^{i\xi\cdot(\mathbb{E}_{s-(t-s)}\theta_r+W_r)}\mathrm{d}r,$$

with the convention that $\mathbb{E}_a=\mathbb{E}$ whenever $a<0$. We start by showing that the limit of the Riemann-Stjeltes sums of A is given by $\int_s^t e^{i\xi\cdot(\theta_r+W_r)}\mathrm{d}r$. Applying the basic inequality (5.39) and assumption (5.40), it holds

$$\begin{aligned}\left\|\int_s^t e^{i\xi\cdot(\theta_r+W_r)}\mathrm{d}r-A_{s,t}\right\|_{L^p_\omega} &\lesssim \left\|\int_s^t e^{i\xi\cdot(\theta_r+W_r)}-e^{i\xi\cdot(\mathbb{E}_{s-(t-s)}\theta_r+W_{s-(t-s)})}\mathrm{d}r\right\|_{L^p_\omega}\\ &\lesssim|\xi|\int_s^t\|\theta_r-\mathbb{E}_{s-(t-s)}\theta_r\|_{L^p_\omega}+\|W_r-W_{s-(t-s)}\|_{L^p_\omega}\\ &\lesssim|\xi|\,\big(|t-s|^{\beta+3/2}+|t-s|^{\tilde{\beta}+1}\big)\end{aligned}$$

which readily implies the claim.

We pass to verifying assumptions *i.-ii.* from Lemma 5.87, By the β-SLND property of W, it holds

$$|A_{s,t}| \lesssim |t-s|\, e^{-c|\xi|^2|t-s|^{2\beta}} \lesssim |\xi|^{-\frac{1-2\varepsilon}{2\beta}}\, |t-s|^{\frac{1}{2}+\varepsilon}$$

where we used the basic inequality $e^{-x^2}x^\alpha \lesssim 1$ for $\alpha = (1-2\varepsilon)/(2\beta)$ and $x = |\xi||t-s|^\beta$. Similarly, recalling that $u = (t+s)/2$, so that $t-u = u-s = (t-s)/2$, it holds

$$\begin{aligned}
\|\mathbb{E}_{s-(t-s)}\delta A_{s,u,t}\|_{L^p_\omega} \leqslant & \left\| \mathbb{E}_{u-(t-u)} \int_u^t e^{i\xi\cdot W_r}(e^{i\xi\cdot \mathbb{E}_{s-(t-s)}\theta_r} - e^{i\xi\cdot\mathbb{E}_{u-(t-u)}\theta_r})\,\mathrm{d}r \right\|_{L^p_\omega} \\
& + \left\| \mathbb{E}_{s-(u-s)} \int_s^u e^{i\xi\cdot W_r}(e^{i\xi\cdot \mathbb{E}_{s-(t-s)}\theta_r} - e^{i\xi\cdot\mathbb{E}_{s-(u-s)}\theta_r})\,\mathrm{d}r \right\|_{L^p_\omega} \\
\leqslant & \int_u^t e^{-c|\xi|^2|r-u+(t-u)|^{2\beta}}\, \|e^{i\xi\cdot\mathbb{E}_{s-(t-s)}\theta_r} - e^{i\xi\cdot\mathbb{E}_{u-(t-u)}\theta_r}\|_{L^p_\omega}\,\mathrm{d}r \\
& + \int_s^u e^{-c|\xi|^2|r-s+(u-s)|^{2\beta}}\, \|e^{i\xi\cdot\mathbb{E}_{s-(t-s)}\theta_r} - e^{i\xi\cdot\mathbb{E}_{s-(u-s)}\theta_r}\|_{L^p_\omega}\,\mathrm{d}r \\
\leqslant & e^{-c|\xi|^2|t-u|^{2\beta}}\, |\xi| \int_u^t \|\mathbb{E}_{s-(t-s)}\theta_r - \mathbb{E}_{u-(t-u)}\theta_r\|_{L^p_\omega}\,\mathrm{d}r \\
& + e^{-c|\xi|^2|t-u|^{2\beta}} |\xi| \int_s^u \|\mathbb{E}_{s-(t-s)}\theta_r - \mathbb{E}_{s-(u-s)}\theta_r\|_{L^p_\omega}\,\mathrm{d}r \\
\lesssim & e^{-\tilde{c}|\xi|^2|t-s|^{2\beta}}\, |\xi|\, |t-s|^{\beta+3/2+\varepsilon};
\end{aligned}$$

above we used assumption (5.40) and basic properties of conditional expectation as follows:

$$\|\mathbb{E}_{u-(t-u)}[\mathbb{E}_{s-(t-s)}\theta_r - \theta_r]\|_{L^p_\omega} \leqslant \|\theta_r - \mathbb{E}_{s-(t-s)}\theta_r\|_{L^p_\omega} \lesssim_p |r-s+(t-s)|^{\beta+1/2+\varepsilon}.$$

Applying the basic inequality $e^{-x^2}x^\alpha \lesssim 1$, this time for $\alpha = 1 + 1/(2\beta)$, we find

$$\|\mathbb{E}_{s-(t-s)}\delta A_{s,u,t}\|_{L^p_\omega} \lesssim |\xi|^{1-\alpha}\, |t-s|^{\beta+3/2+\varepsilon-\alpha\beta} = |\xi|^{-\frac{1}{2\beta}}\, |t-s|^{1+\varepsilon}.$$

The assumptions of Lemma 5.87 are therefore satisfies and by (5.42) we can deduce that

$$\left\| \int_s^t e^{i\xi\cdot(\theta_r+W_r)}\mathrm{d}r \right\|_{L^p_\omega} \lesssim_{\varepsilon,p} |t-s|^{1/2+\varepsilon}\, |\xi|^{-\frac{1-2\varepsilon}{2\beta}} \quad \forall \xi\in\mathbb{R}^d, s<t.$$

The conclusion then follows by the arbitrariness of $\varepsilon > 0, p < \infty$ and Lemma 5.86. □

As already mentioned in Remark 5.84, in the case $\beta < 1/2$, a simple condition to verify that (5.40) holds is to impose $\mathbb{E}[\|\theta\|^p_{C^{\beta+1/2+\varepsilon}}] < \infty$.[5.7] More interestingly, for $\beta > 1/2$, there exist processes θ satisfying (5.40) with limited Hölder regularity. To illustrate this, from now on we will work with the W being a (generalized) fBm of Hurst parameter $H \in (1/2, \infty) \setminus \mathbb{N}$ (which was defined in Section 5.1.3) and look at SDEs of the form

$$X_t = x_0 + \int_0^t b_s(X_s)\mathrm{d}s + W_t. \tag{5.43}$$

Lemma 5.88. *Let W be a fBm of parameter $H \in (1/2, \infty) \setminus \mathbb{N}$, $x_0 \in \mathbb{R}^d$ and $b \in L^\infty_t C^\alpha_x$ with*

$$\alpha > 1 - \frac{1}{2H}; \tag{5.44}$$

5.7. Actually, the same argument holds true also for $\beta > 1$, up to further Taylor expanding: indeed

$$\|\theta_t - \mathbb{E}_s\theta_t\|_{L^p} \lesssim \left\| \theta_t - \sum_{k\leqslant\lfloor\beta\rfloor} \theta_s^{(k)}\frac{(t-s)^k}{k!} \right\|_{L^p} \lesssim \mathbb{E}[\|\theta\|^p_{C^\beta}]^{1/p}|t-s|^\beta.$$

suppose X is a solution to (5.43). Then $X=\theta+W$ with θ satisfying the assumptions of Theorem 5.83 for $\beta=H$; in particular, X is $\mathbb{P}$-a.s. ρ-irregular for any $\rho<1/(2H)$.

Proof. Observe that under (5.44), $\alpha>0$, so equation (5.43) is meaningful in the classical sense. Let us assume for simplicity $x_0=0$ (true up to shifting and relabelling b); then θ satisfies

$$\theta_t=\int_0^t b_r(\theta_r+W_r)\mathrm{d}r.$$

Suppose we already know that $\|\theta_t-\mathbb{E}_s\theta_t\|_{L^p}\lesssim|t-s|^\gamma$ for some $\gamma>0$ (this is certainly the case for $\gamma=1$, since $b\in L^\infty_{t,x}$); then, by the properties of conditional expectation, we have

$$\begin{aligned}
\|\theta_t-\mathbb{E}_s\theta_t\|_{L^p_\omega} &\leqslant 2\left\|\theta_t-\theta_s-\int_s^t b_r(\mathbb{E}_s\theta_r+\mathbb{E}_sW_r)\mathrm{d}r\right\|_{L^p_\omega}\\
&\lesssim \int_s^t \|b_r(\theta_r+W_r)-b_r(\mathbb{E}_s\theta_r+\mathbb{E}_sW_r)\|_{L^p_\omega}\,\mathrm{d}r\\
&\lesssim \|b\|_{L^\infty C^\alpha}\int_s^t (\||\theta_r-\mathbb{E}_s\theta_r|^\alpha\|_{L^p_\omega}+\||W_r-\mathbb{E}_sW_r|^\alpha\|_{L^p_\omega})\,\mathrm{d}r\\
&\lesssim \|b\|_{L^\infty C^\alpha}\int_s^t (\|\theta_r-\mathbb{E}_s\theta_r\|_{L^p_\omega}^\alpha+|r-s|^{\alpha H})\,\mathrm{d}r\\
&\lesssim \|b\|_{L^\infty C^\alpha}|t-s|^{\alpha(\gamma\wedge H)+1}.
\end{aligned}$$

In particular, as long as $\gamma<H$, we can improve it to a new exponent $\gamma'=\alpha\gamma+1$. The affine map $\gamma\mapsto\alpha\gamma+1$ admits a unique fixed point $\bar{\gamma}=(1-\alpha)^{-1}$ and by (5.44) $\bar{\gamma}>H$; this implies that, after a finite amount of iterations of this procedure, the estimate will stabilize to

$$\|\theta_t-\mathbb{E}_s\theta_t\|_{L^p}\lesssim|t-s|^{\alpha H+1}.$$

Finally, again by (5.44), it's easy to see that $\alpha H+1>H+1/2$. □

Remark 5.89. We left a few details in the above setup vague on purpose. Since, as already mentioned, we are restricting ourselves to the case $\alpha>0$, it's easy to construct weak solutions (X,W) to (5.43) by standard tightness arguments; in particular, X might not be adapted to the filtration generated by W, but we can find a common filtration $\mathcal{F}_t$ under which both processes are adapted and W is a $\mathcal{F}_t$-fBm (so that it is H-SLND w.r.t. $\mathcal{F}_t$). It is far less obvious to see that actually, under condition (5.44), equation (5.43) admits a strong, pathwise unique solution; this is not a consequence of the results from Chapter 3, since the drift b enjoys limited time regularity. The claim instead follows from the strategy adopted in [154], although therein the author only considers time-independent drifts; a further generalization will be presented in the upcoming [142].

5.5 Bibliographical comments

As already mentioned, almost all of the material presented is taken from [143]; there are two notable exceptions: Section 5.3.4, whose prevalence result is unpublished (although the proof is strongly based on [152]) and Section 5.4.5, based on the ongoing project [142]. Let me mention that Theorem 5.83 is mostly non-optimal, as condition (5.40) can be further refined (but the presentation becomes quite technical). I also decided to exclude some technical parts, like Section 4.4 from [143], or the proofs of the statements involving strong ρ-irregularity appearing in Section 5.4.4.

As already mentioned, while with Max we were completing the first draft of [143], the work [170] appeared, which clearly shares several key intuitions (the role played by the local nondeterminism property, the relation with occupation measures, the infinitely regularising effect of some continuous functions); in some sense, [170] merges the contents of Chapters 3 and 5. This simplification allows for a short, elegant and self-contained exposition, but comes at a considerable price; indeed, one loses the fine structure of the original problem and many statements become non optimal (e.g. in terms of the required smallness of the parameter H). Also, the authors in [170] do not realize that they can fully work outside a probabilistic framework through the concept of prevalence.

On the historical side, let me mention that the problem of regularity of the local times of Gaussian processes is more than fifty years old (yet still very active!), with pioneering works by Berman [36, 37], Pitt [240] and Kahane [186]; see the reviews [152] and [272] for further references. Although the authors therein did not study the concept of ρ-irregularity (it hadn't been introduced yet!), there is a striking similarity between the stochastic estimates first presented in [57] and those already appearing in Chapter 18 from [186] (which were not motivated by regularisation by noise, but rather the study of the Hausdorff dimension of images of fractional Brownian motion).

Concerning the results on regularisation by noise for PDEs presented in Section 5.2.2, notable precursors are given by the works [88, 89, 34] for the study of modulated dispersive equations and [205, 156] for scalar conservation laws; all these works results however only consider w sampled as a Brownian motion and crucially relied on its stochastic properties. The realization that pathwise properties of the process can be exploited to obtain path-by-path results (and largely extend the class of allowed processes w) instead is due to [66, 65, 64], see also [151] for the case of Hamilton–Jacobi equation.

In a different direction, the regularity of local times of Volterra-Lévy processes has been recently studied in [167]; the techniques presented is Section 5.3 allow to show that these processes are ρ-irregular as well. In the upcoming work [142], together with Máté Gerencsér we will show ρ-irregularity of a large class of solutions to SDEs driven by β-SLND Gaussian processes.

There are yet many interesting problems to explore related to ρ-irregularity, besides Conjectures 5.72 and 5.75. For instance, it is a completely open problem is to understand whether a similar (useful) concept could be introduced for paths on manifolds. On another note, one could try to study ρ-irregularity of random *fields* and try to apply it in regularisation by noise for PDEs.[5.8]

Let me conclude with a more philophical discussion, which is also propedeutical for the upcoming Chapter 6. Like regularity can be measured in many different ways and at several different scales, the same holds for irregularity. While ρ-irregularity seems to be special (a posteriori, maybe even *magical*), given the plethora of applications it finds, it is of course not the only possible such notion; it also doesn't completely recover other "irregularity" properties of the path (like the sharp Hölder regularity of local times from Section 5.3.4, which is not a consequence of Lemma 5.22).

Depending on the problem in consideration, one might need to introduce a tailor-made notion of irregularity to tackle it. Notable examples are given by the works [172, 173], which develop the concept of *variability* of paths (still partially related to ρ-irregularity, see the discussion in Section 4.5 from [172]) and the *scaling condition* introduced in Section 3.1 of [64].

Let me also mention the upcoming works [174, 248]; on one hand, the authors further explore connections between notions of roughness, also provide a negative answer to Conjecture 5.75; on the other, another concept of irregularity, based on *small ball probability* estimates is introduced, which is still deeply tied to the joint space-time regularity of μ^w, but is also easier to check for stochastic processes which are not of Gaussian nature.

Although slightly artificial, there is in fact a systematic way to introduce quantitative irregularity properties, based on "inverse norms"; in a broad sense, this class includes the aforementioned scaling condition from [64] and the upcoming *Wei's irregularity condition* (see Definition 6.26 and Remark 6.27).

Definition 5.90. *Given a measurable path $w\colon [0,T]\to\mathbb{R}^d$, $\alpha\in(0,1]$, $\beta\in(0,1)$, we set*

$$I_w^{\alpha,\beta} := \int_{[0,T]^2} \frac{|t-s|^{\alpha\beta-1}}{|w_t-w_s|^\beta}\,\mathrm{d}s\mathrm{d}t \tag{5.45}$$

possibly taking value $+\infty$.

The intuition is that $I_w^{\alpha,\beta}$ represents an *inverse Gagliardo–Niremberg seminorm*. Let me shortly discuss some connections between the finiteness of $I_w^{\alpha,\beta}$, the irregularity of w and other concepts of irregularity, as well as criteria for stochastic processes.

5.8. Fabian Harang communicated to me privately his attempts to pursue this line of research in the case 2D dispersive type of equations with additive fractional Brownian sheet.

Set $I_w^{\alpha,\beta} = \int_{[0,T]} F(t)\,\mathrm{d}t$, where

$$F(t) := \int_{[0,T]} \frac{|t-s|^{\alpha\beta-1}}{|w_t - w_s|^\beta}\,\mathrm{d}s.$$

Lemma 5.91. *Let $w:[0,T]\to\mathbb{R}^d$ satisfy $F(t)<\infty$. Then w does not satisfy an approximate α-Hölder condition around t, i.e.*

$$\limsup_{s\to t} \frac{|w_t - w_s|}{|t-s|^\alpha} = +\infty. \tag{5.46}$$

In particular, if $I_w^{\alpha,\beta}<\infty$, then w satisfies (5.46) at a.e. $t\in[0,T]$.

Proof. We are actually gonna show a slightly stronger statement, in which the limsup is replaced by the approximate limit superior (see [152]). Let t be such that $F(t)<\infty$ and let $\mathcal{L}_T$ denote the Lebesgue measure on $[0,T]$. Then for a given $N>0$ it holds

$$\begin{aligned}\frac{1}{2\varepsilon}\mathcal{L}_T\{s\in(t-\varepsilon,t+\varepsilon): |w_{t-s}|\leqslant N|t-s|^\alpha\} &= \frac{1}{2\varepsilon}\mathcal{L}_T\left\{s\in(t-\varepsilon,t+\varepsilon): \frac{|t-s|^{\alpha\beta}}{|w_t-w_s|^\beta} > N^{-\beta}\right\}\\ &\leqslant \frac{N^\beta}{2\varepsilon}\int_{(t-\varepsilon,t+\varepsilon)} \frac{|t-s|^{\alpha\beta}}{|w_t-w_s|^\beta}\,\mathrm{d}s\\ &\leqslant \frac{N^\beta}{2}\int_{(t-\varepsilon,t+\varepsilon)} \frac{|t-s|^{\alpha\beta-1}}{|w_t-w_s|^\beta}\,\mathrm{d}s;\end{aligned}$$

Since $F(t)<\infty$, the last quantity is infinitesimal as $\varepsilon\to 0$:

$$\lim_{\varepsilon\to 0}\int_{(t-\varepsilon,t+\varepsilon)} \frac{|t-s|^{\alpha\beta-1}}{|w_t-w_s|^\beta}\,\mathrm{d}s = 0.$$

As the reasoning holds for any N, it follows that for any t such that $F(t)<\infty$ it holds

$$\mathrm{ap}-\limsup_{s\to t}\frac{|w_t-w_s|}{|t-s|^\alpha} = +\infty.$$

The second claim follows from the fact that, if $I_w^{\alpha,\beta}<\infty$, then $F(t)<\infty$ for a.e. $t\in[0,T]$. □

We can readily find simple conditions for a process X to satisfy $I_X^{\alpha,\beta}<\infty$. For simplicity, let us consider a centered, $\mathbb{R}$-valued Gaussian process X such that

$$\mathbb{E}[|X_t - X_s|^2] \geqslant C|t-s|^{2\delta} \quad \forall (s,t)\in[0,T]^2 \tag{5.47}$$

for some $C>0$. Then for any $\beta<1$ it holds

$$\begin{aligned}\mathbb{E}\left[\int_{[0,T]^2}\frac{|t-s|^{\alpha\beta-1}}{|X_t-X_s|^\beta}\,\mathrm{d}s\mathrm{d}t\right] &= \int_{[0,T]^2}|t-s|^{\alpha\beta-1}\mathbb{E}[|X_t-X_s|^{-\beta}]\,\mathrm{d}s\mathrm{d}t\\ &\sim_\beta \int_{[0,T]^2}|t-s|^{\alpha\beta-1}\,\mathrm{Var}(X_t-X_s)^{-\beta/2}\,\mathrm{d}s\mathrm{d}t\\ &\lesssim \int_{[0,T]^2}|t-s|^{\alpha\beta-1-\delta\beta}\,\mathrm{d}s\mathrm{d}t<\infty\end{aligned}$$

whenever $\delta<\alpha$, regardless the value of β. As a consequence, for $\mathbb{P}$-a.e. $\omega\in\Omega$, the path $X(\omega)$ does not satisfy an approximate Hölder condition of order $\delta>\alpha$ on a set of points t of full Lebesgue measure.

The next two lemmas give explicit conditions for the finiteness $I_w^{\alpha,\beta}$.

Lemma 5.92. *Let w be such that $\ell^w\in C_t^\gamma C_x^0$. Then $F(t)<\infty$ for all $t\in[0,T]$ and $\alpha\in(0,1)$, $\beta\in(0,d)$ such that*

$$\alpha\beta+\gamma>1;$$

in this case, there exists a constant $C = C(\alpha, \beta, \gamma)$ *such that*

$$F(t) \leqslant C \|\ell^w\|_{C^\alpha C^0} \|w\|_{L^\infty}^{d-\beta} (1 + T^\gamma) \quad \forall t \in [0, T].$$

Proof. Let us fix $t \in [0, T]$. We adopt the following convention: whenever I is a finite union of intervals, we denote by ℓ_I^w the local time on I, i.e. $\ell_I^w = \sum_i \ell_{s_i, t_i}^w$ where $I = \cup_i [s_i, t_i]$. Moreover, for $n \in \mathbb{N}$, let us set $I_n := \{s \in [0, T] : |s - t| \in (2^{-n-1}, 2^{-n}]\}$, which is always the union of two intervals of size 2^{-n-1}. Then we can decompose $F(t)$ as follows:

$$\begin{aligned} F(t) &= \int_{\{s \in [0,T] : |s-t| > 1\}} \frac{|t-s|^{\alpha\beta - 1}}{|w_t - w_s|^\beta} \mathrm{d}s + \sum_{n \in \mathbb{N}} \int_{s \in I_n} \frac{|t-s|^{\alpha\beta-1}}{|w_t - w_s|^\beta} \mathrm{d}s \\ &\lesssim \int_{\{s \in [0,T] : |s-t| > 1\}} |w_t - w_s|^{-\beta} \mathrm{d}s + \sum_n 2^{n(1-\alpha\beta)} \int_{s \in I_n} |w_t - w_s|^{-\beta} \mathrm{d}s. \end{aligned}$$

We can now estimate the intervals appearing in the series as follows:

$$\begin{aligned} \int_{I_n} |w_t - w_s|^{-\beta} \mathrm{d}s &= \int_{\mathbb{R}^d} |w_t - x|^{-\beta} \ell_{I_n}^w(x) \mathrm{d}x \\ &\leqslant \|\ell_{I_n}^w\|_{C^0} \int_{B(w_t, 2\|w\|_{L^\infty})} |w_t - x|^{-\beta} \mathrm{d}x \\ &\lesssim \|\ell^w\|_{C^\gamma C^0} \|w\|_{L^\infty}^{d-\beta} 2^{-n\gamma}; \end{aligned}$$

similarly, the first integral can be estimated by

$$\int_{\{s \in [0,T] : |s-t| > 1\}} |w_t - w_s|^{-\beta} \mathrm{d}s \leqslant \int_{\mathbb{R}^d} |w_t - x|^{-\beta} \ell_T^w(x) \mathrm{d}x \lesssim T^\gamma \|\ell^w\|_{C^\gamma C^0} \|w\|_{L^\infty}^{d-\beta}.$$

Overall, we obtain

$$F(t) \lesssim_\beta \|\ell^w\|_{C^\alpha C^0} \|w\|_{L^\infty}^{d-\beta} \left[T^\gamma + \sum_n 2^{n(1-\alpha\beta-\gamma)} \right] < \infty$$

under the condition $\alpha\beta + \gamma > 1$. □

Lemma 5.93. *Let w be (γ, ρ)-irregular. Then $F(t) < \infty$ for all $t \in [0, T]$ and $\alpha \in (0, 1)$, $\beta \in (0, d)$ s.t.*

$$\beta < \rho, \quad \alpha + \beta\gamma > 1;$$

in this case there exists a constant $C = C(\alpha, \beta, \gamma, \rho)$ *such that*

$$F(t) \leqslant C \, \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} (1 + T^\gamma) \quad \forall t \in [0, T].$$

Proof. The proof is analogous to the one above, the main difference being how to estimate the integrals appearing in the series associated to $F(t)$. This time we use Parseval identity and the fact that $\mathcal{F}(|\cdot|^{-\beta}) = c_{\beta,d} |\cdot|^{\beta-d}$ (cf. Proposition 1.29 from [19]), to get

$$\begin{aligned} \int_{I_n} |w_t - w_s|^{-\beta} \mathrm{d}s &= \int_{\mathbb{R}^d} |w_t - x|^{-\beta} \mu_{I_n}^w(\mathrm{d}x) \\ &\sim_\beta \int_{\mathbb{R}^d} |\xi|^{\beta-d} e^{i\xi \cdot w_t} \overline{\hat{\mu}_{I_n}^w(\xi)} \mathrm{d}\xi \\ &\leqslant \|\hat{\mu}_{I_n}^w\|_{\mathcal{F}L^{\rho,\infty}} \int_{\mathbb{R}^d} |\xi|^{\beta-d} (1 + |\xi|)^{-\rho} \mathrm{d}\xi \\ &\lesssim_{\beta,\rho} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} 2^{-n\gamma} \end{aligned}$$

under the condition $\beta < \rho$, so that the integral is convergent. The estimate for the integral over $\{s \in [0, T] : |t - s| > 1\}$ is analogous. Summing over, we find

$$F(t) \lesssim_{\beta,\rho} \|\Phi^w\|_{\mathcal{W}^{\gamma,\rho}} \left[T^\gamma + \sum_n 2^{n(1-\alpha\beta-\gamma)} \right] < \infty$$

under the condition $\alpha\beta + \gamma > 1$. □

So far, to the best of my knowledge, the condition $I_w^{\alpha,\beta} < \infty$ has not appeared in the literature, thus it is hard to say whether it can have any practical purpose. Nontheless, I presented it here to emphasize that (γ, ρ)-irregularity is only one option (not necessarily the best one!) and not the end of the story. There is likely a “mathematical panorama of irregularities” out there waiting to be discovered.

Chapter 6

Inviscid mixing for shear flows

As the title of the chapter suggests, here we will stray away from the standard regularisation by noise setting (even more generally, from the probabilistic framework). We ask the reader a bit of patience, as it will become increasingly clear why there is a natural *fil rouge* and a common philosophy connecting the results given here with the ones from previous parts of the thesis (especially Chapter 5).

In order to properly explain the motivations for this chapter, we need go far back to one of the most fundamental classes of PDEs in fluid mechanics, namely advection-diffusion equations

$$\partial_t f + b\cdot\nabla f = \nu\Delta f. \tag{6.1}$$

For the moment, let us assume the equation to be set on a compact, smooth, d-dimensional manifold M without boundaries and let b be a smooth, divergence free vector field; ∇ is the covariant derivative, Δ the Laplace-Beltrami operator, $\nu\geqslant 0$ and an initial condition $f|_{t=0}=f_0$ is given.

Usually f appearing in (6.1) represents the density of a quantity (e.g. chemical, temperature, etc.) which is subject to diffusion and transported by the field b. For simplicity we will restrict here to the case of a *passive scalar* f, i.e. in the situation where b is given and does not depend on f; let us stress however that the case of *active scalars* is the most interesting in fluid dynamics, and often people study (6.1) as a simplified toy problem for more complicated situations.

Since b is divergence free, the mean $\bar{f} := \int f(x)\,\mathrm{d}x$ of the solution to (6.1) is constant; here $\mathrm{d}x$ denotes the normalized volume form on M and the L^p-norms appearing in the sequel are understood w.r.t. it. A natural question, both for theoretical reasons and in view of applications, is to understand how fast the passive scalar f solving (6.1) is *mixed* by the dynamics, in the sense of getting close to its mean $\bar{f}=\bar{f}_0$. In fact, a simple computation based on (6.1) and $\nabla\cdot b=0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f-\bar{f}\|_{L^2}^2 = -2\nu\|\nabla(f-\bar{f})\|_{L^2}^2; \tag{6.2}$$

applying in the above the Poincarè inequality $\|f-\bar{f}\|_{L^2}^2\leqslant\lambda^{-1}\|\nabla(f-\bar{f})\|_{L^2}^2$, where λ is the first eigenvalue of $-\Delta$, and Grönwall's lemma then yields

$$\|f_t-\bar{f}\|_{L^2}\leqslant e^{-\frac{\nu}{\lambda}t}\|f_0-\bar{f}\|_{L^2}\quad\forall t\geqslant 0; \tag{6.3}$$

namely, an exponentially fast decay to 0. Formula (6.3) presents both several advantages and drawbacks: on one hand, $\|f_t-\bar{f}\|_{L^2}^2$ has a natural interpretation as a variance, thus justifies its use for a quantitative description of mixing; on the other, in the computation b does not play any role (it cancels out when deriving (6.2) due to $\nabla\cdot b=0$). Most importantly, in physical and engineering applications $\nu\ll 1$ and thus a decay of the form (6.3) is hardly of practical use; in the limit case $\nu=0$ it holds $\|f_t-\bar{f}\|_{L^2}=\|f_0-\bar{f}\|_{L^2}$ for all $t\geqslant 0$, again due to $\nabla\cdot b=0$.

The above discussion then raises the question, first addressed in [214]: what is an efficient way of quantifying mixing? Taking $\nu=0$ in (6.1), we are left with the transport equation

$$\partial_t f + b\cdot\nabla f = 0 \tag{6.4}$$

which can be solved explicitly by $f_t(x)=f_0(\Phi_t^{-1}(x))$, where $\{\Phi_t\}_{t\geqslant 0}$ is the flow generated by b and Φ_t^{-1} denotes is inverse as a function from M to itself. In this way we find a pleasant ambiguity with ergodic theory: since $\nabla\cdot b=0$, Φ_t preserves the volume measure and one can rather ask whether Φ_t is mixing in the sense that

$$\lim_{t\to\infty}\mathrm{vol}(\Phi_t^{-1}(A)\cap B)=\mathrm{vol}(A)\,\mathrm{vol}(B)\quad\forall A,B\in\mathcal{B}(M).$$

In fact, this property has a nice characterization at the Eulerian level (6.4): the flow Φ_t is mixing in the ergodic theory sense if and only if, for any $f_0 \in L^2$, the associated solution f to (6.4) is such that $f_t - \bar{f} \rightharpoonup 0$ as $t \to \infty$, where $\rightharpoonup$ denotes weak convergence in L^2. The result is classical, see e.g. [74]; it is also worth mentioning [233], which discusses physical (kinetic) mechanisms for mixing and addresses the problem using concepts and methods of dynamical systems theory, as well as [193] for a more complete discussion of the relation between weak convergence and mixing.

The next natural step in this chain of considerations is the realization (first due to [204]) that, since we are working on a compact manifold and $\| f_t - \bar{f} \|_{L^2}$ is constant, $f_t - \bar{f} \rightharpoonup 0$ if and only if $\| f_t - \bar{f} \|_{H^{-s}} \to 0$ for any $s < 0$.[6.1] In particular, we can use the smallness of $\| f_t - \bar{f} \|_{H^{-s}}$ as a way to measure how mixed is f_t. This doesn't however fully answer the problem, especially if the final goal is to devise an optimal mixer b for a given initial state f_0; quoting the review [261]: "Any negative Sobolev norm will capture mixing in the sense of ergodic theory, so there is no profound difference in using either norm. However, the rate at which different norms decrease will in general be different, so different optimal solutions can be obtained."

In fact, there are in the literature two common choices: i) the norm $\|\cdot\|_{H^{-1/2}}$, motivated by the pioneering works [214, 213]; ii) the norm $\|\cdot\|_{H^{-1}}$, mostly due to its simplicity of use (the evolution of $\| f_t - \bar{f} \|_{H^{-1}}$ can be computed almost explicitly from (6.1) and $\|\cdot\|_{H^{-1}}$ has a rather simple dual interpretation), see [261] and the references therein.

We are now ready to come full circle back to the PDE (6.1); we already mentioned that, in the energy balance (6.2), the presence of b doesn't play any role, which questions the optimality of (6.3). The answer, both deep and surprisingly elegant, was obtained in [73]: in order for the presence of b to affect the exponential decay in (6.3), in a way that can be quantified precisely by the property of *relaxation enhancing flows*, it suffices for b to be mixing in the sense of ergodic theory![6.2] The aforementioned property (as given in Definition 1.1 from [73]) may be equivalently rephrased in the following sense (see equation (1.2) from [79]):

Definition 6.1. *The vector field b is **diffusion enhancing** if there exist $C, \nu_0 > 0$ and a continuous, increasing function $r: (0, \nu_0] \to (0, 1)$ such that any solution f to (6.1) satisfies*

$$\| f_t - \bar{f} \|_{L^2} \leqslant C e^{-r(\nu) t} \| f_0 - \bar{f} \|_{L^2} \quad \forall t \geqslant 0 \tag{6.5}$$

and moreover

$$\lim_{\nu \to 0} \frac{\nu}{r(\nu)} = 0. \tag{6.6}$$

Condition (6.6) encodes the idea that the *rate* r is sublinear; the idea behind Definition 6.5 is that, when $\nu \ll 1$, the *relevant time scale* of the dynamics (in the sense of $\| f_t - \bar{f} \|_{L^2}$ staying macroscopic) is given by $t \sim r(\nu)^{-1}$. By condition (6.6), $r(\nu)^{-1} \ll \nu^{-1}$, the latter being the standard *diffusive time scale* suggested by (6.3), which also holds in the absence of b.

To make the context less abstract, let us mention a real-life example of diffusion enhancement: whenever we pour sugar in a cup of coffee and stir it, what we are actually doing is accelerating the dissolution of the sugar by means of a mechanical (mixing) action. The sugar is naturally subject to diffusion, so the coffee would eventually become sweet even without stirring; but in order for it to do so in a reasonable amount of time (before the coffee gets cold), we need to use our spoon (thus change the relevant time-scale of the dynamics to accomodate the human one).

We can now finally come to the main purpose of this chapter, which is based on the work [144]. The theoretical questions we will try to address here (in a very simplified setting, that will be shortly presented) are the following: i) are *generic* vector fields b mixing and/or diffusion enhancing? ii) what is the optimal rate we can hope to achieve? iii) is there a link between the mixing properties of b and its *roughness*? iv) what is the correct way to measure roughness in this setting?

6.1. Here H^{-s} can be defined either by duality of by spectral theory, e.g. $g \in H^{-s}$ iff $(1 - \Delta)^{-s/2} g \in L^2$. We will soon specialize to the case $M = \mathbb{T}^d$ d-dimensional torus, where H^{-s} can be defined in a more standard way in terms of Fourier series.

6.2. Actually there is an exact characterization in terms of the spectum of $b \cdot \nabla$ in $H^1(M)$ and a sufficient condition is given by *weakly mixing* b, see Theorem 1.2 and Corollary 1.3 from [73]; here for simplicity we will only work in the mixing framework.

We hope the reader will now start to realize why this chapter is a pertinent addition to this thesis. Indeed, *genericity* will be understood in the sense of prevalence, allowing us to give probabilistic proofs and a key role in our analysis will be played by ρ-irregular fields (spoiler: this will not be the only notion of roughness we wil employ! think of Section 5.5).

Moreover, we will obtain here a nice counterpart of the principle "*the rougher the noise, the better the regularisation*", which can be summarized as "*the rougher the drift, the faster the mixing*".

As already mentioned, we will not however address questions i)-iv) in the fully general setting of the PDE (6.1) (or its counterpart (6.4)), but we will restrict ourselves to the paragdimatic (yet undoubtly simpler) setting of 2-dimensional shear flows on $\mathbb{T}^2$ (see the upcoming eq. (6.7)).

Structure of the chapter. In Section 6.1 we introduce our rigorous analytical setup and provide the statements of our main results, specifically Theorems 6.2, 6.6 and 6.7.

Sections 6.2 and 6.3 contain the proofs of Theorems 6.6 and 6.7 and are designed in a similar manner: in both cases we will first prove the lower bound, then introduce the concept of ρ-irregularity (resp. Wei's condition) and explain its connection to the upper bound, as well as to the irregularity of u; finally, we will show by probabilistic means that a.e. $u\in B^{\alpha}_{1,\infty}$ satisfies such property. The end of Section 6.3 also contains the proof of Theorem 6.2.

Finally, Section 6.4 contains further comments on future research directions and a broader discussion on existing literature.

Notations and conventions. We will mostly adopt the same notations and conventions as in previous chapters, up to the following few exceptions.

As already mentioned, $\mathbb{T}^d$ will denote the d-dimensional torus, which whenever needed for simplicity will be identified with $[-\pi,\pi]^d$ with periodic boundary condition.

Since we will both consider function spaces like $L^2(\mathbb{T};\mathbb{C})$ or $L^2(\mathbb{T}^2;\mathbb{R})$ (similarly for H^s), we will quite often write them explicitly instead of resorting to the shortcut notation L^2 (which however will still be used when there is no confusion). We denote by $d_{\mathbb{T}^d}(x,y)$ the canonical distance on the flat torus $\mathbb{T}^d$, namely $d_{\mathbb{T}^d}(x,y)=\inf_{k\in\mathbb{Z}^d}|x+2\pi k-y|$, where $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}^d$. With a slight abuse, we will keep writing $|x|$ for $x\in\mathbb{T}^d$ to denote $d_{\mathbb{T}^d}(x,0)$.

Operator norms for linear operators between two Banach spaces E, F will be denoted by $\|\cdot\|_{E\to F}$, while $\mathcal{L}$ will be sometimes used to denote the Lebesgue measure on $\mathbb{T}^d$.

Finally, we will keep using the notation $\mathbb{E}_s[X]$ to indicate conditioning w.r.t. a reference filtration $\mathcal{F}_s$, whenever there is no ambiguity (otherwise we will use the full expression $\mathbb{E}[X|\mathcal{F}_s]$).

6.1 Analytical setting and main results

We are interested in the long time behavior of solutions f to

$$\begin{cases} \partial_t f + u\partial_x f = \nu\Delta f \\ f|_{t=0}=f_0, \quad \int_{\mathbb{T}} f_0(x,y)\mathrm{d}x = 0 \end{cases} \tag{6.7}$$

on the 2-dimensional flat torus $\mathbb{T}^2$. The PDE (6.7) is an advection-diffusion equation associated to a shear flow $u=u(y)\colon\mathbb{T}\to\mathbb{R}$, $f\colon\mathbb{R}_{\geqslant 0}\times\mathbb{T}^2\to\mathbb{R}$ with initial condition $f_0\in L^2(\mathbb{T}^2)$ and where $\nu\in[0,1]$ is the diffusion coefficient. Indeed, defining $b\colon\mathbb{T}^2\to\mathbb{R}^2$ as $b(x,y):=(u(y),0)^T$, equation (6.7) becomes a particular instance of (6.1); note that such b is a divergence-free vector field and a stationary solution to 2D Euler equations, thus (6.7) describes the action a perfect fluid on the passive scalar f.

Exactly for this reason, shear flows have received a lot of attention in the literature, in connection to the aforementioned problem of understanding the interaction between mixing and diffusion in fluid mechanics and the transfer of energy from large to small scales for the scalar f. In particular, shear flows are sufficiently simple to allow explicit calculations, while presenting a highly non trivial behavior, as already observed by Kelvin in [187] in the case of the Couette flow $u(y)=y$.

Observe that for continuous u, eq. (6.7) can be solved explicitly by Feynman–Kac formula, giving

$$f_t(x,y)=\mathbb{E}\left[f_0\left(x-\int_0^t u(y+\sqrt{2\nu}\,B_s^2)\mathrm{d}s+\sqrt{2\nu}B_t^1,y+\sqrt{2\nu}\,B_t^2\right)\right] \tag{6.8}$$

where $B=(B^1,B^2)$ is a standard 2D Brownian motion. In the case $\nu=0$ we obtain

$$f_t(x,y)=f_0(x-t\,u(y),y). \tag{6.9}$$

Both formulas (6.8) and (6.9) can then be extended to the case $u\in L^1(\mathbb{T})$,[6.3] in which case eq. (6.7) must be understood in the weak sense, and generate continuous semigroups $e^{t(-u\partial_x+\nu\Delta)}$ on $L^2(\mathbb{T}^2)$. Yet, they do not provide any immediate insight on the long time behavior of the solution f, in particular on the decay in time of quantities like $\|f_t\|_{H^{-s}}$ and $\|f_t\|_{L^2}$.

Following the line of research initiated in [271, 71], we consider rough shear flows, in the sense of requiring $u\in B^\alpha_{1,\infty}(\mathbb{T})$ for some $\alpha\in(0,1)$. Here $B^\alpha_{1,\infty}(\mathbb{T})$ denote the Besov–Nikolskii spaces, see Appendix A.2 for their definition.

We are interested in understanding the behavior of *generic* $u\in B^\alpha_{1,\infty}(\mathbb{T})$, a problem explicitly left open in [71]. For this purpose, we will again adopt the measure-theoretic notion of prevalence, which is recalled Section A.3. As usual, the expression "for almost every $\varphi\in E$", where E is a function space, will be understood in the sense of prevalence.

The next statement summarizes the main findings of [144].

Theorem 6.2. *Let $\alpha\in(0,1)$. The following hold:*

i. For almost every $u\in B^\alpha_{1,\infty}(\mathbb{T})$ we have inviscid mixing in the scale $H^{1/2}(\mathbb{T}^2)$, in the following sense: for any $\tilde\alpha>\alpha$, there exists $C=C(\alpha,\tilde\alpha,u)$ such that, for any $f_0\in H^{1/2}(\mathbb{T})$ satisfying $\int_{\mathbb{T}} f(x,\cdot)\mathrm{d}x\equiv 0$, it holds

$$\|e^{-tu\partial_x}f_0\|_{H^{-1/2}}\leqslant Ct^{-\frac{1}{2\tilde\alpha}}\|f_0\|_{H^{1/2}}\quad\forall t\geqslant 0.$$

ii. For almost every $u\in B^\alpha_{1,\infty}(\mathbb{T})$ we have enhanced dissipation in the following sense that: for any $\tilde\alpha>\alpha$ there exist $C_i=C(\alpha,\tilde\alpha,u)$ such that, for any $f_0\in L^2(\mathbb{T})$ satisfying $\int_{\mathbb{T}}f(x,\cdot)\mathrm{d}x\equiv 0$, it holds

$$\|e^{t(-u\partial_x+\Delta)}f_0\|_{L^2}\leqslant C_1\exp\left(-C_2\,t\,\nu^{\frac{\tilde\alpha}{\tilde\alpha+2}}\right)\|f_0\|_{L^2}\quad\forall t\geqslant 0,\nu\in[0,1].$$

In the above statement, the condition $\int_{\mathbb{T}}f(x,\cdot)\mathrm{d}x\equiv 0$ is necessary, as it naturally ensures that f witnesses the effect of the transport operator $u\partial_x$; indeed $g_t(y):=\int_{\mathbb{T}}f_t(x,y)\mathrm{d}x$ must solve the standard heat equation $\partial_t g=\nu\partial_y^2 g$ and thus cannot exhibit any mixing/enhanced dissipation effect (it is similar to considering the decay of $f-\bar f$ in the context of (6.5)).

There is no obvious a priori reason to work with the spaces $B^\alpha_{1,\infty}(\mathbb{T})$ (e.g. in [71] the authors deal with $C^\alpha(\mathbb{T})=B^\alpha_{\infty,\infty}(\mathbb{T})$), rather they arise naturally in our analysis. One of the main intuitions of the work [144] is the identification of such spaces as the correct one for studying generic inviscid mixing and enhanced dissipation properties of shear flows. At the same time, let us mention that the only truly relevant parameter is $\alpha\in(0,1)$: indeed statements similar to those of Theorem 6.2 can be given for the (smaller) spaces $B^\alpha_{p,q}(\mathbb{T})$ for any choice of $p,q\in[1,\infty]$, see Remark 6.8 below.

Remark 6.3. Before moving further, let us heuristically motivate the connection between Points *i.* and *ii.* of Theorem 6.2 and why it is natural to expect the exponent $\nu^{\alpha/(\alpha+2)}$ to appear, given the decay $\|f_t\|_{H^{-1/2}}\lesssim t^{-1/(2\alpha)}$. In fact, the argument can be given in the general framework from the introduction: let f^ν be a solution to (6.1) with $\nu>0$, $\bar f=0$ and $b\colon\mathbb{T}^d\to\mathbb{R}^d$ be a divergence free vector field; then f^ν satisfies the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_t^\nu\|_{L^2}^2=-2\nu\|\nabla f_t^\nu\|_{L^2}^2.$$

6.3. For $u\in L^1(\mathbb{T})$, the formal expression $\int_0^t u(y+\sqrt{2\nu}\,B_s^2)\mathrm{d}s$ in (6.8) can be made rigorous using the local time of B^2; alternatively, equation (6.7) can be solved by applying the Fourier transform in the x-variable and solving the family of equations for $f^k=P_kf$, see the beginning of Appendix B from [144] for more details.

Now assume the solution f to the transport equation $\partial_t f + b\cdot\nabla f = 0$ to satisfy $\|f_t\|_{\dot{H}^{-s}} \lesssim t^{-\alpha s}$ for suitable parameters $\alpha>0, s\in(0,1]$ (for $s>1$, one may reduce to $s=1$ by Riesz–Thorin interpolation theorem). For $\nu \ll 1$ and sufficiently short times, we expect f^ν and f to stay close and therefore f^ν to exhibit the same decay as f. By the interpolation inequality

$$\|f^\nu\|_{L^2} \lesssim \|f^\nu\|_{\dot{H}^{-s}}^{\frac{1}{1+s}} \|\nabla f^\nu\|_{L^2}^{\frac{s}{s+1}},$$

we can then deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_t^\nu\|_{L^2}^{-\frac{2}{s}} \sim \nu \|f_t^\nu\|_{L^2}^{-2\left(\frac{s+1}{s}\right)} \|\nabla f_t^\nu\|_{L^2}^2 \gtrsim \nu\, \|f_t^\nu\|_{\dot{H}^{-s}}^{-\frac{1}{s}} \gtrsim \nu t^{\frac{2}{\alpha}}. \tag{6.10}$$

Assume for simplicity $\|f_0\|_{L^2}=1$ and define $\tau>0$ to be the first time such that $\|f_t^\nu\|_{L^2}=1/2$. Integrating (6.10) over $[0,\tau]$, we obtain

$$1 \sim 2^{\frac{2}{s}} - 1 \gtrsim \nu \int_0^\tau t^{\frac{2}{\alpha}} \mathrm{d}t \sim \nu \tau^{1+\frac{2}{\alpha}} = \left(\nu^{\frac{\alpha}{\alpha+2}}\tau\right)^{\frac{\alpha}{\alpha+2}}.$$

Namely, in order for the energy $\|f_t^\nu\|_{L^2}$ to be reduced by half by the dynamics, we need to wait for at most $\tau \lesssim \nu^{-\alpha/(\alpha+2)}$. Iterating the argument on intervals $[n\tau, n(\tau+1)]$ would then produce an asymptotic decay at least of the form $\exp(-Ct\nu^{\frac{\alpha}{\alpha+2}})$.

Although the above argument is clearly heuristic, it predicts the correct exponent $\frac{\alpha}{\alpha+2}$ and works for any choice of the parameter $s>0$ (in particular for $s=1/2$ as in Theorem 6.2) and not only for $s=1$, which is the case receiving the most attention in the literature.

Unfortunately, there are only few rigorous quantitative results connecting explicitly inviscid mixing and enhanced dissipation properties (see [76] and the references therein) and they appear not to be optimal in the case of shear flows[6.4]. For instance for $s\in(0,1]$, an application of Corollary 2.3 from [76] would only predict a decay

$$\|f_t\|_{L^2} \leqslant \exp(-C\nu^{q_s} t)\|f_0\|_{L^2}, \qquad q_s := \frac{\alpha(1+s)}{\alpha+s+\alpha s};$$

in particular $q_1 = \frac{2\alpha}{2\alpha+1}$ while $q_{1/2} = \frac{3\alpha}{3\alpha+1}$.

Relation with existing literature. Understanding the interaction between mixing and diffusion is one of the most fundamental problems in fluid mechanics, dating back to the works of Kelvin [187] and Reynolds [245].

As already mentioned, in the pioneering work [73], such relation has been formalized mathematically by introducing the concept of relaxation enhancing flow; the result has been recently revisited in a more quantitative fashion in the works [76, 109]. The use of weak norms H^{-s} in order to quantify mixing of passive scalars was first introduced in [214] for $s=1/2$, the general case being due to [204]; see also the review [261] and the recent work [231].

Shear flows and circular flows in particular have been recently studied by several authors, employing a variety of technique, including stationary phase methods and hypocoercivity schemes [32, 75, 77], spectral methods [271, 149] and stochastic analysis [79]. Roughly speaking, the main known results for (6.7) are the following:

- If $u\in C^{n+1}$ has a finite number of critical points with maximal order n, then enhanced dissipation holds with $r(\nu)\sim\nu^{\frac{n}{n+2}}(1+\log\nu^{-1})^{-1}$, see Theorem 1.1 in [32].
- There exist $u\in C^\alpha$, $\alpha\in(0,1)$, for which enhanced dissipation holds with $r(\nu)\sim\nu^{\frac{\alpha}{\alpha+2}}$, see Theorem 5.1 from [271].
- The above results are sharp, up to logarithmic corrections, in the sense that for $u\in C^{n+1}$ (resp. $u\in C^\alpha$) the best possible rate is $r(\nu)\sim\nu^{\frac{n}{n+2}}$ (resp. $r(\nu)\sim\nu^{\frac{\alpha}{\alpha+2}}$), see Theorem 4 in [79]; the proof is based on the Lagrangian Fluctuation Dissipation relation introduced in [99], [100].

6.4. Actually, in order to apply Corollary 2.3 from [76], one would need to verify the abstract condition (2.4) therein, which tipically requires u to be Lipschitz. Leaving aside this technicality and assuming the result is still applicable, it doesn't yield the optimal rate anyway.

Let us also mention the remarkable stable mixing estimate obtained in [75] for u satisfying Assumption (H) therein[6.5]. Motivated by the above results, the authors of [71] explore the mixing and enhanced dissipation properties of rough shear flows, namely u sharply α–Hölder for $\alpha \in (0,1)$. In particular, they construct a Weierstrass-type flow u such that the following hold (see Theorem 1.1 in [71]):

1. enhanced dissipation holds with rate $r(\nu) \sim \nu^{\frac{\alpha}{\alpha+2}}$, confirming the results from [271];
2. along suitable sequences $t_n \to \infty$, inviscid mixing holds on H^{-1} with rate $r(t) \sim t^{1/\alpha}$:
$$\|e^{-t_n u \partial_x} f_0\|_{H^{-1}} \lesssim t_n^{-\frac{1}{\alpha}} \|f_0\|_{H^1}.$$
3. however, to the authors' surprise, there exist other sequences $\tilde{t}_n \to \infty$ on which inviscid mixing only holds with rate $r(t) \sim t$, in the sense that
$$\|e^{-t_n u \partial_x} f_0\|_{H^{-1}} \gtrsim t_n^{-1} \|f_0\|_{H^1}.$$

In particular, the inviscid mixing rate $r(t) \sim t$ is the same attained by suitable Lipschitz functions; the authors wonder whether such a discrepancy between Points 2. and 3. is to be expected for generic flows $u \in C^\alpha$, see the paragraph "Perspectives", p.3 in [71].

The main aim of the present work is to give a negative answer to the above question, while letting a more natural picture emerge in the context of generic rough shear flows. Theorem 6.2 shows that for generic $u \in B^\alpha_{1,\infty}$ (similarly for $u \in C^\alpha$, see Remark 6.8) inviscid mixing holds on $H^{-1/2}$ with rate $r(t) \sim t^{1/2\alpha}$, uniformly over all $t \geqslant 0$. Such a decay is also the best possible, see Theorem 6.6 below. On the other hand, Theorem 6.2 confirms the enhanced dissipation rate $r(\nu) \sim \nu^{\alpha/(\alpha+2)}$, already identified in [271, 71], as a property of generic shear flows.

A complete picture is however still missing; for instance, the question whether generic $u \in B^\alpha_{1,\infty}$ satisfy inviscid mixing on H^{-1} with rate $r(t) \sim t^{-1/\alpha}$ is still open and goes beyond the methods presented here. It also raises the question (supported by the works [214] and [102]) whether H^{-1} is indeed the correct way of measuring mixing and whether $H^{-1/2}$ is instead a better one.

Structure of the proof. As done frequently in the literature, in order to prove Theorem 6.2 for the PDE (6.7), we will pass to study its hypoelliptic counterpart

$$\partial_t f + u \partial_x f = \nu \partial_y^2 f \tag{6.11}$$

again under the assumption $\int_{\mathbb{T}} f_0(x,y) \mathrm{d}x = 0$ for all $y \in \mathbb{T}$.

For $k \in \mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$, define the Fourier transform in the x-variable as

$$(P_k f)(y) := \int_{\mathbb{T}} f(x,y) e^{-ikx} \, \mathrm{d}x$$

so that any $f: \mathbb{T}^2 \to \mathbb{R}$ has a decomposition $f(x,y) = \sum_k (P_k f)(y) e^{ikx}$. If f solves (6.11), then for each $k \in \mathbb{Z}_0$ the function $f^k_t := P_k f_t$ solves the one dimensional complex valued PDE (harmonic oscillator)

$$\partial_t f^k + iku f^k = \nu \partial_y^2 f^k. \tag{6.12}$$

For $k \in \mathbb{Z}_0$, $\nu \geqslant 0$ and $u \in L^1(\mathbb{T})$, the PDE (6.12) has an associated semigroup on $L^2(\mathbb{T};\mathbb{C})$, which we denote by $e^{t(-iku + \nu \partial_y^2)}$; observe that the parameter k, up to its sign, may be removed by the rescaling $\tilde{t} = t|k|$, $\tilde{\nu} = \nu / |k|$. In this way, the study of asymptotic behavior of f^k may be reduced to that of $f^{\pm 1}$, which motivates the following definitions.

Recall that whenever referring to a *rate* $r: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, we mean a continuous, increasing map.

Definition 6.4. *A velocity field $u \in L^1(\mathbb{T})$ is said to be **mixing** on the scale $H^s(\mathbb{T};\mathbb{C})$, $s \geqslant 0$, with rate r, if there exist a constant $C > 0$ such that*

$$\|e^{-itku}\|_{H^s \to H^{-s}} \leqslant \frac{C}{r(t|k|)} \quad \forall k \in \mathbb{Z}_0, \ t \geqslant 1. \tag{6.13}$$

6.5. As a side remark, it is extremely interesting to observe the similarity between the assumption from Proposition 1.4 from [66] and Assumption (H) from [75].

Definition 6.5. *A velocity field $u \in L^1(\mathbb{T})$ is said to be* ***diffusion enhancing*** *on $L^2(\mathbb{T};\mathbb{C})$ with rate r if there exists a constant $C > 0$ such that*

$$\left\| e^{t(-iku+\nu\partial_y^2)} \right\|_{L^2 \to L^2} \leqslant C \exp\left(-r\left(\frac{\nu}{|k|}\right)|k|t \right) \quad \forall k \in \mathbb{Z}_0,\ \nu \in (0,1],\ t \geqslant 1. \tag{6.14}$$

The following theorems, which are the main results of this chapter, provide sharp inviscid mixing and enhanced diffusion statements for generic shear flows. In particular, they describe precisely the behavior of solutions to (6.7) at each Fourier level P_k.

Theorem 6.6. (Inviscid case $\boldsymbol{\nu = 0}$) *Let $\alpha \in (0,1)$.*

- *a) Lower bound. Suppose that $u \in B^{\alpha}_{1,\infty}(\mathbb{T})$ is mixing on the scale $H^{1/2}(\mathbb{T};\mathbb{C})$ with rate r, in the sense of Definition 6.4; then necessarily $r(t) \lesssim t^{\frac{1}{2\alpha}}$.*
- *b) Upper bound. Almost every $u \in B^{\alpha}_{1,\infty}(\mathbb{T})$ satisfies the following property: for any $\tilde{\alpha} > \alpha$, u is mixing on the scale $H^{1/2}(\mathbb{T};\mathbb{C})$ with rate $r(t) \gtrsim t^{\frac{1}{2\tilde{\alpha}}}$.*

Theorem 6.7. (Dissipative case $\boldsymbol{\nu > 0}$) *Let $\alpha \in (0,1)$.*

- *a) Lower bound. Suppose that $u \in B^{\alpha}_{1,\infty}(\mathbb{T})$ is diffusion enhancing with rate r, in the sense of Definition 6.5; then necessarily $r(\nu) \lesssim \nu^{\frac{\alpha}{\alpha+2}}$.*
- *b) Upper bound. Almost every $u \in B^{\alpha}_{1,\infty}(\mathbb{T})$ satisfies the following property: for any $\tilde{\alpha} > \alpha$, u is diffusion enhancing with rate $r(\nu) \gtrsim \nu^{\tilde{\alpha}/(\tilde{\alpha}+2)}$.*

Theorems 6.6 and 6.7 will be proven respectively in Sections 6.2 and 6.3, which are structured in a very similar way. Roughly speaking, the strategy we adopt in proving upper and lower bounds may be summarized in three main steps:

1. In both cases, the lower bound follows from estimates which explicitly employ the regularity assumption $u \in B^{\alpha}_{1,\infty}$; in the case $\nu > 0$, we need to preliminary establish a Lagrangian Fluctuation-Dissipation relation for the PDE (6.12) (see Proposition 6.24) similarly in spirit to what was done in [79].
2. The upper bound is satisfied by any u enjoying a suitable analytic property, which encodes its *irregularity*. It turns out that the right properties are given respectively by ρ-irregularity (already introduced in Chapter 5) for $\nu = 0$ and by Wei's irregularity condition (see Definition 6.26) for $\nu > 0$. A shear flow u satisfying any of such properties necessarily enjoys only limited regularity in the scales $B^{\alpha}_{1,\infty}$ (see Lemmas 6.17 and 6.30), confirming that these are the correct spaces to work with.
3. Finally, we show that a.e. $u \in B^{\alpha}_{1,\infty}$ is ρ-irregular (resp. satisfies Wei's condition), see Section 6.2.3 (resp. Section 6.3.4). As before, this is achieved by probabilistic methods, using the law of fractional Brownian motion to construct a measure witnessing the prevalence of such properties.

Remark 6.8. Let us stress that points a) of Theorems 6.6-6.7 holds for *all* $u \in B^{\alpha}_{1,\infty}$, not only generic elements. Since $\mathbb{T}$ is finite, we have the embeddings $B^{\alpha}_{p,q} \hookrightarrow B^{\alpha}_{1,\infty}$ for any $p, q \in [1,\infty]$, thus the lower bound is true for all $u \in B^{\alpha}_{p,q}$ as well. On the other hand, the proofs of points b) of Theorems 6.6-6.7 can be easily readapted to provide the same statements for almost every $u \in B^{\alpha}_{p,q}$, for any choice of $p, q \in [1,\infty]$.

In particular, one could always work with the spaces $C^{\alpha} = B^{\alpha}_{\infty,\infty}$ if desired. There are however several reasons for working with $B^{\alpha}_{1,\infty}$, or more generally $B^{\alpha}_{p,q}$, instead of C^{α}.

Mathematically, such spaces include genuinely discontinuous functions, as well as (possibly continuous) functions of finite p-variation for any $p \in [1,\infty]$: it holds

$$B^{1/p}_{p,1} \hookrightarrow V^p_c \hookrightarrow V^p \hookrightarrow B^{1/p}_{p,\infty},$$

see Proposition 4.3 from [207], Proposition 2.3 from [135] for more details.

Physically, a simple way to explain singularities in fully developed turbulence is by means of structure functions (see e.g. [131]), which are closely related to the finite difference characterization of Besov spaces $B^\alpha_{p,\infty}$. Turbulence is also believed to be closely connected to multifractality (again we refer to the appendix of [131]), a feature which is absent from generic $u \in C^\alpha$ (which are monofractal) but instead manifested by almost every $u \in B^\alpha_{p,q}$, see [184, 130, 129].

Our results show that the only relevant parameter in understanding mixing and enhanced dissipation rates for a.e. $u \in B^\alpha_{p,q}$ is $\alpha \in (0,1)$, regardless of the values of p, q; thus there is no apparent connection between mixing and multifractal features of u, at least in the context of shear flows.

6.2 Inviscid mixing

This section contains the proof of Theorem 6.6, which is split in several steps.

Recall the setting: in order to study the transport equation $\partial_t f + u\partial_x f = 0$, we pass to Fourier modes $f^k_t(y) = (P_k f_t)(y)$, each one solving $\partial_t f^k + iku f^k = 0$; namely, $f^k_t(y) = e^{-iktu(y)} f^k_0(y)$. It is thus natural to take a slightly more general perspective and study properties of maps of the form $y \mapsto e^{i\xi u(y)} g(y)$ with $\xi \in \mathbb{R}$, $g \in H^s(\mathbb{T})$.

All the proofs of this section (and the upcoming Section 6.3) are heavily based on fine properties of Besov spaces, for which we refer the reader to Appendix A.2.

6.2.1 Lower bounds in terms of regularity

We show here that the regularity of u, measured in the Besov–Nikolskii scale $B^\alpha_{1,\infty}$, necessarily implies a lower bound on the decay of solutions in the $H^{-1/2}$-norm. The proof is partly inspired by that of Proposition 3.2 from [71].

Lemma 6.9. *Let $u \in B^\alpha_{1,\infty}(\mathbb{T})$ for some $\alpha \in (0,1)$. Then for any $g \in H^1(\mathbb{T})$ there exists a constant $C = C(\alpha, g)$ such that*

$$\|e^{i\xi u} g\|_{H^{-1/2}} \geqslant C(1 + \|u\|_{B^\alpha_{1,\infty}})^{-\frac{1}{2\alpha}} |\xi|^{-\frac{1}{2\alpha}} \quad \forall |\xi| \geqslant 1. \tag{6.15}$$

Proof. Fix ξ with $|\xi| \geqslant 1$ and set $\bar{g} := e^{i\xi u} g$; we claim that $\bar{g} \in B^{\alpha/2}_{2,\infty}$. By Sobolev and Besov embeddings, $g \in L^\infty \cap B^{\alpha/2}_{2,\infty}$; $e^{i\xi u} \in L^\infty$, so it's enough to show that $e^{i\xi u} \in B^{\alpha/2}_{2,\infty}$. By the basic estimate $|e^{ia} - e^{ib}| \leqslant \sqrt{2}\, |a - b|^{1/2}$, it holds

$$\begin{aligned}\|e^{i\xi u(\cdot + y)} - e^{i\xi u(\cdot + \tilde{y})}\|_{L^2} &\lesssim |\xi|^{1/2} \|u(\cdot + y) - u(\cdot + \tilde{y})\|_{L^1}^{1/2} \\ &\lesssim |\xi|^{1/2} \|u\|_{B^\alpha_{1,\infty}}^{1/2} d_{\mathbb{T}}(y, \tilde{y})^{\alpha/2}.\end{aligned}$$

By the equivalent characterization of Besov–Nikolskii spaces, this implies

$$\|e^{i\xi u}\|_{B^{\alpha/2}_{2,\infty}} \lesssim 1 + |\xi|^{1/2} \|u\|_{B^\alpha_{1,\infty}}^{1/2} \lesssim (1 + \|u\|_{B^\alpha_{1,\infty}})^{1/2} |\xi|^{1/2}$$

and so by Proposition A.12 in Appendix A.2 we conclude that $\bar{g} \in B^{\alpha/2}_{2,\infty}$ with

$$\|\bar{g}\|_{B^{\alpha/2}_{2,\infty}} \lesssim \|g\|_{H^1} (1 + \|u\|_{B^\alpha_{1,\infty}})^{1/2} |\xi|^{1/2}. \tag{6.16}$$

Clearly $\|\bar{g}\|_{L^2} = \|g\|_{L^2}$. Using the interpolation inequality from Corollary A.10 in Appendix A.2 (for the choice $s_1 = 1/2$, $s_2 = \alpha/2$) we obtain

$$\|g\|_{L^2} = \|\bar{g}\|_{L^2} \lesssim \|\bar{g}\|_{H^{-1/2}}^{\frac{\alpha}{1+\alpha}} \|\bar{g}\|_{B^{\alpha/2}_{2,\infty}}^{\frac{1}{1+\alpha}}. \tag{6.17}$$

Rearranging now the terms in (6.17) and applying the estimate (6.16) we find

$$\begin{aligned}\|\bar{g}\|_{H^{-1/2}} &\gtrsim \|\bar{g}\|_{B^{\alpha/2}_{2,\infty}}^{-\frac{1}{\alpha}} \|g\|_{L^2}^{1+\frac{1}{\alpha}} \\ &\gtrsim \|g\|_{L^2}^{1+\frac{1}{\alpha}} \|g\|_{H^1}^{-\frac{1}{\alpha}} (1 + \|u\|_{B^\alpha_{1,\infty}})^{-\frac{1}{2\alpha}} |\xi|^{-\frac{1}{2\alpha}}\end{aligned} \tag{6.18}$$

where the hidden constant in (6.18) only depends on α. Using the definition of $\bar{g}$ and relabelling the constant to include the g-dependent terms yields the conclusion. □

Corollary 6.10. *Let $u\in B^{\alpha}_{1,\infty}(\mathbb{T})$ be mixing with rate r, in the sense of Definition 6.4. Then there exists a constant $C=C(\alpha,u)$ such that*

$$r(t)\leqslant C t^{\frac{1}{2\alpha}}.$$

Proof. Choose any $g\in H^1(\mathbb{T})$ with $\|g\|_{H^{1/2}}=1$; then by Definition 6.4 applied for the choice $k=1$ and Lemma 6.9 for $\xi=-t$, it holds

$$\frac{1}{r(t)}\gtrsim \|e^{-itu}\|_{H^{1/2}\to H^{-1/2}}\geqslant \|e^{-itu}g\|_{H^{-1/2}}\gtrsim_{\alpha,g}(1+\|u\|_{B^{\alpha}_{1,\infty}})^{-\frac{1}{2\alpha}}\,t^{-\frac{1}{2\alpha}};$$

up to relabelling constants, this yields the conclusion. □

Remark 6.11. In fact, the statement of Lemma 6.9 can be generalized as follows. For $\alpha\in(0,1)$, $u\in B^{\alpha}_{1,\infty}(\mathbb{T})$, $g\in H^1(\mathbb{T})$ and any $s>0$ there exists a constant $C(\alpha,g,s)$ such that

$$\|e^{i\xi u}g\|_{H^{-s}}\geqslant C(1+\|u\|_{B^{\alpha}_{1,\infty}})^{-\frac{s}{\alpha}}\,|\xi|^{-\frac{s}{\alpha}}\quad\forall|\xi|\geqslant 1.$$

Then arguing as in Corollary 6.10 one can conclude that the best possible rate for inviscid mixing on the scale $H^s(\mathbb{T})$ is $r(t)\sim t^{-s/\alpha}$. Taking $s=1$ provides the rate $t^{-1/\alpha}$, which is in line with Proposition 3.2 from [71].

6.2.2 Upper bounds in terms of ρ-irregularity

The concept of ρ-irregularity was introduced in Section 5.1.1 for paths defined on $[0,T]$; for the reader's convenience, we recall here the definition, modified in order to allow paths defined on $\mathbb{T}$.

Definition 6.12. *Let $\gamma\in[0,1)$, $\rho>0$; a measurable map $u:\mathbb{R}\to\mathbb{R}$ is said to be* (γ,ρ)-irregular *if there exists a constant $C>0$ such that*

$$\left|\int_I e^{i\xi u(z)}\mathrm{d}z\right|\leqslant C|I|^{\gamma}\,|\xi|^{-\rho}\quad\forall\xi\in\mathbb{R}, I\subset\mathbb{R}\text{ with }|I|\leqslant 2\pi \tag{6.19}$$

where I stands for an interval of $\mathbb{R}$ and $|I|$ denotes its length[6.6]*. A map $u:\mathbb{T}\to\mathbb{R}$ is said to be (γ,ρ)-irregular if it is so once it is identified with a 2π-periodic map $u:\mathbb{R}\to\mathbb{R}$. In both cases, the optimal constant C in (6.19) is denoted by $\|\Phi^u\|_{\gamma,\rho}$. We say that u is* ρ-irregular *for short if there exists $\gamma>1/2$ such that it is (γ,ρ)-irregular.*

The property of ρ-irregularity may be rephrased in the following form, more suited for the purposes of this chapter.

Lemma 6.13. *Let $u:\mathbb{T}\to\mathbb{R}$ be (γ,ρ)-irregular, then*

$$\|e^{i\xi u}\|_{B^{\gamma-1}_{\infty,\infty}}\lesssim\|\Phi^u\|_{\gamma,\rho}\,|\xi|^{-\rho}\quad\forall\xi\in\mathbb{R}.$$

Proof. For $\bar{y}\in[-\pi,\pi]$ and $\xi\in\mathbb{R}$, define the function

$$v^{\xi}(\bar{y})=\int_{-\pi}^{\bar{y}}e^{i\xi u(y)}\mathrm{d}y-\left(\frac{\bar{y}+\pi}{2\pi}\right)\int_{-\pi}^{\pi}e^{i\xi u(y)}\mathrm{d}y;$$

by periodicity, it can be identified with a function on $\mathbb{T}$. Then, by definition of (γ,ρ)-irregularity, it holds $\|v^{\xi}\|_{C^{\gamma}}\lesssim\|\Phi^u\|_{\gamma,\rho}\,|\xi|^{-\rho}$; by Proposition A.8, we deduce that

$$\begin{aligned}\|e^{i\xi u}\|_{B^{\gamma-1}_{\infty\infty}}&=\left\|(v^{\xi})'+\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\xi u(y)}\mathrm{d}y\right\|_{B^{\gamma-1}_{\infty,\infty}}\\&\lesssim\|v^{\xi}\|_{C^{\gamma}}+\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}e^{i\xi u(y)}\mathrm{d}y\right|\lesssim\|\Phi^u\|_{\gamma,\rho}\,|\xi|^{-\rho}\end{aligned}$$

6.6. The condition $|I|\leqslant 2\pi$ appearing in (6.19) is merely out of convenience, since our endgoal is to work on $\mathbb{T}$ which we identify with $[-\pi,\pi]$, but it could have been replaced with $|I|\leqslant M$ for any fixed finite $M>0$.

implying the conclusion. □

The relation between ρ-irregularity and inviscid mixing comes from the next result.

Lemma 6.14. *Let $u: \mathbb{T} \to \mathbb{R}$ be (γ, ρ)-irregular for some $\gamma > 1/2$. Then there exists a constant $C = C(\gamma)$ such that*

$$\|e^{i\xi u} g\|_{H^{-1/2}} \leqslant C \, \|\Phi^u\|_{\gamma,\rho} \, |\xi|^{-\rho} \|g\|_{H^{1/2}} \quad \forall \xi \neq 0, \, g \in H^{1/2}. \tag{6.20}$$

As a consequence, u is mixing on the scale $H^{1/2}$ with rate $r(t) = t^{\rho}$, in the sense of Definition 6.4.

Proof. By assumption $\gamma + 1/2 > 1$, thus by Proposition A.11 and Lemma 6.13 it holds

$$\begin{aligned}\|e^{i\xi u} g\|_{B^{\gamma-1}_{2,\infty}} &\lesssim \|e^{i\xi u}\|_{B^{\gamma-1}_{\infty,\infty}} \|g\|_{B^{1/2}_{2,\infty}}\\ &\lesssim \|\Phi^u\|_{\gamma,\rho} \, |\xi|^{-\rho} \|g\|_{B^{1/2}_{2,2}}\\ &= \|\Phi^u\|_{\gamma,\rho} \, |\xi|^{-\rho} \|g\|_{H^{1/2}}.\end{aligned}$$

Again by the hypothesis $\gamma - 1 > -1/2$ and so by Besov embeddings $B^{\gamma-1}_{2,\infty} \hookrightarrow H^{-1/2}$, yielding the first claim. Applying estimate (6.20) for $k \in \mathbb{Z}_0$, $\xi = -tk$ gives

$$\|e^{-itku}\|_{H^{1/2} \to H^{-1/2}} \leqslant \frac{C \, \|\Phi^u\|_{\gamma,\rho}}{(t|k|)^{\rho}}$$

and thus the conclusion. □

Remark 6.15. Going through the same proof as in Lemma 6.14, one can show that if u is (γ, ρ)-irregular with $\gamma > 1 - s$, then it is mixing on the scale H^s with rate $r(t) = t^{-\rho}$. In the case $\gamma = 0$ an even simpler proof, based on duality and integration by parts, provides mixing on the scale H^1 with the same rate. In fact, since $H^1(\mathbb{T}; \mathbb{C})$ is an algebra, by integration by parts it holds

$$\begin{aligned}|\langle e^{i\xi u} f, g\rangle| &= \left| \int_{-\pi}^{\pi} e^{i\xi u(y)} \, f(y) g(y) \mathrm{d}y \right|\\ &\leqslant |(fg)(-\pi)| \left| \int_{-\pi}^{\pi} e^{i\xi u(y)} \mathrm{d}y \right| + \int_{-\pi}^{\pi} |(fg)'(y)| \left| \int_{-\pi}^{y} e^{i\xi u(z)} \mathrm{d}z \right| \mathrm{d}y\\ &\lesssim (\|fg\|_{L^\infty} + \|(fg)'\|_{L^1}) \, \|\Phi^u\|_{0,\rho} \, |\xi|^{-\rho}\\ &\lesssim \|f\|_{H^1} \|g\|_{H^1} \|\Phi^u\|_{0,\rho} \, |\xi|^{-\rho}\end{aligned}$$

which by duality implies $\|e^{i\xi u} f\|_{H^{-1}} \lesssim \|f\|_{H^1} \|\Phi^u\|_{0,\rho} \, |\xi|^{-\rho}$ and so the claim.

As we have already seen, the property of ρ-irregularity implies roughness of u, in a sense that can be quantified precisely. In particular, the notion of θ-Hölder roughness (cf. Definition 5.68) carries immediately to the periodic setting (one only needs to be careful by replacing $|y - z|$ by the canonical distance on the torus $d_{\mathbb{T}}(y, z)$). In fact, since we are in the one dimensional case, Definition 5.68 admits the following equivalent characterization: $u: \mathbb{T} \to \mathbb{R}$ is α-Hölder rough with modulus of α-Hölder roughness $L_\alpha(u)$ if and only if

$$L_\alpha(u) = \inf_{y \in \mathbb{T}, \delta > 0} \sup_{z \in B_\delta(y)} \frac{|u(z) - u(y)|}{\delta^\alpha} > 0. \tag{6.21}$$

Arguing as in the proofs of Theorem 5.64, Corollary 5.66, Corollary 5.69, one can then show the following result.

Proposition 6.16. *Let $u: \mathbb{T} \to \mathbb{R}$ be (γ, ρ)-irregular and set $\alpha^* := (1 - \gamma)/\rho$. Then:*

a) u is α-Hölder rough for any $\alpha > \alpha^*$ with $L_\alpha(u) = +\infty$.

b) u has infinite p-variation on any subinterval $I \subset \mathbb{T}$ and for any $p > 1/\alpha^*$.

The parameter α^* is also linked to the regularity of u in the Besov–Nikolskii scales $B^{\alpha}_{1,\infty}$; although the result is also true for $u: [0, T] \to \mathbb{R}^d$, we preferred to keep this result here, rather than in the general discussion in Section 5.4.2.

Lemma 6.17. *Let $u:\mathbb{T}\to\mathbb{R}$ be (γ,ρ)-irregular and set $\alpha^*=(1-\gamma)/\rho$; then u does not belong to $B^\alpha_{1,\infty}$ for any $\alpha>\alpha^*$.*

Proof. Fix $\alpha>\alpha^*$ and choose $\tilde\alpha\in(\alpha^*,\alpha)$; going through the same computations as in Theorem 5.64 and Corollary 5.66, it can be shown that

$$\lim_{\varepsilon\to 0^+}\inf_{y\in\mathbb{T}}\varepsilon^{-1}\lambda_1(h\in(0,\varepsilon):|u(y+h)-u(y)|\geqslant\varepsilon^{\tilde\alpha})=1,$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$. It follows that, for all $\varepsilon>0$ sufficiently small, it must hold

$$\begin{aligned}\pi\leqslant&\int_{\mathbb{T}}\varepsilon^{-1}\lambda_1(h\in(0,\varepsilon):|u(y+h)-u(y)|\geqslant\varepsilon^{\tilde\alpha})\,\mathrm{d}y\\ \leqslant&\int_{\mathbb{T}}\varepsilon^{-1-\tilde\alpha}\int_0^\varepsilon|u(y+h)-u(y)|\mathrm{d}h\,\mathrm{d}y\\ =&\varepsilon^{-1-\tilde\alpha}\int_0^\varepsilon\|u(\cdot+h)-u(\cdot)\|_{L^1}\,\mathrm{d}h\\ \leqslant&\varepsilon^{\alpha-\tilde\alpha}\,[\![u]\!]_{B^\alpha_{1,\infty}}\end{aligned}$$

where in the second passage we used Markov's inequality. Since $\alpha>\tilde\alpha$, letting $\varepsilon\to 0^+$ we conclude that $[\![u]\!]_{B^\alpha_{1,\infty}}=+\infty$. □

Remark 6.18. If u is ρ-irregular, then Lemma 6.17 implies that u does not belong to $B^\alpha_{1,\infty}$ for any $\alpha>(2\rho)^{-1}$. Conversely, if $u\in B^\alpha_{1,\infty}$, then it can only be ρ-irregular for parameters ρ satisfying $\rho\leqslant(2\alpha)^{-1}$.

6.2.3 Prevalence statements and proof of Theorem 6.6

Given the results of Sections 6.2.1–6.2.2, it is natural to wonder whether generic elements of $B^\alpha_{1,\infty}$ are "almost as irregular as possible", in the sense of being ρ-irregular for any $\rho<(2\alpha)^{-1}$; given the results from Chapter 5, the reader will not be surprised that we can give a positive answer. Yet, the proof requires a few simple preparations.

Differently from before, in this section we will denote functions on $\mathbb{T}$ by φ, while the letter u shall be used for the canonical process associated to the measure μ involved in the proofs of prevalence statements (which, up to technical modifications, will be the usual fBm law μ^H).

We identify the torus $\mathbb{T}$ with the interval $[-\pi,\pi]$, up to $-\pi\sim\pi$; thus any measurable function $\varphi:\mathbb{T}\to\mathbb{R}$ can be identified with $\varphi:[-\pi,\pi]\to\mathbb{R}$ such that $\varphi(-\pi)=\varphi(\pi)$. Any such φ is in a 1-1 correspondence with a pair (φ_1,φ_2) of measurable functions defined on $[0,\pi]$, given by $\varphi_1(y):=\varphi(y)$, $\varphi_2(y):=\varphi(-y)$; they satisfy the constraint $\varphi_1(\pi)=\varphi_2(\pi)$.

Lemma 6.19. *A measurable function $\varphi:\mathbb{T}\to\mathbb{R}$ is (γ,ρ)-irregular if and only if the functions φ_1, $\varphi_2:[0,\pi]\to\mathbb{R}$ are so.*

Proof. The proof is elementary. Given $I\subset[-\pi,\pi]$, setting $I_1=I\cap[0,\pi]$, $I_2=I\cap[-\pi,0]$ it holds $\max\{|I_1|,|I_2|\}\leqslant|I|\leqslant 2\max\{|I_1|,|I_2|\}$, so that

$$\max\{\|\Phi^{\varphi_1}\|_{\gamma,\rho},\|\Phi^{\varphi_2}\|_{\gamma,\rho}\}\leqslant\|\Phi^\varphi\|_{\gamma,\rho}\leqslant 2\max\{\|\Phi^{\varphi_1}\|_{\gamma,\rho},\|\Phi^{\varphi_2}\|_{\gamma,\rho}\}.$$ □

Conversely, given a measurable $\tilde\varphi:[0,\pi]\to\mathbb{R}$, we can associate it another function $\varphi=T\tilde\varphi:\mathbb{T}\to\mathbb{R}$ by setting $T\tilde\varphi(y)=\tilde\varphi(|y|)$, which corresponds to $(T\tilde\varphi)_1=(T\tilde\varphi)_2=\tilde\varphi$.

It immediately follows from Lemma 6.19 that $T\tilde\varphi$ is (γ,ρ)-irregular if and only if $\tilde\varphi$ is so; it is also easy to check that, if $\tilde\varphi\in B^\alpha_{1,\infty}(0,\pi)\cap L^\infty(0,\pi)$, then $T\tilde\varphi\in B^\alpha_{1,\infty}(\mathbb{T})$.

Next, we give the following result, which is a simple analogue of Point i. of Theorem 5.29.

Proposition 6.20. *Let $\alpha\in(0,1)$; then a.e. $\varphi\in B^\alpha_{1,\infty}(0,\pi)$ is ρ-irregular for every $\rho<(2\alpha)^{-1}$.*

Proof. Given $\rho > 0$, define the set $\mathcal{A}_\rho = \{\varphi \in B^\alpha_{1,\infty}(0,\pi) : \varphi$ is ρ-irregular$\}$; arguing as in the proof of Lemma 5.10, it's easy to check that $\mathcal{A}_\rho$ defines a Borel set. If we show that $\mathcal{A}_\rho$ is prevalent in $B^\alpha_{1,\infty}(0,\pi)$ for any $\rho < (2\alpha)^{-1}$, then the same holds for

$$\mathcal{A} := \left\{\varphi \in B^\alpha_{1,\infty}(0,\pi) : \varphi \text{ is } \rho\text{-irregular for every } \rho < \frac{1}{2\alpha}\right\} = \bigcap_{n=1}^{\infty} \mathcal{A}_{\frac{1}{2\alpha} - \frac{1}{n}}$$

which implies the conclusion.

Now fix $\rho < (2\alpha)^{-1}$ and choose $H \in (0,1)$ such that $H > \alpha$, $\rho < (2H)^{-1}$; denote by μ^H the law of fractional Brownian motion on $C([0,\pi])$ and by $u = \{u_y, y \in [0,\pi]\}$ the associated canonical process. Since μ^H is supported on $C^{H-\varepsilon}([0,\pi])$ for any $\varepsilon > 0$ and $H > \alpha$, it is also a tight probability measure on $B^\alpha_{1,\infty}(0,\pi)$. Since u is a H-SLND process and $\rho < (2H)^{-1}$, it follows from Point a) of Proposition 5.53 that, for any measurable $\varphi: [0,\pi] \to \mathbb{R}$, μ^H-a.s. $\varphi + u$ is ρ-irregular. Taking $\varphi \in B^\alpha_{1,\infty}(0,\pi)$ we deduce that

$$\mu^H(\varphi + \mathcal{A}_\rho) = 1 \quad \forall \varphi \in B^\alpha_{1,\infty}(0,\pi),$$

namely that μ^H witnesses the prevalence of $\mathcal{A}_\rho$ in $B^\alpha_{1,\infty}(0,\pi)$. □

Corollary 6.21. *Let $\alpha \in (0,1)$, then a.e. $\varphi \in B^\alpha_{1,\infty}(\mathbb{T})$ is ρ-irregular for any $\rho < (2\alpha)^{-1}$.*

Proof. The proof that the set $\mathcal{A} := \{\varphi \in B^\alpha_{1,\infty}(\mathbb{T}) : \varphi$ is ρ-irregular for any $\rho < 1/2\alpha\}$ is Borel in the topology of $B^\alpha_{1,\infty}(\mathbb{T})$ is again identical to that of Lemma 5.10.

Now let μ be a measure on $B^\alpha_{1,\infty}(0,\pi)$ witnessing the prevalence statement of Proposition 6.20; since in that proof we can take $\mu = \mu^H$ for suitable $H > \alpha$, we may assume μ to take values in $B^\alpha_{1,\infty}(0,\pi) \cap L^\infty(0,\pi)$. Therefore we can define a measure on $B^\alpha_{1,\infty}(\mathbb{T})$ by $\nu = T_\sharp \mu$, where $(T\tilde{\varphi})(y) = \tilde{\varphi}(|y|)$ for $y \in [0,\pi]$; also recall the notation φ_1, φ_2 from Lemma 6.19. For any $\varphi \in B^\alpha_{1,\infty}(\mathbb{T})$ it holds

$$\begin{aligned}\nu(\varphi + \mathcal{A}) &= \mu\left(\left\{u \in B^\alpha_{1,\infty}(0,\pi) : Tu + \varphi \text{ is } \rho\text{-irregular for any } \rho < \frac{1}{2\alpha}\right\}\right)\\ &= \mu\left(\bigcap_{i=1}^{2} \left\{u \in B^\alpha_{1,\infty}(0,\pi) :\ u + \varphi_i \text{ is } \rho\text{-irregular for any } \rho < \frac{1}{2\alpha}\right\}\right) = 1\end{aligned}$$

where we used the fact that the μ witnesses the prevalence of the statement of Proposition 6.20 and that the intersection of sets of full measure is still of full measure. Thus ν witnesses the prevalence of the set $\mathcal{A}$. □

We are now ready to complete the

Proof of Theorem 6.6. The lower bound comes from Corollary 6.10, while the upper bound from a combination of Lemma 6.14 and Corollary 6.21. □

Remark 6.22. In this section we have always focused on u belonging to the scales $B^\alpha_{1,\infty}(\mathbb{T})$ with $\alpha \in (0,1)$. If one is instead interested in the mixing properties of generic $u \in C(\mathbb{T})$, much faster rates are available, given the results from Chapter 5. Indeed, for any $\beta > 1$, it's possible to construct $\tilde{u}^\beta \in C([0,\pi])$ satisfying

$$\left|\int_{y_1}^{y_2} e^{i\xi u^\beta(z)} \mathrm{d}z\right| \lesssim_{\gamma,\beta} |y_2 - y_1|^\gamma \exp\left(-C_{\gamma,\beta} |\xi|^{\frac{2}{1+\beta}}\right) \quad \forall \xi \in \mathbb{R}, \quad 0 \leqslant y_1 \leqslant y_2 \leqslant \pi \tag{6.22}$$

and so by symmetrization the same holds for $u^\beta := T\tilde{u}^\beta$. Such $\tilde{u}^\beta$ are given by typical realization of the process X^β as defined in Proposition 5.51; in fact, one could use the law of such process to prove that almost every $u \in C(\mathbb{T})$ satisfies (6.22) for any $\beta > 1$. Arguing as in the proof of Lemma 6.14, one can deduce that such u are exponentially mixing, in the sense that they satisfy the estimate

$$\|e^{i\xi u} g\|_{H^{-1}} \lesssim \exp\left(-C_{\gamma,\beta} |\xi|^{\frac{2}{1+\beta}}\right) \|g\|_{H^1} \quad \forall g \in H^1(\mathbb{T};\mathbb{C}) \tag{6.23}$$

and so that

$$\|e^{-tu\partial_x} f\|_{L^2_x H^{-1}_y} \lesssim \exp\left(-C_{\gamma,\beta} t^{\frac{2}{1+\beta}}\right) \|f\|_{L^2_x H^1_y} \tag{6.24}$$

for all $f \in H^1(\mathbb{T}^2)$ satisfying $P_0 f \equiv 0$.

6.3 Enhanced dissipation

This section contains the proof of Theorem 6.7, which is split in several steps.

Recall the setting: we want to study the asymptotic behavior of the family of complex-valued PDEs (6.12), equivalently obtain upper and lower bounds on

$$\|e^{tL_{k,\nu}}\|_{L^2(\mathbb{T};\mathbb{C})\to L^2(\mathbb{T};\mathbb{C})} \quad \text{as } t\to\infty,$$

where $L_{k,\nu} := -iku + \nu\partial_y^2$.

6.3.1 Lower bounds in terms of regularity

We show here that if u has regularity of degree $\alpha \in (0,1)$, as measured in a suitable Besov–Nikolskii scale, then the its best possible diffusion enhancing rate is $r(\nu) \sim \nu^{\alpha/(2+\alpha)}$. The precise statement goes as follows.

Proposition 6.23. *Let $u \in B^{\alpha}_{1,\infty}(\mathbb{T})$ be diffusion enhancing with rate r, in the sense of Definition 6.5; then there exists a constant $C > 0$ such that*

$$r(\nu) \leqslant C\nu^{\frac{\alpha}{\alpha+2}}$$

for all $\nu \in (0,1]$.

In order to provide estimates for $e^{tL_{k,\nu}}$, it is convenient to study more generally the properties of solutions $g\colon \mathbb{T} \to \mathbb{C}$ to

$$\partial_t g + i\xi u g = \nu \partial_{yy} g \tag{6.25}$$

in function of the parameters $\xi \in \mathbb{R}$, $\nu \in (0,1)$ and the shear flow u.

The proof of Proposition 6.23 follows a similar strategy to [79] and is based on deriving a Lagrangian Fluctuation-Dissipation relation (FDR) for the PDE (6.25), which is a result of independent interest.

Proposition 6.24. *Let $u \in L^1(\mathbb{T})$, g be a solution to (6.25) with initial data $g_0 \in L^2(\mathbb{T};\mathbb{C})$; for any $(t,y) \in \mathbb{R}_{\geqslant 0} \times \mathbb{T}$, define the complex random variable*

$$Z_t^y = \exp\left(-i\xi \int_0^t u(y + \sqrt{2\nu} B_s)\,\mathrm{d}s\right) g_0(y + \sqrt{2\nu} B_t)$$

where B is a standard real-valued Brownian motion. Then we have the following Lagrangian FDR:

$$\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 = \int_{\mathbb{T}} \mathrm{Var}(Z_t^y)\,\mathrm{d}y. \tag{6.26}$$

Proof. Without loss of generality, we can assume u and g_0 to be smooth, as identity (6.26) in the general case follows from an approximation argument (the definition of Z_t^y is meaningful for any $u \in L^1(\mathbb{T})$ thanks to the properties of the local time of a Brownian motion). By the Feynman–Kac formula, the solution g to (6.25) is given by $g_t(y) = \mathbb{E}[Z_t^y]$. Moreover since u is real valued, we have the energy balance

$$\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 = 2\nu \int_0^t \|\partial_y g_s\|_{L^2}^2\,\mathrm{d}s;$$

and more generally, the map $(t,x) \mapsto |g|^2(t,x)$ satisfies

$$\partial_t |g|^2 = \nu \partial_y^2 |g|^2 - 2\nu |\partial_y g|^2.$$

Now let h to be a solution of $\partial_t h = \nu \partial_y^2 h$ with initial data $h_0 = |g_0|^2$. It holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} [|g|^2 - h]\,\mathrm{d}x = -2\nu \|\partial_y g\|_{L^2}^2,$$

which implies that

$$\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 = 2\nu\int_0^t \|\partial_y g\|_{L^2}^2 = \int_{\mathbb{T}} [h_t(y) - |g_t(y)|^2] \mathrm{d}y.$$

Finally, since by Feynman–Kac, $h(t,y) = \mathbb{E}[|g_0(y+\sqrt{2\nu}B_t)|^2]$, we obtain

$$\begin{aligned}\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 &= \int_{\mathbb{T}} (\mathbb{E}[|g_0(y+\sqrt{2\nu}B_t)|^2] - |\mathbb{E}[Z_t^y]|^2)\, \mathrm{d}x \\ &= \int_{\mathbb{T}} (\mathbb{E}[|Z_t^y|^2] - |\mathbb{E}[Z_t^y]|^2) \mathrm{d}x\end{aligned}$$

which gives the conclusion. □

Lemma 6.25. *Let $g_0 \in H^1(\mathbb{T};\mathbb{C})$, $u \in B^\alpha_{1,\infty}(\mathbb{T})$ for some $\alpha \in (0,1)$ and $\xi \in \mathbb{R}$. Then there exists $C = C(\alpha) > 0$ such that the solution g to (6.25) satisfies*

$$\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 \leqslant C\|g_0\|_{H^1}^2 \Big(\nu t + [\![u]\!]_{B^\alpha_{1,\infty}} |\xi|\, \nu^{\frac{\alpha}{2}} t^{1+\frac{\alpha}{2}}\Big) \quad \forall t, \nu > 0.$$

Proof. Recall the elementary identity $2\mathrm{Var}(X) = \mathbb{E}[|X - \tilde{X}|^2]$ for $\tilde{X}$ being an i.i.d. copy of X. In our setting, we can take

$$\tilde{Z}_t^y = \exp\Big(-i\xi\int_0^t u(y+\sqrt{\nu}\tilde{B}_s)\mathrm{d}s\Big) g_0(y+\sqrt{\nu}\tilde{B}_t),$$

where $\tilde{B}$ is another Brownian motion independent of B. Therefore

$$\begin{aligned}\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 &= \frac{1}{2}\int_{\mathbb{T}} \mathbb{E}[|Z_t^y - \tilde{Z}_t^y|^2]\, \mathrm{d}y \\ &\leqslant \mathbb{E}\Big[\int_{\mathbb{T}} |g_0(y+\sqrt{\nu}B_t) - g_0(y+\sqrt{\nu}\tilde{B}_t)|^2 \mathrm{d}y\Big] \\ &\quad + \|g_0\|_{L^\infty}^2\, \mathbb{E}\Big[\int_{\mathbb{T}} \Big|e^{-i\xi\int_0^t u(y+\sqrt{\nu}B_s)\mathrm{d}s} - e^{-i\xi\int_0^t u(y+\sqrt{\nu}\tilde{B}_s)\mathrm{d}s}\Big|^2 \mathrm{d}y\Big].\end{aligned}$$

Using the inequality $|e^{i\xi a} - e^{i\xi b}| \leqslant \sqrt{2}\,|\xi|^{1/2}|b-a|^{1/2}$ and the characterization of Besov spaces in terms of finite differences (see Appendix A.2), we deduce that

$$\begin{aligned}\|g_0\|_{L^2}^2 - \|g_t\|_{L^2}^2 &\lesssim \mathbb{E}[\|g_0(\cdot+\sqrt{\nu}\,B_t) - g_0(\cdot+\sqrt{\nu}\,\tilde{B}_t)\|_{L^2}^2] \\ &\quad + \|g_0\|_{L^\infty}^2 |\xi| \mathbb{E}\Big[\int_{\mathbb{T}}\int_0^t |u(y+\sqrt{\nu}B_s) - u(y+\sqrt{\nu}\tilde{B}_s)| \mathrm{d}s\,\mathrm{d}y\Big] \\ &\lesssim \|g_0\|_{H^1}^2 \Big[\nu\, \mathbb{E}[|B_t - \tilde{B}_t|^2] + |\xi| \int_0^t \mathbb{E}[\|u(\cdot+\sqrt{\nu}B_s) - u(\cdot+\sqrt{\nu}\tilde{B}_s)\|_{L^1}]\, \mathrm{d}s\Big] \\ &\lesssim \|g_0\|_{H^1}^2 \Big[\nu t + [\![u]\!]_{B^\alpha_{1,\infty}} |\xi|\, \nu^{\frac{\alpha}{2}} \int_0^t \mathbb{E}[|B_s - \tilde{B}_s|^\alpha]\, \mathrm{d}s\Big];\end{aligned}$$

computing the last expectation yields the conclusion. □

We are now ready to complete the

Proof of Proposition 6.23. The proof goes along the same lines as Lemma 2 from [79]. We argue by contradiction. Assume there exists no such constant C, then it must hold

$$\limsup_{\nu\to 0^+} \nu^{-\frac{\alpha}{\alpha+2}} r(\nu) = +\infty. \tag{6.27}$$

Now fix $g_0 \in H^1$ with $\|g_0\|_{L^2} = 1$; by Definition 6.5 and Lemma 6.25 applied to $\xi = 1$ we deduce that there exist constants $C_1, C_2 > 0$ such that, for any $\nu \leqslant 1$ and $t \geqslant 1$, it holds

$$\begin{aligned}1 - C_1 e^{-r(\nu)t} &\leqslant 1 - \|e^{tL_{1,\nu}}\|_{L^2}^2 \leqslant 1 - \|g_t\|_{L^2}^2 \\ &\leqslant C_2 \|g_0\|_{H^1}^2 \Big(\nu t + [\![u]\!]_{B^\alpha_{1,\infty}} \nu^{\frac{\alpha}{2}} t^{1+\frac{\alpha}{2}}\Big) \\ &\leqslant C_2 \|g_0\|_{H^1}^2 (1 + [\![u]\!]_{B^\alpha_{1,\infty}}) \nu^{\frac{\alpha}{2}} t^{1+\frac{\alpha}{2}}.\end{aligned}$$

Let $\nu_n \downarrow 0$ be a sequence realizing the limsup in (6.27) and choose

$$t_n = \big(r(\nu_n)\, \nu_n^{\alpha/(\alpha+2)}\big)^{-1/2};$$

then we obtain

$$1 - C_1 \exp\Big(-\Big(\nu_n^{-\frac{\alpha}{\alpha+2}} r(\nu)\Big)^{1/2}\Big) \lesssim_{g_0, u} \Big(\nu_n^{-\frac{\alpha}{\alpha+2}} r(\nu)\Big)^{-\frac{\alpha+2}{4}}.$$

Taking the limit as $n \to \infty$ on both sides, we find $1 \leqslant 0$, which is absurd. □

6.3.2 Wei's irregularity condition

A major role in the analysis of dissipation enhancement by rough shear flows is played by the following condition, first introduced in [271] (although described in a slightly different manner).

Definition 6.26. *We say that $u \in L^1(0,T)$ satisfies **Wei's condition** with parameter $\alpha > 0$ if, setting $\psi(y) = \int_0^y u(z) \mathrm{d}z$, it holds*

$$\Gamma_\alpha(u) := \left[\inf_{\delta \in (0,1), \bar{y} \in [0, T-\delta]} \delta^{-2\alpha-3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y}+\delta} |\psi(y) - c_1 - c_2 y|^2 \, \mathrm{d}y\right]^{1/2} > 0. \tag{6.28}$$

A similar definition holds for $u \in L^1_{\mathrm{loc}}(\mathbb{R})$; $u \in L^1(\mathbb{T})$ is said to satisfy Wei's condition once it is identified with a 2π-periodic map on $\mathbb{R}$.

Remark 6.27. Note that the condition is independent of the choice of the primitive ψ. Denoting by $\mathcal{P}_1$ the set of all polynomials of degree at most one, for $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ the definition is equivalent to

$$\Gamma_\alpha(u) = \left(\inf_{I \subset \mathbb{R}, |I| < 1} |I|^{-2\alpha-3} \inf_{P \in \mathcal{P}_1} \int_I |\psi(y) - P(y)|^2 \, \mathrm{d}y\right)^{1/2} > 0;$$

this highlights its "complementarity" to the seminorm $[\![\psi]\!]_{\mathcal{L}_1^{2,2\alpha+3}}$ associated to the higher order Campanato space $\mathcal{L}_1^{2,2\alpha+3}$, as defined in [54] (think of the analogy with Definition 5.90!). Observe that Γ_α is homogeneous, i.e. $\Gamma_\alpha(\lambda u) = \lambda \Gamma_\alpha(u)$ for all $\lambda \geqslant 0$.

The importance of condition (6.28) comes from the following result.

Theorem 6.28. *Let $u \in L^1(\mathbb{T})$ be such that $\Gamma_\alpha(u) > 0$ for some $\alpha > 0$. Then there exist positive constants C_1, C_2, depending on α and $\Gamma_\alpha(u)$, such that*

$$\|e^{t L_{k,\nu}}\|_{L^2 \to L^2} \leqslant C_1 \exp\Big(-C_2\, \nu^{\frac{\alpha}{\alpha+2}} |k|^{\frac{2}{\alpha+2}} t\Big) \quad \forall \nu \in (0,1), k \in \mathbb{Z}_0, t \geqslant 0. \tag{6.29}$$

Namely, u is diffusion enhancing with rate $r(x) \sim x^{\alpha/(\alpha+2)}$, in the sense of Definition 6.5.

Proof. The statement comes from Theorem 5.1 from [271]; therein u is required to be continuous, but this restriction is not necessary, see Appendix B from [144] for the proof. □

Following the same approach as in Section 6.2, we proceed to show that the condition $\Gamma_\alpha(u)$ implies irregularity of u; we start by relating it to the property of α-Hölder roughness, in the sense of its characterization (6.21).

Lemma 6.29. *Let $u \in L^1(\mathbb{T})$ be such that $\Gamma_\alpha(u) > 0$ for some $\alpha > 0$. Then u is α-Hölder rough and it holds $L_\alpha(u) \geqslant \Gamma_\alpha(u)$.*

Proof. Fix $\delta > 0$, $\bar{y} \in [-\pi, \pi]$; it holds

$$\begin{aligned}
\inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y}+\delta} |\psi(y) - c_1 - c_2 y|^2 \, \mathrm{d}y &\leqslant \int_{\bar{y}}^{\bar{y}+\delta} |\psi(y) - \psi(\bar{y}) - \psi'(\bar{y})(y - \bar{y})|^2 \mathrm{d}y \\
&\leqslant \int_{\bar{y}}^{\bar{y}+\delta} \left(\int_{\bar{y}}^{y} |u(z) - u(\bar{y})| \mathrm{d}z\right)^2 \mathrm{d}y \\
&\leqslant \delta^{2\alpha+3} \left(\sup_{z \in B_\delta(\bar{y})} \frac{|u(z) - u(\bar{y})|}{\delta^\alpha}\right)^2.
\end{aligned}$$

As the inequality holds for all δ and $\bar{y}$, we obtain $\Gamma_\alpha(u)^2 \leqslant L_\alpha(u)^2$ and the conclusion. □

We can also relate Wei's condition to regularity in the Besov–Nikolskii scales $B^\alpha_{1,\infty}$.

Lemma 6.30. *Let $u \in L^1(\mathbb{T})$ be such that $\Gamma_\alpha(u) > 0$ for some $\alpha \in (0,1)$. Then u does not belong to $B^{\tilde{\alpha}}_{1,\infty}$ for any $\tilde{\alpha} > \alpha$ and does not belong to $B^\alpha_{1,q}$ for any $q < \infty$.*

Proof. For any $\bar{y} \in [-\pi,\pi]$ and $\delta > 0$ it holds

$$\begin{aligned}\delta^{2\alpha+3}\,\Gamma_\alpha(u)^2 &\leqslant \int_{\bar{y}}^{\bar{y}+\delta}\left|\int_{\bar{y}}^{y}[u(z)-u(\bar{y})]\mathrm{d}z\right|^2\mathrm{d}y\\ &\leqslant \int_{\bar{y}}^{\bar{y}+\delta}\left(\int_{\bar{y}}^{\bar{y}+\delta}|u(z)-u(\bar{y})|\mathrm{d}z\right)^2\mathrm{d}y\end{aligned}$$

thus implying that

$$\inf_{\bar{y}\in\mathbb{T}}\int_0^\delta |u(\bar{y}+h)-u(\bar{y})|\mathrm{d}h \geqslant \delta^{1+\alpha}\,\Gamma_\alpha(u) \qquad \forall \delta\in(0,1). \tag{6.30}$$

Now fix $\tilde{\alpha} > \alpha$; starting from (6.30) and arguing as in the proof of Lemma 6.17 (with ε replaced by δ), one obtains

$$2\pi\,\Gamma_\alpha(u) \leqslant \delta^{\tilde{\alpha}-\alpha}[\![u]\!]_{B^{\tilde{\alpha}}_{1,\infty}},$$

which implies the first claim by letting $\delta \to 0^+$. Integrating (6.30) over $\bar{y} \in \mathbb{T}$ yields

$$\int_0^\delta \|u(\cdot+h)-u(\cdot)\|_{L^1}\,\mathrm{d}h \geqslant \delta^{1+\alpha}\,\Gamma_\alpha(u) \quad \forall\delta\in(0,1); \tag{6.31}$$

now assume by contradiction that $u \in B^\alpha_{1,q}$ for some $q < \infty$, then by its equivalent characterization (see Appendix A.2) and the uniform integrability of $h \mapsto h^{-1-\alpha q}\|u(\cdot+h)-u(\cdot)\|^q_{L^1}$ it must hold

$$\lim_{\delta\to 0^+}\int_0^\delta \frac{\|u(\cdot+h)-u(\cdot)\|^q_{L^1}}{|h|^{1+\alpha q}}\,\mathrm{d}h = 0. \tag{6.32}$$

On the other hand, by estimate (6.31) and Jensen's inequality, it holds

$$\begin{aligned}\int_0^\delta \frac{\|u(\cdot+h)-u(\cdot)\|^q_{L^1}}{|h|^{1+\alpha q}}\,\mathrm{d}h &\geqslant \delta^{-1-\alpha q}\int_0^\delta \|u(\cdot+h)-u(\cdot)\|^q_{L^1}\,\mathrm{d}h\\ &\geqslant \delta^{-q(1+\alpha)}\left(\int_0^\delta \|u(\cdot+h)-u(\cdot)\|_{L^1}\,\mathrm{d}h\right)^q\\ &\geqslant \Gamma_\alpha(u)^q > 0\end{aligned}$$

uniformly in $\delta \in (0,1)$, contradicting (6.32). □

Remark 6.31. It follows from Lemma 6.30 and the construction presented in Section 2 from [71] that, for any $\alpha \in \mathbb{Q}$ as in Lemma 2.1 therein, there exists a Weierstrass-type function which belongs to $C^\alpha(\mathbb{T})$, satisfies Wei's condition with parameter α and does not belong to $B^\alpha_{p,q}$ for any $p \in [1,\infty]$, $q \in [1,\infty)$, nor to any $B^{\tilde{\alpha}}_{p,q}$ with $\tilde{\alpha} > \alpha$.

In light of Theorem 6.28, in order to show that almost every shear flow u enhances dissipation, it will suffice to show that almost every u satisfies Wei's condition. We therefore need to find sufficient conditions in order for $\Gamma_\alpha(u) > 0$ to hold. We start with the following simple fact, whose proof is almost identical to that of Lemma 6.19.

Lemma 6.32. *A map $u: \mathbb{T} \to \mathbb{R}$ satisfies $\Gamma_\alpha(u) > 0$ if and only if the maps $u_i: [0,\pi] \to \mathbb{R}$ defined by $u_1(x) = u(x)$, $u_2(x) = u(-x)$ do so.*

In this way, we can reduce the task to identifying sufficient conditions for functions defined on a standard interval $[0,\pi]$. For any $\delta > 0$, we denote by Δ^2_δ the discrete Laplacian operator $\Delta^2_\delta f(y) = f(y+2\delta) - 2f(y+\delta) + f(y)$.

Lemma 6.33. *For any $\alpha > 0$ and any $(\bar{y}, \delta)$ it holds*

$$\delta^{-2\alpha-3} \inf_{c_1, c_2} \int_{\bar{y}}^{\bar{y}+3\delta} |\psi(y) - c_1 - c_2 y|^2 \mathrm{d}y \geqslant \frac{1}{12} \left(\int_{\bar{y}}^{\bar{y}+\delta} |\Delta_\delta^2 \psi(y)|^{-\frac{1}{1+\alpha}} \mathrm{d}y \right)^{-2(1+\alpha)}. \tag{6.33}$$

Proof. First observe that $\Delta_\delta^2(c_1 + c_2 y) \equiv 0$ for any c_1, c_2 and that for any f it holds

$$\int_{\bar{y}}^{\bar{y}+3\delta} |f(y)|^2 \mathrm{d}y \geqslant \frac{1}{12} \int_{\bar{y}}^{\bar{y}+\delta} |\Delta_\delta^2 f(y)|^2 \,\mathrm{d}y.$$

Next, applying Jensen's inequality for $g(x) = x^{-\frac{1}{2(1+\alpha)}}$, which is convex on $(0, \infty)$, it holds

$$\left(\frac{1}{\delta} \int_{\bar{y}}^{\bar{y}+\delta} |\Delta_\delta^2 f(y)|^2 \,\mathrm{d}y \right)^{-\frac{1}{2(1+\alpha)}} \leqslant \frac{1}{\delta} \int_{\bar{y}}^{\bar{y}+\delta} |\Delta_\delta^2 f(y)|^{-\frac{1}{1+\alpha}} \mathrm{d}y.$$

Algebraic manipulations of this inequality and the choice $f(y) = \psi(y) - c_1 - c_2 y$ yield (6.33). $\square$

In view of Lemma 6.33, given $\alpha > 0$ and an integrable $u \colon [0, \pi] \to \mathbb{R}$, we define

$$G_\alpha(y, \delta) := \int_{\bar{y}}^{\bar{y}+\delta} |\Delta_\delta^2 \psi(y)|^{-\frac{1}{1+\alpha}} \,\mathrm{d}y, \tag{6.34}$$

where ψ is as usual a primitive of u (defined up to constant). The next result reduces the verification of Wei's condition to controlling a countable family of quantities associated to G_α.

Lemma 6.34. *For any $\alpha \in (0, 1)$ and $\varepsilon > 0$, define $\beta := \alpha + \varepsilon(1 + \alpha)$ and*

$$K_{\alpha, \varepsilon}(u) := \sup_{n \in \mathbb{N}, 1 \leqslant k \leqslant 2^n - 1} 2^{-n\varepsilon} G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1}).$$

Then there exists a constant $C = C(\alpha, \varepsilon)$ such that

$$\Gamma_\beta(u) \geqslant C\, (K_{\alpha, \varepsilon}(u))^{-1-\alpha}.$$

Proof. First observe that, for any $\beta \in (0, 1)$,

$$|\Gamma_\beta(u)|^2 \sim_\beta \inf_{\delta \in (0, 1/3), \bar{y} \in [0, 1-3\delta]} \delta^{-2\beta-3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y}+3\delta} |\psi(y) - c_1 - c_2 y|^2 \,\mathrm{d}y$$

so to conclude it suffices to provide a lower bound on the latter for our choice of β. Fix $(\bar{y}, \delta)$ and choose $n \in \mathbb{N}$ and $k \in \{1, \ldots, 2^n - 1\}$ such that

$$\delta \in (\pi 2^{-n}, \pi 2^{-n+1}], \quad \bar{y} \in [\pi(k-1) 2^{-n}, \pi k 2^{-n}]$$

so that $[\bar{y}, \bar{y} + 3\delta] \supseteq [\tilde{y}, \tilde{y} + 3\tilde{\delta}]$ for the choice $\tilde{y} = \pi k 2^{-n}$, $\tilde{\delta} = \pi 2^{-n-1}$. As a consequence,

$$\begin{aligned}
&\delta^{-2\beta-3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y}+3\delta} |\psi(y) - c_1 - c_2 y|^2 \,\mathrm{d}y \\
&\gtrsim_\beta \tilde{\delta}^{-2\beta-3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\tilde{y}}^{\tilde{y}+3\tilde{\delta}} |\psi(y) - c_1 - c_2 y|^2 \,\mathrm{d}y \\
&\gtrsim \tilde{\delta}^{-2(\beta-\alpha)} \left(\int_{\tilde{y}}^{\tilde{y}+\tilde{\delta}} |\Delta_{\tilde{\delta}}^2 \psi(y)|^{-\frac{1}{1+\alpha}} \,\mathrm{d}y \right)^{-2(1+\alpha)} \\
&= (\tilde{\delta}^\varepsilon\, G_\alpha(\tilde{y}, \tilde{\delta}))^{-2(1+\alpha)}
\end{aligned}$$

where in the second passage we employed inequality (6.33) and then the definition of β. Overall we deduce by the definition of K and the choice of $(\tilde{y}, \tilde{\delta})$ that

$$\delta^{-2\beta-3} \inf_{c_1, c_2 \in \mathbb{R}} \int_{\bar{y}}^{\bar{y}+3\delta} |\psi(z) - c_1 - c_2 z|^2 \,\mathrm{d}z \gtrsim_\beta K_{\alpha, \varepsilon}(u)^{-2(1+\alpha)};$$

taking the infimum over (δ, y) yields the conclusion. □

6.3.3 Sufficient conditions for stochastic processes

In order to establish prevalence statements, we will sample u as a suitable stochastic process. Lemma 6.34 readily gives the following intermediate, general result.

Proposition 6.35. *Let $u\colon [0,\pi]\to\mathbb{R}$ be an integrable stochastic process, $\psi = \int_0^{\cdot} u_s\, \mathrm{d}s$ and suppose that there exist $\lambda, \kappa > 0$, $\alpha\in(0,1)$ such that*

$$\sup_{\delta\in(0,1),\bar{y}\in[0,\pi-\delta]} \mathbb{E}[\exp(\lambda G_\alpha(\bar{y},\delta))] \leqslant \kappa$$

for G as defined in (6.34). Then for any $\beta>\alpha$ it holds $\mathbb{P}(\Gamma_\beta(u)>0)=1$.

Proof. By virtue of Lemma 6.34, for $\beta=\alpha+\varepsilon(1+\alpha)$ it holds

$$\mathbb{P}(\Gamma_\beta(u)>0) \geqslant \mathbb{P}(K_{\alpha,\varepsilon}(u)<\infty),$$

so to conclude it suffices to show that $\mathbb{P}(K_{\alpha,\varepsilon}(u)<\infty)=1$ for all $\varepsilon>0$. Given λ as in the hypothesis, define the random variable

$$J := \sum_{n\in\mathbb{N}} 2^{-2n} \sum_{k=1}^{2^n-1} \exp(\lambda G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1})).$$

By assumption $\mathbb{E}[J]<\infty$, so that $\mathbb{P}(J<\infty)=1$. For any n,k it holds

$$G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1}) \leqslant \frac{1}{\lambda}\log(2^{2n}J) \lesssim \frac{n}{\lambda}(1+\log J)$$

which implies that

$$Y := \sup_{n\in\mathbb{N},1\leqslant k\leqslant 2^{-n}-1} \frac{1}{n} G_\alpha(\pi k 2^{-n}, \pi 2^{-n-1}) \lesssim \frac{1}{\lambda}(1+\log J) < \infty \quad \mathbb{P}\text{-a.s.}$$

Finally, for any $\varepsilon>0$ it holds $K_{\alpha,\varepsilon}(u)\lesssim_\varepsilon Y$, which yields the conclusion. □

We want to apply Proposition 6.35 to a specific family of Gaussian processes (i.e. those satisfying suitable local nondetermism property, similarly to Chapter 5); this requires a few preparations, in terms of the three Lemmas 6.36-6.38 below.

The next elementary lemma often appears in the probabilistic literature in connection to so called Krylov or Khasminskii type of estimates, see Lemma 1.1 from [241] for a slightly more general statement. For the sake of completeness, we give the proof; we also invite the reader to compare this result with the one from Lemma 5.44, where the integrand is not required to be nonnegative.

Lemma 6.36. *Let X be a real valued, nonnegative stochastic process, defined on an interval $[t_1,t_2]$, adapted to a filtration $\{\mathcal{F}_s\}_{s\in[t_1,t_2]}$; suppose there exists a deterministic $C>0$ such that*

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \mathbb{E}_s\left[\int_s^t X_r\right] \leqslant C \quad \forall s\in[t_1,t_2].$$

Then for any $\lambda\in(0,1)$ it holds

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{C}\int_{t_1}^{t_2} X_r\,\mathrm{d}r\right)\right] \leqslant (1-\lambda)^{-1}.$$

Proof. Up to rescaling X, we may assume $C=1$. It holds

$$\mathbb{E}\left[\exp\left(\lambda\int_{t_1}^{t_2} X_r\,\mathrm{d}r\right)\right] = \sum_{n=0}^\infty \frac{\lambda^n}{n!}\mathbb{E}\left[\left(\int_{t_1}^{t_2} X_r\,\mathrm{d}r\right)^n\right] = \sum_{n=0}^\infty \lambda^n I_n$$

where

$$I_n = \mathbb{E}\left[\int_{t_1<r_1<\ldots<r_n<t_2} X_{r_1}\cdot\ldots\cdot X_{r_n}\,\mathrm{d}r_1\ldots\mathrm{d}r_n\right].$$

By the assumptions and the non-negativity of X, it holds

$$\begin{aligned}I_n &= \int_{t_1<r_1<\ldots<r_{n-1}<t_2} \mathbb{E}\bigg[X_{r_1}\cdot\ldots\cdot X_{r_{n-1}}\int_{r_{n-1}}^t X_{r_n}\mathrm{d}r_n\bigg]\mathrm{d}r_1\ldots\mathrm{d}r_{n-1}\\ &= \int_{t_1<r_1<\ldots<r_{n-1}<t_2} \mathbb{E}\bigg[X_{r_1}\cdot\ldots\cdot X_{r_{n-1}}\mathbb{E}_{r_{n-1}}\bigg[\int_{r_{n-1}}^t X_{r_n}\mathrm{d}r_n\bigg]\bigg]\mathrm{d}r_1\ldots\mathrm{d}r_{n-1}\\ &\leqslant \int_{t_1<r_1<\ldots<r_{n-1}<t_2} \mathbb{E}[X_{r_1}\cdot\ldots\cdot X_{r_{n-1}}]\mathrm{d}r_1\ldots\mathrm{d}r_{n-1} = I_{n-1}\end{aligned}$$

which iteratively implies $I_n\leqslant 1$. Therefore we obtain

$$\mathbb{E}\bigg[\exp\bigg(\lambda\int_s^t X_u\,\mathrm{d}u\bigg)\bigg]\leqslant\sum_{n=0}^\infty \lambda^n = (1-\lambda)^{-1}. \qquad \square$$

In the next statement, $\mathcal{N}(m,\sigma^2)$ denotes the law of a standard Gaussian variable with mean m and variance σ^2.

Lemma 6.37. *Let $Z\sim\mathcal{N}(m,\sigma^2)$ be a real valued Gaussian variable. Then for any $\theta\in(0,1)$ there exists $c_\theta>0$ such that*

$$\mathbb{E}[|Z|^{-\theta}]\leqslant c_\theta\,\sigma^{-\theta}.$$

Proof. Set $Z=\sigma N+m$, then $\mathbb{E}[|Z|^{-\theta}]=\sigma^{-\theta}\,\mathbb{E}[|N-x|^{-\theta}]$ for $x=-m/\sigma$; therefore is suffices to show that

$$\sup_{x\in\mathbb{R}}\mathbb{E}[|N-x|^{-\theta}]=\sup_{x\in\mathbb{R}}\int|x-y|^{-\theta}p(y)\,\mathrm{d}y=\||\cdot|^{-\theta}*p\|_{L^\infty}<\infty$$

where p stands for the Gaussian density $p(x)=(2\pi)^{-1/2}\exp(-|x|^2/2)$. By Young's inequality it holds

$$\begin{aligned}\||\cdot|^{-\theta}*p\|_{L^\infty} &\leqslant \|(|\cdot|^{-\theta}\mathbb{1}_{|\cdot|<1})*p\|_{L^\infty}+\|(|\cdot|^{-\theta}\mathbb{1}_{|\cdot|\geqslant 1})*p\|_{L^\infty}\\ &\leqslant \||\cdot|^{-\theta}\mathbb{1}_{|\cdot|<1}\|_{L^1}\|p\|_{L^\infty}+\||\cdot|^{-\theta}\mathbb{1}_{|\cdot|\geqslant 1}\|_{L^\infty}\|p\|_{L^1}\\ &\leqslant (2\pi)^{-1/2}\||\cdot|^{-\theta}\mathbb{1}_{|\cdot|<1}\|_{L^1}+1<\infty\end{aligned}$$

which gives the conclusion. $\square$

Lemma 6.38. *Let $Y\colon[0,\pi]\to\mathbb{R}$ be a $(1+H)$-SLND Gaussian process with constant C_Y, in the sense of Definition 5.25. Then for any $\alpha>H$ there exists $\lambda=\lambda(\alpha,H,C_Y)>0$ such that*

$$\mathbb{E}\bigg[\exp\bigg(\lambda\int_{\bar{y}}^{\bar{y}+\delta}|\Delta_\delta^2 Y_y|^{-\frac{1}{1+\alpha}}\,\mathrm{d}y\bigg)\bigg]\leqslant 2\quad\forall\delta\in(0,1),\bar{y}\in[0,\pi-\delta].$$

Proof. The result follows Lemmas 6.36 and 6.37 applied to the process $X_y=|\Delta_\delta^2\psi_y|^{-\frac{1}{1+\alpha}}$.

Indeed, denote by $\mathcal{F}_y$ the natural filtration generated by ψ and set $\mathcal{G}_y:=\mathcal{F}_{y+2\delta}$. It is clear that $\Delta_\delta^2\psi_y=Y_{y+2\delta}-2Y_{y+\delta}+Y_y$ is $\mathcal{G}_y$-adapted; for any $[z,y]\subset[\bar{y},\bar{y}+\delta]$ it holds

$$\mathrm{Var}(\Delta_\delta^2 Y_y|\mathcal{G}_z)=\mathrm{Var}(Y_{y+2\delta}|\mathcal{F}_{z+2\delta})\geqslant C_Y|y-z|^{2(1+H)}.$$

Therefore we have a decomposition $\Delta_\delta^2 Y_y=\mathbb{E}_z[\Delta_\delta^2 Y_y]+(\Delta_\delta^2 Y_y-\mathbb{E}_z[\Delta_\delta^2 Y_y])=:\mathbb{E}_z[\Delta_\delta^2 Y_y]+Z_{z,y}$ with $\mathbb{E}_z[\Delta_\delta^2 Y_y]$ adapted to $\mathcal{G}_z$ and $Z_{z,y}$ Gaussian and independent of $\mathcal{G}_z$; thus

$$\mathbb{E}\bigg[\int_u^{\bar{y}+\delta}|\Delta_\delta^2 Y_y|^{-\frac{1}{1+\alpha}}\mathrm{d}y\bigg|\mathcal{G}_z\bigg]=\int_z^{\bar{y}+\delta}\mathbb{E}\Big[|Z_{z,y}+\cdot|^{-\frac{1}{1+\alpha}}\Big](\mathbb{E}_z[\Delta_\delta^2 Y_y])\,\mathrm{d}y.$$

By Lemma 6.37, since $\mathrm{Var}(Z_{z,y})\geqslant C_Y|y-z|^{2(1+H)}$ and $\theta=(1+\alpha)^{-1}\in(0,1)$, it holds

$$\sup_{x\in\mathbb{R}}\mathbb{E}\Big[|Z_{z,y}+x|^{-\frac{1}{1+\alpha}}\Big]\lesssim_\alpha \mathrm{Var}(Z_{z,y})^{-\frac{1}{2(1+\alpha)}}\lesssim_{\alpha,H,C_Y}|y-z|^{-\frac{1+H}{1+\alpha}}$$

and thus

$$\mathbb{E}\left[\int_z^{\bar{y}+\delta} |\Delta_\delta^2 X_y|^{-\frac{1}{1+\alpha}} \mathrm{d}y \middle| \mathcal{G}_z\right] \lesssim \int_z^{\bar{y}+\delta} |y-z|^{-\frac{1+H}{1+\alpha}} \mathrm{d}z$$
$$\lesssim \int_0^1 |r|^{-\frac{1+H}{1+\alpha}} \mathrm{d}r \sim C(\alpha, H, C_Y)$$

where the estimate is uniform over $z \in [\bar{y}, \bar{y}+\delta]$, $\bar{y} \in \mathbb{T}$ and $\delta \in (0,1)$. Choosing

$$\lambda = \frac{1}{2C(\alpha, H, C_Y)},$$

we obtain the conclusion by applying Lemma 6.36. □

Corollary 6.39. *Let $X: [0,\pi] \to \mathbb{R}$ be a Gaussian process such that*

$$Y_y = \int_0^y X_z \,\mathrm{d}z$$

is $(1+H)$-SLND for some $H \in (0,1)$. Then

$$\mathbb{P}(\Gamma_\alpha(X) > 0) = 1$$

for any $\alpha > H$.

Proof. It follows immediately combining Lemma 6.38 and Proposition 6.35. □

6.3.4 Prevalence statements and proof of Theorems 6.7, 6.2

Similarly to Section 6.2.3, we define for $\tilde{\varphi}: [0,\pi] \to \mathbb{R}$ the map $(T\tilde{\varphi})(y) = \tilde{\varphi}(|y|)$; conversely for φ: $\mathbb{T} \to \mathbb{R}$, $\varphi_1(y) := \varphi(y)$, $\varphi_2(y) := \varphi(-y)$. Recall that if $\tilde{\varphi} \in B^\alpha_{1,\infty} \cap L^\infty$, then $T\tilde{\varphi} \in B^\alpha_{1,\infty}$.

We are now ready to provide a prevalence statement which is of interest on its own.

Theorem 6.40. *Let $\alpha \in (0,1)$; then a.e. $\varphi \in B^\alpha_{1,\infty}(0,\pi)$ satisfies $\Gamma_\beta(\varphi) > 0$ for all $\beta > \alpha$.*

Proof. Fix $\alpha \in (0,1)$ and define $\mathcal{A} := \{\varphi \in B^\alpha_{1,\infty}(0,\pi) : \Gamma_\beta(\varphi) > 0$ for all $\beta > \alpha\}$; it holds

$$\mathcal{A} = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \mathcal{A}_{n,m} := \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \left\{\varphi \in B^\alpha_{1,\infty}(0,\pi) : \Gamma_\beta(\varphi) \geqslant \frac{1}{m} \text{ for } \beta = \alpha + \frac{1}{n}\right\}.$$

The sets $\mathcal{A}_{n,m}$ are closed in the topology of $B^\alpha_{1,\infty}(0,\pi)$ (the map $\varphi \mapsto \Gamma_\beta(\varphi)$ is upper semicontinuous in the topology of $L^1(0,\pi)$), thus $\mathcal{A}$ is Borel measurable. In order to conclude, it is enough to show that for any $\beta > \alpha$, the set $\mathcal{A}_\beta := \{\varphi \in B^\alpha_{1,\infty}(0,\pi) : \Gamma_\beta(\varphi) > 0\}$ (which is Borel by the same line of argument) is prevalent.

Now fix $\beta > \alpha$ and choose $H \in (\alpha, \beta)$; denote by μ^H the law of fractional Brownian motion of parameter H on $C([0,\pi])$ and by $u = \{u_y\}_{y \in [0,\pi]}$ the associated canonical process. Since μ^H is supported on $C^{H-\varepsilon}([0,\pi])$ for any $\varepsilon > 0$ and $H > \alpha$, it is tight on $B^\alpha_{1,\infty}(0,\pi)$. As shown in Section 5.1.3 (as well as Example 5.49), the associated process $\psi = \int_0^\cdot u(y)\mathrm{d}y$ is $(1+H)$-SLND and so is $f + \psi$ for any measurable $f: [0,\pi] \to \mathbb{R}$.

In particular, for a given $\varphi \in B^\alpha_{1,\infty}(0,\pi)$, taking $f = \int_0^\cdot \varphi(y)\mathrm{d}y$, it follows from Corollary 6.39 and the choice $\beta > H$ that

$$\mu^H(\varphi + \mathcal{A}_\beta) = \mu^H(\{u \in B^\alpha_{1,\infty}(0,\pi) : \Gamma_\beta(u+\varphi) > 0\}) = 1.$$

As the reasoning holds for any $\varphi \in B^\alpha_{1,\infty}(0,\pi)$, we deduce that μ^H witnesses the prevalence of $\mathcal{A}_\beta$ and we obtain the conclusion. □

Corollary 6.41. *Almost every $\varphi \in B^\alpha_{1,\infty}(\mathbb{T})$ satisfies $\Gamma_\beta(\varphi) > 0$ for all $\beta > \alpha$.*

Proof. The proof is almost identical to that of Corollary 6.21, only this time employing the measure $\nu = T_\sharp\mu$ for $(T\tilde{\varphi})(x) = \tilde{\varphi}(|x|)$, μ being a measure witnessing the prevalence statement from Theorem 6.40; as in Corollary 6.21, we may assume μ to be a tight probability measure on $B^\alpha_{1,\infty}(0,\pi) \cap L^\infty(0,\pi)$, so that ν is tight on $B^\alpha_{1,\infty}(\mathbb{T})$. □

At this point we have all the ingredient to close the dissipative case.

Proof of Theorem 6.7. The lower bound comes from Proposition 6.23, while the upper bound from a combination of Theorem 6.28 and Corollary 6.41. □

The main result of the paper, Theorem 6.2, is now a direct consequence of Theorems 6.6 and 6.7. In fact, let us record here a slightly sharper estimate. Given $f\in L^2(\mathbb{T}^2)$, for any $s\in\mathbb{R}$ define

$$\|f\|_{L^2_x H^s_y}^2 := \sum_{k\in\mathbb{Z}} \|P_k f\|_{H^s(\mathbb{T};\mathbb{C})}^2 = \sum_{(k,\eta)\in\mathbb{Z}^2} (1+|\eta|^2)^s\, |\hat{f}(k,\eta)|^2;$$

it's clear that, for $s\geqslant 0$, $\|f\|_{L^2_x H^s_y}\leqslant \|f\|_{H^s(\mathbb{T}^2)}$ and $\|f\|_{H^{-s}(\mathbb{T}^2)}\leqslant \|f\|_{L^2_x H^{-s}_y}$.

Theorem 6.42. *Almost every $u\in B^{\alpha}_{1,\infty}(\mathbb{T})$ satisfies the following property: for any $\tilde{\alpha}>\alpha$, there exists $C=C(\alpha,\tilde{\alpha},u)$ such that, for any $f_0\in H^{1/2}(\mathbb{T}^2)$ with $P_0 f_0\equiv 0$, it holds*

$$\|e^{tu\partial_x} f_0\|_{L^2_x H^{-1/2}_y} \leqslant C t^{-\frac{1}{2\tilde{\alpha}}} \|f_0\|_{L^2_x H^{1/2}_y}. \tag{6.35}$$

Almost every $u\in B^{\alpha}_{1,\infty}(\mathbb{T})$ satisfies the following property: for any $\tilde{\alpha}>\alpha$ there exist $C_i=C(\alpha,\tilde{\alpha},u)$ such that, for any $f_0\in L^2(\mathbb{T}^2)$ with $P_0 f_0\equiv 0$, it holds

$$\|e^{-t(u\partial_x-\nu\Delta)} f\|_{L^2(\mathbb{T}^2)} \leqslant C_1 \exp\Big(-C_2\, t\, \nu^{\frac{\tilde{\alpha}}{\tilde{\alpha}+2}}\Big) \|f_0\|_{L^2(\mathbb{T}^2)}. \tag{6.36}$$

Proof. By Theorem 6.6 b), for a.e. $u\in B^{\alpha}_{1,\infty}(\mathbb{T})$ and any $\tilde{\alpha}>\alpha$ it holds

$$\|e^{-tu\partial_x} f_0\|_{L^2_x H^{-1/2}_y}^2 = \sum_{k\in\mathbb{Z}_0} \|P_k(e^{-tu\partial_x} f_0)\|_{H^{-1/2}}^2 \lesssim \sum_{k\in\mathbb{Z}_0} (t|k|)^{-\frac{1}{\tilde{\alpha}}} \|P_k f_0\|_{H^{-1/2}}^2 \lesssim t^{-\frac{1}{\tilde{\alpha}}} \|f_0\|_{L^2_x H^{-1/2}_y}^2$$

proving (6.35). Denote $\mathcal{L}_\nu = -u\partial_x + \nu\partial_y^2$, so that $-u\partial_x + -\nu\Delta = \mathcal{L}_\nu + \nu\partial_x^2$, where the operators $\mathcal{L}_\nu$ and $\nu\partial_x^2$ commute; also observe that $P_k(e^{t\partial_x^2} f) = e^{-tk^2} P_k f$.

Combining these facts with Theorem 6.7, for a.e. $u\in B^{\alpha}_{1,\infty}(\mathbb{T})$ and any $\tilde{\alpha}>\alpha$ it holds

$$\begin{aligned}
\|e^{t(-u\partial_x+\nu\Delta)} f\|_{L^2}^2 &= \sum_{k\in\mathbb{Z}_0} \|P_k(e^{t\partial_x^2} e^{t\mathcal{L}_\nu} f)\|_{L^2}^2 = \sum_{k\in\mathbb{Z}_0} e^{-2tk^2} \|P_k(e^{t\mathcal{L}_\nu} f)\|_{L^2}^2 \\
&\lesssim \sum_{k\in\mathbb{Z}_0} \exp\Big(-2t|k|^2 - Ct\nu^{\frac{\alpha}{\alpha+2}} |k|^{\frac{2}{\alpha+2}}\Big) \|P_k f\|_{L^2}^2 \\
&\lesssim \exp\Big(-Ct\nu^{\frac{\alpha}{\alpha+2}}\Big) \sum_{k\in\mathbb{Z}_0} \|P_k f\|_{L^2}^2
\end{aligned}$$

which yields (6.36). □

6.4 Open problems and further references

6.4.1 Closing remarks and future directions

We have shown in this chapter that generic rough shear flows satisfy both inviscid mixing and enhanced dissipation properties, with rates sharply determined by their regularity $\alpha\in(0,1)$ as measured in the Besov scale $B^{\alpha}_{1,\infty}$. In the enhanced dissipation case, this confirms the intuition from [71]; instead in the inviscid mixing one, it shows that the behavior presented by Weierstrass-type functions constructed therein is not generic in the sense of prevalence. Our results provide a connection to the property of ρ-irregularity, which was never observed before in this context, and highlight the importance of working with mixing scales H^{-s} with $s\neq 1$ (especially $H^{-1/2}$). We conclude by presenting a few additional remarks and open problems arising from this work.

Remark 6.43. We are currently unable to establish a clear connection between the properties of ρ-irregularity and Wei's condition. Lemma 6.14 and the trivial estimate $\|f_t\|_{H^{-1}} \leqslant \|f_t\|_{H^{-1/2}}$ imply that, for $\alpha \in (0, 1/2)$, the shear flows $u \in C^\alpha$ constructed in [71] satisfy $\Gamma_\alpha(u) > 0$ but are not ρ-irregular with $\rho \sim (2\alpha)^{-1}$; thus one implication does not hold in general. Heuristically, this fact is similar to the existence of flows with small dissipation time which are not mixing, like the cellular flows presented in [183]. The above argument also implies the existence of Weierstrass type functions which are not ρ-irregular, for suitable values ρ. We believe this problem was open in the probabilistic community, although never been explicitly addressed in the literature.

Given the above remark, it is reasonable to expect the property of ρ-irregularity to be strictly stronger than Wei's condition, although we are not able to prove it.

Conjecture 6.44. *If $u: \mathbb{T} \to \mathbb{R}$ is ρ-irregular, then it satisfies Wei's condition for any $\alpha < (2\rho)^{-1}$.*

Even without establishing a direct connection to Wei's condition, it would be desirable to show directly that ρ-irregular functions are also diffusion enhancing, in line with the heuristic argument presented in Remark 6.3. Since such functions are mixing, they are indeed "qualitatively" diffusion enhancing by [73]; however, the quantitative results from [76] do not provide the sharp rate which is known to hold for typical fBm trajectories.

Conjecture 6.45. *If $u: \mathbb{T} \to \mathbb{R}$ is ρ-irregular, then it is diffusion enhancing with $r(\nu) \sim \nu^{\alpha/(\alpha+2)}$ for any $\alpha < (2\rho)^{-1}$, in the sense of Defiition 6.5.*

Conjectures 6.44 and 6.45 are the most natural ones to address, in order to obtain a cleaner picture on what are the correct notions of irregularity to use in connection to mixing properties. They are not however necessarily the most interesting ones.

As already seen in Chapter 5, the property of (γ, ρ)-irregularity can be formulated in terms of (the Fourier transform of) the occupation measure of u (since we are on the torus, we will denote it by $\mu_{\mathbb{T}}^u$); closely related to it, there is also the local time of u, namely the Radon–Nikodym derivative $\mathrm{d}\mu_{\mathbb{T}}^u / \mathrm{d}\lambda_{\mathbb{T}}$. This gives rise to the following task:

Problem. Find a link between the mixing properties of u and the regularity of its local time.

In a different direction, although there are valid reasons to measure mixing by the weak norm $H^{-1/2}$, it would be desirable to extend the results to other scales, especially the most often considered H^{-1}. Given Remark 6.15, this can be reduced to the task of finding $(0, \rho)$-irregular functions with arbitrarily large ρ (in particular, scaling would suggest $u \in C^\alpha$ with $\rho \sim 1/\alpha$). As already mentioned in Remark 5.70, it is however an open problem to provide examples of $(0, \rho)$-irregular functions u, for any $\rho > 1$. Instead there are several examples of $u: [0, \pi] \to \mathbb{R}$ which are $(0, 1)$-irregular, including the choice $u(y) = y$, see Proposition 5.6.

Remark 6.46. Finally, recall that the property of ρ-irregularity holds for generic vector-valued functions $u: [0, 1] \to \mathbb{R}^d$ (resp. $u: \mathbb{T} \to \mathbb{R}^d$), for any $d \in \mathbb{N}$. In particular, similar statements to part *i.* of Theorem 6.2 can be established for "higher dimensional" shear flows of the form

$$\partial_t f + \bar{u} \cdot \nabla f = \nu \Delta f \tag{6.37}$$

for $f: \mathbb{T}^{d+1} \to \mathbb{R}$, $\bar{u}(x_1, \ldots, x_{d+1}) := (u(x_{d+1}), 0)^T$; observe that for $d = 2$, $\bar{u}$ is a stationary solution to 3D Euler equations. In light of [73], the vector field $\bar{u}$ constructed by a ρ-irregular u is diffusion enhancing; thus can be applied in the study of suppression of blow-up by mixing phenomena, similarly to what was done e.g. in [188, 33, 183, 78].

Instead of (6.37), one might consider the case of *parallel shear flows*, i.e. PDEs of the form

$$\partial_t f + v(y) \partial_x f = \nu \Delta f \tag{6.38}$$

where now $f = f(t, x, y)$, $x \in \mathbb{R}$ (or $x \in \mathbb{T}$) and $y \in \mathbb{T}^d$ (or more generally $y \in \Omega$, Ω smooth subset of $\mathbb{R}^d$, in which case f is prescribed suitable boundary conditions). This case has been examined, by means of resolvent estimates, in the recent work [149] (at least for fairly smooth v).

When $\nu = 0$, our analysis on inviscid mixing estimates would still work, if we were to introduce a suitable definition of ρ-irregularity for fields $v: \mathbb{R}^d \to \mathbb{R}$. So far, examples of ρ-irregular fields have not been given in the literature, although it is reasonable to expect typical realisation of fractional Brownian sheet to fullfill this kind of notion. An analogue of Wei's condition and a clear link between it and a diffusion enhancement property is instead completely open.

6.4.2 Bibliographical comments

As already illustrated at the beginning of this chapter, the study of the mixing and enhanced dissipation properties of vector fields lies at the intersection of several disciplines: ergodic theory, PDEs and functional analysis, but also engineering applications (devising optimal stirring strategies for chemicals) and theoretical physics (understanding energy cascade mechanisms and more generally turbulence). There are several important mathematical contributions I haven't discussed properly, due to the large literature and its countless ramifications; I try to mention some of them here in terms of different "thematic blocks".

Universal, generic and random mixers. If we go away from the shear flow setting, where the best possible mixing rates are of polynomial type (cf. Theorem 6.6), it is by now well established that, even in the case of an incompressible Lipschitz drift b, the associated flow can mix exponentially fast. A natural questions in this context is whether one can find *universal exponential mixers* (in the sense of mixing exponentially fast all initial data $f_0 \in H^s$ for some $s \geqslant 0$). The work [102] answers affirmatively the question in any dimension $d \geqslant 2$ and any $s > 0$, while showing the impossibility of universal mixers (even not exponential) for $s = 0$.

In a different direction, the work [40] studies for $d = 2$ (more precisely on $[0,1]^2$) whether the properties of being exponential, strong or weakly mixing are generic (in the Baire sense, w.r.t. to a suitable topology on $L^1_t \mathrm{BV}_x$), or at least form a dense set, for time-periodic vector fields.

Another natural assumption is to either consider a random flow, by taking b itself solution to a fundamental (stochastic) equation of fluid dynamics like Navier-Stokes, or to deal with a stochastic flow (tipically the so called *isotropic Brownian flows* first introduced in [181, 274]), the latter being strictly connected to the Kraichnan model of turbulence [62, 104]. The first option has recently seen tremendous progress thanks to the works by Bedrossian and collaborators, see [30, 31] and the references therein; regarding the second one, a classical result can be found in [98], while novel advances are given in [157].

Functional mixing and geometric mixing. In order to measure "mixedness" of a given density f, we adopted the use of negative Sobolev norms $\|\cdot\|_{H^{-s}}$, due to their natural connection to ergodic theory. This is however only one possible option, usually referred to as *functional mixing*, with an alternative given by the so called *geometric mixing* property, first introduce by Bressan in the work [48] and associated to his famous cost-rearrangement conjecture. The terminology in the literature is not completely settled: for instance the definitions adopted in [2, 3] are rather different from the ones in [284] (which we will shortly comment more in depth).

Relation with propagation of regularity. In the analytic community, the importance of the concept of geometric mixing comes from its close connection to the general principle that "propagation of regularity implies lower bounds on mixing" (also testified by our Theorems 6.6-6.7). Indeed, the first work to give a partial positive answer to Bressan's rearrangement conjecture was [81], which derived it as a simple corollary of their general results on Lusin-Lipschitz type regularity of generalized Lagrangian flows. Since then, several papers have sharpened these arguments, including [253], [182] (cf. Lemma 2.3), [197] and [50] (cf. Lemma 3.9). Related to Bressan's conjecture(s), let us finally mention the recent work [39].

Let me conclude by commenting further on the relation between geometric and functional mixing. I follow the definitions from [284, 102]; here $Q=(0,1)^d$ is the unit cube, which can be endowed with either slip, no-slip or periodic boundary conditions (the latter reducing to $\mathbb{T}^d$)

Definition 6.47. (Definition 1.1 from [102]) *Let $f\in L^\infty(Q)$ be mean-zero on Q and let ε, $\kappa\in(0,1/2]$. We say that f is κ-mixed up to scale ε if*

$$\frac{1}{|B_\varepsilon(y)|}\left|\int_{B_\varepsilon(y)} f(x)\mathrm{d}x\right| \leqslant \kappa\|f\|_\infty \quad \forall y\in Q.$$

The smallest such ε is the (κ-dependent) geometric mixing scale of f.

As explained in the comments in [284] right after Corollary 1.5, there is a deep link between the geometric mixing scale and the mix-norm defined in [214] (which is in turn equivalent to the functional norm $\|\cdot\|_{H^{-1/2}}$ by the results therein). In particular we have the following:

- if f is κ-mixed up to scale ε, then $\|f\|_{H^{-1/2}}\lesssim\sqrt{\varepsilon+\kappa^2}\,\|f\|_{L^\infty}$;
- conversely, if $\|f\|_{H^{-1/2}}\lesssim\kappa^{3/4}\varepsilon^{3/2}\|f\|_{L^\infty}$, then f is κ-mixed up to scale ε.

These two facts provide the justification why in [102] the quantity $\|f\|^2_{H^{-1/2}}/\|f\|^2_{L^\infty}$ is defined as the functional mixing scale of f. It would be interesting in the future to understand if a suitable analogue of our Theorem 6.6 can be given in terms of geometric mixing rates and whether the above link between the two notions can be sharpened.

Appendix A
Miscellanea

A.1 Fractional Brownian Motion

Fractional Brownian motion (henceforth fBm) is a fundamental fractional process, first introduced by Kolmogorov [189] in the study of turbulence and later rediscovered by Mandelbrot, Van Ness [211]. We recall in this appendix several classical facts involving fBm, which can be found in [225, 236].

A one dimensional fBm $(W_t)_{t\geqslant 0}$ of Hurst parameter $H\in(0,1)$ is a centered, continuous Gaussian process with covariance

$$\mathbb{E}[W_t W_s]=\frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}).$$

A d-dimensional fBm of parameter $H\in(0,1)$ is a Gaussian process with i.i.d. coordinates distributed as one dimensional fBms with the same parameter.

When $H=1/2$, fBm coincides with standard Brownian motion; however for $H\neq 1/2$, it is not a semimartingale nor a Markov process, see [247]. Still, it shares many properties of Brownian motion, such as stationarity, reflexivity and self-similarity. Sharp results on the support of μ^H, the law of fBm of parameter $H\in(0,1)$, go back to [68] (see also [268] for a modern proof which extends to the vector valued case): it holds

$$\mu^H(C^{H-\varepsilon}([0,T];\mathbb{R}^d))=1\quad\forall\varepsilon>0,\quad \mu^H(B^H_{p,\infty}(0,T;\mathbb{R}^d))=1\quad\forall p\in[1,\infty),T\in(0,\infty),$$

while

$$\mu^H(C^H([0,T];\mathbb{R}^d))=0,\quad \mu^H(B^H_{p,q}(0,T;\mathbb{R}^d))=0\quad\forall p,q\in[1,\infty),T\in(0,\infty);$$

here $B^H_{p,q}$ denote Besov spaces as defined in Appendix A.2. In particular, fBm trajectories are sharply not H-Hölder continuous, but by Ascoli–Arzelà μ^H is a tight probability measure on $B^{H-\varepsilon}_{p,\infty}(0,T;\mathbb{R}^d)$ for any $\varepsilon>0$, $p\in[1,\infty]$ and $T\in(0,\infty)$.

A very useful property of fBm is that it admits representations in terms of stochastic integrals. Given a two-sided Brownian motion $\{B_t\}_{t\in\mathbb{R}}$, a fBm of parameter $H\neq 1/2$ can be constructed by

$$W_t=c_H\int_{-\infty}^t\big[(t-r)_+^{H-1/2}-(-r)_+^{H-1/2}\big]\,\mathrm{d}B_r \tag{A.1}$$

where $c_H=\Gamma(H+1/2)^{-1}$ is a suitable renormalising constant and Γ denotes the Gamma function. Such a representation is usually called *non canonical* as the filtration $\mathcal{F}_t=\sigma(B_s: s\leqslant t)$ is strictly larger than the one generated by W; set $\mathbb{E}_s X:=\mathbb{E}[X|\mathcal{F}_s]$. Expression (A.1) is useful as it immediately shows that, for any pair $0\leqslant s<t$, W_t decomposes into the sum $W_t=(W_t-\mathbb{E}_s W_t)+\mathbb{E}_s W_t$, where

$$W_t-\mathbb{E}_s W_t=c_H\int_s^t(t-r)^{H-1/2}\mathrm{d}B_r,\quad \mathbb{E}_s W_t=c_H\int_{-\infty}^s\big[(t-r)_+^{H-1/2}-(-r)_+^{H-1/2}\big]\,\mathrm{d}B_r.$$

In particular, $\mathbb{E}_s W_t$ is $\mathcal{F}_s$-measurable, while $W_t-\mathbb{E}_s W_t$ is independent of $\mathcal{F}_s$ and

$$\mathrm{Var}(W_t-\mathbb{E}_s W_t)=\tilde{c}_H|t-s|^{2H}$$

for $\tilde{c}_H=c_H^2/(2H)$. This also implies that

$$\mathrm{Var}(W_t|\sigma(W_r,r\leqslant s))\geqslant\mathrm{Var}(W_t|\mathcal{F}_s)=\mathrm{Var}(W_t-\mathbb{E}_s W_t)=\tilde{c}_H|t-s|^{2H}. \tag{A.2}$$

Equation A.2 is a **strong local nondeterminism** property; loosely speaking, it means that for any $s<t$, the increment $W_t - W_s$ contains a part which is independent of the history of the path up to time s, thus making its trajectories "intrinsically chaotic". The local nondeterminism (LND) property was first introduced by Berman in [37] in the study of local times of Gaussian processes; it plays a major role in the regularisation by noise effect of fBm trajectories on SDEs (see the thorough discussion in Section 5.1.3).

Another useful integral representation of fBm is based on fractional calculus, which we quickly introduce and for which we refer the interested reader to [249]. For simplicity, from now on we only work with $d=1$, but everything can be immediately extended to $\mathbb{R}^d$ by reasoning componentwise.

Given $f \in L^1(0,T)$ and $\alpha>0$, the fractional integral of order α of f is defined as

$$(I^\alpha f)_\cdot = \frac{1}{\Gamma(\alpha)} \int_0^\cdot (t-s)^{\alpha-1} f_s \, \mathrm{d}s. \tag{A.3}$$

For $\alpha \in (0,1)$ and $p>1$, the map I^α is an injective bounded operator on $L^p = L^p(0,T)$; we denote by $I^\alpha(L^p)$ the image of L^p under the I^α, which is a Banach space endowed with the norm $\|f\|_{I^\alpha(L^p)} := \|g\|_{L^p}$ if $f = I^\alpha g$. On this domain, I^α admits an inverse, the fractional derivative of order α, given by

$$(D^\alpha f)_t = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f_s}{(t-s)^\alpha} \mathrm{d}s = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_t}{t^\alpha} + \alpha \int_0^t \frac{f_t - f_s}{(t-s)^{\alpha+1}} \mathrm{d}s \right). \tag{A.4}$$

With this notation in mind, a fBm of Hurst parameter $H \in (0,1)$ can be constructed starting from a standard Brownian motion B on the interval $[0,T]$ by setting $W = K_H(\mathrm{d}B)$, where the operator K_H is defined as

$$K_H f = \begin{cases} I^1 s^{H-1/2} I^{H-1/2} s^{1/2-H} h & \text{if } H \geqslant 1/2 \\ I^{2H} s^{1/2-H} I^{1/2-H} s^{H-1/2} h & \text{if } H \leqslant 1/2 \end{cases}$$

where s^β denotes the multiplication operator by the function $s \mapsto s^\beta$. It can be shown that this definition of W is meaningful and that the operator K_H corresponds to a Volterra kernel $K_H(t,s)$, so that the above representation is equivalent to

$$W_t = \int_0^t K_H(t,s) \, \mathrm{d}B_s. \tag{A.5}$$

The explicit expression for K_H in the case $H>1/2$ is given by

$$K_H(t,s) = c_H \, s^{1/2-H} \int_s^t (u-s)^{H-3/2} \, u^{H-1/2} \, \mathrm{d}u; \tag{A.6}$$

in the case $H<1/2$ it is more complicated and can be found in [225]. It can be shown that the processes B and W generate the same filtration, which makes (A.5) a **canonical representation**; moreover K_H is invertible, so that for any given fBm W on a probability space, one construct an associated standard Bm by setting $B_\cdot = \int_0^\cdot (K_H^{-1} W)_s \, \mathrm{d}s$. The inverse operator K_H^{-1} is given by

$$K_H^{-1} f = \begin{cases} s^{H-1/2} D^{H-1/2} s^{1/2-H} f' & \text{if } H>1/2 \\ s^{1/2-H} D^{1/2-H} s^{H-1/2} D^{2H} f & \text{if } H<1/2 \end{cases}. \tag{A.7}$$

Given a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbb{P})$, we say that a process W is an $\mathcal{F}_t$-fBm if it is a fBm under $\mathbb{P}$ and the associated B is an $\mathcal{F}_t$-Bm in the usual sense.

The importance of the representation (A.5) lies in the following version of Girsanov theorem for fractional Brownian motion.

Theorem A.1. (Girsanov) *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space, W be an $\mathcal{F}_t$-fBm of parameter $H \in (0,1)$ and h be an $\mathcal{F}_t$-adapted process with continuous trajectories s.t. $h_0 = 0$. Let B be the Bm such that $W = K_H \, \mathrm{d}B$. Suppose that $K_H^{-1} h \in L_t^2$ with probability 1 and that*

$$\mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} \right] = 1, \tag{A.8}$$

where the variable $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$ *is given by*

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = \exp\left(-\int_0^T (K_H^{-1}h)_s\,\mathrm{d}B_s - \frac{1}{2}\int_0^T |(K_H^{-1}h)_s|^2\,\mathrm{d}s\right). \tag{A.9}$$

Then the shifted process $\tilde{W} := W + h$ *is an* $\mathcal{F}_t$*-fBm with parameter* H *under the probability* $\mathbb{Q}$*. A sufficient condition in order for (A.8) to hold is given by Novikov's condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |(K_H^{-1}h)_s|^2\,\mathrm{d}s\right)\right] < \infty. \tag{A.10}$$

Proof. The result is taken from [226], Theorem 2, with the exception of the final part which is just classical Novikov condition; in the original statement from [226], the process h is taken of the form $h_\cdot = \int_0^\cdot u_s\,\mathrm{d}s$, but this doesn't play any role in the proof, which indeed holds also in the case h is not of bounded variation. □

A.2 Function spaces

We recall here the definition and main properties of the Besov spaces $B^s_{p,q}$, which have been used frequently throught this thesis. For simplicity, we will only state the results on $\mathbb{R}^d$, where Besov spaces can be defined by means of Littlewood–Paley blocks as done in the monograph [19]. The same results transfer to the analogous spaces on $\mathbb{T}^d$ by a clever use of Poisson summation formula, see [163], [222]; alternatively, periodic Besov spaces have been treated in Chapter 3 of [251]. Besov spaces on general open domains $O \subset \mathbb{R}^d$ can be defined by means of finite differences, see e.g. [201] or the classical paper [256] for spaces on an interval $I \subset \mathbb{R}$. Finite different characterizations are robust enough to generalize to functions taking values in a metric space, see [207].

To avoid confusion, in this section $\mathcal{B}_r$ will denote the open ball in $\mathbb{R}^d$ of radius $r > 0$, $\bar{\mathcal{B}}_r$ its closure. Closed annuli on $\mathbb{R}^d$ are then of the form $\mathcal{A} = \bar{\mathcal{B}}_R \setminus \mathcal{B}_r$ with $0 < r < R$. Let us also recall that $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ denotes the Schwarz space, $\mathcal{S}'$ the space of tempered distributions; the Fourier transform of f will be denoted by either $\mathcal{F}f$ or $\hat{f}$.

Definition A.2. *Let* $\mathcal{A}$ *be the annulus* $\bar{\mathcal{B}}_{8/3} \setminus \mathcal{B}_{3/4}$*. A* ***dyadic pair*** *is a couple of functions* (χ, φ) *such that* $\chi \in C^\infty_c(\mathcal{B}_{4/3})$*,* $\varphi \in C^\infty_c(\mathcal{A})$ *and such that*

$$\chi(\xi) + \sum_{j=0}^{\infty} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^d$$

as well as

$$|j - j'| \geqslant 2 \Rightarrow \mathit{supp}\,\varphi(2^{-j}\cdot) \cap \mathit{supp}\,\varphi(2^{-j'}\cdot) = \emptyset.$$

Given such a dyadic pair, we define the operator Δ_{-1} *by* $\Delta_{-1}f = \mathcal{F}^{-1}(\chi\mathcal{F}f)$ *and similarly* Δ_j *for* $j \geqslant 0$ *by* $\Delta_j f = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f)$*.*

Before introducing Besov spaces, let us recall some fundamental inequalities involving functions with compactly supported Fourier transform, which come very handy when applied to Littlewood–Paley blocks Δ_j.

Lemma A.3. ([19], Lemma 2.1, Bernstein estimates) *Let* $\mathcal{A}$ *be an annulus and* $\mathcal{B}$ *be a ball. There exists a constant* C *such that, for any* $k \in \mathbb{N}$*,* $p, q \in [1, \infty]$ *with* $q \geqslant p$*,* $\lambda \geqslant 0$ *and any* $u \in L^p$*, it holds*

$$\mathrm{Supp}\,\hat{u} \subset \lambda\mathcal{B} \quad \Rightarrow \quad \|D^k u\|_{L^q} \leqslant C^{k+1}\lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p},$$

$$\mathrm{Supp}\,\hat{u} \subset \lambda\mathcal{A} \quad \Rightarrow \quad \|D^k u\|_{L^q} \geqslant C^{-k-1}\lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p}.$$

Lemma A.4. ([19], Lemma 2.4) *Let* $\mathcal{A}$ *be an annulus. There exist positive constants* c, C *such that for any* $p \in [1, \infty]$*,* $t > 0$*,* $\lambda \in \mathbb{N}$ *and any* $u \in L^p$*, it holds*

$$\mathrm{Supp}\,\hat{u} \subset \lambda\mathcal{A} \quad \Rightarrow \quad \|P_t u\|_{L^p} \leqslant C e^{-ct\lambda^2}\|u\|_{L^p}.$$

Definition A.5. *For $s\in\mathbb{R}$, $(p,q)\in[1,\infty]^2$, the (inhomogeneous)* ***Besov space*** $B^s_{p,q}(\mathbb{R}^d)=B^s_{p,q}$ *is defined as the set of all tempered distributions $f\in\mathcal{S}'$ such that*

$$\|f\|_{B^s_{p,q}}:=\left(\sum_{j=0}^{\infty}2^{sjq}\|\Delta_j f\|_{L^p}^q\right)^{1/q}<\infty$$

with the usual convention when $q=\infty$.

$B^s_{p,q}$ endowed with $\|\cdot\|_{B^s_{p,q}}$ is a Banach space and enjoys the Fatou property, see Theorem 2.72 from [19]; its definition does not depend on the chosen (χ,φ), in the sense that different pairs yield the same space of distributions with equivalent norms. If $p,q\neq\infty$, $B^s_{p,q}$ is separable and C^∞_c is dense in it; if $p,q\in(1,\infty)$, $B^s_{p,q}$ is reflexive and its dual can be identified with $B^{-s}_{p',q'}$.

Let us mention some basic facts that can be easily checked using the definition of $B^s_{p,q}$. For any $\varepsilon>0$ and any $p,q\in[1,\infty]$, $B^s_{p,q}$ continuously embeds in $B^{s-\varepsilon}_{p,1}$; for any $p\in[1,\infty]$, we have the embeddings

$$B^0_{p,1}\hookrightarrow L^p\hookrightarrow B^0_{p,\infty},$$

where the second inclusion comes from Young's inequality for convolutions: for all $j\geqslant 0$ it holds

$$\|\Delta_j f\|_{L^p}=\|\mathcal{F}^{-1}(\varphi(2^{-j}\cdot))*f\|_{L^p}\leqslant\|\mathcal{F}^{-1}(\varphi(2^{-j}\cdot))\|_{L^1}\|f\|_{L^p}=\|\mathcal{F}^{-1}(\varphi)\|_{L^1}\|f\|_{L^p}.$$

Similarly, $\mathcal{M}\hookrightarrow B^0_{1,\infty}$, $\mathcal{M}$ denoting the set of finite signed measures on $\mathbb{R}^d$. For $p\in[2,\infty)$ we actually have the sharper embedding $B^0_{p,2}\hookrightarrow L^p$, which also induces the dual embedding $L^{p'}\hookrightarrow B^0_{p',2}$, see Theorem 2.40 from [19].[A.1]

Besov spaces are handy to use due to their many properties, including functional embeddings, behavior under translation and derivation, and interpolation inequalities.

Proposition A.6. ([19], Prop. 2.71, Besov embeddings) *Let $1\leqslant p_1\leqslant p_2\leqslant\infty$, $1\leqslant q_1\leqslant q_2\leqslant\infty$. Then for any $s\in\mathbb{R}$, the space $B^s_{p_1,q_1}$ continuously embeds in $B^{s^*}_{p_2,q_2}$, where*

$$s^*=s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right).$$

Lemma A.7. *Let $f\in B^s_{p,q}$ and set $\tau^v f=f(\cdot+v)$ for all $v\in\mathbb{R}^d$. Then for any $\alpha\in[0,1]$ it holds*

$$\|\tau^x f-\tau^y f\|_{B^{s-\alpha}_{p,q}}\lesssim|x-y|^\alpha\quad\forall x,y\in\mathbb{R}^d.$$

Proof. For any $j\in\mathbb{N}$, by Bernstein estimates and interpolation inequalities it holds

$$\begin{aligned}\|\Delta_j(\tau^x f-\tau^y f)\|_{L^p}&=\|(\tau^x-\tau^y)\Delta_j f\|_{L^p}\\&\leqslant|x-y|^\alpha\,\|(\tau^x-\tau^y)D\Delta_j f\|_{L^p}^\alpha\,\|(\tau^x-\tau^y)\Delta_j f\|_{L^p}^{1-\alpha}\\&\lesssim|x-y|^\alpha\,2^{j\alpha}\,\|\Delta_j f\|_{L^p}.\end{aligned}$$

Therefore

$$\|\tau^x f-\tau^y f\|^q_{B^{s-\alpha}_{p,q}}=\sum_j 2^{(s-\alpha)jq}\,\|(\tau^x-\tau^y)\Delta_j f\|^q_{L^p}\lesssim|x-y|^\alpha\sum_j 2^{sjq}\,\|\Delta_j f\|^q_{L^p}$$

which yields the conclusion. □

Proposition A.8. ([222], Prop. A.5) *Let $s\in\mathbb{R}$, $p,q\in[1,\infty]$, $i\in\{1,\ldots,n\}$. Then the map $f\mapsto\partial_i f$ is a continuous linear operator from $B^s_{p,q}$ to $B^{s-1}_{p,q}$.*

A.1. In fact, this follows from a much more general embedding result. Similarly to the Besov spaces $B^s_{p,q}$, one can use dyadic pairs to define the Triebel–Lizorkin spaces $F^s_{p,q}$, see [263]. Standard Sobolev spaces $W^{k,p}$ with $k\in\mathbb{N}$, $p\in(1,\infty)$ coincide with $F^k_{p,2}$ and for $p\in[2,\infty)$, $s\in\mathbb{R}$ by Minkowski's inequality one has $B^s_{p,2}\hookrightarrow F^s_{p,2}$, $F^s_{p',2}\hookrightarrow B^s_{p',2}$.

Proposition A.9. ([19], Thm. 2.80, Interpolation inequalities) *A constant C exists which satisfies the following properties. For any $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, $\theta \in (0,1)$ and $p, q \in [1, \infty]$, setting $s_\theta = \theta s_1 + (1-\theta) s_2$, it holds*

$$\|f\|_{B^{s_\theta}_{p,q}} \leqslant \|f\|^{\theta}_{B^{s_1}_{p,q}} \|f\|^{1-\theta}_{B^{s_2}_{p,q}}, \tag{A.11}$$

$$\|f\|_{B^{s_\theta}_{p,1}} \leqslant \frac{C}{s_2 - s_1} \left(\frac{1}{\theta} + \frac{1}{1-\theta} \right) \|f\|^{\theta}_{B^{s_1}_{p,\infty}} \|f\|^{1-\theta}_{B^{s_2}_{p,\infty}}. \tag{A.12}$$

Proof. The result is well known, see e.g. Theorem 2.80 from [19]; let us provide a simple proof of inequality (A.12) without tracking the exact dependence of the constants on $s_2 - s_1$, θ. By linearity, we may assume $\|f\|_{B^{s_2}_{p,\infty}} = 1$; then for any $N \geqslant 0$ it holds

$$\begin{aligned}
\|f\|_{B^{s_\theta}_{p,1}} &= \sum_{j<N} 2^{j(\theta s_1 + (1-\theta) s_2)} \|\Delta_j f\|_{L^p} + \sum_{j \geqslant N} 2^{j(\theta s_1 + (1-\theta) s_2)} \|\Delta_j f\|_{L^p} \\
&\leqslant \|f\|_{B^{s_1}_{p,\infty}} \sum_{j<N} 2^{j(1-\theta)(s_2 - s_1)} + \|f\|_{B^{s_2}_{p,\infty}} \sum_{j \geqslant N} 2^{-j\theta(s_2 - s_1)} \\
&\lesssim \|f\|_{B^{s_1}_{p,\infty}} 2^{N(1-\theta)(s_2 - s_1)} + 2^{-N\theta(s_2 - s_1)}.
\end{aligned}$$

Choosing N such that $\|f\|_{B^{s_1}_{p,\infty}} \sim 2^{-N(s_2 - s_1)}$ the conclusion then follows. □

Let us stress the power of inequality (A.12): by means of Besov norms $B^s_{p,q}$ with $q = \infty$, we are actually able to control an intermediate Besov norm with $q = 1$ (and thus by embeddings also for any other $q \in [1, \infty)$, or Triebel-Lizorkin norms $F^{s_\theta}_{p,q}$). In particular, the so called Agmon inequality (see Lemma 13.2 from [1]) may be regarded as a particular subcase of (A.12); another simple consequence of Proposition A.9, which is quite useful in Chapter 6, is the following.

Corollary A.10. *For any $s_1, s_2 > 0$ there exists a constant $C(s_1, s_2)$ such that*

$$\|f\|_{L^2} \leqslant C \, \|f\|^{s_2/(s_1+s_2)}_{H^{-s_1}} \, \|f\|^{s_1/(s_1+s_2)}_{B^{s_2}_{2,\infty}} \quad \forall f \in B^{s_2}_{2,\infty}. \tag{A.13}$$

Proof. Applying Proposition A.9 for the choice $p = 2$, $\theta = s_2/(s_1 - s_2)$ and using Besov embeddings, we find

$$\|f\|_{L^2} \leqslant \|f\|_{B^0_{2,1}} \lesssim \|f\|^{\theta}_{B^{-s_1}_{2,\infty}} \|f\|^{1-\theta}_{B^{s_2}_{2,\infty}} \lesssim \|f\|^{\theta}_{H^{-s_1}} \|f\|^{1-\theta}_{B^{s_2}_{2,\infty}}. \qquad □$$

Another advantage of Besov spaces is that they allow to define the product between distributions, at least whenever the sum of their regularities is positive. The key tool in the proof of such results is the so called Bony's paradecomposition.

Proposition A.11. ([222], Prop. A.7) *Let $s_1, s_2 \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that*

$$s_1 < 0 < s_2, \quad s_1 + s_2 > 0, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2};$$

then $(f, g) \mapsto fg$ is a well-defined continuous bilinear map from $B^{s_1}_{p_1,q} \times B^{s_2}_{p_2,q}$ to $B^{s_1}_{p,q}$.

Besov spaces of positive regularity also enjoy a nice algebra structure. Observe that when $sp > d$, by Besov embedding the intersection with L^∞ is redundant.

Proposition A.12. ([19], Cor. 2.86) *For any $s > 0$ and $p, q \in [1, \infty]$, the space $B^s_{p,q} \cap L^\infty$ is an algebra and there exists a constant $C = C(s)$ such that*

$$\|fg\|_{B^s_{p,q}} \leqslant C (\|f\|_{L^\infty} \|g\|_{B^s_{p,q}} + \|f\|_{B^s_{p,q}} \|g\|_{L^\infty}) \quad \forall f, g \in B^s_{p,q} \cap L^\infty.$$

We also need to recall the action of the heat flow P_t on Besov spaces. The statement is classical and can be proved easily using Lemma A.4; see Proposition 5, p. 2414 of [221], for the proof in a more general context.

Lemma A.13. *For any $s\in\mathbb{R}$, $\rho>0$, $p,q\in[1,\infty]$ and for any $f\in B^s_{p,q}$, $t>0$ it holds*

$$\|P_t f\|_{B^{s+\rho}_{p,q}} \lesssim t^{-\rho/2}\|f\|_{B^s_{p,q}}.$$

Finally, let us mention that Besov spaces include several other classical function spaces (whenever we say that two spaces coincide, we enforce equivalence of the respective norms):

- For $s\in\mathbb{R}$, $B^s_{2,2}$ coincides with the fractional Sobolev space H^s.
- For $s\in(0,\infty)\setminus\mathbb{N}$, $B^s_{\infty,\infty}$ coincides with the Hölder space C^s, i.e. the space of bounded functions with bounded, $\{s\}$-Hölder continuous derivatives up to order $\lfloor s\rfloor$.
- For $s\in(0,1)$ and $p\in[1,\infty)$, the space $B^s_{p,\infty}$, often referred to as Besov–Nikolskii space, can be characterized by the equivalent norm

$$\|f\tilde{\|}_{B^s_{p,\infty}} := \|f\|_{L^p} + \sup_{x\neq y\in\mathbb{R}^d}\frac{\|f(\cdot+x)-f(\cdot+y)\|_{L^p}}{|x-y|^s}.$$

- For $s\in(0,1)$, $p,q\in[1,\infty)$ the space $B^s_{p,q}$ has equivalent norm

$$\|f\tilde{\|}_{B^s_{p,q}} := \|f\|_{L^p} + \left(\int_{\mathbb{R}^d}\left(\frac{\|f(\cdot+x)-f(\cdot)\|_{L^p}}{|x|^s}\right)^q\frac{1}{|x|^d}\,\mathrm{d}x\right)^{1/q}.$$

  When $p=q\in(1,\infty)$, the above integral quantity is a Gagliardo–Niremberg seminorm and $B^s_{p,p}$ coincides with the fractional Sobolev space $W^{s,p}$, see e.g. [95].

We conclude this section by discussing another (less canonical) class of functions which was used in Chapter 5, namely the Fourier–Lebesgue spaces.

Definition A.14. *Let $s\in\mathbb{R}$, $p\in[1,\infty]$; Fourier–Lebesgue space $\mathcal{F}L^{s,p}=\mathcal{F}L^{s,p}(\mathbb{R}^d)$ is defined as*

$$\mathcal{F}L^{s,p}=\{f\in\mathcal{S}'(\mathbb{R}^d):\langle\xi\rangle^s|\hat{f}(\xi)|\in L^p(\mathbb{R}^d)\}$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. It is a Banach space endowed with the norm

$$\|f\|_{\mathcal{F}L^{s,p}}=\|\langle\cdot\rangle^s\hat{f}\|_{L^p}.$$

It follows immediately from the definition that we could replace $\langle\cdot\rangle$ with any other locally bounded function with the same behaviour at infinity, e.g. $(1+|\cdot|)$; $\langle\cdot\rangle$ is usually considered as it is the Fourier symbol associated to the operator $(I-\Delta)^{1/2}$. Here is a list of relations of Fourier–Lebesgue spaces with other known functional spaces:

- For any $s\in\mathbb{R}$, $\mathcal{F}L^{s,2}$ coincides the classical fractional Sobolev space $H^s=(I-\Delta)^{s/2}L^2$.
- By the Hausdorff–Young inequality (see Proposition 2.2.16 from [160]), for $p\in[1,2]$ we have the embedding $L^p\hookrightarrow\mathcal{F}L^{0,p'}$; similarly for the Bessel spaces $L^{s,p}=(I-\Delta)^{-s/2}L^p$ we have $L^{s,p}\hookrightarrow\mathcal{F}L^{s,p'}$ and conversely $\mathcal{F}L^{s,p}\hookrightarrow L^{s,p'}$ (always only for $p\in[1,2]$).
- In the case $f\in L^1$ the result is slightly stronger, namely $\hat{f}$ is uniformly continuous, bounded and $\hat{f}(\xi)\to0$ as $\xi\to\infty$ by the Riemann–Lebesgue lemma; if f is a finite measure on $\mathbb{R}^d$, then $\hat{f}$ is still uniformly continuous and bounded.
- We have the embedding $\mathcal{F}L^{0,1}\hookrightarrow C^0$ and more generally $\mathcal{F}L^{s,1}\hookrightarrow C^s$, where for $s=n\in\mathbb{N}$ we mean the classical C^n space, while for s fractional or negative $C^s=B^s_{\infty,\infty}$.

There are also embeddings in different scales of Fourier–Lebesgue spaces.

Lemma A.15. *For any $q<p$ and any $\varepsilon>0$ it holds*

$$\mathcal{F}L^{s,p}\hookrightarrow\mathcal{F}L^{s-d\left(\frac{1}{q}-\frac{1}{p}\right)-\varepsilon,q}.$$

Proof. For any $q<p$ and $\delta>0$ we have

$$\|f\|_{\mathcal{F}L^{s-\delta,q}}=\left(\int_{\mathbb{R}^d}(\langle\xi\rangle^s|\hat{f}(\xi)|)^q\langle\xi\rangle^{-\delta q}\mathrm{d}\xi\right)^{1/q}\leqslant\|f\|_{\mathcal{F}L^{s,p}}\left(\int_{\mathbb{R}^d}\langle\xi\rangle^{-\delta\frac{pq}{p-q}}\right)^{\frac{1}{q}-\frac{1}{p}}$$

where the integral is convergent if and only if $-\delta pq/(p-q)<-d$, namely

$$\delta > d\left(\frac{1}{q}-\frac{1}{p}\right). \qquad \square$$

The above statement can be combined with other embeddings like the ones mentioned above. For instance we have $\mathcal{F}L^{s,\infty}\hookrightarrow\mathcal{F}L^{s-d/2-\varepsilon,2}=H^{s-d/2-\varepsilon}$ and $\mathcal{F}L^{s,\infty}\hookrightarrow\mathcal{F}L^{s-d-\varepsilon,1}\hookrightarrow C^{\alpha-d-\varepsilon}$.

One of the main motivations to introduce Fourier–Lebesgue spaces is that they behave nicely under convolution, due to the properties of Fourier transform.

Lemma A.16. *Let $f\in\mathcal{F}L^{\alpha,p}$, $g\in\mathcal{F}L^{\beta,q}$ with $\frac{1}{p}+\frac{1}{q}\leqslant 1$. Then $f*g\in\mathcal{F}L^{\alpha+\beta,r}$ where $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ and*

$$\|f*g\|_{\mathcal{F}L^{\alpha+\beta,r}}\leqslant\|f\|_{\mathcal{F}L^{\alpha,p}}\|g\|_{\mathcal{F}L^{\beta,q}}.$$

Proof. By the properties of Fourier transform $\widehat{f*g}=\hat{f}\,\hat{g}$, therefore

$$\|f*g\|_{\mathcal{F}L^{\alpha+\beta,r}}=\left(\int_{\mathbb{R}^d}(\langle\xi\rangle^{\alpha}|\hat{f}(\xi)|)^r(\langle\xi\rangle^{\beta}|\hat{g}(\xi)|)^r\,\mathrm{d}\xi\right)^{1/r}\leqslant\|f\|_{\mathcal{F}L^{\alpha,p}}\|g\|_{\mathcal{F}L^{\beta,q}}$$

where in the last passage we used the generalised Hölder inequality $\|\varphi\psi\|_{L^r}\leqslant\|\varphi\|_{L^p}\|\psi\|_{L^q}$ for r, p and q as above. $\square$

As a consequence, any bounded Fourier symbol acts continuously on $\mathcal{F}L^{\alpha,p}$, for any choice of α and p; we also have $\mathcal{F}L^{\alpha,p}*\mathcal{F}L^{\beta,\infty}\hookrightarrow\mathcal{F}L^{\alpha+\beta,p}$.

We conclude this appendix by recalling a classical fact on the properties of convolutions with tempered distributions. To this end, we first recall that the following notion.

Definition A.17. *A function $\varphi\in C^{\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ is said to be slowly increasing if all of its derivatives grow at most polynomially at infinity, namely if for any $\alpha\in\mathbb{N}^d$ there exists $N(\alpha)$ such that*

$$|\partial^{\alpha}\varphi(x)|\leqslant\langle x\rangle^{N(\alpha)}\quad\forall x\in\mathbb{R}^d.$$

Proposition A.18. (Proposition 9.10 from [127]) *If $\varphi\in\mathcal{S}$ and $\psi\in\mathcal{S}'$, then $\varphi*\psi$ is a slowly increasing $C^{\infty}_{\mathrm{loc}}$ function.*

A.3 Prevalence

Prevalence [232] is a notion of "Lebesgue measure zero sets" in infinite dimensional complete metric vector spaces. Such sets cannot be naively defined, due to the fact that there cannot exist σ-additive, translation invariant measures in infinite dimensional spaces. It was first introduced by Christensen in [67] in the context of abelian Polish groups and later rediscovered independently by Hunt, Sauer and Yorke in [179] for complete metric vector spaces. A key advantage of prevalence, with respect to other notions of genericity, is that it allows the use of probabilistic methods in the proof.

Prevalence has been used in different contexts in order to study the properties of generic functions belonging to spaces of suitable regularity. For instance, it was proved in [178] that almost every continuous function is nowhere differentiable, while in [129, 130] the multifractal nature of generic Sobolev functions was shown. Recently, prevalence has also attracted a lot of attention in the study of dimension of graphs and images of continuous functions, see among others [128, 26].

Here we follow the setting and the terminology given in [179] even if, for our purposes, working on a Banach space E will always suffice.

Definition A.19. *Let E be a complete metric vector space. A Borel set $A\subset E$ is said to be* ***shy*** *if there exists a measure μ such that:*

i. There exists a compact set $K\subset E$ such that $0<\mu(K)<\infty$.

ii. For every $v\in E$, $\mu(v+A)=0$.

In this case, the measure μ is said to be ***transverse*** *to A. More generally, a subset of E is shy if it is contained in a shy Borel set. The complement of a shy set is called a* ***prevalent*** *set.*

Sometimes it is said more informally that the measure μ "witnesses" the prevalence of A^c.

It follows immediately from part *i.* of the definition that, if needed, one can assume μ to be a compactly supported probability measure on E. If E is separable, then any probability measure on E is tight and therefore *i.* is automatically satisfied.

The following properties hold for prevalence:

1. If E is finite dimensional, then a set A is shy if and only if it has zero Lebesgue measure.
2. If A is shy, then so is $v+A$ for any $v\in E$.
3. Prevalent sets are dense.
4. If $\dim(E)=+\infty$, then compact subsets of E are shy.
5. Countable union of shy sets is shy; conversely, countable intersection of prevalent sets is prevalent.

All proofs can be found in [179]; let us only give a short proof of Point 1.

If $\mathcal{L}^d(A)=0$, where $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$, then Definition A.19 is satisfied for $\mu=\mathcal{L}^d$ due its the translation invariance property. Conversely, if A is shy, then there exists a probability measure μ transverse to it; then combining Definition A.19 with the translation invariance of $\mathcal{L}^d$ and Fubini theorem, it holds

$$\mathcal{L}^d(A)=\mu(\mathbb{R}^d)\times\mathcal{L}^d(A)=(\mu*\mathcal{L}^d)(A)=\int_{\mathbb{R}^d}\mu(x+A)\,\mathrm{d}x=0.$$

We will say that a statement holds for **almost every** $v\in E$ whenever the set of elements of E for which the statement holds is prevalent. Property 1. states that this convention is consistent with the finite dimensional case.

Definition A.19 might appear to depend on a given measure μ transverse to A; as the next lemma shows, once such a measure can be found, infinitely many others can be produced from it, thus making the property to some extend independent of the specific measure under consideration.

Lemma A.20. *Let μ be a measure tranverse to a Borel set $A\subset E$. Then for any compactly supported $\nu\in\mathcal{M}(E)$, $\nu*\mu$ is transverse to A.*

Proof. By Fubini, for any $v\in E$, it holds

$$(\nu*\mu)(v+A)=\int_E\mu(v+w+A)\,\nu(\mathrm{d}w)=\int_E 0\,\nu(\mathrm{d}w)=0.$$

Similarly, it's easy to check the existence of a compact set $K\subset E$ such that $(\nu*\mu)(K)<\infty$. □

In the context of a function space E, it is natural to consider as probability measure the law induced by an E-valued stochastic process. Namely, given a stochastic process W defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, taking values in a separable Banach space E, in order to show that a property $\mathcal{P}$ holds for a.e. $f\in E$, it suffices to show that

$$\mathbb{P}(f+W\text{ satisfies property }\mathcal{P})=1\qquad\forall f\in E.$$

Clearly, we are assuming that the set $A=\{w\in E: w\text{ satisfies property }\mathcal{P}\}$ is Borel measurable; if E is not separable, then we need to additionally require the law of W to be tight, so as to satisfy Point *i.* of Definition A.19.

As a consequence of Properties 4. and 5., the set of all possible realisations of a probability measure on a separable Banach space is a shy set, as it is contained in a countable union of compact sets (this is true more generally for any tight measure on a Banach space). This highlights the difference between a statement of the form

"Property $\mathcal{P}$ holds for a.e. f"

and, for instance,

"Property $\mathcal{P}$ holds for all Brownian trajectories",

where this last statement corresponds to $\mu(\text{Property } \mathcal{P} \text{ holds}) = 1$, μ being the Wiener measure on $C([0,1])$. Indeed, the second statement doesn't provide any information regarding whether the property might be prevalent or not. Intuitively, the elements satisfying a prevalence statement are "many more" than just the realisations of the Wiener measure.

A.4 Stochastic integration in Banach spaces

In this appendix we recall several results on abstract stochastic integration; in view of application to Chapter 3, we will only recall results for martingale type $\mathfrak{p}$ spaces, but the modern theory is far reaching and allows for the more general setting of UMD Banach spaces.

All the material presented here is taken from [266, 267]. For simplicity we restrict to the case of W being a real valued Bm (the extension to the d-dim. case being straightforward), but one could consider $\mathcal{H}$-cylindrical Brownian motion, $\mathcal{H}$ being an abstract Hilbert space. This gives rise to γ-Radonifying norms $\gamma(\mathcal{H}, E)$; in the simple case $\mathcal{H} = \mathbb{R}$ it holds $\|\cdot\|_{\gamma(H,E)} = \|\cdot\|_E$.

Definition A.21. *Let $\mathfrak{p} \in [1,2]$. A Banach space E has **martingale type** $\mathfrak{p}$ if there exists a constant $C \geqslant 0$ such that for all finite E-valued martingale difference sequences $(d_n)_{n=1}^N$ it holds*

$$\mathbb{E}\left[\left\|\sum_{n=1}^N d_n\right\|_E^{\mathfrak{p}}\right] \leqslant C^{\mathfrak{p}} \sum_{n=1}^N \mathbb{E}[\|d_n\|_E^{\mathfrak{p}}].$$

The least admissible constant is denoted by $C_{\mathfrak{p},E}$.

Examples of martingale type spaces are the following:

- Every Banach space has martingale type 1.
- Every Hilbert space has martingale type 2.
- A closed subspace of a Banach space of martingale type $\mathfrak{p}$ has still martingale type $\mathfrak{p}$.
- If E has martingale type $\mathfrak{p}$ and $(S, \mathcal{A}, \mu)$ is a measure space, then $L^r(S; E)$ with $r \in [1, \infty)$ has martingale type $\mathfrak{p} \wedge r$; in particular Lebesgue spaces $L^p(\mathbb{R}^d)$ have martingale type $p \wedge 2$.
- Let (E_0, E_1) be an interpolation couple such that E_i has martingale type $\mathfrak{p}_i \in [1,2]$, let $\theta \in (0,1)$ and consider $\mathfrak{p} \in [1,2]$ such that $1/\mathfrak{p} = (1-\theta)/\mathfrak{p}_0 + \theta/\mathfrak{p}_1$. Then both the complex and real interpolation spaces E_θ and $\tilde{E}_\theta$ have martingale type $\mathfrak{p}$.

For the last two examples, see Propositions 7.1.3 and 7.1.4 from [180]. It follows that Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ with $p \in [2, \infty)$ have martingale type 2 as they can be identified with closed subspaces of $L^p(\mathbb{R}^d)^{\otimes n}$ for suitable n. Besov spaces $B^s_{p,q}$ with $p, q \in [2, \infty)$ have martingale type 2, as can be shown alternatively by: a) by constructing them as interpolation spaces (see for instance Section 17.3 from [201]); b) identifying $B^s_{p,q}$ by means of its Littlewood-Paley decomposition with $\ell^q(\mathbb{N}, \mu; L^p(\mathbb{R}^d))$, where $\mu(\{j\}) = 2^{-sqj}$.

Let W be a real valued $\mathcal{F}_t$-Bm on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, $\{\mathcal{F}_t\}_{t\geqslant 0}$ being a filtration satisfying the usual conditions. For martingale type 2 spaces it is possible to define stochastic integrals analogously to the Euclidean case: for an adapted elementary process $\phi: \mathbb{R}_+ \times \Omega \to E$, namely of the form

$$\phi(t,\omega) = \sum_{i=1}^{n-1} x_i \mathbb{1}_{(t_i, t_{i+1}] \times F_i}(t,\omega)$$

where $0 \leqslant t_1 < t_2 < \cdots < t_n$, $x_i \in E$, $F_i \in \mathcal{F}_{t_i}$, we set

$$\int_0^{\cdot} \phi \mathrm{d}W := \sum_{i=1}^{n-1} x_i \mathbb{1}_{F_i}(W_{\cdot \wedge t_{i+1}} - W_{\cdot \wedge t_i}).$$

Using the martingale type 2 property it is then possible to show that the L^2 norm of the process defined in this way is controlled by $\|\phi\|_{L^2(\mathbb{R}_+ \times \Omega, E)}$, see Theorem 4.6 from [267]. By standard approximation procedures, together with Doob's maximal inequality, the following analogue of standard Itô integration can then be proven.

Theorem A.22. *Let $\phi: \mathbb{R}_+ \times \Omega \to E$ be a progressively measurable process satisfying*

$$\|\phi\|^2_{L^2(\mathbb{R}_+ \times \Omega, E)} = \mathbb{E}\bigg[\int_0^{+\infty} \|\phi_t\|_E^2 \,\mathrm{d}t\bigg] < \infty.$$

Then $\int \phi \mathrm{d}W$ is well defined as an E-valued martingale with paths in $C_b(\mathbb{R}_+; E)$ and satisfies

$$\mathbb{E}\Bigg[\sup_{t \geqslant 0} \bigg\| \int_0^t \phi_s \,\mathrm{d}W_s \bigg\|_E^2\Bigg] \leqslant 4 C_{2,E}^2 \,\mathbb{E}\bigg[\int_0^{+\infty} \|\phi_t\|_E^2 \,\mathrm{d}t\bigg]. \tag{A.14}$$

Remark A.23. It follows immediately from the definition for simple processes and the usual approximation procedure that, for any ϕ as above and any deterministic $\varphi^* \in E^*$, the following identity holds

$$\langle \varphi^*, \int_0^{\cdot} \phi_s \,\mathrm{d}W_s \rangle = \int_0^{\cdot} \langle \varphi^*, \phi_s \rangle \,\mathrm{d}W_s \tag{A.15}$$

where the integral on the r.h.s. is a standard real valued stochastic integral.

In the setting of martingale type 2 spaces a one-sided Burkholder's inequality is available; we state it with the optimal asymptotic behaviour of the constants, which is needed in the estimates in Section 3.1.3. It was first shown by Seidler in [252].

Theorem A.24. (Theorem 4.7 from [267]) *Let E be martingale type 2. Then for any progressively measurable process $\phi: \mathbb{R}_+ \times \Omega \to E$ and $p \in (0, \infty)$ there exists a constant $\tilde{C}_{p,E}$ such that*

$$\mathbb{E}\Bigg[\sup_{t \geqslant 0} \bigg\| \int_0^t \phi_s \,\mathrm{d}W_s \bigg\|_E^2\Bigg] \leqslant \tilde{C}_{p,E}^p \,\mathbb{E}\Bigg[\bigg(\int_0^{\infty} \|\phi_s\|_E^2 \,\mathrm{d}s\bigg)^{p/2}\Bigg]. \tag{A.16}$$

In particular, it is possible to choose $\tilde{C}_{p,E}$ such that $\tilde{C}_{p,E} \leqslant C_E \sqrt{p}$ for any $p \geqslant 2$, where C_E is a universal constant that only depends on the space E.

Finally we recall the Azuma-Hoeffiding inequality in martingale type $\mathfrak{p}$ spaces, which played a key role in Section 3.1.4.

Theorem A.25. *Let E be a Banach space of martingale type $\mathfrak{p} \in (1, 2]$, and $\{X_n\}_{n \geqslant 1}$ be an E-valued martingale sequence satisfying $\|X_{n+1} - X_n\|_E \leqslant c_n$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$, where $\{c_n\}_{n \geqslant 1}$ are deterministic constansts. Then there exist μ, K depending on E such that*

$$\mathbb{E}\Bigg[\exp\Bigg(\mu \frac{\|X_\infty\|_E^{\mathfrak{p}}}{\sum_n c_n^{\mathfrak{p}}}\Bigg)\Bigg] \leqslant K. \tag{A.17}$$

Proof. The result is a useful rewriting of Theorem 1.3 from [208], where it is stated in terms of $\mathfrak{p}$-uniformly smooth Banach spaces; it was shown by Pisier [239] that martingale type $\mathfrak{p}$ spaces admit an equivalent $\mathfrak{p}$-uniformly smooth norm.

By linearity, we can assume $\sum_n c_n^{\mathfrak{p}} = 1$; by Theorem 1.3 from [208], there exists $\kappa > 0$ such that

$$\mathbb{E}[\exp(\mu \|X_\infty\|_E^{\mathfrak{p}})] = \int_0^{+\infty} \mathbb{P}(\exp(\mu \|X_\infty\|_E^{\mathfrak{p}}) > a) \,\mathrm{d}a \leqslant 1 + \int_1^{+\infty} 2 a^{-\frac{\kappa}{\mu}} \,\mathrm{d}a =: K$$

where the last quantity is finite as soon as we take $\mu < \kappa$. □

A.5 Space-time regularity of random fields

The Garsia-Rodemich-Rumsay lemma [150] is a very powerful tool to analyse the local regularity of vector fields starting from macroscopic integral quantities. The following version is taken from [258], Exercise 2.41; see also Appendix B from [84].

Theorem A.26. *Let $p,\psi\colon[0,+\infty)\to[0,+\infty)$ be strictly increasing functions, $\bar{B}(a,r)$ denote the closed ball of radius $r>0$ around $a\in\mathbb{R}^d$ and let E be a Banach space; assume $f\colon\bar{B}(a,r)\to E$ is a continuous map such that*

$$B:=\int_{B(a,r)\times B(a,r)}\psi\left(\frac{\|f(x)-f(y)\|_E}{p(|x-y|)}\right)\mathrm{d}x\mathrm{d}y<\infty. \tag{A.18}$$

Then there exists a dimensional constant κ_d such that

$$\|f(x)-f(y)\|_E\leqslant\int_0^{2|x-y|}\psi^{-1}\left(\frac{\kappa_d B}{u^{2d}}\right)p(\mathrm{d}u)\quad\forall x,y\in\bar{B}(a,r). \tag{A.19}$$

Starting from Theorem A.26, one can derive several criteria for establishing regularity of stochastic processes and vector fields, as well as Gaussian tails of the associated norms.

Corollary A.27. *Let $\{X_t\}_{t\in[0,T]}$ be a continuous, E-valued stochastic process such that*

$$\mathbb{E}\left[\exp\left(\mu\frac{\|X_{s,t}\|_E{}^2}{|t-s|^{2\alpha}}\right)\right]\leqslant K\quad\forall s,t\in[0,T]. \tag{A.20}$$

Then there exists $c>0$, depending only on α, such that

$$\mathbb{E}\left[\exp\left(c\mu\left|\sup_{s\neq t}\frac{\|X_{s,t}\|_E}{\phi_\alpha(t-s)}\right|^2\right)\right]\leqslant KT^2 \tag{A.21}$$

for the choice $\phi_\alpha(t-s)=|t-s|^\alpha\sqrt{|\log|t-s||}$.

Proof. Up to rescaling, we may assume $\mu=1$. We apply Theorem A.26 for the choice $d=1$, $\psi=\exp(x^2)$, $p=|t-s|^\alpha$; for B as defined in (A.18), we deduce that

$$\|X_{s,t}\|_E\leqslant\alpha\int_0^{2|t-s|}\sqrt{\log(\kappa_d)+\log B-2d\log u}\;u^{\alpha-1}\,\mathrm{d}u\lesssim_\alpha\sqrt{\log B}\,\omega(t-s)$$

and so that

$$\sup_{s\neq t}\frac{\|X_{s,t}\|_E}{\phi_\alpha(t-s)}\leqslant\tilde{c}\,\sqrt{\log B}$$

for some $\tilde{c}=\tilde{c}_\alpha$. Taking $c=\tilde{c}^{-2}$ and using the hypothesis (A.20) in the definition of B, we obtain

$$\mathbb{E}\left[\exp\left(c\left|\sup_{s\neq t}\frac{\|X_{s,t}\|_E}{\phi_\alpha(t-s)}\right|^2\right)\right]\leqslant\mathbb{E}[B]\leqslant KT^2.$$ □

Remark A.28. In the case of a Gaussian process X, Corollary A.27 provides a quantitative version of Fernique's theorem [111]. If for any $\mu>0$ there exists K_μ such that X satisfies assumption (A.20), then we can infer the stronger conclusion that

$$\mathbb{E}\left[\exp\left(\mu\left|\sup_{s\neq t}\frac{\|X_{s,t}\|_E}{\phi_\alpha(t-s)}\right|^2\right)\right]<\infty\quad\forall\mu>0.$$

A similar consideration also applies to the upcoming Corollary A.30.

In the remainder of this Section, we are going to apply Theorem A.26 to deduce joint space-time regularity of suitable vector fields; in particular we will estimate seminorms of the form $[\![\cdot]\!]_{\beta,\lambda}$, $[\![\cdot]\!]_{\beta,R}$, $[\![\cdot]\!]_{\alpha,\beta,\lambda}$ as defined in Section 1.1.1. Our approach is not the only possible, see [177], Section 2.5 from [21] or Section 3.2 from [139] for some alternatives.

Corollary A.29. *Suppose we are given a continuous random vector field $F:\mathbb{R}^d\to\mathbb{R}$ such that*

$$\mathbb{E}\left[\exp\left(\mu\,\frac{|F(x)-F(y)|^2}{|x-y|^{2\beta}}\right)\right]\vee\mathbb{E}[\exp(\mu|F(x)|^2)]\leqslant K\quad x,y\in\mathbb{R}^d. \tag{A.22}$$

Then for any $\lambda>0$ and $\varepsilon>0$, $\mathbb{P}$-a.s. it holds $F\in C_x^{\beta-\varepsilon,\lambda}$; moreover there exists constants $c,C>0$ depending on $d,\beta,\varepsilon,\lambda$ such that

$$\mathbb{E}[\exp(c\mu\|F\|_{\beta-\varepsilon,\lambda}^2)]\leqslant CK. \tag{A.23}$$

Proof. It is clear from assumption (A.22) and the definition of $\|F\|_{\beta-\varepsilon,\lambda}$ that we only need to estimate the corresponding seminorm $[\![F]\!]_{\beta-\varepsilon,\lambda}$; as before, we can assume $\mu=1$.

As in the proof of Corollary A.27, an application of Theorem A.26 for the choice $\psi(x)=\exp(x^2)$, $p(x)=x^\beta$ on $\bar{B}(0,R)$ with fixed R gives the existence of $\tilde{c}=\tilde{c}(d,\beta,\varepsilon)$ such that

$$\mathbb{E}[\tilde{c}\,[\![F]\!]_{\beta-\varepsilon,R}^2]\leqslant KR^{2d}\quad\forall R\geqslant 1; \tag{A.24}$$

we have only introduce the parameter $\varepsilon>0$ for simplicity, to get rid of logarithmic corrections.

Observe that by definition

$$[\![F]\!]_{\beta-\varepsilon,\lambda}=\sup_{R\geqslant 1}R^{-\lambda}\,[\![F]\!]_{\beta-\varepsilon,R}\sim\sup_{n\geqslant 1}2^{-n\lambda}[\![F]\!]_{\beta-\varepsilon,2^n}. \tag{A.25}$$

Define the random variable

$$J=\sum_n 2^{-n(2d+1)}\exp(\lambda[\![F]\!]_{\beta-\varepsilon,2^n}^2);$$

by estimate (A.24), it holds $\mathbb{E}[J]\lesssim K$. Moreover (A.25) and simple manipulations show that

$$[\![F]\!]_{\beta-\varepsilon,\lambda}\sim\sup_{n\geqslant 1}2^{-n\lambda}[\![F]\!]_{\beta-\varepsilon,2^n}\lesssim\sup_{n\geqslant 1}2^{-n\lambda}\sqrt{\log J+n}\lesssim_\lambda\sqrt{1+\log J};$$

up to relabelling the hidden constants, we conclude that $\mathbb{E}[\exp(c\,[\![F]\!]_{\beta-\varepsilon,\lambda}^2)]\leqslant\mathbb{E}[J]\lesssim K$. $\square$

Recall the incremental notation $F_{s,t}(x):=F(t,x)-F(s,x)$ from Chapter 1.

Corollary A.30. *Suppose we are given a continuous random field $F(t,x):[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ such that $F(0,\cdot)\equiv 0$; assume there exist $\mu,K>0$ such that*

$$\mathbb{E}\left[\exp\left(\mu\,\frac{|F_{s,t}(x)-F_{s,t}(y)|^2}{|t-s|^{2\alpha}\,|x-y|^{2\beta}}\right)\right]\vee\mathbb{E}\left[\exp\left(\mu\,\frac{|F_{s,t}(x)|^2}{|t-s|^{2\alpha}}\right)\right]\leqslant K\quad\forall s,t\in[0,T],x,y\in\mathbb{R}^d. \tag{A.26}$$

Then for any $\varepsilon>0$ and $\lambda>0$, $\mathbb{P}$-a.s. it holds $F\in C_t^{\alpha-\varepsilon}C_x^{\beta-\varepsilon,\lambda}$; there exist constants c,C, depending on $d,\varepsilon,\alpha,\beta,\lambda$ such that, for ϕ_α as defined in Corollary A.27, it holds

$$\mathbb{E}\left[\exp\left(c\mu\left|\sup_{s\neq t}\frac{\|F_{s,t}\|_{\beta-\varepsilon,\lambda}}{\phi_\alpha(t-s)}\right|^2\right)\right]\leqslant CKT^2. \tag{A.27}$$

Proof. For any $s<t$, applying Corollary A.29 for the choice $\tilde{F}(x)=F_{s,t}/|t-s|^\alpha$, we deduce the existence of $c,C>0$ such that

$$\mathbb{E}\left[\exp\left(c\mu\frac{\|F_{s,t}\|_{\beta-\varepsilon,\lambda}^2}{|t-s|^{2\alpha}}\right)\right]\leqslant CK;$$

up to relabelling the constants, the conclusion then follows by applying Corollary A.27 for the choice $E=C^{\beta-\varepsilon,\lambda}$. $\square$

Remark A.31. Corollary A.30 admits several extensions; for instance, if instead of (A.26) we had assumed more generally that, for any $R\geqslant 1$, it holds

$$\mathbb{E}\left[\exp\left(\mu\frac{|F_{s,t}(x)-F_{s,t}(y)|^2}{|t-s|^{2\alpha}\,|x-y|^{2\beta}R^{2\lambda}}\right)\right]\vee\mathbb{E}\left[\exp\left(\mu\frac{|F_{s,t}(x)|^{\;\;2}}{|t-s|^{2\alpha}\,R^{2\lambda}}\right)\right]\leqslant K\quad\forall s,t\in[0,T],x,y\in\bar{B}_R(0),$$

then similar computations would show that for any $\varepsilon > 0$ $\mathbb{P}$-a.s. it holds $F \in C_t^{\alpha-\varepsilon} C_x^{\beta-\varepsilon,\lambda+\varepsilon}$ and

$$\mathbb{E}\left[\exp\left(c\mu\left|\sup_{s\neq t}\frac{\|F_{s,t}\|_{\beta-\varepsilon,\lambda+\varepsilon}}{\phi_\alpha(t-s)}\right|^2\right)\right] \lesssim KT^2.$$

A.6 Nonlinear Young lemmas

We collect in this appendix some basic tools concerning nonlinear Young integrals, as treated in Chapter 1. For the sake of brevity, we do not include the (elementary) proofs, which can be found in [141].

Let us shortly recall the setting: V is a separable Banach space, endowed with its Borel σ-algebra; the space $C_2^{\alpha,\beta}V$ is defined as in Section 1.1.1 and endowed with the σ-algebra induced by the norm $\|\cdot\|_{\alpha,\beta}$. Recall that by the Lemma 1.1, the map $\mathcal{J}: C_2^{\alpha,\beta}V \to C_t^\alpha V$ is linear and continuous.

Lemma A.32. (Lemma A.1 from [141]) *Let V as above, $(S, \mathcal{A}, \mu)$ a measure space and consider a measurable map $\Gamma: S \to C_2^{\alpha,\beta}V$, $\theta \mapsto \Gamma(\theta)$, such that*

$$\int_S \|\Gamma(\theta)\|_{\alpha,\beta}\,\mu(\mathrm{d}\theta) < \infty.$$

Then the map $\mathcal{J} \circ \Gamma: S \to C_t^\alpha V$ is measurable and it holds

$$\mathcal{J}\left(\int_S \Gamma(\theta)\mu(\mathrm{d}\theta)\right) = \int_S \mathcal{J}(\Gamma(\theta))\mu(\mathrm{d}\theta). \tag{A.28}$$

Lemma A.33. (Lemma A.2 from [141]) *Let $\{\Gamma^n\}_n \subset C_2^{\alpha,\beta}V$ satisfying $\sup_n \|\delta\Gamma^n\|_\beta \leqslant R$ and $\lim_n \|\Gamma^n\|_\alpha \to 0$. Then $\mathcal{J}\Gamma^n \to 0$ in $C_t^\alpha V$ and for all n big enough it holds*

$$[\![\mathcal{J}\,\Gamma^n]\!]_\alpha \lesssim_{T,\beta} (1+R)\,\|\Gamma^n\|_\alpha^{(\beta-1)/(\beta-\alpha)}. \tag{A.29}$$

The next simple interpolation estimate concerns the effect of translations on $C_V^{n+\beta}$.

Lemma A.34. (Lemma A.3 from [141]) *Let $f \in C_V^{n+\beta}$, $z_1, z_2 \in V$. Then for any $\eta \in (0,1)$ with $\eta < n+\beta$ it holds*

$$\|f(\,\cdot\,+z_1) - f(\,\cdot\,+z_2)\|_{n+\beta-\eta} \lesssim \|z_1 - z_2\|_V^\eta \|f\|_{n+\beta}.$$

Finally we recall a useful density result; here approximation in $C_t^{\alpha-}C_{V,W}^{\beta-}$ stands for approximation in $C_t^{\alpha-\varepsilon}C_{V,W}^{\beta-\varepsilon}$, for any $\varepsilon > 0$.

Lemma A.35. (Lemma A.4 from [141]) *Any $A \in C_t^\alpha C_{V,W}^\beta$ can be approximated in $C_t^{\alpha-}C_{V,W}^{\beta-}$ by a sequence A^n such that $\partial_t A^n$ exists and is continuous.*

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