Torsion classes, wide subcategories and maximal green sequences

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Abstract

In this thesis, the relationship between torsion classes and wide subcategories in module categories of finite dimensional algebras is studied. Of particular interest are wide intervals, which are interval subsets of the lattice of torsion classes that enclose a wide subcategory in a certain way. For a fixed wide subcategory, the set of wide intervals is described and new results relating this set with the lattice of torsion classes of some other finite dimensional algebra are obtained. In the second half, we study maximal green sequences for finite dimensional algebras. The well-known relationship to torsion classes allows us to integrate the terminology of wide intervals. Among several smaller results, we put this theory to use to prove a purely representation theoretic version of the Target before Source conjecture for quivers with potential.

Contents

Introduction

Representation theory is the study of linear actions of algebraic objects on vector spaces. In the nineteenth century, it started with the study of representations of finite groups and since then has grown into a big field of active research. Apart from group representations, important objects of study are representations of Lie algebras and general associative algebras. The field has connections to different areas all over mathematics, including analysis, combinatorics and geometry. In this thesis, we study representations of finite dimensional associative algebras, in particular actions on finite dimensional vector spaces. In the language of ring theory, representations of an algebra A are simply modules over A. The A-modules $\mathrm{mod}(A)$ form a so called abelian k-linear category, and this category is the central object of interest in modern representation theory of finite dimensional algebras.

An important family of finite dimensional algebras comes from path algebras of acyclic quivers. An acyclic quiver is an oriented graph Q without cycles, i.e. a tree, and its path algebra kQ is the finite dimensional algebra with basis given by all paths in the quiver and multiplication induced by concatenation of paths. The representation theory of quivers is connected to Lie algebras: For example, Gabriel [Gab72] proved that there is a bijection between indecomposable representations of Dynkin quivers and positive roots of the corresponding Lie algebras. Kac [Kac80] generalized this bijection to arbitrary finite quivers without loops. Since the set of positive roots of the associated Lie algebra does not depend on the orientation of arrows in the quiver, these classification results hint at a strong relationship between quivers that are orientations of the same diagram.

This relationship has been worked out by Bernšteĭn–Gel'fand-Ponomarev [BGP73] and was generalized by Brenner–Butler [BB80]. Given a quiver Q with a sink vertex i, i.e. arrows only end in i but do not start there, the module category $\mathrm{mod}(kQ)$ is decomposed into two parts $(\mathcal{T}, \mathcal{F})$. A similar decomposition $(\mathcal{X}, \mathcal{Y})$ also works if i is a source vertex. In both cases all indecomposable representations are either in the left or the right part. The main theorem now concerns the situation where we flip all arrows adjacent to a sink vertex i, and compare the decompositions $(\mathcal{T}, \mathcal{F})$ where i is a sink and $(\mathcal{X}, \mathcal{Y})$ where i has become a source after flipping. One can construct functors that define equivalences of categories $\mathcal{T} \simeq \mathcal{Y}$ and $\mathcal{F} \simeq \mathcal{X}$. In particular, one obtains bijections between the sets of indecomposable representations.

The decompositions $(\mathcal{T}, \mathcal{F})$ encountered in the Brenner–Butler theorem are examples of so called torsion pairs. The left part $\mathcal{T}$ is called a torsion class and the right part $\mathcal{F}$ is called a torsion-free class. It turns out that in many cases there are bijections between sets of these types of subcategories of $\operatorname{mod}(A)$ and several algebraically and combinatorially defined objects. For example, Ingalls and Thomas [IT09] showed that for a Dynkin quiver Q the set of torsion classes in $\operatorname{mod}(kQ)$ is in bijection with the set of noncrossing partitions associated to Q. Following the introduction of cluster algebras by Fomin–Zelevinsky [FZ02], it was observed by several authors, e.g. [DWZ08], [IT09], that there are bijections between sets of certain subcategories of $\operatorname{mod}(A)$ and the defining clusters of cluster algebras. Therefore the study of $\operatorname{mod}(A)$, certain types of subcategories of it and their interactions has become an important part in the study of representation theory of finite dimensional algebras.

Torsion classes are subcategories of a module category that are closed under forming factor and extension modules. The set of torsion classes forms a lattice. If we consider a Dynkin quiver Q and its path algebra, the Hasse quiver of its lattice of torsion classes can be canonically identified with the combinatorially defined oriented exchange graph of Q defined in [BDP14]. The oriented exchange graph is an orientation of the cluster exchange graph defined by Fomin–Zelevinsky in their study of cluster algebras [FZ03]. A maximal green sequence for Q is a finite path from the maximal element, i.e. the full module category, to the minimal element $\{0\}$ in the Hasse quiver of torsion classes. In the oriented exchange graph, a maximal green sequence corresponds to a maximal sequence of certain *green mutations* of the quiver Q. Originally, maximal green sequences were defined by [Kel11] as such maximal sequences of green mutations.

In this thesis, we study the lattice of torsion classes and maximal green sequences for an arbitrary finite dimensional algebra A. Of importance is the interplay of torsion classes and wide subcategories. A wide subcategory is a full exact abelian subcategory that is closed under extensions. Torsion classes and wide subcategories are both determined by the *bricks* they contain. A brick is a module whose endomorphism ring is a division algebra. The precise relationship between torsion classes, wide subcategories and bricks has been worked out in recent years by many authors, cf. e.g. [IT09], [MŠ17], [DIR+17], [Asa18], [AP19]. A central result is the *Ingalls–Thomas correspondence*, which states that the map that sends a wide subcategory $\mathcal{W}$ to the minimal torsion class $\mathsf{T}(\mathcal{W})$ that contains $\mathcal{W}$ is injective and also constructs an explicit one-sided inverse α_{T}, so that we have $\alpha_{\mathsf{T}} \circ \mathsf{T} = \mathrm{id}$.

$$\begin{array}{rcl} \operatorname{wide}(A) & \hookrightarrow & \operatorname{tors}(A) \\ \mathcal{W} & \longmapsto & \mathsf{T}(\mathcal{W}) \\ \alpha_{\mathsf{T}}(\mathcal{T}) & \longleftarrow\!\!\shortmid & \mathcal{T} \end{array}$$

The result goes back to work of Ingalls and Thomas [IT09] in the context of quiver representations and has been generalized to arbitrary finite dimensional algebras by Marks and Šťovíček [MŠ17]. Since it is an important theorem for our work, we give a self-contained proof in Chapter 1.

Building on the generalized Ingalls–Thomas correspondence for finite dimensional algebras, we prove a relative version formulated using *wide intervals.* A wide interval is an interval subset $[\mathcal{T}, \mathcal{U}] \subseteq \mathrm{tors}(A)$ of the lattice of torsion classes such that the enclosed category $\mathcal{W} := \mathcal{T}^{\perp} \cap \mathcal{U}$ is a wide subcategory of $\mathrm{mod}(A)$. Wide intervals were first studied systematically in [DIR⁺17], called *polygons* there and only studied in the functorially finite case. They were studied in greater generality in [AP19]. In our approach we fix a wide subcategory $\mathcal{W}$ and consider the set $\mathrm{intv}(\mathcal{W})$ of all wide intervals that enclose the fixed $\mathcal{W}$ and call intervals $[\mathcal{T}, \mathcal{F}] \in \mathrm{intv}(\mathcal{W})$ of this form $\mathcal{W}$*-intervals.* The main result is the following theorem.

Theorem 0.1 (cf. Theorem 2.10) *The map* I *defined below is an injection with one-sided inverse* $\hat{\alpha}$*, i.e. we have* $\hat{\alpha} \circ \mathsf{I} = \mathrm{id}$.

$$
\begin{array}{rcl}
\mathrm{intv}(\mathcal{W}) & \underset{\mathsf{I}}{\overset{\hat{\alpha}}{\rightleftarrows}} & \left\{ \mathcal{V} \in \mathrm{wide}(A) \;\middle|\; \begin{array}{c} \mathcal{W} \subseteq \mathcal{V} \textit{ is a Serre} \\ \textit{subcategory in } \mathcal{V} \end{array} \right\} \\
[\mathcal{T}, \mathcal{U}] & \longmapsto & \alpha_{\mathsf{T}}(\mathcal{U}) \\
[\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W}, \mathsf{T}(\mathcal{V})] & \longleftarrow\!\shortmid & \mathcal{V}
\end{array}
$$

The theorem shows that there is a canonical injection from the set of wide subcategories of $\mathrm{mod}(A)$ that have $\mathcal{W}$ as a Serre subcategory into the set of wide intervals. A Serre subcategory is a wide subcategory that is also a torsion class. In the case $\mathcal{W} = \{0\}$, this recovers the generalized Ingalls–Thomas correspondence, since a $\{0\}$-interval is just a singleton set $[\mathcal{T}, \mathcal{T}] = \{\mathcal{T}\}$.

We further consider the set of $\mathcal{W}$-intervals $\mathrm{intv}(\mathcal{W})$ for a fixed wide subcategory $\mathcal{W}$ in Chapter 3. The set $\mathrm{intv}(\mathcal{W})$ inherits a poset structure from the lattice of torsion classes. We show that it also has a unique maximal and minimal element and give explicit formulas for these elements. If the wide subcategory is actually a Serre subcategory of $\mathrm{mod}(A)$, we show the following theorem.

Theorem 0.2 (cf. Theorem 3.4) *Let* $\mathcal{W} \subseteq \mathrm{mod}(A)$ *be a Serre subcategory. Then the partially ordered set* $\mathrm{intv}(\mathcal{W})$ *of* $\mathcal{W}$*-intervals is canonically isomorphic to the lattice of torsion classes for the corner algebra* eAe *with respect to some idempotent* e.

In general, the poset $\mathrm{intv}(\mathcal{W})$ is not isomorphic to the lattice of torsion classes over some finite dimensional algebra. However, we prove that under additional assumptions the poset $\mathrm{intv}(\mathcal{W})$ is isomorphic to a full subposet of the lattice of torsion classes over some finite dimensional algebra B. We call a wide subcategory that satisfies these additional assumptions *admissible.* We remark that admissibility includes left-finiteness of $\mathcal{W}$ and the vanishing of a certain Ext^2-space.

Theorem 0.3 (cf. Theorem 3.20) *Let $\mathcal{W} \subseteq \mathrm{mod}(A)$ be an* admissible *wide subcategory. Then there is a finite dimensional algebra B, which is obtained from A by τ-tilting reduction, such that the partially ordered set of $\mathcal{W}$-intervals in A is isomorphic to an interval sublattice of* $\mathrm{tors}(B)$*.*

The theorem holds in particular for all functorially finite wide subcategories over a hereditary algebra. We remark that the proofs for Theorem 0.2 and Theorem 0.3 make use of abstract localization techniques for categories.

The terminology of wide intervals is central to our approach to the study of maximal green sequences. For us, a maximal green sequence for a finite dimensional algebra is a finite sequence of bricks $(S_1, S_2, \dots, S_m)$ with $\mathrm{Hom}_A(S_i, S_j) = 0$ for $i < j$ such that no other brick can be inserted under this Hom-vanishing condition. Work of Nagao [Nag13], Igusa [Igu19] and Demonet–Keller [DK19] relates this definition to the original one for quivers we mentioned earlier, making use of the theory of Jacobian algebras for quivers with potential and their mutations.

We start by recalling and proving new general facts about maximal green sequences for general finite dimensional algebras. Based on the notion of wide intervals, which have been called polygons by other authors before, we define polygonal flips and deformations. The concept has already been applied successfully to tame quivers by Hermes–Igusa [HI19]. Using polygonal flips, it is possible to construct maximal green sequences from others, and it is an open question to find all finite dimensional algebras with one polygonal deformation class: By definition, an algebra has one polygonal deformation class if all maximal green sequences can be turned into each other by a sequence of polygonal flips. It is known that τ-tilting finite algebras, i.e. those with only finitely many isomorphism classes of bricks, and path algebras over tame quivers belong to this class, the latter is the main theorem in the aforementioned work of Hermes–Igusa [HI19].

Any maximal green sequence $(S_1, \dots, S_m)$ defines a canonical filtration for each module, where the subfactors are the bricks S_i in order, possibly with multiplicity. The existence of such a filtration implies various things, for example, a maximal green sequence must contain all simple modules. We investigate the possible orders in which the simple modules can appear. For this, we consider certain modules that can not appear in a maximal green sequence, therefore must have a non-trivial filtration. For the order of simple modules, it turns out that we should look at thin uniserial modules.

Theorem 0.4 (cf. Theorem 4.26) *Let $M \in \mathrm{mod}(A)$ be a thin uniserial module with composition series $S_\bullet = (S_1, \dots, S_m)$ in reverse order, i.e. S_1 is the top of M and S_m is the socle of M. If M does not appear in any maximal green sequence, then $S_\bullet$ is not a subsequence of any maximal green sequence for A.*

We use this theorem to prove non-existence of maximal green sequence for finite dimensional algebras $A = kQ/I$ that are certain admissible quotients of the following two quivers.

$$
\begin{tikzcd}
 & 2 \arrow[dl, shift left] \arrow[dl, shift right] & \\
1 \arrow[rr, shift left] \arrow[rr, shift right] & & 3 \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 2 \arrow[dl, shift left] \arrow[dr, shift left] & \\
1 \arrow[ur, shift left] \arrow[rr, shift left] & & 3 \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

Note that the non-existence of maximal green sequence for the so called Markov quiver on the left has already been proven using different methods by several authors, e.g. Muller [Mul16] and Brüstle–Smith–Treffinger [BST19].

To really make full use of filtrations induced by a maximal green sequence, bricks that can not appear in a maximal green sequence are of special interest. For example, our discussion of the order of simple modules made use of these. We show that a brick S that appears in a maximal green sequences has certain rigidity properties, for example, the wide subcategory $\mathrm{Filt}(S)$ is functorially finite and in case the ground field is algebraically closed its orbit in the corresponding module variety is open. In general such bricks do not have to be rigid or τ-rigid, we provide simple counterexamples. However, if we restrict to hereditary algebras, e.g. path algebras of quivers, a brick in a maximal green sequence will always be rigid.

For a finite dimensional algebra A, one can ask for the existence of a maximal green sequence. Assuming maximal green sequences exist, the next question is finiteness of the set of maximal green sequences. For algebras that only have finitely many maximal green sequences, a problem considered by several authors is the determination of the maximal length ℓ_0 of maximal green sequences. In the case of path algebras of quivers, there is the *no-gap conjecture* by Brüstle–Dupont–Pérotin [BDP14], who conjecture that the set of lengths of maximal green sequences forms an integer interval. Assuming the conjecture, one can determine the maximal length ℓ_0 algorithmically, as it is the largest length for which one can find a maximal green sequence. Note that for a fixed length ℓ there can only by finitely many maximal green sequences of length ℓ. Using computer experiments, Brüstle–Dupont–Pérotin further ask whether the maximal length is invariant under source-sink reflections of quivers. The conjecture has been proven for quivers of tame representation type [KN20]. We give a counterexample involving quivers of wild representation type with four vertices that shows that the maximal length ℓ_0 is not invariant under source-sink reflection, cf. Example 4.33. The two quivers in question are shown below.

$$
Q = \begin{tikzcd}
 & 2 \arrow[dl] & & \\
1 & & 3 \arrow[ll] \arrow[ul] & 4 \arrow[l]
\end{tikzcd}
\qquad
Q' = \begin{tikzcd}
 & 2 & & \\
1 \arrow[ur] \arrow[rr] & & 3 \arrow[ul] & 4 \arrow[l]
\end{tikzcd}
$$

We show that for Q the integer interval of lengths of maximal green sequences is $[4, 9]$, while for Q' it is $[4, 8]$. Note that Q' is obtained from Q by source-sink reflection at the vertex 1. Our proof is not based on computer assisted calculations, in particular, it does not assume the no-gap conjecture to be true.

In the last chapter, we return to the cluster combinatorial origins of maximal green sequences. After recalling the general theory of quivers with potential and their mutations following Derksen–Weyman–Zelevinsky [DWZ08] with a particular focus on the case of finite dimensional Jacobian algebras, we give an elementary proof of a tilting theorem for neighboring Jacobian algebras. Here, two Jacobian algebras are called neighbors if their quivers with potentials are direct mutations of each other. A similar elementary proof has already been given by Igusa [Igu19], however, only if the Jacobian algebra has finite representation type. The main application of the tilting theorem is a representation theoretic rotation lemma. The rotation lemma states that a maximal green sequence for a quiver gives maximal green sequences of the same length for all quivers that appear while one mutates along the maximal green sequence. It has been proven combinatorially for valued quivers by Brüstle–Hermes–Igusa–Todorov [BHIT17] and representation theoretically in the finite representation type case by Igusa. We remark that this does not imply that the existence of maximal green sequence is stable under mutation of quivers, a counterexample has been constructed by Muller [Mul16].

Building on the rotation lemma, we give a representation theoretic version of another result from [BHIT17], namely the Target before Source conjecture. This conjecture originates from the physics literature [Xie16] and has been proven for acyclic valued quivers by Brüstle–Hermes–Igusa–Todorov in [BHIT17]. It states that for an acyclic quiver that has more than arrow between two vertices, a maximal green sequence must mutate at the target before it mutates at the common source of these arrows. Our result only works for quivers with potential without valuation, however it is more general in this case: We do not need to require that the full quiver Q is acyclic, we only need that none of the two or more multiple arrows appear in a cycle of Q.

Theorem 0.5 (cf. Theorem 5.25) *Let (Q, W) be a quiver Q with non-degenerate potential W and consider the Jacobian algebra $\mathcal{P}(Q, W)$ over an algebraically closed field k. Let $i, j \in Q_0$ be vertices such that there are two or more arrows from j to i in the quiver Q. Suppose that no arrows from j to i appear in a cycle of W. Then for any maximal green sequence $S_\bullet$ with associated mutation sequence $k_\bullet = (k_1, \dots, k_m)$, the vertex i appears before the vertex j in $k_\bullet$.*

In the proof we use the representation theoretic rotation lemma and a general lemma for finite dimensional algebras with multiple arrows in their presentation $A = kQ/I$ as an admissible quotient of a path algebra. Then we apply our previous results on the order of simple modules.

Our representation theoretic proof of the rotation lemma used a tilting type theorem for neighboring Jacobian algebras. The same strategy can be used to prove rotation lemmas for other tilting theorems. In the last Section 5.4 we work out a rotation lemma for maximal green sequences using classical tilting theory in the sense of Brenner–Butler. It is possible that one can further generalize the strategy to prove different rotation lemmas using other types of tilting theorems.

Chapter 1

The lattice of torsion classes

We recall and present results on the structure of the lattice of torsion classes for a finite dimensional algebra over a field. A common theme is the interplay between torsion classes and wide subcategories. This interplay goes back to work of Ingalls–Thomas [IT09] in the context of quiver representations. It has been generalized to arbitrary finite dimensional algebras in work of Marks–Šťovíček [MŠ17].

The functorially finite torsion classes form a subposet $\text{f-tors}(A) \subseteq \text{tors}(A)$ of the full lattice of torsion classes. In general, $\text{f-tors}(A)$ is not a lattice, [IRTT15]. However, τ-tilting theory [AIR14] provides an effective method for studying functorially finite torsion classes. The relation to wide subcategories and their associated semibricks in the functorially finite case has been studied in [Asa18]. More recently, in [AP19], [DIR$^+$17], [BCZ19] results have been obtained about the full lattice of torsion classes.

Let A be a finite dimensional algebra over a field k. We denote by $\text{mod}(A)$ the category of finite dimensional left A-modules.

1.1 Preliminary definitions

We recall some well-known results on the lattice of torsion classes for a finite dimensional algebras. For corresponding results in the general case of torsion classes in abelian categories, see e.g. [BR07, Chapter 1.1].

Definition 1.1

(1) A full subcategory $\mathcal{T} \subseteq \text{mod}(A)$ is called a *torsion class* if for every right exact sequence in $\text{mod}(A)$

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

$X', X'' \in \mathcal{T}$ implies that $X \in \mathcal{T}$. We denote the set of torsion classes in $\text{mod}(A)$ by $\text{tors}(A)$.

(2) Dually, a full subcategory $\mathcal{F} \subseteq \text{mod}(A)$ is called a *torsion-free class* if for every left

exact sequence in $\mathrm{mod}(A)$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X''$$

$X', X'' \in \mathcal{F}$ implies that $X \in \mathcal{F}$. We denote the set of torsion-free classes in $\mathrm{mod}(A)$ by $\mathrm{torf}(A)$.

(3) An exact abelian subcategory $\mathcal{W} \subseteq \mathrm{mod}(A)$ is called a *wide subcategory* if for every short exact sequence in $\mathrm{mod}(A)$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

$X', X'' \in \mathcal{W}$ implies that $X \in \mathcal{W}$. We denote the set of wide subcategories in $\mathrm{mod}(A)$ by $\mathrm{wide}(A)$.

Remark 1.2 The abstract study of torsion theories in abelian categories goes back to Dickson [Dic66]. In Definition 1.1 we give a more compact version that works well for the abelian category $\mathrm{mod}(A)$ of finite dimensional modules. In other words, a torsion class is a full subcategory of $\mathrm{mod}(A)$ that is closed under extensions and factor modules, while a torsion-free class is a full subcategory of $\mathrm{mod}(A)$ that is closed under extensions and submodules.

By definition, a wide subcategory of $\mathrm{mod}(A)$ is an abelian category with exact inclusion functor $I\colon \mathcal{W} \to \mathrm{mod}(A)$ that is closed under forming extensions in $\mathrm{mod}(A)$.

Definition 1.3 Let $\mathcal{X} \subseteq \mathrm{mod}(A)$ be a full subcategory.

(1) We let

$$\mathsf{T}(\mathcal{X}) := \bigcap_{\substack{\mathcal{T} \in \mathrm{tors}(A) \\ \mathcal{X} \subseteq \mathcal{T}}} \mathcal{T}$$

be the torsion class generated by $\mathcal{X}$, i.e. the smallest torsion class of $\mathrm{mod}(A)$ that contains the subcategory $\mathcal{X}$.

(2) Dually, we let

$$\mathsf{F}(\mathcal{X}) := \bigcap_{\substack{\mathcal{F} \in \mathrm{torf}(A) \\ \mathcal{X} \subseteq \mathcal{F}}} \mathcal{F}$$

be the torsion-free class generated by $\mathcal{X}$, i.e. the smallest torsion-free class of $\mathrm{mod}(A)$ that contains $\mathcal{X}$.

Definition 1.4 Let $\mathcal{X} \subseteq \mathrm{mod}(A)$ be a full subcategory. Then we define the right perpendicular category to $\mathcal{X}$

$$\mathcal{X}^{\perp} := \{Y \in \mathrm{mod}(A) \mid \mathrm{Hom}_A(\mathcal{X}, Y) = 0\}$$

and the left perpendicular category to $\mathcal{X}$

$$^{\perp}\mathcal{X} := \{Y \in \operatorname{mod}(A) \mid \operatorname{Hom}_A(Y, \mathcal{X}) = 0\}.$$

By forming perpendicular categories to torsion and torsion-free categories, we obtain a bijection between the set of torsion classes $\operatorname{tors}(A)$ and the set of torsion-free classes $\operatorname{torf}(A)$.

Proposition 1.5 ([IRTT15, Proposition 1.1]) *The following defines a bijection of sets.*

$$\begin{array}{rcl} \operatorname{tors}(A) & \longleftrightarrow & \operatorname{torf}(A) \\ \mathcal{T} & \longmapsto & \mathcal{T}^{\perp} \\ {}^{\perp}\mathcal{F} & \longleftarrow\!\shortmid & \mathcal{F} \end{array}$$

If $\mathcal{T}$ is a torsion class, the pair $(\mathcal{T}, \mathcal{T}^{\perp})$ is called a *torsion pair.* We have the following well-known characterization of torsion pairs. For a corresponding result for general abelian categories see [BR07, Proposition 1.2].

Proposition 1.6 *A pair of full subcategories* $(\mathcal{T}, \mathcal{F})$ *in* $\operatorname{mod}(A)$ *is a torsion pair if and only if* $\operatorname{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$ *and for each* $X \in \operatorname{mod}(A)$ *there is a short exact sequence*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0 \tag{$*$}$$

with $X' \in \mathcal{T}$ *and* $X'' \in \mathcal{F}$.

Proof. Suppose that $(\mathcal{T}, \mathcal{F})$ is a torsion pair. Then $\mathcal{F} = \mathcal{T}^{\perp}$, hence we have $\operatorname{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$. Let $M \in \operatorname{mod}(A)$ and let $N', N'' \subseteq M$ be submodules. Then we have a short exact sequence

$$0 \longrightarrow N' \cap N'' \longrightarrow N' \oplus N'' \longrightarrow N' + N'' \longrightarrow 0.$$

In particular, if N' and N'' are in the torsion class $\mathcal{T}$, their direct sum $N' \oplus N''$ is in $\mathcal{T}$ and therefore also their sum $N' + N''$ is in $\mathcal{T}$.

Since we only consider finite dimensional modules in $\operatorname{mod}(A)$, this implies that each A-module $X \in \operatorname{mod}(A)$ has a unique maximal submodule $X' \subseteq X$ with $X' \in \mathcal{T}$. We can construct X' as the sum of all submodules of X that are in $\mathcal{T}$. Since X is finite dimensional, we can always realize this the possibly infinite sum as a finite sum of submodules. With the submodule $X' \subseteq X$, we consider the short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0.$$

Let $Y \in \mathcal{T}$ be some other module in $\mathcal{T}$ and let $f \colon Y \to X'$ be a morphism. We form the following pullback.

$$
\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0
\end{tikzcd}
$$

Then $X' \in \mathcal{T}$ and $Y \in \mathcal{T}$, hence $E \in \mathcal{T}$. The image of g is a factor of E, so we have $\operatorname{im}(g) \in \mathcal{T}$. But the image is also a submodule of X, hence we must have $\operatorname{im}(g) \subseteq X'$, as X' is the unique maximal submodule in $\mathcal{T}$. But then g factors over X', which means composition in the pullback square is zero, hence $f = 0$. We conclude that $X'' \in \mathcal{T}^{\perp} = \mathcal{F}$.

On the other hand, suppose we are given a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod}(A)$ with $\operatorname{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$ and a short exact sequence $(*)$ for each module $X \in \operatorname{mod}(A)$. We have to show that $\mathcal{T}$ is a torsion class and that $\mathcal{F} = \mathcal{T}^{\perp}$. Note that $\operatorname{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$ implies that $\mathcal{F} \subseteq \mathcal{T}^{\perp}$. Let $X \in \mathcal{T}^{\perp}$. Then by assumption we have a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$. But $X \in \mathcal{T}^{\perp}$, hence $X' = 0$ and therefore $X = X'' \in \mathcal{F}$. So we have $\mathcal{F} = \mathcal{T}^{\perp}$. Similarly it follows that $\mathcal{T} = {}^{\perp}\mathcal{F}$, and combining both equations we have $\mathcal{T} = {}^{\perp}\mathcal{F} = {}^{\perp}(\mathcal{T}^{\perp})$. By left-exactness of Hom, for any full subcategory $\mathcal{X} \subseteq \operatorname{mod}(A)$ the left-perpendicular ${}^{\perp}\mathcal{X}$ is a torsion class, hence $\mathcal{T} = {}^{\perp}(\mathcal{T}^{\perp})$ implies that $\mathcal{T}$ is a torsion class. It follows that $(\mathcal{T}, \mathcal{F})$ is a torsion pair. $\square$

We recall the definition of a lattice.

Definition 1.7 Let $(P, \leq)$ be a partially ordered set and let $X \subseteq P$.

(1) If the set $\{y \in P \mid y \leq x, \forall x \in X\}$ has a unique maximal element, we call it the *meet* of X and denote it by $\bigwedge X$ or $\bigwedge_{x \in X} x$.

(2) If the set $\{y \in P \mid x \leq y, \forall x \in X\}$ has a unique minimal element, we call it the *join* of X and denote it by $\bigvee X$ or $\bigvee_{x \in X} x$.

(3) We call $(P, \leq)$ a *lattice* if every finite subset $X \subseteq P$ has a join and a meet.

(4) We call $(P, \leq)$ a *complete lattice* if every subset $X \subseteq P$ has a join and a meet.

Then we have the following proposition on the lattice structure of $\operatorname{tors}(A)$ and $\operatorname{torf}(A)$.

Proposition 1.8 ([IRTT15, Proposition 1.3])

(1) *The sets* $\operatorname{tors}(A)$ *and* $\operatorname{torf}(A)$ *are complete lattices and the bijection from Proposition* 1.5 *is an order-reversing isomorphism of lattices.*

(2) *For torsion classes* $\mathcal{T}_i \in \operatorname{tors}(A)$, $i \in I$, *we have the following equations for meet and join.*

$$\bigwedge_{i \in I} \mathcal{T}_i = \bigcap_{i \in I} \mathcal{T}_i \quad \textit{and} \quad \bigvee_{i \in I} \mathcal{T}_i = \mathsf{T}\Big(\bigcup_{i \in I} \mathcal{T}_i\Big) = {}^{\perp}\Big(\bigcap_{i \in I} \mathcal{T}_i^{\perp}\Big)$$

(3) *For torsion-free classes* $\mathcal{F}_i \in \operatorname{torf}(A)$, $i \in I$, *we have the following equations for meet and join.*

$$\bigwedge_{i\in I} \mathcal{F}_i = \bigcap_{i\in I} \mathcal{F}_i \quad \text{and} \quad \bigvee_{i\in I} \mathcal{F}_i = \mathsf{F}\Big(\bigcup_{i\in I} \mathcal{F}_i\Big) = \Big(\bigcap_{i\in I} {}^{\perp}\mathcal{F}_i\Big)^{\perp}$$

Definition 1.9 Let $\mathcal{X} \subseteq \operatorname{mod}(A)$ be a full subcategory.

(1) We denote by $\operatorname{add}(\mathcal{X})$ the additive closure of $\mathcal{X}$, i.e. the full subcategory on all direct summands of direct sums of objects in $\mathcal{X}$.

(2) Let $\operatorname{Fac}(\mathcal{X})$ denote the full subcategory on all factor modules of objects in the additive closure $\operatorname{add}(\mathcal{X})$ of $\mathcal{X}$.

(3) Let $\operatorname{Sub}(\mathcal{X})$ denote the full subcategory on all submodules of objects in the additive closure $\operatorname{add}(\mathcal{X})$ of $\mathcal{X}$.

(4) Let $\operatorname{Filt}(\mathcal{X})$ denote the full subcategory on all objects $Y \in \operatorname{mod}(A)$ that have a finite filtration

$$0 = Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \ldots \subsetneq Y_\ell = Y$$

with subfactors $Y_i/Y_{i-1} \in \mathcal{X}$ for $i \in \{1, \ldots, \ell\}$.

Then we have the following well-known lemma, giving more explicit formulas for the torsion(-free) classes generated by a full subcategory $\mathcal{X}$.

Lemma 1.10 ([MŠ17, Lemma 3.1]) *Let* $\mathcal{X} \subseteq \operatorname{mod}(A)$ *be a full subcategory. Then the following hold.*

(a) *We have* $\mathsf{T}(\mathcal{X}) = \operatorname{Filt}(\operatorname{Fac}(\mathcal{X}))$.

(b) *We have* $\mathsf{F}(\mathcal{X}) = \operatorname{Filt}(\operatorname{Sub}(\mathcal{X}))$.

Proof. By duality, it suffices to show (a). Torsion classes are closed under extensions and factor modules, hence the asserted equality follows if we show that $\operatorname{Filt}(\operatorname{Fac}(\mathcal{X}))$ is a torsion class. By construction, this subcategory is closed under extensions, so we have to show that it is closed under factor modules.

Let $X \in \operatorname{Filt}(\operatorname{Fac}(\mathcal{X}))$. We use induction on the dimension of X to show that all factor modules of X are in $\operatorname{Filt}(\operatorname{Fac}(\mathcal{X}))$. By construction, there is a short exact sequence

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0$$

with $0 \neq X' \in \operatorname{Fac}(\mathcal{X})$ and $X'' \in \operatorname{Filt}(\operatorname{Fac}(\mathcal{X}))$. By induction any factor module of X'' is in $\operatorname{Filt}(\operatorname{Fac}(\mathcal{X}))$. Suppose that $h\colon X \twoheadrightarrow Y$ is a surjection in $\operatorname{mod}(A)$. Then we have the following commutative diagram with short exact rows.

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d, "h"] & X'' \arrow[r] \arrow[d, "h''"] & 0 \\
0 \arrow[r] & X' \arrow[r, "hf"] & Y \arrow[r] & Y'' \arrow[r] & 0
\end{tikzcd}$$

Then h'' is surjective and therefore $Y'' \in \mathrm{Filt}(\mathrm{Fac}(\mathcal{X}))$. It follows that $Y \in \mathrm{Filt}(\mathrm{Fac}(\mathcal{X}))$. □

1.2 The Ingalls–Thomas correspondence

To a wide subcategory $\mathcal{W} \subseteq \mathrm{mod}(A)$ we can assign the torsion class $\mathsf{T}(\mathcal{W})$ and the torsion-free class $\mathsf{F}(\mathcal{W})$, see Definition 1.3. We also have a bijection between torsion classes and torsion-free classes, see Proposition 1.5.

$$\begin{tikzcd}
\mathrm{wide}(A) \arrow[d, "\mathsf{T}"] & \mathrm{wide}(A) \arrow[d, "\mathsf{F}"] \\
\mathrm{tors}(A) \arrow[r, shift left, "()^{\perp}"] \arrow[r, phantom, "\sim"] & \mathrm{torf}(A) \arrow[l, shift left, "{}^{\perp}()"]
\end{tikzcd}$$

In this section, we review the existence and explicit construction of retractions to T and F. In particular, it follows that both T and F are injective maps. As a corollary we obtain that if there are only finitely many torsion(-free) classes in $\mathrm{mod}(A)$, the maps T and F are bijections. The results in this section can be found in [MŠ17] but go back to results in [IT09] for the quiver case. We remark that the finite dimensional algebras with only finitely many torsion classes are the so called τ-tilting finite algebras, see [DIJ19].

Definition 1.11 Let A be a finite dimensional algebra.

(a) Let $\mathcal{T} \subseteq \mathrm{mod}(A)$ be a torsion class in $\mathrm{mod}(A)$. We define $\alpha_{\mathsf{T}}(\mathcal{T})$ as the full subcategory on the objects

$$\alpha_{\mathsf{T}}(\mathcal{T}) := \{Y \in \mathcal{T} \mid \forall (f\colon X \to Y) \in \mathcal{T}\colon\ \ker(f) \in \mathcal{T}\}\,.$$

(b) Let $\mathcal{F} \subseteq \mathrm{mod}(A)$ be a torsion-free class in $\mathrm{mod}(A)$. We define $\alpha_{\mathsf{F}}(\mathcal{F})$ as the full subcategory on the objects

$$\alpha_{\mathsf{F}}(\mathcal{F}) := \{X \in \mathcal{F} \mid \forall (f\colon X \to Y) \in \mathcal{F}\colon\ \mathrm{coker}(f) \in \mathcal{F}\}\,.$$

Lemma 1.12 ([MŠ17, Lemma 3.2]) *Let A be a finite dimensional algebra.*

(1) *Let $\mathcal{T} \subseteq \mathrm{mod}(A)$ be a torsion class. Let $X \in \alpha_{\mathsf{T}}(\mathcal{T})$ and let $Y \subseteq X$ be a submodule with $Y \in \mathcal{T}$. Then $Y \in \alpha_{\mathsf{T}}(\mathcal{T})$.*

(2) *Let* $\mathcal{F} \subseteq \mathrm{mod}(A)$ *be a torsion-free class. Let* $X \in \alpha_{\mathsf{F}}(\mathcal{F})$ *and let* $X \twoheadrightarrow Y$ *be a factor module with* $Y \in \mathcal{F}$*. Then* $Y \in \alpha_{\mathsf{F}}(\mathcal{F})$*.*

Proof. By duality, it suffices to show (1). Let $f \colon W \to Y$ be a morphism in $\mathcal{T}$. Let $\iota \colon Y \to X$ denote the inclusion map and let $\tilde{f} := \iota f \colon W \to X$ be the composite of f with the inclusion map. Then $\tilde{f}$ is a morphism in $\mathcal{T}$ and since $X \in \alpha_{\mathsf{T}}(\mathcal{T})$ we have $\ker(\tilde{f}) \in \mathcal{T}$. But $\ker(\tilde{f}) = \ker(f)$, hence $\ker(f) \in \mathcal{T}$ and thus $Y \in \alpha_{\mathsf{T}}(\mathcal{T}.)$ □

Lemma 1.13 ([IT09, Proposition 2.12]) *Let A be a finite dimensional algebra.*

(1) *Let* $\mathcal{T} \subseteq \mathrm{mod}(A)$ *be a torsion class. Then* $\alpha_{\mathsf{T}}(\mathcal{T}) \subseteq \mathrm{mod}(A)$ *is a wide subcategory.*

(2) *Let* $\mathcal{F} \subseteq \mathrm{mod}(A)$ *be a torsion-free class. Then* $\alpha_{\mathsf{F}}(\mathcal{F}) \subseteq \mathrm{mod}(A)$ *is a wide subcategory.*

Proof. By definition we have $\alpha_{\mathsf{T}}(\mathcal{T}) \subseteq \mathcal{T}$. Let $g \colon X \to Y$ be a morphism in $\mathrm{mod}(A)$ with $X, Y \in \alpha_{\mathsf{T}}(\mathcal{T})$.

Since $\mathrm{coker}(g)$ is a factor of $Y \in \alpha_{\mathsf{T}}(\mathcal{T}) \subseteq \mathcal{T}$, we have $\mathrm{coker}(g) \in \mathcal{T}$. Let $f \colon W \to \mathrm{coker}(g)$ be a morphism in $\mathcal{T}$. Then we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \ker(f) \arrow[r, equal] \arrow[d] & \ker(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{im}(g) \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "h"] \arrow[dr, phantom, very near start, "\lrcorner"] & W \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathrm{im}(g) \arrow[r] & Y \arrow[r] & \mathrm{coker}(g) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{im}(g)$ is a factor of $X \in \mathcal{T}$, we have $\mathrm{im}(g) \in \mathcal{T}$ and therefore also $Z \in \mathcal{T}$ as $W \in \mathcal{T}$. Hence $h \colon Z \to W$ is a morphism in $\mathcal{T}$. But $Y \in \alpha_{\mathsf{T}}(\mathcal{T})$, hence $\ker(f) \in \mathcal{T}$. It follows that $\mathrm{coker}(g) \in \alpha_{\mathsf{T}}(\mathcal{T})$.

Since $g \colon X \to Y$ is a morphism in $\mathcal{T}$ and $Y \in \alpha_{\mathsf{T}}(\mathcal{T})$, we also have that $\ker(g) \in \mathcal{T}$. Hence $\ker(g) \in \alpha_{\mathsf{T}}(\mathcal{T})$ follows from Lemma 1.12.(1).

Finally, suppose we are given a short exact sequence in $\mathrm{mod}(A)$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $X', X'' \in \alpha_{\mathsf{T}}(\mathcal{T})$. Then we have $X \in \mathcal{T}$. Let $f \colon Y \to X$ be a morphism in $\mathcal{T}$. With a pullback we obtain the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, equal] \arrow[d] & \ker(f) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z \arrow[r, "g"] \arrow[d, "h"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[r] \arrow[d, "f"] & \operatorname{coker}(g) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0
\end{tikzcd}$$

Since $Y \in \mathcal{T}$, we also have that $\operatorname{coker}(g) \in \mathcal{T}$. Note that $\operatorname{coker}(g)$ is a submodule of $X'' \in \alpha_{\mathsf{T}}(\mathcal{T})$, hence $\operatorname{coker}(g) \in \alpha_{\mathsf{T}}(\mathcal{T})$ by Lemma 1.12.(1). Hence we have $Z \in \mathcal{T}$ by definition of $\alpha_{\mathsf{T}}(\mathcal{T})$. But then the morphism $h \colon Z \to X'$ is a morphism in $\mathcal{T}$ with $X' \in \alpha_{\mathsf{T}}(\mathcal{T})$, hence $\ker(f) \in \mathcal{T}$ again by the definition of $\alpha_{\mathsf{T}}(\mathcal{T})$. Finally, we can conclude that $X \in \alpha_{\mathsf{T}}(\mathcal{T})$.

We have shown that $\alpha_{\mathsf{T}}(\mathcal{T})$ is closed under kernels, cokernels and extensions in $\operatorname{mod}(A)$, hence it is a wide subcategory. The statement for $\alpha_{\mathsf{F}}(\mathcal{F})$ follows by duality. □

Lemma 1.14 *Let A be a finite dimensional algebra and let $\mathcal{W} \in \operatorname{wide}(A)$ be a wide subcategory.*

(1) *For any morphism $f \colon X \to Y$ with $X \in \mathsf{T}(\mathcal{W})$ and $Y \in \mathcal{W}$ we have* $\ker(f) \in \mathsf{T}(\mathcal{W})$.

(2) *For any morphism $f \colon X \to Y$ with $X \in \mathcal{W}$ and $Y \in \mathsf{F}(\mathcal{W})$ we have* $\operatorname{coker}(f) \in \mathsf{F}(\mathcal{W})$.

Proof. Again, (2) is the dual statement to (1), so we only give a proof for (1).

Since $\mathsf{T}(\mathcal{W}) = \operatorname{Filt}(\operatorname{Fac}(\mathcal{W}))$ by 1.10, there is a short exact sequence

$$0 \longrightarrow X' \xrightarrow{\ i\ } X \xrightarrow{\ r\ } X'' \longrightarrow 0$$

with $X' \in \operatorname{Fac}(\mathcal{W})$ and $X'' \in \mathsf{T}(\mathcal{W})$ and $X' \neq 0$. Let $Y' := \operatorname{im}(fi)$ be the image of the composite $fi \colon X' \to Y$. By adding kernels and cokernels we obtain the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r, "i"] \arrow[d] & X \arrow[r, "r"] \arrow[d, "f"] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] & Y'' \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

By assumption $Y \in \mathcal{W}$, so $Y' \in \operatorname{Sub}(\mathcal{W})$. But $X' \in \operatorname{Fac}(\mathcal{W})$ and therefore also $Y' \in \operatorname{Fac}(\mathcal{W})$. But then Y' is the image of a morphism in $\mathcal{W}$, so $Y' \in \mathcal{W}$. It follows that $Y'' \in \mathcal{W}$. Since

$X' \neq 0$, the dimension of X'' is strictly smaller than the dimension of X, so we can apply induction on the morphism $X'' \to Y''$ and conclude that $Z'' \in \mathsf{T}(\mathcal{W})$.

It remains to show that $Z' \in \mathrm{Fac}(\mathcal{W})$. By assumption, there is a surjection $W \to X'$ with $W \in \mathcal{W}$. Pulling back along the injection $Z' \to X$ gives the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V' \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & Y' \arrow[r, equal] \arrow[d] & Y' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $W \in \mathcal{W}$ and $Y' \in \mathcal{W}$, we also have $V \in \mathcal{W}$ since $\mathcal{W}$ is wide. But then it follows that $Z' \in \mathrm{Fac}(\mathcal{W}) \subseteq \mathsf{T}(\mathcal{W})$. $\square$

Lemma 1.15 *Let A be a finite dimensional algebra and let $\mathcal{W} \subseteq \mathrm{mod}(A)$ be a wide subcategory. Then $\mathsf{F}(\mathcal{W}) \cap \mathsf{T}(\mathcal{W}) = \mathcal{W}$.*

Proof. It is obvious that $\mathcal{W} \subseteq \mathsf{F}(\mathcal{W}) \cap \mathsf{T}(\mathcal{W})$. Let $X \in \mathsf{F}(\mathcal{W}) \cap \mathsf{T}(\mathcal{W})$. Since $\mathsf{F}(\mathcal{W}) = \mathrm{Filt}(\mathrm{Sub}(\mathcal{W}))$, cf. Lemma 1.10.(2), there are $0 \neq X' \in \mathrm{Sub}(\mathcal{W})$ and $X'' \in \mathsf{F}(\mathcal{W}))$ that fit in a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0.$$

By assumption, $X'' \in \mathsf{T}(\mathcal{W}) \cap \mathsf{F}(\mathcal{W})$ always has strictly smaller dimension than X and therefore we can apply induction and conclude that $X'' \in \mathcal{W}$. But then by Lemma 1.14.(1) we have $X' \in \mathsf{T}(\mathcal{W})$.

But since $\mathsf{T}(\mathcal{W}) = \mathrm{Filt}(\mathrm{Fac}(\mathcal{W}))$ by Lemma 1.10.(1) there is a short exact sequence

$$0 \longrightarrow Y' \longrightarrow X' \longrightarrow Y'' \longrightarrow 0$$

with $Y' \in \mathsf{T}(\mathcal{W})$ and $0 \neq Y'' \in \mathrm{Fac}(\mathcal{W})$. Since $Y' \in \mathsf{T}(\mathcal{W}) \cap \mathrm{Sub}(\mathcal{W}) \subseteq \mathsf{T}(\mathcal{W}) \cap \mathsf{F}(\mathcal{W})$ has strictly smaller dimension than X' and therefore has strictly smaller dimension than X, we know by induction that $Y' \in \mathcal{W}$. Hence by Lemma 1.14.(2) we know that $Y'' \in \mathsf{F}(\mathcal{W})$.

If $Y' \neq 0$, the module Y'' also has smaller dimension than X', so we have $Y'' \in \mathcal{W}$. If $Y' = 0$, we have $X' = Y'' \in \mathrm{Sub}(\mathcal{W}) \cap \mathrm{Fac}(\mathcal{W})$ and therefore Y'' is the image of a morphism in $\mathcal{W}$. But $\mathcal{W}$ is exact abelian, hence $Y'' \in \mathcal{W}$. So in both cases we have $Y' \in \mathcal{W}$ and

$Y'' \in \mathcal{W}$, and therefore $X' \in \mathcal{W}$ since $\mathcal{W}$ is wide. We have already shown that $X'' \in \mathcal{W}$, hence $X \in \mathcal{W}$ since $\mathcal{W}$ is wide. $\square$

Proposition 1.16 (cf. [IT09, Proposition 2.14], [MŠ17, Proposition 3.3]) *Let A be a finite dimensional algebra. Then the following hold.*

(1) *The assignments T and α_{T} define maps*

$$\operatorname{wide}(A) \underset{\alpha_{\mathsf{T}}}{\overset{\mathsf{T}}{\rightleftarrows}} \operatorname{tors}(A)$$

with $\alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W})) = \mathcal{W}$ for all wide subcategories $\mathcal{W} \in \operatorname{wide}(A)$. In particular, the map $\mathsf{T}\colon \operatorname{wide}(A) \to \operatorname{tors}(A)$ is injective.

(2) *The assignments F and α_{F} define maps*

$$\operatorname{wide}(A) \underset{\alpha_{\mathsf{F}}}{\overset{\mathsf{F}}{\rightleftarrows}} \operatorname{torf}(A)$$

with $\alpha_{\mathsf{F}}(\mathsf{F}(\mathcal{W})) = \mathcal{W}$ for all wide subcategories $\mathcal{W} \in \operatorname{wide}(A)$. In particular, the map $\mathsf{F}\colon \operatorname{wide}(A) \to \operatorname{torf}(A)$ is injective.

Proof. We show (1) as (2) is the dual statement. The maps are well-defined by Lemma 1.13, so it suffices to show that for all wide subcategories $\mathcal{W} \in \operatorname{wide}(A)$ we have $\alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W})) = \mathcal{W}$.

Let $W \in \mathcal{W}$. Suppose that $f\colon X \to W$ is a morphism in $\mathsf{T}(\mathcal{W})$. Then $\ker(f) \in \mathsf{T}(\mathcal{W})$ by Lemma 1.14.(1), hence $X \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$.

Conversely, let $X \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$. Then by definition $X \in \mathsf{T}(\mathcal{W})$. By Lemma 1.10.(1) we have $\mathsf{T}(\mathcal{W}) = \operatorname{Filt}(\operatorname{Fac}(\mathcal{W}))$, so we can find a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $0 \neq X' \in \operatorname{Fac}(\mathcal{W})$ and $X'' \in \mathsf{T}(\mathcal{W})$. Then $X' \in \operatorname{Fac}(\mathcal{W}) \subseteq \mathsf{T}(\mathcal{W})$ is a submodule of $X \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$ which is again in $\mathsf{T}(\mathcal{W})$, so by Lemma 1.12.(1) also $X' \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$. Choose a surjection $f\colon W \to X'$ with $W \in \mathcal{W} \subseteq \mathsf{T}(\mathcal{W})$. Since $X' \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$, we have $\ker(f) \in \mathsf{T}(\mathcal{W})$. On the other hand, we have $\ker(f) \in \operatorname{Sub}(\mathcal{W}) \subseteq \mathsf{F}(\mathcal{W})$. It follows that $\ker(f) \in \mathsf{F}(\mathcal{W}) \cap \mathsf{T}(\mathcal{W}) = \mathcal{W}$ using Lemma 1.15. Therefore f is the cokernel of a morphism in $\mathcal{W}$, hence $X' \in \mathcal{W}$ since $\mathcal{W}$ is abelian.

By assumption, X'' has strictly smaller dimension than $X \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$ and since we have $X' \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$ also $X'' \in \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$ since $\alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$ is wide, cf. Lemma 1.13. So we can apply induction on the dimension of X and conclude that $X'' \in \mathcal{W}$. Hence we have shown that $X' \in \mathcal{W}$ and $X'' \in \mathcal{W}$, and therefore $X \in \mathcal{W}$ since $\mathcal{W}$ is wide. $\square$

Example 1.17 (cf. [MŠ17, Example 3.4]) Let $A = kQ$ be the path algebra over the Kronecker quiver $Q = (1 \leftleftarrows 2)$. Since a preinjective module has a non-zero morphism only to a preinjective module, the set of all preinjective modules is a torsion class.

Let $I \in \operatorname{mod}(A)$ be an indecomposable preinjective module. Then there is a unique indecomposable preinjective module I' such that there are irreducible surjective morphisms $I' \to I$. But the kernel of each such surjective morphism is a quasi-simple regular module, so not in $\mathcal{T}$. Hence $\alpha_{\mathsf{T}}(\mathcal{T}) = \{0\}$.

Hence $\mathcal{T}$ is not in the image of $\mathsf{T}\colon \operatorname{wide}(A) \to \operatorname{tors}(A)$, in particular, the map is not a bijection.

1.3 Functorial finiteness

Now we consider the maps between wide subcategories and torsion(-free) classes studied in the previous sections restricted to functorially finite subcategories. τ-tilting theory, introduced by Adachi–Iyama–Reiten [AIR14], gives useful tools to understand the poset of functorially finite torsion(-free) classes. The Ingalls–Thomas correspondence in the functorially finite case for general finite dimensional algebras is due to Marks–Šťovíček [MŠ17] and the connection to τ-tilting theory was studied by Asai [Asa18].

Our starting point is given by the following maps constructed in Section 1.2.

$$\begin{tikzcd}
\operatorname{wide}(A) \arrow[d, shift left, "\mathsf{T}"] & & \operatorname{wide}(A) \arrow[d, shift left, "\mathsf{F}"] \\
\operatorname{tors}(A) \arrow[u, shift left, "\alpha_{\mathsf{T}}"] \arrow[rr, shift left, "()^{\perp}", "\sim"'] & & \operatorname{torf}(A) \arrow[u, shift left, "\alpha_{\mathsf{F}}"] \arrow[ll, shift left, "{}^{\perp}()"]
\end{tikzcd}$$

The maps satisfy $\alpha_{\mathsf{T}} \circ \mathsf{T} = \mathrm{id}$ and $\alpha_{\mathsf{F}} \circ \mathsf{F} = \mathrm{id}$, cf. Proposition 1.16. In particular, T and F are injective. The following definition collects some notions from τ-tilting theory that we will be relevant for us.

For a finite dimensional algebra A, we denote by τ the Auslander–Reiten translation on $\operatorname{mod}(A)$. Dually, τ^{-1} denotes the inverse Auslander–Reiten translation on $\operatorname{mod}(A)$. Note that we have $\tau M = 0$ if and only if M is projective and $\tau^{-1}M = 0$ if and only if M is injective.

Recall that for a module $M \in \operatorname{mod}(A)$ we write $|M|$ for the number of pairwise non-isomorphic indecomposable direct summands. In particular, $|A|$ equals the number of isoclasses of indecomposable projective modules, which also equals the number of isoclasses of simple A-modules.

Definition 1.18 ([AIR14, Definitions 0.1 and 0.3]) Let A be a finite dimensional algebra.

(1) A module $M \in \operatorname{mod}(A)$ is called *τ-rigid* if $\operatorname{Hom}_A(M, \tau M) = 0$. Dually, $N \in \operatorname{mod}(A)$ is called *τ^{-1}-rigid* if $\operatorname{Hom}_A(\tau^{-1}N, N) = 0$.

(2) A τ-rigid module M is called *τ-tilting* if $|M| = |A|$. Dually, a τ^{-1}-rigid module N is called *τ^{-1}-tilting* if $|N| = |A|$.

(3) A τ-rigid module M is called *support τ-tilting* if there is a projective module P with $\mathrm{Hom}_A(P, M) = 0$ and $|M| + |P| = |A|$. Dually, a τ^{-1}-rigid module N is called *support τ^{-1}-tilting* if there is an injective module I with $\mathrm{Hom}_A(N, I) = 0$ and $|N| + |I| = |A|$.

(4) We write $\mathrm{s}\tau\text{-tilt}(A)$ for the set of isomorphism classes of basic support τ-tilting modules. We write $\mathrm{s}\tau^{-1}\text{-tilt}(A)$ for the set of isomorphism classes of basic support τ^{-1}-tilting modules.

(5) We write $\text{f-tors}(A)$ for the subposet of functorially finite torsion classes, similarly $\text{f-torf}(A)$ for the functorially finite torsion-free classes.

(6) Let $\mathcal{T}$ be a torsion class. A module $M \in \mathcal{T}$ is called Ext-*projective* if $\mathrm{Ext}^1_A(M, \mathcal{T}) = 0$. We denote by $\mathsf{P}(\mathcal{T})$ the direct sum of a representative of each isomorphism class of Ext-projective modules in $\mathcal{T}$. Dually, a module $N \in \mathcal{F}$ in a torsion-free class $\mathcal{F}$ is called Ext-*injective* if $\mathrm{Ext}^1_A(\mathcal{F}, N) = 0$. We denote by $\mathsf{I}(\mathcal{F})$ the direct sum of a representative of each isomorphism class of Ext-injective modules in $\mathcal{F}$.

(7) We write $\text{f-wide}(A)$ for the set of functorially finite torsion classes.

To formulate the results of [Asa18], we also need the following elementary definition.

Definition 1.19 Let A be a finite dimensional algebra. A set of of bricks $\mathcal{S} \subseteq \mathrm{mod}(A)$ is called a *semibrick* if $\mathrm{Hom}_A(S', S'') = 0$ for all pairwise distinct $S' \neq S'' \in \mathcal{S}$. We denote the set of isomorphism classes of semibricks in $\mathrm{mod}(A)$ by $\mathrm{sbrick}(A)$.

The Jordan–Hölder theorem in $\mathrm{mod}(A)$ and Schur's lemma imply the following well-known relationship between wide subcategories and semibricks.

Proposition 1.20 ([Rin76, Theorem 1.2]) *Let A be a finite dimensional algebra. Then the assignment $\mathcal{W} \to \mathcal{S}(\mathcal{W})$ that associates to a wide subcategory $\mathcal{W} \in \mathrm{wide}(A)$ its set of simple objects $\mathcal{S}(\mathcal{W}) \subseteq \mathcal{W}$ gives the following bijection.*

$$\begin{array}{rcl} \mathrm{wide}(A) & \longleftrightarrow & \mathrm{sbrick}(A) \\ \mathcal{W} & \longmapsto & \mathcal{S}(\mathcal{W}) \\ \mathrm{Filt}(\mathcal{S}) & \longleftarrow\!\shortmid & \mathcal{S} \end{array}$$

We also make the following definition, which is inspired by [Asa18, Definition 1.2] but formulated in terms of wide subcategories instead of semibricks.

Definition 1.21 Let A be a finite dimensional algebra and let $\mathcal{W} \in \mathrm{wide}(A)$ be a wide subcategory. Then $\mathcal{W}$ is called

(1) *left-finite* if $\mathsf{T}(\mathcal{W})$ is functorially finite. We denote the set of left-finite wide subcategories by lf-wide(A). A semibrick $\mathcal{S} \subseteq \mathrm{mod}(A)$ is called left-finite if the wide subcategory $\mathrm{Filt}(\mathcal{S})$ is left-finite.

(2) *right-finite* if $\mathsf{F}(\mathcal{W})$ is functorially finite. We denote the set of right-finite wide subcategories by rf-wide(A). A semibrick $\mathcal{S} \subseteq \mathrm{mod}(A)$ is called right-finite if the wide subcategory $\mathrm{Filt}(\mathcal{S})$ is right-finite.

To formulate the results of [Asa18], we make the following definition.

Definition 1.22 Let $M \in \mathrm{mod}(A)$ be a module and let $B := \mathrm{End}_A(M)$. Then the B-radical $\mathrm{rad}_B(M)$ and the B-socle $\mathrm{soc}_B(M)$ of M are defined as the following A-modules.

$$\mathrm{rad}_B(M) = \sum_{f \in \mathrm{rad}(B)} \mathrm{im}(f) \quad \text{and} \quad \mathrm{soc}_B(M) = \bigcap_{f \in \mathrm{rad}(B)} \ker(f).$$

By construction, we have that $\mathrm{rad}_B(M) \in \mathrm{Fac}(M)$ and $\mathrm{soc}_B(M) \in \mathrm{Sub}(M)$.

Recall that a quiver is called ℓ-regular for some $\ell \in \mathbb{N}$ if for each vertex v the sum of the number of arrows that start in v with the number of arrows that end in v equals ℓ. We have the following results.

Proposition 1.23 *Let A be a finite dimensional algebra. Then the following hold.*

(1) [AIR14, Theorem 2.7] *We have the following mutually inverse bijections.*

$$\begin{array}{rcl} \text{f-tors}(A) & \longleftrightarrow & \text{s}\tau\text{-tilt}(A) \\ \mathcal{T} & \longmapsto & \mathsf{P}(\mathcal{T}) \\ \mathrm{Fac}(M) & \longleftarrow\!\shortmid & M \end{array} \quad \textit{and} \quad \begin{array}{rcl} \text{f-torf}(A) & \longleftrightarrow & \text{s}\tau^{\text{-}1}\text{-tilt}(A) \\ \mathcal{F} & \longmapsto & \mathsf{I}(\mathcal{F}) \\ \mathrm{Sub}(N) & \longleftarrow\!\shortmid & N \end{array}$$

In particular, the Hasse quiver of functorially finite torsion classes is $n = |A|$-regular.

(2) [MŠ17, Proposition 3.9] *The bijections from Proposition* 1.16 *restrict to the following mutually inverse bijections.*

$$\text{lf-wide}(A) \underset{\alpha_{\mathsf{T}}}{\overset{\mathsf{T}}{\rightleftarrows}} \text{f-tors}(A) \quad \textit{and} \quad \text{rf-wide}(A) \underset{\alpha_{\mathsf{F}}}{\overset{\mathsf{F}}{\rightleftarrows}} \text{f-torf}(A)$$

Moreover, left-finite (resp. right-finite) wide subcategories are functorially finite.

(3) [Asa18, Proposition 1.9] *Let M be a support τ-tilting module. Then*

$$\alpha_{\mathsf{T}}\big(\mathrm{Fac}(M)\big) = \mathrm{Filt}(M/\operatorname{rad}_B(M))$$

and the indecomposable direct summands of $M/\operatorname{rad}_B(M)$ form a left-finite semibrick.

Dually, let N be a support τ^{-1}-tilting module. Then

$$\alpha_{\mathsf{F}}\big(\mathrm{Sub}(N)\big) = \mathrm{Filt}(\mathrm{soc}_B(M))$$

and the indecomposable direct summands of $\mathrm{soc}_B(M)$ *form a right-finite semibrick.*

An important class of finite dimensional algebras is given by those with only finitely many support τ-tilting modules, hence they were given a name and first studied in [DIJ19].

Definition 1.24 A finite dimensional algebra A is called *τ-tilting finite* if $|\mathrm{s}\tau\text{-tilt}(A)| < \infty$.

Then we have the following equivalent characterizations of τ-tilting finiteness. Our formulation of Proposition 1.25 summarizes several results in [DIJ19].

Proposition 1.25 ([DIJ19]) *Let A be a finite dimensional algebra. Then the following are equivalent.*

(1) *A is τ-tilting finite.*

(2) *Every torsion class in* $\mathrm{mod}(A)$ *is functorially finite.*

(3) *Every torsion-free class in* $\mathrm{mod}(A)$ *is functorially finite.*

(4) *There are only finitely many torsion classes in* $\mathrm{mod}(A)$.

(5) *There are only finitely many torsion-free classes in* $\mathrm{mod}(A)$.

(6) *There are only finitely many isomorphism classes of bricks in* $\mathrm{mod}(A)$.

(7) *There are only finitely many wide subcategories in* $\mathrm{mod}(A)$.

Proof. The equivalence of (1-3) is [DIJ19, Theorem 3.8]. With Proposition 1.23.(1), an algebra with finitely many torsion(-free) classes has finitely many support τ-tilting modules, so (4) implies (1) and (5) implies (1). Again using Proposition 1.23 we see that (1-3) imply (4) and (5). The equivalence of (1) and (6) is [DIJ19, Theorem 4.2].

Finally, the implication (6) $\Rightarrow$ (7) follows from Proposition 1.20. Conversely, Proposition 1.23.(2) gives an injection from the set of functorially torsion classes, which are in bijection to support τ-tilting modules by Proposition 1.23.(1), to the set of wide subcategories. So finiteness of the set of wide subcategories implies τ-tilting finiteness, i.e. (7) implies (1). $\square$

For τ-tilting finite algebras, the relationship between the maps T, F and $\alpha_{\mathsf{T}}, \alpha_{\mathsf{F}}$ restricted to functorially finite torsion(-free) classes becomes simpler.

Corollary 1.26 *Let A be a τ-tilting finite algebra. Then the we have the following mutually inverse bijection.*

$$\begin{tikzcd} \mathrm{wide}(A) \arrow[d, shift left, "\mathsf{T}"] & \mathrm{wide}(A) \arrow[d, shift left, "\mathsf{F}"] \\ \mathrm{tors}(A) \arrow[u, shift left, "\alpha_{\mathsf{T}}"] \arrow[r, shift left, "()^{\perp}"] \arrow[r, phantom, "\sim"] & \mathrm{torf}(A) \arrow[u, shift left, "\alpha_{\mathsf{F}}"] \arrow[l, shift left, "{}^{\perp}()"] \end{tikzcd}$$

Proof. For general finite dimensional algebras we have $\alpha_{\mathsf{T}} \circ \mathsf{T} = \mathrm{id}$ and $\alpha_{\mathsf{F}} \circ \mathsf{F} = \mathrm{id}$. But since A is τ-tilting finite, all torsion(-free) classes are functorially finite, cf. Proposition 1.25. Hence α_{T} and α_{F} are injective by Proposition 1.23, which implies that they are isomorphisms. $\square$

Examples of τ-tilting finite algebras are obviously the representation finite algebras. However, the class of τ-tilting finite algebras also includes representation infinite algebras, e.g. preprojective algebras of Dynkin type [Miz14].

For general finite dimensional algebras, functorially finite wide subcategories need not to be left-finite, see [Asa18, Example 3.13], but the wide subcategory constructed there is still right-finite. However, using Asai's example one can easily construct a functorially finite subcategory that is neither right nor left-finite, cf. the following example.

Example 1.27 We consider the algebra $A = kQ/I$ given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2 \xleftarrow{\gamma} 3 \underset{\varepsilon}{\overset{\delta}{\leftleftarrows}} 4$$

and relations $\alpha\gamma = 0$ and $\gamma\delta = 0$. Note that A is a string algebra, in fact, it is gentle. We consider the string module

$$S = M(\beta\gamma\varepsilon) = \begin{smallmatrix} 4 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{smallmatrix}.$$

We compute its Auslander–Reiten translate to be

$$\tau S = M(\varepsilon) = \begin{smallmatrix} 4 & \\ & 3 \end{smallmatrix}.$$

We have $\dim_k \mathrm{Hom}_A(S, \tau S) = 1$, in particular, S is not τ-rigid. But every morphism $S \to \tau S$ factors over the injective module $I(1)$ at 1.

$$\begin{smallmatrix} 4 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{smallmatrix} \hookrightarrow \begin{smallmatrix} 4 & & & & \\ & 3 & & & \\ & & 2 & & 2 \\ & & & 1 & \end{smallmatrix} \twoheadrightarrow \begin{smallmatrix} 4 & \\ & 3 \end{smallmatrix}$$

In particular, in the injective stable category all morphisms $S \to \tau S$ are zero, therefore $0 = D\,\overline{\mathrm{Hom}}_A(S, \tau S) = \mathrm{Ext}^1_A(S, S) = 0$ by Auslander–Reiten duality. It follows that $\mathrm{Filt}(S)$ is functorially finite.

However, we *claim* that neither the torsion class $\mathsf{T}(S)$ nor the torsion-free class $\mathsf{F}(S)$ generated by S are functorially finite. Note that A is canonically isomorphic to its dual and S is preserved under this canonical isomorphism. Hence $\mathsf{T}(S)$ is not functorially finite if and only if $\mathsf{F}(S)$ is not functorially finite.

To show that $\mathsf{T}(S)$ is not functorially finite, we show that the torsion class does not contain an Ext-projective generator. Such an Ext-projective generator has to be τ-rigid by Proposition 1.23.(1). This implies that the indecomposable direct summands have to be string modules.

Consider the set of string modules in $\mathsf{T}(S) = \mathrm{Filt}(\mathrm{Fac}(S))$. It is easy to see that the set of strings for which there is a string module in $\mathsf{T}(S)$ is given by

$$\left\{\beta\gamma\varepsilon(\delta^{-1}\varepsilon)^{\ell},\ \gamma\varepsilon(\delta^{-1}\varepsilon)^{\ell},\ \varepsilon(\delta^{-1}\varepsilon)^{\ell},\ (\delta^{-1}\varepsilon)^{\ell}\right\}$$

where $\ell \geq 0$. An Ext-projective generator in $\mathsf{T}(S)$ must have a string module of the following form as a direct summand.

$$M(\beta\gamma\varepsilon(\delta^{-1}\varepsilon)^{\ell}) = \begin{smallmatrix} 4 & \cdots & & 4 & & & \\ & & 3 & \cdots & 3 & & \\ & & & & & 2 & \\ & & & & & & 1 \end{smallmatrix}$$

The other families of strings listed earlier result in string modules which are not supported on all vertices.

But we can compute the Auslander–Reiten translate of each string module $M(\beta\gamma\varepsilon(\delta^{-1}\varepsilon)^{\ell})$ for $\ell \geq 0$ to be

$$\tau M(\beta\gamma\varepsilon(\delta^{-1}\varepsilon)^{\ell}) = M(\varepsilon(\delta^{-1}\varepsilon)^{\ell}) = \begin{smallmatrix} 4 & \cdots & & 4 & \\ & & 3 & \cdots & 3 \end{smallmatrix}.$$

In particular, all the string modules $M(\beta\gamma\varepsilon(\delta^{-1}\varepsilon)^{\ell})$ for $\ell \geq 0$ are not τ-rigid, hence can not be a direct summand of an Ext-projective generator in $\mathsf{T}(S)$. It follows that $\mathsf{T}(S)$ is not functorially finite. $\square$

For the remainder of this Section 1.3 we want to consider the results of Proposition 1.23 for the class of hereditary algebras in more detail.

First of all, for a hereditary algebra A a module $M \in \mathrm{mod}(A)$ is τ-rigid if and only if it is rigid, i.e. if and only $\mathrm{Ext}^1_A(M, M) = 0$, if and only if it is τ^{-1}-rigid.

Concerning left-finite and right-finite wide subcategories, we have the following result of Asai.

Proposition 1.28 ([Asa18, Proposition 3.11]) *Let A be a hereditary algebra. Then* $\text{f-wide}(A) = \text{lf-wide}(A) = \text{rf-wide}(A)$, *i.e. a wide subcategory $\mathcal{W} \subseteq \mathrm{mod}(A)$ is functorially finite if and only if it is left-finite if and only if it is right-finite.*

So in order to determine all left-finite or right-finite wide subcategories for hereditary algebras it suffices to determine the functorially finite wide subcategories. We want to show how this can be done concretely using the sets of simple objects in wide subcategories, i.e. using their associated semibricks.

For this, we use the framework of exceptional sequences for hereditary algebras, cf. e.g. [Rin94]. For convenience, we recall the definition.

Definition 1.29 Let A be a hereditary algebra. A sequence of finite dimensional A-modules $(M_1, \ldots, M_\ell)$ is called *exceptional* if $\mathrm{Hom}_A(M_j, M_i) = 0$ for $i < j$ and $\mathrm{Ext}^1_A(M_j, M_i) = 0$ for $i \leq j$.

An exceptional sequence $(M_1, \ldots, M_\ell)$ is called *strongly exceptional* if $\mathrm{Ext}^1_A(M_i, M_j) = 0$ for all i, j. It is called *orthogonal* if $\mathrm{Hom}_A(M_i, M_j) = 0$ for all $i \neq j$.

Then we have the following extension of Asai's result in Proposition 1.28.

Proposition 1.30 *Let A be a hereditary algebra and let $\mathcal{S} \subseteq \mathrm{mod}(A)$ be a semibrick. Then the following are equivalent.*

(1) *The wide subcategory $\mathcal{W} := \mathrm{Filt}(\mathcal{S})$ is functorially finite.*

(2) *$\mathcal{S}$ is left-finite.*

(3) *$\mathcal{S}$ is right-finite.*

(4) *The elements of $\mathcal{S}$ can be arranged into an orthogonal exceptional sequence.*

Proof. The equivalence of (1-3) is Asai's result, cf. Proposition 1.28. To finish the proof, we show that (2) implies (4) and that (4) implies (1).

(2) $\Rightarrow$ (4) By definition, $\mathcal{S}$ is left-finite if and only if $\mathsf{T}(\mathcal{S})$ if functorially finite. So there is a support τ-tilting module $M \in \mathrm{mod}(A)$ with $\mathsf{T}(\mathcal{S}) = \mathrm{Fac}(M)$, cf. Proposition 1.23.(1). Since A is hereditary, M is rigid, i.e. $\mathrm{Ext}^1_A(M, M) = 0$. The Happel–Ringel lemma [HR82] implies that we can arrange the indecomposable direct summands M_i of M such that $\mathrm{Hom}_A(M_j, M_i) = 0$ for $i < j$. Moreover, each M_i is a brick. This implies that no morphism in $\mathrm{rad}(B)$, where $B = \mathrm{End}_A(M)$, has image in M_1. So M_1 is a direct summand of $M/\mathrm{rad}_B(M)$. But by Proposition 1.23.(3), $\mathcal{S}$ is given by the indecomposable direct summands of $M/\mathrm{rad}_B(M)$, so $M_1 \in \mathcal{S}$. Since M is Ext-projective in $\mathsf{T}(\mathcal{S}) = \mathrm{Fac}(M)$, we have $\mathrm{Ext}^1_A(M_1, \mathcal{S}) = 0$.

By [Asa18, Proposition 3.12] also $\mathcal{S} \setminus \{M_1\}$ is a left-finite semibrick in $\mathrm{mod}(A)$, so by induction on the cardinality $\mathcal{S}$ we can arrange its element into an exceptional sequence. But $\mathrm{Ext}^1_A(M_1, \mathcal{S} \setminus \{M_1\}) = 0$, so by putting M_1 at the end of that sequence we can arrange all elements $\mathcal{S}$ into an exceptional sequence.

(4) $\Rightarrow$ (1) We use induction on the cardinality of $\mathcal{S}$. If $\mathcal{S} = \{S\}$ contains only one element, $\mathcal{W} = \mathrm{Filt}(S)$ is semisimple since S is rigid, hence functorially finite.

Now suppose $\mathcal{S}$ has two or more elements and suppose that $(S_1, \ldots, S_\ell)$ are the elements of $\mathcal{S}$ arranged into an exceptional sequence. Let $\mathcal{W} := \mathrm{Filt}(\mathcal{S})$. Then $\mathrm{Ext}^1_A(S_\ell, S_i) = 0$ for all i, so in particular $\operatorname{Ext}^1_A(S_\ell, \mathcal{W}) = 0$. Hence S_ℓ is a projective simple object in $\mathcal{W}$, which means that if S_ℓ is a submodule of any module, it must be a direct summand. But then we can find for any object $X \in \mathcal{W}$ a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

where $X' \in \operatorname{Filt}(\mathcal{S} \setminus \{S_\ell\})$ and $X'' \in \operatorname{add}(S_\ell)$. By induction, we know that $\operatorname{Filt}(\mathcal{S} \setminus \{S_\ell\})$ is functorially finite and for $\operatorname{add}(S_\ell)$ functorial finiteness is obvious. Together, this implies that $\mathcal{W}$ is functorially finite [SS93, Theorem 2.6]. $\square$

Chapter 2

Wide intervals

In this chapter, A is a finite dimensional algebra over a field k.

2.1 Definitions and known results

We use the terminology of *wide intervals* from [AP19]. The notion is a generalization of the τ-perpendicular category [Jas15] and the so called polytopes from [DIR$^+$17] away from the functorially finite setting.

Definition 2.1 (cf. [AP19, Definition 4.1]) Let $\mathcal{U}, \mathcal{T} \in \operatorname{tors}(A)$ be torsion classes. Then the interval set

$$[\mathcal{T}, \mathcal{U}] := \{\mathcal{V} \in \operatorname{tors}(A) \mid \mathcal{T} \subseteq \mathcal{V} \subseteq \mathcal{U}\}$$

is called a *wide interval* if $\mathcal{U} \cap \mathcal{T}^{\perp} \subseteq \operatorname{mod}(A)$ is a wide subcategory.

Let $\mathcal{W} \in \operatorname{wide}(A)$ be a wide subcategory in $\operatorname{mod}(A)$. We say that the wide interval $[\mathcal{T}, \mathcal{U}]$ is a $\mathcal{W}$*-interval* if $\mathcal{U} \cap \mathcal{T}^{\perp} = \mathcal{W}$.

By definition, the elements of a $\mathcal{W}$-interval $[\mathcal{T}, \mathcal{U}]$ are torsion classes in $\operatorname{mod}(A)$. It is easy to see that for each torsion class $\mathcal{V} \in [\mathcal{T}, \mathcal{U}]$ the intersection $\mathcal{T}^{\perp} \cap \mathcal{V}$ is a torsion class in the wide subcategory $\mathcal{W}$. In fact, this extends to a bijection, which is one of the main results in [AP19].

Theorem 2.2 ([AP19, Theorem 4.2]) *Let $\mathcal{W} \in \operatorname{wide}(A)$ be a wide subcategory and let $[\mathcal{T}, \mathcal{U}]$ be a $\mathcal{W}$-interval in $\operatorname{tors}(A)$. Then we have the following bijection.*

$$\begin{array}{rcl} [\mathcal{T}, \mathcal{U}] & \longleftrightarrow & \operatorname{tors}(\mathcal{W}) \\ \mathcal{V} & \longmapsto & \mathcal{T}^{\perp} \cap \mathcal{V} \\ \mathcal{T} * \mathcal{X} & \longleftarrow\!\shortmid & \mathcal{X} \end{array}$$

In the theorem, we denote by $\mathcal{T} * \mathcal{X}$ the set of all objects Y that fit into a short exact sequence

$$0 \longrightarrow T \longrightarrow Y \longrightarrow X \longrightarrow 0$$

with $T \in \mathcal{T}$ and $X \in \mathcal{X}$.

Corollary 2.3 *Let* $[\mathcal{T}, \mathcal{U}]$ *be a* $\mathcal{W}$*-interval in* $\operatorname{tors}(A)$*. Then we have*

$$\mathcal{T} = \mathcal{U} \cap {}^{\perp}\mathcal{W} \quad \textit{and} \quad \mathcal{U} = \mathcal{T} * \mathcal{W}.$$

Proof. The second equation follows by setting $\mathcal{X} := \mathcal{W} = \mathcal{T}^{\perp} \cap \mathcal{U} \in \operatorname{tors}(\mathcal{W})$ and using the map from Theorem 2.2. For the first one, note that $\mathcal{W} \subseteq \mathcal{T}^{\perp}$ and therefore $\mathcal{T} \subseteq {}^{\perp}\mathcal{W}$, so that $\mathcal{U} \cap {}^{\perp}\mathcal{W} \in [\mathcal{T}, \mathcal{U}]$. But $\mathcal{T}^{\perp} \cap (\mathcal{U} \cap {}^{\perp}\mathcal{W}) = \mathcal{W} \cap {}^{\perp}\mathcal{W} = \{0\} = \mathcal{T}^{\perp} \cap \mathcal{T}$, hence $\mathcal{T} = \mathcal{U} \cap {}^{\perp}\mathcal{W}$ by the theorem. $\square$

Definition 2.4 Let $\mathcal{W} \in \operatorname{wide}(A)$ be a wide subcategory in $\operatorname{mod}(A)$. Let $\operatorname{intv}(\mathcal{W})$ denote the set of $\mathcal{W}$-intervals in $\operatorname{tors}(A)$. We define a partial order on $\operatorname{intv}(A)$ by

$$[\mathcal{T}, \mathcal{U}] \leq [\mathcal{T}', \mathcal{U}'] \quad :\Longleftrightarrow \quad \mathcal{T} \subseteq \mathcal{T}' \quad \Longleftrightarrow \quad \mathcal{U} \subseteq \mathcal{U}'.$$

Note that the second equivalence follows from Corollary 2.3.

Example 2.5 Let $A = kQ/I$ be the algebra given by the path algebra kQ with the quiver

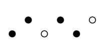
$$\begin{tikzcd}
 & 2 \arrow[dl] & \\
1 \arrow[rr] & & 3 \arrow[ul]
\end{tikzcd}$$

modulo the admissible ideal generated by all paths of length 2. Note that A is a self-injective Nakayama algebra. The Auslander–Reiten quiver of A looks as follows.

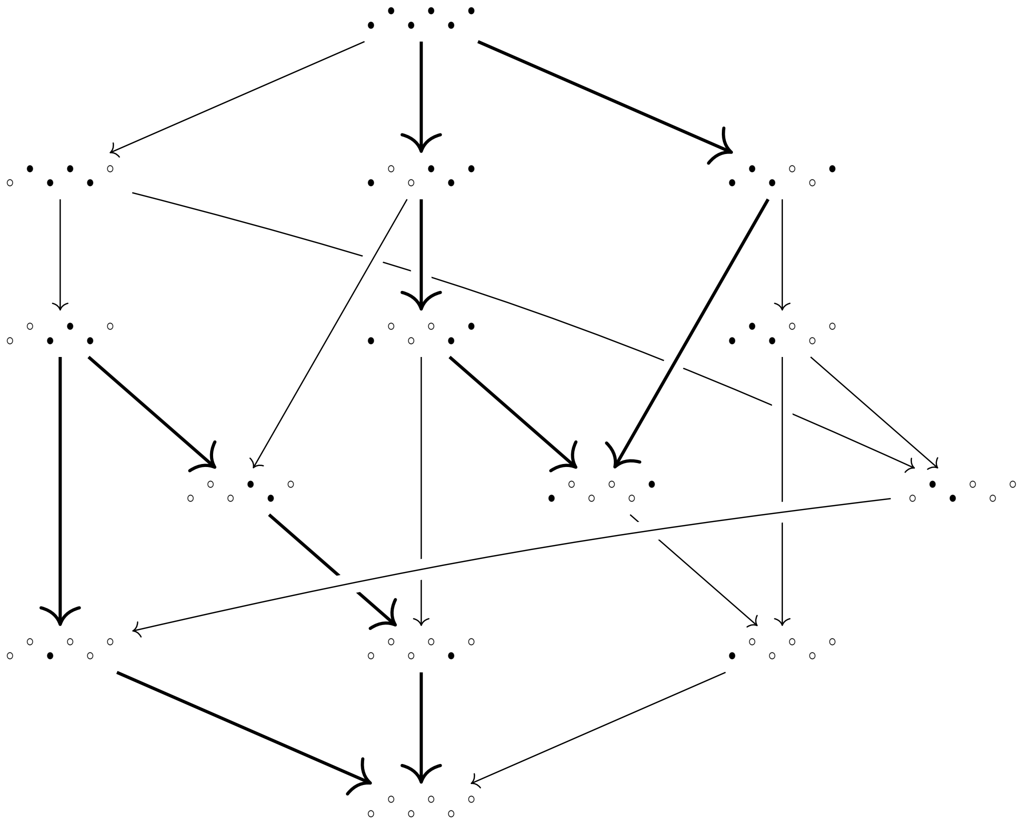
$$\begin{tikzcd}
 & {\substack{2\\1}} \arrow[dr] & & {\substack{3\\2}} \arrow[dr] & & {\substack{1\\3}} \arrow[dr] & \\
1 \arrow[ur] & & 2 \arrow[ur] \arrow[ll, dashed] & & 3 \arrow[ur] \arrow[ll, dashed] & & 1 \arrow[ll, dashed]
\end{tikzcd}$$

Both appearances of the simple module 1 are identified.

We use the following shorthand graphical notation for additive subcategories of $\operatorname{mod}(A)$, which directly corresponds to the shape of the Auslander–Reiten quiver, e.g. we let

 $:= \operatorname{add}\{\, 1,\ \substack{2\\1},\ \substack{3\\2},\ 3 \,\}.$

In words, a filled out circle indicates that the indecomposable at this position in the Auslander–Reiten quiver is in the additive subcategory. Now we are in a position to efficiently write down the full Hasse quiver of the lattice $\operatorname{tors}(A)$ of torsion classes in $\operatorname{mod}(A)$.



With bold arrows, we marked the two $\mathcal{W}$-intervals for the wide subcategory $\mathcal{W} = \mathrm{Filt}(\,2\,,\,3\,)$ generated by the simple modules at 2 and 3. This wide subcategory is equivalent to the module category of an A_2-quiver. There are only two $\mathcal{W}$-intervals for this wide subcategory. Clearly, the one in the top right is greater than the one in the bottom left with respect to the partial order on the set $\mathrm{intv}(\mathcal{W})$ of $\mathcal{W}$-intervals.

Remark 2.6 Each brick $S \in \mathrm{mod}(A)$ in the module category of a finite dimensional algebra defines a wide subcategory $\mathcal{W} = \mathrm{Filt}(S)$ with only one simple object, and no proper non-zero subcategory of $\mathcal{W}$ is again wide in $\mathrm{mod}(A)$. By [DIR⁺17, Theorem 3.3] an interval $[\mathcal{T}, \mathcal{U}]$ is a wide interval for some wide subcategory with only one object if and only if $\mathcal{U} \to \mathcal{T}$ is an arrow in the Hasse quiver of torsion classes. The unique brick S such that $\mathcal{U} \cap \mathcal{T}^{\perp} = \mathrm{Filt}(S)$ is called the *brick label* for this arrow, cf. [DIR⁺17, Definition 3.5]. In this language, the set of $\mathrm{Filt}(S)$-intervals $\mathrm{intv}(\mathrm{Filt}(S))$ for a brick S can be identified with the set of arrows in the Hasse quiver of torsion classes that have S as their brick label.

2.2 Minimal and maximal intervals

Having defined the partially ordered set of $\mathcal{W}$-intervals $\mathrm{intv}(\mathcal{W})$ for a wide subcategory $\mathcal{W} \in \mathrm{wide}(A)$ in Definition 2.1, we now start to show some of its properties. The goal here is to show that $\mathrm{intv}(\mathcal{W})$ has unique canonical maximal and minimal elements. For this, we need a characterization of the upper torsion class $\mathcal{U}$ for $\mathcal{W}$-intervals $[\mathcal{T}, \mathcal{U}]$ that is based on the Ingalls–Thomas correspondence, which we studied in the previous chapter.

First we recall some preliminary well-known definitions and observations.

Definition 2.7 (Serre subcategory, cf. [Ser53]) Let $\mathcal{A}$ be an abelian category. A full non-empty subcategory $\mathcal{X} \subseteq \mathcal{A}$ is called a *Serre subcategory* if for every short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

in $\mathcal{A}$ we have $X \in \mathcal{X}$ if and only if $X' \in \mathcal{X}$ and $X'' \in \mathcal{X}$.

In other words, a Serre subcategory is a full exact subcategory closed under extensions, factor objects and subobjects.

For abelian categories in which every object has finite length, Serre subcategories are fairly rare among wide subcategories and easily classified.

Proposition 2.8 *Let $\mathcal{A}$ be an abelian category in which every object has finite length. Let $\mathcal{S} \subseteq \mathcal{A}$ be the set of simple objects in $\mathcal{A}$, i.e. a set of representatives of isoclasses of objects in $\mathcal{A}$ that have length* 1*. Then there is a bijection*

$$\begin{aligned} \{\mathcal{X} \subseteq \mathcal{A} \mid \mathcal{X} \textit{ is a Serre subcategory}\} &\longleftrightarrow \mathcal{P}(\mathcal{S}) \\ \mathcal{X} &\longmapsto \mathsf{S}(\mathcal{X}) \\ \mathrm{Filt}(\mathcal{S}') &\longleftarrow\!\shortmid \mathcal{S}', \end{aligned}$$

where $\mathsf{S}(\mathcal{X})$ denotes a complete set of representatives of isoclasses of simple objects in $\mathcal{X}$ and $\mathcal{P}(\mathcal{S})$ denotes the power set of $\mathcal{S}$.

Proof. The definition of a Serre subcategory immediately implies that for any object $X \in \mathcal{X}$ in a Serre subcategory $\mathcal{X} \subseteq \mathcal{A}$, all its simple composition factors must also lie in $\mathcal{X}$. On the other hand, given a set of simple objects $\mathcal{S}'$ in $\mathcal{A}$, the category $\mathrm{Filt}(\mathcal{S}')$ is a Serre subcategory. $\square$

Now recall the following maps constituting the Ingalls–Thomas correspondence as presented in Proposition 1.16.

$$\mathrm{wide}(A) \underset{\alpha_{\mathsf{T}}}{\overset{\mathsf{T}}{\rightleftarrows}} \mathrm{tors}(A)$$

The map α_{T} sends a torsion class $\mathcal{T} \in \mathrm{tors}(A)$ to the wide subcategory

$$\alpha_{\mathsf{T}}(\mathcal{T}) = \{Y \in \mathrm{mod}(A) \mid \forall (f : X \to Y) \in \mathcal{T},\, \ker(f) \in \mathcal{T}\}\,.$$

Also recall that the Ingalls–Thomas correspondence states that $\alpha_{\mathsf{T}} \circ \mathsf{T} = \mathrm{id}$, in particular, $\mathsf{T}\colon\ \mathrm{wide}(A) \to \mathrm{tors}(A)$ is injective.

The following result due to Asai–Pfeifer [AP19] classifies the set of wide intervals $[\mathcal{T}, \mathcal{U}]$

for a fixed upper torsion class $\mathcal{U}$ in terms of Serre subcategories of $\alpha_{\mathsf{T}}(\mathcal{U})$.

Theorem 2.9 (cf. [AP19, Theorem 6.7]) *Let A be a finite dimensional algebra and let $\mathcal{U} \in \operatorname{tors}(A)$ be a torsion class. Then the following is a bijection.*

$$\begin{aligned}
\left\{ \mathcal{T} \in \operatorname{tors}(A) \;\middle|\; \begin{matrix} [\mathcal{T}, \mathcal{U}] \textit{ is} \\ \textit{wide interval} \end{matrix} \right\} &\longleftrightarrow \left\{ \begin{matrix} \mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U}) \\ \textit{is a Serre subcategory} \end{matrix} \right\} \\
\mathcal{T} &\longmapsto \mathcal{U} \cap \mathcal{T}^{\perp} \\
\mathcal{U} \cap {}^{\perp}\mathcal{W} &\longleftarrow\!\shortmid \mathcal{W}
\end{aligned}$$

In particular, for a Serre subcategory $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$ the wide interval $[\mathcal{U} \cap {}^{\perp}\mathcal{W}, \mathcal{U}]$ is a $\mathcal{W}$-interval.

The preceding result classifies all wide intervals $[\mathcal{T}, \mathcal{U}]$ for a fixed upper torsion class $\mathcal{U}$. However, we are interested in classifying $\mathcal{W}$-intervals for a fixed wide subcategory $\mathcal{W}$. Any two distinct $\mathcal{W}$-intervals must have distinct upper torsion class. Our first result towards such a classification is the following generalization of the Ingalls–Thomas correspondence as stated in Proposition 1.16. Note that the following theorem reduces to the original Ingalls–Thomas correspondence if one considers the zero wide subcategory $\mathcal{W} = \{0\}$, since $\{0\}$-intervals are just singleton sets of torsion classes $\{\mathcal{T}\} = [\mathcal{T}, \mathcal{T}]$.

Theorem 2.10 *Let A be a finite dimensional algebra and let $\mathcal{W} \in \operatorname{wide}(A)$ be a wide subcategory in $\operatorname{mod}(A)$. Then we have the following maps.*

$$\begin{aligned}
\operatorname{intv}(\mathcal{W}) &\;\underset{\mathsf{I}}{\overset{\hat{\alpha}}{\rightleftarrows}}\; \left\{ \mathcal{V} \in \operatorname{wide}(A) \;\middle|\; \begin{matrix} \mathcal{W} \subseteq \mathcal{V} \textit{ is a Serre} \\ \textit{subcategory in } \mathcal{V} \end{matrix} \right\} \\
[\mathcal{T}, \mathcal{U}] &\longmapsto \alpha_{\mathsf{T}}(\mathcal{U}) \\
[\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W}, \mathsf{T}(\mathcal{V})] &\longleftarrow\!\shortmid \mathcal{V}
\end{aligned}$$

The maps satisfy $\hat{\alpha} \circ \mathsf{I} = \mathrm{id}$.

Proof. We first show that the maps in both directions are well-defined.

Let $[\mathcal{T}, \mathcal{U}] \in \operatorname{intv}(\mathcal{W})$ be a $\mathcal{W}$-interval in $\operatorname{tors}(A)$. To show well-definedness of the map from left to right, we have to show that the wide subcategory $\mathcal{W}$ is a Serre subcategory in the wide subcategory $\alpha_{\mathsf{T}}(\mathcal{U})$. So suppose we are given a short exact sequence in $\alpha_{\mathsf{T}}(\mathcal{U})$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0.$$

Since $\mathcal{W}$ is wide, $X', X'' \in \mathcal{W}$ implies that $X \in \mathcal{W}$. On the other hand, let $X \in \mathcal{W}$. Since $[\mathcal{T}, \mathcal{U}]$ is a $\mathcal{W}$-interval, we have $\mathcal{U} \cap \mathcal{T}^{\perp} = \mathcal{W}$. It follows that $X \in \mathcal{T}^{\perp}$ and therefore that $X' \in \mathcal{T}^{\perp}$, which follows from the fact that a right-perpendicular subcategory is a torsion-free class. But $X' \in \alpha_{\mathsf{T}}(\mathcal{U}) \subseteq \mathcal{U}$, and therefore $X' \in \mathcal{U} \cap \mathcal{T}^{\perp} = \mathcal{W}$. Since $\mathcal{W}$ is wide, we also have $X'' \in \mathcal{W}$ and so $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$ is a Serre subcategory.

For well-definedness of the map in the other direction, let $\mathcal{V} \in \mathrm{wide}(A)$ such that $\mathcal{W} \subseteq \mathcal{V}$ is a Serre subcategory. We have to show that the interval $[\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W}, \mathsf{T}(\mathcal{V})]$ is a $\mathcal{W}$-interval, i.e. that $\mathsf{T}(\mathcal{V}) \cap (\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W})^{\perp} = \mathcal{W}$. Note that since $\mathcal{W} \subseteq \mathcal{V}$, we have $\mathcal{W} \subseteq \mathsf{T}(\mathcal{V})$. Since $\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W} \subseteq {}^{\perp}\mathcal{W}$, we have $\mathcal{W} \subseteq ({}^{\perp}\mathcal{W})^{\perp} \subseteq (\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W})^{\perp}$. Hence $\mathcal{W} \subseteq \mathsf{T}(\mathcal{V}) \cap (\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W})^{\perp}$. Now let $0 \neq X \in \mathsf{T}(\mathcal{V}) \cap (\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W})^{\perp}$. Then $X \notin {}^{\perp}\mathcal{W}$, hence there is a non-zero morphism $f\colon X \to Y$ with $Y \in \mathcal{W}$. The image $\mathrm{im}(f)$ is in $\mathsf{T}(\mathcal{V})$ and Y is in $\mathcal{V} = \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{V}))$, hence $\mathrm{im}(f) \in \mathcal{V}$ by Lemma 1.12. Since $\mathcal{W} \subseteq \mathcal{V}$ is Serre, this implies that $0 \neq \mathrm{im}(f) \in \mathcal{W}$. By definition of α_{T}, we have $\ker(f) \in \mathsf{T}(\mathcal{V})$ and hence with $X' := \ker(f)$ we have a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow \mathrm{im}(f) \longrightarrow 0$$

with $0 \neq \mathrm{im}(f) \in \mathcal{W}$ and $X', X \in \mathsf{T}(\mathcal{V}) \cap (\mathsf{T}(\mathcal{V}) \cap {}^{\perp}\mathcal{W})^{\perp}$. In particular, the length of X' is strictly smaller than the length of X. Hence we can use induction on the length of X to conclude that $X \in \mathcal{W}$.

Note that Proposition 1.16 implies that $\alpha_{\mathsf{T}} \circ \mathsf{T} = \mathrm{id}$. Hence we have $\hat{\alpha} \circ \mathsf{I} = \mathrm{id}$. □

Theorem 2.11 *Let A be a finite dimensional algebra and let $\mathcal{S} \in \mathrm{sbrick}(A)$ be a semibrick. Let $\mathcal{T}, \mathcal{U} \in \mathrm{tors}(A)$ be torsion classes.*

Then $[\mathcal{T}, \mathcal{U}] \in \mathrm{intv}(\mathrm{Filt}(\mathcal{S}))$ if and only if $\mathcal{S} \subseteq \mathcal{U}$, $\mathcal{T} = \mathcal{U} \cap {}^{\perp}\mathcal{S}$ and for all $X \in \mathcal{U}$ and $S \in \mathcal{S}$ every non-zero morphism $f\colon X \to S$ is surjective with $\ker(f) \in \mathcal{U}$.

Proof. We have to show two directions. First, let $[\mathcal{T}, \mathcal{U}] \in \mathrm{intv}(\mathrm{Filt}(\mathcal{S}))$ be a $\mathrm{Filt}(\mathcal{S})$-interval. By Corollary 2.3 we have $\mathcal{T} = \mathcal{U} \cap {}^{\perp}\mathcal{S}$. Let $X \in \mathcal{U}$ and $S \in \mathcal{S}$ and let $f\colon X \to S$ be a non-zero morphism. The image $\mathrm{im}(f)$ is in $\mathcal{U} \cap \mathsf{F}(\mathcal{S})$. However, we have $\mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{S}) \subseteq {}^{\perp}\mathsf{F}(\mathcal{S})$, therefore $\mathsf{F}(\mathcal{S}) \subseteq (\mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{S}))^{\perp}$ and thus

$$\mathcal{U} \cap \mathsf{F}(\mathcal{S}) \subseteq \mathcal{U} \cap (\mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{S}))^{\perp} = \mathcal{U} \cap \mathcal{T}^{\perp} = \mathrm{Filt}(\mathcal{S}).$$

It follows that $\mathrm{im}(f) = S$, i.e. f is surjective. Now consider the canonical short exact sequence associated to the torsion pair $(\mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{S}), (\mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{S}))^{\perp})$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $X' \in \mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{S})$ and $X'' \in (\mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{S}))^{\perp}$. Then we have $X'' \in \mathcal{U} \cap (\mathcal{U}^{\perp}\mathsf{F}(\mathcal{S}))^{\perp} = \mathrm{Filt}(S)$. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d, "h"] & X \arrow[r] \arrow[d, equal] & X'' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \ker(f) \arrow[r] & X \arrow[r, "f"] & S \arrow[r] & 0
\end{tikzcd}$$

Since $f \neq 0$ and $X'' \in \mathrm{Filt}(\mathcal{S})$, the morphism g is surjective and a morphism in $\mathrm{Filt}(\mathcal{S})$, hence $\ker(g) \in \mathrm{Filt}(\mathcal{S})$. Moreover, the Snake lemma implies that we have the following short exact sequence.

$$0 \longrightarrow X' \longrightarrow \ker(f) \longrightarrow \ker(g) \longrightarrow 0$$

By assumption, $X' \in \mathcal{U}$ and $\ker(g) \in \mathrm{Filt}(\mathcal{S}) \subseteq \mathcal{U}$, hence $\ker(f) \in \mathcal{U}$. For the other direction, note that the assumptions imply that $S \in \alpha_{\mathsf{T}}(\mathcal{U})$ for all $S \in \mathcal{S}$. Since every morphism from $\mathcal{U}$ to $\mathcal{S}$ is surjective, the bricks $S \in \mathcal{S}$ are actually simple objects in the wide subcategory $\alpha_{\mathsf{T}}(\mathcal{U})$, hence $\mathrm{Filt}(\mathcal{S}) \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$ is a Serre subcategory. Now Theorem 2.10 implies that $[\mathcal{T}, \mathcal{U}]$ is a $\mathrm{Filt}(\mathcal{S})$-interval. $\square$

Theorem 2.12 (Minimal and maximal $\mathcal{W}$-intervals) *Let A be a finite dimensional algebra and let $\mathcal{W} \in \mathrm{wide}(A)$ be a wide subcategory.*

The partially ordered set $\mathrm{intv}(\mathcal{W})$ of $\mathcal{W}$-intervals has a unique minimal element given by $[\mathsf{T}(\mathcal{W}) \cap {}^{\perp}\mathcal{W}, \mathsf{T}(\mathcal{W})]$ and a unique maximal element given by $[{}^{\perp}\mathsf{F}(\mathcal{W}), {}^{\perp}(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp})]$.

Proof. The interval $[\mathsf{T}(\mathcal{W}) \cap {}^{\perp}\mathcal{W}, \mathsf{T}(\mathcal{W})]$ is a $\mathcal{W}$-interval by Theorem 2.10. For the interval $[{}^{\perp}\mathsf{F}(\mathcal{W}), {}^{\perp}(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp})]$, note that we have to show that

$$\mathsf{F}(\mathcal{W}) \cap {}^{\perp}(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}) = \mathcal{W}.$$

But k-duality translates this statement to the statement for the interval $[\mathsf{T}(\mathcal{W}) \cap {}^{\perp}\mathcal{W}, \mathsf{T}(\mathcal{W})]$ over the opposite algebra.

To show that these two intervals form the unique minimal and maximal elements, we have to show that for any $\mathcal{W}$-interval $[\mathcal{T}, \mathcal{U}]$ we have

$$\mathsf{T}(\mathcal{W}) \subseteq \mathcal{U} \subseteq {}^{\perp}(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}).$$

The first inclusion is obvious since $\mathcal{U} \cap \mathcal{T}^{\perp} = \mathcal{W}$ implies $\mathcal{W} \subseteq \mathcal{U}$ and therefore $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{U}$. The second inclusion is equivalent to $\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp} \subseteq \mathcal{U}^{\perp}$. Let $X \in \mathcal{U}$ and $Y \in \mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}$. We have to show that $\mathrm{Hom}_A(X, Y) = 0$. Let $f \colon X \to Y$ be a morphism. The image is a factor of X, hence $\mathrm{im}(f) \in \mathcal{U}$. However, it is a submodule of Y, hence $\mathrm{im}(f) \in \mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}$. So

$$\begin{aligned} \mathrm{im}(f) \in \mathcal{U} \cap \mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp} &\subseteq \mathcal{U} \cap (\mathcal{U} \cap {}^{\perp}\mathsf{F}(\mathcal{W}))^{\perp} \cap \mathcal{W}^{\perp} \\ &= \mathcal{U} \cap \mathcal{T}^{\perp} \cap \mathcal{W}^{\perp} \\ &= \mathcal{W} \cap \mathcal{W}^{\perp} \\ &= \{0\}. \end{aligned}$$

Therefore $\mathrm{Hom}_A(X, Y) = 0$, i.e. $\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp} \subseteq \mathcal{U}^{\perp}$. $\square$

Corollary 2.13 *Let A be a finite dimensional algebra and let $\mathcal{W} \in \mathrm{wide}(A)$ be a wide subcategory. Then $|\mathrm{intv}(\mathcal{W})| = 1$ if and only if $\mathsf{T}(\mathcal{W})^{\perp} \subseteq \mathsf{F}(\mathcal{W})$.*

Proof. By Theorem 2.12 we have exactly one $\mathcal{W}$-interval if and only if the minimal and maximal $\mathcal{W}$-intervals coincide. This happens if and only if $\mathsf{T}(\mathcal{W}) = {}^{\perp}(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp})$, which is equivalent to $\mathsf{T}(\mathcal{W})^{\perp} = \mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}$. Note that $\mathcal{W}^{\perp} = \mathsf{T}(\mathcal{W})^{\perp}$, hence the minimal and maximal $\mathcal{W}$-intervals coincide if and only if $\mathcal{T}(\mathcal{W})^{\perp} = \mathsf{F}(\mathcal{W}) \cap \mathsf{T}(\mathcal{W})^{\perp}$ if and only if $\mathcal{T}(\mathcal{W})^{\perp} \subseteq \mathsf{F}(\mathcal{W})$. $\square$

Corollary 2.14 *Let A be a finite dimensional algebra and let $\mathcal{W} \in \mathrm{wide}(A)$ be a Serre subcategory. Then the minimal $\mathcal{W}$-interval is given by $[\{0\}, \mathcal{W})]$ and the maximal $\mathcal{W}$-interval is given by $[{}^{\perp}\mathcal{W}, \mathrm{mod}(A)]$.*

Proof. Serre subcategories are torsion and torsion-free classes, i.e. we have $\mathcal{W} = \mathsf{T}(\mathcal{W}) = \mathsf{F}(\mathcal{W})$. The result follows immediately from Theorem 2.12. $\square$

At this point, we record the following lemma for future reference.

Lemma 2.15 *Let A be a finite dimensional algebra and let $\mathcal{W} \in \mathrm{wide}(A)$ be a wide subcategory in $\mathrm{mod}(A)$. Let $\mathcal{T} \in \mathrm{mod}(A)$ be a torsion class.*
Then $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{T} \subseteq {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big)$ if and only if $\mathcal{T} \cap \mathsf{F}(\mathcal{W}) = \mathcal{W}$.

Proof. Applying the map $- \cap \mathsf{F}(\mathcal{W})$ to the chain $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{T} \subseteq {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big)$ gives

$$\mathcal{T}(\mathcal{W}) \cap \mathsf{F}(\mathcal{W}) \subseteq \mathcal{T} \cap \mathsf{F}(\mathcal{W}) \subseteq {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big) \cap \mathsf{F}(\mathcal{W}).$$

We have $\mathcal{W} \subseteq \mathsf{T}(\mathcal{W}) \cap \mathsf{F}(\mathcal{W})$. By Lemma 1.15 we actually have equality $\mathcal{W} = \mathsf{T}(\mathcal{W}) \cap \mathsf{F}(\mathcal{W})$. Now Theorem 2.12 shows that $\big[{}^{\perp}\mathsf{F}(\mathcal{W}), {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big)\big]$ is a $\mathcal{W}$-interval, hence by definition

$$\mathcal{W} = {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big) \cap \big({}^{\perp}\mathsf{F}(\mathcal{W})\big)^{\perp} = {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big) \cap \mathsf{F}(\mathcal{W}).$$

It follows that $\mathcal{W} \subseteq \mathcal{T} \cap \mathsf{F}(\mathcal{W}) \subseteq \mathcal{W}$, i.e. $\mathcal{W} = \mathcal{T} \cap \mathsf{F}(\mathcal{W})$.
On the other hand, suppose that $\mathcal{W} = \mathcal{T} \cap \mathsf{F}(\mathcal{W})$. Then $\mathcal{W} \subseteq \mathcal{T}$, hence $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{T}$. Let $f \colon X \to Y$ be a morphism in $\mathrm{mod}(A)$ with $X \in \mathcal{T}$ and $Y \in \mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}$. Then the image of f is a factor of X, hence $\mathrm{im}(f) \in \mathcal{T}$. But the image is also a subobject of Y, hence $\mathrm{im}(f) \in \mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}$. Therefore

$$\mathrm{im}(f) \in \mathcal{T} \cap \mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp} = \mathcal{W} \cap \mathcal{W}^{\perp} = \{0\}.$$

So $f = 0$ must be the zero morphism and we have $\mathcal{T} \subseteq {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big)$. $\square$

Example 2.16 We consider the path algebra $A = kQ$ where Q is the linearly oriented A_4-quiver

$$Q = (1 \leftarrow 2 \leftarrow 3 \leftarrow 4).$$

We consider the wide subcategory $\mathcal{W}$ generated by the representations $\begin{smallmatrix}1\end{smallmatrix}$ and $\begin{smallmatrix}3\\2\end{smallmatrix}$, i.e. $\mathcal{W} = \operatorname{Filt}\left(\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}\right)$. The torsion and torsion-free classes generated by $\mathcal{W}$ are given by

$$\mathsf{T}(\mathcal{W}) = \operatorname{add}\left\{\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}\right\} \quad \text{and} \quad \mathsf{F}(\mathcal{W}) = \operatorname{add}\left\{\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}\right\}.$$

The perpendicular categories are given by

$$^{\perp}\mathsf{F}(\mathcal{W}) = {}^{\perp}\mathcal{W} = \operatorname{add}\left\{\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix}\right\} \quad \text{and} \quad \mathsf{T}(\mathcal{W})^{\perp} = \mathcal{W}^{\perp} = \operatorname{add}\left\{\begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix}\right\}.$$

So Theorem 2.12 states that the minimal $\mathcal{W}$-interval is

$$\left[\operatorname{add}\{\begin{smallmatrix}3\end{smallmatrix}\}, \operatorname{add}\left\{\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}\right\}\right]$$

and the maximal $\mathcal{W}$-interval is

$$\left[\operatorname{add}\left\{\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix}\right\}, \operatorname{add}\left\{\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix}\right\}\right].$$

Let $[\mathcal{T}, \mathcal{U}]$ be some other $\mathcal{W}$-interval. We claim that $[\mathcal{T}, \mathcal{U}]$ is either the minimal or the maximal $\mathcal{W}$-interval, i.e. $|\operatorname{intv}(\mathcal{W})| = 2$. To see this, note that $\mathcal{U}$ is bounded between the upper torsion classes of the minimal and maximal $\mathcal{W}$-interval, i.e.

$$\operatorname{add}\left\{\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}\right\} \subseteq \mathcal{U} \subseteq \operatorname{add}\left\{\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix}\right\}.$$

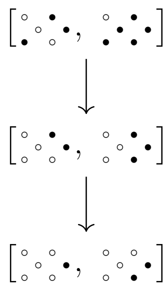
Since torsion classes are closed under factor modules, for each module in a torsion class $\mathcal{U}$ also its top must be in the torsion class $\mathcal{U}$. So if the first inclusion is proper, $\mathcal{U}$ has to contain the simple module $\begin{smallmatrix}4\end{smallmatrix}$, but then the right inclusion becomes an equality since torsion classes also have to be closed under extensions.

Example 2.17 Let $A = k(1 \leftarrow 2 \rightarrow 3)$. We recall the Auslander–Reiten quiver of $\operatorname{mod}(A)$.

$$\begin{tikzcd}
1 \arrow[dr] & & \begin{smallmatrix}2&\\&3\end{smallmatrix} \arrow[dr] & \\
& \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \arrow[ur] \arrow[dr] & & 2 \\
3 \arrow[ur] & & \begin{smallmatrix}&2\\1&\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

We use the shape of the Auslander–Reiten quiver to denote full additive subcategories, e.g. we write $\begin{smallmatrix}\circ&&\bullet&\\&\circ&&\bullet\\\circ&&\bullet&\end{smallmatrix}$ for the torsion class $\operatorname{add}\{\begin{smallmatrix}2&\\&3\end{smallmatrix}, \begin{smallmatrix}&2\\1&\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}\}$. With this notation, the Hasse quiver of torsion classes $\mathsf{Q}(\operatorname{tors}(A))$ with brick labeling is depicted in Figure 2.1. From the complete Hasse quiver with brick labeling, we can read off the Hasse quiver of $\operatorname{Filt}(S)$-intervals for any brick S, since $\operatorname{Filt}(S)$-intervals are precisely arrows in $\mathsf{Q}(\operatorname{tors}(A))$ with label S, cf.

Remark 2.6. For $S := {}_1{}^{2}$, we obtain the following Hasse quiver for intv(Filt(S)).



This quiver is not ℓ-regular for any ℓ, in particular, it is not the Hasse quiver of torsion classes of some finite dimensional algebra A.

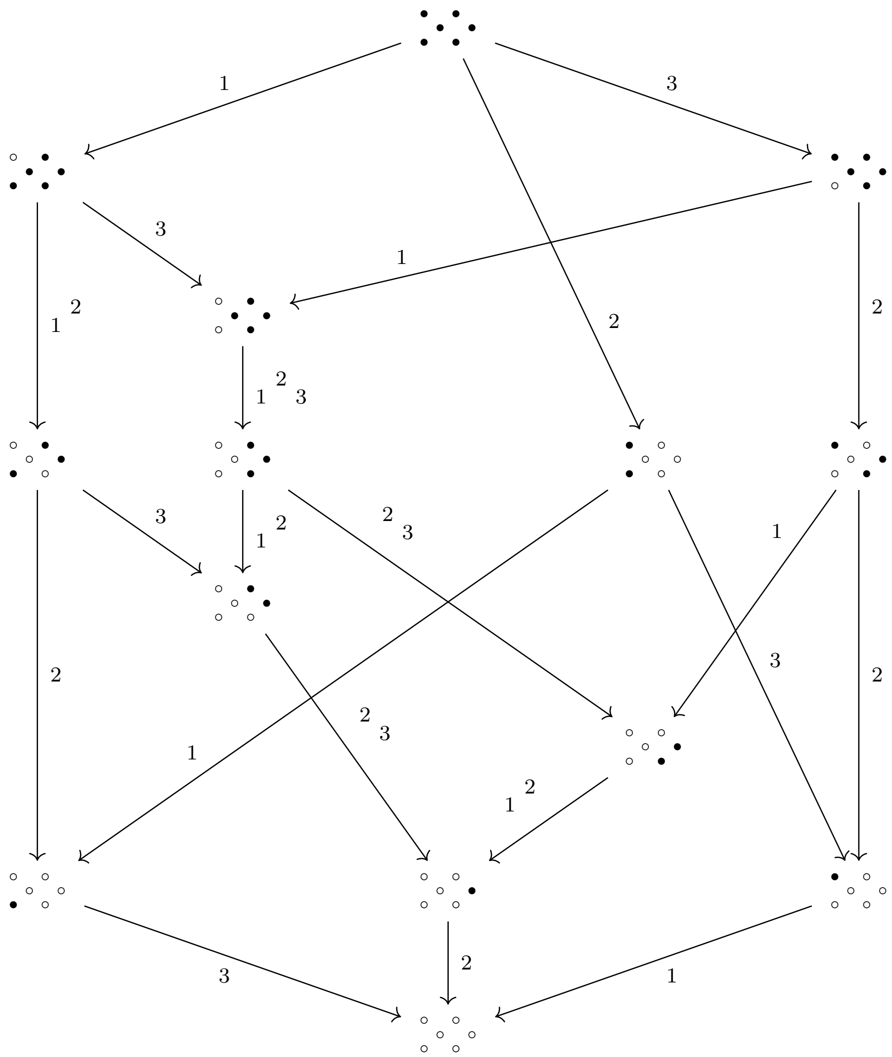


Figure 2.1: The complete Hasse quiver of torsion with brick labeling for $A = k(1 \leftarrow 2 \rightarrow 3)$

Example 2.18 In this example, k is an algebraically closed field. We consider the algebra $A = kQ/I$ given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2 \xleftarrow{\gamma} 3$$

modulo the ideal generated by the admissible relation $\alpha\gamma = 0$. We consider the brick $S := \begin{smallmatrix}3\\2\\1\end{smallmatrix}$.

The dual of this algebra and its Hasse quiver of functorially finite torsion classes have been studied by Asai in [Asa18]. In particular, Asai's observations imply that the brick $S = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ is left-finite, but not right-finite, i.e. the torsion class $\mathsf{T}(S)$ is functorially finite while the torsion-free class $\mathsf{F}(S)$ is not.

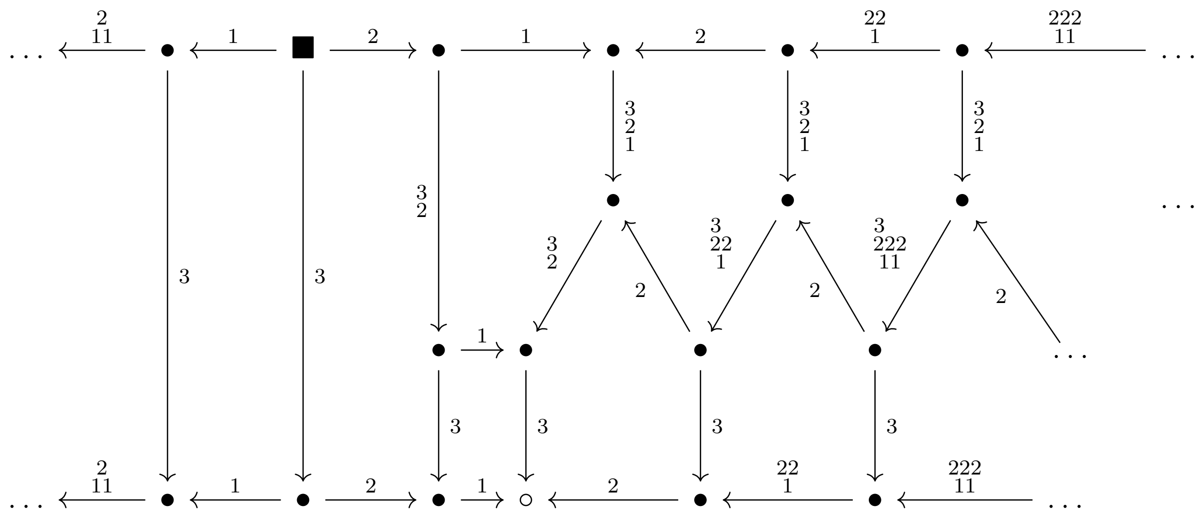
In view of Filt(S)-intervals, this shows that the minimal Filt(S)-interval

$$\left[\mathsf{T}(S) \cap {}^{\perp}S,\, \mathsf{T}(S)\right]$$

is an interval between functorially finite torsion classes while the maximal Filt(S)-interval

$$\left[{}^{\perp}\mathsf{F}(S),\, {}^{\perp}(\mathsf{F}(S) \cap S^{\perp})\right]$$

is not. To further visualize this fact, we take a look at the Hasse quiver of functorially finite torsion free classes with brick labeling below. Recall that Filt(S)-intervals correspond to arrows in this Hasse quiver with brick label $S = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$.



In this picture, the torsion class given by the full module category is denoted by a black square ■, the zero torsion class is denoted by an empty circle ∘.

We can see that the Hasse quiver of Filt(S)-intervals between functorially finite torsion classes forms an increasing unbounded chain. Hence the maximal element is not between functorially finite torsion classes, which also follows from the fact that $\mathsf{F}(S)$ is not functorially finite, i.e. that S is not right-finite.

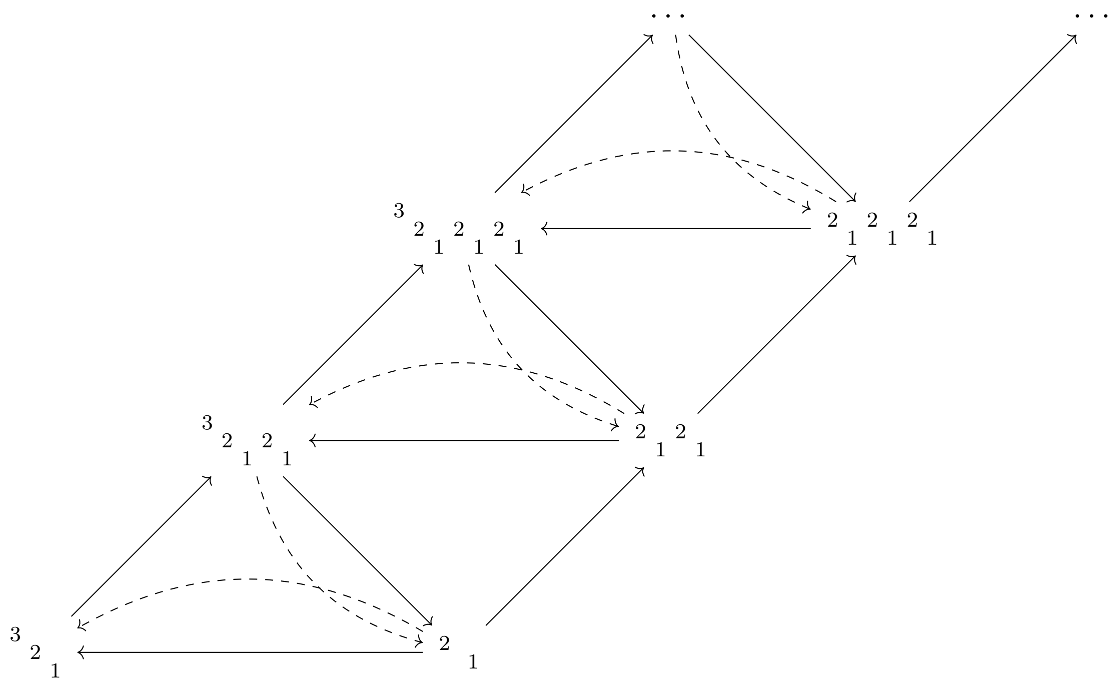
We give a few comments on the Filt(S)-intervals for this algebra between torsion classes that are not functorially finite. First, we focus on the maximal Filt(S)-interval, which according to Theorem 2.12 is given by

$$\left[{}^{\perp}\mathsf{F}(S),\ {}^{\perp}(\mathsf{F}(S) \cap S^{\perp})\right].$$

Note that A is a string algebra. In the following, we freely borrow notation and use results from [BR87].

The submodules of S are the simple module 1 and the string module $M(\beta) = \begin{smallmatrix}2\\1\end{smallmatrix}$. Hence $\mathsf{F}(S) = \mathrm{Filt}(\mathrm{Sub}(S))$ contains all the projectives 1 and $M(\alpha^{-1}\beta) = \begin{smallmatrix}2\\11\end{smallmatrix}$ and thus for all $\ell \geq 1$ the string modules $M((\alpha^{-1}\beta)^{\ell}) = \begin{smallmatrix}2\cdots2\\11\cdots1\end{smallmatrix}$, which form the τ^{-1}-orbits of the two projectives in $\mathsf{F}(S)$.

Moreover, as $\mathsf{F}(S)$ contains both $S = M(\beta\gamma) = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ and $M(\beta) = \begin{smallmatrix}2\\1\end{smallmatrix}$ it contains for all $\ell \geq 1$ the string modules $M(\beta(\alpha^{-1}\beta)^{\ell}\gamma) = \begin{smallmatrix}3\\2\cdots2\\1\cdots1\end{smallmatrix}$ and $M(\beta(\alpha^{-1}\beta)^{\ell}) = \begin{smallmatrix}2\cdots2\\1\cdots1\end{smallmatrix}$. These indecomposable modules form a component of the Auslander–Reiten quiver of A.



The dashed arrows indicate the action of the Auslander–Reiten translation τ. In particular, this component is stable under τ^{-1}. So $\mathsf{F}(S)$ consists of the τ^{-1}-translates of all projective modules, which sit in two irreducible components of the Auslander–Reiten quiver of A.

The orthogonal category ${}^{\perp}\mathsf{F}(S)$ is the torsion class that contains all injectives and their τ-translates (i.e. the preinjectives) as well as the homogeneous tubes with mouths given by the band modules with band $\alpha\beta^{-1}$ and the string module $M(\alpha)$. The torsion-free class $\mathsf{F}(S) \cap \mathsf{T}(S)^{\perp}$ contains everything in $\mathsf{F}(S)$ but the modules $\begin{smallmatrix}3\\2\cdots2\\1\cdots1\end{smallmatrix}$, so the orthogonal category $^{\perp}(\mathsf{F}(S) \cap \mathsf{T}(S)^{\perp})$ contains everything in $^{\perp}\mathsf{F}(S)$ and also the brick $S = \substack{3\\2\\1}$. So the maximal Filt(S)-interval is given by

$$\left[\operatorname{add}\left(\{\text{preinjectives}\} \cup \left\{\begin{matrix}\text{homogeneous tubes over} \\ \text{band modules and } M(\alpha)\end{matrix}\right\}\right),\right.$$
$$\left.\operatorname{add}\left(\{\text{preinjectives}\} \cup \left\{\begin{matrix}\text{homogeneous tubes over} \\ \text{band modules and } M(\alpha)\end{matrix}\right\} \cup \left\{\begin{smallmatrix}3&&\\&2&\\&&1\end{smallmatrix}\right\}\right)\right].$$

Note that instead of taking all homogeneous tubes over band modules and $M(\alpha)$, we can take any subset of these tubes in both the upper and lower torsion class and still get a Filt(S)-interval. In this way, we obtain a set of Filt(S)-intervals between torsion classes that are not functorially finite parameterized by the power set $\mathcal{P}(\mathsf{P}^1(k))$ of the projective line $\mathsf{P}^1(k)$.

In the next section, we show how one can use reduction techniques for finite dimensional algebras to show that the unbounded ascending chain of Filt(S)-intervals between functorially finite torsion classes together with this set of intervals parameterized by $\mathcal{P}(\mathsf{P}^1(k))$ is the complete set of Filt(S)-intervals in tors(A).

Proposition 2.19 *Let A be a finite dimensional algebra and let $\mathcal{W}$ be a wide subcategory. Suppose that $[\mathcal{T}, \mathcal{U}]$ is a $\mathcal{W}$-interval in* tors(A). *Then the following hold.*

(1) *If $\mathcal{U}$ is functorially finite, then $\mathcal{T}$ is functorially finite.*

(2) *If $\mathcal{T}$ is functorially finite, then $\mathcal{U}$ is functorially finite.*

Proof. We follow the proof of the similar [DIR$^+$17, Proposition 4.20]. By duality, we only show (1). Let $\mathcal{W} = \text{Filt}(\mathcal{S})$ where $\mathcal{S}$ is the semibrick of simple objects in $\mathcal{W}$. For each $S \in \mathcal{S}$ we have a wide interval $[\mathcal{U} \cap {}^{\perp}S, \mathcal{U}]$ by Theorem 2.10. Using [DIR$^+$17, Theorem 3.3] and [DIJ19, Theorem 1.3] it follows that all torsion classes $\mathcal{U} \cap {}^{\perp}S$ for $S \in \mathcal{S}$ are functorially finite, so by Proposition 1.23 there are basic support τ-tilting modules M_S with an associated projective module P_S such that $\mathcal{U} \cap {}^{\perp}S = \text{Fac}(M_S)$ and $|M_S| + |P_S| = |A|$.

By [DIR$^+$17, Theorem 4.3] the pairs (M_S, P_S) admit a maximal common direct summand (N, Q) with $|N| + |Q| = |A| - |\mathcal{S}|$. Now let $\mathcal{T}(N, Q) = \text{Fac}(N)$ and $\mathcal{U}(N, Q) = {}^{\perp}(\tau N) \cap Q^{\perp}$. As in the proof of [DIR$^+$17, Proposition 4.20], the interval $[\mathcal{T}(N, Q), \mathcal{U}(N, Q)]$ is a $\mathcal{W}$-interval with $\mathcal{U}(N, Q) = \mathcal{U}$, so we have a $\mathcal{W}$-interval $[\mathcal{T}(N, Q), \mathcal{U}]$. However, by Corollary 2.13 this implies that $\mathcal{T} = \mathcal{T}(N, Q) = \text{Fac}(N)$, hence $\mathcal{T}$ is functorially finite. $\square$

Remark 2.20 Let A be a hereditary algebra. By Proposition 1.30, a wide subcategory $\mathcal{W}$ is left-finite if and only if it is right-finite. Hence the minimal $\mathcal{W}$-interval is an interval between functorially finite torsion classes if and only if the maximal $\mathcal{W}$-interval is an interval between functorially finite torsion classes. This is not true for general finite dimensional algebras, as we have seen in Example 2.18.

Chapter 3

$\mathcal{W}$-intervals and localization

Let A be a finite dimensional algebra over some field k. Let $n := |A|$ denote the number of non-isomorphic indecomposable projective A-modules, which is also the number of pairwise non-isomorphic simple A-modules.

In this chapter, we study the relationship between the partially ordered set of $\mathcal{W}$-intervals introduced in the previous chapter and localization of module categories in an abstract category-theoretical sense. We will show that under certain assumptions so called *localization functors* induce isomorphisms between the poset of $\mathcal{W}$-intervals $\operatorname{intv}(\mathcal{W})$ and an interval subset $[\{0\}, \mathcal{U}_{\max}] \subseteq \operatorname{tors}(B)$ of the lattice of torsion classes over some finite dimensional algebra B.

The chapter consists of three sections. In the first section, we consider the classical case of Serre localization. However, we need to assume that the wide subcategory $\mathcal{W}$ is a Serre subcategory. In the second section, we make use of the theory of ring epimorphisms and τ-tilting reduction to get more general, but weaker results. Finally, the third section considers hereditary algebras, for which the situation is more straightforward.

3.1 For Serre subcategories

For our first result, we only consider wide subcategories $\mathcal{W} \subseteq \operatorname{mod}(A)$ that are also Serre subcategories, i.e. $\mathcal{W}$ is also both a torsion and a torsion-free class. By Proposition 2.8, such Serre subcategories are given by $\operatorname{Filt}(\mathcal{S})$ where $\mathcal{S}$ is a subset of the simple A-modules. In particular, we may choose an idempotent $e = e^2 \in A$ such that

$$Ae \twoheadrightarrow Ae/\operatorname{rad}(Ae) \simeq \bigoplus_{S \in \mathcal{S}} S$$

is a projective cover of the direct sum of simple modules in the wide subcategory $\mathcal{W}$.

Our aim is to show Theorem 3.4 below, which gives a bijection between the partially ordered set of $\mathcal{W}$-intervals and the partially ordered set of torsion classes in the module category $\operatorname{mod}((1-e)A(1-e))$ over the corner algebra $(1-e)A(1-e)$.

To prove this result, we make use of the notion of an adjoint pair of functors. In particular, we are interested in a very specific adjoint pair, which appears in the lower right corner of the following canonical recollement of module categories, cf. e.g. [PV14, Example 2.10].

$$\begin{tikzcd}[column sep=huge]
\operatorname{mod}(A/AeA) \arrow[r, "\text{incl.}"] & \operatorname{mod}(A) \arrow[l, bend right, "A/AeA\otimes_A -"'] \arrow[l, bend left, "\operatorname{Hom}_A(A/AeA,-)"] \arrow[r, "eA\otimes_A -"] & \operatorname{mod}(eAe) \arrow[l, bend right, "Ae\otimes_{eAe}-"'] \arrow[l, bend left, "\operatorname{Hom}_{eAe}(eA,-)"]
\end{tikzcd}$$

We identify $\operatorname{mod}(A/AeA)$ via the inclusion functor as a full subcategory in $\operatorname{mod}(A)$. We summarize some properties of this adjoint pair in the following proposition.

Proposition 3.1 *Let $e = e^2 \in A$ be an idempotent. Then there is the following adjoint pair of functors.*

$$\begin{tikzcd}[column sep=huge]
\operatorname{mod}(A) \arrow[r, shift left, "eA\otimes_A -"] \arrow[r, phantom, "\perp"] & \operatorname{mod}(eAe) \arrow[l, shift left, "\operatorname{Hom}_{eAe}(eA,-)"]
\end{tikzcd}$$

The left adjoint $eA \otimes_A -$ is exact with kernel $\ker(eA \otimes_A -) = \operatorname{mod}(A/AeA)$ *and the right adjoint* $\operatorname{Hom}_{eAe}(eA, -)$ *is fully faithful. Moreover, the counit of this adjoint pair* $\varepsilon\colon eA \otimes_A \operatorname{Hom}_{eAe}(eA, -) \to \operatorname{id}$ *is a natural isomorphism.*

Proof. This follows e.g. from basic well-known properties of the recollement shown before, see e.g. [PV14]. $\square$

In general, functors whose right adjoint is fully faithful are called *localization functors*, see e.g. [Kra10]. With this terminology, Proposition 3.1 shows the canonical example of an exact localization functor between module categories of finite dimensional algebras.

As the kernel of an exact functor, $\operatorname{mod}(A/AeA) \subseteq \operatorname{mod}(A)$ is a Serre subcategory of $\operatorname{mod}(A)$, so is of the form $\operatorname{Filt}(\mathcal{S})$ where $\mathcal{S}$ is some set of simple A-modules. In fact, we have

$$\operatorname{mod}(A/AeA) = \operatorname{Filt}\{S \in \operatorname{mod}(A) \mid S \text{ is simple with } eS = 0\}.$$

For a torsion class $\mathcal{T} \in \operatorname{tors}(eAe)$ we use the following shorthand notation

$$\mathcal{T}^* := \{X \in \operatorname{mod}(A) \mid eA \otimes_A X \in \mathcal{T}\} \subseteq \operatorname{mod}(A)$$

for its preimage under the functor $eA \otimes_A -\colon \operatorname{mod}(A) \to \operatorname{mod}(eAe)$. Since the tensor product functor $eA \otimes_A -$ is right-exact, the preimage $\mathcal{T}^*$ is a torsion class in $\operatorname{mod}(A)$. As the counit of the adjunction in Proposition 3.1 is a natural isomorphism, the assignment $\mathcal{T} \mapsto \mathcal{T}^*$ defines an injective map $\operatorname{tors}(eAe) \to \operatorname{tors}(A)$.

Let $\eta\colon \operatorname{id} \to \operatorname{Hom}_{eAe}(eA, eA \otimes_A -)$ be the unit of the adjunction. Since the right adjoint is fully faithful, the morphism $eA \otimes_A \eta$ is an isomorphism, cf. e.g. [Kra10, Proposition 2.4.1].

Lemma 3.2 *Let $e = e^2 \in A$ be an idempotent. Let $\mathcal{T} \subseteq \operatorname{mod}(A)$ be a torsion class. The following are equivalent.*

(1) *We have $\mathcal{T} = (eA \otimes_A \mathcal{T})^*$.*

(2) *We have $\operatorname{mod}(A/AeA) \subseteq \mathcal{T}$ and for every surjection $f\colon X \to Y$ with $X \in \mathcal{T}$ and $Y \in \operatorname{mod}(A/AeA)$ we have $\ker(f) \in \mathcal{T}$.*

Proof. (1) ⇒ (2) Note that $\operatorname{mod}(A/AeA) = \ker(eA \otimes_A -) = \{0\}^* \subseteq (eA \otimes_A \mathcal{T})^* = \mathcal{T}$. Let $f\colon X \to Y$ be a surjective morphism with $X \in \mathcal{T}$ and $Y \in \operatorname{mod}(A/AeA)$, i.e. we have a short exact sequence

$$0 \longrightarrow K \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$$

with $K = \ker(f)$. Since $Y \in \operatorname{mod}(A/AeA) = \ker(eA \otimes_A -)$ and $eA \otimes_A -$ is exact we have $eA \otimes_A K \simeq eA \otimes_A X$ in $\operatorname{mod}(eAe)$. But $eA \otimes_A X \in eA \otimes_A \mathcal{T}$ and so is $eA \otimes_A K \in eA \otimes_A \mathcal{T}$. Hence $\ker(f) = K \in (eA \otimes_A \mathcal{T})^* = \mathcal{T}$.

(2) ⇒ (1) We clearly have $\mathcal{T} \subseteq (eA \otimes_A \mathcal{T})^*$. For the converse, let $X \in (eA \otimes_A \mathcal{T})^*$. So there is some $Y \in \mathcal{T}$ such that $eA \otimes_A X \simeq eA \otimes_A Y$. With the unit η_Y we have the following right-exact sequence.

$$Y \xrightarrow{\eta_Y} \operatorname{Hom}_{eAe}(eA, eA \otimes_A Y) \longrightarrow \operatorname{coker}(\eta_Y) \longrightarrow 0$$

As $eA \otimes_A \eta_Y$ is an isomorphism and $eA \otimes_A -$ is exact, we have $\operatorname{coker}(\eta_Y) \in \ker(eA \otimes_A -) = \operatorname{mod}(A/AeA)$. So we have $\operatorname{coker}(\eta_Y) \in \mathcal{T}$ by assumption. Since $Y \in \mathcal{T}$, this implies that $\operatorname{Hom}_{eAe}(eA, eA \otimes_A Y) \in \mathcal{T}$ and therefore also $\operatorname{Hom}_{eAe}(eA, eA \otimes_A X) \in \mathcal{T}$.

With the unit η_X we have the following exact sequence.

$$0 \longrightarrow \ker(\eta_X) \longrightarrow X \xrightarrow{\eta_X} \operatorname{Hom}_{eAe}(eA, eA \otimes_A X) \longrightarrow \operatorname{coker}(\eta_X) \longrightarrow 0$$

As $\operatorname{Hom}_{eAe}(eA, eA \otimes_A X) \in \mathcal{T}$ and $\operatorname{coker}(\eta_X) \in \operatorname{mod}(A/AeA)$, we have $\operatorname{im}(\eta_X) \in \mathcal{T}$ by assumption. Since $eA \otimes_A \eta_X$ is an isomorphism and $eA \otimes_A -$ is exact we have $\ker(\eta_X) \in \operatorname{mod}(A/AeA) \subseteq \mathcal{T}$. Hence X is an extension of $\ker(\eta_X) \in \mathcal{T}$ and $\operatorname{im}(\eta_X) \in \mathcal{T}$, and therefore $X \in \mathcal{T}$. □

In general, the image of a torsion class under an exact functor is not a torsion class, consider for example the inclusion functor $\operatorname{mod}(k) \hookrightarrow \operatorname{mod}(A)$ defined by $k \mapsto M$ where $M \in \operatorname{mod}(A)$ satisfies $\operatorname{Hom}_A(M, M) \simeq k$ and $\operatorname{Ext}^1_A(M, M)$, but M is not simple. However, in the situation of the exact localization functor $eA \otimes_A -$ from Proposition 3.1 we have the following lemma.

Lemma 3.3 *Let $e = e^2 \in A$ be an idempotent. Let $\mathcal{T} \subseteq \operatorname{mod}(A)$ be a torsion class with $\operatorname{mod}(A/AeA) \subseteq \alpha_{\mathsf{T}}(\mathcal{T})$. Then the essential image $eA \otimes_A \mathcal{T}$ of $\mathcal{T}$ under the functor*

$eA \otimes_A -\colon \operatorname{mod}(A) \to \operatorname{mod}(eAe)$ *is a torsion class in* $\operatorname{mod}(eAe)$.

Proof. We have to show that the essential image $eA \otimes_A \mathcal{T}$ is closed under extensions and factor modules.

Suppose that $Y \in eA \otimes_A \mathcal{T}$ and that $f\colon Y \twoheadrightarrow Z$ is a surjection in $\operatorname{mod}(eAe)$. Applying the left-exact right adjoint $\operatorname{Hom}_{eAe}(eA, -)$ we obtain the right-exact sequence

$$\operatorname{Hom}_{eAe}(eA, Y) \xrightarrow{f_*} \operatorname{Hom}_{eAe}(eA, Z) \longrightarrow C \longrightarrow 0 \tag{$*$}$$

with the cokernel $C := \operatorname{coker}(f_*) = \operatorname{coker}(\operatorname{Hom}_{eAe}(eA, f))$. The counit ε of the adjunction from Proposition 3.1 is a natural isomorphism, hence $eA \otimes_A f_* \simeq f$ is surjective. However, $eA \otimes_A -$ is exact, which implies that $eA \otimes_A C \simeq 0$. So $C \in \ker(eA \otimes_A -)$. But the kernel of $eA \otimes_A -$ is given by $\operatorname{mod}(A/AeA)$, hence $C \in \alpha_{\mathsf{T}}(\mathcal{T}) \subseteq \mathcal{T}$ by assumption.

Since $Y \in eA \otimes_A \mathcal{T}$, there is some $X \in \mathcal{T}$ such that $eA \otimes_A X \simeq Y$. Using the unit $\eta\colon \operatorname{id} \to \operatorname{Hom}_{eAe}(eA, eA \otimes_A -)$ of the adjunction we have a right exact sequence

$$X \xrightarrow{\eta_X} \operatorname{Hom}_{eAe}(eA, eA \otimes_A X) \longrightarrow \operatorname{coker}(\eta_X) \longrightarrow 0$$

Because $eA \otimes_A \eta_X$ is an isomorphism, cf. e.g. [Kra10, Proposition 2.4.1], the same argument we used for C above shows that $\operatorname{coker}(\eta_X) \in \mathcal{T}$. But also $X \in \mathcal{T}$, hence $\operatorname{Hom}_{eAe}(eA, eA \otimes_A X) \simeq \operatorname{Hom}_{eAe}(eA, Y) \in \mathcal{T}$. Now in the right-exact sequence $(*)$ both outer terms are in $\mathcal{T}$, hence the middle term $\operatorname{Hom}_{eAe}(eA, Z) \in \mathcal{T}$. Since the counit ε of the adjunction is a natural isomorphism, this shows that $Z \simeq eA \otimes_A \operatorname{Hom}_{eAe}(eA, Z) \in eA \otimes_A \mathcal{T}$, i.e. the essential image $eA \otimes_{\mathcal{T}}$ is closed under factor objects.

It remains to show that $eA \otimes_A \mathcal{T}$ is closed under forming extensions. Suppose that we have a short exact sequence in $\operatorname{mod}(eAe)$

$$0 \longrightarrow Y' \longrightarrow Y \xrightarrow{f} Y'' \longrightarrow 0$$

with $Y', Y'' \in eA \otimes_A \mathcal{T}$. Applying the left-exact right adjoint $\operatorname{Hom}_{eAe}(eA, -)$ gives the exact sequence

$$\operatorname{Hom}_{eAe}(eA, Y') \longrightarrow \operatorname{Hom}_{eAe}(eA, Y) \xrightarrow{f_*} \operatorname{Hom}_{eAe}(eA, Y'') \longrightarrow \operatorname{coker}(f_*) \longrightarrow 0$$

with $f_* = \operatorname{Hom}_{eAe}(eA, f)$. As before, we have that $eA \otimes_A f_* \simeq f$ is surjective and therefore $\operatorname{coker}(f_*) \in \ker(eA \otimes_A -) \subseteq \alpha_{\mathsf{T}}(\mathcal{T})$. Moreover, since $Y'' \in eA \otimes_A \mathcal{T}$ the same argument we used for Y earlier shows that $\operatorname{Hom}_{eAe}(eA, Y'') \in \mathcal{T}$. By the definition of $\alpha_{\mathsf{T}}(\mathcal{T})$ this implies that $\operatorname{im}(f_*) \in \mathcal{T}$. Since $Y' \in eA \otimes_A \mathcal{T}$, we have $\operatorname{Hom}_{eAe}(eA, Y') \in \mathcal{T}$, hence the right-exact sequence

$$\operatorname{Hom}_{eAe}(eA, Y') \longrightarrow \operatorname{Hom}_{eAe}(eA, Y) \longrightarrow \operatorname{im}(f_*) \longrightarrow 0$$

shows that $\operatorname{Hom}_{eAe}(eA, Y) \in \mathcal{T}$. As before, using that the counit ε is a natural isomorphism this implies that $Y \simeq eA \otimes_A \operatorname{Hom}_{eAe}(eA, Y) \in eA \otimes_A \mathcal{T}$, i.e. the essential image $eA \otimes_A \mathcal{T}$ is closed under extensions. □

Let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be a Serre subcategory. As explained earlier, we can find an idempotent $e = e^2 \in A$ such that $Ae \twoheadrightarrow Ae/\operatorname{rad}(Ae) \simeq \bigoplus_{S \in \mathcal{S}} S$ is a projective cover of the direct sum of the simple objects $S \in \mathcal{S}$ in $\mathcal{W}$. On the other hand, every idempotent $e = e^2 \in A$ defines a set of simple modules and therefore a Serre subcategory by taking the direct summands of $Ae/\operatorname{rad}(Ae)$. For each idempotent e also $1-e$ defines an idempotent and from now on we consider the adjoint pair from Proposition 3.1 relative to the idempotent $1-e$. In particular, for a torsion class $\mathcal{T} \subseteq \operatorname{mod}((1-e)A(1-e))$ we write

$$\mathcal{T}^* := \{X \in \operatorname{mod}(A) \mid (1-e)A \otimes_A X \in \mathcal{T}\} \subseteq \operatorname{mod}(A)$$

for the preimage of $\mathcal{T}$ under the tensor product functor $(1-e)A \otimes_A -$. As before, this defines an injective map $\operatorname{tors}((1-e)A(1-e)) \to \operatorname{tors}(A)$, $\mathcal{T} \mapsto \mathcal{T}^*$.

Theorem 3.4 *Let A be a finite dimensional algebra. Let $e = e^2 \in A$ be an idempotent and let $\mathcal{W} \subseteq \operatorname{mod}(A)$ the Serre subcategory spanned by the simple modules obtained by taking the indecomposable direct summands of $Ae/\operatorname{rad}(Ae)$. Then there is the following bijection of partially ordered sets.*

$$\begin{aligned} \operatorname{tors}((1-e)A(1-e)) &\longleftrightarrow \operatorname{intv}(\mathcal{W}) \\ \mathcal{T} &\longmapsto [\mathcal{T}^* \cap {}^{\perp}\mathcal{W}, \mathcal{T}^*] \\ (1-e)A \otimes_A \mathcal{U} &\longleftarrow [\mathcal{T}, \mathcal{U}] \end{aligned}$$

Proof. First note that

$$\begin{aligned} \mathcal{W} &= \operatorname{Filt}\{S \in \operatorname{mod}(A) \text{ simple} \mid eS \neq 0\} \\ &= \operatorname{Filt}\{S \in \operatorname{mod}(A) \text{ simple} \mid (1-e)S = 0\} \\ &= \ker((1-e)A \otimes -) \\ &= \operatorname{mod}(A/A(1-e)A). \end{aligned}$$

For a $\mathcal{W}$-interval $[\mathcal{T}, \mathcal{U}] \in \operatorname{intv}(\mathcal{W})$, Theorem 2.10 implies that $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$, hence $\operatorname{mod}(A/A(1-e)A) \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$. So Lemma 3.3 shows that $(1-e)A \otimes \mathcal{U}$ is a torsion class in $\operatorname{mod}((1-e)A(1-e))$. Hence the map from right to left $\operatorname{intv}(\mathcal{W}) \to \operatorname{tors}((1-e)A(1-e))$ is well-defined.

Since the counit $\varepsilon\colon (1-e)A \otimes_A \operatorname{Hom}_{(1-e)A(1-e)}((1-e)A, -)$ of the adjunction from Proposition 3.1 is a natural isomorphism, the functor $(1-e)A\otimes -\colon \operatorname{mod}(A) \to \operatorname{mod}((1-e)A(1-e))$ is surjective on objects. This implies that for a torsion class $\mathcal{T} \in \operatorname{tors}((1-e)A(1-e))$ we have

$$((1-e)A \otimes_A \mathcal{T}^*)^* = \mathcal{T}^*,$$

since $\mathcal{T}^*$ is precisely the preimage of $\mathcal{T}$ under the functor $(1-e)A\otimes_A -$. But then Lemma 3.2 shows that $\mathcal{W} = \mathrm{mod}(A/A(1-e)A) \subseteq \mathcal{T}^*$ and that for every surjection $f\colon X \to Y$ with $X \in \mathcal{T}^*$ and $Y \in \mathcal{W}$ we have $\ker(f) \in \mathcal{T}^*$. In particular, if we let $\mathcal{S} \subseteq \mathcal{W}$ be the semibrick of simple objects in $\mathcal{W}$, Theorem 2.11 implies that $[\mathcal{T}^* \cap {}^{\perp}\mathcal{W}, \mathcal{T}^*]$ is a $\mathcal{W} = \mathrm{Filt}(\mathcal{S})$-interval in $\mathrm{mod}(A)$. This shows that the map from left to right is well-defined too.

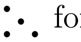
Finally, we have noted earlier that the assignment $\mathcal{T} \to \mathcal{T}^*$ is an injection. Since $\mathcal{W}$-intervals are determined by their upper torsion class, this shows that the map from left to right is an injection. Moreover, Theorem 2.11 together with Lemma 3.2 show that for a $\mathcal{W}$-interval $[\mathcal{T}, \mathcal{U}]$ we have that $((1-e)A \otimes_A \mathcal{U})^* = \mathcal{U}$, which shows that the injective map from left to right is also a retraction, so in particular a two-sided inverse to the map going from right to left. $\square$

Example 3.5 Consider the path algebra $A := k(1 \leftarrow 2 \leftarrow 3)$. Let $e = e_2$ be the idempotent corresponding to the second vertex. The algebra $B := (1-e)A(1-e)$ is isomorphic to the path algebra of the A_2-quiver, i.e. $B \simeq k(1 \leftarrow 2)$. On indecomposables, the functor $(1-e)A \otimes - \colon \mathrm{mod}(A) \to \mathrm{mod}((1-e)A(1-e))$ is given by

$$(1-e)A \otimes_A {\scriptstyle 1} = {\scriptstyle 1},\ (1-e)A \otimes_A {\scriptstyle\begin{smallmatrix}2\\1\end{smallmatrix}} = {\scriptstyle 1},\ (1-e)A \otimes_A {\scriptstyle\begin{smallmatrix}3\\2\\1\end{smallmatrix}} = {\scriptstyle\begin{smallmatrix}2\\1\end{smallmatrix}},$$
$$(1-e)A \otimes_A {\scriptstyle 2} = 0,\ (1-e)A \otimes_A {\scriptstyle\begin{smallmatrix}3\\2\end{smallmatrix}} = {\scriptstyle 2},\ (1-e)A \otimes_A {\scriptstyle 3} = {\scriptstyle 2}.$$

Recall the shape of the Auslander–Reiten quiver of A, shown in the following diagram.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] & & 2 \arrow[ur] & & 3
\end{tikzcd}$$

As in the previous Example 2.17 we use this shape to denote full additive subcategories, e.g. torsion classes, of $\mathrm{mod}(A)$, i.e. we write  for the full subcategory $\mathrm{add}\left\{ \begin{smallmatrix}3\\2\\1\end{smallmatrix}, {\scriptstyle 2}, \begin{smallmatrix}3\\2\end{smallmatrix}, {\scriptstyle 3} \right\}$. The Hasse quiver $\mathsf{Q}(\mathrm{tors}(A))$ with brick labeling is depicted in Figure 3.1, We have marked the upper torsion class of each $\mathrm{Filt}({\scriptstyle 2})$-interval, i.e. of each arrow with brick label ${\scriptstyle 2}$. Notice that there are precisely five $\mathrm{Filt}({\scriptstyle 2})$-intervals, and we can see that the partially ordered set of $\mathrm{Filt}({\scriptstyle 2})$-intervals is isomorphic to the lattice of torsion classes for a A_2-quiver, i.e. the smallest non-modular lattice N_5.

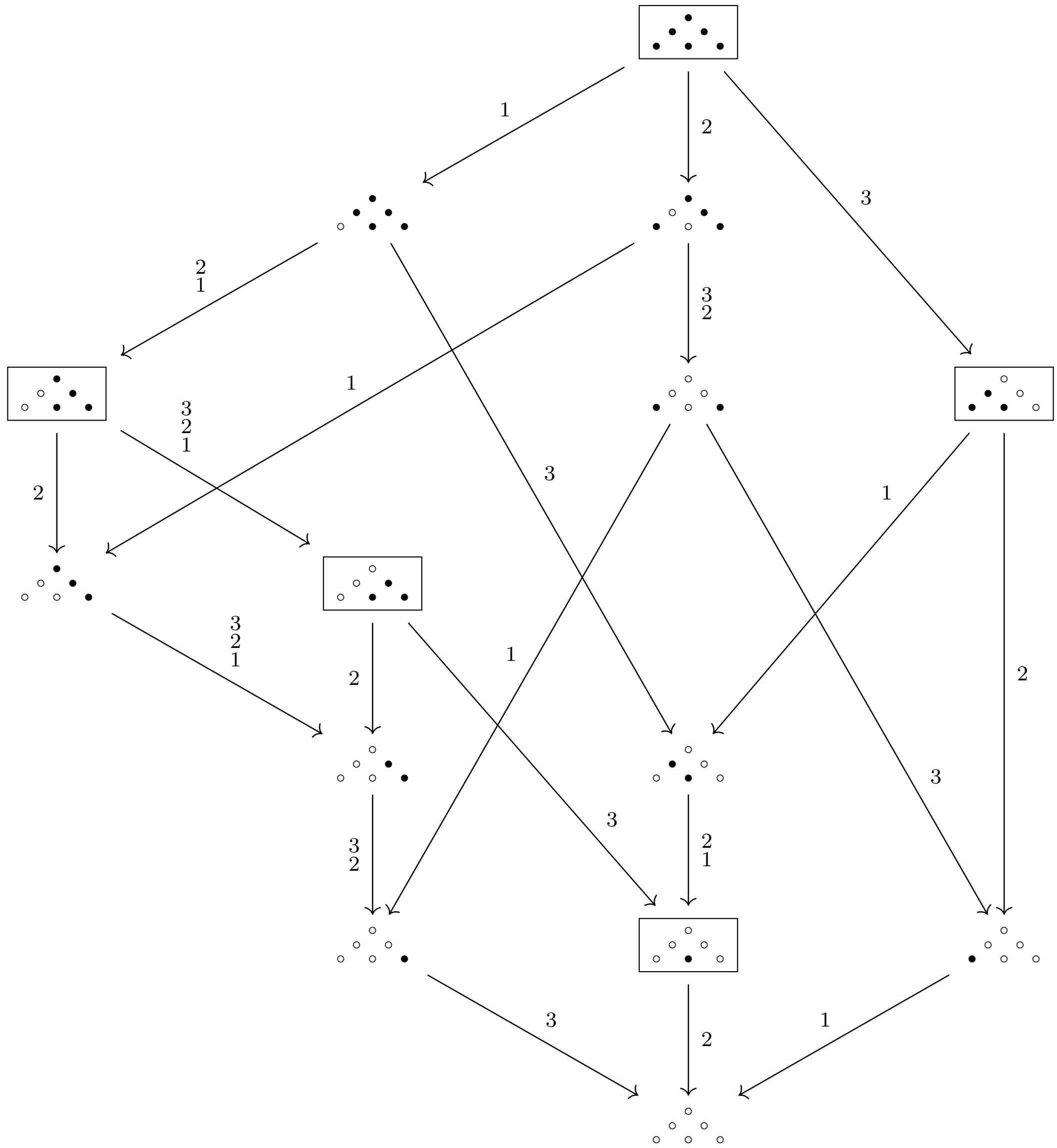


Figure 3.1: The Hasse quiver of torsion classes for $k(1 \leftarrow 2 \leftarrow 3)$ with the upper torsion classes of Filt(2)-intervals in boxes.

3.2 For admissible wide subcategories using reduction

In the previous section we have shown that the partially ordered set of $\mathcal{W}$-intervals for certain wide subcategories in $\mathrm{mod}(A)$ is isomorphic to the lattice of torsion classes over some other algebra B. Of particular importance for the proof has been the existence of an adjoint pair of functors

$$\mathrm{mod}(A) \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathrm{mod}(B)$$

where the right adjoint R is fully faithful, i.e. L is a localization functor. In the previous section, the localization functor L was exact and the wide subcategory $\mathcal{W}$ was its kernel $\ker(L) = \mathcal{W}$, which implied that $\mathcal{W}$ was required to be a Serre subcategory in $\mathrm{mod}(A)$.

The goal in this section is to study the situation in which the right adjoint R is exact, i.e.

R is an exact embedding of module categories $R\colon \operatorname{mod}(B) \hookrightarrow \operatorname{mod}(A)$. In this situation, the localization L is usually not exact, but still right-exact, i.e. its kernel is a torsion class. We will construct an adjoint pair such that the kernel $\ker(L) = \mathsf{T}(\mathcal{W})$ is the torsion class generated by $\mathcal{W}$ and under an additional assumption prove Theorem 3.20, describing the partially ordered set of $\mathcal{W}$-intervals in $\operatorname{mod}(A)$ using the lattice of torsion classes in $\operatorname{mod}(B)$.

We start by introducing some notation and results necessary to define the class of *admissible wide subcategories* for which we can prove our theorem, then construct the adjoint pair using the notion of *ring epimorphisms* and finally prove its properties required for the theorem.

Admissible wide subcategories

Recall the maps $\alpha_{\mathsf{T}}\colon \operatorname{tors}(A) \to \operatorname{wide}(A)$ and $\alpha_{\mathsf{F}}\colon \operatorname{torf}(A) \to \operatorname{wide}(A)$ from torsion(-free) classes to wide subcategories, which are part of the Ingalls–Thomas correspondence, see Definition 1.11.

Definition 3.6 Let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be a wide subcategory. We define the *right complementary wide subcategory* of $\mathcal{W}$ as the wide subcategory

$$\mathcal{W}^{\circ} := \alpha_{\mathsf{F}}\big(\mathsf{T}(\mathcal{W})^{\perp}\big) \subseteq \operatorname{mod}(A)$$

and the *left complementary wide subcategory* of $\mathcal{W}$ as the wide subcategory

$${}^{\circ}\mathcal{W} := \alpha_{\mathsf{T}}\big({}^{\perp}\mathsf{F}(\mathcal{W})\big) \subseteq \operatorname{mod}(A).$$

Recall that a wide subcategory $\mathcal{W}$ is called left-finite (resp. right-finite) if the torsion class $\mathsf{T}(\mathcal{W})$ (resp. the torsion-free class $\mathsf{F}(\mathcal{W})$) generated by $\mathcal{W}$ is functorially finite.

Lemma 3.7 *Let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be a wide subcategory. Then the following hold.*

(1) *If $\mathcal{W}$ is left-finite, then $\mathcal{W}^{\circ}$ is right-finite and we have $\mathcal{W} = {}^{\circ}(\mathcal{W}^{\circ})$.*

(2) *If $\mathcal{W}$ is right-finite, then ${}^{\circ}\mathcal{W}$ is left-finite and we have $\mathcal{W} = ({}^{\circ}\mathcal{W})^{\circ}$.*

Proof. We show (1), as (2) follows by duality.

Since $\mathcal{W}$ is left-finite the torsion class $\mathsf{T}(\mathcal{W})$ is functorially finite by definition. This implies that $\mathsf{T}(\mathcal{W})^{\perp}$ is a functorially finite torsion-free class, cf. [Sma84]. By Proposition 1.23.(2) the map α_{F} sends functorially finite torsion-free classes to right-finite wide subcategories, i.e. $\mathcal{W}^{\circ}$ is right-finite.

In fact, Proposition 1.23.(2) also states that the map α_{F} and F are mutually inverse bijections between the sets of right-finite wide subcategories and functorially finite torsion-free classes, and dually, α_{T} and T are mutually inverse bijections between the sets of left-finite wide subcategories and functorially finite torsion classes. Hence we have

$$\begin{aligned}
{}^{\circ}(\mathcal{W}^{\circ}) &= \alpha_{\mathsf{T}}\big({}^{\perp}\mathsf{F}(\mathcal{W}^{\circ})\big)\\
&= \alpha_{\mathsf{T}}\Big({}^{\perp}\mathsf{F}(\alpha_{\mathsf{F}}\big(\mathsf{T}(\mathcal{W})^{\perp}\big)\Big)\\
&= \alpha_{\mathsf{T}}\big({}^{\perp}\big(\mathsf{T}(\mathcal{W})^{\perp}\big)\big)\\
&= \alpha_{\mathsf{T}}\big(\mathsf{T}(\mathcal{W})\big)\\
&= \mathcal{W}.
\end{aligned}$$

□

Remark 3.8 Let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be a left-finite wide subcategory. It follows from Theorem 2.9 that the simple objects in $\mathcal{W} = \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$ are precisely the brick labels of arrows starting at $\mathsf{T}(\mathcal{W})$ in $\operatorname{tors}(A)$, cf. also Remark 2.6. Looking at the dual of this theorem, we note that the simple objects $\mathcal{W}^{\circ} = \alpha_{\mathsf{F}}(\mathsf{T}(\mathcal{W})^{\perp})$ are the brick labels of arrows ending in $\mathsf{T}(\mathcal{W})$. So we can visualize the relationship between $\mathcal{W}$ and $\mathcal{W}^{\circ}$ as follows.

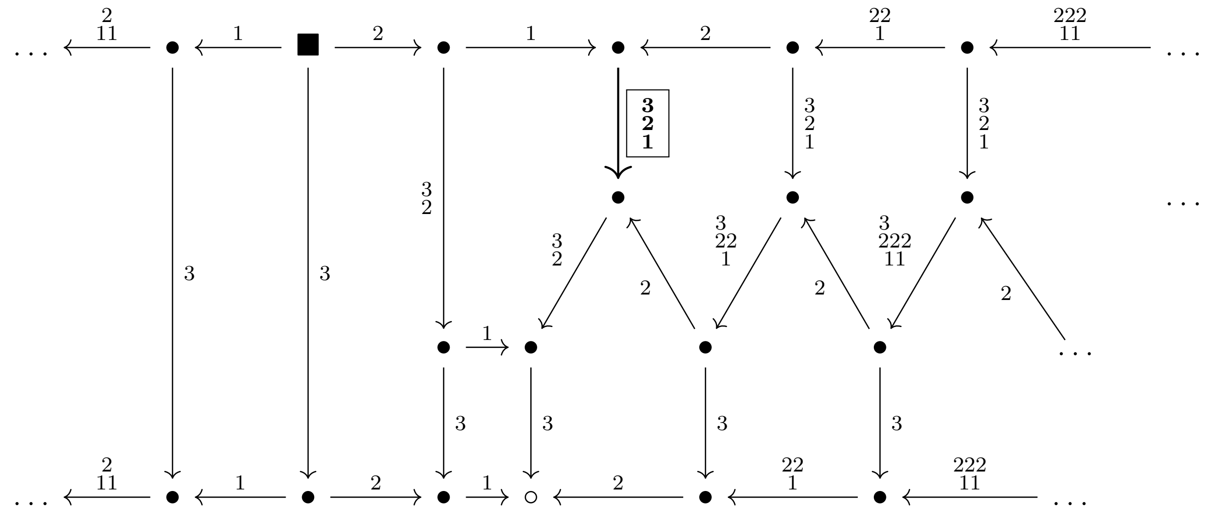
$$\begin{tikzcd}
{} \arrow[dr] & \mathcal{W}^{\circ} & {} \arrow[dl] \\
& \mathsf{T}(\mathcal{W}) \arrow[dl] \arrow[dr] & \\
{} & \mathcal{W} & {}
\end{tikzcd}$$

However, the upper torsion class $\mathcal{U}$ of the $\mathcal{W}^{\circ}$-interval $[\mathsf{T}(\mathcal{W}), \mathcal{U}]$ with lower torsion class given by $\mathsf{T}(\mathcal{W})$ does not need to be $\mathsf{T}(\mathcal{W}^{\circ})$.

For example, take the algebra $A = kQ/I$ with quiver

$$\begin{tikzcd}
1 & 2 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"] & 3 \arrow[l, "\gamma"']
\end{tikzcd}$$

module the admissible ideal I generated by the relation $\alpha\gamma = 0$. We already studied this algebra in Example 2.18. In particular, we showed the Hasse quiver of torsion classes with brick labeling for this algebra, for convenience again depicted below.



Consider in particular the marked arrow with brick label $S = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$, i.e. the $\mathrm{Filt}(S)$-interval given by the two torsion classes incident to the marked arrow. It is easy to see that this is the minimal $\mathrm{Filt}(S)$-interval, i.e. the upper torsion class of this arrow is given by $\mathsf{T}(\mathrm{Filt}(S))$. The complementary wide subcategory $\mathrm{Filt}(S)^{\circ}$ is spanned by the two simple modules 1 and 2. However, the upper torsion class of the $\mathrm{Filt}(1,2)$-interval incident to $\mathsf{T}(\mathrm{Filt}(S))$ is the full module category $\mathrm{mod}(A) \neq \mathsf{T}(\mathrm{Filt}(1,2))$.

For left-finite $\mathcal{W} \in \mathrm{wide}(A)$, the right complementary wide subcategory can also be described using τ-tilting theory, making a connection to τ-tilting reduction in the sense of [Jas15].

Definition 3.9 ([Jas15]) Let $U \in \mathrm{mod}(A)$ be a τ-rigid module, i.e. $\mathrm{Hom}_A(U, \tau U) = 0$. Then the subcategory $U^{\perp} \cap {}^{\perp}(\tau U) \subseteq \mathrm{mod}(A)$ is called the *τ-perpendicular category* with respect to U.

Proposition 3.10 ([Jas15, Theorem 1.4], see also [DIR+17, Theorem 4.12]) *Let A be a finite dimensional algebra and let $U \in \mathrm{mod}(A)$ be a τ-rigid module. Then the τ-perpendicular category $U^{\perp} \cap {}^{\perp}(\tau U)$ is a functorially finite wide subcategory of* $\mathrm{mod}(A)$.

Proposition 3.11 *Let $\mathcal{W} \subseteq \mathrm{mod}(A)$ be a left-finite wide subcategory. Let $U \in \mathrm{mod}(A)$ be the basic τ-rigid module such that $\mathrm{Fac}(U) = \mathsf{T}(\mathcal{W})$ and $|U|$ is minimal. Then one has $\mathcal{W}^{\circ} = U^{\perp} \cap {}^{\perp}(\tau U)$.*

Proof. By Proposition 3.10 the interval $[\mathrm{Fac}(U), {}^{\perp}(\tau U)]$ is an $U^{\perp} \cap {}^{\perp}(\tau U)$-interval. By [DIR+17, Proposition 4.17] there are precisely $|U|$-many arrows in the Hasse quiver $\mathsf{Q}(\mathrm{tors}(A))$ of torsion classes that start at $\mathrm{Fac}(U) = \mathsf{T}(\mathcal{W})$. In the language of [DIR+17], the interval $[\mathrm{Fac}(U), {}^{\perp}(\tau U)]$ is an ℓ-polytope with $\ell = |A| - |U|$. Hence by [DIR+17, Proposition 4.19] the simple objects in $U^{\perp} \cap {}^{\perp}(\tau U)$ are the brick labels of arrows in this interval ending in $\mathsf{T}(\mathcal{W})$. However, since $\mathsf{T}(\mathcal{W})$ is functorially finite, the sum of incoming and outgoing arrows equals $|A|$, hence the simple objects in $U^{\perp} \cap {}^{\perp}(\tau U)$ are the brick labels of all arrows ending in $\mathsf{T}(\mathcal{W})$. But in Remark 3.8 we noticed that these are precisely the simple objects in the right complementary wide subcategory $\mathcal{W}^{\circ}$, hence $U^{\perp} \cap {}^{\perp}(\tau U) = \mathcal{W}^{\circ}$. $\square$

Corollary 3.12 *Let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be a left-finite wide subcategory. Then we have* $\operatorname{Ext}^1_A(\mathcal{W}, \mathcal{W}^{\circ}) = 0$.

Proof. Let $U \in \operatorname{mod}(A)$ be the basic τ-rigid module from Proposition 3.11. Let $X \in \mathcal{W}$. Since $\mathcal{W} \subseteq \mathsf{T}(\mathcal{W}) = \operatorname{Fac}(U)$ and $\mathcal{W} = \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$, cf. Proposition 1.16, we can find $V \in \operatorname{add}(U)$ such that there is a short exact sequence

$$0 \longrightarrow X' \longrightarrow V \longrightarrow X \longrightarrow 0$$

with $X' \in \operatorname{Fac}(U) = \mathsf{T}(\mathcal{W})$. Let $Y \in \mathcal{W}^{\circ} = U^{\perp} = {}^{\perp}(\tau U)$. Applying the functor $\operatorname{Hom}_A(-, Y)$ to the short exact sequence we get the exact sequence

$$\operatorname{Hom}_A(X', Y) \longrightarrow \operatorname{Ext}^1_A(X, Y) \longrightarrow \operatorname{Ext}^1_A(V, Y)$$

Since $X' \in \operatorname{Fac}(U)$ and $Y \in U^{\perp}$, we have $\operatorname{Hom}_A(X', Y) = 0$. Since $Y \in {}^{\perp}(\tau U)$ and $V \in \operatorname{add}(U)$ we have $\operatorname{Hom}_A(Y, \tau V) = 0$. So by Auslander–Reiten duality we have $\operatorname{Ext}^1_A(V, Y) = 0$. Hence it follows from exactness that we have $\operatorname{Ext}^1_A(X, Y) = 0$. $\square$

Definition 3.13 We call a wide subcategory $\mathcal{W} \subseteq \operatorname{mod}(A)$ *admissible* if $\mathcal{W}$ is left-finite and $\operatorname{Ext}^2_A(\mathcal{W}, \mathcal{W}^{\circ}) = 0$.

Clearly, for hereditary algebras all left-finite wide subcategories are admissible. In the particular case of hereditary algebras Proposition 1.30 gives several equivalent characterizations of left-finite wide subcategories.

Ring epimorphisms

A canonical way to obtain localization functors with exact right adjoints is given by ring epimorphisms, i.e. epimorphisms in the category of rings. Let $\varphi\colon A \to B$ be a morphism of rings, to match our usual setup we always assume that φ is a morphism between finite dimensional k-algebras A and B. The morphism φ defines an exact restriction functor $\varphi_*\colon \operatorname{mod}(B) \to \operatorname{mod}(A)$. The restriction functor φ_* is fully faithful if and only if φ is a ring epimorphism. In this case, we usually identify $\operatorname{mod}(B)$ as a full subcategory in $\operatorname{mod}(A)$ via φ_*.

We also need the following equivalence relation on the set of ring epimorphisms starting at a fixed k-algebra A. Two ring epimorphisms $\varphi\colon A \to B$ and $\psi\colon A \to C$ are called *equivalent* if there is a ring isomorphism $\eta\colon B \to C$ such that $\eta \circ \varphi = \psi$.

The important result for us is the following proposition, which appears in this form in [MŠ17, Proposition 4.1], combining results of Geigle–Lenzing [GL91] and Schofield [Sch85], see also [Iya03, Theorem 1.6.1].

Proposition 3.14 *Let A be a finite dimensional algebra over a field k. Then we have the following bijection.*

$$\left\{ \begin{array}{c} \textit{Equivalence classes of ring epim.} \\ \varphi\colon A \to B \textit{ with } \dim(B) < \infty \\ \textit{and } \operatorname{Tor}_1^A(B,B) = 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Functorially finite} \\ \textit{wide subcategories in } \operatorname{mod}(A) \end{array} \right\}$$

$$(\varphi\colon A \to B) \longmapsto \operatorname{mod}(B) \subseteq \operatorname{mod}(A)$$

Note that a ring epimorphism $f\colon A \to B$ allows us to define restriction functors for the categories of left and right modules, so we can consider B as an A-A-bimodule. In particular, the restriction functor becomes representable as $f_* = \operatorname{Hom}_B(B, -)\colon \operatorname{mod}(B) \to \operatorname{mod}(A)$, where we consider $B = {}_BB_A$ as a B-A-bimodule. It follows that a ring epimorphism gives rise to a localization functor $B \otimes_A -\colon \operatorname{mod}(A) \to \operatorname{mod}(B)$, i.e. we have an adjoint pair

$$\begin{tikzcd} \operatorname{mod}(A) \arrow[r, shift left, "B\otimes_A -"] \arrow[r, phantom, "\perp"] & \operatorname{mod}(B) \arrow[l, shift left, "\operatorname{Hom}_B(B,-)"] \end{tikzcd}$$

with exact fully faithful right adjoint $\operatorname{Hom}_B(B, -)$.

In the following proposition, we study this adjoint pair in the context of the right complementary wide subcategory of some wide subcategory $\mathcal{W}$, cf. Definition 3.6.

Proposition 3.15 *Let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be a left-finite wide subcategory. Then there is a finite dimensional algebra B, a ring epimorphism $\varphi\colon A \to B$ and an adjoint pair of functors*

$$\begin{tikzcd} \operatorname{mod}(A) \arrow[r, shift left, "B\otimes_A -"] \arrow[r, phantom, "\perp"] & \operatorname{mod}(B) \arrow[l, shift left, "\operatorname{Hom}_B(B,-)"] \end{tikzcd}$$

such that the restriction functor $f_* = \operatorname{Hom}_B(B, -)$ is fully faithful exact with essential image $\operatorname{im}(f_*) = \mathcal{W}^\circ \subseteq \operatorname{mod}(A)$ given by the right complementary wide subcategory of $\mathcal{W}$. Moreover, for each choice of $\varphi\colon A \to B$ as before, the kernel of the localization functor is given by $\ker(B \otimes_A -) = \mathsf{T}(\mathcal{W})$.

Proof. Since $\mathcal{W}$ is left-finite, the right complementary wide subcategory $\mathcal{W}^\circ$ is right-finite, cf. Lemma 3.7. Since right-finite wide subcategories are also functorially finite, cf. Proposition 1.23.(2), we can find a finite dimensional algebra B with a ring epimorphism $\varphi\colon A \to B$ such that the essential image of the restriction functor f_* equals $\mathcal{W}^\circ$, cf. Proposition 3.14. It remains to show that $\ker(B \otimes_A -) = \mathsf{T}(\mathcal{W})$.

Let $X \in \mathsf{T}(\mathcal{W})$ and let $\eta_X\colon X \to \operatorname{Hom}_B(B, B \otimes_A X)$ be the unit of the adjunction. Since

$\mathrm{Hom}_B(B, B \otimes_A X) \in \mathcal{W}^\circ = \alpha_{\mathsf{F}}(\mathsf{T}(\mathcal{W})^\perp) \subseteq \mathsf{T}(\mathcal{W})^\perp$, we must have $\eta_X = 0$. But $B \otimes_A -$ is a localization functor, so $B \otimes_A \eta_X$ is an isomorphism, cf. e.g. [Kra10, Proposition 2.4.1]. Therefore $B \otimes_A X = 0$.

We have shown that $\mathsf{T}(\mathcal{W}) \subseteq \ker(B \otimes_A -)$. As tensor functors are right-exact, the kernel $\ker(B \otimes_A -)$ is a torsion class in $\mathrm{mod}(A)$. Assume that we have a proper inclusion $\mathsf{T}(\mathcal{W}) \subsetneq \ker(B \otimes_A -)$. By assumption, $\mathsf{T}(\mathcal{W})$ is functorially finite, so $\mathsf{T}(\mathcal{W}) = \mathrm{Fac}(T)$ for some support τ-tilting module $T \in \mathrm{mod}(A)$. By [DIJ19, Theorem 3.1] there is a mutation $T' \in \mathrm{mod}(A)$ of T such that $\mathsf{T}(\mathcal{W}) = \mathrm{Fac}(T) \subsetneq \mathrm{Fac}(T') \subseteq \ker(B \otimes_A -)$. There is no other torsion class between $\mathsf{T}(\mathcal{W})$ and $\mathrm{Fac}(T')$, hence we have $\mathrm{Fac}(T') \cap \mathsf{T}(\mathcal{W})^\perp = \mathrm{Filt}(S)$ for some brick $S \in \mathrm{mod}(A)$ by [DIR+17, Theorem 3.3]. It follows that $[\mathsf{T}(\mathcal{W}), \mathrm{Fac}(T')]$ is a wide interval in $\mathrm{tors}(A)$, or equivalently, $[\mathrm{Fac}(T')^\perp, \mathsf{T}(\mathcal{W})^\perp]$ is a wide interval in $\mathrm{torf}(A)$. By the dual of Theorem 2.9 this implies that $\mathrm{Filt}(S)$ is a Serre subcategory of $\alpha_{\mathsf{F}}(\mathsf{T}(\mathcal{W})^\perp) = \mathcal{W}^\circ$. Hence $S \in \mathcal{W}^\circ$, i.e. S is in the essential image of f_*. But then we must have $B \otimes_A S \neq 0$, contradiction. $\square$

For the remainder of this section, we fix an admissible wide subcategory $\mathcal{W} \subseteq \mathrm{mod}(A)$, cf. Definition 3.13. By the previous Proposition 3.15 we can find a finite dimensional algebra B and a ring epimorphism $\varphi\colon A \to B$ such that there is an adjoint pair

$$\begin{tikzcd} \mathrm{mod}(A) \arrow[r, shift left, "B\otimes_A -"] \arrow[r, phantom, "\perp"] & \mathrm{mod}(B) \arrow[l, shift left, "\mathrm{Hom}_B(B,-)"] \end{tikzcd}$$

with $\mathrm{im}(f_*) = \mathcal{W}^\circ$ and $\ker(B \otimes_A -) = \mathsf{T}(\mathcal{W})$. We also fix the data of B and the ring epimorphism for the remainder of this section.

For a torsion class $\mathcal{T} \in \mathrm{tors}(B)$ in $\mathrm{mod}(B)$ we write

$$\mathcal{T}^* := \{X \in \mathrm{mod}(A) \mid B \otimes_A X \in \mathcal{T}\} \subseteq \mathrm{mod}(A)$$

for the preimage of $\mathcal{T}$ under the localization functor $B \otimes_A -$.

Lemma 3.16 *Let $\mathcal{U} \in \mathrm{tors}(A)$ be a torsion class with $(B \otimes_A \mathcal{U})^* = \mathcal{U}$ and $\mathcal{U} \cap \mathsf{F}(\mathcal{W}) = \mathcal{W}$. Then $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$ and $\mathcal{W}$ is a Serre subcategory of $\alpha_{\mathsf{T}}(\mathcal{U})$.*

Proof. By definition of α_{T}, cf. Definition 1.11, we have to show that for every morphism $f\colon Y \to X$ with $X \in \mathcal{W}$ and $Y \in \mathcal{U}$ also $\ker(f) \in \mathcal{U}$. The image $\mathrm{im}(f)$ is a factor of $Y \in \mathcal{U}$ and a subobject of $X \in \mathcal{W}$, hence $\mathrm{im}(f) \in \mathcal{U} \cap \mathsf{F}(\mathcal{W}) = \mathcal{W}$. So without loss of generality we can assume that f is surjective so that we have a short exact sequence

$$0 \longrightarrow K \longrightarrow Y \xrightarrow{f} X \longrightarrow 0$$

where $K := \ker(f)$. Applying the localization functor $B \otimes_A -$ to this sequence gives the exact sequence

$$\operatorname{Tor}_1^A(B,X) \longrightarrow B\otimes_A K \longrightarrow B\otimes_A Y \longrightarrow B\otimes_A X \longrightarrow 0$$

Since $X \in \mathcal{W} \subseteq \mathsf{T}(\mathcal{W}) = \ker(B\otimes_A -)$, we have $B\otimes_A X = 0$. On the other hand, Tensor-Hom-adjunction gives a natural isomorphism

$$D(B\otimes_A -) \simeq \operatorname{Hom}_A(-, DB),$$

which implies that there is an isomorphism $D\operatorname{Tor}_1^A(B,X) \simeq \operatorname{Ext}_A^1(X,DB)$. However, $X\in\mathcal{W}$ and $DB \in \mathcal{W}^\circ$, hence by Corollary 3.12 we have $\operatorname{Ext}_A^1(X,DB) = 0$. But then exactness shows that we have an isomorphism $B\otimes_A K \simeq B\otimes_A Y \in B\otimes_A \mathcal{U}$. Hence $K = \ker(f) \in (B\otimes_A \mathcal{U})^* = \mathcal{U}$.

It remains to show that $\mathcal{W}$ is a Serre subcategory of $\alpha_{\mathsf{T}}(\mathcal{U})$. Let

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

be a short exact sequence in $\alpha_{\mathsf{T}}(\mathcal{U})$. We have to show that $X\in\mathcal{W}$ if and only if $X'\in\mathcal{W}$ and $X''\in\mathcal{W}$. Since $\mathcal{W}$ is wide, $X', X'' \in \mathcal{W}$ implies that $X\in\mathcal{W}$. For the other direction, if $X\in\mathcal{W}$, then $X'\in\mathsf{F}(\mathcal{W})$, in fact $X' \in \mathsf{F}(\mathcal{W})\cap\mathcal{U}$ since $\alpha_{\mathsf{T}}(\mathcal{U}) \subseteq \mathcal{U}$. But then by assumption $X' \in \mathcal{U}\cap\mathsf{F}(\mathcal{W}) = \mathcal{W}$. This also implies that $X''\in\mathcal{W}$ since $\mathcal{W}$ is wide and hence closed under forming cokernels. □

Lemma 3.17 *Let* $X\in\mathsf{T}(\mathcal{W})$ *with* $\operatorname{Tor}_1^A(B,X) = 0$. *Then* $X\in\mathcal{W}$.

Proof. Let $f\colon Y\to X$ be a morphism in $\mathsf{T}(\mathcal{W})$. Since $\mathcal{W} = \alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W}))$ by Proposition 1.16, it suffices to show that $\ker(f)\in\mathsf{T}(\mathcal{W})$. Since $\mathsf{T}(\mathcal{W})$ is closed under factor objects, we may assume without loss of generality that f is surjective so that we have a short exact sequence

$$0 \longrightarrow K \longrightarrow Y \longrightarrow X \longrightarrow 0$$

with $K = \ker(f)$. We apply the localization functor $B\otimes_A -$ to this sequence and obtain the following exact sequence.

$$\operatorname{Tor}_1^A(B,X) \longrightarrow B\otimes_A K \longrightarrow B\otimes_A Y \longrightarrow B\otimes_A X \longrightarrow 0$$

By assumption, $\operatorname{Tor}_1^A(B,X) = 0$. Since $Y\in\mathsf{T}(\mathcal{W}) = \ker(B\otimes_A -)$, also $B\otimes_A Y = 0$ and therefore by exactness $B\otimes_A K = 0$. In conclusion, we have $K\in\ker(B\otimes_A -) = \mathsf{T}(\mathcal{W})$ and therefore $X\in\alpha_{\mathsf{T}}(\mathsf{T}(\mathcal{W})) = \mathcal{W}$. □

Lemma 3.18 *Let* $\mathcal{U}\in\operatorname{tors}(A)$ *be a torsion class in* $\operatorname{mod}(A)$ *that satisfies* $\mathsf{T}(\mathcal{W})\subseteq\mathcal{U}$. *Then* $\operatorname{Hom}_B(B, B\otimes_A\mathcal{U}) \subseteq \mathcal{U}$ *and* $B\otimes_A\mathcal{U}$ *is a torsion class in* $\operatorname{mod}(B)$.

Proof. Let $X \in \mathcal{U}$ and let $\eta_X\colon X \to \mathrm{Hom}_B(B, B \otimes_A X)$ be the counit of the adjunction from Proposition 3.15. From a standard category-theoretic argument, cf. e.g. [Kra10, Proposition 2.4.1] it follows that $B \otimes_A \eta_X$ is an isomorphism. Now apply the right-exact functor $B \otimes_A -$ to the right-exact sequence

$$X \xrightarrow{\eta_X} \mathrm{Hom}_B(B, B \otimes_A X) \longrightarrow \mathrm{coker}(\eta_X) \longrightarrow 0$$

to obtain a right-exact sequence

$$B \otimes_A X \xrightarrow{B \otimes_A \eta_X} B \otimes_A \mathrm{Hom}_B(B, B \otimes_A X) \longrightarrow B \otimes_A \mathrm{coker}(\eta_X) \longrightarrow 0.$$

But since $B \otimes_A \eta_X$ is an isomorphism, we must have $B \otimes_A \mathrm{coker}(\eta_X) = 0$. So it follows that $\mathrm{coker}(\eta_X) \in \ker(B \otimes_A -) = \mathsf{T}(\mathcal{W})$. Using the assumptions we now have $X \in \mathcal{U}$ and $\mathrm{coker}(\eta_X) \in \mathsf{T}(\mathcal{W}) \subseteq \mathcal{U}$, hence also $\mathrm{Hom}_B(B, B \otimes_A X) \in \mathcal{U}$.

To show that $B \otimes_A \mathcal{U}$ is a torsion class in $\mathrm{mod}(B)$, take a right-exact sequence in $\mathrm{mod}(B)$

$$Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$$

with $Y', Y'' \in B \otimes_A \mathcal{U}$. It suffices to show that $Y \in B \otimes_A \mathcal{U}$. Applying the exact functor $\mathrm{Hom}_B(B, -)$ gives the right-exact sequence

$$\mathrm{Hom}_B(B, Y') \longrightarrow \mathrm{Hom}_B(B, Y) \longrightarrow \mathrm{Hom}_B(B, Y'') \longrightarrow 0$$

where $\mathrm{Hom}_B(B, Y'), \mathrm{Hom}_B(B, Y'') \in \mathrm{Hom}_B(B, B \otimes_A \mathcal{U}) \subseteq \mathcal{U}$. Hence we also have $\mathrm{Hom}_B(B, Y) \in \mathcal{U}$. But the counit $\varepsilon\colon B \otimes_A \mathrm{Hom}_B(B, -) \to \mathrm{id}$ of the adjunction is a natural isomorphism, cf. Proposition 3.15, hence we have $Y \simeq B \otimes_A \mathrm{Hom}_B(B, Y) \in B \otimes_A \mathcal{U}$. It follows that $B \otimes_A \mathcal{U}$ is a torsion class in $\mathrm{mod}(B)$. $\square$

Lemma 3.19 *Let $\mathcal{U} \in \mathrm{tors}(A)$ be a torsion class in $\mathrm{mod}(A)$ that satisfies $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{U}$. Suppose that $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$. Then we have $\mathcal{U} = (B \otimes_A \mathcal{U})^*$.*

Proof. Clearly we have $\mathcal{U} \subseteq (B \otimes_A \mathcal{U})^*$. So let $X \in (B \otimes_A \mathcal{U})^*$, which implies that there is a $Y \in \mathcal{U}$ such that $B \otimes_A X \simeq B \otimes_A Y$. With the counit η_X of the adjoint pair from Proposition 3.15 we have the following exact sequence.

$$0 \longrightarrow \ker(\eta_X) \longrightarrow X \xrightarrow{\eta_X} \mathrm{Hom}_B(B, B \otimes_A X) \longrightarrow \mathrm{coker}(\eta_X) \longrightarrow 0$$

Since $B \otimes_A -$ is right-exact and $B \otimes_A \eta_X$ is an isomorphism, we have $B \otimes_A \mathrm{coker}(\eta_X) = 0$. Hence $\mathrm{coker}(\eta_X) \in \ker(B \otimes_A -) = \mathsf{T}(\mathcal{W})$.

The morphism $\eta_X\colon X \to \mathrm{Hom}_B(B, B \otimes_A X)$ factors over its image $\mathrm{im}(\eta_X)$. As $B \otimes_A -$ is right-exact and therefore preserves surjections, the factorization gives isomorphisms $B \otimes_A X \simeq B \otimes_A \mathrm{im}(\eta_X) \simeq B \otimes_A \mathrm{Hom}_B(B, B \otimes_A X)$. Write $X' := \mathrm{Hom}_B(B, B \otimes_A X)$. If we apply $B \otimes_A -$ to the short exact sequence

$$0 \longrightarrow \operatorname{im}(\eta_X) \longrightarrow \operatorname{Hom}_B(B, B \otimes_A X) \longrightarrow \operatorname{coker}(\eta_X) \longrightarrow 0$$

we obtain the exact sequence

$$\operatorname{Tor}_2^A(B, \operatorname{coker}(\eta_X)) \to \operatorname{Tor}_1^A(B, \operatorname{im}(\eta_X)) \to \operatorname{Tor}_1^A(B, X') \longrightarrow$$

$$\to \operatorname{Tor}_1^A(B, \operatorname{coker}(\eta_X)) \longrightarrow B \otimes_A \operatorname{im}(\eta_X) \longrightarrow B \otimes_A X' \longrightarrow 0.$$

We have $X' \in \mathcal{W}^\circ$ and B is projective in $\mathcal{W}^\circ$, hence $\operatorname{Tor}_1^A(B, X') = 0$. Since the inclusion $\operatorname{im}(\eta_X) \to X' = \operatorname{Hom}_B(B, B \otimes_A X)$ becomes an isomorphism under $B \otimes_A -$, exactness implies that $\operatorname{Tor}_1^A(B, \operatorname{coker}(\eta_X)) = 0$. Now we use Lemma 3.17 to conclude that we have $\operatorname{coker}(\eta_X) \in \mathcal{W}$.

We have $B \otimes_A X \simeq B \otimes_A Y$ with $Y \in \mathcal{U}$, and therefore by Lemma 3.18

$$X' = \operatorname{Hom}_B(B, B \otimes_A X) \simeq \operatorname{Hom}_B(B, B \otimes_A Y) \in \operatorname{Hom}_B(B, B \otimes_A \mathcal{U}) \subseteq \mathcal{U}.$$

Since $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$ by assumption, this implies that $\operatorname{im}(\eta_X) \in \mathcal{U}$. Now we use that $\mathcal{W}$ is admissible, i.e. that $\operatorname{Ext}_A^2(\mathcal{W}, \mathcal{W}^\circ) = 0$. Since $\operatorname{coker}(\eta_X) \in \mathcal{W}$ this implies that $\operatorname{Ext}_A^2(\operatorname{coker}(\eta_X), DB) = 0$. Using the natural isomorphism $D(B \otimes_A -) \simeq \operatorname{Hom}_B(-, DB)$ coming from Tensor-Hom-adjunction we obtain an isomorphism $D \operatorname{Tor}_2^A(B, \operatorname{coker}(\eta_X)) \simeq \operatorname{Ext}_A^2(\operatorname{coker}(\eta_X), DB) = 0$. Hence exactness implies that also $\operatorname{Tor}_1^A(B, \operatorname{im}(\eta_X))$, as we have already noted earlier that $\operatorname{Tor}_1^A(B, X') = 0$.

Now we turn our attention to the short exact sequence

$$0 \longrightarrow \ker(\eta_X) \longrightarrow X \longrightarrow \operatorname{im}(\eta_X) \longrightarrow 0.$$

Again we apply the functor $B \otimes_A -$ and obtain the following exact sequence.

$$\operatorname{Tor}_1^A(B, \operatorname{im}(\eta_X)) \to B \otimes_A \ker(\eta_X) \to B \otimes_A X \to B \otimes_A \operatorname{im}(\eta_X) \to 0$$

We have noted earlier that the surjection $X \to \operatorname{im}(\eta_X)$ becomes an isomorphism under $B \otimes_A -$. Moreover, we have seen that $\operatorname{Tor}_1^A(B, \operatorname{im}(\eta_X)) = 0$, hence $B \otimes_A \ker(\eta_X) = 0$. Therefore $\ker(\eta_X) \in \ker(B \otimes_A -) = \mathsf{T}(\mathcal{W})$. By assumption, $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{U}$, so it follows that both $\ker(\eta_X) \in \mathcal{U}$ and $\operatorname{im}(\eta_X) \in \mathcal{U}$ hold. We conclude that $X \in \mathcal{U}$. $\square$

Theorem 3.20 *Let A be a finite dimensional algebra and let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be an admissible wide subcategory. Let $\mathcal{W}^\circ \subseteq \operatorname{mod}(A)$ be the right complementary wide subcategory of $\mathcal{W}$. Let B be a finite dimensional algebra with ring epimorphism $\varphi \colon A \to B$ such that the essential image of φ equals $\mathcal{W}^\circ$. Write $\mathcal{U}_{\max} := {}^\perp\big(\mathsf{F}(\mathcal{W}) \cap \mathsf{T}(\mathcal{W})^\perp\big) \in \operatorname{tors}(A)$.*

Then the following maps define mutually inverse bijections of partially ordered sets.

$$\begin{aligned}\big\{\mathcal{T} \in \operatorname{tors}(B) \mid \mathcal{T} \subseteq B \otimes_A \mathcal{U}_{\max}\big\} &\longleftrightarrow \operatorname{intv}(\mathcal{W})\\ \mathcal{T} &\longmapsto [\mathcal{T}^* \cap {}^{\perp}\mathcal{W}, \mathcal{T}^*]\\ B \otimes_A \mathcal{U} &\longleftarrow [\mathcal{T}, \mathcal{U}]\end{aligned}$$

Here we write $\mathcal{T}^* := \{X \in \operatorname{mod}(A) \mid B \otimes_A X \in \mathcal{T}\}$ *for the preimage of a torsion class* $\mathcal{T} \in \operatorname{tors}(B)$ *under the functor* $B \otimes_A -\colon \operatorname{mod}(A) \to \operatorname{mod}(B)$.

Proof. First we note that $B \otimes_A -$ is a localization functor, which means that the counit $\varepsilon\colon B \otimes_A \operatorname{Hom}_B(B, -) \to \operatorname{id}$ is an isomorphism. This implies that the functor $B \otimes_A -$ induces a surjection on isomorphism classes of objects. In particular, it follows that for a torsion class $\mathcal{T} \in \operatorname{tors}(B)$ we have $(B \otimes_A \mathcal{T}^*)^* = \mathcal{T}^*$ and that the assignment $\mathcal{T} \to \mathcal{T}^*$ from torsion classes in $\operatorname{mod}(B)$ to torsion classes in $\operatorname{mod}(A)$ is injective.

Secondly, by Theorem 2.12 the interval $[{}^{\perp}\mathsf{F}(\mathcal{W}), {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big)]$ is a $\mathcal{W}$-interval in $\operatorname{mod}(A)$, note that $\mathcal{U}_{\max} = {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big)$. Hence we have $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{U}_{\max}$ and $\mathcal{W}$ is a Serre subcategory of $\alpha_{\mathsf{T}}(\mathcal{U}_{\max})$ by Theorem 2.9. So Lemma 3.19 implies that $(B \otimes_A \mathcal{U}_{\max})^* = \mathcal{U}_{\max}$.

Now we show well-definedness of both maps. Let $\mathcal{T} \in \operatorname{tors}(B)$ be a torsion class with $\mathcal{T} \subseteq B \otimes_A \mathcal{U}_{\max}$. To show well-definedness, we must verify that $[\mathcal{T}^* \cap {}^{\perp}\mathcal{W}, \mathcal{T}^*]$ is a $\mathcal{W}$-interval. In view of Theorem 2.10 it suffices to show that $\mathcal{W}$ is a Serre subcategory of $\mathcal{T}^*$. By the previous remarks we have $(B \otimes_A \mathcal{T}^*)^* = \mathcal{T}^*$. Moreover, $\mathcal{T} \subseteq B \otimes_A \mathcal{U}_{\max}$ implies that $\mathcal{T}^* \subseteq (B \otimes_A \mathcal{U}_{\max})^* = \mathcal{U}_{\max}$. Since $\mathsf{T}(\mathcal{W}) = \ker(B \otimes_A -) = \{0\}^*$ we also have $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{T}^*$. So using Lemma 2.15 we have $\mathcal{T}^* \cap \mathsf{F}(\mathcal{W}) = \mathcal{W}$. But then the assumptions of Lemma 3.16 are satisfied, so $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{T}^*)$ is indeed a Serre subcategory.

On the other hand, let $[\mathcal{T}, \mathcal{U}]$ be a $\mathcal{W}$-interval. By Theorem 2.12 we have $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{U} \subseteq \mathcal{U}_{\max}$. Hence Lemma 3.18 shows that the image $B \otimes_A \mathcal{U}$ is indeed a torsion class in $\operatorname{mod}(B)$ and we have $B \otimes \mathcal{U} \subseteq B \otimes_A \mathcal{U}_{\max}$.

It remains to show that both maps are mutually inverse bijections. First, we already noted that the assignment $\mathcal{T} \mapsto \mathcal{T}^*$ for torsion classes $\mathcal{T} \in \operatorname{tors}(B)$ is injective. Moreover, since for each $\mathcal{W}$-interval $[\mathcal{T}, \mathcal{U}]$ we always have $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{U}$ and $\mathcal{W} \subseteq \alpha_{\mathsf{T}}(\mathcal{U})$, cf. Theorem 2.10, we have that $\mathcal{U} = (B \otimes_A \mathcal{U})^*$ by Lemma 3.19. But $\mathcal{W}$-intervals are uniquely determined by their upper torsion class, cf. Corollary 2.3, so going the map from right to left and then back right composes to the identity. But the map from left to right is injective, hence both maps are mutually inverse bijections. □

Corollary 3.21 *Let* A, $\mathcal{W}$, $\varphi\colon A \to B$ *and* $\mathcal{U}_{\max}$ *as in Theorem* 3.20. *Write*

$$\mathcal{T}_{\max} := \mathsf{T}(\mathcal{W}) * \operatorname{Hom}_B(B, B \otimes_A \mathcal{U}_{\max}).$$

Then the following maps define mutually inverse bijections of partially ordered sets.

$$\begin{array}{rcl} [\mathsf{T}(\mathcal{W}), \mathcal{T}_{\max}] & \longleftrightarrow & \mathrm{intv}(\mathcal{W}) \\ \mathcal{T} & \longmapsto & [(B \otimes_A (\mathcal{W}^{\perp} \cap \mathcal{T}))^* \cap {}^{\perp}\mathcal{W}, (B \otimes_A (\mathcal{W}^{\perp} \cap \mathcal{T}))^*] \\ \mathsf{T}(\mathcal{W}) * \mathrm{Hom}_B(B, B \otimes_A \mathcal{U}) & \longleftarrow\!\shortmid & [\mathcal{T}, \mathcal{U}] \end{array}$$

Proof. The fully faithful exact functor $\mathrm{Hom}_B(B, -)\colon \mathrm{mod}(B) \to \mathrm{mod}(A)$ induces an equivalence between $\mathrm{mod}(B)$ and $\mathcal{W}^{\circ}$, hence a bijection between torsion classes in $\mathrm{mod}(B)$ and $\mathcal{W}^{\circ}$. Since $\mathcal{W}^{\circ} = \alpha_{\mathsf{F}}(\mathsf{T}(\mathcal{W})^{\circ})$, the dual of Theorem 2.9 implies that $[\mathsf{T}(\mathcal{W}), \mathsf{T}(\mathcal{W}) * \mathcal{W}^{\circ}]$ is a $\mathcal{W}^{\circ}$-interval. Now the result follows from Theorem 2.2, which gives a bijection between torsion classes in the interval $[\mathsf{T}(\mathcal{W}), \mathsf{T}(\mathcal{W}) * \mathcal{W}^{\circ}]$ and torsion classes in $\mathcal{W}^{\circ}$. $\square$

Note that in general the interval $[\mathsf{T}(\mathcal{W}), \mathcal{T}_{\max}]$ is not a wide interval.

The following example shows that a bijection like the one in Theorem 3.20 does not hold if we drop the admissibility assumption on the wide subcategory $\mathcal{W} \in \mathrm{wide}(A)$, i.e. if we do not require that $\mathrm{Ext}^2_A(\mathcal{W}, \mathcal{W}^{\circ}) = 0$.

Example 3.22 Let $A = kQ/I$ be the finite dimensional algebra given by the cyclic quiver Q with three vertices

$$\begin{tikzcd} & 2 \arrow[dl, "\alpha"'] & \\ 1 \arrow[rr, "\gamma"] & & 3 \arrow[ul, "\beta"'] \end{tikzcd}$$

modulo the admissible ideal generated by the relation $\gamma\alpha = 0$. The algebra A is a Nakayama algebra of global dimension $\mathrm{gldim}(A) = 2$. As A is Nakayama, all modules are uniserial and uniquely determined by their composition series.

We consider the simple module $\scriptstyle 2$ at the second vertex and the wide subcategory $\mathcal{W} = \mathrm{Filt}({\scriptstyle 2}) = \mathrm{add}({\scriptstyle 2})$ generated by it. Let $e := e_2$ be the primitive idempotent of A corresponding to the second vertex such that we have ${\scriptstyle 2} = Ae/\mathrm{rad}(Ae)$. Hence we can apply Theorem 3.4 and obtain a bijection

$$\mathrm{tors}((1-e)A(1-e)) \simeq \mathrm{intv}(\mathcal{W}).$$

The algebra $(1-e)A(1-e)$ is isomorphic to the endomorphism ring of the projective A-module

$$A(1-e) = \begin{smallmatrix} 1 \\ 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}.$$

We can compute that $(1-e)A(1-e)$ is isomorphic to the algebra $B = kQ'/I'$ given by the cyclic quiver with two vertices

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

modulo the admissible ideal generated by the relation $\alpha\beta = 0$. The algebra B is again a Nakayama algebra of global dimension $\mathrm{gldim}(B) = 2$. The Hasse quiver of torsion classes $\mathsf{Q}(\mathrm{tors}(B))$ in $\mathrm{mod}(B)$ looks as follows.

$$\begin{tikzcd}[column sep=small]
 & \mathrm{mod}(B) \arrow[dl] \arrow[dr] & \\
\mathrm{add}\{ \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix} \} \arrow[d] & & \mathrm{add}\{ \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\end{smallmatrix} \} \arrow[d] \\
\mathrm{add}\{ \begin{smallmatrix}2\end{smallmatrix} \} \arrow[dr] & & \mathrm{add}\{ \begin{smallmatrix}1\end{smallmatrix} \} \arrow[dl] \\
 & \{0\} &
\end{tikzcd}$$

In particular, there are six torsion classes in $\mathrm{mod}(B)$ and therefore six $\mathcal{W}$-intervals in $\mathrm{mod}(A)$.

Since A is a Nakayama algebra, it is representation-finite and therefore all torsion classes are functorially finite. To compute the right-perpendicular wide subcategory, we use the description of $\mathcal{W}^\circ$ as a τ-perpendicular category from Proposition 3.11. We compute that $\tau\, \begin{smallmatrix}2\end{smallmatrix} = \begin{smallmatrix}1\end{smallmatrix}$, in particular, $\begin{smallmatrix}2\end{smallmatrix}$ is τ-rigid, hence we can set $U := \begin{smallmatrix}2\end{smallmatrix}$ and the proposition implies that $\mathcal{W}^\circ = U^\perp \cap {}^\perp(\tau U)$, which equals

$$\begin{smallmatrix}2\end{smallmatrix}^\perp \cap {}^\perp \begin{smallmatrix}1\end{smallmatrix} = \mathrm{add}\left\{ \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}\, \begin{smallmatrix}1\\3\end{smallmatrix} \right\} \cap \mathrm{add}\left\{ \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix} \right\} = \mathrm{add}\left\{ \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix} \right\}.$$

The injective module with socle $\begin{smallmatrix}3\end{smallmatrix}$ is given by $\begin{smallmatrix}1\\3\end{smallmatrix}$, hence we have $\Omega^{-1}(\begin{smallmatrix}3\end{smallmatrix}) = \begin{smallmatrix}1\end{smallmatrix}$ for the first cosyzygy of $\begin{smallmatrix}3\end{smallmatrix}$. Using this, we use dimension shifting to compute

$$\mathrm{Ext}^2_A(\begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}) = \mathrm{Ext}^1_A(\begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\end{smallmatrix}) \neq 0,$$

where the last Ext^1_A is non-zero since there is an arrow from 2 to 1. It follows that $\mathrm{Ext}^2_A(\mathcal{W}, \mathcal{W}^\circ) \neq 0$, i.e. the wide subcategory $\mathcal{W}$ is not admissible. The conclusion of Theorem 3.20 now fails: We have seen that $\mathcal{W}^\circ$ has three indecomposables and $\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ is a projective generator for $\mathcal{W}^\circ$. Its endomorphism ring is isomorphic to the path algebra over a A_2-quiver $C = k(1 \leftarrow 2)$. The Hasse quiver of torsion classes $\mathsf{Q}(\mathrm{tors}(C))$ in $\mathrm{mod}(C)$ looks as follows.

$$\begin{tikzcd}
 & & \mathrm{mod}(C) \arrow[dll] \arrow[ddr] & \\
\mathrm{add}\{\begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}\} \arrow[dd] & & & \\
 & & & \mathrm{add}\{1\} \arrow[ddl] \\
\mathrm{add}\{\begin{smallmatrix}2\end{smallmatrix}\} \arrow[drr] & & & \\
 & & \{0\} &
\end{tikzcd}$$

In particular, there are only five torsion classes in $\mathrm{mod}(C)$. But we have already seen that there are six $\mathcal{W}$-intervals in $\mathrm{mod}(A)$, so we can not find a bijection between the poset of $\mathcal{W}$-intervals and a subposet of the lattice of torsion classes for some finite dimensional algebra whose module category is equivalent to $\mathcal{W}^{\circ}$.

Example 3.23 We return to one of our running examples so far, the algebra $A = kQ/I$ with quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2 \xleftarrow{\gamma} 3$$

modulo the ideal generated by the admissible relation $\alpha\gamma = 0$. We assume that k is algebraically closed for this example. We consider the brick $S := \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ and the wide subcategory $\mathcal{W} = \mathrm{Filt}(S)$ generated by it. In Example 2.18 we have seen that the poset of $\mathcal{W}$-intervals contains one ascending chain and some other component containing the intervals between torsion classes that are not functorially finite. Now we can show that these two components give the full poset $\mathrm{intv}(\mathcal{W})$ of $\mathcal{W}$-intervals by applying Theorem 3.20. The brick S is actually the projective A-module corresponding to the third vertex. In particular, S is τ-rigid and we have $\mathsf{T}(\mathcal{W}) = \mathrm{Fac}(S)$. Using Proposition 3.11 we compute the right complementary subcategory $\mathcal{W}^{\circ}$ as a τ-perpendicular category as follows.

$$\mathcal{W}^{\circ} = S^{\perp} \cap {}^{\perp}(\tau S) = S^{\perp} \cap {}^{\perp}(0) = S^{\perp} = \mathrm{Fac}\left(\begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\,1\end{smallmatrix}\right)$$

In particular, $\mathcal{W}^{\circ}$ is equivalent to the module category of the path algebra over the Kronecker quiver $B = k(1 \leftleftarrows 2)$. As representations of the quiver Q, the representations of $\mathcal{W}^{\circ}$ are the representations for A that are supported only the first and second vertex. In fact, we have a ring epimorphisms $\varphi\colon A \to B$ that simply sends the idempotent at the third vertex to 0 and thereby the image of the restriction functor $\mathrm{im}(\varphi_*) = \mathcal{W}^{\circ}$ equals the right complementary wide subcategory. Note that since $\mathcal{W}$ is projective we clearly have $\mathrm{Ext}^2_A(\mathcal{W}, \mathcal{W}^{\circ}) = 0$, in particular all assumptions of Theorem 3.20 are satisfied.

In Example 2.18, we determined the upper torsion class of the maximal $\mathcal{W}$-interval as

$$\begin{aligned}\mathcal{U}_{\max} &= {}^{\perp}\left(\mathsf{F}(\mathcal{W}) \cap \mathsf{T}(\mathcal{W})^{\perp}\right) \\ &= \operatorname{add}\left(\{\text{preinjectives}\} \cup \begin{Bmatrix}\text{homogeneous tubes over} \\ \text{band modules and } M(\alpha)\end{Bmatrix} \cup \left\{ \begin{smallmatrix} 3 & & \\ & 2 & \\ & & 1 \end{smallmatrix} \right\}\right).\end{aligned}$$

The functor $B \otimes_A -$ sends preinjective A-modules to preinjective B-modules, band modules over A to band modules over B, and finally the modules $M(\alpha)$ and S to the two quasi-simple non-rigid bricks in $\operatorname{mod}(B)$ are not band modules. Hence $B \otimes_A \mathcal{U}_{\max}$ is the torsion class in $\operatorname{mod}(B)$ given by

$$B \otimes_A \mathcal{U}_{\max} = \operatorname{add}\Big(\{\text{preinjective } B\text{-modules}\} \cup \{\text{regular } B\text{-modules}\}\Big).$$

All torsion classes in $\operatorname{mod}(B)$ contained in $B \otimes_A \mathcal{U}_{\max}$ are either of the form $\mathsf{T}(\mathcal{X})$, where $\mathcal{X}$ is some subset of the regular bricks, or of the form $\operatorname{Fac}(I)$ for some preinjective indecomposable module I. The regular bricks are parameterized by the projective line $\mathsf{P}^1(k)$, the preinjective indecompossable modules can be parameterized by the natural numbers, see e.g. the well-known Auslander–Reiten quiver for the Kronecker algebra B. Hence Theorem 3.20 shows indeed that the Hasse quiver $\mathsf{Q}(\operatorname{intv}(\mathcal{W}))$ of $\mathcal{W} = \operatorname{Filt}(S)$-intervals in $\operatorname{mod}(A)$ has the following shape.

$$\begin{tikzcd} \mathsf{Q}(\mathcal{P}(\mathsf{P}^1(k))) \\ \vdots \arrow[d] \\ \bullet \arrow[d] \\ \bullet \arrow[d] \\ \bullet \end{tikzcd}$$

Here we used the notation $\mathcal{P}(\mathsf{P}^1(k))$ for the power set of the projective line.

3.3 For hereditary algebras

In the previous Theorem 3.20 we determined the partially ordered set of $\mathcal{W}$-intervals $\operatorname{intv}(\mathcal{W})$ in the lattice of torsion classes for some finite dimensional algebra A under some additional assumptions on $\mathcal{W}$. One of them was *admissibility* in the sense of Definition 3.13, where we require that $\mathcal{W}$ is left-finite and satisfies $\operatorname{Ext}^2_A(\mathcal{W}, \mathcal{W}^{\circ}) = 0$ with its right complementary $\mathcal{W}^\circ$. If we restrict ourselves to hereditary algebras, the second condition is trivially satisfied. Moreover, we investigated left-finite wide subcategories, or equivalently, semibricks, for hereditary algebras earlier in Proposition 1.30.

We recall the following classical definition of (right-)perpendicular categories for hereditary algebras, which has been studied [GL91] and [Sch91].

Definition 3.24 Let A be a finite dimensional hereditary algebra and let $\mathcal{X} \subseteq \operatorname{mod}(A)$ be some set of objects. Then the *right perpendicular category* with respect to $\mathcal{X}$ is defined as the full subcategory

$$\mathcal{X}^{\perp_{0,1}} := \{Y \in \operatorname{mod}(A) \mid \operatorname{Hom}_A(\mathcal{X}, Y) = \operatorname{Ext}^1_A(\mathcal{X}, Y) = 0\}.$$

Proposition 3.25 ([GL91], [Sch91]) *Let A be a finite dimensional hereditary algebra and let $\mathcal{E} = (E_1, \ldots, E_\ell)$ be an exceptional sequence for A, cf. Definition 1.29.*

Then the perpendicular category $\mathcal{E}^{\perp_{0,1}}$ is equivalent to the module category of a finite dimensional hereditary algebra with $|A| - \ell$ isomorphism classes of simple objects.

Before we can formulate our simplified version of Theorem 3.20 for hereditary algebras, we need to make the following connection between perpendicular categories and right complementary wide subcategories.

Proposition 3.26 *Let A be a finite dimensional hereditary algebra and let $\mathcal{W} \subseteq \operatorname{mod}(A)$ be a functorially finite wide subcategory with set of simple objects $\mathcal{S} \subseteq \mathcal{W}$. Then $\mathcal{W}^\circ = \mathcal{S}^{\perp_{0,1}}$.*

Proof. Since A is hereditary Auslander–Reiten duality gives an isomorphism of vector spaces $\operatorname{Hom}_A(M, \tau N) \simeq \operatorname{Ext}^1_A(N, M)$ for all A-modules $M, N \in \operatorname{mod}(A)$. In particular a module $M \in \operatorname{mod}(A)$ is τ-rigid if and only if it is rigid.

As in Proposition 3.11, let $U \in \operatorname{mod}(A)$ be the rigid module with the minimal number of pairwise non-isomorphic direct summands such that $\operatorname{Fac}(U) = \mathsf{T}(\mathcal{S})$. Then by the proposition we have $\mathcal{W}^\circ = U^\perp \cap {}^\perp(\tau U) = U^{\perp_{0,1}}$. It remains to show that $U^{\perp_{0,1}} = \mathcal{S}^{\perp_{0,1}}$.

By Proposition 1.30, we can arrange the elements of $\mathcal{S}$ into an exceptional sequence. However, using the braid group action on the set of such exceptional sequences, Ringel has shown that there is a rigid module U with $\operatorname{Fac}(U) = \mathsf{T}(\mathcal{S})$ that satisfies $U \in \operatorname{Filt}(\mathcal{S}) = \mathcal{W}$, see [Rin94, Theorem 5]. Hence $\mathcal{S}^{\perp_{0,1}} \subseteq U^{\perp_{0,1}}$.

On the other hand, if $X \in U^{\perp_{0,1}}$, then $X \in \mathcal{S}^\perp = \mathcal{W}^\perp$ since $\mathcal{S} \in \operatorname{Fac}(U)$. Moreover, for each $S \in \mathcal{S}$ we can find some $U' \in \operatorname{add}(U)$ such that there is a surjection $g\colon U' \to S$. But both $U', S \in \mathcal{W}$, hence also $\ker(g) \in \mathcal{W}$ and we have a short sequence

$$0 \longrightarrow \ker(g) \longrightarrow U' \longrightarrow S \longrightarrow 0$$

to which we can apply $\mathrm{Hom}_A(-,X)$ and get the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_A(S,X) \longrightarrow \mathrm{Ext}^1_A(U',X) \longrightarrow \mathrm{Ext}^1_A(\ker(g),X) \longrightarrow 0$$

in which the middle is zero by assumption. Hence $\mathrm{Ext}^1_A(S,X) = 0$. $\square$

Taking these results together, we arrive at the following simplified version of Theorem 3.20 for hereditary algebras.

Corollary 3.27 *Let A be a finite dimensional hereditary algebra. Let $\mathcal{W} \subseteq \mathrm{mod}(A)$ be a functorially finite wide subcategory with set of simple objects $\mathcal{S} \subseteq \mathcal{W}$. Let B be a hereditary algebra with ring epimorphism $\varphi\colon A \to B$ such that $\mathrm{mod}(B) \simeq \mathrm{im}(\varphi_*) = \mathcal{S}^{\perp_{0,1}}$. Let $\mathcal{U}_{\max} = {}^{\perp}\big(\mathsf{F}(\mathcal{W}) \cap \mathcal{W}^{\perp}\big)$.*

Then the following maps define mutually inverse bijections of partially ordered sets.

$$\begin{aligned}
\big\{\mathcal{T} \in \mathrm{tors}(B) \mid \mathcal{T} \subseteq B \otimes_A \mathcal{U}_{\max}\big\} &\longleftrightarrow \mathrm{intv}(\mathcal{W}) \\
\mathcal{T} &\longmapsto [\mathcal{T}^* \cap {}^{\perp}\mathcal{W}, \mathcal{T}^*] \\
B \otimes_A \mathcal{U} &\longleftarrow\!\shortmid [\mathcal{T}, \mathcal{U}]
\end{aligned}$$

Example 3.28 Let k be an algebraically closed field and let $A = kQ$ be the path algebra over the following quiver of type $\tilde{A}_2$.

$$\begin{tikzcd}
 & 2 \arrow[dl] & \\
1 & & 3 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

Note that A is of tame representation type. We consider the regular exceptional module $S = \begin{smallmatrix} 3 & \\ & 1 \end{smallmatrix}$ and the semibrick $\mathcal{S} = \{S\}$. We compute the perpendicular category $\mathcal{S}^{\perp_{0,1}}$ as described in [Sch91]. One calculates that $\dim_k \mathrm{Ext}^1_A(S,A) = 1$ and that we have a short exact sequence

$$0 \longrightarrow \begin{array}{c} 1 \\ \oplus \\ \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix} \\ \oplus \\ \begin{smallmatrix} & 3 & \\ 2 & & 1 \\ 1 & & \end{smallmatrix} \end{array} \longrightarrow \begin{array}{c} 1 \\ \oplus \\ \begin{smallmatrix} & 3 & \\ 2 & & 1 \\ 1 & & \end{smallmatrix} \\ \oplus \\ \begin{smallmatrix} & 3 & \\ 2 & & 1 \\ 1 & & \end{smallmatrix} \end{array} \longrightarrow \begin{smallmatrix} 3 & \\ & 1 \end{smallmatrix} \longrightarrow 0.$$

Hence the results of Schofield imply that $T = 1 \oplus \begin{smallmatrix} & 3 & \\ 2 & & 1 \\ 1 & & \end{smallmatrix}$ is a projective generator in $\mathcal{S}^{\perp_{0,1}}$. We see that $B := \mathrm{End}_A(T)$ is isomorphic to the path algebra over the Kronecker quiver.

We need to determine the torsion class $\mathcal{U}_{\max} = {}^{\perp}(\mathsf{F}(S) \cap S^{\perp}) \subseteq \mathrm{mod}(A)$. Since $S = \begin{smallmatrix} 3 & \\ & 1 \end{smallmatrix}$ is exceptional, $\mathsf{F}(S) = \mathrm{Sub}(S) = \mathrm{add}\{1, \begin{smallmatrix} 3 & \\ & 1 \end{smallmatrix}\}$ and $\mathsf{T}(S) = \mathrm{Fac}(S) = \mathrm{add}\{\begin{smallmatrix} 3 & \\ & 1 \end{smallmatrix}, 3\}$. Therefore

$\mathcal{U}_{\max} = {}^{\perp}\,{}_{1}$. This is a tilting torsion class, the only indecomposable A-module that is not in $\mathcal{U}_{\max}$ is ${}_{1}$. Since ${}_{1} \in S^{\perp_{0,1}}$, we have $B \otimes {}_{1} \simeq {}_{1}$. It follows that $B \otimes \mathcal{U}_{\max}$ contains every indecomposable B-module but $B \otimes {}_{1}$.

The algebra B is the path algebra $k\tilde{Q}$ over the Kronecker quiver

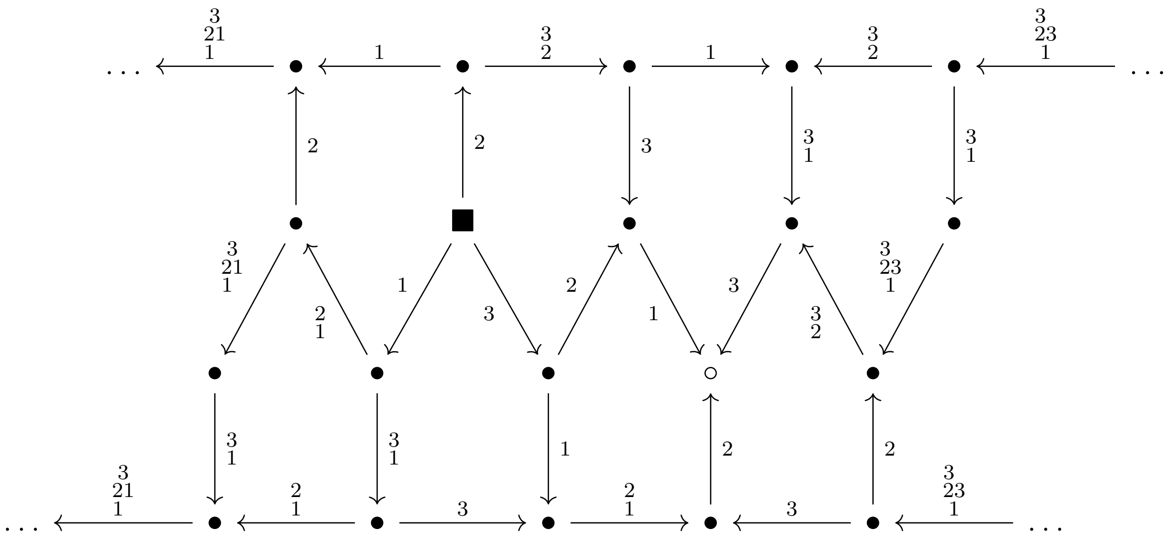
$$\tilde{Q} = \left(1 \leftleftarrows 2 \right).$$

The full Hasse quiver of torsion classes for the Kronecker quiver is shown in e.g. [DIR+17, Example 3.6]. Note that the earlier observations imply that $B \otimes \mathcal{U}_{\max} = \mathrm{Fac}({}_{1}{}^{2}{}_{1})$. Therefore by Corollary 3.27 the Hasse quiver of $\mathrm{Filt}(\mathcal{S})$-intervals $\mathsf{Q}(\mathrm{intv}(\mathrm{Filt}(\mathcal{S})))$ in $\mathrm{tors}(A)$ is isomorphic to the following subquiver of $\mathsf{Q}(\mathrm{tors}(B))$.

$$\mathrm{Fac}({}_{1}{}^{2}{}_{1}) \longrightarrow \mathrm{Fac}({}_{1}{}^{2}{}_{1}{}^{2}{}_{1}) \cdots\cdots \mathcal{R} \cdots\cdots \mathrm{Fac}({}^{2}{}_{1}{}^{2}) \longrightarrow \mathrm{add}({}_{2}) \longrightarrow \{0\}.$$

Here, $\mathcal{R}$ denotes the connected component of $\mathrm{tors}(B)$ coming from the torsion classes that are not functorially finite. These are of the form $\mathsf{T}(\mathcal{X})$, where $\mathcal{X}$ is a subset of the isomorphism classes of non-rigid bricks in $\mathrm{mod}(B)$.

To further illustrate this example, the following diagram shows the Hasse quiver of functorially finite torsion classes $\mathsf{Q}(\text{f-tors}(A))$ for this algebra with brick labeling. As in the previous Example 2.18, we save some space by not using labels for torsion classes.



The maximal torsion class $\mathrm{mod}(A)$ is denoted by a black square, the minimal torsion class $\{0\}$ by an empty circle. The poset of $\mathcal{W}$-intervals between functorially finite torsion class with $\mathcal{W} = \mathrm{Filt}(\substack{3\\1})$ can be identified as a subposet of the above poset by looking at the set of arrows with label $\substack{3\\1}$. We can see the two chains of $\mathcal{W}$-intervals in the bottom left going left and in the top right going right, which agrees with our description of the full poset $\mathrm{intv}(\mathcal{W})$ obtained from Corollary 3.27 earlier.

As an application of Corollary 3.27, we want to consider bricks that appear only finitely many as the brick label of some arrow in the Hasse quiver of torsion classes. More specifically, we give a description of these bricks for path algebra over Euclidean quivers.

Let $Q = (Q_0, Q_1)$ be a Euclidean quiver, i.e. the underlying unoriented graph is of type $\tilde{A}_n$ for $n \geq 1$, $\tilde{D}_n$ for $n \geq 4$ or $\tilde{E}_n$ for $n = 6, 7, 8$. These are precisely the quivers for which the path algebra kQ has tame representation type, see e.g. [Rin84].

Let $\langle -, - \rangle_Q$ denote the Euler form for Q, which for $a, b \in \mathbb{N}Q_0$ is given by the formula

$$\langle a, b \rangle_Q = \sum_{i \in Q_0} a_i b_i - \sum_{(\alpha\colon i \to j) \in Q_1} a_{s(\alpha)} b_{t(\alpha)}.$$

Since Q is Euclidean, there is a unique indivisible dimension vector $\eta \in \mathbb{N}Q_0$ with $\langle a, a \rangle_Q = 0$. A dimension vector $a \in \mathbb{N}Q_0$ is called indivisible if $\gcd\{a_i \mid i \in Q_0\} = 1$. We call η the *null root* of Q.

For a module $M \in \mathrm{mod}(kQ)$ let $|M| \in \mathbb{N}Q_0$ denote its dimension vector. Then it is well-known that for modules $M, N \in \mathrm{mod}(kQ)$ we have

$$\langle |M|, |N| \rangle_Q = \dim_k \mathrm{Hom}_A(M, N) - \dim_k \mathrm{Ext}^1_A(M, N).$$

For details on the theory of tame algebras and associated integral forms we refer to Ringel's book [Rin84].

Let τ denote the Auslander–Reiten translation on $\mathrm{mod}(kQ)$. Recall that an indecomposable module $M \in \mathrm{mod}(kQ)$ is called *preprojective* if $M = \tau^{-\ell} P$ for some projective kQ-module P and $\ell \geq 0$. Dually, it is called *preinjective* if $M = \tau^{\ell} I$ for some injective kQ-module I and $\ell \geq 0$. An indecomposable module that is neither preprojective nor preinjective is called *regular*.

Lemma 3.29 *Let Q be an Euclidean quiver with $n = |Q_0|$ vertices and let k be an algebraically closed field. Let $S \in \mathrm{mod}(kQ)$ be a stone, i.e. a brick S with $\mathrm{Ext}^1_A(S, S) = 0$. Then there is a quiver Q' with $n - 1$ vertices and an equivalence of categories $S^{\perp_{0,1}} \simeq \mathrm{mod}(kQ')$. The quiver Q' is a disjoint union of quivers of finite Dynkin type if and only if S is not regular. Moreover, if S is regular, then there are infinitely many isoclasses of bricks $M \in S^{\perp_{0,1}}$ with $|M| = \eta$ and $\mathrm{Hom}_A(M, S) = 0$.*

Proof. The equivalence of categories $S^{\perp_{0,1}} \simeq \mathrm{mod}(kQ')$ for a quiver Q' with $n - 1$ vertices follows from results of Geigle–Lenzing [GL91], this particular statement was proven by Schofield [Sch91, Theorem 2.3].

Since kQ is of tame representation type and $\mathrm{mod}(kQ')$ is equivalent to an exact full subcategory of $\mathrm{mod}(kQ)$, the path algebra kQ' is either of tame or of finite representation type. Let $s_i \in \mathbb{N}Q_0$ denote the dimension vectors of the simple objects in $S^{\perp_{0,1}} \subseteq \mathrm{mod}(kQ)$.

By [Sch91, Theorem 2.4] there is an additive map given by

$$\begin{array}{rccc} \Phi\colon & \mathbb{N}Q'_0 & \longrightarrow & \mathbb{N}Q_0 \\ & e_i & \longmapsto & s_i \end{array}$$

where $e_i \in \mathbb{N}Q'_0$ denote the dimension vectors of the simple 1-dimensional kQ'-modules. Moreover, the map is an isometry with respect to the Euler form. It follows that kQ' is tame if and only if the null root $\eta \in \mathbb{N}Q_0$ is in the image of Φ.

Suppose that S is regular. The indecomposable kQ-modules decompose into the set of preprojective, regular and preinjective modules. The set of regular modules forms a stable separating tubular $\mathsf{P}_1(k)$-family of some type, cf. [Rin84, Theorem 3.6.(5)]. In particular, S sits at the mouth of some tube of rank $n \geq 2$ and there are infinitely many tubes $\mathcal{T}$ of rank $n = 1$ that satisfy $\operatorname{Hom}_A(S, \mathcal{T}) = \operatorname{Ext}^1_A(S, \mathcal{T}) = 0$ and also $\operatorname{Hom}_A(\mathcal{T}, S) = \operatorname{Ext}^1_A(\mathcal{T}, S) = 0$. The dimension vector of the quasi-simple module at the mouth of each such tube $\mathcal{T}$ is the null root η, hence η is in the image of Φ, i.e. kQ' is tame. Moreover, it follows that there are infinitely many bricks $M \in \operatorname{mod}(kQ)$ of dimension vector $|M| = \eta$ with $\operatorname{Hom}_A(M, S) = 0$.

Now suppose that S is preprojective or preinjective. We only deal with the case when S is preprojective, as the other one is dual. Let $S = \tau^{-\ell}P$ for some projective indecomposable P. Let $\varphi\colon \mathbb{Z}Q_0 \to \mathbb{Z}Q_0$ be the Coxeter matrix for kQ, see [Rin84, §2.4] for the definition and properties. Then $|S| = \Phi^{-\ell}|P|$ and since $\Phi\eta = \eta$ we have

$$\langle |S|, \eta\rangle = \langle \Phi^{-\ell}|P|, \eta\rangle = \langle |P|, \Phi^{\ell}\eta\rangle = \langle |P|, \eta\rangle.$$

However, since Q is Euclidean the null root η has the full quiver as its support. In particular, if $P = P(i)$ is the projective at the i-th vertex, then $\langle |P|, \eta\rangle = \eta_i \neq 0$. It follows that the null root is not in the image of Φ, which consists of the dimension vectors of modules in $S^{\perp_{0,1}}$. Hence Q' must be a disjoint union of quivers of finite Dynkin type. $\square$

Theorem 3.30 *Let Q be an Euclidean quiver and let k be an algebraically closed field. Let $S \in \operatorname{mod}(kQ)$ be a stone, i.e. a brick with $\operatorname{Ext}^1_A(S,S) = 0$.*

Then there are infinitely many arrows in the Hasse quiver $\mathsf{Q}(\operatorname{tors}(kQ))$ of torsion classes in $\operatorname{mod}(kQ)$ with brick label S if and only if S is regular.

Proof. If S is not regular, then $S^{\perp_{0,1}}$ is equivalent to the module category of a representation-finite path algebra, see Lemma 3.29. Hence Corollary 3.27 implies that there are only finitely many $\operatorname{Filt}(S)$-intervals, i.e. arrows in $\mathsf{Q}(\operatorname{tors}(kQ))$ with label S.

If S is regular, then $S^{\perp_{0,1}}$ is equivalent to the module category $\operatorname{mod}(kQ')$ of a path algebra of tame representation type, in particular not of finite type. By Lemma 3.29 there are infinitely many bricks $M \in \operatorname{mod}(kQ)$ of dimension vector $\eta \in \mathbb{N}Q_0$, the null root of Q, that are in $S^{\perp_{0,1}}$ and satisfy $\operatorname{Hom}_A(M,S) = 0$. Hence these bricks are in $\mathcal{U}_{\max} = {}^{\perp}(\mathsf{F}(S) \cap S^{\perp})$. It follows that we have an infinite family of torsion classes given by $\mathsf{T}(S, M)$ for each such brick M and each such torsion class is contained in $\mathcal{U}_{\max}$. As in Theorem 3.27, we have a functor $kQ' \otimes_{kQ} -\colon \operatorname{mod}(kQ) \to \operatorname{mod}(kQ')$ whose right adjoint is the fully faithful exact inclusion $\operatorname{Hom}_{kQ'}(kQ, -)$. Since $M \in S^{\perp_{0,1}}$, we have $\operatorname{Hom}_{kQ'}(kQ, kQ' \otimes_{kQ} M) \simeq M$. Hence the infinite family of torsion classes $\mathsf{T}(S, M) \subseteq \mathcal{U}_{\max}$ gets mapped to the infinite family of torsion classes $\mathsf{T}(kQ' \otimes_{kQ} M) \subseteq kQ' \otimes_{kQ} \mathcal{U}_{\max}$ under the functor $kQ' \otimes_{kQ} -$. But then Corollary 3.27 implies that there are infinitely many $\operatorname{Filt}(S)$-intervals in $\operatorname{tors}(kQ)$, i.e. infinitely many arrows with S as their brick label in the Hasse quiver $\mathsf{Q}(\operatorname{tors}(kQ))$. $\square$

Chapter 4

Maximal green sequences

4.1 Definition, first properties and examples

Definition 4.1 Let A be a finite dimensional algebra. A *forward* Hom-*orthogonal sequence of bricks* for A is a finite sequence of bricks $S_i \in \mathrm{mod}(A)$

$$S_\bullet = (S_1, S_2, \dots, S_m)$$

with $\mathrm{Hom}_A(S_i, S_j) = 0$ for $i < j$. A forward Hom-orthogonal sequence $S_\bullet$ of bricks is called a *maximal green sequence* for A if it is not a proper subsequence of some other forward Hom-orthogonal sequence of bricks for A.

We have the following simple observation.

Proposition 4.2 *Let A be a finite dimensional and let $S_\bullet = (S_1, \dots, S_m)$ be a maximal green sequence for A. Then S_1 and S_m are simple A-modules.*

Proof. Assume that S_1 is not simple. Let $S \in \mathrm{mod}(A)$ be a simple module that is in the top of S_1. Then $\mathrm{Hom}_A(S, S_1) = 0$ since S_1 is a brick. Moreover, we have $\mathrm{Hom}_A(S, S_i) = 0$ for $i > 1$, since $\mathrm{Hom}_A(S_1, S_i) = 0$ for all $i > 1$ and S_1 surjects onto S. Hence $(S, S_1, \dots, S_m)$ is also a forward Hom-orthogonal sequence of bricks, which means $S_\bullet$ is not a maximal green sequence, *contradiction.*

A dual argument shows that S_m also has to be simple. $\square$

Example 4.3 Consider the path algebra over the A_2-quiver $1 \leftarrow 2$. A maximal green sequence has to start and end with a simple module. But the module category $\mathrm{mod}(kQ)$ only has one non-simple module, hence there are two maximal green sequences, namely

$$\left({\scriptstyle 1}\,, {\scriptstyle 2} \right) \quad \text{and} \quad \left({\scriptstyle 2}\,, {\scriptstyle \substack{2\\1}}\,, {\scriptstyle 1} \right).$$

Example 4.4 Now consider the path algebra over the Kronecker quiver, i.e. $1 \leftleftarrows 2$. We still have the maximal green sequence

$$\left({\scriptstyle 1}\, , {\scriptstyle 2} \right),$$

as every non-simple module in kQ must have ${\scriptstyle 1}$ in its socle, so we can put no module between the two simples. However, looking at the well-known Auslander–Reiten quiver for the Kronecker quiver, we see that we can arrange all bricks in $\mathrm{mod}(kQ)$ in a sequence

$$\left({\scriptstyle 2}\, , {\scriptstyle {}^2{}_1{}^2}, \ldots, \ldots, \ldots, {\scriptstyle {}_1{}^2{}_1}\, , {\scriptstyle 1} \right),$$

which starts with all preinjective modules, then all regular bricks, then all preprojective modules opposite to the direction of all arrows in the Auslander–Reiten quiver. Moreover, every finite sequence of bricks with no homomorphisms from left to right will be a subsequence of this infinite sequence. It follows that there is only one maximal green sequence, namely $\left({\scriptstyle 1}\, , {\scriptstyle 2} \right)$.

In the previous example, we have seen a representation-infinite and also τ-tilting infinite finite dimensional algebra where only two bricks appear in maximal green sequences. However, if A is τ-tilting finite, cf. Definition 1.24, each brick of $\mathrm{mod}(A)$ appears in a maximal green sequence.

Proposition 4.5 *Let A be a τ-tilting finite algebra and let $S \in \mathrm{mod}(A)$ be a brick. Then there is a maximal green sequence $S_\bullet = (S_1, \ldots, S_m)$ with $S = S_i$ for some $i \in \{1, \ldots, m\}$.*

Proof. Since A is τ-tilting finite, there are only finitely many isomorphism classes of bricks, cf. Proposition 1.25. Hence the sequence (S) can only be refined finitely many times to a sequence of bricks $(S_1, \ldots, S_m)$ with $S = S_i$ for some i and $\mathrm{Hom}_A(S_i, S_j) = 0$ for $i < j$. If it can not be refined any further, the sequence is a maximal green sequence. □

4.2 Equivalent characterizations

We review a special case of a correspondence between maximal sequences, possibly infinite, of bricks with no non-zero homomorphisms in one direction and maximal chains of torsion classes, which can be found in [DK19, Appendix A]. Since our focus will be on maximal green sequences, which are by definition finite sequences of bricks, we sketch the construction for this correspondence only in this case.

Definition 4.6 Let A be a finite dimensional algebra. A *chain of torsion classes* in $\mathrm{mod}(A)$ is a sequence of torsion classes $\mathcal{T}_\bullet := (\mathcal{T}_0, \mathcal{T}_1, \ldots)$, where $\mathcal{T}_i \in \mathrm{tors}(A)$ for each i, that forms a strictly ascending chain

$$\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \ldots .$$

The set of chains of torsion classes is partially ordered by the usual subsequence relation. A chain of torsion classes will be called *maximal* if it is maximal with respect to the partial order on chains of torsion classes.

The following theorem appears in greater generality in [DK19, Theorem A.3], where also the case of infinite sequences of bricks and torsion classes is considered. However, we will not need this generality here and therefore skip it, as it requires some additional notation and definitions.

Theorem 4.7 (cf.[DK19, Theorem A.3]) *Let A be a finite dimensional algebra. The following defines a bijection between the set of maximal green sequences and the set of finite maximal chains of torsion classes in* $\operatorname{mod}(A)$.

$$(S_1, \dots, S_m) \mapsto \big(\{0\}, \operatorname{Filt}(S_1), \operatorname{Filt}(S_1, S_2), \dots, \operatorname{Filt}(S_1, S_2, \dots, S_{m-1}), \operatorname{mod}(A)\big)$$

The inverse of this map sends a finite maximal chain of torsion classes $(\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_m)$ to the maximal green sequence $(S_1, \dots, S_n)$ where S_i is the unique (isoclass of a) brick in the wide subcategory $\mathcal{T}_i \cap \mathcal{T}_{i-1}^{\perp}$ for $i \in \{1, \dots, m\}$.

Remark 4.8 We make the following remarks.

(1) Since a finite maximal chain of torsion classes has to start with $\{0\}$ and end with $\operatorname{mod}(A)$, both of which are functorially finite, [DIJ19, Theorem 3.1] implies that all torsion classes in a finite maximal chain of torsion classes must be functorially finite.

(2) A finite maximal chain of torsion classes is nothing but a finite (inverse, i.e. against the arrow direction) walk in the Hasse quiver of torsion classes from the minimal element $\{0\}$ to the maximal element $\operatorname{mod}(A)$. To each arrow $\mathcal{T} \to \mathcal{U}$ in the quiver of torsion classes Demonet–Iyama–Reading–Reiten–Thomas associate its *brick label* as the unique brick in $\mathcal{T} \cap \mathcal{U}^{\perp}$, cf. [DIR$^+$17, Theorem 3.3]. In this language, the bijection from Theorem 4.7 maps a maximal chain of torsion classes, i.e. a walk through the Hasse quiver of torsion classes, to the sequence of brick labels of the arrows the walk passes through.

(3) Let $M \in \operatorname{mod}(A)$ be an A-module. Any finite chain of torsion classes in $\operatorname{mod}(A)$

$$\{0\} = \mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \dots \subsetneq \mathcal{T}_m = \operatorname{mod}(A)$$

defines a filtration

$$\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_m = M$$

of M, where $M_j \in \mathcal{T}_j$ for $j \in \{0, \dots, m\}$. The filtration is defined inductively by taking torsion parts: Suppose that the filtration is already defined for all $i \in \{j, \dots, m\}$. Then M_j is defined by the canonical short exact sequence associated to the torsion pair $(\mathcal{T}_{j-1}, \mathcal{T}_{j-1}^{\perp})$

$$0 \longrightarrow M_{j-1} \longrightarrow M_j \longrightarrow M_j' \longrightarrow 0,$$

so that $M_{j-1} \in \mathcal{T}_{j-1}$ and $M'_{j-1} \in \mathcal{T}_j \cap \mathcal{T}_{j-1}^{\perp}$. If the finite chain of torsion classes is maximal, i.e. comes from a maximal green sequence $(S_1, \ldots, S_m)$, this implies that any A-module $M \in \operatorname{mod}(A)$ admits a filtration

$$\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_m = M$$

with subfactors $M_j/M_{j-1} \in \operatorname{Filt}(S_j)$.

Let $\mathcal{T} \in \operatorname{tors}(A)$ be a torsion class. For the following theorem, we need to recall the definition of $\alpha_{\mathsf{T}}(\mathcal{T})$ cf. Definition 1.11.

$$\alpha_{\mathsf{T}}(\mathcal{T}) := \{Y \in \operatorname{mod}(A) \mid \forall (f\colon X \to Y) \in \mathcal{T} : \ker(f) \in \mathcal{T}\}$$

The full subcategory $\alpha_{\mathsf{T}}(\mathcal{T})$ is in fact a wide subcategory of $\operatorname{mod}(A)$, cf. Lemma 1.13.

Theorem 4.9 *Let $S_\bullet = (S_1, \ldots, S_m)$ be a maximal green sequence for a finite dimensional algebra A. Let $(\mathcal{T}_0, \ldots, \mathcal{T}_m)$ be the chain of torsion classes corresponding to $S_\bullet$ as in Theorem 4.7, i.e. we have $\mathcal{T}_i = \operatorname{Filt}(S_1, \ldots, S_i)$. For $i \in \{1, \ldots, m\}$, let $\mathcal{S}(\alpha_{\mathsf{T}}(\mathcal{T}_i))$ denote the set of isoclasses of simple objects in the wide subcategory $\alpha_{\mathsf{T}}(\mathcal{T}_i)$. Then for each $i \in \{1, \ldots, m\}$ we have*

$$S_i \in \mathcal{S}(\alpha_{\mathsf{T}}(\mathcal{T}_i)) \subseteq \{S_1, \ldots, S_i\}.$$

Proof. By Theorem 4.7 we have $\operatorname{Filt}(S_i) = \mathcal{T}_i \cap \mathcal{T}_{i-1}^{\perp}$, in particular, $[\mathcal{T}_{i-1}, \mathcal{T}_i]$ is a wide interval. Now Theorem 2.9 implies that $\operatorname{Filt}(S_i)$ is a Serre subcategory of $\alpha_{\mathsf{T}}(\mathcal{T}_i)$. But then S_i must be a simple object in $\alpha_{\mathsf{T}}(\mathcal{T}_i)$, i.e. we have $S_i \in \mathcal{S}(\alpha_{\mathsf{T}}(\mathcal{T}_i))$, cf. Proposition 2.8.

Now let $S \in \mathcal{S}(\alpha_{\mathsf{T}}(\mathcal{T}_i))$ be a simple object in $\alpha_{\mathsf{T}}(\mathcal{T})$. Then we have $S \in \mathcal{T}_i$ by definition. Let $j \in \{0, \ldots, i-1\}$ be maximal such that $S \notin \mathcal{T}_j$. From the torsion pair $(\mathcal{T}_j, \mathcal{T}_j^{\perp})$ we obtain the short exact sequence

$$0 \longrightarrow S' \longrightarrow S \longrightarrow S'' \longrightarrow 0.$$

with $S' \in \mathcal{T}_j$ and $S'' \in \mathcal{T}_j^{\perp}$. Since $S \notin \mathcal{T}_j$, we have $S'' \neq 0$. But S'' is a factor of $S \in \mathcal{T}_i$, hence also $S'' \in \mathcal{T}_i$.

Let $X \in \mathcal{T}_i$ and let $f\colon X \to S''$ be a morphism. We form the pullback of f along the surjection $S \to S''$ and by adding kernels obtain the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & K \arrow[r, equal] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S' \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & S' \arrow[r] & S \arrow[r] & S'' \arrow[r] & 0
\end{tikzcd}$$

Since $S' \in \mathcal{T}_j \subseteq \mathcal{T}_i$ and $X \in \mathcal{T}_i$, we also have $Y \in \mathcal{T}_j$. Since $S \in \alpha_{\mathsf{T}}(\mathcal{T}_i)$, this implies that $K \in \mathcal{T}_i$ by definition of α_{T}. But then $S'' \in \alpha_{\mathsf{T}}(\mathcal{T}_i)$. However, S is a simple object in the wide subcategory $\alpha_{\mathsf{T}}(\mathcal{T}_i)$ and $S'' \neq 0$ is a non-zero factor, so we must have $S \simeq S''$ and $S' = 0$. In particular, we have $S \in \mathcal{T}_j^{\perp}$.

Since $j < i$ is maximal with $S \notin \mathcal{T}_j$, we have $S \in \mathcal{T}_{j+1}$. But we have just shown that $S \in \mathcal{T}_j^{\perp}$. Hence we have $\operatorname{Filt}(S_{j+1}) = \mathcal{T}_{j+1} \cap \mathcal{T}_j^{\perp} = \operatorname{Filt}(S)$. So the brick S appears in the maximal green sequence with index $j + 1 \leq i$, i.e. $S \in \{S_1, \ldots, S_i\}$. $\square$

Corollary 4.10 *Let $S_\bullet = (S_1, \ldots, S_m)$ be a maximal green sequences for a finite dimensional algebra A. Then all simple A-modules appear in $S_\bullet$.*

Proof. From the definition of α_{T} it is clear that $\alpha_{\mathsf{T}}(\operatorname{mod}(A)) = \operatorname{mod}(A)$, hence the statement follows immediately from Theorem 4.9. $\square$

We also record the following proposition on bricks that appear adjacent to each other in a maximal green sequence. If $S_1, S_2 \in \operatorname{mod}(A)$ are two such bricks, recall that we have $\operatorname{Hom}_A(S_1, S_2) = 0$ by definition.

Proposition 4.11 *Let A be a finite dimensional algebra and suppose we are a given a maximal green sequence $S_\bullet = (S_1, \ldots, S_n)$ for A. Then for all $i \in \{1, \ldots, n-1\}$, any non-zero morphism $f\colon S_{i+1} \to S_i$ is either injective or surjective.*

Proof. With the notation $\mathcal{T}_i := \operatorname{Filt}(S_1, \ldots, S_i)$ from Theorem 4.7 and the subsequent Remark 4.8, we have

$$\operatorname{Filt}(S_i, S_{i+1}) = \mathcal{T}_{i+1} \cap \mathcal{T}_{i-1}^{\perp}.$$

Let $f\colon S_{i+1} \to S_i$ be a non-zero morphism. Then the image $\operatorname{im}(f)$ is a factor of $S_{i+1} \in \mathcal{T}_{i+1}$ and a subobject of $S_i \in \mathcal{T}_{i-1}^{\perp}$. Hence it follows that $\operatorname{im}(f) \in \mathcal{T}_{i+1} \cap \mathcal{T}_{i-1}^{\perp} = \operatorname{Filt}(S_i, S_{i+1})$. However, the length of $\operatorname{im}(f)$ is bounded by the length of S_i and the length of S_{i+1}. So we either have $\operatorname{im}(f) \simeq S_i$ and f is surjective or we have $\operatorname{im}(f) \simeq S_{i+1}$ and f is injective. $\square$

Let A be a finite dimensional algebra that has maximal green sequences. In general, even for the case $A = kQ$ for some acyclic quiver Q, it is not known whether the set of maximal green sequences is always a finite set. Clearly, if A is τ-tilting finite there can only be finitely many maximal green sequences. In [BHIT17, Section 5] the authors show that if $A = kQ$ and Q is an orientation of a Euclidian (i.e. affine) Dynkin diagram or has at most three vertices the set of all maximal green sequences for A is finite.

In the remainder of this section, we give two elementary characterizations of algebras that only have finitely many maximal green sequences.

Proposition 4.12 *Let A be a finite dimensional algebra and suppose that there is an $\ell \in \mathbb{N}$ such that all maximal green sequences $S_\bullet = (S_1, \ldots, S_m)$ for A have length $m \leq \ell$. Then A has only finitely many maximal green sequences.*

Proof. Recall that maximal green sequences $S_\bullet$ are in bijection with paths in the Hasse quiver of functorially finite torsion classes from the maximal elements $\operatorname{mod}(A)$ to the minimal element $\{0\}$, cf. Theorem 4.7 and Remark 4.8. But the Hasse quiver of functorially finite torsion classes is an n-regular quiver, cf. Proposition 1.23, where $n = |A|$ is the number of isomorphism classes of simple modules. In particular, the number of paths starting at the maximal element of length ℓ is bounded by n^ℓ, hence bounded length implies that there are only finitely many maximal green sequences. $\square$

If the field k is algebraically closed, we also have the following characterization of algebras with only finitely many maximal green sequences. The characterization has a certain similarity to the Brauer–Thrall I theorem.

Proposition 4.13 *Let A be a finite dimensional algebra over an algebraically closed field k. Suppose that there is a dimension $d_{\max} \in \mathbb{N}$ such that for all maximal green sequences $S_\bullet = (S_1, \ldots, S_m)$ for A the dimension of each brick is bounded by $\dim(S_i) \leq d_{\max}$ for all $i \in \{1, \ldots, m\}$. Then A has only finitely many maximal green sequences.*

Proof. By Proposition 4.25 below the orbit of a brick S that appears in a maximal green sequence in the module variety $\operatorname{mod}(A, d)$ with of modules with dimension vector $d = \underline{\dim}(S)$ is open, hence for each dimension vector $d \in \mathbb{N}Q_0$ there can only be finitely many bricks S of dimension vector d that appear in maximal green sequences. In particular, if the k-dimension of bricks in maximal green sequences is bounded by $d_{\max}$, only finitely many bricks can appear in maximal green sequences and therefore only finitely many maximal green sequences exist for A. $\square$

4.3 Polygonal deformations and flips

We describe a mutation-like operation on the set maximal green sequences, so called *polygonal deformations.* The notion comes from lattice theory, cf. [Rea16]. In the context of quiver mutation, it was applied in [HI19] to show the no-gap conjecture for tame hereditary algebras, i.e. that the set of lengths of maximal green sequences forms an integer interval. In our context of torsion classes, wide subcategories and bricks polygonal deformations have been studied in [DIR$^+$17], although not applied to maximal green sequences. Hence we focus here only on the application to maximal green sequences in the representation-theoretic setting for arbitrary finite dimensional algebras.

We start with the following redefinition of wide intervals, the connection to the previous Definition 2.1 will be made afterwards.

Definition 4.14 Let A be a finite dimensional algebra and let $S_\bullet = (S_1, \dots, S_m)$ be a maximal green sequence for A. For indices $1 \leq i < j \leq m$ we call the interval subsequence $S_{[i,j]} := (S_i, S_{i+1}, \dots, S_j)$ a *wide interval* if the subcategory

$$\mathrm{Filt}(S_{[i,j]}) := \mathrm{Filt}(S_i, S_{i+1}, \dots, S_j)$$

generated by it is a wide subcategory in $\mathrm{mod}(A)$.

Proposition 4.15 *Let A be a finite dimensional algebra and let $S_\bullet = (S_1, \dots, S_m)$ be a maximal green sequence for A. Let $1 \leq i < j \leq m$ and let*

$$\mathcal{T} := \mathrm{Filt}(S_1, \dots, S_{i-1}) \quad \textit{and} \quad \mathcal{U} := \mathrm{Filt}(S_1, \dots, S_j).$$

Note that $\mathcal{T}$ and $\mathcal{U}$ are torsion classes by Theorem 4.7.

Then $S_{[i,j]} := (S_i, S_{i+1}, \dots, S_j)$ is a wide interval in $S_\bullet$ if and only if $[\mathcal{T}, \mathcal{U}]$ is a wide interval in $\mathrm{tors}(A)$*.*

Proof. Maximal green sequences give rise to a filtration of each module $M \in \mathrm{mod}(A)$ with the S_i as subfactors, cf. Remark 4.8. Hence $\mathcal{U} \cap \mathcal{T}^\perp = \mathrm{Filt}(S_i, S_{i+1}, \dots, S_j) = \mathrm{Filt}(S_{[i,j]})$. So $S_{[i,j]}$ is a wide interval in $S_\bullet$ if and only if $[\mathcal{T}, \mathcal{U}]$ is a wide interval in $\mathrm{tors}(A)$. □

Definition 4.16 Let A be a finite dimensional algebra. A wide interval $[\mathcal{T}, \mathcal{U}]$ in $\mathrm{tors}(A)$ is called a *polygon* if the wide subcategory $\mathcal{U} \cap \mathcal{T}^\perp$ has finitely many torsion classes and only two isomorphism classes of simple objects.

Remark 4.17 Suppose that $[\mathcal{T}, \mathcal{U}]$ is a polygon such that the wide subcategory $\mathcal{W} = \mathcal{U} \cap \mathcal{T}^\perp$ is functorially finite. Then $\mathcal{W}$ is equivalent to the category of finite dimensional modules over a finite dimensional algebra $B_\mathcal{W}$ with two isomorphism classes simple modules. This follows e.g. from [MŠ17, Proposition 4.1].

By Theorem 2.2 the subposet $[\mathcal{T}, \mathcal{U}]$ of $\mathrm{tors}(A)$ is equivalent to the lattice $\mathrm{tors}(B_\mathcal{W})$ of torsion classes in $\operatorname{mod}(B_{\mathcal{W}})$. Since $[\mathcal{T}, \mathcal{U}]$ is a polygon, $B_{\mathcal{W}}$ is τ-tilting finite, cf. Proposition 1.25, hence the Hasse quiver of $\operatorname{tors}(B_{\mathcal{W}})$ is 2-regular. But then $\mathsf{Q}(\operatorname{tors}(B_{\mathcal{W}}))$ is of the following form.

$$\begin{tikzcd}
 & \operatorname{mod}(B_{\mathcal{W}}) \arrow[dl] \arrow[dr] & \\
\vdots & & \vdots \\
 & \{0\} \arrow[from=ul] \arrow[from=ur] &
\end{tikzcd}$$

Here, the two paths from $\operatorname{mod}(B_{\mathcal{W}})$ to $\{0\}$ are finite, not necessarily of the same length. In particular, $\operatorname{mod}(B_{\mathcal{W}})$ has precisely two maximal green sequences.

Definition 4.18 (Polygonal deformation) Let A be a finite dimensional algebra. Let $S_\bullet = (S_1, \ldots, S_m)$ and $S'_\bullet = (S'_1, \ldots, S'_{m'})$ be maximal green sequences for A.

Suppose there are non-negative integers $s, t \geq 0$ with $s + t < m, m'$ such that

- $S_i = S'_i$ for $1 \leq i \leq s$,
- $S_{m-j} = S'_{m'-j}$ for $0 \leq j < t$,
- the set $[\mathcal{T}, \mathcal{U}]$ is a polygon in $\operatorname{tors}(A)$, where $\mathcal{T} := \operatorname{Filt}(S_1, \ldots, S_s) = \operatorname{Filt}(S'_1, \ldots, S'_s)$ and $\mathcal{U} := \operatorname{Filt}(S_1, \ldots, S_{m-t}) = \operatorname{Filt}(S_1, \ldots, S_{m'-t})$,.

Then we say that $S_\bullet$ and $S'_\bullet$ are related by a *polygonal flip*.

The diagram in Figure 4.3 visualizes the relationship between two maximal green sequences that are related by a polygonal flip.

Using the notion of wide intervals in maximal green sequences from Definition 4.14 we can define a mutation-like operation on maximal green sequences, thereby construction maximal green sequences from others.

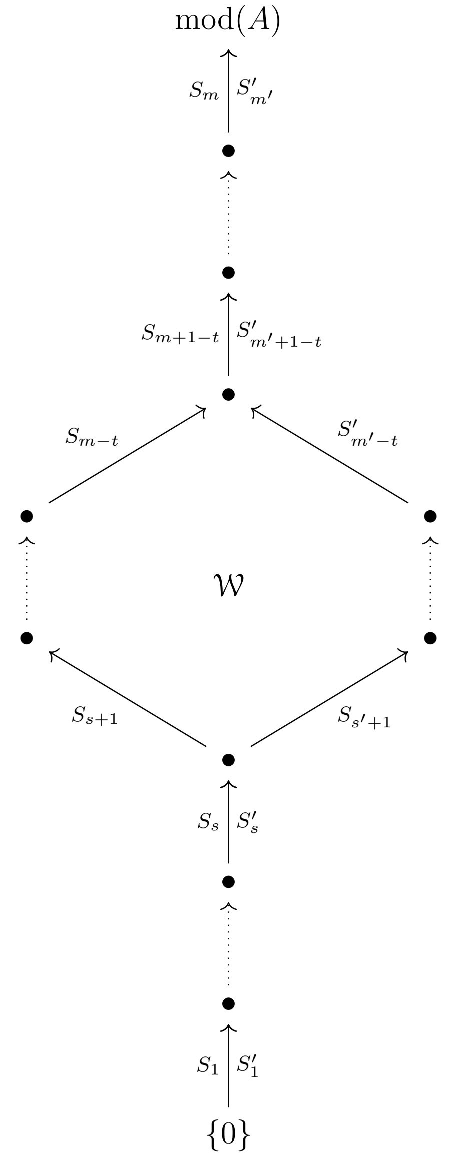


Figure 4.1: Two maximal green sequences that are related by a polygonal flip at $\mathcal{W}$.

Remark 4.19 (Polygonal flips) Let A be a finite dimensional algebra and let $S_\bullet = (S_1, \ldots, S_m)$ be a maximal green sequence for A. Let $1 \leq i < j \leq m$ be indices. Suppose that $S_{[i,j]}$ is a wide interval in the sense of Definition 4.14 such that the associated interval $[\mathcal{T}, \mathcal{U}]$ in tors(A) is a polygon. The torsion classes $\mathcal{T}$ and $\mathcal{U}$ are given by

$$\mathcal{T} := \mathrm{Filt}(S_1, \ldots, S_{i-1}) \quad \text{and} \quad \mathcal{U} := \mathrm{Filt}(S_1, \ldots, S_j),$$

cf. Proposition 4.15. The subsequence of bricks $(S_i, \ldots, S_j)$ is a maximal green sequence in the wide subcategory $\mathcal{W} = \mathcal{U} \cap \mathcal{T}^\perp$, which is equivalent to the module category of a τ-tilting finite algebra with two simple modules. Hence there is a unique second maximal green sequence $(\tilde{S}_1, \ldots, \tilde{S}_\ell)$ in $\mathcal{W}$, and therefore gives a second maximal green sequence for A, i.e.

$$\hat{S}_\bullet = (S_1, \ldots, S_{i-1}, \tilde{S}_1, \ldots, \tilde{S}_\ell, S_{j+1}, \ldots, S_m).$$

Looking at the visualization of polygonal deformations from earlier, the maximal green sequence $\hat{S}_\bullet$ results from $S_\bullet$ by a *polygonal flip* at the polygon associated to the interval $[\mathcal{T}, \mathcal{U}]$.

Example 4.20 We consider the path algebra $A = kQ$ over a Euclidean quiver of type $\tilde{A}_2$.

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ul] \arrow[ll] \end{tikzcd}$$

Since the quiver is acyclic, we can arrange the simple modules into a maximal green sequence, i.e.

$$(1, 2, 3)$$

is a maximal green sequence for A. Since all simples have to appear in a maximal green sequence, and no other ordering of the simple modules gives rise to one, this is the unique maximal green sequence for A of minimal length 3. This maximal green sequence has two wide intervals which define polygons in $\operatorname{tors}(A)$, namely

$$(\underline{1, 2}, 3) \quad \text{and} \quad (1, \underline{2, 3}).$$

In the first case, the wide subcategory $\operatorname{Filt}(1, 2)$ is equivalent to the module category over a path algebra over an A_2 quiver, hence the other maximal green sequence in $\operatorname{Filt}(1, 2)$ is given by $(2, \begin{smallmatrix}2\\1\end{smallmatrix}, 1)$. In the second case, $\operatorname{Filt}(2, 3)$ is again equivalent to the module category over an A_2 quiver. It follows that performing the two possible polygonal flips on $(1, 2, 3)$ gives the maximal green sequences

$$(2, \begin{smallmatrix}2\\1\end{smallmatrix}, 1, 3) \quad \text{and} \quad (1, 3, \begin{smallmatrix}3\\2\end{smallmatrix}, 2).$$

Now we continue with the first sequence. Again, this sequence has two wide intervals which define polygons in $\operatorname{tors}(A)$, namely

$$(\underline{2, \begin{smallmatrix}2\\1\end{smallmatrix}, 1}, 3) \quad \text{and} \quad (2, \begin{smallmatrix}2\\1\end{smallmatrix}, \underline{1, 3}).$$

Flipping at the first wide interval gets us back to where we started. In the second case, the wide subcategory $\operatorname{Filt}(1, 3)$ is again equivalent to the module category over an A_2 quiver, so flipping here gives us

$$(2, \begin{smallmatrix}2\\1\end{smallmatrix}, 3, \begin{smallmatrix}3\\1\end{smallmatrix}, 1).$$

This maximal green sequence has two wide intervals, but only one of them defines a polygon: The wide subcategory associated to the wide interval

$$(2, \underline{\begin{smallmatrix}2\\1\end{smallmatrix}, 3}, \begin{smallmatrix}3\\1\end{smallmatrix}, 1)$$

is equivalent to the module category over a Kronecker quiver, which is not τ-tilting finite and has only one maximal green sequence. So we can not perform a polygonal flip here. Flipping at the other wide interval gets us back to the previous maximal green sequence.

We remark that the results in [HI19] imply that all maximal green sequences for A can be obtained by polygonal flips from $(1, 2, 3)$. The authors show that for all Euclidean quivers, i.e. the quivers for which the path algebra has tame representation type, any two maximal green sequences are connected by a sequence of polygonal flips.

4.4 Properties of bricks in maximal green sequences

We collect a few basic properties of bricks appearing in maximal green sequences. We start with the following general lemma, which appears to be fairly well-known, cf. e.g. [MŠ17, Section 3].

Lemma 4.21 *Let A be a finite dimensional algebra and let $\mathcal{T}, \mathcal{U} \in \text{f-tors}(A)$ be functorially finite torsion classes with $\mathcal{T} \subseteq \mathcal{U}$. Then $\mathcal{U} \cap \mathcal{T}^{\perp} \subseteq \text{mod}(A)$ is functorially finite.*

Proof. By definition, a subcategory $\mathcal{X} \subseteq \text{mod}(A)$ is functorially finite if every module $M \in \text{mod}(A)$ admits left and right $\mathcal{X}$-approximations.

Let $M \in \text{mod}(A)$. Since $\mathcal{U}$ is functorially finite, there is a left $\mathcal{U}$-approximation $f \colon M \to U$. The torsion pair $(\mathcal{T}, \mathcal{T}^{\perp})$ gives a canonical short exact sequence

$$0 \longrightarrow U' \longrightarrow U \xrightarrow{g} U'' \longrightarrow 0$$

with $U' \in \mathcal{T}$ and $U'' \in \mathcal{T}^{\perp}$. We claim that the composite $gf \colon M \to U''$ is a left $\mathcal{U} \cap \mathcal{T}^{\perp}$-approximation of M. Let $X \in \mathcal{U} \cap \mathcal{T}^{\perp}$. Then

$$\text{Hom}_A(gf, X) = \text{Hom}_A(f, X)\,\text{Hom}_A(g, X) \colon\ \text{Hom}_A(U'', X) \to \text{Hom}_A(M, X)$$

is surjective. But since $X \in \mathcal{T}^{\perp}$, the morphism $\text{Hom}_A(g, X)$ is an isomorphism and since f is a left $\mathcal{U}$-approximation and $X \in \mathcal{U}$ the morphism $\text{Hom}_A(f, X)$ is surjective, hence $\text{Hom}_A(gf, X)$ is surjective which implies that gf is a left $\mathcal{U} \cap \mathcal{T}^{\perp}$-approximation. The case of right approximation follows by duality. □

Proposition 4.22 (Functorial finiteness) *Let $(S_1, \ldots, S_m)$ be a maximal green sequence for a finite dimensional algebra A. Then $\text{Filt}(S_i)$ is functorially finite.*

Proof. Using Theorem 4.7 there are torsion classes $\mathcal{T}_{i-1} \subseteq \mathcal{T}_i$ such that $\text{Filt}(S_i) = \mathcal{T}_i \cap \mathcal{T}_{i-1}^{\perp}$. By Remark 4.8.(1) the torsion classes $\mathcal{T}_{i-1}$ and $\mathcal{T}_i$ are functorially finite. Hence Lemma 4.21 implies the result. □

We obtain the following corollary for string algebras. The result was obtained using different methods, namely string combinatorics and τ-tilting theory, in [GS20, Lemma 3.1].

Corollary 4.23 *Let A be a string algebra and let $(S_1, \ldots, S_m)$ be a maximal green sequence for A. Then each S_i is a string module.*

Proof. Let $S \in \mathrm{mod}(A)$ be a brick that is a band module. Then $\mathrm{Filt}(S)$ is a homogeneous tube of rank 1, cf. [BR87, Section 3], so $\mathrm{Filt}(S)$ not functorially finite. Hence the brick can not appear in a maximal green sequence by Proposition 4.22. $\square$

Lemma 4.24 *Let $(S_1, \ldots, S_m)$ be a maximal green sequence for a finite dimensional algebra A. Then for each $i \in \{1, \ldots, m\}$, the brick S_i is the unique brick in $\mathrm{mod}(A)$ such that $\mathrm{Hom}_A(S_j, S_i) = 0$ for $j < i$ and $\mathrm{Hom}_A(S_i, S_j) = 0$ for $j > i$.*

Proof. Let $M \in \mathrm{mod}(A)$ be a brick with $\mathrm{Hom}_A(S_j, M) = 0$ for $j < i$ and $\mathrm{Hom}_A(M, S_j) = 0$ for $j > i$. The maximal green sequence induces a filtration of M with subfactors given by the S_j in order, cf. Remark 4.8. But the Hom-vanishing implies that $M \in \mathrm{Filt}(S_i)$ and since M is a brick we must have $M = S_i$. $\square$

Proposition 4.25 (Open orbit) *Suppose that the ground field k is algebraically closed. Let A be a finite dimensional algebra and let $(S_1, \ldots, S_m)$ be a maximal green sequence for A and let d_i be the dimension vector of S_i for each $i \in \{1, \ldots, m\}$. Then the orbit of S_i in the module variety $\mathrm{mod}(A, d_i)$ of d_i-dimensional A-modules is open.*

Proof. In $\mathrm{mod}(A, d_i)$ consider the following subset.

$$U_i := \{M \mid \mathrm{End}_A(M) = 1\} \cap \bigcap_{j<i} \{M \mid \mathrm{Hom}_A(S_j, M) = 0\} \cap \bigcap_{i<j} \{M \mid \mathrm{Hom}_A(M, S_j) = 0\}$$

It is well-known, see e.g. [CBS02, Lemma 4.3], that the dimension of Hom-sets is upper-semicontinuous. This implies that all m sets taking part in the intersection for U_i are open, hence U_i is open. But by Lemma 4.24 the set U_i contains a unique isomorphism class, namely the one of S_i. Hence the orbit of S_i is open. $\square$

4.5 The order of simple modules

By Corollary 4.10 a maximal green sequence for a finite dimensional algebra A contains all simple A-modules, so it is natural to ask in which order the simples appear. If the algebra A is τ-tilting finite and therefore only has finitely many isomorphism classes of bricks, we can start with an arbitrary ordering of the simples and extend to a maximal green sequence.

Recall that a module $M \in \mathrm{mod}(A)$ is called *uniserial* if it has a unique composition series. We call a module $M \in \mathrm{mod}(A)$ *thin* if in a composition series for M each simple A-module appears at most once as a composition factor.

Theorem 4.26 *Let A be a finite dimensional algebra and let $M \in \mathrm{mod}(A)$ be a thin uniserial module that does not appear in any maximal green sequence. Suppose that $\tilde{S}_\bullet = (S(i_1), S(i_2), \ldots, S(i_\ell))$ are the simple composition factors of M in the reverse order, i.e. M has $S(i_1)$ as the simple top, $S(i_2)$ is the simple top of the radical of M and so on. Then $\tilde{S}_\bullet$ is not a subsequence of any maximal green sequence for A.*

Proof. Let $S_\bullet = (S_1, \ldots, S_m)$ be a maximal green sequence and assume that $\tilde{S}_\bullet$ is a subsequence of $S_\bullet$. For $p \in \{1, \ldots, \ell\}$ let $j_p \in \{1, \ldots, m\}$ be the index such that $S_{j_p} = S(i_p)$. Note that we have $j_1 < \ldots < j_\ell$.

By Remark 4.8, the module M has a unique filtration

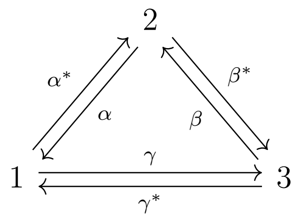
$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_m = M \tag{$*$}$$

with subfactors $M_q/M_{q-1} \in \mathrm{Filt}(S_q)$ for $q \in \{1, \ldots, m\}$. Since M is thin we must have $M_q/M_{q-1} = S_q$ whenever $M_q/M_{q-1} \neq 0$. Let r be the number of non-zero subfactors and let $q_1 < \ldots < q_r \in \{1, \ldots, m\}$ be the indices such that $S_{q_i} := M_{q_i}/M_{q_i-1} \neq 0$. Since M does not appear in the maximal green sequence we must have $r \geq 2$.

Since M is uniserial, all bricks S_{q_i} appearing as subfactors in the filtration of M are again uniserial, so have simple socles and tops. In particular, there are indices $p', p'' \in \{1, \ldots, \ell\}$ such that $\mathrm{soc}(S_{q_r}) = S(i_{p'}) = S_{j_{p'}}$ and $\mathrm{top}(S_{q_1}) = S(i_{p''}) = S_{j_{p''}}$. We have $p' \neq p''$ because M is not simple. Then $\mathrm{Hom}_A(S_{j_{p'}}, S_{q_r}) \neq 0$, therefore $q_r \leq j_{p'}$, and $\mathrm{Hom}_A(S_{q_1}, S_{j_{p''}}) \neq 0$, therefore $j_{p''} \leq q_1$. Since $r \neq 1$ it follows that $j_{p''} < j_{p'}$.

Refining the filtration $(*)$ for M shows that in a composition series for M the simple module $S(i_{p''})$ comes below the simple module $S(i_{p'})$. But then the ordering of the simple composition factors in $\tilde{S}_\bullet$ implies that $p' < p''$ and therefore $j_{p'} < j_{p''}$, contradiction. $\square$

Example 4.27 For simplicity, assume that the field k is algebraically closed. Let $A = kQ/I$ be a string algebra with quiver



and relations $\alpha\alpha^* = \alpha^*\alpha = \beta\beta^* = \beta^*\beta = \gamma^*\gamma = \gamma\gamma^* = 0$, as well as additional relations contained in $\mathrm{rad}^3(A)$ such that A is finite dimensional.

We consider the following six bands $w_1, \dots, w_6$ for A and the band modules $M(w_i, \lambda, 1)$ where $\lambda \in k^\times$, whose composition series are listed below.

band	comp. series of $M(w_i, \lambda, 1)$
$w_1 = \alpha\beta(\gamma^*)^{-1}$	$\begin{smallmatrix}3&&\\&2&\\&&1\end{smallmatrix}$
$w_2 = \beta\gamma(\alpha^*)^{-1}$	$\begin{smallmatrix}1&&\\&3&\\&&2\end{smallmatrix}$
$w_3 = \gamma\alpha(\beta^*)^{-1}$	$\begin{smallmatrix}2&&\\&1&\\&&3\end{smallmatrix}$
$w_4 = \gamma^*\beta^*\alpha^{-1}$	$\begin{smallmatrix}2&&\\&3&\\&&1\end{smallmatrix}$
$w_5 = \beta^*\alpha^*\gamma^{-1}$	$\begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix}$
$w_6 = \alpha^*\gamma^*\beta^{-1}$	$\begin{smallmatrix}3&&\\&2&\\&&1\end{smallmatrix}$

Note that all band modules are uniserial and thin. However, each permutation of the set of simple modules appears as an ordered sequence of composition factors. Since band modules do not appear in any maximal green sequence for A, cf. Corollary 4.23, we conclude that there are no maximal green sequences for A.

We remark that our argument does not rely on the fact that A is a string algebra. In fact, consider the algebra $B = kQ/I$ which is the given by the path algebra over the same quiver Q

$$\begin{tikzcd}
& 2 \arrow[dl, shift left, "\alpha"] \arrow[dr, shift left, "\beta^*"] & \\
1 \arrow[ur, shift left, "\alpha^*"] \arrow[rr, shift left, "\gamma"] & & 3 \arrow[ul, shift left, "\beta"] \arrow[ll, shift left, "\gamma^*"]
\end{tikzcd}$$

but this modulo the preprojective relations $\alpha\alpha^* + \gamma^*\gamma = 0$, $\beta\beta^* + \alpha^*\alpha = 0$, $\gamma\gamma^* + \beta^*\beta = 0$ and additional relations contained in $\mathrm{rad}^3(B)$ such that B is finite dimensional. Hence A is a certain finite dimensional quotient of the infinite dimensional preprojective algebra of type $\tilde{A}_2$.

Note that all thin uniserial band modules from the six infinite families for the previous string algebra A are also modules for B. In particular, not all of them can appear in maximal green sequences, which follows from Proposition 4.25. So the same argument we used for the string algebra A shows that B does not have any maximal green sequences.

We note the following consequence to Theorem 4.26.

Theorem 4.28 (Simple at target before simple at source) *Let k be an algebraically closed field. Let $A = kQ/I$ be a finite dimensional k-algebra given by the path algebra over a* *quiver Q modulo an ideal I generated by admissible relations. If $i, j \in Q_0$ are vertices, $i \neq j$, such that there are two or more arrows from i to j, then in any maximal green sequence for A the simple module $S(j)$ comes before the simple module $S(i)$.*

Proof. Let $Q' = (1 \leftleftarrows 2)$ denote the Kronecker quiver. By assumption, Q' is a subquiver of Q, and the inclusion maps 1 to the vertex $j \in Q_0$ and 2 to $i \in Q_0$. Let $d \in \mathbb{N}Q_0$ be the dimension vector for A with $d_i = 1$ and $d_j = 1$ but $d_\ell = 0$ for $\ell \neq i, j$. Since I is generated by admissible relations, the closed subset of the module variety $\mathcal{A} \subseteq \operatorname{mod}(A, d)$ defined by

$$\mathcal{A} := \{(M_\alpha)_{\alpha \in Q_1} \in \operatorname{mod}(A, d) \mid M_\alpha = 0 \text{ for } \alpha \notin Q'\}$$

is isomorphic to the module variety $\operatorname{mod}(kQ', (1,1))$ of representations for the Kronecker quiver of dimension vector $(1,1)$. But $\operatorname{mod}(kQ', (1,1))$ contains an infinitely family uniserial band modules with non-open orbits, hence so does $\mathcal{A}$. So we can find a brick $M \in \operatorname{mod}(A, d)$ with a non-open orbit, which implies that M does not appear in a maximal green sequence, cf. Proposition 4.25.
So we have found a thin and uniserial module M with composition series $\begin{smallmatrix} S(i) \\ S(j) \end{smallmatrix}$ that does not appear in any maximal green sequence. Hence Theorem 4.26 implies that in a maximal green sequence for A the simple $S(j)$ must come before the simple $S(i)$. □

From the theorem we obtain at once the following corollary, which generalizes [BST19, Theorem 4.14] to algebras with arbitrarily many simple modules. Moreover, our proof does not require a detailed combinatorial analysis of the wall and chamber structure for the algebra.

Corollary 4.29 *Let k be an algebraically closed field and let $A = kQ/I$ be a finite dimensional k-algebra with quiver Q and ideal I generated by admissible relations. Suppose that Q has the following so called* Markov quiver *as a subquiver.*

$$\begin{tikzcd} & 2 \arrow[dl, shift left] \arrow[dl, shift right] & \\ 1 \arrow[rr, shift left] \arrow[rr, shift right] & & 3 \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd}$$

Then A admits no maximal green sequences.

Proof. Suppose that Q has the Markov quiver with the vertex labels as above as a subquiver and assume that there is a maximal green sequence $S_\bullet$ for A. Then Theorem 4.28 requires that in $S_\bullet$ the simple module $S(1)$ comes before $S(2)$, which in turn comes before $S(3)$, which must come before $S(1)$, contradiction. □

4.6 Counterexamples

We collect a few properties bricks in maximal green sequences generally do not have by giving explicit counterexamples.

Example 4.30 (Not τ-rigid, not left-finite, not right-finite) Consider the string algebra $A = kQ/I$ which already appeared in Example 1.27 with quiver

$$\begin{tikzcd} 1 & 2 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"] & 3 \arrow[l, "\gamma"'] & 4 \arrow[l, shift right, "\delta"'] \arrow[l, shift left, "\varepsilon"] \end{tikzcd}$$

and relations $\alpha\gamma = 0$ and $\gamma\delta = 0$.

We use polygonal flips, cf. Remark 4.8, to construct a maximal green sequence that contains a brick that is neither τ-rigid, left-finite nor right-finite. Since the quiver Q is acyclic, we can arrange the simple modules into a maximal green sequence as follows.

$$\big(1, 2, 3, 4\big)$$

The wide subcategory $\mathrm{Filt}(2, 3)$ is equivalent to the module category of an A_2-quiver, so performing a polygonal flip here gives the following maximal green sequence.

$$\big(1, 3, \begin{smallmatrix}3\\2\end{smallmatrix}, 2, 4\big)$$

Both of the wide subcategories $\mathrm{Filt}(1, 3)$ and $\mathrm{Filt}(2, 4)$ are semisimple, hence a polygonal flip at these two intervals swaps these simples and we obtain

$$\big(3, 1, \begin{smallmatrix}3\\2\end{smallmatrix}, 4, 2\big).$$

Now note that the wide subcategory $\mathrm{Filt}(1, \begin{smallmatrix}3\\2\end{smallmatrix})$ is again equivalent to the module category of an A_2-quiver in $\mathrm{mod}(A)$. Note that this would not be true without the relation $\alpha\gamma = 0$. We flip there and get

$$\Big(3, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, 1, 4, 2\Big).$$

We swap again at the semisimple wide subcategory $\mathrm{Filt}(1, 4)$

$$\Big(3, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, 4, 1, 2\Big)$$

and then notice that the wide subcategory $\mathrm{Filt}(\begin{smallmatrix}3\\2\\1\end{smallmatrix}, 4)$ is again equivalent to the module category of an A_2-quiver in $\mathrm{mod}(A)$. So we perform a final polygonal flip and obtain the following maximal green sequence for A.

$$\Bigg(3, \begin{smallmatrix}3\\2\end{smallmatrix}, 4, \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, 1, 2\Bigg)$$

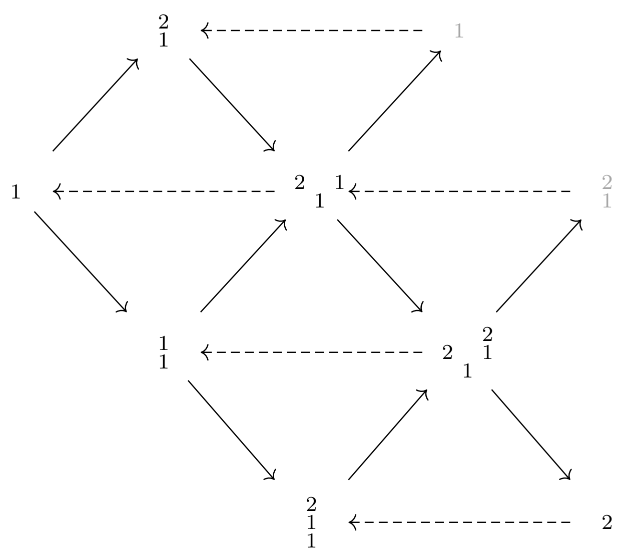
In Example 1.27 we have shown that the brick $S = \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}$ is not left-finite. By duality, it follows that it is also not right-finite. Note that this implies that S is not τ-rigid.

Example 4.31 (Not rigid) Recall that all simple module have to appear in a maximal green sequences, cf. Corollary 4.10. So any algebra $A = kQ/I$ given by a quiver modulo an ideal generated by admissible relations such that the quiver Q has loops gives rise to a simple non-rigid module, i.e. a simple $S \in \operatorname{mod}(A)$ with $\operatorname{Ext}^1_A(S,S) \neq 0$.

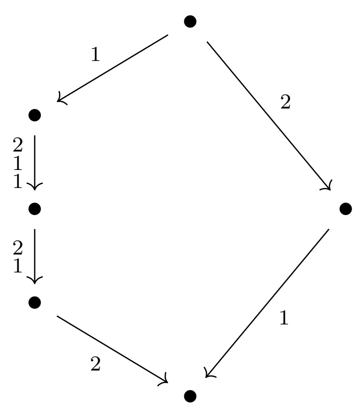
However, expanding on this consider the algebra $A = kQ/I$ with quiver

$$\varepsilon \circlearrowright 1 \longleftarrow 2$$

and relation $\varepsilon^2 = 0$. The algebra A belongs to the class of algebras $H(C, D, \Omega)$ associated to a generalized Cartan matrix of type B_2 with D minimal introduced in [GLS17, §13.6]. It is a representation-finite gentle algebra with the following Auslander–Reiten quiver.



The Hasse quiver of torsion classes with brick labeling looks as follows.



However, the brick $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ is not rigid.

4.7 Maximal lengths of maximal green sequences

In this section, we study the lengths of maximal green sequences. In particular, in case there are only finitely many maximal green sequences, we study the maximal length of a maximal green sequence. We focus on path algebras over acyclic quivers Q.

Assume that we labeled the vertices of the quiver Q with numbers $\{1, \ldots, n\}$ such that there is no path from i to j whenever $i < j$. Then we can arrange the simple modules into a maximal green sequence, namely

$$\big(S(1), S(2), \ldots, S(n)\big).$$

It follows that kQ always has a maximal green sequence of length $n = |Q_0|$, and this is the minimal length of a maximal green sequence by Corollary 4.10. The *No-Gap* conjecture of Brüstle–Dupont–Pérotin motivates the importance of studying the maximal length of maximal green sequences for quivers.

Conjecture 4.32 (No-Gap conjecture, [BDP14]) Let Q be an acyclic quiver. Then the set of lengths of maximal green sequences forms an integer interval $[n, \ell_0]$ where $n = |Q_0|$ is the number of vertices.

The No-Gap conjecture has been proven for path algebras of finite, cf. [GM19], and tame, cf. [HI19], representation type, i.e. for orientations of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 and the extended Dynkin diagrams $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

We note that the No-Gap conjecture has significance for computing the set of maximal green sequences. An algorithm that finds maximal green sequences by walking paths in the Hasse quiver of torsion classes or an isomorphic quiver can stop its search at the first length larger than $n = |Q_0|$ for which it did not find a path from the maximal to the minimal element.

For quivers of finite type, the maximal length ℓ_0 is known to be the number of isomorphism class of indecomposable representations, for quivers of type $\tilde{A}_n$ it was determined in [AI20] to be

$$\ell_0 = \frac{n(n+1)}{2} + ab,$$

where a and b are the numbers of arrows in clockwise and counter-clockwise orientation if we draw the $\tilde{A}_n$ in the plane. In particular, ℓ_0 is invariant under source-sink reflections. The invariance of ℓ_0 under source-sink reflection has been proven for all quivers of tame representation type in [KN20]. Moreover, the authors prove that $\ell_0 = 2n^2 - 2n - 2$ in case of $\tilde{D}_n$ and $\ell_0 = 78$ for $\tilde{E}_6$, $\ell_0 = 159$ for $\tilde{E}_7$ and $\ell_0 = 390$ for $\tilde{E}_8$.

The authors ask the question if the maximal length of maximal green sequence is invariant under source-sink reflection for all finite acyclic quivers Q.

Example 4.33 Consider the following two wild quivers Q and Q'.

$$Q = \begin{tikzcd} & 2 \arrow[dl] & & \\ 1 & & 3 \arrow[ul] \arrow[ll] & 4 \arrow[l] \end{tikzcd} \qquad Q' = \begin{tikzcd} & 2 & & \\ 1 \arrow[ur] \arrow[rr] & & 3 \arrow[ul] & 4 \arrow[l] \end{tikzcd}$$

We claim that the maximal length of maximal green sequences for kQ is $\ell_0 = 9$, while for kQ' we have $\ell_0 = 8$. Since Q' is obtained from Q by source-sink reflection at 1, this shows that ℓ_0 is not source-sink reflection invariant for general acyclic quivers.

In the following, we provide a full proof for this claim, based on the following technical lemma, which might be of independent interest.

We write $[M : S] := \ell$ for the multiplicity ℓ of the simple module A-module S in a composition series for $M \in \operatorname{mod}(A)$.

Lemma 4.34 *Let A be a finite dimensional algebra. Let $(S_1, \ldots, S_m)$ be a maximal green sequence. Let $i \in Q_0$ be a vertex and let $\ell \in \{1, \ldots, m\}$ be minimal such that $[S_\ell : S(i)] \neq 0$. Then S_ℓ is isomorphic to a submodule of the indecomposable injective module $I(i)$ at i.*

Proof. Since $[S_j : S(i)] = 0$ for all $j < \ell$ we have $\operatorname{Hom}_A(S_j, I(i)) = 0$. But the injective $E := I(i)$ has a filtration

$$\{0\} = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_m = E$$

where $E_j \in \operatorname{Filt}(S_1, \ldots, S_j)$, cf. Theorem 4.7 and Remark 4.8. Hence $\operatorname{Hom}_A(S_j, E) = 0$ for $j < \ell$ implies that $\operatorname{Hom}_A(E_j, E) = 0$ and therefore $E_j = 0$ for $j < \ell$.

We claim that $E_\ell \neq 0$. Suppose that $E_\ell = 0$. Then the filtration only has non-zero subfactors E_j/E_{j-1} for $j > \ell$, hence $E \in \operatorname{Filt}(S_{\ell+1}, \ldots, S_m)$. But since $\operatorname{Hom}_A(S_\ell, S_j) = 0$ for $j > \ell$ this implies $\operatorname{Hom}_A(S_\ell, E) = 0$, contradiction. So $E_\ell \neq 0$, which implies that $0 \neq E_\ell = E_\ell/E_{\ell-1} \in \operatorname{Filt}(S_\ell)$, hence S_ℓ is isomorphic to a submodule of $E = I(i)$. $\square$

We will also make use of the following theorem.

Theorem 4.35 ([Mul16], see also [DK19]) *Let Q be an acyclic quiver and let $Q' \subseteq Q$ be a full subquiver. Let $S_\bullet = (S_1, \ldots, S_m)$ be a maximal green sequence for* $\operatorname{mod}(kQ)$. *Let $S'_\bullet = (S'_1, \ldots, S'_{m'})$ be the subsequence of $S_\bullet$ on all bricks that are supported only on Q'. Then $S'_\bullet$ is a maximal green sequence for* $\operatorname{mod}(kQ')$.

Lemma 4.36 *Let Q be the following quiver of extended Dynkin type $\tilde{A}_2$.*

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ul] \arrow[ll] \end{tikzcd}$$

Then the following is a complete list of maximal green sequence for kQ.

$$\left(\scriptstyle 1,\ 2,\ 3\right),\ \left(\scriptstyle 2,\ \begin{smallmatrix}2\\1\end{smallmatrix},\ 1,\ 3\right),\ \left(\scriptstyle 1,\ 3,\ \begin{smallmatrix}3\\2\end{smallmatrix},\ 2\right),\ \left(\scriptstyle 2,\ \begin{smallmatrix}2\\1\end{smallmatrix},\ 3,\ \begin{smallmatrix}3\\1\end{smallmatrix},\ 1\right),\ \left(\scriptstyle 3,\ \begin{smallmatrix}3\\1\end{smallmatrix},\ 1,\ \begin{smallmatrix}3\\2\end{smallmatrix},\ 2\right)$$

Proof. Since kQ is tame, all maximal green sequences are polygonal deformations of each other, cf. [HI19]. Hence we can read off the list from parts of the Hasse quiver of functorially finite torsion classes for kQ we have shown earlier in Example 3.28. $\square$

In what follows, we make use of the Euler form for the quivers Q and Q' from Example 4.33. For Q, it is for $a, b \in \mathbb{Z}Q_0$ given by

$$\langle a, b\rangle_Q = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 - a_1b_2 - a_1b_3 - a_2b_3 - a_3b_4$$

and for Q'

$$\langle a, b\rangle_{Q'} = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 - a_2b_1 - a_2b_3 - a_3b_1 - a_3b_4.$$

It is well-known that for representations $M, N \in \operatorname{mod}(kQ)$ with dimension vectors $\underline{\dim}(M) = a$ and $\underline{\dim}(N) = b$ we have

$$\langle a, b\rangle_Q = \dim_k \operatorname{Hom}_A(M, N) - \dim_k \operatorname{Ext}^1_A(M, N),$$

cf. e.g. [Rin84]. Let $S \in \operatorname{mod}(kQ)$ be a brick that appears in a maximal green sequence and let $d = \underline{\dim}(S)$ be its dimension vector. By Proposition 4.22 the wide subcategory $\operatorname{Filt}(S)$ is functorially finite, which implies that S must be rigid since A is kQ is hereditary, cf. Proposition 1.30. But then we must have $\langle S, S\rangle_Q = 1$.

Recall that a module $M \in \operatorname{mod}(A)$ for a finite dimensional algebra A is called *sincere* if $[M : S] \neq 0$ for all simple A-modules S.

Lemma 4.37 *Let A be the path algebra over the quiver Q or Q' from Example* 4.33*. Then no sincere bricks appear in maximal green sequences for A.*

Proof. First consider $A = kQ$. Suppose that $S_\bullet = (S_1, \dots, S_m)$ is a maximal green sequence for A and suppose that there is $r \in \{1, \dots, m\}$ such that the brick S_r is sincere. Let $\ell \in \{1, \dots, m\}$ be minimal such that $[S_\ell : S(1)] \neq 0$. By the previous Lemma 4.34 the rigid brick S_ℓ is isomorphic to a submodule of $I(1) = \begin{smallmatrix}4&&\\3&&4\\2&&3\\&1&\end{smallmatrix}$. Looking at the possible dimension vectors d of submodules of $I(1)$ and discarding those with $\langle d, d\rangle_Q \neq 1$ we only have the following six cases, where we repeatedly use without further comment the existence of filtrations induced by $S_\bullet$, cf. Theorem 4.7 and Remark 4.8.

Case $S_\ell \simeq 1$: Since S_r is sincere and the vertex 1 is a sink for Q, the simple 1 is a submodule of S_r. But then we have $\mathrm{Hom}_A(S_\ell, S_r) \neq 0$ and hence we have a contradiction, since we assumed $\ell < r$.

Case $S_\ell \simeq \begin{smallmatrix}2\\1\end{smallmatrix}$: Note that $P(2) = \begin{smallmatrix}2\\1\end{smallmatrix}$ is the projective module at the vertex 2. By assumption we have $\ell < r$, therefore $\mathrm{Hom}_A(S_\ell, S_r) = 0$. But then $\mathrm{Hom}_A(P(2), S_r) = 0$, which implies that S_r is not supported on 2, contradiction to S_r being sincere.

Case $S_\ell \simeq \begin{smallmatrix}3\\1\end{smallmatrix}$: We look at the subsequence of $S_\bullet$ with bricks only supported on the first three vertices. Using Theorem 4.35 and Lemma 4.36, and the assumption that $[S_j : S(1)] = 0$ for $j < \ell$ we conclude that $(3, \begin{smallmatrix}3\\1\end{smallmatrix}, 1, \begin{smallmatrix}3\\2\end{smallmatrix}, 2)$ must be a subsequence of $S_\bullet$ and all other bricks S satisfy $[S : S(4)] \neq 0$. Consider the following representation of dimension vector $(1, 1, 1, 1)$.

$$M = \begin{tikzcd} & k \arrow[dl, "0"'] & & \\ k & & k \arrow[ll, "1"] \arrow[ul, "1"'] & k \arrow[l, "1"'] \end{tikzcd}$$

Then $M = \begin{smallmatrix}&4&\\&3&\\1&&2\end{smallmatrix}$ is not rigid, hence must have a non-trivial filtration with subfactors in $S_\bullet$. However, the only rigid submodules that are bricks are the simple modules 1 and 2. Looking at the subsequence $(3, \begin{smallmatrix}3\\1\end{smallmatrix}, 1, \begin{smallmatrix}3\\2\end{smallmatrix}, 2)$, we see that the filtration starts with $\{0\} \subseteq 1$. Now the sincere brick S_r has the simple 1 in its socle and the simple 4 in its top, so in the maximal green sequence the simple 4 must come before the some 1. We claim that this implies that $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ must be in the maximal green sequence $S_\bullet$. If not, the filtration of M has to be $\{0\} \subseteq 1 \subseteq \begin{smallmatrix}3\\2\\1\end{smallmatrix} \subseteq M$, which implies that 1 comes before the last subfactor 4. So the injective $I(2) = \begin{smallmatrix}4\\3\\2\end{smallmatrix}$ is in $S_\bullet$, and it must come after the simple 1, which comes after S_r. But then $\mathrm{Hom}_A(S_r, I(2)) = 0$, which implies that S_r is not supported at 2, contradiction.

Case $S_\ell \simeq \begin{smallmatrix}4\\3\\1\end{smallmatrix}$: Since $[S_j : S(1)] = 0$ for $j < \ell$ the filtration of the injective $I(1)$ induced by $S_\bullet$ must begin with $\{0\} \subseteq S_\ell$. The factor is $I(1)/S_\ell = \begin{smallmatrix}4\\3\\2\end{smallmatrix}$. We claim that $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ must be a brick in $S_\bullet$. If not, the filtration for $I(1)$ contains either 4 or $\begin{smallmatrix}4\\3\end{smallmatrix}$ as a subfactor, hence one of these bricks must appear in $S_\bullet$ after S_ℓ. But this is not possible since these two are also factors of S_ℓ. So $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ appears as a brick in $S_\bullet$ after S_ℓ. We *claim* that $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ must appear after 1 in $S_\bullet$. For this, look at the non-rigid brick $M = \begin{smallmatrix}&4&\\&3&\\1&&2\end{smallmatrix}$ from the previous case. If $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ comes before 1, the filtration of M for $S_\bullet$ can not start with $\{0\} \subseteq 1$, since $M/1 = \begin{smallmatrix}4\\3\\2\end{smallmatrix}$. Hence it must start with $\{0\} \subseteq 2$. But the factor then is $S_\ell = \begin{smallmatrix}4\\3\\1\end{smallmatrix}$, hence 2 must come before S_ℓ. But $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ comes after S_ℓ, so we must have $\mathrm{Hom}_A(2, \begin{smallmatrix}4\\3\\2\end{smallmatrix}) = 0$, which is obviously false, thereby proving the *claim.* In summary, we have $(\begin{smallmatrix}4\\3\\1\end{smallmatrix}, 1, \begin{smallmatrix}4\\3\\2\end{smallmatrix})$ as a subsequence. But the sincere module S_r comes before 1, hence we have $\mathrm{Hom}_A(S_r, \begin{smallmatrix}4\\3\\2\end{smallmatrix}) = \mathrm{Hom}_A(S_r, I(2)) = 0$. But then S_r is not supported at 2, contradiction.

Case $S_\ell \simeq \begin{smallmatrix}3&&\\2&&3\\&1&\end{smallmatrix}$: This brick can not appear since it does not appear as a brick in a maximal green sequence for the path algebra over the full subquiver on the first three vertices, cf. Theorem 4.35 and Lemma 4.36.

Case $S_\ell \simeq I(1) = \begin{smallmatrix}4&&\\3&&4\\2&&3\\&1&\end{smallmatrix}$: We consider the following representation.

$$N = \begin{tikzcd} & k \arrow[dl, "1"'] & & \\ k & & k \arrow[ll, "0"] \arrow[ul, "1"'] & k \arrow[l, "1"'] \end{tikzcd}$$

Note that $N = \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}$ is not rigid and that $N \subseteq S_\ell$. Since N is not rigid, it must have a non-trivial filtration with subfactors in $S_\bullet$. Since $N \subseteq S_\ell$, all non-zero subfactors must be bricks S_j with $j > \ell$. But every proper factor module N' of N is injective and satisfies $\mathrm{Hom}_A(I(1), N') \neq 0$, hence such a filtration can not exist, contradiction.

Since we reached a contradiction for all six cases, we conclude that no sincere bricks appear in maximal green sequences for kQ.

Now we turn to the second quiver, so let $A = kQ'$. Our strategy is the same as for Q, so let $S_\bullet = (S_1, \ldots, S_m)$ be a maximal green sequence for A and suppose that there is an $r \in \{1, \ldots, m\}$ such that the brick S_r is sincere. Let $\ell \in \{1, \ldots, m\}$ be minimal such that $[S_\ell, S(2)] \neq 0$. By the previous Lemma 4.34 the rigid brick S_ℓ is isomorphic to a submodule of $I(2) = \begin{smallmatrix}&1&&4\\1&&3&\\&2&&\end{smallmatrix}$. Again we look at rigid indecomposable submodules of $I(2)$, already discarding those whose dimension vector d has $\langle d, d\rangle_{Q'} \neq 1$. This leaves us with the following six cases.

Case $S_\ell \simeq \begin{smallmatrix}2\end{smallmatrix}$: Since S_r is sincere at the vertex 2 is the unique sink in Q', the simple $\begin{smallmatrix}2\end{smallmatrix}$ is in the socle of S_r, hence $\mathrm{Hom}_A(S_\ell, S_r) \neq 0$, contradiction.

Case $S_\ell \simeq \begin{smallmatrix}1\\2\end{smallmatrix}$: The subsequence of $S_\bullet$ of bricks only supported on the first three vertices must be one of the five maximal green sequences from Lemma 4.36, cf. Theorem 4.35. However, note that we have to permute the vertices accordingly, so the subsequence is one of

$$\left(\begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}1\end{smallmatrix}\right),\ \left(\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\end{smallmatrix}\right),\ \left(\begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}\right),\ \left(\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}\right).$$

But the brick S_ℓ is the first one supported at 2, so the subsequence has to be the last one in the list, i.e. $\left(\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}\right)$. The sincere brick S_r has $\begin{smallmatrix}2\end{smallmatrix}$ in its socle and $\begin{smallmatrix}4\end{smallmatrix}$ in its top, so $\begin{smallmatrix}4\end{smallmatrix}$ must come before $\begin{smallmatrix}2\end{smallmatrix}$. Since $\begin{smallmatrix}3\\2\end{smallmatrix}$ does not appear in $S_\bullet$, the brick $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ must appear in $S_\bullet$, as otherwise the filtration of $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ induced by $S_\bullet$ has to be $\{0\} \subseteq \begin{smallmatrix}2\end{smallmatrix} \subseteq \begin{smallmatrix}4\\3\\2\end{smallmatrix}$, which implies that the injective $\begin{smallmatrix}4\\3\end{smallmatrix}$ comes after S_r, hence $\mathrm{Hom}_A(S_r, I(3)) = 0$ and thus S_r can not be sincere. So $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ appears in $S_\bullet$ and has to appear before $\begin{smallmatrix}2\end{smallmatrix}$. Now consider the representation

$M \in \operatorname{mod}(kQ')$ of dimension vector $(1,1,1,1)$ defined as follows.

$$M = \begin{tikzcd} & 2 & & \\ 1 \arrow[ur, "0"] \arrow[rr, "1"] & & 3 \arrow[ul, "1"'] & 4 \arrow[l, "1"'] \end{tikzcd}$$

The representation $M = \begin{smallmatrix}1&&4\\&3&\\&2&\end{smallmatrix}$ is indecomposable but not rigid, hence does not appear in a maximal green sequence. But the only submodules that are rigid bricks are $\begin{smallmatrix}2\end{smallmatrix}$, $\begin{smallmatrix}3\\2\end{smallmatrix}$ and $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$. Since $\begin{smallmatrix}3\\2\end{smallmatrix}$ does not appear in $S_\bullet$ and $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$ comes before $\begin{smallmatrix}2\end{smallmatrix}$, the filtration induced by $S_\bullet$ is $\{0\} \subseteq \begin{smallmatrix}4\\3\\2\end{smallmatrix} \subseteq M$. But then $\begin{smallmatrix}1\end{smallmatrix}$ has to come after $\begin{smallmatrix}4\\3\\2\end{smallmatrix}$, contradiction.

Case $S_\ell \simeq \begin{smallmatrix}3\\2\end{smallmatrix}$: Since $\begin{smallmatrix}3\\2\end{smallmatrix} = P(2)$ is projective, and S_r comes after S_ℓ, we have $\operatorname{Hom}_A(P(2), S_r) = 0$, hence S_r can not be supported on 2. Contradiction since S_r is assumed to be sincere.

Case $S_\ell \simeq \begin{smallmatrix}4\\3\\2\end{smallmatrix}$: Again, $\begin{smallmatrix}4\\3\\2\end{smallmatrix} = P(3)$ is projective, and the same argument as before shows that in this case S_r can not be supported on 3, contradiction.

Case $S_\ell \simeq \begin{smallmatrix}&&1\\1&&3\\&2&\end{smallmatrix}$: This representation is supported on the first three vertices, hence has to be in a maximal green sequence for a quiver of type $\tilde{A}_2$ by Theorem 4.35. But by Lemma 4.36, it does not appear in such a maximal green sequence, contradiction.

Case $S_\ell \simeq I(2) = \begin{smallmatrix}&&1&&4\\1&&&3&\\&2&&&\end{smallmatrix}$: We consider the non-rigid representation $M = \begin{smallmatrix}1&&4\\&3&\\&2&\end{smallmatrix}$ from the previous case. It must have a filtration with subfactors in $S_\bullet$. But since $M \subseteq I(2)$, the filtration can only have subfactors S_j with $j > \ell$. But every proper factor module M is also a factor module of $I(2)$, hence such a filtration can not exist, contradiction.

We reached contradictions in all six cases, hence we can conclude that no sincere bricks appear in maximal green sequence for kQ'. □

Using Lemma 4.37, we can list all bricks that may appear in maximal green sequences for $A = kQ$ or $A = kQ'$, by simply listing all rigid bricks of all proper full subquivers of finite type and using Theorem 4.35 with Lemma 4.36 to get all bricks supported on the unique tame subquiver of type $\tilde{A}_2$. This leaves us with the following list of 10 bricks for $A = kQ$.

$$\left\{ \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix}, \begin{smallmatrix}4\\3\\1\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}4\\3\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix} \right\} \subseteq \operatorname{mod}(kQ)$$

For $A = kQ'$, the same idea also gives us a list with 10 bricks.

$$\left\{ \begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, \begin{smallmatrix}4\\3\end{smallmatrix}, \begin{smallmatrix}4&&1\\&3&\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix} \right\} \subseteq \operatorname{mod}(kQ')$$

Now our claim from Example 4.33 follows from the following lemma.

Lemma 4.38 *Let Q and Q' be the quivers from Example* 4.33*. For $A = kQ$, the maximal green sequence of maximal length has length* 9*. For $A = kQ'$, the maximal green sequence of maximal length has length* 8*.*

Proof. Note that both quivers Q and Q' have a full subquiver of type $\tilde{A}_2$, but by Lemma 4.36 no maximal green sequence for a quiver of type $\tilde{A}_2$ contains all three bricks of length 2. Hence Theorem 4.35 implies that a maximal green sequence Q or Q' also can not contain all three of these length 2 modules. But a maximal green sequence can only contain the 10 bricks we listed for each quiver above, hence there are no maximal green sequence of length 10 or higher.

For $A = kQ$ however, we have the following maximal green sequence of length 9.

$$\left(4,\ \substack{4\\3},\ 3,\ \substack{4\\3\\1},\ \substack{3\\1},\ 1,\ \substack{4\\3\\2},\ \substack{3\\2},\ 2\right)$$

This maximal green sequence can be constructed using polygonal flips from the maximal green sequence $(1,\ 2,\ 3,\ 4)$ as follows, cf. Remark 4.19.

$$\begin{aligned}
\left(1,\ \underline{2,\ 3},\ 4\right) &\rightsquigarrow \left(1,\ 3,\ \substack{3\\2},\ \underline{2,\ 4}\right)\\
&\rightsquigarrow \left(1,\ 3,\ \underline{\substack{3\\2},\ 4},\ 2\right)\\
&\rightsquigarrow \left(\underline{1,\ 3},\ 4,\ \substack{4\\3\\2},\ \substack{3\\2},\ 2\right)\\
&\rightsquigarrow \left(3,\ \substack{3\\1},\ \underline{1,\ 4},\ \substack{4\\3\\2},\ \substack{3\\2},\ 2\right)\\
&\rightsquigarrow \left(3,\ \underline{\substack{3\\1},\ 4},\ 1,\ \substack{4\\3\\2},\ \substack{3\\2},\ 2\right)\\
&\rightsquigarrow \left(\underline{3,\ 4},\ \substack{4\\3\\1},\ \substack{3\\1},\ 1,\ \substack{4\\3\\2},\ \substack{3\\2},\ 2\right)\\
&\rightsquigarrow \left(4,\ \substack{4\\3},\ 3,\ \substack{4\\3\\1},\ \substack{3\\1},\ 1,\ \substack{4\\3\\2},\ \substack{3\\2},\ 2\right)
\end{aligned}$$

For $A = kQ'$, we first show that there is no maximal green sequence of length 9. Assume that there is such a sequence. A maximal green sequence can not contain all three bricks of length 2 supported on the first three vertices. Looking at the full list of all ten rigid bricks that can appear in a maximal green sequence, we see that a maximal green of length 9 must contain the following subsequence.

$$\left(1,\ \begin{smallmatrix}4&&1\\&3&\end{smallmatrix},\ \substack{4\\3},\ \substack{4\\3\\2},\ 3\right)$$

In particular, the brick 2 must appear after $\substack{4\\3\\2}$. But we need at least two length 2 bricks supported on the first three vertices, so by Theorem 4.35 and using Lemma 4.36 we must have

$$\left(1,\ \substack{1\\2},\ 2,\ \substack{1\\3},\ 3\right)$$

as the subsequence of bricks supported on the first three vertices, in particular, the brick $\substack{3\\2}$ does not appear in a maximal green sequence. Now consider the representation

$M \in \operatorname{mod}(kQ')$ of dimension vector $(1,1,1,1)$ defined as follows.

$$M = \begin{tikzcd} & 2 & \\ 1 \arrow[ur, "0"] \arrow[rr, "1"] & & 3 \arrow[ul, "1"'] & 4 \arrow[l, "1"'] \end{tikzcd}$$

The representation $M = \begin{smallmatrix} 1 & & 4 \\ & 3 & \\ & 2 & \end{smallmatrix}$ is indecomposable but not rigid, hence does not appear in a maximal green sequence. The only rigid submodules are $\begin{smallmatrix} 2 \end{smallmatrix}$, $\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$ and $\begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix}$. But $\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$ is not in the maximal green sequence and $\begin{smallmatrix} 2 \end{smallmatrix}$ comes after $\begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix}$, so the filtration is $\{0\} \subseteq \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} \subseteq M$, which requires that $\begin{smallmatrix} 1 \end{smallmatrix}$ comes after $\begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix}$, contradiction.

Finally, we can construct a maximal green sequence of length 8 using polygonal flips starting from $\left(\begin{smallmatrix} 2 \end{smallmatrix}, \begin{smallmatrix} 1 \end{smallmatrix}, \begin{smallmatrix} 3 \end{smallmatrix}, \begin{smallmatrix} 4 \end{smallmatrix} \right)$ as follows.

$$\begin{aligned}
\left(\begin{smallmatrix} 2 \end{smallmatrix}, \underline{\begin{smallmatrix} 3 \end{smallmatrix}, \begin{smallmatrix} 1 \end{smallmatrix}}, \begin{smallmatrix} 4 \end{smallmatrix} \right) &\rightsquigarrow \left(\begin{smallmatrix} 2 \end{smallmatrix}, \begin{smallmatrix} 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}, \underline{\begin{smallmatrix} 3 \end{smallmatrix}, \begin{smallmatrix} 4 \end{smallmatrix}} \right) \\
&\rightsquigarrow \left(\begin{smallmatrix} 2 \end{smallmatrix}, \begin{smallmatrix} 1 \end{smallmatrix}, \underline{\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 4 \end{smallmatrix}}, \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 3 \end{smallmatrix} \right) \\
&\rightsquigarrow \left(\underline{\begin{smallmatrix} 2 \end{smallmatrix}, \begin{smallmatrix} 1 \end{smallmatrix}}, \begin{smallmatrix} 4 \end{smallmatrix}, \begin{smallmatrix} 4 & & 1 \\ & 3 & \end{smallmatrix}, \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 3 \end{smallmatrix} \right) \\
&\rightsquigarrow \left(\begin{smallmatrix} 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 2 \end{smallmatrix}, \begin{smallmatrix} 4 \end{smallmatrix}, \begin{smallmatrix} 4 & & 1 \\ & 3 & \end{smallmatrix}, \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 3 \end{smallmatrix} \right)
\end{aligned}$$

We remark that the construction of a maximal green sequence of maximal length using polygonal flips also shows that the no-gap conjecture holds for both kQ and kQ'. $\square$

Chapter 5

The rotation lemma and applications

A useful result involving the existence of maximal green sequences is the *rotation lemma*. A general version of this result in the context for valued quivers and their mutations was proven by Brüstle–Hermes–Igusa–Todorov.

Theorem 5.1 ([BHIT17, Theorem 3]) *Let Q be a valued quiver with vertices $\{1, \ldots, n\}$ and let $(k_0, k_1, \ldots, k_{m-1})$ be a maximal green sequence for Q. Then there is a permutation σ on the set of vertices of Q, which only depends on the valued quiver Q, such that the sequence*

$$(k_1, k_2, \ldots, k_{m-1}, \sigma^{-1}(k_0))$$

is a maximal green sequence on the valued quiver $\mu_{k_0}Q$ obtained by mutating at k_0.

In other words, the existence of a maximal green sequence for some valued quiver Q gives the existence of maximal green sequences of the same length for all valued quivers that appear while one mutates along the maximal green sequence for Q.

Our aim is to give representation-theoretic versions of the rotation lemma for the representation-theoretic definition of maximal green sequences from Definition 4.1. The first version will be a more direct equivalent of the special case of Theorem 5.1 for non-valued quivers using finite-dimensional Jacobian algebras, which is a result already obtained by Igusa [Igu19] for Jacobian algebras of finite representation type. The second version requires a restriction on the maximal green sequence we start with, but can be formulated for any finite dimensional algebra and makes use of classical tilting theory in the sense of Brenner–Butler.

5.1 Quivers with potential, Jacobian algebras and their mutations

We give a short introduction to quivers with potentials and their Jacobian algebras. We will focus on the special case where the Jacobian algebras are finite-dimensional. This leads to a few simplifications of the general theory which goes back to work of Derksen–Weyman–Zelevinsky [DWZ08].

In the following, Q will always denote a finite quiver with set of vertices Q_0 and set of arrows Q_1. By a *cycle* in Q we mean a finite path in Q that starts and ends at the same vertex. We denote by $\widehat{kQ}$ the completion of the path algebra kQ with respect to the ideal generated by all arrows and let $\mathfrak{m} \subseteq \widehat{kQ}$ be the arrow ideal in $\widehat{kQ}$, the ideal of infinite linear combinations of paths of non-zero length in Q. By construction, $\widehat{kQ}$ is complete with respect to the ideal $\mathfrak{m}$, which is the Jacobson radical of $\widehat{kQ}$. For a subset $U \subseteq \widehat{kQ}$, its $\mathfrak{m}$-adic closure in $\widehat{kQ}$ is defined as

$$\overline{U} := \bigcap_{j \geq 1} (U + \mathfrak{m}^j).$$

If $I \subseteq \widehat{kQ}$ is an ideal, the closure $\overline{I} \subseteq \widehat{kQ}$ will again be an ideal of $\widehat{kQ}$.

We denote by $[\widehat{kQ}, \widehat{kQ}]$ the *additive commutator* of the (possibly infinite-dimensional) completed path algebra $\widehat{kQ}$ over Q, i.e. $[\widehat{kQ}, \widehat{kQ}] := \langle ab - ba \mid a, b \in \widehat{kQ} \rangle_k$

Definition 5.2 A *potential* on Q is an element $W \in \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$ that is a (possibly infinite) linear combination of equivalence classes of cycles in Q.

We call the pair (Q, W) a *quiver with potential.*

We remark that for a cycle $c = \alpha_\ell \cdots \alpha_1 \in \widehat{kQ}$ in Q the equivalence class of c in $\widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$ consists of all cyclic permutations of c, i.e. of all cycles of the form $\alpha_i \cdots \alpha_1 \alpha_\ell \cdots \alpha_{i+1}$ for $i \in \{1, \ldots, \ell\}$.

Definition 5.3 Let $|c| \in \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$ be the equivalence class of a cycle c in Q and let $\alpha \in Q_1$ be an arrow. The *cyclic derivative* of $|c|$ with respect to α is defined as

$$\partial_\alpha(|c|) := \sum_{\substack{\text{paths } c' \text{ with} \\ |c'\alpha| = |c|}} c' = \sum_{\substack{\text{paths } c' \text{with} \\ |\alpha c'| = |c|}} c'$$

Note that the cyclic derivative is an element in $\widehat{kQ}$. Extending linearly, we obtain a linear map ∂_α from the vector space of potentials on Q to the completed path algebra $\widehat{kQ}$.

Definition 5.4 Let (Q, W) be a quiver with potential. The *Jacobian ideal* is defined as the $\mathfrak{m}$-closure of the ideal generated by all cyclic derivatives of W.

$$\mathcal{J}(Q,W) = \overline{\langle \partial_\alpha W \mid \alpha \in Q_1 \rangle}$$

The *Jacobian algebra* associated to (Q,W) is defined as the quotient of the completed path algebra by the Jacobian ideal.

$$\mathcal{P}(Q,W) := \widehat{kQ}/\mathcal{J}(Q,W)$$

Note that we always have $\mathcal{J}(Q,W) \subseteq \mathfrak{m}^2$.

Remark 5.5 Assume now that the Jacobian algebra $A := \mathcal{P}(Q,W)$ of a quiver with potential (Q,W) is finite dimensional. Under the surjective algebra morphism $f\colon \widehat{kQ} \twoheadrightarrow A$ the Jacobson radical $\mathfrak{m}$ of $\widehat{kQ}$ is mapped to the Jacobson radical $\mathfrak{r} \subseteq A$, i.e. we have $f(\mathfrak{m}) \subseteq \mathfrak{r}$. It follows that for all $\ell \geq 1$ we have $f(\mathfrak{m}^\ell) \subseteq \mathfrak{r}^\ell$. But since A is finite-dimensional, it has finite radical length and therefore $\mathfrak{r}^\ell = 0$ for some $\ell \geq 1$. Hence if the Jacobian algebra is finite dimensional we have $\mathfrak{m}^\ell \subseteq \mathcal{J}(Q,W) \subseteq \mathfrak{m}^2$.

The Jacobian ideal $\mathcal{J}(Q,W)$ is defined as the $\mathfrak{m}$-adic closure of the ideal $J = \langle \partial_\alpha W \mid \alpha \in Q_1 \rangle$ generated by the cyclic derivatives of the potential. Hence we have

$$\mathfrak{m}^\ell \subseteq \mathcal{J}(Q,W) = \bigcap_{j \geq 1} (J + \mathfrak{m}^j),$$

so $\mathfrak{m}^\ell \subseteq J + \mathfrak{m}^j$ for all $j \geq 1$. The canonical embedding $kQ \hookrightarrow \widehat{kQ}$ induces an isomorphism $kQ/\mathfrak{a}^\ell \xrightarrow{\sim} \widehat{kQ}/\mathfrak{m}^\ell$, where $\mathfrak{a} \subseteq kQ$ is the ideal generated by all arrows. The factor algebra $\widehat{kQ}/\mathfrak{m}^\ell$ is finite dimensional, which means that the descending chain of ideals

$$(J+\mathfrak{m})/\mathfrak{m}^\ell \supseteq (J+\mathfrak{m}^2)/\mathfrak{m}^\ell \supseteq (J+\mathfrak{m}^3)/\mathfrak{m}^\ell \supseteq \dots$$

must become stationary, which implies that the descending chain

$$J+\mathfrak{m} \supseteq J+\mathfrak{m}^2 \supseteq J+\mathfrak{m}^3 \supseteq \dots$$

becomes stationary. But then $\mathcal{J}(Q,W) = J + \mathfrak{m}^r$ for some $r \geq 1$ and we can assume $r \geq \ell$. Moreover, this implies that we may write the Jacobian algebra as the following factor algebra

$$\mathcal{P}(Q,W) = \widehat{kQ}/\mathcal{J}(Q,W) \simeq \left(\widehat{kQ}/\mathfrak{m}^\ell\right) \Big/ \left((J+\mathfrak{m}^r)/\mathfrak{m}^\ell\right).$$

In particular, we see that the finite dimensional Jacobian algebra $A = \mathcal{P}(Q,W)$ can also be written as the factor algebra of the regular path algebra kQ without completion modulo an admissible ideal I. Tracing the isomorphisms, we see that we can write I as

$$I = (J+\mathfrak{m}^r) \cap kQ + \mathfrak{a}^\ell = (J+\mathfrak{m}^\ell) \cap kQ.$$

Note that even in the case of a potential W with only finitely many summands the ideal $I \subseteq kQ$ does not have to be the ideal generated by the cyclic derivatives of W, the construction implies that we may have to add at least some nilpotency relations.

For an explicit example, we take the Markov quiver Q

$$\begin{tikzcd} & 2 \arrow[dl, shift right, "\beta_1"'] \arrow[dl, shift left, "\alpha_1"] & \\ 1 \arrow[rr, shift left, "\alpha_3"] \arrow[rr, shift right, "\beta_3"'] & & 3 \arrow[ul, shift right, "\alpha_2"'] \arrow[ul, shift left, "\beta_2"] \end{tikzcd}$$

with potential $W = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 - \alpha_1\beta_2\alpha_3\beta_1\alpha_2\beta_3$, which appears in work of Labardini-Fragoso, see [LF09, Example 8.2]. In particular, it is shown that the Jacobian algebra $\mathcal{P}(Q, W)$ is finite dimensional and that all paths of length 7 are zero. By our earlier considerations, this implies that $\mathcal{P}(Q, W)$ is a factor algebra of $kQ/\mathfrak{a}^7$. In [Pla13, Example 4.3], a complete set of generating relations for an ideal I such that $\mathcal{P}(Q, W) \simeq kQ/I$ is shown, and this set of relations indeed only contains the cyclic derivatives of W and nilpotency relations.

For the following definition, note that a morphism of algebras $\varphi\colon A \to B$ induces a morphism of algebras $\overline{\varphi}\colon A/[A, A] \to B/[B, B]$, as for $a, a' \in A$

$$\varphi(a'a - aa') = \varphi(a')\varphi(a) - \varphi(a)\varphi(a') \in [B, B].$$

Definition 5.6 Let (Q, W) and (Q', W') be quivers with potential. Then (Q, W) and (Q', W') are called *right-equivalent* if $Q_0 = Q'_0$ and if there is an isomorphism of algebras $\psi\colon \widehat{kQ} \to \widehat{kQ'}$ such that $\psi(e_i) = e_i$ for $i \in Q_0$ and $\overline{\psi}(W) = W'$ with the induced morphism $\overline{\psi}\colon \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}] \to \widehat{kQ'}/[\widehat{kQ'}, \widehat{kQ'}]$ as above.

Definition 5.7 A quiver with potential (Q, W) is called *reduced* if W contains no equivalence classes of loops or 2-cycles. The quiver with potential is called *trivial* if $\mathcal{P}(Q, W)$ is semisimple.

Theorem 5.8 (Splitting theorem, cf. [DWZ08, Theorem 4.6]) *Let (Q, W) be a quiver with potential and assume that Q has no loops. Then there exist a trivial $(Q_\mathsf{t}, W_\mathsf{t})$ and a reduced $(Q_\mathsf{r}, W_\mathsf{r})$ quiver with potential with $(Q_\mathsf{t})_0 = (Q_\mathsf{t})_0 = Q_0$ such that (Q, W) is right-equivalent to $(Q_\mathsf{t} \cup Q_\mathsf{r}, W_\mathsf{t} + W_\mathsf{r})$.*

The trivial and reduced parts $(Q_\mathsf{t}, W_\mathsf{t})$ and $(Q_\mathsf{r}, W_\mathsf{r})$ are determined up to right-equivalence and the embedding of quivers $Q_\mathsf{r} \hookrightarrow Q_\mathsf{t} \cup Q_\mathsf{r}$ induces an isomorphism of Jacobian algebras $\mathcal{P}(Q_\mathsf{r}, W_\mathsf{r}) \xrightarrow{\sim} \mathcal{P}(Q_\mathsf{t} \cup Q_\mathsf{r}, W_\mathsf{t} + W_\mathsf{r})$.

Definition 5.9 Let (Q, W) be a quiver with potential and assume that Q has no loops and no 2-cycles. Let $j \in Q_0$. Then the *premutation* $\tilde{\mu}_j(Q, W)$ of (Q, W) at the vertex j is the quiver with potential $(\tilde{Q}, \tilde{W}) := \tilde{\mu}_j(Q, W)$ defined as follows.

- The quiver $\tilde{Q}$ has the same set of vertices $\tilde{Q}_0 = Q_0$ as Q. The set of arrows is defined as

$$\begin{aligned}\tilde{Q}_1 :=&\{\alpha : i_1 \to i_2 \mid \alpha \in Q_1 \text{ and } i_1 \neq j \neq i_2\} \\ &\cup \{\alpha^* : j \to i_1 \mid (\alpha\colon i_1 \to j) \in Q_1\} \\ &\cup \{\beta^* : i_2 \to j \mid (\beta\colon j \to i_2) \in Q_1\} \\ &\cup \{[\beta\alpha] : i_1 \to i_2 \mid (\alpha\colon i_1 \to j), (\beta\colon j \to i_2) \in Q_1\}.\end{aligned}$$

- The potential $\tilde{W}$ is given by

$$\tilde{W} := U + \Delta_j(Q),$$

  where we obtain U by choosing a lift $\overline{W} \in \widehat{kQ}$ of W such that no path in $\overline{W}$ starts at j, which is possible since Q has no 2-cycles. Then we replace each occurrence of the product $\beta\alpha$ by the new arrow $[\beta\alpha]$ and thereby obtain an element $\overline{U} \in \widehat{k\tilde{Q}}$. Define U as the equivalence class of $\overline{U}$ in $\widehat{k\tilde{Q}}/[\widehat{k\tilde{Q}}, \widehat{k\tilde{Q}}]$.

  The second summand is defined as the equivalence class of the following sum of 3-cycles in $\tilde{Q}$.

$$\Delta_j(Q) := \sum_{(\alpha:i_1\to j),(\beta:j\to i_2)\in Q_1} [\beta\alpha]\alpha^*\beta^*$$

The *mutation* $\mu_j(Q, W)$ of the quiver with potential (Q, W) at the vertex $j \in Q_0$ is defined as the reduced part of the premutation $(\tilde{Q}, \tilde{W})$, i.e. using the notation from Theorem 5.8 it is defined by $\mu_j(Q, W) := (\tilde{Q}_{\mathsf{r}}, \tilde{W}_{\mathsf{r}})$.

Proposition 5.10 ([DWZ08, Corollary 6.6]) *Let (Q, W) be a quiver with potential and let $j \in Q_0$ be a vertex such that mutation of (Q, W) at j is defined.*

Then the Jacobian algebra $\mathcal{P}(Q, W)$ is finite dimensional if and only if the Jacobian algebra $\mathcal{P}(\mu_j(Q, W))$ is finite dimensional.

Definition 5.11 Let Q be a quiver without loops and without 2-cycles. A potential W on Q is called *non-degenerate* if for any sequence $j_1, \ldots, j_\ell \in Q_0$ of vertices the mutated quiver with potential $\mu_{j_1}(\mu_{j_2}(\ldots(\mu_{j_\ell}(Q, W))))$ has no 2-cycles.

Note that we need to assume non-degeneracy of the potential so that we can perform iterated mutations on quivers with potentials.

Definition 5.12 Let $|c| \in \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$ be the equivalence class of a cycle c in Q and let

$\alpha, \beta \in Q_1$ be arrows. We define the *second derivative* of $|c|$ with respect to α as

$$\partial_{\beta,\alpha}(|c|) := \sum_{\substack{\text{paths } c'' \text{ with} \\ |c''\beta\alpha|=|c|}} c'' = \sum_{\substack{\text{paths } c'' \text{ with} \\ |\alpha c''\beta|=|c|}} c''$$

As with the cyclic derivative, we extend $\partial_{\beta,\alpha}$ linearly to obtain a linear map $\partial_{\beta,\alpha}$ from the vector space of potentials on Q to the completed path algebra $\widehat{kQ}$.

Lemma 5.13 *Let $W \in \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$ be a potential on Q and let $\beta \in Q_1$ be an arrow. Then the following hold.*

$$\begin{aligned} \partial_\beta(W) &= \sum_{\alpha \in Q_1} \alpha \partial_{\beta,\alpha}(W) \\ &= \sum_{\gamma \in Q_1} \partial_{\gamma,\beta}(W)\gamma \end{aligned}$$

Proof. Both follow directly from the definitions by considering a single cycle c in Q and using linearity, e.g.

$$\sum_{\alpha \in Q_1} \alpha \partial_{\beta,\alpha}(|c|) = \sum_{\alpha \in Q_1} \sum_{\substack{\text{paths } c'' \text{ with} \\ |\alpha c''\beta|=|c|}} \alpha c'' = \sum_{\substack{\text{paths } c' \text{ with} \\ |c'\beta|=|c|}} c' = \partial_\beta(|c|). \qquad \square$$

Definition 5.14 ([DWZ08, Section 10]) Let (Q, W) be a quiver Q with non-degenerate potential W. Let $j \in Q_0$ be and let $(\tilde{Q}, \tilde{W}) := \tilde{\mu}_j(Q, W)$ be the premutation of (Q, W) at the vertex j. We define a map

$$G\colon \operatorname{mod}(\mathcal{P}(Q, W)) \to \operatorname{mod}(\mathcal{P}(\tilde{Q}, \tilde{W}))$$

on objects as follows. Let $M \in \operatorname{mod}(\mathcal{P}(Q, W))$, considered as a representation of the quiver Q. For an element $r \in \widehat{kQ}$ let $M(r)\colon M \to M$ be the action of r on M. If r is a linear combination of paths from some vertex j_1 to some vertex j_2, we can consider this action as a linear map $M(r)\colon M_{j_1} \to M_{j_2}$.

With this notation, we can define the following linear maps, where π_α and ι_β denote projection and inclusion maps of direct summands.

$$\begin{aligned}
f &:= \sum_{\substack{\alpha\in Q_1\\ t(\alpha)=j}} M(\alpha)\pi_\alpha &&: \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=j}} M_{s(\alpha)} \to M_j\\
g &:= \sum_{\substack{\beta\in Q_1\\ s(\alpha)=j}} \iota_\beta M(\beta) &&: M_j \to \bigoplus_{\substack{\beta\in Q_1\\ s(\beta)=j}} M_{t(\beta)}\\
h &:= \sum_{\substack{\alpha,\beta\in Q_1\\ t(\alpha)=j=s(\beta)}} \iota_\alpha M(\partial_{\beta,\alpha}(W))\pi_\beta &&: \bigoplus_{\substack{\beta\in Q_1\\ s(\beta)=j}} M_{t(\beta)} \to \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=j}} M_{s(\alpha)}
\end{aligned}$$

Using Lemma 5.13 we have

$$\begin{aligned}
fh &= \Big(\sum_{\substack{\alpha\in Q_1\\ t(\alpha)=j}} M(\alpha)\pi_\alpha\Big)\Big(\sum_{\substack{\alpha,\beta\in Q_1\\ t(\alpha)=j=s(\beta)}} \iota_\alpha M(\partial_{\beta,\alpha}(W))\pi_\beta\Big) = \sum_{\substack{\alpha,\beta\in Q_1\\ t(\alpha)=j=s(\beta)}} M(\alpha\partial_{\beta,\alpha}(W))\pi_\beta\\
&= \sum_{\substack{\beta\in Q_1\\ s(\beta)=j}} \sum_{\alpha\in Q_1} M(\alpha\partial_{\beta,\alpha}(W))\pi_\beta = \sum_{\substack{\beta\in Q_1\\ s(\beta)=j}} M(\partial_\beta(W))\pi_\beta = 0
\end{aligned}$$

and similarly $hg = 0$.

We are now able to define $\tilde{M} := GM$ as the representation of $\tilde{Q}$ with

- For all vertices $j \neq i \in Q_0$ we set $\tilde{M}_i := M_i$ and on the vertex j we set

$$\tilde{M}_j := \frac{\ker(f)}{\operatorname{im}(h)} \oplus \operatorname{im}(h) \oplus \frac{\ker(h)}{\operatorname{im}(g)}.$$

- For an $\gamma \in Q_1 \cap \tilde{Q}_1$ of Q that neither starts nor ends in j, we set $\tilde{M}(\gamma) := M(\gamma)$. Let $\alpha, \beta \in Q_1$ be arrows with $t(\alpha) = j = s(\beta)$. We have to define $\tilde{M}$ on the arrows $\alpha^*, \beta^*, [\beta\alpha] \in \tilde{Q}_1$. We set

$$\tilde{M}([\beta\alpha]) := M(\beta\alpha) = M(\beta)M(\alpha)\colon \tilde{M}_{s(\alpha)} \to \tilde{M}_{t(\beta)}$$

Now choose a retraction $\rho\colon \bigoplus_{\beta\in Q_1\colon s(\beta)=j} M_{t(\beta)} \to \ker(h)$ to the inclusion of the kernel of h and a section $\sigma\colon \ker(f)/\operatorname{im}(h) \to \ker(f)$ to the canonical factor map onto the quotient $\ker(f)/\operatorname{im}(h)$. Then we set

$$\tilde{M}(\alpha^*) := \begin{pmatrix} \iota_\alpha\sigma & \iota_\alpha & 0 \end{pmatrix} : \tilde{M}_j \to \tilde{M}_{s(\alpha)}$$

and

$$\tilde{M}(\beta^*) := \begin{pmatrix} 0 \\ -h\iota_\beta \\ -p\rho\iota_\beta \end{pmatrix} : \tilde{M}_{t(\beta)} \to \tilde{M}_j$$

with the canonical factor map $p\colon \ker(h)/\operatorname{im}(g)$.

Let $S(j)$ denote the simple one-dimensional representation supported at the vertex j. We can define the stable category $\operatorname{mod}(\mathcal{P}(Q,W))/\operatorname{add}(S(j))$ as the additive factor category modulo the ideal of all morphisms that factor through $\operatorname{add}(S(j))$.

Theorem 5.15 ([BIRS11, Theorem 7.1]) *The map G from the previous definition induces an equivalence of categories between the stable categories* $\operatorname{mod}(\mathcal{P}(Q,W))/\operatorname{add}(S(j))$ *and* $\operatorname{mod}(\mathcal{P}(\tilde{Q},\tilde{W}))/\operatorname{add}(S(j))$.

In other words, the module categories of Jacobian algebras that are related by a single mutation are *nearly Morita equivalent.*

5.2 The rotation lemma for Jacobian algebras

Let (Q,W) be a quiver with non-degenerate potential such that the Jacobian algebra $\mathcal{P}(Q,W)$ is finite dimensional. Suppose that $(S_1,S_2,\dots,S_m)$ is a maximal green sequence for $\mathcal{P}(Q,W)$. Then S_1 is a simple $\mathcal{P}(Q,W)$-module, so $S_1 \simeq S(j)$ for some vertex $j \in Q_0$. To categorify the combinatorial rotation lemma, cf. Theorem 5.1, we need to find a map or preferably a functor $F\colon \operatorname{mod}(\mathcal{P}(Q,W)) \to \operatorname{mod}(\mathcal{P}(\tilde{Q},\tilde{W}))$, where $(\tilde{Q},\tilde{W}) = \mu_j(Q,W)$ is the mutation of (Q,W) at j, such that the rotated sequence

$$(FS_2, FS_3, \dots, FS_m, S(j))$$

is a maximal green sequence for $\mathcal{P}(\tilde{Q},\tilde{W})$.

The following example shows that the map G from Definition 5.14 can not be used to categorify the rotation lemma.

Example 5.16 Consider the linearly oriented A_3-quiver

$$Q = \; 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$$

which clearly only has the trivial potential $W=0$. The Jacobian algebra $\mathcal{P}(Q,W)$ is the path algebra over Q and has a maximal green sequence

$$\left({\scriptstyle 2}\,,\, {\scriptstyle \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}}\,,\, {\scriptstyle 1}\,,\, {\scriptstyle 3} \right).$$

The first module in this sequence is the simple module at 2. Mutating (Q,W) at this vertex gives the quiver with potential $(\tilde{Q}, \tilde{W}) = \mu_2(Q, W)$ with

$$\tilde{Q} = \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

and $\tilde{W} = \gamma\beta\alpha$. It follows that the Jacobian algebra $\mathcal{P}(\tilde{Q}, \tilde{W})$ is isomorphic to the quotient of the path algebra $k\tilde{Q}$ modulo all paths of length 2.

However, if we apply the map G, which gives a map on objects from $\mathrm{mod}(\mathcal{P}(Q, W))$ to $\mathrm{mod}(\mathcal{P}(\tilde{Q}, \tilde{W}))$, to the bricks in the maximal green sequence for kQ and rotate accordingly, we obtain the sequence

$$\left(G\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right), G\left(\begin{smallmatrix} 1 \end{smallmatrix}\right), G\left(\begin{smallmatrix} 3 \end{smallmatrix}\right), \begin{smallmatrix} 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 2 \end{smallmatrix}\right).$$

But this is not a maximal green sequence for $\mathcal{P}(\tilde{Q}, \tilde{W})$, as it does not contain every simple module, cf. Corollary 4.10.

Definition 5.17 Let (Q, W) be a quiver with non-degenerate potential. Let $j \in Q_0$ be some vertex. Let $(\tilde{Q}, \tilde{W}) = \tilde{\mu}_j(Q, W)$ be the premutation of (Q, W) at j. We define two functors

$$\begin{aligned} F_j^+ : \quad & \mathrm{mod}(\mathcal{P}(Q, W)) \longrightarrow \mathrm{mod}(\mathcal{P}(\tilde{Q}, \tilde{W})) \\ F_j^- : \quad & \mathrm{mod}(\mathcal{P}(Q, W)) \longrightarrow \mathrm{mod}(\mathcal{P}(\tilde{Q}, \tilde{W})) \end{aligned}$$

as follows. Let $M \in \mathrm{mod}(\mathcal{P}(Q, W))$, which we will consider as a representation of Q. As in Definition 5.14, we define the following linear maps.

$$\begin{array}{rcllrcl} f & := & \displaystyle\sum_{\substack{\alpha \in Q_1 \\ t(\alpha) = j}} M(\alpha)\pi_\alpha & : & \displaystyle\bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha) = j}} M_{s(\alpha)} & \to & M_j \\ g & := & \displaystyle\sum_{\substack{\beta \in Q_1 \\ s(\alpha) = j}} \iota_\beta M(\beta) & : & M_j & \to & \displaystyle\bigoplus_{\substack{\beta \in Q_1 \\ s(\beta) = j}} M_{t(\beta)} \\ h & := & \displaystyle\sum_{\substack{\alpha, \beta \in Q_1 \\ t(\alpha) = j = s(\beta)}} \iota_\alpha M(\partial_{\beta,\alpha}(W))\pi_\beta & : & \displaystyle\bigoplus_{\substack{\beta \in Q_1 \\ s(\beta) = j}} M_{t(\beta)} & \to & \displaystyle\bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha) = j}} M_{s(\alpha)} \end{array}$$

The maps satisfy $fh = 0$ and $hg = 0$. Now we construct two representations M_i^+ and M_i^- of the mutated quiver $\tilde{Q}$ as follows.

- On vertices $j \neq i \in Q_0$ we set $M_i^+ := M_i$ and $M_i^- := M_i$. On the vertex $j \in Q_0$ we set

$$\begin{aligned} M_j^+ &:= \operatorname{coker}(g) = \bigoplus_{\beta \in Q_1 \colon s(\beta)=j} \operatorname{coker}(M_\beta) \\ M_j^- &:= \ker(f) = \bigoplus_{\alpha \in Q_1 \colon t(\alpha)=j} \ker(M_\alpha). \end{aligned}$$

- If $\gamma \in Q_1 \cap \tilde{Q}_1$ is an arrow of Q that neither starts nor ends in j, we set $M^+(\gamma) := M(\gamma)$ and $M^-(\gamma) := M(\gamma)$. For arrows $\alpha, \beta \in Q_1$ with $t(\alpha) = j = s(\beta)$ we have to define M^+ and M^- on the arrows $[\beta\alpha], \alpha^*, \beta^* \in \tilde{Q}_1$.

  For this, we define linear maps f^-, g^-, f^+, g^+ using the following diagram, where

  $$M_{\text{in}} := \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=j}} M_{s(\alpha)} \qquad \text{and} \qquad M_{\text{out}} := \bigoplus_{\substack{\beta \in Q_1 \\ s(\beta)=j}} M_{t(\beta)}$$

  and all triangles are supposed to be anti-commutative.

$$\begin{tikzcd}
 & & & M_j \arrow[dr, "g"] & & & \\
M_{\text{out}} \arrow[rr, shift left, "h"] \arrow[dr, "f^-"'] & & M_{\text{in}} \arrow[ll, shift left, "gf"] \arrow[ur, "f"] & & M_{\text{out}} \arrow[ll, "h"'] \arrow[rr, shift left, "h"] \arrow[dr, "f^+"'] & & M_{\text{in}} \arrow[ll, shift left, "gf"] \\
 & M_j^- \arrow[ur, "g^-"'] & & & & M_j^+ \arrow[ur, "g^+"'] \arrow[dr] & \\
0 \arrow[ur] & & & & & & 0
\end{tikzcd}$$

  For g^-, note that $M_j^- = \ker(f)$ by definition. Since $fh = 0$ there is a unique $f^- \colon M_{\text{out}} \to M_j^-$ such that $-h = g^- f^-$. Similarly, f^+ is the cokernel of g and since $hg = 0$ we can find $g^+ \colon M_j^+ \to M_{\text{in}}$ with $-h = g^+ f^+$.

  Now we set

$$\begin{aligned} M^+([\beta\alpha]) &:= M(\beta)M(\alpha) & \qquad M^-([\beta\alpha]) &:= M(\beta)M(\alpha) \\ M^+(\alpha^*) &:= \pi_\alpha g^+ & M^-(\alpha^*) &:= \pi_\alpha g^- \\ M^+(\beta^*) &:= f^+ \iota_\beta & M^+(\beta^*) &:= f^- \iota_\beta. \end{aligned}$$

So far, we have defined M^+ and M^- as representations of $\tilde{Q}$. To show that they are indeed representations of the Jacobian algebra $\mathcal{P}(\tilde{Q}, \tilde{W})$, we need to show that $M^+(\partial_\gamma(\tilde{W})) = 0$ and $M^-(\partial_\gamma(\tilde{W})) = 0$ for all arrows $\gamma \in \tilde{Q}_1$.

Recall that the potential of the premutation is given by

$$\tilde{W} := U + \Delta_j(Q)$$

where we get U from W by replacing each occurrence of $\beta\alpha$ in some rotation of a cycle by $[\beta\alpha] \in \tilde{Q}_1$ whenever possible and $\Delta_j(Q)$ is the sum of all new 3-cycles in $\tilde{Q}$ of the form $[\beta\alpha]\alpha^*\beta^*$.

If we take a cyclic derivative $\partial_\gamma(\tilde{W})$ with respect to an arrow $\gamma \in Q_1 \cap \tilde{Q}_1$ we have $M^+(\partial_\gamma \tilde{W}) = M(\partial_\gamma(W)) = 0$ since $M^+([\beta\alpha]) = M(\beta)M(\alpha)$.

Now consider a new arrow in $\tilde{Q}$ of type $[\beta\alpha] \in \tilde{Q}_1$ where $\alpha, \beta \in Q_1$ with $t(\alpha) = j = s(\beta)$. Then by definition of $\partial_{\beta,\alpha}$ the cyclic derivative $\partial_{[\beta\alpha]}(U)$ equals the second derivative $\partial_{\beta,\alpha}(W)$ after each occurrence of a product $\beta\alpha$ is replaced by $[\beta\alpha]$. Moreover, $\partial_{[\beta\alpha]}(\Delta_j(Q)) = \alpha^*\beta^*$ and therefore

$$\begin{aligned} M^+(\partial_{[\beta\alpha]}(\tilde{W})) &= M(\partial_{\beta,\alpha}(W)) + M^+(\alpha^*)M^+(\beta^*) \\ &= \pi_\alpha h \iota_\beta + \pi_\alpha g^+ f^+ \iota_\beta \\ &= \pi_\alpha (h + g^+ f^+) \iota_\beta \\ &= 0. \end{aligned}$$

Finally, we have

$$\begin{aligned} M^+(\partial_{\alpha^*}(\tilde{W})) &= \sum_{\beta \in Q_1 :\, s(\beta) = j} M^+(\beta^*[\beta\alpha]) \\ &= \sum_{\beta \in Q_1 :\, s(\beta) = j} f^+ \iota_\beta M(\beta) M(\alpha) \\ &= f^+ g M(\alpha) \\ &= 0, \end{aligned}$$

since f^+ is the cokernel of g and we have

$$\begin{aligned} M^+(\partial_{\beta^*}(\tilde{W})) f^+ &= \sum_{\alpha \in Q_1 :\, t(\alpha) = j} M^+([\beta\alpha]\alpha^*) f^+ \\ &= \sum_{\alpha \in Q_1 :\, t(\alpha) = j} M(\beta) M(\alpha) \pi_\alpha g^+ f^+ \\ &= \sum_{\alpha \in Q_1 :\, t(\alpha) = j} -M(\beta) M(\alpha) \pi_\alpha h \\ &= -M(\beta) f h \\ &= 0, \end{aligned}$$

since $fh = 0$. Note that f^+ as the cokernel of g is surjective, hence the result above implies that $M^+(\partial_{\beta^*}) = 0$.

The previous calculations show that M^+ and M^- are indeed representations of $\mathcal{P}(\tilde{Q}, \tilde{W})$. Finally, note that both definitions are functorial in M, since they are defined using kernels and cokernels and their universal properties. Hence we actually have defined two functors F_j^+ and F_j^- as follows.

$$\begin{aligned} F_j^+\colon \quad \mathrm{mod}(\mathcal{P}(Q,W)) &\longrightarrow \mathrm{mod}(\mathcal{P}(\tilde{Q},\tilde{W})) \\ M &\longmapsto M^+ \\ F_j^-\colon \quad \mathrm{mod}(\mathcal{P}(Q,W)) &\longrightarrow \mathrm{mod}(\mathcal{P}(\tilde{Q},\tilde{W})) \\ M &\longmapsto M^- \end{aligned}$$

Our main application is the following tilting theorem.

Proposition 5.18 *Let (Q, W) be a quiver with non-degenerate potential, let $j \in Q_0$ and let $F_j^+, F_j^-\colon \mathrm{mod}(\mathcal{P}((Q,W)) \to \mathrm{mod}(\mathcal{P}(\tilde{\mu}_j(Q,W)))$ be the functors from Definition 5.17. Let $S(j)$ be the simple one-dimensional representation supported at j. We write $S(j)$ for the simple representation in both $\mathrm{mod}(\mathcal{P}(Q,W))$ and $\mathrm{mod}(\mathcal{P}(\tilde{\mu}_j(Q,W)))$.*

Then F_j^+ and F_j^- induce the following equivalences of additive categories.

$$\mathrm{mod}(\mathcal{P}(Q,W)) \supseteq S(j)^\perp \xrightarrow[\sim]{F_j^+} {}^\perp S(j) \subseteq \mathrm{mod}(\mathcal{P}(\tilde{\mu}_j(Q,W)))$$

$$\mathrm{mod}(\mathcal{P}(Q,W)) \supseteq {}^\perp S(j) \xrightarrow[\sim]{F_j^-} S(j)^\perp \subseteq \mathrm{mod}(\mathcal{P}(\tilde{\mu}_j(Q,W)))$$

Proof. Note that the linear map f^+ in Definition 5.17 is a cokernel and therefore always surjective, hence we have $F_j^+ M \in {}^\perp S(j)$ for all representation $M \in \mathrm{mod}(\mathcal{P}(Q,W))$.

A representation $M \in \mathrm{mod}(\mathcal{P}(Q,W))$ is in the perpendicular category $S(j)^\perp$ if and only if the linear map g from Definition 5.17 is injective. But then g is the kernel of f^+. Hence we can construct a new functor $G_j^-\colon \mathrm{mod}(\mathcal{P}(\tilde{\mu}_j(Q,W))) \to \mathrm{mod}(\mathcal{P}(Q,W))$ analogous to F_j^- by forming the kernel of the map $M_{\mathrm{out}} \to M_j^+$, which will be a quasi-inverse to F_j^+. It follows that F_j^+ is an equivalence of additive categories between $S(j)^\perp$ in $\mathrm{mod}(\mathcal{P}(Q,W))$ and ${}^\perp S(j)$ in $\mathrm{mod}(\mathcal{P}(\tilde{\mu}_j(Q,W)))$.

The statement for F_j^- follows by duality. □

Remark 5.19 The equivalences $F_j^+\colon S(j)^\perp \to {}^\perp S(j)$ and $F_j^-\colon {}^\perp S(j) \to S(j)^\perp$ from the previous Proposition 5.18 already appeared in work of Keller–Yang, cf. [KY11], see also [Kel12]. For a vertex $j \in Q_0$, Keller–Yang construct two derived equivalences $\Phi_j^\pm$ between the so called *Ginzburg dg algebras* $\Gamma := \Gamma(Q,W)$ associated to a quiver with potential. The finite dimensional derived category $\mathcal{D}_{\mathrm{fd}}(\Gamma)$, i.e. the full subcategory on all objects with finite dimensional total homology, has a canonical t-structure with heart $\mathcal{A} \subseteq \mathcal{D}_{\mathrm{fd}}(\Gamma)$ equivalent to $\mathrm{mod}(\mathcal{P}(Q, W))$.

Let $\tilde{\Gamma} := \Gamma(\mu_j(Q, W))$ be the Ginzburg dg algebra of the quiver of potential obtained from (Q, W) after mutating at j and let $\tilde{\mathcal{A}} \subseteq \mathcal{D}_{\mathrm{fd}}(\tilde{\Gamma})$ be the heart of the canonical t-structure, which is equivalent to $\mathrm{mod}(\mathcal{P}(\mu_j(Q, W)))$. Following [Kel12, Section 7.5], $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are related as follows.

The equivalences $\Phi_j^{\pm} \colon \mathcal{D}_{\mathrm{fd}}(\Gamma) \to \mathcal{D}_{\mathrm{fd}}(\tilde{\Gamma})$ send $\mathcal{A}$ to two new hearts $\mu_j^{\pm}(\mathcal{A}) \subseteq \mathcal{D}_{\mathrm{fd}}(\tilde{\Gamma})$ of two t-structures. In fact, the hearts $\mu_j^{\pm}(\mathcal{A})$ are so called *tilted hearts* of $\mathcal{A}$ corresponding to the torsion pairs $(\mathrm{add}(S(S_j)), S(j)^{\perp})$ and $({}^{\perp}S(j), \mathrm{add}(S(j)))$ in $\mathcal{A} \simeq \mathrm{mod}(\mathcal{P}(Q, W))$. In particular, we have as subcategories of $\tilde{\mathcal{A}} \simeq \mathrm{mod}(\mathcal{P}(\mu_j(Q, W)))$

$$\mu_j^{+}(\mathcal{A}) \cap \tilde{\mathcal{A}} = {}^{\perp}S(j) \quad \text{and} \quad \mu_j^{-}(\mathcal{A}) \cap \tilde{\mathcal{A}} = S(j)^{\perp}$$

These equations imply the existence of equivalences $F_j^{\pm}$ as in Proposition 5.18.

Lemma 5.20 *Let A be a finite dimensional algebra and suppose we are given a forward* Hom*-orthogonal sequence $S_\bullet = (S_1, \ldots, S_m)$ of bricks S_i, i.e. for $i < j$ we have* $\mathrm{Hom}_A(S_i, S_j) = 0$. *Let S be a simple A-module that does not appear as an S_i in the sequence $S_\bullet$. Then there is an $j \in \{0, \ldots, m\}$ such that*

$$(S_1, \ldots, S_j, S, S_{j+1}, \ldots, S_m)$$

is also a forward Hom*-orthogonal sequence of bricks.*

Proof. Let $j \in \{0, \ldots, m\}$ be minimal such that $\mathrm{Hom}_A(S, S_i) = 0$ for $i > j$. If $j = 0$ we are done, as we can put S to the front.

If $j \geq 1$, then by minimality there is a non-zero morphism $f \colon S \to S_j$. Since S is simple, f is injective but not surjective as S does not appear in $S_\bullet$.

Let $i \leq j$ and assume that there is a non-zero morphism $g \colon S_i \to S$. If we have $i = j$, then $fg \colon S_j \to S_j$ is o non-zero endomorphism that is not an isomorphism, since it factors over S. Since S_j is a brick, we have a contradiction. If $i < j$, then $fg \colon S_i \to S_j$ is a non-zero morphism and again we have a contradiction, this time to forward Hom-orthogonality of $S_\bullet$. □

The following theorem has been proven by Igusa for Jacobian algebras of finite representation type, cf. [Igu19, Corollary 2.15]. Our theorem below shows that the result holds in greater generality, namely for finite dimensional Jacobian algebras with non-degenerate potential.

Theorem 5.21 *Let (Q, W) be a quiver with non-degenerate potential. Assume that the Jacobian algebra $\mathcal{P}(Q, W)$ is finite dimensional. Let $(S_1, \ldots, S_m)$ be a maximal green* *sequence for $\mathcal{P}(Q,W)$. By Proposition 4.2 both S_1 and S_m are simple modules, so there are $j^+, j^- \in Q_0$ such that $S_1 = S(j^+)$ and $S_m = S(j^-)$. Then the following hold.*

(1) *The sequence of bricks $(F^+_{j^+}S_2, F^+_{j^+}S_3, \ldots, F^+_{j^+}S_m, S(j^+))$ is a maximal green sequence for $\mathcal{P}(\tilde{\mu}_{j^+}(Q,W))$.*

(2) *The sequence of bricks $(S(j^-), F^-_{j^-}S_1, F^-_{j^-}S_2, \ldots, F^-_{j^-}S_{m-1})$ is a maximal green sequence for $\mathcal{P}(\tilde{\mu}_{j^-}(Q,W))$.*

Proof. We give a full proof for (1), as the statement (2) follows from (1) by duality.

By assumption, the sequence $(S(j^+), S_2, \ldots, S_m)$ is a maximal green sequence for $\mathcal{P}(Q,W)$. In particular, $S_i \in S(j^+)^\perp$ for $i \geq 2$. So the equivalence $F^+_{j^+}$ maps each $S_i \in S(j^+)^\perp$ to $F^+_{j^+}S_i \in {}^\perp S(j^+)$ for $i \geq 2$. In particular, it follows that the sequence

$$(F^+_{j^+}S_2, F^+_{j^+}S_3, \ldots, F^+_{j^+}S_m, S(j^+))$$

is a forward Hom-orthogonal sequence of bricks for $\mathcal{P}(\tilde{\mu}_{j^+}(Q,W))$.

It remains to show that the sequence can not be extended by an additional brick $\tilde{S}$. We assume that we can extend the sequence by a brick $\tilde{S}$ and distinguish between the following two cases.

Case 1: There is some $i \in \{1, \ldots, m\}$ such that

$$(F^+_{j^+}S_2, \ldots, F^+_{j^+}S_i, \tilde{S}, F^+_{j^+}S_{i+1}, \ldots, F^+_{j^+}S_m, S(j^+))$$

is a forward Hom-orthogonal sequence of bricks. But then $\tilde{S} \in {}^\perp S(j^+)$, so we can use a quasi-inverse $G^+_{j^+}$ to $F^+_{j^+}$ to obtain the following forward Hom-orthogonal sequence for $\mathcal{P}(Q,W)$.

$$(S(j^+), S_2, \ldots, S_i, G^+_{j^+}\tilde{S}, S_{i+1}, \ldots, S_m)$$

But this contradicts the assumption that $(S(j^+), S_2, \ldots, S_m)$ is a maximal green sequence.

Case 2: The sequence

$$(F^+_{j^+}S_2, F^+_{j^+}S_3, \ldots, F^+_{j^+}S_m, S(j^+), \tilde{S}) \tag{$*$}$$

is forward Hom-orthogonal. We consider two subcases.

Subcase 2.1: $\tilde{S}$ is a simple $\mathcal{P}(\tilde{\mu}_{j^+}(Q,W))$-module.

It follows that $\operatorname{Hom}_A(S(j^+), \tilde{S}) = \operatorname{Hom}_A(\tilde{S}, S(j^+)) = 0$, so we can permute $S(j^+)$ and $\tilde{S}$ in the sequence $(*)$ and apply Case 1 to obtain a contradiction.

Subcase 2.2: $\tilde{S}$ is not simple.

We claim that all simple $\mathcal{P}(\tilde{\mu}_{j^+}(Q,W))$-modules already appear in the sequence $(*)$. If not, we can extend $(*)$ by the missing simple module using Lemma 5.20. But then we obtain a contradiction to Case 1 or Case 2.1, depending on the position of the inserted missing simple module.

Hence the claim implies that $\operatorname{Hom}_A(S, \tilde{S}) = 0$ for all simple $\mathcal{P}(\tilde{\mu}_{j^+}(Q, W))$-modules S. But this implies $\tilde{S} = 0$, contradiction. $\square$

We remark that although the previous theorem is formulated and proved using the premutations $\tilde{\mu}_{j^+}(Q, W)$ and $\tilde{\mu}_{j^-}(Q, W)$, we actually obtain rotated maximal green sequence for the full mutations $\mu_{j^+}(Q, W)$ and $\mu_{j^-}(Q, W)$ using isomorphisms $\mathcal{P}(\tilde{\mu}_j(Q, W)) \simeq \mathcal{P}(\mu_j(Q, W))$ which exist by Theorem 5.8.

5.3 Applications

We fix a quiver Q with non-degenerate potential W such that the Jacobian algebra $\mathcal{P}(Q, W)$ is finite dimensional. Moreover, we assume that the field k is algebraically closed.

Definition 5.22 Let $S_\bullet = (S_1, S_2, \ldots, S_m)$ be a maximal green sequence for $\mathcal{P}(Q, W)$. The *mutation sequence* associated to $S_\bullet$ is the sequence

$$(k_1, k_2, \ldots, k_m)$$

of vertices $k_i \in Q_0$ defined inductively as follows.

- $k_1 \in Q_0$ is the unique vertex such that $S_1 = S(k_1)$. Note that S_1 is a simple $\mathcal{P}(Q, W)$-module by Proposition 4.2.
- For $j > 1$, the vertex k_j is the unique vertex such that

  $$F^+_{k_{j-1}} F^+_{k_{j-2}} \cdots F^+_{k_1} S_j = S(k_j).$$

  Note that by repeated application of the rotation lemma, cf. Theorem 5.21, the left-hand side is the first brick of a maximal green sequence for the Jacobian algebra over the mutated quiver with potential

  $$\mu_{k_{j-1}} \mu_{k_{j-2}} \cdots \mu_{k_1}(Q, W),$$

  and therefore a simple module. Note that quivers with potential that are mutations of each other have by definition the same set of vertices.

Remark 5.23 The mutation sequences from Definition 5.22 recover the original definition of maximal green sequences in terms of green mutation of framed quivers, cf. [Kel11]. The exact relationship has been worked out by Nagao [Nag13], see also [DK19], [Kel12]. Nagao assigns to a maximal green mutation sequence a chain of torsion classes which gives a maximal green sequence with respect to our definition by the correspondence from Remark 4.8.

Example 5.24 We consider the quiver with potential (Q, W) where Q is linearly oriented A_3-quiver and $W = 0$.

$$Q = \ 1 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 3$$

The Jacobian algebra $\mathcal{P}(Q, W)$ is the path algebra over Q. It has the following maximal green sequence.

$$S_\bullet = \left(2 \, , \, \begin{smallmatrix}2\\1\end{smallmatrix} , \, 1 \, , \, 3 \right)$$

We want to construct the associated mutation sequence (k_1, k_2, k_3, k_4). The first brick is the simple module at the second vertex, hence $k_1 = 2$.

For k_2, we have to rotate $S_\bullet$ once. For this, we have to mutate at $k_1 = 2$, which gives the quiver with potential $\mu_2(Q, W) = (Q^{(2)}, W^{(2)})$ with quiver

$$Q^{(2)} = \quad \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

and potential $W^{(2)} = \gamma\beta\alpha$. The rotated maximal green sequence equals

$$S_\bullet^{(2)} = \left(F_2^+ \begin{smallmatrix}2\\1\end{smallmatrix} , F_2^+ \, 1 \, F_2^+ \, 3 \, , \, 2 \right) = \left(1 \, , \, \begin{smallmatrix}1\\2\end{smallmatrix} , \, 3 \, , \, 2 \right).$$

The first brick is the simple at the first vertex, hence $k_2 = 1$. Now we rotate again, hence we have to mutate at the first vertex and obtain $\mu_1\mu_2(Q, W) = (Q^{(3)}, W^{(3)})$ with quiver

$$Q^{(3)} = \ 2 \xrightarrow{\alpha} 1 \xrightarrow{\gamma} 3$$

and potential $W^{(3)} = 0$. The rotated maximal green sequence equals

$$S_\bullet^{(3)} = \left(F_1^+ \begin{smallmatrix}1\\2\end{smallmatrix} , F_1^+ \, 3 \, , F_1^+ \, 2 \, , \, 1 \right) = \left(2 \, , \, 3 \, , \, \begin{smallmatrix}2\\1\end{smallmatrix} , \, 1 \right).$$

Hence $k_3 = 2$. Finally, we rotate again by mutating at the second vertex and obtain the quiver with potential $\mu_2\mu_1\mu_2(Q, W) = (Q^{(4)}, W^{(4)})$ with quiver

$$Q^{(4)} = \ 2 \xleftarrow{\alpha} 1 \xrightarrow{\gamma} 3$$

and potential $W^{(4)} = 0$. The rotated maximal green sequence is

$$S_\bullet^{(4)} = \left(F_2^+ \, 3 \, , F_2^+ \begin{smallmatrix}2\\1\end{smallmatrix} , F_2^+ \, 1 \, , \, 2 \right) = \left(3 \, , \, 1 \, , \, \begin{smallmatrix}1\\2\end{smallmatrix} , \, 2 \right)$$

which implies that $k_4 = 3$. Hence the association between maximal green sequence and mutation sequence is

$$S_\bullet = \left(2 \, , \, \begin{smallmatrix}2\\1\end{smallmatrix} , \, 1 \, , \, 3 \right) \quad \longleftrightarrow \quad (2, 1, 2, 3).$$

Theorem 5.25 (Target before Source for mutation sequences) *Let (Q, W) be a quiver Q with non-degenerate potential W and consider the Jacobian algebra $\mathcal{P}(Q, W)$ over an algebraically closed field k.*

Let $i, j \in Q_0$ be vertices such that there are two or more arrows from j to i in the quiver Q. Suppose that no arrows from j to i appear in a cycle of W.

Then for any maximal green sequence $S_\bullet$ with associated mutation sequence $k_\bullet = (k_1, \ldots, k_m)$, the vertex i appears before the vertex j in $k_\bullet$.

We defer the proof of the theorem until after the following lemma, which holds for a general finite dimensional algebra A over an algebraically closed field k.

As in the previous chapter, we denote by $[M : S]$ the multiplicity of the isomorphism class of the simple module S in a composition series of M.

Lemma 5.26 *Let $A = kQ/I$ be a finite dimensional algebra over an algebraically closed field k that is the quotient of the path algebra over a quiver Q modulo an ideal I generated by admissible relations.*

Let $i, j \in Q_0$ vertices such that are two or more arrows from j to i in the quiver Q. Suppose that no arrow from j to i appears in a generating relation for I. Let $M, N \in \operatorname{mod}(A)$ be bricks such that $[M : S(i)] = [N : S(j)] = 1$ and $[M : S(j)] = [N : S(i)] = 0$.

Then in all maximal green sequences $S_\bullet$ for A that contain both M and N the brick M must appear before the brick N.

Proof. Let $Q_1'' = \{\alpha_1, \ldots, \alpha_\ell\} \in Q_1$ be the set of arrows from j to i and let $Q'' = (Q_0, Q_1'')$ be the quiver with the same vertices as Q but only the arrows in Q_1''. Let $Q' = (Q_0, Q_1 \setminus Q_1'')$ be the quiver with the same vertices as Q_1 but with all the other arrows. Note that as k-algebras we have $kQ' \simeq kQ/(\alpha_1, \ldots, \alpha_\ell)$. However, since by assumption $I \cap (\alpha_1, \ldots, \alpha_\ell) = \{0\}$, the generating relations for I are all relations for Q', so we can consider $A' := kQ'/I$.

Let $d \in \mathbb{N}Q_0$ be a dimension vector for Q and let $G := \prod_{v \in Q_0} \mathrm{GL}_{d_v}(k)$. Then the G-variety of d-dimensional A-modules can be written as the following product of G-varieties.

$$\operatorname{mod}(A, d) \simeq \operatorname{mod}(kQ'', d) \times \operatorname{mod}(A', d) \qquad (*)$$

It follows that if $\operatorname{mod}(A, d)$ has an open orbit, also $\operatorname{mod}(kQ'', d)$ has an open orbit since the canonical projection map is open. In particular, if $d_i = d_j = 1$, there are no open orbits in $\operatorname{mod}(A, d)$ since $(1, 1)$ is not a real Schur root for the ℓ-Kronecker quiver $Q_{\text{kron}}^{(\ell)}$ when $\ell \geq 2$, which implies that $\operatorname{mod}(kQ'', d) \simeq \operatorname{rep}(Q_{\text{kron}}^{(\ell)}, (1, 1))$ has no open orbits.

By assumption, both M and N are representations of A', hence so is $M \oplus N$. Let $d = \underline{\dim}(M \oplus N)$ be its dimension vector. Choose a non-semisimple representation $R \in \operatorname{mod}(kQ'', d)$, i.e. a non-zero point in the variety, and let $E \in \operatorname{mod}(A, d)$ be the representation which corresponds to the point $(R, M \oplus N)$ under the bijection $(*)$. Then

E is indecomposable and fits into a short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0.$$

Since $d_i = d_j = 1$ and $d = \underline{\dim}(E)$ the dimension vector of E, the orbit of E in the module variety $\operatorname{mod}(A, d)$ is not open. Hence E can not appear as a brick in the maximal green sequence $S_\bullet$ by Proposition 4.25.

Now assume that there is a maximal green sequence $S_\bullet = (S_1, \ldots, S_m)$ such that $S_p = M$ and $S_q = N$ and N appears before M, i.e. $q < p$.

$$S_\bullet = (S_1, \ldots, S_{q-1}, N, S_{q+1}, \ldots S_{p-1}, M, S_{p+1}, \ldots, S_m)$$

Since E does not appear in this maximal green sequence, E has a filtration

$$\{0\} = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_m = E$$

with subfactors $E_r/E_{r-1} \in \operatorname{Filt}(S_r)$ for $r \in \{1, \ldots, m\}$ and at least two subfactors are not zero. However, for $r \geq p+1$ we have $\operatorname{Hom}_A(N, S_r) = \operatorname{Hom}_A(M, S_r) = 0$ and therefore $\operatorname{Hom}_A(E, S_r) = 0$. For $r \leq q-1$ we have $\operatorname{Hom}_A(S_r, N) = \operatorname{Hom}_A(S_r, M) = 0$ and therefore $\operatorname{Hom}_A(S_r, E) = 0$. But the non-zero subfactor E_r/E_{r-1} with r maximal is a factor of E and the non-zero subfactor with r minimal is a submodule of E. It follows that the non-zero subfactors E_r/E_{r-1} in the filtration of E satisfy $r \in \{q, \ldots, p\}$.

Let $(r_1, \ldots, r_t)$ be the ordered sequence of indices for which $E_r/E_{r-1} \neq 0$. Then $q \leq r_1 \leq \ldots \leq r_t \leq p$. Assume that $r_1 = q$. Then N is a submodule of E but since $\operatorname{Hom}_A(M, N) = 0$ and $\operatorname{Hom}_A(N, N) \simeq k$ is a brick this implies that N is a direct summand of E, contradiction. Similarly, if we assume that $r_t = p$ holds M is a factor module of E, which again implies using $\operatorname{Hom}_A(M, N) = 0$ and $\operatorname{Hom}_A(M, M) \simeq k$ that M is a direct summand of E, contradiction. Hence we have $q+1 \leq r_1 < r_2 < \ldots < r_t \leq p-1$. Also note that this implies that for all $s \in \{1, \ldots, t\}$ we have $\operatorname{Hom}_A(E_{r_s}, M) = 0$.

We need the following fact. Let $X, Y \in \operatorname{mod}(A')$ and suppose we are given a short exact sequence in $\operatorname{mod}(A)$

$$0 \longrightarrow X \xrightarrow{f} Z \xrightarrow{g} Y \longrightarrow 0.$$

Let $\iota\colon W \to Z$ be an injection with $W \in \operatorname{mod}(A')$ and $\operatorname{Hom}_A(W, X) = 0$. Then $g\iota\colon W \to Y$ is also an injection. To prove this fact, simply note that A-module morphisms from W to Z naturally can be identified as a subspace of A'-module morphism from W to $X \oplus Y$.

Now note that for all $s \in \{1, \ldots, t\}$, the orbit of S_{r_s} in $\operatorname{mod}(A, d)$ is open by Proposition 4.25, where $d = \underline{\dim}(S_{r_s})$. However, since S_{r_s} is a subfactor of E and $[E : S(i)] = [E : S(j)] = 1$, we have $d_i \leq 1$ and $d_j \leq 1$. But we have seen earlier that there are no open orbits if $d_i = d_j = 1$. In particular, S_{r_s} must be an A'-module.

We claim that $[E_{r_s} : S(i)] = 0$ for all $s \in \{1, \ldots, t\}$. Assume we can choose a $s \in \{1, \ldots, t\}$ minimal such that $[E_{r_s} : S(i)] \neq 0$. Then $E_{r_{s-1}} \subseteq E$ is an A'-module, where we let $E_{r_0} := 0$ for convenience. By the previous fact we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & E_{r_{s-1}} \arrow[r, equal] \arrow[d] & E_{r_{s-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & E/E_{r_{s-1}} \arrow[r] \arrow[d] & N/E_{r_{s-1}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $E_{r_s}/E_{r_{s-1}} \in \mathrm{Filt}(S_{r_s})$ and $S_{r_s} \in A'$ with $\mathrm{Hom}_A(S_{r_s}, M) = 0$, the previous fact applied to the injection $E_{r_s}/E_{r_{s-1}} \hookrightarrow E/E_{r_{s-1}}$ implies that $E_{r_s}/E_{r_{s-1}}$ is actually isomorphic to a submodule of $N/E_{r_{s-1}}$. But by assumption $[E_{r_{s-1}} : S(i)] = [N : S(i)] = 0$, therefore $[E_{r_s}/E_{r_{s-1}} : S(i)] = 0$. But then also $[E_{r_s} : S(i)] = 0$, contradiction.

Hence for all $s \in \{1, \ldots, t\}$ we have $[E_{r_s} : S(i)] = 0$. But $E_{r_t} = E$, thus $[E : S(i)] = 0$, contradiction since by construction we have $[E : S(i)] = 1$. $\square$

The following example shows that the lemma does not hold if we do not require that no arrow from j to i appears in a relation for I.

Example 5.27 Let $A = kQ/I$ be the string algebra with quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2 \xleftarrow{\gamma} 3$$

and relation $\beta\gamma = 0$. We use polygonal flips, cf. Remark 4.19, to construct a maximal green sequence starting from the maximal green sequence

$$S_\bullet^{(1)} = (1, 2, 3).$$

Since $\mathrm{Filt}(2, 3)$ is equivalent to $\mathrm{mod}(kA_2)$, where A_2 is the A_2-quiver, flipping at this wide interval gives the maximal green sequence

$$S_\bullet^{(2)} = \left(1,\ 3,\ \begin{smallmatrix}3\\2\end{smallmatrix},\ 2\right).$$

The category Filt$(\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix})$ is semisimple, so flipping here simply permutes the bricks and we obtain the maximal green sequence

$$S_\bullet^{(3)} = \left(\begin{smallmatrix}3\end{smallmatrix},\ \begin{smallmatrix}1\end{smallmatrix},\ \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix},\ \begin{smallmatrix}2\end{smallmatrix}\right).$$

Now we consider the wide subcategory Filt$(\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix})$, which is again equivalent to the category of representation over the A_2-quiver. So we flip and obtain the maximal green sequence

$$S_\bullet^{(4)} = \left(\begin{smallmatrix}3\end{smallmatrix},\ \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix},\ \begin{smallmatrix} & & 3 \\ & 2 & \\ 1 & & \end{smallmatrix},\ \begin{smallmatrix}1\end{smallmatrix},\ \begin{smallmatrix}2\end{smallmatrix}\right).$$

Let $i = 1$ and $j = 2$, so there are two arrows from j to i. Let $M := \begin{smallmatrix}1\end{smallmatrix}$ and $N := \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix}$, so $[M : S(i)] = [N : S(j)] = 1$ while $[M : S(j)] = [N : S(i)] = 0$. But in the maximal green sequence $S_\bullet^{(4)}$ the brick N appears before the brick M. In fact, without the relation $\beta\gamma = 0$, we could still have performed the first two polygonal flips until $S_\bullet^{(3)}$. However, now the wide subcategory Filt$(\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix} & 3 \\ 2 & \end{smallmatrix})$ is equivalent to the representations of a Kronecker quiver, so does not allow for a polygonal flip. Note that in $S_\bullet^{(3)}$ the brick M appears before the brick N.

Proof of Theorem 5.25. Let $j \in Q_0$ and let $A = \mathcal{P}(Q, W)$ be a finite dimensional Jacobian algebra for a quiver with Q with non-degenerate potential W. We repeatedly make use of the following fact, which follows at once from the definition of the functors F_j^+ from Definition 5.17: For $M \in \operatorname{mod}(A)$ we have $[M : S(i)] = [F_j^+ M : S(i)]$ for all $j \neq i \in Q_0$.

So in the situation of the theorem let $S_\bullet = (S_1, \dots, S_m)$ be a maximal green sequence for A with associated mutation sequence $k_\bullet = (k_1, \dots, k_m)$. Let $t \in \{1, \dots, m\}$ be minimal such that $k_t = j$. Assume that for all $r < t$ we have $k_r \neq i$, so we assume that j appears before the vertex i. Let $F := F_{k_{t-1}}^+ F_{k_{t-2}}^+ \cdots F_{k_1}^+$. Repeated application of the rotation lemma, cf. Theorem 5.21 gives the following maximal green sequence for a Jacobian algebra over the corresponding mutated quiver with potential.

$$\Big(FS_t,\ FS_{t+1}, \dots,\ FS_m,$$
$$F_{k_{t-1}}^+ \cdots F_{k_2}^+ S(k_1),\ F_{k_{t-1}}^+ \cdots F_{k_3}^+ S(k_2),\ \dots,\ F_{k_{t-1}}^+ S(k_{t-2}),\ S(k_{t-1})\Big)$$

Moreover, the definition of the associated mutation sequence, Definition 5.22, gives $FS_t = S(j)$. The maximal green sequence has to contain all simple modules by Corollary 4.10. In particular, one brick has to be isomorphic to $S(i)$. But since $k_r \neq i$ for all $r < t$ and the functors $F_{k_r}^+$ can not produce a composition factor isomorphic to $S(i)$, we must have $S(i) = FS_s$ for some $s \in \{t + 1, \dots, m\}$.

But F is the composite of functors $F_{k_r}^+$ with $r < t$ and for $r < t$ we have $k_r \neq i$ and $k_r \neq j$. Hence the equations $[S(i) : S(i)] = [S(j) : S(j)] = 1$ and $[S(i) : S(j)] = [S(j) : S(i)] = 0$ imply that $[S_s : S(i)] = [S_t : S(j)] = 1$ and $[S_s : S(i)] = [S_t : S(j)] = 0$, since we have $FS_s = S(i)$ and $FS_t = S(j)$.

Finally, since we assume that no arrows from j to i are part of a cycle in Q, these arrows can not be part of any relation of $A = \mathcal{P}(Q, W)$ if we consider a A as the path algebra over Q modulo an ideal generated by admissible relations, simply because no arrows from j to i can appear in W and the additional nilpotency relations that come from writing $\mathcal{P}(Q, W)$ as the factor algebra of the regular path algebra can be chosen of arbitrary sufficiently large length ℓ, cf. Remark 5.5.

Therefore the assumptions of Lemma 5.26 are satisfied with $M = S_s$ and $N = S_t$ and we conclude that M must appear before N. But $t < s$, contradiction. $\square$

The Target before Source theorem for mutation sequences is motivated by the physics literature, see e.g. [Xie16, Section 4.2] and has been proven in the context of mutations for acyclic valued quivers in [BHIT17, Theorem 1]. Note that we do not require the quiver to be acyclic. However, the following example, which also appears in [BHIT17], shows that the Target before Source theorem does not hold in full generality if there are multiple parallel arrows that are part of a cycle in Q.

Example 5.28 Let (Q, W) be the quiver with potential with quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, shift right, "\gamma"'] \arrow[ll, shift left, "\delta"] \end{tikzcd}$$

with potential $W = \gamma\beta\alpha$. Then

$$S_\bullet = (2,\ \substack{2\\3},\ \substack{2\\3\\1},\ 1,\ 3)$$

is a maximal green sequence for $\mathcal{P}(Q, W)$. Note that this sequence can be obtained by rotating the unique maximal green sequence of length 5 that ends at the simple 2 for the quiver

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3. \arrow[ll] \arrow[ul] \end{tikzcd}$$

Note that such a maximal green sequences for this quiver of type $\tilde{A}_2$ can be constructed using polygonal flips as demonstrated earlier in Example 4.20. The mutation sequence $k_\bullet$ that corresponds to the maximal green sequence $S_\bullet$ is

$$k_\bullet = (2, 3, 1, 3, 2).$$

In particular, this mutation sequence mutates at the source 3 of the double arrow in Q before it mutates at the target 1.

5.4 A rotation lemma using classical tilting theory

Let A be a finite dimensional algebra. In the following, our notation follows [AI12, Section 2.7].

Let $e \in A$ be a primitive idempotent and let $S := Ae/\mathrm{rad}(Ae)$ be the corresponding simple A-module. We assume that

- S is not injective,
- $\mathrm{Ext}^1_A(S, S) = 0$ and
- $\mathrm{pdim}(\tau^{-1}S) \leq 1$.

In this case, the *Brenner–Butler (BB) tilting module* [BB80] with respect to S is defined as

$$T_S := \tau^{-1}S \oplus A(1-e).$$

We let $B := \mathrm{End}_A(T_S)$ be its endomorphism ring. The BB tilting module T_S is a so called (classical) tilting module, i.e. it satisfies

- $\mathrm{pdim}(T_S) \leq 1$,
- $\mathrm{Ext}^1_A(T_S, T_S) = 0$ and
- $|T_S| = |A|$.

If S is projective, the BB tilting module is an APR tilting module in the sense of [APR79]. From the viewpoint of τ-tilting theory, we have the following lemma.

Lemma 5.29 *Let A be a finite dimensional algebra and $T_S \in \mathrm{mod}(A)$ be a BB tilting module for some simple A-module S. Then T_S is a τ-tilting module.*

Proof. Since the Auslander–Reiten translation of projective A-modules vanishes, we have

$$\mathrm{Hom}_A(T_S, \tau T_S) = \mathrm{Hom}_A(T_S, S).$$

The only projective module that maps to S is Ae, hence

$$\mathrm{Hom}_A(T_S, S) = \mathrm{Hom}_A(\tau^{-1}S, S).$$

Now we use the fact that S is simple, which implies that

$$\mathrm{Hom}_A(\tau^{-1}S, S) = \mathrm{Ext}^1_A(S, S),$$

cf. [AS81, Corollary 5.7]. Hence $\mathrm{Hom}_A(T_S, \tau T_S) = 0$. By assumption, we have $|T_S| = |A|$, hence T_S is a support τ-tilting module, in fact, it is a τ-tilting module. $\square$

Now we recall the Brenner–Butler theorem, cf. [BB80]. Let $T \in \mathrm{mod}(A)$ be a classical tilting module for a finite dimensional algebra A and let $B := \mathrm{End}_A(T)$ be its endomorphism ring. In $\mathrm{mod}(A)$, we consider the torsion pair $(\mathcal{T}, \mathcal{F})$ where

$$\mathcal{T} = \mathrm{Fac}(T) = \{X \in \mathrm{mod}(A) \mid \mathrm{Ext}^1_A(T, X) = 0\}$$

and

$$\mathcal{F} = \mathcal{T}^\perp = \{X \in \mathrm{mod}(A) \mid \mathrm{Hom}_A(T, X) = 0\}.$$

On the other hand, in $\mathrm{mod}(B)$, we consider the torsion pair $(\mathcal{X}, \mathcal{Y})$ where

$$\mathcal{X} = \{Y \in \mathrm{mod}(B) \mid T \otimes_B Y = 0\}$$

and

$$\mathcal{Y} = \{Y \in \mathrm{mod}(B) \mid \mathrm{Tor}_1^B(T, Y) = 0\}.$$

The Brenner–Butler theorem states that there are equivalences of additive categories between categories forming these torsion pairs, as follows.

Theorem 5.30 (Brenner–Butler, cf. [BB80]) *Let $T \in \mathrm{mod}(A)$ be a tilting module, let $B := \mathrm{End}_A(B)$ its endomorphism ring and let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ be the torsion pairs in* $\mathrm{mod}(A)$ *and* $\mathrm{mod}(B)$ *as above. Then there are the following mutually inverse equivalences of additive categories.*

$$\mathrm{mod}(A) \supseteq \mathcal{T} \underset{T \otimes_B -}{\overset{\mathrm{Hom}_A(T,-)}{\rightleftarrows}} \mathcal{Y} \subseteq \mathrm{mod}(B)$$

$$\mathrm{mod}(A) \supseteq \mathcal{F} \underset{\mathrm{Tor}_1^B(T,-)}{\overset{\mathrm{Ext}^1_A(T,-)}{\rightleftarrows}} \mathcal{X} \subseteq \mathrm{mod}(B)$$

Finally, let us return to the case when the tilting module $T = T_S$ is a BB tilting module for some simple A-module S. The support τ-tilting module T_S differs from the regular A-module A in only one indecomposable summand, hence they are mutations of each other in the sense of τ-tilting theory. But then $\mathrm{mod}(A) \to \mathrm{Fac}(T_S) = \mathcal{T}$ is an arrow in the Hasse quiver of torsion classes, which implies that $\mathcal{F} = \mathrm{mod}(A) \cap \mathcal{T}^\perp = \mathrm{Filt}(S) = \mathrm{add}(S)$.

Now we can state our representation-theoretic rotation lemma.

Theorem 5.31 (Rotation lemma) *Let $S_\bullet = (S_1, \dots, S_{m-1}, S)$ be a maximal green sequence for some finite dimensional algebra A. Suppose that S satisfies the requirements for a Brenner–Butler tilting module T_S. Let $B := \mathrm{End}_A(T_S)$ be its endomorphism ring, $S'_i := \mathrm{Hom}_A(T_S, S_i)$ for $i \in \{1, \dots, m-1\}$ and $S' := \mathrm{Ext}^1_A(T_S, S)$. Then*

$$S'_\bullet := (S', S'_1, \dots, S'_{m-1})$$

is a maximal green sequence for B.

Proof. Using the notation of Theorem 5.30, we have $\mathcal{F} = \mathrm{add}(S)$. With the interpretation of maximal green sequences as chains of torsion classes, cf. Theorem 4.7 and the subsequent Remark 4.8, we have a torsion pair

$$(\mathrm{Filt}(S_1, \dots, S_{m-1}), \mathrm{add}(S)) = (\mathrm{Filt}(S_1, \dots, S_{m-1}), \mathcal{F})$$

In particular, we have $\mathcal{T} = \mathrm{Filt}(S_1, \dots, S_{m-1})$.

Now Theorem 5.30 shows that $\mathrm{Hom}_A(T_S, -)$ defines an equivalence between $\mathcal{T} \subseteq \mathrm{mod}(A)$ and $\mathcal{Y} \subseteq \mathrm{mod}(B)$ and $\mathrm{Ext}^1_A(T_S, -)$ defines an equivalence between $\mathcal{F} = \mathrm{add}(S) \subseteq \mathrm{mod}(A)$ and $\mathcal{Y} = \mathrm{add}(S') \subseteq \mathrm{mod}(B)$ and that $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathrm{mod}(B)$. All this implies that the sequence $S'_\bullet$ is forward Hom-orthogonal, i.e. we have $\mathrm{Hom}_B(S', S'_i) = 0$ and $\mathrm{Hom}_B(S'_i, S'_j) = 0$ for $1 \leq i < j \leq m-1$.

Assume that $S'_\bullet$ is not a maximal green sequence. We consider two cases. Note that the rest of the proof is analogous to the proof of Theorem 5.21.

Case 1: There is a brick $\tilde{S}$ such that

$$(S', S'_1, \dots, S'_{i-1}, \tilde{S}, S'_i, \dots, S'_{m-1})$$

is forward Hom-orthogonal for some $i \in \{1, \dots, m\}$. But then we use the inverse functors $T_S \otimes_B -$ and $\mathrm{Tor}^B_1(T_S, -)$ from Theorem 5.30 to conclude that

$$(S_1, \dots, S_{i-1}, T_S \otimes_B \tilde{S}, S_i, \dots, S_{m-1}, S)$$

is a forward Hom-orthogonal sequence of bricks in $\mathrm{mod}(A)$. But this is a contradiction to $S_\bullet$ being a maximal green sequence.

Case 2: There is a brick $\tilde{S}$ such that the sequence

$$(\tilde{S}, S', S'_1, \dots, S'_{m-1})$$

is forward Hom-orthogonal. We consider two subcases.

Subcase 2.1: $\tilde{S}$ is a simple B-module.

It follows that $\mathrm{Hom}_A(S', \tilde{S}) = \mathrm{Hom}_A(\tilde{S}, S') = 0$, so we can permute S' and $\tilde{S}$ in the sequence $(*)$ and apply Case 1 to obtain a contradiction.

Subcase 2.2: $\tilde{S}$ is not simple.

We claim that all simple B-modules already appear in the sequence $(*)$. If not, we can extend $(*)$ by the missing simple module using Lemma 5.20. But then we obtain a contradiction to Case 1 or Case 2.1, depending on the position of the inserted missing simple module.

Hence the claim implies that $\mathrm{Hom}_A(S, \tilde{S}) = 0$ for all simple B-modules S. But this implies $\tilde{S} = 0$, contradiction. $\square$

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