Analysis of Elliptic and Parabolic Partial Differential Equations Based on Carleson Measures

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To my family, for their love and constant support.

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Contents

Chapter 1

Introduction

Advances of recent years in harmonic analysis and the theory of function spaces have contributed to major developments in the theory of partial differential equations. Carleson measures were first introduced by L. Carleson in 1962 to study a problem of analytic interpolation and have undoubtedly played a significant role, in particular through their connection with the boundedness of classical operators in harmonic analysis and the characterization of function spaces in various geometrical settings by means of elliptic and parabolic linear operators.

This thesis provides a novel approach to the study of certain boundary value problems (and initial value problems) for nonlinear equations by exploiting the knowledge generated by the associated Carleson measure characterization of admissible data classes. In contrast to existing methods available in the literature which are mostly based on direct methods of calculus of variations or energy methods in general, the techniques developed in this thesis are essentially nonvariational, require much weaker assumptions on the data and produce stronger results. In particular, its peculiarity lies in that it is suitable for the analysis of low regularity data problems. By low regularity data, we mean those satisfying a smoothness property weaker than that needed in order for energy methods to be implemented. In practice, whilst Carleson measures mainly appear in the analysis of problems exhibiting a critical behavior reflected in the scaling symmetry and the nature of nonlinearity in the equation, these techniques can be suitably adapted to study subcritical and supercritical semilinear equations in various domains. The main questions of interest are:

- Existence and uniqueness of solutions
- Continuous dependence of solutions on the data
- Regularity of solutions.

1.1 Motivation

Consider the Cauchy problem for the viscous Hamilton-Jacobi equation of the form

$$\begin{cases} \partial_t u - \Delta u = |\nabla u|^2 \ \text{ in } \ \Omega \times (0,\infty) \\ u(x,0) = u_0(x), \qquad x \in \Omega \end{cases} \tag{1.1.1}$$

and the corresponding stationary equation

$$-\Delta u = |\nabla u|^2 \ \text{ in } \ \Omega. \tag{1.1.2}$$

Both equations are scaling invariant with respect to the transformations

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t), \ \ (x,t) \in \Omega_\lambda \times (0,\infty) \ \text{ and } \ u_\lambda(x) = u(\lambda x)$$

respectively, for any $\lambda > 0$ where $\Omega_\lambda = \{x \in \mathbb{R}^n : \lambda x \in \Omega\}$. This observation has the consequence that the nonlinear term in (1.1.2) cannot be weakened under any rescaling argument. If a weak solution $u \in W^{1,2}(\Omega)$ exists, then it does not enjoy a better smoothness property by classical elliptic regularity theory. Moreover, supplementing (1.1.2) with zero Dirichlet boundary condition, it is easy to check that

$$u_1(x) = 0 \quad \text{ and } \quad u_2(x) = \log\log(|x|^{-1})$$

are both solutions to (1.1.2) in $\Omega = B_{1/e}(0) \subset \mathbb{R}^2$. While u_1 is regular, u_2 is a weak solution with isolated singularity at the origin. Thus, weak solutions of (1.1.2) are neither nonunique nor regular. Similar behavior also pertain to weak solutions of Problem (1.1.1), also known as the Kardar-Parisi-Zhang equation [KPZ86] arising in the theory of growth and roughening of surfaces. These type of problems are called critical and their intrinsic properties (nature of nonlinearity, scaling and translation invariance) seem to be the main reason why energy methods or classical compactness arguments fail to provide a satisfactory answer to questions such as existence, uniqueness and regularity theory. Hence, their analysis requires new analytical tools.

A recurring challenge in investigating the well-posedness theory in the context of low regularity data comes from the definition of an adequate notion of solutions. For parabolic problems, a Duhamel formulation based on semigroup theory leads to the so-called mild solutions which seem to be the correct notion of solutions to work with. The situation is a little different in the elliptic setting. There is a popular weaker concept of solutions known as very weak solutions. It is obtained via a variational formulation wherein less stringent regularity conditions on the solution and thus the data are needed. However, in practice, their existence is established by means of duality arguments and functional analytical approaches mainly relying on the available theory for generalized weak solutions, hence always confined in the framework of Sobolev spaces. It is a fairly general principle, that given the Dirichlet problem for a semilinear elliptic equation, its solution can be represented as a sum of two sub-problems, the corresponding linear equation and the inhomogeneous equation. This is always possible depending on the regularity of the domain where the problem is posed. This type of solution, by analogy to mild solutions is the notion of solutions we adopt. In either case, we fully exploit the regularizing properties of solutions to the associated linear equation generated by the prescribed rough data (via Carleson measures or extrinsic characterization of boundary classes) in order to identify the correct functional setting where solutions are sought for. This procedure, in combination with the use of suitable Banach fixed point arguments leads to sharp results, in the sense of optimal solution space for the prescribed data class.

As an illustration, three selected problems are discussed into three separate chapters.

The first chapter deals with the well-posedness of the Dirichlet problem for the weakly harmonic maps equations from a smooth domain into a closed Riemannian manifold. It presents a new approach to the harmonic maps problem in two and higher dimensions for Dirichlet data having infinite energy. We show that under a mere smallness requirement on the Dirichlet data in the space of bounded mean oscillations or the space of measurable essentially bounded functions, there exists a unique small weakly harmonic map which is locally infinitely smooth. Whilst this regularity result may fail in absence of this smallness condition, the solvability persists for large Dirichlet data provided the domain is bounded and there exists a smooth stable weakly harmonic map. The main finding improves the result by S. Hildebrant, H. Kaul and K. Widman as well as that of M. Struwe both requiring $\frac{1}{2}$-regularity on the data in addition to the smallness of the energy.

The second part is concerned with the applicability of the method in the context of initial value problems (based on a non-trivial adaptation of the ideas introduced by H. Koch and D. Tataru in their work on the well-posedness for the Navier-Stokes equations). The chemotaxis Navier-Stokes system describing bacterial swimming within viscous incompressible fluid environments and the double chemotaxis system coupling the Navier-Stokes equations and the Keller-Segel system are investigated. Carefully analyzing each coupling term in the system and using the scaling invariance feature of unknowns, we identify classes of initial data, each of which is defined extrinsically via Carleson measures or their fractional analogues. Local well-posedness results are then established for large data while sharp global existence theory is obtained under a smallness condition on the initial data. Moreover, uniqueness criterion of solutions is also studied.

The third chapter is devoted to the study of the forced steady-state Navier-Stokes equations. Supplementing the system with an inhomogeneous Dirichlet boundary condition, the solvability of the resulting problem is open in its full generality when the fluid region is unbounded. The upper half-space in arbitrary dimensions is a simple example of an unbounded domain which serves as a reference (practically, flows in many unbounded domains locally via flattening reduces to the half-space). The stationary Navier-Stokes system does not fall into the category of "critical" problems as described above and in contrast to the unsteady Navier-Stokes, Carleson type measures do not naturally appear in this case. Nevertheless, some of our main ideas still prevail. The scaling feature and the local integrability requirement on the solution suggest the choice of Dirichlet data in a class of homogeneous Sobolev spaces with smoothness parameter $s = -1/2$. Existence, uniqueness and local Hölder regularity of solutions are then established under a smallness condition on the data. These conclusions follows from a thorough analysis of the Stokes system subject to boundary value in homogeneous Triebel-Lizorkin spaces with negative amount of smoothness and external force in tent spaces for which we derive new solvability statements, generalizing known results.

We contend that the techniques developed in the present thesis complement the existing methods in the literature and can be employed to analyze many other examples of elliptic and parabolic problems.

1.2 Contributions

This thesis is essentially based on two publications and one preprint. A detailed list of articles is indicated below.

- Gael Yomgne Diebou and Herbert Koch. "Dirichlet problem for weakly harmonic maps with rough data". *Comm. Partial Differential Equations*, **47**:7 (2022), 1504-1535.

The main results of this article may be summarized as follows. Given a smooth closed Riemannian manifold N, any map in $L^\infty(\partial\Omega, N)$ or in $BMO(\partial\Omega, N)$ with small norm gives rise to a unique weakly harmonic map in Ω having locally finite energy. To produce this conclusion, an equivalent reformulation of the problem is introduced and consists of a fixed point equation with extended nonlinear term and a separated geometric constraint. The intrinsic features of the original problem combined with the study of the corresponding linear system, motivated by Carleson measures both suggest the appropriate framework wherein a contraction mapping argument is carried out. This generates a bounded, locally smooth solution to the extended problem which lives in a small neighborhood of the manifold N. A maximum principle argument is invoked to show that this solution satisfies the geometric condition. In addition, via a perturbative approach the smallness assumption on the data can be removed whenever the domain is bounded.

- Gael Yomgne Diebou. "Well-posedness for chemotaxis-fluid models in arbitrary dimensions". To appear in *Nonlinearity*. ArXiv:2111.04792.

This article investigates the Cauchy problem for a strongly coupled system of equations describing the chemotaxis driven processes of cells swimming in presence of an incompressible viscous fluid. Prescribing initial data in an almost optimal class, local and global mild solutions satisfying the expected properties (mass conservation for the density of cells and the nonnegativity perservation for the first two unknowns: oxygen concentration and cells density) are constructed. Moreover, a decay in time criterion guarantees the uniqueness of these global solutions. In the process, the nature of the coupling terms is used to deduce what assumptions on each unknown is needed in order for the system to be meaningful. Combining this with the scaling and translation invariance features of the system leads to the choice of admissible initial data classes, some of which are new and identified using the knowledge generated by Carleson type measures arising in the study of the linear counterpart of the system.

- Gael Yomgne Diebou. "Existence and regularity of solutions to stationary Navier-Stokes equations arising from irregular data". *Submitted for publication*.

In this article, the forced incompressible stationary Navier-Stokes flow in the region $\mathbb{R}^n_+$, $n \geq 2$ is analyzed. Existence of a unique solution satisfying a global integrabilty property measured in a scale of tent spaces is established for small data in the homogenous Sobolev space with $-\frac{1}{2}$-degree of smoothness. Moreover, the velocity field is shown to be locally Hölder continuous while the pressure belongs to $L^p_{loc}(\mathbb{R}^n_+)$ for every $p \in (1, \infty)$. The main approach is based on the analysis of the inhomogeneous Stokes system for which we derive a new solvability result involving Dirichlet data in Triebel-Lizorkin classes with negative amount of smoothness.

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Chapter 2

Weakly harmonic maps into closed Riemannian manifolds

2.1 Statement of main results

In this chapter, we are interested in the solvability of the Dirichlet problem for the weakly harmonic maps equation subject to rough data at the boundary. We will consider two types of domains namely, $\Omega \subset \mathbb{R}^n$, $n \geq 2$ bounded with $C^{1,\alpha}$ boundary or the half-space

$$\Omega = \mathbb{R}^n_+ := \{x = (x', x_n),\ x' \in \mathbb{R}^{n-1},\ x_n > 0\}.$$

Let N be a smooth closed Riemannian manifold. As a result of Nash's embedding theorem we can assume without any restriction that N isometrically embeds into $\mathbb{R}^m$ for some positive integer m. Denote by $\dot{W}^{1,2}(\mathbb{R}^n_+, N)$ the space of functions $u : \mathbb{R}^n_+ \to \mathbb{R}^m$ whose first order (distributional) derivatives belong to the Lebesgue space $L^2(\mathbb{R}^n_+, \mathbb{R}^{nm})$ and satisfy the constraint $u(x) \in N$ a.e. $x \in \mathbb{R}^n_+$. With $x_1, ..., x_n$ representing a coordinate system on $\mathbb{R}^n_+$, one can associate to any Sobolev map $u \in \dot{W}^{1,2}(\mathbb{R}^n_+, N)$ an energy density defined as

$$e(u) = \sum_{i=1}^{n} \partial_{x_i} u \cdot \partial_{x_i} u = |\nabla u|^2 .$$

The Dirichlet energy functional of u is then given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n_+} e(u) dx.$$

Consider a tubular neighborhood U of the manifold N in $\mathbb{R}^m$ on which the nearest point projection map $\mathcal{P}_N : U \to N$ is well-defined and smooth (cf. Section 2.2.3 below). For any test function $\phi \in C^\infty_0(\mathbb{R}^n_+, \mathbb{R}^m)$ and for $s > 0$ small enough, critical points of the functional

E are maps u in $\dot{W}^{1,2}(\mathbb{R}^n_+, N)$ such that the first variation of the energy satisfies the identity

$$\frac{\partial}{\partial s}\bigg|_{s=0} E(\mathcal{P}_N(u+s\phi)) = 0$$

where $\mathcal{P}_N(u+s\phi)$ belongs to $\dot{W}^{1,2}(\mathbb{R}^n_+, N)$. The Euler-Lagrange system associated to this variational problem reads

$$\Delta u + \Gamma(u)(\nabla u, \nabla u) = 0 \tag{2.1.1}$$

in the sense of distributions where Δ is the Laplace operator for the n-dimensional Euclidean space and $\Gamma(q) : T_qN \times T_qN \to (T_qN)^\perp$ is the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^m$ with

$$\Gamma(u)(\nabla u, \nabla u) = \sum_{i=1}^{n} \Gamma(u)\left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\right).$$

Solutions of Syst. (2.1.1) are called weakly harmonic maps. Supplementing this equation with the boundary condition

$$u = f \quad \text{on} \quad \partial\mathbb{R}^n_+, \tag{2.1.2}$$

we aim at addressing the well-posedness issue for the boundary value problem (2.1.1)-(2.1.2) when f assumes minimal translation and scale invariant regularity assumption in the sense made precise at (2.1.6). The question of existence of harmonic maps plays an important role in differential geometry, Teichmüller theory [Jos84] and in surface matching problem (in computer vision). In hydrodynamics theory, weakly harmonic maps into the 2-sphere are fundamental objects in the modelling of flows of nematic liquid crystals.

The Dirichlet problem for weakly harmonic maps in various geometrical settings have been studied in many works. When the source manifold is a compact connected Riemannian manifold of class C^3, Hildebrandt and his collaborators [HKW77] showed the existence of small solutions in the energy space provided the image of the boundary is contained in a small geodesic ball of N. Moreover, they further proved that if the radius of the ball is strictly bounded above by $\frac{\pi}{2\kappa^{1/2}}$ where $\kappa \geq 0$ is an upper-bound for the sectional curvature of N, then solutions are C^2-regular in the interior. Uniqueness of these small solutions was independently obtained by Jäger and Kaul in [JK79]. When Ω is the unit Euclidean ball in $\mathbb{R}^3$, Struwe [Str98] established solvability in the Sobolev class $H^{1,2}(\Omega, N)$ for data f having small energy. In particular, only solutions obeying the restriction

$$\sup_{x\in\Omega, r>0}\left(r^{-1}\int_{B_r(x)\cap\Omega} |\nabla u|^2 dy\right) < \varepsilon \tag{2.1.3}$$

for $\varepsilon > 0$ sufficiently small are unique. This result was generalized to arbitrary dimension ($n \geq 3$) in [Mos01]. The study of rotationally symmetric weakly harmonic maps with finite energy and their stability is the main subject of the article [JK83]. Regarding regularity, observe that the nonlinearity in Eq. (2.1.1) belongs to L^1 whenever u has finite energy. Thus a boostrapping argument will not improve the initial regularity of the solution. We quote Helein's unconditional regularity results for two-dimensional sources and general targets [Hél91a, Hél91b] which use the special structure of the equation and the celebrated Wente's inequality. See also [Jos84, Sch83] for the higher regularity of continuous weakly harmonic maps.

We observe that most of the aforementioned solvability results employ direct methods of calculus of variations under higher regularity condition on boundary data. In order to lower the requirement in smoothness, one needs new techniques. A starting point is to identify a befitting notion of solutions allowing for low regularity data. Formally, the Dirichlet problem (2.1.1)-(2.1.2) can be reformulated using Green identities so that the resulting equation reads

$$u(x) = \mathcal{H}f(x) + \mathcal{N}(\Gamma(u)(\nabla u, \nabla u))(x), \quad u(x) \in N \ a.e.\ x \in \mathbb{R}^n_+ \tag{2.1.4}$$

where $\mathcal{H}f$ is the Poisson extension of f and $\mathcal{N}$ the Newtonian potential, see Section 2.2 below for more details. Observe that $v = \mathcal{H}f$ makes sense as an absolutely convergent integral under the weaker condition

$$I := \int_{\mathbb{R}^{n-1}} |f(x')|(1+|x'|^n)^{-1} dx' < \infty \tag{2.1.5}$$

and v solves the Laplace equation in $\mathbb{R}^n_+$. If $\mathcal{M}$ denotes the centered Hardy-Littlewood maximal function, then it can be verified that $I \le C\mathcal{M}f(z)$ for some constant $C > 0$ depending on a fixed point $z \in \mathbb{R}^{n-1}$. Thus, whether or not (2.1.5) holds can be verified using the mapping properties of $\mathcal{M}$. It is worth pointing out that the latter fully characterizes the solvability of the Dirichlet problem for linear elliptic systems of second-order with constant complex coefficients in half-space [MMMM16]. Moreover, if u formally solves Eq. (2.1.1) then, the rescaled map

$$u_\lambda(x) := u(\lambda x), \ \ x \in \mathbb{R}^n_+, \ \lambda > 0 \tag{2.1.6}$$

is another solution since the second fundamental form $\Gamma(u)(\cdot,\cdot)$ has a quadratic growth in the gradient of u. Hence, we seek for classes of functions defined on $\mathbb{R}^{n-1}$, enjoying both (2.1.5) and the scaling law (2.1.6) and such that there exists a suitable notion of trace associated to harmonic functions in half-space. Natural candidates include the John-Nirenberg's space $BMO(\mathbb{R}^{n-1})$ and the class of measurable essentially bounded functions on $\mathbb{R}^{n-1}$. Indeed, it was established by Fefferman [Fef72] that $f \in BMO(\mathbb{R}^{n-1})$ if its Poisson extension $v = \mathcal{H}f(x)$ satisfies

$$\sup_{x' \in \mathbb{R}^{n-1}, x_n > 0} \left(x_n^{-(n-1)} \int_{B_{x_n}(x')} \int_0^{x_n} s|\nabla v(y,s)|^2 dy ds \right)^{1/2} < \infty. \tag{2.1.7}$$

This condition was later shown to characterize all harmonic functions whose trace belong to $BMO(\mathbb{R}^{n-1})$, see [FJN76, FS72]. This motivates the consideration of the following functional setting.

Definition 2.1.1. Let $n \geq 2$ and $m > 1$. Call $\mathbf{X}$ the space of functions $u : \mathbb{R}^n_+ \to \mathbb{R}^m$ such that

$$\|u\|_{\mathbf{X}} = \|u\|_{L^\infty(\mathbb{R}^n_+)} + |u|_{\mathbf{X}} < \infty \tag{2.1.8}$$

where the semi-norm $|\cdot|_{\mathbf{X}}$ reads

$$|u|_{\mathbf{X}} = \sup_{x_n>0} x_n \|\nabla u\|_{L^\infty(\mathbb{R}^{n-1})} + \sup_{(x',x_n)\in\mathbb{R}^n_+} \left(x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla u|^2 dy_n dy' \right)^{1/2}.$$

When endowed with the norm (2.1.8), $\mathbf{X}$ is a Banach space and one easily verifies that it is scaling invariant with respect to (2.1.6). Such functional frameworks turn out to be suitable for the analysis of certain critical boundary value problems and their use certainly goes beyong the context of harmonic maps. Our motivation comes from Koch & Tataru's work [KT01] on the well-posedness for the Navier-Stokes equations where similar spaces with parabolic scaling and ideas leading to their consideration were first introduced. This approach was subsequently employed in many other works [GdL19, KL12, KL15, Yom21, Wan11], just to mention a few. While these articles exclusively deal with parabolic problems, the present work also aims at showing how elliptic boundary value problems subject to low regularity data can be analyzed via similar methods. Observe that the first term in Eq. (2.1.4) is harmonic and has a well-defined trace at the boundary while the second term is continuous up to the boundary when $u \in \mathbf{X}$. In what follows, the boundary value problem should be understood in this sense.

We are ready to state our main results.

Theorem 2.1.2. *Assume that $f \in L^\infty(\mathbb{R}^{n-1}, N)$. There exists a positive number $\varepsilon := \varepsilon(n, N)$ such that if $\|f\|_{L^\infty(\mathbb{R}^{n-1})} \leq \varepsilon$, then the Dirichlet problem* (2.1.1)-(2.1.2) *is uniquely solvable in a small closed ball of* $\mathbf{X}$.

The *BMO*-Dirichlet problem for (2.1.1) is equally well-posed in the following sense.

Theorem 2.1.3. *There exists $\varepsilon_0 > 0$ such that for any map f in $BMO(\mathbb{R}^{n-1}, N)$ satisfying $\|f\|_{BMO(\mathbb{R}^{n-1})} \leq \varepsilon_0$, the BMO-Dirichlet problem* (2.1.1)-(2.1.2) *admits a solution $u \in \mathbf{X}$. Furthermore, this solution is unique in a small closed ball of* $\mathbf{X}$,

$$B^{\mathbf{X}}_{c\varepsilon_0}(v) = \{u \in \mathbf{X} : \|u - v\|_{\mathbf{X}} \leq c\varepsilon_0\}$$

for some constant $c > 0$ depending on N. Here v is the harmonic extension of f.

This existence result seems sharp in the sense that $BMO(\mathbb{R}^{n-1})$ is the largest translation and scaling invariant (with respect to (2.1.6)) space so that the first iteration of the fixed point map is well-defined.

Remark 2.1.4. Although our solvability results have been stated for $\mathbb{R}^n_+$, analogous conclusions for the geometrical setting given by bounded smooth domains remain valid in natural analogues of $\mathbf{X}$ (see Theorem 2.4.5 for more details). Moreover,

- Our method also infers the solvability of the problem $-\Delta u = |\nabla u|^2 + F$ in Ω (with $F \in \mathbf{Y}$ with small norm if $\Omega = \mathbb{R}^n_+$ or $F \in \mathbf{Z}$ (c.f. Section 2.4) with small norm if Ω is bounded) subject to small Dirichlet data in $L^\infty(\partial\Omega)$ or $BMO(\partial\Omega)$. Its applicability is not restricted to the dimension, it works well in the case $n = 2$.

- Unlike the results quoted earlier, our boundary data are allowed to have unbounded energies. However, it is not clear whether the smallness assumptions on the size of the boundary value can be relaxed. For bounded energy data, it is known that uniqueness fails in absence of appropriate smallness condition (like (2.1.3) for weakly harmonic maps from the unit ball in $\mathbb{R}^3$). A similar observation was made in [JK79] for smooth harmonic maps.

Despite these evidences about the necessity of having a size restriction condition on the boundary data, one may still ask the question whether or not f in Theorem 2.1.2 and 2.1.3 can be prescribed "large" in the L^∞-norm or BMO semi-norm, respectively. We come back to this particular question in the last part of this chapter.

Remark 2.1.5. Solutions constructed in the above theorems are locally smooth. Indeed, Theorem 2.1.2 tells us that ∇u is bounded in $\mathbb{R}^{n-1} \times (\eta, \infty)$ for any $\eta > 0$ since $u \in \mathbf{X}$. Thus $\Delta u \in L^\infty_{loc}$ and $u \in W^{2,p}_{loc}$ for any $p < \infty$ by the standard L^p-theory for elliptic equations. Once again, by using (2.1.1), we arrive at $\Delta u \in W^{1,p}_{loc}$ which infers $u \in W^{3,p}_{loc}$. In a repetitive way, one obtains that u belongs to the Sobolev space $W^{k,p}_{loc}$ for all $k = 1, 2, 3, \ldots$. Applying Sobolev embedding theorem ultimately yields u in C^∞_{loc}. This regularity result, however, is another consequence of smallness – there are everywhere discontinuous weakly harmonic maps [Riv95].

2.2 Preliminaries

2.2.1 The homogeneous theory

For a point x of $\mathbb{R}^n_+$, we write $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in (0, \infty)$. Let f be a locally integrable function on $\mathbb{R}^{n-1}$. For a subset $E \subset \mathbb{R}^{n-1}$, denote by $|E|$ its Lebesgue measure and let $f_E = \fint_E f\,dx' = |E|^{-1} \int_E f\,dx'$ the integral mean of f. We say that f belongs to $BMO(\mathbb{R}^{n-1})$ if

$$\|f\|_{BMO(\mathbb{R}^{n-1})} = \sup_B |B|^{-1} \int_B |f(x') - f_B|\,dx'$$

where the supremum is taken over all balls in $\mathbb{R}^{n-1}$. In what follows, we do not always distinguish between $BMO(\mathbb{R}^{n-1}, \mathbb{R}^m)$ and $BMO(\mathbb{R}^{n-1})$ and it should be clear from the context. Let $T(B) = B_r(x') \times (0, r)$ be the Carleson region above the boundary ball $B_r(x') \subset \mathbb{R}^{n-1}$.

A measure μ in $\mathbb{R}^n_+$ is termed Carleson if $\mathcal{C}(\mu) = \sup_B |B|^{-1}\mu(T(B))$ is finite. The role of these measures in connection to linear elliptic boundary value problems was first observed in [Fef72, FS72]: $BMO(\mathbb{R}^{n-1})$ is the trace space of harmonic functions v in $\mathbb{R}^n_+$ for which $x_n|\nabla v|^2 dx' dx_n$ is a Carleson measure. Moreover, we have the equivalence

$$\sup_{x'\in\mathbb{R}^{n-1}, x_n>0} \left(x_n^{-(n-1)} \int_{B_{x_n}(x')} \int_0^{x_n} s|\nabla v(y',s)|^2 dy' ds \right)^{1/2} \approx \|f\|_{BMO(\mathbb{R}^{n-1})}. \tag{2.2.1}$$

Now consider the Laplace equation $\Delta v = 0$ in $\mathbb{R}^n_+$ with $v\big|_{\partial\mathbb{R}^n_+} = f$. The Poisson extension

$$v(x) = \mathcal{H}f(x) := [P_{x_n} * f](x') \tag{2.2.2}$$

is the unique solution which decays at infinity. Recall that P_{x_n} is explicitly given by

$$P_{x_n}(x') = x_n^{-(n-1)} P(x'/x_n) = c_n \frac{x_n}{(|x'|^2 + x_n^2)^{n/2}}$$

where $c_n > 0$ is a normalizing constant such that $\int_{\mathbb{R}^{n-1}} P_{x_n}(x')dx' = 1$. We collect in the following lemma the boundedness properties of $\mathcal{H}$.

Lemma 2.2.1. *Let $f = (f_1, ..., f_m)$ defined on $\mathbb{R}^{n-1}$. Then, $\mathcal{H}f \in \mathbf{X}$ for all $f \in L^\infty(\mathbb{R}^{n-1})$ and*

$$\|\mathcal{H}f\|_{\mathbf{X}} \le C\|f\|_{L^\infty(\mathbb{R}^{n-1})}. \tag{2.2.3}$$

Moreover, if $f \in BMO(\mathbb{R}^{n-1})$, then there exists a positive constant $C > 0$ independent of f such that

$$|\mathcal{H}f|_{\mathbf{X}} \le C\|f\|_{BMO(\mathbb{R}^{n-1})}. \tag{2.2.4}$$

Proof. Note that the estimate (2.2.3) is invariant by scaling and translation; hence it suffices to consider the case $x' = 0$ and $x_n = 1$, that is,

$$|v(0,1)| + |\nabla v(0,1)| + \left\|y_d^{1/2}\nabla v\right\|_{L^2(B_1(0)\times(0,1))} \le C\|f\|_{L^\infty(\mathbb{R}^{n-1})}; \quad v = \mathcal{H}f.$$

Observe that v is harmonic in $\mathbb{R}^n_+$ so that the bound

$$|v(0,1)| + |\nabla v(0,1)| \le C\|f\|_{L^\infty(\mathbb{R}^{n-1})}$$

follows from standard local elliptic estimates. To establish the third bound, one proceeds as follows. Let $B_2(0)$ be a ball in $\mathbb{R}^{n-1}$ and denote by χ its characteristic function. Decompose f into a local and global part, $f = \chi f + (1-\chi)f = f_1 + f_2$. Taking into consideration the harmonic extension of each part, set $v = v_1 + v_2$ and write

$$\begin{aligned} \left\|y_d^{1/2}\nabla v\right\|_{L^2(B_1(0)\times(0,1))} &\le \left\|y_d^{1/2}\nabla v_1\right\|_{L^2(B_1(0)\times(0,1))} + \left\|y_d^{1/2}\nabla v_2\right\|_{L^2(B_1(0)\times(0,1))} \\ &= I_1 + I_2 \end{aligned}$$

We prove that both I_1 and I_2 satisfy the desired estimate. For the first integral, we use integration by parts assuming that f_1 is continuous with compact support. Let w be a harmonic function in $\mathbb{R}^n_+$ with $w(x', 0) = f_1$. We have

$$\begin{aligned} 0 = \int_{\mathbb{R}^n_+} \Delta w(x_n w) dx &= -\int_{\mathbb{R}^n_+} (\nabla w \cdot e_d w + x_n |\nabla w|^2) dx \\ &= -\int_{\mathbb{R}^n_+} \left(\frac{\partial w}{\partial x_n} w + x_n |\nabla w|^2 \right) dx \end{aligned}$$

from which it follows that

$$\begin{aligned} \int_{\mathbb{R}^n_+} x_n |\nabla w|^2 dx &= -\int_{\mathbb{R}^{n-1}} \int_0^\infty \partial_d w \cdot w dx_d dx' \\ &= \frac{1}{2} \int_{\mathbb{R}^{n-1}} f_1^2(x') dx'. \end{aligned}$$

Thus

$$I_1 \le C \|f_1\|_{L^2(\mathbb{R}^{n-1})} \le C \|f\|_{L^\infty(\mathbb{R}^{n-1})}. \tag{2.2.5}$$

On the other hand, using the kernel decay property

$$|\partial^\alpha P_{x_n}(x')| \le C_n |x|^{1-|\alpha|-n} \text{ for all } x = (x', x_n) \in \overline{\mathbb{R}^n_+} \setminus \{0\}, \ \alpha \in \mathbb{N}^n \tag{2.2.6}$$

one obtains

$$I_2^2 = \left\| y_n^{1/2} \nabla v_2 \right\|^2_{L^2(B_1(0)\times(0,1))} = \int_{\mathbb{R}^n_+} y_n |\nabla v_2|^2 \chi_{B(0)\times(0,1)} dy_n dy'.$$

But

$$\begin{aligned} |\nabla v_2(y', y_n)| &\le \int |\nabla P_{y_n}(y' - z')| |f_2(z')| dz' \\ &\le \int_{\mathbb{R}^{n-1}\setminus B_2(0)} |\nabla P_{y_n}(y' - z')| |f(z')| dz' \\ &\le C \int_{\mathbb{R}^{n-1}\setminus B_2(0)} (y_n + |y' - z'|)^{-n} |f(z')| dz'. \end{aligned}$$

Observe that for $y' \in B_1(0)$ and $0 \le y_n \le 1$, it holds that

$$(y_n + |y' - z'|)^{-n} \le C(1 + |z'|^n)^{-1}, \quad z' \in \mathbb{R}^{n-1} \setminus B_2(0).$$

Thus

$$\begin{aligned} |\nabla v_2(y', y_n)| &\le C \|f\|_{L^\infty(\mathbb{R}^{n-1})} \int_{\mathbb{R}^{n-1}\setminus B_2(0)} (1 + |z'|^n)^{-1} dz' \\ &\le C \|f\|_{L^\infty(\mathbb{R}^{n-1})}. \end{aligned}$$

Upon squaring the previous inequality, multiplying by y_n and integrating over the cylinder $B_1(0)\times(0,1)$ one obtains the bound

$$\int_{B_1(0)}\int_0^1 y_n|\nabla v_2(y',y_n)|^2\,dy_n\,dy' \le C\|f\|^2_{L^\infty(\mathbb{R}^{n-1})}$$

which in turn completes the proof of the first statement (2.2.3).
Next, we prove the estimate

$$\sup_{x_n>0} x_n\|\nabla v(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \le C\|f\|_{BMO(\mathbb{R}^{n-1})}. \tag{2.2.7}$$

Since v is harmonic in $\mathbb{R}^n_+$, then so is the gradient ∇v and by the mean value theorem, we find that

$$|\partial_i v(x)| \le Cr^{-n}\int_{B_r(x)}|\partial_i v(y)|\,dy, \quad i=\{1,\cdots,n\} \tag{2.2.8}$$

for any ball $B_r(x)$ with $\overline{B_r(x)}\subset\mathbb{R}^n_+$, $r>0$, $x\in\mathbb{R}^n_+$. Let $B_{t/4}(x)\subset\mathbb{R}^n_+$ be the ball with center at x and radius $t/4=x_n/3$. It follows that $B_{t/4}(x)\subset Q_t(x')\times[t/2,t]$ where $Q_t(x')$ is the cube in $\mathbb{R}^{n-1}$ with center at x' and side-length $t>0$. We may appeal to (2.2.8) and write

$$\begin{aligned}
|\partial_i v(x)| &\le C_n\left(\fint_{B_{t/4}(x)}|\partial_i v|^2\,dy\right)^{1/2}\\
&\le C_n t^{-n/2}\left(\int_{Q_t(x')}\int_{t/2}^t|\nabla v|^2\,dy_n\,dy'\right)^{1/2}\\
&\le C_n t^{-(n+1)/2}\left(\int_{Q_t(x')}\int_{t/2}^t y_n|\nabla v|^2\,dy_n\,dy'\right)^{1/2}\\
&\le C_n t^{-1}\left(t^{1-n}\int_{Q_t(x')}\int_0^t y_n|\nabla v(y,y_n)|^2\,dy_n\,dy'\right)^{1/2},
\end{aligned}$$

which in turn implies (2.2.7) and shows (2.2.4) in view of (2.2.1). □

Remark 2.2.2. Estimate (2.2.7) can alternatively be derived from the integral representation of v and the cancellation property

$$\int_{\mathbb{R}^{n-1}}\nabla^\ell P_{x_n}(x'-y')\,dy' = 0 \ \text{ for all } \ (x',x_n)\in\mathbb{R}^n_+, \ \text{ for all } \ \ell\in\mathbb{N}. \tag{2.2.9}$$

2.2.2 The Poisson equation

This section is devoted to the study of the Poisson equation with source term in the space $\mathbf{Y}$ collecting functions F defined on $\mathbb{R}^n_+$ such that the quantity $\|F\|_{\mathbf{Y}}$ is finite,

$$\|F\|_{\mathbf{Y}} := \sup_{x_n>0} x_n^2 \|F(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \sup_{(x',x_n)\in\mathbb{R}^n_+} x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |F| dy_n dy'.$$

It is clear that $\mathbf{Y}$ equipped with the above norm is complete, hence Banach. Let F be a measurable function in $\mathbb{R}^n_+$ such that

$$\int_{\mathbb{R}^n_+} \frac{y_n |F(y)|}{(1+|y|)^n} dy < \infty. \tag{2.2.10}$$

A solution u to the Poisson equation $-\Delta u = F$ in $\mathbb{R}^n_+$, $u = 0$ on $\mathbb{R}^{n-1}$ can explicitly be given by the Newtonian potential

$$u(x) = \mathcal{N}F(x) := \int_{\mathbb{R}^n_+} G(x,y)F(y)dy$$

where $G(\cdot,\cdot)$ is the Green kernel for the Laplacian in the upper half-space $\mathbb{R}^n_+$ explicitly given by

$$G(x,y) = \gamma_n \begin{cases} \left[|x-y|^{-(n-2)} - |x-y^*|^{-(n-2)} \right] & \text{if } \ n \geq 3 \\ \log|x-y| - \log|x-y^*| & \text{if } \ n = 2 \end{cases}, \quad \gamma_n = \begin{cases} \frac{1}{(n-2)\sigma_n} & \text{if } \ n \geq 3 \\ \frac{1}{2\pi} & \text{if } \ n = 2 \end{cases}$$

for $x \in \overline{\mathbb{R}^n_+}$, $y \in \mathbb{R}^n_+$, $x \neq y$ where σ_n is the surface area of the unit sphere of $\mathbb{R}^n$ and $y^* = (y_1, ..., -y_n)$ is the reflection of the point y across the hyperplane $\{y_n = 0\}$. From the explicit expression of $G(\cdot,\cdot)$, we can deduce the following upper-bound estimates (see e.g. [Wid67, Lemma 3.5])

1. For every $x, y \in \mathbb{R}^n_+$, $x \neq y$

$$G(x,y) \leq C \min\left\{ \frac{\min(x_n, y_n)}{|x-y|^{n-1}}, \frac{x_n y_n}{|x-y|^n}, \frac{1}{|x-y|^{n-2}} \right\}.$$

2. For every $x, y \in \mathbb{R}^n_+$, $x \neq y$

$$|\nabla G(x,y)| \leq \min\left\{ |x-y|^{1-n}, y_n |x-y|^{-n} \right\}.$$

3. For each $k \in \mathbb{N}^n$,

$$|\nabla^k G(x,y)| \leq |x-y|^{2-|k|-n} \ \text{ for all } \ x, y \in \mathbb{R}^n_+, x \neq y.$$

It should be observed that functions in $\mathbf{Y}$ satisfy (2.2.10). Our next lemma deals with the mapping properties of the Green potential.

Lemma 2.2.3. *The Newtonian potential $\mathcal{N}$ maps $\mathbf{Y}$ boundedly into $\mathbf{X}$ i.e. for any $F \in \mathbf{Y}$, $\mathcal{N}F \in \mathbf{X}$ and in addition, there holds the estimate*

$$\|\mathcal{N}F\|_{\mathbf{X}} \leq C\|F\|_{\mathbf{Y}} \tag{2.2.11}$$

where the constant C only depends on the dimension n.

Proof. Once again, due to the scaling and translation invariance nature of (2.2.11), its validity simplifies to that of the following localized bound

$$|\mathcal{N}F(0,1)| + |\nabla\mathcal{N}F(0,1)| + \left\|y_n^{1/2}|\nabla\mathcal{N}F|\right\|_{L^2(B_1(0)\times(0,\infty))} \leq C\|F\|_{\mathbf{Y}}$$

whose proof is divided into two steps.

Step 1. *The inequality $|\mathcal{N}F(0,1)| \leq C\|F\|_{\mathbf{Y}}$.* By definition of $\mathcal{N}$, we have

$$\begin{aligned}|\mathcal{N}F(0,1)| &\leq \int_{\mathbb{R}^{n-1}}\int_0^\infty G(e_n,y)|F(y)|dy_n'dy, \quad e_n = (0,...0,1)\\ &= I_1 + I_2 + I_3 + I_4\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{B_1(0)}\int_0^{1/2} G(e_n,y)|F(y)|dy_n dy', \quad I_2 = \int_{B_1(0)}\int_{1/2}^{2} G(e_n,y)|F(y)|dy_n dy',\\ I_3 &= \int_{B_1^c(0)}\int_0^{2} G(e_n,y)|F(y)|dy_n dy', \quad I_4 = \int_{\mathbb{R}^{d-1}}\int_2^{\infty} G(e_n,y)|F(y)|dy_n dy'.\end{aligned}$$

In what follows, we repeatedly make use of the above upper-bound estimates on the Green function $G(\cdot,\cdot)$

$$\begin{aligned}I_1 &= \int_{B_1(0)}\int_0^{1/2} G(e_n,y)|F(y)|dy_n dy'\\ &\leq C\int_{B_1(0)}\int_0^{1/2} \frac{y_n}{(|y'|^2+(1-y_n)^2)^{\frac{n-1}{2}}}|F(y)|dy_n dy'\\ &\leq C2^{n-1}\int_{B_1(0)}\int_0^{1/2} y_n|F(y)|dy_n dy'\\ &\leq C\|F\|_{\mathbf{Y}}.\end{aligned}$$

Moving on, we have

$$
\begin{aligned}
I_2 &= \int_{B_1(0)} \int_{1/2}^{2} G(e_n, y)|F(y)|dy_n dy' \\
&\le C \int_{B_1(0)} \int_{1/2}^{2} \frac{|F(y)|dy_n dy'}{(|y'|^2 + (1-y_n)^2)^{\frac{n-2}{2}}} \\
&\le C \sup_{y_n>0} y_n^2 \|F(\cdot, y_n)\|_{L^\infty(\mathbb{R}^{n-1})} \int_{B_1(0)} \int_{1/2}^{2} |y'|^{2-n} dy_n dy' \\
&\le C\|F\|_{\mathbf{Y}} \int_{|y'|\le 1} \int_{1/2}^{2} |y'|^{2-n} dy_n dy' \\
&\le C\|F\|_{\mathbf{Y}}.
\end{aligned}
$$

To estimate I_3, cover $B_1^c(0) = \mathbb{R}^{n-1} \setminus B_1(0)$ with the family of cubes $\{Q_1(z')\}_{z'\in\mathbb{Z}^{n-1}}$ centered at z', $|z'| > 1$ with side length 1. It follows that

$$
\begin{aligned}
I_3 &\le \int_{\mathbb{R}^{n-1}\setminus B_1(0)} \int_0^2 G(e_n, y)|F(y)|dy_n dy' \\
&\le C \sum_{\substack{z'\in\mathbb{Z}^{n-1} \\ |z'|>1}} \int_{Q_1(z')\cap B_1^c(0)} \int_0^2 \frac{y_n|F(y)|dy_n dy'}{(|y'|^2 + (1-y_n)^2)^{n/2}} \\
&\le C \sum_{\substack{z'\in\mathbb{Z}^{n-1} \\ |z'|>1}} |z'|^{-n} \int_{Q_1(z')} \int_0^2 y_n|F(y)|dy_n dy' \\
&\le C\|F\|_{\mathbf{Y}}.
\end{aligned}
$$

Finally, noticing that for all $y_n \ge 2$, $y_n - 1 \ge y_n/2$ we obtain that

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^{n-1}} \int_2^\infty G(e_n, y)|F(y)|dy_n dy' \\
&\le C \int_{\mathbb{R}^{n-1}} \int_2^\infty \frac{y_n|F(y)|}{(|y'|^2 + (y_n-1)^2)^{n/2}} dy_n dy' \\
&\le C\|y_n^2 F\|_{L^\infty(\mathbb{R}^n_+)} \int_{\mathbb{R}^{n-1}} \int_2^\infty \frac{dy_n dy'}{y_n[|y'|^2 + y_n^2]^{n/2}} \\
&\le C\|F\|_{\mathbf{Y}} \left(\int_{\mathbb{R}^{n-1}} (1 + |z'|^2)^{-n/2} dz' \right) \left(\int_2^\infty y_n^{-2} dy_n \right) \\
&\le C\|F\|_{\mathbf{Y}}.
\end{aligned}
$$

The strategy employed above via decomposition of the integral domain can also be used to prove the *local pointwise gradient estimate*. Indeed, the relation

$$\nabla\mathcal{N}F(x)=\int_{\mathbb{R}^n_+}\nabla G(x,y)F(y)dy$$

holds in the sense of distributions so that by utilizing the pointwise bound (2) and splitting the above integral exactly as before in the same regions to get, say J_i, $i=1,2,3,4$, we obtain the desired estimate. We note however, that to estimate J_2, one rather invokes the property that $|\nabla G(x,\cdot)|\in L^{\frac{n}{n-1},\infty}(\mathbb{R}^n_+)$ uniformly for any $x\in\mathbb{R}^n_+$. Here, $L^{p,\infty}(\mathbb{R}^n)$ denotes the Lorentz space defined as the set of measurable functions f such that $\sup_E|E|^{1/p-1}\int_E|f(y)|dy$ is finite where the supremum is taken over all open subsets of $\mathbb{R}^n$. This concludes step 1.

Step 2. *The energy estimate.*
One may proceed here as in the proof of Lemma 2.2.1 using the Green's kernel bounds together with the usual cut-off procedure on F but there is an alternative shorter argument which allows us to derive this energy-type bound. In fact, there are two different estimates leading to the desired L^2-bound, namely

$$\|\mathcal{N}F\|_{L^\infty(\mathbb{R}^n_+)}\le C\|F\|_{\mathbf{Y}},$$

whose validity has already been justified in step 1 and the second bound

$$\left\|x_n^{1/2}|\nabla\mathcal{N}F|\right\|^2_{L^2(\mathbb{R}^n_+)}\le\|\mathcal{N}F\|_{L^\infty(\mathbb{R}^n_+)}\|x_nF\|_{L^1(\mathbb{R}^n_+)}$$

which may be deduced from a priori estimates. We prove the latter estimate by assuming that $F\in\mathbf{Y}$ is smooth and has compact support in $\mathbb{R}^n_+$. Thus $\mathcal{N}F$ is smooth and it is not difficult to justify the formal calculations below. Now multiply the equation $-\Delta\mathcal{N}F=F$ in $\mathbb{R}^n_+$ by $x_n\mathcal{N}F$ and integrate by parts over $\mathbb{R}^n_+$ to obtain

$$-\int_{\mathbb{R}^n_+}(x_n\mathcal{N}F)\cdot\Delta\mathcal{N}F\,dx=\int_{\mathbb{R}^n_+}x_nF\cdot\mathcal{N}F\,dx.$$

The left hand side of this identity further simplifies to

$$\begin{aligned}-\int_{\mathbb{R}^n_+}(x_n\mathcal{N}F)\cdot\Delta\mathcal{N}F\,dx&=\sum_{i=1}^m\sum_{j=1}^n\int_{\mathbb{R}^n_+}\partial_j(x_d\mathcal{N}F_i)\partial_j\mathcal{N}F_i\,dx\\&=\sum_{i=1}^m\sum_{j=1}^n\int_{\mathbb{R}^n_+}\left(x_n|\partial_j\mathcal{N}F_i|^2+\partial_jx_n\mathcal{N}F_i\partial_j\mathcal{N}F_i\right)dx\\&=\int_{\mathbb{R}^n_+}x_n|\nabla\mathcal{N}F|^2dx+\sum_{i=1}^m\int_{\mathbb{R}^n_+}\partial_n(\mathcal{N}F_i)\mathcal{N}F_i\,dx\\&=\int_{\mathbb{R}^n_+}x_n|\nabla\mathcal{N}F|^2dx.\end{aligned}$$

As such, with the aid of Hölder's inequality, this implies

$$\int_{\mathbb{R}^n_+} x_n |\nabla \mathcal{N} F|^2 dx = \int_{\mathbb{R}^n_+} x_n F \cdot \mathcal{N} F dx$$
$$\leq \|\mathcal{N} F\|_{L^\infty(\mathbb{R}^n_+)} \|x_n F\|_{L^1(\mathbb{R}^n_+)}$$

which completes this particular step and finishes the proof of Lemma 2.2.3. □

Next, one shall prove that the solution u satisfies the condition $u(x) \in N$ a.e. $x \in \mathbb{R}^n_+$. This is done by invoking a suitable maximum principle (for unbounded domains). Let L be a uniformly elliptic operator in $\Omega \subset \mathbb{R}^n$ (possibly unbounded). We say that L satisfies the maximum principle if for $u \in W^{2,n}_{loc}(\Omega)$,

$$\|u\|_{L^\infty(\Omega)} < \infty, \;\; Lu \geq 0 \;\text{ in }\; \Omega \;\text{ and }\; \limsup_{x \to P} u(x) \leq 0 \;\text{ for every }\; P \in \partial\Omega$$

implies

$$u \leq 0 \;\text{ in }\; \Omega.$$

It is well-known that if Ω is unbounded such that its complement set $\mathbb{R}^n \setminus \Omega$ contains an open infinite cone, then $L = \Delta + c$ for c non-positive function satisfies the maximum principle (see [BCN97, Lemma 2.1]). A first step towards proving the geometric constraint is to estimate the distance (in the L^∞-norm) between the solution and the target manifold N.

Proposition 2.2.4. *Assume that f is a measurable bounded map into N, compact smooth manifold. There exists a constant $C > 0$ independent of f with*

$$\operatorname{dist}(v(x), N) \leq C \|f\|_{L^\infty(\mathbb{R}^{n-1})} \tag{2.2.12}$$

for all $x \in \mathbb{R}^n_+$. For all $\Lambda > 0$, there exists $C_1 > 0$ depending on Λ and N such that if f belongs to $BMO(\mathbb{R}^{n-1}, N)$, then

$$\operatorname{dist}(v(x), N) \leq C_1 \|f\|_{BMO(\mathbb{R}^{n-1})} + \Lambda \tag{2.2.13}$$

for every $x \in \mathbb{R}^n_+$. Here, v is the Poisson extension of f.

Proof. We only give the proof of (2.2.13) for the first statement (2.2.12) directly follows from the fact that the distance function is evaluated with respect to the sup norm. Pick a real number $\ell > 0$, fix $x = (x', x_n)$ in $\mathbb{R}^n_+$ and put $\overline{f_x} = ⨍_{B_\ell(0)} f(x' - x_n y') dy'$. Owing to the triangle inequality, one has the bound $\operatorname{dist}(v(x), N) \leq \operatorname{dist}(v(x', x_n), \overline{f_x}) + \operatorname{dist}(\overline{f_x}, N)$. Since $f(y') \in N$ for all $y' \in \mathbb{R}^{n-1}$, we find that

$$\operatorname{dist}(\overline{f_x}, N) \leq |\overline{f_x} - f(x' - x_n z')| \;\text{ for any }\; z' \in B_\ell(0),$$

from which we easily deduce the bound

$$\operatorname{dist}(\overline{f_x}, N) \le \|f\|_{BMO(\mathbb{R}^{n-1})}. \tag{2.2.14}$$

Also note that the Poisson kernel for the Laplace operator obeys $P_{x_n}(x') = x_n^{1-n} P(x'/x_n)$ with $|P(x')| \le \dfrac{C}{(|x'|^2+1)^{n/2}}$. This permits us to write

$$v(x) = \int_{\mathbb{R}^{n-1}} P(y') f(x' - x_n y') dy',$$

from which it follows that

$$\begin{aligned}
|v(x) - \overline{f_x}| &= \left| \int_{\mathbb{R}^{n-1}} P(y')[f(x' - x_n y') - \overline{f_x}] dy' \right| \\
&= \left| \left(\int_{B_\ell(0)} + \int_{\mathbb{R}^{n-1} \setminus B_\ell(0)} \right) P(y')[f(x' - x_n y') - \overline{f_x}] dy' \right| \\
&\le C_n \int_{B_\ell(0)} \frac{|f(x' - x_n y') - \overline{f_x}|}{(|y'|^2+1)^{n/2}} dy' + 2\|f\|_{L^\infty(\mathbb{R}^{n-1})} \int_{\mathbb{R}^{n-1} \setminus B_\ell(0)} |P(y')| dy' \\
&\le C_n \ell^{n-1} \|f\|_{BMO(\mathbb{R}^{n-1})} + C_N \int_\ell^\infty \frac{r^{n-2}}{(1+r^2)^{n/2}} dr
\end{aligned}$$

where the second estimate is a simple consequence of the compactness of N. With an appropriate choice of the radius ℓ depending on Λ and N, one can achieve

$$C_N \int_\ell^\infty \frac{r^{n-2}}{(1+r^2)^{n/2}} dr \le \Lambda.$$

Consequently, one has

$$|v(x) - \overline{f_x}| \le C' \|f\|_{BMO(\mathbb{R}^{n-1})} + \Lambda; \quad C' := C'(\Lambda, N),$$

which combined with (2.2.14) gives the desired estimate. □

2.2.3 Reformulation of the problem

The Dirichlet problem for (2.1.1) coupled with a boundary condition (2.1.2) can be recast as

$$u(x) = v(x) + \mathcal{N}(\Gamma(u)(\nabla u, \nabla u))(x); \quad u(x) \in N \ \text{ a.e } \ x \in \mathbb{R}^n_+ \tag{2.2.15}$$

where v represents the harmonic extension of f and $\mathcal{N}$ the Newtonian potential. Thus, for boundary data which are small in the L^∞-norm (and BMO-semi-norm) one can uniquely solve Eq. (2.2.15) via Banach fixed point argument. However, there is an incompatibility which emanates from the fact that u is thought of as an $\mathbb{R}^m$-valued map whereas the second fundamental form $\Gamma(\cdot)$ must be defined on N. To override this, we construct an extension of Γ to the entire space $\mathbb{R}^m$. In this respect, of significance to us is the nearest point projection map whose Hessian is expressed in terms of the second fundamental form Γ. We know (see for instance [Sim96, Appendix to chapter 2, Theorem 1]) that if N is a compact smooth manifold isometrically embedded in $\mathbb{R}^m$, then N has a ρ-neighborhood in $\mathbb{R}^m$ of the form $U_\rho = \{z \in \mathbb{R}^m : \mathrm{dist}(z,N) < \rho\}$ such that the projection $\mathcal{P}_N$ which maps a point $z \in U_\rho$ to the closest point in N is well-defined and smooth. In addition, it satisfies a number of properties which we partially recall below:

($\mathbf{a}_1$) $\mathcal{P}_N(z) \in N,\ \ z - \mathcal{P}_N(z) \in (T_{\mathcal{P}_N(z)}N)^\perp,\ \ |\mathcal{P}_N(z) - z| = dist(z,N)$ for all $z \in U_\rho$

($\mathbf{a}_2$) $|y - z| > dist(z,N)$ for any $y \in N \setminus \{\mathcal{P}_N(z)\}$, for all $z \in U_\rho$

($\mathbf{a}_3$) $\mathcal{P}_N(z+y) = z$ for $z \in N$, $y \in (T_zN)^\perp$, $|y| < \rho$ and $D_V\mathcal{P}_N|_z = P^\perp_{\mathcal{P}_N(z)}(V)$, $z \in U_\rho$, $V \in \mathbb{R}^m$

($\mathbf{a}_4$) $Hess\ \mathcal{P}_N(z)(V_1,V_2) = -\Gamma(z)(V_1,V_2)$ for $z \in N$ and $V_1, V_2 \in T_zN$

where D_V stands for the directional derivative in the direction of V, $P^\perp_{\mathcal{P}_N(z)}$ denotes the orthogonal projection of $\mathbb{R}^m$ onto $T_{\mathcal{P}_N(z)}N$ and $Hess\ \mathcal{P}_N(z)$ denotes the Hessian of $\mathcal{P}_N$ at z. We then extend the second fundamental form Γ as follows: take a smooth extension of the projection $\mathcal{P}_N$, say $\mathcal{P} \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$ so that $\mathcal{P}$ restricted to U_ρ coincides with $\mathcal{P}_N$ and define the extension $\widetilde{\Gamma}$ of Γ by

$$\widetilde{\Gamma}(z)(V,W) = -Hess\ \mathcal{P}(z)(V,W),\ z \in \mathbb{R}^m;\ \ V,W \in T_z\mathbb{R}^m.$$

In the special case $N = \mathbb{S}^{m-1}$, the nearest point projection map $\mathcal{P}_{\mathbb{S}^{m-1}}$ can be realized explicitly. Clearly, the set $U_{\frac{1}{4}} = \{u \in \mathbb{R}^m : \frac{3}{4} \leq |u| \leq \frac{5}{4}\}$ is a neighborhood of $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$ and one may consider $\mathcal{P}_{\mathbb{S}^{m-1}} : U_{\frac{1}{4}} \to \mathbb{S}^{m-1}$, $u \mapsto \dfrac{u}{|u|}$. Now, introduce the operator $\mathcal{S}$ defined by

$$\mathcal{S}u(x) = v(x) + \mathcal{N}[\widetilde{\Gamma}(u)(\nabla u, \nabla u)](x), \quad x \in \mathbb{R}^n_+. \tag{2.2.16}$$

Note that the above formulation contains the information at boundary. Our problem then becomes that of finding a map $u = (u_1, ..., u_m)$ such that

$$u = \mathcal{S}u \ \text{ in } \ \mathbb{R}^n_+;\ \ u \in N \ \text{ a.e. in } \mathbb{R}^n_+. \tag{2.2.17}$$

In the sequel, we study some basic properties of the operator $\mathcal{S}$, especially those required for an eventual application of the Banach fixed point theorem. To this end, since $v \in \mathbf{X}$ thanks to Lemma 2.2.1 consider the closed ball $B^{\mathbf{X}}_\epsilon(v) \subset \mathbf{X}$ centered at v with radius ε,

$$B^{\mathbf{X}}_\varepsilon(v) = \{u \in \mathbf{X} : \|u - v\|_{\mathbf{X}} \leq \varepsilon\}.$$

Lemma 2.2.5. *Assume that the Dirichlet data $f : \mathbb{R}^{n-1} \to N$ satisfies $\|f\|_{L^\infty(\mathbb{R}^{n-1})} \le \varepsilon$. For all u in the ball $B^{\mathbf{X}}_{\varepsilon}(v)$, there exists $C > 0$ depending only on n with $\|u\|_{\mathbf{X}} \le C\varepsilon$.*

The proof of this result immediately follows from Lemma 2.2.1. Likewise, we have

Lemma 2.2.6. *Given $f \in BMO(\mathbb{R}^{n-1}, N)$ with $\|f\|_{BMO(\mathbb{R}^{n-1})^m} \le \varepsilon$, the following estimates hold, namely*

$$|u|_{\mathbf{X}} \le c\varepsilon, \;\; \|u\|_{L^\infty(\mathbb{R}^n_+)} \le C \;\; \textit{for all} \;\; u \in B^{\mathbf{X}}_{\varepsilon}(v)$$

where $c := c(n) > 0$ and $C := C(\varepsilon, N) > 0$.

The next result establishes the mapping properties of $\mathcal{S}$ and arises as a direct consequence of the Lemmas 2.2.5 and 2.2.6.

Lemma 2.2.7. *There exists $\varepsilon' > 0$ such that the operator $\mathcal{S}$ maps $B^{\mathbf{X}}_{\varepsilon'}(v)$ into itself whenever $f \in L^\infty(\mathbb{R}^{n-1})$ satisfies the smallness condition $\|f\|_{L^\infty(\mathbb{R}^{n-1})} \le \varepsilon'$. Furthermore, if $\|f\|_{BMO(\mathbb{R}^{n-1})} \le \varepsilon'$ then $\mathcal{S} : B^{\mathbf{X}}_{\varepsilon'}(v) \to B^{\mathbf{X}}_{\varepsilon'}(v)$ continuously with respect to the semi-norm $|\cdot|_{\mathbf{X}}$.*

Proof. Let $\varepsilon_0 > 0$ and let $u \in B^{\mathbf{X}}_{\varepsilon_0}(v)$, we want to achieve $\|\mathcal{S}u - v\|_{\mathbf{X}} \le \varepsilon_0$. Taking into account (2.2.16) and by using the potential estimate from Lemma 2.2.3, it follows that

$$\begin{aligned}
\|\mathcal{S}u - v\|_{\mathbf{X}} &= \|\mathcal{N}\widetilde{\Gamma}(u)(\nabla u, \nabla u)\|_{\mathbf{X}} \le C\|\widetilde{\Gamma}(u)(\nabla u, \nabla u)\|_{\mathbf{Y}} \\
&= C\Bigg(\sup_{x_n>0} x_n^2 \|\widetilde{\Gamma}(u)(\nabla u, \nabla u)(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \\
&\qquad \sup_{(x',x_n)\in\mathbb{R}^n_+} x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\widetilde{\Gamma}(u)(\nabla u, \nabla u)(y', y_n)| dy_n dy' \Bigg) \\
&\le C\Bigg(\sup_{x_n>0} x_n^2 \|\nabla u(\cdot, x_n)\|^2_{L^\infty(\mathbb{R}^{n-1})} + \\
&\qquad \sup_{(x',x_n)\in\mathbb{R}^n_+} x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla u(y', y_n)|^2 dy_n dy' \Bigg) \\
&\le C\Bigg(\|x_n \nabla u(\cdot, x_n)\|_{L^\infty(\mathbb{R}^n_+)} + \\
&\qquad \sup_{(x',x_n)\in\mathbb{R}^n_+} \left(x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla u(y', y_n)|^2 dy_n dy' \right)^{1/2} \Bigg)^2 \\
&\le C\|u\|^2_{\mathbf{X}} \le C\varepsilon_0^2 \le \varepsilon_0
\end{aligned}$$

due to Lemma 2.2.5 as long as ε_0 is chosen small enough. Mimicking the preceding proof, one obtains the second part of the Lemma (relying in this case on Lemma 2.2.6) whenever f is sufficiently small in BMO. □

Lemma 2.2.8. *Let $\varepsilon' > 0$ be as in Lemma 2.2.7. There exists $\varepsilon_0 \in (0, \varepsilon')$ with the property that if $\|f\|_{L^\infty(\mathbb{R}^{n-1})} \le \varepsilon_0$ then the operator $\mathcal{S} : B^{\mathbf{X}}_{\varepsilon_0}(v) \to B^{\mathbf{X}}_{\varepsilon_0}(v)$ is a contraction map, that is, there exists $\theta \in (0,1)$ with*

$$\|\mathcal{S}u - \mathcal{S}w\|_{\mathbf{X}} \le \theta \|u - w\|_{\mathbf{X}} \text{ for all } u, w \in B^{\mathbf{X}}_{\varepsilon_0}(v).$$

Proof. By linearity of $\mathcal{N}$ and in light of Lemma 2.2.3, we have that

$$\begin{aligned}
\|\mathcal{S}u - \mathcal{S}w\|_{\mathbf{X}} &= \left\|\mathcal{N}[\widetilde{\Gamma}(u)(\nabla u, \nabla u) - \widetilde{\Gamma}(w)(\nabla w, \nabla w)]\right\|_{\mathbf{X}} \\
&\le C\left\|(\widetilde{\Gamma}(u)(\nabla u, \nabla u) - \widetilde{\Gamma}(u)(\nabla w, \nabla w)) + (\widetilde{\Gamma}(u)(\nabla w, \nabla w) - \widetilde{\Gamma}(w)(\nabla w, \nabla w))\right\|_{\mathbf{Y}} \\
&\le C(J_1 + J_2)
\end{aligned}$$

where

$$J_1 = \left\||\nabla(u - w)|(|\nabla u| + |\nabla w|)\right\|_{\mathbf{Y}} \text{ and } J_2 = \left\||u - w||\nabla w|^2\right\|_{\mathbf{Y}}.$$

We estimate J_1 using Hölder's inequality and Lemma 2.2.5 as follows.

$$\begin{aligned}
J_1 &= \sup_{x_n > 0} x_n^2 \left\||\nabla(u - w)|(|\nabla u| + |\nabla w|)(\cdot, x_n)\right\|_{L^\infty(\mathbb{R}^{n-1})} + \\
&\qquad \sup_{x' \in \mathbb{R}^{n-1}, x_n > 0} x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla(u - w)|(|\nabla u| + |\nabla w|) dy_n dy' \\
&\le \sup_{x_n > 0} x_n \|\nabla(u - w)(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \sup_{x_n > 0} x_n \Big(\|\nabla u(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \|\nabla w(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \Big) + \\
&\qquad \sup_{x' \in \mathbb{R}^{n-1}, x_n > 0} \left(x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n (|\nabla u| + |\nabla w|)^2 dy_n dy' \right)^{1/2} \cdot \\
&\qquad \sup_{x' \in \mathbb{R}^{n-1}, x_n > 0} \left(x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla(u - w)|^2 dy_n dy' \right)^{1/2} \\
&\le C(|u|_{\mathbf{X}} + |w|_{\mathbf{X}}) \left(\sup_{x_n > 0} x_n \|\nabla(u - w)(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \right. \\
&\qquad \left. \sup_{x' \in \mathbb{R}^{n-1}, x_n > 0} \left(x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla(u - w)|^2 dy_n dy' \right)^{1/2} \right) \\
&\le C\varepsilon_0 \left(\sup_{x_n > 0} x_n \|\nabla(u - w)(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \right. \\
&\qquad \left. \sup_{x' \in \mathbb{R}^{n-1}, x_n > 0} \left(x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla(u - w)|^2 dy_n dy' \right)^{1/2} \right) \\
&\le C\varepsilon_0 \|u - w\|_{\mathbf{X}}.
\end{aligned}$$

Estimating J_2 does require the use of the inequality $a^2 + b^2 \le (a+b)^2$, $a, b \ge 0$ and Lemma 2.2.5. Indeed,

$$
\begin{aligned}
J_2 &= \left\| |\nabla w|^2 |u-w| \right\|_{\mathbf{Y}} \\
&= \sup_{x_n>0} x_n^2 \left\| |\nabla w|^2 |u-w|(\cdot, x_n) \right\|_{L^\infty(\mathbb{R}^{n-1})} + \sup_{x' \in \mathbb{R}^{n-1}, x_n>0} x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla w|^2 |u-w| dy_n dy' \\
&\le \sup_{x_n>0} x_n^2 \|\nabla w(\cdot, x_n)\|^2_{L^\infty(\mathbb{R}^{n-1})} \|u-w\|_{L^\infty(\mathbb{R}^n_+)} + \\
&\qquad\qquad \|u-w\|_{L^\infty(\mathbb{R}^n_+)} \sup_{x' \in \mathbb{R}^{n-1}, x_n>0} x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla w|^2 dy_n dy' \\
&\le \|u-w\|_{L^\infty(\mathbb{R}^n_+)} \Bigg(\sup_{x_n>0} x_n \|\nabla w(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \\
&\qquad\qquad \sup_{x' \in \mathbb{R}^{n-1}, x_n>0} \left(x_n^{1-n} \int_{B_{x_n}(x')} \int_0^{x_n} y_n |\nabla w|^2 dy_n dy' \right)^{1/2} \Bigg)^2 \\
&\le C \|u-w\|_{L^\infty(\mathbb{R}^n_+)} \|w\|^2_{\mathbf{X}} \\
&\le C\varepsilon_0^2 \|u-w\|_{\mathbf{X}}.
\end{aligned}
$$

Summarizing, we find that

$$
\|\mathcal{S}u - \mathcal{S}w\|_{\mathbf{X}} \le C\varepsilon_0(1+\varepsilon_0)\|u-w\|_{\mathbf{X}}.
$$

One can make $\theta = C\varepsilon_0(1+\varepsilon_0) < 1$ if ε' is chosen sufficiently small. This achieves the proof of Lemma 2.2.8. □

Following the lines of the above proof, we easily deduce the following.

Lemma 2.2.9. *Let $\varepsilon' > 0$ be the number in Lemma 2.2.7. There exists $\varepsilon_1 \in (0, \varepsilon')$ and $\theta_0 \in (0,1)$ such that whenever $f \in BMO(\mathbb{R}^{n-1})$ with $f(x') \in N$ a.e. $x' \in \mathbb{R}^{n-1}$ satisfies $\|f\|_{BMO(\mathbb{R}^{n-1})} \le \varepsilon_1$, the operator $\mathcal{S} : B^{\mathbf{X}}_{\varepsilon_1}(v) \to B^{\mathbf{X}}_{\varepsilon_1}(v)$ is a θ_0-contraction map, that is,*

$$
\|\mathcal{S}u - \mathcal{S}w\|_{\mathbf{X}} \le \theta_0 \|u-w\|_{\mathbf{X}} \ \textit{for all} \ \ u, w \in B^{\mathbf{X}}_{\varepsilon_1}(v).
$$

2.3 Proofs of the main results

This section aims at proving Theorems 2.1.2 and 2.1.3 by making use of the auxiliary results derived in the previous section.

Proof of Theorem 2.1.2

A simple application of the contraction principle establishes the existence and uniqueness of solutions. The main task is to show that the solution u satisfies the constraint $u \in N$.

Proof. In light of Lemmas 2.2.7 and 2.2.8 and the Banach fixed-point Theorem, there exists $\varepsilon_0 := \varepsilon_0(N,n)$ such that for $\|f\|_{L^\infty(\mathbb{R}^{n-1})} \le \varepsilon_0$, Eq. (2.2.17) admits a unique small solution in $\mathbf{X}$. Now, we need to show that this solution lies in N using Proposition 2.2.4. As announced in Section 2.2, we first show that the distance from the solution u to N can be appropriately controlled so that u lives in a tubular neighborhood of N. Let $x \in \mathbb{R}^n_+$, by virtue of Proposition 2.2.4, we have that

$$\begin{aligned}
\operatorname{dist}(u(x),N) &\le \operatorname{dist}(v(x),N) + \sup_{x\in\mathbb{R}^n_+} |u(x)-v(x)| \\
&\le c\|f\|_{L^\infty(\mathbb{R}^{n-1})} + \|u-v\|_{L^\infty(\mathbb{R}^n_+)} \\
&\le c\varepsilon_0 + c'\|u\|^2_{\mathbf{X}} \\
&\le C\varepsilon_0(1+\varepsilon_0).
\end{aligned}$$

This implies that $u \in U_{\rho_0}$ with $C\varepsilon_0(1+\varepsilon_0) < \rho_0$ (which may be chosen small as far as ε_0 is sufficiently small). As a consequence, one obtains the following identity

$$\Delta u = \nabla^2 \mathcal{P}_N(u)(\nabla u, \nabla u) \ \text{ in } \ \mathbb{R}^n_+, \tag{2.3.1}$$

from ($\mathbf{a}_4$) (see section 2.3). Now define for $z \in U_{\rho_0}$, the map $\Upsilon_N(z) = z - \mathcal{P}_N(z)$ and observe that the conclusion immediately follows if $\Upsilon_N(u)$ vanishes identically. The existence theory reveals that the gradient of the solution u to the Dirichlet problem (2.1.1)-(2.1.2) is locally bounded; this qualitative property as pointed out in Remark 2.1.5 entails higher regularity of u. In fact, $u \in C^\infty_{loc} \cap L^\infty(\mathbb{R}^n_+)$, $\Upsilon_N(u)$ is bounded and the following holds in the weak sense

$$\begin{aligned}
\Delta\left(\frac{1}{2}|\Upsilon_N(u)|^2\right) &= \langle \Upsilon_N(u), \Delta\Upsilon_N(u)\rangle + |\nabla\Upsilon_N(u)|^2 \\
&= \Big\langle \Upsilon_N(u), \nabla^2\Upsilon_N(u)(\nabla u,\nabla u)\Big\rangle + \langle \Upsilon_N(u), \nabla\Upsilon_N(u)(\Delta u)\rangle + |\nabla\Upsilon_N(u)|^2 \\
&= -\Big\langle \Upsilon_N(u), \nabla^2\mathcal{P}_N(u)(\nabla u,\nabla u)\Big\rangle - \langle \Upsilon_N(u), \nabla\Upsilon_N(u)(\Delta u)\rangle + |\nabla\Upsilon_N(u)|^2 \\
&= |\nabla\Upsilon_N(u)|^2
\end{aligned}$$

where we have successively used besides the formulas $\nabla\Upsilon_N(z)(p) = (Id - \nabla\mathcal{P}_N(z))(p)$ and $\nabla^2\Upsilon_N(z)(p,q) = -\nabla^2\mathcal{P}_N(z)(p,q)$ for all $p,q \in \mathbb{R}^m$, the identity (2.3.1) together with the properties ($\mathbf{a}_1$) and ($\mathbf{a}_3$). Indeed, it holds that

$$\Upsilon_N(u) \in (T_{\mathcal{P}_N(u)}N)^\perp \quad \text{and} \quad \nabla\mathcal{P}_N(u)(\nabla^2\mathcal{P}_N(u)(\nabla u,\nabla u)) \in T_{\mathcal{P}_N(u)}N$$

for $u \in U_{\rho_0}$. On the other hand, since $\mathcal{P}_N$ coincides with the identity map of N at the boundary, it follows that $\Upsilon_N(u) = 0$ on $\mathbb{R}^{n-1}$. Hence, $G(u) = \frac{1}{2}|\Upsilon_N(u)|^2$ is a bounded subharmonic function in $\mathbb{R}^n_+$, one can apply the maximum principle to obtain the conclusion $u = \mathcal{P}_N(u) \in N$. The proof of Theorem 2.1.2 is now complete. □

Proof of Theorem 2.1.3

Here, we argue similarly as before given that the auxiliary results used in the proof of Theorem 2.1.2 have analogous versions for data sitting in $BMO(\mathbb{R}^{n-1})$.

Proof. Thanks to Lemmas 2.2.7 and 2.2.9, an application of the Banach fixed point Theorem shows that Eq. (2.2.17) has a unique solution u in $\mathcal{B}^{\mathbf{X}}_{C\varepsilon_1}(v) = \{u \in \mathbf{X} : \|u - v\|_{\mathbf{X}} \leq C\varepsilon_1\}$ for some constant $C > 0$ whenever f satisfies the smallness condition $\|f\|_{BMO(\mathbb{R}^{n-1})} \leq \varepsilon_1$. In the next lines, we prove that $u \in N$. In effect, it follows from Proposition 2.2.4 (applied with $\Lambda = \varepsilon_1$) that

$$
\begin{aligned}
\operatorname{dist}(u(x), N) &\leq \operatorname{dist}(v(x), N) + \|u - v\|_{\mathbf{X}} \\
&\leq C_1\varepsilon_1 + \varepsilon_1 + C\varepsilon_1 \\
&\leq C_2\varepsilon_1
\end{aligned}
$$

for any $x \in \mathbb{R}^n_+$. This shows in particular that $u \in U_\rho$ provided $C_2\varepsilon_1 < \rho$. Therefore, as before we can define $\Upsilon_N(u)$ and observe that $\Upsilon_N(u)\big|_{\mathbb{R}^{n-1}} = 0$. Then by a similar argument to that performed above we conclude that $u(x) \in N$ a.e. $x \in \mathbb{R}^n_+$. □

2.4 Large data situation in bounded domains

In this section we attempt to answer the question whether or not the Dirichlet problem for weakly harmonic maps equations subject to "large" data in $BMO(\mathbb{R}^{n-1})$ or $L^\infty(\mathbb{R}^{n-1})$ is solvable (in the sense described in Section 2.1). In such a scenario, it is clear, based on the theory which has been developed earlier that the norm of the solution in our function space $\mathbf{X}$ may grow, leading to a nonexistence result. This motivates the consideration of stable smooth solutions of equation (2.1.1) and more specifically, boundary data which are to a certain sense close to the latter – we shall be more precise regarding this statement in subsequent lines. Existence of stable harmonic maps is not a restricting assumption as exemplified by the class of harmonic maps into targets with nonpositive sectional curvature (see Remark 2.4.2 below). Another class of stable harmonic maps includes local minimizers of the energy E which are smooth under suitable conditions on boundary data. Opting for a perturbation technique the main difficulty comes from the nonlinear geometric constraints in the problem which we bypass by considering an appropriate extension problem and maximum principle arguments as performed in the proof of Theorems 2.1.2 and 2.1.3. However, we will need the source manifold to be bounded unlike the case treated earlier involving the half-space domain. Consider the weakly harmonic maps equation

$$-\Delta u = \Gamma(u)(\nabla u, \nabla u) \ \text{ in } \ \Omega \tag{2.4.1}$$

subject to the Dirichlet boundary condition

$$u|_{\partial\Omega} = f \tag{2.4.2}$$

where $\Omega \subset \mathbb{R}^n$ is a $C^{1,\alpha}$, ($\alpha \in (0,1]$) bounded domain. Our main Theorems in Section 2.1 claims that if f has a small L^∞ or BMO norm, then Problem (2.4.1)-(2.4.2) is solvable in $\mathbf{W}$. Before we define the notion of stability, recall that a weak solution to BVP (2.4.1)-(2.4.2) is a map $u \in L^\infty \cap W^{1,2}_f(\Omega, N)$ such that

$$\int_\Omega \left\{ (\nabla u, \nabla \phi) + \sum_{j=1}^n \Gamma(u)(\partial_j u, \partial_j u) \cdot \phi \right\} dx = 0 \tag{2.4.3}$$

for all $\phi \in L^\infty \cap W^{1,2}_0(\Omega, \mathbb{R}^m)$ where $(\cdot,\cdot)$ denotes the standard scalar product in $\mathbb{R}^{nm}$ and

$$L^\infty \cap W^{1,2}_f(\Omega, N) = \{ v \in L^\infty \cap W^{1,2}(\Omega, \mathbb{R}^m) : v \in N \ \text{ a.e. on } \ \Omega \ \text{ and } \ v\big|_{\partial\Omega} = f \}.$$

Definition 2.4.1. Let u be a nontrivial weak solution of Eq. (2.4.1). We say that u is strictly stable if

$$Q_u(\phi) := \int_\Omega \left\{ |\nabla \phi|^2 + \sum_{j=1}^n R^N(\phi, \partial_j u)\partial_j u \cdot \phi \right\} dx \geq M \|\phi\|^2_{L^2(\Omega)} \tag{2.4.4}$$

for every $\phi \in L^\infty \cap W^{1,2}_0(\Omega, \mathbb{R}^m)$, $\phi \in T_u N$ a.e. and for some $M > 0$. In case $M = 0$, we say that u is stable.

In the above definition $R^N(\cdot,\cdot)\cdot$ denotes the curvature tensor of N which at each point u of N is a trilinear map on $T_u N \times T_u N \times T_u N$ to $T_u N$. The notion of strict stability involving a weighted L^2-norm on the R.H.S. of (2.4.4) appeared in the study of harmonic maps with prescribed set of singularities [HM92]. Note that the integral expression in (2.4.4) up to a change in sign in the second term due to the symmetry feature $(R^N(U,V)W, Z) = -R^N((U,V)Z, W)$ represents the second variation formula of the energy functional associated to (2.4.1), see [Sch83]. Thus stability of u and nonnegativeness of the second variation of the energy are formally two equivalent notions.

Remark 2.4.2. If the target N has a nonpositive sectional curvature (in the sense of distributions) then any weakly harmonic map $u : \Omega \to N$ is stable, that is, u satisfies (2.4.4) with $M = 0$.

2.4.1 Function spaces and linear estimates

Recall the definition of the space of bounded mean oscillations defined on the boundary

$$BMO(\partial\Omega) = \left\{ f \in L^1_{loc}(\partial\Omega) : \|f\|_{BMO(\partial\Omega)} = \sup_{S\subset\partial\Omega} \sigma(S)^{-1} \int_S |f(\xi) - f_S| d\sigma(\xi) \right\}$$

where $S = S_r(\zeta)$, $\zeta \in \partial\Omega$ is the surface ball centered at ζ with radius $r > 0$ and

$$f_S = \fint_S f\, d\sigma = \frac{1}{\sigma(S)} \int_S f\, d\sigma.$$

Call $T(S_r(\zeta)) = \Omega \cap B_r(\zeta)$ the Carleson region associated to the surface ball $S_r(\zeta)$ and let $r_0 > 0$. A measure μ in Ω is termed Carleson if there exists a constant $C > 0$ depending on r_0 such that for all $r \le r_0$, $\mu(T(S_r)) \le C\sigma(S_r)$ with $d\sigma$ being the Lebesgue surface measure. Fabes & Neri in [FN80] showed that harmonic functions on Ω whose traces belong to $BMO(\partial\Omega)$ can also be characterized by means of Carleson measures. Indeed, u is harmonic in Ω and the measure $d\mu(x) = |\nabla u(x)|^2 d(x) dx$ ($d(x)$ is the distance from x to $\partial\Omega$) is Carleson if and only if u is the Poisson integral of $f \in BMO(\partial\Omega)$. In addition,

$$\sup_{S\subset\partial\Omega} \left(\sigma(S)^{-1} \int_{T(S)} d(y) |\nabla u(y)|^2 dy \right)^{1/2} \le C \|f\|_{BMO(\partial\Omega)} \tag{2.4.5}$$

where the supremum runs over all surface balls $S \subset \partial\Omega$. Next, we introduce some function spaces which will be useful in the sequel. For $\xi \in \partial\Omega$ such that $d(x) = |x - \xi|$, we use the shorthand $T(S)$ for the Carleson region associated to the surface ball $S_{d(x)}(\xi)$, i.e. $T(S_{d(x)}(\xi))$.

Definition 2.4.3. We say that $w : \Omega \to \mathbb{R}^m$ belongs to $\mathbf{W}$ if the quantity $\|w\|_{\mathbf{W}}$ is finite where

$$\|w\|_{\mathbf{W}} = \sup_{x\in\Omega} |w(x)| + \sup_{x\in\Omega} d(x) |\nabla w(x)| + \sup_{x\in\Omega} \left(d(x)^{1-n} \int_{T(S)} d(y) |\nabla w(y)|^2 dy \right)^{1/2}.$$

We denote by $\mathbf{Z}$ the space of functions $F : \Omega \to \mathbb{R}^m$ such that

$$\|F\|_{\mathbf{Z}} = \sup_{x\in\Omega} d(x)^2 |F(x)| + \sup_{x\in\Omega} d(x)^{1-n} \int_{T(S)} d(y) |F(y)| dy < \infty.$$

Observe that the function spaces $\mathbf{W}$ and $\mathbf{Z}$ are simply the analogs of $\mathbf{X}$ and $\mathbf{Y}$ in bounded domains respectively, which have been used earlier in Section 1, the fundamental difference being that the distance function to the boundary in this case is a bounded function. Therefore, it is not surprising that some of the results derived in Section 2 persist here. This is the case of the lemma below which provides some relevant information on the solutions to both homogeneous (subject to BMO and L^∞ boundary data) and inhomogeneous (with source term in $\mathbf{Z}$) problems for the Laplacian. In what follows we set $[w]_{\mathbf{W}}$ to be

$$[w]_{\mathbf{W}} := \sup_{x\in\Omega} d(x)|\nabla w(x)| + \sup_{x\in\Omega}\left(d(x)^{1-n}\int_{T(S)} d(y)|\nabla w(y)|^2 dy\right)^{1/2}.$$

Lemma 2.4.4. *Let $F \in \mathbf{Z}$ and u such that $-\Delta u = F$ in Ω with $u|_{\partial\Omega} = f$. The following conclusions hold:*

1. *If $f \in BMO(\partial\Omega)$, then u satisfies*

$$[u]_{\mathbf{W}} \le C(\|f\|_{BMO(\partial\Omega)} + \|F\|_{\mathbf{Z}}).$$

2. *For f in $L^\infty(\partial\Omega)$, u is an element of $\mathbf{W}$ and it holds that*

$$\|u\|_{\mathbf{W}} \le c(\|f\|_{L^\infty(\partial\Omega)} + \|F\|_{\mathbf{Z}}).$$

The generic constants C and c appearing in the above estimates only depend on the dimension and Ω.

Proof. We distinguish between two steps.
Step 1. Assume that $F = 0$. We prove the corresponding two claims of the lemma.
Let $f \in BMO(\partial\Omega, \mathbb{R}^m)$, we would like to establish the bound $[u]_{\mathbf{W}} \le c\|f\|_{BMO(\partial\Omega)}$. Note that from the Carleson measure characterization of $BMO(\partial\Omega)$, one only needs to verify that the estimate

$$\sup_{x\in\Omega} d(x)|\nabla u(x)| \le c\|f\|_{BMO(\partial\Omega)}$$

is valid. Pick $x_0 \in \Omega$, put $R_0 = \frac{d(x_0)}{2}$ and assume that $B_{3R_0}(x_0) \subset\subset \Omega$. By harmonicity of $D_i u$ $(i = 1, 2, \cdots, d)$ and standard interior estimates for the Laplace equation we have

$$\begin{aligned}|\nabla u(x_0)|^2 &\le c \fint_{B_{R_0/4}(x_0)} |\nabla u|^2 dy \\ &\le cR_0^{1-n} d(x_0)^{-1} \int_{B_{R_0/4}(x_0)} |\nabla u|^2 dy.\end{aligned}$$

Let $\xi_0, \xi_y \in \partial\Omega$ such that $d(x_0) = |x_0 - \xi_0|$ and $d(y) = |y - \xi_y|$. Since $|y - x_0| \le d(x_0)/8$, we have $|y - \xi_0| \le 9R_0/4$ and $7d(x_0)/8 \le d(y)$ so that

$$|\nabla u(x_0)|^2 \le c' R_0^{1-n} d(x_0)^{-2} \int_{T(S_{\frac{9R_0}{4}}(\xi_0))} d(y)|\nabla u|^2 dy,$$

from which it follows that

$$d(x_0)^2|\nabla u(x_0)|^2 \le c' R_0^{1-n} \int_{T(S_{9R_0/4}(\xi_0))} d(y)|\nabla u|^2 dy.$$

At this point, as x_0 was chosen arbitrary we simply pass to the supremum over Ω on both sides of the above inequality to deduce the desired estimate. For f bounded, the bounds

$$\|u\|_{L^\infty(\Omega)} \le c\|f\|_{L^\infty(\partial\Omega)} \quad \text{and} \quad \sup_{x\in\Omega} d(x)|\nabla u(x)| \le C\|f\|_{L^\infty(\partial\Omega)} \tag{2.4.6}$$

follows from elliptic interior estimates. Next, we prove the Carleson measure estimate

$$\sup_{x\in\Omega}\left(d(x)^{1-n}\int_{T(S)} d(y)|\nabla u(y)|^2 dy\right)^{1/2} \le C\|f\|_{L^\infty(\Omega)}. \tag{2.4.7}$$

Fix $x \in \Omega$ and let $2S := S_{2d(x)}(\xi)$ be the surface ball with center at $\xi \in \partial\Omega$ and radius $2d(x)$. Denoting by $\mathbf{1}_{2S}$ the characteristic function of $2S$, we make the decomposition $f = \mathbf{1}_{2S}f + (1-\mathbf{1}_{2S})f = f_1 + f_2$ and write correspondingly $u = u_1 + u_2$ for the Poisson extension of f to Ω. We first prove (2.4.7) for u_2 with the aid of the following pointwise decay bound for the Poisson kernel for the Laplacian in Ω:

$$|\nabla_x \mathcal{P}(x,\zeta)| \le \frac{cd(x)}{|x-\zeta|^{n+1}}, \quad \zeta \in \partial\Omega$$

which can be found e.g. in [Ste72]. In effect, let $y \in T(S)$, we have

$$\begin{aligned}
|\nabla u_2(y)| &\le \int |\nabla\mathcal{P}(y,\zeta)||f_2(\zeta)|d\sigma(\zeta) \\
&\le cd(y)\int_{\partial\Omega\setminus(2S)} |y-\zeta|^{-(n+1)}|f(\zeta)|d\sigma(\zeta) \\
&\le cd(y)\|f\|_{L^\infty(\partial\Omega)} \sum_{i=1}^{\infty}\int_{S_i} |y-\zeta|^{-(n+1)} d\sigma(\zeta)
\end{aligned}$$

where $S_i = 2^{i+1}S \setminus 2^iS$. Let $\zeta \in S_i$, we have $2^i d(x) < |\zeta - \xi|$ which by the triangle inequality implies $|y-\zeta| \ge |\xi-\zeta| - |y-\xi| > 2^i d(x) - d(x) \ge 2^{i-1}d(x)$. Hence, we have that

$$\begin{aligned}
|\nabla u_2(y)| &\le cd(y)d(x)^{-2}\|f\|_{L^\infty(\partial\Omega)} \sum_{i=1}^{\infty} 2^{(1-i)(n+1)}2^{(i+1)(n-1)} \\
&\le Cd(x)^{-1}\|f\|_{L^\infty(\partial\Omega)}
\end{aligned}$$

since $d(y) \le d(x)$ whenever $y \in T(S)$. Squaring the above inequality, multiplying both sides by $d(y)$ and integrating over the Carleson region $T(S)$, we obtain

$$\left(d(x)^{1-n}\int_{T(S)} d(y)|\nabla u_2(y)|^2 dy\right)^{1/2} \le C\|f\|_{L^\infty(\partial\Omega)}.$$

This yields the bound we were looking for after taking the supremum over Ω on both sides. To establish the corresponding estimate for u_1, we assume without any restriction that f_1 is supported in $2S$ and that $\|u_1\|_{L^\infty(\Omega)}$ is finite (in view of estimate (2.4.6)). Thus, we have

$$\int_{T(S)} d(y)|\nabla u_1(y)|^2 dy \le C \int_{\Omega} d(y)|\nabla u_1|^2 dy \le C \int_{\partial\Omega} [N^* u_1(\zeta)]^2 d\sigma(\zeta) \tag{2.4.8}$$

where $N^* u_1(\zeta) = \sup_{\Gamma(\zeta)} |u_1(x)|$ is the nontangential maximal function of u_1, $\Gamma(\zeta)$ is the cone in Ω with vertex at $\zeta \in \partial\Omega$. We note that the last estimate in (2.4.8) is due to Dahlberg [Dah80]. Hence, from the mapping properties of the nontangential maximal function in Lebesgue spaces (see [Dah79]) and the fact that Ω is smooth, we find from (2.4.8) that

$$\int_{T(S)} d(y)|\nabla u_1(y)|^2 dy \le C\|f_1\|^2_{L^2(\partial\Omega)} \le C d(x)^{n-1} \|f\|^2_{L^\infty(\partial\Omega)}.$$

This shows that (2.4.7) is valid and completes this part.
Step 2. We prove that any solution u of $-\Delta u = F$ in Ω which vanishes on $\partial\Omega$ satisfies the bound

$$\|u\|_{\mathbf{W}} \le C\|F\|_{\mathbf{Z}}. \tag{2.4.9}$$

Under the condition that Ω satisfies the uniform exterior sphere condition, the Green function G_Ω for Δ satisfies (see [GW82, Theorem 3.3])

$$G_\Omega(x,y) \le C \min\left(\frac{1}{|x-y|^{n-2}}, \frac{d(y)}{|x-y|^{n-1}}, \frac{d(x)d(y)}{|x-y|^n} \right)$$

and

$$|\nabla G_\Omega(x,y)| \le C \min\left(|x-y|^{1-n}, \frac{d(y)}{|x-y|^n} \right).$$

To derive the L^∞-estimate, we may write u as the Green potential of F such that for $x \in \Omega$, we have

$$\begin{aligned} |u(x)| &\le \int_\Omega |G_\Omega(x-y)||F(y)| dy \\ &\le \left\{ \int_{\{y\in\Omega:\ |x-y|\le 2^{-1}d(x)\}} + \int_{\{y\in\Omega:\ |x-y|> 2^{-1}d(x)\}} \right\} |G_\Omega(x-y)||F(y)| dy \\ &= I + II. \end{aligned}$$

Since for any $y \in B_{2^{-1}d(x)}(x)$ we have the inequality $d(y) \geq d(x)/2$, we handle I as follows

$$\begin{aligned} I &\leq c \sup_{x\in\Omega}(d^2(y)|F(y)|) \int_{\Omega\cap B_{2^{-1}d(x)}(x)} d^{-2}(y)|x-y|^{2-n} dy \\ &\leq c\|F\|_{\mathbf{Z}} d^{-2}(x) \int_{B_{2^{-1}d(x)}(x)} |x-y|^{2-n} dy \\ I &\leq c\|F\|_{\mathbf{Z}}. \end{aligned} \tag{2.4.10}$$

In order to estimate the second integral, we cover the set $\left\{y \in \Omega : \ |x-y| > 2^{-1}d(x)\right\}$ with the family of annuli $(A_i)_i$, $A_i = 2^i B_{d(x)}(x) \setminus 2^{i-1}B_{d(x)}(x)$ and use the above pointwise estimate on the Green kernel to arrive at

$$\begin{aligned} II &\leq \sum_{i=0}^{\infty} \int_{A_i} |F(y)| G_\Omega(x-y) dy \\ &\leq C \sum_{i=0}^{\infty} \int_{A_i} \frac{d(x)d(y)|F(y)|}{|x-y|^n} dy \\ &\leq C d^{1-n}(x) \sum_{i=0}^{\infty} 2^{-(i-1)n} \int_{2^i B_{d(x)}(x)} d(y)|F(y)| dy. \end{aligned}$$

It is easy to see that $y \in T(2^{i+1}S)$ whenever $y \in 2^i B_{d(x)}(x)$ so that

$$II \leq \sum_{i=0}^{\infty} 2^{-(i-1)n} 2^{(i+1)(n-1)} (2^{i+1}d(x))^{1-n} \int_{T(2^{i+1}S)} d(y)|F(y)| dy \leq C\|F\|_{\mathbf{Z}}.$$

Combining the latter with (2.4.10) yields the desired L^∞-bound. The estimate of the weighted-sup norm of ∇u is obtained in a similar fashion using the pointwise gradient bounds on the Green kernel, details are omitted. In the same vein, the very last estimate (bound on the Carleson measure norm of u) follows from a much stronger variant which can be obtained via an integration by parts argument (testing the Poisson equation against $d(y)u$) combined with the previous L^∞-estimate. This finishes the proof of Lemma 2.4.4. □

The main result of this section, pertaining to the solvability of the Dirichlet problem (2.4.1) is the following

Theorem 2.4.5. *Let v be a smooth solution of* (2.4.1) *subject to $v\big|_{\partial\Omega} = g \in C^1(\partial\Omega)$ and assume v obeys the strict stability condition* (2.4.4)*. Then there exists $\varepsilon > 0$ such that for any* (*large*) *bounded map $f : \partial\Omega \to \mathbb{R}^m$ satisfying $\|f-g\|_{L^\infty(\partial\Omega)} \leq \varepsilon$, there exists a solution u of the Dirichlet problem* (2.4.1)-(2.4.2) *in $\phi + v + \mathbf{W}$. Moreover, this solution is unique in the ball*

$$B^{\mathbf{W}}_{c_\Omega\varepsilon} = \{u \in \mathbf{W} : \|u - \phi - v\|_{\mathbf{W}} \leq c_\Omega \varepsilon\}$$

for some constant c_Ω depending on the domain only. In particular, the solution u lies in a small neighborhood of v, that is, $\|u-v\|_{\mathbf{W}} \le \tau$ where τ depends on ε and Ω. Here, ϕ denotes the Poisson extension of $(f-g)$ to Ω.

Remark 2.4.6. If f is chosen large in $BMO(\partial\Omega, N)$, then the Poisson extension of $h = f - g$, ϕ_h is also bounded and a similar smallness hypothesis on the BMO-perturbation h yields the existence of a solution u such that $u - v - \phi_h$ is small in $\mathbf{X}$.

The stability condition in Theorem 2.4.5, i.e. (2.4.4) can be replaced by an invertibility condition for the linearized operator associated to $Q_v(\cdot,\cdot)$. Indeed, consider the bilinear form defined for any $\psi, \phi \in W_0^{1,2}(\Omega)$ with $\phi \in T_vN$ by

$$Q_u(\psi,\phi) := \int_\Omega \left\{ (\nabla\psi, \nabla\phi) + \sum_{j=1}^n R^N(\phi, \partial_j u)\partial_j u \cdot \psi \right\} dx.$$

Our next result shows that the conclusion of Theorem 2.4.5 remains valid under a weaker condition.

Proposition 2.4.7. *Let $\phi \in W_0^{1,2}(\Omega)$ with $\phi \in T_vN$. If condition* (2.4.4) *is replaced by the requirement that*

$$Q_v(\psi,\phi) = 0 \quad \forall \psi \in C_0^\infty(\Omega) \Longrightarrow \psi = 0, \tag{2.4.11}$$

then the conclusion of Theorem 2.4.5 remains true.

2.4.2 Idea and structure of the proof of Theorem 2.4.5

We discuss in this part the procedure we adopt in establishing the claims in Theorem 2.4.5. Once again, the plan is to perform a suitable perturbation argument in order to have a setting in which our hypotheses fit. To this end we convert the original equation into a vanishing boundary data problem. Set $h = f - g$ where $g = v\big|_{\partial\Omega}$ and denote by ϕ_h the Poisson extension of h to Ω. Make the ansatz $w = u - v - \phi_h$ and realize that w solves the boundary value problem

$$-\Delta w = \Gamma(v + w + \phi_h)(\nabla(v + w + \phi_h), \nabla(v + w + \phi_h)) - \Gamma(v)(\nabla v, \nabla v) \text{ in } \Omega, \; w\big|_{\partial\Omega} = 0$$

which can be transformed into the following Dirichlet problem

$$\begin{aligned} -\mathscr{L}_v w &= F(v, \phi_h, w) \text{ in } \Omega \\ w &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.4.12}$$

where $-\mathscr{L}_v$ is the operator acting on vector-valued functions defined on Ω and given by

$$\mathscr{L}_v w := \Delta w + \sum_{j=1}^n R^N(w, \partial_j v)\partial_j v$$

while the nonlinearity $F(v,\phi_h,\cdot)$ reads

$$F(v,\phi_h,w) = \Gamma(v+w+\phi_h)(\nabla(v+w+\phi_h),\nabla(v+w+\phi_h)) - \Gamma(v)(\nabla v,\nabla v) + \sum_{j=1}^{n} R^N(w,\partial_j v)\partial_j v.$$

The main focus is now on problem (2.4.12) for it is clear that from its solvability directly follows the statement of Theorem 2.4.5. Assume for a moment that $-\mathscr{L}_v$ defined as an operator from $\mathbf{W}$ to $\mathbf{Z}$ can be inverted so that problem (2.4.12) is reformulated as a fixed point equation

$$\text{find } w \in \mathbf{W}: \; w = (-\mathscr{L}_v)^{-1} \circ \widetilde{F}(v,\phi_h,w) \;\text{ in }\; \Omega$$

where $\widetilde{F}(v,\phi_h,\cdot)$ is a suitable extension of $F(v,\phi_h,\cdot)$ which we define in subsequent lines. This allows us to avoid the geometric constraints which we treat separately. Hence, for so long as the composition $\mathcal{K}(v,\phi_h,\cdot) := (-\mathscr{L}_v)^{-1} \circ \widetilde{F}(v,\phi_h,\cdot)$ with v and ϕ_h as described above can be shown to be a strict contraction mapping, we are done. With other words, this amounts to saying that if altogether the following key estimate

$$\|w\|_{\mathbf{W}} \leq C\|\mathscr{L}_v w\|_{\mathbf{Z}} \tag{2.4.13}$$

combined with the contraction property: $\exists\ \theta_0 \in (0,1)$ such that

$$\|\widetilde{F}(v,\phi_h,w_1) - \widetilde{F}(v,\phi_h,w_2)\|_{\mathbf{Z}} \leq \theta_0\|w_1 - w_2\|_{\mathbf{W}} \tag{2.4.14}$$

for w_1 and w_2 in some ball of $\mathbf{W}$ hold true, then Theorem 2.4.5 readily appears as a consequence of an application of the Banach fixed point theorem. Note that estimate (2.4.14) only makes sense once we know that $\widetilde{F}(v,\phi_h,w)$ is an element of $\mathbf{Z}$ for any $w \in \mathbf{W}$. However, $-\mathscr{L}_v$ does not possess these mapping properties. In contrast, what we do know is that if the nonlinearity $\widetilde{F}(v,\phi_h,\cdot)$ maps into the Sobolev space $W^{-1,2}(\Omega)$, then the Dirichlet problem (2.4.12) is uniquely solvable in $W_0^{1,2}(\Omega)$. Moreover, the inverse operator $(-\mathscr{L}_v)^{-1} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is continuous, that is,

$$\|w\|_{W_0^{1,2}(\Omega)} \leq C\|\mathscr{L}_v w\|_{W^{-1,2}(\Omega)^m}.$$

This is a consequence of the stability condition (2.4.4), since it entails the coercivity of the continuous bilinear form associated to $-\mathscr{L}_v$ on $W_0^{1,2}(\Omega)$ and an application of the Lax-Milgram theorem. On the other hand, it can be checked that neither $W_0^{1,2}(\Omega)$ is a subspace of $\mathbf{W}$ nor $\widetilde{F}(v,\phi_h,w)$ does lie in the Sobolev space $W^{-1,2}(\Omega)$ whenever $w \in W_0^{1,2}(\Omega)$. It rather seems plausible to establish that the nonlinearity in (2.4.12) is well-behaved with respect to the topology of the function space $\mathbf{Z}$. By this, we mean for every $w \in \mathbf{W}$ we have

$$\widetilde{F}(v,\phi_h,w) \in \mathbf{Z}, \tag{2.4.15}$$

which in turn shows that estimate (2.4.14) is legitimate. Summarizing, we shall prove that the nonlinearity $\widetilde{F}(v,\phi_h,\cdot)$ satisfies the needed properties (2.4.14) and (2.4.15) – this will constitute the first part of the proof whereas the second segment aims at showing that the operator $-\mathscr{L}_v$ is invertible and obeys the continuity property (2.4.13). A decisive point in achieving these facts is that one has solvability of $-\mathscr{L}_v w = H$ in $W_0^{1,2}(\Omega)$ for H in the dual space $W^{-1,2}(\Omega)$.

To define $\widetilde{F}(v, \phi_h, \cdot)$, recall the relation between the nearest point projection map and the second fundamental form

$$D^2\mathcal{P}_N(u)(V,W) = -\Gamma(u)(V,W) \tag{2.4.16}$$

whenever $u \in N$ for all $V, W \in T_u N$. Let $\mathcal{P} \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$ be any extension of $\mathcal{P}_N$ such that $\mathcal{P}\big|_{U_\rho} = P_N$ where $U_\rho := \{x \in \mathbb{R}^m : \operatorname{dist}(x, N) < \rho\}$ for some $\rho > 0$ sufficiently small and define $\widetilde{\Gamma}(z)(V,W) = D^2\mathcal{P}(z)(V,W)$ for $z \in \mathbb{R}^m$. Then

$$\widetilde{F}(v,\phi_h,w) = \widetilde{\Gamma}(v+w+\phi_h)(\nabla(v+w+\phi_h), \nabla(v+w+\phi_h)) - \widetilde{\Gamma}(v)(\nabla v, \nabla v) + \sum_{j=1}^{n} R^N(w, \partial_j v)\partial_j v. \tag{2.4.17}$$

Proof of Theorem 2.4.5. **Part 1**. We state a lemma in which one quantifies the statement (2.4.15) and also shows that (2.4.14) is indeed true.

Lemma 2.4.8. *Let v falling under the scope of Theorem 2.4.5 with $v = g$ on $\partial\Omega$ and denote by ϕ_h the Poisson extension of $h = f - g$. Then the map $\widetilde{F}(v, \phi_h, \cdot)$ sends* $\mathbf{W}$ *onto* $\mathbf{Z}$ *and there exists a dimensional constant $C := C(\Omega)$ and $K := K(v)$ such that*

$$\begin{aligned}\|\widetilde{F}(v,\phi_h,w)\|_{\mathbf{Z}} \le C\|w\|_{\mathbf{W}}(1+\|w\|_{\mathbf{W}}) + C\|\phi_h\|_{\mathbf{W}}(1+\|\phi_h\|_{\mathbf{W}}) + \\ C(\|w\|_{\mathbf{W}} + \|\phi_h\|_{\mathbf{W}})(\|w\|_{\mathbf{W}}^2 + \|\phi_h\|_{\mathbf{W}}^2 + K(v)\Big).\end{aligned} \tag{2.4.18}$$

In addition, if $\|h\|_{L^\infty(\Omega)} \le \varepsilon$ for some $\varepsilon > 0$ small, then there exists $\tau := \tau(\Omega, \varepsilon)$ and $\eta \in (0,1)$ such that for all w_1, w_2 in the closed ball $B_\tau^{\mathbf{W}}(0) = \{w \in \mathbf{W} : \|w\|_{\mathbf{W}} \le \tau\}$, we have

$$\|\widetilde{F}(v,\phi_h,w_1) - \widetilde{F}(v,\phi_h,w_2)\|_{\mathbf{Z}} \le \eta\|w_1 - w_2\|_{\mathbf{W}}.$$

Proof of Lemma 2.4.8. Observe that (2.4.17) can further be written as

$$\widetilde{F}(v,\phi_h,w) = F_1(v,\phi_h,w) + F_2(v,\phi_h,w) + F_3(v,\phi_h,w)$$

where

$$F_1(v,\phi_h,w) = \widetilde{\Gamma}(v)(\nabla(w+v+\phi), \nabla(w+v+\phi)) - \widetilde{\Gamma}(v)(\nabla v, \nabla v)$$

$$F_2(v,\phi_h,w) = \sum_{j=1}^{n} R^N(w, \partial_j v)\partial_j v,$$

$$F_3(v,\phi_h,w) = \widetilde{\Gamma}(w+v+\phi_h)(\nabla(w+v+\phi_h), \nabla(w+v+\phi_h)) - \widetilde{\Gamma}(v)(\nabla(w+\phi_h+v), \nabla(w+\phi_h+v))$$

with v and ϕ_h are as described above. Since $\widetilde{\Gamma}(\cdot,\cdot)$ and $R^N(\cdot,\cdot)\cdot$ are smooth maps, one can easily verify that

$$|F_1(v,\phi_h,w)| \le c_1(|\nabla v||\nabla(w+\phi_h)| + |\nabla(w+\phi_h)|^2) \tag{2.4.19}$$

where $c_1 := c_1(\|v\|_{L^\infty(\Omega)})$ and there exist $c_2, c_3 > 0$ with

$$|F_2(v,\phi_h,w)| \le c_2|w||\nabla v|^2, \;\; |F_3(v,\phi_h,w)| \le c_3|w+\phi_h||\nabla(w+v+\phi_h)|^2. \tag{2.4.20}$$

Let $w \in \mathbf{W}$, we have

$$\|\widetilde{F}(v,\phi_h,w)\|_{\mathbf{Z}} = \sup_{x\in\Omega} d(x)^2|\widetilde{F}(v,\phi_h,w)(x)| + \sup_{x\in\Omega} d(x)^{1-n}\int_{T(S)} d(y)|\widetilde{F}(v,\phi_h,w)(y)|dy := I_1 + I_2.$$

Making use of (2.4.19) and (2.4.20), we separately estimate each of the above terms as follows. For the first term,

$$\begin{aligned}
I_1 &\le C\Bigg[\sup_{x\in\Omega} d(x)^2(|\nabla v||\nabla(w+\phi_h)| + |\nabla(w+\phi_h)|^2) + \sup_{x\in\Omega} d(x)^2(|w||\nabla v|^2) + \\
&\qquad\qquad \sup_{x\in\Omega} d^2(x)(|w+\phi_h||\nabla(w+\phi_h+v)|^2)\Bigg] \\
&\le C\Bigg[C(v)\sup_{x\in\Omega} d(x)(|\nabla\phi_h| + |\nabla w|) + \sup_{x\in\Omega} d^2(x)(|\nabla w|^2 + |\nabla\phi_h|^2) + C(v)\|w\|_{L^\infty(\Omega)} + \\
&\qquad\qquad (\|w\|_{L^\infty(\Omega)} + \|\phi_h\|_{L^\infty(\Omega)})\sup_{x\in\Omega} d^2(x)(|\nabla w|^2 + |\nabla\phi_h|^2 + |\nabla v|^2)\Bigg] \\
&\le C\|w\|_{\mathbf{W}}(1+\|w\|_{\mathbf{W}}) + C\|\phi_h\|_{\mathbf{W}}(1+\|\phi_h\|_{\mathbf{W}}) + C(\|w\|_{\mathbf{W}} + \|\phi_h\|_{\mathbf{W}})(\|w\|^2_{\mathbf{W}} + \|\phi_h\|^2_{\mathbf{W}} + C(v)\Big).
\end{aligned}$$

Taking into account the hypotheses on v, it follows that

$$
\begin{aligned}
I_2 &= \sup_{x\in\Omega} d(x)^{1-n} \int_{T(S)} d(y)|\widetilde{F}(v,\phi_h,w)(y)|dy \\
&\le C\Bigg[\sup_{x\in\Omega} d(x)^{1-n} \int_{T(S)} d(y)(|\nabla v||\nabla(w+\phi_h)| + |\nabla(w+\phi_h)|^2)dy + \\
&\sup_{x\in\Omega} d(x)^{1-n} \int_{T(S)} d(y)|w||\nabla v|^2 dy + \sup_{x\in\Omega} d(x)^{1-n} \int_{T(S)} d(y)|w+\phi_h||\nabla(w+\phi_h+v)|^2 dy\Bigg] \\
&\le C\Bigg[C(v)\sup_{x\in\Omega} d(x)^{\frac{1-n}{2}} \left\|d(\cdot)^{1/2}|\nabla w|\right\|_{L^2(T(S))} + \sup_{x\in\Omega} d(x)^{\frac{1-n}{2}} \left\|d(\cdot)^{1/2}|\nabla \phi_h|\right\|_{L^2(T(S))} + \\
&\quad \sup_{x\in\Omega} d(x)^{1-n} \left\|d(\cdot)^{1/2}|\nabla w|\right\|^2_{L^2(T(S))} + \sup_{x\in\Omega} d(x)^{1-n} \left\|d(\cdot)^{1/2}|\nabla \phi_h|\right\|^2_{L^2(T(S))} + \\
&\quad C(v)\|w\|_{L^\infty(\Omega)} + (\|w\|_{L^\infty(\Omega)} + \|\phi_h\|_{L^\infty(\Omega)})\Big(\sup_{x\in\Omega} d(x)^{1-n} \left\|d(\cdot)^{1/2}|\nabla w|\right\|^2_{L^2(T(S))} + \\
&\qquad\qquad \sup_{x\in\Omega} d(x)^{1-n} \left\|d(\cdot)^{1/2}|\nabla \phi_h|\right\|^2_{L^2(T(S))} + C(v)\Big)\Bigg] \\
&\le C\Bigg[\|w\|_{\mathbf{W}}(1+\|w\|_{\mathbf{W}}) + C\|\phi_h\|_{\mathbf{W}}(1+\|\phi_h\|_{\mathbf{W}}) + (\|w\|_{\mathbf{W}} + \|\phi_h\|_{\mathbf{W}})(\|w\|^2_{\mathbf{W}} + \|\phi_h\|^2_{\mathbf{W}} + C(v))\Bigg].
\end{aligned}
$$

Collecting the bounds on I_1 and I_2 and adding them up we conclude on the validity of (2.4.18). Next, we estimate $\widetilde{F}(v,\phi_h,w_1) - \widetilde{F}(v,\phi_h,w_2)$ for $w_1, w_2 \in \mathbf{W}$ under the condition that $h = f - g$ is small in the L^∞-norm. Write

$$\widetilde{F}(v,\phi_h,w_1) - \widetilde{F}(v,\phi_h,w_2) := A + B + C$$

where

$$
\begin{aligned}
A &= \widetilde{\Gamma}(w_1+v+\phi_h)(\nabla(w_1+v+\phi_h), \nabla(w_1+v+\phi_h)) - \\
&\qquad \widetilde{\Gamma}(w_2+v+\phi_h)(\nabla(w_1+v+\phi_h), \nabla(w_1+v+\phi_h)) \\
B &= \widetilde{\Gamma}(w_2+v+\phi_h)(\nabla(w_1+v+\phi_h), \nabla(w_1+v+\phi_h)) - \\
&\qquad \widetilde{\Gamma}(w_2+v+\phi_h)(\nabla(w_2+v+\phi_h), \nabla(w_2+v+\phi_h)) \\
C &= \sum_{i=1}^{n} R^N(w_2 - w_1, \partial_i v)\partial_i v
\end{aligned}
$$

so that it suffices to estimate each of these quantities in $\mathbf{Z}$. For the same reasons as above, we have

$$\begin{aligned}
\|A\|_{\mathbf{Z}} &\le C\sup_{x\in\Omega} d(x)^2|(w_1-w_2)(x)||\nabla(w_1+v+\phi_h)|^2+\\
&\qquad\qquad C\sup_{x\in\Omega} d(x)^{1-n}\int_{T(S)} d(y)|w_1-w_2||\nabla(w_1+v+\phi_h)|^2dy\\
&\le C\|w_1-w_2\|_{L^\infty(\Omega)}\Bigg(\sup_{x\in\Omega} d^2(x)|\nabla(w_1+v+\phi_h)|^2+C(v)+\\
&\qquad \sup_{x\in\Omega} d(x)^{1-n}\left\|d(\cdot)^{1/2}|\nabla w_1|\right\|^2_{L^2(T(S))}+\sup_{x\in\Omega} d(x)^{1-n}\left\|d(\cdot)^{1/2}|\nabla\phi_h|\right\|^2_{L^2(T(S))}\Bigg)\\
&\le C\|w_1-w_2\|_{\mathbf{W}}\Big(\|w_1\|^2_{\mathbf{W}}+\|\phi_h\|^2_{\mathbf{W}}+C(v)\Big) \qquad (2.4.21)
\end{aligned}$$

Observe that B can further be written as

$$\begin{aligned}
&\widetilde{\Gamma}(w_2+v+\phi_h)(\nabla(w_1-w_2),\nabla(w_1+v+\phi_h))+\widetilde{\Gamma}(w_2+v+\phi_h)(\nabla w_2,\nabla(w_1-w_2))+\\
&\qquad\qquad\widetilde{\Gamma}(w_2+v+\phi_h)(\nabla(w_1-w_2),\nabla(v+\phi_h))
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
\|B\|_{\mathbf{Z}} &\le C\sup_{x\in\Omega} d(x)^2(|\nabla(w_1-w_2)(x)||\nabla w_1+v+\phi_h|)+C\sup_{x\in\Omega} d(x)^2(|\nabla(w_1-w_2)||\nabla w_2(x)|+\\
&\qquad C\sup_{x\in\Omega} d(x)^2(|\nabla(w_1-w_2(x)||\nabla(v+\phi_h)|)+\\
&\qquad C\sup_{x\in\Omega} d(x)^{1-n}\int_{T(S)} d(y)|\nabla(w_1-w_2)||\nabla(w_1+v+\phi_h)|dy+\\
&\qquad C\sup_{x\in\Omega} d(x)^{1-n}\int_{T(S)} d(y)|\nabla(w_1-w_2)||\nabla w_2|dy+\\
&\qquad C\sup_{x\in\Omega} d(x)^{1-n}\int_{T(S)} d(y)|\nabla(w_1-w_2)||\nabla(v+\phi_h)|dy\\
&\le C\|w_1-w_2\|_{\mathbf{W}}(\|w_1\|_{\mathbf{W}}+\|w_2\|_{\mathbf{W}}+\|\phi_h\|_{\mathbf{W}}+C(v)) \qquad (2.4.22)
\end{aligned}$$

where we have applied Hölder's inequality to estimate the integral terms. Finally, we have

$$\begin{aligned}
\|C\|_{\mathbf{Z}} &\le C\sup_{x\in\Omega} d(x)^2|\nabla v|^2|w_1-w_2|+C\sup_{x\in\Omega} d(x)^{1-n}\int_{T(S)} d(y)|\nabla v|^2|w_1-w_2|dy\\
&\le CC(v)\|w_1-w_2\|_{\mathbf{W}}. \qquad (2.4.23)
\end{aligned}$$

The generic constant appearing in (2.4.21), (2.4.22) and (2.4.23) depends on diam Ω. Recall that ϕ_h is small in $\mathbf{W}$ since h is small in $L^\infty(\partial\Omega)$ from Lemma 2.4.4. For w_1 and w_2 in the closed ball $B^{\mathbf{W}}_\tau(0)$ of $\mathbf{W}$, we deduce in view of the above bounds on A, B and C that the second part of Lemma 2.4.8 holds true. The proof of Lemma 2.4.8 is now complete. □

Part 2. Here we prove the invertibility of the operator $-\mathscr{L}_v$ together with (2.4.13). Let us introduce the operator $L := \Delta + \ell$ acting on $\mathbb{R}^m$-valued functions defined on Ω where ℓ is a smooth linear map. Consider the zero data Dirichlet boundary value problem

$$\begin{aligned} -Lw(x) &= F(x), \quad x \in \Omega \\ w(x) &= 0, \quad x \in \partial\Omega \end{aligned} \tag{2.4.24}$$

where $F \in \mathbf{Z}$. Remark that the operator $\mathscr{L}_v$ has the form of L with

$$\ell(w) = \sum_{j=1}^{n} R^N(w, \partial_j v)\partial_j v.$$

Now if w^0 is a solution of the Poisson equation $-\Delta w^0 = F$ in Ω with zero data at the boundary, then $w^1 = w - w^0$ solves the Dirichlet problem

$$\begin{aligned} -Lw^1 = F_0 := \sum_{j=1}^{n} R^N(w_0, \partial_j v)\partial_j v \ \text{ in } \ \Omega \\ w^1\big|_{\partial\Omega} = 0. \end{aligned} \tag{2.4.25}$$

At this point, we only need to show that w^1 belongs to $\mathbf{W}$ with a corresponding "good" estimate since the solution w^0 is well understood by now due to Lemma 2.4.4. A first step towards this is the following

Claim 2.4.9. F_0 belongs to $\mathbf{Z} \cap W^{-1,2}(\Omega)$.

Let us momentarily defer the proof of this claim and observe that it implies the existence of a unique $w^1 \in W_0^{1,2}(\Omega)$ solving (2.4.25). The extra information $F_0 \in \mathbf{Z}$ will enable us to improve the regularity of w^1 via an iterative scheme. Set $F_1 = \sum_{j=1}^{n} R^N(w^1, \partial_j v)\partial_j v$ and let w^2 be such that $-\Delta w^2 = F_0$ in Ω. One can easily verify that $F_1 \in L^2(\Omega)$ and that $w^3 = w^1 - w^2$ is a solution to the problem $-\Delta w^3 = F_1$ with zero data on $\partial\Omega$. This implies by elliptic regularity theory $w^3 \in W^{2,2}(\Omega)$. Iterating this procedure, we eventually find $w^1 \in \mathbf{W}$. Hence, $w = w^1 + w^0 \in \mathbf{W}$ and $\|w\|_{\mathbf{W}} \le C\|F\|_{\mathbf{Z}}$. This, in concert with part 1, proves (2.4.13). By the method of continuity, $-\mathscr{L}_v$ is invertible. Moreover, for $w_1, w_2 \in B_\tau^{\mathbf{W}}(0)$, we have in view of (2.4.13) and using Lemma 2.4.8 from part 1,

$$\begin{aligned} \|\mathcal{K}(v,\phi,w_1) - \mathcal{K}(v,\phi,w_2)\|_{\mathbf{W}} &\le C\|\widetilde{F}(v,\phi,w_1 - w_2)\|_{\mathbf{Z}} \\ &\le \varepsilon\|w_1 - w_2\|_{\mathbf{W}} \end{aligned}$$

where $\varepsilon = \varepsilon(\tau)$ can be made small if τ is sufficiently small. Thus, $\mathcal{K}(v,\phi,\cdot)$ is a strict contraction mapping. We now verify that $F_0 \in \mathbf{Z} \cap W^{-1,2}(\Omega)$.

$$\begin{aligned}
\|F_0\|_{\mathbf{Z}} &= \sup_{x\in\Omega} d(x)^2 |F_0(x)| + \sup_{x\in\Omega} d(x)^{1-n} \int_{T(S)} d(y)|F_0(y)|dy \\
&= I + II.
\end{aligned}$$

Using the smoothness of R^N and the fact that $w^0 \in \mathbf{W}$, one finds that

$$\begin{aligned}
I &\le \sup_{x\in\Omega} d(x)^2 \left| \sum_{j=1}^{n} R^N(w^0, \partial_j v)\partial_j v \right| \le C(v)\|w^0\|_{\mathbf{W}} \\
&\le C(v)\|F\|_{\mathbf{Z}}.
\end{aligned}$$

For the solid integral, we have

$$\begin{aligned}
II &\le \sup_{x\in\Omega} d(x)^{1-n} \int_{T(S)} d(y) \left| \sum_{j=1}^{n} R^N(w^0, \partial_j v)\partial_j v \right| dy \\
&\le C(v)\|w^0\|_{L^\infty(\Omega)} \\
&\le C(v)\|w^0\|_{\mathbf{W}} \le C(v)\|F\|_{\mathbf{Z}}.
\end{aligned}$$

Now we establish that $F_0 \in W^{-1,2}(\Omega)$. Let $\varphi \in W_0^{1,2}(\Omega)$, using the integration by parts formula and Hölder's inequality, it follows that

$$\begin{aligned}
\left| \langle F_0, \varphi \rangle_{W^{-1,2}, W_0^{1,2}} \right| &= \left| \int_\Omega \left(\sum_{j=1}^{n} R^N(w^0, \partial_j v)\partial_j v\right) \cdot \varphi dx \right| \\
&\le C(v)\|w^0\|_{\mathbf{W}} \|\varphi\|_{L^2(\Omega)} \\
&\le C(v)\|F\|_{\mathbf{Z}} \|\varphi\|_{W^{1,2}(\Omega)}.
\end{aligned}$$

To conclude the proof of Theorem 2.4.5, one needs to show that $w + \phi_h + v \in N$. Since $v \in N$, it holds that

$$\begin{aligned}
\operatorname{dist}(w + \phi_h + v, N) &\le C\|w + \phi_h\|_{L^\infty(\Omega)} \\
&\le C(\|w\|_{\mathbf{W}} + \|\phi_h\|_{\mathbf{W}}) \le C\varepsilon
\end{aligned}$$

for $\varepsilon > 0$ small. Thus $w + \phi_h + v \in U_{\varepsilon'}$, $\varepsilon' = C\varepsilon$ (with $U_{\varepsilon'}$ as defined above) so that one can define $\Upsilon_N(w + \phi_h + v) = w + \phi_h + v - \mathcal{P}_N(w + \phi_h + v)$. It is clear that $\Upsilon_N(w + \phi_h + v)\big|_{\partial\Omega} = 0$ because $(w + \phi_h + v)\big|_{\partial\Omega} \in N$. Moreover, similar calculations to those performed in Section 2.3 reveals that $\Upsilon_N(w + \phi_h + v)$ is subharmonic in Ω. As a consequence, $\Upsilon_N(w + \phi_h + v) = 0$ in Ω. This achieves the Proof of Theorem 2.4.5. □

Proof of Proposition 2.4.7. We first solve the inhomogeneous linear problem $-\mathscr{L}_v w = H$ in $W_0^{1,2}(\Omega)$ for H in $\mathbf{Z}$. The remaining bit of the proof will just be a reprise of the argument in part 2 of the proof of Theorem 2.4.5. Again, write $\mathscr{L}_v := \Delta + \ell$ with $\ell(\cdot) = \sum_{j=1}^{n} R^N(\cdot, \partial_j v)\partial_j v$ and set $K = (-\Delta)^{-1} \circ \ell$. The operator ℓ is bounded and compact from $W_0^{1,2}(\Omega)$ to $L^2(\Omega)$. On the other hand, the inverse Laplacian $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is also bounded and compact so that K can be realized as a linear bounded compact operator from $W_0^{1,2}(\Omega)$ to $L^2(\Omega)$ and $\widetilde{H} = (-\Delta)^{-1} H \in L^2(\Omega)$ since $(-\Delta)^{-1}$ maps continuously $\mathbf{Z}$ into $\mathbf{W}$. Our problem therefore reduces to that of solving

$$\begin{cases} w + Kw = \widetilde{H} \text{ in } \Omega \\ w|_{\partial\Omega} = 0. \end{cases}$$

By virtue of the hypothesis in Proposition 2.4.7, the trivial solution is the only solution of $w + Kw = 0$ in Ω with vanishing boundary data. Hence, existence of a unique solution for the above problem in $W_0^{1,2}(\Omega)$ is a consequence of the Fredholm alternative. The conclusion then follows from the proof of Theorem 2.4.5 (see part 2) as previously mentioned. □

Chapter 3

Well-posedness of Chemotaxis-fluids models

3.1 Introduction

Micro-organisms (e.g. bacteria) have very limited ability to adapt to fluid environments due to their small size. They respond to detectable change by swimming towards specific regions. The orientation mechanism by which they approach or are repelled from a chemical source is known as chemotaxis. When the fluid is incompressible, upon assuming that swimmers contribute at a very small scale to the swimmers-fluid suspension and that hydrodynamics interactions between swimmers (e.g. cell-cell interaction, which can lead to collective motion, see for instance [CM17] and cited works therein) are negligible, the authors in [TCD$^+$05] proposed the following mathematical model

$$\begin{cases} \partial_t c - D_c \Delta c + nf(c) + u \cdot \nabla c = 0 & \text{in } \Omega \times (0,T) \\ \partial_t n - D_n \Delta n + \rho \operatorname{div}\, (n\chi(c)\nabla c) + u \cdot \nabla n = 0 & \text{in } \Omega \times (0,T) \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + n \nabla \Phi + \nabla p = 0 & \text{in } \Omega \times (0,T) \\ \operatorname{div}\, u = 0 & \text{in } \Omega \times (0,T) \end{cases} \tag{3.1.1}$$

where c is the concentration of oxygen, n is the density of cells, u is the velocity field of the fluid governed by the incompressible Navier-Stokes equations with scalar pressure p and viscosity ν. The time-independent gravitational force exerted from a bacteria onto the fluid is modelled through $\nabla\Phi$. The constant ρ represents the magnitude of chemotaxis and D_c, D_n are diffusion coefficients. The function $f(c)$ models the inactivity level caused by a low supply of oxygen and $\chi(c)$ is a suitable cut-off function (usually determined by experiments). The second equation in (3.1.1) describes the mass balance equation for the cells combining the advection effect (modelled through $u \cdot \nabla n$), the chemotactic effect or the migration towards regions of high concentration of oxygen (modelled by $\operatorname{div}\, (n\chi(c)\nabla c)$) and the diffusion of cells (modelled through $D_n\Delta$).

The analysis of the Cauchy problem for Syst. 3.1.1 seems challenging from a mathematical point of view and a lot of effort over the past recent years have been devoted to the understanding of its dynamics with a particular focus on the existence of local and global solutions as well as their qualitative behaviour (long-time asymptotic, stability, blow-up,...). Some of the main challenges arising in the analysis of Syst. 3.1.1 are inherited from the Navier-Stokes equations.

3.1.1 Known results

When $\Omega \subset \mathbb{R}^N$ is a (sufficiently) smooth bounded domain, upon neglecting the contribution of the convection term $u \cdot \nabla u$ in (3.1.1), Lorz [Lor10] obtained the existence of local-in-time weak solutions for the associated initial boundary value problem with no-flux boundary conditions in dimensions $N = 2, 3$ for $\chi(c) \equiv const.$, $\nu = D_c = D_n = \rho = 1$ under some monotonicity and differentiabilily condition on f. Still in absence of the convection term, global well-posedness in $\mathbb{R}^2$ was proved by Duan, Lorz & Markowich [DLM10] for non-constant smooth χ under a smallness assumption on either the gravitational force $\nabla\Phi$ or the initial concentration c_0. Moreover, in the presence of a convection term, they established the existence of classical solutions in $\mathbb{R}^3$ using uniform a priori estimates under a suitable smallness condition on the initial data in $H^3(\mathbb{R}^3)$ and derived time-decay rate of solutions near constant steady states. For $\Omega \subset \mathbb{R}^3$ smooth and bounded, Winkler [Win16] constructed global weak solutions under some structural and strong smoothness assumptions on f and χ. Under very similar requirements, the same author in [Win17] introduced the notion of eventual energy solutions and proved that the initial-boundary value problem for Syst. 3.1.1 (with homogeneous Neumann conditions) admits at least one such solution. Existence of smooth local solutions in higher order Sobolev space and blow-up issues have been considered by Chae, Kang & Lee [CKL13] for $N = 2, 3$. Their result was later extended by Zhang in [Zha14] in the framework of Besov spaces by means of Fourier localisation technique. Regarding large data global existence, we quote the works [LL11, Win12, ZZ14] and references therein. Other interesting related models with inhomogeneous tensor-valued chemotactic sensitivity can be found in [CL16, Win15]. A popular model considers the choices $f(c) = c$, $\chi(c) = 1$; $D_c = D_n = \rho = \nu = 1$ and $\Omega = \mathbb{R}^N$ turning (3.1.1) into the system

$$\begin{cases} \partial_t c - \Delta c + cn + u \cdot \nabla c = 0 & \text{in } \mathbb{R}^N \times (0, T) \\ \partial_t n - \Delta n + \text{div } (n \nabla c) + u \cdot \nabla n = 0 & \text{in } \mathbb{R}^N \times (0, T) \\ \partial_t u - \Delta u + u \cdot \nabla u + n \nabla \Phi + \nabla p = 0 & \text{in } \mathbb{R}^N \times (0, T) \\ \text{div } u = 0 & \text{in } \mathbb{R}^N \times (0, T). \end{cases} \tag{CNS}$$

where $T \in (0, \infty]$. Recently, small data global existence and large data local existence in critical Besov space have been investigated in [CLY15]. Kozono, Miura & Sugiyama [KMS16] obtained global existence and large time asymptotic behaviour of solutions to

(CNS) for initial data

$$[c_0, n_0, u_0] \in L^\infty(\mathbb{R}^N) \times L^{N/2,\infty}(\mathbb{R}^N) \times L^{N,\infty}(\mathbb{R}^N) \text{ with } \nabla c \in L^{N,\infty}(\mathbb{R}^N)$$

when $N \geq 3$ (and $n_0 \in L^1(\mathbb{R}^2)$ when $N = 2$) having a sufficiently small norm. Their proof relies on heat semigroups estimates in weak-Lebesgue spaces combined with the implicit function theorem. Using a Picard iteration argument, the authors in [YFS19] (see also [FP19] for similar results pertaining to a generalized model) extended the latter result by considering small data

$$[c_0, n_0, u_0] \in L^\infty(\mathbb{R}^N) \times \mathcal{N}^{-s_1}_{p_1,\lambda,\infty}(\mathbb{R}^N) \times \mathcal{N}^{-s_2}_{p_2,\lambda,\infty}(\mathbb{R}^N) \text{ with } \nabla c \in \mathcal{N}^{-s_3}_{p_3,\lambda,\infty}(\mathbb{R}^N)$$

for some parameters $s_j > 0$ depending on $0 \leq \lambda < N$, $N \geq 2$ and $1 < p_j < \infty$, $j = 1, 2, 3$. Here $\mathcal{N}^{-s}_{p,\lambda,\infty}(\mathbb{R}^N)$ stands for the homogeneous Besov-Morrey space (see below for the definition). We note that in the aforementioned results, the extra assumption on the gradient of the initial concentration plays a central role and as we will show later, this hypothesis is unnecessary and can be discarded.

The main purpose of this chapter is the study of the well-posedness for the Cauchy problem (CNS) and its generalized model (cf. Syst. D-CNS in Section 3.2). In particular, we are interested in the existence of small data global-in-time and large data local-in-time solutions in the largest scaling and translation invariant function spaces. The method carried out here is dimension independent so that our results are valid in any space dimension larger or equal to two. Our motivation comes from the work by Koch & Tataru [KT01]. Introducing a new function space (see $\mathbf{X}_3$ below) based on the intrinsic properties of solutions, they investigated the local and global well-posedness issues for the incompressible Navier-Stokes equations (the third equation in (CNS) with $\Phi = 0$). More precisely, they proved existence of a unique small global solution under the conditions that the initial data is divergence-free and has small BMO^{-1}-norm and of local solutions for divergence-free initial data in $\overline{VMO^{-1}}$. As observed by the authors in [CG09], their results seem to be the endpoint case for small data global existence. Coming back to (CNS), there is an additional term in the Navier-Stokes equations which causes further difficulty. We also observe that most of the existing local existence theory are obtained under higher regularity assumptions on the initial data. We show later that initial concentration c_0 which are L^∞-close to a uniformly continuous function give rise to local-in-time solutions. This allows one to prescribe, as a byproduct, initial data c_0 which are small perturbations in $L^\infty(\mathbb{R}^N)$, $N \geq 2$ of constants leading to global existence statements.

3.2 Local and global Existence theory

System CNS is scaling and translation invariant provided $\Phi \in \mathcal{S}'(\mathbb{R}^N)$ is such that $\nabla\Phi$ is homogeneous of degree -1. More precisely, if $[c, n, u]$ solves (CNS) (in a classical sense), then $[c_\delta, n_\delta, u_\delta]$ with

$$c_\delta(x,t) = c(\delta x, \delta^2 t),\ n_\delta(x,t) = \delta^2 n(\delta x, \delta^2 t),\ u_\delta(x,t) = \delta u(\delta x, \delta^2 t) \tag{3.2.1}$$

for all $\delta > 0$ is another solution. On the other hand, a weaker requirement on the unknowns c, n and u for (CNS) to make sense is that

$$\begin{cases} c \in L^\infty_{loc}(\mathbb{R}^N \times [0,\infty)), \nabla c \in L^2_{loc}(\mathbb{R}^N \times (0,\infty)) \\ n \in L^2_{loc}(\mathbb{R}^N \times [0,\infty)) \\ u \in L^2_{loc}(\mathbb{R}^N \times [0,\infty)). \end{cases} \tag{3.2.2}$$

Thus we look for initial data $[c_0, n_0, u_0]$ whose caloric extension $[\widetilde{c}, \widetilde{n}, \widetilde{u}]$ satisfy the scaling and translation invariant analogue of condition (3.2.2), that is,

$$\begin{cases} \sup\limits_{t>0} \|\widetilde{c}(t)\|_{L^\infty(\mathbb{R}^N)} + \sup\limits_{x,R>0} R^{-N} \int_{B_R(x)} \int_0^{R^2} |\nabla \widetilde{c}(t,y)|^2 dt dy < \infty & \text{(3.2.3a)} \\ \sup\limits_{x,R>0} R^{2-N} \int_{B_R(x)} \int_0^{R^2} |\widetilde{n}(t,y)|^2 dt dy < \infty & \text{(3.2.3b)} \\ \sup\limits_{x,R>0} R^{-N} \int_{B_R(x)} \int_0^{R^2} |\widetilde{u}(t,y)|^2 dt dy < \infty. & \text{(3.2.3c)} \end{cases}$$

It is well-known that the finiteness of the second term in (3.2.3a) is equivalent to c_0 being an element of $BMO(\mathbb{R}^N)$ with the equivalence of semi-norms, see for instance [Ste72]. However, the requirement that c be bounded in space and time rules out the choice of c_0 in $BMO(\mathbb{R}^N)$. It rather seems plausible to prescribe the initial data c_0 in a subclass namely, in $L^\infty(\mathbb{R}^N)$. On the other hand, condition (3.2.3c) is equivalent to $u_0 \in BMO^{-1}(\mathbb{R}^N)$, see [KT01]. By analogy to the latter cases, one would like to relate condition (3.2.3b) to some class of functions defined on $\mathbb{R}^N$ in an extrinsic manner.

Definition 3.2.1. Let $N > 2$. A tempered distribution f on $\mathbb{R}^N$ is an element of $\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)$ if its caloric extension $\widetilde{f} = e^{t\Delta} f$ satisfies

$$\|f\|_{\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)} := \sup_{x,R>0} \left(|B(x,R)|^{2/N-1} \int_0^{R^2} \int_{B(x,R)} |\widetilde{f}(y,t)|^2 dy dt \right)^{1/2} < \infty. \tag{3.2.4}$$

Carleson measures characterization of square Campanato spaces (see [JXY16]) suggests that $\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)$ may be regarded as the space of derivatives of distributions in the Campanato class $\mathscr{L}_{2,N-2}(\mathbb{R}^N)$.

Lemma 3.2.2. *A tempered distribution f belongs to $\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)$ if and only if there exists $f_j \in \mathscr{L}_{2,N-2}(\mathbb{R}^N)$, $j = 1, \cdots, N$ such that $f = \sum_{j=1}^N \partial_j f_j$.*

The proof of this lemma is postponed to the Appendix for convenience. Let $r > 0$ and the open ball $B_r(x) = \{y \in \mathbb{R}^N : |y - x| < r\}$. For $1 \le p < \infty$, $0 \le \mu < N$, recall the Morrey space $M_{p,\mu}(\mathbb{R}^N)$ defined as

$$M_{p,\mu}(\mathbb{R}^N) = \{f \in L^p_{loc}(\mathbb{R}^N) : \|f\|_{M_{p,\mu}(\mathbb{R}^N)} < \infty\} \tag{3.2.5}$$

where

$$\|f\|_{M_{p,\mu}(\mathbb{R}^N)} := \sup_{x \in \mathbb{R}^N, r>0} r^{-\frac{\mu}{p}} \|f\|_{L^p(B_r(x))}. \tag{3.2.6}$$

With $\overline{f}_{B_r(x)} := \fint_{B_r(x)} f(y)dy$ denoting the integral mean of f, the Campanato space $\mathscr{L}_{p,\lambda}(\mathbb{R}^N)$, $\lambda \in [0, N+p)$ collects all functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that $\|f\|_{\mathscr{L}_{p,\lambda}(\mathbb{R}^N)}$ is finite where

$$\|f\|_{\mathscr{L}_{p,\lambda}(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N, r>0} \left(r^{-\lambda} \int_{B_r(x)} \left| f(x) - \overline{f}_{B_r(x)} \right|^p \right)^{1/p}. \tag{3.2.7}$$

The expression in (3.2.7) defines a semi-norm on $\mathscr{L}_{p,\lambda}(\mathbb{R}^N)$ and upon identifying functions which differ by a real constant, this space becomes Banach. Campanato spaces unify some classical function spaces: the space $\mathscr{L}_{p,\lambda}(\mathbb{R}^N)$ coincides with the space of bounded mean oscillations $BMO(\mathbb{R}^N)$ when $\lambda = N$, reduces to constants when $\lambda \ge N + p$ and is equivalent to the homogeneous Hölder space $C^{0,\alpha}(\mathbb{R}^N)$, $\alpha = \frac{\lambda - N}{p} \in (0,1)$ whenever $N < \lambda < N + p$. We also define the local Campanato space $\mathscr{L}_{p,\lambda;R}(\mathbb{R}^N)$ (resp. local Morrey space $M_{p,\lambda;R}(\mathbb{R}^N)$) by taking in (3.2.7) (resp. in (3.2.6)) balls of radius R and smaller. The space $\mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N)$ is defined analogously and we use the notation $\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)$ for $\mathscr{L}^{-1}_{2,N-2;\infty}(\mathbb{R}^N)$.

Definition 3.2.3. A tempered distribution f is an element of $BMO^{-1}(\mathbb{R}^N)$ if there exists $f = (f_1, \cdots, f_N)$, $f_j \in BMO(\mathbb{R}^N)$, $j = 1, \cdots, N$ such that $f = \sum_{j=1}^N \partial_j f_j$. This space is equipped with the norm

$$\|f\|_{BMO^{-1}(\mathbb{R}^N)} = \inf \left\{ \sum_{j=1}^N \|f_j\|_{BMO(\mathbb{R}^N)} : f = \sum_{j=1}^N \partial_j f_j \right\}.$$

The local space $BMO^{-1}_R(\mathbb{R}^N)$ is defined similarly as above by replacing the BMO semi-norm by its local version. The Sarason space of vanishing mean oscillations is defined as

$$VMO(\mathbb{R}^N) = \left\{ h \in BMO(\mathbb{R}^N) : \lim_{R \to 0} \|h\|_{BMO_R(\mathbb{R}^N)} = 0 \right\}.$$

We say that $f \in \overline{VMO^{-1}}(\mathbb{R}^N)$ if

$$\lim_{R \to 0} \|f\|_{BMO^{-1}_R(\mathbb{R}^N)} = 0.$$

Let $f \in \mathscr{L}^{-1}_{2,\lambda;1}(\mathbb{R}^N)$. We say that f belongs to $\overline{V\mathscr{L}^{-1}_{2,\lambda}}(\mathbb{R}^N)$ if

$$\lim_{R\to 0} \|f\|_{\mathscr{L}^{-1}_{2,\lambda;R}(\mathbb{R}^N)} = 0.$$

Recall the definition of Besov-Morrey spaces [KY94, Maz03]. Let ψ be a Schwartz function supported in the annulus $1/2 \le |\xi| \le 2$ with values in $[0,1]$ such that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^N \setminus \{0\}.$$

Let $\mathcal{F}$ and $\mathcal{S}'_\infty(\mathbb{R}^N)$ denote respectively, the Fourier transform and the space of Schwartz distribution in $\mathbb{R}^N$ modulo polynomials. Set $\dot\Delta_j f = \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\mathcal{F}f)$, the homogeneous Littlewood-Paley projection of f. The homogeneous Besov-Morrey space $\mathcal{N}^s_{p,\lambda,q}(\mathbb{R}^N)$ for $s \in \mathbb{R}$, $0 \le \lambda < N$ and $p, q \in [1,\infty]$ is defined as

$$\mathcal{N}^s_{p,\lambda,q}(\mathbb{R}^N) := \{f \in \mathcal{S}'_0(\mathbb{R}^N) : \|f\|_{\mathcal{N}^s_{p,\lambda,q}(\mathbb{R}^N)} < \infty\}$$

where

$$\|f\|_{\mathcal{N}^s_{p,\lambda,q}(\mathbb{R}^N)} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} \left(2^{js}\left\|\dot\Delta_j f\right\|_{M_{p,\lambda}(\mathbb{R}^N)}\right)^q\right)^{1/q} < \infty, & q \in [1,\infty) \\ \sup_{j\in\mathbb{Z}} \left(2^{js}\left\|\dot\Delta_j f\right\|_{M_{p,\lambda}(\mathbb{R}^N)}\right), & q = \infty. \end{cases}$$

These spaces were introduced by Kozono and Yamazaki in [KY94] and can be regarded as straightforward extensions of homogeneous Besov spaces. As a matter of fact, one has

$$\mathcal{N}^s_{p,0,q}(\mathbb{R}^N) = \dot{B}^s_{pq}(\mathbb{R}^N), \quad 1 \le p, q \le \infty.$$

Definition 3.2.4. Let $T \in (0,\infty]$. We say that a function $v : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$, $N > 2$ belongs to $\mathbf{X}_{j,T}$, $j = 1,2$ if $\|v\|_{\mathbf{X}_{j,T}}$ is finite,

$$\|v\|_{\mathbf{X}_{1,T}} = \sup_{0<t\le T} \|v(t)\|_{L^\infty(\mathbb{R}^N)} + [v]_{\mathbf{X}_{1,T}}$$

where

$$[v]_{\mathbf{X}_{1,T}} = \sup_{0<t\le T} t^{\frac{1}{2}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^N)} + \sup_{x\in\mathbb{R}^N, 0<R\le T^{\frac{1}{2}}} \left(|B(x,R)|^{-1} \int_0^{R^2} \int_{B(x,R)} |\nabla v(y,t)|^2 dy dt\right)^{\frac{1}{2}},$$

and

$$\|v\|_{\mathbf{X}_{2,T}} = \sup_{0<t\le T} t\|v(t)\|_{L^\infty(\mathbb{R}^N)} + \sup_{x\in\mathbb{R}^N, 0<R\le T^{\frac{1}{2}}} \left(|B(x,R)|^{\frac{2}{N}-1} \int_0^{R^2} \int_{B(x,R)} |v(y,t)|^2 dy dt\right)^{1/2}.$$

The space $\mathbf{X}_{3,T}$ collects all functions $u : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N$ such that

$$\|u\|_{\mathbf{X}_{3,T}} = \sup_{0<t\le T} t^{\frac{1}{2}}\|u(t)\|_{L^\infty(\mathbb{R}^N)} + \sup_{x\in\mathbb{R}^N, 0<R\le T^{\frac{1}{2}}} \left(|B(x,R)|^{-1}\int_0^{R^2}\int_{B(x,R)} |u(y,t)|^2 dy dt\right)^{\frac{1}{2}}$$

is finite. We simply write $\mathbf{X}_j$ instead of $\mathbf{X}_{j,\infty}$ and adopt the notation $[\cdot]_{\mathbf{X}_{j,\infty}} = [\cdot]_{\mathbf{X}_j}$.

We easily verify that each of the spaces $\mathbf{X}_{j,T}$ is a Banach space when endowed with the norm $\|\cdot\|_{\mathbf{X}_{j,T}}$, $j = 1,2,3$ respectively. Note that $\mathbf{X}_3$ is the Koch-Tataru space [KT01]. The above discussion motivates the choice of the class $\mathbf{X}_0$, comprising 3-tuples $[c_0, n_0, u_0]$ such that

$$c_0 \in L^\infty(\mathbb{R}^N), \ \ n_0 \in \mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N) \ \text{ and } \ u_0 \in BMO^{-1}(\mathbb{R}^N,\mathbb{R}^N). \tag{3.2.8}$$

Now for $T \in (0,\infty]$, define the spaces

$$\mathbf{X}_T = \mathbf{X}_{1,T} \times \mathbf{X}_{2,T} \times \mathbf{X}_{3,T} \ \text{ and } \ \mathbf{X}_0 = L^\infty(\mathbb{R}^N) \times \mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N) \times BMO^{-1}(\mathbb{R}^N)$$

with their respective norms

$$\begin{aligned} \big\|[c,n,u]\big\|_{\mathbf{X}_T} &:= \|c\|_{\mathbf{X}_{1,T}} + \|n\|_{\mathbf{X}_{2,T}} + \|u\|_{\mathbf{X}_{3,T}}, \\ \big\|[c_0,n_0,u_0]\big\|_{\mathbf{X}_0} &:= \|c_0\|_{L^\infty(\mathbb{R}^N)} + \|n_0\|_{\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)} + \|u_0\|_{BMO^{-1}(\mathbb{R}^N)}. \end{aligned}$$

When $N = 2$, the space $\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)$ is replaced by the homogeneous Besov space $\dot{B}^{-1}_{2,2}(\mathbb{R}^N)$ (or the negative Sobolev space $H^{-1}(\mathbb{R}^N)$) and we instead define $\mathbf{X}_2$ with the norm

$$\|v\|_{\mathbf{X}_2} = \sup_{t>0} t\|v(t)\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(0,\infty;L^2(\mathbb{R}^2))}.$$

In what follows, $UC(\mathbb{R}^N)$ stands for the space of uniformly continuous (real valued) functions in $\mathbb{R}^N$.

The main results of this paper read as follows.

Theorem 3.2.5 (Local existence). *Let $N \ge 2$. There exists $\varepsilon_0 > 0$ so that for all $R > 0$ and for any $\Phi \in \mathcal{S}'(\mathbb{R}^N)$ with $\nabla\Phi \in M_{2,N-2}(\mathbb{R}^N)$, $[n_0,u_0]$ in $\mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N) \times BMO^{-1}_R(\mathbb{R}^N)$ with $\nabla \cdot u_0 = 0$, for any function $c_0 : \mathbb{R}^N \to \mathbb{R}$ and all $d_0 \in UC(\mathbb{R}^N)$ satisfying*

$$\|c_0 - d_0\|_{L^\infty(\mathbb{R}^N)} + \|n_0\|_{\mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N)} + \|u_0\|_{BMO^{-1}_R(\mathbb{R}^N)} < \varepsilon_0, \tag{3.2.9}$$

there exists $\delta_0 := \delta_0(d_0,\varepsilon_0) > 0$, $T_0 := T_0(\delta_0,R) > 0$ and a unique solution $[c,n,u]$ of (CNS) *in $(\Gamma_{\delta_0} + \mathbf{X}_{1,T_0^2}) \times \mathbf{X}_{2,T_0^2} \times \mathbf{X}_{3,T_0^2}$ provided $\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} < \varepsilon_0$. In particular, for all $[c_0,n_0,u_0]$ in $\overline{UC(\mathbb{R}^N)}^{L^\infty(\mathbb{R}^N)} \times \overline{V\mathscr{L}^{-1}_{2,N-2}}(\mathbb{R}^N) \times \overline{VMO^{-1}}(\mathbb{R}^N)$ with $\nabla \cdot u_0 = 0$ there exists a unique small local solution. Here $\Gamma_s = e^{s^2\Delta}d_0$, $s > 0$ and $\overline{UC(\mathbb{R}^N)}^{L^\infty(\mathbb{R}^N)}$ is the closure of $UC(\mathbb{R}^N)$ in the L^∞-norm.*

Theorem 3.2.6 (Global existence)**.** *Let $N \geq 2$. There exists $\varepsilon > 0$ such that for every 3-tuple $[c_0, n_0, u_0]$ in $\mathbf{X}_0$ with $\nabla \cdot u_0 = 0$ and all $\Phi \in \mathcal{S}'(\mathbb{R}^N)$ with $\nabla\Phi \in M_{2,N-2}(\mathbb{R}^N)$, if it holds that*

$$\left\|[c_0, n_0, u_0]\right\|_{\mathbf{X}_0} + \|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} < \varepsilon, \tag{3.2.10}$$

then there exists a global solution $[c,n,u] \in \mathbf{X}$ of (CNS)*. This solution is unique in the closed ball*

$$B^{\mathbf{X}}_{C\varepsilon} := \left\{[c,n,u] \in \mathbf{X} : \left\|[c,n,u]\right\|_{\mathbf{X}} \leq C\varepsilon\right\}$$

for some constant $C > 0$.

Our next result deals with the uniqueness of global mild solutions constructed in Theorem 3.2.6.

Theorem 3.2.7. *Let $\Phi \in \mathcal{S}'(\mathbb{R}^N)$ and $U_0 = [c_0, n_0, u_0] \in \mathbf{X}_0$ such that $\nabla \cdot u_0 = 0$. Assume that $[c_1, n_1, u_1]$ and $[c_2, n_2, u_2]$ are two global mild solutions of* (CNS) *in $L^\infty_{loc}(0,\infty; L^\infty(\mathbb{R}^N))$ with initial data U_0 and $\nabla\Phi \in M_{2,N-2}(\mathbb{R}^N)$ with sufficiently small norm. If it holds that*

$$\lim_{T\to 0} \left\|[c_1, n_1, u_1]\right\|_{\mathbf{X}_T} = 0, \quad \lim_{T\to 0} \left\|[c_2, n_2, u_2]\right\|_{\mathbf{X}_T} = 0, \tag{3.2.11}$$

then $[c_1, n_1, u_1] = [c_2, n_2, u_2]$ on $\mathbb{R}^N \times [0,\infty)$.

Moving on, we study a more general model known as the double chemotaxis system

$$\begin{cases} \partial_t c - \Delta c + cn + u \cdot \nabla c = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+ \\ \partial_t n - \Delta n + u \cdot \nabla n + \text{div } (n\nabla c) + \text{div } (n\nabla v) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+ \\ \partial_t v - \Delta v + u \cdot \nabla v + \kappa v - n = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+ \\ \partial_t u - \Delta u + u \cdot \nabla u + \nabla p + n\Psi = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+ \\ \text{div } u = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+ \\ c(0) = c_0, n(0) = n_0, v(0) = v_0, u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{D-CNS}$$

where c, n, u, p have the same meaning as before, v is the concentration of chemical attractant, $\kappa \geq 0$ represents the decay rate of the attractant and Ψ is an external force acting on the fluid. For $v = 0$, (D-CNS) is closely related to (CNS). Unlike in the classical parabolic-parabolic Keller-Segel system of chemotaxis [KS70]

$$\begin{cases} \partial_t n - \Delta n = -\text{div } (n\nabla v) \\ \partial_t v - \Delta v = n - \kappa v \end{cases} \quad \text{in } \Omega \times (0,\infty), \tag{3.2.12}$$

the oxygen in the model (3.1.1) is consumed and not produced by the bacteria. Syst. D-CNS couples the Keller-Segel model and the Navier-Stokes equations. We refer the reader to [BBTW15, HV97, Hor03, NSY97] for existence of solutions and blow-up phenomena in bounded two dimensional smooth domains depending on the range of the total mass $\int_\Omega n_0 dy$. See also [CC08] and references therein for the case $\Omega = \mathbb{R}^2$. To the best of our knowledge the blow-up of solutions to (3.2.12) in dimension 3 and higher is an unsolved problem.

Comparing the above system of equations to (CNS), one sees that the new equation with unknown v has no proper scaling if $\kappa \neq 0$. However, we can take advantage of our earlier analysis and the scaling property inherited from the case $\kappa = 0$, i.e. $v_\delta(x,t) = v(\delta^2 t, \delta x)$ to make the choice of v_0 in BMO. In fact the term $-\kappa v$ is linear so that $L_k := -\Delta + \kappa$ can be treated as a perturbation of the Laplacian. In order to state our results in this case we consider for $0 < T \leq \infty$, the function space

$$\mathbf{Z}_T = \Big\{[c,n,v,u] : c \in \mathbf{X}_{1,T},\ n \in \mathbf{X}_{2,T},\ v \in \mathbf{X}_{1,T},\ u \in \mathbf{X}_{3,T}\Big\}$$

equipped with the norm

$$\Big\|[c,n,v,u]\Big\|_{\mathbf{Z}_T} = \|c\|_{\mathbf{X}_{1,T}} + \|n_0\|_{\mathbf{X}_{2,T}} + \|v\|_{\mathbf{X}_{1,T}} + \|u\|_{\mathbf{X}_{3,T}}.$$

The local and global well-posedness result pertaining to (D-CNS) is given in the next theorems.

Theorem 3.2.8 (Local existence)**.** *Let $N \geq 2$. There exists $\varepsilon_0 > 0$ with the following property. For all $R > 0$ and all $[n_0, v_0, u_0] \in \mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N) \times BMO_R(\mathbb{R}^N) \times BMO^{-1}_R(\mathbb{R}^N)$ with $\nabla \cdot u_0 = 0$, for all function $c_0 : \mathbb{R}^N \to \mathbb{R}$ and $d_0 \in UC(\mathbb{R}^N)$, if it holds that*

$$\|c_0 - w_0\|_{L^\infty(\mathbb{R}^N)} + \|n_0\|_{\mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N)} + \|v_0\|_{BMO_R(\mathbb{R}^N)} + \|u_0\|_{BMO^{-1}_R(\mathbb{R}^N)} < \varepsilon_0 \tag{3.2.13}$$

then there exists $\delta_0 := \delta_0(d_0, \varepsilon_0)$, $T_0 := T_0(\delta_0, R)$ and a mild solution $[c,n,v,u]$ of (D-CNS) *such that $[c, n, v - \widetilde{v}_\kappa, u] \in (\Gamma_{\delta_0} + \mathbf{X}_{1,T_0^2}) \times \mathbf{X}_{2,T_0^2} \times \mathbf{X}_{1,T_0^2} \times \mathbf{X}_{3,T_0^2}$ provided $\Psi \in M_{2,N-2}(\mathbb{R}^N)$ with $\|\Psi\|_{M_{2,N-2}(\mathbb{R}^N)} < \varepsilon_0$. In particular, for all*

$$[c_0, n_0, v_0, u_0] \in \overline{UC(\mathbb{R}^N)}^{L^\infty(\mathbb{R}^N)} \times \overline{V\mathscr{L}^{-1}_{2,N-2}}(\mathbb{R}^N) \times VMO(\mathbb{R}^N) \times \overline{VMO^{-1}}(\mathbb{R}^N)$$

with $\nabla \cdot u_0 = 0$, there exists a unique small local solution. For $\kappa > 0$, $\widetilde{v}_{\kappa,R}$ denotes the L_κ-caloric extension of v_0.

Next, consider the space

$$\mathbf{Z}_0 = \Big\{[c_0, n_0, v_0, u_0] : c_0 \in L^\infty(\mathbb{R}^N),\ n_0 \in \mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N),\ v_0 \in BMO(\mathbb{R}^N),\ u_0 \in BMO^{-1}(\mathbb{R}^N)\Big\}$$

with the norm

$$\Big\|[c_0, n_0, v_0, u_0]\Big\|_{\mathbf{Z}_0} = \|c_0\|_{L^\infty(\mathbb{R}^N)} + \|n_0\|_{\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)} + \|v_0\|_{BMO(\mathbb{R}^N)} + \|u_0\|_{BMO^{-1}(\mathbb{R}^N)}.$$

Theorem 3.2.9. *Assume $N \geq 2$. Syst. D-CNS is globally well-posed. There exist $\varepsilon > 0$ and $\vartheta := \vartheta(\varepsilon) > 0$ such that for any $[c_0, n_0, v_0, u_0] \in \mathbf{Z}_0$ with $\nabla \cdot u_0 = 0$ and $\Psi \in M_{2,N-2}(\mathbb{R}^N)$ satisfying*

$$\left\|[c_0, n_0, v_0, u_0]\right\|_{\mathbf{Z}_0} + \|\Psi\|_{M_{2,N-2}(\mathbb{R}^N)} < \varepsilon,$$

there exists a mild solution $[c, n, v, u]$ of (D-CNS)*. This solution is unique in the set*

$$B^{\mathbf{Z}}_{2\vartheta} := \left\{[c, n, v, u] \in \mathbf{Z} : \left\|[c, n, v, u] - [0, 0, \widetilde{v}_\kappa, 0]\right\|_{\mathbf{Z}} \leq 2\vartheta\right\}.$$

Moreover, we have the following uniqueness criterion. Let $[c_1, n_1, v_1, u_1]$, $[c_2, n_2, v_2, u_2]$ be two global mild solutions of (D-CNS) *in $L^\infty_{loc}(0, \infty; L^\infty(\mathbb{R}^N))$ with the same initial data $[c_0, n_0, v_0, u_0]$. If the condition*

$$\lim_{T \to 0} \left\|[c_1, n_1, v_1, u_1]\right\|_{\mathbf{Z}_T} = 0, \quad \lim_{T \to 0} \left\|[c_2, n_2, v_2, u_2]\right\|_{\mathbf{Z}_T} = 0 \tag{3.2.14}$$

is satisfied, then $[c_1, n_1, v_1, u_1] = [c_2, n_2, v_2, u_2]$ on $\mathbb{R}^N \times [0, \infty)$.

From the two previous theorems, we deduce that the Keller-Segel system (3.2.12) is locally well-posed for initial data in $\overline{V\mathscr{L}^{-1}_{2,N-2}}(\mathbb{R}^N) \times VMO(\mathbb{R}^N)$ and globally well-posed whenever $[n_0, v_0] \in \mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N) \times BMO(\mathbb{R}^N)$ with $\|n_0\|_{\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)} + \|u_0\|_{BMO(\mathbb{R}^N)}$ sufficiently small.

Remark 3.2.10. As a direct consequence of Theorem 3.2.5, one deduces the existence of global in time solutions for initial data which are small L^∞-perturbations of constants.

Remark 3.2.11. Comparing our main results with earlier findings, one merely requires the initial data c_0 to belong to $L^\infty(\mathbb{R}^N)$ together with a suitable smallness condition and no assumption on its first order derivative is needed. Moreover, for $2 \leq p < \infty$, $N > 2$ and $0 \leq \lambda < N$, due to the embeddings (see the Appendix for the proofs)

$$\mathcal{N}^{-2s}_{p,\lambda,\infty}(\mathbb{R}^N) \subset \mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N) \subset \dot{B}^{-2}_{\infty,\infty}(\mathbb{R}^N); \;\; s = 1 + \frac{\lambda - N}{2p}, \; (N-\lambda)/2 \leq p < N - \lambda \tag{3.2.15}$$

$$\mathcal{N}^{-s}_{p,\lambda,\infty}(\mathbb{R}^N) \subset BMO^{-1}(\mathbb{R}^N); \quad s = 1 - \frac{N-\lambda}{p}, \;\; p > N - \lambda \tag{3.2.16}$$

our initial data class in Theorem 3.2.6 is larger than those considered in [YFS19]. Likewise, in Theorem 3.2.9, the initial concentration of chemical attractant is taken in $BMO(\mathbb{R}^N)$ and no extra requirement on its first order gradient is necessary unlike in the articles [FP19, KY94]. In fact, their initial data classes are contained in ours when the dimensions is larger or equal to 3. This plainly shows that our global existence results encompasses all those which have been cited before. In 2D, however, the initial concentration n_0 is taken in a smaller class $\dot{B}^{-1}_{22}(\mathbb{R}^2)$ but gives rise to a much natural functional setting. Finally, it is worth pointing out that our local well-posedness results (Theorems 3.2.5 and 3.2.8) are derived under much weaker regularity assumptions as compared to those obtained for instance in [CKL13, Zha14] and related works therein.

Remark 3.2.12 (Self-similar solutions). From the embedding $L^{N/2,\infty}(\mathbb{R}^N) \subset \mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)$, one sees that $\mathscr{L}^{-1}_{2,N-2}(\mathbb{R}^N)$ contains homogeneous distributions of degree -2. Thus, if $[c_0, n_0, u_0]$ is homogeneous of degree 0, -2 and -1, respectively and $\left\|[c_0, n_0, u_0]\right\|_{\mathbf{X}_0}$ is sufficiently small, then as a by-product of Theorem 3.2.6, there exits a unique (forward self-similar) solution $[c, n, u]$ satisfying

$$n(x,t) = \delta^2 n(\delta x, \delta^2 t),\ c(x,t) = c(\delta x, \delta^2 t),\ u(x,t) = \delta u(\delta x, \delta^2 t) \quad \text{for all } \delta > 0. \tag{3.2.17}$$

provided $\Phi \in \mathcal{S}'(\mathbb{R}^N)$, $\nabla\Phi \in M_{2,N-2}(\mathbb{R}^N)$ is homogeneous of degree -1 with the quantity $\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}$ chosen small enough. A similar conclusion persists if in the double chemotaxis Navier-Stokes equation (D-CNS), one takes $\kappa = 0$, v_0 and Ψ homogeneous of degree 0 and -1, respectively.

Finally we point out that our settings may easily be adapted to study the higher regularity of solutions to (CNS) and (D-CNS) which arise from small initial data in $\mathbf{X}_0$ and $\mathbf{Z}_0$ respectively as well as the time decay rate of spatial derivatives.

3.3 Preliminaries and auxiliary results

In this section, we collect key estimates for the homogeneous and the inhomogeneous heat equation. For a suitable function f (e.g. smooth and compactly supported), denote by Sf the operator

$$Sf(x,t) = e^{\Delta t} f(x) = (g_t * f)(x)$$

where $g_t(x) = g(x,t) = \dfrac{e^{\frac{|x|}{4t}}}{(4\pi t)^{\frac{N}{2}}}$. Then Sf solve the heat equation $(\partial_t - \Delta)u = 0$ in $\mathbb{R}^N \times (0,\infty)$, $u(0) = f$ on $\mathbb{R}^N$.

Lemma 3.3.1. *Assume $N > 2$. Let $0 < R \le \infty$. The operator S maps $L^\infty(\mathbb{R}^N)$ to $\mathbf{X}_{1,R^2}$, $\mathscr{L}^{-1}_{2,N-\lambda;R}(\mathbb{R}^N)$ to $\mathbf{X}_{2,R^2}$ and $BMO^{-1}_R(\mathbb{R}^N)$ to $\mathbf{X}_{3,R^2}$ continuously. If $N = 2$, then $Sf \in \mathbf{X}_2$ whenever $f \in \dot{B}^{-1}_{2,2}(\mathbb{R}^2)$ and there exists $C > 0$ independent of f such that*

$$\|Sf\|_{\mathbf{X}_2} \le C\|f\|_{\dot{B}^{-1}_{2,2}(\mathbb{R}^2)}. \tag{3.3.1}$$

Proof. From the Carleson measure characterization of BMO (see e.g. [Ste72]) it is well-known that

$$\sup_{x\in\mathbb{R}^N, r>0} |B(x,r)|^{-1} \int_0^{r^2} \int_{B(x,r)} |\nabla Sf(x,t)|^2 dx dt \approx \|f\|_{BMO(\mathbb{R}^N)}.$$

If $f \in BMO_R(\mathbb{R}^N)$ for some $0 < R \le \infty$, then an analogue of the above inequality holds where the supremum on the left-hand side is taken over all balls of radius R and smaller.

Thus, from the continuous embedding $L^\infty(\mathbb{R}^N) \subset BMO_R(\mathbb{R}^N)$ one gets the estimate

$$\sup_{x\in\mathbb{R}^N, 0<r\le R} |B(x,r)|^{-1} \int_0^{r^2} \int_{B(x,r)} |\nabla S f(x,t)|^2 dxdt \le C\|f\|_{L^\infty(\mathbb{R}^N)}$$

for any $f \in L^\infty(\mathbb{R}^N)$. On the other hand, the estimate

$$\sup_{0<t\le R^2} \|Sf(t)\|_{L^\infty(\mathbb{R}^N)} + \sup_{0<t\le R^2} t^{1/2}\|\nabla S f(t)\|_{L^\infty(\mathbb{R}^N)} \le C\|f\|_{L^\infty(\mathbb{R}^N)}$$

follows straight from the smoothing effect of the heat semigroup. The boundedness property of S from $\mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N)$ to $\mathbf{X}_{2,R^2}$ is a consequence of (3.2.15) and the Carleson characterization of $\mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N)$ (see Definition 3.2.1 and Lemma 3.5.2 in the Appendix) while the proof of the third statement can be found in [LR02]. In the case $N = 2$, we shall show that

$$\sup_{t>0} t\|u(t)\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{\dot{B}^{-1}_{2,2}(\mathbb{R}^2)} \tag{3.3.2}$$

and

$$\|u\|_{L^2(0,\infty;L^2(\mathbb{R}^2))} \le C\|f\|_{\dot{B}^{-1}_{2,2}(\mathbb{R}^2)}. \tag{3.3.3}$$

While the latter estimate follows from the caloric characterization of Besov spaces [Tri83], the former is established as follows. For $x \in \mathbb{R}^N$ and $0 < s < t/2$, one may use semigroup properties and Hölder's inequality to get

$$\begin{aligned}
|e^{t\Delta} f(x)| &= \left|e^{(t-s)\Delta} e^{s\Delta} f(x)\right| \\
&\le C\left(\frac{1}{t}\int_0^{t/2}\int_{\mathbb{R}^2} g(x-y,t-s)|e^{s\Delta} f(y)|^2 dyds\right)^{1/2} \\
&\le C\left(\frac{1}{t}\int_0^{t/2}\int_{\mathbb{R}^2} (t-s)^{-1} e^{-\frac{|x-y|^2}{4(t-s)}}|e^{s\Delta} f(y)|^2 dyds\right)^{1/2} \\
&\le Ct^{-1}\left(\int_0^{t/2}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t-s)}}|e^{s\Delta} f(y)|^2 dyds\right)^{1/2} \\
&\le Ct^{-1}\|e^{s\Delta} f\|_{L^2(0,\infty;L^2(\mathbb{R}^2))} \\
&\le Ct^{-1}\|f\|_{\dot{B}^{-1}_{2,2}(\mathbb{R}^2)}.
\end{aligned}$$

Passing to the supremum on both sides over all $t \in \mathbb{R}_+$ yields the desired bound. □

Let $d_0 \in UC(\mathbb{R}^N)$, for any $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that for all $x, y \in \mathbb{R}^N$ with $|x-y| < \delta_0$, we have $|d_0(x) - d_0(y)| \le \varepsilon_0$.

Lemma 3.3.2. *Let $d_0 \in UC(\mathbb{R}^N)$ and set $\Gamma_{\delta_0} = e^{\delta_0^2 \Delta} d_0$. Then the following estimates hold.*

$$\|\Gamma_{\delta_0}\|_{L^\infty(\mathbb{R}^N)} \le C \tag{3.3.4}$$

$$\|\Gamma_{\delta_0} - d_0\|_{L^\infty(\mathbb{R}^N)} + \delta_0 \|\nabla \Gamma_{\delta_0}\|_{L^\infty(\mathbb{R}^N)} + \delta_0^2 \|\nabla^2 \Gamma_{\delta_0}\|_{L^\infty(\mathbb{R}^N)} \le C\varepsilon_0 \tag{3.3.5}$$

for some constant $C > 0$.

Proof. We refer the reader to [KL12]. □

3.3.1 Bilinear estimates

Let $R_j = \partial_j(-\Delta)^{-1/2}$, $j = 1, \ldots, N$ denote the Riesz transform and $\mathbf{P} = Id - \nabla\Delta^{-1}(\nabla \cdot)$ be the Leray projection onto divergence-free vector fields. Applying $\mathbf{P}$ to the Navier-Stokes equations in (CNS), the resulting equations can be recast (using Duhamel's principle) into the following integral system

$$\begin{cases} c = e^{t\Delta} c_0 - \int_0^t e^{(t-s)\Delta}(cn + u \cdot \nabla c)(\cdot, s) ds \\ n = e^{t\Delta} n_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot (n\nabla c + nu)(\cdot, s) ds \\ u = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes u)(\cdot, s) ds - \int_0^t e^{(t-s)\Delta} \mathbf{P}(n\nabla\Phi)(\cdot, s) ds. \end{cases} \tag{3.3.6}$$

Define the linear map

$$\mathscr{L}_\Phi(n) = \int_0^t e^{(t-s)\Delta} \mathbf{P}(n\nabla\Phi)(\cdot, s) ds$$

and the bilinear maps

$$\begin{aligned} B_1(w, n) &= \int_0^t e^{(t-s)\Delta}(wn)(\cdot, s) ds, \\ B_2(n, w) &= \int_0^t e^{(t-s)\Delta} \nabla \cdot (nw)(\cdot, s) ds, \\ B_3(u, w) &= \int_0^t e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes w)(\cdot, s) ds \end{aligned}$$

whenever the integrals are well-defined. The next lemma establishes the continuity properties of these maps in targeted functions spaces.

Lemma 3.3.3. *Let $N \ge 2$ and $0 < T \le \infty$. Assume that $\Phi \in \mathcal{S}'(\mathbb{R}^N)$ with $\nabla\Phi \in M_{2,N-2}(\mathbb{R}^N)$. The linear operator $\mathscr{L}_\Phi(n) : \mathbf{X}_{2,T} \to \mathbf{X}_{3,T}$ continuously and the bilinear operators $B_j(\cdot, \cdot)$, $j = 1, 2, 3$ are such that*

$$B_1 : \mathbf{X}_{1,T} \times \mathbf{X}_{2,T} \to \mathbf{X}_{1,T},\ B_1 : \mathbf{X}_{3,T} \times \mathbf{X}_{3,T} \to \mathbf{X}_{1,T},\ B_2 : \mathbf{X}_{3,T} \times \mathbf{X}_{3,T} \to \mathbf{X}_{1,T}$$

and $B_3 : \mathbf{X}_{3,T} \times \mathbf{X}_{3,T} \to \mathbf{X}_{3,T}$ *continuously. Moreover, there exists* $C_j > 0$, $j = 1, \cdots, 5$ *such that*

$$\|\mathscr{L}_\Phi(n)\|_{\mathbf{X}_{3,T}} \le C_1 \|n\|_{\mathbf{X}_{2,T}} \|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} \quad \text{for all} \quad n \in \mathbf{X}_{2,T} \tag{3.3.7}$$

$$\|B_1(w,n)\|_{\mathbf{X}_{1,T}} \le C_2 \|w\|_{\mathbf{X}_{1,T}} \|n\|_{\mathbf{X}_{2,T}} \quad \text{for all} \quad w \in \mathbf{X}_{1,T} \text{ and } n \in \mathbf{X}_{2,T} \tag{3.3.8}$$

$$\|B_1(u,w)\|_{\mathbf{X}_{1,T}} \le C_3 \|u\|_{\mathbf{X}_{3,T}} \|w\|_{\mathbf{X}_{3,T}} \quad \text{for all} \quad u, w \in \mathbf{X}_{3,T} \tag{3.3.9}$$

$$\|B_2(n,w)\|_{\mathbf{X}_{2,T}} \le C_4 \|n\|_{\mathbf{X}_{2,T}} \|w\|_{\mathbf{X}_{3,T}} \quad \text{for all} \quad n \in \mathbf{X}_{2,T} \text{ and } w \in \mathbf{X}_{3,T} \tag{3.3.10}$$

$$\|B_3(v,w)\|_{\mathbf{X}_{3,T}} \le C_5 \|v\|_{\mathbf{X}_{3,T}} \|w\|_{\mathbf{X}_{3,T}} \quad \text{for all} \quad v, w \in \mathbf{X}_{3,T}. \tag{3.3.11}$$

Let f be a locally integrable function and $\mathcal{M}f$ its uncentered maximal function defined as

$$\mathcal{M}f(x) = \sup_{B \ni x} |B|^{-1} \int_B |f(y)| dy.$$

Let $p \in (1,\infty)$ with $1/p + 1/p' = 1$. A nonnegative measurable function μ on $\mathbb{R}^N$ belongs to the Muckenhoupt weight class $A_p(\mathbb{R}^N) = A_p$ if

$$[\mu]_{A_p} := \sup_{B \subset \mathbb{R}^N} \left(⨍_B \mu(x) dx \right) \left(⨍_B \mu(x)^{-p'/p} dx \right)^{p/p'} < \infty.$$

Given a weight $\mu \in A_p$ and denoting $L^p_\mu(\mathbb{R}^N) = L^p(\mathbb{R}^N, \mu dx)$, it is well-known (see e.g. [GCdF85]) that $\mu \in A_p$ if and only if $\mathcal{M}$ is bounded on $L^p_\mu(\mathbb{R}^N)$. Given a non-negative measurable function h on $\mathbb{R}_+$ and $\alpha, \beta \in (0,1)$, define the fractional integral operator

$$E(h)(s) = \int_0^s (s-\sigma)^{\alpha-1} \sigma^{-\beta} h(\sigma) d\sigma.$$

The next lemma establishes the boundedness properties of E between weighted-Lebesgue spaces.

Lemma 3.3.4. *Let* $\alpha, \beta \in (0,1)$ *and* $p \in (1,\infty)$ *such that* $-1/p < \alpha - \beta < 1/p'$ *holds. Then* E *maps* $L^p_\nu(\mathbb{R}_+)$ *continuously into* $L^p(\mathbb{R}_+)$ *where* $\nu(s) = s^{(\alpha-\beta)p}$, $s > 0$. *In particular,* E *is bounded on* $L^p(\mathbb{R}_+)$ *(including* $p = \infty$*) if* $\alpha = \beta$.

Proof. The proof of the lemma relies on the following pointwise estimate for the operator E: for $0 < \alpha, \beta < 1$, there exists $C := C(\alpha, \beta) > 0$ such that

$$|E(h)(s)| \le C s^{\alpha-\beta} \mathcal{M}(h)(s). \tag{3.3.12}$$

We have

$$E(h)(s) = \left(\int_0^{s/2} + \int_{s/2}^s \right) (s-\sigma)^{\alpha-1} \sigma^{-\beta} h(\sigma) d\sigma := I + II.$$

By a simple covering argument, it follows that

$$\begin{aligned}
|I| &= \sum_{j=1}^{\infty} \int_{2^{-j-1}s}^{2^{-j}s} (s-\sigma)^{\alpha-1}\sigma^{-\beta}|h(\sigma)|d\sigma \\
&\leq Cs^{\alpha-1}\sum_{j=1}^{\infty}(2^{-j-1}s)^{-\beta}\int_{2^{-j-1}s}^{2^{-j}s}|h(\sigma)|d\sigma \\
&\leq Cs^{\alpha-1}\sum_{j=1}^{\infty}(2^{-j-1}s)^{-\beta+1}\fint_{2^{-j-1}s}^{2^{-j}s}|h(\sigma)|d\sigma \\
&\leq Cs^{\alpha-\beta}\left(\sum_{j=1}^{\infty}2^{-j(1-\beta)}\right)\mathcal{M}(h)(s)
\end{aligned}$$

and

$$\begin{aligned}
|II| &= \sum_{j=1}^{\infty} \int_{s-2^{-j-1}s}^{s-2^{-j}s} (s-\sigma)^{\alpha-1}\sigma^{-\beta}|h(\sigma)|d\sigma \\
&\leq Cs^{-\beta}\sum_{j=1}^{\infty}(2^{-j-1}s)^{\alpha-1}\int_{s-2^{-j}s}^{s-2^{-j-1}s}|h(\sigma)|d\sigma \\
&\leq Cs^{-\beta}\sum_{j=1}^{\infty}(2^{-j-1}s)^{\alpha}\fint_{s-2^{-j}s}^{s-2^{-j-1}s}|h(\sigma)|d\sigma \\
&\leq Cs^{\alpha-\beta}\left(\sum_{j=1}^{\infty}2^{-j\alpha}\right)\mathcal{M}(h)(s).
\end{aligned}$$

An immediate consequence of 3.3.12 is that the mapping properties of E may be deduced from those of $\mathcal{M}$. Since the function $\nu(s) = s^{(\alpha-\beta)p}$ is an A_p-weight under the restriction $-1/p < \alpha - \beta < 1/p'$, one has

$$\|E(h)\|_{L^p(\mathbb{R}_+)} \leq C\|\mathcal{M}h\|_{L^p_\nu(\mathbb{R}_+)} \leq C\|h\|_{L^p_\nu(\mathbb{R}_+)}.$$

□

Proof of Lemma 3.3.3. We estimate B_1, B_2 and $\mathscr{L}_\Phi(n)$ in 3 steps respectively. The required bound on the bilinear map B_3 is known and can be found for instance in [KT01].

Step 1 (Estimates on B_1). We prove that $B_1(\cdot,\cdot)$ is continuous from $\mathbf{X}_{1,T}\times\mathbf{X}_{2,T}$ to $\mathbf{X}_{1,T}$. Mimicking the same steps, we similarly show that $B_1(\cdot,\cdot) : \mathbf{X}_{3,T}\times\mathbf{X}_{3,T} \to \mathbf{X}_{1,T}$ is continuous. The details of the latter case are therefore omitted. Let $w \in \mathbf{X}_{1,T}$ and $n \in \mathbf{X}_{2,T}$. For $(x,t) \in$

$\mathbb{R}^N \times (0,T]$ write

$$B_1(w,n)(x,t) = \int_0^t \int_{\mathbb{R}^N} g(x-y,t-s)(wn)(y,s)dyds := B_{11}(w,n)(x,t) + B_{12}(w,n)(x,t)$$

where

$$\begin{aligned} B_{11}(w,n)(x,t) &= \int_0^{t/2} \int_{\mathbb{R}^N} g(x-y,t-s)(wn)(y,s)dyds, \\ B_{12}(w,n)(x,t) &= \int_{t/2}^{t} \int_{\mathbb{R}^N} g(x-y,t-s)(wn)(y,s)dyds. \end{aligned}$$

Let $B_r^c(x)$ denote the complement of the Euclidean ball $B_r(x) = B(x,r)$ with center at $x \in \mathbb{R}^N$ and radius $r > 0$ and further make the decomposition

$$\begin{aligned} B_{11}(w,n)(x,t) &= \int_0^{t/2} \left(\int_{B_{2\sqrt{t}}(x)} + \int_{B^c_{2\sqrt{t}}(x)} \right) g(x-y,t-s)(wn)(y,s)dyds \\ &= B^1_{11}(w,n)(x,t) + B^2_{11}(w,n)(x,t). \end{aligned}$$

Using Hölder's inequality, one gets

$$\begin{aligned} |B^1_{11}(w,n)(x,t)| &\leq \int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} g(x-y,t-s)|(nw)(y,s)|dyds \\ &\leq C \sup_{0<t\leq T} \|w(t)\|_{L^\infty(\mathbb{R}^N)} \|g\|_{L^2(B_{2\sqrt{t}}(0)\times[t/2,t])} \left(\int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} |n(y,s)|^2 dyds \right)^{\frac{1}{2}} \\ &\leq C\|w\|_{\mathbf{X}_{1,T}} t^{\frac{2-N}{4}} \left(\int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} |n(y,s)|^2 dyds \right)^{\frac{1}{2}} \leq C\|w\|_{\mathbf{X}_{1,T}} \|n\|_{\mathbf{X}_{2,T}}. \end{aligned}$$

On the other hand, if one sets $A_j(x) = B_{(j+1)\sqrt{t}}(x) \setminus B_{j\sqrt{t}}(x)$, then it follows that

$$\begin{aligned}
|B_{11}^2(w,n)(x,t)| &\leq \int_0^{t/2} \int_{B^c_{2\sqrt{t}}(x)} g(x-y,t-s)|(nw)(y,s)|dyds \\
&\leq C \sup_{0<t\leq T} \|w(t)\|_{L^\infty(\mathbb{R}^N)} \sum_{j=2}^{\infty} \int_0^{t/2} \int_{A_j(x)} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{N/2}} |n(y,s)|dyds \\
&\leq C\|w\|_{\mathbf{X}_{1,T}} \sum_{j=2}^{\infty} e^{-\frac{j^2}{2}} \sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N} \int_0^{t/2} \int_{B_{\sqrt{t}}(z)} (t-s)^{-N/2}|n(y,s)|dyds \\
&\leq C\|w\|_{\mathbf{X}_{1,T}} \sum_{j=2}^{\infty} e^{-\frac{j^2}{2}} \sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N} t^{\frac{2-N}{4}} \left(\int_0^{t/2} \int_{B_{\sqrt{t}}(z)} |n(y,s)|^2 dyds \right)^{1/2} \\
&\leq C\|w\|_{\mathbf{X}_{1,T}} \left(\sum_{j=2}^{\infty} j^{N-1} e^{-\frac{j^2}{2}} \right) \sup_{z\in\mathbb{R}^N} \left(t^{2-N} \int_0^{t/2} \int_{B_{\sqrt{t}}(z)} |n(y,s)|^2 dyds \right)^{1/2} \\
&\leq C\|w\|_{\mathbf{X}_{1,T}} \|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

Since $g \in L^1(\mathbb{R}^N)$, it holds that

$$\begin{aligned}
|B_{12}(w,n)(x,t)| &= \int_{t/2}^{t} \int_{\mathbb{R}^N} g(x-y,t-s)|(wn)(y,s)|dyds \\
&\leq C \sup_{0<t\leq T} \|w(t)\|_{L^\infty(\mathbb{R}^N)} \sup_{0<t\leq T} t\|n(t)\|_{L^\infty(\mathbb{R}^N)} t^{-1} \int_0^{t/2} \|g(\cdot,s)\|_{L^1(\mathbb{R}^N)} ds \\
&\leq C\|w\|_{\mathbf{X}_{1,T}} \|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

Moving on, we prove the pointwise gradient bound on B_1. For any $x \in \mathbb{R}^N$, $0 < t \leq T$ and $k_1(x,t) = \nabla_x g(x,t)$ we have

$$\begin{aligned}
\nabla B_1(w,n)(x,t) &= \int_0^t \int_{\mathbb{R}^N} k_1(x-y,t-s)(wn)(y,s)dyds \\
&= \int_0^{t/2} \int_{\mathbb{R}^N} k_1(x-y,t-s)(wn)(y,s)dyds + \\
&\qquad \int_{t/2}^{t} \int_{\mathbb{R}^N} k_1(x-y,t-s)(wn)(y,s)dyds \\
&:= B_1^1(w,n)(x,t) + B_1^2(w,n)(x,t).
\end{aligned}$$

We estimate each of these terms using the fact that $k_1(x,t) = -t^{-1}xg(x,t)$ (recall g is the heat kernel). Indeed,

$$
\begin{aligned}
|B_1^2(w,n)(x,t)| &\le C\|w\|_{\mathbf{X}_1}\|n\|_{\mathbf{X}_2}\int_{t/2}^{t}\int_{\mathbb{R}^N} s^{-1}|k_1(x-y,t-s)|dyds \\
&\le C\|w\|_{\mathbf{X}_1}\|n\|_{\mathbf{X}_2}\int_{t/2}^{t}\int_{\mathbb{R}^N} \frac{|x-y|}{s(t-s)}g(x-y,t-s)dyds \\
&\le Ct^{-1/2}\|w\|_{\mathbf{X}_{1,T}}\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}
$$

Now, note that $\|k_1(\cdot,\cdot)\|_{L^2(B_{2\sqrt{t}}(x)\times[t/2,t])} \le Ct^{-\frac{N}{4}}$ so that by arguing as above, we find

$$
\begin{aligned}
|B_1^1(w,n)(x,t)| \le &\int_0^{t/2}\int_{B_{2\sqrt{t}}(x)}|k_1(x-y,t-s)||(nw)(y,s)|dyds+ \\
&\int_0^{t/2}\int_{B^c_{2\sqrt{t}}(x)}|k_1(x-y,t-s)||(nw)(y,s)|dyds \\
\le &\, C\sup_{0<t\le T}\|w(t)\|_{L^\infty(\mathbb{R}^N)}\|k_1(\cdot,\cdot)\|_{L^2(B_{2\sqrt{t}}(0)\times[t/2,t])}\|n(\cdot,\cdot)\|_{L^2(B_{2\sqrt{t}}(x)\times(0,t/2])}+ \\
&C\sup_{0<t\le T}\|w(t)\|_{L^\infty(\mathbb{R}^N)}\sum_{j=2}^{\infty}\int_0^{t/2}\int_{A_j(x)}|k_1(x-y,t-s)||n(y,s)|dyds \\
\le &\, C\|w\|_{\mathbf{X}_{1,T}}t^{-\frac{N}{4}}\left(\int_0^{t/2}\int_{B_{\sqrt{t}}(x)}|n(y,s)|^2dyds\right)^{1/2}+ \\
&C\|w\|_{\mathbf{X}_{1,T}}\sum_{j=2}^{\infty}(j+1)e^{-\frac{j^2}{2}}\sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N}t^{\frac{-N}{4}}\left(\int_0^{t/2}\int_{B_{\sqrt{t}}(z)}|n(y,s)|^2dyds\right)^{\frac{1}{2}} \\
\le &\, Ct^{-\frac{1}{2}}\|w\|_{\mathbf{X}_{1,T}}\|n\|_{\mathbf{X}_{2,T}}+ \\
&Ct^{-\frac{1}{2}}\|w\|_{\mathbf{X}_{1,T}}\sum_{j=2}^{\infty}(j+1)j^{N-1}e^{-\frac{j^2}{2}}\sup_{z\in\mathbb{R}^N}\left(t^{-\frac{2-N}{2}}\int_0^{t/2}\int_{B_{\sqrt{t}}(z)}|n(y,s)|^2dyds\right)^{\frac{1}{2}} \\
\le &\, Ct^{-\frac{1}{2}}\|w\|_{\mathbf{X}_{1,T}}\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}
$$

To estimate the L^2-gradient norm, write

$$
\begin{aligned}
|B(x,t)|^{-1}\int_0^{t^2}\int_{B(x,t)}|\nabla B_1(n,w)(y,s)|^2dyds &= |B(x,t)|^{-1}\int_0^{t^2}\int_{B(x,t)}|I_1(n,w)(y,s)|^2dyds+ \\
&\quad |B(x,t)|^{-1}\int_0^{t^2}\int_{B(x,t)}|I_2(n,w)(y,s)|^2dyds \\
&= I(w,n)(x,t)+II(w,n)(x,t), \quad 0<t\le T
\end{aligned}
$$

where

$$I_1(w,n)(y,s) = \int_0^s \int_{\mathbb{R}^N} k_1(y-z,s-\sigma)(wn\mathbf{1}_{B(x,2t)})(z,\sigma)dzd\sigma,$$

and

$$I_2(w,n)(y,s) = \int_0^s \int_{\mathbb{R}^N} k_1(y-z,s-\sigma)(wn\mathbf{1}_{B^c(x,2t)})(z,\sigma)dzd\sigma.$$

Since $|k_1(y,s)| \le C(|y|^2+s)^{-\frac{N+1}{2}}$ for $y \in \mathbb{R}^N, s>0$, one may use Young's convolution inequality to obtain

$$\begin{aligned}\|I_1(w,n)(\cdot,s)\|_{L^2(\mathbb{R}^N)} &\le C\int_0^s (s-\sigma)^{-1/2}\|wn\mathbf{1}_{B_{2t}(x)}(\cdot,\sigma)\|_{L^2(\mathbb{R}^N)}d\sigma \\ &\le C\|w\|_{\mathbf{X}_{1,T}} s^{1/2}\int_0^s (s-\sigma)^{-1/2}\sigma^{-1/2}\|n\mathbf{1}_{B_{2t}(x)}(\sigma)\|_{L^2(\mathbb{R}^N)}d\sigma.\end{aligned}$$

Thus, by Lemma 3.3.4 applied with $p=2$; $\alpha=\beta=\frac{1}{2}$, $I(w,n)(x,t)$ may be estimated as follows

$$\begin{aligned}\left|I(w,n)(x,t)\right| &\le |B(x,t)|^{-1}\int_0^{t^2} \|I_1(\cdot,s)\|^2_{L^2(\mathbb{R}^N)}ds \\ &\le C\|w\|^2_{\mathbf{X}_{1,T}}|B(x,t)|^{-1}t^2\int_0^{t^2} \|n\mathbf{1}_{B_{2t}(x)}(\cdot,s)\|^2_{L^2(\mathbb{R}^N)}ds \\ &\le C\|w\|^2_{\mathbf{X}_{1,T}}|B(x,t)|^{\frac{2}{N}-1}\int_0^{t^2}\int_{B_{2t}(x)} |n(y,s)|^2dyds \\ &\le C\|w\|^2_{\mathbf{X}_{1,T}}\|n\|^2_{\mathbf{X}_{2,T}}.\end{aligned}$$

Remark that if $z \in B^c(x,2t)$ and $y \in B(x,t)$, then $|y-z| \ge \dfrac{1}{2}|x-z|$. Thus, for $s \le t^2 < T$, one has

$$
\begin{aligned}
|I_2(w,n)(y,s)| &\le \int_0^s \int_{|x-z|\ge 2t} |k_1(y-z,s-\sigma)||(wn)(z,\sigma)|dzd\sigma \\
&\le C\int_0^s \int_{|x-z|\ge 2t} \frac{|(wn)(z,\sigma)|dzd\sigma}{(|y-z|+(s-\sigma)^{1/2})^{N+1}} \\
&\le C\|w\|_{\mathbf{X}_{1,T}} \int_0^s \int_{|x-z|\ge 2t} |y-z|^{-(N+1)}|n(z,\sigma)|dzd\sigma \\
&\le C\|w\|_{\mathbf{X}_{1,T}} \int_0^{t^2} \sum_{j=2}^{\infty} \int_{B(x,(j+1)t)\setminus B(x,jt)} |x-z|^{-(N+1)}|n(z,\sigma)|dzd\sigma \\
&\le C\|w\|_{\mathbf{X}_{1,T}} t^{-(N+1)} \sum_{j=2}^{\infty} j^{-(N+1)} \sum_{\substack{q\in t\mathbb{Z}^N \\ q\in B(x,(j+1)t)\setminus B(x,jt)}} \int_0^{t^2}\int_{B(q,t)} |n(z,\sigma)|dzd\sigma \\
&\le C\|w\|_{\mathbf{X}_{1,T}} t^{-1} \left(\sum_{j=2}^{\infty} j^{-2}\right) \sup_{q\in\mathbb{R}^N} \left(t^{2-N}\int_0^{t^2}\int_{B(q,t)} |n(z,\sigma)|^2 dzd\sigma\right)^{\frac{1}{2}} \\
&\le Ct^{-1}\|w\|_{\mathbf{X}_{1,T}}\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}
$$

This estimate directly gives the desired bound for $II(w,n)$, namely

$$
\sup_{x\in\mathbb{R}^N, 0<t\le\sqrt{T}} \Big|II(w,n)(x,t)\Big| \le C\|w\|_{\mathbf{X}_{1,T}}\|n\|_{\mathbf{X}_{2,T}}.
$$

Step 2 (The bounds on B_2). Let $n\in\mathbf{X}_{2,T}$ and $w\in\mathbf{X}_{3,T}$, we want to show that

$$
\|B_2(n,w)\|_{\mathbf{X}_{2,T}} \le C\|n\|_{\mathbf{X}_{2,T}}\|w\|_{\mathbf{X}_{3,T}}. \tag{3.3.13}
$$

We first estimate the norm $\sup\limits_{x\in\mathbb{R}^N, 0<t\le\sqrt{T}} [B_2(n,w)]_{x,t}^{1/2}$ where

$$
[B_2(n,w)]_{x,t} := |B(x,t)|^{2/N-1}\int_0^{t^2}\int_{B(x,t)} |B_2(n,w)|^2 dyds.
$$

To this end, split $B_2(n,w)(y,s)$ into two parts

$$
B_{21}(n,w)(y,s) = \int_0^s\int_{\mathbb{R}^N} \nabla g(y-z,s-\sigma)\cdot(nw\mathbf{1}_{B_{2t}(x)})(z,\sigma)dzd\sigma,
$$

and

$$
B_{22}(n,w)(y,s) = \int_0^s\int_{\mathbb{R}^N} \nabla g(y-z,s-\sigma)\cdot(nw\mathbf{1}_{B^c_{2t}(x)})(z,\sigma)dzd\sigma.
$$

Arguing as before, we can show that the following inequality is valid, namely

$$|B_{22}(n,w)(x,t)| \le Ct^{-2}\|n\|_{\mathbf{X}_{2,T}}\|w\|_{\mathbf{X}_{3,T}}$$

from which we immediately get

$$\sup_{x\in\mathbb{R}^N,0<t\le\sqrt{T}}[B_{22}(n,w)]_{x,t} \le C\|n\|^2_{\mathbf{X}_{2,T}}\|w\|^2_{\mathbf{X}_{3,T}}. \tag{3.3.14}$$

Next, using Young's convolution inequality and Lemma 3.3.4 with $\alpha = \beta = 1/2$ it follows that

$$\begin{aligned}\|B_{22}(n,w)(\cdot,s)\|_{L^2(\mathbb{R}^N)} &\le C\int_0^s (s-\sigma)^{-1/2}\|wn\mathbf{1}_{B_{2t}(x)}(\cdot,\sigma)\|_{L^2(\mathbb{R}^N)}d\sigma\\ &\le C\sup_{0<t\le T} t^{\frac{1}{2}}\|w(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}\int_0^s (s-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}\|n\mathbf{1}_{B_{2t}(x)}(\sigma)\|_{L^2(\mathbb{R}^N)}d\sigma\end{aligned}$$

and

$$\begin{aligned}[B_{21}(n,w)]_{x,t} &= |B(x,t)|^{2/N-1}\int_0^{t^2}\int_{B(x,t)}|B_{21}(n,w)(y,s)|^2dyds\\ &\le C|B(x,t)|^{2/N-1}\int_0^{t^2}\|wn\mathbf{1}_{B_{2t}(x)}(\cdot,s)\|^2_{L^2(\mathbb{R}^N)}ds\\ &\le C\|w\|^2_{\mathbf{X}_{3,T}}|B(x,t)|^{2/N-1}\Big\|\|(n\mathbf{1}_{B_{2t}(x)})\|_{L^2(\mathbb{R}^N)}(\cdot)\Big\|_{L^2(0,t^2]}\\ &\le C\|w\|^2_{\mathbf{X}_{3,T}}|B(x,t)|^{\frac{2}{N}-1}\int_0^{t^2}\int_{B(x,2t)}|n(y,s)|^2dyds\\ &\le C\|w\|^2_{\mathbf{X}_{3,T}}\|n\|^2_{\mathbf{X}_{2,T}}.\end{aligned} \tag{3.3.15}$$

Combining (3.3.14) and (3.3.15), we get the desired bound. Next, we show that

$$\sup_{0<t\le T} t\|B_2(n,w)\|_{L^\infty(\mathbb{R}^N)} \le C\|n\|_{\mathbf{X}_{2,T}}\|w\|_{\mathbf{X}_{3,T}}. \tag{3.3.16}$$

Once again, we make the decomposition

$$\begin{aligned}B_2(n,w)(x,t) &= \int_0^t\int_{\mathbb{R}^N}\nabla g(x-y,t-s)\cdot(nw)(y,s)dyds\\ &:= B_2^1(n,w)(x,t) + B_2^2(n,w)(x,t)\end{aligned}$$

where

$$\begin{aligned}B_2^1(n,w)(x,t) &= \int_0^{t/2}\int_{\mathbb{R}^N}\nabla g(x-y,t-s)\cdot(nw)(y,s)dyds,\\ B_2^2(n,w)(x,t) &= \int_{t/2}^t\int_{\mathbb{R}^N}\nabla g(x-y,t-s)\cdot(nw)(y,s)dyds.\end{aligned}$$

To bound $B_2^2(n,w)$, we use the kernel decay bound and proceed as follows

$$\begin{aligned}
|B_2^2(n,w)(x,t)| &\le C\|w\|_{\mathbf{X}_{3,T}}\|n\|_{\mathbf{X}_{2,T}}\int_{t/2}^{t}\int_{\mathbb{R}^N} s^{-3/2}|\nabla g(x-y,t-s)|dyds \\
&\le C\|w\|_{\mathbf{X}_{3,T}}\|n\|_{\mathbf{X}_{2,T}}\int_{t/2}^{t}\int_{\mathbb{R}^N} s^{-3/2}\frac{|x-y|}{(t-s)}g(x-y,t-s)dyds \\
&\le Ct^{-1}\|w\|_{\mathbf{X}_{3,T}}\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

To estimate $B_2^1(n,w)$, we decompose it as a sum of two integrals and use Hölder's inequality:

$$\begin{aligned}
|B_2^1(n,w)(x,t)| \le& \int_0^{t/2}\int_{B_{2\sqrt{t}}(x)} |\nabla g(x-y,t-s)||(nw)(y,s)|dyds+ \\
&\int_0^{t/2}\int_{B^c_{2\sqrt{t}}(x)} |\nabla g(x-y,t-s)||(nw)(y,s)|dyds \\
\le& Ct^{-\frac{N+1}{2}}\left(\int_0^{\frac{t}{2}}\int_{B(x,2\sqrt{t})}|n(y,s)|^2dyds\right)^{\frac12}\left(\int_0^{\frac{t}{2}}\int_{B(x,2\sqrt{t})}|w(y,s)|^2dyds\right)^{\frac12} \\
&+Ct^{-\frac{N+1}{2}}\sum_{j=2}^{\infty}(j+1)e^{-j^2/2}\int_0^{t/2}\int_{B(x,(j+1)\sqrt{t})\setminus B(x,j\sqrt{t})}|nw(y,s)|dyds \\
\le& Ct^{-1}\|w\|_{\mathbf{X}_{3,T}}\|n\|_{\mathbf{X}_{1,T}}+ \\
&Ct^{-\frac{N+1}{2}}\left(\sum_{j=2}^{\infty}(j+1)j^{N-1}e^{-\frac{j^2}{2}}\right)\left(\int_0^{t/2}\int_{B_{\sqrt{t}}(z)}|n(y,s)|^2dyds\right)^{\frac12}\cdot \\
&\left(\int_0^{t/2}\int_{B_{\sqrt{t}}(z)}|w(y,s)|^2dyds\right)^{\frac12} \\
\le& Ct^{-1}\|w\|_{\mathbf{X}_{3,T}}\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

This shows (3.3.16) and finishes Step 2.

Step 3 (*The bounds on* $\mathscr{L}_\Phi(n)$). Here we show that $\mathscr{L}_\Phi$ is continuous on $\mathbf{X}_{3,T}$ for $n \in \mathbf{X}_{2,T}$ and $\nabla\Phi \in M_{2,N-2}(\mathbb{R}^N)$ for $T \in (0,\infty]$. The operator $e^{t\Delta}\mathbf{P}$ is an integral operator whose kernel is given by the Oseen kernel $k_2(t)$ which satisfies the polynomial decay bound (see for instance [LR02, Chapter 11])

$$t^{|\alpha|/2}|\partial^\alpha k_2(x,t)| \le Ct^{-N/2}(1+t^{-1/2}|x|)^{-N-|\alpha|} \quad \text{for all } \alpha\in\mathbb{N}^N,\ x\in\mathbb{R}^N \text{ and } t>0. \tag{3.3.17}$$

Set

$$[\mathscr{L}_\Phi(n)]_{x,t} := \left(|B(x,t)|^{-1}\int_0^{t^2}\int_{B(x,t)}|\mathscr{L}_\Phi(n)(y,s)|^2dyds\right)^{1/2}.$$

We primarily show that $\sup_{x\in\mathbb{R}^N, 0<t\le T^{1/2}} [\mathscr{L}_\Phi(n)]_{x,t} \le C\|n\|_{\mathbf{X}_{2,T}} \|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}$. Define

$$\mathscr{L}_\Phi^1(n)(y,s) = \int_0^s \int_{\mathbb{R}^N} k_2(y-z, s-\sigma)(n(\sigma)\mathbf{1}_{B_{2t}(x)}\nabla\Phi)(z)dzd\sigma,$$

$$\mathscr{L}_\Phi^2(n)(y,s) = \int_0^s \int_{\mathbb{R}^N} k_2(y-z, s-\sigma)(n(\sigma)\mathbf{1}_{B_{2t}^c(x)}\nabla\Phi)(z)dzd\sigma.$$

For $s \le t^2 < T$ and $y \in B(x, 2t)$, we have

$$\begin{aligned}
|\mathscr{L}_\Phi^2(n)(y,s)| &\le \int_0^s \int_{|x-z|\ge 2t} |k_2(y-z, s-\sigma)||(n(\sigma)\nabla\Phi)(z)|dzd\sigma \\
&\le C\int_0^s \int_{|x-z|\ge 2t} \frac{(s-\sigma)^{-N/2}|n(\sigma)\nabla\Phi(z)|}{(1+|y-z|(s-\sigma)^{-1/2})^N} dzd\sigma \\
&\le Ct^{-N}\sum_{j=2}^{\infty} j^{-N} \sum_{\substack{z\in t\mathbb{Z}^N \\ z\in B(x,(j+1)t)\setminus B(x,jt)}} \int_0^{t^2}\int_{B(z,t)} |(n(\sigma)\nabla\Phi)(y)|dzd\sigma \\
&\le Ct^{1-N}\left(\sum_{j=2}^{\infty} j^{-1}\right)\left(\int_{B(z,t)} |\nabla\Phi|^2 dy\right)^{1/2} \left(\int_0^{t^2}\int_{B(z,t)} |n(y,s)|^2 dyds\right)^{\frac{1}{2}} \\
&\le Ct^{-1}\|n\|_{\mathbf{X}_{2,T}}\|\nabla\Phi\|_{M_{2,N-2;\sqrt{T}}(\mathbb{R}^N)}
\end{aligned}$$

where Hölder's inequality was used to obtain the estimate before the last. Hence,

$$\sup_{x\in\mathbb{R}^N, 0<t\le\sqrt{T}} [\mathscr{L}_\Phi^2(n)]_{x,t} \le C\|n\|_{\mathbf{X}_{2,T}}\|\nabla\Phi\|_{M_{2,N-2;\sqrt{T}}(\mathbb{R}^N)}.$$

On the other hand, let $0<\eta<1/2$, $1<\theta<\dfrac{N}{N+2\eta-1}$. Take $1<\theta_0<2$ such that $\dfrac{1}{\theta}+\dfrac{1}{\theta_0}=\dfrac{3}{2}$. Then by Young's inequality we find that

$$\begin{aligned}
\|\mathscr{L}_\Phi^1(n)(s)\|_{L^2(\mathbb{R}^N)} &\le C\int_0^s (s-\sigma)^{\frac{N}{2}(1/\theta-1)}\|n(\sigma)\mathbf{1}_{B(x,2t)}\nabla\Phi\|_{L^{\theta_0}(\mathbb{R}^N)}d\sigma \\
&\le Ct^{2\eta}\int_0^s (s-\sigma)^{\frac{N}{2}(\frac{1}{\theta}-1)}\sigma^{-\eta}\|n(\sigma)\mathbf{1}_{B(x,2t)}\nabla\Phi\|_{L^{\theta_0}(\mathbb{R}^N)}d\sigma.
\end{aligned}$$

This implies (in view of Lemma 3.3.4 with $\alpha = 1 + \frac{N}{2}(\frac{1}{\theta} - 1)$, $\beta = \eta$ and $p = 2$) that

$$\begin{aligned}
[\mathscr{L}^1_\Phi(n)]^2_{x,t} &\leq |B(x,t)|^{-1} \int_0^{t^2} \|\mathscr{L}^1_\Phi(n)(\cdot,s)\|^2_{L^2(\mathbb{R}^N)} ds \\
&\leq Ct^{4\eta}|B(x,t)|^{-1} \int_0^{t^2} s^{2+N(1/\theta-1)-2\eta} \|n(s)\mathbf{1}_{B_{2t}(x)}\nabla\Phi\|^2_{L^{\theta_0}(\mathbb{R}^N)} ds \\
&\leq C\Big(\sup_{0<t\leq T} t\|n(t)\|_{L^\infty(\mathbb{R}^N)}\Big)^2 t^{4\eta-N} \|\nabla\Phi\|^2_{L^{\theta_0}(B(x,2t))} \int_0^{t^2} s^{N(1/\theta-1)-2\eta} ds \\
&\leq C\|n\|^2_{\mathbf{X}_{2,T}} t^{4\eta-N+N(2/\theta_0-1)} \|\nabla\Phi\|^2_{L^2(B(x,2t))} \int_0^{t^2} s^{N(1/\theta-1)-2\eta} ds \\
&\leq C\|\nabla\Phi\|^2_{M_{2,N-2}(\mathbb{R}^N)} \|n\|^2_{\mathbf{X}_{2,T}}.
\end{aligned}$$

Finally, we need to prove the estimate

$$\sup_{0<t\leq T} t^{1/2} \|\mathscr{L}_\Phi(n)(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C\|n\|_{\mathbf{X}_{2,T}} \|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}. \tag{3.3.18}$$

Making use of (3.3.17) and Hölder's inequality we have

$$\begin{aligned}
\Big|\mathscr{L}_\Phi(n)(x,t)\Big| &\leq \int_0^{\frac{t}{2}} \int_{B_{2\sqrt{t}}(x)} \frac{|(n(s)\nabla\Phi)(y)| dy ds}{(|x-y| + (t-s)^{\frac{1}{2}})^N} + \int_0^{\frac{t}{2}} \int_{B^c_{2\sqrt{t}}(x)} \frac{|(n(s)\nabla\Phi)(y)| dy ds}{(|x-y| + (t-s)^{\frac{1}{2}})^N} \\
&\leq Ct^{\frac{-N}{2}} \int_0^{t/2} \left(\int_{B_{2\sqrt{t}}(x)} |n(y,s)|^2 dy \right)^{\frac{1}{2}} \|\nabla\Phi\|_{L^2(B_{2\sqrt{t}}(x))} ds + \\
&\qquad C \sum_{j=2}^{\infty} \int_0^{t/2} \int_{A_j(x)} \frac{|(n(s)\nabla\Phi)(y)| dy ds}{(|x-y| + (t-s)^{1/2})^N} \\
&\leq Ct^{\frac{1-N}{2}} \left(\int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} |n(y,s)|^2 dy ds \right)^{\frac{1}{2}} \|\nabla\Phi\|_{L^2(B_{2\sqrt{t}}(x))} + \\
&\qquad C \sum_{j=2}^{\infty} j^{-N} \sum_{\substack{z\in A_j(x) \\ z\in\sqrt{t}\mathbb{Z}^N}} t^{\frac{1-N}{2}} \left(\int_0^{t/2} \int_{B_{\sqrt{t}}(z)} |n(y,s)|^2 dy ds \right)^{\frac{1}{2}} \|\nabla\Phi\|_{L^2(B_{2\sqrt{t}}(x))} \\
&\leq Ct^{\frac{1-N}{2}} \left(\int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} |n(y,s)|^2 dy ds \right)^{\frac{1}{2}} \|\nabla\Phi\|_{L^2(B_{2\sqrt{t}}(x))} \left(1 + \sum_{j=2}^{\infty} j^{-1} \right).
\end{aligned}$$

This clearly implies that

$$\sup_{0<t\leq T} t^{\frac{1}{2}} \Big\|\mathscr{L}_\Phi(n)(t)\Big\|_{L^\infty(\mathbb{R}^N)} \leq C\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} \|n\|_{\mathbf{X}_{2,T}}.$$

On the other hand, using (3.3.17) with $|\alpha| = 1$ we find that

$$\begin{aligned}
\left|[k_2(\cdot,t) * (n\nabla\Phi)](x)\right| &= \left|\int_{\mathbb{R}^N} k_2(x-y,t)(n\nabla\Phi)(y)dy\right| \\
&\le \int_0^\infty \left|\frac{dk_2(r,t)}{dr}\right|\left(\int_{B_r(x)} |(n\nabla\Phi)(y)|dy\right)dr \\
&\le Ct^{-N/2-1/2}\int_0^\infty (1+t^{-1/2}r)^{-(N+1)} r^{N/2} \|n\nabla\Phi\|_{L^2(B_r(x))} dr \\
&\le Ct^{-1/2}\|n\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} \int_0^\infty \frac{r^{N-1}}{(1+r)^{N+1}} dr.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathscr{L}_\Phi(n)(x,t)| &\le \left|\int_{t/2}^t e^{(t-s)\Delta}\mathbf{P}(n\nabla\Phi)(y,s)ds\right| \\
&\le \int_{t/2}^t \left\|e^{(t-s)\Delta}\mathbf{P}(n\nabla\Phi)(y,s)\right\|_{L^\infty(\mathbb{R}^N)} ds \\
&\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}} \|n(s,\cdot)\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} ds \\
&\le C \sup_{0<t\le T} t\|n(t)\|_{L^\infty(\mathbb{R}^N)} \|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} \int_{t/2}^t (t-s)^{-\frac{1}{2}} s^{-1} ds \\
&\le Ct^{-\frac{1}{2}} \|n\|_{\mathbf{X}_{2,T}} \|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}.
\end{aligned}$$

This gives (3.3.18), concludes Step 3 and thus the proof of Lemma 3.3.3. □

Now we turn to the proof of Theorem 3.2.9. Reformulate (D-CNS) into the following system whose solutions are referred to as mild solutions

$$\begin{cases}
c = e^{t\Delta}c_0 - \int_0^t e^{(t-s)\Delta}(cn + u\cdot\nabla c)(\cdot,s)ds \\
n = e^{t\Delta}n_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot(un + n\nabla c + n\nabla v)(\cdot,s)ds \\
v = e^{-\kappa t}e^{t\Delta}v_0 - \int_0^t e^{-\kappa(t-s)}e^{(t-s)\Delta}(u\cdot\nabla v)(\cdot,s)ds - \int_0^t e^{-\kappa(t-s)}e^{(t-s)\Delta}n(\cdot,s)ds \\
u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla\cdot(u\otimes u)(\cdot,s)ds - \int_0^t e^{(t-s)\Delta}\mathbf{P}(n\Psi)(\cdot,s)ds.
\end{cases} \tag{3.3.19}$$

In order to prove the well-posedness of Syst. 3.3.19, one needs in addition to Lemma 3.3.3 another auxiliary result about the mapping properties of the linear operator $\mathscr{L}$ and the bilinear map B_5 respectively given by

$$\mathscr{L}n(t) = \int_0^t e^{-\kappa(t-s)}e^{(t-s)\Delta}n(\cdot,s)ds, \quad t>0$$

and

$$B_4(u,v)(t) = \int_0^t e^{-\kappa(t-s)} e^{(t-s)\Delta}(u\cdot\nabla v)(\cdot,s)ds, \quad t>0, \quad \kappa>0.$$

Lemma 3.3.5. *Let $\kappa > 0$ and $0 < T \leq \infty$. The linear operator $\mathscr{L}$ is continuous from $\mathbf{X}_{2,T}$ to $\mathbf{X}_{1,T}$ and there exists $C > 0$ such that*

$$\|\mathscr{L} n\|_{\mathbf{X}_{1,T}} \leq C\|n\|_{\mathbf{X}_{2,T}} \tag{3.3.20}$$

for any $n \in \mathbf{X}_{2,T}$. Moreover, if $v \in \mathbf{X}_{1,T}$ and $u \in \mathbf{X}_{3,T}$, then $B_4(u,v) \in \mathbf{X}_{1,T}$ and

$$\|B_4(u,v)\|_{\mathbf{X}_{1,T}} \leq C\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}}. \tag{3.3.21}$$

This Lemma is proved in a similar fashion as before. We include the details for the sake of completeness.

Proof. Let $n \in \mathbf{X}_{2,T}$, we have

$$\mathscr{L} n(x,t) = \left(\int_0^{t/2} + \int_{t/2}^t\right) e^{-\kappa(t-s)} e^{(t-s)\Delta} n(y,s)dyds := J_1(x,t) + J_2(x,t).$$

Only using the property $g(\cdot,t) \in L^1(\mathbb{R}^N)$ for every $t > 0$, we bound J_2 as follows:

$$\begin{aligned}
|J_2(x,t)| &= \int_{t/2}^t e^{-\kappa(t-s)} \int_{\mathbb{R}^N} g(x-y,t-s)|n(y,s)|dyds \\
&\leq C \sup_{t>0} t\|n(t)\|_{L^\infty(\mathbb{R}^N)} \int_{t/2}^t s^{-1} e^{-\kappa(t-s)} ds \\
&\leq C\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

Regarding J_1, we further split it into two terms, $J_1 = J_{11} + J_{12}$ with

$$J_{11}(x,t) = \int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} e^{-\kappa(t-s)} g(x-y,t-s)n(y,s)dyds,$$

$$J_{12}(x,t) = \int_0^{t/2} \int_{B^c_{2\sqrt{t}}(x)} e^{-\kappa(t-s)} g(x-y,t-s)n(y,s)dyds.$$

By Hölder's inequality,

$$\begin{aligned}
|J_{11}(x,t)| &\leq C\left\|e^{-\kappa\cdot} g(\cdot,\cdot)\right\|_{L^2(B_{2\sqrt{t}}(0)\times[t/2,t])} \left(\int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} |n(y,s)|^2 dyds\right)^{\frac{1}{2}} \\
&\leq Ct^{-N/4+1/2} \left(\int_0^{t/2} \int_{B_{2\sqrt{t}}(x)} |n(y,s)|^2 dyds\right)^{\frac{1}{2}} \leq C\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

With $A_j(x)$ as previously defined, we have

$$\begin{aligned}
|J_{12}(x,t)| &\le \int_0^{t/2}\int_{B^c_{2\sqrt{t}}(x)} e^{-\kappa(t-s)} g(x-y,t-s)|n(y,s)|dyds \\
&\le C\sum_{j=2}^{\infty} e^{-\kappa(t-s)} e^{-\frac{j^2}{2}} \sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N} \int_0^{t/2}\int_{B_{\sqrt{t}}(z)} e^{-\kappa(t-s)}(t-s)^{-n/2}|n(y,s)|dyds \\
&\le C\sum_{j=2}^{\infty} e^{-\frac{j^2}{2}} \sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N} t^{\frac{2-N}{4}} e^{-\frac{\kappa}{2t}} \left(\int_0^{t/2}\int_{B_{\sqrt{t}}(z)} |n(y,s)|^2 dyds\right)^{1/2} \\
&\le C\left(\sum_{j=2}^{\infty} j^{N-1} e^{-\frac{j^2}{2}}\right) \sup_{z\in\mathbb{R}^N}\left(t^{2-N}\int_0^{t/2}\int_{B_{\sqrt{t}}(z)} |n(y,s)|^2 dyds\right)^{1/2} \\
&\le C\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

The pointwise gradient estimate $\sup_{0<t\le T} t^{\frac{1}{2}}\|\mathscr{L}n(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\|n\|_{\mathbf{X}_{2,T}}$ follows the same steps using the decay of the kernel $k_1(x,t) = \nabla_x g(t)$. Next, we show the energy-type estimate

$$[\mathscr{L}n]_{Car} := \sup_{x\in\mathbb{R}^N, 0<t\le T}\left(|B(x,\sqrt{t})|^{-1}\int_0^t\int_{B(x,\sqrt{t})} |\nabla\mathscr{L}n(y,s)|^2 dyds\right)^{1/2} \le C\|n\|_{\mathbf{X}_{2,T}}.$$

Let $n\in\mathbf{X}_{2,T}$ and for $(x,t)\in\mathbb{R}^N\times(0,T]$ write (in the sense of distributions)

$$\begin{aligned}
\nabla\mathscr{L}n(y,s) = &\int_0^s\int_{\mathbb{R}^N} e^{-\kappa(s-\sigma)} k_1(y-z,s-\sigma)\mathbf{1}_{B(x,2\sqrt{t})} n(z,\sigma)dzd\sigma + \\
&\int_0^s\int_{\mathbb{R}^N} e^{-\kappa(s-\sigma)} k_1(y-z,s-\sigma)(1-\mathbf{1}_{B(x,2\sqrt{t})}) n(z,\sigma)dzd\sigma \\
:= &I_1(n)(y,s) + I_2(n)(y,s).
\end{aligned}$$

As such, we have that

$$\begin{aligned}
[\mathscr{L}n]_{Car} = &\sup_{x\in\mathbb{R}^N, 0<t\le T}\left(|B(x,\sqrt{t})|^{-1}\int_0^t\int_{B(x,\sqrt{t})} |I_1(n)(y,s)|^2 dyds\right)^{1/2} + \\
&\sup_{x\in\mathbb{R}^N, 0<t\le T}\left(|B(x,\sqrt{t})|^{-1}\int_0^t\int_{B(x,\sqrt{t})} |I_2(n)(y,s)|^2 dyds\right)^{1/2} \\
= &M_1 + M_2.
\end{aligned}$$

Using the following L^2-bound

$$\|I_1(n)(\cdot,s)\|_{L^2(\mathbb{R}^N)} \le C\int_0^s e^{-\kappa(s-\sigma)}(s-\sigma)^{-1/2}\|n\mathbf{1}_{B(x,2\sqrt{t})}(\cdot,\sigma)\|_{L^2(\mathbb{R}^N)},$$

the bound on M_1 follows from an application of Lemma 3.3.4 with $p = 2$, $\alpha = \beta = 1/2$. Indeed,

$$\begin{aligned}
M_1 &\leq \sup_{x\in\mathbb{R}^N, 0<t\leq T} \left(|B(x,\sqrt{t})|^{-1} \int_0^t \|K_1(n)(\cdot,s)\|^2_{L^2(\mathbb{R}^N)} ds \right)^{1/2} \\
&\leq C \sup_{x\in\mathbb{R}^N, 0<t\leq T} \left(|B(x,\sqrt{t})|^{-1} t \int_0^t \|n\mathbf{1}_{B(x,2\sqrt{t})}\|^2_{L^2(\mathbb{R}^N)})]^2(s) ds \right)^{1/2} \\
&\leq C \sup_{x\in\mathbb{R}^N, 0<t\leq T} \left(|B(x,\sqrt{t})|^{-(1-2/N)} \int_0^t \int_{B(x,2\sqrt{t})} |n(y,s)|^2 dy ds \right)^{1/2} \leq C\|n\|_{\mathbf{X}_{2,T}}.
\end{aligned}$$

Also, for $y \in B(x, 2\sqrt{t})$ and $z \in \mathbb{R}^N \setminus B(x, 2\sqrt{t})$, we have $|y - z| \geq |x - z|/2$. Thus for $s \leq t$,

$$\begin{aligned}
|I_2(n)(y,s)| &\leq \int_0^s \int_{\mathbb{R}^N \setminus B(x,2\sqrt{t})} e^{-\kappa(s-\sigma)} |k_1(y-z, s-\sigma)||n(z,\sigma)| dz d\sigma \\
&\leq C \int_0^t \int_{|x-z|\geq 2\sqrt{t}} \frac{e^{-\kappa(s-\sigma)}}{(|x-y| + (s-\sigma)^{1/2})^{N+1}} |n(z,\sigma)| dz d\sigma \\
&\leq C \int_0^t \int_{|x-z|\geq 2\sqrt{t}} \frac{|n(z,\sigma)| dz d\sigma}{|x-y|^{N+1}}.
\end{aligned}$$

Performing a similar covering argument as before we obtain the estimate $M_2 \leq C\|n\|_{\mathbf{X}_{2,T}}$. To conclude the proof we show that

$$\|B_4(u,v)\|_{\mathbf{X}_{1,T}} \leq C\|u\|_{\mathbf{X}_{3,T}} \|v\|_{\mathbf{X}_{1,T}} \tag{3.3.22}$$

for all $u \in \mathbf{X}_{3,T}$ and $v \in \mathbf{X}_{1,T}$. Put

$$\begin{aligned}
N_1(u,v)(x,t) &= \int_0^{t/2} \int_{\mathbb{R}^N} e^{-\kappa(t-s)} g(x-y, t-s)(u\cdot\nabla v)(y,s) dy ds \\
N_2(u,v)(x,t) &= \int_{t/2}^{t} \int_{\mathbb{R}^N} e^{-\kappa(t-s)} g(x-y, t-s)(u\cdot\nabla v)(y,s) dy ds.
\end{aligned}$$

We have

$$\begin{aligned}
|N_2(u,v)(x,t)| &= \int_{t/2}^{t} \int_{\mathbb{R}^N} e^{-\kappa(t-s)} g(x-y, t-s)|(u\cdot\nabla v)(y,s)| dy ds \\
&\leq C \sup_{0<t\leq T} t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \sup_{0<t\leq T} t^{\frac{1}{2}} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^N)} t^{-1} \int_0^{\frac{1}{2}} e^{-\kappa s} \|g(s)\|_{L^1(\mathbb{R}^N)} ds \\
&\leq C\|u\|_{\mathbf{X}_{3,T}} \|v\|_{\mathbf{X}_{1,T}}.
\end{aligned}$$

We estimate N_1 using Hölder's inequality:

$$
\begin{aligned}
|N_1(u,v)(x,t)| &= \int_0^{t/2}\int_{B_{2\sqrt{t}}(x)} e^{-\kappa(t-s)} g(x-y,t-s)|(u\cdot\nabla v)(y,s)|dyds + \\
&\qquad \int_0^{t/2}\int_{\mathbb{R}^N\setminus B_{2\sqrt{t}}(x)} e^{-\kappa(t-s)} g(x-y,t-s)|(u\cdot\nabla v)(y,s)|dyds \\
&\le Ct^{-N/2}\left(\int_0^{t/2}\int_{B_{2\sqrt{t}}(x)} |u(y,s)|^2 dyds\right)^{\frac{1}{2}}\left(\int_0^{t/2}\int_{B_{2\sqrt{t}}(x)} |\nabla v(y,s)|^2 dyds\right)^{\frac{1}{2}} \\
&\qquad + C\sum_{j=2}^{\infty}\int_0^{t/2}\int_{A_j(x)} e^{-\kappa(t-s)} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{N/2}}|(u\cdot\nabla v)(y,s)|dyds \\
&\le C\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}} + \\
&\quad C\sum_{j=2}^{\infty} e^{-\frac{j^2}{2}} \sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N}\int_0^{t/2}\int_{B_{\sqrt{t}}(z)} e^{-\kappa(t-s)}(t-s)^{-\frac{N}{2}}|(u\cdot\nabla v)(y,s)|dyds \\
&\le C\|u\|_{\mathbf{X}_{1,T}}\|v\|_{\mathbf{X}_{1,T}} + \\
&\qquad C\sum_{j=2}^{\infty} e^{-\frac{j^2}{2}} \sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N} t^{\frac{-N}{2}} e^{-\frac{\kappa t}{2}}\left(\int_0^{t/2}\int_{B_{\sqrt{t}}(z)}|u(y,s)|^2 dyds\right)^{\frac{1}{2}}\cdot \\
&\qquad\qquad \left(\int_0^{t/2}\int_{B_{\sqrt{t}}(z)}|\nabla v(y,s)|^2 dyds\right)^{\frac{1}{2}} \\
&\le C\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}} + \left(\sum_{j=2}^{\infty} e^{-\frac{j^2}{2}} j^{N-1}\right)\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}} \le C\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}}.
\end{aligned}
$$

The gradient estimate $\sup_{0<t\le T} t^{1/2}\|\nabla B_4(u,v)\|_{L^\infty(\mathbb{R}^N)} \le C\|u\|_{\mathbf{X}_{1,T}}\|v\|_{\mathbf{X}_{1,T}}$ is obtained in a similar fashion. It remains to establish the L^2-gradient estimate

$$
\begin{aligned}
[B_4(u,v)]_{Car} &= \sup_{x\in\mathbb{R}^N, 0<t\le T} |B(x,\sqrt{t})|^{-1}\int_0^t\int_{B(x,\sqrt{t})}|\nabla B_4(u,v)(y,s)|^2 dyds \\
&\le C\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}}.
\end{aligned} \tag{3.3.23}
$$

Set

$$B_{41}(u,v)(y,s) = \int_0^s\int_{\mathbb{R}^N} e^{-\kappa(t-s)} k_1(y-z,s-\sigma)[(u\mathbf{1}_{B(x,2\sqrt{t})})\cdot\nabla v](z,\sigma)dzd\sigma,$$

and

$$B_{42}(u,v)(y,s) = \int_0^s\int_{\mathbb{R}^N} e^{-\kappa(t-s)} k_1(y-z,s-\sigma)[(u\mathbf{1}_{\mathbb{R}^N\setminus B(x,2\sqrt{t})})\cdot\nabla v](z,\sigma)dzd\sigma.$$

Then, by Young's convolution inequality one has

$$\|B_{41}(u,v)(s)\|_{L^2(\mathbb{R}^N)} \le C\int_0^s e^{-\kappa(t-s)}\Big\|[(u\mathbf{1}_{B(x,2\sqrt{t})})\cdot\nabla v](\cdot,\sigma)\Big\|_{L^2(\mathbb{R}^N)}$$
$$\le C\sup_{0<s\le T} s^{\frac12}\|\nabla v\|_{L^\infty(\mathbb{R}^N)}\int_0^s e^{-\kappa(s-\sigma)}(s-\sigma)^{-\frac12}\sigma^{-\frac12}\|\mathbf{1}_{B(x,2\sqrt{t})}u\|_{L^2(\mathbb{R}^N)}d\sigma$$
$$\le C\|v\|_{\mathbf{X}_{1,T}}\int_0^s (s-\sigma)^{-\frac12}\sigma^{-\frac12}\|\mathbf{1}_{B(x,2\sqrt{t})}u(\sigma)\|_{L^2(\mathbb{R}^N)}d\sigma$$

and thus by Lemma 3.3.4 with $p=2$, $\alpha=\beta=1/2$, it holds that

$$[B_{41}(u,v)]_{Car} = \sup_{x\in\mathbb{R}^N,0<t\le T}|B(x,\sqrt{t})|^{-1}\int_0^t\int_{B(x,\sqrt{t})}|B_{51}(u,v)(y,s)|^2dxdt$$
$$\le C\sup_{x\in\mathbb{R}^N,0<t\le T}|B(x,\sqrt{t})|^{-1}\int_0^t\|B_{52}(u,v)(\cdot,s)\|^2_{L^2(\mathbb{R}^N)}ds$$
$$\le C\|v\|_{\mathbf{X}_{1,T}}\sup_{x\in\mathbb{R}^N,0<t\le T}|B(x,\sqrt{t})|^{-1}\int_0^t\|\mathbf{1}_{B(x,2\sqrt{t})}u(\cdot,s)\|^2_{L^2(\mathbb{R}^N)}ds$$
$$\le C\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}}. \tag{3.3.24}$$

On the other hand,

$$|B_{42}(u,v)(y,s)| = \int_0^s\int_{\mathbb{R}^N\setminus B(x,2\sqrt{t})} e^{-\kappa(t-s)}|k_1(y-z,s-\sigma)|\Big|[u\cdot\nabla v](z,\sigma)\Big|dzd\sigma$$
$$\le C\sum_{j=2}^{\infty}\int_0^s\int_{A_j(x)}\frac{e^{-\kappa(s-\sigma)}(u\cdot\nabla v)(z,\sigma)dzd\sigma}{(|y-z|+(s-\sigma)^{1/2})^{N+1}}$$
$$\le C\sum_{j=2}^{\infty}\int_0^s\int_{A_j(x)}|y-z|^{-(N+1)}e^{-\kappa(s-\sigma)}(u\cdot\nabla v)(z,\sigma)dzd\sigma$$
$$\le C\sum_{j=2}^{\infty}j^{-(N+1)}\sum_{z\in A_j(x)\cap\sqrt{t}\mathbb{Z}^N}t^{\frac{-(N+1)}{2}}\left(\int_0^t\int_{B_{\sqrt{t}}(z)}|(u\cdot\nabla v)(y,s)|dyds\right)^{\frac12}$$
$$\le C\left(\sum_{j=2}^{\infty}j^{-2}\right)t^{-1/2}\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}}$$

where the last estimate follows from Hölder's inequality. This implies that

$$[B_{42}(u,v)]_{Car}\le C\|u\|_{\mathbf{X}_{3,T}}\|v\|_{\mathbf{X}_{1,T}}.$$

Combining this with (3.3.24), we obtain (3.3.23). The proof of Lemma 3.3.5 is now complete. □

3.4 Proofs of main results

We mainly present the proofs of the local well-posedness results since the existence of global-in-time solutions is a direct consequence of Lemmas 3.3.1, 3.3.3 and the contraction mapping principle. Also, the uniqueness criterion in Theorem 3.2.7 and 3.2.9 may be established via similar arguments – we therefore only present the proof of the former.

Proof of Theorem 3.2.5

Let $d_0 \in UC(\mathbb{R}^N)$ and Γ_{δ_0} as in Lemma 3.3.2. Make the ansatz $\overline{c} = c - \Gamma_{\delta_0}$ and observe that $\overline{c}$ solves the Cauchy problem

$$\begin{aligned} \partial_t \overline{c} - \Delta \overline{c} &= -(\overline{c} - \Gamma_{\delta_0})n - u \cdot (\nabla \overline{c} + \nabla \Gamma_{\delta_0}) - \Delta \Gamma_{\delta_0} \ \text{ in } \ \mathbb{R}^N \times \mathbb{R}^+ \\ \overline{c}(0) &= c_0 - \Gamma_{\delta_0} \ \text{ on } \ \mathbb{R}^N. \end{aligned}$$

It is clear that the conclusion of Theorem 3.2.5 now follows from the well-posedness of the following system of equations

$$\begin{cases} \overline{c} = e^{t\Delta}\overline{c}_0 - \displaystyle\int_0^t e^{(t-s)\Delta}[(\overline{c} + \Gamma_{\delta_0})n + u \cdot \nabla(\overline{c} + \nabla \Gamma_{\delta_0}) + \Delta \Gamma_{\delta_0}](\cdot, s)ds \\ n = e^{t\Delta}n_0 - \displaystyle\int_0^t e^{(t-s)\Delta}\nabla \cdot [n\nabla(\overline{c} + \Gamma_{\delta_0}) + nu](\cdot, s)ds \\ u = e^{t\Delta}u_0 - \displaystyle\int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)(\cdot, s)ds - \int_0^t e^{(t-s)\Delta}\mathbf{P}(n\nabla\Phi)(\cdot, s)ds. \end{cases} \tag{3.4.1}$$

Next, define the maps

$$\begin{aligned} \mathbb{F}_1(\overline{c}, n, u, \Gamma_{\delta_0}) &= B_1(\overline{c} + \Gamma_{\delta_0}, n) + B_1(u, \nabla(\overline{c} + \Gamma_{\delta_0})) + \int_0^t e^{(t-s)\Delta}\Delta\Gamma_{\delta_0} ds \\ \mathbb{F}_2(\overline{c}, n, u, \Gamma_{\delta_0}) &= B_2(n, \nabla(\overline{c} + \Gamma_{\delta_0})) + B_2(n, u) \\ \mathbb{F}_3(n, u, \nabla\Phi) &= B_3(u, u) + \mathscr{L}_\Phi(n) \end{aligned}$$

where B_j, $j = 1, 2, 3$ are as in Section 3.3. Combining Lemmas 3.3.2 and 3.3.3 we obtain the following result.

Proposition 3.4.1. *Let $\Gamma_{\delta_0} = e^{\delta_0^2 \Delta} d_0$. Let $R > 0$ and put $T_0 = \min(\delta_0, R)$. Given $[\overline{c}, n, u]$ in $\mathbf{X}_{T_0^2}$ and $\nabla\Phi \in M_{2,N-2}$, one has*

$$\|\mathbb{F}_1(\overline{c}, n, u, \Gamma_{\delta_0})\|_{\mathbf{X}_{1,T_0^2}} \le C_1(\|\overline{c}\|_{\mathbf{X}_{1,T_0^2}} + 1)\|n\|_{\mathbf{X}_{2,T_0^2}} + C_1\|u\|_{\mathbf{X}_{3,T_0^2}}(\|\overline{c}\|_{\mathbf{X}_{1,T_0^2}} + \varepsilon_0) + C_1\varepsilon_0 \tag{3.4.2}$$

$$\|\mathbb{F}_2(\overline{c}, n, u, \Gamma_{\delta_0})\|_{\mathbf{X}_{2,T_0^2}} \le C_2(\|\overline{c}\|_{\mathbf{X}_{1,T_0^2}} + \|u\|_{\mathbf{X}_{3,T_0^2}} + \varepsilon_0)\|n\|_{\mathbf{X}_{2,T_0^2}} \tag{3.4.3}$$

$$\|\mathbb{F}_3(n, u, \nabla\Phi)\|_{\mathbf{X}_{3,T_0^2}} \le C_3(\|u\|^2_{\mathbf{X}_{3,T_0^2}} + \|n\|_{\mathbf{X}_{2,T_0^2}}\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}) \tag{3.4.4}$$

for some constants $C_1, C_2, C_3 > 0$. Moreover, for any $[\overline{c}_1, n_1, u_1] \in \mathbf{X}_{T_0^2}$, it holds that

$$\begin{aligned}
\left\|\mathbb{F}_1(\overline{c}, n, u, \Gamma_{\delta_0}) - \mathbb{F}_1(\overline{c}_1, n_1, u_1, \Gamma_{\delta_0})\right\|_{\mathbf{X}_{1,T_0^2}} &\le C_1(\|\overline{c}\|^2_{\mathbf{X}_{1,T_0^2}} + 1)\|n - n_1\|_{\mathbf{X}_{2,T_0^2}} + \\
&\quad C_1\|n_1\|_{\mathbf{X}_{2,T_0^2}}(\|\overline{c} - \overline{c}_1\|^2_{\mathbf{X}_{1,T_0^2}}) + C_1(\|\overline{c}\|_{\mathbf{X}_{1,T_0^2}} + \varepsilon_0)\|u - u_1\|_{\mathbf{X}_{3,T_0^2}} \\
\left\|\mathbb{F}_2(\overline{c}, n, u, \Gamma_{\delta_0}) - \mathbb{F}_2(\overline{c}_1, n_1, u_1, \Gamma_{\delta_0})\right\|_{\mathbf{X}_{2,T_0^2}} &\le C_2[(\varepsilon_0\|\overline{c}_1\|_{\mathbf{X}_{1,T_0^2}} + \|u\|_{\mathbf{X}_{3,T_0^2}})\|n - n_1\|_{\mathbf{X}_{2,T_0^2}} + \\
&\quad (\|n_1\|_{\mathbf{X}_{2,T_0^2}}\|u - u_1\|_{\mathbf{X}_{3,T_0^2}} + \|n\|_{\mathbf{X}_{2,T_0^2}}\|\overline{c} - \overline{c}_1\|_{\mathbf{X}_{1,T_0^2}})] \\
\left\|\mathbb{F}_3(n, u, \nabla\Phi) - \mathbb{F}_3(n_1, u_1, \nabla\Phi)\right\|_{\mathbf{X}_{3,T_0^2}} &\le C_3(\|u\|_{\mathbf{X}_{3,T_0^2}} + \|u_1\|_{\mathbf{X}_{3,T_0^2}})\|u - u_1\|_{\mathbf{X}_{3,T_0^2}} + \\
&\quad C_3\|n - n_1\|_{\mathbf{X}_{2,T_0^2}}\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}.
\end{aligned}$$

From Lemma 3.3.2, we remark that $\|\overline{c}_0\|_{L^\infty(\mathbb{R}^N)} \le \|c_0 - d_0\|_{L^\infty(\mathbb{R}^N)} + C\varepsilon_0$ for some $C > 0$. Let $R > 0$ fixed and $\overline{\varepsilon}_0 > \varepsilon_0$. Assume that

$$\|c_0 - d_0\|_{L^\infty(\mathbb{R}^N)} + \|n_0\|_{\mathscr{L}^{-1}_{2,N-2;R}(\mathbb{R}^N)} + \|u_0\|_{BMO_R^{-1}(\mathbb{R}^N)} < \overline{\varepsilon}_0,$$

then by Lemma 3.3.1, it holds that

$$\left\|[e^{t\Delta}\overline{c}_0, e^{t\Delta}n_0, e^{t\Delta}u_0]\right\|_{\mathbf{X}_{T_0^2}} \le C_0\overline{\varepsilon}_0. \tag{3.4.5}$$

Let $U = [\overline{c}, n, u]$ and introduce the map

$$\mathbb{F}(U) = (e^{t\Delta}\overline{c}_0 + \mathbb{F}_1(U, \Gamma_{\delta_0}), e^{t\Delta}n_0 + \mathbb{F}_2(U, \Gamma_{\delta_0}), e^{t\Delta}u_0 + \mathbb{F}_3(n, u, \nabla\Phi)).$$

Using Proposition 3.4.1 together with (3.4.5), we have

$$\begin{aligned}
\|\mathbb{F}(U)\|_{\mathbf{X}_{T_0^2}} &\le \|e^{t\Delta}\overline{c}_0, e^{t\Delta}n_0, e^{t\Delta}u_0)\|_{\mathbf{X}_{T_0^2}} + \|\mathbb{F}_1(U, \Gamma_{\delta_0})\|_{\mathbf{X}_{1,T_0^2}} + \|\mathbb{F}_2(U, \Gamma_{\delta_0})\|_{\mathbf{X}_{2,T_0^2}} + \\
&\quad \left\|\mathbb{F}_3(e^{t\Delta}n_0 + \mathbb{F}_2(U, \Gamma_{\delta_0}), u, \nabla\Phi)\right\|_{\mathbf{X}_{3,T_0^2}} \\
&\le (2C_0 + C_1)\overline{\varepsilon}_0 + ((C_1 + C_2 + C_1C_2)\varepsilon_0 + \varepsilon_0^2 C_2C_3)\|U\|_{\mathbf{X}_{T_0^2}} + \\
&\quad (2C_1 + 2C_2 + C_3 + 2\varepsilon_0 C_2(C_3 + C_1))\|U\|^2_{\mathbf{X}_{T_0^2}} \\
&\le \overline{C}\overline{\varepsilon}_0
\end{aligned}$$

for some $\overline{C} > 0$ provided $\|U\|_{\mathbf{X}_{T_0^2}} \le \overline{C}\overline{\varepsilon}_0$ and ε_0 is chosen sufficiently small. On the other hand, we similarly show that $\mathbb{F}$ is a contraction on $B_{\overline{C}\overline{\varepsilon}_0} = \{U \in \mathbf{X}_{T_0^2} : \|U\|_{\mathbf{X}_{T_0^2}} \le \overline{C}\overline{\varepsilon}_0\}$. This implies that $\mathbb{F}$ has a unique fixed point in $B_{\overline{C}\overline{\varepsilon}_0}$ and concludes the proof of Theorem 3.2.5.

Proof of Theorem 3.2.7

Assume that $\Phi \in \mathcal{S}'(\mathbb{R}^N)$ such that $\nabla\Phi \in M_{2,N-2}(\mathbb{R}^N)$ and let $U_j = [c_j, n_j, u_j] \in L^\infty_{loc}(0,\infty; L^\infty(\mathbb{R}^N))$, $j = 1,2$ and set $U = U_1 - U_2$. Through similar ideas as in [Miu05], we first show that $U = 0$ on $\mathbb{R}^N \times [0, T_0)$ for some $T_0 > 0$. By Lemma 3.3.3 and the identities

$$
\begin{aligned}
(c_1 - c_2) &= B_1(c_2, n_2 - n_1)) + B_1(c_2 - c_1, n_1)) + B_1(u_2, \nabla(c_2 - c_1))) + B_1(u_2 - u_1, \nabla c_1) \\
(n_1 - n_2) &= B_2(n_1 - n_2, \nabla c_2) + B_2(n_1, \nabla(c_2 - c_1)) + B_2(n_2, u_2 - u_1) + B_2(u_1, n_2 - n_1) \\
(u_1 - u_2) &= B_3(u_2, u_2 - u_1) + B_3(u_2 - u_1, u_1) + \mathscr{L}_\Phi(n_2 - n_1),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\|U\|_{\mathbf{X}_T} &= \|c - c_1\|_{\mathbf{X}_{1,T}} + \|n_1 - n_2\|_{\mathbf{X}_{2,T}} + \|u_1 - u_2\|_{\mathbf{X}_{2,T}} \\
&\le K_1\Big(\|U_1\|_{\mathbf{X}_T} + \|U_2\|_{\mathbf{X}_T} + \|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}\Big)\|U\|_{\mathbf{X}_T}.
\end{aligned} \tag{3.4.6}
$$

In view of condition (3.2.11), there exists $T_0 > 0$ such that $\|U_1\|_{\mathbf{X}_{T_0}} + \|U_2\|_{\mathbf{X}_{T_0}} \le \dfrac{1}{4K_1}$. Given $\varepsilon \in (0,1)$, if $\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)} \le \dfrac{\varepsilon}{4K_1}$, then (3.4.6) implies that

$$
\|U\|_{\mathbf{X}_{T_0}} \le \frac{1}{2}\|U\|_{\mathbf{X}_{T_0}}.
$$

Hence, $U_1 = U_2$ on $\mathbb{R}^N \times [0, T_0)$. To extend this property to the whole interval $[0,\infty)$, observe that

$$
K_{12} \equiv \sup_{s\in(T_0,T)} \|U_1(s)\|_{L^\infty(\mathbb{R}^N)} + \sup_{s\in(T_0,T)} \|U_2(s)\|_{L^\infty(\mathbb{R}^N)} < \infty
$$

since $U_1, U_2 \in L^\infty_{loc}(0,\infty; L^\infty(\mathbb{R}^N))$. Now set

$$
a(t) = \sup_{T_0<s<t} \|U(s)\|_{L^\infty(\mathbb{R}^N)}, \quad t > T_0.
$$

We claim that there exits $\tau := \tau(T_0)$ such that $U_1 = U_2$ on $\mathbb{R}^N \times [0, T_0 + \tau)$. To see this, compute

$$
\begin{aligned}
|(c_1 - c_2)(x,t)| &= \bigg| \int_0^t \int_{\mathbb{R}^N} g(x-y, t-s)[c_2(n_2 - n_1) + n_1(c_2 - c_1) + \\
&\qquad\qquad u_2\nabla(c_2 - c_1) + (u_2 - u_1)\nabla c_1] dyds \bigg| \\
&\le \int_{T_0}^t \int_{\mathbb{R}^N} g(x-y, t-s)(|c_2(n_2 - n_1)| + |n_1(c_2 - c_1)| + \\
&\qquad\qquad |u_2\nabla(c_2 - c_1)| + |(u_2 - u_1)\nabla c_1|) dyds \\
&\le C_1 K_{12} a(t) \int_{T_0}^t \int_{\mathbb{R}^N} g(x-y, t-s) dyds \\
&\le C_1 K_{12} a(t)(t - T_0).
\end{aligned}
$$

On the other hand, using (3.3.17), we find that

$$
\begin{aligned}
|(u_1-u_2)(x,t)| &= \Big|\int_0^t\int_{\mathbb{R}^N}\nabla k_2(x-y,t-s)[u_2\otimes(u_2-u_1)+(u_2-u_1)\otimes u_1]dyds+ \\
&\qquad \int_0^t\int_{\mathbb{R}^N}k_2(x-y,t-s)(n_2-n_1)\nabla\Phi]dyds\Big| \\
&\le \int_{T_0}^t\int_{\mathbb{R}^N}|\nabla k_2(x-y,t-s)||u_2\otimes(u_2-u_1)|+|(u_2-u_1)\otimes u_1|dyds+ \\
&\qquad \int_{T_0}^t\int_{\mathbb{R}^N}|k_2(x-y,t-s)||(n_2-n_1)\nabla\Phi|dyds \\
&\le C_3K_{12}a(t)\int_{T_0}^t\int_{\mathbb{R}^N}\frac{dy}{[(x-y)^2+(t-s)]^{\frac{N+1}{2}}}ds+ \\
&\qquad C_4\int_{T_0}^t\int_{\mathbb{R}^N}\frac{|(n_2-n_1)(s)\nabla\Phi|dy}{(|x-y|+(t-s)^{1/2})^{-N}}ds \\
&\le C_3K_{12}a(t)\sqrt{t-T_0}+C_4\int_{T_0}^t(t-s)^{-1/2}\|(n_2-n_1)(s)\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}ds \\
&\le C_3K_{12}a(t)\sqrt{t-T_0}+C_4a(t)\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}\int_{T_0}^t(t-s)^{-1/2}ds \\
&\le C_3K_{12}a(t)\sqrt{t-T_0}+C_4a(t)\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)}\sqrt{t-T_0}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
|(n_1-n_2)(x,t)| &= \Big|\int_0^t\int_{\mathbb{R}^N}g(x-y,t-s)\nabla\cdot[(n_2-n_1)\nabla c_2+n_1\nabla(c_2-c_1)+ \\
&\qquad n_2(u_2-u_1)+u_1(n_2-n_1)]dyds\Big| \\
&\le \int_{T_0}^t\int_{\mathbb{R}^N}|\nabla g(x-y,t-s)|\Big(|(n_2-n_1)\nabla c_2|+|n_1\nabla(c_2-c_1)|+ \\
&\qquad |n_2(u_2-u_1)|+|u_1(n_2-n_1)|\Big)dyds \\
&\le C_2K_{12}a(t)\int_{T_0}^t\int_{\mathbb{R}^N}\frac{dy}{[(x-y)^2+(t-s)]^{\frac{N+1}{2}}}ds \\
&\le C_2K_{12}a(t)\left(\int_{T_0}^t(t-s)^{-1/2}ds\right)\left(\int_{\mathbb{R}^N}\frac{dy}{(|y|^2+1)^{\frac{N+1}{2}}}\right) \\
&\le C_2K_{12}a(t)\sqrt{t-T_0}.
\end{aligned}
$$

Summarizing, we have that (setting $C_5 = \max(C_1, C_2 + C_3, C_4\|\nabla\Phi\|_{M_{2,N-2}(\mathbb{R}^N)})$)

$$|U(x,t)| \le C_5 K_{12} a(t)\left((t-T_0) + \frac{K_{12}+1}{K_{12}}(t-T_0)^{\frac{1}{2}}\right)$$

from which it follows that

$$a(T_0+\tau) \le \frac{1}{4} a(T_0+\tau)$$

for $\tau = \theta^2$, $\theta = \sqrt{\left(\frac{K_{12}+1}{K_{12}}\right)^2 + \frac{1}{C_5 K_{12}}} - \frac{K_{12}+1}{K_{12}} > 0$. This shows the claim. Iterating this procedure yields the desired conclusion.

Proof of Theorem 3.2.8

The argument here is similar to that used above. We give the details for the reader's convenience. Let $d_0 \in UC(\mathbb{R}^N)$, Γ_{δ_0} and $\overline{c} = c - \Gamma_{\delta_0}$ as before. Next let $\widetilde{v_\kappa} = e^{-t\kappa} e^{t\Delta} v_0$ and make the change of variable $w = v - \widetilde{v_\kappa}$. Then Syst. (3.3.19) becomes

$$\begin{cases} \overline{c} = e^{t\Delta}\overline{c}_0 - \displaystyle\int_0^t e^{(t-s)\Delta}[(\overline{c} + \Gamma_{\delta_0})n + u \cdot \nabla(\overline{c} + \nabla\Gamma_{\delta_0}) + \Delta\Gamma_{\delta_0}](\cdot, s)ds \\ n = e^{t\Delta} n_0 - \displaystyle\int_0^t e^{(t-s)\Delta}\nabla \cdot [nu + n\nabla(\overline{c} + \Gamma_{\delta_0}) + n\nabla(w + \widetilde{v_\kappa})](\cdot, s)ds \\ w = \displaystyle\int_0^t e^{-(t-s)\kappa} e^{(t-s)\Delta}[u \cdot \nabla(w + \widetilde{v_\kappa}) + n](\cdot, s)ds \\ u = e^{-t\kappa} e^{t\Delta} u_0 - \displaystyle\int_0^t e^{-t\kappa} e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)(\cdot, s)ds - \int_0^t e^{(t-s)\Delta}\mathbf{P}(n\Psi)(\cdot, s)ds. \end{cases} \tag{3.4.7}$$

Observe that if $v_0 \in BMO_R(\mathbb{R}^N)$ for some $0 < R \le \infty$, then

$$[\widetilde{v_\kappa}]_{\mathbf{X}_{1,R^2}} \le C\|v_0\|_{BMO_R(\mathbb{R}^N)}.$$

This easily follows from the Carleson measure characterization of $BMO_R(\mathbb{R}^N)$. Remark that the kernel of the integral operator $e^{-t\kappa}e^{t\Delta}$, $\kappa > 0$ is bounded above by the heat kernel. This in turn implies that $\nabla\widetilde{v_\kappa} \in \mathbf{X}_{3,R^2}$. Hence, setting

$$\begin{aligned} \mathbb{F}_{2,\kappa}(\overline{c}, n, w, u) &= B_2(n, \nabla(\overline{c} + \Gamma_{\delta_0})) + B_2(n, u) + B_2(n, \nabla(w + \widetilde{v_\kappa})) \\ \mathbb{F}_4(n, w, u) &= B_4(u, w + \widetilde{v_\kappa}) + \mathscr{L}(n), \end{aligned}$$

the next Proposition follows from Lemmas 3.3.3 and 3.3.5.

Proposition 3.4.2. *For $R > 0$ and $T_0 = \min(\delta_0, R)$. If $[n, w, u] \in \mathbf{X}_{2,T_0^2} \times \mathbf{X}_{1,T_0^2} \times \mathbf{X}_{3,T_0^2}$ then $\mathbb{F}_{2,\kappa}(\overline{c}, n, w, u) \in \mathbf{X}_{2,T_0^2}$, $\mathbb{F}_4(n, w, u) \in \mathbf{X}_{1,T_0^2}$ and there exists $C_4, C_5 > 0$ such that*

$$\|\mathbb{F}_{2,\kappa}(\overline{c}, n, w, u)\|_{\mathbf{X}_{2,T_0^2}} \le C_5(\|\overline{c}\|_{\mathbf{X}_{1,T_0^2}} + \|u\|_{\mathbf{X}_{3,T_0^2}} + [\widetilde{v}_\kappa]_{\mathbf{X}_{1,T_0^2}} + \|w\|_{\mathbf{X}_{1,T_0^2}} + \varepsilon_0)\|n\|_{\mathbf{X}_{2,T_0^2}} \tag{3.4.8}$$

$$\|\mathbb{F}_4(n, w, u)\|_{\mathbf{X}_{1,T_0^2}} \le C_4([\widetilde{v}_\kappa]_{\mathbf{X}_{1,T_0^2}} + \|w\|_{\mathbf{X}_{1,T_0^2}})\|u\|_{\mathbf{X}_{3,T_0^2}} + C_4\|n\|_{\mathbf{X}_{2,T_0^2}}. \tag{3.4.9}$$

In addition, for any $[\overline{c}_1, n_1, w_1, u_1] \in \mathbf{Z}_{T_0^2}$, it holds that

$$\begin{aligned}\left\|\mathbb{F}_{2,\kappa}(\overline{c}, n, w, u) - \mathbb{F}_{2,\kappa}(\overline{c}_1, n_1, w_1, u_1)\right\|_{\mathbf{X}_{2,T_0^2}} &\le C_5\|n\|_{\mathbf{X}_{2,T_0^2}}\|w - w_1\|_{\mathbf{X}_{1,T_0^2}} + \\ &\quad C_5(\|\overline{c}\|^2_{\mathbf{X}_{1,T_0^2}} + [\widetilde{v}_\kappa]_{\mathbf{X}_{1,T_0^2}} + \|w\|_{\mathbf{X}_{1,T_0^2}} + 1)\|n - n_1\|_{\mathbf{X}_{2,T_0^2}} + \\ &\quad C_5\Big[\|n_1\|_{\mathbf{X}_{2,T_0^2}}(\|\overline{c} - \overline{c}_1\|^2_{\mathbf{X}_{1,T_0^2}}) + (\|\overline{c}\|_{\mathbf{X}_{1,T_0^2}} + \varepsilon_0)\|u - u_1\|_{\mathbf{X}_{3,T_0^2}}\Big] \\ \left\|\mathbb{F}_4(n, w, u) - \mathbb{F}_4(n_1, w_1, u_1)\right\|_{\mathbf{X}_{1,T_0^2}} &\le C_4(\|w_1\|_{\mathbf{X}_{1,T_0^2}} + [w_1]_{\mathbf{X}_{1,T_0^2}})\|u - u_1\|_{\mathbf{X}_{3,T_0^2}} + \\ &\quad C_4\|u\|_{\mathbf{X}_{1,T_0^2}}\|w - w_1\|_{\mathbf{X}_{1,T_0^2}} + C_4\|n - n_1\|_{\mathbf{X}_{2,T_0^2}}.\end{aligned}$$

The remaining part of the proof is done exactly as before. The details are omitted.

3.5 Appendix

This section contains all the deferred proofs which follow as particular cases of more general results.

Definition 3.5.1. Let $N \ge 3$ and $-2 < \lambda \le 2$. We say that a tempered distribution f belongs to $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ if $\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)}$ is finite,

$$\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N, R > 0}\left(|B(x,R)|^{\frac{\lambda}{N}-1}\int_0^{R^2}\int_{B(x,R)}|e^{t\Delta}f(y,s)|^2 dyds\right)^{1/2}.$$

For $\lambda = 0$, $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ is the space $BMO^{-1}(\mathbb{R}^N)$. Recall the characterization of square-Campanato spaces via caloric extension [JXY16]: $f \in \mathscr{L}_{2,N-\lambda}(\mathbb{R}^N)$, $\lambda \in (-2, 2]$ if and only if its caloric extension $u = e^{t\Delta}f \in T^{2,\lambda}$ and $\|u\|_{T^{2,\lambda}} \le C\|f\|_{\mathscr{L}_{2,N-\lambda}(\mathbb{R}^N)}$ where C is a constant independent of f and

$$\|u\|_{T^{2,\lambda}} := \sup_{x \in \mathbb{R}^N, R > 0}\left(|B(x,R)|^{\frac{\lambda}{N}-1}\int_0^{R^2}\int_{B(x,R)}|\nabla u(y,s)|^2 dyds\right)^{1/2}.$$

This extrinsic definition of Campanato spaces suggests that $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N) = \nabla\cdot(\mathscr{L}_{2,N-\lambda}(\mathbb{R}^N))^N$. This is indeed the case as shown below.

Lemma 3.5.2. *Assume that $\lambda \in (-2,2]$ and $N > 2$. A tempered distribution $f \in \mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ if and only if there exists a family $(f_l)_{l=1}^N \subset \mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ such that $f = \sum_{l=1}^N f_l$. Moreover, the following equivalence holds*

$$\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)} \approx \inf\left\{ \sum_{l=1}^N \|f_l\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)} : f = \sum_{l=1}^N \partial_l f_l \right\}. \tag{3.5.1}$$

Proof of Lemma 3.5.2. Let f be a tempered distribution. Assume that there is $f_l \in \mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$, $l = 1,\cdots,N$ with $f = \sum_{l=1}^N f_l$. Using the characterization of Morrey spaces by heat extension we obtain

$$\begin{aligned}\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)} &\le C \sum_{l=1}^N \sup_{x\in\mathbb{R}^N, R>0} \left(|B(x,R)|^{\frac{\lambda}{N}-1} \int_0^{R^2} \int_{B(x,R)} |\partial_j e^{t\Delta} f_l|^2 dy ds \right)^{1/2} \\ &\le C \sum_{l=1}^N \|f_l\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)}.\end{aligned}$$

This shows that $\nabla \cdot (\mathscr{L}_{2,N-\lambda}(\mathbb{R}^N))^N \subset \mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$. The converse follows from the observation that if $f \in \mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ then $f_{j,l} = \partial_j \partial_l (-\Delta)^{-1} f \in \mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$. Indeed, let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function with supp $\varphi \subset B(0,1)$, the Euclidean unit ball and $\int_{\mathbb{R}^N} \varphi dx = 1$. Set $\varphi_R(x) = R^{-N}\varphi(x/R)$ for $R > 0$ and write

$$e^{t\Delta} f_{j,l} = \partial_j \partial_l (-\Delta)^{-1} (e^{t\Delta} f) = f^1_{j,l} + f^2_{j,l}$$

where $f^1_{j,l} = \varphi_R * \partial_j \partial_l (-\Delta)^{-1} (e^{t\Delta} f)$ and $f^2_{j,l} = f_{j,l} - f^1_{j,l}$. Noticing that $\partial_j \partial_l (-\Delta)^{-1}$ is a Fourier multiplier of order 0, one has

$$\begin{aligned}\|f^1_{j,l}\|_{L^\infty(\mathbb{R}^N)} &\le C \|\varphi_R\|_{\dot{B}^{1+\frac{\lambda}{2}}_{1,1}(\mathbb{R}^N)} \|f_{j,l}\|_{\dot{B}^{-(1+\frac{\lambda}{2})}_{\infty,\infty}(\mathbb{R}^N)} \\ &\le C R^{-1-\frac{\lambda}{2}} \|e^{t\Delta} f\|_{\dot{B}^{-(1+\frac{\lambda}{2})}_{\infty,\infty}(\mathbb{R}^N)} \\ &\le C R^{-1-\frac{\lambda}{2}} \|f\|_{\dot{B}^{-(1+\frac{\lambda}{2})}_{\infty,\infty}(\mathbb{R}^N)} \\ &\le C R^{-1-\frac{\lambda}{2}} \|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)}\end{aligned}$$

since the operator $e^{t\Delta}$ maps $\dot{B}^s_{p,q}(\mathbb{R}^N)$ into itself (for $1 \le p,q \le \infty$, $s \in \mathbb{R}$) in addition to $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N) \subset \dot{B}^{-(1+\lambda/2)}_{\infty,\infty}(\mathbb{R}^N)$. The proof of the latter continuous embedding is given below. Using this, it follows that

$$\int_0^{R^2} \int_{B(x,R)} |f^1_{j,l}|^2 dy dt \le C |B(x,R)|^{1-\frac{\lambda}{N}} \|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)}.$$

To estimate $f^2_{j,l}$, we further decompose it into two parts writing $f^2_{j,l} = f^{21}_{j,l} + f^{22}_{j,l}$ with

$$\begin{aligned} f^{21}_{j,l} &= \partial_j\partial_l(-\Delta)^{-1}(\zeta_{x,R}e^{t\Delta}f) - \varphi_R * \partial_j\partial_l(-\Delta)^{-1}(\zeta_{x,R}e^{t\Delta}f) \\ f^{22}_{j,l} &= \partial_j\partial_l(-\Delta)^{-1}[(1-\zeta_{x,R})e^{t\Delta}f] - \varphi_R * \partial_j\partial_l(-\Delta)^{-1}[(1-\zeta_{x,R})e^{t\Delta}f] \end{aligned}$$

where $\zeta_{x,R} = \zeta(R^{-1}(x-\cdot))$ and $\zeta \in C^\infty_0(\mathbb{R}^N)$, supp $\zeta \subset B(0,20)$, $\zeta = 1$ on $B(0,10)$. By Pancherel's identity,

$$\begin{aligned} \int_0^{R^2}\int_{B(x,R)} |\partial_j\partial_l(-\Delta)^{-1}(\zeta_{x,R}e^{t\Delta}f)|^2 dydt &\leq \int_0^{R^2} \|\partial_j\partial_l(-\Delta)^{-1}(\zeta_{x,R}e^{t\Delta}f)\|^2_{L^2(\mathbb{R}^N)}dt \\ &\leq C\int_0^{R^2} \left\|\xi_j\xi_l|\xi|^{-2}\mathcal{F}\left(\zeta_{x,R}e^{t\Delta}f\right)\right\|^2_{L^2(\mathbb{R}^N)}dt \\ &\leq C\int_0^{R^2} \left\|\mathcal{F}\left(\zeta_{x,R}e^{t\Delta}f\right)\right\|^2_{L^2(\mathbb{R}^N)}dt \\ &\leq C\int_0^{R^2} \left\|\zeta_{x,R}e^{t\Delta}f\right\|^2_{L^2(\mathbb{R}^N)}dt. \end{aligned} \tag{3.5.2}$$

On the other hand, invoking Minkowski's inequality we find that

$$\begin{aligned} \int_0^{R}\int_{B_R(x)} |\varphi_R * \partial_j\partial_l(-\Delta)^{-1}(\zeta_{x,R}e^{t\Delta}f)|^2 dydt &\leq \int_0^{R^2} \|\varphi_R * \partial_j\partial_l(-\Delta)^{-1}(\zeta_{x,R}e^{t\Delta}f)\|^2_{L^2}dt \\ &\leq C\int_0^{R^2} \|\partial_j\partial_l(-\Delta)^{-1}(\zeta_{x,R}e^{t\Delta}f)\|^2_{L^2(\mathbb{R}^N)}dt \\ &\leq C\int_0^{R^2} \left\|\zeta_{x,R}e^{t\Delta}f\right\|^2_{L^2(\mathbb{R}^N)}dt. \end{aligned} \tag{3.5.3}$$

Thus, from (3.5.2) and (3.5.3), one deduces that

$$\int_0^{R^2}\int_{B(x,R)} |f^{21}_{j,l}(y,t)|^2 dydt \leq C|B(x,R)|^{1-\lambda/N}.$$

In order to estimate the term $f^{22}_{j,l}$, recall the pointwise estimate (see [LR02, Page 161])

$$\left|f^{22}_{j,l}(y,t)\right| \leq C\int_{|x-z|\geq 10R} \frac{R}{|x-z|^{N+1}}|e^{t\Delta}f(z)|dz, \quad y \in B(x,R). \tag{3.5.4}$$

Set $\widehat{A_k} = B(x, 10R(k+1)) \setminus B(x, 10Rk)$ and observe that (3.5.4) implies

$$\begin{aligned}\int_{B(x,R)} \left|f_{j,l}^{22}(y,t)\right|^2 dy &\leq CR^{N+1} \int_{|x-z|\geq 10R} \frac{1}{|x-z|^{N+1}} |e^{t\Delta} f(z)|^2 dz \\ &\leq CR^{N+1} \sum_{k=1}^{\infty} \int_{\widehat{A_k}} \frac{1}{|x-z|^{N+1}} |e^{t\Delta} f(z)|^2 dz \\ &\leq C \sum_{k=1}^{\infty} k^{-(N+1)} \sum_{\substack{w\in\mathbb{Z}^N \\ \& \\ w\in\widehat{A_k}}} \int_{B(w,R)} |e^{t\Delta} f|^2 dz \\ &\leq C\left(\sum_{k=1}^{\infty} k^{-2}\right) \int_{B(w,R)} |e^{t\Delta} f|^2 dz.\end{aligned}$$

Hence,

$$|B(x,R)|^{1/\lambda - N} \int_0^{R^2} \int_{B(x,R)} \left|f_{j,l}^{22}(y,t)\right|^2 dy dt \leq C.$$

Next, let $f_l = -\partial_l(-\Delta)^{-1} f$. It is clear that $f_l \in \mathscr{L}_{2,N-\lambda}(\mathbb{R}^N)$ for each $l = 1, ..., N$. Moreover, we verify that

$$\mathcal{F}\left(\sum_{l=1}^{N} \partial_l f_l\right)(\xi) = \sum_{l=1}^{N} i\xi_l \mathcal{F}(f_l)(\xi) = \sum_{l=1}^{N} -i\xi_l (i\xi_l)|\xi|^{-2} \mathcal{F}(f)(\xi) = \mathcal{F}(f)(\xi).$$

This achieves the proof of Lemma 3.5.2. □

Proof of the embedding (3.2.15). Let $N \geq 3$, $0 \leq \beta < N$, $0 \leq \lambda \leq 2$ and take $2 \leq p < \frac{2(N-\beta)}{\lambda}$ ($2 \leq p < \infty$ when $\lambda = 0$). Assume that $f \in \mathcal{N}_{p,\beta,\infty}^{-2s}(\mathbb{R}^N)$, $s = \frac{\lambda+2}{4} + \frac{\beta-N}{2p} > 0$. From the characterization of Besov-Morrey spaces (see e.g. [KY94, Maz03]), it holds that

$$\sup_{t>0} t^s \|e^{t\Delta} f\|_{M_{p,\beta}(\mathbb{R}^N)} \approx \|f\|_{\mathcal{N}_{p,\beta,\infty}^{-2s}(\mathbb{R}^N)}. \tag{3.5.5}$$

For $t > 0$, a use of Hölder's inequality yields

$$\begin{aligned}\|e^{t\Delta} f\|_{L^2(B_R(x))}^2 &\leq CR^{\frac{N(p-2)}{p}} \|e^{t\Delta} f(\cdot,t)\|_{L^p(B_R(x))}^2 \\ &\leq CR^{\frac{2\beta+N(p-2)}{p}} t^{-2s} \|f\|_{\mathcal{N}_{p,\beta,\infty}^{-2s}(\mathbb{R}^N)}^2\end{aligned}$$

for any $x \in \mathbb{R}^N$ and $R > 0$ so that

$$\begin{aligned}
\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)} &= \sup_{x\in\mathbb{R}^N, R>0} \left(|B(x,R)|^{\frac{\lambda}{N}-1} \int_0^{R^2} \int_{B(x,R)} |e^{t\Delta} f(y)|^2 dy dt \right)^{1/2} \\
&= \sup_{x\in\mathbb{R}^N, R>0} \left(|B(x,R)|^{\frac{\lambda}{N}-1} \int_0^{R^2} \|e^{t\Delta} f(t)\|^2_{L^2(B_R(x))} dt \right)^{\frac{1}{2}} \\
&\le C \sup_{x\in\mathbb{R}^N, R>0} \left(|B(x,R)|^{\frac{\lambda}{N}-1} R^{\frac{2\beta+N(p-2)}{p}} \int_0^{R^2} t^{-2s} dt \right)^{1/2} \|f\|_{\mathcal{N}^{-2s}_{p,\beta,\infty}(\mathbb{R}^N)} \\
&\le C\|f\|_{\mathcal{N}^{-2s}_{p,\beta,\infty}(\mathbb{R}^N)}.
\end{aligned}$$

The proof of the continuous embedding $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N) \subset \dot{B}^{-(1+\lambda/2)}_{\infty,\infty}(\mathbb{R}^N)$ which holds for any $\lambda \in (-2, 2]$ follows from the definition of $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ and is inspired by [Can04, Proposition 7]. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathcal{S}(\mathbb{R}^N)$ and its dual $\mathcal{S}'(\mathbb{R}^N)$. There exists a constant $C > 0$ such that

$$\left| \langle f, e^{-\frac{|x|^2}{4}} \rangle \right| \le C\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)} \tag{3.5.6}$$

for all $f \in \mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$. By translation invariance of $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$, (3.5.6) implies that

$$\left| (e^{-\frac{|\cdot|^2}{4}} * f) \right| \le C\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)}.$$

Moreover, by the invariance of $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ with respect to the scaling map $f_\lambda(\cdot) = \delta^{\lambda/2+1} f(\delta \cdot)$, $\delta > 0$ it holds that

$$\sup_{t>0} t^{\frac{\lambda}{4}+\frac{1}{2}} \|e^{t\Delta} f\|_{L^\infty(\mathbb{R}^N)} \le C\|f\|_{\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)}$$

which produces the desired bound. □

It is worth pointing out that the membership of a distribution f in $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ may be interpreted in terms of Carleson measures.

Definition 3.5.3. Let $\alpha > 0$. A positive measure μ in $\mathbb{R}^N \times \mathbb{R}^+$ is a (*parabolic*) α-Carleson measure if

$$\sup_{B\subset\mathbb{R}^N} \frac{\mu(T(B))}{|B|^\alpha} < \infty$$

where the supremum is taken over all balls in $\mathbb{R}^N$ and $T(B)$ is the (parabolic) Carleson box $T(B_r(x)) = B_r(x) \times (0, r^2]$ for $x \in \mathbb{R}^N$ and $r > 0$.

By this definition, it is easy to see that f belongs to $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$, $N > 2$ implies that $d\mu(x,t) = |n(x,t)|^2 dx dt$ is a $(1 - \frac{\lambda}{N})$-Carleson measure. Thus, $\mathscr{L}^{-1}_{2,N-\lambda}(\mathbb{R}^N)$ may be identified with the dual of certain tent space, we refer the interested reader to [Ame18].

Chapter 4

Existence and regularity of Solutions to Stationary Navier-Stokes equations arising from irregular data

4.1 Introduction

The steady state (forced) incompressible Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is the following system

$$\begin{cases} -\Delta u + \nabla \pi + u \cdot \nabla u = F \text{ in } \Omega \\ \operatorname{div} u = 0 \text{ in } \Omega \end{cases} \tag{\textbf{NS}}$$

where $u : \Omega \to \mathbb{R}^n$ is the unknown velocity field, $\pi : \Omega \to \mathbb{R}$ is the unknown scalar pressure and $F : \Omega \to \mathbb{R}^n$ is a given external force. This system is supplemented by the boundary condition

$$u = f \text{ on } \partial\Omega \tag{4.1.1}$$

where $f = (f_1, ..., f_n)$ is a prescribed vector field satisfying (in the case Ω smooth bounded) the compatibility condition $\displaystyle\int_{\partial\Omega} f \cdot \mathbf{N} d\sigma(Q) = 0$ with $\mathbf{N} = (N_1, \cdots, N_n)$ being the outer unit normal vector at the boundary.

Probably, the first striking result regarding the solvability of the Dirichlet problem for the Navier-Stokes equations was obtained by Leray [Ler33]. In a bounded three dimensional domain, he showed the existence of a weak solution $(u, \pi) \in W^{1,2}(\Omega) \times L^2(\Omega)$ provided $f \in W^{1/2,2}(\partial\Omega)$ and $F \in W^{-1,2}(\Omega)$. Existence of generalized weak solutions to (**NS**)-(4.1.1), those are $(u, \pi) \in W^{1,q}(\Omega) \times L^q(\Omega)$, $q \in [2, \infty)$ is a consequence of the work of Cattabriga [Cat61] (the reader may consult the monograph [Gal11] for a more complete theory). It is also known that in the most physically relevant dimensions $n = 2, 3$; any weak solution is smooth (see e.g. [Lad69] and [Ser83] in the case of nonhomogeneous data). The L^p-regularity of weak solutions in four dimensions is proved in [Ger79]. As for uniqueness, it seems that a smallness assumption on the given data is necessary and a recent result by Luo [Luo19] predicts that this condition cannot be dropped.

There have been growing interest in recent years in the analysis of the Navier-Stokes equations subject to low regularity data. By this we mean that the data lies in a space whose regularity index is less than that giving rise to generalized weak solutions. Assume that f enjoys a low regularity property, does problem (**NS**)-(4.1.1) admits a solution? In the affirmative case, what are the qualitative properties of such solutions? Prescribing boundary value with low regularity forces one to consider a notion of solution weaker than weak solutions, that is, those which need not to have finite Dirichlet energy. A good candidate, roughly speaking is obtained by testing (**NS**) against a suitable divergence-free smooth vector field and performing two successive integration by parts after which a variational formulation is obtained. This idea to the best of our knowledge, first appeared in [Ama00]. When $\Omega \subset \mathbb{R}^n$, $n = 2,3$ is a C^2 regular bounded domain, the author in [MP00] constructed such a solution in $L^{2n/(n-1)}(\Omega)$ provided $f \in L^2(\partial\Omega)$ (with arbitrarily large norm) and $F \in W^{-1,2n/(n-1)}(\Omega)$. Existence, uniqueness and regularity of very weak solutions (u,π) in the class $L^q(\Omega)\times W^{-1,q}(\Omega)$ have been obtained in [FGS06, GSS05] under certain smallness conditions on $f \in W^{-1/q,q}(\partial\Omega)$ and $F \in W^{-1,r}(\Omega)$ with $1 < r \le q < \infty$, $1/r \le 1/n + 1/q$. These results were generalized in [Kim09] where the author gave a complete theory for very weak solutions of (**NS**)-(4.1.1). In particular, refining the definition of very weak solutions and using some ideas from the preceding references, the author showed the existence of $(u,\pi) \in L^n(\Omega)\times W^{-1,n}(\Omega)$ for arbitrary large data f and forcing term F for $n = 3,4$. In two-dimensions, he proved existence of $(u,\pi) \in L^{q_0}(\Omega)\times W^{-1,1/q_0}(\Omega)$, $2 < q_0 < 3$. Moreover, he investigated the regularity of these solutions and also derived uniqueness results under suitable smallness requirements. The existence theory for very weak solutions in unbounded domains (the half-space, exterior domains, ect.) seems to be more subtle. We refer the reader to [Fin61] for an interesting discussion pertaining to generalized (weak) solutions. In general the same methods (as those employed in e.g. [Kim09]) cannot be carried out because the boundary of the domain is unbounded. We point out, however, the following existing results [ANR08, FS15] for the linear Stokes problem in half-space and [KKP15] in exterior domains.

This chapter aims at establishing the solvability theory for (**NS**) relying on novel ideas. The techniques employed here complement those introduced in Chapter 2. Assuming $\Omega = \mathbb{R}^n_+$, we seek for velocity field of the form $u = v + w$ where v solves the linear Stokes equation with Dirichlet data f while w solves the inhomogeneous Stokes problem with zero boundary data and source term $F + u\cdot\nabla u$. Odqvist [Odq30] proved that v assumes an integral representation, it is the Stokes extension of f to $\mathbb{R}^n_+$ (see Section 4.2). We look for f in a large class of distributions on $\mathbb{R}^{n-1}$ for which v is well-defined and has f as trace in a suitable sense. On one hand, (**NS**) is scaling (and translation) invariant with respect to the maps

$$u_\lambda(x) = \lambda u(\lambda x), \quad \pi_\lambda(x) = \lambda^2 \pi(\lambda x), \quad \lambda > 0$$

for appropriately rescaled external force and boundary data. On the other hand, we want to have u in the local Lebesgue space L^2_{loc} in order to make sense of the equation. From these observations, we are led to the consideration of v in $T^{2(n-1),2}$, a scale of tent spaces introduced by Coifmann, Meyer and Stein in [CMS85]. Thanks to the work by Triebel [Tri83] and others, we know that v must have a distributional trace f in the homogeneous negative Sobolev space $\dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})$. By the same token, the pressure π is sought for in the weighted tent space $T^{2(n-1),2}_{-1/(n-1)}$ (see below for the definition of weighted tent spaces). Tent spaces naturally arise in the analysis of linear elliptic equations and systems, see e.g. [AA11, HMM11] and references therein. We quote the recent work [YK22] where these spaces are used in the context of nonlinear systems.

The main result of this chapter states that there exits a unique solution of (**NS**)-(4.1.1) in a suitable framework under a smallness condition on $f \in \dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})$. A more general statement involving Dirichlet data in homogeneous Triebel-Lizorkin spaces is obtained. The global integrability of solutions is expressed in terms of tent norms and it is further shown that these solutions enjoy a better regularity locally. This latter property is derived from the pointwise decay rate of the velocity field near the boundary. To achieve this, we study the inhomogeneous Stokes problem in $\mathbb{R}^n_+$ (which plays a fundamental role in the analysis of (**NS**) when the flow takes place in an exterior region, a channel or a pipe) and derived key estimates of the solution for prescribed data in the homogeneous Triebel-Lizorkin class with negative amount of smoothness. These estimates are new and generalize those obtained in [FS15].

4.1.1 Tent spaces and functional settings

Throughout, a point $x \in \mathbb{R}^n_+$ will typically be denoted by (x', x_n), $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$. For $R > 0$, $B_R(x')$ is the closed ball with radius $R > 0$ and center at $x' \in \mathbb{R}^{n-1}$. Given $\alpha > 0$, define the cone (nontangential region) with vertex at $x' \in \mathbb{R}^{n-1}$ by

$$\Gamma_\alpha(x') := \{(y', y_n) \in \mathbb{R}^n_+ : |x' - y'| < \alpha y_n\}.$$

We simply use the notation Γ when $\alpha = 1$. Given a ball $B = B_R(x')$, we denote by $T(B) = B_R(x') \times (0, 2R)$ the Carleson box over $B_R(x')$. For $q \in [1, \infty)$, consider the functionals $\mathcal{A}_q$, $\mathscr{C}_q$ defined for F measurable in $\mathbb{R}^n_+$ by

$$\mathcal{A}_q F(x') = \left(\iint_{\Gamma(x')} |F(y', y_n)|^q y_n^{-(n-1)} dy' dy_n \right)^{1/q}, \ \mathcal{A}_\infty F(x') = \operatorname*{ess\,sup}_{(y', y_n) \in \Gamma(x')} |F(y', y_n)| \qquad (4.1.2)$$

$$\mathscr{C}_q F(x') = \sup_{B \ni x'} \left(\iint_{T(B)} |F(y', y_n)|^q dy_n dy' \right)^{1/q}. \qquad (4.1.3)$$

The membership of each of these functionals in a Lebesgue space gives rise to a scale of functions space first introduced by Coifman, Meyer and Stein [CMS85]. We point out here the use of a different normalization in (4.1.2) and (4.1.3). Let $p,q \in [1,\infty)$. The tent space $T^{p,q}$ collects all functions $F \in L^q_{loc}(\mathbb{R}^n_+)$ for which $\mathcal{A}_q F \in L^p(\mathbb{R}^{n-1})$. We equip this space with the norm

$$\|F\|_{T^{p,q}} := \|\mathcal{A}_q F\|_{L^p(\mathbb{R}^{n-1})}. \tag{4.1.4}$$

When $p=\infty$, the space $T^{\infty,q}$ is defined by

$$T^{\infty,q} = \{F \in L^q_{loc}(\mathbb{R}^n_+) : \mathscr{C}_q F \in L^\infty(\mathbb{R}^{n-1})\}.$$

The space $T^{\infty,q}$ is intrinsically linked to Carleson measures. In fact, it is the space of functions $F \in L^q_{loc}(\mathbb{R}^n_+)$ for which $d\mu(y',y_n) = |F|^q dy' dy_n$ is a Carleson measure in $\mathbb{R}^n_+$. For any $p \in (1,\infty]$ and $q \in [1,\infty)$, $T^{p,q}$ is a Banach space having the space of functions in $L^q(\mathbb{R}^n_+)$ with compact support as a dense subspace. This property together with the completeness of $T^{p,q}$ for any $p,q \in [1,\infty)$ follows from Lemma 4.1.1 below.

Lemma 4.1.1. *Let K be a compact set in $\mathbb{R}^n_+$ and assume that $F \in T^{p,q}$ for $p,q \in [1,\infty)$. Then*

$$C_1\|\mathbf{1}_K F\|_{T^{p,q}} \le \|F\|_{L^q(K)} \le C_2\|F\|_{T^{p,q}} \tag{4.1.5}$$

where the constant C_1, C_2 only depend on p,q,n and K.

Out of convenience, we defer the proof of Lemma 4.1.1 to the Appendix.

Remark 4.1.2. We also remark that change of aperture in the cone does not affect the tent norm. In other words, if

$$\mathcal{A}^\alpha_q F(x') := \left(\iint_{\Gamma_\alpha(x')} |F(y',y_n)|^q y_n^{-(n-1)} dy' dy_n \right)^{1/q}, \quad \alpha > 0$$

then

$$\|\mathcal{A}^\alpha_q\|_{L^p(\mathbb{R}^{n-1})} \approx \|\mathcal{A}^\beta_q F\|_{L^p(\mathbb{R}^{n-1})} \tag{4.1.6}$$

where the implicit constant depends on p,q and $\alpha,\beta \in (0,\infty)$. See [CMS85, Proposition 4, p. 309] which remains valid for $q \neq 2$.

For $s \in \mathbb{R}$, we say that $F : \mathbb{R}^n_+ \to \mathbb{R}$ belongs to the weighted tent spaces [Ame18, HMM11], which we denote by $T^{p,q}_s$ if

$$(y',y_n) \mapsto y_n^{-(n-1)s} F(y',y_n) \in T^{p,q}.$$

We easily verify that $\|F\|_{T^{p,q}_s} := \left\| y_n^{(1-n)s} F \right\|_{T^{p,q}}$ defines a norm on $T^{p,q}_s$. Moreover, for $s_1, s_2 \in \mathbb{R}$ such that $s_2 < s_1$ and $1 \le p_1 < p_2 \le \infty$, $q \in (0,\infty]$ the following continuous embedding holds (see [Ame18, Lemma 2.19])

$$T^{p_1 q}_{s_1} \subset T^{p_2 q}_{s_2} \tag{4.1.7}$$

provided $s_2 - s_1 = \dfrac{1}{p_2} - \dfrac{1}{p_1}$. Recall Hölder's inequality in weighted tent spaces.

Lemma 4.1.3. *Let $p_i, q_i, r_i \in [1,\infty)$ and $s_i \in \mathbb{R}$, $i = \{0,1,2\}$ such that $\sum_{i=1}^{2} 1/p_i = 1/p_0$ and $\sum_{i=1}^{2} 1/q_i = 1/q_0$ with the convention $1/\infty = 0$. If $f \in T_{s_1}^{p_1,q_1}$ and $g \in T_{s_2}^{p_2,q_2}$, then $fg \in T_{s_0}^{p_0,q_0}$ and it holds that*

$$\|fg\|_{T_{s_0}^{p_0,q_0}} \leq C\|f\|_{T_{s_1}^{p_1,q_1}}\|g\|_{T_{s_2}^{p_2,q_2}} \tag{4.1.8}$$

provided $s_0 = s_1 + s_2$.

This lemma can be proved via a direct argument – there is also another strategy relying on factorization of tent spaces, see [Hua16]. It is long-established that there is an intrinsic connection between weighted tent spaces and Triebel-Lizorkin spaces which we now recall its definition.

Let us denote by $\mathcal{S}(\mathbb{R}^{n-1})$ the class of Schwartz (smooth rapidly decreasing) functions on $\mathbb{R}^{n-1}$ and $\mathcal{S}'(\mathbb{R}^{n-1})$ its topological dual space endowed with the weak-$\star$ topology. Define the space

$$\mathcal{S}_0(\mathbb{R}^{n-1}) = \{f \in \mathcal{S}(\mathbb{R}^{n-1}) \mid \int x^\gamma f(x)dx = 0, \ \forall \gamma \in \mathbb{N}^n\}$$

which inherits the topology of $\mathcal{S}(\mathbb{R}^{n-1})$ as subspace. This space may be identified with the space of Schwartz functions whose Fourier transforms vanish together with all their derivatives at the origin. Its dual space is denoted by $\mathcal{S}_0'(\mathbb{R}^{n-1})$. Let φ be a cut-off function given by

$$\varphi(\xi) = \begin{cases} 1 \ \text{ if } \ |\xi| \leq 1 \\ \text{smooth } \ \text{if } \ 1 < |\xi| \leq 2 \\ 0 \ \text{ if } \ |\xi| > 2. \end{cases}$$

Let $\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$ and define $\psi_j(\xi) = \psi(2^{-j}\xi)$, $j \in \mathbb{Z}$ so that

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1, \ \ \xi \in \mathbb{R}^{n-1} \setminus \{0\}.$$

Let us denote by $\mathcal{F}f$ the Fourier transform of f on $\mathbb{R}^{n-1}$ and by $\dot{\Delta}_j = \mathcal{F}^{-1}(\psi_j\mathcal{F})$ the homogeneous Littlewood-Paley operator. Let $f \in \mathcal{S}_0'(\mathbb{R}^{n-1})$. For $s \in \mathbb{R}$; $p,q \in [1,\infty)$ we say that f belongs to the Triebel-Lizorkin space $\dot{F}_{p,q}^s(\mathbb{R}^{n-1})$ if

$$\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^{n-1})} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{-jsq} \left| \dot{\Delta}_j f \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n-1})} < \infty.$$

This space is of Banach type and is equivalent to the Sobolev space $\dot{H}^{s,p}(\mathbb{R}^{n-1})$ whenever $q = 2$ and $1 < p < \infty$. Moreover, for $1 \leq q_1, q_2 \leq \infty$ and $-\infty < s_2 < s_1 < \infty$ we have the continuous inclusion

$$\dot{F}_{p_1,q_1}^{s_1}(\mathbb{R}^{n-1}) \subset \dot{F}_{p_2,q_2}^{s_2}(\mathbb{R}^{n-1})$$

provided $p_1, p_2 \in (1,\infty)$ with $s_1 - \frac{n-1}{p_1} = s_2 - \frac{n-1}{p_2}$.

Definition 4.1.4. For $q \in (\frac{n}{n-1}, \infty)$, $n \geq 2$ we define $\mathbf{X}^q$ as the space of vector fields $u : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfying $\|u\|_{\mathbf{X}^q} < \infty$ and $\mathbf{Z}^q := \{\pi : \mathbb{R}^n_+ \to \mathbb{R} \,\big|\, \|\pi\|_{\mathbf{Z}^q} < \infty\}$ where

$$\|u\|_{\mathbf{X}^q} = \sup_{x_n>0} x_n^{\frac{1}{q-1}} \|u(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \|u\|_{T^{p,q}}$$

and

$$\|\pi\|_{\mathbf{Z}^q} := \|\pi\|_{T^{p,q}_{s_0}}, \;\; s_0 = -\frac{1}{n-1}, \; p = (n-1)(q-1)q.$$

Definition 4.1.5. Let $1 \leq \eta < \tau < \infty$. We say that $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ belongs to $\mathbf{Y}^{\tau,\eta}$ if

$$\|F\|_{\mathbf{Y}^{\tau,\eta}} = \sup_{x_n>0} x_n^{\frac{1}{\eta}+\frac{n-1}{\tau}} \|F(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} + \|F\|_{T^{\tau,\eta}}$$

is finite.

Note that either of the expression $\|\cdot\|_{\mathbf{X}^q}$ or $\|\cdot\|_{\mathbf{Z}^q}$ defines a norm on $\mathbf{X}^q$ and $\mathbf{Z}^q$ respectively. It can also be easily verified that they are Banach spaces. For convenience, when $q = 2$, we will specially denote the spaces $\mathbf{X}^q$ and $\mathbf{Z}^q$ by $\mathbf{X}$ and $\mathbf{Z}$, respectively.

4.1.2 Main results

Our first result deals with the well-posedness theory. In what follows, the dimension is assumed larger or equals to 3 unless otherwise stated.

Theorem 4.1.6. *Assume that $F = 0$. (**NS**)-(4.1.1) has a unique solution (u, π) in a small closed ball of $\mathbf{X} \times \mathbf{Z}$ provided the data f has a sufficiently small $[\dot{H}^{-\frac{1}{2}, 2(n-1)}(\mathbb{R}^{n-1})]^n$-norm.*

In presence of the forcing term, our main finding reads as follows.

Theorem 4.1.7. *Let $1 < \eta < \tau < \infty$ such that $\dfrac{1}{\eta} + \dfrac{n-1}{\tau} = 3$. There exist $\varepsilon > 0$ and $\kappa := \kappa(\varepsilon) > 0$ such that for every $f \in [\dot{H}^{-\frac{1}{2}, 2(n-1)}(\mathbb{R}^{n-1})]^n$ and $F \in \mathbf{Y}^{\tau,\eta}$ with $\|f\|_{\dot{H}^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau,\eta}} < \varepsilon$, (**NS**)-(4.1.1) has a solution (u, π) in $\mathbf{X} \times \mathbf{Z}$ which is the only one among those satisfying the condition $\|u\|_{\mathbf{X}} + \|\pi\|_{\mathbf{Z}} \leq 2\kappa$.*

Existence of solutions in $\mathbf{X}^q \times \mathbf{Z}^q$ for any $2 < q < \infty$ is a consequence of Theorem 4.1.7 together with an improved regularity result. In more details, the statement reads as follows.

Theorem 4.1.8. *Let $2 < q < \infty$, $\eta > 0$ as in Theorem 4.1.7 and $1 < \eta_1 < \tau_1 < \infty$. Given f in $\left[\dot{H}^{-\frac{1}{2}, 2(n-1)} \cap \dot{F}^s_{p,q}(\mathbb{R}^{n-1})\right]^n$ and $F \in \mathbf{Y}^{\tau,\eta} \cap \mathbf{Y}^{\tau_1,\eta_1}$, there exist $\varepsilon_q \in (0, \varepsilon)$ and $\kappa_q > 0$ such that if*

$\|f\|_{\dot H^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}\tau,\eta} < \varepsilon_q$ *then there exists a solution* (u,π) *of* (**NS**)-(4.1.1) *in the space* $\mathbf{X}^q \times \mathbf{Z}^q$ *which is unique in the ball*

$$B_{2\kappa_q}(\mathbf{0}) = \{(u,\pi) \in \mathbf{X} \times \mathbf{Z} : \|u\|_{\mathbf{X}} + \|\pi\|_{\mathbf{Z}} \leq 2\kappa_q\}$$

provided $\frac{1}{\eta_1} + \frac{n-1}{\tau_1} = 2 + \frac{1}{q-1}$, $s = -\frac{1}{q}$ *and* $p = (n-1)(q-1)q$.

The uniqueness of the pressure as claimed in the previous results should be understood up to an additive constant. We also record the following regularity result which arises as a consequence of the local boundedness property of the velocity field.

Theorem 4.1.9. *If* $(u,\pi) \in \mathbf{X}\times\mathbf{Z}$ *is the solution of problem* (**NS**)-(4.1.1) *constructed in Theorem 4.1.6 or Theorem 4.1.7, then* $(u,\pi) \in [C^{0,\alpha}_{loc}(\mathbb{R}^n_+)]^n \times L^p_{loc}(\mathbb{R}^n_+)$ *for some* $\alpha \in (0,1)$ *and every* $p \in (1,\infty)$.

Remark 4.1.10. It should be observed that Theorem 4.1.7 in the precise form stated above fails to hold in two dimensions. Indeed, if $n = 2$, then the exponents η and τ will obviously fail to satisfy the required assumption in Theorem 4.1.7. A close inspection reveals that the two-dimensional (unforced) Navier-Stokes equations is well-posed, that is, Theorem 4.1.6 is true in 2 dimensions. Theorem 4.1.8 shows that if f is taken in a slightly more regular space, then the solution (u,π) has a better global integrability property.

4.2 Auxiliary results

This section is devoted to the analysis of the Dirichlet problem for the following system

$$\begin{cases} -\Delta u + \nabla \pi = F + \mathrm{div}\, H \ \text{ in } \ \mathbb{R}^n_+ \\ \mathrm{div}\, u = 0 \ \text{ in } \ \mathbb{R}^n_+ \\ u = f \ \text{ on } \ \partial\mathbb{R}^n_+ \end{cases} \tag{\textbf{S}}$$

for given vector fields f, F and tensor H. Our goal is to prove that (**S**) admits a solution (u,π) in the target space $\mathbf{X}^q \times \mathbf{Z}^q$ whose norm can be estimated by the norms of f, F and H in suitable functions spaces. To this end, for better readability we simply separate the study into two parts: the homogeneous case ($f = 0$) and the inhomogeneous case ($F = 0$, $H = \mathbf{0}$).

4.2.1 Homogeneous Stokes system and linear estimates

Consider the Stokes operator L_S acting on pair of functions $(u,\pi) \in [\mathscr{D}'(\mathbb{R}^n)]^n \times \mathscr{D}'(\mathbb{R}^n)$, $n > 2$ and given by

$$L_S(u,\pi) = \Big(-\Delta u_1 + \partial_1 \pi, \cdots, -\Delta u_n + \partial_n \pi, \sum_{i=1}^n \partial_i u_i \Big).$$

A fundamental solution of the Stokes operator L_S in $\mathbb{R}^n$ is a pair $(\mathbb{E}, \mathbf{b})$ with $\mathbb{E} = (E_{ij})_{i,j=1}^n$ in $\mathcal{M}_{n\times n}[\mathcal{S}'(\mathbb{R}^n)]$ and $\mathbf{b} = (b_1, ..., b_n) \in [\mathcal{S}'(\mathbb{R}^n)]^n$ satisfying coordinate-wise the equations

$$\begin{cases} -\Delta E_{ij} + \partial_i b_j = \delta_{ij}\delta \text{ in } \mathcal{S}'(\mathbb{R}^n), \quad i,j \in \{1,...,n\} \\ \displaystyle\sum_{k=1}^n \partial_k E_{kj} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^n), \quad j \in \{1,...,n\}. \end{cases}$$

Applying the Fourier transform to both sides of each of the above equations yields the explicit expressions

$$E_{ij}(x) = \frac{1}{2\omega_{n-1}}\left[\frac{1}{(n-2)}\frac{\delta_{ij}}{|x|^{n-2}} + \frac{x_i x_j}{|x|^n}\right], \; b_j = \frac{1}{\omega_{n-1}}\frac{x_j}{|x|^n}, \quad i,j \in \{1,...,n\} \tag{4.2.1}$$

defined for $x \in \mathbb{R}^{n-1} \setminus \{0\}$ where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^{n-1}$. Details of explicit computations leading to (4.2.1) can be found in [Mit13, Chap. 10]. On the other hand, when $n = 2$, $\mathbf{E}$ and $\mathbf{b}$ assume the following forms:

$$E_{ij}(x) = \frac{1}{4\pi}\left[\frac{x_i x_j}{|x|^2} - \delta_{ij}\log|x|\right], \; b_j = \frac{x_j}{2\pi|x|^2}, \quad x \in \mathbb{R}^{n-1} \setminus \{0\}, \; i,j \in \{1,...,n\}. \tag{4.2.2}$$

Now, let us consider the homogeneous Stokes system

$$\begin{cases} -\Delta u + \nabla\pi = 0 \;\text{ in }\; \mathbb{R}^n_+ \\ \operatorname{div} u = 0 \;\text{ in }\; \mathbb{R}^n_+ \\ u = f \;\text{ on }\; \partial\mathbb{R}^n_+. \end{cases} \tag{4.2.3}$$

With the convolution being understood in a component wise sense, define

$$\mathcal{H}f(x', x_n) = (\mathcal{K}_{x_n} * f)(x'), \quad \mathcal{E}f(x', x_n) = (\mathrm{k}_{x_n} * f)(x') \tag{4.2.4}$$

where

$$\mathcal{K}_{x_n}(x') = (K_{ij}(x', x_n))_{1\le i,j\le n} \text{ and } \mathrm{k}_{x_n}(x') = (\mathrm{k}_1(x', x_n), ..., \mathrm{k}_n(x', x_n))$$

are commonly referred to as the Odqvist kernels [Odq30] – each entry of the tensors assuming an explicit form in terms of (4.2.1) via the formulas

$$K_{ij}(x) = 2(\partial_{x_n} E_{ij} + \partial_j E_{in} + \delta_{jn} b_i) = \frac{2n}{\omega_{n-1}}\frac{x_n x_i x_j}{|x|^{n+2}} \tag{4.2.5}$$

and

$$\mathrm{k}_j(x) = 4\partial_j b_n = \frac{1}{\omega_{n-1}}\partial_j \frac{4x_n}{|x|^n}. \tag{4.2.6}$$

For the derivation of these kernels, the interested reader may as well consult the articles [Odq30, Sol77]. Note that if f belongs to the weighted Lebesgue space $L^1\Big(\mathbb{R}^{n-1}, \frac{dx'}{(1+|x'|)^n}\Big)$, then $u = \mathcal{H}f$ and $\pi = \mathcal{E}f$ are both meaningful as absolutely convergent integrals and (u,π) is the unique solution of Eq. (4.2.3) decaying at infinity. This is no longer the case if f is merely a generic distribution. In fact, the Stokes extension $\mathcal{H}$ does not map $\mathcal{S}'(\mathbb{R}^{n-1})$ into itself in general (for example, in one dimension $f(x') = x'^2 \in \mathcal{S}'(\mathbb{R})$ but $\mathcal{H}f \notin \mathcal{S}'(\mathbb{R})$). However, it can be shown that if $f \in \mathcal{S}_0'(\mathbb{R}^{n-1})$, then so are $\mathcal{H}f$ and $\mathcal{E}f$. Poisson extensions of Schwartz distributions have been studied by H. Triebel [Tri83] – they characterize almost all scale of Triebel-Lizorkin spaces on $\mathbb{R}^{n-1}$ using tent spaces. In particular, the following equivalence holds true

$$\|f\|_{\dot{F}_{p,q}^{-\frac{1}{q}}(\mathbb{R}^{n-1})} \sim \left\| \mathcal{A}_q[\mathcal{P}_{x_n} * f] \right\|_{L^p(\mathbb{R}^{n-1})}, \quad p = q(q-1)(n-1) \tag{4.2.7}$$

where $1 < q < \infty$ and $\mathcal{P}_{x_n}(x') = c_n x_n (|x'|^2 + x_n^2)^{-\frac{n}{2}}$ (with c_n normalizing constant such that $\mathcal{P}_{x_n}$ has a normalized L^1-norm equals to 1) is the Poisson kernel for the Laplacian in $\mathbb{R}^n_+$.

Lemma 4.2.1. *Let $n \geq 2$, $q \in (\frac{n}{n-1}, \infty)$ and set $p = (n-1)q(q-1)$. There exists a constant $C := C(n,q) > 0$ such that*

$$\|\mathcal{H}f\|_{\mathbf{X}^q} + \|\mathcal{E}f\|_{\mathbf{Z}^q} + \sup_{x_n>0} x_n^{q/(q-1)} \|\mathcal{E}f(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\|f\|_{\dot{F}_{p,q}^{-\frac{1}{q}}(\mathbb{R}^{n-1})} \tag{4.2.8}$$

for all $f \in [\dot{F}_{p,q}^{-\frac{1}{q}}(\mathbb{R}^{n-1})]^n$ where $\mathcal{H}f$ and $\mathcal{E}f$ are defined as in (4.2.4).

We state two more auxiliary results which will be useful in the demonstration of Lemma 4.2.1.

Lemma 4.2.2 (Averaging Lemma)**.** *Assume that $F \in L^q(\mathbb{R}^n_+)$, $q \geq 1$. We have*

$$\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} |F(y', y_n)|^q \frac{dy' dy_n}{y_n^{n-1}} dx' = \mu \int_{\mathbb{R}^n_+} |F(y)|^q dy$$

where $\mu > 0$ only depends on n, the dimension.

The proof of this identity follows from a simple application of Fubini–Tonelli's Theorem.

Lemma 4.2.3. *Let $K \subset \mathbb{R}^n_+$ compact set and $E(K) = \{x' \in \mathbb{R}^{n-1} : K \cap \Gamma(x') \neq \emptyset\}$. Then $E(K)$ is open, its Lebesgue measure $|E(K)|$ is finite and only depends on K.*

Proof. Let $x' \in E(K)$, there exists $(y', y_n) \in \mathbb{R}^n_+$ with $(y', y_n) \in K$ and $y' \in B_{y_n}(x')$. Putting $R = y_n - |x' - y'| > 0$, it plainly follows that $B(x', R) \subset E(K)$. Moving on, we remark that $E(K)$ is actually bounded. Moreover, since K is compact, we may assume without loss of generality that $K = B_\theta(z') \times [a,b]$ for some $a, b, \theta > 0$ with $a < b$ and thus a simple covering argument implies that $|E(K)| \leq C\theta^{n-1}$ for some constant $C > 0$. □

Now we are ready to prove Lemma 4.2.1.

Proof of Lemma 4.2.1. By a direct computation, each coefficient of the matrix $\mathcal{K}$ satisfies the pointwise estimate $\left|\nabla^k K_{ij}(x',x_n)\right| \le c x_n^{-k}\mathcal{P}_{x_n}(x')$, $k = 0,1$ for each $i,j = 1,\cdots,n$. Also, from the explicit expression

$$\mathrm{k}_j(x',x_n) = \frac{4}{\omega_{n-1}}\begin{cases} \frac{x_n x_j}{|x|^{n+2}} & \text{if } \ j = 1,\ldots,n-1 \\ \frac{n x_n^2 - |x|^2}{n|x|^{n+2}} & \text{if } \ j = n \end{cases}$$

we verify that $|\mathrm{k}_j(x',x_n)| \le C x_n^{-1}\mathcal{P}_{x_n}(x')$ for all $(x',x_n) \in \mathbb{R}^n_+$. Let f_j in $\dot{F}_{pq}^{-\frac{1}{q}}(\mathbb{R}^{n-1})$ and set $(\overline{u},\overline{\pi}) = (\mathcal{H}(f),\mathcal{E}(f))$. For $x \in \mathbb{R}^n_+$ fixed, the interior estimate (see e.g. [Sim92]) for the linear Stokes problem together with Lemma 4.2.2 allow us to write

$$\begin{aligned}
|\overline{\pi}(x)|^q &\le C|B_{x_n/2}(x)|^{-1}\int_{B_{x_n/2}(x)} |\overline{\pi}(y',y_n)|^q dy'dy_n \\
&\le C|B_{x_n/2}(x)|^{-1}\int_{B_{x_n/2}(x)} |y_n^{-1}(\mathcal{P}_{y_n} * f)|^q dy'dy_n \\
&\le C|B_{x_n/2}(x)|^{-1}\int_{B_{x_n}(x')\times[x_n/3,2x_n]} |y_n^{-1}(\mathcal{P}_{y_n} * f)|^q dy'dy_n \\
&\le C x_n^{-(n+q)}\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{B_{y_n}(z')} \mathbf{1}_{B_{x_n}(x')\times[x_n/3,2x_n]}(y',y_n)|\mathcal{P}_{y_n} * f|^q \frac{dy'dy_n}{y_n^{n-1}}dz' \\
&\le C x_n^{-(n+q)}\left\|\mathcal{A}_q[\mathcal{P}_{y_n} * f]\right\|^q_{L^p(\mathbb{R}^{n-1})}\left|E\Big(B_{x_n}(x')\times[x_n/3,2x_n]\Big)\right|^{\frac{p-q}{p}} \\
&\le C x_n^{-1-q-\frac{(n-1)q}{p}}\|f\|^q_{\dot{F}^{-1/q}_{p,q}(\mathbb{R}^{n-1})}.
\end{aligned}$$

Observe that we have used Hölder's inequality and Lemma 4.2.3 to get the estimate before the last and the choice $p = q(n-1)(q-1)$ yields the desired bound. From the above remark on the kernel k_j, one has

$$\begin{aligned}
\|\overline{\pi}\|_{T_1^{p,q}} &= \left\|\left(\iint_{\Gamma(\cdot)} |y_n\overline{\pi}(y',y_n)|^q \frac{dy'dy_n}{y_n^{n-1}}\right)^{1/q}\right\|_{L^p(\mathbb{R}^{n-1})} \\
&\le C\left\|\left(\iint_{\Gamma(\cdot)} |\mathcal{P}_{y_n} * f|^q \frac{dy'dy_n}{y_n^{n-1}}\right)^{1/q}\right\|_{L^p(\mathbb{R}^{n-1})} \le C\|f\|_{\dot{F}^{-1/q}_{p,q}(\mathbb{R}^{n-1})}.
\end{aligned}$$

The latter bound is a consequence of the extrinsic characterization (4.2.7). The same observation pertaining to the velocity field gives $\|\overline{u}\|_{T^{p,q}} \le C\|f\|_{\dot{F}^{-1/q}_{p,q}(\mathbb{R}^{n-1})}$. It then remains to establish the bound

$$\sup_{x_n>0} x_n^{\frac{1}{q-1}}\|\overline{u}(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \le C\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})}. \tag{4.2.9}$$

By the mean value property for the velocity field [Sim92, Theorem 4.5] and with the same notation as above

$$|\overline{u}_i(x)| \le \fint_{B_{x_n/2}(x)} |\overline{u}_i(y)|dy + \frac{1}{2}\fint_{B_{x_n/2}(x)} |\overline{\pi}(z)||z_i - x_i|dz := I + II,\ i = 1,2,...,n.$$

Using Hölder's inequality and Lemma 4.2.2, we estimate I as follows:

$$\begin{aligned}
I^q &\le Cx_n^{-(n-1)} \int_{B_{x_n/2}(x)} |\mathcal{P}_{x_n} * f_j|^q dy \\
&\le Cx_n^{-(n-1)} \int_{\mathbb{R}^{n-1}} \iint_{\Gamma(z')} \mathbf{1}_{B_{x_n/2}(x)}(y)|\mathcal{P}_{x_n} * f_j|^q y_n^{-(n-1)} dydz' \\
&\le Cx_n^{-(n-1)} \left\|\mathcal{P}_{x_n} * f_j\right\|^q_{T^{p,q}} \left|E\Big(B_{\frac{x_n}{2}}(x') \times [x_n/3, 2x_n]\Big)\right|^{\frac{p-q}{p}} \\
&\le Cx_n^{q/(q-1)} \|f\|^q_{\dot{F}^{-1/q}_{p,q}(\mathbb{R}^{n-1})}.
\end{aligned} \tag{4.2.10}$$

In order to estimate the integral $II := \frac{1}{2}\fint_{B_{x_n/2}(x)} |\overline{\pi}(z)||z_i - x_i|dz$, we use the fact that if $z \in B_{x_n/2}(x)$, then $B_{x_n/2}(z) \subset B_{x_n}(x)$. Indeed, we have

$$\begin{aligned}
II &\le |B_{x_n/2}(x)|^{-1} \int_{B_{x_n/2}(x)} \left(\fint_{B_{x_n/2}(z)} |\overline{\pi}(y)|dy\right)|z_i - x_i|dz \\
&\le C|B_{x_n}(x)|^{-1} \left(\fint_{B_{x_n}(x)} |\overline{\pi}(y)|^q dy\right)^{1/q} \int_{B_{x_n/2}(x)} |z - x|dz \\
&\le C|B_{x_n}(x)|^{-1} \left(\fint_{B_{x_n}(x)} |\overline{\pi}(y)|^q dy\right)^{1/q} \int_0^{x_n/2} \sigma^n d\sigma \\
&\le x_n^{-\frac{1}{q-1}} \|f\|_{\dot{F}^{-1/q}_{p,q}(\mathbb{R}^{n-1})}
\end{aligned} \tag{4.2.11}$$

Combining (4.2.10) and (4.2.11), we obtain (4.2.9). This achieves the proof of Theorem 4.2.1. □

4.2.2 Inhomogeneous Stokes system

Consider the operators $\mathscr{G}$ and Ψ in $\mathbb{R}^n_+$ respectively defined by

$$\mathscr{G}(F,H)(x) = \int_{\mathbb{R}^n_+} G(x,y)F(y)dy - \int_{\mathbb{R}^n_+} \nabla_y G(x,y)H(y)dy,$$

$$\Psi(F,H)(x) = \int_{\mathbb{R}^n_+} \mathrm{g}(x,y)F(y)dy - \int_{\mathbb{R}^n_+} \nabla_y \mathrm{g}(x,y)H(y)dy$$

whenever the integrals make sense for almost every $x \in \mathbb{R}^n_+$. The kernels $G(x,y) = (G_{ij}(x,y))^n_{i,j=1}$ and $\mathrm{g}(x,y) = (g_j(x,y))^n_{j=1}$, $(x \neq y)$ are the Green tensor for the Stokes operator in $\mathbb{R}^n_+$, that is, coordinates-wise the function satisfying

$$\begin{cases} -\Delta_x G_{ij} + \partial_i g_j = \delta_x \delta_{ij} \ \text{ in } \ \mathbb{R}^n_+ \\ \partial_i G_{ij} = 0 \ \text{ in } \ \mathbb{R}^n_+ \\ G_{ij}(x,\cdot)\big|_{\partial\mathbb{R}^n_+} = 0. \end{cases} \tag{4.2.12}$$

in the sense of distributions where δ_x is the Dirac distribution with mass at $x \in \mathbb{R}^n_+$, Under mild assumptions on F and H, the vector-valued functions $v = \mathscr{G}(F,H)$ and $w = \Psi(F,H)$ satisfy the system of equations

$$\begin{cases} -\Delta v + \nabla w = F + \operatorname{div} H \ \text{ in } \ \mathbb{R}^n_+ \\ \operatorname{div} v = 0 \ \text{ in } \ \mathbb{R}^n_+ \\ v = 0 \ \text{ on } \ \partial\mathbb{R}^n_+ \end{cases} \tag{4.2.13}$$

Refined properties of Green matrices were recently obtained by the authors in [KMT18] relying on ideas introduced earlier in the articles [MPS84] (for $n = 2,3$) and [Gal11] (for the general case). For our purpose we will need the following properties which include sharp pointwise decay bounds. In what follows, $\mathbb{N}^n_0 =$

Lemma 4.2.4. *Let $n \geq 2$. The Green tensor G is symmetric, $G_{ij}(x,y) = G_{ji}(y,x)$ for all $x,y \in \mathbb{R}^n_+$, $x \neq y$ and satisfies together with* g *the pointwise estimates*

$$|G_{ij}(x,y)| \leq C\left(\frac{x_n y_n}{|x-y|^2} + \mathbf{1}_{\{n=2\}} \log(2 + y_n|x-y|^{-1})\right) \tag{4.2.14}$$

$$\left|\nabla^\alpha_x \nabla^\beta_y G_{ij}(x,y)\right| \leq C_N \begin{cases} |x-y|^{-(n-2+N)} \\ \dfrac{x_n y_n}{|x-y|^{n+N}} & \textit{if } \ \alpha_n = \beta_n = 0 \\ \dfrac{x_n}{|x-y|^{n-1+N}} & \textit{if } \ \alpha_n = 0 \end{cases} \tag{4.2.15}$$

for all multi-indices α, β with $|\alpha| + |\beta| = N > 0$. Moreover,

$$\left|\nabla^\alpha \mathrm{g}_j(x,y)\right| \leq C_\alpha |x-y|^{-(n-1)-|\alpha|}, \ j = 1,\dots,n \tag{4.2.16}$$

where the constants are independent of x and y.

These inequalities find their applicability in our next result which deals with the mapping properties of the potentials $\mathscr{G}$ and Ψ. Recall the space $\mathbf{Y}^{\tau,\eta}$ introduced in Section 4.1.

Proposition 4.2.5. *Fix $n \geq 3$ and assume that $q \in (\frac{n}{n-1}, \infty)$. Let $1 < \eta < \tau < \infty$ and $1 \leq \sigma < \Lambda < p < \infty$ satisfy the condition*

$$\frac{1}{\eta} + \frac{n-1}{\tau} = 2 + \frac{1}{q-1} = 1 + \frac{1}{\sigma} + \frac{n-1}{\Lambda}.$$

For all $F \in \mathbf{Y}^{\tau,\eta}$ and $H \in \mathbf{Y}^{\Lambda,\sigma}$ we have $\mathscr{G}(F,H) \in \mathbf{X}^q$, $\Psi(F,H) \in \mathbf{Z}^q$ and it holds that

$$\|\mathscr{G}(F,H)\|_{\mathbf{X}^q} + \|\Psi(F,H)\|_{\mathbf{Z}^q} \leq C(\|F\|_{\mathbf{Y}^{\tau,\eta}} + \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}) \tag{4.2.17}$$

for some constant $C := C(n,q) > 0$ independent of F and H.

Remark 4.2.6. The proof of the above result reveals that elliptic estimates of the form

$$\sup_{x_n>0} x_n^{\frac{1}{q-1}+|\alpha|} \left\|\partial_{x'}^{\alpha} u\right\|_{L^\infty(\mathbb{R}^{n-1})} \leq (\|F\|_{\mathbf{Y}^{\tau,\eta}} + \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}) \tag{4.2.18}$$

are valid for u solution of the Stokes equation (4.2.13) for each multi-index α. However, it is not clear whether vertical derivatives of u enjoy this property. In fact, we are relying heavily on (4.2.15) which seems to fail in the case $\alpha_n \neq 0$ or $\beta_n \neq 0$, see [KMT18, Remark 2.6]. We also point out that in absence of the forcing term F, Proposition 4.2.5 holds true in two dimensions.

The proof of the proposition essentially relies on two auxiliary results, one of which deals with the mapping properties in mixed Lebesgue spaces of the operator G_β defined for $0 < \beta < n$ by

$$G_\beta F(y) = \int_{\mathbb{R}^n_+} \frac{F(z)dz}{|y-z|^{n-\beta}} \tag{4.2.19}$$

whenever the integral exists for almost all $y \in \mathbb{R}^n_+$. For $p,q \in [1,\infty]$, let us denote by $L^pL^q(\mathbb{R}^n_+)$ the mixed Lebesgue space of function $F : \mathbb{R}^n_+ \to \mathbb{R}$ with the property that $x' \mapsto F(x',\cdot) \in L^p(\mathbb{R}^{n-1})$ and $x_n \mapsto F(\cdot,x_n) \in L^q(\mathbb{R}_+)$ and equip it with the norm

$$\|F\|_{L^pL^q(\mathbb{R}^n_+)} = \left\|\|F(\cdot,x_n)\|_{L^q(\mathbb{R}_+,dx_n)}\right\|_{L^p(\mathbb{R}^{n-1})}.$$

Lemma 4.2.7. *Let $0 < \beta < n$ and $1 < \tau < \infty$. Assume that $1 \leq \eta \leq q \leq p < \infty$ are such that*

$$\frac{1}{\eta} < \beta + \frac{1}{q}, \quad \frac{n-1}{p} = \frac{n-1}{\tau} + \frac{1}{\eta} - \frac{1}{q} - \beta. \tag{4.2.20}$$

Then the operator G_β is bounded from $L^\tau L^\eta(\mathbb{R}^n_+)$ into $L^pL^q(\mathbb{R}^n_+)$.

Recall the Riesz potential I_α of order $\alpha \in (0,n-1)$, that is, the convolution operator with the kernel $|x|^{\alpha-(n-1)}$, $x \in \mathbb{R}^{n-1} \setminus \{0\}$.

Proof. Along the lines of the proof of [Yom22, Lemma 2.2], take $F \in L^\tau L^\eta(\mathbb{R}^n_+)$ and let $\widetilde{F}$ be the zero extension of F to $\mathbb{R}^n$. For $1 \le \eta < \infty$, $1 < \tau < \infty$, we have

$$\|G_\beta F\|_{L^p L^q(\mathbb{R}^n_+)} = \Big\| \big\| G_\beta F(y', \cdot) \big\|_{L^q(\mathbb{R}_+)} \Big\|_{L^p(\mathbb{R}^{n-1})}.$$

Let $x' \in \mathbb{R}^{n-1}$ and set $S(x', s) = (|x'|^2 + s^2)^{-\frac{n-\beta}{2}}$, $s > 0$. For $1 \le \theta < \infty$ such that $\frac{1}{\eta} + \frac{1}{\theta} \ge 1$ we use Minkowski's inequality to arrive at

$$\begin{aligned}
\big\| G_\beta F(y', \cdot) \big\|_{L^q(\mathbb{R}_+)} &= \bigg\| \int_{\mathbb{R}^n_+} \frac{|F(z', z_n)| dz' dz_n}{(|y' - z'|^2 + |\cdot - z_n|^2)^{\frac{n-\beta}{2}}} \bigg\|_{L^q(\mathbb{R}_+)} \\
&= \bigg\| \int_{\mathbb{R}^{n-1}} \Big(S(|y' - z'|, \cdot) * |\widetilde{F}|(z') \Big)(y_n) dy' \bigg\|_{L^q(\mathbb{R}_+, dy_n)} \\
&\le C \int_{\mathbb{R}^{n-1}} \big\| (S(|y' - z'|, \cdot) * |\widetilde{F}|)(z', \cdot) \|_{L^q(\mathbb{R}_+)} dz' \\
&\le C \int_{\mathbb{R}^{n-1}} \big\| S(|y' - z'|, \cdot) \big\|_{L^\theta(\mathbb{R}_+)} \|F(z', \cdot)\|_{L^\eta(\mathbb{R}_+)} dz' \\
&\le C [I_{\beta + \frac{1}{\theta} - 1} \|F(\cdot, y_n)\|_{L^\eta(\mathbb{R}_+, dy_n)}](y'), \quad y' \in \mathbb{R}^{n-1}
\end{aligned}$$

where $\frac{1}{q} + 1 = \frac{1}{\theta} + \frac{1}{\eta}$. Thus, if $\frac{n-1}{p} = \frac{n-1}{\tau} - (\beta + \frac{1}{\theta} - 1)$, then by the boundedness of I_α in Lebesgue spaces, we find that

$$\begin{aligned}
\|G_\beta F\|_{L^p L^q(\mathbb{R}^n_+)} &\le C \Big\| I_{\beta - \frac{1}{\theta} - 1} \|F(y', \cdot)\|_{L^\eta(\mathbb{R}_+)} \Big\|_{L^p(\mathbb{R}^{n-1}, dy')} \\
&\le C \Big\| F \Big\|_{L^\tau L^\eta(\mathbb{R}^n_+)}.
\end{aligned}$$

□

Remark 4.2.8. In the sequel, we will need an analogue of Lemma (4.2.7) in weighted mixed Lebesgue spaces of the form

$$\Big\| \big\| G_\beta F(\cdot, y_n) \big\|_{L^q(\mathbb{R}_+, y_n^q dy_n)} \Big\|_{L^p(\mathbb{R}^{n-1})} \le C \Big\| \big\| F(\cdot, y_n) \big\|_{L^\eta(\mathbb{R}_+, y_n^{b\eta} dy_n)} \Big\|_{L^r(\mathbb{R}^{n-1})} \tag{4.2.21}$$

for all functions F such that $(x', x_n) \mapsto x_n^b F \in L^r L^\eta(\mathbb{R}^n_+)$. This is valid under the conditions

$$\begin{cases} 2 + \dfrac{1}{q} = (n-1)\left(\dfrac{1}{r} - \dfrac{1}{p}\right) + \dfrac{1}{\eta} + b - (\beta - 1) \\ 1 < r < p < \infty, \quad b \ge 1 \\ n > \beta + 2 + \frac{1}{q} - \frac{1}{\eta} - b. \end{cases} \tag{4.2.22}$$

In fact, one may use the same strategy as before to prove (4.2.21). If $a \geq 1$ and $\delta > 1$ are such that

$$\frac{1}{\delta} + a = (n-1)\left(\frac{1}{r} - \frac{1}{p}\right) - (\beta - 1),$$

then using the weighted convolution inequality [GFWZ18, Theorem 1.2] for $n = 1$, we obtain

$$\begin{aligned}\left\|G_\beta F(y', \cdot)\right\|_{L^q(\mathbb{R}_+, y_n^q dy_n)} &\leq C \int_{\mathbb{R}^{n-1}} \|S(|y'-z'|, \cdot)\|_{L^\delta(\mathbb{R}_+, y_n^{a\delta} dy_n)} \|F(z', \cdot)\|_{L^\eta(\mathbb{R}_+, y_n^{b\eta})} dz' \\ &\leq I_{\frac{1}{\delta}+a+\beta-1} \|F(\cdot, y_n)\|_{L^\eta(\mathbb{R}_+, y_n^{b\eta} dy_n)}(y'), \quad y' \in \mathbb{R}^{n-1}.\end{aligned}$$

This, in conjunction with (4.2.22) gives the desired bound.

We are now ready to prove Proposition 4.2.5 and we divide the proof in two steps.

Step 1. The bound

$$\|\mathscr{G}(F,H)\|_{\mathbf{X}^q} \leq C(\|F\|_{\mathbf{Y}^{\tau,\eta}} + \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}). \tag{4.2.23}$$

Let $1 < \eta < \infty$ and $1 < \tau < \infty$ such that $\frac{1}{\eta} + \frac{n-1}{\tau} = 2 + \frac{1}{q-1}$. Pick F in $\mathbf{Y}^{\tau,\eta}$ and $H \in \mathbf{Y}^{\Lambda,\sigma}$. We first prove that

$$\sup_{x_n > 0} x_n^{1/(q-1)} \|\mathscr{G}(F,H)(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \leq \|F\|_{\mathbf{Y}^{\tau,\eta}}. \tag{4.2.24}$$

Fix $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$ and write

$$\int_{\mathbb{R}^n_+} G(x', x_n, y) F(y) dy = J_1 + J_2 + J_3 + J_4$$

where

$$J_1 = \int_{B_{x_n}(x')} \int_0^{x_n/2} G(x,y)F(y)dy, \; J_2 = \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} G(x,y)F(y)dy,$$

$$J_3 = \int_{\mathbb{R}^{n-1} \setminus B_{x_n}(x')} \int_0^{2x_n} G(x,y)F(y)dy, \; J_4 = \int_{\mathbb{R}^{n-1}} \int_{2x_n}^\infty G(x,y)F(y)dy.$$

Next, we estimate each of these integrals by means of the pointwise inequalities from

Lemma 4.2.4. Indeed, starting with J_1 and using the summation convention, we have

$$\begin{aligned}
|J_1| &\leq \int_{B_{x_n}(x')} \int_0^{x_n/2} |G_{ij}(x', x_n, y)||F_j(y)|dy \\
&\leq C \int_{B_{x_n}(x')} \int_0^{x_n/2} \frac{|F(y)|}{(|x'-y'|^2 + (x_n - y_n)^2)^{\frac{n-2}{2}}} dy_n dy' \\
&\leq C x_n^{-(n-2)} x_n^{\frac{n}{\eta'}} \left(\int_{B_{x_n}(x')} \int_0^{x_n/2} |F(y)|^\eta dy_n dy' \right)^{1/\eta} \\
&\leq C x_n^{-(n-2)+\frac{n}{\eta'}} \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x') \cap B_{x_n}(x') \times (0, x_n/2)} |F(y)|^\eta y_n^{-(n-1)} dy_n dy' dz' \right)^{1/\eta} \\
&\leq C x_n^{-(n-2)+\frac{n}{\eta'}} \|F\|_{T^{\tau,\eta}} \left| E(B_{x_n/2}(x') \times (0, x_n/2)) \right|^{\frac{\tau-\eta}{\tau\eta}} \\
&\leq C x_n^{2-\frac{n-1}{\tau}-\frac{1}{\eta}} \|F\|_{T^{\tau,\eta}}.
\end{aligned}$$

where we have utilized Hölder's inequality in order to derive the third and fifth bounds in the above chain of estimates and $\frac{1}{\eta'} + \frac{1}{\eta} = 1$. On the other hand,

$$\begin{aligned}
|J_2| &\leq \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} |G_{ij}(x,y)||F_j(y)|dy \\
&\leq C \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} |x-y|^{-(n-2)} |F_j(y)|dy \\
&\leq C \sup_{y_n>0} y_n^{\frac{n-1}{\tau}+\frac{1}{\eta}} \|F(\cdot, y_n)\|_{L^\infty(\mathbb{R}^{n-1})} \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} \frac{y_n^{-\frac{1}{\eta}-\frac{n-1}{\tau}} dy_n dy'}{[|x'-y'|^2 + (x_n - y_n)^2]^{(n-2)/2}} \\
&\leq C x_n^{-\frac{1}{\eta}-\frac{n-1}{\tau}} \|F\|_{\mathbf{Y}^{\tau,\eta}} \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} |x'-y'|^{-(n-2)} dy_n dy' \\
&\leq C x_n^{1-\frac{1}{\eta}-\frac{n-1}{\tau}} \|F\|_{\mathbf{Y}^{\tau,\eta}} \int_{B_{x_n}(x')} |x'-y'|^{-(n-2)} dy \\
&\leq C x_n^{2-\frac{1}{\eta}-\frac{n-1}{\tau}} \|F\|_{\mathbf{Y}^{\tau,\eta}}.
\end{aligned}$$

Similarly as above, by Lemma 4.2.2 and Hölder's inequality, we find that

$$\begin{aligned}
|J_3| &\leq \int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')} \int_0^{2x_n} |G_{ij}(x,y)||F_j(y)|dy \\
&\leq Cx_n \int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')} \int_0^{2x_n} |x-y|^{-(n-1)}|F_j(y)|dy \\
&\leq Cx_n \sum_{k=1}^{\infty} \int_{2^k B_{x_n}(x')\setminus 2^{k-1}B_{x_n}(x')} \int_0^{2x_n} |x-y|^{-(n-1)}|F_j(y)|dy \\
&\leq Cx_n^{2-n+n/\eta'} \sum_{k=1}^{\infty} 2^{-(k-1)(n-1)+\frac{(n-1)k}{\eta'}} \left(\int_{2^k B_{x_n}(x')} \int_0^{2x_n} |F_j(y)|^\eta dy_n dy' \right)^{\frac{1}{\eta}} \\
&\leq Cx_n^{2-\frac{n}{\eta}+\frac{(n-1)(\tau-\eta)}{\tau\eta}} \left[\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(z')} |F_j(y)|^\eta \frac{dy}{y_n^{n-1}} \right)^{\tau/\eta} dz' \right]^{\frac{1}{\tau}} \left(\sum_{k=1}^{\infty} 2^{-\frac{k(n-1)}{\tau}} \right) \\
&\leq Cx_n^{2-\frac{1}{\eta}-\frac{n-1}{\tau}} \|F\|_{\mathbf{Y}^{\tau,\eta}}.
\end{aligned}$$

Again, by using the Green matrix bound 4.2.14, we bound J_4 as follows

$$\begin{aligned}
|J_4| &\leq \int_{\mathbb{R}^{n-1}} \int_{2x_n}^{\infty} |G_{ij}(x,y)||F_j(y)|dy \\
&\leq C \int_{\mathbb{R}^{n-1}} \int_{2x_n}^{\infty} \frac{x_n y_n |F_j(y)|}{|x-y|^n} dy \\
&\leq C \int_{\mathbb{R}^{n-1}} \int_{2x_n}^{\infty} \frac{x_n y_n |F_j(y)| dy_n dy'}{\left[|x'-y'|^2+y_n^2\right]^{\frac{n}{2}}} \\
&\leq C \sup_{y_n>0} y_n^{\frac{1}{\eta}+\frac{n-1}{\tau}} \|F(\cdot,y_n)\|_{L^\infty(\mathbb{R}^{n-1})} \left(\int_{2x_n}^{\infty} x_n y_n^{-\frac{1}{\eta}-\frac{n-1}{\tau}} dy_n \right) \left(\int_{\mathbb{R}^{n-1}} \frac{dz'}{\left[|z'|^2+1\right]^{\frac{n}{2}}} \right) \\
&\leq Cx_n^{2-\frac{1}{\eta}-\frac{n-1}{\tau}} \|F\|_{\mathbf{Y}^{\tau,\eta}}.
\end{aligned}$$

In the same vein, we establish the weighted gradient sup-norm estimate

$$\sup_{x_n>0} x_n^{q/(q-1)} \left\| \int_{\mathbb{R}^n_+} \nabla_y G(\cdot, x_n, y) H(y) dy \right\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}. \tag{4.2.25}$$

Decompose the solid integral in the above estimate into four parts to get

$$L_1 = \int_{B_{x_n}(x')} \int_0^{x_n/2} \nabla_y G(x,y)H(y)dy, \; L_2 = \int_{B_{x_n}(x')} \int_{x_n/2}^{2x_n} \nabla_y G(x,y)H(y)dy,$$

$$L_3 = \int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')}\int_0^{2x_n} \nabla_y G(x,y)H(y)dy,\ L_4 = \int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty} \nabla_y G(x,y)H(y)dy.$$

Suppose $\frac{1}{\sigma}+\frac{n-1}{\Lambda}=\frac{1}{\eta}+\frac{n-1}{\tau}-1$. Utilizing (4.2.15) and Hölder's inequality, we arrive at

$$\begin{aligned}
|L_1| &\le \int_{B_{x_n}(x')}\int_0^{x_n/2} |\nabla_y G(x,y)||H(y)|dy \\
&\le C\int_{B_{x_n}(x')}\int_0^{x_n/2} \frac{|H(y',y_n)|}{(|x'-y'|^2+(x_n-y_n)^2)^{\frac{n-1}{2}}}dy_n dy' \\
&\le C x_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}} \|\mathcal{A}_q H\|_{L^\Lambda(\mathbb{R}^{n-1})} \\
&\le C x_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}} \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}
\end{aligned}$$

Next, noticing that $|\nabla G_{ij}(x,\cdot)|$ belongs to the weak-Lebesgue space $L^{\frac{n}{n-1},\infty}(\mathbb{R}^n_+)$ uniformly for all $x\in\mathbb{R}^n_+$, it follows that

$$\begin{aligned}
|L_2| &\le \int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n} |\nabla_y G(x,y)H(y)|dy \\
&\le C\sup_{x_n>0} x_n^{\frac{1}{\sigma}+\frac{n-1}{\Lambda}}\|H(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\int_{B_{x_n}(x')}\int_{x_n/2}^{2x_n} y_n^{-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}|\nabla_y G(x,y)|dy_n dy' \\
&\le C x_n^{-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\sup_{x_n>0} x_n^{\frac{1}{\sigma}+\frac{n-1}{\Lambda}}\|H(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\|\nabla_y G(x,\cdot)\|_{L^1(B_{x_n}(x')\times[x_n/2,2x_n])} \\
&\le C x_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}.
\end{aligned}$$

Recall here that for any $p>1$ the belonging of f to $L^{p,\infty}(\mathbb{R}^{n-1})$ is equivalent to the condition

$$\sup_{E\subset\mathbb{R}^{n-1}} |E|^{1/p-1}\int_E |f(y)|dy < \infty$$

where the supremum runs over all open set E of $\mathbb{R}^{n-1}$. We argue as before to bound L_3

$$\begin{aligned}
|L_3| &\leq \int_{\mathbb{R}^{n-1}\setminus B_{x_n}(x')}\int_0^{2x_n}|\nabla_y G(x,y)H(y)|dy_n dy' \\
&\leq \sum_{k=1}^{\infty}\int_{2^kB_{x_n}(x')\setminus 2^{k-1}B_{x_n}(x')}\int_0^{2x_n}|\nabla_y G(x,y)||H(y)|dy_n dy' \\
&\leq C\sum_{k=1}^{\infty}\int_{2^kB_{x_n}(x')\setminus 2^{k-1}B_{x_n}(x')}\int_0^{2x_n}|x-y|^{-n+1}|H(y)|dy_n dy' \\
&\leq Cx_n^{1-\frac{n}{\sigma}}\sum_{k=1}^{\infty}2^{-(n-1)k+\frac{k(n-1)}{\sigma'}}\left(\int_{\mathbb{R}^{n-1}}\iint_{\Gamma(z')\cap[2^kB_{x_n}(x')\times(0,2x_n)]}|H(y)|^{\sigma}\frac{dy_n dy'}{y_n^{n-1}}dz'\right)^{\frac{1}{\sigma}} \\
&\leq Cx_n^{-(n-1)+\frac{n}{\sigma'}+\frac{(\Lambda-\sigma)}{\Lambda\sigma}n}\sum_{k=1}^{\infty}2^{-(k-1)(n-1)+\frac{k(n-1)}{\sigma'}+k(n-1)\frac{(\Lambda-\sigma)}{\sigma\Lambda}}\|\mathcal{A}_{\sigma}H\|_{L^{\Lambda}(\mathbb{R}^{n-1})} \\
&\leq Cx_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|H\|_{T^{\Lambda,\sigma}}\sum_{k=1}^{\infty}2^{-\frac{(n-1)}{\Lambda}k} \\
&\leq Cx_n^{2-\frac{1}{\eta}-\frac{n-1}{\tau}}\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}.
\end{aligned}$$

Finally, observe that for $y_n > 2x_n$, we have $y_n - x_n > \frac{1}{2}y_n$ so that by the third bound in (4.2.15), we find that

$$\begin{aligned}
|L_4| &\leq \int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}|\nabla_y G(x,y)||H(y)|dy \\
&\leq C\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}\frac{x_n|H(y)|dy_n dy'}{\left[|x'-y'|^2+(x_n-y_n)^2\right]^{\frac{n}{2}}} \\
&\leq C\int_{\mathbb{R}^{n-1}}\int_{2x_n}^{\infty}\frac{x_n|H(y)|dy_n dy'}{\left[|x'-y'|^2+y_n^2\right]^{\frac{n}{2}}} \\
&\leq C\sup_{x_n>0}x_n^{\frac{1}{\sigma}+\frac{n-1}{\Lambda}}\|H(\cdot,x_n)\|_{L^{\infty}(\mathbb{R}^{n-1})}\left(\int_{\mathbb{R}^{n-1}}\frac{dy'}{(|y'|^2+1)^{\frac{n}{2}}}\right)\left(\int_{2x_n}^{\infty}x_n y_n^{-\frac{1}{\sigma}-\frac{n-1}{\Lambda}-1}dy_n\right) \\
&\leq Cx_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}}\|H\|_{\mathbf{Y}^{\Lambda,\sigma}}.
\end{aligned}$$

Summing up all the above inequalities, one obtains (4.2.24). Next, we show that

$$\|\mathcal{G}(F,H)\|_{T^{nq(q-1),q}} \leq C(\|F\|_{T^{\tau,\eta}}+\|H\|_{T^{\Lambda,\sigma}}). \tag{4.2.26}$$

Write

$$\|\mathcal{G}(F,H)\|_{T^{p,q}} \leq \left\|\int_{\mathbb{R}^n_+}G(\cdot,y)F(y)dy\right\|_{T^{p,q}}+\left\|\int_{\mathbb{R}^n_+}\nabla_y G(\cdot,y)H(y)dy\right\|_{T^{p,q}} := I+II.$$

Fix $x' \in \mathbb{R}^{n-1}$ and $y_n > 0$ and let's decompose $F \in L^{\eta}_{loc}(\mathbb{R}^n_+)$ into three parts

$$F = F\mathbf{1}_{B_{4y_n}(x')\times(0,4y_n]} + F\mathbf{1}_{B_{4y_n}(x')\times(4y_n,\infty)} + F\mathbf{1}_{(\mathbb{R}^{n-1}\setminus B_{4y_n}(x'))\times(0,\infty)} = F_1 + F_2 + F_3$$

and write

$$I := \Sigma_1 + \Sigma_2 + \Sigma_3, \quad \Sigma_i = \left\| \int_{\mathbb{R}^n_+} G(\cdot, y) F_i(y) dy \right\|_{T^{p,q}}, \quad i = 1, 2, 3.$$

We control Σ_3 using the following

Claim 4.2.9. For all $x' \in \mathbb{R}^{n-1}$ and $y_n > 0$, there exists $C > 0$ independent on x' and y_n such that

$$A(x', y_n) \leq C G_2 \left(\fint_{B_{y_n}(\cdot)} |F(z', \cdot)| dz' \right)(x', y_n).$$

Here,

$$A(x', y_n) = \left(\fint_{B_{y_n}(x')} \left| \int_{\mathbb{R}^n_+} G(y, z) F_3(z) dz \right|^q dy' \right)^{\frac{1}{q}}, \quad (x', y_n) \in \mathbb{R}^n_+.$$

Proof. We have

$$
\begin{aligned}
A(x',y_n) &\le \left(\fint_{B_{y_n}(x')} \left(\int_{\mathbb{R}^n_+} |G(y,z)||F_3(z)|dz \right)^q dy' \right)^{1/q} \\
&\le C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\mathbb{R}^{n-1}\setminus B_{4y_n}(x')} \frac{|F(z',z_n)|dz'dz_n}{(|y'-z'|^2+|y_n-z_n|^2)^{\frac{n-2}{2}}} \right)^q dy' \right)^{1/q} \\
&\le C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-z'|>4y_n\}} \frac{|F(z',z_n)|dz'dz_n}{(|y'-z'|^2+|y_n-z_n|^2)^{\frac{n-2}{2}}} \fint_{B_{y_n}(z')} dw \right)^q dy' \right)^{\frac{1}{q}} \\
&\le C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-w|>3y_n\}} \fint_{B_{y_n}(w)} \frac{|F(z',z_n)|dz'dwdz_n}{(|y'-z'|^2+|y_n-z_n|^2)^{\frac{n-2}{2}}} \right)^q dy' \right)^{1/q} \\
&\le C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-w|>3y_n\}} (|x'-w|^2+|y_n-z_n|^2)^{\frac{-(n-2)}{2}} \cdot \right.\right. \\
&\qquad\qquad \left.\left. \fint_{B_{y_n}(w)} \left[\frac{(|x'-w|^2+|y_n-z_n|^2)}{(|y'-z'|^2+|y_n-z_n|^2)} \right]^{\frac{n-2}{2}} |F(z',z_n)|dz'dwdz_n \right)^q dy' \right)^{\frac{1}{q}} \\
&\le C\left(\fint_{B_{y_n}(x')} \left(\int_0^\infty \int_{\{|x'-w|>3y_n\}} (|x'-w|^2+|y_n-z_n|^2)^{\frac{-(n-2)}{2}} \cdot \right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. \fint_{B_{y_n}(w)} |F(z',z_n)|dz'dwdz_n \right)^q dy' \right)^{1/q} \\
&\le C \int_0^\infty \int_{\{|x'-w|>3y_n\}} (|x'-w|^2+|y_n-z_n|^2)^{\frac{-(n-2)}{2}} \left(\fint_{B_{y_n}(w)} |F(z',z_n)|dz' \right) dwdz_n \\
&\le C \int_{\mathbb{R}^n_+} (|x'-w|^2+|y_n-z_n|^2)^{\frac{-(n-2)}{2}} \left(\fint_{B_{y_n}(w)} |F(z',z_n)|dz' \right) dwdz_n \\
&\le C G_2 \left(\fint_{B_{y_n}(\cdot)} |F(z',\cdot)|dz' \right)(x',y_n).
\end{aligned}
$$

□

Applying Lemma 4.2.7 and Jensen's inequality, the above claim clearly implies that

$$
\begin{aligned}
\Sigma_3 = \|A\|_{L^pL^q(\mathbb{R}^n_+)} &\le C \left\| G_2 \left(\fint_{B_{y_n}(\cdot)} |F(z',\cdot)|dz' \right) \right\|_{L^pL^q(\mathbb{R}^n_+)} \\
&\le C \left\| \fint_{B_{y_n}(\cdot)} |F(z',y_n)|dz' \right\|_{L^pL^q(\mathbb{R}^n_+)} \\
&\le C\|F\|_{T^{\tau,\eta}}.
\end{aligned}
$$

To bound Σ_2, we first observe that

$$\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right| \leq Cy_n\mathcal{A}_\eta F(x')\left(\int_{4y_n}^{\infty}\int_{B_{4y_n}(x')} \frac{z_n^{\frac{n-1}{\eta-1}}dz'dz_n}{(|y'-z'|^2+|y_n-z_n|^2)^{\frac{(n-1)\eta'}{2}}}\right)^{\frac{1}{\eta'}}$$
$$\leq Cy_n^{2-\frac{1}{\eta}}\mathcal{A}_\eta F(x'), \quad x' \in B(y', y_n). \tag{4.2.27}$$

On the other hand, this inequality also implies the pointwise bound

$$\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right| \leq Cy_n^{-\frac{1}{q-1}}\|\mathcal{A}_\eta F\|_{L^\tau(\mathbb{R}^{n-1})}, \quad y' \in B(x', y_n). \tag{4.2.28}$$

Let $M > 0$ to be determined later. Using (4.2.27) and (4.2.28), we find that

$$\int_0^\infty \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n \leq \int_0^M \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n +$$
$$\int_M^\infty \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n$$
$$\leq CM^{1+(2-\frac{1}{\eta})q}[\mathcal{A}_\eta F(x')]^q + M^{-\frac{1}{q-1}}\|F\|^q_{T^{\tau,\eta}}.$$

Optimizing this inequality with respect to M, that is taking $M = \left(\frac{\|F\|_{T^{\tau,\eta}}}{\mathcal{A}_\eta F(x')}\right)^{\frac{\tau}{n-1}}$, we arrive at

$$\left(\int_0^\infty \fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_2(z)dz\right|^q dy'dy_n\right)^{\frac{1}{q}} \leq C\|F\|_{T^{\tau,\eta}}^{1-\frac{\tau}{q(n-1)(q-1)}}[\mathcal{A}_\eta F(x')]^{\frac{\tau}{q(n-1)(q-1)}}.$$

Taking the L^p-norm on both sides of the inequality, we conclude that

$$\Sigma_2 \leq C\|F\|_{T^{\tau,\eta}}.$$

Finally, estimating Σ_1 goes through a duality argument. Let $r = (n-1)(q-1)$ and $\varphi \in L^{r'}(\mathbb{R}^{n-1})$, $\varphi \geq 0$ and define the operator

$$M_t\varphi(x') = t^{-(n-1)}\int_{B_t(x')}\varphi(y')dy', \quad t > 0.$$

If $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $L^p(\mathbb{R}^{n-1})$ and its dual $L^{p'}(\mathbb{R}^{n-1})$, then

$$\begin{aligned}\left\langle \mathcal{A}_q^q\left[\int_{\mathbb{R}^n_+} G(\cdot,z)F_1(z)dz\right],\varphi\right\rangle &= \int_{\mathbb{R}^{n-1}}\int_0^\infty\int_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} G(y,z)F_1(z)dz\right|^q \frac{dy'dy_n}{y_n^{n-1}}\varphi(x')dx' \\ &\le C\int_{\mathbb{R}^{n-1}}\int_0^\infty \fint_{B_{y_n}(y')}\varphi(x')dx'[G_2|F|(y',y_n)]^q dy'dy_n \\ &\le C\int_{\mathbb{R}^{n-1}}\int_0^\infty [G_2|F|(y',y_n)]^q M_{y_n}\varphi(y')dy'dy_n \\ &\le C\|G_2F\|^q_{L^pL^q(\mathbb{R}^n_+)}\|M_\cdot\varphi\|_{L^{r'}L^\infty(\mathbb{R}^n_+)} \\ &\le C\|G_2F\|^q_{L^pL^q(\mathbb{R}^n_+)}\|\mathcal{M}\varphi\|_{L^{r'}(\mathbb{R}^{n-1})}.\end{aligned}$$

Applying Lemma 4.2.7, the fact that $\int_0^\infty |F(y',y_n)|^q dy' \le \liminf_{\alpha\to 0}[\mathcal{A}_q^\alpha F(y')]^q$ (which is a consequence of the Lebesgue differentiation Theorem and Fatou lemma), the boundedness of the Hardy-Littlewood maximal function in Lebesgue spaces successively, we obtain

$$\left\langle \mathcal{A}_q^q\left[\int_{\mathbb{R}^n_+} G(\cdot,z)F_1(z)dz\right],\varphi\right\rangle \le \|F\|^q_{T^{\tau,\eta}}\|\varphi\|_{L^{r'}(\mathbb{R}^{n-1})} \quad \forall\varphi\in L^{r'}(\mathbb{R}^{n-1}),$$

from which it plainly follows that

$$\Sigma_1 \le C\|F\|_{T^{\tau,\eta}}.$$

We equally estimate II splitting H into three components exactly as before and follow the same procedure (details are left to the interested reader). This yields

$$\left\|\int_{\mathbb{R}^n_+}\nabla_y G(\cdot,y)H(y)dy\right\|_{T^{p,q}} \le C\|H\|_{T^{\Lambda,\sigma}}.$$

Summarizing, we see that (4.2.23) holds true. This finishes Step 1.

Step 2. The estimate

$$\|\Psi(F,H)\|_{\mathbf{Z}^q} \le C\Big(\|F\|_{\mathbf{Y}^{\tau,\eta}} + \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}\Big) \tag{4.2.29}$$

for all $F\in\mathbf{Y}^{\tau,\eta}$ and $H\in\mathbf{Y}^{\Lambda,\sigma}$. We have

$$\|\Psi(F,H)\|_{T^{p,q}_{s_0}} \le \left\|\int_{\mathbb{R}^n_+} \mathrm{g}(\cdot,y)F(y)dy\right\|_{T^{p,q}_{s_0}} + \left\|\int_{\mathbb{R}^n_+}\nabla_y \mathrm{g}(\cdot,y)H(y)dy\right\|_{T^{p,q}_{s_0}} := III + IV.$$

Let F_1, F_2 and F_3 as above and write correspondingly

$$III \le III_1 + III_2 + III_3, \quad III_i = \left\|\int_{\mathbb{R}^n_+} \mathrm{g}(\cdot,y)F_i(y)dy\right\|_{T^{p,q}_{s_0}}, \quad i=1,2,3.$$

Since (see proof of Claim 4.2.9)

$$\left(\fint_{B_{y_n}(x')}\left|\int_{\mathbb{R}^n_+} g(y,z)F_3(z)dz\right|^q dy'\right)^{\frac{1}{q}} \le cG_1\left(\fint_{B_{y_n}(\cdot)}|F(z',\cdot)|dz'\right)(x',y_n), \quad (x',y_n)\in\mathbb{R}^n_+.$$

Now let $\tau < r < \infty$ such that $\frac{1}{r}+\frac{1}{n-1}\le\frac{1}{\tau}$. Invoking (4.2.21) with $\beta = 1$ and $b = (n-1)(1/\tau - 1/r)$ together with Jensen's inequality we arrive at

$$\begin{aligned}
III_3 &\le C\left\|\left\|G_1\left(\fint_{B_{y_n}(\cdot)}|F(z',\cdot)|dz'\right)\right\|_{L^q(\mathbb{R}_+,y_n^q dy_n)}\right\|_{L^p(\mathbb{R}^{n-1})}\\
&\le C\left(\int_{\mathbb{R}^{n-1}}\left(\int_0^\infty\fint_{B_{y_n}(x')}|y_n^b F(z',y_n)|^\eta dz'dy_n\right)^{\frac{r}{\eta}}dx'\right)^{\frac{1}{r}}\\
&\le C\|F\|_{T^{\tau,\eta}}.
\end{aligned}$$

The last inequality follows from the embedding (4.1.7) (with $s_1 = 0$, $s_2 = -\frac{b}{n-1}$, $q = \eta$, $p_1 = \tau$ and $p_2 = r$). Moving on, we use (4.2.16) and Hölder's inequality to get the pointwise bound

$$\begin{aligned}
\left|\int_{\mathbb{R}^n_+} g(y,z)F_2(z)dz\right| &\le C|G_1F_2(y)|\\
&\le C\mathcal{A}_\eta F(x')\left(\int_{4y_n}^\infty\int_{B_{4y_n}(x')}\frac{z_n^{\frac{n-1}{\eta-1}}dz'dz_n}{(|y'-z'|^2+|y_n-z_n|^2)^{\frac{(n-1)\eta'}{2}}}\right)^{\frac{1}{\eta'}}\\
&\le Cy_n^{1-\frac{1}{\eta}}\mathcal{A}_\eta F(x'), \quad x'\in B_{y_n}(y')
\end{aligned}\tag{4.2.30}$$

from which it follows that

$$\left|\int_{\mathbb{R}^n_+} g(y,z)F_2(z)dz\right| \le Cy_n^{-\frac{q}{q-1}}\|\mathcal{A}_\eta F\|_{L^\tau(\mathbb{R}^{n-1})}, \quad y'\in B(x',y_n). \tag{4.2.31}$$

Therefore, for $M > 0$ to be determined later, we have

$$\begin{aligned}
\int_0^\infty\fint_{B_{y_n}(x')}y_n^q\left|\int_{\mathbb{R}^n_+} g(y,z)F_2(z)dz\right|^q dy'dy_n &\le \int_0^M\fint_{B_{y_n}(x')}y_n^q\left|\int_{\mathbb{R}^n_+} g(y,z)F_2(z)dz\right|^q dy'dy_n +\\
&\quad \int_M^\infty\fint_{B_{y_n}(x')}y_n^q\left|\int_{\mathbb{R}^n_+} g(y,z)F_2(z)dz\right|^q dy'dy_n\\
&\le CM^{1+(2-\frac{1}{\eta})q}[\mathcal{A}_\eta F(x')]^q + M^{-\frac{1}{q-1}}\|F\|^q_{T^{\tau,\eta}}.
\end{aligned}$$

The choice $M = \left(\frac{\|F\|_{T^{\tau,\eta}}}{\mathcal{A}_\eta F(x')}\right)^{\frac{\tau}{n-1}}$ yields

$$\left(\int_0^\infty\fint_{B_{y_n}(x')}y_n^q\left|\int_{\mathbb{R}^n_+} g(y,z)F_2(z)dz\right|^q dy'dy_n\right)^{\frac{1}{q}} \le C\|F\|_{T^{\tau,\eta}}^{1-\frac{\tau}{(n-1)q(q-1)}}[\mathcal{A}_\eta F(x')]^{\frac{\tau}{(n-1)q(q-1)}}.$$

Hence, (after taking the L^p-norm on both sides of the previous inequality)

$$III_2 \leq C\|F\|_{T^{\tau,\eta}}.$$

We also claim that

$$III_1 \leq C\|F\|_{T^{\tau,\eta}}.$$

In fact, setting $VF(y,y_n) = y_n \int_{\mathbb{R}^n_+} \mathrm{g}(y,z)F_1(z)dz$, for all $\phi \in L^r(\mathbb{R}^{n-1})$ we have that

$$\begin{aligned}
\left\langle \mathcal{A}_q^q(VF), \varphi \right\rangle &= \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_{B_{y_n}(x')} \left|VF_1(y',y_n)\right|^q \frac{dy'dy_n}{y_n^{n-1}} \phi(x')dx' \\
&\leq C \int_{\mathbb{R}^{n-1}} \int_0^\infty ⨍_{B_{y_n}(y')} \phi(x')dx'[y_n(G_1|F|)(y',y_n)]^q dy'dy_n \\
&\leq C \int_{\mathbb{R}^{n-1}} \int_0^\infty [y_n(G_1|F|)(y',y_n)]^q M_{y_n}\phi(y')dy'dy_n \\
&\leq C\left\|(y',y_n) \mapsto y_n G_1|F|\right\|^q_{L^pL^q(\mathbb{R}^n_+)} \|M.\phi\|_{L^{r'}L^\infty(\mathbb{R}^n_+)} \\
&\leq C\|F\|^q_{T^{m,\eta}_{-\frac{b}{n-1}}} \|\mathcal{M}\phi\|_{L^{r'}(\mathbb{R}^{n-1})} \\
&\leq C\|F\|^q_{T^{\tau,\eta}} \|\phi\|_{L^{r'}(\mathbb{R}^{n-1})}
\end{aligned}$$

Note that the penultimate inequality follows from Remark 4.2.8 with $m \in (1,r)$ is such that $\frac{1}{\tau} \leq \frac{1}{m} - \frac{1}{n-1}$ and $b = (n-1)(\frac{1}{\tau} - \frac{1}{m})$ while the last bound comes from (4.1.7). Collecting and summing up all the estimates on the Σ_i's, we find that

$$\left\| \int_{\mathbb{R}^n_+} \mathrm{g}(\cdot,y)F(y)dy \right\|_{T^{p,q}} \leq C\|F\|_{T^{\tau,\eta}}.$$

The remaining estimate reads

$$\left\| \int_{\mathbb{R}^n_+} \nabla_y \mathrm{g}(\cdot,y)H(y)dy \right\|_{T^{p,q}} \leq C\|H\|_{T^{\Lambda,\sigma}}.$$

The argument used here is similar to the previous one. In fact, for $(y',y_n) \in \mathbb{R}^n_+$ we write

$$y_n \left| \int_{\mathbb{R}^n_+} \nabla_z \mathrm{g}(y,z)H(z)dz \right| \leq \sum_{k=1}^3 \mathit{\Gamma}_k(y',y_n),$$

with

$$\Gamma_1(y',y_n) = y_n \int_{\mathbb{R}^{n-1}\setminus B_{4y_n}(y')} \int_0^\infty |\nabla_z g(y,z)||H(z)|dz$$
$$\Gamma_2(y',y_n) = y_n \int_{B_{4y_n}(y')} \int_{4y_n}^\infty |\nabla_z g(y,z)||H(z)|dz$$
$$\Gamma_3(y',y_n) = y_n \int_{B_{4y_n}(y')} \int_0^{4y_n} |\nabla_z g(y,z)||H(z)|dz.$$

It is easy to see that $|\Gamma_1(y',y_n)| \leq G_1 H(y',y_n)$ for any $(y',y_n) \in \mathbb{R}^n_+$. Then, by Step 1 and in particular (4.2.26), we deduce the desired estimate. Next, we show that

$$\|\Gamma_2\|_{T^{p,q}} \leq C\|H\|_{T^{\Lambda,\sigma}}. \tag{4.2.32}$$

To achieve this, let us primarily observe that

$$\begin{aligned}
|\Gamma_2(y',y_n)| &\leq C y_n \mathcal{A}_\sigma^{\frac{5}{4}} H(x') \left(\int_{4y_n}^\infty \int_{B_{4y_n}(x')} \frac{z_n^{\frac{n-1}{\sigma-1}} dz' dz_n}{(|y'-z'|^2 + |y_n - z_n|^2)^{\frac{n\sigma'}{2}}} \right)^{\frac{1}{\sigma'}} \\
&\leq C y_n^{1-\frac{1}{\sigma}} \mathcal{A}_\sigma^{\frac{5}{4}} H(x'), \quad x' \in B(y',y_n).
\end{aligned}$$

Taking the Λ-power of both sides of the last inequality and integrating with respect to the variable x'

$$|\Gamma_2(y',y_n)| \leq C y_n^{1-\frac{1}{\sigma}-\frac{n-1}{\Lambda}} \left\| \mathcal{A}_\sigma^{\frac{5}{4}} H \right\|_{L^\Lambda(\mathbb{R}^{n-1})} \leq C\|H\|_{T^{\Lambda,\sigma}}, \quad y' \in B(x',y_n).$$

Let $\delta > 0$. The preceding inequalities imply

$$\begin{aligned}
\int_0^\infty \fint_{B_{y_n}(x')} \left|\Gamma_2(y',y_n)\right|^q dy' dy_n &\leq \left(\int_0^\delta + \int_\delta^\infty \right) \fint_{B_{y_n}(x')} \left|\Gamma_2(y',y_n)\right|^q dy' dy_n \\
&\leq C\delta^{1+\frac{q}{\sigma'}} [\mathcal{A}_\sigma H(x')]^q + \delta^{1-\left(\frac{n-1}{\Lambda}-\frac{1}{\sigma'}\right)q} \|H\|^q_{T^{\Lambda,\sigma}}.
\end{aligned}$$

Optimizing with respect to δ (i.e. choosing $\delta = \left(\|H\|_{T^{\Lambda,\sigma}} / \mathcal{A}_\sigma^{\frac{5}{4}} H(x')\right)^{\frac{\Lambda}{n-1}}$) yields

$$\left(\int_0^\infty \fint_{B_{y_n}(x')} \left|\Gamma_2(y',y_n)\right|^q dy' dy_n \right)^{\frac{1}{q}} \leq C\|H\|_{T^{\Lambda,\sigma}}^{\frac{\Lambda}{n-1}(1+\frac{1}{q}-\frac{1}{\sigma})} \left[\mathcal{A}_\sigma^{\frac{5}{4}} H(x')\right]^{1-\frac{\Lambda}{n-1}(1+\frac{1}{q}-\frac{1}{\sigma})}.$$

Taking the L^p-norm on both sides and using Remark 4.1.2 gives (4.2.32). Finally, the $T^{p,q}$-norm of Γ_3 is controlled from above by a constant multiple of $\|H\|_{T^{\Lambda,\sigma}}$. This is derived from a simple duality argument. The proof of Proposition 4.2.5 is now complete.

We can now summarize the findings obtained above into a single theorem establishing the well-posedness of System (**S**) for boundary data in the scale of Triebel-Lizorkin space with negative amount of smoothness. We say that a pair (u,π) is a solution to (**S**) if u and π satisfy the relations

$$u(x) = \mathcal{H}f(x) + \mathscr{G}(F,H)(x), \quad \pi(x) = \mathcal{E}f(x) + \Psi(F,H)(x), \ \ x \in \mathbb{R}^n_+. \tag{4.2.33}$$

Theorem 4.2.10. *Assume that the positive numbers $\eta, \Lambda, \sigma, p, q$ are as in Proposition 4.2.5. Then for any $f \in [\dot{F}^{-1/q}_{p,q}(\mathbb{R}^{n-1})]^n$, $F \in \mathbf{Y}^{\tau,\eta}$ and $H \in \mathbf{Y}^{\Lambda,\sigma}$, the Stokes system* (**S**) *has a solution $(u,\pi) \in \mathbf{Z}^q \times \in \mathbf{X}^q$ (in the sense made precise in* (4.2.33)*) which obeys*

$$\|u\|_{\mathbf{X}^q} + \|\pi\|_{\mathbf{Z}^q} \leq C(\|f\|_{\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})} + \|F\|_{\mathbf{Y}^{\tau,\eta}} + \|H\|_{\mathbf{Y}^{\Lambda,\sigma}}) \tag{4.2.34}$$

for some constant $C > 0$ independent of f, F and H.

Practically, Theorem 4.2.10 can easily be extended to the case where the vector field u is not necessarily solenoidal, i.e. div $u = \phi$ using the formulation derived in [Sol77, formula 2.32], see also [Cat61] so that our result gives an alternative approach to the Dirichlet problem for the Stokes system (to be compared to [ANR08] wherein the analysis is carried out in weigthed Sobolev spaces).

Remark 4.2.11. These estimates of the velocity field and the pressure in tent and weighted tent framework respectively against boundary data in low regularity spaces are new and generalize well-known results [FS15]. In fact, our boundary class $\dot{F}^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})$ contains the homogeneous Sobolev space $\dot{H}^{s,r}(\mathbb{R}^{n-1})$ for $-1/q < s$ and $\dfrac{n-1}{r} - s = \dfrac{1}{q-1}$.

4.3 Proofs of main results

Here, we give a detailed proof of some of our main findings relying on preliminary results obtained in Section 4.2. The procedure used in the proof of Theorem 4.1.6 and 4.1.7 is the same, thus only the proof of the latter and that of 4.1.8 will be given.

Proof of Theorem 4.1.7. Let $f \in [\dot{H}^{-\frac{1}{2},2(n-1)}(\mathbb{R}^{n-1})]^n$ with $n > 2$ and assume $F \in \mathbf{Y}^{\tau,\eta}$ for $1 < \eta < \tau < \infty$ with $\dfrac{1}{\eta} + \dfrac{n-1}{\tau} = 3$. Equip the Banach space $\mathbf{X} \times \mathbf{Z}$ by the norm $\|\cdot\| := \|\cdot\|_{\mathbf{X}} + \|\cdot\|_{\mathbf{Z}}$ and introduce the operators $\mathscr{L}$ defined by

$$\mathscr{L}(u,\pi) = \Big(\mathcal{H}f + \mathscr{G}[F, u \otimes u],\ \mathcal{E}(f) + \Psi[F, u \otimes u]\Big)$$

where $\mathcal{H}$ and $\mathcal{E}$ are given by (4.2.4). A solution of Eq. (**NS**) according to Definition 4.2.33 is a couple (u,π) satisfying the fixed point equation

$$(u,\pi) = \mathscr{L}(u,\pi) \ \text{ in } \ \mathbb{R}^n_+. \tag{4.3.1}$$

Using a Banach fixed point argument, we wish to show that the latter equation admits a solution in $\mathbf{X}\times\mathbf{Z}$. Let $(u,\pi),(v,\pi')\in\mathbf{X}\times\mathbf{Z}$ two solutions of Eq. (4.3.1) associated to the same Dirichlet data and forcing term and use Proposition 4.2.5 with $q=2$, $(\Lambda,\sigma)=(n-1,1)$ to get

$$\begin{aligned}
\left\|\mathscr{L}(u,\pi)-\mathscr{L}(v,\pi')\right\| &= \left\|\mathscr{G}(0,u\otimes u-v\otimes v)\right\|_{\mathbf{X}}+\left\|\Psi(0,u\otimes u-v\otimes v\right\|_{\mathbf{Z}}\\
&\le C\|u\otimes u-v\otimes v\|_{\mathbf{Y}^{n-1,1}}\\
&\le C(\|u\otimes(u-v)\|_{\mathbf{Y}^{n-1,1}}+\|(u-v)\otimes v\|_{\mathbf{Y}^{n-1,1}})\\
&\le C(\sup_{x_n>0} x_n^2\left\|[u\otimes(u-v)](\cdot,x_n)\right\|_{L^\infty(\mathbb{R}^{n-1})}+\|u\otimes(u-v)\|_{T^{n-1,1}}+\\
&\qquad \sup_{x_n>0} x_n^2\|[(u-v)\otimes v](\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}+\|(u-v)\otimes v\|_{T^{n-1,1}})\\
&\le C\Big(\sup_{x_n>0} x_n\|u(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\sup_{x_n>0} x_n\|(u-v)(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}+\\
&\qquad \sup_{x_n>0} x_n\|(u-v)(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\sup_{x_n>0} x_n\|v(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}+\\
&\qquad\quad \|u\|_{T^{2(n-1),2}}\|u-v\|_{T^{2(n-1),2}}+\|u-v\|_{T^{2(n-1),2}}\|v\|_{T^{2(n-1),2}}\Big)\\
&\le C\|u-v\|_{\mathbf{X}}(\|u\|_{\mathbf{X}}+\|v\|_{\mathbf{X}}).
\end{aligned}\tag{4.3.2}$$

In light of Lemma 4.2.1 (applied with $q=2$) we find that

$$\begin{aligned}
\left\|\mathscr{L}(u,\pi)\right\| &\le K\Big(\|u\|_{\mathbf{X}}^2+\|\mathcal{H}f\|_{\mathbf{X}}+\|\mathscr{G}[F,0]\|_{\mathbf{X}}+\|\mathcal{E}(f)\|_{\mathbf{Z}}+\|\Psi[F,0]\|_{\mathbf{Z}}\Big)\\
&\le K(\|u\|_{\mathbf{X}}^2+\|f\|_{\dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})}+\|F\|_{\mathbf{Y}^{\tau,\eta}}).
\end{aligned}\tag{4.3.3}$$

Now pick $\varepsilon>0$ such that $\|f\|_{\dot{H}^{-1/2,2(n-1)}(\mathbb{R}^{n-1})}+\|F\|_{\mathbf{Y}^{\tau,\eta}}\le\varepsilon$. If ε is sufficiently small, then it readily follows from (4.3.2) and (4.3.3) that $\mathscr{L}$ has a unique fixed point in a closed ball of $\mathbf{X}\times\mathbf{Z}$ centered at the origin with radius $c\varepsilon$ for some $c>0$. □

Proof of Theorem 4.1.8. Let $2<q<\infty$ and put $p=(n-1)q(q-1)$. Further let $1<\eta_1<\tau_1<n-1$ and $1<\sigma<\Lambda$ such that

$$\frac{1}{\eta_1}+\frac{n-1}{\tau_1}=1+\frac{1}{\sigma}+\frac{n-1}{\Lambda}=2+\frac{1}{q-1}.\tag{4.3.4}$$

Assume $f\in\dot{H}^{-\frac{1}{2},2(n-1)}\cap\dot{F}_{p,q}^{-\frac{1}{q}}(\mathbb{R}^{n-1})$ and $F\in\mathbf{Y}^{\tau_1,\eta_1}\cap\mathbf{Y}^{\tau,\eta}$. We remark that the solution found above may be realized as the unique limit in $\mathbf{X}\times\mathbf{Z}$ of the following sequence of approximations given by

$$(u_1,\pi_1)=(\mathcal{H}(f),\mathcal{E}(f)),\ (u_{j+1},\pi_{j+1})=(\mathscr{G}[F,u_j\otimes u_j]+u_1,\Psi[F,u_j\otimes u_j]+\pi_1),\ j=1,2,\cdots$$

Each element of this sequence belongs to $\mathbf{X}^q\times\mathbf{Z}^q$. In fact, since $(u_1,\pi_1)\in\mathbf{X}^q\times\mathbf{Z}^q$ (see Lemma 4.2.1) one may proceed via an induction argument to prove the claim. Choose

(σ,Λ) such that $\frac{1}{\sigma}=\frac{1}{2}+\frac{1}{q}$, $\frac{1}{\Lambda}=\frac{1}{2(n-1)}+\frac{1}{p}$ and invoke Proposition 4.2.5, Hölder's inequality in tent spaces simultaneously to have for each $j=1,\ldots$

$$\begin{aligned}
\|(u_{j+1},\pi_{j+1})\|_{\mathbf{X}^q\times\mathbf{Z}^q} &= \|\mathscr{G}[F,u_j\otimes u_j]+u_1\|_{\mathbf{X}^q}+\|\Psi[F,u_j\otimes u_j]+\pi_j\|_{\mathbf{Z}^q}\\
&\le C\Big(\|u_1\|_{\mathbf{X}^q}+\|\mathscr{G}[F,u_j\otimes u_j]\|_{\mathbf{X}^q}+\|\pi_1\|_{\mathbf{Z}^q}+\|\Psi[F,u_j\otimes u_j]\|_{\mathbf{Z}^q}\Big)\\
&\le C\Big(\|f\|_{\dot F^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})}+\|F\|_{\mathbf{Y}^{\tau_1,\eta_1}}+\|u_j\otimes u_j\|_{\mathbf{Y}^{\Lambda,\sigma}}\Big)\\
&\le C\Big(\|f\|_{\dot F^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})}+\|F\|_{\mathbf{Y}^{\tau_1,\eta_1}}+\|u_j\|_{T^{2(n-1),2}}\|u_j\|_{T^{p,q}}+\\
&\qquad\qquad \sup_{x_n>0} x_n^{\frac{1}{\eta}+\frac{n-1}{\Lambda}}\|u_j\otimes u_j(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\Big)\\
&\le C\Big(\|f\|_{\dot F^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})}+\|F\|_{\mathbf{Y}^{\tau_1,\eta_1}}+\sup_{x_n>0} x_n^{\frac{1}{q-1}}\|u_j(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\cdot\\
&\qquad\qquad \sup_{x_n>0} x_n\|u_j(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}+\|u_j\|_{T^{2(n-1),2}}\|u_j\|_{T^{p,q}}\Big)\\
&\le C\Big(\|f\|_{\dot F^{-\frac{1}{q}}_{p,q}(\mathbb{R}^{n-1})}+\|F\|_{\mathbf{Y}^{\tau_1,\eta_1}}+\|u_j\|_{\mathbf{X}}\|u_j\|_{\mathbf{X}^q}\Big)
\end{aligned}$$

so that if $(u_j,\pi_j)\in\mathbf{X}^q\times\mathbf{Z}^q$, then so is (u_{j+1},π_{j+1}). Next, we show that the latter sequence is Cauchy in $\mathbf{X}^q\times\mathbf{Z}^q$. We estimate $(w_j,q_j)=(u_{j+1}-u_j,\pi_{j+1}-\pi_j)$, $j=1,2,\ldots$

$$\begin{aligned}
\|(w_j,q_j)\|_{\mathbf{X}^q\times\mathbf{Z}^q} &= \left\|\mathscr{G}[0,u_j\otimes u_j-u_{j-1}\otimes u_{j-1}]\right\|_{\mathbf{X}^q}+\left\|\Psi[0,u_j\otimes u_j-u_{j-1}\otimes u_{j-1}]\right\|_{\mathbf{Z}^q}\\
&\le c\|u_j\otimes u_j-u_{j-1}\otimes u_{j-1}\|_{\mathbf{Y}^{\Lambda,\eta}}\\
&\le c\|u_j\otimes w_{j-1}+w_{j-1}\otimes u_{j-1}\|_{\mathbf{Y}^{\Lambda,\eta}}\\
&\le c\|w_{j-1}\|_{\mathbf{X}^q}(\|u_j\|_{\mathbf{X}}+\|u_{j-1}\|_{\mathbf{X}})\\
&\le c\|(w_{j-1},q_{j-1})\|_{\mathbf{X}^q\times\mathbf{Z}^q}(\|u_j\|_{\mathbf{X}}+\|u_{j-1}\|_{\mathbf{X}}).
\end{aligned}$$

Let $\varepsilon>0$ be as in Theorem 4.1.7 and take $0<\varepsilon_q<\varepsilon$. If $\|f\|_{\dot H^{-\frac{1}{2},2(n-1)}}+\|F\|_{\mathbf{Y}^{\tau,\eta}}\le\varepsilon_q$, then the conclusion of Theorem 4.1.7 shows that $\|u_j\|_{\mathbf{X}}\le 2\kappa_q$, $\kappa_q=\kappa_q(\varepsilon_q)$. Whence,

$$\|(w_j,q_j)\|_{\mathbf{X}^q\times\mathbf{Z}^q}\le C\kappa_q\|(w_{j-1},q_{j-1})\|_{\mathbf{X}^q\times\mathbf{Z}^q}.$$

With $\varepsilon_q>0$ chosen sufficiently small with $C\kappa_q<1$, a simple iteration of the previous inequality yields

$$\|(w_j,q_j)\|_{\mathbf{X}^q\times\mathbf{Z}^q}\le (C\kappa_q)^{j-1}\|(w_1,q_1)\|_{\mathbf{X}^q\times\mathbf{Z}^q}$$

thus implying the convergence of the sequence (w_j,q_j) in $\mathbf{X}^q\times\mathbf{Z}^q$. The limit of this sequence solves (**NS**) and by uniqueness, it is the same as that constructed in Theorem 4.1.7. □

4.4 Appendix

Here we sketch the proof of Lemma 4.1.1. Let $K \subset \mathbb{R}^n_+$ be a compact set. Then by Lemma 4.2.3 we know that $E(K) = \{x' \in \mathbb{R}^{n-1} : K \cap \Gamma(x') \neq \emptyset\}$ has a finite Lebesgue measure. Let us denote by $\mathbf{1}_K$ the characteristic function of the compact set K. If $p \leq q$, then via Hölder's inequality, one obtains

$$\begin{aligned}
\|\mathbf{1}_K f\|_{T^{p,q}} &= \left(\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{p/q} dx' \right)^{1/p} \\
&= \left(\int_{E(K)} \left(\iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n dx' \right)^{p/q} dx' \right)^{1/p} \\
&\leq \left(\int_{E(K)} \iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \left| E(K) \right|^{\frac{1}{p} - \frac{1}{q}} \\
&\leq \left| E(K) \right|^{\frac{1}{p} - \frac{1}{q}} \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x') \cap K} |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\leq \left| E(K) \right|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(K)}.
\end{aligned}$$

Moving on, for $q < p$, applying Minkowski's inequality implies

$$\begin{aligned}
\|\mathbf{1}_K f\|_{T^{p,q}} &= \left(\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n \right)^{p/q} dx' \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty \mathbf{1}_{B_{y_n}(y')}(x') \mathbf{1}_K(y', y_n) |f|^q y_n^{1-n} dy_n dy' \right)^{p/q} dx' \right)^{1/p} \\
&\leq C_K \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty \mathbf{1}_K |f|^q dy' dy_n dx' \right)^{1/q} \\
&\leq C_K \|f\|_{L^q(K)}.
\end{aligned}$$

Assuming that $p \le q$, we use Lemma 4.2.2 and Minkowski's inequality simultaneously to get

$$\begin{aligned}
\|f\|_{L^q(K)} &= \left(\int_{\mathbb{R}^n_+} \mathbf{1}_K |f|^q dy' dy_n \right)^{1/q} \\
&\le C \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\le C_K \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} y_n^{\frac{(n-1)p}{q}} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\le C_K \left(\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n \right)^{p/q} dx' \right)^{1/p} \\
&\le C_K \|f\|_{T^{p,q}}.
\end{aligned}$$

When $p > q$, the desired bound follows from Hölder's inequality. Indeed, we have

$$\begin{aligned}
\|f\|_{L^q(K)} &= \left(\int_{\mathbb{R}^n_+} \mathbf{1}_K |f|^q dy' dy_n \right)^{1/q} \\
&\le C \left(\int_{\mathbb{R}^{n-1}} \iint_{\Gamma(x')} \mathbf{1}_K |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\le C \left(\int_{E(K)} \iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n dx' \right)^{1/q} \\
&\le C \left(\int_{\mathbb{R}^{n-1}} \left(\iint_{\Gamma(x')} |f|^q y_n^{1-n} dy' dy_n \right)^{p/q} dx' \right)^{1/p} \left| E(K) \right|^{\frac{1}{q} - \frac{1}{p}} \\
&\le C \left| E(K) \right|^{\frac{1}{q} - \frac{1}{p}} \|f\|_{T^{p,q}}.
\end{aligned}$$

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