Essays on Individual Decisions in Large Collectives

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Contents

Introduction

This dissertation consists of three essays in microeconomic theory on individual decisions in large collectives. An "important concern of microeconomics is how economic units interact to form larger units – markets and industries." (Pindyck and Rubinfeld, 2013, p. 3). The topic of the three essays in this thesis is how individual behavior shapes a large collective and how in return the large collective affects the individual behavior.

Chapter 1, *Voting with Endogenous Timing*, studies the role of timing in common-value elections. In many collective decision processes, people want to coordinate on the best policy decision. However, people might have different information and different beliefs about what the best decision in a given situation is. As private information might not be easily communicable, voting procedures help aggregate information and find the best policy to implement. Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1996) analyze simultaneous voting procedures under the scope of game theory by introducing strategic voters who act rationally. However, if voters act simultaneously, better-informed voters have little possibility to influence other voters. We contribute to the branch of strategic voting literature by exploring the effects of an endogenous timing decision on voting procedures. We construct a model with two voting periods where voters receive private information about the state of the world, and can afterward decide for themselves when to publicly cast their votes.

As an implication, disclosing the own vote in period one allows an agent to inform the voters who have decided to vote in period two and possibly influence their voting decision. In contrast, a voter who votes in period two does not disclose her own vote but observes the votes from period one and makes a better-informed voting decision herself. We show that in welfare-optimal equilibria, agents use their timing to communicate the strength of their private information to the other voters. Voters with more informative private signals vote in period one and voters with less informative private signals wait for period two. This communication allows the collective a better information aggregation than simultaneous voting or voting with an exogenously fixed timing.

Chapter 2, *On-the-Match Search and Match-Specific Productivity Growth*, which is joint work with Sophie Kreutzkamp and Axel Niemeyer, investigates a frictional search-and-matching model with heterogeneous agents that continue searching when matched, similar to the model of Shimer and Smith (2000). The novelty of our model is the combination of productivity growth inside a match with the option for both partners to rematch. As partners become more attuned to each other, the value of persisting partnerships increases over time. Additionally, the possibility that your partner might leave you changes how you perceive the value of the current partnership. Consequently, the partner's rematching behavior affects your own rematching behavior, and vice versa. For this effect to occur, the two-sided rematching of our model is of crucial importance. In contrast to our model, on-the-job search models with one-sided rematching, like for example surveyed by Rogerson, Shimer, and Wright (2005), do not feature this interaction.

Our primary interest lies in understanding how productivity growth and the option to rematch affect the agents' equilibrium behavior and, consequently, the set of steady-state equilibria. Without productivity growth, there exists a coordination problem that leads to a multiplicity of equilibria. We show that even minuscule productivity growth is sufficient to eliminate unreasonable equilibria that typically occur in models with on-the-match search. Therefore, vanishing growth rates can serve as a criterion for equilibrium selection in such models. Further, productivity growth can accelerate sorting in the market. On the one hand, it stabilizes even asymmetric matches between more productive and less productive agents over time, on the other hand, it also destabilizes them: Capital accumulation provides incentives for agents to foster growth in more stable relationships, which are most often between agents of the same productivity type. We show that there exist parameters for which the latter effect dominates in equilibrium. In such settings less productive agents who are matched with more productive agents actively seek to rematch with another less productive agent, which leads to an increased sorting into symmetric matches. As a result, the collective market structure is more assortative and the assertiveness of the market stems from both the rematching behavior of the less productive agents as well as of the more productive agents.

Chapter 3, *Partnership Dissolution in a Search Market with On-the-Match Learning*, analyzes the steady-state equilibria of a search-and-matching market with ex-ante homogeneous agents and on-the-match learning. In many matches, it is not clear upon forming how good of a fit the match actually is. An agent can be a good potential partner for one agent but a bad fit for another one. The goodness of the fit may only be found out while the match persists and agents learn about it over time.

We construct a model where, upon forming a match, the agents draw unobserved types that specify the idiosyncratic profitability of the current match. This aspect is in the spirit of Smith (1995) who models an exchange market where the valuations for every good are idiosyncratic. While being in a match, the agents receive information about their unknown types over time following a Poisson process. As the continuation value of a match does not only depend on the profitability for oneself but also on the partner's rematching behavior, agents have an endogenous interest in their partners finding the match profitable. Applying comparative statics, we show that a faster learning rate is beneficial for the agents if the profitability in a match has a strong positive correlation between partners, and, in contrast, that with a strong negative correlation, a faster learning rate reduces the ex-ante expected payoff of the agents.

While Chapter 1 analyzes a collective decision problem where individual agents have to decide on a policy together, Chapter 2 and Chapter 3 feature the common theme of rematching in a large search market where the individual matching decisions shape the market structure. Notably, in both of the latter chapters, the expected continuation payoff inside a match changes over time as the absence of bad news over a period of time mathematically has similar effects on the present value of a match as capital accumulation has. The major differences in the models are that Chapter 2 assumes a common ranking over heterogeneous agents with constant productivity types and capital accumulation inside the matches, while Chapter 3 considers ex-ante homogeneous agents whose productivity types are idiosyncratic, initially unknown to the agents, and have to be learned over time.

All three essays illustrate the role of individual decisions in large collectives. While a single agent has little to no power to make a change by herself, the aggregation of individual actions fundamentally shapes the collective as a whole, and in return, affects the behavior of the individuals.

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Chapter 1

Voting with Endogenous Timing

1.1 Introduction

In many collective decision processes agents can and do preemptively announce their positions. For instance, when a parliament votes on a bill, participating politicians often publicly disclose their stances beforehand. An example was the second impeachment trial of Donald Trump, where various senators spoke out in favor or against an impeachment before the official vote even began.[1] This kind of information disclosure can affect the decision-making. It discloses information to the other politicians about the own opinion on the bill. The other politicians can react to this information and adjust their behavior accordingly. Additionally, voters can anticipate this effect and may use their own vote to influence the other voters.

In this chapter, we analyze how the possibility to disclose the own action and thereby inform the other voters can affect a voting procedure and the associated information aggregation.[2] We provide a stylized model of sequential voting with common values, two voting periods, and endogenous timing. The voters have to decide between two options, and every voter receives a private signal about which option is more preferable. We model a prior announcement of the own vote as binding.[3] We restrict the analysis to homogeneous preferences throughout this chapter. The voters all agree on the best decision for each state of the world and get the same utility, but their private information about the state have different realizations.

The main trade-off for each voter arises from the timing decision: The voter can vote early and disclose her vote to the other voters. This informs others and

[1] The senators' positions were prominently announced in the media at that time. See for example CNN Politics (2021) or Zurcher, Anthony (2021).

[2] There are various other effects associated to a preemptive disclosure. For example, a prior announcement of the own vote informs the citizens about the political agenda and increases transparency. Politicians can use this for reputation-building as described by Keefer and Vlaicu (2007) who analyze the role of credibility and reputation in democracies.

[3] Even though a public disclosure of the own vote is only a partial commitment, it is strong in the sense that politicians generally care about their reputation, and deviating from an announcement may lead to a loss of reputation.

allows them to make better-informed voting decisions but the voter cannot observe other votes herself. Alternatively, the voter can vote late and first observe the other voters' early votes. This provides additional information to the voter and allows her to make a better-informed voting decision but in return she cannot inform others. The more informative the votes are in period one, the higher is the incentive to vote in period two.

We start by showing the existence of a welfare-optimal equilibrium. Due to the infinite type space and the sequential voting structure, we construct a non-standard metric on the strategy-space for this. Then, we characterize the welfare-optimal equilibria of the two-period voting game. Similarly to simultaneous voting studied by Duggan and Martinelli (2001), the strategies of a welfare-optimal equilibrium follow a cutoff rule. In the first period, voters with more informative signals cast an early vote to influence other voters in their direction. Voters with less informative signals wait for period two to get more information before voting.

We show that the welfare-optimal equilibria of our sequential voting model with endogenous timing welfare-dominate all equilibria of simultaneous voting games and voting games with exogenously fixed voting sequences. More precisely, it is the combination of the timing decision and the voting decision that conveys useful information to the other voters.

In setups where the swing voter's curse[4] occurs, voting with endogenous timing mitigates its negative effect on welfare, even outperforming simultaneous voting with abstention.

Moreover, information is aggregated even under assumptions for which the simultaneous voting model fails to do so. In particular, even in a setting with bounded signals and under the unanimity voting rule, the probability of the correct decision under a welfare-optimal equilibrium of our two-period voting game converges to one as the number of voters grows large.

Our results contrast the result of Dekel and Piccione (2000) who show that in general if the timing is exogenous, the disclosure of the votes alone does not improve the information aggregation of a voting procedure compared to simultaneous voting. The reason is that learning the other agents' votes only changes the probability of being pivotal but not the optimal action upon being pivotal. Instead, if the timing is endogenous, agents cannot only use the vote itself but also the timing of the vote to convey information about the strength of the own signal to the other voters. As a result, endogenizing timing improves the outcome of a voting procedure.

The rest of this chapter is organized as follows. Section 1.1.1 gives an overview over the related literature. Section 1.2 lays out the model with two periods. In Section 1.3, an example illustrates the model and the voter's behavior. Section 1.4

[4]If ties are randomly broken, less informed voters may strictly prefer to abstain rather than to vote. See Feddersen and Pesendorfer (1996) for more details on the swing voter's curse.

contains the main analysis and characterizes the welfare-optimal equilibria. Section 1.5 covers information aggregation and Section 1.6 relates sequential voting to the swing voter's curse under the simple majority voting rule. Section 1.7 shows that voting with endogenous timing welfare-dominates a voting procedure with a fixed voting sequence and Section 1.8 concludes. The proofs of the results can be found in Appendix 1.A.

1.1.1 Related Literature

Our model is related to the Condorcet Jury Theorem and information aggregation in large elections. Condorcet (1785) suggested that for homogeneous preferences, a decision made by a large group of "sincere" voters yields better results than a decision made by an individual alone. This result was later reproduced for strategic voters in simultaneous voting procedures.[5]

Feddersen and Pesendorfer (1998) show that even for large electorates information is not aggregated under the unanimity voting rule for binary signals due to the bounded informativeness. In contrast to their result, the probability of choosing the optimal decision converges to one in our two-period voting model with endogenous timing.

Previous work on sequential voting has mainly focused on exogenously fixed voting sequences. We contribute to this line of research by endogenizing the timing decision in a sequential voting game. This work builds on Schmieter (2019), where the welfare-optimality of cutoff rules is shown for the special case of the unanimity voting rule.

Dekel and Piccione (2000) consider sequential voting with two alternatives where the order of voting is exogenously fixed. In their setting, voters cast their vote in a given order, and every voter observes all actions that have been made prior to her vote. They show that each symmetric equilibrium of the corresponding simultaneous voting game is also an equilibrium of any sequential voting game, regardless of the voting sequence. Furthermore, they prove that under the unanimity voting rule, the set of equilibria of any sequential voting game, regardless of the voting sequence, is equal to the set of equilibria of the corresponding simultaneous voting game. An important implication from their work is that observing the votes of the other voters does not improve the aggregation of information. This is due to the fact that voters condition on the event of being pivotal. Since in their model there exists exactly one event for which a voter is pivotal, this conditioning is equivalent to observing the other agents' votes directly. Thus, learning the votes of other agents does not convey useful information. In particular, learning the earlier voters' actions does not change the behavior of the later voters. However, except for the unanimity voting

[5] Among others, Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1996, 1997, 1998) and Duggan and Martinelli (2001) analyze strategic voting.

rule, they do not show whether a new equilibrium of the sequential voting game might welfare-dominate the equilibria of the simultaneous voting game. Also, their equivalence result under the unanimity voting rule relies on the exogeneity of the voting sequence.

One crucial aspect of Dekel and Piccione (2000) is that they do not allow for any tie-breaking in their model. Instead, they restrict their analysis to n_p-voting rules, where alternative one is adopted if and only if at least n_p voters vote for it and alternative two is chosen otherwise. Their result does for example not carry over to a simple majority voting rule with tie-breaking by a fair coin toss. In particular, their analysis excludes settings where the so-called swing voter's curse occurs: Feddersen and Pesendorfer (1996) show that under the simple majority voting rule with an even number of voters and tie-breaking by a fair coin toss, less informed voters strictly prefer to abstain. As a result, allowing abstention in such simultaneous voting settings increases welfare. We show that the welfare-optimal equilibrium of our two-period model without abstention welfare-dominates all equilibria of the simultaneous voting model with abstention.

Dekel and Piccione (2014) analyze voting with an endogenous timing decision. Compared to our model, they cover three alternatives, private values, and voters have to decide for a voting period before they learn their preferences. In particular, voters in their model have a conflict of interest, and, in contrast to our model, revealing information to other voters can have a negative effect for oneself.

There are various other papers related to sequential voting. Battaglini (2005) adds abstention and costs of voting to the model of Dekel and Piccione (2000) and shows that even arbitrarily small voting costs can break the equivalence of equilibria. Another strand of literature analyzes herding behavior (see for example Fey (1998)), where herding hinders full information aggregation. The difference to sequential voting is that herding features an individual payoff relevant choice for each agent instead of a collective decision. Callander (2002) relates herding to sequential voting and shows that if voters want to vote for the winning candidate, herding occurs with probability one. Eyster and Rabin (2005) introduce the concept of cursed equilibria, where agents underestimate the correlation of other players' information. Piketty (2000) considers two-period voting, where both periods are payoff-relevant. There, agents of different types use the first period to signal information and influence the outcome of the second voting period. In contrast to our model, there are three competing candidates and the voters are confronted with a coordination problem rather than a problem of information aggregation. McLennan (1998) shows that for common interest games, a symmetric strategy that maximizes the expected welfare is a Nash equilibrium. We use this finding multiple times to prove our results.

1.2 The Model

In this section, we introduce our model of sequential voting with two voting periods. There are $N \geq 2$ jurors who vote on whether to convict or acquit a defendant.[6] An unknown state ω describes whether the defendant is innocent, I, or guilty, G. The realization of ω is randomly drawn according to a commonly known prior $q \coloneqq P(\omega = I)$ and $1 - q = P(\omega = G)$ with $q \in (0, 1)$.

Each agent $i \in \{1, \ldots, N\}$ receives a private signal s_i about ω from the closed interval $S \coloneqq [\underline{s}, \bar{s}] \subseteq \mathbb{R}$. Conditional on the state, the signals are drawn independently from each other, according to the cumulative distribution function $F(\cdot|I)$ if the defendant is innocent or $F(\cdot|G)$ if the defendant is guilty. The distribution functions $F(\cdot|I)$ and $F(\cdot|G)$ are absolutely continuous and have piecewise continuous densities $f(\cdot|I)$ and $f(\cdot|G)$ which are strictly positive on S.

We assume that the *likelihood ratio of the signals*, $f(s|I)/f(s|G)$, is weakly decreasing on S. This implies that low signals indicate innocence, while high signals are indicators of guilt. We let the signals be sufficiently informative by assuming that both events

$$\left\{ s \in S \;\middle|\; \frac{f(s|I)}{f(s|G)} < \frac{1-q}{q} \right\} \text{ and } \left\{ s \in S \;\middle|\; \frac{f(s|I)}{f(s|G)} > \frac{1-q}{q} \right\}$$

occur with positive probability. That is, the likelihood ratio of a single signal can dominate the likelihood ratio of the prior in either direction.

Preferences and Timing The defendant can be either acquitted, A, or convicted, C. The agents have common preferences and want to match the outcome with the state. They get a utility of 1 if C is implemented in state G or if A is implemented in state I and a utility of 0 otherwise.

The outcome is determined by a voting procedure with two voting periods, period one (*early*) and period two (*late*). In period one, the agents can either vote for A or C or choose to wait, denoted by W. In period two, the agents who waited observe the aggregated votes from period one and now have to vote for either A or C themselves. Abstention is not allowed in period two. Agents who already voted in period one cannot change their decision anymore and are not allowed to vote a second time.

The *voting rule* is parameterized by a pair $(K, p) \in \{1, 2, \ldots, N-1\} \times [0, 1]$. If strictly less than K voters vote for conviction, then the defendant is acquitted and if strictly more than K voters vote for conviction, then the defendant is convicted. If the number of C-votes is exactly K, then conviction occurs with probability p.[7] This

[6] To simplify the exposition, we frame the model as if it were about a decision at court, but it is in no way restricted to this particular application. The notation mainly follows Duggan and Martinelli (2001).

[7] Note that the voting rules $(K, 0)$ and $(K+1, 1)$ are equivalent.

captures all standard (anonymous) voting rules such as the unanimity voting rule, all super-majority voting rules, and the simple majority voting rule with and without random tie-breaking. For example, for the parameters $(N-1, 0)$, the defendant is only convicted if all N voters vote unanimously for C, i.e., we have the unanimity voting rule. With an even number of voters N, the voting rule $(\frac{N}{2}, \frac{1}{2})$ represents the simple majority voting rule where a tie is broken by a fair coin flip.

Histories, Strategies, and Equilibria A (public) *history* h specifies the past voting actions. Let $h = \emptyset$ denote the empty history at the beginning of period one. A history in period two can be characterized by a pair $h = (n_A, n_C)$ that specifies the number n_A of early A-votes and the number n_C of early C-votes. Let H be the set of all histories.[8]

A *mixed strategy* for voter i is given by the probabilities of voting for A, waiting W, and voting for C for every private signal $s \in S$ and every history $h \in H$. Formally, a mixed strategy is a measurable[9] function

$$\sigma^i : S \times H \to \{(p_A, p_W, p_C) \in [0,1]^3 \mid p_A + p_W + p_C = 1\},$$

with $p_W = 0$ for every history of period two. The triple $\sigma^i(s_i, h) = (p_A, p_W, p_C)$ specifies the probabilities p_Y of playing action Y for each $Y \in \{A, W, C\}$ for every signal $s_i \in S$ and at every history $h \in H$. Let σ^i_A, σ^i_W and σ^i_C be the marginals of σ^i, i.e., the maps to p_A, p_W and p_C, respectively. For convenience, let $\sigma^i(s_i, h) = A$, $\sigma^i(s_i, h) = W$, and $\sigma^i(s_i, h) = C$ denote that the corresponding actions are played with probability 1.

Fix a single voter i, fix a strategy σ^i and the strategies σ^{-i} of the other voters. For these strategies, the expected utility for voter i is given by

$$U(\sigma^i, \sigma^{-i}) = qP(A|I, \sigma^i, \sigma^{-i}) + (1-q)P(C|G, \sigma^i, \sigma^{-i}),$$

where $P(Y|\omega, \sigma^i, \sigma^{-i})$ denotes the probability of outcome Y under state ω given the strategies σ^i and σ^{-i}, i.e., the expected utility is the ex-ante probability of choosing the correct outcome. The strategies $(\sigma^1, \ldots, \sigma^N)$ constitute a mixed Bayesian Nash equilibrium if for every voter i, the strategy σ^i maximizes $U(\sigma^i, \sigma^{-i})$ for fixed σ^{-i}. We restrict attention to symmetric strategies and omit the index i to write σ/σ_Y for the strategies/marginals instead of σ^i/σ^i_Y.

As the expected utility in an equilibrium is identical for every voter, we consider welfare on a per-capita level and call it the *expected welfare* $U(\sigma)$. A *welfare-optimal*

[8]Formally, $H = \big\{(n_A, n_C) \in \{0, 1, \ldots, N-1\}^2 \mid n_A + n_C < N,\ n_C \le K,\ n_A \le N-K\big\} \cup \{\emptyset\}$.

[9]Let $\mathcal{B}(S)$ and $\mathcal{B}([0,1]^3)$ denote the Borel σ-algebras on S and $[0,1]^3$, respectively. Consider the power set $\mathcal{P}(H)$, which is a σ-algebra on the finite set H. A strategy σ^i is required to be measurable with respect to the product σ-algebra $\Sigma = \mathcal{B}(S) \times \mathcal{P}(H)$ and $\mathcal{B}([0,1]^3)$.

equilibrium is an equilibrium that maximizes the welfare, or equivalently, the ex-ante probability of a correct decision. Unless stated otherwise, "equilibrium" refers to symmetric mixed Bayesian Nash equilibrium.[10]

Assumptions For some results, we additionally assume that the following properties hold. Their usage is explicitly stated each time. The first assumption says that the likelihood ratio is strictly decreasing instead of weakly decreasing. That is, no two signals induce the same belief.

Strictly monotone likelihood ratio property ($\mathrm{MLRP}_<$). The likelihood ratio of the signals, $f(s|I)/f(s|G)$, is strictly decreasing on S.

The second assumption states that the informativeness of the signals is unbounded. This is for convenience only and ensures that all cutoffs are in the interior of S, thus avoiding the need to consider corner cases.

Unbounded likelihood ratio (ULR). The likelihood ratio of the signals is unbounded, i.e.,

$$\lim_{s \to \underline{s}} \frac{f(s|I)}{f(s|G)} = \infty$$
$$\lim_{s \to \overline{s}} \frac{f(s|I)}{f(s|G)} = 0.$$

Monotonicity A strategy profile is *monotone* if voting A or C in period one increases the probability of the respective outcome regardless of the state ω. We will see that this class of strategy profiles has multiple desirable properties. First, in monotone equilibria, the agents' votes and beliefs are aligned: If an agent knew which outcome is correct, then she would vote for that outcome. Second, for the number of voters being large, there are monotone equilibria that implement the correct outcome with a probability close to one. Therefore, under the viewpoint of information aggregation, it is without loss to restrict attention to monotone equilibria. Third, monotone equilibria are a generalization of cutoff equilibria of the simultaneous voting game and they will be particularly straightforward to work with.

Formally, monotonicity is defined as follows. Fix a strategy profile σ of the two-period game. Let $P(Y, \omega)$ denote the probability that the defendant is convicted given that the state is ω and given that a voter i votes for $Y \in \{A, W, C\}$ in the

[10] We will later see that in a welfare-optimal Bayesian Nash equilibrium $p_\sigma(Y|\emptyset, \omega) \in (0, 1)$ holds for all $Y \in \{A, W, C\}$ and $\omega \in \{I, G\}$, i.e., agents wait with positive probability. Therefore, all public histories are reached with strictly positive probability, and the beliefs are determined by Bayes' rule. For improved readability, we omit the beliefs and consider Bayesian Nash equilibria instead of perfect Bayesian equilibria throughout this chapter.

first stage and that the remaining $N-1$ voters follow strategy σ. If voter i waits in period one, then she also follows strategy σ in period two. Now, the strategy profile σ is called monotone if the inequalities

$$\begin{aligned} P(A,I) &\le P(W,I) \le P(C,I) \\ P(A,G) &\le P(W,G) \le P(C,G) \end{aligned}$$

hold, i.e., the probability of conviction is monotone increasing in the actions A, W, and C.

An illustration of a monotone strategy profile is the following strategy profile where agents follow cutoff rules: Agents with a strong signal towards innocence vote for acquittal and agents with a strong signal towards guilt vote for conviction in period one. Agents with intermediate signals wait in period one and vote in period two, conditioning on the own signal and the observed votes (see Figure 1.1).

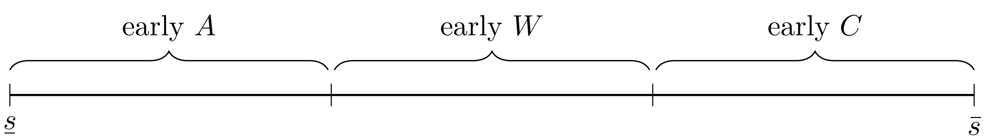

Figure 1.1: Example of a monotone strategy

Monotonicity rules out strategy profiles where the meanings of the votes are reversed, e.g., strategy profiles where voting early for conviction actually decreases the probability of conviction.

One example of a strategy profile that is not monotone is illustrated in Figure 1.2. Voters with low/intermediate signals vote for acquittal/conviction in period one and voters with high signals wait in period one. Voters understand waiting as a strong signal towards guilt and vote in period two according to their updated beliefs. For a given number of early A-votes, the lower the number of early C-votes is, the more the agents update their beliefs towards G. Therefore, voting early for C can actually decrease the probability of C being the outcome. Depending on the parameters, there can exist strategy profiles of this form that constitute equilibria.

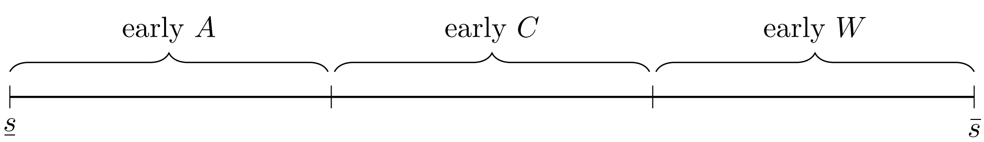

Figure 1.2: Example of a non-monotone strategy

Note that non-monotone equilibria can only exist due to the sequential nature of the voting procedure and are not possible in related simultaneous voting games. For the remainder of this chapter, we restrict our attention to the class of monotone equilibria.

Derived Terms We conclude this section by defining and deriving some technical terms for later use. Fix some strategy σ^i. The probability that voter i votes for $Y \in \{A, W, C\}$ in period one, conditional on the state ω, is obtained by integrating the marginal σ^i_Y over all signals, i.e.,

$$p_{\sigma^i}(Y|\emptyset, \omega) := \int_S \sigma^i_Y(s, \emptyset) dF(s|\omega).$$

Now, assume that waiting occurs with positive probability and consider an agent i who waited in period one. Then, the probability $p_{\sigma^i}(Y|h, \omega)$ of agent i voting for $Y \in \{A, C\}$ at history $h \neq \emptyset$ given that the state is ω is

$$p_{\sigma^i}(Y|h, \omega) := \frac{\int_S \sigma^i_Y(s, h)\sigma^i_W(s, \emptyset) dF(s|\omega)}{p_{\sigma^i}(W|\emptyset, \omega)}.$$

Furthermore, let

$$G_{\sigma^i}(s|\omega) := \frac{\int_{\underline{s}}^{s} \sigma^i_W(s', \emptyset) dF(s'|\omega)}{p_{\sigma^i}(W|\emptyset, \omega)}$$

denote the conditional distribution of signals of agents who waited in period one.

Fix a state ω and a strategy profile σ. Then, the history after period one is trinominally distributed with parameters N and $p_\sigma(Y|\emptyset, \omega)$ for $Y \in \{A, W, C\}$. More precisely, the probability that history $h = (n_A, n_C)$ occurs is

$$P(h|\omega, \sigma) = \frac{N!}{n_A! n_C! (N - n_A - n_C)!} p_\sigma(A|\emptyset, \omega)^{n_A} p_\sigma(C|\emptyset, \omega)^{n_C} p_\sigma(W|\emptyset, \omega)^{N - n_A - n_C}.$$

The period two vote count after history h is then binominally distributed with parameters $N - n_A - n_C$ and $p_\sigma(C|h, \omega)$. The probability that after history h, the total number of C-votes is equal to $k \geq n_C$ is

$$P(k|h, \omega, \sigma) = \binom{N - n_A - n_C}{k - n_C} p_\sigma(C|h, \omega)^{k - n_C} p_\sigma(A|h, \omega)^{N - k - n_A}.$$

Taking the sum over all possible histories yields the ex-ante probability $P(k|\omega, \sigma)$ of a vote count k

$$P(k|\omega, \sigma) = \sum_{h = (n_A, n_C)} P(h|\omega, \sigma) P(k|h, \omega, \sigma).$$

From this, we obtain the probability of conviction in state ω under strategy profile

σ. It is given by the sum of the probabilities of all vote counts where the defendant is convicted

$$P(C|\omega,\sigma) = \sum_{k=K+1}^{N} P(k|\omega,\sigma) \; + \; pP(K|\omega,\sigma).$$

This includes the event of exactly K votes for conviction where the outcome is a conviction with probability p.

1.3 Example with Two Voters

To illustrate the model, we present an example with $N = 2$ voters under the voting rule $(1,0)$, i.e., under the unanimity voting rule, and solve it for one and two periods, respectively.

Let the prior be $q = \frac{1}{2}$ and let the signals be distributed on the unit interval $[0,1]$ according to the conditional density functions

$$f(s|I) = 2 - 2s, \qquad f(s|G) = 2s.$$

Figure 1.3 displays the signal distributions and the likelihood ratio.

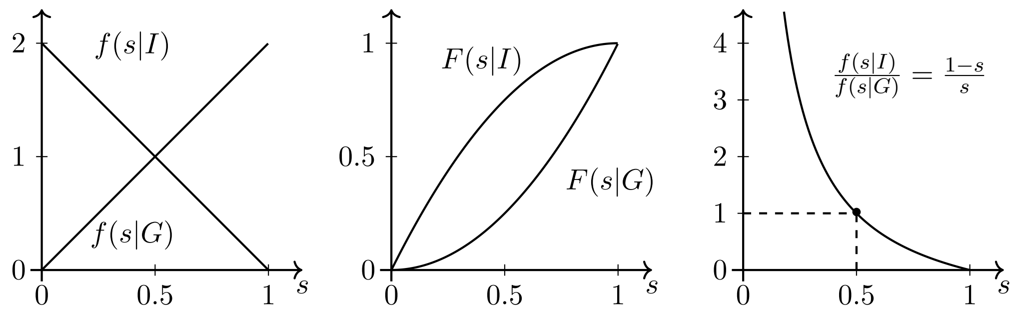

Figure 1.3: Density functions, c.d.f.'s and likelihood ratio

The densities are symmetric in the sense that $f(s|I) = f(1-s|G)$ holds. However, due to the unanimity voting rule, the setup is asymmetric in I and G. A single vote for A suffices for acquittal, while two votes are necessary for conviction. Strategic voters take the voting rule into account and adjust their voting behavior accordingly.

Example 1a: One Period First, consider a single voting period with simultaneous voting. We use the results from Duggan and Martinelli (2001) who show that in their one-period model there is a unique responsive[11] equilibrium. The equilibrium

[11] Duggan and Martinelli (2001) call an equilibrium in their simultaneous voting model a responsive equilibrium if there is no σ_i that chooses one action with probability 1, i.e., for all σ_i, $0 < \int \sigma_i(s)dF(s|G) < 1$ and $0 < \int \sigma_i(s)dF(s|I) < 1$ hold.

follows a cutoff rule, i.e., there is a unique cutoff $\hat{s}$ such that the strategies are almost everywhere equal to

$$\sigma(s) = \begin{cases} A, & \text{for } s \in [0, \hat{s}] \\ C, & \text{for } s \in (\hat{s}, 1]. \end{cases}$$

To calculate the cutoff $\hat{s}$, one has to condition on the event that a voter is pivotal, i.e., the event that a voter's decision could change the outcome. In this example, a voter is pivotal if and only if the other voter votes C. Conditioning on this event, a voter with signal $\hat{s}$ is indifferent between voting for A and voting for C if and only if

$$\frac{f(\hat{s}|I)}{f(\hat{s}|G)} \frac{1 - F(\hat{s}|I)}{1 - F(\hat{s}|G)} \frac{q}{1-q} = 1$$

holds. Solving this for $\hat{s}$ yields $\hat{s} = \frac{1}{3}$ and the ex-ante expected pay-off in this equilibrium is $U_1 \approx 0.796$. Voters with low signals vote for A, while voters with high signals vote for C. Although voters with a signal $s \in (\frac{1}{3}, \frac{1}{2})$ assign a higher probability to state I than to state G, they still vote for C in equilibrium as they try to counteract the bias of the voting rule.

The probabilities of having a signal in the respective intervals conditional on the state are depicted in Figure 1.4.

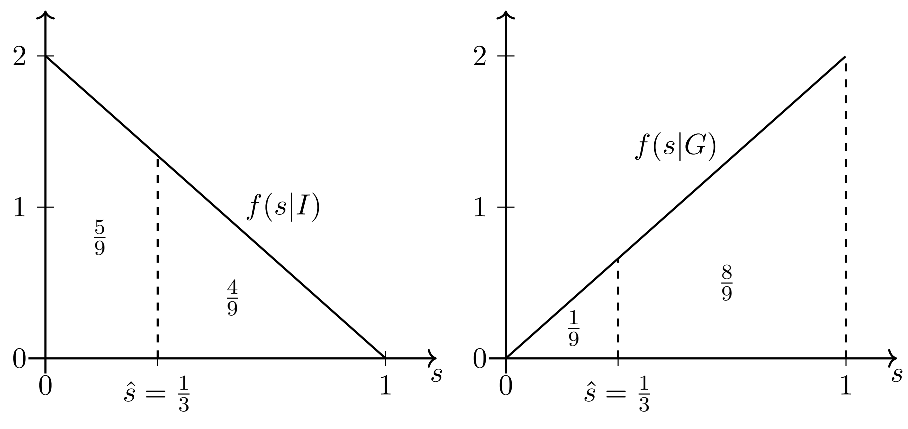

Figure 1.4: Conditional probabilities for Example 1a

Example 1b: Two Periods Now, consider the same example within our two-period model. For this setup, there exist various equilibria. We present a welfare-optimal equilibrium. Recall that $h = (0, 0)$ and $h = (0, 1)$ denote the possible histories in period two with 0 and 1 early C-votes, respectively.

Claim 1.1. A welfare-optimal equilibrium is given by the strategies

$$\sigma(s,\emptyset) = \begin{cases} A, & \text{for } \; s \in [0,\hat{x}] \\ W, & \text{for } \; s \in (\hat{x},\hat{z}] \\ C, & \text{for } \; s \in (\hat{z},1] \end{cases}$$

$$\sigma(s,(0,0)) = \begin{cases} A, & \text{for } \; s \in [0,\hat{y}] \\ C, & \text{for } \; s \in (\hat{y},1] \end{cases}$$

$$\sigma(s,(0,1)) = \begin{cases} A, & \text{for } \; s \in [0,\hat{x}] \\ C, & \text{for } \; s \in (\hat{x},1] \end{cases}$$

with the cutoffs $\hat{x} = \frac{1}{7}$, $\hat{y} = \frac{3}{7}$ and $\hat{z} = \frac{5}{7}$.

The strategies are graphically illustrated in Figure 1.5. There, "late A/C" labels the signals for which a voter votes either A or C in period two depending on the other voter's action as described below.

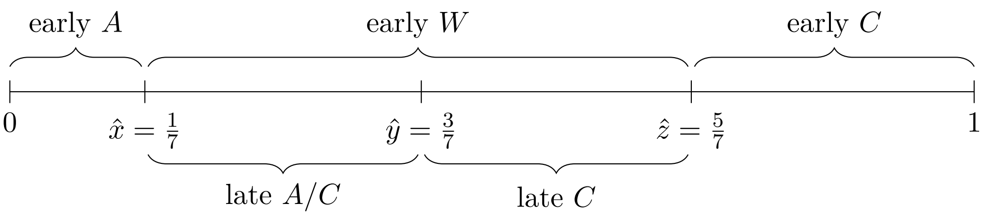

Figure 1.5: Strategy for Example 1b

The probabilities of having a signal in the respective intervals conditional on the state are the integrals of the corresponding densities and they are displayed in Figure 1.6.

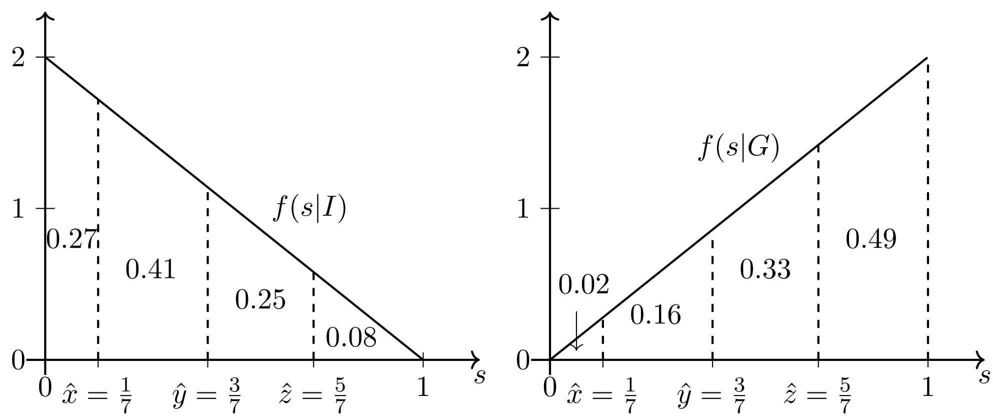

Figure 1.6: Conditional probabilities for Example 1b

In this equilibrium, a voter with a signal $s \leq \frac{1}{7}$ immediately votes for A and ends the game. A voter with a signal $s > \frac{5}{7}$ votes for C in period one. This can be understood as a message for the other agent about the strength of the private signal. A voter i with a signal $s \in (\frac{1}{7}, \frac{3}{7}]$ waits in period one and then votes depending on the other voter j's behavior. If j has voted for C in period one, then i also votes for C in period two. If j has instead waited in period one, then i votes for A. A voter with a signal $s \in (\frac{3}{7}, \frac{5}{7}]$ always waits and then votes C in period one. This way, she votes for C but ensures that the other voter does not misinterpret her voting as a strong indicator of guilt.

Using this voting structure allows the agents to communicate with each other. An agent with a strong signal votes early, and by doing so, she informs the other voter that her signal is highly informative. A voter with a weak signal waits for the other agent to vote and updates her beliefs depending on the outcome of period one. The values of the cutoffs $\hat{x}, \hat{y}$ and $\hat{z}$ are determined by the likelihood ratios

$$\frac{f(\hat{x}|I)}{f(\hat{x}|G)} \frac{1 - F(\hat{z}|I)}{1 - F(\hat{z}|G)} = 1 \tag{1.1a}$$

$$\frac{f(\hat{y}|I)}{f(\hat{y}|G)} \frac{F(\hat{z}|I) - F(\hat{y}|I)}{F(\hat{z}|G) - F(\hat{y}|G)} = 1 \tag{1.1b}$$

$$\frac{f(\hat{z}|I)}{f(\hat{z}|G)} \frac{F(\hat{y}|I) - F(\hat{x}|I)}{F(\hat{y}|G) - F(\hat{x}|G)} = 1. \tag{1.1c}$$

Setting the likelihood ratios equal to 1 identifies the signal strength at which a strategic voter, who conditions on the event of being pivotal, is indifferent between two actions. Consider an agent i with a signal equal to the cutoff $\hat{x}$ who considers voting early A or voting late A/C. She is pivotal with her choice, if and only if the other agent votes for C in period one. If the other agent has a signal lower than $\hat{z}$, then i will vote for A in period two either way. Similarly, an agent with signal $\hat{y}$ is only pivotal if the other agent has a signal $s \in (\hat{y}, \hat{z}]$ and an agent with signal $\hat{z}$ is only pivotal if the other agent has a signal $s \in (\hat{x}, \hat{y}]$. The cutoffs are derived in detail in the appendix.

The defendant is acquitted if at least one voter has a signal below $\hat{x}$ or both voters have signals in $(\hat{x}, \hat{z}]$ with at least one of them being in $(\hat{x}, \hat{y}]$. Otherwise, the defendant is convicted. As a result, with two voting periods, the ex-ante expected payoff, which is the probability of a correct choice, is $U_2 \approx 0.8265$ and it is larger than the ex-ante expected payoff with only one period. In this example, introducing a second period results in a strict welfare improvement.

At least a weak welfare improvement was to be expected since the outcome of the equilibrium of the model with one period can also be implemented by an equilibrium in the model with two periods. To see this claim, note that if all agents vote in period one, then no agent has a strict incentive to wait.[12] We show in the next section that it holds generally that the introduction of the second voting period implies a strict welfare gain for all parameters.

1.4 Welfare-Optimal Equilibrium

In this section, we show the existence of a welfare-optimal equilibrium, we characterize the structure of welfare-optimal equilibria, and we show that there is a strict welfare improvement to a standard voting procedure with only one period. The results hold for all (K, p)-voting rules. In particular, they also apply to the unanimity voting rule and the simple majority voting rule.

First, we formalize the notion of a cutoff equilibrium in the two-period model. An equilibrium is called a cutoff equilibrium if, at every history, the agents' strategies follow (monotone) cutoff rules. More precisely, an equilibrium σ *follows a cutoff rule in period one* if there are cutoffs $\hat{s}_A, \hat{s}_C \in [\underline{s}, \bar{s}]$ with $\hat{s}_A \leq \hat{s}_C$ such that

$$\sigma(s, \emptyset) = \begin{cases} A, & \text{for } \ s \leq \hat{s}_A \\ W, & \text{for } \ \hat{s}_A < s \leq \hat{s}_C \\ C, & \text{for } \ s > \hat{s}_C \end{cases}$$

holds for almost all signals $s \in S$.

Fix a history $h \in H \backslash \{\emptyset\}$ from period two. Then, an equilibrium σ *follows a cutoff rule at history* h if there is some cutoff $\hat{s}_h \in [\underline{s}, \bar{s}]$ such that

$$\sigma(s, h) = \begin{cases} A, & \text{for } \ s \leq \hat{s}_h \\ C, & \text{for } \ s > \hat{s}_h \end{cases}$$

holds for almost all signals $s \in S$.

We call an equilibrium σ a *cutoff equilibrium*, if it follows a cutoff rule in period one and at every history $h \neq \emptyset$.

Our main result shows that (i) there exists an equilibrium that maximizes welfare (in the class of all symmetric monotone equilibria) and (ii) all equilibria that maximize welfare (in the class of all symmetric monotone equilibria) follow cutoff rules. This result generalizes the findings of Duggan and Martinelli (2001) for the simultaneous voting model to the model with two voting periods.

Theorem 1.1. *There exists a welfare-optimal equilibrium. Under (MLRP$_<$) and (ULR), every welfare-optimal equilibrium is a cutoff equilibrium.*

[12] An agent who waits only learns whether or not she is pivotal. Since agents already condition on the event of being pivotal, waiting does not increase the expected payoff of an agent in this situation.

Note that there is a multiplicity of welfare-optimal equilibria. First, changing σ on a set of measure zero does not change the welfare and does still constitute an equilibrium. Moreover, for some parameters, there also exist welfare-optimal cutoff equilibria with different cutoffs simultaneously. Theorem 1.1 uses the assumptions ($\text{MLRP}_<$) and (ULR) to ensure that every action A, W and C is played in period one with positive probability. Relaxing these assumptions allows for setups where degenerate[13] equilibria can be welfare-optimal.

The first part of the theorem states the existence of a welfare-optimal equilibrium. This is proven by the maximality principle. While the strategy-space is not compact under the usual metrics, we construct a specific metric on the set $\mathcal{S}$ of the symmetric monotone strategy profiles. Under this metric, $\mathcal{S}$ is compact, and the function $\Psi : \mathcal{S} \to [0, 1]$ that maps a strategy profile to its induced welfare is continuous. By the maximality principle, there exists a strategy profile which maximizes welfare. McLennan (1998) shows that such a welfare-optimal strategy profile always constitutes an equilibrium.

After having established existence, the remainder of this section is dedicated to proving the second part of Theorem 1.1: First, in Lemma 1.1 we show that in every welfare-optimal equilibrium, both voting periods are used. Then, in Lemma 1.2 and Lemma 1.3 we analyze the equilibrium strategies in period one and two, respectively.

Lemma 1.1. *Assume that ($MLRP_<$) holds and let σ be a welfare-optimal equilibrium. Then, agents wait with a probability strictly between* 0 *and* 1*, i.e.,*

$$0 < p_\sigma(W|\emptyset, \omega) < 1$$

holds for $\omega \in \{I, G\}$.

The first period can be used to differentiate between agents with more informative and less informative signals. This way, any agent who waits can update her prior accordingly and make a better-informed decision, increasing the expected payoff. Equilibria that do not use both periods forfeit this opportunity of communication and, as a result, cannot be welfare-optimal.

For the proof, we start with an equilibrium σ in which only one period is used and construct an equilibrium with higher welfare over several steps. First, we construct a strategy profile σ' with the same welfare as σ: The agents' time of voting can be split between both periods without changing the outcome. To do so, we let voters with more informative signals vote in period one and voters with less informative

[13] Here, with "degenerate", we mean that not all actions are used. For example, without (ULR), there exist parameters for which it is never optimal to vote for A. Similarly, without ($\text{MLRP}_<$), there exist parameters for which the welfare-optimal equilibria do not use both periods (e.g. settings with binary signals and a small number of voters). In such settings, there can exist welfare-optimal equilibria that yield the same outcome as a degenerate cutoff equilibrium but do not follow cutoff rules themselves.

signals vote in period two. The resulting strategy profile σ' is not necessarily an equilibrium but yields the same welfare as σ by construction. Now, starting from σ', we construct a strategy profile σ'' with strictly higher welfare as follows. For some signals, there is a profitable deviation from σ' for an individual voter in period two: As the voting of period one reveals information about the signal strengths of the other voters, a voter in period two can update her prior accordingly and deviate to a more profitable strategy. By McLennan (1998), this individual profitable deviation shows the existence of a symmetric profitable deviation σ'' where every voter plays the individual profitable deviation with a small probability ε. Hence we have shown that an equilibrium in which only one voting period is used is not a welfare-optimal (symmetric) strategy profile. For the second step, we use again an argument by McLennan (1998) who shows that a welfare-optimal equilibrium is also a welfare-optimal symmetric strategy profile. Since σ is not the latter, it can also not be a welfare-optimal equilibrium.

As an immediate implication, there exists an equilibrium of the two-period voting game that yields a strictly higher welfare than all equilibria of the voting game with only one period.

Corollary 1.1. *Under assumption (MLRP$_<$), there exists an equilibrium of the two-period voting game that strictly welfare-dominates all equilibria of the simultaneous voting game.*

Another direct implication of Lemma 1.1 is that in a welfare-optimal equilibrium, the inequalities of the monotonicity conditions are strict, i.e., the probabilities of conviction given that one voter votes A, W, or C, respectively, cannot be equal.

Corollary 1.2. *Under assumption (MLRP$_<$), in any welfare-optimal equilibrium*

$$P(A,\omega) \; < \; P(W,\omega) \; < \; P(C,\omega)$$

holds for $\omega \in \{I, G\}$.

If, in a welfare-optimal equilibrium, waiting led to the same probability of conviction as any other action, then the equilibrium would be outcome-equivalent to an equilibrium without waiting. By Lemma 1.1, this cannot be true for a welfare-optimal equilibrium.

Strategies in Period One Now, we analyze the equilibrium strategies in period one. We show why agents follow cutoff strategies, and we establish equations that characterize these cutoffs.

Fix a strategy profile σ and assume that (MLRP$_<$) holds. Recall that $P(Y,\omega)$ denotes the probability that the defendant is convicted given that the state is ω and given that all voters follow strategy σ except for one voter who instead votes for $Y \in \{A, W, C\}$ in the first stage.

First, we focus on comparing voting C with waiting in period one. The probability that one individual voter changes the outcome with voting C instead of waiting is

$$P(C, \omega) - P(W, \omega).$$

For a given signal s, the conditional probability of being in state G is

$$(1-q)\ \frac{f(s|G)}{qf(s|I) + (1-q)f(s|G)}$$

and therefore, the probability of changing the outcome for the better with voting C instead of waiting is

$$(1-q)\ \frac{f(s|G)}{qf(s|I) + (1-q)f(s|G)}\ \big(P(C,G) - P(W,G)\big). \tag{1.2}$$

Analogously, the probability of changing the outcome for the worse is

$$q\ \frac{f(s|I)}{qf(s|I) + (1-q)f(s|G)}\ \big(P(C,I) - P(W,I)\big). \tag{1.3}$$

The net effect of voting C instead of waiting is strictly positive if and only if the ratio of term (1.3) divided by term (1.2),

$$\frac{q}{1-q}\ \frac{f(s|I)}{f(s|G)}\ \frac{P(C,I) - P(W,I)}{P(C,G) - P(W,G)}, \tag{1.4}$$

is strictly smaller than 1. If the ratio is strictly larger than 1, then an agent stricly prefers waiting to voting C in period one. For a fixed strategy in period two and fixed strategies of the other voters, because of $\frac{f(s|I)}{f(s|G)}$ being strictly monotone, setting term (1.4) equal to 1 yields a unique cutoff $\hat{s}_C$ for which an agent is indifferent between voting C in period one and waiting.

For the second cutoff, $\hat{s}_A$, we analogously get that an agent is indifferent between voting A in period one and waiting if and only if

$$\frac{q}{1-q}\ \frac{f(s|I)}{f(s|G)}\ \frac{P(W,I) - P(A,I)}{P(W,G) - P(A,G)} = 1 \tag{1.5}$$

holds. Note that for general strategy profiles $\hat{s}_A \leq \hat{s}_C$ does not need to hold. However, for welfare-optimal equilbria, agents wait with strictly positive probability and we get that the cutoffs for the first period are ordered as in Figure 1.7.

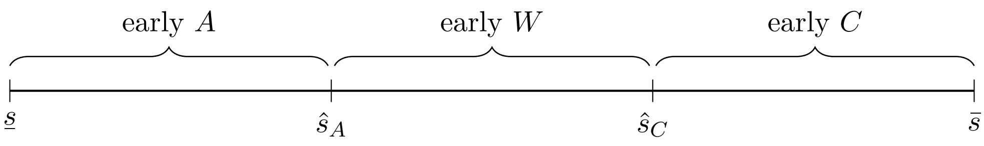

Figure 1.7: Cutoff strategies in period one

This observation is formally derived by the following lemma.

Lemma 1.2. *Under (MLRP$_<$) and (ULR) for every monotone equilibrium σ with a probability of waiting strictly between* 0 *and* 1*, the equilibrium follows a cutoff rule in period one.*

Strategies in Period Two We continue by analyzing the equilibrium strategies in period two. To understand the equilibrium behavior, first, note that every history h in period two induces a single-period game for all agents who waited. In the induced game, the votes of all agents who voted in period one are observed by the remaining agents and thus result in an updated prior.

Fix an equilibrium σ in which agents wait with probability strictly between 0 and 1. Then, the *updated prior at history* $h = (n_A, n_C)$ is given by

$$\rho_{\sigma,h} = \frac{q}{1-q} \left(\frac{p_\sigma(A|\emptyset, I)}{p_\sigma(A|\emptyset, G)} \right)^{n_A} \left(\frac{p_\sigma(C|\emptyset, I)}{p_\sigma(C|\emptyset, G)} \right)^{n_C} \left(\frac{p_\sigma(W|\emptyset, I)}{p_\sigma(W|\emptyset, G)} \right)^{N-n_A-n_C-1} . \tag{1.6}$$

It consists of the ex-ante prior $\frac{q}{1-q}$, the likelihood ratio of n_A agents voting for A in period one,

$$\left(\frac{p_\sigma(A|\emptyset, I)}{p_\sigma(A|\emptyset, G)} \right)^{n_A}, \tag{1.6a}$$

the likelihood ratio of n_C agents voting for C in period one,

$$\left(\frac{p_\sigma(C|\emptyset, I)}{p_\sigma(C|\emptyset, G)} \right)^{n_C}, \tag{1.6b}$$

and the likelihood ratio of the remaining $N - n_A - n_C - 1$ voters waiting in period one,

$$\left(\frac{p_\sigma(W|\emptyset, I)}{p_\sigma(W|\emptyset, G)} \right)^{N-n_A-n_C-1} \tag{1.6c}$$

(excluding one voter, since every voter knows her own signal).

At history h, the one-period game is played with the induced conditional distribution function

$$G_\sigma(s|\omega) = \frac{\int_{\underline{s}}^{s} \sigma_W(s', \emptyset) dF(s'|\omega)}{p_\sigma(W|\emptyset, \omega)}$$

replacing $F(s|\omega)$ and with the updated prior $\rho_{\sigma,h}$ replacing the ex-ante prior $\frac{q}{1-q}$. We call this the *at h induced game G_h* and use σ_{G_h} for the strategy profile played at G_h that is induced by σ.

Under voting rules without tie-breaking[14] and with the assumption of a sufficiently weak prior, Duggan and Martinelli (2001) show that in the simultaneous voting model, there is an almost everywhere unique responsive equilibrium[15] and it follows a cutoff rule.

However, in the two-period voting game, the updated prior $\rho_{\sigma,h}$ is an endogenous object. For fixed parameters, there may be some histories for which the prior is sufficiently weak and other histories for which it is not. Therefore, there can exist induced games G_h with a responsive equilibrium and other induced games without one. Our next result characterizes the period two equilibrium strategies of a welfare-optimal equilibrium. To calculate the cutoffs, we first need the likelihood ratio of being pivotal. If the (K, p)-voting rule allows for tie-breaks, i.e., $p \in (0, 1)$, then there are two events where a single voter is pivotal. In the first event, the voter changes the outcome from A to a tie-break, and in the second event, the voter changes the outcome from a tie-break to C. Without random tie-breaks, i.e., with $p \in \{0, 1\}$, exactly one of these events can occur. For a general (K, p)-voting rule, the likelihood ratio $l_h(\hat{s}_h)$ of a single voter to be pivotal at history h is

$$\frac{pG_\sigma(\hat{s}_h|I)^{(N-K)-n_A}\Big(1-G_\sigma(\hat{s}_h|I)\Big)^{K-n_C-1} + (1-p)G_\sigma(\hat{s}_h|I)^{(N-K)-n_A-1}\Big(1-G_\sigma(\hat{s}_h|I)\Big)^{K-n_C}}{pG_\sigma(\hat{s}_h|G)^{(N-K)-n_A}\Big(1-G_\sigma(\hat{s}_h|G)\Big)^{K-n_C-1} + (1-p)G_\sigma(\hat{s}_h|G)^{(N-K)-n_A-1}\Big(1-G_\sigma(\hat{s}_h|G)\Big)^{K-n_C}}, \tag{1.7}$$

if all other voter follow a cutoff rule with cutoff $\hat{s}_h$.

Now, we are ready to characterize the period-two strategies in a welfare-optimal equilibrium.

Lemma 1.3. *Assume that ($MLRP_<$) and (ULR) hold and fix a welfare-optimal equilibrium σ of the two-period voting game. Then, at every history $h \in H\backslash\{\emptyset\}$, σ follows a cutoff rule with a cutoff $\hat{s}_h$ which is equivalent[16] to the unique solution for*

[14] In our model, these are voting rules of the form $(K, 0)$ or $(K, 1)$.

[15] An equilibrium of the one-period voting game is called *responsive* if both actions, A and C, are played with positive probability.

[16] More precisely, if s' lies between the first-period cutoffs $\hat{s}_A$ and $\hat{s}_C$, then the cutoff $\hat{s}_h$ is equal to s'. If s' is smaller than $\hat{s}_A$ or larger than $\hat{s}_C$, then the induced game G_h has an unresponsive equilibrium that maximizes its welfare and every cutoff $\hat{s}_h < \hat{s}_A$ or $\hat{s}_h > \hat{s}_C$, respectively, yields the same outcome.

s' of the equation

$$\rho_{\sigma,h} \ \cdot \ \frac{f(s'|I)}{f(s'|G)} \ \cdot \ l_h(s') = 1. \tag{1.8}$$

Equation (1.8) consists of the updated prior $\rho_{\sigma,h}$, the likelihood ratio of the own signal $\frac{f(s'|I)}{f(s'|G)}$, and the likelihood ratio of the event of being pivotal in period two conditional on the observations in period one. A voter in period two is indifferent between voting for A or C if and only if the product of these three is equal to 1. If the product is strictly larger than 1, then conditioning on the event of being pivotal, the voter reasons that the state is more likely to be I and, therefore, strictly prefers voting for A in period two. Analogously, if the product is strictly smaller than 1, then the voter strictly prefers voting for C.

We have now seen that the voters follow cutoff strategies in both periods and how these cutoffs are calculated. This concludes the analysis of the structure of the welfare-optimal equilibria.

1.5 Information Aggregation

We now show that the two-period voting procedure aggregates information when the number of voters grows large. Consider a sequence $(a_N)_{N\in\mathbb{N}}$ of voting setups where every a_N has exactly N voters. We say that $(a_N)_{N\in\mathbb{N}}$ *allows information aggregation* if there exists a sequence of equilibria σ_N for a_N, respectively, such that the probability of the correct decision under σ_N converges to 1 as N tends to infinity.

We show that our model with two periods allows for information aggregation even in settings where simultaneous voting and sequential voting with an exogenous voting sequence fail to do so. Feddersen and Pesendorfer (1998) analyze the unanimity voting rule in a simultaneous voting model with binary signals. They prove that even for large electorates, the probability of the correct decision is bounded away from 1. There, an increase in the jury size does not lead to information aggregation. Dekel and Piccione (2000) show that in a sequential voting model with exogenous timing, the equilibria are equivalent to the equilibria of the simultaneous voting model. Therefore, an exogenous voting sequence does also not allow for information aggregation under the unanimity voting rule as long as the likelihood ratio of the signals is bounded. In our two-period voting model, this observation does not hold anymore. Theorem 1.2 shows that, even with only binary signals, information is aggregated for a large jury size regardless of the voting rule.

Theorem 1.2. *Fix a sequence $(a_N)_{N\in\mathbb{N}}$ of voting setups that share the same parameters and only differ in the number of voters N and the voting rule (K,p). Then, regardless of the voting rules along the sequence, there exists a sequence of equilibria for which the probability of a correct decision converges to* 1.

The idea of the proof is to construct a strategy profile σ as follows. For every voting rule there is at least one of the two alternatives that needs a vote share of at least $\frac{1}{2}$ to win. Without loss of generality, let it be C. Fix a cutoff z with $F(z|I) + F(z|G) = 1$ and let voters with signals above the cutoff z vote early for C and the remaining voters wait for period two. The weakly monotone likelihood ratio implies that $F(z|I) > \frac{1}{2}$ and $F(z|G) < \frac{1}{2}$ hold. By the strong law of large numbers, the realized vote share of C-votes converges to the expected vote share. The probability that the game ends in period one with a wrong decision converges to 0. The expected vote share of early C-votes is different for both states, and therefore, the late voters learn the correct state with probability converging to 1. Thus, we have constructed a sequence of strategy profiles for which the probability of a correct decision converges to 1. Now, the result of Theorem 1.2 follows by using an argument by McLennan (1998) that says that under homogeneous preferences, the welfare-optimal symmetric strategy profile is an equilibrium. Therefore, for a sequence of welfare-optimal equilibria, the probability of the correct decision also converges to 1.

Assumptions (MLRP$_<$) and (ULR) are not needed for Theorem 1.2. The result also holds if the informativeness of the signals is bounded. In particular, Theorem 1.2 also applies to the setup of Feddersen and Pesendorfer (1998) who consider a binary signal space.

We conclude this section with Lemma 1.4 giving a bound on the speed of convergence.

Lemma 1.4. *The rate of convergence of the probability of a correct decision in the welfare-optimal equilibrium of the two-period game is at least N^{-1}.*

1.6 The Swing Voter's Curse

In this section, we analyze the so-called swing voter's curse, which occurs under the simple majority voting rule for an even number of voters.[17] In this situation, in the simultaneous voting game, less informed voters strictly prefer to abstain rather than to vote (see Feddersen and Pesendorfer (1996)). The reason for this swing voter's curse is that there exist two different voting situations where a single voter i's decision is pivotal. That is, if the aggregated number of A-votes of the other voters is either one vote more or one vote less than the aggregated number of C-votes of the other voters. In a simultaneous voting game, the swing voter's curse reduces welfare because less informed agents who strictly prefer not to vote have to vote and may be pivotal, changing the outcome to the wrong alternative.

[17] We follow the literature by concentrating our analysis on the swing voter's curse under the simple majority voting rule. Note that the swing voter's curse occurs in our setup also under other voting rules as long as random tie-breaks can occur.

In the simultaneous voting game, this effect can be mitigated, and the welfare can be improved by allowing agents to abstain. However, this way, the information of the less informed voters is lost. We show that the introduction of a second voting period (without allowing for abstention in period two) can utilize the information of such voters and leads to a greater welfare improvement than the possibility to abstain.

First, note that voters in the two-period voting game can mimic abstention of the simultaneous voting model. Voters can effectively abstain by waiting in period one and then voting for the majority outcome from period one in period two (or randomizing with probability $\frac{1}{2}$ if the outcome of the first period is a tie). Therefore, the two-period voting game can achieve the welfare of the welfare-optimal equilibrium of the simultaneous voting game with abstention. The following theorem states that there even exists a strict welfare improvement.

Theorem 1.3. *Assume that N is even and that (MLRP$_<$) and (ULR) hold. Then, under the simple majority voting rule, the welfare-optimal equilibrium of the model with two periods (without abstention) strictly welfare-dominates all equilibria of the simultaneous voting game with abstention.*

The first part of the proof is to construct a strategy profile of the two-period voting game that yields the same welfare as the welfare-optimal equilibrium of the simultaneous voting game with abstention. Then, we show that there is a profitable deviation in the two-period voting game. Using McLennan, 1998, this shows that there is an equilibrium in the two-period voting game, which yields a strictly higher welfare than the simultaneous voting game with abstention.

As a result, the welfare of the welfare-optimal equilibria in the different voting procedures can be ranked as follows:

$$U_{\text{Simultaneous}} \preccurlyeq U_{\text{Abstain}} \preccurlyeq U_{\text{Twoperiods}} \tag{1.9}$$

If (MLRP$_<$) and (ULR) hold, then the inequalities are strict. Without these assumptions, there exist parameters for which the welfare of all three voting procedures is equal: A voting setting with binary signals, symmetric likelihood ratios, and a prior of $\frac{1}{2}$ is an example.

1.7 Endogenous Timing Compared to a Fixed Sequence

In this section, we compare our voting model with endogenous timing to a voting procedure with an exogenously fixed voting sequence. More precisely, we compare it to a setup with two voting periods where for each voter it is exogenously given (and common knowledge) in which period this voter casts her vote. For a more detailed analysis of voting with an exogenously fixed sequence, see Dekel and Piccione (2000).

One substantial difference between an exogenously fixed voting sequence and voting with endogenous timing is the asymmetry between voters that is induced by the fixed timing of voting. Naturally, voters that vote in period one and voters that vote in period two are not ex-ante equal. We show that if we allow for asymmetric strategies in our voting model with endogenous timing, a strict welfare improvement is gained over voting with a fixed sequence.

Theorem 1.4. *Under assumptions (MLRP$_<$) and (ULR), there exists a (potentially) asymmetric equilibrium of the two-period voting game with endogenous timing that strictly welfare-dominates all equilibria of the two-period voting game with an exogenously fixed voting sequence.*

The strict welfare gain is obtained by constructing a profitable deviation. We start with a welfare-optimal equilibrium of the voting game with a fixed sequence. The outcome of this equilibrium can be replicated with endogenous timing. Now, we let a single voter in period one deviate and instead vote in period two with a positive probability. This discloses additional information to the voters in period two and subsequently allows for a profitable deviation. Therefore, an endogenous timing decision yields a strict welfare improvement over a fixed voting sequence.

1.8 Conclusion

In this chapter, we explore a voting model with an endogenous timing decision. We show the existence and characterize the structure of welfare-optimal equilibria. We generalize the well-known result from simultaneous voting models that responsive strategies follow cutoff rules. Moreover, the welfare-optimal equilibria of our model with endogenous timing welfare-dominate the equilibria from simultaneous voting models and voting procedures with a fixed voting sequence. Information is aggregated even with bounded informativeness of the signals under the unanimity voting rule. In the case of a possible random tie-break, sequential voting mitigates the swing voter's curse more effectively than abstention. The endogenous sorting into the two voting periods allows the voters to convey the strength of their private information to each other and ultimately make a better-informed collective decision.

There are various extensions to our model that can be pursued for future research. First, the two periods can be generalized to an arbitrary finite number or a countable infinite number of periods. Adding more periods makes the information transmission of the agents more efficient, resulting in a higher probability of choosing the correct outcome. However, we have shown in this chapter that even under the unanimity voting rule with bounded signals, two voting periods suffice for information aggregation.

Another possible extension is the generalization to a continuous time interval.

Going from two periods to continuous time allows for finer communication between the voters. Depending on the modeling of the strategies, continuous time can allow the agents to perfectly communicate their signals and solves the collective coordination problem completely. Note that allowing the set of possible voting times to be as rich as a real interval is a particularly strong assumption.

Possible other extensions in this line of research could be the addition of voting costs that induce a free-riding problem, making waiting costly, or considering a private value component such that the voter's interests are not perfectly aligned anymore.

1.A Proofs

1.A.1 Proofs for Section 1.3

Proof of Claim 1.1. First, we show that if an equilibrium follows such cutoff rules, then the cutoffs $\hat{x}$, $\hat{y}$, and $\hat{z}$ solve the following system of equations:

$$2\hat{x} + \hat{z} = 1 \tag{1.10}$$

$$2\hat{z} - \hat{z}^2 - 2\hat{y} - 2\hat{y}\hat{z} + 3\hat{y}^2 = 0 \tag{1.11}$$

$$2\hat{y} - \hat{y}^2 - 2\hat{x} + \hat{x}^2 - 2\hat{z}\hat{y} + 2\hat{z}\hat{x} = 0. \tag{1.12}$$

Solving the system numerically then yields the unique solution $\hat{x} = \frac{1}{7}$, $\hat{y} = \frac{3}{7}$ and $\hat{z} = \frac{5}{7}$.

Equation (1.10) is given by setting a voter with signal $\hat{x}$ to be indifferent between voting A and waiting for period two:

$$\begin{aligned}
& \frac{f(\hat{x}|I)}{f(\hat{x}|G)} \frac{P(\mathrm{Piv}_{AW}\,|I)}{P(\mathrm{Piv}_{AW}\,|G)} = 1 \\
\iff \quad & \frac{f(\hat{x}|I)}{f(\hat{x}|G)} \frac{1 - F(\hat{z}|I)}{1 - F(\hat{z}|G)} = 1 \\
\iff \quad & \frac{2 - 2\hat{x}}{2\hat{x}} \frac{1 - 2\hat{z} + \hat{z}^2}{1 - \hat{z}^2} = 1 \\
\iff \quad & (1 - \hat{x})(1 - \hat{z}) = \hat{x}(1 + \hat{z}) \\
\iff \quad & 1 - \hat{x} - \hat{z} + \hat{x}\hat{z} = \hat{x} + \hat{x}\hat{z} \\
\iff \quad & 1 = 2\hat{x} + \hat{z},
\end{aligned}$$

where $P(\mathrm{Piv}_{AW}\,|\omega)$ is the probability of being pivotal with the decision of voting A or waiting. It is equal to the probability that the other voter votes C. If the other voter votes A or waits, then A will be the outcome even if i waits.

Equation (1.11) is given by setting a voter with signal $\hat{y}$ who waited to be indifferent for the case that the other voter also waited. Let $P(\mathrm{Piv}_{AC})$ be the probability that a voter is pivotal with deciding between voting late A or C. Then, one gets

$$\begin{aligned}
& \frac{f(\hat{y}|I)}{f(\hat{y}|G)} \frac{P(\mathrm{Piv}_{AC}\,|I)}{P(\mathrm{Piv}_{AC}\,|G)} = 1 \\
\iff \quad & \frac{f(\hat{y}|I)}{f(\hat{y}|G)} \frac{F(\hat{z}|I) - F(\hat{y}|I)}{F(\hat{z}|G) - F(\hat{y}|G)} = 1 \\
\iff \quad & \frac{2 - 2\hat{y}}{2\hat{y}} \frac{2\hat{z} - \hat{z}^2 - 2\hat{y} + \hat{y}^2}{\hat{z}^2 - \hat{y}^2} = 1
\end{aligned}$$

$$
\begin{aligned}
\iff \quad & (1-\hat{y})(2\hat{z}-\hat{z}^2-2\hat{y}+\hat{y}^2) = \hat{y}(\hat{z}^2-\hat{y}^2) \\
\iff \quad & 2\hat{z}-\hat{z}^2-2\hat{y}+\hat{y}^2-2\hat{y}\hat{z}+\hat{y}\hat{z}^2+2\hat{y}^2-\hat{y}^3 = \hat{y}\hat{z}^2-\hat{y}^3 \\
\iff \quad & 2\hat{z}-\hat{z}^2-2\hat{y}-2\hat{y}\hat{z}+3\hat{y}^2 = 0.
\end{aligned}
$$

Equation (1.12) is given by setting a voter with signal $\hat{z}$ to be indifferent between waiting and voting early C:

$$
\begin{aligned}
& \frac{f(\hat{z}|I)}{f(\hat{z}|G)} \frac{P(\mathrm{Piv}_{WC}\,|I)}{P(\mathrm{Piv}_{WC}\,|G)} = 1 \\
\iff \quad & \frac{f(\hat{z}|I)}{f(\hat{z}|G)} \frac{F(\hat{y}|I)-F(\hat{x}|I)}{F(\hat{y}|G)-F(\hat{x}|G)} = 1 \\
\iff \quad & \frac{2-2\hat{z}}{2\hat{z}} \frac{2\hat{y}-\hat{y}^2-2\hat{x}+\hat{x}^2}{\hat{y}^2-\hat{x}^2} = 1 \\
\iff \quad & (1-\hat{z})(2\hat{y}-\hat{y}^2-2\hat{x}+\hat{x}^2) = \hat{z}(\hat{y}^2-\hat{x}^2) \\
\iff \quad & 2\hat{y}-\hat{y}^2-2\hat{x}+\hat{x}^2-2\hat{z}\hat{y}+2\hat{z}\hat{x} = 0.
\end{aligned}
$$

The probability $P(\mathrm{Piv}_{WC}\,|\omega)$ of being pivotal between waiting and voting C in period one is given by the probability that the other voter has a signal in the interval $(\hat{x}, \hat{y}]$, which means that for voter i voting C early changes the outcome compared to waiting.

To finish the proof of Claim 1.1, it is left to show that there exists a welfare-optimal equilibrium that follows cutoff strategies. This part is deferred to Section 1.4, Theorem 1.1, which shows for the two-period voting model that there is an equilibrium that maximizes welfare and follows such cutoff rules. □

1.A.2 Proofs for Section 1.4

Proof of Theorem 1.1. First, we show the existence of a welfare-optimal equilibrium. We construct a metric $d_{\mathcal{S}}$ on the set $\mathcal{S}$ of the symmetric monotone strategy profiles. Let $Z := H \times \{A, W, C\} \times \{I, G\}$. The distance of two strategies under $d_{\mathcal{S}}$ is given by the sum of the differences of the induced ex-ante probabilities $p_\sigma(Y|h,\omega)$ of playing certain actions:

$$
d_{\mathcal{S}}(\sigma_1, \sigma_2) = \sum_{(h,Y,\omega)\in Z} \big|p_{\sigma_1}(Y|h,\omega) - p_{\sigma_2}(Y|h,\omega)\big|.
$$

Next, we show that the metric space $(\mathcal{S}, d_{\mathcal{S}})$ is compact. To show sequentially compactness, we start with a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of symmetric monotone strategy profiles. Let φ denote the function that maps such a strategy into the space of induced ex-ante probabilities, i.e.,

$$\begin{aligned} \varphi : \mathcal{S} &\to [0,1]^{6|H|} \\ \sigma &\mapsto \left(p_\sigma(Y|h,\omega)\right)_{(h,Y,\omega)\in Z} \end{aligned}$$

Then, as $[0,1]^{6|H|}$ together with the taxicab distance d_1 is a compact space, the sequence of the induced probabilities $(\varphi(\sigma_n))_{n\in\mathbb{N}}$ has a convergent subsequence $(\varphi(\sigma_{n_k}))_{k\in\mathbb{N}}$. Its limit is induced by a strategy profile σ^*.[18] By construction, the subsequence $(\varphi(\sigma_{n_k}))_{k\in\mathbb{N}}$ converges to σ^*. Therefore, in $\mathcal{S}$, every sequence has a converging subsequence and $(\mathcal{S}, d_{\mathcal{S}})$ is a compact metric space.

Now, the function φ is continuous with respect to the metrics $d_{\mathcal{S}}$ and d_1. Furthermore, the function that maps the probabilities of actions to the expected welfare

$$\begin{aligned} \psi : [0,1]^{6|H|} &\to [0,1] \\ \left(p_\sigma(Y|x,\omega)\right)_{(h,Y,\omega)\in Z} &\mapsto U \end{aligned}$$

is continuous with respect to the metrics d_1 on $[0,1]^{6|H|}$ and d_1 on $[0,1]$.

By the continuity of the composition $\psi \circ \varphi$ and the compactness of the strategy space, there exists a welfare-optimal symmetric strategy profile. McLennan (1998) shows that such a welfare-optimal strategy profile constitutes an equilibrium.

The second statement of the theorem says that all welfare-optimal equilibria follow cutoff rules. We prove this through the other results established in Section 1.4. The overview is as follows: First, we fix a welfare-optimal equilibrium. Lemma 1.1 shows that in a welfare-optimal equilibrium, both periods are used. Lemma 1.2 shows that agents in a welfare-optimal equilibrium follow cutoff strategies in period one for almost all signals. Lemma 1.3 shows that agents follow a strategy in period two that is equivalent to a cutoff strategy. Together, these results show that in a welfare-optimal equilibrium, agents follow a cutoff strategy for all signals except for a subset with probability measure zero. Therefore, every welfare-optimal equilibrium is almost everywhere equal to a cutoff equilibrium. □

Proof of Lemma 1.1. Every equilibrium σ with $p_\sigma(W|\emptyset, I) = 1$ yields the same payoff as a corresponding equilibrium with $p_\sigma(W|\emptyset, I) = 0$, i.e., it is of no importance whether all agents wait or no agent waits. Thus, it suffices to fix an equilibrium σ with $p_\sigma(W|\emptyset, I) = 0$, which is optimal in the class of such equilibria and to show that there exists an equilibrium with higher welfare. We show that there exists

[18] To construct such a strategy profile σ^* for a given limit, define the strategy separately for every history h. For a given h, start with cutoff strategies that induce the correct probabilities for state $\omega = I$. Then, adjust the strategy by shifting the probability mass between the actions to obtain the probabilities for state $\omega = G$ without changing the probabilities for $\omega = I$. As the probabilities are the limit probabilities induced by monotone strategy profiles, the limit strategy profile σ^* is also monotone.

an equilibrium with higher welfare by dividing the agents who vote for one action such that some agents with specific signals vote in period one and the agents with other signals vote in period two. Then, a single agent can profitably deviate due to her updated information. Using a result by McLennan (1998), the existence of a welfare-better strategy profile implies the existence of an equilibrium with higher welfare.

First, we construct an equilibrium with $p_\sigma(W|\emptyset, I) = 0$ that is optimal in the class of all such equilibria. As an equilibrium with $p_\sigma(W|\emptyset, I) = 0$ is equivalent to an equilibrium of the simultaneous voting model, we can apply the results from Duggan and Martinelli (2001) to our (more general) (K, p)-voting rule. A welfare-optimal equilibrium is given by the strategy profile

$$\sigma(s,h) = \begin{cases} A, & \text{for } \ s \in [\underline{s}, \hat{s}] \\ C, & \text{for } \ s \in (\hat{s}, \bar{s}] \end{cases}$$

with $\hat{s}$ being the solution of the equation

$$\frac{q}{1-q}\frac{f(\hat{s}|I)}{f(\hat{s}|G)}\, l_h(\hat{s}) = 1$$

where $l_h(\hat{s})$ denotes the term

$$\frac{pF(\hat{s}|I)^{N-K}\big(1-F(\hat{s}|I)\big)^{K-1} + (1-p)F(\hat{s}|I)^{N-K-1}\big(1-F(\hat{s}|I)\big)^{K}}{pF(\hat{s}_h|G)^{N-K}\big(1-F(\hat{s}|G)\big)^{K-1} + (1-p)F(\hat{s}|G)^{N-K-1}\big(1-F(\hat{s}|G)\big)^{K}},$$

which is the likelihood ratio of being pivotal if all other voters follow the cutoff rule with cutoff $\hat{s}$. Now, we modify the strategies without changing the outcome by letting a small part of C-voters vote in period two instead of period one. Fix a positive $\varepsilon < 1 - \hat{s}$ and define a new strategy profile σ' by

$$\sigma'(s,\emptyset) = \begin{cases} A, & \text{for } \ s \in [\underline{s}, \hat{s}] \\ W, & \text{for } \ s \in (\hat{s}, \hat{s}+\varepsilon] \\ C, & \text{for } \ s \in (\hat{s}+\varepsilon, \bar{s}] \end{cases}$$

$$\sigma'(s,h) = \begin{cases} A, & \text{for } \ s \in [\underline{s}, \hat{s}] \\ C, & \text{for } \ s \in (\hat{s}, \bar{s}] \end{cases}, \qquad \text{for all } \ h \neq \emptyset.$$

Now, we fix an agent i, the threshold $\hat{s}$ and the strategies of all other agents. For the case $p \neq 1$, we construct a payoff increasing strategy profile σ_i'' for agent i by letting her wait in period one and updating her prior at one particular history

$h = (N-K-1, K)$. At any other history, i follows the strategy σ. This is given by

$$\sigma''(s,\emptyset) = \quad W$$

$$\sigma''(s,(N-K-1,K)) = \begin{cases} A, & \text{for } \; s \in [\underline{s}, \hat{s}'] \\ C, & \text{for } \; s \in (\hat{s}', \bar{s}]. \end{cases}$$

$$\sigma''(s,h) = \begin{cases} A, & \text{for } \; s \in [\underline{s}, \hat{s}] \\ C, & \text{for } \; s \in (\hat{s}, \bar{s}]. \end{cases} \qquad \text{for all } h \neq \emptyset, \; h \neq (N-K-1,K)$$

with $\hat{s}'$ being the unique solution of the equation

$$\frac{q}{1-q}\,\frac{f(\hat{s}'|I)}{f(\hat{s}'|G)}\left(\frac{F(\hat{s}|I)}{F(\hat{s}|G)}\right)^{N-K-1}\left(\frac{1-F(\hat{s}+\varepsilon|I)}{1-F(\hat{s}+\varepsilon|G)}\right)^{K} = 1. \tag{1.13}$$

Duggan and Martinelli (2001) show that the inequality

$$\frac{1-F(\hat{s}|I)}{1-F(\hat{s}|G)} > \frac{1-F(\hat{s}+\varepsilon|I)}{1-F(\hat{s}+\varepsilon|G)}$$

follows from (MLRP$_<$). As an immediate consequence, the likelihood ratio of being pivotal is different for $\hat{s}$ and $\hat{s}'$. This implies that

$$\hat{s}' < \hat{s}$$

holds, i.e., the cutoffs of σ' and σ'' are different at history $h = (N-K-1, K)$. Since $\hat{s}'$ solves equation (1.13), it is the optimal strategy for agent i given that she observes that exactly K voters vote C in period one. Hence, σ'' is a profitable deviation for player i. For the case $p = 1$, the analogue construction for history $h = (N-K, K-1)$ instead of $h = (N-K-1, K)$ yields the same result.

By a result of McLennan (1998), this implies that there exists a symmetric equilibrium with higher welfare. □

Proof of Corollary 1.1. Every equilibrium of the simultaneous voting game is outcome-equivalent to an equilibrium of the two-period model with $p_\sigma(W|\emptyset, I) = 0$. By Lemma 1.1, there exists an equilibrium with strictly higher welfare. □

Proof of Corollary 1.2. Suppose for contradiction that there exists a welfare-optimal equilibrium σ^* with one of the inequalities being an equality. Without loss of generality, let

$$P(A,\omega) = P(W,\omega)$$

be true. Then, the strategy profile where in the first period all probability mass from waiting is put onto A instead, yields the same expected welfare. By Lemma 1.1, there exists an equilibrium with strictly higher welfare, which contradicts welfare-optimality of σ^*. □

Proof of Lemma 1.2. We show that the best response to any symmetric strategy profile σ follows cutoff rules in period one. Recall that $P(Y, \omega)$ is the probability that the defendant is convicted given that the state is ω and given that one voter votes for $Y \in \{A, W, C\}$ in the first period and all other voters follow strategy σ. The expected payoff of voting A early after receiving signal s is now given by

$$U(A, s) = \frac{f(s|I)}{f(s|I) + f(s|G)}(1 - P(A, I)) + \frac{f(s|G)}{f(s|I) + f(s|G)}P(A, G). \tag{1.14}$$

Similarly, let

$$U(W, s) = \frac{f(s|I)}{f(s|I) + f(s|G)}(1 - P(W, I)) + \frac{f(s|G)}{f(s|I) + f(s|G)}P(W, G) \tag{1.15}$$

and

$$U(C, s) = \frac{f(s|I)}{f(s|I) + f(s|G)}(1 - P(C, I)) + \frac{f(s|G)}{f(s|I) + f(s|G)}P(C, G) \tag{1.16}$$

denote the respective expected payoffs. To see when a voter is indifferent between two options, let x, $a_l < a_h$ and $b_l < b_h$ be real numbers and consider the equation

$$xa_h + (1 - x)b_l = xa_l + (1 - x)b_h, \tag{1.17}$$

which is uniquely solved by

$$x = \frac{b_h - b_l}{(a_h - a_l) + (b_h - b_l)} \in [0, 1].$$

Set any two of the three utility functions (1.14) , (1.15) and (1.16) equal to each other. Then, the resulting equation has the form of equation (1.17) with $x = \frac{f(s|I)}{f(s|I)+f(s|G)}$. Thus, for every pair of utility functions this gives a unique solution for

$$\frac{f(s|I)}{f(s|I) + f(s|G)} \in [0, 1].$$

Furthermore, we know that it lies in the interior $(0, 1)$ by Corollary 1.2.

Let x_{AW} denote the value obtained by setting $U(A, s)$ and $U(W, s)$ to be equal. Then, the utility of voting for A is strictly higher than the utility of voting for W for all signals s with $\frac{f(s|I)}{f(s|I)+f(s|G)} > x_{AW}$ and strictly lower for all signals s with $\frac{f(s|I)}{f(s|I)f(s|G)} < x_{AW}$. In particular a voter is indifferent with a signal s with $\frac{f(s|I)}{f(s|I)+f(s|G)} = x_{AW}$. The same holds for x_{WC} and x_{AC}, which are defined the same way.

By monotonicity, one can rewrite

$$\begin{aligned} P(W,I) &= P(A,I) + \varepsilon_1 \\ P(C,I) &= P(A,I) + \varepsilon_1 + \varepsilon_2 \\ P(W,G) &= P(A,I) + \delta_1 \\ P(C,G) &= P(A,I) + \delta_1 + \delta_2 \end{aligned}$$

for $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 > 0$. Thus, one gets

$$\begin{aligned} x_{AW} &= \frac{\delta_1}{\varepsilon_1 + \delta_1} \\ x_{AC} &= \frac{\delta_1 + \delta_2}{\varepsilon_1 + \varepsilon_2 + \delta_1 + \delta_2} \\ x_{WC} &= \frac{\delta_2}{\varepsilon_2 + \delta_2}. \end{aligned}$$

In particular, x_{AC} is a convex combination of x_{AW} and x_{WC}. By (MLRP$_<$), the term $\frac{f(s|I)}{f(s|I)+f(s|G)}$ is strictly decreasing in s. By (ULR), for all $x \in (0,1)$ there exists a unique

$$s_x = \sup_s \left\{ \frac{f(s|I)}{f(s|I) + f(s|G)} \leq x \right\}.$$

Now, exactly one of the three (in-)equalities

$$s_{x_{AW}} < s_{x_{AC}} < s_{x_{WC}}, \tag{1.18}$$

$$s_{x_{AW}} > s_{x_{AC}} > s_{x_{WC}}, \tag{1.19}$$

$$s_{x_{AW}} = s_{x_{AC}} = s_{x_{WC}} \tag{1.20}$$

holds. If either (1.19) or (1.20) holds, then the equilibrium cannot be welfare-optimal by Lemma 1.1 since there is no set of signals with a positive measure for which W is a strictly best response. Thus, inequality (1.18) holds, which implies that the equilibrium follows a cutoff rule in period one. □

Proof of Lemma 1.3. Consider the history $h = (n_A, n_C)$ in period two after n_A voters voted early A and n_C voters voted early C. Let

$$A_{\sigma,n} = \left\{ s \;\middle|\; \left(\frac{f(s|I)}{f(s|G)}\right)^N > \frac{1}{\rho_{\sigma,n}} \right\}$$

denote the set of all signals s such that the likelihood ratio raised to the power of

N overcomes the updated prior. Similarly, define

$$B_{\sigma,n} = \left\{ s \;\middle|\; \frac{f(s|I)}{f(s|G)} < \frac{1}{\rho_{\sigma,n}} \right\}$$

to be the set of all signals whose likelihood ratio is smaller than the updated prior. Taking the idea from the proof of Lemma 2 in Duggan and Martinelli (2001), there exists a responsive equilibrium in the induced game G_h if and only if the inequalities

$$\int_{A_{\sigma,n}} \sigma_W(s|\emptyset)\mu(ds) > 0$$

and

$$\int_{B_{\sigma,n}} \sigma_W(s|\emptyset)\mu(ds) > 0$$

hold, i.e., if the probability that an agent has a signal which is stronger than the prior in either direction is positive.

Consider now the case that this condition is satisfied at h. Even though assumption (A4) in Duggan and Martinelli (2001) does not necessarily hold in our two-period model, the assumptions necessary for their Theorem 1 are fulfilled and its conclusion applies to the induced game G_h. Hence, there exists an almost everywhere unique responsive strategy profile that is an equilibrium of G_h with cutoff s' given as the solution of

$$\rho_{\sigma,h} \cdot \frac{f(s'|I)}{f(s'|G)} \cdot l_h(s') = 1.$$

By Lemma 1.1 all histories are reached with positive probability. As an unresponsive equilibrium in a one-period voting game yields a lower welfare than the unique responsive equilibrium, we get that in a welfare-optimal equilibrium, the unique responsive equilibrium is played in every induced game of periode two where one exists.

At the histories where no responsive equilibrium exists, the welfare-optimal unresponsive equilibrium is played in period two, i.e., either all voters vote A or all voters vote C. $\square$

1.A.3 Proofs for Section 1.5

Proof of Theorem 1.2. As a consequence of McLennan (1998), it is sufficient to show that there exists a sequence of strategy profiles for which the probability of a correct decision converges to one.

For our construction, let z be a cutoff with the symmetric property $F(z|I) + F(z|G) = 1$. By the intermediate value theorem, such a z exists. Let $r := F(z|I) =$

$1 - F(z|G)$. Intuitively, treating the two intervals $[0, z]$ and $(z, 1]$ like two discrete signals that indicate innocence/guilt, respectively, r is the probability that an agent receives a correct signal. The number of correct signals is binomially distributed with parameters N and r. Note that $r > \frac{1}{2}$ holds as the likelihood ratio is weekly decreasing and not everywhere constant.

For each N, we now construct a strategy profile σ_N. Fix a setup a_N with voting rule (K, p). At least one of the two alternatives needs at least half of the votes to be implemented with positive probability. First, consider the case that this C needs at least $N/2$ votes, i.e., $K \geq N/2$ holds. Define the strategy profile σ_N by

$$\sigma_N(s, \emptyset) = \begin{cases} W, & \text{for } \; s \leq z \\ C, & \text{for } \; s > z \end{cases}$$
$$\sigma_N(s, h) = \quad A, \qquad \text{for all } h \neq \emptyset \text{ with } n_C < N/2$$
$$\sigma_N(s, h) = \quad C, \qquad \text{for all } h \neq \emptyset \text{ with } n_C \geq N/2.$$

The outcome of σ_N is C if and only if at least $N/2$ voters receive a signal $s \in (z, 1]$.

Now, consider the second case that A needs at least $N/2$ votes to be implemented with positive probability. Analogously, we construct σ_N by

$$\sigma_N(s, \emptyset) = \begin{cases} A, & \text{for } \; s \leq z \\ W, & \text{for } \; s > z \end{cases}$$
$$\sigma_N(s, h) = \quad A, \qquad \text{for all } h \neq \emptyset \text{ with } n_A \geq N/2$$
$$\sigma_N(s, h) = \quad C, \qquad \text{for all } h \neq \emptyset \text{ with } n_A < N/2.$$

Again, the outcome of σ_N is C if and only if at least $N/2$ voters receive a signal $s \in (z, 1]$. For both cases and for both states, the probability of a wrong decision is bounded above by the probability that a binomially distributed random variable $X_{(N,r)}$ with parameters N and r takes a value less or equal to $N/2$ (i.e., at least half of the voters receive the wrong signal).

By the weak law of large numbers, the realized vote share of C-voters in period one converges to the expected vote share $1 - F(z|\omega_N)$ in probability, which implies that the correct outcome is implemented with probability approaching 1. Thus, we have constructed a sequence $(a_N)_{N \in \mathbb{N}}$ of strategy profiles such that, regardless of the sequence of voting rules along the setups, the probability of an incorrect choice converges to 0 as N converges to infinity. □

Proof of Lemma 1.4. Consider our construction for the proof of Theorem 1.2. The probability of a wrong decision is bounded above by the probability that a binomially distributed random variable $X_{(N,r)}$ with parameters N and r takes a value less or equal $N/2$. By Chebyshev's inequality this probability is at most

$$\begin{aligned}
&P\left(X_{(N,r)} \leq \frac{N}{2}\right) \\
&\leq P\left(|X_{(N,r)} - rN| \geq N\Big(r - \frac{1}{2}\Big)\right) \\
&\leq \frac{r(1-r)N}{N^2\big(r-\frac{1}{2}\big)^2} \\
&= \frac{r(1-r)}{\big(r-\frac{1}{2}\big)^2} \cdot \frac{1}{N} \\
&= \mathcal{O}(N^{-1}).
\end{aligned}$$

Thus, we have constructed a bounding sequence that converges to zero at rate N^{-1}. Therefore, for the probability of a wrong decision under the strategy profiles (σ_N), the rate of convergence is at least N^{-1}. As the probability of a wrong decision is even smaller in a welfare-optimal equilibrium, this constitutes a bound for the rate of convergence of the probability of a correct decision for the sequence of welfare-optimal monotone equilibria. □

1.A.4 Proofs for Section 1.6

Proof of Theorem 1.3. Consider the welfare-optimal equilibrium of the simultaneous voting game. Feddersen and Pesendorfer (1996) show that this equilibrium follows a cutoff rule. The probability of a single voter abstaining is non-zero. Hence, for a fixed N, there is a positive probability that all voters abstain. For the sequential voting game, construct the strategy profile σ as follows. In the first period, the strategy is given by the strategy profile of the simultaneous voting game, except that agents wait instead of abstaining. In period two, all agents vote for the outcome that gained a simple majority in period one. If the result of the first period results is a tie, all agents who waited then vote for each alternative with equal probability. This strategy profile is outcome-equivalent to the welfare-optimal equilibrium of the simultaneous voting game.

Now, change the voting strategies such that at the history $h = (0, 0)$ where every agent waited, the welfare-optimal cutoff strategy of the induced game G_h is played. This event occurs with positive probability, and the welfare-optimal equilibrium of the induced game in period two yields a strictly higher welfare than a coin flip. Since this strictly increases the probability of the correct decision, there exists a strategy profile of the two-period model with strictly higher welfare than all equilibria of the simultaneous voting model. By McLennan (1998), there also exists an equilibrium with strictly higher welfare. □

1.A.5 Proofs for Section 1.7

Proof of Theorem 1.4. Consider a welfare-optimal equilibrium of the two-period voting game with a fixed voting sequence. The outcome of this equilibrium can be replicated by an asymmetric strategy profile σ of the two-period voting game with endogenous timing. If all voters vote in the same voting period, then the same argument as in Corollary 1.1 implies the existence of a profitable deviation. Therefore, we consider the situation that there is at least one voter in each period.

Fix a single voter i who votes in period one. Let $\varepsilon > 0$ be sufficiently small and define a deviation for voter i as follows: For a signal s with $\sigma(s) = A$ and $F(s|I),\ F(s,G) \leq \varepsilon$, voter i waits in period one and votes for A in period two instead. If the other voters in period two observe a history h where this event occurred, they play the welfare-optimal equilibrium of the induced simultaneous voting game. By the assumptions (MLRP$_<$) and (ULR), the induced prior at h is different compared to the induced prior where i votes early. Thus, the equilibrium of the induced simultaneous voting game yields a strict welfare gain.

Note that the existence proof for Theorem 1.1 for a welfare-optimal symmetric equilibrium also shows the existence of a welfare-optimal asymmetric equilibrium as the number of voters is finite, and the space of all asymmetric strategy profiles is therefore also compact. $\square$

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Chapter 2

On-the-Match Search and Match-Specific Productivity Growth

Joint with Sophie Kreutzkamp and Axel Niemeyer

2.1 Introduction

We study decentralized match formation as in Shimer and Smith (2000) with two new features: Matches become more productive the longer they last and matched partners continue to meet others.[1] The combination of these two features is new and captures many scenarios: For example, employees look for better jobs, and firms try to replace underperforming employees. Architects, lawyers, and physicians may part ways with their business partners upon getting to know someone more capable. Professional athletes in badminton, skibob, or tennis may opt for more talented partners. Marital relationships end in divorce and remarriage. In many of these situations, it takes time for partners to become attuned to one another so that partnerships become increasingly more productive the longer they persist. Our primary goal is to investigate how such match-specific productivity growth affects the agents' matching decisions and which matching patterns emerge as a result.

There are two types of agents in our model, agents with a high productivity and agents with a low productivity, and a continuum of agents for each type. Time is continuous, and meetings between agents are Poisson events. If two agents meet and agree to match, they form a pair. Paired agents produce a flow of output that they divide in a fixed proportion; that is, we assume utility to be non-transferable reflecting sticky wages or non-monetary partnerships. This output depends on both

[1] See also Smith (2006) and Burdett and Coles (1997) for decentralized search models.

agents' productivity types and the novel feature is that the output is an increasing function of match tenure.[2] We study steady-state equilibria, that is, we require each agent's matching decisions to maximize her expected lifetime utility, and we demand that the distribution of both matched and unmatched agents is in steady-state.

A straightforward implication of productivity growth is a lock-in effect: Agents become less likely to dissolve partnerships the longer they last. In particular, asymmetric pairs, consisting of one more productive and one less productive agent, may gladly stay together if neither agent finds a more suitable partner in due time. On one hand such lock-in effects stabilize asymmetric pairs when productivity growth is introduced to the model. On the other hand, our analysis uncovers subtle other effects that oppose the lock-in effects.

To see why, observe that agents must make the following fundamental trade-off when deciding with whom to (re-)match: They weigh the instantaneous utility enjoyed from partners with higher productivity against the stability of a match, that is, the rate at which they expect to be abandoned by their partner. With productivity growth, becoming single not only hurts because it entails a period without production but also because one could have instead accumulated productivity in a more stable match. In other words, productivity growth is an additional incentive for agents to seek stability. This effect can stretch to the point where less productive agents leave more productive partners to find stability with other less productive agents, although their match with productive agents would become stable in the long run.

A fundamental problem in models with on-the-match search is the possible multiplicity of equilibria. Which relationships are stable crucially depends on which relationships agents believe to be stable. For example, even in a match of two highly productive agents, if one agent believes that her partner plans to abandon her, she may have an incentive to do so first. In other words, instability can be a self-fulfilling prophecy. We show that there is a continuum of steady-state equilibria when productivity does not grow, regardless of the model parameters. Moreover, for any direction of match-to-match transitions, say, productive agents seeking less productive partners, there exists a steady-state equilibrium featuring these transitions.

Perhaps surprisingly, we show that miscoordination among highly productive agents ceases to be an equilibrium when productivity does increase in match duration, regardless of how minuscule this growth is. A rough intuition is as follows: A productive agent knows that there is a point in time far into the future where enough productivity has accumulated such that a productive partner will never abandon her

[2] In many scenarios, one could also imagine that productivity is declining or has even more complicated paths. We do not anaylze such settings but note that they would be interesting as well.

from then on, regardless of her own strategy. Thus, it is optimal for her not to abandon her partner even slightly before this point in time. Her partner will, in turn, stick to the match even earlier. This unraveling halts only in the present. In other words, when productivity grows, there is no reason for a pair of highly productive agents to believe that their match is unstable.

Based on this observation, we can show that the steady-state equilibria under productivity growth take a simple form: Equilibrium strategies are cutoff strategies. That is, the agents only accept to rematch with other agents if their current match has not lasted for too long. In particular, there are essentially only two types of equilibria: Agents with low productivity either prefer highly productive partners or other agents with low productivity. In particular, this shows that sorting occurs in equilibrium. Further, this result provides a rationale for selecting equilibria in a model without productivity growth based on whether they are limit equilibria in our model with productivity growth as the growth rate vanishes.

This chapter proceeds as follows. The remainder of this section discusses the related literature. Section 2.2 presents our model with on-the-match search and match-specific productivity growth, and Section 2.3 develops steady-state equilibria. Section 2.4 analyzes partial equilibrium behavior, and Section 2.5 investigates the existence and uniqueness of steady-state equilibria. Section 2.6 discusses a limit model with a productivity growth of zero. Section 2.7 discusses the effects of productivity growth. Section 2.8 concludes. The proofs of all results are found in Appendix 2.A.

2.1.1 Related Literature

On-the-job search is pervasive in labor markets (Fallick and Fleischman, 2004), and search-theoric models have long taken this fact into account (see the survey by Rogerson, Shimer, and Wright, 2005). However, on-the-job search and on-the-match search, as considered in this chapter, are two separate notions. In a typical model with one-sided on-the-job search, firms create jobs through vacancy posting, and once a vacancy is filled, a firm cannot fire or replace its worker while the worker can search for better jobs. On-the-match search, by contrast, postulates that all agents – not only the agents on one side of the market – continue to search while being in a match. It thus captures replacement hiring in labor markets as well as divorce and remarriage in the marriage market (for empirical evidence, see Burgess, Lane, and Stevens (2000) and Stevenson and Wolfers (2007), respectively).

Recall that the agents' main trade-off with two-sided on-the-match search is between productivity and stability. However, with one-sided on-the-job search, stability is not a concern because the market side that searches in the first place is not abandoned. For this reason, search-and-matching models with heterogeneous workers, heterogeneous firms, and on-the-job search (see Dolado, Jansen, and Jimeno

(2009), Gautier, Teulings, and Vuuren (2010), and Lentz (2010)), are quite different from ours.

In recent work, Bartolucci and Monzon (2019) and Goldmanis and Ray (2019) argue that on-the-match search can lead to positive assortative matching even without complementarities in the agents' productivities. Both papers restrict attention to a specific steady-state equilibrium in which agents replace their partners if and only if they meet someone more productive. This equilibrium is only one of many and may not generally exist. Thus, our scope is quite different from theirs as we explicitly tackle equilibrium multiplicity through exploring match-specific productivity growth.

Cornelius (2003) also studies equilibrium multiplicity in a model that is similar to our limit model without productivity growth. In contrast to us, she places some restrictions on agents' strategies, which rule out some equilibria in the first place. We do not make these restrictions and instead show that productivity growth eliminates the additional equilibria. Burdett, Imai, and Wright (2004) identify a related source of equilibrium multiplicity that arises when search intensity is endogenous: If an agent's partner puts more effort into search, the agent may be inclined to do the same.

Productivity growth has been first formally considered in Pissarides (1994) to account for the empirical fact that job-to-job transitions decrease with tenure in the labor market (see Hall, 1982). This feature has later been reconsidered in Schwartz (2020). The scope of both papers is different from our model: Pissarides (1994) studies on-the-job search with homogenous workers. While Schwartz (2020) does, like us, include binary types of workers, he only solves his model numerically and is primarily concerned with optimal unemployment insurance. In particular, the idea to use productivity growth as a device for equilibrium selection in models with on-the-match search is novel to our approach.

One could endogenize productivity growth, say, by modeling firms' training decisions. This endogenization would yield a model that we believe to be untractable up to numeric solutions. For such approaches, see Lentz and Roys (2015) and Flinn, Gemici, and Laufer (2017).

2.2 The Model

Agents Time is continuous. At every instant, there is a continuum of agents in the market. Each agent has a permanent productivity type that is either L(ow) or H(igh). For brevity, we call an agent of type $i \in \{L, H\}$ an *i-agent*. Each agent is either single or matched with another agent. We refer to a match in which both agents have the same productivity type as *symmetric*, and we refer to a match in which their productivity types differ as *asymmetric*.

We assume that new single i-agents enter the market at a constant rate η_i, that is, in an interval of time dt a mass $\eta_i dt$ of new single i-agents enters the market. Moreover, each agent exits the market with Poisson rate δ so that the lifetime of agents is random and exponentially distributed. If a matched agent exits the market, her partner becomes single.

Search Search is undirected and time-consuming but otherwise costless. Meetings between agents follow a quadratic search technology, that is, each agent (matched or not) meets other agents uniformly at random with Poisson rate λm, where m is the total mass of agents in the market and λ is a parameter that captures the underlying degree of search frictions.[3] When two agents meet, they observe each others' types and then decide whether or not to accept forming a match. If they both accept to form a match, their current partnerships (if they are matched) are dissolved, and their former partners become single.

Payoffs Without loss of generality, we normalize the flow utility of a single agent to 0. When an i-agent is matched with a j-agent she obtains a flow utility $u_{ij}(t) > 0$, where t is the time that the current match persists. Therefore, the total utility that an i-agent receives in a match with a j-agent that is dissolved after persisting for time s is

$$\int_0^s u_{ij}(t)\ dt.$$

We assume that utility is non-transferable across agents. To ascribe meaning to productivity types, we assume that H-partners provide a higher flow payoff than L-partners, i.e., $u_{iL}(t) < u_{iH}(t)$ holds for all $i \in \{L, H\}$ and $t \geq 0$.

The exit rate δ implicitly acts as a discount rate on future flow utility as agents no longer gain utility when they exit the market. For simplicity, we assume no further discounting beyond the exit rate. Throughout, we assume that the maximal expected continuation payoff of a match is finite, that is,

$$\forall_{i,j\in\{L,H\}} \quad \int_0^\infty u_{ij}(t)e^{-2\delta t}\ dt < \infty.$$

To study match-specific productivity growth, we assume that $u_{ij}(\cdot)$ is strictly increasing for all $i, j \in \{L, H\}$. We also want productivity growth to have bite by eventually locking agents into their matches. To guarantee eventual lock-in, assume

[3] In particular, the search velocity is independent of whether an agent is single or matched. This assumption simplifies our analysis because it will render accepting any match a dominant strategy for single agents. Relaxing this assumption does not change the gist of our results.

that flow utilities grow arbitrarily large, that is,

$$\lim_{t\to\infty} u_{ij}(t) = \infty$$

holds for all $i, j \in \{L, H\}$. We believe this assumption to be mild as it makes no statement about when agents lock into their matches, or equivalently, how fast productivity grows. We only assume that agents lock into their matches *eventually*. For example, the point of lock-in may be orders of magnitude higher than the agents' average lifetime of $1/\delta$.

2.3 The Concept of Steady-State Equilibrium

In this section, we define the concept of a steady-state equilibrium. A steady-state equilibrium requires that the distribution of masses is time-invariant given the agents' acceptance strategies, and that the agents' acceptance strategies form an equilibrium given the steady-state masses.

A *stationary acceptance strategy* for an i-agent is a measurable function

$$p_i : \{\emptyset, L, H\} \times \{L, H\} \times [0, \infty) \to [0, 1],$$

where $p_i(j, k, t)$ is the probability that an i-agent matched with a j-agent for time t accepts to rematch with a k-agent upon meeting. For notational convenience, we use $\emptyset$ as a placeholder for the non-existing partner of a single agent. It will be a result in the analysis that $p_i(\emptyset, \cdot, \cdot) = 1$ for all $i \in \{L, H\}$, that is, all single agents will accept to match with anyone they meet. Throughout, we restrict attention to type-symmetric equilibria in which agents of the same type use the same strategy.

Balance Conditions Suppose that the agents follow stationary acceptance strategies (p_L, p_H) and fix an instant of time. Let m_{ij} denote the current mass of i-agents that are matched with j-agents, and let $m_{i\emptyset}$ denote the mass of single i-agents. Furthermore, let $m_{ij}(t)$ be the mass density of i-agents who have been matched with a j-agent for exactly time t so that $m_{ij} = \int_0^\infty m_{ij}(t)dt$ holds.[4] We shall now develop the balance conditions that ensure masses to be in steady-state given the strategies (p_L, p_H). Let

$$m_{i,j,k} = \int_0^\infty m_{ik}(t) p_i(k, j, t) dt$$

[4]Note that $m_{LH} = m_{HL}$ and $m_{LH}(t) = m_{HL}(t)$ hold for all t as all asymmetric pairs consist of one H-agent and one L-agent.

be the mass of i-agents who agree to match with a j-agent if they are currently matched with a k-agent, and let

$$M_{i,j} = \sum_{k \in \{L,H,\emptyset\}} m_{i,j,k}$$

denote the mass of i-agents who accept to match with a j-agent. The quadratic search technology then implies an inflow

$$I_{ij} = \lambda M_{i,j} M_{j,i},$$

of i-agents into matches with j-agents. For the inflow of single i-agents, we have

$$I_{i\emptyset} = \eta_i + \delta \sum_{j \in \{L,H\}} m_{ij} + \lambda \sum_{j \in \{L,H\}} \int_0^\infty m_{ij}(t) \sum_{k \in \{L,H\}} p_j(i,k,t) M_{k,j} dt.$$

The right-hand side comprises the exogenous inflow η_i, the endogenous inflow of i-agents whose partner exits the market (second term), and the endogenous inflow of i-agents who are left by their partner for another agent (third term).

Let $q_{ij}(t)$ denote the probability that a match between an i-agent and a j-agent lasts for at least t units of time, given the stationary strategies (p_L, p_H) and given the masses

$$\mathcal{M} = \Big(\big(m_{L\emptyset}(t)\big)_{t \geq 0}, \big(m_{H\emptyset}(t)\big)_{t \geq 0}, \big(m_{LL}(t)\big)_{t \geq 0}, \big(m_{LH}(t)\big)_{t \geq 0}, \big(m_{HH}(t)\big)_{t \geq 0} \Big).$$

In other words, q_{ij} is the survival function with respect to match duration. For notational convenience, we again write $q_{i\emptyset}(t)$ for the probability that a single i-agent stays single for at least t units of time. We formally derive these survival functions in Section 2.A.1 in the appendix.

We are now equipped to state the *pointwise balance conditions*. We say that a tuple $(\mathcal{M}, p_L, p_H)$ of fixed masses and strategies satisfies the pointwise balance conditions, if

$$m_{ij}(t) = I_{ij} q_{ij}(t) \tag{2.1}$$

holds for all $i \in \{L, H\}$, $j \in \{L, H, \emptyset\}$, and $t \geq 0$. It says that the mass density of matches with duration t is equal to the induced inflow I_{ij} times the induced survival probability $q_{ij}(t)$. The pointwise balance conditions ensure that the environment is indeed stationary.

Definition 2.1. The tuple $(\mathcal{M}, p_L, p_H)$ is a *steady-state* if it satisfies the pointwise balance conditions (equation (2.1)).

It is instructive to integrate both sides of equation (2.1) with respect to time to obtain aggregate balance conditions of the form

$$\forall i \in \{L, H\} \; \forall j \in \{L, H, \emptyset\} \; : \; m_{ij} = I_{ij} \int_0^\infty q_{ij}(t) dt. \tag{2.2}$$

The aggregate balance conditions state that the mass of i-agents matched with j-agents must equal the total inflow of such agents multiplied by the expected match duration[5] (and analogously for single agents).

Value Functions Previously, we have defined what it means for the masses to be in steady-state, given a profile of stationary strategies (p_L, p_H). Now, we shall define what it means for a profile of stationary strategies (p_L, p_H) to be a best response, given a vector of masses $\mathcal{M}$.

Let $i \in \{L, H\}$ and $j \in \{\emptyset, L, H\}$. Then, for given $(\mathcal{M}, p_L, p_H)$, let $V_{ij}(t_0)$ denote the expected lifetime utility of an i-agent who is matched with a j-agent for time t_0 (or for $j = \emptyset$ the lifetime utility of an agent who is single for time t_0). The expected lifetime utility of matched agents under the assumption that the masses do not change over time is recursively given by

$$\begin{aligned} V_{ij}(t_0) = \int_{t_0}^\infty \frac{q_{ij}(t)}{q_{ij}(t_0)} u_{ij}(t) dt + \int_{t_0}^\infty \frac{q_{ij}(t)}{q_{ij}(t_0)} \sum_{k \in \{L,H\}} \left(\lambda p_i(j,k,t) M_{k,i} V_{ik}(0)\right) \, dt \\ + \int_{t_0}^\infty \frac{q_{ij}(t)}{q_{ij}(t_0)} \Big(\delta + \sum_{k \in \{L,H\}} \left(\lambda p_j(i,k,t) M_{k,j}\right)\Big) V_{i\emptyset}(0) \, dt. \end{aligned} \tag{2.3}$$

All three integrals are obtained by integration by parts. The first integral is the conditional expected future flow payoff inside the current match given that the match already lasted for a time t_0. The second term is the expected continuation utility if i rematches with another agent and the third term is the expected continuation utility if her partner dissolves the match by rematching herself or exiting the market.

For singles, the expected lifetime utility is given by

$$V_{i\emptyset}(t_0) = \int_{t_0}^\infty \frac{q_{i\emptyset}(t)}{q_{i\emptyset}(t_0)} \sum_{k \in \{L,H\}} \left(\lambda p_i(\emptyset, k, t) M_{k,i} V_{ik}(0)\right) \, dt. \tag{2.4}$$

Since singles receive no flow utility and cannot be left by a partner, their expected lifetime utility only contains the expected continuation utility upon rematching.

Masses and strategies constitute a partial equilibrium if the agents' acceptance decisions maximize their expected lifetime utility, given others' strategies (p_L, p_H) and given masses $\mathcal{M}$ that are constant over time.

[5]Since $q_{ij}(\cdot)$ is equal to the cdf function F of the time that a match is together, the integral over the match survival function with respect to time is identical to the expected match duration.

Definition 2.2. The tuple $(\mathcal{M}, p_L, p_H)$ is a *partial equilibrium* if

$$V_{ij}(t) > V_{ik}(0) \Rightarrow p_i(j,k,t) = 0$$
$$V_{ij}(t) < V_{ik}(0) \Rightarrow p_i(j,k,t) = 1$$

hold for all $i, k \in \{L, H\}$, $j \in \{L, H, \emptyset\}$ and $t \geq 0$.

Note that this particular definition demands that the agents' acceptance decisions are optimal conditional on being accepted. It thus rules out partial equilibria in weakly dominated strategies where i-agents never accept to match with j-agents because j-agents never accept to match with i-agents.

In a partial equilibrium, the strategies are optimal for given time-invariant masses while in a steady-state the masses are time-invariant given the strategies. Together, steady-state and partial equilibrium constitue our equilibrium concept of a steady-state equilibrium.

Definition 2.3. The tuple $(\mathcal{M}, p_L, p_H)$ is a *steady-state equilibrium* if it is a steady-state and a partial equilibrium.

2.4 Partial Equilibrium Analysis

Our first result characterizes partial equilibrium strategies. Recall that $p_i(j,k,t)$ is the probability that an i-agent who is in a match with a j-agent for time t accepts to form a new match with a k-agent upon meeting.

Lemma 2.1. *Suppose that (p_L, p_H) are strategies that constitute a partial equilibrium. Then, the following statements hold:*

(a) Single agents always accept to match, i.e., $p_i(\emptyset, \cdot, \cdot) = 1$ holds for all $i \in \{L, H\}$.

(b) Agents do not accept to match with a new partner who has the same type as the current partner, i.e., $p_i(j,j,t) = 0$ holds for all $i, j \in \{L, H\}$ and $t > 0$.

(c) Agents in an HH-match never accept to match with a new partner, i.e., $p_H(H, \cdot, \cdot) = 0$ holds.

(d) The agents follow cutoff strategies, i.e., for every $(i,j,k) \in \{L, H\}^3$, there exists $t_{ijk} \in [0, \infty)$ such that

$$p_i(j,k,t) = \begin{cases} 1, & \text{if } \; t < t_{ijk}; \\ 0, & \text{if } \; t > t_{ijk} \end{cases}$$

holds.

The first two statements come as no surprise. Statement (a) says that single agents accept any match, since continuing to search on-the-match has the same velocity λ. Statement (b) says that agents do not leave a j-agent for another j-agent as they accumulate productivity in a match and a match that already existed for time t is more desirable than a newly formed matched with the same type of partner.

When a matched agent meets someone, her acceptance decision hinges on the rematching strategies of her current and potential future partner. In particular, distrust in one's partner's loyalty can become a self-fulfilling prophecy: It would be conceivable that there might be equilibria where partners do not leave one another up to a point in time where they suddenly start accepting rematches because they rationally expect their partner to do the same. However, it turns out that such behavior can never be equilibrium behavior. In statement (c), we establish that no H-agent dissolves a symmetric match unless she exits the market.

We are then left with three cases where the agents' equilibrium matching decisions are still undetermined: First, the rematching decision of H-agents from asymmetric to symmetric pairs. Second and third, the rematching decisions of L-agents from symmetric to asymmetric pairs, and vice versa. In statement (d), we establish that the agents employ cutoff strategies with respect to match duration. That is, the agents accept to rematch until reaching a particular point in time after which they remain loyal: Applied to the three remaining cases, we obtain that H-agents never leave asymmetric pairs after t_{HLH} and L-agents never leave symmetric (asymmetric) pairs after t_{LLH} (t_{LHL}). We omit k in t_{ijk} and denote the three relevant cutoffs by t_{HL}, t_{LL}, and t_{LH}, respectively.

The intuition why agents never rematch their partners after some point in time is as follows: Since the flow utility grows arbitrary large, there exists a (potentially very large) point in time, after which both agents in a match strictly prefer to stay in the current match - regardless of the partner's actions. Anticipating this lock-in, both partners will refrain from rematching even slightly earlier as the probability of one's partner meeting another agent vanishes in short timeframes while the productivity advantage over a new match does not. Further backward induction eventually yields a cutoff where at least one of the partners is indifferent between rematching and locking into the match. For a pair of two H-agents, this backward iteration halts only at the time of matching.

We now show that in asymmetric pairs, the L-agent reaches her cutoff point first.

Corollary 2.1. *In every partial equilibrium,* $t_{LH} \leq t_{HL}$.

Intuitively, t_{LH} cannot be larger than t_{HL} because once an H-agent stays in an asymmetric match forever, the L-agent in this match has no incentive to rematch because she can never be better of than she is in a stable asymmetric relationship.

Cutoff strategies immediately imply that valuation functions of matched agents are strictly increasing with respect to match duration.

Corollary 2.2. *In every partial equilibrium, $V_{ij}(t)$ is strictly increasing in t for all $i, j \in \{L, H\}$.*

Intuitively, longer match durations are desirable because the agents accumulate match-specific productivity and because cutoff strategies render the probability of being left by one's partner weakly decreasing over time.

An H-agent arrives at her cutoff t_{HL} when being indifferent between a newly formed stable symmetric match and a stable asymmetric match that has accumulated enough productivity.

Corollary 2.3. *In every partial equilibrium, $t_{HL} \in (0, \infty)$ is the unique solution to the equation*

$$V_{HL}(t_{HL}) = V_{HH}(0)$$

or, equivalently,

$$\int_{t_{HL}}^{\infty} u_{HL}(t) e^{-2\delta(t - t_{HL})} dt = \int_{0}^{\infty} u_{HH}(t) e^{-2\delta t} dt. \tag{2.5}$$

This solution is the same across all partial equilibria.

The fact that a unique t_{HL} solves equation (2.5) immediately follows from Corollary 2.2. The cutoff t_{HL} only depends on the primitives of the model because no L-agent leaves an LH-pair after t_{HL} and therefore, H-agents trade-off two matches in which they are not left by the partner against one another. Hence, t_{HL} is the same across all partial equilibria and, a fortiori, across all steady-state equilibria.

L-agents in asymmetric matches anticipate being potentially left anytime before t_{HL}. However, they also anticipate their match to be stable anytime after t_{HL}. With this thought in mind, the cutoff t_{LH} is such that L-agents they are indifferent between rematching with another L-agent or staying in an asymmetric match that will not be stable until t_{HL}. The cutoff t_{LL} after which L-agents stay in symmetric matches is also determined by backwards induction. It turns out that L-agents either rematch into symmetric pairs, into asymmetric pairs, or not at all.

Corollary 2.4. *In every partial equilibrium, either*

$$t_{LH} > 0 \;\wedge\; t_{LL} = 0 \;\wedge\; V_{LH}(t_{LH}) = V_{LL}(0) \text{ or} \tag{2.6}$$

$$t_{LH} = 0 \;\wedge\; t_{LL} = 0 \;\wedge\; V_{LH}(0) = V_{LL}(0) \text{ or} \tag{2.7}$$

$$t_{LH} = 0 \;\wedge\; t_{LL} > 0 \;\wedge\; V_{LH}(0) = V_{LL}(t_{LL}) \tag{2.8}$$

holds.

The reason for this case distinction is simple: If an L-agent actively rematches out of a match, she will not seek to rematch back into the same kind of match since the continuation value is increasing in time if all agents play cutoff strategies. The case where L-agents stick to any match is non-generic as it requires them to be exactly indifferent between symmetric and asymmetric matches and we prove that such an equilibrium would disappear under a small perturbation of parameters.

Which type of partner an L-agent prefers in equilibrium hinges on her trading-off higher flow utility with H-agents against conceivably higher stability in symmetric partnerships. In other words, while matching with H-agents is productive in the short run, the likelihood of being left by an H-agent may be higher than that of being left by an L agent in the long run. The higher this likelihood, the more likely an agent becomes single and consequently unproductive, and the more likely an agent is to lose accumulated productivity that one could have instead nurtured in a safer match.

With fixed masses $\mathcal{M}$, by Corollary 2.3, the likelihood of being left by an H-agent is the same across all partial equilibria, but the likelihood of being left by an L-agent can differ between equilibria: If all other L-agents seek symmetric matches, then symmetric matches are inherently stable for a given L-agent, which may lead her to not rematch with H-agents as well. If all other L-agents seek asymmetric matches, then symmetric matches are inherently unstable for a given L-agent, which may too lead her to seek matches with H-agents.

These two kinds of partial equilibria may both exist for the same set of parameters as each can be self-fulfilling. Using the case distinction from Corollary 2.3, we finally close this section by showing existence of a partial equilibrium.

Lemma 2.2. *For any masses $\mathcal{M}$, there exists a partial equilibrium $(\mathcal{M}, p_L, p_H)$. In each of the three classes identified in Corollary 2.4 there exists at most one partial equilibrium $(\mathcal{M}, p_L, p_H)$.*

To summarize this section, the content of Lemma 2.1 says that the triple of cutoffs (t_{HL}, t_{LH}, t_{LL}) characterizes partial equilibrium strategies.[6] Corollaries 2.1, 2.3, and 2.4 establish that partial equilibria can be classified solely based on the rematching decisions of L-agents. Finally, Lemma 2.2 verifies the existence of partial equilibria for given masses $\mathcal{M}$ and equilibrium uniqueness for each of the three respective cases.

We now take these insights to the steady-state analysis where we account for the balance conditions that ensure partial equilibrium behavior to preserve the masses $\mathcal{M}$ which we so far only assumed to be fixed. Note already that the steady-state analysis will complicate the basic intuition regarding the agents' trade-offs established in this section: Strategies no longer only affect the stability of matches but

[6] The equilibrium behavior at $t = 0$ and at the cutoffs is not determined. However, matching decisions on a finite number of points in time do not affect equilibrium masses and utilities. Therefore, it suffices to characterize equilibrium strategies up to a nullset.

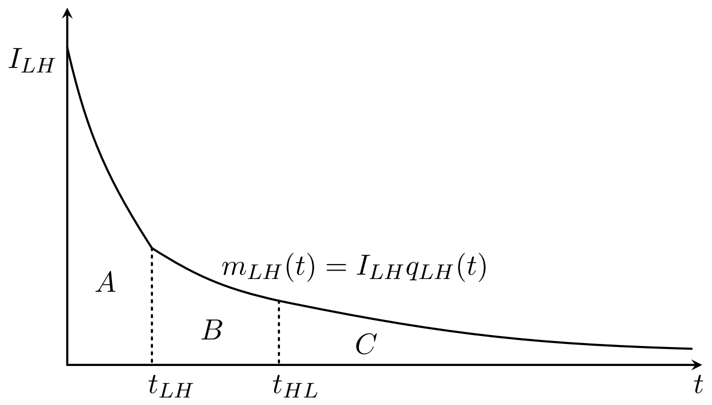

Figure 2.1: The steady-state mass density $m_{LH}(t)$ of asymmetric pairs decreases exponentially over time; the rate changes at the cutoffs t_{LH} and t_{HL}.

they also determine who (or rather, how many agents) will be available for matching. Consequently, strategies endogenize the cost of being left by one's partner, or, conversely, the value of stability.

2.5 Steady-State Analysis

Lemma 2.1 yields a concise characterization of partial equilibrium behavior, allowing us to simplify the balance conditions (equation (2.1)) considerably. For example, asymmetric pairs can only be in one of three different states: First, if the two agents are matched for less than t_{LH}, then both accept to rematch with other agents, and therefore, the match dissolves at rate $2\delta + \lambda(M_{L,L} + M_{H,H})$. Second, if the agents have been matched for longer than t_{LH} but for shorter than t_{HL}, then only H-agents agree to rematch with other agents, and, consequently, the match dissolves at rate $2\delta + \lambda M_{H,H}$. Third, for durations longer than t_{HL}, the match is stable and henceforth only dissolves at rate 2δ.

Hence, the survival function q_{LH} decreases exponentially on each of the three intervals, albeit with different rates. This observation is illustrated in Figure 2.1. Let A, B, and C denote the total masses of asymmetric pairs in the respective states. Similarly, LL-pairs can only be in one of two states, depending on whether the match duration exceeds t_{LL}, yielding two aggregate masses, D and E, instead of three.

The balancing of the in- and outflow regarding these aggregate masses is enough to ensure that the entire mass density $m_{ij}(t)$ is time-invariant. More formally, for every partial equilibrium $(\mathcal{M}, p_L, p_H)$ that satisfies the state-wise aggregate balance conditions, there exist unique masses $\mathcal{M}'$ such that the state-wise aggregate masses are the same in $\mathcal{M}$ and $\mathcal{M}'$, and such that $(\mathcal{M}', p_L, p_H)$ is a steady-state equilibrium.

Consider $A = \int_0^{t_{LH}} m_{LH}(t)\ dt$ as a concrete example. We obtain[7] the aggregate balance condition for A by integrating both sides of the point-wise balance conditions for $m_{ij}(t)$ from 0 to t_{LH}. The resulting aggregate balance condition is

$$I_{LH} = (2\delta + \lambda(M_{L,L} + M_{H,H}))\, A + I_{LH} q_{LH}(t_{LH}). \tag{2.9}$$

The left-hand side of equation (2.9) is the inflow I_{LH} of agents into asymmetric pairs which is, by definition, the inflow into A. The right-hand side of equation (2.9) is the outflow from A, which comprises the outflow by match dissolution (first term), and the outflow into B from matches that last for longer than t_{LH} (second term).

The simplified balance conditions allow us to prove the following result.

Theorem 2.1. *There exists a steady-state equilibrium. Moreover, for every pair* $(m_{L\emptyset}, m_{H\emptyset})$ *of single masses, there exists a unique pair of inflows* (η_L, η_H) *and a unique tuple* $(\mathcal{M}, p_L, p_H)$ *consistent with these single masses such that* $(\mathcal{M}, p_L, p_H)$ *is a steady-state equilibrium.*

Theorem 2.1 establishes the existence of a steady-state equilibrium. It will be an immediate corollary of Theorems 2.4 and 2.5 that uniqueness cannot generally be guaranteed in our model. However, Theorem 2.1 provides an "inverse" uniqueness result that has its own appeal: Observing the single masses in our model is sufficient to *uniquely* back out the mass densities of matched agents, the agents equilibrium behavior, and even the inflows that must have led to these single masses.

In the proof, we distinguish cases based on partial equilibrium behavior as in Corollary 2.4. Consider, for instance, the case when L-agents prefer to match with L-agents. Here, we fix $t_{LL} = 0$ and can therefore completely describe strategies (p_L, p_H) via the cutoff t_{LH}.

Consider the composite mapping

$$T : t_{LH} \mapsto \mathcal{M} \mapsto t_{LH}$$

where the first mapping returns steady-state masses $\mathcal{M}$ for a given cutoff t_{LH}, and the second mapping returns a partial equilibrium cutoff t_{LH} for given masses $\mathcal{M}$. We first establish that this mapping is well-defined by showing that, given the cutoff t_{LH}, the state-wise balance conditions have a unique solution $\mathcal{M}$.[8] We then show that T is continuous to finally conclude the existence of a fixed point by appealing to Brouwer's fixed point theorem. A fixed point corresponds to a steady-state equilibrium if the cutoff t_{LH} is non-zero. We show that a fixed point corresponds to a steady-state equilibrium in at least one of our three cases from Corollary 2.4, establishing existence.

[7]For the detailed aggregate balance conditions and their derivation, see Section 2.A.3 in the appendix.

[8]Recall that Corollaries 2.2 and 2.4 already establish that t_{LH} is unique given $\mathcal{M}$.

For the uniqueness part, we fix the single masses $(m_{L\emptyset}, m_{H\emptyset})$, allowing us to ignore their balance conditions and to consequently ignore the inflows (η_L, η_H). Given this, we show that T is strictly decreasing, establishing uniqueness for fixed $(m_{L\emptyset}, m_{H\emptyset})$: The state-wise balance conditions imply that a higher cutoff t_{LH} leads to a larger mass A and a smaller mass $A+B$. However, the partial equilibrium conditions imply that a larger mass A and a smaller mass $A+B$ lead to a smaller cutoff t_{LH}. Finally, we show that there exist unique inflows (η_L, η_H) that solve the balance conditions of the singles masses $(m_{L\emptyset}, m_{H\emptyset})$.

As a sanity check, let us briefly mention that we can uniquely predict equilibrium behavior if the search technology is either very efficient or very inefficient.

Lemma 2.3. *Fix all parameters except for λ. Then, there exists $\bar{\lambda} > 0$ such that for all $0 < \lambda < \bar{\lambda}$, in all steady-state equilibria with search parameter λ we have $t^*_{LL} > 0$. Conversely, as $\lambda \to \infty$, we get that $m_{HH} \to \frac{\eta_H}{\delta}$, $m_{LL} \to \frac{\eta_L}{\delta}$, and $m_{L\emptyset}, m_{H\emptyset}, m_{LH} \to 0$ hold.*

Intuitively, at low meeting rates, all matches are stable enough for agents to make their rematching decisions based on flow utility alone. In particular, L-types do not leave their H-type partners because the fear that their partner would meet someone better is low. When agents encounter others quickly, the balance conditions imply that all agents are matched symmetrically and that no agent remains single.

2.6 Limit Model

In this section, we discard the assumption of productivity growth and instead assume that productivity in a match is time-invariant, that is, $u_{ij}(t) = u_{ij}$ for all $i, j \in \{L, H\}$ and $t \geq 0$. We investigate this alternate model for two reasons. First, we use it to benchmark the effects of productivity growth, which we then discuss in Section 2.7. Second, the limit model is exemplary for models with on-the-match search, where equilibrium multiplicity is pervasive. Productivity growth turns out to eliminate some of these equilibria and can consequently serve as a criterion for equilibrium selection.

Under productivity growth, the agents' decision problems in a given match may depend on the match duration duration, even if their partner's strategies do not. Without productivity growth, this decision problem is time-invariant if the partner's strategy is time-invariant as well. We thus restrict attention to stationary acceptance strategies for i-agents that are measurable functions

$$p_i : \{L, H, \emptyset\} \times \{L, H\} \to [0, 1]$$

where $p_i(j, k)$ is the probability that an i-agent accepts to match with a k-agent when currently matched with a j-agent (or single).

Given that agents play time-invariant strategies, their value functions are also time-invariant, that is, $V_{ij}(t) = V_{ij}$ for all $i, j \in \{L, H\}$ and $t \geq 0$. Moreover, the aggregate balance conditions (equation (2.2)) are sufficient to ensure a steady-state, that is, we no longer need to worry about balancing the mass density regarding match duration but only the total masses of agents in every match and those of single agents. An exact derivation of this fact and the corresponding balance conditions can be found in Section 2.A.4 in the appendix. The definition of steady-state equilibrium is otherwise analogous to that used in the analysis with productivity growth.

Partial Equilibrium Behavior We now turn to analyzing the agents' partial equilibrium behavior. For the same reasons as under productivity growth, single agents accept to match with agents of any type (cf. Lemma 2.1, statement (a)). However, matched agents no longer need to decline matching with agents of the same type as their partner since their continuation utility is the same regardless of this decision (cf. Lemma 2.1, statement (b)).

Under productivity growth, the L-agents alone determine the nature of equilibrium through their rematching decisions as H-agents never leave symmetric matches voluntarily (cf. Lemma 2.1, statement (c)). This statement is no longer true when productivity is constant over time: One can now rationalize beliefs that lead H-agents to rematch out of symmetric pairs. It will be convenient to let $\bar{i} \in \{L, H\} \setminus \{i\}$ denote the respective opposite type of i. Partial equilibrium requires that i-agents matched with j-agents accept to rematch with agents of the opposite type $\bar{j}$ than their partner only if $V_{i\bar{j}} \geq V_{ij}$. In particular, mixing and even rematching from both j to $\bar{j}$ and from $\bar{j}$ to j can happen only if $V_{i\bar{j}} = V_{ij}$.

Recall that the respective equilibrium behavior is due to the agents weighing flow utility against stability in a match. In the limit model, it hurts less to become single as one cannot lose accumulated productivity. This simplification allows us the capture the above trade-off in an intuitively accessible formula. Let

$$r_{i,j\to\emptyset} = \delta + \sum_{k\in\{L,H\}} \lambda p_j(i,k) M_{k,j}$$

denote the rate at which an i-agent becomes single because her j-partner either exits the market or rematches. Moreover, let

$$r_{i,\emptyset\to k} = \lambda p_i(\emptyset, k) M_{k,i}.$$

denote the rate at which a single i-agent matches with a k-agent.

Lemma 2.4. *$V_{i\bar{j}} > V_{ij}$ holds if and only if*

$$\frac{u_{i\bar{j}}}{u_{ij}} > \frac{\delta + \sum_{k\in\{L,H\}} r_{i,\emptyset\to k} + r_{i,\bar{j}\to\emptyset}}{\delta + \sum_{k\in\{L,H\}} r_{i,\emptyset\to k} + r_{i,j\to\emptyset}}$$

holds.

In other words, an i-agent accepts to rematch with a type different than her partner's if the relative gain in flow utilities (left-hand side) exceeds the relative gain in the costs of instability (right-hand side), that is, the rate at which i is left by her potential new partner $r_{i,\bar{j}\to\emptyset}$ versus her old partner $r_{i,j\to\emptyset}$.

Note the following comparative statics: If the exit rate δ or the rate of finding a partner when being single $\sum_{k\in\{L,H\}} r_{i,\emptyset\to k}$ is high, then the right-hand side is close to 1. In either case, stability has no value, and agents make their rematching decisions almost exclusively based on flow payoffs. Conversely, if these rates are low, then the rate of being abandoned by one's partner dominates the right-hand side, and stability becomes an essential consideration.

The same argument as in Lemma 2.2 establishes the existence of partial equilibria for fixed masses $\mathcal{M}$. However, we cannot restrict the set of partial equilibria any further because lock-in points, as used in the proof of Lemma 2.1, do not exist without productivity growth. Thus, all nine classes of equilibria, based on the pairwise comparisons between V_{LL} and V_{LH} as well as between V_{HL} and V_{LH}, remain. Moreover, in each of these classes, agents can rematch with other agents of the same type as their partner with positive probability.

Equilibrium Multiplicity We shall now explore the existence of steady-state equilibria in the classes mentioned above. Our first result shows that, for any probabilities by which agents accept to rematch with agents of the same type as their current partner, there exists a steady-state equilibrium.

Theorem 2.2. *For any $(p_i(j,j))_{(i,j)\in\{L,H\}^2} \in [0,1]^4$, there exists a steady-state equilibrium.*

One might be inclined to consider this kind of equilibrium multiplicity as contrived because, after all, one can adopt a tie-breaking rule to deal with indifferences. This measure is not an easy way out because the mere existence of steady-state equilibria may hinge on mixed strategies, where the agents are, by definition, indifferent as to whether or not to rematch. To see why, let us briefly review the basic argument behind the proof of Theorem 2.2. As a first step, we establish the following lemma, which is akin to the "fundamental matching lemma" in Shimer and Smith (2000).

Lemma 2.5. *For any pair of strategies (p_L, p_H), there exist masses $\mathcal{M}$ solving the balance conditions. Moreover, the set of solutions is continuous in (p_L, p_H).*

Now, for the sake of contradiction, suppose that no pure strategy equilibrium exists. For example, if $p_L(L, H) = 1$ and $p_L(H, L) = 0$, that is, if L-agents rematch with H-agents but not with L-agents, then we would need $V_{LL} > V_{LH}$ to not have a pure strategy equilibrium. If $p_L(L, H) = 0$ and $p_L(H, L) = 1$, we would need to have $V_{LH} > V_{LL}$. Using Lemma 2.5 and an intermediate value theorem, we can show that there must exist mixed strategies and induced steady-state masses such that $V_{LH} = V_{LL}$ – a mixed strategy equilibrium exists.

This existence argument requires that the agents play specific mixed strategies. Consequently, it would not go through under any tie-breaking rule. Theorem 2.4 will indeed confirm that, for some parameter constellations, there exist no meaningful pure strategy steady-state equilibria in the limit model.

Next, we examine more closely the equilibrium multiplicity that stems from the behavior of H-agents, which is not present when productivity grows.

Theorem 2.3. *Fixing all other parameters, there exists a non-empty open set $\mathcal{U} \subset \mathbb{R}^4_+$ of flow utilities such that there exists a steady-state equilibrium in each of the following three cases for every $(u_{LL}, u_{LH}, u_{HL}, u_{HH}) \in \mathcal{U}$:*

1. *$V_{LL} > V_{LH}$ and $V_{HH} > V_{HL}$,*
2. *$V_{LH} > V_{LL}$ and $V_{HH} > V_{HL}$, and*
3. *$V_{LH} > V_{LL}$ and $V_{HL} > V_{HH}$.*

Theorem 2.3 establishes the non-genericity of situations where, among multiple other pure-strategy equilibria, there exists an equilibrium in which H-agents abandon other H-agents. Intuitively, if L-agents commit to matches with H-agents, but H-agents expect to be abandoned by other H-agents, then they may seek stability in asymmetric relationships and forego the higher flow utility in symmetric but unstable relationships. Simultaneously, though, symmetric relationships among H-agents are stable if both partners believe their relationship to be stable. The belief about (not) being abandoned thus turns out to be a self-fulfilling prophecy.

The two previous findings underline the difficulty of accurately predicting agents' behavior in the limit model. Since more sophisticated models with on-the-match search and heterogeneous agents also embed our limit model, these observations are valid more generally.

Productivity Growth as a Selection Criterion In the analysis with productivity growth, we have established that agents do not rematch with the same types of agents as their current partners and that H-agents do not rematch out of symmetric matches (cf. Lemma 2.1). Our previous results show that these assertions cease to be valid in the limit model. Nonetheless, we shall now pay particular attention to those equilibria where agents do behave as under productivity growth because one could argue that this is a realistic assumption on the agents' behavior.

Theorem 2.4. *Among steady-state equilibria* $(\mathcal{M}, p_L, p_H)$ *with* $p_i(j,j) = 0$ *for all* $i, j \in \{L, H\}$, $p_H(L,H) = 1$, *and* $p_H(H,L) = 0$, *there exists at most one in each of the following two cases:*

1. $p_L(L,H) = 1$ *and* $p_L(H,L) = 0$, *and*
2. $p_L(L,H) = 0$ *and* $p_L(H,L) = 1$.

Moreover, there exist $\underline{u}, \bar{u} \geq 1$ *such that the first steady-state equilibrium exists if and only if* $\frac{u_{HL}}{u_{LL}} \geq \underline{u}$ *holds and the second steady-state equilibrium exists if and only if* $\frac{u_{HL}}{u_{LL}} \leq \bar{u}$ *holds.*

Theorem 2.4 states that there is at most one equilibrium of each of those two kinds of pure strategy equilibria that also occur in the model with productivity growth. Moreover, it characterizes when these two equilibria exist: The equilibrium in which L-agents rematch with H-agents exists if and only if L-agents are sufficiently more productive with H-agents than with other L-agents, so that the higher flow utility in asymmetric pairs dominates the agents' preference for stability. The equilibrium in which L-agents rematch with other L-agents exists if and only if L-agents are not significantly less productive with L-agents, so that stability becomes the more important consideration.

In particular, one can find parameters such that the two existence regions overlap, in which case both steady-state equilibria co-exist. One can also find parameters such that the two existence region do not overlap, in which case only mixed strategy equilibria exist under the restrictions in Theorem 2.4. Intuitively, if all other L-agents prefer symmetric matches, then there are many single H-agents in steady-state. Thus, a given L-agent may prefer to match with an H-agent because she expects to soon find a new partner upon becoming single. Conversely, if all other L-agents prefer asymmetric matches, then there are few single H-agents in steady-state. Thus, a given L-agent may prefer to match with another L-agent as the rate with which she is left by an L-agent for an H-agent is low.

We shall now establish a correspondence between the above-mentioned equilibria of the limit model and limit equilibria in the model with productivity growth, as the growth rate vanishes. This result justifies the equilibrium selection in Theorem 2.4; minuscule growth is sufficient to eliminate all other equilibria. For this, let us first define precisely what a limit equilibrium is.

Definition 2.4. $(\mathcal{M}^*, t^*_{LH}, t^*_{LL}, t^*_{HL})$ is a *limit equilibrium* if there exist sequences

$$\left(\left(u^n_{ij}(t)\right)_{i,j\in\{L,H\},\, t\geq 0}\right)_{n\in\mathbb{N}} \qquad \text{and} \qquad \left(\mathcal{M}^n, t^n_{LH}, t^n_{LL}, t^n_{HL}\right)_{n\in\mathbb{N}}$$

such that the following three properties hold:

1. $u^n_{ij}(0) = u_{ij}$ and $\lim_{m\to\infty} u^m_{ij}(t) = u_{ij}$ for all $i, j \in \{L, H\}$, $n \in \mathbb{N}$, and $t > 0$,
2. $(\mathcal{M}^n, t^n_{LH}, t^n_{LL}, t^n_{HL})$ is a steady-state equilibrium when flow utilities are given

by $\left(u_{ij}^n(t)\right)_{i,j\in\{L,H\},\,t\geq 0}$, and

3. $(\mathcal{M}^n, t_{LH}^n, t_{LL}^n, t_{HL}^n) \to (\mathcal{M}^*, t_{LH}^*, t_{LL}^*, t_{HL}^*)$ pointwise as $n \to \infty$.

In other words, if we (1) fix initial flow utilities u_{ij} and let subsequent flow utilities converge to this value, and if we (2) pick a sequence of steady-state equilibria corresponding to this sequence of flow utilities, then (3), the limit of the steady-state masses and partial equilibrium cutoffs (if it exists) is said to be a limit equilibrium. With slight abuse of notation, we allow the limit cutoffs t_{ij}^* to take any value in $\mathbb{R}_+ \cup \{\infty\}$.

We now need a notion that captures when a limit equilibrium is in fact an equilibrium in the limit model.

Definition 2.5. A limit equilibrium $(\mathcal{M}^*, t_{LH}^*, t_{LL}^*, t_{HL}^*)$ *corresponds* to a steady-state equilibrium $(\mathcal{M}, p_L, p_H)$ in the limit model if

$$\forall i,k \in \{L,H\}\ \forall j \in \{L,H,\emptyset\}\ :\ m_{ij}^* = m_{ij}\ \wedge\ m_{i,k,j}^* = m_{i,k,j}$$

holds.

In other words, we declare two such equilibria as corresponding if they have the same steady-state masses and the same masses of agents that accept a rematch with other agents.

Theorem 2.5. *If a steady-state equilibrium $(\mathcal{M}, p_L, p_H)$ in the limit model corresponds to a limit equilibrium, then $p_H(L,H) = 1$, $p_H(H,L) = 0$, $p_i(j,j) = 0$, and $p_L(L,H) \cdot p_L(H,L) = 0$ for all $i,j \in \{L,H\}$. Every pure strategy steady-state equilibrium $(\mathcal{M}, p_L, p_H)$ in the limit model with $p_H(L,H) = 1$, $p_H(H,L) = 0$, and $p_i(j,j) = 0$ for all $i,j \in \{L,H\}$ corresponds to a limit equilibrium.*

Theorem 2.5 confirms that (1) only those equilibria that fulfill the selection criteria in Theorem 2.4 can be limit equilibria, and (2) that every such equilibrium in pure strategies indeed corresponds to a limit equilibrium. In particular, this implies that multiple equilibria can exist in our model with productivity growth. Mixed strategy equilibria are dicier to handle, which also explains why we have stated Definition 2.5 in terms of acceptance masses and not in terms of strategies: Recall from Theorem 2.4 that a pure strategy equilibrium need not exist under the above selection criteria. At the same time, Corollary 2.4 implies that mixed strategy equilibria are non-generic under productivity growth. To reconcile these two seemingly contradictory facts, note that, to approximate a mixed strategy equilibrium in the limit model, the cutoffs of L-agents must consequently converge to a finite limit as productivity vanishes. In other words, agents mix via match duration and not by explicitly playing mixed strategies.

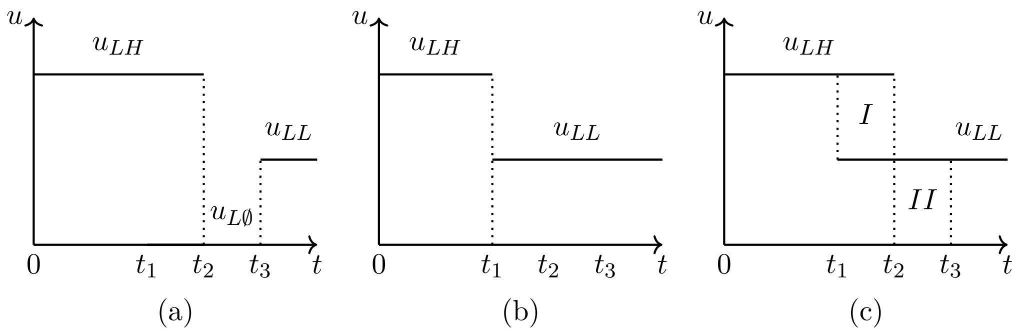

Figure 2.2: An example of flow payoffs without productivity growth for an L-agent if she (a) stays with her H-partner or (b) rematches for an L-agent. Figure (c) shows the difference in payoffs.

Equipped with Theorem 2.5, we now not only have a tool for selecting equilibria in the limit model, but we can also compare what happens to the limit model when productivity growth is introduced. We shall discuss this comparison in the next section.

2.7 Discussion

A surprising result is that L-agents may prefer other L-agents over H-agents even though matching with the latter yields a higher flow utility. In many matching models, acceptance strategies are monotone, i.e., higher types are more accepted by other agents. A positive sorting (with non-transferable utility) is generally driven by the fact that highly productive agents search for other highly productive agents and do not accept agents with a low productivity type. In contrast to that, in our model L-agents may actually search for L-agents even after being matched to H-agents. Thus, positive sorting does not only come from the H-agents' decisions but also from the L-agents.

To better understand the effects of match-specific productivity growth, we first analyze the trade-off that agents face with constant productivity. Then, we look at the changes to this trade-off when productivity grows.

As an example, let's consider a consumption path, where an an L-agent is matched with an H-agent and they meet the following other agents: First, at t_1, the L-agent meets another L-agent. Then, at $t_2 > t_1$ the H-agent meets another H-agent and accepts to match. Finally, at $t_3 > t_2$ the L-agent meets another L-agent. Without productivity growth, the flow utility for an L-agent who stays with her H-partner at t_1 is given by Figure 2.2a. It is u_{LH} until t_2, then 0 until t_3 and u_{LL} afterwards. The flow utility of an L-agent who does accept the first other L-agent is given by Figure 2.2b. It is u_{LH} until t_1 and u_{LL} afterward. The difference in utility between both decisions is given by the area between these two curves and is

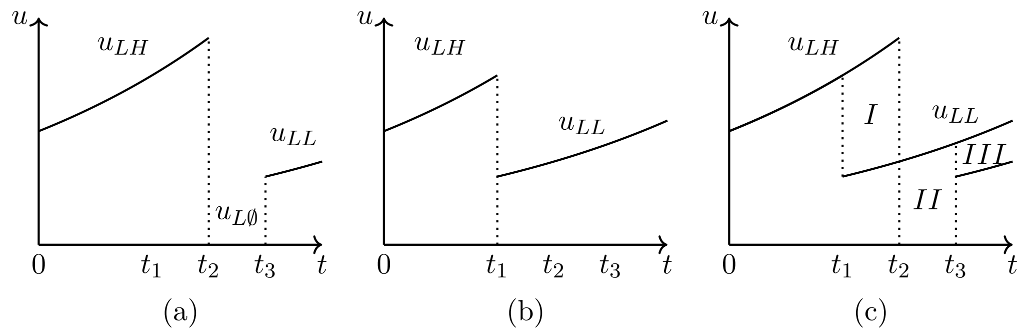

Figure 2.3: An example of flow payoffs with capital accumulation for an L-agents if she (a) stays with her H-partner or (b) rematches for an L-agent. Figure (c) shows the difference in payoffs.

displayed in Figure 2.2c. From t_1 to t_2 an L-agent who stays with her H-partner receives a higher flow payoff (Area I). However she is left at t_2 and gets no payoff instead of u_{LL} until t_3 (Area II). If the L-agent could foresee all the future meetings, then she would choose to stay with her H-partner at t_1 if and only if $I > II$ holds.

Now, meetings (and market exits) occur at a given rate following a Poisson Process. Therefore, an agent compares the difference in the flow payoffs to the difference in the matching rates. As seen in the last section, the higher flow payoff of an H-partner dominates for the decision of an i-agent if and only if

$$\frac{u_{iH}(0)}{u_{iL}(0)} \geq \frac{\delta + r_{i,\emptyset\to L} + r_{i,\emptyset\to H} + r_{i,H\to\emptyset}}{\delta + r_{i,\emptyset\to L} + r_{i,\emptyset\to H} + r_{i,L\to\emptyset}}$$

holds.

Now, let's consider the same example with productivity growth. Figure 2.3 displays the flow payoff as before. The main difference is that the flow payoff is strictly increasing during a match. If the L-agent decides to stay with her partner at t_1 and is left at t_2, she is not only single until t_3, but she also has to start over with accumulating productivity growth. If she instead accepts to rematch at t_1, her flow payoff increases from t_1 on. Therefore, with productivity growth, her flow payoff is different in both decisions even after time t_3. The difference after time t_3 is given by area III in Figure 2.3c. If she could foresee the future meetings, then she would choose to stay with her H-partner at t_1 if and only if $I > II + III$ holds. In particular, being left reduces the L-agents' payoffs for a longer time than in the example without productivity growth. This illustrates how, with productivity growth, agents have an additional incentive to value stability in a match: Even though matches eventually become stable, the loss upon being left is more severe.

2.8 Conclusion

We have characterized the steady-state equilibria in our decentralized matching model with productivity growth and in a limit model with constant productivity. As a surprising result, there are equilibria where L-agents prefer to rematch to another L-agent over staying with an H-agent - even though the flow payoff in a match with an H-agent is higher. The reason for this is the stability that another L-agent might provide in a match and the forgone utility upon being left.

Another insight is that productivity growth does not allow for equilibria with coordination failure. In the limit model without productivity growth, there can be multiplicity of equilibria including equilibria where matched H-agents leave each other driven by the belief that their partners are also searching to replace them. Even with the slightest productivity growth, the equilibrium multiplicity in the limit model ceases to exist. Based on this, we have presented an equilibrium selection criterium for the model without productivity growth. Equilibria of the limit model that have a coordination failure cannot be approximated as a limit of equilibria with vanishing productivity growth, and vice versa.

2.A Proofs

2.A.1 Proofs for Section 2.3

Survival Functions In this paragraph, we formally derive the survival functions q_{ij} for pairs and $q_{i\emptyset}$ for singles.

First, we derive the outflow of agents out of pairs. For $i, j \in \{L, H\}$ let $\Pi(i, j)$ be the set of permutations of the ordered set (i, j). For $i \neq j$ there are two permutations[9] and for $i = j$ there is one permutation[10].

Then, the outflow of agents out of ij-pairs that have been together for time t is

$$O_{ij}(t) = 2 \cdot \delta \sum_{\pi \in \Pi(i,j)} m_{\pi(ij)}(t) + 2 \cdot \lambda \sum_{\pi \in \Pi(i,j)} m_{\pi(ij)}(t) \sum_{k \in \{L,H\}} p_{\pi(i)}(\pi(j), k, t) M_{k,\pi(i)}, \tag{2.10}$$

and the outflow of unmatched i-agents that have been single for time t is

$$O_{i\emptyset}(t) = \delta m_{i\emptyset}(t) + \lambda m_{i\emptyset}(t) p_i(\emptyset, i, t) \int_0^\infty m_{i\emptyset}(s) p_i(\emptyset, i, s) ds + \lambda m_{i\emptyset}(t) \sum_{k \in \{L,H\}} p_i(\emptyset, k, t) M_{k,i}. \tag{2.11}$$

Since rematchings and market exits arrive according to independent Poisson processes, and since pairs break up when the first of these events occurs, they are dissolved by a Poisson process at rate

$$\frac{O_{ij}(t)}{\sum_{\pi \in \Pi(i,j)} m_{\pi(ij)}(t)}.$$

Thus, the probability that such a pair does not dissolve before time t is

$$q_{ij}(t) = e^{-\int_0^t \frac{O_{ij}(s)}{\sum_{\pi \in \Pi(i,j)} m_{\pi(ij)}(s)} ds}$$

for pairs. Similar, for singles, the rate is given by

$$\frac{O_{i\emptyset}(t)}{m_{i\emptyset}(t)}$$

and the survival probability is

$$q_{i\emptyset}(t) = e^{-\int_0^t \frac{O_{i\emptyset}(s)}{m_{i\emptyset}(s)} ds}.$$

[9]The first one is defined by $\pi(ij) = (ij)$, $\pi(i) = i$ and $\pi(j) = j$. The second one is defined by $\pi'(ij) = (ji)$, $\pi'(i) = j$ and $\pi'(j) = i$.

[10]The unique permutation π in $\Pi(i, i)$ is defined by $\pi(ii) = (ii)$ and $\pi(i) = i$.

2.A.2 Proofs for Section 2.4

Proof of Lemma 2.1. For single agents it is a strictly dominant strategy to always accept to match. Therefore, in all steady-state equilibria

$$\forall\ t \geq 0,\ i,j \in \{L,H\}\ :\ p_i(\emptyset,j,t) = 1$$

holds. For the next step, we need that agents follow cutoff strategies. First, we restrict attention to the asymmetric pairs.

Claim. All agents in asymmetric pairs follow cutoff strategies with cutoffs $t_{LH} < t_{HL}$. Furthermore, no agent in an asymmetric pair accepts to match with a partner of the same type.

Proof of Claim. The proof proceeds in four steps:

1. Derive a lower and upper bound on the continuation payoffs.
2. Show that there exists some point in time t^*_{HL} such that no agent rematches at any $t > t^*_{HL}$.
3. Show that there exists some $t^*_{LH} < t^*_{HL}$ such that the agent of type L does not rematch and the agent of type H rematches at any $t \in (t^*_{LH}, t^*_{HL})$.
4. Show that at any $t < t^*_{LH}$, both agents rematch.

Step 1: First, let's derive a lower and an upper bound on the continuation payoff of the agents in an asymmetric pair: For $i \neq j$ define

$$\underline{V}_i(j,t,t') := e^{-(2\delta+\lambda(M_{ij}+M_{ji}))(t'-t)} \cdot V_i(j,t')$$

and

$$\begin{aligned}\bar{V}_i(i,t,t') := &\int_t^{t'} u_i(j,t'')e^{-2\delta(t''-t)}\ dt'' \\ &+ \frac{(\delta+\lambda M_{ji})\cdot V_i(\emptyset,0)+\lambda M_{ij}\cdot V_i(i,0)}{2\delta}\cdot\left(1-e^{-2\delta(t'-t)}\right) \\ &+ e^{-2\delta(t'-t)}\cdot V_i(j,t'),\end{aligned}$$

and note that

$$\underline{V}_i(j,t,t') \leq V_i(j,t) \leq \bar{V}_i(j,t,t')$$

holds.

Step 2: The continuation payoff of an agent of type $i \in \{L,H\}$ who is matched with a j-agent with $j \neq i$ for time t and who stays in the current match is bounded below by

$$V_i(j,t) \geq \int_t^\infty e^{-(2\delta+\lambda(M_i+M_j))(t'-t)} \cdot u_i(j,t')\ dt'$$

no matter what strategies the two agents pursue from time t onwards. This lower bound is strictly increasing in t and by assumption it converges to infinity as $t \to \infty$, while the payoff from leaving at time t and rematching into another match is at most $\max\{V_i(i,0), V_i(j,0)\}$ which is bounded above as it is constant in t. Hence, there exists some finite point in time T_i such that an i-agent strictly prefers staying to leaving at any $t \geq T_i$ for any possible strategies s_L and s_H. Let $T = \max\{T_L, T_H\}$ and note that it is a strictly dominant strategy to stay in the LH-pair from time T onwards for both types.

Now, consider the sequence $(t_n)_{n\in\mathbb{N}}$, which is defined as follows:

$$t_0 = T,$$
$$t_n = \max\{t_n^L, t_n^H\} \text{ for all } n > 0,$$

with

$$t_n^i = \inf\left\{t \in \mathbb{R}_+ | \underline{V}_i(j,t,t_{n-1}) \geq \max\{V_i(i,0), V_i(j,0)\}\right\}.$$

In words, this procedure iteratively constructs points in time such that no agent wants to rematch thereafter, respectively. Note that $t_n^L < t_n^H$ holds for all n. By construction, (t_n) is monotonically decreasing and bounded below (by 0). Consequently, it converges. The limit is equal to t^*_{HL}, which is the (unique) solution to the equation

$$\int_{t^*_{HL}}^\infty u_H(L,t)e^{-2\delta t}dt = \int_0^\infty u_H(H,t)e^{-2\delta t}dt,$$

that is, t^*_{HL} is the point in time at which the agent of type H is indifferent between staying in and leaving the LH-pair if both agents stay after t^*_{HL}. To see that t^*_{HL} is the limit of (t_n), note that $t_n^H \geq t^*_{HL}$ holds for all n. This implies that $\lim_{n\to\infty} t_n \geq t^*_{HL}$ holds. So assume to the contrary that $\lim_{n\to\infty} t_n > t^*_{HL}$ holds. But then, since $\lim_{n\to\infty} t_n^H \geq \lim_{n\to\infty} t_n^L$ holds, we would have $V_H(L, \lim_{n\to\infty} t_n) = V_H(H,0)$, which is a contradiction.

Step 3: First, there exists some non-empty interval $[\tilde{t}, t_H)$ on which the agent of type H rematches and $V_{HL}(t) < V_{HL}(, t_H) = V_{HH}(0)$ holds for all $t \in [\tilde{t}, t_H)$. In order to verify this claim, note that there is some $\tilde{t} < t_H$ satisfying $\underline{U}_L(LH, \tilde{t}, t_H) \geq V_{LL}(0)$, i.e., the agent of type L does not rematch on $[\tilde{t}, t_H)$. But then, the agent of type H must rematch on $[\tilde{t}, t_H)$: The continuation payoff of the agent of type H in the LH-pair at time $t \in [\tilde{t}, t_H)$ conditional on that the agent of type L does not rematch afterwards is given by

$$
\begin{aligned}
V_{HL}(t) = &\int_t^{t_H} e^{-(2\delta(\tau-t)+\lambda M_H(S_H(\tau)-S_H(t)))} \cdot e^{r\tau} d\tau \\
&+ \int_t^{t_H} [\delta V_{h\emptyset}(0) + \lambda M_H s_H(\tau) \cdot V_{HH}(0)] \cdot e^{-(2\delta(\tau-t)+\lambda M_H(S_H(\tau)-S_H(t)))} d\tau \\
&+ e^{-(2\delta(t_H-t)+\lambda M_H(S_H(t_H)-S_H(t)))} \cdot V_{HH}(0).
\end{aligned}
$$

The corresponding derivative is

$$
\begin{aligned}
\frac{dV_{HL}(t)}{dt} &= (2\delta + \lambda M_H s_H(t)) \cdot V_{HL}(t) - e^{rt} - (\delta V_{H\emptyset}(0) + \lambda M_H s_H(t) V_{HH}(0) \\
&= \begin{cases} 2\delta V_H(LH,t) - (e^{rt} + \delta V_{H\emptyset}(0)) + \lambda M_H(V_H(LH,t) - V_H(HH)), & \\ & \text{if } V_H(LH,t) < V_H(HH) \\ 2\delta V_H(LH,t) - (e^{rt} + \delta V_{H\emptyset}(0)), & \text{if } V_H(LH,t) \geq V_H(HH) \end{cases}
\end{aligned}
$$

where the entries in the case distinction follow from the optimality of the strategy p_H. Moreover, it holds that $\frac{dV_{HL}(t_H)}{dt} \geq 0$. Hence, we obtain that $\frac{dV_{HL}(t)}{dt} > 0$ whenever $V_{HL}(t) \geq V_{HH}(0)$ for every $t \in [\tilde{t}, t_H)$. Consequently, continuity of the derivative yields $V_{HL}(t) < V_{HH}(0)$ for all $t \in [\tilde{t}, t_H)$. (If $V_{HL}(t) \geq V_{HH}(0)$ for some $t \in [\tilde{t}, t_H)$, then $V_{HL}(\cdot)$ would be strictly increasing on $[t, t_H]$ which contradicts $V_{HL}(t_H) = V_{HH}(0)$.)

As in the previous step, consider the sequence $(u_n)_{n\in\mathbb{N}}$ given by

$$
\begin{aligned}
u_0 &= \tilde{t}, \\
u_n &= \max\{u_n^L, u_n^H\} \text{ for all } n > 0,
\end{aligned}
$$

with

$$u_n^L = \inf\{u \in \mathbb{R}_+ | \underline{U}_L(LH, u, u_{n-1}) \geq V_{LL}(0)\}$$

and

$$u_n^H = \inf\{u \in \mathbb{R}_+ | \bar{U}_H(LH, u', u_{n-1}) \leq V_{HH}(0) \text{ for all } u' \in [u, u_{n-1}]\}.$$

This sequence contains points in time after which the agent of type L does not rematch and the agent of type H always rematches up to t_H. Again, this sequence is monotonically decreasing and bounded, and thus convergent. Its limit is the largest solution t_L in the interval $[0, t_H]$ of the equation

$$
\begin{aligned}
\int_{t_L}^{t_H} e^{-(2\delta+\lambda M_H)(\tau-t_L)} \cdot e^{r\tau} d\tau + \int_{t_L}^{t_H} (\delta + \lambda M_H) \cdot V_{L\emptyset}(0) \cdot e^{-(2\delta+\lambda M_H)(\tau-t_L)} d\tau \\
+ e^{-(2\delta+\lambda M_H)(t_H-t_L)} \cdot V_{LH}(t_H) = V_{LL}(0),
\end{aligned}
$$

whenever it exists, and $t_L \equiv 0$, otherwise. In words, t_L is the largest point in time at which the agent of type L is indifferent between staying and leaving if she stays after t_L and the agent of type H stays from time t_H on. In particular, it follows that $t_L < t_H$ holds because the right hand side of the above equation attains the value $V_{LH}(t_H) > V_{LL}(0)$ if $t_L = t_H$ holds.

To check that t_L must be the limit of the sequence (u_n), note that $u_n^L \geq t_L$ holds for all n yielding $\lim_{n\to\infty} u_n \geq t_L$. So suppose that $\lim_{n\to\infty} u_n > t_L$ holds. But then, we would have $V_{HL}(\lim_{n\to\infty} u_n) = V_{HH}(0)$. However, since the agent of type L does not rematch on $[\lim_{n\to\infty} u_n, \infty)$, we have that $V_{HL}(\lim_{n\to\infty} u_n) < V_{HH}(0)$ holds by the same reasoning as why $V_{HL}(\tilde{t}) < V_{HH}(0)$ – a contradiction.

Step 4: The proof of this step works similar as for the two cases before: First, there exists some $\tilde{u} < t_L$ such that both agents will rematch on $[\tilde{u}, t_L)$ and $U_{ij}(\tilde{u}) < U_{ii}(0)$ holds for each $i, j \in \{L, H\}$ with $i \neq j$. Then, we consider the sequence $(v_n)_{n\in\mathbb{N}}$ with

$$\begin{aligned} v_0 &= \tilde{u}, \\ v_n &= \max\{v_n^L, v_n^H\} \text{ for all } n > 0, \end{aligned}$$

with

$$v_n^i = \inf\{v \in \mathbb{R}_+ | \bar{U}_i(LH, v', v_{n-1}) \leq U_{ii}(0) \text{ for all } v' \in [v, v_{n-1}]\}$$

for each i. This sequence iteratively constructs points in time after which both agents rematch (at least) up to t_L, and it converges to 0. Thus, both types of agents accept to rematch with probability 1 for all $t \in [0, t^*_{LH})$. This finishes the proof of the claim. □

Continuation of the proof of Lemma 2.1. Analogously, we get that L-agents in an LL-pair follow cutoff strategies and do not rematch to an L-agent. For the rematching behavior for an agent in an HH-pair, assume for contradiction that an H-agent in an H-pair rematches until some cutoff $t_0 > 0$ and stays afterwards. Then, the payoff of staying is

$$V_{HH}(t_0) = \int_{t_0}^{\infty} u_{HH}(t) e^{-2\delta t} dt + \frac{1}{2} V_{H\emptyset}(0),$$

and the payoff of rematching with an L-agent is bounded above by

$$V_{HL}(0) = \int_{0}^{\infty} u_{HL}(t) e^{-2\delta t} dt + \frac{1}{2} V_{H\emptyset}(0),$$

which is the expected payoff if no agent leaves an LH-pair. Note that for this to be an upper bound we need that $V_{HL}(0) > V_{HH}(0)$ holds by our assumption. Now, $V_{HH}(t_0)$ is strictly larger than $V_{HL}(0)$. Thus, an H-agent with an H-partner is not indifferent at t_0, but strictly prefers to stay. Therefore, there is an ε-ball around t_0

such that for all t in it, an H-agent prefers to stay. Hence, a strategy with cutoff t_0 cannot be optimal and we get that

$$p_H(H, \cdot, \cdot) = 0$$

holds. Together, this implies that for $t > 0$ all agents strictly prefer to stay in a pair compared to rematching to a partner of the same type as their current partner, i.e., we get that

$$\forall\ t > 0,\ i, j \in \{L, H\}\ :\ p_i(j, j, t) = 0$$

holds. $\square$

Proof of Corollary 2.1. It is shown in step (3) of the proof of Lemma 2.1 that $t_{LH} \leq t_{HL}$ has to hold in every partial equilibrium. $\square$

Proof of Corollary 2.2. By assumption, the flow utility is strictly increasing over time and by Lemma 2.1, the agents' strategies are weakly decreasing. Therefore $V_{ij}(t)$ is strictly increasing. $\square$

Proof of Corollary 2.3. By Lemma 2.1, no H-agent rematches from an HH-pair. Thus, the payoff in such a pair is

$$V_{HH}(0) = \int_0^\infty u_{HH}(t) e^{-2\delta t} dt + \frac{1}{2} V_{H\emptyset}(0),$$

where the integral is the expected flow payoff and the term $\frac{1}{2} V_{H\emptyset}(0)$ comes from the probability $\frac{1}{2}$ that the partner dies before oneself multiplied by the expected utility of being single. By $t^*_{LH} \leq t^*_{HL}$, no L-agent rematches after t^*_{HL}. Thus, the continuation payoff at the cutoff is

$$V_{HL}(t^*_{HL}) = \int_{t^*_{HL}}^\infty u_{HL}(t) e^{-2\delta t} dt + \frac{1}{2} V_{H\emptyset}(0).$$

At the cutoff, the agent is indifferent. This leads to equation (2.5) as desired. For existence and uniqueness of a solution, note that by assumption we have $u_{HL}(t) \leq u_{HH}(t)$ and therefore

$$\int_0^\infty u_{HL}(t) e^{-2\delta t} dt \leq \int_0^\infty u_{HH}(t) e^{-2\delta t} dt$$

holds which implies $V_{HL}(0) < V_{HH}(0)$. Furthermore, $V_{HL}(t_0)$ is continuous and strictly increasing in t_0. Also, by assumption,

$$\lim_{t_0 \to \infty} V_{HL}(t_0) = \infty$$

holds. Thus, by the intermediate value theorem there exists a unique solution to equation (2.5). □

Proof of Corollary 2.4. In every steady-state equilibrium exactly one of the three cases

$$\begin{aligned} &\text{Case 1:} \quad V_{LL}(0) < V_{LH}(0)\\ &\text{Case 2:} \quad V_{LL}(0) = V_{LH}(0)\\ &\text{Case 3:} \quad V_{LL}(0) > V_{LH}(0) \end{aligned}$$

holds. By Corollary 2.2, we have

$$\forall\, t > 0,\ i \in \{L, H\} \ :\ V_{Li}(t) > V_{Li}(0).$$

Thus, if an agent prefers to match with an agent of a specific type at $t = 0$, she strictly prefers to stay with such a partner afterwards. Together with the fact that an agent is indifferent at a cutoff, the three possible cases for a partial equilibrium are:

$$\begin{aligned} &\text{Case 1:} \quad t_{LH} > 0 \ \wedge\ t_{LL} = 0 \ \wedge\ V_{LH}(t_{LH}) = V_{LL}(0)\\ &\text{Case 2:} \quad t_{LH} = 0 \ \wedge\ t_{LL} = 0 \ \wedge\ V_{LH}(0) = V_{LL}(0)\\ &\text{Case 3:} \quad t_{LH} = 0 \ \wedge\ t_{LL} > 0 \ \wedge\ V_{LH}(0) = V_{LL}(t_{LL}). \end{aligned}$$

□

Proof of Lemma 2.2. Fix the masses $\mathcal{M}$ and let $i \in \{L, H\}$. Then, let $W_{Li}^{L\to H}$ be the highest attainable utility of an L-agent who has just matched with an i-agent when H-agents behave as specified in Corollary 2.3, a positive mass of L-agents rematches into asymmetric pairs, and no L-agent rematches into symmetric pairs. Define $W_{Li}^{H\to L}$ analogously for the case that a positive mass of L agents rematches into symmetric pairs and no L-agents rematch into asymmetric pairs. We have

$$W_{LL}^{L\to H} \leq W_{LL}^{H\to L} \ \wedge\ W_{LH}^{L\to H} \leq W_{LH}^{H\to L}$$

as the masses are constant and the decisions of other L-agents only matter in a symmetric match. Moreover,

$$W_{LL}^{H\to L} - W_{LL}^{L\to H} \geq W_{LH}^{H\to L} - W_{LH}^{L\to H}$$

holds, as the decision of other L-agents not to leave a symmetric match always benefits a given L-agent and this benefit is immediate in a symmetric match but only implicit in an asymmetric match via the continuation utility from a symmetric match.

Suppose there exists no partial equilibrium as in equation (2.6) or equation (2.8). Then,

$$W_{LH}^{L\to H} \leq W_{LL}^{L\to H} \;\wedge\; W_{LL}^{H\to L} \leq W_{LH}^{H\to L}$$

holds, which is only possible if

$$W_{LL}^{L\to H} = W_{LL}^{H\to L} = W_{LH}^{L\to H} = W_{LH}^{H\to L}$$

holds, so that a zero mass of L-agents rematch and every L-agent is indeed indifferent when given the choice between matching with an L-agent or H-agent. Then, equation (2.7) holds and there exists such a partial equilibrium. This establishes existence. Uniqueness in every class follows from Corollary 2.4, the fact that the masses are fixed, and the monotonicity of V: For a partial equilibrium that satisfies equation (2.6) and for fixed masses there is at most one cutoff t^*_{LH} that makes an L-agent indifferent. For a partial equilibrium that satisfies equation (2.8) note that for fixed masses the utility $V_{LL}(t)$ in a mixed pair is weakly decreasing in t^*_{LL}. Again, there is at most one cutoff t^*_{LL} that makes an L-agent indifferent. □

2.A.3 Proofs for Section 2.5

Aggregate Balance Conditions Let $A/B/C$ denote the aggregated mass of asymmetric pairs where "L and H" / "only H" / "no one" want to rematch into symmetric pairs. Since the agents follow cutoff strategies, integrating over the pointwise masses yields

$$\begin{aligned}
A &= I_{LH}\int_0^{t_{LH}} q_{LH}(t)dt\\
B &= I_{LH}\int_{t_{LH}}^{t_{HL}} q_{LH}(t)dt\\
C &= I_{LH}\int_{t_{HL}}^{\infty} q_{LH}(t)dt
\end{aligned}$$

with different exponential decay on the three intervalls:

$$\begin{aligned}
t \leq t^*_{LH}: &\quad q_{LH}(t) = e^{-(\lambda M_{L,L}+\lambda M_{H,H}+2\delta)t}\\
t^*_{LH} \leq t \leq t^*_{HL}: &\quad q_{LH}(t) = e^{-\lambda M_{L,L}t^*_{LH}-(M_{H,H}+2\delta)t}\\
t^*_{HL} \leq t: &\quad q_{LH}(t) = e^{-\lambda M_{L,L}t^*_{LH}-\lambda M_{H,H}t^*_{HL}-2\delta t}.
\end{aligned}$$

Analogously, for LL-pairs, let

$$D = I_{LL} \int_0^{t_{LL}} q_{LL}(t)dt$$
$$E = I_{LL} \int_{t_L L}^{\infty} q_{LL}(t)dt$$

with exponential decay

$$t \leq t^*_{LL}: \quad q_{LL}(t) = e^{-(2\lambda M_{H,L}+2\delta)t}$$
$$t^*_{LL} \leq t: \quad q_{LL}(t) = e^{-2\lambda M_{H,L} t^*_{LL} - 2\delta t}.$$

denote the aggregate masses of agents, where in D both agents want to rematch into asymmetric pairs and in E both agents prefer to stay in the current match.

By definition of A, B, C, D, and E, we have

$$M_{L,L} = (A + m_{L\emptyset})$$
$$M_{H,H} = (A + B + m_{H\emptyset})$$
$$M_{H,L} = (m_{H\emptyset}).$$

Note, that by Corollary 2.2, $A = 0$ or $D = 0$ holds. Lemma 2.6 presents the five aggregate balance conditions that are obtained by calculating the integrals.

Lemma 2.6. *Every steady-state equilibrium satisfies the state-wise aggregate balance conditions*

$$(2\delta + 2\lambda A + \lambda B + \lambda m_{L\emptyset} + \lambda m_{H\emptyset})A = I_{LH}(1 - e^{-(\lambda(A+m_{L\emptyset})+\lambda(A+B+m_{H\emptyset})+2\delta)t^*_{LH}}) \quad (2.12)$$

$$(2\delta + \lambda A + \lambda B + \lambda m_{H\emptyset})B = I_{LH} e^{-(\lambda(A+m_{L\emptyset})+\lambda(A+B+m_{H\emptyset})+2\delta)t^*_{LH}} \cdot (1 - e^{-(\lambda(A+B+m_{H\emptyset})+2\delta)(t^*_{HL}-t^*_{LH})}) \quad (2.13)$$

$$2\delta C = I_{LH} e^{-(\lambda(A+m_{L\emptyset})+\lambda(A+B+m_{H\emptyset})+2\delta)t^*_{LH}} e^{-(\lambda(A+B+m_{H\emptyset})+2\delta)(t^*_{HL}-t^*_{LH})} \quad (2.14)$$

$$(2\delta + 2\lambda m_{H\emptyset})D = I_{LL}(1 - e^{-(2\lambda m_{H\emptyset}+2\delta)t^*_{LL}}) \quad (2.15)$$

$$2\delta E = I_{LL} e^{-(2\lambda m_{H\emptyset}+2\delta)t^*_{LL}} \quad (2.16)$$

with $I_{LH} = \lambda(m_{L\emptyset} + D)m_{H\emptyset}$ *and* $I_{LL} = \lambda(m_{L\emptyset} + A)^2$. *Conversely, for every partial equilibrium* $(\mathcal{M}, p_L, p_H)$ *that satisfies the state-wise aggregate balance conditions, there exist unique masses* $\mathcal{M}'$ *such that the state-wise aggregate masses are identical*[11] *in* $\mathcal{M}$ *and* $\mathcal{M}'$, *and such that* $(\mathcal{M}', p_L, p_H)$ *is a steady-state equilibrium.*

Proof. Every steady-state equilibrium satisfies the pointwise balance conditions. In-

[11] In particular A, B, C, D, E, $m_{L\emptyset}$, and $m_{H\emptyset}$ are the same. For a steady-state equilibrium, the remaining mass m_{HH} is uniquely given by the other masses.

tegrating those yields the aggregate balance conditions.

For the converse statement fix some masses $\mathcal{M}$ that satisfy the state-wise aggregate balance conditions. Then, for given A, B, C, D, E, $m_{L\emptyset}$, and $m_{H\emptyset}$, let the pointwise masses in $\mathcal{M}'$ be defined by the pointwise balance conditions:

$$\begin{aligned}
t \leq t^*_{LH}: \quad & m_{LH}(t) = \lambda(m_{L\emptyset} + D)m_{H\emptyset} \cdot e^{-(\lambda M_{L,L} + \lambda M_{H,H} + 2\delta)t} \\
t^*_{LH} \leq t \leq t^*_{HL}: \quad & m_{LH}(t) = \lambda(m_{L\emptyset} + D)m_{H\emptyset} \cdot e^{-\lambda M_{L,L} t^*_{LH} - (M_{H,H} + 2\delta)t} \\
t^*_{HL} \leq t: \quad & m_{LH}(t) = \lambda(m_{L\emptyset} + D)m_{H\emptyset} \cdot e^{-\lambda M_{L,L} t^*_{LH} - \lambda M_{H,H} t^*_{HL} - 2\delta t} \\
t \leq t^*_{LL}: \quad & m_{LL}(t) = \lambda(m_{L\emptyset} + A)^2 \cdot e^{-(2\lambda M_{H,L} + 2\delta)t} \\
t^*_{LL} \leq t: \quad & m_{LL}(t) = \lambda(m_{L\emptyset} + A)^2 \cdot e^{-2\lambda M_{H,L} t^*_{LL} - 2\delta t} \\
& m_{HH}(t) = \lambda(m_{H\emptyset} + A + B)^2 \cdot e^{-2\delta t} \\
& m_{L\emptyset}(t) = I_{L\emptyset} \cdot e^{-(M_{L,L} + M_{H,L} + \delta)t} \\
& m_{H\emptyset}(t) = I_{H\emptyset} \cdot e^{-(M_{L,H} + M_{H,H} + \delta)t}.
\end{aligned}$$

By construction $\mathcal{M}'$ satisfies the state-wise aggregate balance conditions and has the same aggregate masses as $\mathcal{M}$. Furthermore, as it satisfies the pintwise balance conditions, $\mathcal{M}'$ is a steady-state equilibrium. □

Proof of Theorem 2.1. The proof proceeds as follows: First, fix some single masses m_L and m_H. Then, let $\mathcal{S}_L$ denote the set of all strategies with $p_L(L, H, t) = 0$ and $p_i(\emptyset, j, t) = 1$ for all t and $i, j \in \{L, H\}$. Analogously, let $\mathcal{S}_H$ be the set of all strategies with $p_L(H, L, t) = 0$ and $p_i(\emptyset, j, t) = 1$ for all t and $i, j \in \{L, H\}$. That is, with either restriction, the agents are only allowed to rematch in one particular direction and singles always have to accept. We show that for both restricted strategy spaces there exists a unique steady-state equilibrium where the strategies are mutually optimal among all strategies in the restricted set. Then, we show that there are only three cases: (1) The unique equilibrium under the restriction on $\mathcal{S}_L$ is also an equilibrium in the whole stratege space $\mathcal{S}$ and the unique equilibrium among $\mathcal{S}_H$ is not, (2) the other way round, and (3) in both restricted equilibria, L-agents do not rematch and both equilibria are the same. Therefore, there exists a unique steady-state equilibrium.

First, consider the case that the strategy space is restricted to $\mathcal{S}_L$, i.e., L-agents are not allowed to leave an LL-pair. Adding the aggregate balance conditions (2.12), (2.13), and (2.14) together implies an aggregate steady-state condition, i.e., for the total mass of mixed pairs, the inflow equals the outflow:

$$2\delta(A + B + C) + \lambda A^2 + \lambda(A + B)^2 + \lambda m_{H\emptyset}(A + B) + m_{L\emptyset}A = \lambda m_{L\emptyset}m_{H\emptyset}.$$

The next lemma shows that identical cutoffs imply identical masses.

Lemma 2.7. *Assume that there are two steady-state equilibria restricted on $\mathcal{S}_L$ for which t^*_{LH}, t^*_{HL}, $m_{L\emptyset}$, and $m_{H\emptyset}$ are identical. Then, the masses A and B are also equal in both equilibria.*

Proof. Case 1: $t^*_{LH} > 0$. Assume for contradiction that there are two different steady-state equilibria with $(A_1, B_1) \neq (A_2, B_2)$. Without loss of generality assume that $A_1 \leq A_2$ holds.

The masses A_i and B_i satisfy the equations

$$A_i = I_{LH} \int_0^{t^*_{LH}} e^{-(2\lambda A_i + \lambda B_i + \lambda m_{L\emptyset} + \lambda m_{H\emptyset} + 2\delta)t} dt \tag{2.17}$$

$$B_i = I_{LH} e^{-(2\lambda A_i + \lambda B_i + \lambda m_{L\emptyset} + \lambda m_{H\emptyset} + 2\delta)t^*_{LH}} \int_{t^*_{LH}}^{t^*_{HL}} e^{-(\lambda A_i + \lambda B_i + \lambda m_{H\emptyset} + 2\delta)(t - t^*_{LH})} dt. \tag{2.18}$$

If $A_1 \leq A_2$ and $B_1 \leq B_2$ hold with at least one inequality being strict, then equations (2.17) and (2.18) would imply $A_1 > A_2$ and $B_1 > B_2$. By the same argument, $A_1 = A_2$ and $B_1 > B_2$ is not possible.

Thus, $A_1 < A_2$ and $B_1 > B_2$ hold. Now rewrite (A_2, B_2) as $(A_1 + a, B_1 - b)$ and with some $a, b > 0$. The decay for A_2 needs to be smaller than the decay for A_1, i.e.,

$$2\lambda(A_1 + a) + \lambda(B_1 - b) + m_{L\emptyset} + m_{H\emptyset} + 2\delta < 2\lambda A_1 + \lambda B_1 + m_{L\emptyset} + m_{H\emptyset} + 2\delta$$
$$\Leftrightarrow 2a < b.$$

Thus, we get

$$e^{-(2\lambda A_2 + \lambda B_2 + \lambda(m_{L\emptyset} + m_{H\emptyset}) + 2\delta)t^*_{LH}} > e^{-(2\lambda A_1 + \lambda B_1 + \lambda(m_{L\emptyset} + m_{H\emptyset}) + 2\delta)t^*_{LH}},$$

and hence, the decay for B_2 from t^*_{LH} to t^*_{HL} must be larger than for B_1. Therefore,

$$\lambda(A_1 + a) + \lambda(B_1 - b) + \lambda m_{H\emptyset} + 2\delta > \lambda A_1 + \lambda B_1 + \lambda m_{H\emptyset} + 2\delta$$
$$\Leftrightarrow a > b$$

holds. This is a contradiction. Thus, for $t^*_{LH} > 0$ there can be at most one solution for A and B holding everything else fixed.

Case 2: $t^*_{LH} = 0$. By $t^*_{LH} = 0$, we get $A = 0$. Now we show that the equation

$$\delta B + \lambda B^2 + \lambda m_{H\emptyset} B = \eta_{LH}(1 - e^{-(\lambda B + m_{h\emptyset} + 2\delta)t^*_{HL}})$$

has a unique solution in B: The LHS is convex is B and the RHS is concave in B. At $B = 0$, the RHS is larger than the LHS. Thus, both sides cross at most once. They cross at least once by the intermediate value theorem, since for large B the

LHS is larger than 1, which is an upper bound for the RHS. □

Lemma 2.8. *Assume that there are two steady-states* $(t^*_{LH,1}, A_1, B_1) \neq (t^*_{LH,2}, A_2, B_2)$ *restricted on* $\mathcal{S}_L$ *and the variables* I_{LH}*,* t^*_{HL}*,* $m_{L\emptyset}$*, and* $m_{H\emptyset}$ *are the same in both equilibria. Without loss of generality, let* $t^*_{LH,1} \leq t^*_{LH,2}$*. Then,* $t^*_{LH,1} < t^*_{LH,2}$ *and*

$$A_1 < A_2 \quad \wedge \quad B_1 > B_2 \quad \wedge \quad A_1 + B_1 > A_2 + B_2$$

hold.

Proof. By Lemma 2.7, the inequality $t^*_{LH,1} < t^*_{LH,2}$ is strict, because otherwise both steady-states are equal. The remainder of the proof proceeds by a case distinction.

Case 1: $A_1 \geq A_2$ and $A_1 + B_1 \geq A_2 + B_2$ holds. Then,

$$\lambda A_1 + \lambda(A_1 + B_1) + \lambda(m_{L\emptyset} + m_{H\emptyset}) + 2\delta \geq \lambda A_2 + \lambda(A_2 + B_2) + \lambda(m_{L\emptyset} + m_{H\emptyset}) + 2\delta$$

holds, i.e., the decay of A_i is bigger in the first steady-state. Recall, that A_i is equal to the integral

$$I_{LH} \int_0^{t^*_{LH}} e^{-(\lambda A_i + \lambda(A_i+B_i) + \lambda(m_{L\emptyset}+m_{H\emptyset})+2\delta)t} dt.$$

The larger decay in the first steady-state together with a smaller bound $t^*_{LH,1} < t^*_{LH,2}$, implies $A_1 < A_2$. This is a contradiction to our case assumption.

Case 2: $A_1 \geq A_2$ and $A_1 + B_1 < A_2 + B_2$. Then, $B_1 < B_2$ holds. Furthermore, by $t^*_{LH,1} < t^*_{LH,2}$, the only possibility for $A_1 \geq A_2$ is that the decay of A_i is bigger in the second steady-state, i.e., $2A_1 + B_1 < 2A_2 + B_2$. Now, A_i and B_i both have a larger decay in the second steady-state and we have $t^*_{LH,1} < t^*_{LH,2}$. Recall that

$$\begin{aligned} A_i + B_i &= I_{LH} \int_0^{t^*_{LH}} e^{-(\lambda(A_i+m_{L\emptyset}+\lambda(A_i+B_i+m_{H\emptyset})+2\delta)t} dt \\ &\quad + I_{LH} \int_{t^*_{LH}}^{t^*_{HL}} e^{-\lambda(A_i+m_{L\emptyset})t^*_{LH} - (\lambda(A_i+B_i+m_{H\emptyset})+2\delta)t} dt \end{aligned}$$

holds. Applying our observations to this term shows that $A_1 + B_1 > A_2 + B_2$ has to hold and this is a contradiction.

Case 3: $A_1 < A_2$ and $A_1 + B_1 \leq A_2 + B_2$. As in the previous case, this implies $2A_1 + B_1 > 2A_2 + B_2$ and leads to a contradiction.

The only remaining case is $A_1 < A_2$ and $A_1 + B_1 > A_2 + B_2$. This implies that $B_1 > B_2$ also holds. □

Lemma 2.9. *Assume that the strategies are in* $\mathcal{S}_L$*. For fixed* $V_{L\emptyset}(0)$*,* A*, and* B *the function* $W(t_{LH}) := V_{LH}(t_{LH}) - V_{LL}(0)$ *is strictly increasing in* t_{LH} *on the intervall* $[0, t^*_{HL}]$*.*

Proof. We have

$$
\begin{aligned}
W(t_{LH}) = &\int_{t_{LH}}^{t^*_{HL}} u_{LH}(t) e^{-(2\delta+\lambda(B^*+m_H))(t-t_{LH})}\, dt \\
&+ \int_{t^*_{HL}}^{\infty} u_{LH}(t) e^{-(2\delta+\lambda(B^*+m_H))(t^*_{HL}-t_{LH})-2\delta(t-t^*_{HL})}\, dt \\
&+ \frac{\delta+\lambda(A+B+m_H)}{2\delta+\lambda(A+B+m_H)}\Big(1-e^{-(2\delta+\lambda(B^*+m_H))(t^*_{HL}-t_{LH})} \\
&\qquad\qquad + \frac{1}{2}e^{-(2\delta+\lambda(B^*+m_H))(t^*_{HL}-t_{LH})}\Big)V_{L\emptyset}(0) \\
&- \int_0^{\infty} u_{LL}(t)e^{-2\delta t}\, dt - \frac{1}{2}V_{L\emptyset}(0)
\end{aligned}
$$

Now, the derivate of $W(t_{LH})$ with respect to t_{LH} is strictly positive. □

Corollary 2.5. *Fix everything except for t^*_{LH}. Then, there is a unique t^*_{LH} that satisfies the equilibrium condition restricted on $\mathcal{S}_L$. If $W(0) < 0$ holds, then $t^*_{LH} \in (0, t^*_{HL})$ and else, $t^*_{LH} = 0$.*

Proof. This statement follows directly from the previous lemma. □

Lemma 2.10. *Fix a strategy profile in $\mathcal{S}_L$. Fix A, B and let $t_0 < t^*_{HL}$ such that $W(t_0) > 0$ holds. Then, $W(t_0, A+B) = W(t_0)$ with treating A and B as variables is decreasing in $A+B$ at $(t_0, A+B)$.*

Proof. First, note that $V_{LL}(0)$ does not depend on $A+B$. We fix some $\varepsilon > 0$ and show that $V_{LH}(t_0, A+B) > V_{LH}(t_0, A+B+\varepsilon)$ holds, where again A and B are treated as variables for the expected continuation payoffs.

By assumption, $V_{LL}(0) \geq V_{L\emptyset}(0)$ holds. Now, the utility of player L with a given stock $A+B$ conditional on accepting a rematch with probability ε/A until time t^*_{HL} is larger than $V_{LH}(t_0, A+B+\varepsilon)$, since under this strategy, exit rates and flow utility is equal, but the payoff among exiting is larger. However, by $W(t_0) > 0$, rematching with probability zero is optimal and thus we get that $V_{LH}(t_0, A+B)$ is even larger. □

Lemma 2.11. *Consider two steady-states $(t^*_{LH,1}, A_1, B_1) \neq (t^*_{LH,2}, A_2, B_2)$ with $t^*_{LH,1} < t^*_{LH,2}$ and $A_1 + B_1 > A_2 + B_2$. Then, at least one of them does not satisfy the equilibrium condition restricted on $\mathcal{S}_L$.*

Proof. First, $W(t, A+B)$ is continuous in t and in $A+B$. Then, suppose for contradiction that both, $(t^*_{LH,1}, A_1, B_1)$ and $(t^*_{LH,2}, A_2, B_2)$, are steady-state equilibria. Then,

$$W(t^*_{LH,1}, A_1 + B_1) = 0 \text{ and } W(t^*_{LH,2}, A_2 + B_2) = 0.$$

By Lemma 2.9, we get $W(t^*_{LH,2}, A_1 + B_1) > 0$. Now, let

$$\varepsilon := \inf\{x \in \mathbb{R} | W(t^*_{LH,2}, A_1 + B_1 - x) = 0\}.$$

Then, by continuity

$$W(t^*_{LH,2}, A_1 + B_1 - \varepsilon) = 0$$

and

$$W(t^*_{LH,2}, A_1 + B_1 - \varepsilon) = W(t_{L,2}, A_1 + B_1) - \int_{A_1+B_1-\varepsilon}^{A_1+B_1} \frac{dW(t_2, A+B)}{d(A+B)} d(A+B)$$

hold. By Lemma 2.10, the integral is at most zero. Thus,

$$W(t^*_{LH,2}, A_1 + B_1 - \varepsilon) \geq W(t^*_{LH,2}, A_1 + B_1) > 0$$

holds. This is a contradiction. □

Lemma 2.12. *Holding I_{LH}, $V_{L\emptyset}(0)$, and $V_{H\emptyset}(0)$ fixed, there exists a unique steady-state equilibrium on the restricted set $\mathcal{S}_L$.*

Proof. First, we show existence. The steady-state equations imply unique masses (A, B) for every given t^*_{LH}. The function $\phi_1(t^*_{LH}) : [0, t^*_{HL}] \to \mathbb{R}^2$ that maps every t^*_{LH} to such a pair (A, B) is well-defined and continuous. The equilibrium condition implies a unique $t^*_{LH} \in [0, t^*_{HL}]$ for every pair (A, B). The corresponding function $\phi_2 : \mathbb{R}^2 \to [0, t^*_{HL}]$ is again well-defined and continuous. The function $\phi = \phi_2 \circ \phi_1$ is therefore also continuous and defined on a compact interval. By Brouwer's fixed-point theorem, there exists a fixed point of ϕ. Now, if t^*_{LH} is a fixed point of ϕ, then $\big(t^*_{LH}, \phi_1(t^*_{LH})\big)$ is a steady-state equilibrium by construction. The uniqueness is shown by Lemmas 2.7 and 2.11. □

Lemma 2.13. *Holding I_{LH}, $V_{L\emptyset}(0)$, and $V_{H\emptyset}(0)$ fixed, there is a unique steady-state equilibrium on the restricted set $\mathcal{S}_H$.*

Proof. This proof works analogously to the proof of Lemma 2.12. □

Let $V_{L\emptyset}(0, (A, B))$ be the continuation payoff of a single agent of type L in the partial equilibrium with masses (A, B). The following lemma states that this continuation payoff increases when A increases and $A + B$ remains constant or when A remains constant and $A + B$ decreases:

Lemma 2.14. *Let (A, B) and (A', B') be such that $A > A'$ and $A + B < A' + B'$. Then, it holds that*

$$V_{L\emptyset}(0, (A, B)) > V_{L\emptyset}(0, (A', B + A - A')) > V_{L\emptyset}(0, (A', B')).$$

Proof. In order to verify the first inequality, consider the partial equilibrium with masses (A, B). Take an agent of type L, and suppose that this agent can meet other agents of type L only at rate $\lambda(A' + m_L) < \lambda(A + m_L)$ while being in his current match (where he may be single or paired with another agent of either type H or L), but meets them at rate $\lambda(A + m_L)$ in any subsequent match. Compared to the case where she can meet other agents of type L always at rate $\lambda(A+m_L)$, she is worse off because she can rematch less whenever she wants to do so. Hence, her continuation payoff at the beginning of her current match decreases, that is,

$$\begin{aligned} &V_{L\emptyset}(0, [\lambda(A' + m_L), \lambda(A + m_L), \ldots]) < V_{L\emptyset}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots]), \\ &V_{LH}(0, [\lambda(A' + m_L), \lambda(A + m_L), \ldots]) < V_{LH}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots]), \\ &V_{LL}(0, [\lambda(A' + m_L), \lambda(A + m_L), \ldots]) = V_{LL}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots]), \end{aligned} \tag{2.19}$$

where the vector in squared brackets contains the agent's meeting rates in the current and all future matches. (In particular, the continuation payoffs are equal if the agent is currently matched with another agent of type L because the agent never wants to rematch while being in the LL-pair.)

Suppose now that the agent of type L can rematch at rate $\lambda(A' + m_L)$ in her current and the subsequent match, and at rate $\lambda(A+m_L)$ in all matches thereafter. In comparison with the case where she can meet other agents of type L always at rate $\lambda(A + m_L)$, she is now worse off for two reasons: First, she can rematch less whenever he wants to do so in her current match. Second, her continuation payoff when the current match dissolves is smaller due to equations (2.19). This yields

$$\begin{aligned} &V_{\emptyset}(0, [\lambda(A' + m_L), \lambda(A' + m_L), \lambda(A + m_L), \ldots]) < V_{L\emptyset}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots]), \\ &V_{LH}(0, [\lambda(A' + m_L), \lambda(A' + m_L), \lambda(A + m_L), \ldots]) < V_{LH}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots]), \\ &V_{LL}(0, [\lambda(A' + m_L), \lambda(A' + m_L), \lambda(A + m_L), \ldots]) < V_{LL}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots]), \end{aligned}$$

Iterating forward, we obtain that

$$V_{L\emptyset}(0, [\lambda(A' + m_L), \lambda(A' + m_L), \ldots]) < V_{L\emptyset}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots])$$

holds, which leads to the desired inequality because of

$$V_{L\emptyset}(0, [\lambda(A + m_L), \lambda(A + m_L), \ldots]) = V_{L\emptyset}(0, (A, B))$$

and

$$V_{L\emptyset}(0, [\lambda(A' + m_L), \lambda(A' + m_L), \ldots]) = V_{L\emptyset}(0, (A', B + A - A'))$$

due to the fact that the agents of type H meet other agents of the same type at rate $\lambda(m_H + A + B) = \lambda(m_H + A' + (B + A - A'))$ in both partial equilibria with masses (A, B) and $(A', B + A - A')$, respectively.

To show the second inequality, consider an agent of type L in the partial equilibrium with masses (A', B'), and suppose that her next of partner of type H (including her current match) meets other agents of type H only at rate $\lambda(m_H + A' + (B + A - A')) = \lambda(m_H + A + B) < \lambda(m_H + A' + B')$, and all his subsequent partners of type H can meet other agents of that type at rate $\lambda(m_H + A' + B')$. Note that the agent of type L benefits if her first next of partner of type H rematches less because her continuation payoff from remaining in the match with this agent exceeds the continuation payoff from becoming single. Consequently, her continuation payoff at the beginning of the current match increases, i.e.,

$$V_{L\emptyset}(0, [\lambda(m_H + A + B), \lambda(m_H + A' + B'), \ldots]) > V_{L\emptyset}(0, [\lambda(m_H + A' + B'), \lambda(m_H + A' + B'), \ldots])$$
$$V_{LH}(0, [\lambda(m_H + A + B), \lambda(m_H + A' + B'), \ldots]) > V_{LH}(0, [\lambda(m_H + A' + B'), \lambda(m_H + A' + B'), \ldots]),$$
$$V_{LL}(0, [\lambda(m_H + A + B), \lambda(m_H + A' + B'), \ldots]) > V_{LL}(0, [\lambda(m_H + A' + B'), \lambda(m_H + A' + B'), \ldots]),$$

where the vector in squared brackets now contains the meeting rates of the agent's partners of type H in her current and all future matches. As before, forward iteration and the fact that the agents of type L meet other agents of the same type at rate $\lambda(m_L + A')$ in both partial equilibria with masses $(A', B + A - A')$ and (A', B'), respectively, leads to

$$\begin{aligned} V_{L\emptyset}(0, (A', B + A - A')) &= V_{L\emptyset}(0, [\lambda(m_H + A + B), \lambda(m_H + A + B), \ldots]) \\ &> V_{L\emptyset}(0, [\lambda(m_H + A' + B'), \lambda(m_H + A' + B'), \ldots]) \\ &= V_{L\emptyset}(0, (A', B')). \end{aligned}$$

This finishes the proof. □

Let $V_{LL}(0, (A, B))$ denote the continuation payoff of an agent of type L when she is matched into an LL-pair in the partial equilibrium with masses (A, B), and let $V_{LH}(t, (A, B))$ be her continuation payoff when she is matched with an agent of type H for time t in that equilibrium. In the following, we argue that decision optimality requires t_L to decrease in equilibrium when A increases and $A + B$ decreases:

Lemma 2.15. *Consider two partial equilibria with masses (A, B) and (A', B') and respective cutoffs t_{LH} and t'_{LH} such that $A > A'$ and $A + B < A' + B'$. Then,*

$$V_{LH}(t_{LH}, (A, B)) - V_{LL}(0, (A, B)) > V_{LH}(t_{LH}, (A', B')) - V_{LL}(0, (A', B'))$$

holds.

Proof. Let $V_{LH}(t_{LH}, (A, B), \hat{V})$ and $V_{LL}(0, (A, B), \hat{V})$ denote the respective continuation payoffs of an agent of type L if she received continuation payoff $\hat{V}$ upon becoming single in her current match. It follows that

$$V_{LH}(t_{LH}, (A, B)) - V_{LL}(0, (A, B))$$

$$
\begin{aligned}
&= V_{LH}(t_{LH}, (A, B), V_{L\emptyset}(0, (A, B))) - V_{LL}(0, (A, B), V_{L\emptyset}(0, (A, B))) \\
&> V_{LH}(t_{LH}, (A, B), V_{L\emptyset}(0, (A', B'))) - V_{LL}(0, (A, B), V_{L\emptyset}(0, (A', B'))) \\
&> V_{LH}(t_{LH}, (A', B'), V_{L\emptyset}(0, (A', B'))) - V_{LL}(0, (A', B'), V_{L\emptyset}(0, (A', B'))) \\
&= V_{LH}(t_{LH}, (A', B')) - V_{LL}(0, (A', B')),
\end{aligned}
$$

holds. To see the first inequality, recall that $V_{L\emptyset}(0, (A, B)) > V_{L\emptyset}(0, (A', B'))$ holds by Lemma 2.14 and note that the agent becomes single with higher probability in an LH-pair at time t_L than in an LL-pair because in the former the partner of type H rematches on the interval $[t_L, t_H]$ while no agent rematches in the LL-pair. Therefore, the difference $V_{LH}(t_{LH}, (A, B), \hat{V}) - V_{LL}(0, (A, B), \hat{V})$ is increasing in $\hat{V}$. The second inequality is due to

$$V_{LL}(0, (A, B), V_{L\emptyset}(0, (A', B'))) = V_{LL}(0, (A', B'), V_{L\emptyset}(0, (A', B')))$$

and

$$V_{LH}(t_{LH}, (A, B), V_{L\emptyset}(0, (A', B'))) > V_{LH}(t_{LH}, (A', B'), V_{L\emptyset}(0, (A', B'))).$$

In particular, the equality holds true because no agent rematches in an LL-pair. The inequality follows from similar arguments as those used in the proof of Lemma 2.14: Given a continuation payoff amounting to $V_{L\emptyset}(0, (A', B'))$ when becoming single, the agent of type L is better off in an LH-pair at time t_{LH} if her partner of type H can rematch only at a smaller rate. □

To be indifferent at $t = 0$ between LL and LH with never rematching, we need that for $t^*_{LL} = t^*_{LH} = 0$ an L-agent is indifferent between meeting both types, i.e.,

$$V_{LL}(0) = V_{LH}(0)$$

has to hold. Let B^* denote the mass of mixed pairs conditional on L-agents never rematching and H-agents using the cutoff determined as in Corollary 2.3. We know

$$
\begin{aligned}
V_{LL}(0) &= \int_0^\infty u_{LL}(t) e^{-2\delta t}\, dt + \frac{1}{2} V_{L\emptyset}(0) \\
V_{LH}(0) &= \int_0^{t^*_{HL}} u_{LH}(t) e^{-(2\delta + \lambda(B^* + m_H))t}\, dt \\
&\quad + \int_{t^*_{HL}}^\infty u_{LH}(t) e^{-(2\delta + \lambda(B^* + m_H))t^*_{HL} - 2\delta(t - t^*_{HL})}\, dt \\
&\quad + \Big(\frac{\delta + \lambda(A + B + m_H)}{2\delta + \lambda(A + B + m_H)} (1 - e^{-(2\delta + \lambda(B^* + m_H))t^*_{HL}} \\
&\qquad\qquad + \frac{1}{2} e^{-(2\delta + \lambda(B^* + m_H))t^*_{HL}} \Big) \Big) V_{L\emptyset}(0)
\end{aligned}
$$

$$V_{L\emptyset}(0) = \frac{\lambda(A+m_L)}{\delta+\lambda(A+m_L+m_H)}V_{LL}(0) + \frac{\lambda m_H}{\delta+\lambda(A+m_L+m_H)}V_{LH}(0)$$

Plugging $V_{L\emptyset}(0)$ into the other two expressions yields together with some algebra two equations of the form

$$V_{LL}(0) = a + bV_{LH}(0)$$
$$V_{LH}(0) = c + dV_{LL}(0)$$

for some a, b, c, and d. These imply

$$V_{LL}(0) = \frac{a+bc}{1+cd}$$
$$V_{LH}(0) = c + d\frac{a+bc}{1+cd}.$$

Setting $V_{LL}(0)$ and $V_{LH}(0)$ equal, yields after some algebra

$$a = \frac{c+2bcd-bc}{1+d}$$

with

$$a = \Big(1+\frac{\lambda(A+m_L)}{2(\delta+\lambda(A+m_L+m_H))}\Big)^{-1}\int_0^\infty u_{LL}(t)e^{-2\delta t}\,dt$$

and b, c, and d do not depend on $u_{LL}(t)$.

Therefore, we get a cutoff Q^* such that an L-agent is indifferent at $t=0$ if and only if $Q=Q^*$ holds for

$$Q = \int_0^\infty u_{LL}(t)e^{-2\delta}t\,dt.$$

Lemma 2.16. *If there is an equilibrium with $t^*_{LH}>0$, then we have $Q>Q^*$.*

Proof. If there exists such an equilibrium, then an L-agent with an H-partner is indifferent at t^*_{LH}, i.e.,

$$V_{LH}(t^*LH,(A,B,t^*_{LH})) - V_{LL}((A,B,t^*_{LH})) = 0$$

at that equilibrium. By monotonicty, we get

$$V_{LH}(0,(A,B,t^*_{LH})) - V_{LL}((A,B,t^*_{LH})) < 0.$$

Changing (A,B) to the by $t^*_{LH}=0$ induced masses $(0,B^*)$ implies by Lemma 2.15

$$V_{LH}(0,(0,B^*,0)) - V_{LL}((0,B^*,0)) < 0$$

and therefore, we get $Q > Q^*$. $\square$

Lemma 2.17. *If there is a steady-state equilibrium with $t^*_{LL} > 0$, then we have $Q < Q^*$.*

Proof. If there exists such an equilibrium, then an L-agent with an L-partner is indifferent at t^*_{LL}, i.e.,

$$V_{LL}(t^*_{LL}, (A, B, t^*_{LL})) - V_{LH}(0, (A, B, t^*_{LL})) = 0$$

holds. By monotonicity, we get

$$V_{LL}(0, (A, B, t^*_{LL})) - V_{LH}((A, B, t^*_{LL})) < 0.$$

Changing (A, B) to the by $t^*_{LL} = 0$ induced masses $(0, B^*)$ implies:

$$V_{LL}(0, (0, B^*, 0)) - V_{LH}((0, B^*, 0)) < 0$$

and therefore, we get $Q < Q^*$. $\square$

Lemma 2.18. *For every (m_L, m_H) there exists a steady-state equilibrium.*

Proof. Consider the two games with restricted strategies: (1) with the strategies restricted to $\mathcal{S}_L$ and (2) with the strategies restricted to $\mathcal{S}_H$. For both games a unique equilibrium exists by Lemmas 2.12 and 2.13. Consider an equilibrium of game (1). Then, the only possibility that it is not an equilibrium of the whole game is that $t^*_{LH} = 0$ and

$$V_{LL}(0, (0, B^*, 0)) < V_{LH}(0, (0, B^*, 0))$$

hold. Similarly, the only equilibrium of game (2) that is not an equilibrium of the whole game has $t^*_{LL} = 0$ and

$$V_{LL}(0, (0, B^*, 0)) > V_{LH}(0, (0, B^*, 0)).$$

Both cases cannot occur simultaneously. Hence there exists at least one equilibrium of the whole game. $\square$

Let $Q \coloneqq \int_0^\infty u_{LL}(t) e^{-2\delta} t \, dt$ denote the expected flow payoff in an LL-pair where no agents rematches.

Lemma 2.19. *There exists a cutoff Q^* such that*

$$\begin{aligned} t^*_{LL} > 0 \;\wedge\; t^*_{LH} = 0 \quad &\Leftrightarrow \quad Q < Q^* \\ t^*_{LL} = 0 \;\wedge\; t^*_{LH} = 0 \quad &\Leftrightarrow \quad Q = Q^* \\ t^*_{LL} = 0 \;\wedge\; t^*_{LH} > 0 \quad &\Leftrightarrow \quad Q > Q^* \end{aligned}$$

hold, i.e., the comparison between Q and the cutoff Q^* uniquely determines the type of equilibrium.

Proof. Corollary 2.4 says that there are three possible types of equilibria: Lemmas 2.16 and 2.17 show that $Q < Q*$, $Q = Q*$, and $Q > Q^*$ are necessary conditions for the existence for the existence of equilibria of type 1, 2, and 3. In particular, these are mutually exclusive. Lemmas 2.12 and 2.13 show that within each type, there is always a unique equilibrium. Finally, Lemma 2.18 states that there always exists an equilibrium. □

This finishes the proof of Theorem 2.1. □

Proof of Lemma 2.3. Note that the total mass of i-agents in a steady-state equilibrium is determined by the aggregate balance condition, i.e., inflow is equal to outflow. In particular, the total mass of i-agents is $\frac{\eta_i}{\delta}$ and does not depend on λ. Therefore, the rate at which an agent meets other agents also converges to zero. Hence, the payoff with an H-partner converges to

$$V_{ij}(t) = \int_t^\infty u_{ij}(t')e^{-2\delta(t'-t)}dt'$$

and for a single the continuation payoff converges to zero.

With capital accumulation, we get $V_{iL}(0) < V_{iH}(0)$ and the cutoffs $t^*_{LL}, t^*_{HL} \in (0, \infty)$ are given by

$$\int_{t^*_{iL}}^\infty u_{ij}(t')e^{-2\delta(t'-t)}dt' = \int_0^\infty u_{ij}(t')e^{-2\delta(t'-t)}dt'.$$

Therefore, if λ is sufficiently small, all agents prefer matching with H-types.

The cutoff t^*_{HL} does not depend on λ. As λ grows large, the probability that an H-agent meets another H-agent before t^*_{HL} converges to 1. The balance conditions imply the convergence of the corresponding masses. □

2.A.4 Proofs for Section 2.6

Balance Conditions in the Limit Model Without capital accumulation, we obtain from (2.10) and (2.11) that the rate

$$\frac{O_{ij}(t)}{\sum_{\pi\in\Pi(i,j)} m_{\pi(ij)}(t)}$$

is constant in t for all i, j due to the time invariance of the agents' strategies. This yields that

$$q_{ij}(t) = e^{-\frac{O_{ij}}{\sum_{\pi\in\Pi(i,j)} m_{\pi(ij)}}t}$$

for all i, j and all $t \geq 0$, where

$$O_{ij} \equiv \int_0^\infty O_{ij}(t)dt$$

is the total outflow of agents out of ij-pairs. As a consequence, the balance conditions (2.1) are satisfied if and only if the aggregate balance conditions (2.2) hold. Note that the latter ones can be written as

$$\sum_{\pi\in\Pi(i,j)} m_{\pi(ij)} = \int_0^\infty I_{ij} \cdot e^{-\frac{O_{ij}}{\sum_{\pi\in\Pi(i,j)} m_{\pi(ij)}}\cdot t} dt = I_{ij} \cdot \frac{\sum_{\pi\in\Pi(i,j)} m_{\pi(ij)}}{O_{ij}}$$

hold, which is equivalent to

$$I_{ij} = O_{ij}. \tag{2.20}$$

Cumulating the Balance Conditions In the limit model, we obtain

$$M_{i,j} = \sum_{k\in\{L,H,\emptyset\}} p_i(k, j)m_{ik}.$$

The inflow of agents in ij-pairs with $i, j \neq \emptyset$ is as before given by

$$I_{ij} = 2 \cdot \lambda M_{i,j} M_{j,i},$$

and the inflow single i-agents is now

$$I_{i\emptyset} = \eta_i + \delta \sum_{j\in\{L,H\}} m_{ij} + \lambda \sum_{j\in\{L,H\}} m_{ij} \sum_{k\in\{L,H\}} p_j(i, k)M_{k,j}.$$

The outflow of agents out of ij-pairs with $i, j \neq \emptyset$ is

$$O_{ij} = 2 \cdot \delta \sum_{\pi\in\Pi(i,j)} m_{\pi(ij)} + 2 \cdot \lambda \sum_{\pi\in\Pi(i,j)} m_{\pi(ij)} \sum_{k\in\{L,H\}} p_{\pi(i)}(\pi(j), k)M_{k,\pi(i)}, \tag{2.21}$$

and the outflow of single i-agents is given by

$$O_{i\emptyset} = \delta m_{i\emptyset} + \lambda \left(p_i(\emptyset, i) m_{i\emptyset}\right)^2 + \lambda m_{i\emptyset} \sum_{k\in\{L,H\}} p_i(\emptyset, k) M_{k,i}. \tag{2.22}$$

Note that

$$I_{i\emptyset} + I_{ii} + \frac{1}{2} \cdot I_{i\bar{i}} = O_{i\emptyset} + O_{ii} + \frac{1}{2} \cdot O_{i\bar{i}} \tag{2.23}$$

is equivalent to

$$\eta_i = \delta \left(m_{i\emptyset} + m_{ii} + m_{i\bar{i}}\right), \tag{2.24}$$

that is, the inflow of i-agents into the market equals the outflow of i-agents out of the market.[12]

Proof of Lemma 2.4. The expected continuation payoffs of an i-agent (2.4) and (2.3) can be written as

$$V_i(j) = \frac{\sum_{\pi\in\Pi(i,j)} m_{\pi(ij)}}{O_{ij}} \left(u_{ij}(0) + \sum_{k\in\{i,\bar{i},\emptyset\}} r_{i,j\to k} \cdot V_i(k) \right)$$

for all $j \in \{i, \bar{i}, \emptyset\}$, where $u_{i\emptyset}(0) = 0$ and $r_{i,\emptyset\to\emptyset} = 0$. Using (2.21), (2.22), and the definition of the switching rates $r_{i,j\to k}$ yields

$$V_i(j) = \frac{u_{ij}(0) + \sum_{k\in\{i,\bar{i},\emptyset\}} r_{i,j\to k} \cdot V_i(k)}{\delta + \sum_{k\in\{i,\bar{i},\emptyset\}} r_{i,j\to k}}$$

These three equations can be expressed as the linear system of equations

$$\underbrace{\begin{pmatrix} R_{\emptyset\emptyset} & R_{\emptyset i} & R_{\emptyset\bar{i}} \\ R_{i\emptyset} & R_{ii} & R_{i\bar{i}} \\ R_{\bar{i}\emptyset} & R_{\bar{i}i} & R_{\bar{i}\bar{i}} \end{pmatrix}}_{=\boldsymbol{R}} \cdot \begin{pmatrix} V_i(\emptyset) \\ V_i(i) \\ V_i(\bar{i}) \end{pmatrix} = \begin{pmatrix} 0 \\ u_{ii}(0) \\ u_{i\bar{i}}(0) \end{pmatrix},$$

where

$$\boldsymbol{R} = \begin{pmatrix} \delta + r_{i,\emptyset\to i} + r_{i,\emptyset\to\bar{i}} & -r_{i,\emptyset\to i} & -r_{i,\emptyset\to\bar{i}} \\ -r_{i,i\to\emptyset} & \delta + r_{i,i\to\emptyset} + r_{i,i\to\bar{i}} & -r_{i,i\to\bar{i}} \\ -r_{i,\bar{i}\to\emptyset} & -r_{i,\bar{i}\to i} & \delta + r_{i,\bar{i}\to\emptyset} + r_{i,\bar{i}\to i} \end{pmatrix}.$$

It can be shown that $\det(\boldsymbol{R}) > 0$. Hence, the system of equation has a unique

[12] The cumulative balance conditions (2.24) can also be derived for the model with capital accumulation by adding up the aggregate balance conditions as in (2.23).

solution, which is given by

$$V_i(\emptyset) = \frac{1}{\det(\boldsymbol{R})}\left(\left[R_{\emptyset\bar{i}}R_{\bar{i}i} - R_{\emptyset i}R_{\bar{i}\bar{i}}\right]u_{ii}(0) + \left[R_{\emptyset i}R_{i\bar{i}} - R_{\emptyset\bar{i}}R_{ii}\right]u_{i\bar{i}}(0)\right) \tag{2.25}$$

$$V_i(i) = \frac{1}{\det(\boldsymbol{R})}\left(\left[R_{\emptyset\emptyset}R_{\bar{i}\bar{i}} - R_{\emptyset\bar{i}}R_{\bar{i}\emptyset}\right]u_{ii}(0) + \left[R_{\emptyset\bar{i}}R_{i\emptyset} - R_{\emptyset\emptyset}R_{i\bar{i}}\right]u_{i\bar{i}}(0)\right) \tag{2.26}$$

$$V_i(\bar{i}) = \frac{1}{\det(\boldsymbol{R})}\left(\left[R_{\emptyset i}R_{\bar{i}\emptyset} - R_{\emptyset\emptyset}R_{\bar{i}i}\right]u_{ii}(0) + \left[R_{\emptyset\emptyset}R_{ii} - R_{\emptyset i}R_{i\emptyset}\right]u_{i\bar{i}}(0)\right). \tag{2.27}$$

From the equations (2.26) and (2.27), it follows that $V_i(i) \geq V_i(\bar{i})$ if and only if

$$\frac{u_{ii}(0)}{u_{i\bar{i}}(0)} \geq \frac{\delta + r_{i,\emptyset\to i} + r_{i,\emptyset\to\bar{i}} + r_{i,i\to\emptyset}}{\delta + r_{i,\emptyset\to i} + r_{i,\emptyset\to\bar{i}} + r_{i,\bar{i}\to\emptyset}}.$$

□

Proof of Lemma 2.5. Fix some (p_L, p_H). Let $\mathcal{M}$ be the set of mass tuples $\boldsymbol{m} = (m_{L\emptyset}, m_{H\emptyset}, m_{LL}, m_{LH}, m_{HL}, m_{HH}) \in \mathbb{R}^6_+$ satisfying the cumulative balance conditions (2.24), and note that the set $\mathcal{M}$ is non-empty and compact. Moreover, it is convex: For any $\boldsymbol{m}, \boldsymbol{m'} \in \mathcal{M}$ and any $\alpha \in [0,1]$, it holds that $\alpha\boldsymbol{m} + (1-\alpha)\boldsymbol{m'} \in \mathcal{M}$ as

$$\begin{aligned}
&\delta\left((\alpha m_{i\emptyset} + (1-\alpha)m'_{i\emptyset}) + (\alpha m_{ii} + (1-\alpha)m'_{ii}) + (\alpha m_{i\bar{i}} + (1-\alpha)m'_{i\bar{i}})\right) \\
&\qquad = \alpha\cdot\delta(m_{i\emptyset} + m_{ii} + m_{i\bar{i}}) + (1-\alpha)\cdot\delta(m'_{i\emptyset} + m'_{ii} + m'_{i\bar{i}}) \\
&\qquad = \alpha\eta_i + (1-\alpha)\eta_i = \eta_i
\end{aligned}$$

for each $i \in \{L, H\}$.

From the aggregate balance equations (2.20), one can construct a mapping $T : \mathcal{M} \to \mathcal{M}$ in the following way: For each $\boldsymbol{m} \in \mathcal{M}$, let

$$T(\boldsymbol{m}) = \begin{pmatrix} T_{L\emptyset}(\boldsymbol{m}) \\ T_{H\emptyset}(\boldsymbol{m}) \\ T_{LL}(\boldsymbol{m}) \\ T_{LH}(\boldsymbol{m}) \\ T_{HL}(\boldsymbol{m}) \\ T_{HH}(\boldsymbol{m}) \end{pmatrix} = \begin{pmatrix} \mu^I_{L\emptyset} & - \mu^O_{L\emptyset} & + m_{L\emptyset} \\ \mu^I_{H\emptyset} & - \mu^O_{H\emptyset} & + m_{H\emptyset} \\ \mu^I_{LL} & - \mu^O_{LL} & + m_{LL} \\ \mu^I_{LH} & - \mu^O_{LH} & + m_{LH} \\ \mu^I_{HL} & - \mu^O_{HL} & + m_{HL} \\ \mu^I_{HH} & - \mu^O_{HH} & + m_{HH} \end{pmatrix}$$

This mapping is well-defined, particularly because it holds that

$$\delta(T_{i\emptyset}(\boldsymbol{m}) + T_{ii}(\boldsymbol{m}) + \frac{1}{2}\cdot T_{i\bar{i}}(\boldsymbol{m}) = \delta(m_{i\emptyset} + m_{ii} + m_{i\bar{i}}) = \eta_i$$

for each $i \in \{L, H\}$ and all $\boldsymbol{m} \in \mathcal{M}$ implying that $T(\mathcal{M}) \subseteq \mathcal{M}$. Moreover, the mapping is continuous because it is a polynomial map.

So by Brouwer's fixed-point theorem, it has a fixed point, that is, there is a solution to the system of the aggregate balance equations (2.20).

To see the continuity result, let M denote the set of mass tuples $\boldsymbol{m} \in \mathbb{R}^5_+$, and let P denote the set of strategy tuples $\boldsymbol{p} = (p_L, p_H)$. Consider now the correspondence $\Gamma : P \rightrightarrows M$ defined by the solutions to the aggregate balance conditions (2.20), i.e,

$$\Gamma(\boldsymbol{p}) \equiv \{\boldsymbol{m} \in M | \boldsymbol{m} \text{ solves (2.20) for all} (i,j) \in \{L,H\} \times \{L,H,\emptyset\} \text{ given } \boldsymbol{p}.\}.$$

This correspondence is upper hemicontinuous: Take any $\boldsymbol{p} \in P$, any sequence $(\boldsymbol{p}_n)$ in P converging to $\boldsymbol{p}$ and any sequence $(\boldsymbol{m}_n)$ with $\boldsymbol{m}_n \in \Gamma(\boldsymbol{p}_n)$ for all n. Note that the sequence $(\boldsymbol{m}_n)$ is bounded as it lies in $\mathcal{M}$. Hence, it has a convergent subsequence $(\boldsymbol{m}_{n_k})$ with limit $\boldsymbol{m}$. Since $\boldsymbol{m}_{n_k}$ solves the balance conditions (2.20) given $\boldsymbol{p}_{n_k}$ for each k, and since the sum/product of sequences converges to the sum/product of their limits, it follows that $\boldsymbol{m}$ solves the equations (2.20) given $\boldsymbol{p}$, that is, $\boldsymbol{m} \in \Gamma(\boldsymbol{p})$.

To see that this sequence is also lower hemicontinuous, take some $\boldsymbol{p} \in P$ and some $\boldsymbol{m} \in \Gamma(\boldsymbol{p})$. Recall that the balance conditions (2.20) are polynomial in the masses $\boldsymbol{m}$, and note that small changes in $\boldsymbol{p}$ only lead to small changes in the coefficients. Hence, for any $\boldsymbol{p'}$ sufficiently close to $\boldsymbol{p}$, there exists some $\boldsymbol{m'}$ sufficiently close to $\boldsymbol{m}$ that solves the equations given $\boldsymbol{p'}$. □

Proof of Theorem 2.2. Fix some $(p_i(j,j))_{(i,j)\in\{L,H\}^2} \in [0,1]^4$. In order to show the existence of a steady-state equilibrium, recall from Lemma 2.5 that for any strategy profile, there exist masses such that the balance conditions are satisfied. As a consequence, equilibrium existence can only fail if the agents' strategies are not optimal. In particular, we need that $V_i(j) \leq V_i(k)$ if $p_i(j,k) > 0$ and $V_i(j) \geq V_i(k)$ if $p_i(j,k) = 0$ hold.

For each $(p_H(L,H), p_H(H,L)) \in [0,1]^2$, let $\Gamma^L(p_H(L,H), p_H(H,L))$

$$\equiv \left\{ (p_L(L,H), p_L(H,L)) \in [0,1]^2 \;\middle|\; \begin{array}{c} p_L(L,H) \in \arg\max_{p\in[0,1]} \; p \cdot V_L(H) + (1-p) \cdot V_L(L) \\ \text{and} \\ p_L(H,L) \in \arg\max_{p\in[0,1]} \; (1-p) \cdot V_L(H) + p \cdot V_L(L) \end{array} \right\}.$$

Verbally, the correspondence Γ^L describes the set of best responses of the L-agents regarding their rematching behavior to other types of agents in a match given that behavior of the H-agents.

First of all, let's verify that this correspondence is nonempty-valued: Assume for contradiction that

$$\Gamma^L(p_H(L,H), p_H(H,L)) = \emptyset$$

holds for some $(p_H(L,H), p_H(H,L)) \in [0,1]^2$. So in particular, this means that $(0,1) \notin \Gamma^L(p_H(L,H), p_H(H,L))$, that is, $V_L(H) > V_L(L)$ at the point $(0,1)$. Analogously, $(1,0) \notin \Gamma^L(p_H(L,H), p_H(H,L))$ holds, i.e., $V_L(H) < V_L(L)$ at the point $(1,0)$.

Take now any path from the point $(p_L(L,H), p_L(H,L)) = (0,1)$ to the point $(1,0)$. For each strategy profile (p_L, p_H) that corresponds to a point on this path, select a solution $\boldsymbol{m}(p_L, p_H)$ to the respective balance conditions in such a way that the constructed path of solutions is continuous along the path of strategies. This is possible since the set of solutions is continuous by Lemma 2.5. Since the L-agents' value function is jointly continuous in the masses and the agents' strategies (cf. equations (2.4) and (2.3)), there must exist some point $(p_L^*(L,H), p_L^*(H,L))$ on the path between $(0,1)$ and $(1,0)$ at which $V_L(H) = V_L(L)$ by the intermediate value theorem. Hence,

$$(p_L^*(L,H), p_L^*(H,L)) \in \Gamma^L(p_H(L,H), p_H(H,L))$$

holds – a contradiction.

Second, the correspondence Γ^L is continuous: Fix some $(p_H(L,H), p_H(H,L))$ and some sequence $((p_H(L,H)_n, p_H(H,L)_n))$ converging to $(p_H(L,H), p_H(H,L))$ and any sequence $((p_L(L,H)_n, p_L(H,L)_n))$ with

$$(p_L(L,H)_n, p_L(H,L)_n) \in \Gamma^L(p_H(L,H)_n, p_H(H,L)_n)$$

for all n. This sequence is bounded because it lies in the set $[0,1]^2$, it thus has a subsequence

$$((p_L(L,H)_{n_k}, p_L(H,L)_{n_k}))$$

converging to some point $(p_L(L,H), p_L(H,L))$. Moreover, this subsequence has a further subsequence

$$((p_L(L,H)_{n_{k_l}}, p_L(H,L)_{n_{k_l}}))$$

which converges to the point $(p_L(L,H), p_L(H,L))$ where either it holds true that $V_L(H) > V_L(L)$ at all points $((p_H(L,H)_{n_{k_l}}, p_H(H,L)_{n_{k_l}}), (p_L(L,H)_{n_{k_l}}, p_L(H,L)_{n_{k_l}}))$, $V_L(H) < V_L(L)$ holds at all these points, or $V_L(H) = V_L(L)$ holds at all these points.

If $V_L(H) > V_L(L)$ holds true everywhere along the subsubsequece, then we have

$$(p_L(L,H)_{n_{k_l}}, p_L(H,L)_{n_{k_l}}) = (1,0)$$

for all l implying that $(p_L(L,H), p_L(H,L)) = (1,0)$ holds. By the joint continuity of V_L in the agents' strategies and the masses, we obtain $V_L(H) \geq V_L(L)$ at the limit point $((p_H(L,H), p_H(H,L)), (p_L(L,H), p_L(H,L)))$. As a result, we get that

$$(p_L(L,H), p_L(H,L)) \in \Gamma^L(p_H(L,H), p_H(H,L))$$

holds. The proof for the case with $V_L(H) < V_L(L)$ along the subsubsequece works analogously.

Finally, if $V_L(H) = V_L(L)$ everywhere along the subsubsequece, then the continuity of V_L yields that $V_L(H) = V_L(L)$ also holds in the limit. Hence,

$$(p_L(L,H), p_L(H,L)) \in \Gamma^L(p_H(L,H), p_H(H,L))$$

holds because $(p_L(L,H), p_L(H,L))$ is optimal. Consequently, the correspondence is upper hemicontinuous.

Take now some $(p_H(L,H), p_H(H,L))$ and

$$(p_L(L,H), p_L(H,L)) \in \Gamma^L(p_H(L,H), p_H(H,L)).$$

If $V_L(H) > V_L(L)$ at the point $((p_H(L,H), p_H(H,L)), (p_L(L,H), p_L(H,L)))$, then for any $(p'_H(L,H), p'_H(H,L))$ sufficiently close to $(p_H(L,H), p_H(H,L))$ and the corresponding tuple (p'_L, p'_H) with $(p'_L(L,H), p'_L(H,L)) = (1,0)$, there exists some $\boldsymbol{m}(p'_L, p'_H)$, which is sufficiently close to $\boldsymbol{m}(p_L, p_H)$ by Lemma 2.5. So by the continuity of V_L, at the point $((p'_H(L,H), p'_H(H,L)), (p'_L(L,H), p'_L(H,L)))$ we have that $V_L(H) \geq V_L(L)$ holds true so that we can conclude that

$$(p'_L(L,H), p'_L(H,L)) \in \Gamma^L(p'_H(L,H), p'_H(H,L))$$

holds. For the case where at the point $((p_H(L,H), p_H(H,L)), (p_L(L,H), p_L(H,L)))$ $V_L(H) < V_L(L)$ holds, the argument works analogously.

Last, if $V_L(H) = V_L(L)$ at the point $((p_H(L,H), p_H(H,L)), (p_L(L,H), p_L(H,L)))$, then for any $(p'_H(L,H), p'_H(H,L))$ sufficiently close to $(p_H(L,H), p_H(H,L))$, we can find some $(p'_L(L,H), p'_L(H,L))$ sufficiently close to $(p_L(L,H), p_L(H,L))$ such that $V_L(H) = V_L(L)$ at $((p'_H(L,H), p'_H(H,L)), (p'_L(L,H), p'_L(H,L)))$. Thus, Γ^L is lower hemicontinuous.

So far, we know that for any strategy of the H-agents, there exists some strategy for the L-agents and some masses that satisfy the balance conditions such that the L-agents' strategy is optimal. So suppose now that no strategy for the H-agents is optimal given any respective best response of the L-agents and any corresponding masses. This means that $V_H(H) > V_H(L)$ at the point $(p_H(L,H), p_H(H,L) = (0,1)$ and $V_H(H) < V_H(L)$ at the point $(p_H(L,H), p_H(H,L) = (1,0)$ given any best response of the L-agents and any corresponding masses, respectively.

Take now any path between these two points, and note that by the intermediate value theorem, we can find some point $(p^*_L(L,H), p^*_L(H,L))$ on that path such that $V_H(H) = V_H(L)$ at $(p^*_L(L,H), p^*_L(H,L))$ given some optimal strategy of the L-agents and the corresponding masses. The existence of this point is guaranteed because V_H and Γ^L are continuous.

This completes the proof because we have shown that there exist strategies and market masses such that the strategies are mutual best responses and the balance conditions hold. □

Proof of Theorem 2.3. Suppose first that $u_{iL}(0) = u_{iH}(0)$ holds for all $i \in \{L, H\}$.

From Lemma 2.4, one can infer that $V_i(j) > V_i(\bar{j})$ is equivalent to $r_{i,j\to\emptyset} < r_{i,\bar{j}\to\emptyset}$, which holds if

$$p_j(i, L) = 0 \text{ and } p_j(i, H) = 0 \quad \text{and} \quad \left(p_{\bar{j}}(i, L) > 0 \text{ or } p_{\bar{j}}(i, H) > 0\right).$$

Consider now the following two cases:

Case 1: $V_L(L) > V_L(H)$ and $V_H(L) < V_H(H)$: Optimality thus requires that $p_L(L, H) = p_H(H, L) = 0$ and $p_L(H, L) = p_H(L, H) = 1$. Note that we need $r_{L,L\to\emptyset} < r_{L,H\to\emptyset}$ and $r_{H,L\to\emptyset} > r_{H,H\to\emptyset}$. This is sustained in equilibrium if $p_L(L, L) = p_L(L, H) = 0$, $p_H(L, H) = 1$, $p_H(H, H) = p_H(H, L) = 0$ and $p_L(H, H) > 0$.

Case 2: $V_L(L) < V_L(H)$ and $V_H(L) > V_H(H)$: Here, we need that $r_{L,L\to\emptyset} > r_{L,H\to\emptyset}$ and $r_{H,L\to\emptyset} < r_{H,H\to\emptyset}$. This is sustained in equilibrium if $p_H(L, L) = p_H(L, H) = 0$, $p_L(L, H) = 1$, $p_L(H, H) = p_L(H, L) = 0$ and $p_H(H, L) = 1$.

It follows that for all flow payoffs $(u_{iL}(0), u_{iH})_{i\in\{L,H\}}$ so that $\frac{u_{iH}(0)}{u_{iL}(0)}$ is sufficiently close to 1 for all i, there is some equilibrium from each of the two above cases by continuity of the value functions V_L, V_H in the flow payoffs.

Finally, consider the case with $V_L(L) < V_L(H)$ and $V_H(L) < V_H(H)$. By choosing $p_i(L, L) = 1$ and $p_i(H, H) = 0$ for all $i \in \{L, H\}$, this turns out to be an equilibrium for all flow payoffs with $u_{iH}(0) > u_{iL}(0)$.

So there is an open, nonempty set of flow payoffs such that an equilibrium from all three above cases exists. □

Proof of Theorem 2.4. First of all, let's verify the uniqueness results:

Step 1: For an equilibrium with $LL \to LH$ and $HL \to HH$, the balance conditions (2.20) reduce to

$$2\lambda\left(m_{L\emptyset}\right)^2 = 2\delta m_{LL} + 2\lambda m_{LL} m_{H\emptyset} \tag{2.28}$$

$$2\lambda\left(m_{H\emptyset} + m_{HL}\right)^2 = 2\delta m_{HH} \tag{2.29}$$

$$2\lambda m_{H\emptyset}\left(m_{L\emptyset} + m_{HL}\right) = 2\delta\left(m_{LH} + m_{HL}\right) + 2\lambda m_{HL}\left(m_{H\emptyset} + m_{HL}\right) \tag{2.30}$$

$$\begin{aligned}\eta_L + \delta\left(m_{LL} + m_{LH}\right) + \lambda m_{LL} m_{H\emptyset} + \lambda m_{LH}\left(m_{H\emptyset} + m_{HL}\right)&\\ = \delta m_{L\emptyset} + \lambda m_{L\emptyset}\left(2m_{L\emptyset} + m_{H\emptyset}\right)&\end{aligned} \tag{2.31}$$

$$\eta_H + \delta\left(m_{HH} + m_{HL}\right) = \delta m_{H\emptyset} + \lambda m_{H\emptyset}\left(m_{L\emptyset} + m_{LL} + 2m_{H\emptyset} + m_{HL}\right). \tag{2.32}$$

Then, it can be shown that the balance conditions, or equivalently the equations (2.28)-(2.32) and the cumulative balance conditions (2.24), have a unique solution $\boldsymbol{m}$: Note first that for fixed $m_{LH} = m_{HL}$, the masses m_{HH} and $m_{H\emptyset}$ as well as m_{LL}

and $m_{L\emptyset}$ can be uniquely determined from (2.24) for $i = H$ and (2.29) as well as (2.24) for $i = L$ and (2.28), respectively. Consequently, if there existed two different solutions $\boldsymbol{m} \neq \boldsymbol{m'}$, it would hold that $m_{HL} \neq m'_{HL}$. Without loss of generality, suppose that $m_{HL} > m'_{HL}$. But then, the two cumulative balance conditions (2.24) yield that $m_{L\emptyset} + m_{LL} < m'_{L\emptyset} + m'_{LL}$ and $m_{H\emptyset} + m_{HH} < m'_{H\emptyset} + m'_{HH}$, that is, $m_{L\emptyset} < m'_{L\emptyset}$ or $m_{LL} < m'_{LL}$ and $m_{H\emptyset} < m'_{H\emptyset}$ or $m_{H\emptyset} < m'_{H\emptyset}$. From (2.29), , one can then conclude that $m_{H\emptyset} < m'_{H\emptyset}$. With that, we can infer from (2.28) that $m_{L\emptyset} < m'_{L\emptyset}$. But since (2.30) is equivalent to

$$2\lambda m_{H\emptyset} m_{L\emptyset} = 2\delta \left(m_{LH} + m_{HL}\right) + 2\lambda m_{HL} m_{HL},$$

one obtains that

$$\begin{aligned}
2\lambda m_{H\emptyset} m_{L\emptyset} &= 2\delta \left(m_{LH} + m_{HL}\right) + 2\lambda m_{HL} m_{HL} \\
&> \delta \left(m'_{LH} + m'_{HL}\right) + 2\lambda m'_{HL} m'_{HL} \\
&= 2\lambda m'_{H\emptyset} m'_{L\emptyset} \\
&> 2\lambda m_{H\emptyset} m_{L\emptyset}
\end{aligned}$$

holds, which is a contradiction.

Step 2: For an equilibrium with $LL \leftarrow LH$ and $HL \rightarrow HH$, the balance conditions (2.20) reduce to

$$2\lambda \left(m_{L\emptyset} + m_{LH}\right)^2 = 2\delta m_{LL} \tag{2.33}$$

$$2\lambda \left(m_{H\emptyset} + m_{HL}\right)^2 = 2\delta m_{HH} \tag{2.34}$$

$$\begin{aligned}
2\lambda m_{L\emptyset} m_{H\emptyset} &= 2\delta m_{LH} + 2\lambda m_{LH} \left(m_{L\emptyset} + m_{LH}\right) \\
&\quad + 2\lambda m_{HL} \left(m_{H\emptyset} + m_{HL}\right)
\end{aligned} \tag{2.35}$$

$$\eta_L + \delta(m_{LL} + m_{LH}) + \lambda m_{LH} \left(m_{H\emptyset} + m_{HL}\right) = \delta m_{L\emptyset} + \lambda m_{L\emptyset} \left(2m_{L\emptyset} + m_{LH} + m_{H\emptyset}\right) \tag{2.36}$$

$$\eta_H + \delta(m_{HH} + m_{HL}) + \lambda m_{HL} \left(m_{L\emptyset} + m_{LH}\right) = \delta m_{H\emptyset} + \lambda m_{H\emptyset} \left(m_{L\emptyset} + 2m_{H\emptyset} + m_{HL}\right). \tag{2.37}$$

Then, it can be shown that the equations (2.33)-(2.37) and the cumulative balance conditions (2.24), have a unique solution $\boldsymbol{m}$: Again, note first that for fixed $m_{LH} = m_{HL}$, the masses m_{LL} and $m_{L\emptyset}$ as well as m_{HH} and $m_{H\emptyset}$ can be uniquely determined from (2.24) for $i = L$ and (2.33) as well as (2.24) for $i = H$ and (2.34), respectively. So if there existed two solutions $\boldsymbol{m} \neq \boldsymbol{m'}$, it would hold that $m_{HL} \neq m'_{HL}$. Without loss of generality, suppose that $m_{HL} > m'_{HL}$. Then, the cumulative balance conditions (2.24) give us $m_{L\emptyset} + m_{LL} < m'_{L\emptyset} + m'_{LL}$ and $m_{H\emptyset} + m_{HH} < m'_{H\emptyset} + m'_{HH}$. From (2.33) and (2.34), it results that $m_{L\emptyset} < m'_{L\emptyset}$ and $m_{H\emptyset} < m'_{H\emptyset}$. Moreover, one can conclude that

$$m_{L\emptyset} \cdot m_{HL} < m'_{L\emptyset} \cdot m'_{HL} \quad \text{or} \quad m_{H\emptyset} \cdot m_{HL} < m'_{H\emptyset} \cdot m'_{HL}$$

holds by (2.35). Indeed, (2.36) and (2.37) imply that both inequalities must hold since $m_{LL} + m_{LH} > m'_{LL} + m'_{LH}$ and $m_{HH} + m_{HL} > m'_{HH} + m'_{HL}$. Furthermore, equation (2.36) entails that $m_{LH}\left(m_{H\emptyset} m_{HL}\right) < m'_{LH}\left(m'_{H\emptyset} m'_{HL}\right)$.

Since $m_{LH} > m'_{LH}$, we thus have

$$m_{H\emptyset} + m_{HL} < m'_{H\emptyset} + m'_{HL}.$$

Equation (2.24) for $i = H$ then yields that $m_{HH} > m'_{HH}$, whereas equation (2.29) leads to $m_{HH} < m'_{HH}$ – a contradiction.

Let's now prove the existence result: For that, recall from Lemma 2.5 that for any strategy profile, there exist masses such that the balance conditions are satisfied. As a consequence, equilibrium existence can only fail if the agents' strategies are not optimal. In particular, we need that $V_H(H) \geq V_H(L)$ in any equilibrium with $HL \to HH$ and $p_i(j,j) = 0$ for all i, j. To see that this holds true, note that

$$\begin{aligned} r_{H,L\to\emptyset} &= \delta + \lambda p_L(H,L)(m_{L\emptyset} + p_L(H,L) m_{LH}) \\ r_{H,H\to\emptyset} &= \delta \end{aligned}$$

implying that

$$\frac{\delta + r_{H,\emptyset\to L} + r_{H,\emptyset\to H} + r_{H,H\to\emptyset}}{\delta + r_{H,\emptyset\to L} + r_{H,\emptyset\to H} + r_{H,L\to\emptyset}} \leq 1 \leq \frac{u_{HH}(0)}{u_{HL}(0)}.$$

If the L-agents behave according to $LL \to LH$, we obtain

$$\begin{aligned} r_{L,\emptyset\to L} &= \lambda m_{L\emptyset} \\ r_{L,\emptyset\to H} &= \lambda m_{H\emptyset} \\ r_{L,L\to\emptyset} &= \delta + \lambda m_{H\emptyset} \\ r_{L,H\to\emptyset} &= \delta + \lambda(m_{H\emptyset} + m_{LH}). \end{aligned}$$

Thus, we have

$$\frac{u_{LH}(0)}{u_{LL}(0)} \geq \frac{\delta + r_{L,\emptyset\to L} + r_{L,\emptyset\to H} + r_{L,H\to\emptyset}}{\delta + r_{L,\emptyset\to L} + r_{L,\emptyset\to H} + r_{L,L\to\emptyset}}$$

if $\frac{u_{LH}(0)}{u_{LL}(0)}$ is sufficiently large. In particular, this holds true if

$$\frac{u_{LH}(0)}{u_{LL}(0)} \geq \underline{u} \equiv 1 + \lambda \eta_H.$$

Finally, if the L-agents behave according to $LL \leftarrow LH$, we obtain

$$\begin{aligned} r_{L,\emptyset\to L} &= \lambda\left(m_{L\emptyset} + m_{LH}\right) \\ r_{L,\emptyset\to H} &= \lambda m_{H\emptyset} \\ r_{L,L\to\emptyset} &= \delta \\ r_{L,H\to\emptyset} &= \delta + \lambda(m_{H\emptyset} + m_{LH}). \end{aligned}$$

If $\frac{u_{LH}(0)}{u_{LL}(0)} = 1$, we have

$$\frac{u_{LH}(0)}{u_{LL}(0)} = 1 < \frac{\delta + r_{L,\emptyset\to L} + r_{L,\emptyset\to H} + r_{L,H\to\emptyset}}{\delta + r_{L,\emptyset\to L} + r_{L,\emptyset\to H} + r_{L,L\to\emptyset}}.$$

Hence, there is some $\bar{u} > 1$ such that

$$\frac{u_{LH}(0)}{u_{LL}(0)} \leq \frac{\delta + r_{L,\emptyset\to L} + r_{L,\emptyset\to H} + r_{L,H\to\emptyset}}{\delta + r_{L,\emptyset\to L} + r_{L,\emptyset\to H} + r_{L,L\to\emptyset}}$$

if $\frac{u_{LH}(0)}{u_{LL}(0)} \leq \bar{u}$. □

Proof of Theorem 2.5. To prove the first claim of this theorem, fix some limit equilibrium and some corresponding sequence of equilibria

$$\left(\left(u_{ij}^n(t)\right)_{i,j\in\{L,H\},\,t\geq 0}\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\mathcal{M}^n, t_{LH}^n, t_{LL}^n, t_{HL}^n\right)_{n\in\mathbb{N}}.$$

Recall from Lemma 2.1 that

$$\begin{aligned} m_{H,L,H}^n &= 0 \\ m_{H,H,L}^n &= \int_0^{t_{HL}^n} m_{HL}(t)^n \, dt \end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, it can be shown that $t_{HL}^n \to \infty$ as $n \to \infty$: Suppose not, i.e., $t_{HL}^* < \infty$. Since

$$\int_0^\infty u_{HL}^n(t + t_{HL}^n)e^{-2\delta t}dt = \int_0^\infty u_{HH}^n(t)e^{-2\delta t}dt,$$

holds for all n by Corollary 2.3, this equality must be preserved in the limit if t_{HL}^* is finite, that is,

$$\int_0^\infty u_{HL}^* e^{-2\delta t}dt = \int_0^\infty u_{HH}^* e^{-2\delta t}dt.$$

But this is a contradiction because $u_{HL} < u_{HH}$. So in the limit, we get that $m_{H,L,H}^* = 0$ and $m_{H,H,L}^* = m_{HL}^*$. Then, it follows by Definition 2.5 that $p_H(H,L) = 0$ and $p_H(L,H) = 1$.

Next, we can infer from Corollary 2.4 that $t_{LL}^n \cdot t_{LH}^n = 0$ for all n implying that $t_{LL}^* \cdot t_{LH}^* = 0$ in the limit. Consequently, we obtain that $m_{L,H,L}^* = 0$ or $m_{L,L,H}^* = 0$, and thus $p_L(H, L) = 0$ or $p_L(L, H) = 0$.

By the same reasoning, we get that $p_i(j, j) = 0$ for all $i, j \in \{L, H\}$ because $m^n_{i,j,j} = 0$ holds for all n, and therefore $m^*_{i,j,j} = 0$.

For the second statement, fix a pure strategy steady-state equilibrium in the limit model with $p_H(L, H) = 1$, $p_H(H, L) = 0$, and $p_i(j, j) = 0$ for all $i, j \in \{L, H\}$. We now construct a corresponding limit equilibrium. First, fix a sequence $(r_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. Then, define a sequence of utility functions by

$$u^n_{ij}(t) = u_{ij}e^{r_n t}$$

that is strictly decreasing and satisfies condition 1 in Definition 2.4.

Case 1: In the limit equilibrium, we have $p_L(L, H) = 0$. Then, for n sufficiently large, the equilibria along the sequence are unique with $t_{LL} = 0$ and the masses induced by the balance conditions converge to the masses of the limit equilibrium. By continuity, the cutoff t_{LH} converges to zero and the limit equilibrium corresponds to the limit equilibrium of our sequence.

Case 2: In the limit equilibrium, we have $p_L(H, L) = 0$. Then, analogously to case 1, for n sufficiently large, the equilibria along the sequence are unique with $t_{LH} = 0$ and the masses induced by the balance conditions converge to the masses of the limit equilibrium. By continuity, the cutoff t_{HL} converges to zero and the limit equilibrium corresponds to the limit of the equilibria of our sequence. □

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Chapter 3

Partnership Dissolution in a Search Market with On-the-Match Learning

3.1 Introduction

This chapter studies a one-sided matching market with search frictions, on-the-match learning, and on-the match rematching. A continuum of ex-ante homogeneous agents meet each other following a Poisson process and have to decide whether to form a match or not upon meeting. Inside a match, unknown and potentially correlated types are drawn for each agent that specify whether or not that match is beneficial for the corresponding agent. An agent receives an unobserved and constant positive flow payoff if her current match is beneficial for her and a flow payoff of zero otherwise. In particular, a match can be beneficial for one of the two agents and not beneficial for the other one. Agents whose match is not beneficial receive public bad news about their current match according to a Poisson process.

As in the previous chapter, a central assumption is that both agents in a match can search for a new partner. This fundamentally shapes the set of possible equilibria. Not only the payoff inside the match but also the endogenous risk of being left by the partner are important factors for the agents' decision-making. The rematching behavior of the partner affects the continuation value of the current match and, as a consequence, affects the own behavior as well. This establishes an endogenous interest of the agents in the match value for the partner.

Possible applications for this analysis are professional relationships between business partners, athletes who search for a duo partner, or scientists searching for a co-author. In these applications it is plausible that a partner does not find her current match valuable anymore and that she tries to find a more fitting partner. One can abstract from the one-sidedness of the search market and obtain the same results for a corresponding two-sided model. In particular, the trade-offs in this chapter also apply to applications with two market sides like job markets or marriage markets.

We analyze the agents' rematching behavior, the market structure, and the welfare effect of the speed of learning. First, we provide the existence and uniqueness of a steady-state equilibrium. The equilibrium behavior is as follows: For single agents and agents who received bad news about the profitability for themselves it is a dominant strategy to search for a partner. In matches where both agents have not received bad news, the agents do not search for a new partner as not receiving bad news for a period of time makes their belief about their current match type more optimistic than the belief about a potential new match. There are three counteracting effects that determine the equilibrium behavior of an agent whose partner received bad news: The first effect is that the longer the match persists without receiving bad news herself, the more optimistic is the agent about her own type. The second effect is that the agent takes into account that her partner tries to replace her after receiving bad news. The third effect is that due to the partner's bad news the agent updates her belief about her own type. This third effect can change the belief for better or for worse: Depending on whether there is a positive or negative correlation between the unknown types in a pair, bad news for the partner is bad or good news for oneself. Together, these three effects imply that even if an agent herself has not received bad news, the bad news for the partner can cause the agent to try to replace the partner to avoid the risk of becoming single. As a result, there are three cases of how agents whose partners have received bad news behave in equilibrium. First, an agent searches for a new partner if her current partner received bad news. Second, an agent stays in the current match even if her current partner received bad news. Third, an agent whose partner received bad news follows a cutoff strategy, i.e., she searches for a new partner until the current match persists for a certain time and stops searching afterward.

We use comparative statics to show that a faster learning rate is ex-ante beneficial for the agents entering the market if the agents' goals are aligned, i.e., if there is a positive and sufficiently strong correlation of the unknown types in a pair. If both agents in a match are likely to have the same type, then learning this type benefits both of them. Conversely, if there is a sufficiently strong negative correlation, then a faster learning rate ex-ante hurts the agents. If it is likely that there is exactly one agent who profits from the match, then this agent is worse off by a faster learning rate of the types and this utility loss dominates the utility gain of the partner. In particular, with sufficiently strong negative correlation, agents in a pair would strictly prefer that both of them would commit to never rematch if they could do so, which would correspond to a learning rate of zero, i.e., no learning at all.

This section is concluded with an overview of the related literature. The rest of this chapter is organized as follows. Section 3.2 presents the model. In Section 3.3, we define the equilibrium concept of a steady-state equilibrium. In Section 3.4, existence and uniqueness is shown for the subclass of monotone steady-state equilibria. We show in Section 3.5 that the previous restriction to monotone equilibria is without loss. Section 3.6 uses comparative statics to analyze the effect of a faster learning rate on the agents' welfare. Section 3.7 concludes. The proofs can be found in Appendix 3.A.

3.1.1 Related Literature

Our model builds on the search framework developed by Burdett and Coles (1997), Shimer and Smith (2000), Smith (2006), and Chapter 2 of this thesis. In these models, having a high type results in a higher flow utility for all potential partners. In this chapter, agents are ex-ante homogeneous and draw a new type each time they form a new match. Therefore, here, a high long-term potential does not persist outside of the current match.

The assumption of ex-ante homogeneous agents that has led to this chapter was inspired by Smith (1995) who presents a search-and-exchange market for ex-ante homogeneous goods where the valuations for the goods are drawn independently at each meeting. In Smith (1995), agents who meet can exchange goods and separate afterward while in our model matched agents form a pair, and their future utilities also crucially depend on their partners' rematching decisions.

On-the-job search in labor markets has been widely analyzed before. Pissarides (1994) introduces search equilibria with on-the-job search. An important assumption is that only workers can search for new jobs while being employed. For a survey on search models of the labor market, see Rogerson, Shimer, and Wright (2005). In contrast to on-the-job search, in our model, both agents in a match can continue searching which is central to our results. Both agents in a pair can rematch and the resulting equilibrium strategies have to be optimal given the partner's future rematching decisions.

A related strand of literature has studied partnership dissolution where two agents jointly own an asset. Cramton, Gibbons, and Klemperer (1987) show that an ex-post efficient dissolution is possible if the shares of the asset are sufficiently even. Fieseler, Kittsteiner, and Moldovanu (2003) study interdependent valuations and analyze when efficient trade can occur. In recent work, Loertscher and Wasser (2019) study partnership dissolution with interdependent values and derive optimal ownership structures. Van Essen and Wooders (2016) introduce a dynamic auction format to dissolve partnerships. While this strand of literature analyzes how to dissolve a partnership efficiently, we endogenize the question of when to dissolve a partnership by modeling a search market and embedding the partnerships into the market. Also, in our model, there is no jointly owned asset to be divided for the dissolution of a partnership. Fershtman and Szabadi (2020) study a related question and also consider an endogenous partnership dissolution. In contrast to our model, they analyze a single pair of agents with private information about the joint desirability of the partnership who are not ex-ante sure whether or not to dissolve their partnership.

3.2 The Model

We construct a one-sided search model with continuous time and non-transferable utility where agents learn and search on-the-match. There is a continuum of ex-ante homogeneous agents in the market. Every agent is either single or in a match with another agent. New agents enter the market as singles at a constant rate $\eta > 0$ and agents in the market meet each other following a quadratic meeting technology with parameter $\lambda > 0$, that is, each agent meets an agent from a mass m in the market uniformly at random with Poisson rate λm. When two agents meet, both of them have to simultaneously decide whether to accept or decline forming a new match. If both agents agree, they form a new pair and leave their respective partners (if they are matched) who become singles.

After two agents form a pair, a hidden binary type (h or l) is drawn for each of them that indicates the desirability of the current match. The probabilities for the types are $p_{hh} > 0$ for (h, h), $p_{hl} > 0$ for (l, h) and (h, l), respectively, and $p_{ll} > 0$ for (l, l). This allows the types to be correlated. We call pairs where the types are (l, h) or (h, l) mixed pairs.

An h-agent gains an unobserved constant flow utility of $w > 0$ while being matched with her partner. A single agent or a matched l-agent gains a flow utility of 0. All agents discount future payoffs at rate r. For example, an h-agent whose match is dissolved after time t_0 receives an (unobserved) aggregated payoff of

$$\int_0^{t_0} w \cdot e^{-rt} dt = \frac{w}{r}(1 - e^{-rt_0})$$

in that match.

If the hidden type of an agent is l, then the agent will receive bad news about the current match due to a Poisson process at rate γ. The occurrence of bad news is publicly observable by both agents in that match. If the hidden type of an agent is h, then the agent will never receive bad news. Therefore, bad news fully reveal that the type of the corresponding agent is l. For a matched agent, we denote the (public) information about whether or not bad news occured in the current match by $(S, S') \in \{B, U\} \times \{B, U\}$, where $S = B$ if and only if the agent has received bad news herself and $S' = B$ if and only if the partner has received bad news (B standing for "bad news" and U standing for "unknown"). For the remainder of this chapter, we use lower-case letters like $i, j \in \{l, h\}$ for hidden types and upper-case letters like $S, S' \in \{U, B\}$ for the public information.

Beliefs If an agent is in a match without receiving bad news for a period of time, she adjusts her belief accordingly. Let $P(ij|SS', t)$ denote the belief that the hidden types are ij given that the information is (S, S') and given that the pair is together for time t. For the information (B, B) we know that the type is ll for sure. The other conditional beliefs of the agents about the hidden types can be calculated by the Bayesian rule. For the information (U, U) we get the beliefs

$$P(hh|UU, t) = \frac{p_{hh}}{p_{hh} + 2p_{hl}e^{-\gamma t} + p_{ll}e^{-2\gamma t}},$$

$$P(hl|UU, t) = P(lh|UU, t) = \frac{p_{hl}e^{-\gamma t}}{p_{hh} + 2p_{hl}e^{-\gamma t} + p_{ll}e^{-2\gamma t}},$$

$$P(ll|UU, t) = \frac{p_{ll}e^{-2\gamma t}}{p_{hh} + 2p_{hl}e^{-\gamma t} + p_{ll}e^{-2\gamma t}},$$

and for the information (U, B) we get the beliefs

$$P(hl|UB, t) = \frac{p_{hl}}{p_{hl} + p_{ll}e^{-\gamma t}},$$

$$P(ll|UB, t) = \frac{p_{ll}e^{-\gamma t}}{p_{hl} + p_{ll}e^{-\gamma t}}.$$

Figure 3.1 illustrates the change of beliefs over time.

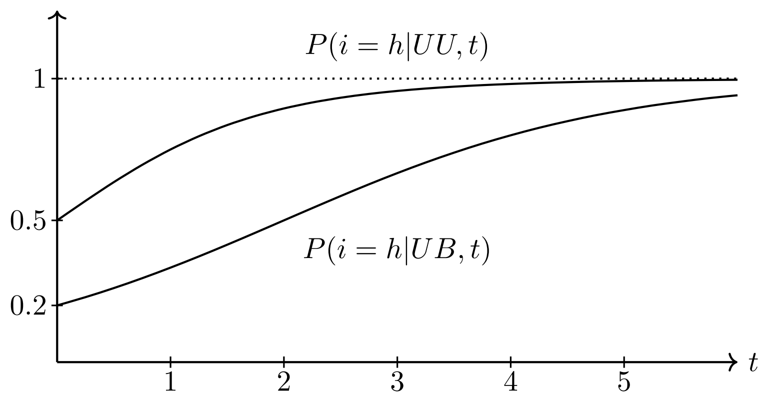

Figure 3.1: Conditional beliefs in pairs that have not received bad news for time t are strictly increasing. The parameters used for the graphs are $\gamma = \ln(2)$, $p_{hh} = 0.4$, $p_{hl} = 0.1$, and $p_{ll} = 0.4$.

The beliefs for the information (B, U) are analogous to the ones for (U, B). Note that receiving no bad news over a period of time is generally good news for the agents, as the belief of having an h-type is increasing over time. In particular,

$$P(i = h|UU, t) = P(hh|UU, t) + P(hl|UU, t) \quad \text{and}$$
$$P(i = h|UB, t) = P(hl|UB, t)$$

are both strictly increasing in t. Analogously, the belief that the partner has an h-type is also increasing over time. For the remainder of this chapter, when we write that the beliefs are increasing over time, we refer to the beliefs $P(i = h|UU, t)$, $P(i = h|UB, t)$, and $P(j = h|UU, t)$ being strictly increasing in t.

Strategies and Masses Note that the belief at time t is independent of the time at which bad news occurred, since bad news fully reveal the state of the corresponding agent and without bad news the conditional probability of having an h-type only depends on the total amount of time without bad news. Therefore, we restrict our attention to symmetric Markov strategies that only condition on the current information type. A Markov strategy for an agent is a measurable function

$$\varphi : \{\emptyset, UU, UB, BU, BB\} \times [0, \infty) \to [0, 1]$$

where $\varphi(SS', t)$ is the probability that an agent who is in a match for exactly time t and for whom the information in the current match is (S, S') accepts to rematch upon meeting another agent. Similarly, $\varphi(\emptyset, t)$ is the probability that an agent who is single for exactly time t agrees to match.

Let $m_{ij,SS'}(t)$ denote mass of agents in pairs which are together for exactly time $t \in [0, \infty)$, who have type i, whose partner has type j, and whose information is SS'. Let Θ denote

$$\big\{\emptyset, (hh, UU), (hl, UU), (hl, UB), (lh, UU), (lh, BU), (ll, UU), (ll, UB), (ll, BU), (ll, BB)\big\},$$

i.e., the set of all indices of matches that can occur, where $\emptyset$ denotes singles. Let

$$\mathcal{M} := \big(m_\theta(t)\big)_{t \geq 0, \theta \in \Theta}$$

denote the vector of all masses. For $\theta \in \Theta$ let

$$m_\theta := \int_0^\infty m_\theta(t) dt \in \overline{\mathbb{R}}_+$$

denote the aggregated mass of such agents in the market.[1] The aggregated mass of

[1]Note that, in general, the aggregated mass could be infinite.

agents who accept forming a new match is given by

$$m_0 := \int_0^\infty m_\emptyset(t)\varphi(\emptyset,t)\ dt \ + \sum_{(ij,SS')\in\Theta\setminus\{\emptyset\}} \int_0^\infty m_{(ij,SS')}(t)\varphi(SS',t)\ dt.$$

The term is derived by integrating over all masses of agents times their respective probability of accepting. In particular, λm_0 is the rate of the Poisson process with which an individual agent meets accepting agents.

Survival Probabilities For fixed masses and for $\theta \in \Theta$ let $q_\theta(t_0,t_1)$ denote the survival probability from t_0 to t_1. More precisely, $q_{(ij,SS')}(t_0,t_1)$ is the probability that an ij-pair with information (S,S') that is together for exactly time t_0 is still together after time t_1 without changing its information. Similarly, $q_\emptyset(t_0,t_1)$ is the respective probability that a single who is single for time t_0 is single for time t_1. Formally, the probabilities are

$$\begin{aligned} q_\emptyset(t_0,t_1) &= \exp\Big(-\int_{t_0}^{t_1} \lambda m_0\varphi(\emptyset,t)\ dt\Big), \\ q_{(ij,SS')}(t_0,t_1) &= \exp\Big(-\int_{t_0}^{t_1} \lambda m_0\varphi(SS',t) + \lambda m_0\varphi(S'S,t) + \big(\mathbb{1}_{i=l}\mathbb{1}_{S=U} + \mathbb{1}_{j=l}\mathbb{1}_{S'=U}\big)\gamma\ dt\Big), \end{aligned}$$

since agents form new matches following an inhomogeneous Poisson process with the corresponding rate $\lambda m_0\varphi(\cdot,t)$. The term $\mathbb{1}_{i=l}\mathbb{1}_{S=U} + \mathbb{1}_{j=l}\mathbb{1}_{S'=U} \in \{0,1,2\}$ denotes the number of agents in the match who can still receive bad news.

Without knowing the hidden types, the expected survival probability of a match with information (S,S') is

$$q_{SS'}(t_0,t_1) = \sum_{i,j\in\{h,l\}} P(ij|SS',t_0)\ q_{(ij,SS')}(t_0,t_1),$$

where $P(ij|SS',t_0)$ denotes the belief of the pair having types ij.

Continuation Payoffs Fix a vector of masses $\mathcal{M}$ and a strategy φ with $m_0 < \infty$. Assume for now that the masses do not change over time. Then, the expected continuation payoffs are well-defined. Let $V(\emptyset,t_0)$ be the expected continuation payoff of an agent who are single for exactly time t_0 and let $V(SS',t_0)$ be the expected continuation payoffs of an agent that is in a match for exactly time t_0 and whose information is (S,S'). In particular, $V(UU,0)$ is equal to the expected utility of forming a new match.

The expected continuation payoff $V(\cdot,t_0)$ can be constructed from the following components: All future flow payoffs in the current match are discounted by

$$q_{SS'}(t_0,t)e^{-r(t-t_0)},$$

i.e., by the survival rate multiplied by the discount factor for time t. The expected flow payoff in the current match is equal to w times the belief of having an h-type:

$$w\big(P(hh|SS',t) + P(hl|SS',t)\big).$$

The rate of accepting a new match multiplied by the corresponding continuation payoff is

$$\lambda m_0 \varphi(SS',t)V(UU,0)$$

and the rate of becoming single multiplied by the continuation payoff of being single is

$$\lambda m_0 \varphi(S'S,t)V(\emptyset,0).$$

The term

$$\mathbb{1}_{S=U}\big(P(lh|SS',t) + P(ll|SS',t)\big)\gamma V(BS,t)$$

describes the rate at which the agent oneself receives bad news times the continuation payoff after that event. Similarly,

$$\mathbb{1}_{S'=U}\big(P(hl|SS',t) + P(ll|SS',t)\big)\gamma V(SB,t)$$

is the rate at which the partner receives bad news times the continuation payoff.

Now, the expected continuation payoff $V(SS',t_0)$ can be calculated recursively by integrating over the expected future payoffs as follows: For all $t_0 \geq 0$,

$$\begin{aligned}
V(SS',t_0) = \int_{t_0}^{\infty} q_{SS'}(t_0,t)e^{-r(t-t_0)} \Big(& w\big(P(hh|SS',t) + P(hl|SS',t)\big) \\
& + \lambda m_0 \varphi(SS',t)V(UU,0) + \lambda m_0 \varphi(S'S,t)V(\emptyset,0) \\
& + \mathbb{1}_{S=U}\big(P(lh|SS',t) + P(ll|SS',t)\big)\gamma V(BS,t) \\
& + \mathbb{1}_{S'=U}\big(P(hl|SS',t) + P(ll|SS',t)\big)\gamma V(SB,t)\Big)\, dt, \\
V(\emptyset,t_0) = \int_{t_0}^{\infty} q_{\emptyset}(t_0,t)e^{-r(t-t_0)} & \lambda m_0 \varphi(\emptyset,t)V(UU,0)\, dt
\end{aligned}$$

holds.

3.3 Steady-State Equilibria

We split our equilibrium concept into two parts. The first part is the mutual optimality of the strategies. The second part requires the masses to satisfy certain balance conditions.

We now begin with the first part, the optimality.

Definition 3.1. The pair $(\mathcal{M}, \varphi)$ constitutes a *partial equilibrium* if m_0 is finite and φ is mutually optimal taking the masses as given, i.e., if for all public information $SS' \in \{\emptyset, UU, UB, BU, BB\}$ and $t \geq 0$

$$V(UU, 0) < V(SS', t) \;\Rightarrow\; \varphi(SS', t) = 0$$
$$V(UU, 0) > V(SS', t) \;\Rightarrow\; \varphi(SS', t) = 1$$

holds.

Taking the masses and the strategies of the other agents as given and constant over time, as well as the own strategy in the future[2], fixes the continuation payoffs $V(UU, 0)$ for accepting to form a new match and $V(SS', t)$ of not accepting. If the expected value of a new match is strictly larger than the continuation payoff of the current state, then an agent accepts. Conversely, if the expected value of a new match is strictly smaller, then an agent rejects.

Note that this equilibrium concept includes sequential rationality. In particular, agents are not allowed to play non-optimal even on a measure null set or if the partner would accept them with probability 0. When an agent is indifferent, i.e., at a time t with $V(UU, 0) = U(SS', t)$, then she can accept with any probability $q \in [0, 1]$.

In any partial equilibrium, singles and agents who received bad news always accept to form a new match. As their current state yields a flow payoff of 0, their only possible payoff comes from forming a new match. Consequently, as the equilibrium strategy of an agent with bad news is constant, the corresponding partner has to follow a monotone equilibrium strategy.

Lemma 3.1. *In all partial equilibria $\varphi(\emptyset, t) = 1$, $\varphi(BU, t) = 1$, and $\varphi(BB, t) = 1$ hold for all t. Furthermore, agents who have not received bad news, but whose partners have received bad news follow a cutoff strategy, that is, they accept to rematch until some cutoff $t^* \in [0, \infty]$ and they do not accept afterwards.*

For the second part of our equilibrium concept, the masses need to satisfy balance conditions for every state. In short, for each type $\theta \in \Theta$, the masses have to be equal to the inflow times the survival probability, i.e.,

$$m_\theta(t) = \text{Inflow}(\theta) \;\cdot q_\theta(0, t)$$

has to hold with $\text{Inflow}(\theta)$ being the inflow of new agents into state θ due the matching process or new market entries of singles. More precisely:

[2]By the One-Shot Deviation Principle, it is sufficient the require pointwise optimality of the strategies.

Definition 3.2. A partial equilibrium $(\mathcal{M}, \varphi)$ is a *steady-state equilibrium* if for all $t \in [0, \infty)$ the pointwise balance conditions

$$
\begin{aligned}
m_{\emptyset}(t) &= (\eta + \lambda m_0(m_0 - m_{\emptyset}))q_{\emptyset}(0,t) \\
m_{hh,UU}(t) &= p_{hh}\lambda m_0^2 \; q_{hh,UU}(0,t) \\
m_{hl,UU}(t) &= 2p_{hl}\lambda m_0^2 \; q_{hl,UU}(0,t) \\
m_{ll,UU}(t) &= p_{ll}\lambda m_0^2 \; q_{ll,UU}(0,t) \\
m_{hl,UB}(t) &= \int_0^t \gamma m_{hl,UU}(t') \; q_{hl,UB}(t',t) \; dt' \\
m_{ll,UB}(t) &= \int_0^t 2\gamma m_{ll,UU}(t') \; q_{ll,UB}(t',t) \; dt' \\
m_{ll,BB}(t) &= \int_0^t \gamma m_{ll,UB}(t') \; q_{ll,BB}(t',t) \; dt'
\end{aligned}
$$

hold.

The balance conditions state that the masses $\mathcal{M}$ together with the strategy φ and the quadratic meeting technology imply the same masses $\mathcal{M}$ again. This ensures that the masses remain stationary in equilibrium.

3.4 Characterization of Monotone Equilibria

The belief of having a high type as well as the belief of the partner having a high type both increase over the time in a match. Therefore, agents become more optimistic the longer a match persists without bad news. In the following, we analyze monotone equilibria where agents willingness to accept to rematch decreases as their beliefs increase. Later we will show that there exist in fact no non-monotone steady-state equilibria. Therefore, it is without loss to restrict attention to monotone equilibria.

Definition 3.3. A partial equilibrium/steady-state equilibrium $(\mathcal{M}, \varphi)$ is called *monotone* if the acceptance probability $\varphi(SS', t)$ is weakly decreasing in t for every information (S, S').

Since single agents and agents who have received bad news always accept to rematch, the only equilibrium behaviors to be specified are the ones for agents with information UU and UB. The next lemma says that in a monotone equilibrium, agents with information UU never accept to match with a new partner.

Lemma 3.2. *In all monotone partial equilibria $\varphi(UU, t) = 0$ holds for all $t > 0$.*

The reason is that the continuation payoff in a match without bad news is strictly increasing over time as the beliefs get more optimistic and the probability of being left by the partner is non-increasing.

Knowing the agents' equilibrium behavior simplifies the balance conditions. More precisely, for a partial equilibrium to satisfy the infinite set of pointwise balance conditions it is necessary and sufficient to satisfy a finite number of aggregate balance conditions. This reduction of the balance conditions to a finite set of equations is a substantial simplification. In particular, for any given cutoff t^* one can obtain the masses numerically.

Lemma 3.3. *A monotone steady-state equilibrium satisfies $(\mathcal{M}, \varphi)$ the aggregated balance conditions. For generic parameters*[3] *the aggregated balance conditions are:*

$$\begin{aligned}
m_\emptyset &= \frac{\eta + \lambda m_0^2}{2\lambda m_0} \\
m_{hh,UU} &= \infty \\
m_{hl,UU} &= \frac{2p_{hl}\lambda m_0^2}{\gamma} \\
m_{ll,UU} &= \frac{p_{ll}\lambda m_0^2}{2\gamma} \\
m_{hl,UB,\le t^*} &= p_{hl}m_0\Big(1 + \frac{2\lambda m_0}{\gamma - 2\lambda m_0}e^{-\gamma t^*} - \frac{\gamma}{\gamma - 2\lambda m_0}e^{-2\lambda m_0 t^*}\Big) \\
m_{hl,UB,>t^*} &= p_{hl}m_0\Big(\frac{2\gamma}{\gamma - 2\lambda m_0}e^{-2\lambda m_0 t^*} - \frac{4\lambda m_0}{\gamma - 2\lambda m_0}e^{-\gamma t^*}\Big) \\
m_{ll,UB,\le t^*} &= p_{ll}m_0\Big(\frac{2\lambda m_0}{\gamma + 2\lambda m_0} + 2\lambda m_0 e^{-2\gamma t^*} + \frac{4\gamma\lambda m_0}{\gamma + 2\lambda m_0}e^{-(2\lambda m_0+\gamma)t^*}\Big) \\
m_{ll,UB,>t^*} &= p_{ll}m_0\Big(\frac{4\gamma\lambda m_0}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)}e^{-(2\lambda m_0+\gamma)t^*} - \frac{2\lambda m_0(\gamma + 2\lambda m_0)}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)}e^{-2\gamma t^*}\Big) \\
m_{ll,BB} &= p_{ll}m_0\Big(\frac{\gamma - 2\lambda m_0}{2(\gamma + 2\lambda m_0)} - \frac{\lambda m_0(\gamma + 2\lambda m_0 - 2\gamma^2 + 2\gamma\lambda m_0 + 4\lambda^2 m_0^2)}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)}e^{-2\gamma t^*} \\
&\qquad - \frac{2\gamma\lambda m_0(\gamma + 2\lambda m_0 + 2\gamma^2 - 2\gamma\lambda m_0 - 4\lambda^2 m_0^2)}{(\gamma + 2\lambda m_0)(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)}e^{-(2\lambda m_0+\gamma)t^*}\Big) \\
m_0 &= m_\emptyset + m_{hl,UB,\le t^*} + \frac{1}{2}m_{hl,UB,>t^*} + m_{hl,UB,\le t^*} + \frac{1}{2}m_{ll,UB,>t^*} + m_{ll,BB}
\end{aligned}$$

Conversely, if the aggregated masses of a monotone partial equilibrium $(\mathcal{M}, \varphi)$ satisfy the aggregated balance conditions, then there exists a unique monotone steady-state equilibrium $(\mathcal{M}', \varphi)$ that has the same aggregated masses as $(\mathcal{M}, \varphi)$.

These aggregate balance conditions are obtained by integrating the pointwise balance conditions. As a direct consequence, the aggregate balance conditions are necessary for the pointwise balance conditions. The last equation gives the mass of all agents who are willing to accept a new match. The factor $\frac{1}{2}$ of the masses

[3]With generic, we here mean that $\gamma \neq \lambda m_0$ and $\gamma \neq 2\lambda m_0$ hold. This is without loss as the statement of Lemma 3.3 also holds for non-generic parameters but with different terms for the masses.

$m_{hl,UB,>t^*}$ and $m_{ll,UB,>t^*}$ accounts for the fact that only half of the agents in such pairs are willing to rematch. The crucial part of Lemma 3.3 is that the aggregate balance conditions are also sufficient. The proof idea is that the aggregate masses determine the mass m_0 of agents who search for a new partner and the mass m_0 determines the pointwise masses.

As a consequence of Lemma 3.3, we get the following equations which correspond to the more commonly known balance conditions of the form “Inflow equals Outflow”.

Corollary 3.1. *For any $(\mathcal{M}, \varphi)$ that satisfies the aggregated balance conditions, the equations*

$$2p_{hl}\lambda m_0^2 = \lambda m_0(2m_{hl,UB,\le t^*} + m_{hl,UB,>t^*})$$

$$p_{ll}\lambda m_0^2 = \lambda m_0(2m_{ll,UB,\le t^*} + m_{ll,UB,>t^*} + 2m_{ll,BB})$$

hold.

Corollary 3.1 says that the total inflow rate of agents into hl-pairs ($2p_{hl}\lambda m_0^2$) is equal to the total outflow rate ($2\lambda m_0$ times the the number of agents who accept to rematch). The analogue also holds for ll-pairs while Lemma 3.3 shows the same result for singles.

In a monotone steady-state equilibrium, every pair except for hh-pairs eventually dissolves. This allows us to calculate the mass m_0 of agents who accept forming a new match, without knowing the specific equilibrium cutoff t^*.

Lemma 3.4. *In every monotone steady-state equilibrium*

$$m_0 = \sqrt{\frac{\eta}{\lambda p_{hh}}}$$

holds.

Lemma 3.4 uniquely determines the rate λm_0 for a given choice of parameters. Intuitively, the balance conditions imply that the inflow into the market is equal to the rate at which agents enter an absorbing state, i.e., the rate at which hh-pairs meet. Thus, $\eta = p_{hh}\lambda m_0^2$ holds. Formally, adding the aggregated balance conditions yields this result.

Our next lemma shows that for the given m_0 there is a unique cutoff t^* with a corresponding monotone steady-state equilibrium. This gives uniqueness in the class of monotone steady-state equilibria. When we talk about uniqueness, we formally mean that the masses in the steady-state equilibrium are uniquely given and the the strategies are unique up to a (finite) measure zero set of points (more precisely, in equilibrium only the agents' acceptance probabilities $V(UU, 0)$ and $V(UB, t^*)$ upon being indifferent can be arbitrary).

Lemma 3.5. *There exists a unique monotone steady-state equilibrium.*

This uniqueness result on the class of all monotone steady-state equilibria is in fact without loss as we will show in the next section.

3.5 Equilibrium Uniqueness

We now analyze the structure of non-monotone partial equilibria and show that those equilibria do not satisfy the balance conditions. Therefore there can only exist monotone steady-state equilibria and by this we get uniqueness for the class of all steady-state equilibria.

The next proposition characterizes the non-monotone partial equilibria.

Proposition 3.1. *In any non-monotone partial equilibrium there exists a* $t_0 \in [0, \infty)$ *with*

$$V(UU,t) > V(UU,0) \text{ for all } t \in (0,t_0) \text{ and}$$
$$V(UU,t) = V(UU,0) \text{ for all } t \geq t_0.$$

Furthermore, $\varphi(UU,t)$ *is strictly increasing after* t_0*.*

In the first part of the proof it is shown that $V(UU,t)$ cannot go below $V(UU,0)$. The second part of the proof shows that once $V(UU,t) = V(UU,0)$ holds for any $t > 0$, then it also holds for all larger t. A key argument for both parts is that agents become more optimistic over time and if all other circumstances are equal for two different points in time, then the later point needs to have a higher continuation payoff. Figure 3.2 illustrates the continuation payoff $V(UU,t)$ for non-monotone partial equilibria.

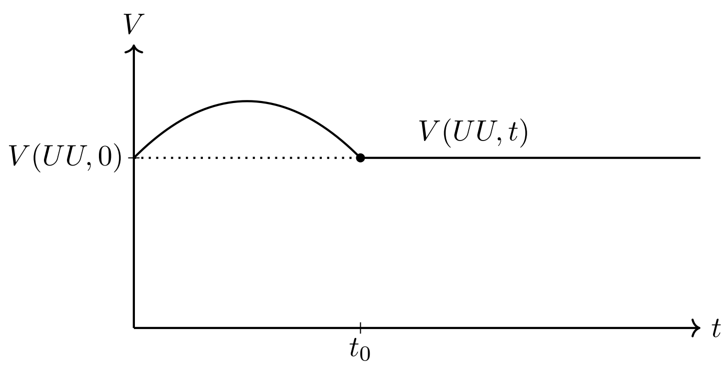

Figure 3.2: Continuation payoff for non-monotone partial equilibria

As long as $V(UU,t)$ is larger than $V(UU,0)$, agents with information (U,U) do not accept to rematch. After t_0, the continuation payoff of not accepting is equal to the payoff of accepting. Therefore, agents follow a mixed strategy after t_0 and they mix with strictly increasing probability to keep their partners indifferent. Since both agents in such a match would strictly prefer that both agents do not accept to rematch, this can be interpreted as a coordination failure.

The next theorem states that a non-monotone equilibrium cannot be a steady-state equilibrium.

Theorem 3.1. *There exists a unique steady-state equilibrium and it is monotone.*

This shows that our restriction to monotone equilibria and our analysis of them are without loss of generality. In particular, in the last section, we have analyzed the equilibrium structure of the unique steady-state equilibrium.

3.6 The Role of Learning

In this section, we investigate the impact of the learning rate on the agents. We use comparative statics to analyze the welfare effects of a faster (or slower) learning rate γ.

The following lemma considers the case of a strong positive correlation[4]. If p_{hl} is close to 0, then agents have most likely the same type. In particular, bad news for the partner is also bad news for oneself. Therefore, a faster learning rate benefits both partners as both get the opportunity to leave an unprofitable match.

Lemma 3.6. *Fix $p_{hh} > 0$ and let p_{hl} converge to 0. Then, the expected equilibrium utility $V(\emptyset, 0)$ upon entering the market converges to*

$$V^*(\emptyset, 0) = \frac{\lambda m_0}{r + \lambda m_0} \cdot \frac{1 - p_{ll}}{r(1 - p_{ll}\frac{2\gamma\lambda m_0}{(r+2\gamma)(r+\lambda m_0)})} \cdot w.$$

For p_{hl} sufficiently small, $V(\emptyset, 0)$ is strictly increasing in γ.

In contrast to the previous lemma, now consider the case of a strong negative correlation, i.e., p_{hl} being close to $\frac{1}{2}$. Then, there is most likely one "winner" with an h-type and one "loser" with an l-type in each match. Learning who has a low type in a match allows that agent to find a new match but imposes a negative externality on the partner. We show that the negative externality on an h-agent is larger than the gain of rematching for an l-agent. More precisely, upon forming a match, the two partners would increase their ex-ante expected payoff if they could commit to never leaving. As a consequence, a faster learning rate γ decreases the expected

[4]We consider the correlation between the hidden types in a match conditional on being in an non-obsorbing state hl, lh, or ll. In our limit analysis, to prevent the equilibrium mass m_0 of agents who search for a match from diverging to ∞, we fix the probability $p_{hh} > 0$ of entering an absorbing state.

utility in equilibrium, and agents would be better of by learning at a slower pace, or not learning at all.

Lemma 3.7. *Fix $p_{hh} > 0$ and let p_{ll} converge to 0. Then, the expected equilibrium utility $V(\emptyset, 0)$ upon entering the market converges to*

$$V^{**}(\emptyset, 0) \approx \frac{\lambda m_0}{r + \lambda m_0} \cdot \frac{\frac{p_{hh}}{r} + p_{hl} \frac{r+\lambda m_0+\gamma}{(r+\gamma)(r+\lambda m_0)}}{1 - p_{hl} \frac{\gamma \lambda^2 m_0^2}{(r+\gamma)(r+\lambda m_0)^2} - p_{hl} \frac{\gamma \lambda m_0}{(r+\gamma)(r+\lambda m_0)}} \cdot w.$$

For p_{ll} sufficiently small, $V(\emptyset, 0)$ is strictly decreasing in γ.

As a result, the effect of a faster learning rate is ambiguous. It depends on the correlation, whether faster learning increases or decreases the welfare of agents. If the agents' goals are aligned (strong positive correlation), then faster learning is beneficial. In contrast, with a strong negative correlation, slow learning is more beneficial as the agents prefer not to know who wins and who loses in a match, to prevent the match from being dissolved.

3.7 Conclusion

In this chapter, we analyze a search model with on-the-match search and on-the-match learning. While being matched, agents learn about the idiosyncratic value of the current match. Not only the own valuation but also the partner's valuation of the match are of importance for an agent as the partner's rematching behavior affects the present value of a persisting match. This leads to an endogenous interest in the match being profitable for the partner.

We show the existence and uniqueness of a steady-state equilibrium. In equilibrium, agents follow cutoff strategies. Further, we provide an infinite set of pointwise balance conditions that ensures the stationarity of the masses for each time t that a match persists and we prove the equivalence to a finite set of aggregate balance conditions. For the welfare effects of learning, the correlation between the types in a match is of importance. With a strong positive correlation, faster learning increases the ex-ante expected payoff while with a strong negative correlation, the ex-ante payoff decreases with a faster learning rate. In the latter case, committing together to never dissolve a match is ex-ante preferred by both agents.

An interesting direction of further research would be the extension to other information structures. For instance, if h-agents received good news over time, instead of l-agents receiving bad news, then the beliefs inside a match grow more pessimistic the longer a match persists without news. This would change the rematching behavior of agents in a sense that agents in newly formed matches immediately search for a new partner as even an ε of time without good news decreases the present value of the current match below the value of a newly formed match. For more general information structures, like the occurrence of multiple different types of news, or the beliefs following a Brownian motion, the drift of the belief would be of importance to the agents' equilibrium behavior.

3.A Proofs

3.A.1 Proofs for Section 3.3

Proof of Lemma 3.1. For single agents and agents with bad news it is a dominant strategy to always accept. An agent in a pair could copy the strategy of a single/agent with bad news and receive a strictly higher payoff.

For the equilibrium behaviour of an agent with information (U, B), note that the partner always accepts to rematch, i.e., her acceptance probability is constant over time. Since the belief $P(hl|UB, t)$ is strictly increasing in t, the continuation payoff $V(UB, t)$ is also strictly increasing in t. Thus, $V(UB, t)$ crosses $V(UU, 0)$ at most once. □

3.A.2 Proofs for Section 3.4

Proof of Lemma 3.2. The equilibrium behavior of agents with information (U, U) follows from the fact that at $t = 0$ an agent is indifferent between accepting to rematch and staying in the current match. By monotonicity, the acceptance probability of the partner is non-increasing. Therefore, the continuation payoff $V(UU, t)$ is strictly increasing over time. Since the continuation payoff at time $t = 0$ is identical to the continuation payoff of accepting, agents with information (U, U) never accept for $t > 0$. □

Proof of Lemma 3.3. Integrating the pointwise balance equation for singles

$$m_\emptyset(t) = (\eta + \lambda m_0(m_0 - m_\emptyset))e^{-\lambda m_0 t}$$

over t yields

$$m_\emptyset = (\eta + \lambda m_0(m_0 - m_\emptyset)) \cdot \frac{1}{\lambda m_0},$$

which is equivalent to

$$m_\emptyset = \frac{\eta + \lambda m_0^2}{2\lambda m_0}.$$

Integrating the pointwise balance equation for hh-pairs

$$m_{hh,UU}(t) = p_{hh}\lambda m_0^2 \cdot 1$$

over t yields

$$m_{hh,UU} = \infty.$$

Integrating the pointwise balance equation for $m_{hl,UU}$

$$m_{hl,UU}(t) = 2p_{hl}\lambda m_0^2 e^{-\gamma t}$$

over t yields

$$m_{hl,UU} = \frac{2p_{hl}\lambda m_0^2}{\gamma}.$$

Integrating the pointwise balance equation for $m_{ll,UU}$

$$m_{ll,UU}(t) = p_{ll}\lambda m_0^2 e^{-2\gamma t}$$

over t yields

$$m_{ll,UU} = \frac{p_{ll}\lambda m_0^2}{2\gamma}.$$

Integrating the three remaining pointwise balance conditions over t yields the integrals

$$\begin{aligned}
m_{hl,UB,\leq t^*} &= \gamma 2p_{hl}\lambda m_0^2 \int_0^{t^*}\int_0^t \exp\big(-\gamma t' - 2\lambda m_0(t-t')\big)\,dt'dt \\
m_{hl,UB,>t^*} &= \gamma 2p_{hl}\lambda m_0^2 \int_{t^*}^{\infty}\int_0^t \exp\big(-\gamma t' - \lambda m_0(t-t') - \lambda m_0 \max(t^*-t',0)\big)\,dt'dt \\
m_{ll,UB,\leq t^*} &= 2\gamma p_{ll}\lambda m_0^2 \int_0^{t^*}\int_0^t \exp\big(-2\gamma t' - (2\lambda m_0+\gamma)(t-t')\big)\,dt'dt \\
m_{ll,UB,>t^*} &= 2\gamma p_{ll}\lambda m_0^2 \int_{t^*}^{\infty}\int_0^t \exp\big(-2\gamma t' - (\lambda m_0+\gamma)(t-t') - \lambda m_0 \max(t^*-t',0)\big)\,dt'dt \\
m_{ll,BB} &= 2\gamma^2 p_{ll}\lambda m_0^2 \int_0^{\infty}\int_0^t\int_0^{t'} \exp\big(-2\gamma t'' - (\lambda m_0+\gamma)(t'-t'') - \lambda m_0 \max(\min(t^*,t')-t'',0) \\
&\qquad - 2\lambda m_0(t-t')\big)\,dt''dt'dt,
\end{aligned}$$

where the survival functions $q_{ij,SS'}(t',t)$ are substituted by the corresponding exponential functions given by the equilibrium strategies:

$$\begin{aligned}
q_{hl,UB}(t',t) &= \exp\big(-\lambda m_0(t-t') - \lambda m_0 \max(\min(t^*,t)-t',0)\big) \\
q_{ll,UB}(t',t) &= \exp\big(-(\lambda m_0+\gamma)(t-t') - \lambda m_0 \max(\min(t^*,t)-t',0)\big) \\
q_{ll,BB}(t',t) &= \exp\big(-2\lambda m_0(t-t')\big).
\end{aligned}$$

Solving these integrals gives the aggregate balance conditions

$$\begin{aligned}
m_{hl,UB,\le t^*} &= p_{hl} m_0 \Big(1 + \frac{2\lambda m_0}{\gamma - 2\lambda m_0} e^{-\gamma t^*} - \frac{\gamma}{\gamma - 2\lambda m_0} e^{-2\lambda m_0 t^*}\Big) \\
m_{hl,UB,>t^*} &= p_{hl} m_0 \Big(\frac{2\gamma}{\gamma - 2\lambda m_0} e^{-2\lambda m_0 t^*} - \frac{4\lambda m_0}{\gamma - 2\lambda m_0} e^{-\gamma t^*}\Big) \\
m_{ll,UB,\le t^*} &= p_{ll} m_0 \Big(\frac{2\lambda m_0}{\gamma + 2\lambda m_0} + 2\lambda m_0 e^{-2\gamma t^*} + \frac{4\gamma\lambda m_0}{\gamma + 2\lambda m_0} e^{-(2\lambda m_0 + \gamma) t^*}\Big) \\
m_{ll,UB,>t^*} &= p_{ll} m_0 \Big(\frac{4\gamma\lambda m_0}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-(2\lambda m_0 + \gamma) t^*} - \frac{2\lambda m_0(\gamma + 2\lambda m_0)}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-2\gamma t^*}\Big) \\
m_{ll,BB} &= p_{ll} m_0 \Big(\frac{\gamma - 2\lambda m_0}{2(\gamma + 2\lambda m_0)} - \frac{\lambda m_0(\gamma + 2\lambda m_0 - 2\gamma^2 + 2\gamma\lambda m_0 + 4\lambda^2 m_0^2)}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-2\gamma t^*} \\
&\qquad - \frac{2\gamma\lambda m_0(\gamma + 2\lambda m_0 + 2\gamma^2 - 2\gamma\lambda m_0 - 4\lambda^2 m_0^2)}{(\gamma + 2\lambda m_0)(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-(2\lambda m_0 + \gamma) t^*}\Big).
\end{aligned}$$

Thus, if the pointwise balance conditions are satisfied, so are the aggregated balance conditions.

The final equation

$$m_0 = m_\emptyset + m_{hl,UB,\le t^*} + \frac{1}{2} m_{hl,UB,>t^*} + m_{hl,UB,\le t^*} + \frac{1}{2} m_{ll,UB,>t^*} + m_{ll,BB},$$

follows from the fact that the set of all agents who want to rematch consists of the following: All singles, all agents who received bad news, and all agents whose partner has received bad news and who are in match for a time less than t^*. For $t > t^*$, only half of the agents in pairs with information UB are willing to rematch, which implies that only half of the masses $m_{hl,UB,>t^*}$ and $m_{ll,UB,>t^*}$ counts towards m_0.

It remains to show that the aggregate balance conditions are sufficient for the pointwise balance equations. For this, we construct pointwise masses $\mathcal{M}$ as follows: First, the aggregate masses uniquely determine the masses m_0 of agents who search for a new partner. Second, the masses m_0, $m_\emptyset$ and the strategies uniquely determine the pointwise masses

$$\begin{aligned}
m_\emptyset(t) &= (\eta + \lambda m_0(m_0 - m_\emptyset)) q_\emptyset(0, t) \\
m_{hh,UU}(t) &= p_{hh} \lambda m_0^2 \; q_{hh,UU}(0, t) \\
m_{hl,UU}(t) &= 2 p_{hl} \lambda m_0^2 \; q_{hl,UU}(0, t) \\
m_{ll,UU}(t) &= p_{ll} \lambda m_0^2 \; q_{ll,UU}(0, t).
\end{aligned}$$

Finally, the remaining pointwise masses are uniquely determined by

$$m_{hl,UB}(t) = \int_0^t \gamma m_{hl,UU}(t')\ q_{hl,UB}(t',t)\ dt'$$

$$m_{ll,UB}(t) = \int_0^t 2\gamma m_{ll,UU}(t')\ q_{ll,UB}(t',t)\ dt'$$

$$m_{ll,BB}(t) = \int_0^t \gamma m_{ll,UB}(t')\ q_{ll,BB}(t',t)\ dt'.$$

Therefore, there exists a unique $(\mathcal{M}', \varphi)$ that has the same aggregated masses as $(\mathcal{M}, \varphi)$ and satisfies the pointwise balance conditions. $\square$

Proof of Corollary 3.1. This corollary follows from summing the aggregate balance conditions together. Adding two times

$$m_{hl,UB,\leq t^*} = p_{hl} m_0 \Big(1 + \frac{2\lambda m_0}{\gamma - 2\lambda m_0} e^{-\gamma t^*} - \frac{\gamma}{\gamma - 2\lambda m_0} e^{-2\lambda m_0 t^*}\Big)$$

plus

$$m_{hl,UB,>t^*} = p_{hl} m_0 \Big(\frac{2\gamma}{\gamma - 2\lambda m_0} e^{-2\lambda m_0 t^*} - \frac{4\lambda m_0}{\gamma - 2\lambda m_0} e^{-\gamma t^*}\Big)$$

yields

$$2m_{hl,UB,\leq t^*} + m_{hl,UB,>t^*} = 2p_{hl} m_0.$$

Multiplying this by λm_0 yields the first equation. Analogously, adding two times

$$m_{ll,UB,\leq t^*} = p_{ll} m_0 \Big(\frac{2\lambda m_0}{\gamma + 2\lambda m_0} + 2\lambda m_0 e^{-2\gamma t^*} + \frac{4\gamma\lambda m_0}{\gamma + 2\lambda m_0} e^{-(2\lambda m_0 + \gamma) t^*}\Big)$$

plus

$$m_{ll,UB,>t^*} = p_{ll} m_0 \Big(\frac{4\gamma\lambda m_0}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-(2\lambda m_0 + \gamma) t^*} - \frac{2\lambda m_0(\gamma + 2\lambda m_0)}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-2\gamma t^*}\Big)$$

plus two times

$$\begin{aligned} m_{ll,BB} = p_{ll} m_0 \Big(&\frac{\gamma - 2\lambda m_0}{2(\gamma + 2\lambda m_0)} - \frac{\lambda m_0(\gamma + 2\lambda m_0 - 2\gamma^2 + 2\gamma\lambda m_0 + 4\lambda^2 m_0^2)}{(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-2\gamma t^*} \\ &- \frac{2\gamma\lambda m_0(\gamma + 2\lambda m_0 + 2\gamma^2 - 2\gamma\lambda m_0 - 4\lambda^2 m_0^2)}{(\gamma + 2\lambda m_0)(\gamma + \lambda m_0)(\gamma - 2\lambda m_0)} e^{-(2\lambda m_0 + \gamma) t^*}\Big) \end{aligned}$$

yields

$$2m_{ll,UB,\leq t^*} + m_{ll,UB,>t^*} + 2m_{ll,BB} = p_{ll} m_0.$$

Multiplying by λm_0 yields the second equation. □

Proof of Lemma 3.4. For the proof, we substitute the masses by the aggregate balance conditions in the term that specifies m_0: First, we multiply

$$m_0 = m_\emptyset + m_{hl,UB,\leq t^*} + \frac{1}{2}m_{hl,UB,>t^*} + m_{hl,UB,\leq t^*} + \frac{1}{2}m_{ll,UB,>t^*} + m_{ll,BB}$$

from Lemma 3.3 by $2\lambda m_0$ to obtain

$$\begin{aligned}2\lambda m_0^2 = 2\lambda m_0 m_\emptyset &+ \lambda m_0(2m_{hl,UB,\leq t^*} + m_{hl,UB,>t^*}) \\ &+ \lambda m_0(2m_{hl,UB,\leq t^*} + m_{ll,UB,>t^*} + 2m_{ll,BB}).\end{aligned}$$

Now, substituting the aggregate balance conditions

$$\begin{aligned}2\lambda m_0 m_\emptyset &= \eta + \lambda m_0^2 \\ 2p_{hl}\lambda m_0^2 &= \lambda m_0(2m_{hl,UB,\leq t^*} + m_{hl,UB,>t^*}) \\ p_{ll}\lambda m_0^2 &= \lambda m_0(2m_{ll,UB,\leq t^*} + m_{ll,UB,>t^*} + 2m_{ll,BB})\end{aligned}$$

from Lemma 3.3 and Corollary 3.1 yields

$$2\lambda m_0^2 = \eta + \lambda m_0^2 + 2p_{hl}\lambda m_0^2 + p_{ll}\lambda m_0^2.$$

By $p_{hh} = (1 - 2p_{lh} - p_{ll})$, we get that

$$\eta = p_{hh}\lambda m_0^2$$

holds and therefore, the aggregate balance conditions imply that m_0 has to be

$$m_0 = \sqrt{\frac{\eta}{\lambda p_{hh}}}.$$

□

Proof of Lemma 3.5. First, we show the existence of a monotone steady-state equilibrium. As shown before, accepting is optimal for singles and agents with bad news. Furthermore, $\varphi(UU, t) = 0$ is not only necessary for all monotone partial equilibrium, but also a best response to itself. For the existence, it remains to show that there exists a cutoff $t^* \in [0, \infty]$ and a corresponding steady-state equilibrium such that it is optimal for agents with information (U, B) to accept to rematch until cutoff t^* and reject to rematch afterwards. Let

$$W(t) := V(UB, t) - V(UU, 0)$$

denote the difference in the expected utility of staying in the current match minus rematching given that all agents follow such a monotone strategy with cutoff t. The difference $W(t)$ is continuous by construction, strictly increasing in t, and bounded. It converges to $W(\infty) := \lim_{t\to\infty} W(t)$. If $W(t) \geq 0$ for all t, then it is never optimal to accept to rematch for an agent with information (U, B) and the cutoff $t^* = 0$ is optimal. If $W(t) \leq 0$ for all t, then it is always optimal to accept to rematch for an agent with information (U, B) and the cutoff $t^* = \infty$ is optimal. If neither of these two cases holds, then there exist $t_0 < t_1$ with $W(t_0) < 0 < W(t_1)$. By the intermediate value theorem, there is an interior cutoff $t^* \in (0, \infty)$ with $W(t^*) = 0$, i.e., agents are indifferent at the cutoff, they strictly prefer to rematch for all $t < t^*$ and strictly prefer to stay in the current match for all $t > t^*$. Thus, there always exists a monotone steady-state equilibrium.

For uniqueness, suppose for contradiction that there are two monotone steady-state equilibria with different cutoffs $t_1^* < t_2^*$. Consider a single agent and take the other agents' strategies as given. In particular, the cutoff t^* of the other agents has no influence on the own continuation payoff, since m_0 is the same in both equilibria and the agents with bad news receive the same continuation payoff as a single agent. The continuation payoff $V(UU, 0)$ of forming a new match cannot be the same in both equilibria. Otherwise, the continuation payoff for all future matches would be equal in both equilibria and since $V(UB, t)$ is strictly increasing in both equilibria, this contradicts $t_1^* \neq t_2^*$. Thus, we get that $V(UU, 0)$ is different in both equilibria and since the cutoff choice t^* of the other agents does not change the own expected payoff, in at least one of the two equilibria the own choice of the cutoff is not optimal. □

3.A.3 Proofs for Section 3.5

Proof of Proposition 3.1. For this proof, we first need the following lemma that shows that the continuation payoff at time t_1 is higher than at time t_0 if the following three conditions are all satisfied: (1) the continuation payoff is higher at $t_1 + \varepsilon$ than at $t_0 + \varepsilon$ for some $\varepsilon > 0$, (2) the partner rematches less often after t_1 than after t_0, and (3) $t_1 > t_0$, i.e., the beliefs are more optimistic at t_1.

Lemma 3.8. *Fix any partial equilibrium and two points in time $t_0 < t_1$. If there exists an $\varepsilon > 0$ with*

$$V(UU, t_0 + \varepsilon) \leq V(UU, t_1 + \varepsilon)$$

such that for almost all $\xi \in (0, \varepsilon)$

$$\varphi(UU, t_0 + \xi) \geq \varphi(UU, t_1 + \xi)$$

holds, then we get

$$V(UU, t_0) < V(UU, t_1).$$

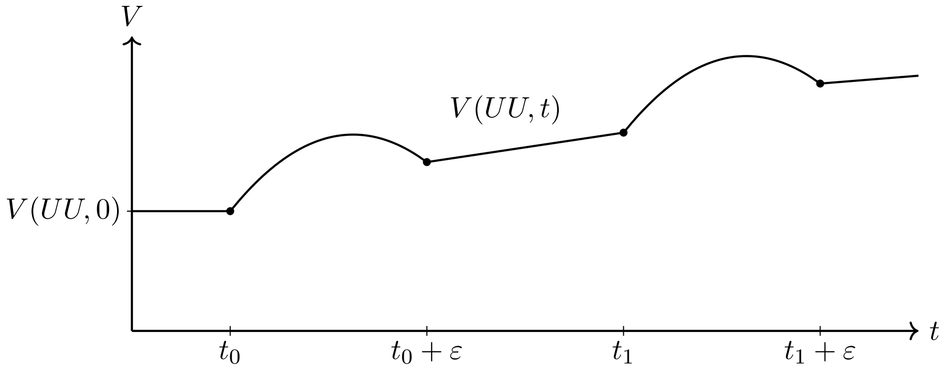

Figure 3.3: Continuation payoff for Lemma 3.8

Proof. Fix a partial equilibrium. The continuation payoff at time t can be expressed as a function of the continuation payoff at time $t+\varepsilon$, the own strategy, the partner's strategy, and the belief (which affects the expected rate at which bad news arrive). The continuation payoff is decreasing in the acceptance probability of the partner and it is increasing in the future continuation payoff at time $t+\varepsilon$. Furthermore, the continuation payoff strictly increases as agents become more optimistic over time. Therefore, $V(UU, t_0) < V(UU, t_1)$ holds. $\square$

Now, to prove Proposition 3.1, fix a non-monotone partial equilibrium. In the following we use Lemma 3.8 to systematically exclude various cases of how $V(UU, t)$ might behave until only one possible equilibrium type remains. Then, we conclude that all non-monotone partial equilibria must be of the form as described in Proposition 3.1.

The first claim shows that the continuation payoff cannot go below $V(UU, 0)$ without going up again.

Claim 3.1. There is no t_0 with $V(UU, t) < V(UU, 0)$ for all $t > t_0$.

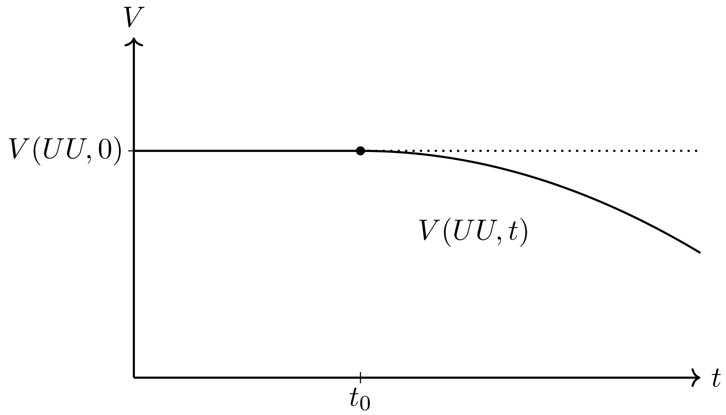

Figure 3.4: Continuation payoff for Claim 3.1

Proof. Assume for contradiction that there exists such a t_0. Let

$$t_1 = \inf_t \{\forall t' > t \ : \ V(UU, t') < V(UU, 0)\}$$

be the infimum over all such times. After t_1, agents strictly prefer to accept to rematch. Thus, the partner's acceptance probability does not change after t_1. Since agents get more optimistic over time, $V(UU, t')$ is strictly increasing on the interval (t_1, ∞). This is a contradiction, since by construction and continuity $V(UU, t_0) = V(UU, 0)$ holds. □

The next claim shows that the continuation payoff cannot go below $V(UU, 0)$ and up again. Together with the last claim, this implies that $V(UU, t)$ is always at least as large as $V(UU, 0)$.

Claim 3.2. There are no $t < t'$ with $V(UU, t) < V(UU, t') = V(UU, 0)$.

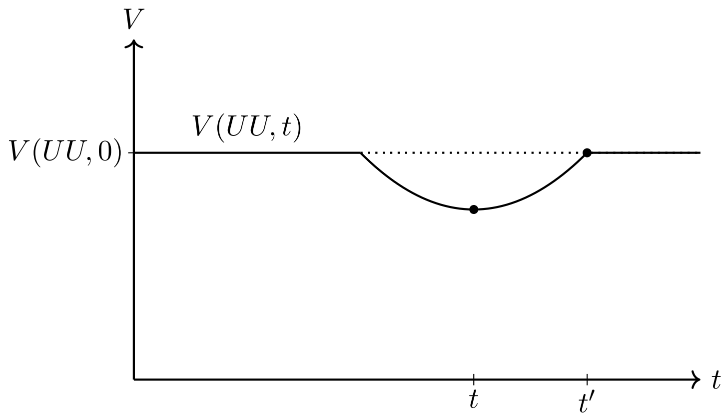

Figure 3.5: Continuation payoff for Claim 3.2

Proof. Assume for contradiction that there exist such $t < t'$. Now, we construct an open interval (t_0, t_1) with positive length such that the continuation payoff is smaller than $V(UU, 0)$ at the interval and the interval is maximal under set-inclusion. More precisely, we define

$$t_0 = \inf_{t'} \{\forall t'' \in (t', t) \ : \ V(UU, t'') < V(UU, 0)\}$$

and

$$t_1 = \sup_{t'} \{\forall t'' \in (t, t') \ : \ V(UU, t'') < V(UU, 0)\}.$$

Then, (t_0, t_1) is such an interval. By continuity the agents are indifferent at the boundary points, i.e., the equality $V(UU, t_0) = V(UU, t_1) = V(UU, 0)$ holds. Now, we compare the continuation payoff at the two times t_0 and $\frac{t_0+t_1}{2}$. Agents are more optimistic at $\frac{t_0+t_1}{2}$, the acceptance probability of the partner is 1 immediately after both times, and the future continuation payoff is higher after $\frac{t_0+t_1}{2}$. We apply Lemma 3.8 to t_0 and $\frac{t_0+t_1}{2}$ with $\varepsilon = \frac{t_1-t_0}{2}$ and we get $V(UU, t_0) < V(UU, \frac{t_0+t_1}{2})$ which contradicts our construction. □

The next claim shows that after $t = 0$ the continuation payoff cannot go above $V(UU, 0)$ without going down again.

Claim 3.3. There exists no $t_0 > 0$ with $V(UU, t_0) = V(UU, 0)$ such that for all larger $t > t_0$ $V(UU, t) > V(UU, 0)$ holds.

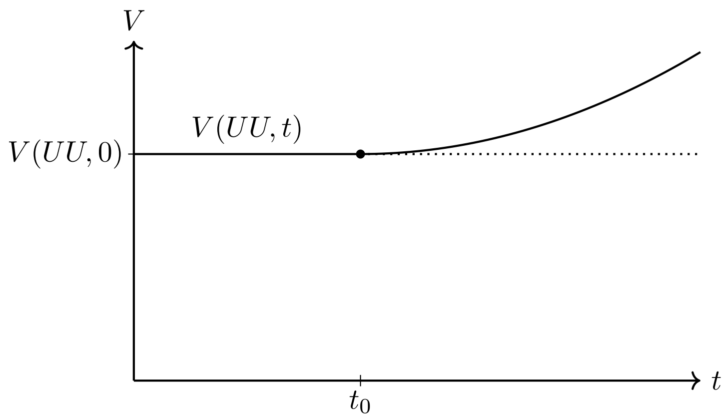

Figure 3.6: Continuation payoff for Claim 3.3

Proof. Assume for contradiction that there exists a $t_0 > 0$ such that agents with information (U, U) do not accept to rematch afterwards. Without loss, let t_0 be the minimum of all such times. Then, the agents are more optimistic at t_0 than at $t = 0$ and the partner will always reject to rematch after t_0. We apply Lemma 3.8 to 0 and t_0 with $\varepsilon = t_0$ and get that the continuation payoff $V(UU, t_0)$ is strictly larger than $V(UU, 0)$. This is a contradiction to the minimality of t_0. □

Finally, the last claim says that after $t = 0$ the continuation payoff cannot go above $V(UU, 0)$ and reach $V(UU, 0)$ again afterward. Together with the previous claim, this implies that if $V(UU, t) = V(UU, 0)$ holds for some $t > 0$, then the same equality also holds for all $t' > t$.

Claim 3.4. There exist no three points $0 < t_0 < t_1 < t_2$ such that $V(UU, t_0) = V(UU, 0)$, $V(UU, t_1) > V(UU, 0)$, and $V(UU, t_2) = V(UU, 0)$ hold.

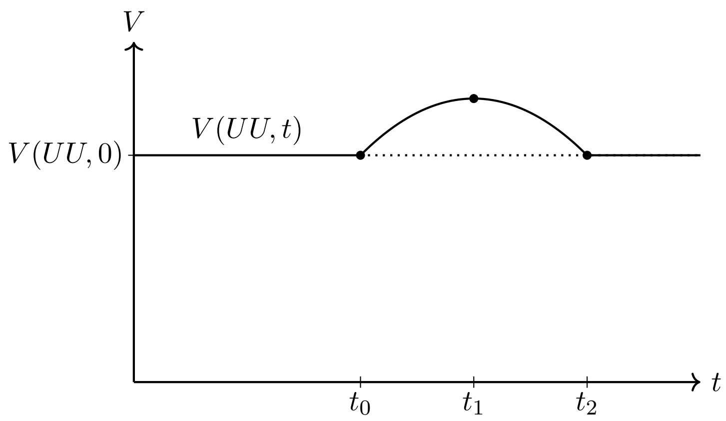

Figure 3.7: Continuation payoff for Claim 3.4

Proof. Assume for contradiction that there exist such three points. Without loss let the distance $t_2 - t_0$ be minimal of all such tuples. Then, $V(UU, t) > V(UU, 0)$ holds for all interior points $t \in (t_0, t_2)$. Now, we distinguish two cases.

Case 1: $t_2 - t_0 \geq t_0$, i.e., the length of the interval (t_0, t_2) is larger than the length of the interval $(0, t_0)$. Then, we apply Lemma 3.8 to 0 and t_0 with $\varepsilon = t_0$. Agents are more optimistic at t_0, agents are never left by their partner in the interval $(t_0, 2t_0)$, and the continuation payoff at $2t_0$ is strictly larger than at t_0. Thus, we get that $V(UU, t_0) \geq V(UU, 0)$ holds which is a contradiction.

Case 2: $t_2 - t_0 < t_0$, i.e., the length of the interval (t_0, t_2) is smaller than the length of the interval $(0, t_0)$. Let $\hat{t} := t_0 - (t_2 - t_0)$. We apply Lemma 3.8 to $\hat{t}$ and t_0 with $\varepsilon = t_2 - t_0$ to compare the continuation payoffs $V(UU, \hat{t})$ and $V(UU, t_0)$. The beliefs are higher at t_0, the partner does not accept to rematch at the interval (t_0, t_2), and the continuation payoff at the end of the interval (t_0, t_2) is equal to the continuation payoff at the end of the interval $(\hat{t}, t_0)$. Therefore, $V(UU, \hat{t}) < V(UU, t_0) = V(UU, 0)$ holds. This is a contradiction, since we have shown that $V(UU, t) \geq V(UU, 0)$ has to hold for all t. □

Continuation of the proof of Proposition 3.1. Now, we know that if we have $V(UU,t) = V(UU,0)$ for any $t > 0$, then this equality also holds for all $t' > t$. If this equality would only hold for $t = 0$, then we would have a monotone equilibrium. Therefore, in any non-monotone partial equilibrium exists a $t \in [0,\infty)$ such that $V(UU,t') = V(UU,0)$ holds for all $t' > t$. Let

$$t_0 = \inf_t \{\forall t' > t \ : \ V(UU,t') = V(UU,0)\}$$

be the earliest time after which $V(UU,t)$ is constant. By the previous claims, we get $V(UU,t) > V(UU,0)$ for all $t \in (0,t_0)$. Therefore, the partial equilibrium is exactly as characterized in Proposition 3.1. □

Proof of Theorem 3.1. In any non-monotone partial equilibrium, there is a time t_0 after which $\varphi(UU,t)$ is strictly increasing. Thus, the probability that an hh-pair stays together for at least time t converges to 0 as t approaches ∞. Since agents who received bad news accept to rematch with probability 1, there is no absorbing state, i.e., every match is eventually dissolved.

Since the balance conditions imply that the inflow of agents into absorbing states is equal to the inflow into the market, a non-monotone partial equilibrium does not satisfy the balance conditions. Therefore, all steady-state equilibria have to be monotone. Since there exists a unique monotone steady-state equilibrium, we get uniqueness among all steady-state equilibria. □

3.A.4 Proofs for Section 3.6

Proof of Lemma 3.6. First, we show that t^* tends to ∞ as p_{hl} vanishes. Let $\hat{t}$ denote the time at which

$$P(hl|UB,t) = P(i = h|UU,0) \qquad \Leftrightarrow$$

$$\frac{p_{hl}}{p_{hl} + p_{ll}e^{-\gamma t}} = \frac{p_{hh}}{p_{hh} + 2p_{hl}e^{-\gamma t} + p_{ll}e^{-2\gamma t}} + \frac{p_{hl}e^{-\gamma t}}{p_{hh} + 2p_{hl}e^{-\gamma t} + p_{ll}e^{-2\gamma t}}$$

holds. For agents whose partner received bad news, it is a dominant strategy to search at all times before $\hat{t}$. Thus, for the equilibrium cutoff $t^* > \hat{t}$ holds. Now, consider a sequence where p_{hl} converges to 0. Along this sequence, the time $\hat{t}$ tends to ∞. Therefore, t^* also tends to ∞.

Next, we consider the limit of such a sequence, i.e., let $p_{hl} = 0$ and $t^* = \infty$ hold. Let $V(ij)$ denote the expected continuation payoff of forming a new pair with type i oneself and type j for the partner. Let $V(x)$ denote the expected continuation payoff of forming a new pair before the types are realized. Recall that $V(\emptyset, 0)$ denotes the expected continuation payoff of becoming single. Then, by integrating the expected future utilities, we get for the continuation payoffs:

$$\begin{aligned} V(x) &= p_{hh}V(hh) + p_{ll}V(ll) \\ V(hh) &= \frac{w}{r} \\ V(ll) &= \frac{2\gamma}{r+2\gamma}V(\emptyset,0) \\ V(\emptyset,0) &= \frac{\lambda m_0}{r+\lambda m_0}V(x). \end{aligned}$$

Taking these together, we get

$$V(x) = p_{hh}\frac{w}{r} + p_{ll}\frac{2\gamma}{r+2\gamma}\frac{\lambda m_0}{r+\lambda m_0}V(x)$$

and therefore

$$V^*(\emptyset,0) = \frac{\lambda m_0}{r+\lambda m_0} \cdot \frac{1-p_{ll}}{r(1-p_{ll}\frac{2\gamma\lambda m_0}{(r+2\gamma)(r+\lambda m_0)})} \cdot w$$

holds. Taking the derivative with respect to the learning rate γ shows that $d\frac{V^*(\emptyset,0)}{d\gamma} > 0$ holds.

Now, by the continuity of the payoff functions, the payoff function $V(\emptyset,0)$ converges to $V^*(\emptyset,0)$ as p_{hl} vanishes. As the derivative of $V^*(\emptyset,0)$ is strictly positive, $d\frac{V(\emptyset,0)}{d\gamma} > 0$ is also positive for p_{hl} sufficiently small. □

Proof of Lemma 3.7. Analogously to the proof of Lemma 3.6, for p_{ll} sufficiently small, an agent whose partner received bad news stays in the match ($t^* = 0$). Next, consider the limit of such a sequence, i.e., $p_{ll} = 0$. The continuation payoffs are:

$$\begin{aligned} V(x) &= p_{hh}V(hh) + p_{hl}V(hl) + p_{lh}V(lh) \\ V(hh) &= \frac{w}{r} \\ V(hl) &= \frac{w}{r+\gamma} + \frac{\gamma}{r+\gamma}\Big(\frac{w}{r+\lambda m_0} + \frac{\lambda m_0}{r+\lambda m_0}V(\emptyset,0)\Big) \\ V(lh) &= \frac{\gamma}{r+\gamma}V(\emptyset,0) \\ V(\emptyset,0) &= \frac{\lambda m_0}{r+\lambda m_0}V(x). \end{aligned}$$

Together, we get

$$V(x) = \frac{p_{hh}\frac{w}{r} + p_{hl}\frac{w}{r+\gamma} + p_{hl}\frac{\gamma}{r+\gamma}\frac{w}{r+\lambda m_0}}{1 - p_{hl}\frac{\gamma}{r+\gamma}\frac{\lambda m_0}{r+\lambda m_0}\frac{\lambda m_0}{r+\lambda m_0} - p_{hl}\frac{\gamma}{r+\gamma}\frac{\lambda m_0}{r+\lambda m_0}}$$

and thus, the expected equilibrium utility upon entering the market is given by

$$V^{**}(\emptyset, 0) \approx \frac{\lambda m_0}{r + \lambda m_0} \cdot \frac{\frac{p_{hh}}{r} + p_{hl}\frac{r+\lambda m_0+\gamma}{(r+\gamma)(r+\lambda m_0)}}{1 - p_{hl}\frac{\gamma\lambda^2 m_0^2}{(r+\gamma)(r+\lambda m_0)^2} - p_{hl}\frac{\gamma\lambda m_0}{(r+\gamma)(r+\lambda m_0)}} \cdot w$$

which is strictly decreasing in γ. By the continuity of the payoff functions, the payoff function $V(\emptyset, 0)$ converges to $V^{**}(\emptyset, 0)$ as p_{ll} vanishes. As the derivative of $V^{**}(\emptyset, 0)$ is strictly negative, $d\frac{V(\emptyset,0)}{d\gamma} > 0$ is also negative for p_{ll} sufficiently small. $\square$

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