Flows of Viscous Fluids: Fluctuations for Stochastic Homogenisation in Perforated Domains, and Non-Newtonian Thin-Film Models

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Summary

This thesis concerns problems arising in the study of flows of viscous fluids. In the first part, we discuss the interaction of the flow of a viscous fluid with a random array of moving particles in the limit of many particles and a total Stokes drag of order one. The second part of this thesis analyses the dynamics of thin liquid films of non-Newtonian fluids driven by capillary forces.

The description of an effective theory of a random array of moving particles in a viscous fluid is known as stochastic homogenisation in perforated domains. If the particles move slowly, inertial effects can be neglected and the fluid flow can be described as a Stokes flow in a perforated domain. Due to the viscous nature of the fluid, the interaction of the particles through the fluid is of long range. The Brinkman equations describe the effective theory for the fluid flow in the limit of many particles so that the collective Stokes drag is of order one. The rigorous derivation of the Brinkman equations from a Stokes flow in a perforated domain has been an active area of research.

This thesis addresses the quantitative study of the homogenisation result for the Stokes flow in perforated domains. For a random configuration of particles and velocities, the fluctuations around the limit are analysed. In the physical setting of three space dimensions, the fluctuation field is derived explicitly, and convergence rates for an approximation of the velocity fields in the perforated domains are shown. Furthermore, this thesis takes a first glance at a connection between stochastic homogenisation in perforated domains and stochastic partial differential equations.

The dynamic behaviour of thin liquid films of viscous fluids is derived from an asymptotic expansion in terms of the film height of a free-boundary Navier–Stokes system in the lubrication approximation. If the dynamics of the thin film are determined only from viscous forces and surface tension, the evolution of the film height is, to leading order, described by a fourth-order nonlinear degenerate-parabolic partial differential equation. In the second part of this thesis, the long-time behaviour and stability of this thin-film equation is studied for different non-Newtonian rheologies. That the evolution of the film height only depends on viscous and capillary forces points towards a gradient-flow structure of the dynamics. The decay rates depend on the fluid rheology. This topic is addressed in the final chapter of this thesis, where the gradient-flow structure of thin films of non-Newtonian power-law fluids with general mobilities is studied.

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Part I

Fluctuations for Homogenisation in Perforated Domains

1 | Introduction

Abstract

Homogenisation theory in perforated domains strives to derive effective equations for the interaction of particles in different media such as viscous fluids. One model describing the velocity of a viscous fluid containing many particles is given by the Dirichlet problem for the Stokes equations in a perforated domain. In the case of charged inclusions in a material, the electrostatic potential of this material is described by the Dirichlet problem for the Poisson equation in a perforated domain.

If the size of the particles is scaled inversely to the number of particles such that the total Stokes drag (or total capacitance) remains of order one, one obtains convergence to a solution to a limit equation involving a 'strange term'. Different techniques for the derivation of this effective theory are described heuristically in this chapter. It is explained how the resulting approximation can be modified to capture higher orders of the homogenisation, giving an outlook on the main results of the first part. Moreover, it is illustrated how this is connected with stochastic partial differential equations. The chapter concludes with an overview of the mathematical literature on the homogenisation problem for electrostatic and hydrostatic equations in perforated domains.

The study of the interaction of moving particles with a viscous fluid constitutes an important area of modern mathematics. Such systems of (small) particles moving in a fluid have many essential applications in technology and physics and can be observed everywhere in nature.

While the effect of one single particle on the fluid is, by Newton's third law, proportional to its mass and therefore relatively small, many small particles can have a significant and complex influence on the (dynamical) behaviour of the viscous fluid. The complexity of this effect is not only due to the number of particles but also due to their long-range interactions.

There is a similar effect in the electrostatics of materials with small charged inclusions. As in the case of the fluid, one small inclusion only affects the electrostatic potential slightly. Many inclusions induce similar complex effects.

In both cases, the analysis and numerics of the fluid flow with many particles or the electrostatic potential of a material with many inclusions becomes mathematically inaccessible when the number of particles or inclusions is large. In many applications, only the macroscopic effect of the particles on the fluid or the inclusions on the material are of interest. This results in the study of the effective average velocity of the fluid or the effective electrostatic potential of the material.

This macroscopic description, i.e. an effective theory for the fluid flow or for the electrostatic potential, can be derived in the limit of many particles or inclusions. For this limiting process which is called *homogenisation*, only a few physical parameters turn out to be relevant: the viscosity and the mass density of the fluid or the conductivity of the material, as well as the average mass density of the particles or inclusions, their volume fraction and their average velocity or charge.

The first part of this thesis deals with a particular effect induced by particles in a fluid or by inclusions in a material: we consider the critical scaling for the Dirichlet problem of the Poisson and Stokes equations in a domain $\Omega_m \subset \mathbb{R}^d$, $d \geq 2$, perforated by m tiny (spherical) holes, described for example in [CM82a] and [All90a].

This critical scaling is characterised by the emergence of a 'strange' term signifying the collective effects of the particles. It can be explained as the collective effect of the Stokes drag given by each particle or the capacity of each inclusion. Consider a single spherical particle with radius $R > 0$. Then the Stokes drag F_d or the effect of viscosity on the particle (or vice versa, the effect of the particle on the viscous fluid) is of order $F_d \sim R^{d-2}(V-u)$, [Sto51]. Here, V denotes the velocity of the particle and u the velocity of the undisturbed fluid. In a system with m particles, the total Stokes drag is of order one if the product mR_m^{d-2}, with the number of particles m and the particle radius R_m, is of order one. Note that the limit of the particle number $m \to \infty$ also leads to a vanishing volume fraction.

For the description of the electrostatic potential in materials with small inclusions the same effect can be observed by considering the capacitance. The capacitance describes the amount of electric charge of the inclusion compared to the electric potential in the material surrounding it [Max73].

We study the limiting behaviour for both the Poisson equation (for the electrostatic case) and Stokes equations (for fluid flows) in this scaling regime described by inclusions with a collective effect of the capacitance or Stokes drag of order one. While the behaviour of the corresponding equations in perforated domains in the limit of many (randomly distributed) small particles has been studied since the pioneering works [Hru79], [CM82a] and [All90a], there are fewer and more restricted results on the quantitative behaviour of the limiting process, [FOT85] and [Rub86].

In the critical scaling, the collective effect of all particles results in the appearance of an additional term in the equation obtained in the limit of particle number $m \to \infty$. If the radius of the particles were much smaller than the critical radius, i.e. if $R_m^{d-2}m \ll 1$, then this collective effect disappears and one recovers the original equation in the limit.

On the other hand, if the particles are much larger, i.e. $R_m^{d-2}m \gg 1$, the collective effect of the particles dominates. Then the limiting equations are solely described by the (static) evolution of the particles. In this case, a rescaled version of the homogenisation problem converges to Darcy's law (see e.g. [All90b] or [Giu21a]). The remainder of this thesis will only be concerned with the case of the critical scaling for the radii of the particles.

This introductory chapter of the first part continues with a concrete formulation of the problem for the Poisson and Stokes equations in Section 1.1. Section 1.2 consists of a phenomenological treatise of different mathematical methods to study the limiting behaviour for the Poisson equation in the simple geometry of spherical particles distributed on a lattice. Using a blow-up argument, we derive an explicit approximation which is modified in Section 1.3 to include the corrections due to the fluctuations. A characterisation of the fluctuations for the Poisson and Stokes equations in three dimensions for randomly distributed spheres with random velocities has been obtained in [HJ22]. A reprint of this paper can be found in Appendix A. The fluctuations field can be described as a solution to a linear stochastic partial differential equation. This link will be explored further in Section 1.4. Finally, in Section 1.5, we give a general overview of the literature on homogenisation results in perforated domains.

Chapter 2 consists of a summary of the main result of the first part of this thesis. It concerns the study of fluctuations for the homogenisation of the Stokes equations in $\mathbb{R}^3$, perforated by particles with random positions and velocities. A reprint of the whole paper can be found in Appendix A.

In Chapter 3, the link between the stochastic homogenisation problem and stochastic partial differential equations is investigated in more detail. As preliminary steps, the homogenisation result is extended to the semilinear Poisson equation, and the solution to the stochastic Helmholtz equation is derived as a homogenisation limit. These observations lead to the conjecture that one can obtain the elliptic Φ^4_d-theory from the theory of stochastic homogenisation in perforated domains.

1.1 Formulation of the problem

We will now give the precise formulation of the homogenisation problem in a domain $\Omega \subset \mathbb{R}^d$ for general space dimension $d \geq 2$. We consider a fixed number of spherical particles $m \in \mathbb{N}$ with a fixed radius $R_m > 0$ and centres $X_1, \dots, X_m \in \Omega$. Note that the positions $X_i = X_i^{(m)}$ also depend on m, but we suppress the corresponding index for simplicity of notation. For the remainder of this thesis, we will restrict the analysis to the case of spherical particles. In the case of non-spherical particles, additional effects might occur depending on the geometry and corresponding dynamics. For results for non-spherical holes see for example [CM82a] and [HMS19].

Furthermore, we denote by d_m the minimal distance between the particles

$$d_m = \min_{i \neq j \in \{1,\dots,m\}} |X_i - X_j|.$$

We will assume that $m^{-1/d} \sim d_m \gg R_m \sim m^{-1}$. To avoid technicalities with particles very close to the boundary, we assume that $\operatorname{dist}\big((X_i)_{i=1,\dots,m}, \partial\Omega\big) > 2R_m$. With these preliminaries, we denote the perforated domain by

$$\Omega_m = \Omega \setminus \bigcup_{i=1}^m \overline{B_{R_m}(X_i)}.$$

For the description of the electrostatic potential in the domain Ω_m, we choose charges $Q_1, \dots, Q_m \in \mathbb{R}$ on each of the inclusions. Given a source term $f \in H^{-1}(\Omega)$ (or $f \in \dot{H}^{-1}(\Omega)$ in the case where Ω is unbounded), the electrostatic potential $u_m \colon \Omega \to \mathbb{R}$ with isotropic conductivity matrix $a = \mathrm{Id}$ in the domain with inclusions $B_{R_m}(X_i)$ and corresponding charges Q_i, $i = 1, \dots, m$, is described by the Poisson equation

$$\begin{cases} -\Delta u_m = f & \text{in } \Omega_m, \\ u_m = Q_i & \text{in } B_{R_m}(X_i),\ i = 1, \dots, m, \\ u_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.1}$$

Then, equation (1.1.1) has a unique weak solution $u_m \in H^1(\Omega)$ (or $u_m \in \dot{H}^1(\Omega)$ for unbounded Ω) by the standard theory of elliptic partial differential equations.

To obtain a macroscopic description, we now assume that the distribution of charges is given by a macroscopic object. In the simplest (non-trivial) case, we may assume that there is a function $Q \in H^1(\Omega)$ such that

$$Q_i = \fint_{B_{R_m}(X_i)} Q(x)\,\mathrm{d}x := \frac{1}{|B_{R_m}(X_i)|} \int_{B_{R_m}(X_i)} Q(x)\,\mathrm{d}x =: [Q]_m(X_i).$$

(We could also give up on the charges being constant and just assign $u_m = Q$ in $B_{R_m}(X_i)$, $i = 1, \dots, m$. This particular choice of charges will play a role in Chapter 3 though.)

Using the newly introduced functions $[Q]_m \in H^1(\Omega)$ (which we may modify close to the boundary so that $[Q] \in H^1_0(\Omega)$ by multiplication with a cut-off), we find that $u_m - [Q]_m$ solves the Poisson equation in the perforated domain Ω_m given by

$$\begin{cases} -\Delta(u_m - [Q]_m) = f + \Delta[Q]_m & \text{in } \Omega_m, \\ u_m - [Q]_m = 0 & \text{in } B_{R_m}(X_i),\ i = 1, \dots, m, \\ u_m - [Q]_m = 0 & \text{on } \partial\Omega. \end{cases}$$

By the standard a priori estimate, we conclude that

$$\|u_m - [Q]_m\|_{H^1} \lesssim \|f\|_{H^{-1}(\Omega)} + \|[Q]_m\|_{H^1(\Omega)} \lesssim \|f\|_{H^{-1}(\Omega)} + \|Q\|_{H^1(\Omega)},$$

and hence u_m is uniformly bounded in $H^1_0(\Omega)$. We conclude that u_m has a weak accumulation point $u \in H^1_0(\Omega)$.

For the case of the static description of a fluid flow around moving particles, we choose velocities $V_1, \dots, V_m \in \mathbb{R}^d$. Then, the Stokes flow $v_m \colon \Omega \to \mathbb{R}^d$ around the particles $B_{R_m}(X_i)$ with velocities V_i is given, if we neglect inertial effects, by the Stokes equations in the perforated domain

$$\begin{cases} -\Delta v_m + \nabla p_m = f & \text{in } \Omega_m, \\ \operatorname{div} v_m = 0 & \text{in } \Omega_m, \\ v_m = V_i & \text{in } B_{R_m}(X_i),\ i = 1, \dots, m, \\ v_m = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $p_m \colon \Omega \to \mathbb{R}$ denotes the pressure and $f \in H^1(\Omega; \mathbb{R}^d)$ is a force term (or $f \in \dot{H}^1(\Omega; \mathbb{R}^d)$ for the case of unbounded Ω). Again, by the standard theory, 1.1 has a unique weak solution $v_m \in H^1(\Omega; \mathbb{R}^d)$ ($v_m \in \dot{H}^1(\Omega; \mathbb{R}^d)$ for unbounded Ω).

If we assume that the velocities are given by a macroscopic function $V \in H^1(\Omega; \mathbb{R}^d)$ such that

$$V_i = \fint_{B_{R_m}(X_i)} V(x)\,\mathrm{d}x =: [V]_m(X_i),$$

then the sequence $(v_m)_m$ is uniformly bounded in $H^1_0(\Omega; \mathbb{R}^d)$ and has a weak accumulation point $v \in H^1_0(\Omega; \mathbb{R}^d)$.

The task at hand is to understand these accumulation points. This is achieved by showing that the accumulation points satisfy a screened (macroscopic) version of the original equation. The following section will explain different variants of the identification of the limit equation. Before we can get there, we need to fix some more assumptions on the distribution of the inclusions or particles, cf. [NV04b].

Definition 1.1.1. *We call a set of configurations for the centres of the holes (or particles) $(X_i^m)_{i \in I_m\}} \subset \mathbb{R}^d$, $d \geq 3$, with a finite or countable index set $I_m \subset \mathbb{N}$ and $m \in \mathbb{N}$ admissible if there is a constant $C_0 > 0$ such that*

(i) The number of particles in a cube $Q \subset \mathbb{R}^d$ does not exceed $C_0 m|Q|$.

(ii) Particles are well separated:

$$\min_{i \neq j} |X_i - X_j| \geq \frac{1}{C_0} m^{-1/d}.$$

(iii) Particles are homogeneously distributed on average:

$$\max_{|Q|\leq} \max_{X_i \in Q} \frac{1}{m} \sum_{\substack{X_j \in Q \\ j \neq i}} \frac{1}{|X_i - X_j|^{d-2}} \leq \delta_{|Q|},$$

with $\delta_{|Q|} \to 0$ *as* $|Q| \to 0$.

(iv) collective capacity of order one: there is a measure μ *such that*

$$\sum_{i \in I_m} \frac{1}{m^{\frac{1}{d-2}}} \mathcal{H}^{d-1}|_{\partial B_{R_m}(X_i)} \longrightarrow \mu$$

in an appropriate sense.

These assumptions are satisfied for the particular case of the centres given by the lattice $m^{-1/d}\mathbb{Z}^d \subset \mathbb{R}^d$ (for the construction of the measure μ see below). We will analyse this case phenomenologically in the following section. They are also satisfied with probability converging to one in the case of particles that are randomly distributed either by a Poisson point process or independently and identically with respect to a continuous density.

1.2 Homogenisation in perforated domains

We now study different techniques for the analysis of the limiting problem of equation (1.1.1) phenomenologically. The study for the Stokes equations is similar. To do this, we assume that $\Omega = \mathbb{R}^d$ and that the holes are given on a lattice $m^{-1/d}\mathbb{Z}^d$. Notice that in an open domain $U \subset \mathbb{R}^d$ of order one there are roughly m holes so that we set

$$R_m = m^{-\frac{1}{d-2}}$$

to guarantee that $R_m^{d-2} m = 1$.

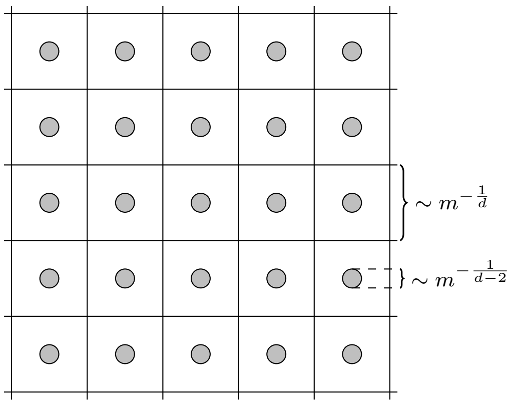

Figure 1.1: Particle configuration on the lattice.

We also choose charges Q_i on the hole $B_{R_m}(X_i)$ and assume that these are macroscopically given by a function $Q \in H^1(\mathbb{R}^d)$ with $Q_i = [Q]_m(X) = \fint_{B_{R_m}(X)} Q \, \mathrm{d}x$ in $B_{R_m}(X)$, $X \in m^{-1/d}\mathbb{Z}^d$. Then, as discussed previously, the electrostatic potential u_m is described by the Poisson equation

$$\begin{cases} -\Delta u_m = f & \text{in } \mathbb{R}^d \setminus \bigcup_{X\in m^{-1/d}\mathbb{Z}^d} B_{R_m}(X), \\ u_m = [Q]_m(X) & \text{in } B_{R_m}(X) \text{ for every } X \in m^{-1/d}\mathbb{Z}^d. \end{cases}$$

We consider $v_m = u_m - [Q]_m$, then v_m solves the equation

$$\begin{cases} -\Delta v_m = f + \Delta[Q]_m = \tilde{f} & \text{in } \mathbb{R}^d \setminus \bigcup_{X\in m^{-1/d}\mathbb{Z}^d} B_{R_m}(X), \\ u_m = 0 & \text{in } B_{R_m}(X) \text{ for every } X \in m^{-1/d}\mathbb{Z}^d. \end{cases} \tag{1.2.1}$$

As discussed before, we know that there is a weak solution $v_m \in \dot{H}^1(\mathbb{R}^d)$ to (1.2.1) and, by standard energy methods, we know that there is a weak accumulation point $v \in \dot{H}^1(\mathbb{R}^d)$.

There are several different methods to derive the corresponding equation for the limit point. The first method, which was developed in [CM82a; CM97] and applied to fluid flows in [All90a], is based on the work of Tartar [Tar76] on so-called correctors. Here a suitable test function vanishing in the particle is constructed that converges weakly in $\dot{H}^1(\mathbb{R}^d)$ to the constant function 1. This function w_m carries the information on the capacity of the holes.

Since the early work of Smoluchowski [Smo11], a second method devised for studying the limit is the method of reflections. This method constructs an approximation of the solutions u_m iteratively in terms of a series starting from $-\Delta v = \tilde{f}$ and correcting the errors made in each particle from the previous approximation. The method of reflection has been rigorously studied in [HV18].

A third method is to derive a monopole approximation for u_m. Since $-\Delta u_m - f$ is supported on $\partial B_{R_m}(X)$, we can try to approximate $-\Delta u_m - f$ as a sum $\hat{u}_m = \sum_{X\in m^{-1/d}\mathbb{Z}^d} q_X \delta_X^m$ over each monopole, taking the contribution of every single particle into account. The additional ingredient is to assume that the approximation is good on each particle. This is achieved by choosing the corresponding charges q_X so that $⨍_{B_{R_m}(X)} \hat{u}_m \, \mathrm{d}\mathcal{H}^{d-1} = Q_X$ for every $X \in m^{-1/d}\mathbb{Z}^d$.

Finally, the limit equation can also be derived utilising a blow-up method as in [Gér22] and [HJ22]. Here, one uses the accumulation point of the sequence u_m to describe the behaviour far away from each particle. This gives an explicit approximation for the charge in each hole only in terms of the accumulation point and the position and charge of each hole. In [HJ22] a refined version of this method has been used to study the fluctuations.

We now give more details on each of these methods in the case of the Poisson equation (1.2.1) on $\mathbb{R}^d$ with particles on the lattice.

Method of Oscillating Test Functions

The first method we introduce here is the method of oscillating test functions. In order to be able to test the Poisson equation (1.2.1) on a perforated domain, we want to use test functions of the form $w_m\varphi$ for $\varphi \in C_c^\infty(\mathbb{R}^d)$, where $w_m \equiv 0$ in $B_{R_m}(X)$ for every $X \in m^{-1/d}\mathbb{Z}^d$. Then the weak formulation becomes

$$\int_{\mathbb{R}^d} \nabla v_m \nabla (w_m\varphi) \, \mathrm{d}x = \int_{\mathbb{R}^d} \tilde{f} w_m \varphi \, \mathrm{d}x.$$

We make the following assumptions for the sequence $(w_m)_m$, cf. [CM82a]:

(A1) $w_m \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$;

(A2) $w_m = 0$ in $B_{R_m}(X)$ for every $X \in m^{-1/d}\mathbb{Z}^d$;

(A3) $w_m \rightharpoonup 1$ weakly in $H^1_{\text{loc}}(\mathbb{R}^d)$;

(A4) there is a measure $\mu \in W^{-1,\infty}_{\text{loc}}(\mathbb{R}^d)$ on $\mathbb{R}^d$ such that for every sequence $\tilde{v}_m$ with $\tilde{v}_m = 0$ in $B_{R_m}(X)$ for every $X \in m^{-1/d}\mathbb{Z}^d$ and $\tilde{v}_m \rightharpoonup \tilde{v}$ weakly in $H^1(\Omega)$, it holds

$$\langle -\Delta w_m, \varphi \tilde{v}_m \rangle \longrightarrow \langle \mu, \varphi \tilde{v} \rangle$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Observe that μ is explicitly given by the formula

$$\langle \mu, \varphi \rangle = \lim_{m\to\infty} \int_{\mathbb{R}^d} |\nabla w_m|^2 \varphi \,\mathrm{d}x,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, cf. [CM82a, Proposition 1.1].

With these assumptions, we may conclude that

$$\int_{\mathbb{R}^d} \tilde{f} w_m \varphi \,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^d} \tilde{f} \varphi \,\mathrm{d}x$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla v_m \nabla (w_m \varphi) \,\mathrm{d}x &= \langle -\Delta w_m, \varphi w_m \rangle - \int_\Omega v_m \nabla w_m \nabla \varphi \,\mathrm{d}x - \int_\Omega v_m w_m \Delta \varphi \,\mathrm{d}x \\ &\longrightarrow \langle \mu, \varphi v \rangle - \int_\Omega v \Delta \varphi \,\mathrm{d}x.\end{aligned}$$

Here, we used (A4) and (A3), which also implies that $\nabla w_m \rightharpoonup 0$ in $L^2_{\text{loc}}(\mathbb{R}^d)$. We conclude that v is a weak solution to the equation

$$-\Delta v + \mu v = f \quad \text{in } \mathbb{R}^d,$$

provided such a sequence of oscillating test functions exists.

The construction of the sequence w_m uses the specific geometry of the positions of the centre of the balls. Here, the geometry of the lattice comes in useful for this phenomenological discussion. The method has been used for more general (random) distributions of particles for example in [DG94] [CM09a], [CCL15], [CCL16], [GHV18] and [GH19a].

In our setting of the lattice, it suffices to construct w_m on a fundamental domain given by the cube C with sidelength $m^{-1/d}$ and centre 0.

We define the function w_m then by the periodic continuation of the solution to

$$\begin{cases} -\Delta w_m = 0 & \text{in } B_{m^{-1/d}/2}(0) \setminus B_{R_m}(0), \\ w_m = 0 & \text{in } B_{R_m}(0), \\ w_m = 1 & \text{in } C \setminus B_{m^{-1/d}/2}(0). \end{cases}$$

Since the volume of the ball $B_{R_m}(0)$ is very small compared to the volume of the cube C, one can show that $w_m \rightharpoonup 1$ in $H^1(C)$. Recall the definition of the capacity of a set

$$\text{Cap}(K) := \int_{\mathbb{R}^d \setminus K} |\nabla w|^2 \,\mathrm{d}x,$$

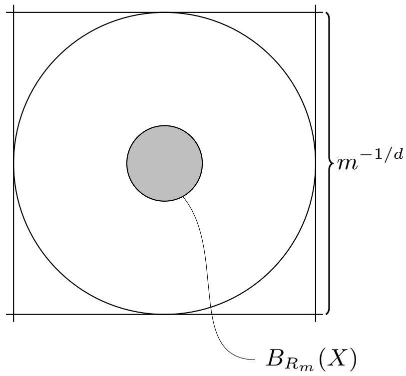

Figure 1.2: Fundamental domain of one particle.

where $w = 0$ on K and $w \to 1$ as $|x| \to \infty$. We observe by rescaling the ball $B_{R_m}(0)$ to the ball with radius one and then sending $m \to \infty$ that

$$\int_{\mathbb{R}^d} |\nabla w_m|^2 \varphi \,\mathrm{d}x \longrightarrow \mathrm{Cap}(B_1(0)) \int_{\mathbb{R}^d} \varphi \,\mathrm{d}x = (d-2)\mathcal{H}^{d-1}(\partial B_1(0)) \int_{\mathbb{R}^d} \varphi \,\mathrm{d}x. \quad (1.2.2)$$

Hence, $\mu = (d-2)\mathcal{H}(B_1(0))$ and the limit equation is given by

$$-\Delta v + (d-2)\omega_d v = \tilde{f} \quad \text{in } \mathbb{R}^d,$$

with $\omega_d = \mathcal{H}^{d-1}(\partial B_1(0))$. Since $[Q]_m \to Q$ in $\mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$, we conclude that $u_m = v_m + [Q]_m$ converges weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ to the function u, which solves

$$-\Delta u + (d-2)\omega_d(u - Q) = f \quad \text{in } \mathbb{R}^d.$$

Method of Reflections

The method of reflections was first used by Smoluchowski [Smo11] to calculate the interaction of particles in a Stokes flow. There are many historical and recent results, most of them numerical, using the method of reflections. For an overview, we refer the reader to [LLS21]. In the context of homogenisation in perforated domains, it was rigorously applied in [HV18].

We again study the equation (1.2.1)

$$\begin{cases} -\Delta v_m = \tilde{f} & \text{in } \mathbb{R}^d \setminus \bigcup_{X \in m^{-1/d}\mathbb{Z}^d} B_{R_m}(X), \\ u_m = 0 & \text{in } B_{R_m}(X) \text{ for every } X \in m^{-1/d}\mathbb{Z}^d. \end{cases}$$

If we neglect the particles, a first approximation could be given by

$$-\Delta \Phi_0 = \tilde{f} \quad \text{in } \mathbb{R}^d.$$

We consequently need to correct Φ_0 at each particle. This is known as reflection, and for $X \in m^{-1/d}\mathbb{Z}^d$ we define $\Phi_{1,X}$ via

$$\begin{cases} -\Delta \Phi_{1,X} = 0 & \text{in } \mathbb{R}^d \setminus B_{R_m}(X), \\ \Phi_{1,X} = -\Phi_0 & \text{in } B_{R_m}(X). \end{cases}$$

We then define $\Phi_1 = \sum_{X\in m^{-1/d}\mathbb{Z}^d} \Phi_{1,X}$ and consider $\Phi_0 + \Phi_1$. Now $\Phi_0 + \Phi_{1,X}$ solves (1.2.1) if there was only one particle. Since there are many particles, we have to continue correcting and define

$$\begin{cases} -\Delta\Phi_{k,X} = 0 & \text{in } \mathbb{R}^d \setminus B_{R_m}(X), \\ \Phi_{k,X} = -\sum_{Y\neq X} \Phi_{k-1,Y} & \text{in } B_{R_m}(X), \end{cases}$$

and $\Phi_k = \sum_{X\in m^{-1/d}\mathbb{Z}^d} \Phi_{k,X}$. To make this method rigorous, one now has to prove convergence of the sequence $\sum_{k=0}^N \Phi_k$. Two main problems stand out: even Φ_k consists of infinitely many terms, and to obtain convergence a good decay rate of $\Phi_{k,X}$ is needed. Next, the convergence as $N \to \infty$ has to be shown. We will not discuss this further here since we will not use the method of reflections in this thesis. Note though, that one way out is to replace $-\Delta$ with $-\Delta + \lambda$ for $\lambda > 0$, which guarantees exponential decay of the fundamental solution. Using a version of the method of reflections, Figari, Orlandi and Teta [FOT85] (for the Poisson equation) and Rubinstein [Rub86] (for the Stokes equations) analysed the fluctuations for the stochastic homogenisation in a perforated domain with finitely many particles with zero Dirichlet boundary conditions and under the technical assumption that λ is very big. They assumed the particles to be independent and identically distributed given a continuous density.

The complete characterisation of the fluctuations for the Poisson and Stokes equations with random Dirichlet boundary conditions will be discussed in Chapter 2 and Appendix A with a different approximation using the blow-up method introduced below.

The monopole method

Next, we introduce the monopole method to find an approximation. This approximation was first used in [NV04b] and[NV04a]. For the case of a single particle and $\tilde{f} = 0$ for simplicity

$$\begin{cases} -\Delta u_0 = 0 & \text{in } \mathbb{R}^d \setminus B_1(0), \\ u_0 = Q_0 & \text{on } B_1(0), \end{cases}$$

the solution is explicitly given by

$$u_0 = (-\Delta)^{-1} \left(q_0 \delta_{\partial B_1(0)}\right),$$

where $\delta_{\partial B_1(0)}$ denotes the normalised Hausdorff measure on $\partial B_1(0)$ and q_0 is a charge depending only on Q_0. Now we turn to the case of many particles and again study the equation (1.2.1)

$$\begin{cases} -\Delta u_m = 0 & \text{in } \mathbb{R}^d \setminus \bigcup_{X\in m^{-1/d}\mathbb{Z}^d} B_{R_m}(X), \\ u_m = [Q]_m X & \text{in } B_{R_m}(X) \text{ for every } X \in m^{-1/d}\mathbb{Z}^d. \end{cases}$$

Now, we make the ansatz that the solution is well-approximated by the sum of the (rescaled) monopole solutions over all holes

$$u_m \approx \hat{u}_m = (-\Delta)^{-1} \left[\sum_{X\in m^{-1/d}\mathbb{Z}^d} R_m q_X \delta_X \right] = \frac{1}{(d-2)\omega_d} \sum_{X\in m^{-1/d}\mathbb{Z}^d} \frac{R_m q_X}{|x-X|^{d-2}},$$

for some charges $q_X \in \mathbb{R}$ to be determined. Since $u_m = [Q]_X$ and $\hat{u}_m$ is supposed to approximate u_m, it makes sense to assume that

$$\fint_{\partial B_{R_m}(X)} \hat{u}_m \, \mathrm{d}\mathcal{H}^{d-1} = [Q]_m(X).$$

Now assume that the charges are macroscopic so that they are given by a field $q\colon \mathbb{R}^d \to \mathbb{R}$ with $q_X = q(X)$. Then, on the one hand,

$$\begin{aligned}
\hat{u}_m(x) &= \frac{1}{(d-2)\omega_d} \frac{1}{m} \sum_{X \in m^{-1/d}\mathbb{Z}^d} \frac{q_X}{|x-X|^{d-2}} \\
&= \frac{1}{(d-2)\omega_d} \frac{1}{m} \sum_{X \in m^{-1/d}\mathbb{Z}^d} \frac{q(X)}{|x-X|^{d-2}} \\
&\approx \frac{1}{(d-2)\omega_d} \int_{\mathbb{R}^d} \frac{q(y)}{|x-y|^{d-2}} \, \mathrm{d}y \\
&= (-\Delta)^{-1} q(x),
\end{aligned}$$

by interpreting $\frac{1}{m}\sum_{X\in m^{-1/d}\mathbb{Z}^d}$ as a Riemann sum, since there are of order m particles in a domain of order one. On the other hand, evaluating $\hat{u}_m$ on $\partial B_{R_m}(X)$ for a given X, it is

$$\begin{aligned}
\fint_{\partial B_{R_m}(X)} \hat{u}_m \, \mathrm{d}\mathcal{H}^{d-1} &= \frac{1}{(d-2)\omega_d} q(X) + \sum_{Y \neq X} \fint_{\partial B_{R_m}(X)} \frac{1}{(d-2)\omega_d} \frac{1}{m} \frac{q(Y)}{|x-Y|^{d-2}} \, \mathrm{d}\mathcal{H}^{d-1} \\
&\approx \frac{1}{(d-2)\omega_d} q(X) + \fint_{\partial B_{R_m}(X)} (-\Delta)^{-1} q \, \mathrm{d}\mathcal{H}^{d-1} \\
&\approx \frac{1}{(d-2)\omega_d} q(X) + (-\Delta)^{-1} q(X).
\end{aligned}$$

But this implies that

$$[Q]_m(X) \approx \frac{1}{(d-2)\omega_d} q(X) + (-\Delta)^{-1} q(X).$$

Assuming this equation to hold true everywhere in $\mathbb{R}^d$ and using that $q \approx -\Delta u$, $[Q]_m \approx Q$, we find

$$(d-2)\omega_d Q = -\Delta u + (d-2)\omega_d u \quad \text{in } \mathbb{R}^d.$$

The Blow-up Method

Very recently, a fourth method to study the limiting behaviour was discovered in [Gér22] and [HJ22]. The critical observation is that from $-\Delta u_m = f$ outside of the balls and $-\Delta u_m = 0$ inside the balls, we may write

$$-\Delta u_m = f \mathbf{1}_{\mathbb{R}^d \setminus \bigcup_{X \in m^{-1/d}\mathbb{Z}^d} B_{R_m}(X)} + \sum_{X \in m^{-1/d}\mathbb{Z}^d} q_X,$$

for charges q_X that are supported on $\partial B_{R_m}(X)$. The charge q_X is uniquely determined by the problem

$$\begin{cases} -\Delta v_X = f & \text{in } B_{m^{-1/d}/2}(X) \setminus B_{R_m}(X), \\ v_X = [Q](X) & \text{in } B_{R_m}(X), \\ v_X = u_m & \text{on } \partial B_{m^{-1/d}/2}(X) \end{cases}$$

for every $X \in m^{-1/d}\mathbb{Z}^d$.

Then, since $m^{-1/d}/2$ is much larger than R_m, we approximate this equation by

$$\begin{cases} -\Delta v_X = 0 & \text{in } \mathbb{R}^d \setminus B_{R_m}(X), \\ v_X = [Q](X) & \text{in } B_{R_m}(X), \\ v_X \to u(X_i) & \text{as } |x - X| \to \infty, \end{cases}$$

where v is the accumulation point of v_m. This is the blow-up argument at the core of this method. The approximation makes sense because, macroscopically speaking, $B_{m^{-1/d}/2}(X)$ is still very small. So we can assume that $v_m \approx u(X)$ is a good approximation on $\partial B_{m^{-1/d}/2}(X)$, and the source term f does not play a role in determining the corresponding charge. The solution to 1.2 then solves

$$-\Delta v_x = q_X \quad \text{in } \mathbb{R}^d,$$

where q_X is given by

$$q_X = R_m([Q](X) - v(X))\delta_X^m,$$

with δ_X^m denoting the normalised Hausdorff measure on $\partial B_{R_m}(X)$.

Combining this, we get the ansatz

$$\tilde{u}_m = (-\Delta)^{-1}\left[f - \sum_{X \in m^{-1/d}\mathbb{Z}^d} R_m\big(u(X) - [Q](X)\big)\delta_X^m\right]. \tag{1.2.3}$$

If we believe that $\tilde{u}_m \approx u_m$, we can conclude that

$$u_m \approx \tilde{u}_m = (-\Delta)^{-1}\left[f - \sum_{X \in m^{-1/d}\mathbb{Z}^d} R_m\big(u(X) - [Q](X)\big)\delta_X^m\right] \rightharpoonup G[f - \omega_d(v - Q)],$$

by observing that

$$\sum_{X \in m^{-1/d}\mathbb{Z}^d} R_m\big(u(X) - [Q](X)\big)\delta_X^m \rightharpoonup (d-2)\omega_d(u - Q)$$

by the same argument as in (1.2.2). Since $u_m \rightharpoonup u$, we find that u solves

$$u = G[f - (d-2)\omega_d(u - Q)],$$

or equivalently

$$-\Delta u + (d-2)\omega_d(u - Q) = f.$$

The Stokes equations

In this section, we have introduced four formal derivations of the limit system for the Poisson equation. In the case of the Stokes equations in three dimensions

$$\begin{cases} -\Delta u_m + \nabla p_m = f & \text{in } \mathbb{R}^3 \setminus \bigcup_{X \in \frac{1}{m^{\frac{1}{3}}}\mathbb{Z}^3} B_{R_m}(X), \\ \operatorname{div} u_m = 0 & \text{in } \mathbb{R}^3 \setminus \bigcup_{X \in \frac{1}{m^{\frac{1}{3}}}\mathbb{Z}^3} B_{R_m}(X), \\ u_m = [V]_m(X) & \text{in } B_{R_m}(X) \text{ for every } X \in \frac{1}{m^{\frac{1}{3}}}\mathbb{Z}^3, \end{cases}$$

the same methods can be applied to show that the corresponding limit equations are the Brinkman equations

$$\begin{cases} -\Delta u + 6\pi(u - V) + \nabla p = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

This equation was first derived in [Bri49].

1.3 Fluctuations around the limit

All the approximations introduced in the previous section allow deriving the limit equation for the homogenisation problem (1.2.1) with more or less technical efforts and potentially additional assumptions in the case of randomly distributed particles.

In the case of m holes being independently and identically distributed given a continuous distribution ρ, the screening effect of the limit equation depends on ρ, i.e. the limit equation is given almost surely by

$$-\Delta u + (d-2)\omega_d \rho(u - Q) = f \quad \text{in } \mathbb{R}^d.$$

This equation is deterministic, and so the limiting process can be interpreted as a law of large number. If one also assumes that both the holes and charges are independently and identically distributed according to $f \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R})$, then with the definitions

$$\rho := \int_{\mathbb{R}} f(x, \mathrm{d}q),$$
$$j := \int_{\mathbb{R}} q f(x, \mathrm{d}q)$$

the limiting equation has the form

$$-\Delta u + (d-2)\omega_d(\rho u - j) = f \quad \text{in } \mathbb{R}^d.$$

Understanding the limiting process raises the natural question of higher orders of convergence. In the stochastic setting, this is directly linked to the understanding of the fluctuations around the limit. The approximations derived in the previous setting are not fine enough to see the fluctuations.

It turns out that the approximation $\tilde{u}_m$ obtained from the blow-up method can be refined to cover the fluctuations. This refined approximation will be used in Chapter 2 and Appendix A to derive the central limit theorem scaling for the fluctuations and an explicit formula for the covariance of the Gaussian field describing the fluctuations in three dimensions and under minor technical assumptions on the distribution f. This result holds true both for the Poisson and Stokes equations. Since this result will only be obtained in three dimensions, we continue with the discussion only for $d = 3$ and $R_m = \frac{1}{4\pi m}$. We have rescaled the radius to avoid the factor $\omega_3 = 4\pi$ in the limiting equation.

The crucial idea to obtain the refined approximation is the observation that the fluctuations are still given by a macroscopic object ξ_m, i.e.

$$u_m = u + m^{-1/2}\xi_m + o(m^{-1/2}),$$

where ξ_m is a random function. Inserting this in the approximation $\tilde{u}_m$ derived in (1.2.3) and adapting this to the case of n particles and charges $(X_i, Q_i)_{i=1,\dots,m} \sim f$, we obtain

$$\tilde{u}_m := (-\Delta)^{-1} \left[f - \frac{1}{m} \sum_{i=1}^{m} (u(X_i) - Q_i + m^{-1/2} \xi_m(X_i)) \delta_{X_i}^m \right].$$

We need to define ξ_m. Assuming that we already know that

$$(-\Delta)^{-1} \left[f - \frac{1}{m} \sum_{i=1}^{m} (u(X_i) - Q_i) \right] \approx u = (-\Delta)^{-1} \left[f + j - \rho u \right],$$

we get

$$\begin{aligned} u + m^{-1/2}\xi_m \approx u_m \approx \tilde{u}_m &= (-\Delta)^{-1} \left[f - \frac{1}{m} \sum_{i=1}^{m} (u(X_i) - Q_i + m^{-1/2} \xi_m(X_i)) \delta_{X_i}^m \right] \\ &\approx u + (-\Delta)^{-1} \left[\rho u - j - \frac{1}{m} \sum_{i=1}^{m} (u(X_i) - Q_i) \right] \\ &\quad - (-\Delta)^{-1} \left[\frac{1}{m} \sum_{i=1}^{m} m^{-1/2} \xi_m(X_i) \delta_{X_i}^m \right]. \end{aligned}$$

We cannot use the interpretation of $\frac{1}{m} \sum_{i=1}^{m} m^{-1/2} \xi_m(X_i) \delta_{X_i}^m$ since the resulting approximation is not fine enough. Instead, we may use that $\tilde{u}_m(X_i) = Q_i$ in $B_{R_m}(X_i)$ to obtain

$$\begin{aligned} \tilde{u}_m(X_i) \approx u(X_i) &+ (-\Delta)^{-1} \left[\rho u - j \right] (X_i) - u(X_i) + Q_i - m^{-1/2} \xi_m(X_i) \\ &- (-\Delta)^{-1} \left[\frac{1}{m} \sum_{j \neq i} (u - Q_j + \xi_m(X_j)) \delta_{X_j}^m \right] (X_i). \end{aligned}$$

After requiring $\tilde{u}_m(X_i) = Q_i$, this leads to

$$\begin{aligned} & m^{-1/2} \xi_m(X_i) + (-\Delta)^{-1} \left[m^{-1/2} \rho \xi_m \right] (X_i) \\ &\approx m^{-1/2} \xi_m(X_i) + (-\Delta)^{-1} \left[\frac{1}{m} \sum_{j \neq i} m^{-1/2} \xi_m(X_j) \delta_{X_j}^m \right] (X_i) \\ &= (-\Delta)^{-1} \left[\rho u - j - \frac{1}{m} \sum_{j \neq i} (u(X_j) - Q_j) \delta_{X_j}^m \right] (X_i). \end{aligned}$$

Assuming that equality between the first and last term holds in $\mathbb{R}^d$, leads us to the definition of ξ_m by

$$\xi_m = (-\Delta + \rho)^{-1} \left[\rho u - j - \frac{1}{m} \sum_{j \neq i} (u(X_j) - Q_j) \delta_{X_j} \right].$$

Note that we have replaced $\delta_{X_i}^m$ formally by δ_{X_i} and that $u(X_i)$ is generally not defined since $u \in H^1_{\text{loc}}(\mathbb{R}^3)$ only. In Appendix A, where these technicalities will be addressed, we will replace $u(X_i)$ by the mean of u over $B_{R_m}(X_i)$ to get a well-defined approximation.

We will show in Appendix A that indeed a well-defined version ξ_m is a good approximation for the fluctuation field. Then the fluctuations can be computed by noting that for $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d)$, it holds

$$\begin{aligned}
&\mathbb{E}\left[\xi_m(\varphi_1)\xi_m(\varphi_2)\right] \\
&= m^{-1}\mathbb{E}_m \Bigg[\left(\varphi_1, \sum_{i=1}^m (-\Delta+\rho)^{-1}\left(\rho u - j - (u(X_i) - Q_i)\delta_{X_i}\right)\right)_{L^2(\mathbb{R}^3)} \\
&\qquad\qquad \left(\varphi_2, \sum_{j=1}^m (-\Delta+\rho)^{-1}\left(\rho u - j - (u(X_j) - Q_j)\delta_{X_j}\right)\right)_{L^2(\mathbb{R}^3)}\Bigg] \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3} ((u(x)-v)\cdot((-\Delta+\rho)^{-1}\varphi_1)(x))((u(x)-v)\cdot((-\Delta+\rho)^{-1}\varphi_2)(x)) f(\mathrm{d}x,\mathrm{d}v) \\
&\quad - \left(\rho u - j, (-\Delta+\rho)^{-1}\varphi_1\right)_{L^2(\mathbb{R}^3)} \left(\rho u - j, (-\Delta+\rho)^{-1}\varphi_2\right)_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

1.4 A link to singular stochastic PDEs

The study of fluctuations for the stochastic homogenisation problem of the Poisson or Stokes equations in a perforated domain is linked to stochastic partial differential equations. The fluctuation field for the stochastic homogenisation of the Poisson equation, derived in [HJ22], is the solution to the stochastic partial differential equation

$$-\Delta u + \rho u = \zeta,$$

where ζ is given by

$$\zeta = \left(\int (v-u)^2 f(\cdot,\mathrm{d}v)\right) W$$

and W is a type of white noise.

In Chapter 3, we show that the solution to the linear stochastic PDE

$$(-\Delta+1)u = \Xi \quad \text{in } \mathbb{T}^3$$

can be obtained as the homogenisation limit of the Poisson equation in a perforated domain with large random charges on the holes. We motivate this by the following heuristic: consider holes $\{X_1,\dots,X_m\} = m^{-1/d}\mathbb{Z}^d \cap \mathbb{T}^d$, the radius $R_m = \frac{1}{d\omega_d} m^{-\frac{1}{d-2}}$ and the equation

$$\begin{cases} -\Delta u_m = 0 & \text{in } \mathbb{T}^d \setminus \bigcup_{i=1}^d B_{R_m}(X_i), \\ u_m = m^{1/2} Q_i & \text{in } B_{R_m}(X_i). \end{cases}$$

Assume that the random charges Q_i are given independently and identically distributed by a normal Gaussian. Then, it holds

$$\frac{1}{m}\sum_{i=1}^m m^{1/2} Q_i \delta_{X_i} \longrightarrow \Xi \tag{1.4.1}$$

in law in distributions, where Ξ denotes white noise in $\mathbb{T}^d$. So, Ξ is the isonormal Gaussian process on $L^2(\mathbb{T}^d)$ with mean zero and covariance given by

$$\mathbb{E}[\Xi[\varphi_1]\Xi[\varphi_2]] = \int_{\mathbb{T}^d} \varphi_1\varphi_2 \,\mathrm{d}x.$$

To derive the limit formally, we again make the ansatz

$$u_m \approx \frac{1}{m}\sum_{i=1}^{m}(-\Delta)^{-1}\left[q_i\delta_{X_i}\right]$$

inspired by the monopole and blow-up method. The blow-up method suggests that if there exists a limit u, then $q_i \approx m^{1/2}Q_i - (u)_i$. By (1.4.1), we may then formally conclude that u is a solution to

$$(-\Delta + 1)u = \Xi \quad \text{in } \mathbb{T}^d.$$

1.5 Previous results and further literature

Homogenisation in perforated domains

The mathematical theory of homogenisation in perforated domains can be traced back to the 1940s. Then, for elliptic equations and fluid flows, the collective effect of rarefied sets of inclusions or particles became an area of heuristic and rigorous study. As seminal works in this area, one has to regard the derivation of the Brinkman equations as an effective equation for a swarm of particles in the Stokes flow with a collective effect coming from the Stokes drag by Brinkman [Bri49]. The spectrum of the Laplace operator in a domain with tiny inclusions has been studied as early as in [Sam48].

Building on these seminal results, both the electro- and hydrostatic problem are analysed mathematically at least since the 1970s. Different methods to study the limiting behaviour of the homogenisation problem have been developed and applied in both cases.

For the homogenisation of the Poisson equation, the first rigorous results were obtained in [Hru72], [MK74], [Hru77] and [Hru79] for elliptic equations of higher order. In [MK74] (see [MK08] for an English version), the authors allow for particles randomly distributed without overlapping and with random sizes. The homogenisation limit is then derived by using projection operators in Hilbert spaces.

In [PV80] the corresponding problem for the linear heat equation is studied first. There, probabilistic tools such as the survival time of a Brownian path are used to derive the limit.

Building on the energy introduced by Tartar [Tar09], the method of oscillating test function is derived and used in [CM82a], [CM82b] (see [CM97] for an English version). An earlier version in a special case was already studied in [CP79]. This method is used in many extensions of the results obtained in [CM82a]: see [DG94], [CCL15], [CCL16] and [GHV18] for applications to the homogenisation problem under different assumptions of the distribution of the holes, radii and charges. In [CM09a], the method of oscillating test functions is applied to an obstacle problem on the lattice with obstacles of random shapes. A similar limit of the corresponding obstacle problems involving a collective term is derived.

The method of reflections is applied rigorously to the Poisson equation in [HV18].

Similar screening phenomena are also obtained in the series of papers [Nie99], [NV04b], [NV04a] and [NV06]. There, a dynamical version of the homogenisation problem is considered. The dynamical behaviour of the holes is given in terms of the solution u_m. In [NV04b] and [NV04a], the monopole approximation is introduced to obtain a good approximation employing the maximum principle. In unbounded domains, additional (exponential) screening properties have to be derived [NV06] to obtain the limit.

From the suggestion of the Brinkman equations as an effective equation for the flow around a swarm of particles [Bri49], the first rigorous results in the case of particles with zero velocity on a cubic lattice are obtained in [Bri86], [Lév83] and [San82]. The method of oscillating test functions is applied in different regimes to the Stokes and Navier-Stokes equations with stationary particles in [All90a] and [All90b]. The case of randomly distributed particles with random radii but zero velocity is studied in [Rub86] and [GH19a]. This was also studied via the method of reflections in [Höf21].

The case of particles with non-zero velocity was studied first in [DGR08] for the stationary Stokes and Navier-Stokes equations under a minimal distance condition for the particles. Physically, this corresponds to particles moving very slowly through a fluid. The minimal distance condition is weakened in [Hil18]. The case of particles being randomly distributed is studied in [CH20].

In [HMS19], the more general case of particles of different shapes that are translating and rotating is discussed. In this case, additional terms appear in the homogenisation limit. In [FNN16], the homogenisation problem for the evolutionary Navier-Stokes equations is studied. There, it is assumed that the distance of the particles is still much larger than their diameter. The blow-up method in the case of the Stokes equations is first discussed in [Gér22] and [HJ22].

Besides the qualitative study and somewhat inspired by similar results for the case of homogenisation in random media (see [AKM17] and [DGO20] for two seminal results and the references therein), the quantitative analysis of the limiting process is an important area of interest. A good understanding of convergence rates and approximations might play an essential role in the rigorous derivation of the Vlasov-(Navier)-Stokes equations (see [Bou+15] and the references therein for the modelling of the Vlasov-Stokes equations).

The earliest results on quantitative analysis go back to the study of fluctuations for both the Poisson [FOT85] and Stokes equations [Rub86] via the method of reflections. In both cases, an additional large mass has to be added from the start to obtain the convergence of the method of reflections. Error estimates for the oscillating-test-function method were first obtained in [KM89]. More recently, results on convergence rates and higher-order estimates for homogenisation both of the Poisson and Stokes equations were obtained in [Giu21b], [Fep22], [FJ21] and [Fep21]. The study of fluctuations of the Poisson and Stokes equations with randomly distributed particles and random velocities is obtained as part of this thesis and can be found in [HJ22] or Appendix A.

Stochastic partial differential equations

Singular stochastic partial differential equations have recognised a lot of attention in recent years. The seminal papers that deal with the (parabolic) Φ^4_3-model are [Hai14], [GIP15], [Kup16] and [OW19]. In all these papers, different perturbative renormalisation techniques are used to find a non-Gaussian limit as universality class of the parabolic Φ^4_3-model. The elliptic Φ^4_d-model for dimensions $d = 4, 5$ is studied in [GH19b]. The connection between stochastic homogenisation and stochastic PDEs is, as of yet, unexplored territory.

2 Convergence Rates and Fluctuations for the Stokes–Brinkman Equations as Homogenisation Limit in Perforated Domains

In this chapter, the results obtained in the paper

[HJ22] R. M. Höfer and J. Jansen. "Convergence rates and fluctuations for the Stokes-Brinkman equations as homogenization limit in perforated domains". In: *arXiv:2004.04111 [math]* (2022)

will be summarised. A reprint of the paper can be found in Appendix A.

The research undertaken in the article in question is a collaboration with R. Höfer. All authors and, in particular, the author of this thesis, have contributed significant parts to each section of the work.

2.1 Introduction

Many mathematical models deal with the study of the interaction of moving particles with the flow of a viscous, incompressible fluid. One of the general mathematical goals is the rigorous derivation of a macroscopic model describing the effective dynamics of the system in the limit of many small particles. A special case, neglecting the particle evolution in time and studying the static picture, is the derivation of the Brinkman equations

$$-\Delta u + (\rho u - j) + \nabla p = h, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3. \tag{2.1.1}$$

from a Stokes flow around many small spherical particles

$$\begin{cases} -\Delta u_m + \nabla p_m = h, \quad \operatorname{div} u_m = 0 & \text{in } \Omega_m, \\ u_m = V_i & \text{in } B_{R_m}(X_i),\ i = 1, \dots, m, \end{cases}$$

where

$$\Omega_m = \mathbb{R}^3 \setminus \bigcup_{i=1}^m B_{R_m}(X_i).$$

Here, the particle radius adheres to the critical scaling $R_m = \frac{1}{6\pi m}$ in which the total Stokes drag exerted from the particles is of order one. Furthermore, we assume that the pairs of centres and velocities of the particles (X_i, V_i) are independently and identically distributed according to $f \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfying the assumptions

(H1) $\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(\mathrm{d}x, \mathrm{d}v) < \infty$;

(H2) the distribution of the centres $\rho(\cdot) := \int_{\mathbb{R}^3} f(\cdot, \mathrm{d}v) \in W^{1,\infty}(\mathbb{R}^3)$ is compactly supported;

(H3) the flux is given by $j(\cdot) := \int_{\mathbb{R}^3} v f(\cdot, \mathrm{d}v) \in H^1(\mathbb{R}^3)$.

The additional term $\rho u - j$ appearing in (2.1.1) accounts precisely for the collective, macroscopic effect of the drag force of the particles on the fluid. It is well-known from the theory of stochastic homogenisation that $u_m \rightharpoonup u$ weakly in $\dot{H}^1(\mathbb{R}^3)$, see e.g. [Hil18].

While this can be interpreted as a law-of-large-number-type result, the study of the fluctuations for this limiting problem is a natural question since this also corresponds to a sharper understanding of convergence rates for the limiting process.

2.2 Main results

The main result of the paper gives the complete characterisation of the fluctuations under the assumptions given above in three dimensions. The fluctuations are described by a Gaussian field with explicit covariance.

Theorem 2.2.1 (=Theorem A.1.2). *Let $h \in \dot{H}^{-1}(\mathbb{R}^3)$ and let u_m and u be defined as in* (A.1.3) *and* (A.1.4).

(i) For any $\beta < 1/2$ and any compact set $K \subset \mathbb{R}^3$

$$m^\beta \|u_m - u\|_{L^2(K)} \longrightarrow 0 \quad \textit{in probability.}$$

(ii) For every $g \in L^2(\mathbb{R}^3)$ with compact support,

$$\xi_m[g] := m^{1/2}(g, u_m - u) \longrightarrow \xi[g]$$

in distribution, where ξ is a Gaussian field with mean zero and covariance

$$\begin{aligned}\mathbb{E}[\xi[g_1]\xi[g_2]] = &\int_{\mathbb{R}^3\times\mathbb{R}^3} \big((u(x) - v)\cdot(Ag_1)(x)\big)\big((u(x) - v)\cdot(Ag_2)(x)\big) f(\mathrm{d}x, \mathrm{d}v) \\ &- (\rho u - j, Ag_1)_{L^2} (\rho u - j, Ag_2)_{L^2}\end{aligned}$$

for all $g_1, g_2 \in L^2(\mathbb{R}^3)$ with compact support.

The convergence rate in part (i) of the theorem is optimal in view of part (ii). By interpolating the convergence in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ with the a-priori energy estimates, one obtains convergence in $H^s_{\mathrm{loc}}(\mathbb{R}^3)$ for any $s < 1$ with rate $m^{-\beta+s/2}$.

The proof of this result relies on a new approximation for the sequence $(u_m)_m$. This approximation is obtained by a refinement of the blow-up argument discussed in Section 1.2 also capturing the fluctuations. Therefore, define the approximation $\tilde{u}_m$ by

$$\tilde{u}_m := G\left[h - \frac{1}{m}\sum_{i=1}^m (u - V_i + m^{-\frac{1}{2}}\xi_m)_i \delta_i^m\right],$$

where G denotes the solution operator to the Stokes equations in R^3 and δ_i^m denotes the normalised Hausdorff measure on the sphere $\partial B_{R_m}(X_i)$. We derive that a natural candidate for the approximation of the fluctuation field is given by

$$\xi_m = AG^{-1}\Theta_m$$
$$m^{-\frac{1}{2}}\Theta_m := G(\rho u - j) - \frac{1}{m}\sum_{i=1}^m G^m\left(((u)_i - V_i)\delta_i^m\right).$$

Here A denotes the solution operator to the Brinkman equations (2.1.1). With this explicit approximation $\tilde{u}_m$, it is now possible to derive the fluctuation field ξ by the standard central limit theorem. Note that $\tilde{u}_m$ only relies on the homogenised solution u to (2.1.1), the particle positions and their velocities.

As a first step in the proof, we show that $\tilde{u}_m$ is a good approximation for u in the sense of the following theorem.

Theorem 2.2.2 (=Theorem A.3.1). *For all* $\varepsilon > 0$ *and all* $\beta < 1$

$$\lim_{m\to\infty} \mathbb{P}_m\left[m^\beta \|u_m - \tilde{u}_m\|_{\dot{H}^1(\mathbb{R}^3)} > \varepsilon\right] \to 0.$$

This theorem might be of more general interest regarding the rigorous derivation of the Vlasov-Stokes equations.

The method of the paper is restricted to the case of three and possibly two space dimensions since in four or more space dimensions the error $\|u_m - u\|_{L^2_{\text{loc}}(\mathbb{R}^d)}$ turns out to be of critical or supercritical order. This is reflected by the regularity of the fluctuation field which is only a distribution in dimensions larger or equal than four. Furthermore, the result can be extended to the case of random radii $R_i^m = r_i R_m$ provided the r_i are independent bounded random variables that are also independent of $(Z_i)_{i=1,\dots,m}$ with $\mathbb{E}[r_i] = 1$.

While the above is framed in the context of the Stokes equations, the same result is valid in the case of the Poisson equation

$$\begin{cases} -\Delta u_m + = h & \text{in } \Omega_m, \\ u_m = Q_i & \text{in } B_{R_m}(X_i),\ i = 1,\dots,m, \end{cases}$$

under the same conditions for the random distribution of charges Q_i on the spherical inclusions $B_{R_m}(X_i)$ with

$$R_m = \frac{1}{4\pi m}.$$

In this case, the corresponding homogenised equation is given by

$$-\Delta u + \rho u - j = h \quad \text{in } \mathbb{R}^3.$$

3 Stochastic PDEs as Homogenisation Limits in Perforated Domains

Abstract

In this chapter, the link between stochastic homogenisation in perforated domains and stochastic partial differential equations is investigated. The study of the fluctuation field suggests that solutions to linear elliptic stochastic partial differential equations can be obtained as limits of the corresponding homogenisation problems with large boundary values. This hints at a possible connection also for the nonlinear Φ^4_d-model.

In the first part of this section, we give a sketch of the homogenisation of the semi-linear Poisson equation in a perforated domain with deterministic boundary conditions in the torus $\mathbb{T}^d$, $d \geq 3$. Furthermore, we prove that the solution to the linear stochastic partial differential equation $(-\Delta+\lambda)u = \Xi$ in $\mathbb{T}^3$ is the limit of a homogenisation problem in perforated domains with charges of order $m^{1/2}$. In the second part, we conjecture that under a specific choice of the probability space, this convergence can be improved to pathwise convergence in the space of optimal regularity in any dimension. Building on this, it is conjectured that the elliptic Φ^4_d-model can be obtained as a homogenisation limit.

3.1 Introduction

In this chapter, we explore the link from stochastic homogenisation in perforated domains to stochastic partial differential equations. The fluctuation field obtained in [HJ22] for the homogenisation of the Poisson equation in a randomly perforated domain is the solution to the homogenised equation including a white Gaussian noise ζ coming from the interactions of particles.

$$-\Delta + \lambda u = \zeta, \tag{3.1.1}$$

$\lambda > 0$. One can also take the opposite perspective and find that solutions to elliptic stochastic partial differential equations can be obtained as homogenisation limits with large charges on the inclusions.

We explore this and explain the rigorous results available so far. A general theory is, as of yet, undiscovered, but the available heuristics point out a path into this new territory. While the study of fluctuations gives first results on the description of the linear theory, one of the main areas of interest are nonlinear (singular) stochastic partial differential equations such as the elliptic Φ^4_d-model, see [GH19b]. Due to the long-range interactions of the particles in the perforated medium, it is a natural conjecture that these non-Gaussian universality classes can be obtained from stochastic homogenisation. Unfortunately, a rigorous argument is at the moment not available.

This introductory section now proceeds with a discussion of semilinear elliptic equations in perforated domains with deterministic boundary data. It is argued that the nonlinear structure remains untouched in the critical scaling of the perforated domain. Furthermore, we explain the known results on the linear elliptic stochastic PDE that one can obtain from the study of fluctuations in [HJ22].

In Section 3.2, we explain the key ideas that are necessary to obtain a pathwise theory of the convergence to the linear elliptic stochastic equation (3.1.1). We close the section with the main conjectures and present an idea of a proof. These conjectures are also the starting point for the development of the corresponding nonlinear Φ^4_d-theory from perturbative arguments. The key features of the Φ^4_d-model are discussed in Section 3.3. This section closes with the main conjecture of the construction of a solution to the elliptic Φ^4_d-model via stochastic homogenisation in perforated domains.

Semilinear elliptic equations with deterministic boundary data

The results obtained in this thesis for the problem of homogenisation in perforated domains have considered linear equations. The derivation of an effective theory transfers to the case of semilinear equations as we will demonstrate in the following. We discuss these semilinear equations on the d-dimensional torus to allow for general charge distributions on the holes and to avoid the technical problem of the holes intersecting the boundary of the domain.

Fix $d \geq 3$ and let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus. We identify $\mathbb{T}^d$ with $[0,1]^d$.

We consider a deterministic and countable set $\Phi \subset \mathbb{R}^d$ of particles so that Definition 1.1.1 is satisfied, e.g. $\Phi = \mathbb{Z}^d$. More generally, as in [GHV18], Φ could be given by a generic configuration of a stationary point process for which the average number of points in a domain of order one is bounded and which is strongly mixing (see the assumptions of [GHV18] for more details).

Define the perforated domain

$$\Omega_m = \mathbb{T}^d \setminus \bigcup_{X \in m^{-1/d}\Phi \cap [0,1)^d} B_{R_m}(X),$$

where the radius of the holes is given as before by the critical scaling

$$R_m = \frac{1}{(d-2)\omega_d} m^{-\frac{1}{d-2}},$$

and $\omega_d = \mathcal{H}^{d-1}(\partial B_1(0))$ is the surface area of the unit sphere.

Let $p < \frac{d+2}{d-2} = 2^* - 1$, where $2^* = \frac{2d}{d-2}$ denotes the critical Sobolev exponent. Fix a distribution of charges $q \in H^1(\mathbb{T}^d)$. We study the equation

$$\begin{cases} -\Delta u_m + |u_m|^{p-1} u_m = 0 & \text{in } \Omega_m, \\ u_m \text{ is periodic }, & \\ u_m = q & \text{in } B_{R_m}(X) \text{ for every } X \in m^{-1/d}\Phi. \end{cases} \tag{3.1.2}$$

There is no obstruction in also adding a source term $f \in H^1(\mathbb{T}^d)'$. We omit this source term to focus on the main novelty. Note that (3.1.2) is the Euler–Lagrange equation of the functional

$$\mathcal{F}_m(u) := \int_{\mathbb{T}^d} |\nabla u|^2 + |u|^{p+1} \, \mathrm{d}x,$$

defined on the set

$$\mathfrak{X}_m = \left\{ u \in H^1(\mathbb{T}^d) : u = q \text{ in } B_{R_m}(X) \text{ for every } X \in m^{-1}\Phi \right\}.$$

Since $p \leq 2^* - 1$ and using $u = q$ as a competitor, this implies the uniform bound

$$\|u_m\|_{H^1(\mathbb{T}^d)} \lesssim \mathcal{F}_m(u_m) = \min_{u \in \mathfrak{X}_m} \leq \mathcal{F}_m(q) \lesssim \|q\|_{H^1(\mathbb{T}^d)}.$$

This uniform bound implies that there is $u_h \in H^1(\mathbb{T}^d)$ a weak accumulation point of the sequence $(u_m)_{m\in\mathbb{N}}$. To study the limit behaviour via the method of oscillating test functions, we introduced the definitions $g(s) = |s|^{p-1}s$ and $v_m = u_m - q$. Then, the semilinear homogenisation problem (3.1.2) is equivalent to the equation

$$\begin{cases} -\Delta v_m + g(v_m + q) = \Delta q & \text{in } \Omega_m, \\ v_m \text{ is periodic }, \\ v_m = 0 & \text{in } B_{R_m}(X) \text{ for every } X \in m^{-1/d}\Phi. \end{cases}$$

Since the sequence $(v_m)_m$ is also uniformly bounded in $H^1(\mathbb{T}^d)$, it also has a weak accumulation point v_h.

By the assumptions on the set of centres of the holes Φ, there exists a sequence of oscillating test functions $(w_m)_m \subset H^1(\mathbb{T}^d)$. Recall that $(w_m)_m$ has the following properties

(A1) $w_m \in H^1(\mathbb{T}^d)$;

(A2) $w_m = 0$ in $B_{R_m}(X)$ for every $X \in m^{-\frac{1}{d}}\Phi$;

(A3) $w_m \rightharpoonup 1$ weakly in $H^1(\mathbb{T}^d)$;

(A4) there is a measure $\mu \in W^{-1,\infty}(\mathbb{T}^d)$ on $\mathbb{T}^d$ such that for every sequence $\tilde{v}_m$ with $\tilde{v}_m = 0$ in $B_{R_m}(X)$ for every $X \in m^{-1/d}\Phi$ and $\tilde{v}_m \rightharpoonup \tilde{v}$ weakly in $H^1(\mathbb{T}^d)$, it holds

$$\langle -\Delta w_m, \varphi \tilde{v}_m \rangle \longrightarrow \langle \mu, \varphi \tilde{v} \rangle$$

for all $\varphi \in C^\infty(\mathbb{T}^d)$.

For the proof of these properties for the case of $\Phi = \mathbb{Z}^d$, we refer the reader to [CM82a]. For the case of a generic configuration of certain point processes, see [GHV18]. In both cases, it holds $\mu = 1$ due to the explicit choice of the radii R_m and the stationarity of the point process.

Now, choose $\varphi \in C^\infty(\mathbb{T}^d)$. We use $w_m\varphi \in H^1_0(\Omega_m)$ as a test function to obtain

$$\int_{\mathbb{T}^d} \nabla v_m \cdot \nabla(w_m\varphi) + g(v_m + q) w_m \varphi \,\mathrm{d}x = -\int_{\mathbb{T}^d} \nabla(w_m\varphi) \cdot \nabla q \,\mathrm{d}x.$$

We now take the limit: as in the linear case, it holds

$$\int_{\mathbb{T}^d} \nabla v_m \cdot \nabla(w_m\varphi)\,\mathrm{d}x \longrightarrow \int \nabla v_h \cdot \nabla\varphi + v_h\varphi \,\mathrm{d}x \text{ and}$$
$$\int_{\mathbb{T}^d} \nabla(w_m\varphi) \cdot \nabla q \,\mathrm{d}x \longrightarrow \int_{\mathbb{T}^d} \nabla\varphi \cdot \nabla q \,\mathrm{d}x$$

as $m \to \infty$. For the nonlinear term, we may use the compactness of the Sobolev embedding $H^1(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$ for $q < 2^*$ and the continuity of g to obtain that $g(v_m + q)$ converges strongly to $g(v_h + q)$ in $L^{\frac{2^*}{2^*-1}}(\mathbb{T}^d) = L^{\frac{2d}{d+2}}(\mathbb{T}^d)$. With this, we directly conclude that

$$\int_{\mathbb{T}^d} g(v_m + q) w_m \varphi \,\mathrm{d}x \longrightarrow \int_{\mathbb{T}^d} g(v_h + q)\varphi \,\mathrm{d}x$$

as $m \to \infty$, since $w_m \rightharpoonup 1$ in $H^1(\mathbb{T}^d)$ implies that $w_m \rightharpoonup 1$ in $L^{2^*}(\mathbb{T}^d)$ and $\frac{1}{2^*} + \frac{2^*-1}{2^*} = 1$.

We conclude that v_h is a weak solution to

$$-\Delta v_h + v_h + g(v_h + q) = \Delta q \quad \text{in } \mathbb{T}^d.$$

Writing $u_h = v_h + q$, we find that u_h is a weak solution to

$$-\Delta u_h + (u_h - q) + |u_h|^{p-1} u_h = 0 \quad \text{in } \mathbb{T}^d.$$

We summarise this result in the following theorem.

Theorem 3.1.1. *Let $d \geq 3$, $\Phi = \mathbb{Z}^d$, $q \in H^1(\mathbb{T}^d)$ and $f \in L^2(\mathbb{T}^d)$. Let $p < \frac{d+2}{d-2}$. Then, the weak solution $u_m \in H^1(\mathbb{T}^d)$ to the semilinear Poisson equation in the perforated domain*

$$\begin{cases} -\Delta u_m + |u_m|^{p-1} u_m = f & \textit{in } \mathbb{T}^d \setminus \bigcup_{X \in m^{-1/d}\mathbb{Z}^d} B_{R_m}(X), \\ u_m \textit{ is periodic}, & \\ u_m = q & \textit{in } B_{R_m}(X) \textit{ for every } X \in m^{-1/d}\mathbb{Z}^d \end{cases} \tag{3.1.3}$$

converges weakly in $H^1(\mathbb{T}^d)$ to a solution $u_h \in H^1(\mathbb{T}^d)$ of the semilinear Poisson equation

$$-\Delta u_h + (u_h - q) + |u_h|^{p-q} u_h = f \quad \textit{in } \mathbb{T}^d.$$

The same result holds true almost surely for the choice of holes given by a stationary point process with additional assumptions as in [GHV18].

In the last section of this chapter, we conjecture that the fluctuations for the homogenisation limit of the semilinear equation (3.1.3) in a randomly perforated domain in dimension $3 \leq d \leq 5$ are non-Gaussian and there is a link to the study of elliptic singular stochastic partial differential equations.

The linear elliptic stochastic PDE

In [HJ22], the fluctuations of the linear homogenisation problem for the Poisson equation in a randomly perforated domain are studied. The explicit form of the fluctuation field implies that, formally, the fluctuation field is given by the solution to the stochastic partial differential equation

$$-\Delta u + \rho u = \zeta \quad \text{in } \mathbb{R}^3,$$

where ζ is given by

$$\zeta = \left(\int (v - u)^2 f(\cdot, \mathrm{d}v) \right) W,$$

where W is a type of white Gaussian noise.

The same method that was developed in [HJ22] can be applied in the setting of the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Assume that $Z_i = (X_i, Q_i)$ are independently and identically distributed according to $f = \rho \otimes \eta \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R})$. For simplicity, we assume that

$$\rho = \int_{\mathbb{R}} f(\cdot, \mathrm{d}v) = 1,$$

i.e. the centres of the holes are uniformly distributed. Furthermore, assume that

$$j = \int_{\mathbb{R}} q f(\cdot, \mathrm{d}q) = 0$$

and

$$\int_{\mathbb{T}^3 \times \mathbb{R}} |q|^2 f(\mathrm{d}x, \mathrm{d}q) = 1.$$

We set the radius of the holes to

$$R_m = \frac{1}{4\pi m},$$

and consider the equation given by

$$\begin{cases} (-\Delta + 1)u_m = 0 & \text{in } \mathbb{T}^3 \setminus \bigcup_{i=1}^m B_{R_m}(X_i), \\ u_m = m^{1/2} Q_i & \text{in } B_{R_m}(X_i),\ i = 1, \dots, m. \end{cases}$$

By the standard theory of homogenisation in perforated domains, we know that $v_m = m^{-1/2} u_m$ converges to a weak solution to the equation

$$(-\Delta + 2)v = 0 \quad \text{in } \mathbb{T}^3.$$

Then, $v = 0$. By Theorem A.1.2 and for every $\varphi \in L^2(\mathbb{T}^d)$, we obtain that

$$\int_{\mathbb{T}^d} u_m \varphi \, \mathrm{d}x = m^{1/2} \int_{\mathbb{T}^d} (v_m - v)\varphi \, \mathrm{d}x \longrightarrow \xi[\varphi],$$

converges in law to the fluctuation field ξ. Furthermore, ξ is the Gaussian field with mean zero and covariance given by

$$\begin{aligned} \mathbb{E}[\xi[\varphi_1]\xi[\varphi_2]] &= \int_{\mathbb{T}^d \times \mathbb{R}} |v|^2 (-\Delta + 2)^{-1}\varphi_1(x)(-\Delta + 2)^{-1}\varphi_2(x) f(\mathrm{d}x, \mathrm{d}q) \\ &= \int_{\mathbb{T}^d} (-\Delta + 2)^{-1}\varphi_1(x)(-\Delta + 2)^{-1}\varphi_2(x)\, \mathrm{d}x. \end{aligned}$$

But then, ξ is the solution to the equation

$$(-\Delta + 2)\xi = \Xi \quad \text{in } \mathbb{T}^3,$$

where Ξ denotes the standard white noise on $\mathbb{T}^3$, that is the Gaussian isonormal process on $L^2(\mathbb{T}^3)$ with mean zero and variance given by

$$\mathbb{E}[\Xi(\varphi_1)\Xi(\varphi_2)] = \int_{\mathbb{T}^3} \varphi_1 \varphi_2 \, \mathrm{d}x.$$

3.2 The linear stochastic Poisson equation as homogenisation limit

In the introduction we proved that we can obtain solutions to the stochastic Poisson equation

$$(-\Delta + 2)u = \Xi \quad \text{in } \mathbb{T}^3,$$

where Ξ denotes white noise on $\mathbb{T}^3$, as the homogenisation limit of the Poisson equation in a perforated domain with large charges on each inclusion given by

$$\begin{cases} (-\Delta + 1)u_m = 0 & \text{in } \mathbb{T}^3 \setminus \bigcup_{i=1}^m B_{R_m}(X_i), \\ u_m = m^{1/2} Q_i & \text{in } B_{R_m}(X_i),\ i = 1, \dots, m \end{cases}$$

from the study of the fluctuation field in [HJ22]. Given the heuristical discussion of the problem in Section 1.4, the restriction to three dimensions seems unnatural. Secondly, the convergence is very weak: we only argued that u_m converges in law weakly in $L^2(\mathbb{T}^d)$, that is we have the convergence

$$\langle u_m, \varphi \rangle_{L^2(T^d)} \to \langle u, \varphi \rangle_{L^2(\mathbb{T}^d)}$$

in law for every $\varphi \in L^2(\mathbb{T}^d)$

To achieve the goal of constructing solutions to singular stochastic partial differental equations as homogenisation limits, this convergence for the linear problem is not strong enough to use a perturbative approach as in [GH19b]. It is desirable to achieve a pathwise description. Therefore, recall that white noise satisfies $\Xi \in H^{-\frac{d}{2}-}(\mathbb{T}^d)$ almost surely [Ver10]. Hence, since $(-\Delta + 2)^{-1}$ maps $H^s(\mathbb{T}^d)$ to $H^{s+2}(\mathbb{T}^d)$, we find that $u \in H^{(-\frac{d}{2}+2)-}(\mathbb{T}^d)$ almost surely. To obtain a pathwise theory, we would like to show that

$$u_m \longrightarrow u \quad \text{in } H^s(\mathbb{T}^d) \text{ almost surely}$$

for every $s < -\frac{d}{2} + 2$. This is natural in view of Proposition A.3.3. Crucially, $-\frac{d}{2} + 2 > 0$ only if $d \leq 3$. This is the main obstruction for the result in [HJ22] to work in dimensions larger or equal than four, since the fluctuations are not in $L^2_{\text{loc}}(\mathbb{R}^d)$ any longer.

The work on the pathwise theory is not entirely completed. The following discussion leads to two conjectures. The first conjecture concerns the pathwise theory for the linear case. We discuss briefly the obstructions and give a sketch of an approach. In the following section, we introduce the corresponding nonlinear Φ^4_d-theory and conjecture that we obtain the non-Gaussian limit via a homogenisation scheme.

To obtain a pathwise theory, one needs to make additional assumptions. For once, the convergence in law is natural when choosing the charges on different probability spaces for a different number of particles m. To upgrade to a pathwise convergence result, we must work on a common probability space, which is naturally the probability space of white noise. Secondly, even there, we have to make a careful choice of the charges to even have hope of getting the convergence almost surely.

We restrict this discussion to the case of three space dimensions. We denote by

$$\Lambda_m = m^{-1/3}\mathbb{Z}^3 \cap [0,1)^d$$

the lattice inside the torus consisting of m centres for the holes $X_1, \dots, X_m$. For $X_i \in \Lambda_m$, we denote by

$$B_i^m = B_{R_m}(X_i)$$

the ball with radius

$$R_m = \frac{1}{4\pi m}$$

and centre X_i. Furthermore, we denote, for $X_i \in \Lambda_m$ by C_i^m the fundamental domain with centre $x_i \subset C_i^m$, i.e.

$$C_i^m = x_i + C_m \quad \text{and} \quad C_m = \left(-\frac{1}{2m^{1/3}}, \frac{1}{2m^{1/3}}\right).$$

While the assumption of the particles lying on the lattice is merely to simplify the computations, we need to choose the charges more carefully: let $\eta \in C_c^\infty(B_{1/2}(0))$ a standard mollifier, that is $0 \le \eta \le 1$ and $\int_{B_{1/2}(0)} \eta \,\mathrm{d}x = 1$. Then, define

$$\eta_m(x) = m\eta(m^{1/3}x),$$

so that $\eta_m(\cdot - X_i)$ is supported in C_i^m and satisfies $\int_{C_i^m} \eta_m(x - X_i)\,\mathrm{d}x = 1$. Let

$$\Xi_m = \eta_m * \Xi$$

the mollification of white noise. Then, for almost all realisations of Ξ, Ξ_m is a smooth function.

We define the charge on the hole by

$$\sigma_i^{(m)} = \frac{1}{\sqrt{m}} \int_{C_i^{(m)}} \Xi_m \,\mathrm{d}x.$$

Note that $\sigma_i^{(m)}$ is a normal Gaussian field: $\mathbb{E}[\sigma_i^{(m)}] = 0$ and

$$\mathbb{E}[\sigma_i^{(m)}\sigma_j^{(m)}] = \begin{cases} \frac{1}{m}\int_{\mathbb{R}^3} |\eta_m(x)|^2\,\mathrm{d}x = 1 & \text{if } i = j, \\ 0 & \text{if } |X_i - X_j| \ge m^{-4/d}. \end{cases}$$

Note that we allow for short range correlations of the charges to simplify the notation.

Define $u_m \in H^1(\mathbb{T}^3)$ as the unique weak solution to

$$\begin{cases} (-\Delta + 1)u_m &= 0 & \text{in } \Omega_m := \mathbb{T}^3 \setminus \bigcup_{i=1}^m B_i^m, \\ u_m &= m^{\frac{1}{2}}\sigma_i^{(m)} & \text{in } B_i^m. \end{cases}$$

For a fixed realisation of the noise Ξ, we define $u \in H^s(\mathbb{T}^3)$, $s < \frac{1}{2}$, to be the solution to

$$(-\Delta + 2)u = \Xi \quad \text{in } \mathbb{T}^3$$

and $\hat{u}_m \in C^\infty(\mathbb{T}^d)$ to be the solution to

$$(-\Delta + 2)\hat{u}_m = \Xi_m \quad \text{in } \mathbb{T}^3.$$

We conjecture that u_m converges to u almost surely in $H^s(\mathbb{T}^d)$ for every $s < -\frac{d}{2} + 2$.

Conjecture 3.2.1. *There exists a subsequence of u_m (not relabelled) such that for every $s < \frac{1}{2}$ it holds*

$$u_m \longrightarrow u \quad \textit{almost surely in } H^s(\mathbb{T}^3).$$

To prove this conjecture, we will need to obtain a new approximation $\tilde{u}_m$ for u_m. We will demonstrate here that the natural approximation $\tilde{u}_m$ obtained via the blow-up method gives a good approximation for $\hat{u}_m$. This approximation does not seem suitable to obtain the necessary bounds for $u_m - \tilde{u}_m$ since the methods introduced in [HJ22] are giving information on the level of $H^1(\mathbb{T}^d)$, where we do not expect convergence. To obtain bounds in weaker spaces such as $L^\infty(\mathbb{T}^d)$, a natural strategy is to apply the maximum principle and use a similar argument as in [NV04b]. This fine-scale approximation has not been obtained as part of this thesis but is a future direction of research.

Inspired by the blow-up method, we define

$$\tilde{u}_m = G\left[\frac{1}{\sqrt{m}}\sum_{i=1}^m \sigma_i^{(m)}\delta_{X_i} - \left(\int_{C_i^m} \hat{u}_m \,\mathrm{d}x\right)\delta_{X_i}\right].$$

Here, we denote $A = (-\Delta + 2)^{-1}$ the solution operator to the homogenised equation and $G = (-\Delta+1)^{-1}$ the solution operator to the Poisson equation in $\mathbb{T}^3$.

Furthermore, as an intermediate candidate, we also introduce

$$\tilde{v}_m = A\left[\frac{1}{\sqrt{m}}\sum_{i=1}^m \sigma_i^{(m)}\delta_{X_i}\right]. \tag{3.2.1}$$

We prove the following first step for a proof of the conjecture.

Proposition 3.2.2. *For every* $0 \le s < \frac{1}{2}$ *it holds*

$$\mathbb{E}\left[\|\tilde{u}_m - u\|_{H^s(\mathbb{T}^d)}\right] \longrightarrow 0$$

as $m \to \infty$.

Proof. In [GH19b], it is shown that

$$\mathbb{E}\left[\|\hat{u}_m - u\|^2_{H^s(\mathbb{T}^d)}\right] \longrightarrow 0$$

as $m \to \infty$ for every $0 \le s < \frac{1}{2}$. It now suffices to control $\tilde{u}_m - \hat{u}_m$ in $L^2((\Omega,\mathbb{P}); H^s(\mathbb{T}^3))$. First, we use the operator identity

$$A = G - GA$$

to observe that

$$\tilde{u}_m = \tilde{v}_m + \frac{1}{m}\sum_{i=1}^m G\left[\sqrt{m}\sigma_i^m A\delta_{X_i} - \left(\int_{C_i^m} A\Xi_m \,\mathrm{d}x\right)\delta_{X_i}\right].$$

In a first step, we estimate $\tilde{v}_m - \hat{u}_m$. We may use that A is a bounded operator from

$H^s(\mathbb{T}^3)$ to $H^{s-2}(\mathbb{T}^3)$ for every $s \in [0, 1/2)$.

$$
\begin{aligned}
&\mathbb{E}\left[\|\tilde{v}_m - \hat{u}_m\|^2_{H^s(\mathbb{T}^3)}\right] \\
&= \mathbb{E}\left[\left\|A\left(\frac{1}{\sqrt{m}}\sum_{i=1}^m \sigma_i^{(m)}\delta_{X_i} - \Xi_m\right)\right\|^2_{H^s(\mathbb{T}^3)}\right] \\
&\lesssim \mathbb{E}\left[\left\|\frac{1}{\sqrt{m}}\sum_{i=1}^m \sigma_i^{(m)}\delta_{X_i} - \Xi_m\right\|^2_{H^{s-2}(\mathbb{T}^3)}\right] \\
&= \mathbb{E}\left[\left(\sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\left|\frac{1}{\sqrt{m}}\sum_{i=1}^m \sigma_i^{(m)}\varphi(X_i) - \int_{\mathbb{T}^d}\Xi_m\varphi \,\mathrm{d}x\right|\right)^2\right] \\
&= \mathbb{E}\left[\left(\sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\left|\frac{1}{m}\left[\sum_{i=1}^m \int_{C_i^{(m)}}\Xi_m\left(\varphi(X_i)-\varphi\right)\,\mathrm{d}x\right]\right|\right)^2\right].
\end{aligned}
$$

For almost every realisation of the noise, we get the estimate

$$
\begin{aligned}
&\sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\left|\frac{1}{m}\left[\sum_{i=1}^m \int_{C_i^{(m)}}\Xi_m\left(\varphi(X_i)-\varphi\right)\,\mathrm{d}x\right]\right| \\
&\lesssim \frac{1}{m}\sum_{i=1}^m \sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\int_{C_i^{(m)}}|\Xi_m|\,|\varphi(X_i)-\varphi|\,\mathrm{d}x.
\end{aligned}
$$

Since $H^{2-s}(\mathbb{T}^3) \hookrightarrow C^{\frac{1}{2}-s}(\mathbb{T}^3)$ for every $0 \le s < \frac{1}{2}$, it holds

$$
\begin{aligned}
\sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\int_{C_i^{(m)}}|\Xi_m|\,|\varphi(X_i)-\varphi|\,\mathrm{d}x &\le \int_{C_i^{(m)}}|\Xi_m||x-X_i|^{\frac{1}{2}-s}\,\mathrm{d}x \\
&\lesssim m^{-\frac{1}{d}\left(\frac{1}{2}-s\right)}\int_{C_i^{(m)}}|\Xi_m|\,\mathrm{d}x.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\left|\frac{1}{m}\left[\sum_{X_i\in\Lambda_m}\int_{C_i^{(m)}}\Xi_m\left(\varphi(X_i)-\varphi\right)\,\mathrm{d}x\right]\right|\right)^2\right] \\
&\le \mathbb{E}\left[m^{-\frac{1}{d}(1-2s)}\left(\frac{1}{m}\sum_{X_i\in\Lambda_m}\int_{C_i^{(m)}}|\Xi_m|\,\mathrm{d}x\right)^2\right] \\
&\lesssim m^{-\frac{1}{d}(1-2s)}\frac{1}{m^2}\mathbb{E}\left[\left(\int_{\mathbb{T}^d}|\Xi_m|\,\mathrm{d}x\right)^2\right] \\
&\lesssim m^{-\frac{1}{d}(1-2s)}\frac{1}{m^2}\mathbb{E}\left[\int_{\mathbb{T}^d}|\Xi_m|^2\,\mathrm{d}x\right] \\
&\lesssim m^{-\frac{1}{d}(1-2s)}\frac{1}{m},
\end{aligned}
$$

where in the final step we have used

$$
\mathbb{E}\left[|\Xi_m|^2(x)\right] = \int |\eta_m(x)|^2\,\mathrm{d}x = m\int|\eta|^2\,\mathrm{d}x.
$$

We also have to estimate the remainder. We may use that also G is a bounded linear operator from $H^{s-2}(\mathbb{T}^d)$ to $H^s(\mathbb{T}^d)$ and use a similar strategy as before.

$$\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m G\left[\sqrt{m}\sigma_i^m A\delta_{X_i} - \left(\int_{C_i^m} A\Xi_m \,\mathrm{d}x\right)\delta_{X_i}\right]\right\|^2_{H^s(\mathbb{T}^3)}\right]$$
$$= \mathbb{E}\left[\left(\sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\left|\frac{1}{m}\sum_{i=1}^m\left[\int_{C_i^m}\Xi_m(A\varphi)(X_i)\,\mathrm{d}x - \int_{C_i^m}(A\Xi_m)\varphi(X_i)\,\mathrm{d}x\right]\right|\right)^2\right].$$

Now, we may use that, for almost every realisation of white noise, it holds

$$\begin{aligned}&\sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\left|\frac{1}{m}\sum_{i=1}^m\left[\int_{C_i^m}\Xi_m(A\varphi)(X_i)\,\mathrm{d}x - \int_{C_i^m}(A\Xi_m)\varphi(X_i)\,\mathrm{d}x\right]\right|\\ &\le \sup_{\|\varphi\|_{H^{2-s}(\mathbb{T}^3)}=1}\left(\left|\frac{1}{m}\sum_{i=1}^m\left[\int_{C_i^m}\Xi_m(A\varphi)(X_i)\,\mathrm{d}x - \int_{C_i^m}\Xi_m A\varphi\right]\right|\right.\\ &\qquad\left.+\left|\sum_{i=1}^m\left[\int_{C_i^m}(A\Xi_m)\varphi\,\mathrm{d}x - \int_{C_i^m}(A\Xi_m)\varphi(X_i)\right]\right|\right).\end{aligned}$$

We conclude as before, using that $A\varphi$ is Hölder continuous, that

$$\begin{aligned}&\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m G\left[\sqrt{m}\sigma_i^m A\delta_{X_i} - \left(\int_{C_i^m} A\Xi_m \,\mathrm{d}x\right)\delta_{X_i}\right]\right\|^2_{H^s(\mathbb{T}^3)}\right]\\ &\le m^{-\frac{1}{d}(1-2s)}\frac{1}{m^2}\mathbb{E}\left[\int_{\mathbb{T}^d}|\Xi_m|^2 + |A\Xi_m|^2\,\mathrm{d}x\right]\\ &\le m^{-\frac{1}{d}(1-2s)-1}.\end{aligned}$$

This concludes the proof. □

As discussed, both approximations appearing in the previous proof are not good enough in $L^\infty\left(\bigcup_{i=1}^m \partial B_{R_m}(X_i)\right)$ to use the maximum principle and obtain the missing bounds. While the previous discussion was restricted to the case $d = 3$, the preceding proposition can easily be adapted to the case of general dimensions $d \ge 3$. This leads us to the following conjecture.

Conjecture 3.2.3. *Let $d \ge 3$ and $R_m = \frac{1}{(d-2)\omega_d} m^{-\frac{1}{d-2}}$. Consider the solution to*

$$\begin{cases} -\Delta u_m = 0 & \text{in } \mathbb{T}^d \setminus \bigcup_{X_i \in m^{-1/d}\mathbb{Z}^d} B_{R_m}(X_i),\\ u_m = m^{1/2}\sigma_i^m & \text{in } B_{R_m}(X_i),\ X_i \in m^{-1/d}\mathbb{Z}^d,\end{cases}$$

where $\sigma_i^m = \frac{1}{\sqrt{m}}\int_{C_i^m}\Xi_m\,\mathrm{d}x$ as before. Then

$$u_m \longrightarrow u \quad \text{in } H^s(\mathbb{T}^d) \text{ almost surely}$$

for every $s < -\frac{d}{2} + 2$.

3.3 The elliptic Φ^4_d-model via stochastic homogenisation

The study of singular stochastic partial differential equations has been an area of intensive study in recent years. One model equation is the Φ^4_d-model. While usually considered in its parabolic form, the elliptic model shares most of the interesting features but in 2 more space dimensions. The elliptic Φ^4_d-model on the torus $\mathbb{T}^d$ consists of the study of the equation

$$(-\Delta + 2)v + v^3 = \Xi \quad \text{in } \mathbb{T}^d.$$

In dimensions $d \leq 5$, a rescaling argument hints that solutions look locally as the solution to the linear equation

$$(-\Delta + 2)\Phi = \Xi \quad \text{in } \mathbb{T}^d.$$

The main difficulty arises from the low regularity of $\Phi \in H^{(-\frac{d}{2}+2)-}(\mathbb{T}^d)$. Hence, in dimension $d = 4, 5$ it is only a distribution and a perturbative approach around the solution to the linear problem fails since there is no canonical interpretation of the nonlinear terms. To regularise the equation, consider $\Xi_m = \eta_m * \Xi$ as before,

$$(-\Delta + 2)\Phi_m = \Xi_m \quad \text{in } \mathbb{T}^d,$$

and

$$(-\Delta + 2)v_m + v_m^3 = \Xi_m \quad \text{in } \mathbb{T}^d.$$

These functions are well-defined, but the natural approach $v_m = \Phi_m + \psi_m$ fails at first sight, since the covariance of Φ_m^3 diverges as $m \to \infty$ in dimensions $d = 4, 5$. This is the reason for the renormalisation argument: if we replace u_m^3 by $u_m^3 - r_m u_m$ and send $r_m \to +\infty$ as $m \to \infty$ in the correct way, one finds convergence of the sequence u_m to a non-Gaussian limit $u \in \mathcal{D}'(\mathbb{T}^d)$ in distributions for almost every realisation of the noise, see [GH19b]. Formally, u then solves

$$(-\Delta + 2)u + u^3 - \infty u = \Xi \quad \text{in } \mathbb{T}^d \tag{3.3.1}$$

in dimensions $d = 4, 5$ and, in dimension $d = 3$,

$$(-\Delta + 2)u + u^3 = \Xi \quad \text{in } \mathbb{T}^3. \tag{3.3.2}$$

In Section 3.2 we have conjectured a pathwise approximation for Φ via solutions to the homogenisation problem

$$\begin{cases} (-\Delta + 1)u_m = 0 & \text{in } \mathbb{T}^d \setminus \bigcup_{i=1}^m B_{R_m}(X_i), \\ u_m = m^{1/2}\sigma_i^m & \text{in } B_{R_m}(X_i),\ i = 1, \dots, m. \end{cases}$$

It is a natural question to study the limiting behaviour of the semilinear homogenisation problem

$$\begin{cases} (-\Delta + 1)u_m + u_m^3 = 0 & \text{in } \mathbb{T}^d \setminus \bigcup_{i=1}^m B_{R_m}(X_i), \\ u_m = m^{1/2}\sigma_i^m & \text{in } B_{R_m}(X_i). \end{cases}$$

We conjecture that one can obtain the renormalised solutions to (3.3.2) and (3.3.1) as a limit to a renormalised sequence u_m for almost every realisation of white noise in distributions. There are two main reasons for the conjecture: one can show that for the approximation $\tilde{v}_m$ obtained in (3.2.1) it holds

$$\text{Var}\left[\tilde{v}_m(x)^3\right] \longrightarrow +\infty$$

in dimensions $d = 4, 5$ for every $x \in \mathbb{T}^d$ with the same divergence rate as for Var $\left[\Phi_m(x)^3\right]$. In dimension $d = 3$, this variance remains bounded.

The second reason to believe that the sequence $(u_m)_m$ has a non-Gaussian limit comes from the long-range interactions of the holes which is an indicator for a non-Gaussian universality class.

We finish this short discussion with the main conjecture.

Conjecture 3.3.1. *Let $d \in \{3, 4, 5\}$. Then there is a sequence $r_m \geq 0$ with $r_m \to +\infty$ if $d = 4, 5$ such that, for almost every realisation of white noise, the solutions u_m to*

$$\begin{cases} (-\Delta + 1)u_m + u_m^3 - r_m u_m = 0 & \textit{in } \mathbb{T}^d \setminus \bigcup_{i=1}^m B_{R_m}(X_i), \\ u_m = m^{1/2}\sigma_i^m & \textit{in } B_{R_m}(X_i). \end{cases}$$

converge in distributions to a distribution $u \in \mathcal{D}'(\mathbb{T}^d)$ which formally solves

$$\begin{cases} (-\Delta + 2)u + u^3 - \infty u = 0 & \textit{in } \mathbb{T}^d. \end{cases}$$

Part II

Non-Newtonian Thin-Film Equations

4 | Introduction

Abstract

Thin films of incompressible, non-Newtonian, viscous fluids are ubiquitous in nature and technology. Mathematically they are described by degenerate-parabolic equations of second- or fourth-order depending on the influence of gravitational or capillary forces. These equations are derived from a free-boundary Navier–Stokes system via the lubrication approximation. In this chapter, the underlying physical and mathematical effects are described, starting from the role of fluid rheology over a description of the lubrication approximation to the role of the contact angle between fluid and solid and the slip conditions on the fluid-solid interface. The chapter concludes with an outlook on the main results of this part and gives an overview of the mathematical literature on Newtonian and non-Newtonian thin-film equations.

In the study of the dynamical behaviour of fluids, free-boundary problems arise naturally in the description of physical phenomena observed in nature or applied in engineering. These free-boundary problems are usually constituted by coupled systems of partial differential equations in a bulk domain filled by a fluid and equations on a moving boundary. The dynamical behaviour of the boundary is then described by the interaction of the fluid with the media surrounding it. Due to the complex processes that determine the coupling, the resulting equations are inherently nonlinear and have a usually rich dynamical and geometrical structure.

Thin fluid films are a famous example of such free-boundary problems. These are films of incompressible, viscous fluids with a thickness ranging from nanometres to a few micrometres spread on a solid. Both the interface between the fluid film and the surrounding air and the contact line between fluid, solid and air are free boundaries.

Thin fluid films arise naturally in physics, chemistry and biophysics, geology and engineering. The tear film in the human eye and the fluid on the inside of the alveoli in mammal lungs are examples of thin liquid films. Film coating processes like the application of paint or adhesives are used in many technological applications. Thin films of lubricants are used in engineering to protect surfaces or reduce friction. Given the enormous length scales, even lava flows above and underwater can be considered thin films. The ubiquity of thin films in nature and technology also reflects the number of different dynamics and models in physics and mathematics. Typically, the resulting equations depend on the fluid's viscosity, the relation between the acting forces, and additional effects such as thermal effects or the presence of surfactants (see [CM09b] for a review of lubrication theory under the influence of different effects).

Mathematically, thin fluid films are modelled via an asymptotic expansion. The starting point of this expansion is the description of the free-boundary problem via the Navier–Stokes equations in the fluid bulk combined with additional boundary conditions. In the limit of high viscosity (or rather in the limit of a low Reynolds number) and using the lubrication approximation, a closed equation for the height of the thin film can be derived.

The idea of this expansion dates back to Reynolds [Rey86]. Depending on the dominance of gravitational or capillary forces, the resulting equation that describes the film height dynamics is usually quasilinear, degenerate-parabolic and of second- or fourth-order.

In this part of the thesis, we investigate the dynamical behaviour of capillary-driven thin films for fluids with non-Newtonian rheology. In this introductory Chapter 4, we continue with the mathematical modelling of such fluid films on domains with lateral boundaries in Section 4.1. We will discuss the typical constitutive laws for the viscosity in Section 4.2 and give insight into the lubrication approximation in Section 4.3. In Section 4.4, we focus on the behaviour of the triple junction between fluid, solid and air. The movement of this contact point also depends on the boundary condition at the fluid-solid interface. We discuss several slip conditions and the no-slip paradox for Newtonian and certain non-Newtonian fluids in Section 4.5. In Section 4.6, we introduce the energy-dissipation mechanism for the thin-film equation. This mechanism is vital both for the construction of weak solutions and the study of long-time behaviour. The natural energy is also the starting point for studying the gradient-flow structure of thin films. We explore this in Section 4.7. Finally, we give an overview of the literature in Section 4.8.

Chapter 5 consists of a summary of the first main result of this part of the thesis. It concerns the stability and long-time behaviour of power-law and Ellis-law thin films close to a steady state. A reprint of the whole paper can be found in Appendix B.

Finally, in Chapter 6 the gradient-flow structure of power-law thin films for general mobilities is studied. Via a minimising movement scheme, it is shown that positive solutions to the power-law thin-film equation are given by a gradient flow. General weak solutions can then be approximated as limits of gradient-flow solutions to a modified thin-film equation.

4.1 Formulation of the problem

Thin-film models arise in many different forms depending on which forces, effects, and geometries are taken into account. The underlying principle for the derivation of the thin-film model is an asymptotic expansion starting from a full free-boundary Navier–Stokes system. For the sake of clarity of presentation, we focus on the case of one incompressible, viscous, non-Newtonian fluid confined between lateral boundaries and located on top of a flat solid bottom with no-slip condition. Furthermore, we assume the dominance of capillary over gravitational forces and ignore the latter altogether. We ignore thermal effects during the modelling. Finally, we assume that the fluid is homogeneous in one spatial direction.

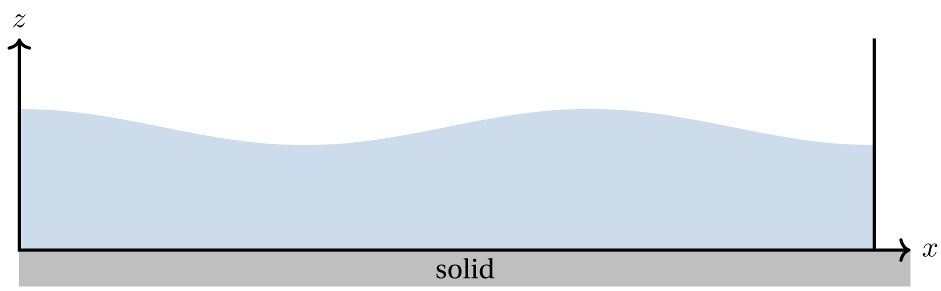

Figure 4.1: Homogeneous (one-dimensional) thin fluid film on solid bottom.

Fix an interval $\Omega = (a, b) \subset \mathbb{R}$. Denote the domain occupied by the fluid at time t by $\Omega(t) \subset \mathbb{R}^2$. We will assume that the free interface is modelled as the graph of a function $h(t, x) > 0$, so that

$$\Omega(t) = \{(x, z) \in \Omega \times \mathbb{R}_{\geq 0} : 0 < z < h(t, x)\}.$$

The velocity field of the fluid is denoted by $\mathbf{u} = (u, v) \colon \Omega(t) \to \mathbb{R}^2$, the fluid pressure by $p \colon \Omega(t) \to \mathbb{R}$ and the Cauchy stress tensor of the fluid by $S(p, \mathbf{u}) \colon \Omega(t) \to \mathbb{R}^{2\times 2}_{\text{sym}}$. The dynamics of the fluid are described by non-Newtonian Navier–Stokes equations

$$\begin{cases} \rho\left(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) &= \operatorname{div} S(p, \mathbf{u}) \quad \text{in } \Omega(t), \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega(t). \end{cases} \tag{4.1.1}$$

Here, ρ denotes the constant density of the fluid. We assume that $S(p, \mathbf{u}) = -p\mathrm{Id} + \sigma(\epsilon) = -p\mathrm{Id} + 2\mu(|\epsilon|)\epsilon$, where $\mu(|\epsilon|)$ denotes the dynamic viscosity of the fluid, $\epsilon = \mathcal{E}u = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ the rate-of-strain tensor and $|\epsilon| = \sqrt{2\,\mathrm{Tr}(|\mathcal{E}u|^2)}$. Then we have $\operatorname{div} S(p, \mathbf{u}) = -\nabla p + \operatorname{div} \mu(|\mathcal{E}\mathbf{u}|)\mathcal{E}\mathbf{u}$. The Cauchy stress tensor and the rheology of the fluid are discussed in more detail in Section 4.2.

The equation for the behaviour of the fluid in the bulk is complemented by boundary conditions both on the solid bottom $\{(x, z) \in \bar{\Omega}(t) : z = 0\}$ and on the free boundary that is described by the graph of the function h denoted by $\Gamma(t) = \{(x, z) \in \bar{\Omega}(t) : z = h(t, x)\}$.

First, on the solid bottom, we prescribe a slip condition describing how fluid particles move with respect to the solid. If adhesive forces dominate cohesive forces on the fluid-solid interface, there is no slip between the fluid and the solid. In this case, we have

$$\mathbf{u} = 0 \quad \text{on } \{(x, z) \in \bar{\Omega}(t) : z = 0\}. \tag{4.1.2}$$

Different slip conditions and their effect on the dynamics of the contact line are discussed in Section 4.5.

We prescribe two more boundary conditions on the free surface $\Gamma(t)$. Denote by

$$\mathbf{t} = \frac{1}{\sqrt{1 + |\partial_x h(t, x)|^2}}(1, \partial_x h(t, x))$$

the unit tangent of the free surface $\Gamma(t)$ and by

$$\mathbf{n} = \frac{1}{\sqrt{1 + |\partial_x h(t, x)|^2}}(-\partial_x h(t, x), 1)$$

its outer unit normal. The first one, the so-called kinematic boundary condition, guarantees that particles that are on the boundary remain at the boundary. Let $V_{\mathbf{n}}$ denote the normal velocity of the interface $\Gamma(t)$. We require

$$\mathbf{u} \cdot \mathbf{n} = V_{\mathbf{n}} \quad \text{on } \Gamma(t).$$

Observing that $V_{\mathbf{n}} = (\partial_t (x, h(t, x))) \cdot \mathbf{n} = \frac{\partial_t h(t,x)}{\sqrt{1+|\partial_x h(t,x)|^2}}$, this equation can be expressed explicitly as

$$\partial_t h(t, x) + u\partial_x h(t, x) = v \quad \text{on } \Gamma(t).$$

The forces exerted on the boundary $\Gamma(t)$ are solely given by the capillary effects coming from surface tension. We assume that the surface tension $\gamma > 0$ of the fluid is constant. Denote by κ the mean curvature of Γ. Then the stress-balance condition at the boundary reads

$$S(p, \mathbf{u})\mathbf{n} = \gamma\kappa\mathbf{n} \quad \text{on } \Gamma(t). \tag{4.1.3}$$

Observe that this implies in particular that there are no tangential forces at the boundary, since $\mathbf{t} \cdot S(p, \mathbf{u})\mathbf{n} = 0$ on $\Gamma(t)$. The equations (4.1.1)–(4.1.3) constitute the complete free-boundary Navier–Stokes system describing the dynamics of the fluid.

Finally, there are two additional constraints for the film height h. Observe that by incompressibility, the mass of the fluid is conserved, i.e. $|\Omega(t)| = |\Omega(0)|$ for every $t > 0$. This implies that the typical (average) film height

$$\bar{h} := \int_\Omega h(t, x)\, \mathrm{d}x, \quad t \geq 0,$$

is constant. Furthermore, the contact angle of the fluid with the lateral wall is another free parameter. This contact angle depends on the thermodynamic equilibrium between the three phases – liquid, solid and air – at the triple junction $\partial\Gamma(t)$. For convenience, we assume that the contact angle is zero, i.e.

$$\partial_x h(t, x) = 0 \quad \text{for } x \in \partial\Omega,$$

and comment on the underlying physics and mathematical consequences in Section 4.4.

4.2 Fluid rheology

A viscous fluid, as opposed to an ideal fluid, is a fluid in which the internal friction between the molecules significantly affects the fluid motion. Internal friction is the force the fluid exerts on itself to resist deformation. Hence, viscous fluids can resist distortion within a characteristic time scale. The mechanical energy exerted on the system is dissipated in the form of heat and cannot be recovered like in elastic materials [Rao14; She18].

A Newtonian fluid is a fluid where the stress $\sigma(\epsilon)$ depends linearly on the strain rate $\epsilon = \mathcal{E}\mathbf{u}$. The constant of proportionality $\mu_0 > 0$ between the stress and strain rate is the viscosity of the fluid. Typical fluids with a Newtonian behaviour are water or usual lubrication oils.

Many fluids have a different behaviour, though. These fluids are called non-Newtonian fluids, and many different effects can occur. We focus on such fluids for which the viscosity $\mu = \mu(|\epsilon|)$ is solely dependent on the strain rate.

Classically, two classes of non-Newtonian fluids with this behaviour can be distinguished: dilatant or shear-thickening fluids, where the fluid becomes more viscous under the exertion of a higher strain. For these fluids, the viscosity is increasing in $|\epsilon|$. A typical dilatant fluid is the mixture of corn starch in water. The second class consists of pseudoplastic or shear-thinning fluids that become less viscous under higher shear rates. The viscosity decreases in $|\epsilon|$. For example, a shear-thinning behaviour can be found in many paints. For the mathematical modelling, constitutive laws on the relation between strain and viscosity are needed. These can, for example, be derived empirically from experimental data. Two important examples in modelling and applications are Ostwald–de Waele fluids and Ellis fluids.

Ostwald–de Waele or power-law fluids are fluids for which the viscosity depends on the rate of strain via a power law relation

$$\mu(|\epsilon|) = K|\epsilon|^{\frac{1}{\alpha}-1}, \quad \alpha \in (0,\infty).$$

$K > 0$ is the consistency index. Such fluids have first been described in [Ost25] and [Wae23]. For flow-behaviour exponents $\alpha > 1$, the fluid is shear-thinning since then $\mu'(|\epsilon|) < 0$. If $0 < \alpha < 1$, the fluid is shear-thickening, and for $\alpha = 1$ the case of a Newtonian fluid is included in this model. In this case, K is equal to the viscosity μ_0 of the Newtonian fluid. The power-law model is widely used in fluid dynamics, albeit it does not cover the observation of constant viscosities at low or high strain rates in (real world) applications.

A model that remedies the issue of the lack of description of such constant viscosities for low strain rates for shear-thinning fluids is the Ellis constitutive law, [MB65],

$$\frac{1}{\mu(|\epsilon|)} = \frac{1}{\mu_0}\left(1 + \left|\frac{\sigma(\epsilon)}{\sigma_{1/2}}\right|^{\alpha-1}\right), \quad \alpha \geq 1.$$

Here, $\mu_0 > 0$ denotes the constant viscosity for small strain rates, and $0 < \sigma_{1/2} < \infty$ denotes the characteristic stress at which the viscosity is reduced to $\mu_0/2$. For $\alpha = 1$ or $\sigma_{1/2} \to \infty$, we recover again the case of Newtonian fluid rheology.

There are many more models of non-Newtonian fluids. For example might the viscous behaviour only appear beyond a certain yield stress. This behaviour is found for example in molten chocolate. The Herschel–Bulkley and the Casson model are typical examples of constitutive laws for yield stress non-Newtonian rheologies [Rao14].

To continue with the lubrication approximation, we assume that

$$\sigma(\epsilon) = \mu(|\epsilon|)\epsilon$$

and that the function $s \mapsto \mu(|s|)s$, $s \in \mathbb{R}$, is monotonically increasing. The relevance of this assumption becomes clear in the lubrication approximation. Note that this property holds true both for power-law and Ellis fluids.

4.3 Lubrication approximation

If the height of the fluid film is very small, then the dynamics of the system can be simplified via an asymptotic expansion with respect to the aspect ratio $\varepsilon = \frac{\bar{h}}{L}$. The limit of vanishing aspect ratio was first studied in 1886 by Reynolds in [Rey86]. We first transform the system of equations (4.1.1)–(4.1.3) into a system of dimensionless equations in dimensionless variables.

Dimensionless variables and the leading-order system

In order to apply asymptotic analysis, we have to non-dimensionalise the system of equations. We will denote by L the characteristic length scale, by $\bar{h}$ the characteristic height of the film. Moreover, u_0 denotes the characteristic horizontal velocity, v_0 the characteristic vertical velocity and p_0 the characteristic pressure. By t_0, we denote the macroscopic time scale of the system and by t_{char} the characteristic time scale of the non-Newtonian fluid. The parameter

$$\varepsilon = \frac{\bar{h}}{L}$$

denotes the aspect ratio of the film. We now introduce the dimensionless variables and unknowns

$$\begin{array}{llll}
\tilde{x} = \dfrac{x}{L}, & \tilde{z} = \dfrac{z}{\bar{h}}, & \tilde{t} = \varepsilon^3 \dfrac{t}{t_0}, & \tilde{h} = \dfrac{h}{\bar{h}}, \\
\tilde{u} = \dfrac{u}{u_0}, & \tilde{v} = \dfrac{v}{v_0}, & \tilde{p} = \dfrac{p}{p_0}, & \tilde{\rho} = 1, \\
\mathrm{Re} = \dfrac{\rho u_0 L}{\mu_0}, & \tilde{\gamma} = \gamma \dfrac{1}{u_0 \mu_0}, & \tilde{\mu}(\tau|\tilde{\varepsilon}|) = \dfrac{1}{\mu_0}\mu(t_{\text{char}}|\varepsilon|) & \tau = \dfrac{t_{\text{char}}}{t_0}\varepsilon^3.
\end{array}$$

Consider the conservation of mass equation $\operatorname{div} \mathbf{u} = 0$. After rescaling, this equation is given by

$$\frac{u_0}{L}\partial_{\tilde{x}}\tilde{u} + \frac{w_0}{\bar{h}}\partial_{\tilde{z}}\tilde{v} = 0 \quad \text{in } \tilde{\Omega}(\tilde{t}).$$

The conservation of mass equation balances if we assume that

$$\frac{u_0}{L} = \frac{w_0}{\bar{h}}.$$

This means that the vertical velocity is very small compared to the velocity in horizontal direction, which implies that the thin film remains thin in times of order one. Furthermore, we make the assumptions

$$u_0 = \frac{L\varepsilon^3}{t_0}, \quad v_0 = \frac{\bar{h}\varepsilon^3}{t_0} = \frac{L\varepsilon^4}{t_0}.$$

Then, we obtain the Reynolds number and the fluid pressure

$$\mathrm{Re} = \frac{\rho L^2 \varepsilon^3}{t_0 \mu_0} \quad \text{and} \quad p_0 = \frac{\mu_0 u_0 L}{\bar{h}^2} = \frac{\mu_0 \varepsilon}{t_0}.$$

Note that since the fluid occupies a very thin layer, we are in the regime of laminar flows so that the assumption of a small Reynolds number is formally justified. First, we compute

$$t_{\text{char}}\mathcal{E}(\mathbf{u}) = \tau \begin{pmatrix} \varepsilon\partial_{\tilde{x}}\tilde{u} & \frac{1}{2}\left(\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\partial_{\tilde{x}}\tilde{v}\right) \\ \frac{1}{2}\left(\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\partial_{\tilde{x}}\tilde{v}\right) & \varepsilon\partial_{\tilde{z}}\tilde{v} \end{pmatrix} =: \tau\tilde{\mathcal{E}}(\tilde{\mathbf{u}}).$$

With these choices, the system of equations in the bulk can be rewritten as

$$\begin{cases} \varepsilon^6 \frac{L}{t_0^2}\left(\partial_{\tilde{t}}\tilde{u} + \tilde{u}\partial_{\tilde{x}}\tilde{u} + \tilde{v}\partial_{\tilde{z}}\tilde{u}\right) &= \frac{\mu_0}{Lt_0}\left(-\varepsilon\partial_{\tilde{x}}\tilde{p} + 2\varepsilon^3\partial_{\tilde{x}}\left[\tilde{\mu}\partial_{\tilde{x}}\tilde{u}\right] + \varepsilon\partial_{\tilde{z}}\left[\tilde{\mu}\partial_{\tilde{z}}\tilde{u}\right] + \varepsilon^3\partial_{\tilde{z}}\left[\tilde{\mu}\partial_{\tilde{x}}\tilde{v}\right]\right), \\ \varepsilon^7 \frac{L}{t_0^2}\left(\partial_{\tilde{t}}\tilde{v} + \tilde{u}\partial_{\tilde{x}}\tilde{v} + \tilde{v}\partial_{\tilde{z}}\tilde{v}\right) &= \frac{\mu_0}{Lt_0}\left(-\partial_{\tilde{z}}\tilde{p} + 2\varepsilon^2\partial_{\tilde{z}}\left[\tilde{\mu}\partial_{\tilde{z}}\tilde{v}\right] + \varepsilon^2\partial_{\tilde{x}}\left[\tilde{\mu}\partial_{\tilde{z}}\tilde{u}\right] + \varepsilon^4\partial_{\tilde{x}}\left[\tilde{\mu}\partial_{\tilde{x}}\tilde{v}\right]\right), \\ \partial_{\tilde{x}}\tilde{u} + \partial_{\tilde{z}}\tilde{v} &= 0. \end{cases}$$

in $\tilde{\Omega}(\tilde{t})$. Since we are only interested in the leading-order system, dividing the first equation by ε and then sending $\varepsilon \to 0$, we obtain the system of equations

$$\begin{cases} \partial_{\tilde{x}}\tilde{p} &= \partial_{\tilde{z}}\left[\tilde{\mu}(\tau|\partial_{\tilde{z}}\tilde{u}|)\partial_{\tilde{z}}\tilde{u}\right], \\ \partial_{\tilde{z}}\tilde{p} &= 0, \\ \partial_{\tilde{x}}\tilde{u} + \partial_{\tilde{z}}\tilde{v} &= 0. \end{cases} \tag{4.3.1}$$

Observe that we used here that, as $\varepsilon \to 0$, $|\tilde{\mathcal{E}}\tilde{u}| \to |\partial_{\tilde{z}}\tilde{u}|$. While (4.3.1) describes the leading-order system in the bulk, we have to non-dimensionalise the boundary conditions and pass to the limit. For the no-slip condition (4.1.2) we obtain

$$\tilde{\mathbf{u}} = 0 \quad \text{on } \{(\tilde{x}, \tilde{z}) : \tilde{z} = 0\}.$$

Note that the kinematic boundary condition remains invariant under rescaling since its rescaled version is given by

$$\varepsilon^3 \frac{\bar{h}}{t_0}\left(\partial_{\tilde{t}}\tilde{h} + \tilde{u}\partial_{\tilde{x}}\tilde{h}\right) = \varepsilon^3 \frac{\bar{h}}{t_0}\tilde{v} \quad \text{on } \tilde{\Gamma}(\tilde{t}).$$

Finally, the stress-balance condition (4.1.3) becomes

$$\begin{cases} \partial_{\tilde{z}}\tilde{u} &= 0 & \text{on } \tilde{\Gamma}(\tilde{t}), \\ \tilde{p} &= -\tilde{\gamma}\partial_{\tilde{x}}^2\tilde{h} & \text{on } \tilde{\Gamma}(\tilde{t}). \end{cases}$$

Derivation of the thin-film equation from the leading-order system

Now that we obtained the complete leading-order system, we can resolve this system to derive a closed equation for the film height h. In the following, we drop the tildes introduced in the previous subsection.

$$\begin{cases} \partial_x p &= \partial_z[\mu(\tau|\partial_z u|)\partial_z u] & \text{in } \Omega(t), \\ \partial_z p &= 0 & \text{in } \Omega(t), \\ \partial_x u + \partial_z v &= 0 & \text{in } \Omega(t), \\ u = v &= 0 & \text{on } z = 0, \\ \partial_t h + u\partial_x h &= v & \text{on } z = h, \\ \partial_z u &= 0 & \text{on } z = h, \\ p &= -\gamma\partial_x^2 h & \text{on } z = h. \end{cases} \tag{4.3.2}$$

First, we note that, using the incompressibility condition, the kinematic boundary condition can be rewritten as

$$\partial_t h(t,x) + \partial_x\left[\int_0^{h(t,x)} u(t,x,z)\,\mathrm{d}z\right] = 0.$$

Hence, it suffices to determine u in terms of h. Since the pressure is constant in z-direction, we obtain that

$$p = -\gamma\partial_x^2 h \quad \text{in } \Omega(t).$$

Together with the first equation of (4.3.2), this leads to

$$-\gamma\partial_x^3 h = \partial_z[\mu(\tau|\partial_z u|)\partial_z u] \quad \text{in } \Omega(t).$$

Since $\partial_z u$ vanishes at $z = 0$, we may integrate from z to h to find

$$\mu(\tau|\partial_z u|)\partial_z u = \gamma(h - z)\partial_x^3 h \quad \text{in } \Omega(t).$$

Since the function $s \mapsto \mu(|s|)s$ is monotonically increasing, we may find a left-inverse $\Psi\colon \mathbb{R} \to \mathbb{R}$ such that $\Psi(\mu(|s|)s) = s$. So, we can resolve the equation for $\partial_z u$ to obtain

$$\partial_z u = \frac{1}{\tau}\Psi\left(\tau\gamma(h-z)\partial_x^3 h\right) \quad \text{in } \Omega(t).$$

Using that u vanishes on the solid bottom, we obtain

$$u = \frac{1}{\tau}\int_0^z \Psi\left(\tau\gamma(h-\zeta)\partial_x^3 h\right)\,\mathrm{d}\zeta \quad \text{in } \Omega(t).$$

Using that

$$\int_0^{h(t,x)}\int_0^z \Psi\left(\tau\gamma(h-\zeta)\partial_x^3 h\right)\,\mathrm{d}\zeta = \int_0^{h(t,x)} \zeta\Psi(\tau\gamma\zeta\partial_x^3 h)\,\mathrm{d}\zeta \quad \text{in } \Omega(t)$$

and inserting this into the kinematic boundary condition, we obtain a closed equation for the film height given by

$$\partial_t h + \partial_x\left[\frac{1}{\tau}\int_0^{h(t,x)} \zeta\Psi(\tau\gamma\zeta\partial_x^3 h)\,\mathrm{d}\zeta\right] = 0.$$

On $\partial\Omega$, we have already required that

$$\partial_x h = 0 \quad \text{on } \partial\Omega.$$

Furthermore, the condition that the mass of the fluid is conserved over time leads to the condition

$$\partial_t \int_\Omega h(t,x)\,\mathrm{d}x = 0, \quad t > 0.$$

Using the divergence theorem, this leads to

$$\int_{\partial\Omega}\int_0^{h(t,x)} \zeta\Psi(\tau\gamma\zeta\partial_x^3 h)\,\mathrm{d}\zeta\,\mathrm{d}\mathcal{H}^0 = 0, \quad t \geq 0.$$

We assume that there is no fluid flow through the lateral boundary. So we require that

$$\int_0^{h(t,x)} \zeta\Psi(\tau\gamma\zeta\partial_x^3 h)\,\mathrm{d}\zeta \quad \text{on } \partial\Omega.$$

Observe that the resulting equation is of fourth-order, degenerate-parabolic, nonlinear and complemented by two boundary conditions, so that the resulting system is a complete description for the dynamics of the free surface at leading order.

In the case of power-law and Ellis-law fluids, Ψ is explicitly given. Recall that for power-law fluids, we have $\mu(|s|)s = K|s|^{1/\alpha-1}s$, so that

$$\Psi = \frac{1}{K^\alpha}|s|^{\alpha-1}s.$$

Then, for power-law fluids, the thin-film equation with a no-slip condition at the solid bottom is given by (after rescaling in time)

$$\begin{cases} \partial_t h + \partial_x\left[h^{\alpha+2}|\partial_x^3 h|^{\alpha-1}\partial_x^3 h\right] &= 0 \quad \text{in } \Omega, \\ \partial_x h = h^{\alpha+2}|\partial_x^3 h|^{\alpha-1}\partial_x^3 h &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

For Ellis fluids with a no-slip condition, we obtain the thin-film equation

$$\begin{cases} \partial_t h + \partial_x \left[h^3 \left(\gamma_E + |h\partial_x^3 h|^{\alpha-1}\right) \partial_x^3 h\right] &= 0 \quad \text{in } \Omega, \\ \partial_x h = h^3 \left(\gamma_E + |h\partial_x^3 h|^{\alpha-1}\right) \partial_x^3 &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here $\gamma_E > 0$ denotes a physical parameter that depends on the constant surface tension, the flow-behaviour exponent $\alpha > 1$ and the characteristic stress $\tau_{1/2}$.

This thesis deals with capillary-driven thin fluid films, i.e. films where the dynamics are governed by the capillary forces on the free surface. If the dynamics are governed instead by gravitational forces, the resulting equations are of second order. For more complicated dynamics, we refer the reader to the review article [ODB97].

A rigorous derivation for the thin-film equation from the Navier–Stokes systems has been obtained in the special case of the Hele-Shaw flow in [GO03].

4.4 Young's law and the contact angle

We have assumed so far that the thin liquid film is enclosed between two lateral walls. Additionally, we have assumed that the angle between the fluid and the lateral wall is $\pm\frac{\pi}{2}$, that is $\partial_x h(t,x) = 0$ for $x \in \partial\Omega$.

Typically though, thin films are not restricted between lateral boundaries but move freely on ideal surfaces. In this case, the boundary of the thin film is given by $\partial\{h > 0\}$, and the dynamics of this contact line between fluid and solid is part of the problem.

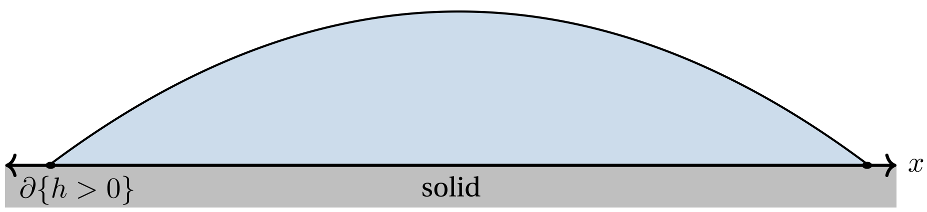

Figure 4.2: Thin droplet on a solid bottom.

The speed V of the contact line is given by the equation, since, if $s(t)$ denotes the contact point at time t and $V(t) = \dot{s}(t)$, it holds

$$0 = \frac{d}{dt} h(t, s(t)) = \partial_t h(t, s(t)) + V(t) h(t, s(t)),$$

and hence

$$V(t) = \lim_{\substack{x \to s(t) \\ x \in \{h>0\}}} \frac{1}{h(t,x)} \frac{1}{\tau} \int_0^{h(t,x)} \zeta \Psi(\tau\gamma\zeta\partial_x^3 h) \, \mathrm{d}\zeta.$$

The contact angle between the fluid and the solid is then given by Young's law [Bon+09; Gen85]. Young's law states that the contact angle θ is given by the equilibrium of the three surface tensions at the triple junction

$$\gamma_{\text{gs}} = \gamma_{\text{ls}} + \cos(\theta)\gamma_{\text{gl}}. \tag{4.4.1}$$

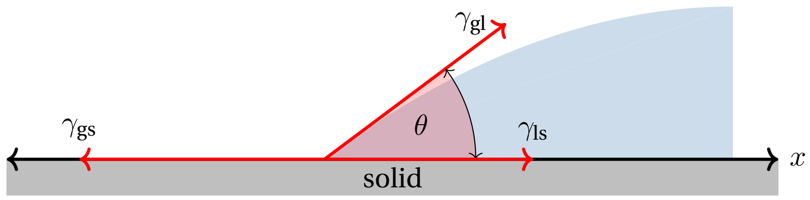

Figure 4.3: Surface tension equilibrium at the contact point.

Here, γ_{gs} denotes the surface tension between gas and solid, γ_{ls} denotes the surface tension between liquid and solid and γ_{gl} denotes the surface tension between gas and liquid. Note that (4.4.1) only has a solution $\theta > 0$ if $\gamma_{\text{gs}} < \gamma_{\text{ls}} + \gamma_{\text{gl}}$. In this case, the contact angle is strictly positive, and the corresponding regime is called *partial wetting*. If, on the other hand, $\gamma_{\text{gs}} \geq \gamma_{\text{ls}} + \gamma_{\text{gl}}$, then $\theta = 0$. A global equilibrium configuration between the surface tensions is not attained. This forces the liquid to spread and cover the solid bottom eventually. This regime is hence called *complete wetting*.

Defining $k = \arctan(\theta)$, the complete free-boundary thin-film equation is then given by the system

$$\begin{cases} \partial_t h + \partial_x \left[\frac{1}{\tau}\int_0^h \zeta\Psi(\tau\gamma\zeta\partial_x^3 h)\,\mathrm{d}\zeta\right] &= 0 & \text{in } \{(t,x) : h(t,x) > 0\}, \\ h &= 0 & \text{on } \partial\{(t,x) : h(t,x) > 0\}, \\ \partial_x h &= k & \text{on } \partial\{(t,x) : h(t,x) > 0\}, \\ \lim\limits_{x\to\partial\{h>0\}} \frac{1}{h}\frac{1}{\tau}\int_0^h \zeta\Psi(\tau\gamma\zeta\partial_x^3 h)\,\mathrm{d}\zeta &= V(t) & \text{for } t > 0 \end{cases}$$

for the unknowns (h, V), where h is again the film height and V denotes the speed of the interface $\partial\{(t,x) : h(t,x) > 0\}$.

It should be noted that Young's law describes an asymptotic regime very close to the contact point. Thus, θ is often called the *microscopic contact angle*. This microscopic contact angle is stationary since it is described by an equilibrium configuration via the capillary forces of the fluid.

In contrast to the microscopic contact angle, at larger scales, one observes a different, dynamic contact angle. In recent years, this *macroscopic contact angle* has been studied in the partial and complete wetting regime. Using a travelling-wave ansatz, the adherence of the macroscopic contact angle to the Cox–Voinov law (see [Cox86; Voi77]) to leading order has been shown in the case of Newtonian thin-films with general slip length (see below), [GW22] . In the case of complete wetting, it has been shown that the macroscopic contact angle to leading order follows Tanner's law (see [Tan79]), [GGO16]. For a derivation of the thin-film equation with a dynamic contact angle following Shikmurzaev's approach [Shi93], we refer the reader to [GNV22].

4.5 Slip conditions and the no-slip paradox

In the modelling, we have so far assumed the special case of the no-slip condition, that is

$$\mathbf{u} = 0 \quad \text{on } z = 0.$$

The fluid molecules are stuck to the surface of the solid. For Newtonian and shear-thickening fluids, the no-slip condition leads to the so-called no-slip paradox. This means that infinite energy would be needed for the contact line between fluid and solid to move, cf. [DD74], [HS71]. To remedy the no-slip paradox, the Navier-slip model [Nav23] is used. In this case, one allows for a free slippage between the solid and the fluid, proportional to the vertical change of the horizontal velocity. This new boundary condition at the solid bottom is

$$u = \lambda \partial_z u, \quad v = 0 \quad \text{on } z = 0.$$

The constant of proportionality $\lambda > 0$ is called the slip length. Reviewing the lubrication approximation, one arrives at a slightly changed equation for non-Newtonian thin-films. In the case of Navier-slip, the equation reads

$$\partial_t h + \frac{1}{\tau}\partial_x \left[\lambda h \Psi(\tau\gamma h \partial_x^3 h) + \int_0^{h(t,x)} \zeta \Psi(\tau\gamma\zeta\partial_x^3 h(t,x))\,\mathrm{d}\zeta \right] = 0.$$

In the case of the power-law thin-film equation, this reduces to

$$\partial_t h + \partial_x \left[\left(\lambda h^{\alpha+1} + h^{\alpha+2} \right) |\partial_x^3 h|^{\alpha-1} \partial_x^3 h \right] = 0.$$

Usually, the film height is assumed to be much smaller than the slip length, so that $\lambda h^{\alpha+1} \gg h^{\alpha+2}$, and one drops the term with $h^{\alpha+2}$ from the equation. For a rigorous justification of the Navier-slip condition, see [JM01].

A generalised version, see e.g. [Gen85] and [Bon+09], of the Navier-slip condition is given by

$$u = \lambda^{3-n} h^{n-2} \partial_z u \quad \text{on } z = 0.$$

In this case and for fluids with non-Newtonian power-law rheology, one obtains the thin-film equation

$$\partial_t h + \partial_x \left[\left(\lambda^{3-n} h^{\alpha-1+n} + h^{\alpha+2} \right) |\partial_x^3 h|^{\alpha-1} \partial_x^3 h \right] = 0.$$

4.6 Energy-dissipation mechanism, steady states and long-time asymptotics

The results of this thesis deal with the case of a thin fluid film confined between two lateral walls. In the case of power-law fluids and a no-slip condition at the solid bottom, the corresponding thin-film equation has the form

$$\begin{cases} \partial_t h + \partial_x \left[h^{\alpha+2} |\partial_x^3 h|^{\alpha-1} \partial_x^3 h \right] = 0 & \text{in } \Omega, \\ \partial_x h = h^{\alpha+2} |\partial_x^3 h|^{\alpha-1} \partial_x^3 h = 0 & \text{on } \partial\Omega, \\ h(0,x) = h_0(x) & \text{in } \Omega, \end{cases} \tag{4.6.1}$$

where $h\colon (0,T) \times \Omega \to \mathbb{R}_{\geq 0}$. Note that the average height of the film is conserved

$$\int_\Omega h(t,x)\,\mathrm{d}x = \int_\Omega h_0(x)\,\mathrm{d}x$$

for all $t \in (0,T)$. Testing the equation with $\partial_x^2 h$, we obtain an energy-dissipation formula for strong solutions to (4.6.1)

$$\frac{1}{2}\int_\Omega |\partial_x h(t,x)|^2\,\mathrm{d}x + \int_0^t \int_\Omega h^{\alpha+2} |\partial_x^3 h|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega |\partial_x h_0|^2\,\mathrm{d}x.$$

For weak solutions obtained from a regularisation scheme, the energy-dissipation formula continues to hold as an inequality. The energy-dissipation mechanism also directly implies that the only non-negative steady-state solutions to (4.6.1) are given by positive constants $h(t,x) = \bar{h}_0 := \fint_\Omega h_0(x)\,\mathrm{d}x$ (see Theorem B.3.5). If the thin film is strictly bounded below, cf. Proposition B.4.1, explicit decay rates of the energy can be deduced from the energy-dissipation and a Łojasiewicz–Simon-type inequality. While in [AG04] only the qualitative result of convergence to the steady state for thin films of shear-thinning power-law fluids is proved, Theorem B.6.1 provides a polynomial decay rate of the form

$$\|h(t,x) - \bar{h}_0\|_{H^1(\Omega)} \leq \frac{C\varepsilon}{\left(1 + C\varepsilon^{\alpha-1}t\right)^{\frac{1}{\alpha-1}}},$$

provided h_0 is initially close to $\bar{h}_0$ in $H^1(\Omega)$. In the case of shear-thickening power-law fluids, it is shown in Theorem B.5.1 that thin films convergence in finite time in $H^1(\Omega)$ to the steady state, assuming that, initially, the profile is close to the equilibrium. Similar results have also been obtained recently for thin films in the cylindrical Taylor–Couette setting in [LV22] and [LPV22].

In the case of Ellis fluids, which are by their nature shear-thinning, the thin-film equation contains an additional Newtonian summand

$$\begin{cases} \partial_t h + \partial_x \left[h^3 \left(\gamma_E + |h\partial_x^3 h|^{\alpha-1}\right)\partial_x^3 h\right] &= 0 & \text{in } \Omega, \\ \partial_x h = h^3 \left(\gamma_E + |h\partial_x^3 h|^{\alpha-1}\right)\partial_x^3 h &= 0 & \text{on } \partial\Omega, \\ h(0,x) &= h_0(x) & \text{on } \Omega, \end{cases}$$

and the equation only degenerates in the film height but not in the third spatial derivative. The corresponding energy-dissipation formula is given by

$$\frac{1}{2}\int_\Omega |\partial_x h(t,x)|^2\,\mathrm{d}x + \int_0^t \int_\Omega \gamma_E h^3 |\partial_x^3 h|^2 + h^{\alpha+2}|\partial_x^3 h|^{\alpha+2}\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega |\partial_x h_0|^2\,\mathrm{d}x.$$

Again, the only non-negative steady states to the Ellis-law thin-film equation are positive constants [LM20]. Close to the equilibrium, the Newtonian effects dominate the dynamics so that the long-time behaviour follows that of the Newtonian thin-film equation for which exponential stability has been observed in [BP96]. In Theorem B.7.5 it is proved that the same exponential stability holds for Ellis fluids, that is

$$\|h(t,x) - \bar{h}_0\|_{H^1(\Omega)} \leq Ce^{-\lambda t},$$

provided h_0 is close to $\bar{h}_0$ in $H^1(\Omega)$.

4.7 Thin-film equations via gradient flows

The physically dominant forces for the thin fluid films investigated in this thesis are capillary forces, i.e. the surface forces given by the surface tension of the fluid. Recall that in the modelling they were introduced via the stress-balance condition at the free surface of the thin film

$$\tau(p,\mathbf{u})\mathbf{n} = \gamma\kappa\mathbf{n} \quad \text{on } \Gamma(t),$$

where $\gamma > 0$ denotes the surface tension and κ the mean curvature. The more the surface is bent locally, the bigger become the local stresses on the surface. These stresses govern the dynamics of the thin film. Surface tension forces the fluid into an equilibrium configuration which is described as a minimum of the length of the thin film (or surface area in two spatial dimensions)

$$\int_\Omega \sqrt{1+|\partial_x h(t,x)|^2}\,\mathrm{d}x,$$

under the constraints given by the contact angle with the lateral boundary or the solid bottom. Note that up to first-order, the length of the film is given by

$$\int_\Omega \sqrt{1+|\partial_x h(t,x)|^2}\,\mathrm{d}x \sim |\Omega| + \frac{1}{2}\int_\Omega |\partial_x h(t,x)|^2\,\mathrm{d}x.$$

This also gives a natural interpretation of the energy used in the previous Section 4.6 as an approximation of the length of the film. It is well-known since the work of Almgren [Alm96] that the Newtonian Hele-Shaw flow

$$\begin{cases} \partial_t h + \partial_x\left[h\partial_x^3 h\right] &= 0 & \text{in } \Omega,\\ \partial_x h = h\partial_x^3 h &= 0 & \text{on } \partial\Omega,\\ h(0,x) &= h_0(x) & \text{in } \Omega \end{cases}$$

is given by a gradient flow with respect to the surface energy

$$\int_\Omega \sqrt{1+|\partial_x h(t,x)|^2}\,\mathrm{d}x.$$

In [GO01] it is shown that the Hele-Shaw flow is a gradient flow with respect to the Dirichlet energy and the metric tensor given by

$$g_h(v_1,v_2) = \int_\Omega h j_1 j_2\,\mathrm{d}x,$$

where $v_1 + \partial_x(hj_1) = 0$ and $v_1 + \partial_x(hj_2) = 0$ with $j_1 = j_2 = 0$ on $\partial\Omega$. $v_1, v_2\colon \Omega \to \mathbb{R}$ are tangent to the film height, that is $\int_\Omega v_1\,\mathrm{d}x = \int_\Omega v_2\,\mathrm{d}x = 0$.

Formally, the Newtonian thin-film equation

$$\begin{cases} \partial_t h + \partial_x\left[h^n\partial_x^3 h\right] &= 0 & \text{in } \Omega,\\ \partial_x h = h^n\partial_x^3 h &= 0 & \text{on } \partial\Omega,\\ h(0,x) &= h_0(x) & \text{in } \Omega \end{cases}$$

with a general mobility $m(h) = h^n$ should then be a gradient flow with respect to the Dirichlet energy and the metric tensor

$$g_h(v_1,v_2) = \int_\Omega \frac{j_1 j_2}{h^n}\,\mathrm{d}x,$$

where $v_1 + \partial_x j_1 = 0$, $j_1 = 0$ on $\partial\Omega$ and $v_2 + \partial_x j_2 = 0$, $j_2 = 0$ in $\partial\Omega$ hold. This metric tensor degenerates for superlinear mobilities, cf. Proposition 6.2.1.

Changing the corresponding dissipation by introducing a regularisation of the mobility $m_\delta(h) \geq \delta$, we construct solutions to the regularised thin-film equation

$$\begin{cases} \partial_t h + \partial_x\left[m_\delta(h)|\partial_x^3 h|^{\alpha-1}\partial_x^3 h\right] &= 0 & \text{in } \Omega,\\ \partial_x h = m_\delta(h)|\partial_x^3 h|^{\alpha-1}\partial_x^3 h &= 0 & \text{on } \partial\Omega,\\ h(0,x) &= h_0(x) & \text{in } \Omega, \end{cases}$$

in Chapter 6, even in the case of non-Newtonian power-law fluids via a minimising movement scheme for the Dirichlet energy with a dissipation functional given by

$$\inf_{\substack{\partial_t u + \operatorname{div} j = 0\\ j\cdot n = 0 \text{ in } \partial\Omega}} \int_\Omega \frac{|j|^{\frac{\alpha+1}{\alpha}}}{m_\delta(u)^{\frac{1}{\alpha}}}\,\mathrm{d}x.$$

4.8 Previous results and further literature

Since the seminal work of Reynolds on lubrication theory [Rey86], thin-film equations have become an important area of mathematical study. In the case of dominance of gravitational forces, the dynamics of the thin film are described by the porous-medium equation (or variances thereof in the case of power-law and Ellis fluids). For this nonlinear degenerate-parabolic equation of second-order, the theory of existence of solutions is seminally studied in [ZK50], [OKJ58] and [Sab61] (see also the monograph [Vaz06] for more references).

NEWTONIAN THIN-FILM EQUATIONS

While the porous-medium equation is a degenerate-parabolic second-order equation, the dynamics of the thin film under the dominance of capillary forces are given by the nonlinear degenerate-parabolic fourth-order equation

$$\partial_t h + \partial_x \left[m(h) \partial_x^3 h \right] = 0,$$

as we have seen in the introduction. While for the porous-medium equation the non-negativity of solutions follows from the maximum principle, such tools are not available for fourth-order equations. Besides the existence of weak solutions to the Newtonian thin-film equation on a domain enclosed by lateral boundaries with general mobility function $m(u) = |u|^n$, $n \geq 1$, the problem of non-negativity is addressed in the seminal paper [BF90] via the introduction of a notion of entropy that guarantees control of certain norms of second derivatives via what is now known as Bernis' estimates. Furthermore, for $n \geq 4$ the uniqueness of non-negative weak solutions is established.

The seminal result of Bernis and Friedman sparked an intensified study of the properties of such solutions. Source-type solutions are studied in [BPW92] and [FB97] for the higher-dimensional case. The finite speed of propagation of the contact line is studied in [Ber96b] and [Ber96a]. Moreover, [BBD95] and [BP96] study regularity, the behaviour of the support and long-time behaviour. In [BBD95] non-uniqueness of non-negative solutions is proved in the case of mobility exponent $n < 3$. A mechanism for non-uniqueness by the self-similar lifting of isolated zeros of the thin-film equation is described in [CKV18]. A waiting-time phenomenon for solutions to the thin-film equation is observed in [DGG01]. Optimal bounds for waiting times are obtained in [Fis14].

The existence of solutions in the case of higher space dimensions is studied in [Ber+98], [PGG98] and [Grü05]. Note that the concept of solutions in higher dimensions is even weaker due to the more limited compactness. In [PGG98], a concept of strong solutions for the higher-dimensional thin-film equation is introduced that is linked to the Bernis' estimates obtained in [BF90] and [Ber96c] for the one-dimensional thin-film equation. For convex domains, strong solutions are shown to exist.

The case of two stratified thin films of immiscible Newtonian fluids is studied in [EMM13], [EM14] and [BG19]. The more involved geometry of a thin film in the setting of Taylor–Couette flows for Newtonian fluids is investigated in [PV20].

A rigorous theory of the lubrication approximation for the Hele-Shaw flow with mobility exponent $n = 1$ is established in the series of papers [Ott00], [GO01] and [GO03]. In [KM15] the lubrication approximation for the Hele-Shaw flow is performed rigorously from Darcy's flow. A rigorous justification of the Hele-Shaw flow in thin threads is obtained in [MP12]. The case of the no-slip boundary condition is rigorously studied in [GP08] from a Stokes flow with surface tension.

These rigorous derivation results formed one entry point into the mathematical study of the full free-boundary problem for Newtonian thin films. In this case, the corresponding system of equations for the film height $h\colon (0,T)\times\mathbb{R}\to\mathbb{R}_{\geq 0}$ and the speed $V\colon (0,T)\times\partial\{h>0\}\to\mathbb{R}$ of the interface $\partial\{h>0\}$ is given by

$$\begin{cases} \partial_t h + \partial_x\left[h^n\partial_x^3 h(t,x)\right] &= 0 \quad \text{in } \{(t,x) : h(t,x)>0\}, \\ h &= 0 \quad \text{on } \partial\{(t,x) : h(t,x)>0\}, \\ \partial_x h &= k \quad \text{on } \partial\{(t,x) : h(t,x)>0\}, \\ \lim\limits_{x\to\partial\{h>0\}} h^{n-1}\partial_x^3 h(t,x) &= V(t) \quad \text{for } t>0. \end{cases}$$

The case $k\neq 0$ of non-zero contact angle is referred to as partial wetting, while the case of zero contact angle $k=0$ is the complete wetting regime. Travelling-wave solutions in different slippage and contact-angle regimes for the Newtonian thin-film equation are studied in [BKO93], [BSB03] and [CG11], as well as in [GKK18] for the case of spin-coating.

From this starting point of travelling-wave solutions, there is a large amount of literature on classical solutions both in the partial and complete wetting regime for different mobility exponents. In the one-dimensional case and using a boundary-layer analysis, this includes the articles [Knü08], [GKO08; Bri+16], [GK10], [GGO13], [Gia+14], [Gna15] and [Gna16] for the case of complete wetting. Here, the contact line dynamics are described asymptotically via a travelling-wave ansatz and then matched to the parabolic behaviour in the bulk. The partial wetting regime is discussed in [Knü11], [Knü15]. The case of two spatial dimensions has so far been mostly studied perturbatively around special solutions [Joh15], [GP18], [Deg17] and [Sei18].

A rigorous investigation of the behaviour of the macroscopic contact angle is conducted in [GGO16], where it is shown that, in the complete wetting regime, the macroscopic contact angle follows Tanner's law to leading order. Similarly, in [GW22], the adherence of the macroscopic contact angle to the Cox–Voinov law to leading order is shown.

The gradient-flow structure of the Hele-Shaw flow has already been observed by Almgren [Alm96]. More recently, a mathematical study of the gradient-flow structure of the Newtonian thin-film equation with mobility exponents $n\leq 1$ has been conducted in [LMS12], using a regularisation and a minimising movement scheme. While in this case the dissipation functional is convex simultaneously in the film height h and the flux j, this fails in the case of superlinear mobility m. This case is studied in detail in Chapter 6 of this work, even in the non-Newtonian power-law case for all flow-behaviour exponents $\alpha>0$. Furthermore, numerical schemes for the gradient-flow structure for general mobilities in the Newtonian case and on different geometries are investigated in[GR00], [RV13] and [Van+17]. The study of the thin-film equation with a dynamic contact angle via discretisations is studied in [PH21].

In [DMS05], it is observed that, taking thermal fluctuations into account, the spreading of thin droplets does not follow Tanner's law and a stochastic version of the thin-film equation is derived. Inspired by the effects of thermal fluctuations in film rupture [ASL04], Grün, Mecke and Rauscher [GMR06] derive a different stochastic thin-film equation with an additional interface potential. Results on existence and positivity of solutions in different stochastic frameworks, such as martingale solutions, are studied in [FG18], [Cor18], [GG20], [Dar+21], [Sau21], [MG21] and [GK21].

Finally, thin films with surfactants acting on the surface tension of the fluids have been an area of mathematical investigation. The corresponding dynamics change due to the Marangoni effect, and the resulting model consists of a coupled system of a thin-film equation with the concentration of the surfactant. This coupled model is studied in [GW06], [Esc+12], [EL18], [Bru17] , [BG20] and [Bru16].

Non-Newtonian thin-film equations

The literature on non-Newtonian thin-film equations is sparser, and few phenomena have yet been studied in detail. Since [WS94] and [AG02] it has been known that shear-thinning Ellis fluids do not exhibit the contact line paradox for fluids with no-slip boundary condition. General asymptotical regimes of the doubly nonlinear power-law thin-film equation are studied in [Kin01a] and [Kin01b].

The seminal rigorous work on shear-thinning power-law thin-film equations in one dimension

$$\begin{cases} \partial_t h + \partial_x \left[h^n |\partial_x^3 h|^{\alpha-1} \partial_x^3 h\right] &= 0, \quad \text{in } \Omega, \\ \partial_x h = h^n |\partial_x^3 h|^{\alpha-1} \partial_x^3 h &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

for $\alpha > 1$, is the work by Ansini and Giacomelli [AG04]. Note that this equation is doubly-degenerate since it degenerates both in the film height h and the third spatial derivative of h. They use a Galerkin approximation with a double regularisation scheme to guarantee both global-in-time existence of weak solutions and non-negativity via a refined entropy approach. Furthermore, many qualitative properties such as long-time behaviour, finite speed of propagation and a waiting-time phenomenon are studied, and it is shown that many results from the Newtonian case transfer to the case of shear-thinning power-law fluids.

Local-in-time strong solutions for the Ellis-law thin-film equation are studied in the framework of semigroups in [LM20]. Furthermore, the thin-film equation for power-law fluids in the setting of a two-phase Taylor–Couette geometry is studied in [LPV22] for shear-thickening rheology and in [LV22] for shear-thinning rheology. Besides existence of weak solutions, the long-time asymptotics of solutions close to a steady state are analysed using the energy-dissipation inequality.

The literature on gradient flows for non-Newtonian thin-film equations is even sparser. We mention [BB20] and [BB22] for works on asymptotic profiles in gradient flows of fourth-order evolution equations.

5 Long-Time Behaviour and Stability for Quasilinear Doubly Degenerate-Parabolic Equations of Higher Order

This chapter is a summary of the results obtained in the paper

[JLN22] J. Jansen, C. Lienstromberg, and K. Nik. "Long-time behaviour and stability for quasilinear doubly degenerate parabolic equations of higher order". In: *arXiv:2204.08231 [math]* (2022)

A reprint of the paper can be found in Appendix B.

The research undertaken in the paper in question is a collaboration with C. Lienstromberg and K. Nik. All authors and, in particular, the author of this thesis have contributed significant parts to each section of the work.

5.1 Introduction

As discussed in Section 4.7, the capillary forces implemented in thin-film model compel the length of the thin-film to equilibrate under the constraints given by the contact angle at the lateral boundary. This effect is regardless of the rheology of the fluid. Since [AG04, Theorem 1] it has been known that for the shear-thinning power-law thin-film equation

$$\begin{cases} \partial_t u + \partial_x \left[u^{\alpha+2}|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\right] = 0, & t>0,\ x\in\Omega, \\ \partial_x u = u^{\alpha+2}|\partial_x^3 u|^{\alpha-1}\partial_x^3 u = 0, & t>0,\ x\in\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{5.1.1}$$

with $\alpha > 1$, solutions converge uniformly to their average $\bar{u}_0 = \fint_\Omega u(t,x)\,\mathrm{d}x$ for non-negative initial data $u_0 \in H^1(\Omega)$ with $|x|^{3/2(\alpha+1)}\partial_x u_0 \in L_2(\Omega)$. Note that the average film height is conserved by the equation, so $\bar{u}_0$ does not depend on time.

For the Newtonian thin-film equation, the long-time behaviour is studied qualitatively in [BF90] and quantitatively in [BP96]. In the latter paper, exponential decay to the equilibrium configuration $\bar{u}_0$ in $H^1(\Omega)$ is shown. Using semigroup theory, see e.g. [Lun12] or [HI11], in the Newtonian case, the exponential decay can even be shown in smaller function spaces.

While in the shear-thinning case at least the qualitative theory has been studied before, in the shear-thickening case $\alpha < 1$, no results have been obtained previously.

Due to the Newtonian plateau at low stresses, Ellis-law fluids close to equilibrium are expected to behave like Newtonian fluids. The corresponding thin-film equation with the flow-behaviour exponent $\alpha > 1$ is given by

$$\begin{cases} \partial_t u + \partial_x \left[u^3\big[1+|u\partial_x^3 u|^{\alpha-1}\big]\partial_x^3 u\right] &= 0, \qquad t>0,\ x\in\Omega, \\ \partial_x u = u^3\big[1+|u\partial_x^3 u|^{\alpha-1}\big]\partial_x^3 u &= 0, \qquad t>0,\ x\in\partial\Omega, \\ u(0,x) &= u_0(x), \quad x\in\Omega. \end{cases}$$

5.2 Main results

Power-law fluids

Now we turn to the main results. Weak solutions had previously been only studied for shear-thinning power-law fluids using a Galerkin approximation. Since we are interested in the behaviour close to steady states, the first result concerns the existence of positive weak solutions also for shear-thickening fluids. Via regularisation of the nonlinearity involving $\partial_x^3 h$, standard semigroup theory for the regularised equation, the energy-dissipation mechanism and Minty's trick, positive weak solutions are constructed.

Theorem 5.2.1 (=B.3.2). *Fix $\alpha > 0$. Given a positive initial value $u_0 \in W^{4\rho}_{\alpha+1,B}(\Omega)$, $4\rho > 3 + 1/(\alpha+1)$, with $u_0(x) > 0,\ x \in \bar{\Omega}$, there exists a time $T > 0$ such that problem* (5.1.1) *admits at least one positive weak solution*

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$$

on $(0,T)$ in the sense of Definition B.3.1. Moreover, such a solution has the following properties:

(i) (Positivity) u is bounded away from zero, i.e. there is a constant $C_T > 0$ such that

$$0 < C_T \leq u(t,x), \quad 0 \leq t \leq T,\ x \in \bar{\Omega}.$$

(ii) (Conservation of mass) u conserves its mass in the sense that

$$\|u(t)\|_{L_1(\Omega)} = \|u_0\|_{L_1(\Omega)}, \quad 0 \leq t \leq T.$$

(iii) (Energy-dissipation identity) Energy is dissipated along solutions

$$E[u](t) + \int_0^t D[u](s)\,ds = E[u_0]$$

for almost every $t \in [0,T]$.

Here

$$E[u](t) = \frac{1}{2}\int_\Omega |\partial_x u(t,x)|^2\,\mathrm{d}x$$

denotes the energy and

$$D[u](t) = \int_\Omega u(t,x)^{\alpha+2}|\partial_x^3 u(t,x)|^{\alpha+1}\,\mathrm{d}x$$

the dissipation functional at time t. Using bootstrapping, such local solutions can be extended to maximal positive solutions. Due to the degeneracy of the equation, solutions cannot be expected to be unique, though.

Starting from the energy-dissipation mechanism, a Łojasiewicz–Simon-type inequality is derived for strictly positive solutions, implying that

$$\frac{d}{dt}E[u](t) = -D[u](t) \leq -C\big(E[u](t)\big)^{\frac{\alpha+1}{2}}.$$

Using this and a bootstrapping arguments, global solutions and explicit convergence rates to the equilibrium are proved, provided the initial datum is close to equilibrium.

Theorem 5.2.2 (=Theorems B.5.1 and B.6.1). *Fix $\alpha > 0$. Then there exists an $\varepsilon > 0$ such that, for all positive initial values $u_0 \in H^1(\Omega)$ with $\|u_0 - \bar{u}_0\|_{H^1(\Omega)} \leq \varepsilon$, problem* (B.1.1) *possesses at least one global positive weak solution*

$$u \in C\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1,loc}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha},loc}\big((0,\infty); (W^1_{\alpha+1,B}(\Omega))'\big),$$

satisfying the boundary condition $u_x = 0$ on $\partial\Omega$ pointwise for almost every $t \geq 0$. Moreover, this global solution has the following asymptotic behaviour:

(i) In the shear-thickening case $0 < \alpha < 1$, there exists a positive but finite time $0 < t^* < \infty$ such that

$$u(t,\cdot) \longrightarrow \bar{u}_0 \text{ in } H^1(\Omega), \text{ as } t \to t^*, \quad \text{and} \quad u(t,x) = \bar{u}_0, \quad t \geq t^*,\ x \in \Omega.$$

(ii) In the shear-thinning case $1 < \alpha < \infty$, there exists a constant $C > 0$ such that

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \leq \frac{C\varepsilon}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{1}{\alpha-1}}}, \quad 0 \leq t < \infty.$$

(iii) In the Newtonian case $\alpha = 1$, there exist positive constants $C, \gamma > 0$ such that

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \leq Ce^{-\gamma t}, \quad 0 \leq t < \infty.$$

Ellis fluids

Finally, the asymptotic behaviour of Ellis-law fluids are investigated. It is found that close to equilibrium the Newtonian behaviour dominates, and the exponential convergence rate to equilibrium present in Newtonian fluids is replicated for Ellis fluids.

Theorem 5.2.3 (=Theorem B.7.5). *Fix $1 < \alpha < \infty$. There exists $\varepsilon > 0$ such that, for all positive initial values $u_0 \in H^1(\Omega)$ with $\|u_0 - \bar{u}_0\|_{H^1(\Omega)} < \varepsilon$, there is a global positive weak solution*

$$u \in C\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1,\mathrm{loc}}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha},loc}\big((0,\infty); (W^1_{\alpha+1,B}(\Omega))'\big).$$

Moreover, there are $\lambda > 0$ *and a constant* $C > 0$ *such that*

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \leq Ce^{-\lambda t}\|u_0\|_{H^1(\Omega)}.$$

Furthermore, we find that the dissipation decreases exponentially along the solution in the following L_1*-in-time sense:*

$$\int_{t/2}^{t} D[u](s)\,ds \leq Ce^{-2\lambda t}\|u_0\|^2_{H^1(\Omega)}.$$

6 The Gradient-Flow Structure of the Thin-Film Equation

Abstract

We study the gradient-flow structure of doubly-degenerate parabolic problems of fourth-order in one spatial dimension, describing, for instance, the dynamics of capillary-driven thin fluid films with non-Newtonian power-law rheology. We construct a formal gradient system and show that the corresponding weighted Wasserstein distance degenerates for physical mobilities. We then set up a minimising movement scheme with a modified mobility function. Using the Aubin–Lions–Simon lemma to gain compactness, we show that the time-discrete flow of the minimising movement scheme converges to a solution to the corresponding modified thin-film equation that satisfies an energy-dissipation equality. Finally, we show that solutions to the modified equation converge, for all flow-behaviour exponents, to a weak solution to the thin-film equation. In the case of Newtonian fluids, we use entropy methods to show that, under mild additional conditions, these weak solutions are non-negative for all times.

The research undertaken in this chapter is a collaboration with P. Gladbach and C. Lienstromberg. All authors and, in particular, the author of this thesis have contributed significant parts to each section of the work.

6.1 Introduction

We consider a thin fluid film in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with Lipschitz boundary. Furthermore, we assume the fluid to be Non-Newtonian with power-law rheology, that is, the constitutive law for the viscosity of the fluid is given by

$$\mu(|\epsilon|) = \mu_0 |\epsilon|^{\frac{1}{\alpha}-1},$$

(see also Section 4.2). Here ϵ denotes the strain rate, $\mu_0 > 0$ is the characteristic viscosity, and $\alpha > 0$ is the flow-behaviour exponent. The local viscous stress of the fluid is given by

$$\sigma(\epsilon) = \mu(|\epsilon|)\epsilon = \mu_0 |\epsilon|^{\frac{1}{\alpha}-1}\epsilon.$$

The fluid is shear-thickening if the flow-behaviour exponent satisfies $\alpha < 1$. It is Newtonian if $\alpha = 1$, and it is shear-thinning if $\alpha > 1$.

For such fluids, after lubrication approximation, a generalised thin-film equation in the domain Ω with initial condition $u_0 \geq 0$ for thin films with zero boundary angle is given by

$$\begin{cases} \partial_t u + \operatorname{div}\left(u^n |\nabla \Delta u|^{\alpha-1} \nabla \Delta u\right) = 0, & t > 0,\ x \in \Omega, \\ \nabla u \cdot n = u^n |\nabla \Delta u|^{\alpha-1} \nabla \Delta u \cdot n = 0, & t > 0,\ x \in \partial\Omega, \\ u(0) = u_0, & x \in \Omega. \end{cases} \tag{6.1.1}$$

Here $m(u) = u^n$ is the mobility of the thin film. Physically, the mobility is explicitly given once one specifies slippage conditions at the solid-fluid interface. For example, if we prescribe a no-slip condition, we obtain $m(u) = u^{\alpha+2}$. Recall from Section 4.5 that in the case of the Navier-slip condition, we obtain $m(u) = \lambda u^{\alpha+1} + u^{\alpha+2}$, where $\lambda > 0$ is the characteristic slip length. Note that, in both cases, the mobility is superlinear regardless of the flow-behaviour exponent $\alpha > 0$ since the mobility exponent n is greater than one.

By testing the equation with the function that is constantly equal to one, we find that solutions to (6.1.1) conserve their mass:

$$\bar{u}(t) := \fint_\Omega u(t, x)\,\mathrm{d}x = \fint_\Omega u_0(x)\,\mathrm{d}x, \quad t \geq 0.$$

It is well-known (see [Alm96], [Ott00], [GO03], where the latter two assume that $d = 1$) that for Newtonian fluids in the setting of Hele-Shaw flows, given by flow-behaviour exponent $\alpha = 1$ and mobility exponent $n = 1$, the thin-film equation is a gradient flow with respect to the energy given by

$$E[u](t) = \int_\Omega |\nabla u(t, x)|^2\,\mathrm{d}x.$$

A numerical gradient flow scheme — discrete both in time and space — for the Newtonian thin-film equation with general mobility exponents in one and two space dimensions is studied in [GR00]. Numerical schemes for more advanced geometries are studied in [RV13] and [Van+17].

In [LMS12], the Newtonian thin-film equation with mobility exponents $n \in (0, 1]$ in dimension $d = 1$ is studied as a gradient flow in weighted Wasserstein spaces. All these results have in common that the dissipation potential turns out to be jointly convex in the film height and the flux, as we will see later.

The problem of lack of convexity for physical mobility exponents $n \geq 1$ can be overcome considering first a modified mobility function m_δ with $m_\delta \geq \delta$ in $\mathbb{R}$. Using a minimising movement scheme in the space $\{u \in H^1(\Omega) : \bar{u} = \bar{u}_0\}$, solutions to the modified thin-film equation will be constructed for general flow-behaviour exponents $\alpha > 0$ in one space dimension. In particular, this shows that positive solutions to the one-dimensional power-law thin-film equation are given by a gradient flow. Furthermore, as one sends $m_\delta \to m$, these solutions converge to a weak solution of (6.1.1) in the sense of [BF90].

Gradient systems

In $\mathbb{R}^d$, $d \geq 1$, a gradient flow of a convex energy $E\colon \mathbb{R}^d \to \mathbb{R}$ is given by a solution to

$$\frac{d}{dt} x(t) = -\nabla E[x](t). \tag{6.1.2}$$

Gradient flows are curves of steepest descent for the energy since

$$\frac{d}{dt} E[x](t) = \nabla E[x](t) \partial_t x(t) = -|\nabla E[x](t)|^2.$$

Under the assumption of sufficient convexity of E, solutions to (6.1.2) can then be characterised in terms of an energy-dissipation equality: an absolutely continuous curve $x\colon [0,\infty) \to \mathbb{R}^d$ is a solution to (6.1.2) with initial value $x(0) = x_0 \in \mathbb{R}^d$ if and only if

$$E[x](t) + \frac{1}{2}\int_0^t \left|\frac{d}{dt}x(s)\right|^2 \mathrm{d}s + \frac{1}{2}\int_0^t |\nabla E[x](s)|^2 \,\mathrm{d}s = E[x_0]$$

holds for every $t > 0$, [ABS21].

A more general approach to gradient flows is provided by the theory of gradient systems. To recall the basic notions in a formal manner, we mainly follow [Mie16].

Let X be a space in which there are notions of a formal tangent space TX and formal cotangent space T^*X and a dual pairing $\langle\cdot,\cdot\rangle$ between TX and T^*X. For example, X could be a Riemannian manifold or a Banach manifold.

Definition 6.1.1 (Gradient system). *A gradient system on a space X consists of a tuple (E,Ψ), where $E\colon X \to \mathbb{R}\cup\{+\infty\}$ is an energy functional and $\Psi\colon X \times TX \to [0,\infty]$ is a dissipation potential that is assumed to be lower semicontinuous, proper and convex with $\Psi(u,0) = 0$ for every $u \in X$.*

We formally define the dual dissipation potential to be

$$\Psi^*(u,\xi) = \sup\{\langle \xi, v\rangle - \Psi(v) : v \in T_uX\}$$

as the formal Legendre transform of Ψ. Then formally

$$\Psi(u,v) + \Psi^*(u,\xi) \geq \langle u, v\rangle \quad \text{for all } u \in X,\ v \in T_uX \text{ and } \xi \in T_u^*X. \tag{6.1.3}$$

Definition 6.1.2 (Gradient flow). *Given a gradient system (E,Ψ) on a space X, a solution to a gradient flow is an absolutely continous curve $u\colon [0,\infty) \to X$ such that one of the following three equivalent formulations are satisfied*

(i) $\partial_t u = \partial_\xi \Psi^*(u, -DE[u])$;

(ii) $0 = \partial_v \Psi(u, \partial_t u) + DE[u]$;

(iii) $\Psi(u,\partial_t u) + \Psi^*(u,-DE[u]) = \langle -DE[u], \partial_t u\rangle$.

Note that it is immediate from (iii) that formally gradient flows in gradient systems satisfy the energy-dissipation equality

$$E[u(t)] + \int_0^t \Psi(u,\partial_t u) + \Psi^*(u,-DE[u])\,\mathrm{d}s = E[u(0)].$$

Note that formally (assuming that the chain rule is applicable) by (6.1.3), the energy-dissipation equality is equivalent to the energy-dissipation inequality

$$E[u(t)] + \int_0^t \Psi(u,\partial_t u) + \Psi^*(u,-DE[u])\,\mathrm{d}s \leq E[u(0)].$$

Heuristic for the dissipation potential

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be open, bounded and connected with smooth boundary. The concrete setting for the power-law thin-film equation is the following. As discussed previously, the first-order approximation of the length of the surface

$$E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x$$

is a natural energy. Since the mass is conserved, a natural choice for the space X is given by

$$X := \left\{ u \in H^1(\Omega) : \fint_\Omega u \, \mathrm{d}x = \bar{u}_0, \; u \geq 0 \right\}$$

for a fixed mass $\bar{u}_0 > 0$. Elements of the formal tangent space TX to X then have the form

$$TX := \left\{ v \in H^1(\Omega) : \fint_\Omega v \, \mathrm{d}x = 0 \right\}.$$

There is a natural representation of such vector functions in terms of vector fields, that is

$$TX \simeq \{ \operatorname{div} j : j \cdot n = 0 \text{ on } \partial\Omega \},$$

where n denotes the outer unit normal to $\partial\Omega$. To derive an explicit form of Ψ, note that by convexity, the condition in Definition 6.1.2 (ii) is equivalent to

$$\partial_t u \in \operatorname*{arg\,min}_{v \in TX} \Psi(u, v) + \langle DE[u], v \rangle = \operatorname*{arg\,min}_{\substack{j:\, divj = v, \\ j \cdot n = 0 \text{ on } \partial\Omega}} \Psi(u, \operatorname{div} j) + \langle DE[u], \operatorname{div} j \rangle.$$

We now make the ansatz that the dissipation potential is given by a local object $\psi(u, j)$ so that

$$\Psi(u, v) = \inf_{\substack{j:\, v + \operatorname{div} j = 0, \\ j \cdot n = 0 \text{ on } \partial\Omega}} \int_\Omega \psi(u, j) \, \mathrm{d}x.$$

Then it holds

$$\begin{aligned} &\min_{v \in TX} \inf_{\substack{j:\, v + \operatorname{div} j = 0, \\ j \cdot n = 0 \text{ on } \partial\Omega}} \int_\Omega \psi(u, j) \, \mathrm{d}x + \langle DE[u], -\operatorname{div} j \rangle \\ &= \min_{\substack{j:\, v + \operatorname{div} j = 0, \\ j \cdot n = 0 \text{ on } \partial\Omega}} \int_\Omega \psi(u, j) \, \mathrm{d}x + \langle DE[u], -\operatorname{div} j \rangle, \end{aligned}$$

since we can exchange both infima.

We make the ansatz that

$$\psi(u, j) = \frac{|j|^p}{m(u)^q}$$

for some $p > 1$, $q > 0$ to be determined. Then

$$\operatorname{div} j \in \operatorname*{arg\,min}_{\substack{v + \operatorname{div} j = 0, \\ j \cdot n = 0 \text{ on } \partial\Omega}} \int_\Omega \frac{|j|^p}{m(u)^q} \, \mathrm{d}x + \langle \nabla DE[u], j \rangle$$

if and only if j minimises

$$\int_\Omega \frac{|j|^p}{m(u)^q} \, \mathrm{d}x + \langle \nabla DE[u], j \rangle$$

under the constraint $j \cdot n = 0$ in $\partial\Omega$. Since $DE[u] = -\Delta u$, we conclude that

$$\frac{|j|^{p-2} j}{m(u)^q} = \nabla \Delta u,$$

and hence

$$j = m(u)^{\frac{q}{p-1}} |\nabla \Delta u|^{\frac{1}{p-1}-1} \nabla \Delta u.$$

Since $\partial_t u + \operatorname{div} j = 0$, we choose $p = \frac{\alpha+1}{\alpha}$ and $q = \frac{1}{\alpha}$ to obtain that the pair (u, j) solves the thin-film equation

$$\partial_t u + \operatorname{div}\left(m(u)|\nabla \Delta u|^{\alpha-1} \nabla \Delta u\right) = 0.$$

To summarise, we conclude that we choose the dissipation potential

$$\Psi(u, \partial_t u) = \inf_{\substack{\partial_t u + \operatorname{div} j = 0 \\ j \cdot n = 0 \text{ on } \partial\Omega}} \int_\Omega \frac{|j|^{\frac{\alpha+1}{\alpha}}}{m(u)^{\frac{1}{\alpha}}} \, \mathrm{d}x. \tag{6.1.4}$$

This approach of gradient flows would fit in the context of gradient flows on metric spaces [AGS08] and the Otto calculus [JKO98] if

$$d_m^{\frac{\alpha+1}{\alpha}}(u_0, u_1) := \inf \left\{ \int_0^1 \int_\Omega \frac{|j(t,x)|^{\frac{\alpha+1}{\alpha}}}{m(u(t,x))^{\frac{1}{\alpha}}} \, \mathrm{d}x \, \mathrm{d}t \; : \; \partial_t u + \operatorname{div} j = 0, \right.$$
$$\left. j \cdot n = 0 \text{ on } \partial\Omega, \; u \geq 0, u(0,x) = u_0(x), \; u(1,x) = u_1(x) \right\}$$

were a metric. We prove in Proposition 6.2.1 that, for the physically relevant case of superlinear mobilities, d_m degenerates. In the case of concave mobilities $m(s) = s^\alpha$, $0 < \alpha \leq 1$, this metric is used in [LMS12] to write the Newtonian thin-film equation as a gradient flow. In the latter case, the integrand is convex as a function of (u, j), while the integrand lacks this convexity in the case of superlinear mobilities.

Main results

After we prove in Proposition 6.2.1 that with the Dirichlet energy and the dissipation potential

$$\Psi(u, \partial_t u) = \inf_{\substack{\partial_t u + \operatorname{div} j = 0 \\ j \cdot n = 0 \text{ on } \partial\Omega}} \int_\Omega \frac{|j|^{\frac{\alpha+1}{\alpha}}}{m(u)^{\frac{1}{\alpha}}} \, \mathrm{d}x,$$

the corresponding metric degenerates, and we cannot find a gradient flow via Otto calculus in metric spaces, we regularise the mobility m. We introduce $m_\delta \colon \mathbb{R} \to [\delta, 1/\delta]$ so that $m_\delta(s) \geq \delta$ for $s \in \mathbb{R}$ and $m_\delta(s) = \delta$ for $s \leq 0$. We set up a minimising movement scheme on $H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega; \mathbb{R}^d)$. Therefore, if at time t we are in the state u^*, we determine u at time $t + h$ and the flux to be the minimiser of the functional

$$\mathcal{F}_{u^*}^{h,\delta}(u, j) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + h \frac{\alpha}{\alpha+1} \int_\Omega \frac{|j|^{\frac{\alpha+1}{\alpha}}}{m_\delta(u^*)^{\frac{1}{\alpha}}} \, \mathrm{d}x,$$

where the minimisation runs over pairs $(u, j) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega; \mathbb{R}^d)$ that satisfy

$$\begin{cases} \frac{u - u^*}{h} + \operatorname{div} j = 0 & \text{in } \Omega, \\ j \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

and we choose an initial datum $u_0 \in H^1(\Omega)$ with $u_0 \geq 0$.

In the case of one space dimension $\Omega \subset \mathbb{R}$, to gain enough compactness, we prove that a subsequence of the corresponding linear interpolation $\hat{u}^{h,\delta}$ converges, as $h \to 0$, to a weak solution

$$u^\delta \in C_b\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,\infty); \big(W^1_{\alpha+1}(\Omega)\big)'\big)$$

to the modified thin-film equation

$$\begin{cases} \partial_t u^\delta + \operatorname{div}\big(m_\delta(u^\delta)\partial_x^3 u^\delta\big) = 0, & t>0,\ x \in \Omega, \\ \partial_x u^\delta = \partial_x^3 u^\delta = 0, & t>0,\ x \in \partial\Omega, \\ u^\delta(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

which satisfies the energy-dissipation equality

$$\int_\Omega \frac{1}{2}|\partial_x u_t^\delta|^2 \,\mathrm{d}x + \int_s^t \int_\Omega m_\delta(u_\tau^\delta)|\partial_x^3 u_\tau^\delta|^{\alpha+1} \,\mathrm{d}x \,\mathrm{d}\tau = \int_\Omega \frac{1}{2}|\partial_x u_s^\delta|^2 \,\mathrm{d}x.$$

This already implies that positive solutions to the thin-film equation are given by the corresponding gradient flow. We then show that every accumulation point

$$u \in L^\infty\big([0,\infty); H^1(\Omega)\big) \cap C^{\frac{1}{5\alpha+3},\frac{1}{2}}\big([0,\infty] \times \bar{\Omega}\big)$$

with $\partial_x^3 u \in L_{\alpha+1,\mathrm{loc}}(\{u>0\})$ and $\partial_t u \in L_{\alpha+1}\big([0,\infty); \big(W^1_{\alpha+1}(\Omega)\big)'\big)$ of the sequence u^δ, as $\delta \to 0$, is a weak solution to the thin-film equation

$$\begin{cases} \partial_t u^\delta + \operatorname{div}\big(m(u^\delta)\partial_x^3 u^\delta\big) = 0, & t>0,\ x \in \Omega, \\ \partial_x u^\delta = m(u)\partial_x^3 u^\delta = 0, & t>0,\ x \in \partial\Omega, \\ u^\delta(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

in the sense that

$$\int_0^\infty \langle \partial_t u, \varphi \rangle_{W^1_{\alpha+1}} \,\mathrm{d}t - \iint_{\{u>0\}} m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u \cdot \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}t = 0$$

holds for all $\varphi \in L_{\alpha+1}\big([0,\infty); W^1_{\alpha+1}(\Omega)\big)$. Furthermore, u satisfies the energy-dissipation inequality

$$\int_\Omega \frac{1}{2}|\partial_x u_t|^2 \,\mathrm{d}x + \int_0^t \int_{\{u_s>0\}} m(u)|\partial_x^3 u|^{\alpha+1} \,\mathrm{d}x \,\mathrm{d}s \leq \int_\Omega \frac{1}{2}|\partial_x u_0|^2 \,\mathrm{d}x, \quad t \in [0,\infty).$$

These solutions may not be non-negative for all times. Using the entropy method used in [BF90], we deduce that in the case of Newtonian fluids and initial datum with finite entropy, the weak solutions are non-negative for all times.

Notation

Throughout this chapter, we assume that $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is an open and bounded domain with Lipschitz boundary. Mostly, $\Omega \subset \mathbb{R}$ will be an interval. We use, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, the notation $W^k_p(\Omega)$ for the standard Sobolev space with norm

$$\|v\|_{W^k_p(\Omega)} = \left(\sum_{0 \leq |\alpha| \leq k} \|\partial_\alpha v\|^p_{L_p(\Omega)} \right)^{1/p}.$$

Let $\Omega \subset \mathbb{R}$ an interval. To account for the Neumann-type boundary conditions of the solutions to the thin-film equation, we further introduce the notation

$$W^k_{p,B}(\Omega) = \begin{cases} \{v \in W^k_p(\Omega); v_x = v_{xxx} = 0 \text{ on } \partial\Omega\}, & k = 4, \\ \{v \in W^k_p(\Omega); v_x = 0 \text{ on } \partial\Omega\}, & 2 \le k \le 3, \\ W^k_p(\Omega), & 0 \le k \le 1. \end{cases}$$

The spaces $W^k_{p,B}(\Omega)$, $k \in \{0, \dots, 4\}$, are closed linear subspaces of $W^k_p(\Omega)$.

OUTLINE OF THIS CHAPTER

The structure of this chapter is as follows: in Section 6.2, we study the Benamou–Brenier action functional derived previously and show that the corresponding distance degenerates in the (physically relevant) case of superlinear mobility functionals.

Due to the need to take a different approach to the gradient-flow structure, we set up a minimising movement scheme for a modified thin-film equation in Section 6.3 and study the properties of the interpolations of the time-discrete flow. Using the De Giorgi technique, we derive an optimal discrete energy-dissipation inequality and use this to prove a-priori bounds and show convergence to a limit.

In Section 6.4, we investigate the limit obtained in Section 6.3. We show that it satisfies the energy-dissipation equality and solves the modified thin-film equation.

While from the results of Section 6.4 it can already be deduced that positive solutions to the power-law thin-film equation have a gradient-flow structure, in Section 6.5 we study the limit of $\delta \to 0$ and prove that the solutions to the modified thin-film equation converge to weak solutions to the power-law thin-film equation. In the Newtonian case, we further show that these solutions are non-negative for all times.

6.2 Benamou–Brenier action with superlinear mobility

In this section, we define for two non-negative functions $u_0, u_1 \in L_1(\Omega)$, $u_0, u_1 \ge 0$, with $\int_\Omega u_0 \,\mathrm{d}x = \int_\Omega u_1 \,\mathrm{d}x$, the Benamou–Brenier action functional depending on a continuous mobility function $m : [0,\infty) \to [0,\infty)$,

$$d_m^{\frac{\alpha+1}{\alpha}}(u_0, u_1) := \inf\left\{ \int_0^1 \int_\Omega \frac{|j(t,x)|^{\frac{\alpha+1}{\alpha}}}{m(u(t,x))^{\frac{1}{\alpha}}} \,\mathrm{d}x \,\mathrm{d}t \;:\; \partial_t u + \operatorname{div} j = 0, \right.$$
$$\left. j \cdot n = 0 \text{ on } \partial\Omega,\ u \ge 0, u(0,x) = u_0(x),\ u(1,x) = u_1(x) \right\}$$

If the mobility $m(u) = u^n$ is superlinear, we show that, since this functional lacks convexity simultaneously in (u, j), $d_m \equiv 0$. So, the natural distance for the setting of gradient flows in metric spaces degenerates, and we have to resort to a different approach to obtain a gradient-flow scheme.

Proposition 6.2.1. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded, and connected, with Lipschitz boundary. Assume that $m\colon [0,\infty) \to [0,\infty)$ satisfies*

$$\lim_{u \to \infty} \frac{m(u)}{u} = \infty \quad \text{and} \quad m^{-1}(0) = \{0\}.$$

Then

$$d_m^{\frac{\alpha+1}{\alpha}}(u_0,u_1)=0$$

for all pairs $u_0, u_1 \in L^1(\Omega)$ *with* $\int_\Omega u_0\,\mathrm{d}x = \int_\Omega u_1\,\mathrm{d}x$ *and* $\inf_\Omega u_0, \inf_\Omega u_1 > 0$.

Remark 6.2.2. The proof uses a construction that first concentrates mass locally to achieve a very high density, allowing it to be transported very effectively over large distances, and then dissipates the mass to meet the terminal values. A similar effect occurs in [San07].

The opposite effect occurs in dynamic entropic optimal transport and certain mean field games, where at intermediate times the mass is more spread out than at the end points, see e.g. [BCS17].

Recall the Benamou–Brenier formula for the Wasserstein distance, cf. [BB00],

$$W_{\frac{\alpha+1}{\alpha}}^{\frac{\alpha+1}{\alpha}}(u_0,u_1)=\inf\Bigg\{\int_0^1\int_\Omega \frac{|j(t,x)|^{\frac{\alpha+1}{\alpha}}}{u(t,x)^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}t \,:\, \partial_t u+\operatorname{div} j=0,$$
$$j\cdot n=0 \text{ on } \partial\Omega,\ u\geq 0, u(0,x)=u_0(x), u(1,x)=u_1(x)\Bigg\}.$$

Proof. Define $\delta := \frac{1}{2}\min(\operatorname{ess\,inf}_\Omega u_0, \operatorname{ess\,inf}_\Omega u_1)$. For $\eta>0$ define the grid

$$Z_\eta := \{z\in \eta\mathbb{Z}^d\cap\Omega \,:\, \operatorname{dist}(x,\partial\Omega)\geq\eta\}.$$

Also define $l_\eta>0$ as the longest length of a shortest curve in $\bar\Omega$ connecting any point $x\in\Omega$ with some $z\in Z_\eta$, that is

$$l_\eta := \sup_{x\in\Omega}\inf_{z\in Z_\eta}\inf\{L(\gamma) \,:\, \gamma\subset\Omega \text{ is a } C^1\text{-curve connecting } x \text{ and } z\}.$$

By a compactness argument we then have $\lim_{\eta\to 0} l_\eta = 0$.

Define $\tilde u_0 := u_0-\delta\geq\delta$, $\tilde u_1 := u_1-\delta\geq\delta$. Then there are measures $\mu_{0,\eta},\mu_{1,\eta}\in\mathcal{M}_+(Z_\eta)$ of the form

$$\mu_{0,\eta}=\sum_{z\in Z_\eta}\alpha_z\delta_z,\quad \mu_{1,\eta}=\sum_{z\in Z_\eta}\beta_z\delta_z$$

that satisfy

$$W_{\frac{\alpha+1}{\alpha}}^{\frac{\alpha+1}{\alpha}}(\mu_{0,\eta},\tilde u_0)\leq l_\eta^{\frac{\alpha+1}{\alpha}}\int_\Omega\tilde u_0\,\mathrm{d}x \quad\text{and}\quad W_{\frac{\alpha+1}{\alpha}}^{\frac{\alpha+1}{\alpha}}(\mu_{1,\eta},\tilde u_1)\leq l_\eta^{\frac{\alpha+1}{\alpha}}\int_\Omega\tilde u_1\,\mathrm{d}x.$$

Now choose a coupling $\Gamma\in\mathcal{M}_+(Z_\eta\times Z_\eta)$ of $\mu_{0,\eta}$ and $\mu_{1,\eta}$, e.g. the product measure $\mu_{0,\eta}\otimes\mu_{1,\eta}$. Define $c_\eta := \min_{z,z'\in Z_\eta\,:\,\Gamma(z,z')>0}\Gamma(z,z')>0$ since the infimum ranges only over finitely many points.

Also for any $z,z'\in Z_\eta$ there exists a C^1-curve $\gamma_{z,z'}:[0,1]\to\Omega$ connecting z and z'. Let $L_\eta>0$ be the maximal length of such a curve and $d_\eta>0$ the minimal distance from any point on any such curve to $\partial\Omega$.

Finally, we define for $M>0$

$$u_{0,\eta,M} := \delta+\sum_{z\in Z_\eta}\frac{M^d\alpha_z}{\omega_d\eta^d}\mathbb{1}_{B(z,\frac{\eta}{M})} \quad\text{and}\quad u_{0,\eta,M} := \delta+\sum_{z\in Z_\eta}\frac{M^d\beta_z}{\omega_d\eta^d}\mathbb{1}_{B(z,\frac{\eta}{M})}.$$

We now wish to estimate $d_m^{\frac{\alpha+1}{\alpha}}(u_0,u_1)$. Since d_m satisfies the triangle inequality, it is enough to estimate $d_m^{\frac{\alpha+1}{\alpha}}(u_0,u_{0,\eta,M})$, $d_m^{\frac{\alpha+1}{\alpha}}(u_{0,\eta,M},u_{1,\eta,M})$ and $d_m^{\frac{\alpha+1}{\alpha}}(u_{1,\eta,M},u_1)$. We start with the first and last term since they can be treated analogously. To do so, we note that for any pair (u,j) we have

$$\int_0^1\int_\Omega \frac{|j(t,x)|^{\frac{\alpha+1}{\alpha}}}{m(u(t,x))^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}t \le \left[\operatorname*{ess\,sup}_{(t,x)\in\operatorname{supp} j}\frac{u(t,x)^{\frac{1}{\alpha}}}{m(u(t,x))^{\frac{1}{\alpha}}}\right]\int_0^1\int_\Omega \frac{|j(t,x)|^{\frac{\alpha+1}{\alpha}}}{u(t,x)^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}t. \quad (6.2.1)$$

Observe that by construction it holds

$$W_{\frac{\alpha+1}{\alpha}}(\mu_{0,\eta},u_{0,\eta,M}-\delta)\le \frac{\eta}{M}\int_\Omega \tilde u_0\,\mathrm{d}x.$$

Combining this with the triangle inequality, we obtain

$$\begin{aligned} W_{\frac{\alpha+1}{\alpha}}^{\frac{\alpha+1}{\alpha}}(\tilde u_0,u_{0,\eta,M}-\delta) &\le \Big(W_{\frac{\alpha+1}{\alpha}}(\tilde u_0,\mu_{0,\eta})+W_{\frac{\alpha+1}{\alpha}}(\mu_{0,\eta},u_{0,\eta,M}-\delta)\Big)^{\frac{\alpha+1}{\alpha}}\\ &\le (l_\eta+\frac{\eta}{M})^{\frac{\alpha+1}{\alpha}}\int_\Omega \tilde u_0\,\mathrm{d}x. \end{aligned}$$

Hence, for any $\varepsilon>0$, there is a distributional solution $j\in L_{\frac{\alpha+1}{\alpha}}([0,1]\times\Omega;\mathbb{R}^d)$ to the continuity equation

$$\begin{cases} \partial_t\tilde u+\operatorname{div} j=0, & t>0,\ x\in\Omega,\\ j\cdot n=0, & t>0,\ x\in\partial\Omega,\\ \tilde u(0,x)=\tilde u_0(x), & x\in\Omega,\\ \tilde u(1,x)=u_{0,\eta,M}(x)-\delta, & x\in\Omega, \end{cases}$$

with

$$\int_0^1\int_\Omega \frac{|j|^{\frac{\alpha+1}{\alpha}}}{\tilde u^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}t\le W_{\frac{\alpha+1}{\alpha}}^{\frac{\alpha+1}{\alpha}}(\tilde u_0,u_{0,\eta,M}-\delta)+\varepsilon\le (l_\eta+\frac{\eta}{M})^{\frac{\alpha+1}{\alpha}}\int_\Omega \tilde u_0\,\mathrm{d}x+\varepsilon.$$

We see that $u(t,x):=\tilde u(t,x)+\delta$ and j together solve the continuity equation with initial and terminal values $u(0,x)=u_0(x)$ and $u(1,x)=u_{0,\eta,M}(x)$. Moreover, by (6.2.1), we have

$$\int_0^1\int_\Omega \frac{|j(t,x)|^{\frac{\alpha+1}{\alpha}}}{m(u(t,x))^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}t\le \sup_{s\ge\delta}\frac{s^{\frac{1}{\alpha}}}{m(s)^{\frac{1}{\alpha}}}\left[\Big(l_\eta+\frac{\eta}{M}\Big)^{\frac{\alpha+1}{\alpha}}\int_\Omega \tilde u_0\,\mathrm{d}x+\varepsilon\right],$$

so that

$$\limsup_{\eta\to 0}\sup_{M\ge 1} d_m^{\frac{\alpha+1}{\alpha}}(u_0,u_{0,\eta,M})\le\varepsilon,$$

for every $\varepsilon>0$ arbitrary. We conclude that

$$\limsup_{\eta\to 0}\sup_{M\ge 1} d_m^{\frac{\alpha+1}{\alpha}}(u_0,u_{0,\eta,M})=0$$

and likewise

$$\limsup_{\eta\to 0}\sup_{M\ge 1} d_m^{\frac{\alpha+1}{\alpha}}(u_1,u_{1,\eta,M})=0.$$

Next, we estimate $d_m^{\frac{\alpha+1}{\alpha}}(u_{0,\eta,M}, u_{1,\eta,M})$. To this end, we define for $M \geq \frac{\eta}{d_\eta}$

$$u(t,x) := \delta + \sum_{z,z'} \Gamma(z,z') \frac{M^d}{\omega_d \eta^d} \mathbb{1}_{B(\gamma_{z,z'}(t), \frac{\eta}{M})}(x),$$

and

$$j(t,x) := \sum_{z,z'} \Gamma(z,z') \frac{M^d}{\omega_d \eta^d} \mathbb{1}_{B(\gamma_{z,z'}(t), \frac{\eta}{M})}(x) \dot{\gamma}_{z,z'}(t).$$

This is clearly a curve connecting $u_{0,\eta,M}$ and $u_{1,\eta,M}$, and

$$\operatorname*{ess\,inf}_{(x,t)\in \operatorname{supp} j} u(t,x) \geq c_\eta \frac{M^d}{\omega_d \eta^d} \longrightarrow \infty$$

as $M \to \infty$, for every fixed $\eta > 0$. By the superlinear growth condition on m, it follows that

$$\operatorname*{ess\,sup}_{(x,t)\in \operatorname{supp} j} \frac{u(t,x)^{\frac{1}{\alpha}}}{m(u(t,x))^{\frac{1}{\alpha}}} \longrightarrow 0$$

as $M \to \infty$, for every fixed $\eta > 0$. By (6.2.1), we have that

$$d_m^{\frac{\alpha+1}{\alpha}}(u_{0,\eta,M}, u_{1,\eta,M}) \leq \left[\operatorname*{ess\,sup}_{(x,t)\in \operatorname{supp} j} \frac{u(t,x)^{\frac{1}{\alpha}}}{m(u(t,x))^{\frac{1}{\alpha}}} \right] L_\eta^{\frac{\alpha+1}{\alpha}} \int_\Omega \tilde{u}_0(x)\, \mathrm{d}x \longrightarrow 0$$

as $M \to \infty$, for every fixed $\eta > 0$.

Finally, we concatenate the curves connecting u_0 with $u_{0,\eta,M}$, $u_{0,\eta,M}$ with $u_{1,\eta,M}$ and $u_{1,\eta,M}$ with u_1, and estimate

$$\begin{aligned} &d_m^{\frac{\alpha+1}{\alpha}}(u_0, u_1) \\ &\leq \limsup_{\eta\to 0} \lim_{M\to\infty} C \left(d_m^{\frac{\alpha+1}{\alpha}}(u_0, u_{0,\eta,M}) + d_m^{\frac{\alpha+1}{\alpha}}(u_{0,\eta,M}, u_{1,\eta,M}) + d_m^{\frac{\alpha+1}{\alpha}}(u_{1,\eta,M}, u_1) \right) \\ &= 0. \end{aligned}$$

This concludes the proof. □

Remark 6.2.3. If u_0 and u_1 are smooth, the connecting curve (u, j) can be chosen smooth in space-time via mollification and the dominated convergence theorem.

6.3 Minimising movement scheme for the modified thin-film equation

In the previous section, we studied the distance we obtained heuristically for a gradient flow approach for the thin-film equation. Since this distance degenerates, we now modify the mobility of the thin film to obtain a non-degenerate distance function and use a minimising movement scheme in order to show that this modified thin-film equation is indeed a gradient flow. We also fix the space dimension $d = 1$ and for simplicity only consider thin films that are homogeneous in y-direction.

We fix a uniformly continuous mobility function $m\colon \mathbb{R} \to [0,\infty)$ such that $m(s) = 0$ for all $s \leq 0$. Moreover, we fix $\delta > 0$ and consider a uniformly continuous $m_\delta\colon \mathbb{R} \to [\delta, 1/\delta]$ with the following properties

(i) $m_\delta(s) \geq \delta,\ s \in \mathbb{R}$, and $m_\delta(s) = \delta,\ s \leq 0$,

(ii) $m_\delta(s) \geq m(s),\ s \in \mathbb{R}$,

(iii) $m_\delta \to m$ locally uniformly on $\mathbb{R}$.

Fix $\delta > 0$. We start setting up the minimising movement scheme. That is, we fix a time step size $h > 0$. If at a time t, we are at u^*, we define the next iteration, that is the approximation at time $t + h$, to be a minimiser of the functional

$$\mathcal{F}^{h,\delta}_{u^*}(u, j) = \frac{1}{2}\int_\Omega |\partial_x u|^2 \,\mathrm{d}x + h\frac{\alpha}{\alpha+1}\int_\Omega \frac{|j|^{\frac{\alpha+1}{\alpha}}}{m_\delta(u^*)^{\frac{1}{\alpha}}}\,\mathrm{d}x, \tag{6.3.1}$$

where the minimisation runs over pairs $(u, j) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$ that satisfy

$$\begin{cases} \operatorname{div} j + \frac{u-u^*}{h} = 0 & \text{in } \Omega, \\ j = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that we sligthly changed the dissipation potential that we obtained in (6.1.4): we fix u^* in the second integral. This avoids problems with the lack of convexity. Heuristically, this is no major change since solutions turn out to be continuous in time, and hence $m_\delta(u(t + h))$ and $m_\delta(u(t))$ are very close.

Before we set up the minimising movement scheme, we first show existence, uniqueness and regularity properties of a minimiser of the corresponding functional.

Definition 6.3.1. *Let* $\Omega \subset \mathbb{R}$ *a bounded interval,* $u^* \in H^1(\Omega)$ *and* $h > 0$. *We say that a pair* $(u, j) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$ *solves the* flow equation

$$\begin{cases} \operatorname{div} j + \frac{u-u^*}{h} = 0 & \text{in } \Omega, \\ j = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3.2}$$

if the equation

$$-\int_\Omega j \cdot \partial_x\varphi \,\mathrm{d}x + \frac{1}{h}\int_\Omega (u - u^*)\varphi \,\mathrm{d}x = 0 \tag{6.3.3}$$

is satisfied for all $\varphi \in C^1(\bar{\Omega})$.

Remark 6.3.2 (Conservation of mass and Poincaré inequality). For every $u^* \in H^1(\Omega)$ a solution $(u, j) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega; \mathbb{R}^d)$ to the flow equation (6.3.2) conserves its mass in the sense that

$$\bar{u} := \fint_\Omega u \,\mathrm{d}x = \fint_\Omega u^* \,\mathrm{d}x,$$

where $\bar{u}$ denotes the average of u. Indeed, this follows immediately from (6.3.3) with the choice $\varphi \equiv 1$. In particular, u satisfies the Poincaré inequality

$$\|u - \bar{u}\|_{L_2(\Omega)} \leq C \|\partial_x u\|_{L_2(\Omega)}, \tag{6.3.4}$$

where $C > 0$ is a positive constant that depends only on Ω and $\bar{u} = \bar{u}^*$.

The following proposition guarantees existence and uniqueness of minimisers of the functional $\mathcal{F}^{h,\delta}_{u^*}$ for a given initial datum $u^* \in H^1(\Omega)$.

Proposition 6.3.3. *Let $\Omega \subset \mathbb{R}$ a bounded interval, $u^* \in H^1(\Omega)$. Fix $h,\ \delta > 0$. There exists a unique minimiser $(u^{h,\delta}, j^{h,\delta}) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$ of the functional $\mathcal{F}^{h,\delta}_{u^*}$ defined in* (6.3.1)*, where the minimisation runs over all pairs $u \in H^1(\Omega)$, $j \in L_{\frac{\alpha+1}{\alpha}}(\Omega)$ that solve the flow equation* (6.3.2) *in the sense of Definition 6.3.1.*
If the mobility m_δ is Lipschitz continuous, then the minimiser has the additional regularity $(u^{h,\delta}, j^{h,\delta}) \in W^3_{\alpha+1,B}(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$ and solves the elliptic boundary-value problem

$$\begin{cases} \frac{u^{h,\delta}-u^*}{h} + \operatorname{div} j^{h,\delta} = 0 & \text{in } \Omega, \\ j^{h,\delta} = m_\delta(u^*)|\partial_x^3 u^{h,\delta}|^{\alpha-1}\partial_x^3 u^{h,\delta} & \text{in } \Omega, \\ \partial_x u^{h,\delta} = 0 & \text{on } \partial\Omega, \\ j^{h,\delta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3.5}$$

In other words, Proposition 6.3.3 characterises the minimiser of (6.3.1) as a solution $(u^{h,\delta}, j^{h,\delta}) \in W^3_{\alpha+1,B}(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$ to the degenerate-elliptic boundary-value problem

$$\begin{cases} \frac{u-u^*}{h} + \operatorname{div}\left(m_\delta(u^*)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\right) = 0 & \text{in } \Omega, \\ \partial_x u \cdot n = 0 & \text{on } \partial\Omega, \\ \partial_x^3 u \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

where we use that $m_\delta(u^*) \geq \delta > 0$.

Proof. This is a strictly convex minimisation problem with a linear constraint. Existence of a unique minimiser follows from the direct method of the calculus of variations. Indeed, since $H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$ is a reflexive Banach space and the set of all pairs

$$(u,j) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega) \quad \text{satisfying} \quad \begin{cases} \operatorname{div} j + \frac{u-u^*}{h} = 0 & \text{in } \Omega, \\ j = 0 & \text{on } \partial\Omega \end{cases}$$

is a closed, non-empty and affine subspace, containing the point $(u^*, 0)$, and the functional in (6.3.1) is non-negative, there exists a minimising sequence. In view of the Poincaré inequality (6.3.4) and the fact that $m_\delta(u^*) \geq \delta > 0$, we may extract a subsequence that converges weakly in $H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$. The functional is strictly convex in $(u,j) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$, whence we obtain weak lower semicontinuity and therewith the existence of a minimiser $(u^{h,\delta}, j^{h,\delta}) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$ solving the flow equation.

We are left with proving uniqueness. To this end, assume that (u_1, j_1) and (u_2, j_2) are two minimisers of (6.3.1). By the strict convexity of $v \mapsto |v|^2$, $v \in \mathbb{R}$, and $k \mapsto |k|^{\frac{\alpha+1}{\alpha}}$, $k \in \mathbb{R}$, we deduce that

$$\partial_x u_1 = \partial_x u_2 \quad \text{and} \quad j_1 = j_2 \quad \text{a.e. in } \Omega.$$

Since $\bar{u}_2 = \bar{u}_2$, it follows that $(u_1, j_1) = (u_2, j_2)$.

To derive the Euler–Lagrange equation for the minimiser $(u^{h,\delta}, j^{h,\delta}) \in H^1(\Omega) \times L_{\frac{\alpha+1}{\alpha}}(\Omega)$, we first consider a solenoidal vector field $k \in L_{\frac{\alpha+1}{\alpha}}(\Omega)$ with $\langle k, \partial_x \varphi\rangle = 0$ for all $\varphi \in C^1(\bar{\Omega})$. Then, we take the first variation

$$0 = \frac{d}{d\varepsilon}\mathcal{F}^{h,\delta}_{u^*}\left(u^{h,\delta}, j^{h,\delta} + \varepsilon k\right)_{|\varepsilon=0} = h\int_\Omega \frac{|j|^{\frac{1-\alpha}{\alpha}} jk}{m_\delta(u^*)^{\frac{1}{\alpha}}}\,\mathrm{d}x,$$

for all solenoidal vector fields $k \in L_{\frac{\alpha+1}{\alpha}}(\Omega;\mathbb{R}^d)$. Using the Helmholtz decomposition [Sol77], this implies that

$$|j|^{\frac{1-\alpha}{\alpha}} j = m_\delta(u^*)^{\frac{1}{\alpha}} \partial_x \psi$$

for some $\psi \in W^1_{\alpha+1}(\Omega)$.

Now, pick $w \in C^1(\bar{\Omega})$ with average $\bar{w} = 0$ and let $k = \partial_x \Phi$, where $\Phi \in W^1_{\alpha+1}(\Omega)$ solves the Neumann problem

$$\begin{cases} -\partial_x^2 \Phi = \frac{w}{h} & \text{in } \Omega, \\ \partial_x \Phi \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3.6}$$

Since then

$$\frac{u^{h,\delta} + \varepsilon w - u^*}{h} + \operatorname{div}\left(j^{h,\delta} + \varepsilon k\right) = 0,$$

we may take the first variation

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \mathcal{F}^{h,\delta}_{u^*}\left(u^{h,\delta} + \varepsilon w, j^{h,\delta} + \varepsilon k\right)|_{\varepsilon=0} \\ &= \int_\Omega \partial_x u^{h,\delta} \partial_x w \,\mathrm{d}x + h \int_\Omega \partial_x \psi k \,\mathrm{d}x \\ &= \int_\Omega \partial_x u^{h,\delta} \partial_x w \,\mathrm{d}x + \int_\Omega \psi w \,\mathrm{d}x. \end{aligned}$$

In the last step, we have used (6.3.6) and the fact that $\psi \in W^1_{\alpha+1}(\Omega)$ is a valid test function. Since this equation holds true for arbitrary $w \in C^1(\bar{\Omega})$ with average $\bar{w} = 0$, we obtain in particular that

$$\psi = \partial_x^2 u^{h,\delta} + C,$$

for some constant C, in the sense of distributions. This shows that $u^{h,\delta} \in H^1(\Omega)$ satisfies

$$\langle \partial_x u^{h,\delta}, \partial_x v\rangle = -\langle (\psi - C), v\rangle \quad \text{for all } v \in H^1(\Omega).$$

Using [ADN59, Theorem 3.3] yields

$$u^{h,\delta} \in W^3_{\alpha+1,B}(\Omega) \quad \text{and} \quad \partial_x u^{h,\delta} = 0 \quad \text{a.e. on } \partial\Omega.$$

Summarising, we find that $u^{h,\delta} \in W^3_{\alpha+1,B}(\Omega)$ satisfies the boundary-value problem

$$\begin{cases} \psi = \partial_x^2 u^{h,\delta} + C & \text{in } \Omega, \\ \partial_x u^{h,\delta} \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, the minimiser $(u^{h,\delta}, j^{h,\delta})$ solves $j^{\frac{1-\alpha}{\alpha}} j^{h,\delta} = m_\delta(u^*)^{\frac{1}{\alpha}} \partial_x^3 u^{h,\delta}$. Hence, since the function $s \mapsto s^{\frac{1-\alpha}{\alpha}} s$ is invertible, we obtain

$$j^{h,\delta} = m_\delta(u^*)|\partial_x^3 u^{h,\delta}|^{\alpha-1} \partial_x^3 u^{h,\delta}.$$

Inserting this in the Euler–Lagrange equation, we obtain

$$\frac{u - u^*}{h} + \operatorname{div}\left(m_\delta(u^*)\partial_x^3 u^{h,\delta}\right) = 0.$$

This completes the proof. □

The Euler–Lagrange equation (6.3.5) of the functional $\mathcal{F}^{h,\delta}_{u^*}$ is in fact a time-discretised version of the modified thin-film equation. We can exploit this and define a minimising movement scheme with step size $h > 0$ as follows. We pick an initial value $u_0 \in H^1(\Omega)$ and define recursively $u_0^{h,\delta} := u_0$, and

$$\big(u^{h,\delta}_{(k+1)h}, j^{h,\delta}_{(k+1)h}\big) := \operatorname*{arg\,min}_{(u,j)} \mathcal{F}_{u_k^{h,\delta}}(u,j),$$

where the minimisation runs over all pairs $u \in H^1(\Omega)$, $j \in L_{\frac{\alpha+1}{\alpha}}(\Omega;\mathbb{R}^d)$ that solve the flow equation (6.3.2) with $u^* = u^{h,\delta}_{kh}$.

Comparing $\big(u^{h,\delta}_{(k+1)h}, j^{h,\delta}_{(k+1)h}\big)$ to $(u^{h,\delta}_{kh}, 0)$, we get the weak energy-dissipation inequality

$$\begin{aligned}
\mathcal{F}^{h,\delta}_{u_k^{h,\delta}}(u_k^{h,\delta}, 0) &= \int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{kh}\right|^2 \mathrm{d}x \\
&\geq \int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{(k+1)h}\right|^2 \mathrm{d}x + h\frac{\alpha}{\alpha+1}\int_\Omega \frac{\left|j^{h,\delta}_{(k+1)h}\right|^{\frac{\alpha+1}{\alpha}}}{m_\delta\big(u^{h,\delta}_{kh}\big)^{\frac{1}{\alpha}}}\,\mathrm{d}x \\
&= \mathcal{F}^{h,\delta}_{u_k^{h,\delta}}\big(u^{h,\delta}_{(k+1)h}, j^{h,\delta}_{(k+1)h}\big).
\end{aligned} \tag{6.3.7}$$

This inequality may even be improved. Indeed, using the elementary identity

$$|x|^2 - |y|^2 = 2y\cdot(x-y) + |x-y|^2, \quad x,y \in \mathbb{R}^d,$$

we deduce the energy-dissipation formula

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{kh}\right|^2 \mathrm{d}x \\
&= \int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{(k+1)h}\right|^2 \mathrm{d}x - \int_\Omega \partial_x^2 u^{h,\delta}_{(k+1)h}\big(u^{h,\delta}_{kh} - u^{h,\delta}_{(k+1)h}\big)\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{2}\left|\partial_x\big(u^{h,\delta}_{kh} - u^{h,\delta}_{(k+1)h}\big)\right|^2 \mathrm{d}x \\
&= \int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{(k+1)h}\right|^2 \mathrm{d}x + h\int_\Omega \partial_x^3 u^{h,\delta}_{(k+1)h}\cdot j^{h,\delta}_{(k+1)h}\,\mathrm{d}x + \int_\Omega \frac{1}{2}\left|\partial_x\big(u^{h,\delta}_{kh} - u^{h,\delta}_{(k+1)h}\big)\right|^2 \mathrm{d}x \\
&= \int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{(k+1)h}\right|^2 \mathrm{d}x + h\int_\Omega \frac{\left|j^{h,\delta}_{(k+1)h}\right|^{\frac{\alpha+1}{\alpha}}}{m_\delta\big(u^{h,\delta}_{kh}\big)^{\frac{1}{\alpha}}}\,\mathrm{d}x + \int_\Omega \frac{1}{2}\left|\partial_x\big(u^{h,\delta}_{kh} - u^{h,\delta}_{(k+1)h}\big)\right|^2 \mathrm{d}x.
\end{aligned}$$

This implies

$$\int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{kh}\right|^2 \mathrm{d}x \geq \int_\Omega \frac{1}{2}\left|\partial_x u^{h,\delta}_{(k+1)h}\right|^2 \mathrm{d}x + h\int_\Omega \frac{\left|j^{h,\delta}_{(k+1)h}\right|^{\frac{\alpha+1}{\alpha}}}{m_\delta\big(u^{h,\delta}_{kh}\big)^{\frac{1}{\alpha}}}\,\mathrm{d}x. \tag{6.3.8}$$

Now, we define the different interpolations of the minimising movement scheme which are used in the following. First, the piecewise constant interpolation $\big(\bar{u}^{h,\delta}, \bar{j}^{h,\delta}\big)$ is given by

$$\begin{cases} \bar{u}^{h,\delta}_t = u^{h,\delta}_{kh}, & t \in [kh,(k+1)h) \\ \bar{j}^{h,\delta}_t = j^{h,\delta}_{(k+1)h}, & t \in (kh,(k+1)h). \end{cases}$$

Note that $\bar{u}_t^{h,\delta}$ takes the value of $u^{h,\delta}$ at the current time, while $\bar{\jmath}^{h,\delta}$ takes the value of $j^{h,\delta}$ after one time step in the future. Moreover, the piecewise affine interpolation $\hat{u}^{h,\delta}$ is given by

$$\hat{u}_{(k+s)h}^{h,\delta} = (1-s)u_{kh}^{h,\delta} + s u_{(k+1)h}^{h,\delta}, \quad k \in \mathbb{N},\ s \in [0,1).$$

Lemma 6.3.4 (Discrete energy-dissipation inequality). *(i) The pair $\left(\bar{u}_t^{h,\delta}, \bar{\jmath}_t^{h,\delta}\right)$ satisfies the energy-dissipation inequality*

$$\int_\Omega \frac{1}{2}\left|\partial_x \bar{u}_t^{h,\delta}\right|^2 \mathrm{d}x + \frac{\alpha}{\alpha+1}\int_s^t \int_\Omega \frac{\left|\bar{\jmath}_\tau^{h,\delta}\right|^{\frac{\alpha+1}{\alpha}}}{2m_\delta\left(\bar{u}_\tau^{h,\delta}\right)^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}\tau \tag{6.3.9}$$

$$+\frac{1}{\alpha+1}\int_s^t\int_\Omega m_\delta\left(\bar{u}_\tau^{h,\delta}\right)\left|\partial_x^3 \bar{u}_{\tau+h}^{h,\delta}\right|^{\alpha+1}\mathrm{d}x\,\mathrm{d}\tau \le \int_\Omega \frac{1}{2}\left|\partial_x \bar{u}_s^{h,\delta}\right|^2 \mathrm{d}x$$

for all $kh = s < t = lh$ and $k, l \in \mathbb{N}_0$.

(ii) The pair $\left(\hat{u}_t^{h,\delta}, \bar{\jmath}_t^{h,\delta}\right)$ solves the continuity equation

$$\begin{cases} \partial_t \hat{u}_t^{h,\delta} + \operatorname{div} \bar{\jmath}_t^{h,\delta} = 0, & t>0,\ x \in \Omega,\\ \partial_x \bar{u}_t^{h,\delta} = 0, & t>0,\ x\in\partial\Omega,\\ \bar{\jmath}_t^{h,\delta} = 0, & t>0,\ x\in\partial\Omega, \end{cases}$$

in sense of distributions, that is, the equation

$$\int_0^T \int_\Omega \partial_t \hat{u}_t^{h,\delta}\varphi - \bar{\jmath}_t^{h,\delta}\cdot\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t = 0$$

holds true for all $\varphi \in C^\infty\left([0,T]\times\bar{\Omega}\right)$ and all $T>0$.

Proof. **(i)** The energy-dissipation inequality may be derived from (6.3.8), using that

$$h\int_\Omega \frac{\left|j_{(k+1)h}^{h,\delta}\right|^{\frac{\alpha+1}{\alpha}}}{m_\delta\left(u_{kh}^{h,\delta}\right)^{\frac{1}{\alpha}}}\,\mathrm{d}x = \frac{\alpha}{\alpha+1}\int_{kh}^{(k+1)h}\int_\Omega \frac{\left|\bar{\jmath}_\tau^{h,\delta}\right|^{\frac{\alpha+1}{\alpha}}}{m_\delta\left(\bar{u}_\tau^{h,\delta}\right)^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+\frac{1}{\alpha+1}\int_{kh}^{(k+1)h}\int_\Omega m_\delta\left(\bar{u}_\tau^{h,\delta}\right)\left|\partial_x^3\bar{u}_{\tau+h}^{h,\delta}\right|^{\alpha+1}\mathrm{d}x\,\mathrm{d}\tau.$$

Here we used that from (6.3.5) we know that

$$\bar{\jmath}_\tau^{h,\delta} = m_\delta\left(\bar{u}_\tau^{h,\delta}\right)\left|\partial_x^3 \bar{u}_{\tau+h}^{h,\delta}\right|^{\alpha-1}\partial_x^3\bar{u}_{\tau+h}^{h,\delta}.$$

(ii) This follows immediately from

$$\partial_t \hat{u}_t^{h,\delta} = u_{(k+1)h}^{h,\delta} - u_{kh}^{h,\delta}, \quad t \in (kh,(k+1)h),\ x\in\Omega,$$

and the constraints

$$\begin{cases} \frac{u_{(k+1)h}^{h,\delta}-u_{kh}^{h,\delta}}{h} + \operatorname{div} j_{(k+1)h}^{h,\delta} = 0 & \text{in } \Omega\\ \partial_x u_{kh}^{h,\delta} = 0 & \text{on } \partial\Omega\\ j_{(k+1)h}^{h,\delta} = 0 & \text{on } \partial\Omega, \end{cases}$$

cf. (6.3.5).

□

The previous result holds true in general dimension $d \geq 1$ on bounded Lipschitz domains, replacing ∂_x^3 by $\nabla\partial_x^2$ and ∂_x by ∇ and interpreting the boundary conditions to hold true in normal direction and in the sense of traces. However, in order to evaluate the limit $h \to 0$ for $\bar{j}^{h,\delta}$, we have to guarantee that $m_\delta(\bar{u}_t^{h,\delta})$ converges strongly. This requires that the sequence $\bar{u}^{h,\delta}$ converges uniformly to a continuous function which is only valid in dimension $d = 1$.

Lemma 6.3.5 (Uniform a-priori estimates). *Let $d = 1$. For each $\delta > 0$ and each $T > h > 0$, the families $\left(\bar{u}^{h,\delta}, \bar{j}^{h,\delta}\right)_h$ and $\left(\hat{u}^{h,\delta}\right)_h$ satisfy the regularity properties*

(i) $\bar{u}^{h,\delta} \in L_\infty\big([0,T]; H^1(\Omega)\big)$;

(ii) $\bar{u}^{h,\delta} \in L_{\alpha+1}\big([h,T]; W^3_{\alpha+1,B}(\Omega)\big)$;

(iii) $\bar{j}^{h,\delta} \in L_{\frac{\alpha+1}{\alpha}}\big([0,T] \times \Omega\big)$;

(iv) $\partial_t \hat{u}^{h,\delta} \in L_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1}(\Omega)\big)'\big)$;

(v) $\hat{u}^{h,\delta} \in L_{\alpha+1}\big([h,T]; W^3_{\alpha+1,B}(\Omega)\big) \cap L_\infty([0,T]; H^1(\Omega))$,

with bounds that are uniform in h. That is, there exists a constant $C > 0$, independent of h such that the families $\left(\bar{u}^{h,\delta}, \bar{j}^{h,\delta}\right)_h$ and $\left(\hat{u}^{h,\delta}\right)_h$ are, for each $h > 0$, bounded by C in the respective norms. In particular, there exist subsequences $\left(\bar{u}^{h,\delta}, \bar{j}^{h,\delta}\right)_h$ and $\left(\hat{u}^{h,\delta}\right)_h$ (not relabeled) and a limit function

$$\left(u^\delta, j^\delta\right) \in \left[L_2\big((0,T]; H^3_N(\Omega)\big) \cap H^1\big([0,T]; H^{-1}_N(\Omega)\big)\right] \times L_2\big([0,T]; L_2(\Omega; T\Omega)\big)$$

such that

$$\begin{cases} \bar{u}^{h,\delta} \rightharpoonup u^\delta & \text{weakly in } L_{\alpha+1}\big([\varepsilon,T]; W^3_{\alpha+1,B}(\Omega)\big); \\ \bar{j}^{h,\delta} \rightharpoonup j^\delta & \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big([0,T]\times\Omega\big); \\ \hat{u}^{h,\delta} \rightharpoonup u^\delta & \text{weakly in } L_{\alpha+1}\big([\varepsilon,T]; W^3_{\alpha+1,B}(\Omega)\big); \\ \partial_t \hat{u}^{h,\delta} \rightharpoonup u^\delta & \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big); \\ \hat{u}^{h,\delta} \longrightarrow u^\delta & \text{strongly in } C\big([0,T]; C^\rho(\bar{\Omega})\big) \end{cases}$$

for all $0 < \varepsilon < T$ and all $0 < \rho < \frac{1}{2}$. Furthermore, it holds that $u^\delta \in C([0,T]; H^1(\Omega))$.

The proof strongly relies on the energy-dissipation equality (6.3.9).

Proof. Step 1: Uniform a-priori estimates.
(i) and **(ii)** First, the energy-dissipation equality (6.3.9) immediately implies that

$$\int_\Omega \left|\partial_x \bar{u}_t^{h,\delta}\right|^2 dx \leq \int_\Omega \left|\partial_x u_0^{h,\delta}\right|^2 dx, \quad t \in [0,T],$$

where the right-hand side is bounded due to the regularity of the initial value. That is,

$$\partial_x \bar{u}^{h,\delta} \in L_\infty\big([0,T]; L_2(\Omega)\big).$$

Moreover, since the mobility m_δ is bounded below, there exists a constant $C_{\delta,\alpha} > 0$ such that

$$\int_h^T \int_\Omega \left|\partial_x^3 \bar{u}_t^{h,\delta}\right|^{\alpha+1} \mathrm{d}t\,\mathrm{d}x \leq C_{\delta,\alpha} \int_\Omega \left|\partial_x u_0^{h,\delta}\right|^2 dx,$$

that is $\partial_x^3 \bar{u}^{h,\delta} \in L_{\alpha+1}\big([h,T]; L_{\alpha+1}(\Omega)\big)$ and, consequently,

$$\bar{u}^{h,\delta} \in L_{\alpha+1}\big([h,T]; W^3_{\alpha+1,B}(\Omega)\big)$$

with a uniform bound only dependent on $\|u_0\|_{H^1(\Omega)}$.
(iii) This follows from the energy-dissipation equality (6.3.9), using the upper bound on the mobility m_δ.
(iv) The regularity of the flux $\bar{j}^{h,\delta}$ obtained in (iii) implies in particular that $\operatorname{div} \bar{j}^{h,\delta} \in L_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1}(\Omega)\big)'\big)$. Therefore, the continuity equation $\partial_t \hat{u}^{h,\delta} = -\operatorname{div} \bar{j}^{h,\delta}$, obtained in Lemma 6.3.4 (ii), yields

$$\partial_t \hat{u}^{h,\delta} \in L_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big),$$

where the corresponding uniform bound follows by (iii).
(v) By definition of the piecewise affine interpolation $\hat{u}^{h,\delta}$ we have that

$$\|\hat{u}^{h,\delta}_t\|^{\alpha+1}_{W^3_{\alpha+1,B}(\Omega)} \le C \left(\|\bar{u}^{h,\delta}_t\|^{\alpha+1}_{W^3_{\alpha+1,B}(\Omega)} + \|\bar{u}^{h,\delta}_{t+h}\|^{\alpha+1}_{W^3_{\alpha+1,B}(\Omega)} \right), \quad t > 0.$$

Integration with respect to t and the a-priori bound obtained in (i) yield the desired bound. The same argument applies to the H^1-bound.

Step 2: Compactness.
Combining the uniform bounds proved in step 1, an application of the Eberlein–Šmulian theorem provides the existence of the subsequences and weak accumulation points

$$\begin{cases} \bar{u}^{h,\delta} \rightharpoonup u^\delta & \text{weakly in } L_{\alpha+1}\big([\varepsilon,T]; W^3_{\alpha+1,B}(\Omega)\big); \\ \bar{j}^{h,\delta} \rightharpoonup j^\delta & \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big); \\ \hat{u}^{h,\delta} \rightharpoonup u^\delta & \text{weakly in } L_{\alpha+1}\big([\varepsilon,T]; W^3_{\alpha+1,B}(\Omega)\big) \\ \partial_t\hat{u}^{h,\delta} \rightharpoonup \partial_t u^\delta & \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big). \end{cases}$$

By the Aubin–Lions–Simon lemma [Sim86], we find that $\hat{u}^{h,\delta}$ converges strongly (up to a subsequence) in $C([\varepsilon,T]; C^\rho(\bar{\Omega}))$ for every $\varepsilon > 0$ and $0 < \rho < \frac{1}{2}$. Moreover, by [Ber88, Remark 3.4], every function $u^\delta \in L_{\alpha+1}\big((0,T]; W^3_{\alpha+1,B}(\Omega)\big)$ with $\partial_t \hat{u}^\delta \in L_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big)$ satisfies $u^\delta \in C([0;T]; H^1(\Omega))$ and we also conclude $u^\delta(0) = u_0$.

Step 3: Uniqueness of the limit function.
We have claimed above that both $\hat{u}^{h,\delta}$ and $\bar{u}^{h,\delta}$ converge to the same limit u^δ. Indeed, we observe that

$$\begin{aligned} \hat{u}^{h,\delta}_{(k+s)h} - \bar{u}^{h,\delta}_{(k+s)h} &= (1-s)u^{h,\delta}_{kh} + s u^{h,\delta}_{(k+1)h} - u^{h,\delta}_{kh} \\ &= s\big(u^{h,\delta}_{(k+1)h} - u^{h,\delta}_{kh}\big) \\ &= sh \operatorname{div} j^{h,\delta}_{(k+1)h} \\ &= s\,h \operatorname{div} \bar{j}^{h,\delta}_{(k+s)h}, \quad k \in \mathbb{N},\ s \in [0,1). \end{aligned}$$

This implies

$$\begin{aligned} \left\| \hat{u}^{h,\delta}_{(k+s)h} - \bar{u}^{h,\delta}_{(k+s)h} \right\|_{L_{\frac{\alpha+1}{\alpha}}((0,T];(W^1_{\alpha+1,B}(\Omega)'))} &\le h \left\| \operatorname{div} \bar{j}^{h,\delta} \right\|_{L_{\frac{\alpha+1}{\alpha}}((0,T];(W^1_{\alpha+1,B}(\Omega)'))} \\ &\le h \left\| \bar{j}^{h,\delta} \right\|_{L_{\frac{\alpha+1}{\alpha}}((0,T)\times\Omega)} \le C\,h. \end{aligned}$$

This proves that the limit functions coincide. □

Proposition 6.3.6 (Uniform convergence of the energy). *Fix $\delta > 0$. There is a subsequence (not relabelled) of $(\bar{u}^{h,\delta})_h$ such that*

$$\bar{u}^{h,\delta} \longrightarrow u^\delta \quad \textit{strongly in } L_{\alpha+1,\mathrm{loc}}\big([0,\infty); H^1(\Omega)\big)$$

and

$$E[\bar{u}^{h,\delta}] \longrightarrow E[u^\delta] \quad \textit{uniformly on compact subsets of } [0,\infty).$$

The proof of the second part of Proposition 6.3.6 relies on the following result from basic calculus which we prove here for the convenience of the reader.

Lemma 6.3.7. *Let $f_k\colon [0,T] \to \mathbb{R}$ be a sequence of non-increasing real-valued functions. Assume that $f_k(t) \to f(t)$ pointwise for $t \in [0,T]$, where $f\colon [0,T] \to \mathbb{R}$ is a continuous function. Then $f_k \to f$ uniformly.*

Proof. Fix $\varepsilon > 0$. Since f is continuous on the compact interval $[0,T]$, f is uniformly continuous. So there is $\delta > 0$ and a subdivision $0 = t_0 < t_1 < \ldots < t_n = T$ with $t_{i+1} - t_i < \delta$ for every $i = 0, \ldots, n-1$, such that

$$|f(t) - f(s)| < \frac{\varepsilon}{2} \quad \text{for all } t, s \in [t_i, t_{i+1}] \text{ and all } i = 0, \ldots, n-1.$$

Since $f_n(t_i) \to f(t_i)$, we find $N \in \mathbb{N}$ such that

$$|f_n(t_i) - f(t_i)| \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq N \text{ and } i = 0, \ldots, n.$$

We claim that $|f_n(t) - f(t)| \leq \varepsilon$ for all $n \geq N$ and $t \in [0,T]$. Fix $t \in [0,T]$, then there is $i \in \{1, \ldots, n\}$ such that $t \in [t_i, t_{i+1}]$. Since f_n is non-increasing, we know that

$$f_n(t_{i+1}) \leq f_n(t) \leq f_n(t_i)$$

and hence

$$\begin{aligned}
|f_n(t) - f(t)| &\leq \max\{|f_n(t_{i+1}) - f(t)|, |f_n(t_i) - f(t)|\} \\
&\leq \max\{|f_n(t_{i+1}) - f(t_{i+1})|, |f_n(t_i) - f(t_i)|\} \\
&\quad + \max\{|f(t_{i+1}) - f(t)|, |f(t_i) - f(t)|\} \\
&\leq \varepsilon
\end{aligned}$$

for every $n \geq N$. This proves the lemma. □

Now we turn to the proof of the uniform convergence of the energy. The compactness result follows from a modification of the Aubin–Lions–Simon lemma to piecewise constant functions which can be found in [DJ12].

Proof of Proposition 6.3.6. $\bar{u}^{h,\delta}$ is a sequence of piecewise constant functions. In order to obtain compactness in $L_{\alpha+1,\mathrm{loc}}\big([0,\infty]; H^1(\Omega)\big)$, we apply [DJ12, Theorem 1]. Note that $W^3_{\alpha+1,B}(\Omega)$ embeds compactly in $H^1(\Omega)$ and that the embedding of $H^1(\Omega)$ into $\big(W^1_{\alpha+1,B}(\Omega)\big)'$ is continuous. Furthermore, by Lemma 6.3.5, we know that $(\bar{u}^{h,\delta})_h$ is uniformly bounded in $L_{\alpha+1}\big([\varepsilon,T]; W^3_{\alpha+1,B}(\Omega)\big)$ and that

$$\begin{aligned}
\frac{1}{h}\|\bar{u}^{h,\delta}_{t+h} - \bar{u}^{h,\delta}_t\|_{L_{\frac{\alpha+1}{\alpha}}([0,T];(W^1_{\alpha+1}(\Omega))')} &= \|\operatorname{div} j^{h,\delta}_{t+h}\|_{L_{\frac{\alpha+1}{\alpha}}([0,T];(W^1_{\alpha+1}(\Omega))')} \\
&\leq C\|j^{h,\delta}_{t+h}\|_{L_{\frac{\alpha+1}{\alpha}}([0,T]\times\Omega)} \leq C
\end{aligned}$$

is also uniformly bounded in h. Hence, there is a subsequence (not relabelled) which converges in $L_{\alpha+1}\big([0,T];H^1(\Omega)\big)$. Taking $T = T_n = n$ and a diagonal subsequence ensures that $\bar{u}^{h,\delta} \to u^{h,\delta}$ in $L_{\alpha+1,\text{loc}}\big([0,\infty);H^1(\Omega)\big)$.

For the uniform convergence of the energy, we apply Lemma 6.3.7. By the strong convergence in $L_{\alpha+1,\text{loc}}\big([0,\infty);H^1(\Omega)\big)$ we know that there is a subsequence (not relabelled) such that

$$\bar{u}^{h,\delta}(t) \longrightarrow u^\delta(t) \quad \text{in } H^1(\Omega) \text{ for almost every } t \in [0,\infty).$$

Since $u^\delta \in C\big([0,\infty);H^1(\Omega)\big)$ and hence the limit function is defined for every $t \in [0,\infty)$, we may assume (after potentially modifying $\bar{u}^{h,\delta}$ on a set of measure zero that $\bar{u}^{h,\delta}(t)$ converges to $u^\delta(t)$ for every $t \in [0,\infty)$. This guarantees that $E[u^{h,\delta}](t) \to E[u](t)$ for every $t \in [0,\infty)$. By continuity of the limit function, we also obtain that $t \mapsto E[u^\delta](t)$ is continuous. Finally, monotonicity of $t \mapsto E[\bar{u}^{h,\delta}](t)$ follows from the construction of $\bar{u}^{h,\delta}$ as minimising movement scheme via the weak energy-dissipation inequality (6.3.7). Hence, we may apply Lemma 6.3.7 to obtain uniform convergence on compact subsets. □

Thanks to Lemma 6.3.5, Proposition 6.3.6, and by the uniform continuity of the mobility m_δ, we are able to preserve the energy-dissipation inequality (6.3.9) in the limit $h \to 0$ for every $0 \le s,t < \infty$ based on lower semicontinuity of the dissipation.

Proposition 6.3.8 (Energy dissipation inequality). *Any weak limit point*

$$(u^\delta, j^\delta) \in \Big[L_{\alpha+1}\big([0,T];W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,T];W^1_{\alpha+1,B}(\Omega)'\big)\Big] \times L_{\frac{\alpha+1}{\alpha}}\big([0,T]\times\Omega\big)$$

of the family $(\hat{u}^{h,\delta}, \bar{j}^{h,\delta})_h$ has the following properties.

(i) For all $0 \le s < t \le T$, the energy-dissipation inequality

$$\begin{aligned}\int_\Omega \frac{1}{2}|\partial_x u^\delta_t|^2\,\mathrm{d}x + \frac{\alpha}{\alpha+1}\int_s^t\int_\Omega \frac{|j^\delta_\tau|^{\frac{\alpha+1}{\alpha}}}{m_\delta(u^\delta_\tau)^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}\tau& \\ + \frac{1}{\alpha+1}\int_s^t\int_\Omega m_\delta(u^\delta_\tau)|\partial_x^3 u^\delta_\tau|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau \le \int_\Omega \frac{1}{2}|\partial_x u^\delta_s|^2\,\mathrm{d}x&\end{aligned} \tag{6.3.10}$$

is satisfied.

(ii) The pair $\big(u^\delta, j^\delta\big)$ solves the continuity equation

$$\begin{cases} \partial_t u^\delta + \operatorname{div} j^\delta = 0, & t>0,\ x\in\Omega,\\ \partial_x u^\delta = 0, & t>0,\ x\in\partial\Omega,\\ j^\delta = 0, & t>0,\ x\in\partial\Omega,\end{cases}$$

in sense of distributions, that is, the equation

$$\int_0^T \langle \partial_t u^\delta_t, \varphi\rangle_{W^1_{\alpha+1}}\,\mathrm{d}t - \int_0^T\int_\Omega j^\delta_t\cdot\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t = 0$$

holds true for all $\varphi \in L_{\alpha+1}\big([0,T];W^1_{\alpha+1}(\Omega)\big)$ and all $T>0$.

Proof. **(i)** From the uniform convergence of $E[\bar{u}^{h,\delta}]$ to $E[u^\delta]$ proved in Proposition 6.3.6 and continuity of $t \mapsto E[u^\delta](t)$, we obtain

$$\int_\Omega \frac{1}{2}\left|\partial_x \bar{u}^{h,\delta}_{t_h}\right|^2 dx \longrightarrow \int_\Omega \frac{1}{2}\left|\partial_x u^\delta_t\right|^2 dx \quad \text{as } h\to 0 \text{ and } t_h \to t \tag{6.3.11}$$

for any $0 \leq t \leq T$. Let now $s_h = \lfloor s/h \rfloor h$ and $t_h = \lceil t/h \rceil h$. Then, in view of (6.3.9) we know that

$$\int_\Omega \frac{1}{2} \left| \partial_x \bar{u}^{h,\delta}_{t_h} \right|^2 \mathrm{d}x + \frac{\alpha}{\alpha+1} \int_{s_h}^{t_h} \int_\Omega \frac{\left| \bar{j}^{h,\delta}_\tau \right|^{\frac{\alpha+1}{\alpha}}}{2 m_\delta \big(\bar{u}^{h,\delta}_\tau \big)^{\frac{1}{\alpha}}} \mathrm{d}x \, \mathrm{d}\tau + \frac{1}{\alpha+1} \int_s^t \int_\Omega m_\delta \big(\bar{u}^{h,\delta}_\tau \big) \left| \partial_x^3 \bar{u}^{h,\delta}_{\tau+h} \right|^{\alpha+1} \mathrm{d}x \, \mathrm{d}\tau \leq \int_\Omega \frac{1}{2} \left| \partial_x \bar{u}^{h,\delta}_{s_h} \right|^2 \mathrm{d}x.$$

Taking the lim inf on both sides and using (6.3.11) which guarantees convergence of the energy, we obtain

$$\int_\Omega \frac{1}{2} \left| \partial_x u^\delta_t \right|^2 dx + \liminf_{h\to 0} \left[\frac{\alpha}{\alpha+1} \int_{s_h}^{t_h} \int_\Omega \frac{\left| j^{h,\delta}_\tau \right|^{\frac{\alpha+1}{\alpha}}}{m_\delta (u^{h,\delta}_\tau)^{\frac{1}{\alpha}}} \mathrm{d}x \, \mathrm{d}\tau + \frac{1}{\alpha+1} \int_{s_h}^{t_h} \int_\Omega m_\delta (u^{h,\delta}_\tau) |\partial_x^3 u^{h,\delta}_\tau|^{\alpha+1} \mathrm{d}x \, \mathrm{d}\tau \right] \leq \int_\Omega \frac{1}{2} \left| \partial_x u^\delta_s \right|^2 dx.$$

It remains to show

$$\int_s^t \int_\Omega \frac{|j^\delta_\tau|^2}{2 m_\delta (u^\delta_\tau)} + \frac{m_\delta \big(u^\delta_\tau \big) |\partial_x^3 u^\delta_\tau|^2}{2} \mathrm{d}x \, \mathrm{d}\tau \leq \liminf_{h\to 0} \left[\frac{\alpha}{\alpha+1} \int_{s_h}^{t_h} \int_\Omega \frac{\left| \bar{j}^{h,\delta}_\tau \right|^{\frac{\alpha+1}{\alpha}}}{m_\delta (u^{h,\delta}_\tau)^{\frac{1}{\alpha}}} \mathrm{d}x \, \mathrm{d}\tau + \frac{1}{\alpha+1} \int_{s_h}^{t_h} \int_\Omega m_\delta (u^{h,\delta}_\tau) |\partial_x^3 u^{h,\delta}_\tau|^{\alpha+1} \mathrm{d}x \, \mathrm{d}\tau \right].$$

Since $s_h \leq s < t \leq t_h$ for every $h > 0$ and by non-negativity of the integrand, it suffices to prove

$$\int_s^t \int_\Omega \frac{|j^\delta_\tau|^2}{2 m_\delta (u^\delta_\tau)} + \frac{m_\delta \big(u^\delta_\tau \big) |\partial_x^3 u^\delta_\tau|^2}{2} \mathrm{d}x \, \mathrm{d}\tau \leq \liminf_{h\to 0} \left[\frac{\alpha}{\alpha+1} \int_s^t \int_\Omega \frac{\left| \bar{j}^{h,\delta}_\tau \right|^{\frac{\alpha+1}{\alpha}}}{m_\delta (u^{h,\delta}_\tau)^{\frac{1}{\alpha}}} \mathrm{d}x \, \mathrm{d}\tau + \frac{1}{\alpha+1} \int_s^t \int_\Omega m_\delta (u^{h,\delta}_\tau) |\partial_x^3 u^{h,\delta}_\tau|^{\alpha+1} \mathrm{d}x \, \mathrm{d}\tau \right]. \tag{6.3.12}$$

In Lemma 6.3.5 we showed that

$$\hat{u}^{h,\delta} \longrightarrow u^\delta \quad \text{strongly in } C\big([0,T] \times \bar{\Omega}\big).$$

In virtue of the Arzelà–Ascoli theorem, this implies equicontinuity and thus also the uniform convergence

$$\bar{u}^{h,\delta} \longrightarrow u^\delta \quad \text{as } h \to 0,$$

since $\bar{u}^{h,\delta}$ is a piecewise constant approximation of $\hat{u}^{h,\delta}$. Using the uniform continuity of the mobility function m_δ, we find that

$$m_\delta(\bar{u}^{h,\delta}) \longrightarrow m_\delta(u^\delta) \quad \text{uniformly as } h \to 0.$$

Since $m_\delta(s) \geq \delta,\ s \in \mathbb{R}$, by assumption, and since

$$\bar{j}^{h,\delta} \rightharpoonup j^\delta \quad \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big([0,T]\times\Omega\big)$$

by Lemma 6.3.5, this implies by weak lower semicontinuity of the norm

$$\int_s^t \int_\Omega \frac{|j_\tau^\delta|^{\frac{\alpha+1}{\alpha}}}{m_\delta(u_\tau^\delta)^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}\tau \leq \liminf_{h\to 0} \int_s^t \int_\Omega \frac{|\bar{j}_\tau^{h,\delta}|^{\frac{\alpha+1}{\alpha}}}{m_\delta(u_\tau^{h,\delta})^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}\tau.$$

For the second term in (6.3.12) we use that

$$\partial_x^3 \bar{u}_{\cdot+h}^{h,\delta} \rightharpoonup \partial_x^3 u^\delta \quad \text{weakly in } L_{\alpha+1}\big([0,T]\times\Omega\big),$$

since $\partial_x^3 \bar{u}_{\cdot+h}^{h,\delta}$ is uniformly bounded in $L_{\alpha+1}\big([0,T]\times\Omega\big)$ and converges to $\partial_x^3 u^\delta$ in the sense of distributions. Combining this with the uniform convergence of $m_\delta(\bar{u}^{h,\delta})$ to $m_\delta(u^\delta)$ and by weak lower semicontinuity of the norm, we deduce that

$$\int_s^t \int_\Omega m_\delta\big(u_\tau^\delta\big)|\partial_x^3 u_\tau^\delta|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau \leq \liminf_{h\to 0} \int_s^t \int_\Omega m_\delta\big(\bar{u}_\tau^{h,\delta}\big)\left|\partial_x^3 \bar{u}_{\tau+h}^{h,\delta}\right|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau.$$

(ii) That the continuity equation is satisfied for all $\varphi \in C^\infty\big([0,T]\times\bar{\Omega}\big)$ in the limit $h \to 0$ follows from Lemma 6.3.4 (ii) and the weak convergence results of Lemma 6.3.5. By density, we may extend this to $\varphi \in L_{\alpha+1}\big([0,T];W^1_{\alpha+1}(\Omega)\big)$. This completes the proof. □

6.4 Energy-dissipation equality and the modified thin-film equation

In this section, we want to study the limiting equation in the case $\delta > 0$. We want to prove that if a pair

$$(u^\delta, j^\delta) \in \Big[L_{\alpha+1}\big([0,T];W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,T];\big(W^1_{\alpha+1,B}(\Omega)\big)'\big)\Big] \times L_{\frac{\alpha+1}{\alpha}}\big([0,T]\times\Omega\big)$$

satisfies the energy-dissipation inequality (6.3.10) and the continuity equation

$$\begin{cases} \partial_t u^\delta + \operatorname{div} j^\delta = 0, & t>0,\ x\in\Omega,\\ \partial_x u^\delta = 0, & t>0,\ x\in\partial\Omega,\\ j^\delta = 0, & t>0,\ x\in\partial\Omega, \end{cases}$$

then, u^δ is a solution to the regularised power-law thin-film equation

$$\begin{cases} \partial_t u^\delta + \operatorname{div}\big(m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta\big) = 0, & t>0,\ x\in\Omega,\\ \partial_x u^\delta \cdot n = 0, & t>0,\ x\in\partial\Omega,\\ \partial_x^3 u^\delta \cdot n = 0, & t>0,\ x\in\partial\Omega \end{cases} \tag{6.4.1}$$

which satsifies the energy-dissipation equality

$$\int_\Omega \frac{1}{2}|\partial_x u_t^\delta|^2\,\mathrm{d}x + \int_s^t \int_\Omega m_\delta(u_\tau^\delta)|\partial_x^3 u_\tau^\delta|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau = \int_\Omega \frac{1}{2}|\partial_x u_s^\delta|^2\,\mathrm{d}x.$$

To this end, note first that every smooth solution (v, k) to the continuity equation

$$\begin{cases} \partial_t v + \operatorname{div} k = 0, & t > 0,\ x \in \Omega, \\ \partial_x v = 0, & t > 0,\ x \in \partial\Omega, \\ k = 0, & t > 0,\ x \in \partial\Omega \end{cases} \tag{6.4.2}$$

satisfies the reverse of inequality (6.3.10). Indeed, integrating by parts twice yields

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\partial_x v_t|^2 \,\mathrm{d}x = -\int_\Omega k_t \cdot \partial_x^3 v_t \,\mathrm{d}x = -\int_\Omega \frac{k_t}{m_\delta(v_t)^{\frac{1}{\alpha+1}}} \cdot m_\delta(v_t)^{\frac{1}{\alpha+1}} \partial_x^3 v_t \,\mathrm{d}x.$$

Applying Young's inequality $a \cdot b \le \frac{\alpha}{\alpha+1}|a|^{\frac{\alpha+1}{\alpha}} + \frac{1}{\alpha+1}|b|^{\alpha+1}$ and integrating with respect to time, we obtain

$$\begin{aligned} \int_\Omega \frac{1}{2}|\partial_x v_t|^2 \,\mathrm{d}x + \frac{\alpha}{\alpha+1}\int_s^t \int_\Omega \frac{|k_\tau|^{\frac{\alpha+1}{\alpha}}}{m_\delta(v_\tau)^{\frac{1}{\alpha}}} \,\mathrm{d}x\,\mathrm{d}\tau \\ + \frac{1}{\alpha+1}\int_s^t \int_\Omega m_\delta(v_\tau)|\partial_x^3 v_\tau|^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}\tau \ge \int_\Omega \frac{1}{2}|\partial_x v_s|^2 \,\mathrm{d}x. \end{aligned} \tag{6.4.3}$$

We have equality in (6.4.3) if and only if Young's inequality holds with equality, i.e. if and only if

$$\frac{|k_\tau|^{\frac{\alpha+1}{\alpha}}}{m_\delta(v_\tau)^{\frac{1}{\alpha}}} = m_\delta(v_\tau)|\partial_x^3 v_\tau|^{\alpha+1} \quad \text{a.e. in } (0,\infty)\times\Omega.$$

The following proposition shows that the reverse energy-dissipation inequality is already satisfied for solutions (v, k) to (6.4.2) in the regularity class

$$\Big[L_{\alpha+1}\big([0,T]; W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big)\Big] \times L_{\alpha+1}\big([0,T]\times\Omega\big).$$

In addition, we prove that weak solutions to the regularised thin-film equation (6.4.1) are characterised by equality in (6.4.3).

Proposition 6.4.1. *If*

$$(v,k) \in \Big[L_{\alpha+1}\big([0,T]; W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big)\Big] \times L_{\alpha+1}\big([0,T]\times\Omega\big)$$

satisfies the continuity equation (6.4.2) *in the sense that*

$$\int_0^\infty \langle \partial_t v_t, \varphi\rangle_{W^1_{\alpha+1}} \,\mathrm{d}t - \int_0^\infty \int_\Omega k_t \cdot \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{6.4.4}$$

for all $\varphi \in L_{\alpha+1}\big([0,\infty); W^1_{\alpha+1}(\Omega)\big)$*, then*

$$\begin{aligned} \int_\Omega \frac{1}{2}|\partial_x v_t|^2 \,\mathrm{d}x + \frac{\alpha}{\alpha+1}\int_s^t \int_\Omega \frac{|k_\tau|^{\frac{\alpha+1}{\alpha}}}{m_\delta(v_\tau)^{\frac{1}{\alpha}}} \,\mathrm{d}x\,\mathrm{d}\tau \\ + \frac{1}{\alpha+1}\int_s^t \int_\Omega m_\delta(v_\tau)|\partial_x^3 v_\tau|^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}\tau \ge \int_\Omega \frac{1}{2}|\partial_x v_s|^2 \,\mathrm{d}x \end{aligned} \tag{6.4.5}$$

holds for all $0 \leq s < t < \infty$. *Moreover, equality holds if and only if* v *is a weak solution to the regularised thin-film equation*

$$\begin{cases} \partial_t v + \partial_x\big(m_\delta(v)|\partial_x^3 v|^{\alpha-1}\partial_x^3 v\big) = 0, & t > 0,\ x \in \Omega, \\ \partial_x v = 0, & t > 0,\ x \in \partial\Omega, \\ \partial_x^3 v = 0, & t > 0,\ x \in \partial\Omega, \end{cases}$$

in the sense that $v \in L_{\alpha+1}\big([0,T]; W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big)$ *satisfies the equation*

$$\int_0^\infty \langle \partial_t v, \varphi\rangle_{W^1_{\alpha+1}} \,\mathrm{d}t - \int_0^\infty \int_\Omega m_\delta(v)|\partial_x^3 v|^{\alpha-1}\partial_x^3 v \cdot \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{6.4.6}$$

for all $\varphi \in L_{\alpha+1}\big([0,\infty); W^1_{\alpha+1}(\Omega)\big)$.

Proof. We prove that the Dirichlet energy is absolutely continuous in time, i.e. that for all $0 \leq s < t < \infty$, we have

$$\int_\Omega \frac{1}{2}\left|\partial_x v_t\right|^2 \,\mathrm{d}x - \int_\Omega \frac{1}{2}\left|\partial_x v_s\right|^2 \,\mathrm{d}x = -\int_s^t \int_\Omega k_\tau \cdot \partial_x^3 v_\tau \,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.4.7}$$

As a first step, we show that

$$\int_\Omega \frac{1}{2}\left|\partial_x v_t\right|^2 \,\mathrm{d}x - \int_\Omega \frac{1}{2}\left|\partial_x v_s\right|^2 \,\mathrm{d}x = -\int_s^t \langle \partial_\tau v_\tau, \partial_x^2 v_\tau\rangle_{W^1_{\alpha+1}} \,\mathrm{d}\tau \tag{6.4.8}$$

for all $v \in L_{\alpha+1}\big([0,T]; W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,T]; \big(W^1_{\alpha+1,B}(\Omega)\big)'\big)$ and $0 \leq s < t < \infty$. To this end, we mollify in time by introducing, for $\varepsilon > 0$,

$$v^\varepsilon = \eta_\varepsilon * v \in C^\infty\big([\varepsilon,\infty); W^3_{\alpha+1,B}(\Omega)\big).$$

Then, v^ε satisfies

$$\int_\Omega \frac{1}{2}\left|\partial_x v_t^\varepsilon\right|^2 \,\mathrm{d}x - \int_\Omega \frac{1}{2}\left|\partial_x v_s^\varepsilon\right|^2 \,\mathrm{d}x = -\int_s^t \langle \partial_\tau v_\tau^\varepsilon, \partial_x^2 v_\tau^\varepsilon\rangle_{W^1_{\alpha+1}} \,\mathrm{d}\tau \tag{6.4.9}$$

for all $0 < \varepsilon \leq s < t < \infty$. Moreover, we know that, for every $s > 0$,

$$\begin{cases} v^\varepsilon \longrightarrow v & \text{strongly in } C\big([s,\infty); H^1(\Omega)\big) \\ \partial_x^2 v^\varepsilon \longrightarrow \partial_x^2 v & \text{strongly in } L_{\alpha+1}\big([s,\infty); W^1_{\alpha+1}(\Omega)\big) \\ \partial_t v^\varepsilon \longrightarrow \partial_t v & \text{strongly in } L_{\frac{\alpha+1}{\alpha}}\big([s,\infty); W^1_{\alpha+1}(\Omega)'\big) \end{cases}$$

as $\varepsilon \to 0$. Here, the first convergence stated follows again from the generalised Lions–Magenes theorem, [Ber88, Remark 3.4], that is

$$v \in L_{\alpha+1}\big([0,T]; W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,T]; W^1_{\frac{\alpha+1}{\alpha},B}(\Omega)'\big)$$

implies that v has a continuous representative

$$\tilde{v} \in C([0,T]; H^1(\Omega))$$

with $\tilde{v} = v$ almost everywhere. Hence, we may take the limit in (6.4.9) to obtain (6.4.8) for $0 < s < t < \infty$. The case $s = 0$ follows by taking the limit $s \searrow 0$.

Now we note that

$$\int_s^t \langle \partial_\tau v_\tau, \partial_x^2 v_\tau \rangle_{W^1_{\alpha+1}} \,\mathrm{d}\tau = \int_s^t \int_\Omega k_\tau \cdot \partial_x^3 v_\tau \,\mathrm{d}x \,\mathrm{d}\tau$$

by testing the continuity equation (6.4.4) with $\partial_x^2 v_\tau \chi_{[s,t]} \in L_{\alpha+1}\big([0,\infty); W^1_{\alpha+1}(\Omega)\big)$, where $\chi_{[s,t]}$ denotes the characteristic function of the interval $[s,t]$. By an application of Young's inequality, we have

$$-k_\tau \cdot \partial_x^3 v_\tau \geq -\frac{\alpha}{\alpha+1} \frac{|k_\tau|^{\frac{\alpha+1}{\alpha}}}{m_\delta(v_\tau)^{\frac{1}{\alpha}}} - \frac{1}{\alpha+1} m_\delta(v_\tau) |\partial_x^3 v_\tau|^{\alpha+1} \quad \text{a.e. in } [0,\infty) \times \Omega,$$

which, together with (6.4.7), proves (6.4.5). Finally, equality in (6.4.5) holds for all $0 \leq s < t < \infty$ if and only if

$$k_\tau = -m_\delta(v_\tau) |\partial_x^3 v_\tau|^{\alpha-1} \partial_x^3 v_\tau \quad \text{a.e. in } [0,\infty) \times \Omega.$$

Inserting this in the continuity equation (6.4.4) proves (6.4.6). □

Since by Proposition 6.3.8 any accumulation point (u^δ, j^δ) of the family $(\hat{u}^{h,\delta}, \bar{j}^{h,\delta})_h$ satisfies the conditions of Proposition 6.4.1, we find that u^δ is a weak solution to the regularised thin-film equation (6.4.1). For flow-behaviour exponents $\alpha \neq 1$, this equations degenerates in the third derivative and hence we cannot claim uniqueness of solutions. For the Newtonian case $\alpha = 1$ though, uniqueness of solutions is well-known by standard parabolic theory.

Theorem 6.4.2. *Given $u_0 \in H^1(\Omega)$, there exists*

$$u^\delta \in C_b\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,\infty); W^1_{\alpha+1}(\Omega)'\big)$$

such that a subsequence of $(\hat{u}^{h,\delta}, \bar{j}^{h,\delta})_h$ converges as follows:

$$\begin{cases} \hat{u}^{h,\delta} \rightharpoonup u^\delta & \textit{weakly in } L_{\alpha+1}\big([0,\infty); W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,\infty); \big(W^1_{\alpha+1}(\Omega)\big)'\big), \\ \hat{u}^{h,\delta} \longrightarrow u^\delta & \textit{strongly in } C_{\mathrm{loc}}\big([0,\infty); C^\rho(\bar{\Omega})\big) \textit{ for every } 0 \leq \rho < \frac{1}{2}, \\ \bar{j}^{h,\delta} \rightharpoonup j^\delta & \textit{weakly in } L_{\frac{\alpha+1}{\alpha}}\big([0,\infty) \times \Omega\big). \end{cases}$$

Furthermore, it holds

$$j^\delta = m_\delta(u^\delta) \partial_x^3 u^\delta \quad \textit{a.e. in } [0,\infty) \times \Omega.$$

Moreover, u^δ is the weak solution to the initial-boundary-value problem

$$\begin{cases} \partial_t u^\delta + \operatorname{div}\big(m_\delta(u^\delta) \partial_x^3 u^\delta\big) = 0, & t > 0,\ x \in \Omega, \\ \partial_x u^\delta = \partial_x^3 u^\delta = 0, & t > 0,\ x \in \partial\Omega, \\ u^\delta(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{6.4.10}$$

and satisfies the energy-dissipation equality

$$\int_\Omega \frac{1}{2} |\partial_x u_t^\delta|^2 \,\mathrm{d}x + \int_s^t \int_\Omega m_\delta(u_\tau^\delta) |\partial_x^3 u_\tau^\delta|^{\alpha+1} \,\mathrm{d}x \,\mathrm{d}\tau = \int_\Omega \frac{1}{2} |\partial_x u_s^\delta|^2 \,\mathrm{d}x \tag{6.4.11}$$

for all $0 \leq s < t < \infty$. Furthermore, if $\alpha = 1$, i.e. if the fluid is Newtonian, there is exactly one accumulation point u^δ of the sequence $(\hat{u}^{h,\delta})$ and u^δ is the unique weak solution to (6.4.10).

Proof. **(i)** The convergence results and the regularity of the limit function u^δ have been proved in Lemma 6.3.5.

(ii) The energy-dissipation equality is satisfied in view of Proposition 6.3.8 and Proposition 6.4.1.

(iii) That u^δ satisfies the thin-film equation has been shown in Proposition 6.4.1.

(iv) Uniqueness of weak solutions – and thus of the limit point – follows from standard parabolic theory [Paz83], using that $m_\delta(s) \geq \delta$ for all $s \in \mathbb{R}$. □

UNIQUENESS AND NON-NEGATIVITY IN THE NEWTONIAN CASE

Next, we show that the region where u^δ is negative is small. To this end, we define an entropy as in [BF90]. Let $A > \max_{(t,x)\in[0,\infty)\times\bar\Omega} |u^\delta|$ for all $\delta \in (0,1)$. Then we define

$$g_\delta(s) = -\int_s^A \frac{1}{m_\delta(r)}\,\mathrm{d}r \quad \text{and} \quad G_\delta(s) = -\int_s^A g_\delta(r)\,\mathrm{d}r.$$

We also define

$$g(s) = -\int_s^A \frac{1}{m(r)}\,\mathrm{d}r \quad \text{and} \quad G(s) = -\int_s^A g(r)\,\mathrm{d}r.$$

Testing the regularised thin-film equation with $g_\delta(u^\delta)$, one can show that (cf. [BF90, eq. (4.17)])

$$\int_\Omega G_\delta(u_t^\delta)\,\mathrm{d}x + \int_0^t \int_\Omega |\partial_x^2 u_s^\delta|^2\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega G_\delta(u_0)\,\mathrm{d}x. \tag{6.4.12}$$

Lemma 6.4.3. *Given $u_0 \in H^1(\Omega)$, let $u^\delta \in C_b\big([0,\infty); H^1(\Omega)\big) \cap L_2\big((0,\infty); H_B^3(\Omega)\big) \cap H^1\big([0,\infty); H_B^{-1}(\Omega)\big)$ be the unique solution to the regularised thin-film equation as obtained in Theorem 6.4.2. Then u^δ satisfies*

$$\int_{\{u_\delta<0\}} \frac{|u^\delta|^2}{2\delta}\,\mathrm{d}x \leq \int_\Omega G_\delta(u_0)\,\mathrm{d}x \leq \int_\Omega G(u_0)\,\mathrm{d}x$$

for all $t > 0$.

Proof. For $s < 0$ we have that $m_\delta(s) = \delta$. This implies $g_\delta(s) \geq s/\delta$ and hence $G_\delta(s) \geq s^2/(2\delta)$. Together with (6.4.12) this yields the first inequality. The second inequality follows since $m_\delta(s) \geq m(s)$ for all $s \in \mathbb{R}$. □

6.5 The limit $\delta \to 0$: weak solutions to the thin-film equation

Now we investigate the limit as $\delta \to 0$. We show that, for a positive initial datum $u_0 \in H^1(\Omega)$, $u_0 > 0$, every accumulation point u of the family u^δ is a weak solution to the power-law thin-film equation

$$\begin{cases} \partial_t u + \operatorname{div}\big(m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\big) = 0, & t \in (0, \tilde T_{u_0,\delta}],\ x \in \Omega, \\ \partial_x u = m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u = 0, & t \in (0, \tilde T_{u_0,\delta}],\ x \in \partial\Omega, \\ u(0) = u_0, & x \in \Omega \end{cases} \tag{6.5.1}$$

in the sense of [BF90]. Here, typically $m(u) = u^n$ for some $n > 1$. Note that in the non-Newtonian case, the existence time of a physical, that is non-negative, solution is obstructed as the solution might become negative. In the Newtonian case, we can obtain non-negativity as long as we have control over the entropy for the initial value.

By the generic choice of m_δ in the previous sections, we already obtain the gradient flow structure of positive solutions to (6.5.1). Therefore, let $u_0 \in H^1(\Omega)$ satisfy $u_0 > \eta > 0$ for some $\eta > 0$. Also choose δ very small and m_δ such that $m_\delta(u) = u^n$ for every $u > 2\delta$. If $2\delta < \eta$ and by continuity of the solutions u^δ found in Theorem 6.4.2, there is a time $T_{u_0,\delta} > 0$ such that $\min_{(t,x)\in[0,T_{u_0,c}]\times\bar{\Omega}} u^\delta(t,x) > 2\delta$. In particular, u^δ is a weak solution to (6.5.1) for $t \in (0, T_{u_0,\delta})$.

We extend this result and show that every accumulation point u of the sequence u^δ is a solution to the thin-film equation (6.5.1) on the positivity set

$$\{u > 0\} := \{(t,x) \in [0,\infty) \times \Omega : u(t,x) > 0\}$$

.

To pass to the limit, we rely on uniform bounds. First, we show that from the energy-dissipation equality (6.4.11) for u^δ, we obtain a uniform Hölder bound in $C^{\frac{1}{5\alpha+3},\frac{1}{2}}([0,T] \times \bar{\Omega})$. The proof follows an argument given by [GR00] or [Ott00]. By the Arzelà–Ascoli theorem, this will guarantee uniform convergence of $m_\delta(u^\delta)$ to $m(u)$. This also guarantees that, if the initial datum $u_0 \in H^1(\Omega)$ is strictly positive $u_0 > 0$, there is a maximal time $\tau(u_0) > 0$ such that $[0, \tau(u_0)] \times \bar{\Omega} \subset \{u > 0\}$. We will show that $\tau(u_0) = +\infty$ for Newtonian fluids.

Lemma 6.5.1. *Given an initial datum $u_0 \in H^1(\Omega)$, let*

$$u^\delta \in C_b\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big) \cap W^1_{\frac{\alpha+1}{\alpha}}\big([0,\infty); W^1_{\alpha+1}(\Omega)'\big)$$

and

$$j^\delta \in L_{\frac{\alpha+1}{\alpha}}\big([0,\infty) \times \Omega\big)$$

satisfy the energy-dissipation equality

$$\int_\Omega \frac{1}{2}|\partial_x u_t^\delta|^2 \,\mathrm{d}x + \frac{\alpha+1}{\alpha}\int_0^t \int_\Omega \frac{|j^\delta|^{\frac{\alpha+1}{\alpha}}}{2m_\delta(u_\tau^\delta)^{\frac{1}{\alpha}}} \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{\alpha+1}\int_0^t \frac{m_\delta(u_\tau^\delta)|\partial_x^3 u_\tau^\delta|^{\alpha+1}}{2} \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_\Omega \frac{1}{2}|\partial_x u_0|^2 \,\mathrm{d}x$$

for all $t > 0$. Then u^δ is uniformly bounded in $C^{\frac{1}{5\alpha+3},\frac{1}{2}}([0,T] \times \bar{\Omega})$ for all $T > 0$.

Proof. We already know that $u^\delta \in C_b\big([0,\infty); H^1(\Omega)\big)$. In view of the Sobolev embedding theorem, we thus find that $u^\delta \in C_b\big([0,\infty); C^{1/2}(\bar{\Omega})\big)$.

It remains to prove the Hölder continuity in time. The proof follows the lines of [GR00, Lemma 4.2] or [Ott00, Lemma 3.1]. There the result is proven for Newtonian fluids. Consider (U^δ, J^δ) such that U^δ is the even extension of u^δ and J^δ is the odd extension of j^δ about $\partial\Omega$. Moreover, let η_ε be a standard mollifier in space and consider for $x \in \bar{\Omega}$ and $0 \le s < t \le T$

$$\begin{aligned}\left|u^\delta(t,x) - u^\delta(s,x)\right| &\le \left|U^\delta(t,x) - \eta_\varepsilon * U^\delta(t,x)\right| + \left|\eta_\varepsilon * U^\delta(t,x) - \eta_\varepsilon * U^\delta(s,x)\right| \\ &\quad + \left|\eta_\varepsilon * U^\delta(s,x) - U^\delta(s,x)\right| \\ &= (I) + (II) + (III).\end{aligned}$$

Since $u^\delta(t,\cdot) \in C^{\frac{1}{2}}(\bar\Omega)$, we find that (I) and (III) satisfy

$$(I) + (III) \le \varepsilon^{1/2}\left(\Big[U^\delta(t,\cdot)\Big]_{C^{1/2}(\bar\Omega)} + \Big[U^\delta(s,\cdot)\Big]_{C^{1/2}(\bar\Omega)}\right) \le 2\varepsilon^{1/2}\|\partial_x u_0\|_{L_2(\Omega)}.$$

Testing the continuity equation with $\varphi(\tau,y) = \mathbb{1}_{[s,t]}(\tau)\eta_\varepsilon(y-x)$, the second term may be estimated as follows using the bound $\|\eta_\varepsilon'\|_{L^{\alpha+1}(\Omega)} \le C\varepsilon^{-(2\alpha+1)/(\alpha+1)}$ together with the Hölder inequality

$$\begin{aligned}
&\left|\eta_\varepsilon * U^\delta(t,x) - \eta_\varepsilon * U^\delta(s,x)\right| \\
&= \left|\int_s^t \int_{\mathbb{R}} \eta_\varepsilon'(x-y) J^\delta(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\right| \\
&\le C\varepsilon^{-(2\alpha+1)/(\alpha+1)}|t-s|^{1/(\alpha+1)}\|J^\delta\|_{L_{\frac{\alpha+1}{\alpha}}((0,T)\times\Omega)} \\
&\le C\varepsilon^{-(2\alpha+1)/(\alpha+1)}|t-s|^{1/(\alpha+1)}\left(\int_0^T\int_\Omega \frac{|j^\delta|^{\frac{\alpha+1}{\alpha}}}{m_\delta(u^\delta)^{\frac{1}{\alpha}}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\alpha}{\alpha+1}} \|m_\delta(u^\delta)\|^{1/(\alpha+1)}_{L_\infty((0,T)\times\Omega)} \\
&\le C\varepsilon^{-(2\alpha+1)/(\alpha+1)}|t-s|^{1/(\alpha+1)}\|\partial_x u_0\|_{L_2(\Omega)},
\end{aligned}$$

where we use that $m_\delta(u^\delta)$ is uniformly bounded. Combining the two estimates, we obtain

$$\left|u^\delta(t,x) - u^\delta(s,x)\right| \le 2\varepsilon^{1/2}\|\partial_x u_0\|_{L_2(\Omega)} + C\varepsilon^{-(2\alpha+1)/(\alpha+1)}|t-s|^{1/(\alpha+1)}\|\partial_x u_0\|_{L_2(\Omega)}$$

for every $\varepsilon > 0$. Optimising in ε, we may choose $\varepsilon = |t-s|^{2/(5\alpha+3)}$ and obtain

$$\left|u^\delta(t,x) - u^\delta(s,x)\right| \le 2\varepsilon^{1/2}\|\partial_x u_0\|_{L_2(\Omega)} + C\varepsilon^{-3/2}|t-s|^{1/2}\|\partial_x u_0\|_{L_2(\Omega)} \le C|t-s|^{1/(5\alpha+3)}.$$

Note that the generic constant $C > 0$ depends only on the initial datum u_0. This concludes the proof. □

Applying the Arzelà–Ascoli theorem, Lemma 6.5.1 implies that there exists an accumulation point $u \in C^{\frac{1}{5\alpha+3},\frac{1}{2}}([0,\infty)\times\bar\Omega)$ of the sequence $(u^\delta)_\delta$ such that a (non-relabeled) subsequence satisfies

$$u^\delta \longrightarrow u \quad \text{in } C^{\sigma,\rho}([0,\infty)\times\bar\Omega)$$

for every $0 \le \sigma < \frac{1}{5\alpha+3}$ and $0 \le \rho < \frac{1}{2}$.

Fix $u_0 \in H^1(\Omega)$ with $u_0 > 0$. For every $\eta > 0$, denote by $\{u > \eta\} := \{(t,x) \in [0,\infty)\times\bar\Omega : u(t,x) > \eta\}$.

In order to prove that u is a weak solution to (6.5.1) on the set $\{u \ge 0\}$, we need further uniform bounds at least locally on the positivity set.

Proposition 6.5.2. *Let $u_0 \in H^1(\Omega)$ with $u_0 > 0$ in $\bar\Omega$. Let $(u^\delta)_\delta$ be the sequence of weak solutions to the regularised thin-film equation obtained in Theorem 6.4.2. Then there is $\delta_0 > 0$ small enough such that we have the following uniform bounds for every $0 < \delta < \delta_0$:*

(i) $(u^\delta)_\delta$ is uniformly bounded in $L_\infty\big((0,\infty); H^1(\Omega)\big)$;

(ii) $(\partial_x^3 u^\delta)_\delta$ is uniformly bounded in $L_{\alpha+1}(\{u > \eta\})$ for any $\eta > 0$ with $u_0 > \eta$;

(iii) $(\partial_t u^\delta)$ is uniformly bounded in $L_{\frac{\alpha+1}{\alpha}}\big([0,\infty); W^1_{\alpha+1,B}(\Omega)'\big)$;

(iv) $(\partial_t\partial_x u^\delta)$ *is uniformly bounded in* $L_{\frac{\alpha+1}{\alpha}}\big([0,\infty);\big(W^1_{\alpha+1,0}(\Omega)\cap W^2_{\alpha+1}(\Omega)\big)'\big)$.

Proof. Fix $\eta > 0$. Given an accumulation point u of the sequence u^δ, we know that $u^\delta(t,x)\geq\frac{\eta}{2}$ for every $(t,x)\in\{u>\eta\}$ for every $\delta<\delta_0$, where δ_0 is chosen small enough. Since u^δ satisfies the energy-dissipation identity (6.4.11), we obtain with the choice $s=0$

$$\int_\Omega \frac{1}{2}|\partial_x u_t^\delta|^2\,\mathrm{d}x + \int_0^t\int_\Omega m_\delta(u_\tau^\delta)|\partial_x^3 u_\tau^\delta|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau = \int_\Omega\frac{1}{2}|\partial_x u_0|^2\,\mathrm{d}x. \tag{6.5.2}$$

(i) From (6.5.2), we directly obtain

$$\int_\Omega \frac{1}{2}|\partial_x u_t^\delta|^2\,\mathrm{d}x \leq \int_\Omega\frac{1}{2}|\partial_x u_0|^2\,\mathrm{d}x$$

for every $0<\delta<\delta_0$ and every $t\geq 0$. We conclude that $(u^\delta)_\delta$ is uniformly bounded in $L_\infty\big((0,\infty);H^1(\Omega)\big)$.
(ii) Since $u^\delta>\frac{\eta}{2}$ on the set $\{u>\eta\}$ and m_δ converges to $m(u)$ locally uniformly, there is a constant $c_\eta>0$ independent of δ such that $m_\delta(u^\delta)>c_\eta$ on the set $\{u>\eta\}$. We conclude

$$\begin{aligned}\iint_{\{u>\eta\}}|\partial_x^3 u^\delta|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}t &\leq \frac{1}{c_\eta}\iint_{\{u>\eta\}} m_\delta(u_\tau^\delta)|\partial_x^3 u^\delta|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \int_\Omega\frac{1}{2}|\partial_x u_0|^2\,\mathrm{d}x.\end{aligned}$$

Hence, we obtain the desired uniform bound.
(iii) Since u^δ is a weak solution to (6.4.10), we have

$$\int_0^T\langle\partial_t u_t^\delta,\varphi\rangle_{W^1_{\alpha+1}}\,\mathrm{d}t = \int_0^T\int_\Omega m_\delta(u_t^\delta)|\partial_x^3 u_t^\delta|^{\alpha-1}\partial_x^3 u_t^\delta\partial_x\varphi_t\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi\in L_{\alpha+1}\big([0,T];W^1_{\alpha+1}(\Omega)\big)$ and all $T>0$. Applying the Hölder inequality and using that $(u^\delta)_\delta$ is uniformly bounded in $L_\infty\big([0,\infty)\times\Omega\big)$, we conclude

$$\begin{aligned}\left|\int_0^T\langle\partial_t u_t^\delta,\varphi\rangle_{W^1_{\alpha+1}}\,\mathrm{d}t\right| &\leq \int_0^T\int_\Omega m_\delta(u_t^\delta)|\partial_x^3 u^\delta|^\alpha|\partial_x\varphi|\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C\left(\int_0^\infty\int_\Omega m_\delta(u_t^\delta)|\partial_x^3 u_t^\delta|^{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}}\left(\int_0^T|\partial_x\varphi|^{\alpha+1}\right)^{\frac{1}{\alpha+1}}\\ &\leq C(u_0)\|\partial_x\varphi\|_{L_{\alpha+1}((0,T)\times\Omega)}.\end{aligned}$$

This proves that $(\partial_t u^\delta)_\delta$ is uniformly bounded in $L_{\frac{\alpha+1}{\alpha}}\big([0,\infty);\big(W^1_{\alpha+1,B}(\Omega)\big)'\big)$.
(iv) This follows similarly as in (iii) by the same duality argument. □

With these uniform bounds, we are now in the position to show the convergence to a weak solution to the power-law thin-film equation on the set $\{u\geq 0\}$.

Proposition 6.5.3. *Given an initial datum* $u_0\in H^1(\Omega)$, $u_0>0$, *the following holds true. There exists a subsequence of* $(u^\delta)_\delta$ *(not relabeled) and a limit*

$$u\in L_\infty\big([0,\infty);H^1(\Omega)\big)\cap C^{\frac{1}{5\alpha+3},\frac{1}{2}}\big([0,\infty]\times\bar\Omega\big)$$

with $\partial_x^3 u\in L_{\alpha+1,\mathrm{loc}}(\{u>0\})$ *and* $\partial_t u\in L_{\alpha+1}\big([0,\infty);\big(W^1_{\alpha+1}(\Omega)\big)'\big)$ *such that we have convergence in the following sense:*

(i) $u^\delta \to u$ *strongly in* $C^{\sigma,\rho}\big([0,\infty)\times\bar\Omega\big)$ *for all* $0 \le \sigma < \frac{1}{5\alpha+3}$ *and* $0 \le \rho < \frac{1}{2}$*;*

(ii) $\partial_x^3 u^\delta \rightharpoonup \partial_x^3 u$ *weakly in* $L_{\alpha+1,\mathrm{loc}}(\{u>0\})$*;*

(iii) $m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u \rightharpoonup m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u$ *weakly in* $L_{\frac{\alpha+1}{\alpha}}\big([0,\infty)\times\Omega\big)$*;*

(iv) $\partial_t u^\delta \rightharpoonup \partial_t u$ *weakly in* $L_{\alpha+1}\big([0,\infty);\big(W^1_{\alpha+1}(\Omega)\big)'\big)$*.*

Proof. The proof is divided into several steps.
(i) This follows directly from Lemma (6.5.1) combined with the Arzelà–Ascoli theorem.
(ii) Let $K \subset \{u>0\}$ a compact set. Since u is continuous, there is $n>0$ such that $K \subset \{u > 1/n\}$. By Proposition 6.5.2 (ii), we find a subsequence which converges weakly to v in $L_{\alpha+1}(\{u>1/n\})$. By the convergence from (i), we may identify $v = \partial_x^3 u$ and obtain $\partial_x^3 u^\delta \rightharpoonup \partial_x^3 u$ in $L_{\alpha+1}(K)$ as $\delta \to 0$.
(iii) Since the sequence u^δ is uniformly bounded in $L_\infty\big([0,\infty)\times\Omega\big)$, (i) and the local uniform convergence of m_δ to m imply that $m_\delta(u^\delta)$ converges uniformly to $m(u)$ in $[0,\infty)\times\Omega$. To obtain the desired weak convergence, let $\varphi \in L_{\alpha+1}\big([0,\infty)\times\Omega\big)$. We may then split the integral

$$\int_0^\infty\int_\Omega m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta\varphi\,\mathrm{d}x\,\mathrm{d}t = \iint_{\{u>\eta\}} m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta\varphi\,\mathrm{d}x\,\mathrm{d}t + \iint_{\{u\le\eta\}} m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

For the first integral, we use that $m_\delta(u^\delta)\to m(u)$ uniformly and that $|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta \rightharpoonup |\partial_x^3 u|^{\alpha-1}\partial_x^3 u$ weakly in $L_{\frac{\alpha+1}{\alpha}}(\{u>\eta\})$ by (ii) to obtain convergence

$$\iint_{\{u>\eta\}} m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta\varphi\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \iint_{\{u>\eta\}} m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\varphi\,\mathrm{d}x\,\mathrm{d}t$$

as $\delta\to 0$ for every $\eta>0$. We now show that the second integral is small. By the Hölder inequality we find that

$$\begin{aligned}&\iint_{\{u\le\eta\}} m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\le \|\varphi\|_{L_{\alpha+1}((0,\infty)\times\Omega)}\|m_\delta(u^\delta)\|^{\frac{1}{\alpha+1}}_{L_\infty(\{u\le\eta\})}\left(\int_0^\infty m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\alpha}{\alpha+1}}\\ &\le C\|m_\delta(u^\delta)\|^{\frac{1}{\alpha+1}}_{L_\infty(\{u\le\eta\})}\|\varphi\|_{L_{\alpha+1}((0,\infty)\times\Omega)},\end{aligned}$$

where C depends only on $\|u_0\|_{H^1(\Omega)}$. Furthermore, we may estimate

$$\|m_\delta(u^\delta)\|^{\frac{1}{\alpha+1}}_{L_\infty(\{u\le\eta\})} \le C\left(\|m_\delta(u^\delta)-m(u)\|^{\frac{1}{\alpha+1}}_{L_\infty(\{u\le\eta\})} + \|m(u)\|^{\frac{1}{\alpha+1}}_{L_\infty(\{u\le\eta\})}\right).$$

By continuity of m it holds $\|m(u)\|^{\frac{1}{\alpha+1}}_{L_\infty(\{u\le\eta\})}\to 0$, as $\eta\to 0$. Combining this with the convergence $\|m_\delta(u^\delta)-m(u)\|^{\frac{1}{\alpha+1}}_{L_\infty(\{u\le\eta\})}\to 0$, as $\delta\to 0$, we obtain

$$\begin{aligned}\int_0^\infty\int_\Omega m_\delta(u^\delta)|\partial_x^3 u^\delta|^{\alpha-1}\partial_x^3 u^\delta\varphi\,\mathrm{d}x\,\mathrm{d}t &\longrightarrow \iint_{\{u>0\}} m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^\infty\int_\Omega m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\varphi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

as $\delta \to 0$, where we used that $m(s) = 0$ for every $s \leq 0$.
(iv) This is an immediate consequence of the corresponding uniform bound obtained in Proposition 6.5.2. □

We now combine the convergence results obtained in the previous proposition to state the main result of this section.

Theorem 6.5.4. *Fix a positive initial datum $u_0 \in H^1(\Omega),\ u_0 > 0$. Let u be any accumulation point of the sequence $(u^\delta)_\delta$, as obtained in Proposition 6.5.3. Then u is a weak solution to the thin-film equation on the set $\{u > 0\}$*

$$\begin{cases} \partial_t u + \operatorname{div}\big(m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\big) = 0, & (t,x) \in \{u>0\}, \\ \partial_x u = m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega \end{cases}$$

in the sense that u satisfies the equation

$$\int_0^\infty \langle \partial_t u, \varphi \rangle_{W^1_{\alpha+1}} \, \mathrm{d}t - \iint_{\{u>0\}} m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u \cdot \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all $\varphi \in L_{\alpha+1}\big([0,\infty); W^1_{\alpha+1}(\Omega)\big)$ and the energy-dissipation inequality

$$\int_\Omega \frac{1}{2}|\partial_x u_t|^2 \, \mathrm{d}x + \int_0^t \int_{\{u_s>0\}} m(u)|\partial_x^3 u|^{\alpha+1} \, \mathrm{d}x \, \mathrm{d}s \leq \int_\Omega \frac{1}{2}|\partial_x u_0|^2 \, \mathrm{d}x, \quad t \in [0,\infty).$$

The concept of weak solutions obtained in Theorem 6.5.4 is 'very weak'. It is the same concept of weak solutions that is used in [BF90] and, if $\Omega = (0,1)$, it allows for steady-state solutions of the form $u(x) = [1/4 - x^2]_+ + [1/4 - (x-1)^2]_+$.

Observe that we do not claim that the solutions obtained remain non-negative. In fact, in the case of non-Newtonian fluids, we will show that, for positive initial data, there is a maximal time $\tau(u_0) > 0$ up to which u is a solution to the thin-film equation in $[0,\tau(u_0)) \times \Omega$. We do not obtain the non-negativity results on solutions in the shear-thinning case $\alpha > 1$ that are shown in [AG04], since the additional regularisation needed there to use entropy arguments breaks the gradient flow scheme. In the Newtonian case $\alpha = 1$ though, we obtain non-negative solutions for initial data with finite entropy.

Corollary 6.5.5. *Fix a positive initial datum $u_0 \in H^1(\Omega),\ u_0 > 0$. Let u be any accumulation point of the sequence $(u^\delta)_\delta$ as obtained in Proposition 6.5.3. Then there is $\tau(u_0) > 0$ such that u is a weak solution to the thin-film equation*

$$\begin{cases} \partial_t u + \operatorname{div}\big(m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u\big) = 0, & t \in (0,\tau(u_0)),\ x \in \Omega, \\ \partial_x u = m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega \end{cases}$$

in the sense that

$$\int_0^{\tau(u_0)} \langle \partial_t u, \varphi \rangle_{W^1_{\alpha+1}} \, \mathrm{d}t - \int_0^{\tau(u_0)} \int_\Omega m(u)|\partial_x^3 u|^{\alpha-1}\partial_x^3 u \cdot \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all $\varphi \in L_{\alpha+1}((0,\tau(u_0)); W^1_{\alpha+1}(\Omega))$ and the energy-dissipation inequality

$$\int_\Omega \frac{1}{2}|\partial_x u_t|^2 \, \mathrm{d}x + \int_0^t \int_{\{u_s>0\}} m(u)|\partial_x^3 u|^{\alpha+1} \, \mathrm{d}x \, \mathrm{d}s \leq \int_\Omega \frac{1}{2}|\partial_x u_0|^2 \, \mathrm{d}x, \quad t \in [0,\tau(u_0)).$$

holds. Moreover, it holds

$$\lim_{t\nearrow\tau(u_0)}\min_{x\in\bar\Omega} u(t,x)=0.$$

Furthermore, if $\alpha=1$ *and* $u_0\in H^1(\Omega)$ *with* $u_0\geq 0$ *satisfies* $\int_\Omega G(u_0)<\infty$*, then* u *is a global-in-time weak solution to the Newtonian thin-film equation*

$$\begin{cases}\partial_t u+\operatorname{div}\big(m(u)\partial_x^3 u\big)=0, & t\in(0,\infty),\ x\in\Omega,\\ \partial_x u=m(u)\partial_x^3 u=0, & t>0,\ x\in\partial\Omega,\\ u(0,x)=u_0(x), & x\in\Omega.\end{cases}$$

Proof of Theorem 6.5.4. In Theorem 6.4.2 we showed that u^δ is a weak solution to the modified thin-film equation, that is u^δ satisfies

$$\int_0^\infty\langle\partial_t u^\delta,\varphi\rangle_{W^1_{\alpha+1}}\,\mathrm{d}t-\int_0^\infty\int_\Omega m_\delta(u^\delta)|\partial_x^3u^\delta|^{\alpha-1}\partial_x^3u^\delta\cdot\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t=0$$

for all $\varphi\in L_{\alpha+1}\big([0,\infty);W^1_{\alpha+1}(\Omega)\big)$. Using Proposition 6.5.3 (iii) and (iv), we obtain

$$\int_0^\infty\langle\partial_t u^\delta,\varphi\rangle_{W^1_{\alpha+1}}\,\mathrm{d}t\longrightarrow\int_0^\infty\langle\partial_t u,\varphi\rangle_{W^1_{\alpha+1}}\,\mathrm{d}t,$$

$$\int_0^\infty\int_\Omega m_\delta(u^\delta)|\partial_x^3u^\delta|^{\alpha-1}\partial_x^3u^\delta\cdot\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\longrightarrow\iint_{\{u>0\}}m(u)|\partial_x^3u|^{\alpha-1}\partial_x^3u\cdot\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

as $\delta\to0$ for all $\varphi\in L_{\alpha+1}\big([0,\infty);W^1_{\alpha+1}(\Omega)\big)$. This proves that u is a weak solution to (6.5.1) on $\{u>0\}$.

To obtain the energy-dissipation inequality, note that from Proposition 6.5.2 (i) and (iv) it follows that $\partial_x u^\delta(t)\rightharpoonup\partial_x u(t)$ in $L_2(\Omega)$. Hence, by lower semicontinuity, we obtain

$$\begin{aligned}&\int_\Omega\frac12|\partial_x u_t|^2\,\mathrm{d}x+\int_0^t\int_{\{u_s>0\}}m(u)|\partial_x^3u|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}s\\ &\leq\liminf_{\delta\to0}\left[\int_\Omega\frac12|\partial_x u_t^\delta|^2\,\mathrm{d}x+\int_s^t\int_\Omega m_\delta(u_\tau^\delta)|\partial_x^3u_\tau^\delta|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau\right]\\ &\leq\int_\Omega\frac12|\partial_x u_0|^2\,\mathrm{d}x,\quad t\in[0,\tau(u_0)).\end{aligned}$$

This completes the proof. □

We now prove Corollary 6.5.5.

Proof of Corollary 6.5.5. Define $\tau(u_0)=\min\{t>0:\text{there is }x\in\Omega\text{ with }u(t,x)=0\}$. Then, since u is continuous and $u_0>0$, it holds $\tau(u_0)>0$. The other assertions follow directly from Theorem 6.5.4.

For $\alpha=1$, we showed in Lemma 6.4.3 that

$$\int_{\{u_\delta<0\}}\frac{|u^\delta|^2}{2\delta}\,\mathrm{d}x\leq\int_\Omega G_\delta(u_0)\,\mathrm{d}x\leq\int_\Omega G(u_0)\,\mathrm{d}x.$$

This shows that if $\int_\Omega G(u_0)<\infty$, we have

$$\limsup_{\delta\to0}\int_{\{u_\delta<0\}}|u^\delta|^2\,\mathrm{d}x=0.$$

Combining this with the uniform convergence of $u^\delta\to u$, we may conclude that $|\{u<0\}|=0$ and hence $u\geq0$ in $[0,\infty)\times\Omega$. □

Remark 6.5.6. (i) Let $\alpha = 1, m(s) = s^n$ with $n \geq 4$ and $u_0 > 0$. Then $u > 0$ in $[0,\infty) \times \Omega$. This follows from the Hölder continuity of u and the fact that $\int_\Omega G(u(t,x))\,\mathrm{d}x < \infty$ for all $t \geq 0$; cf. [BF90].

(ii) Let $\alpha = 1$, $u_0 \in H^1(\Omega)$ with $u_0 \geq 0$, and let $(u_{0,\delta})_\delta \subset H^1(\Omega)$ be a sequence with $u_{0,\delta} > 0$ and $u_{0,\delta} \to u_0$ in $H^1(\Omega)$. Using the above scheme with initial datum $u_{0,\delta}$, the corresponding sequence $(u^\delta, j^\delta)_\delta$ converges to a non-negative weak solution u to (6.5.1) in the sense of Proposition 6.5.4.

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Appendices

A Convergence Rates and Fluctuations for the Stokes–Brinkman Equations as Homogenisation Limit in Perforated Domains

Abstract

We study the homogenization of the Dirichlet problem for the Stokes equations in $\mathbb{R}^3$ perforated by m spherical particles. We assume the positions and velocities of the particles to be identically and independently distributed random variables. In the critical regime, when the radii of the particles are of order m^{-1}, the homogenization limit u is given as the solution to the Brinkman equations. We provide optimal rates for the convergence $u_m \to u$ in L^2, namely $m^{-\beta}$ for all $\beta < 1/2$. Moreover, we consider the fluctuations. In the central limit scaling, we show that these converge to a Gaussian field, locally in $L^2(\mathbb{R}^3)$, with an explicit covariance.

Our analysis is based on explicit approximations for the solutions u_m in terms of u as well as the particle positions and their velocities. These are shown to be accurate in $\dot{H}^1(\mathbb{R}^3)$ to order $m^{-\beta}$ for all $\beta < 1$. Our results also apply to the analogous problem regarding the homogenization of the Poisson equations.

A.1 Introduction

Numerous applications regarding the dynamics of suspensions and aerosols call for macro- and mesoscopic models which couple the particle evolution to the fluid. One of the most well-known models are the so-called Vlasov-Navier-Stokes equations for spherical, non-Brownian inertial particles. If the fluid inertia is neglected, they reduce to the so-called Vlasov-Stokes equations which take the dimensionless form

$$\left\{ \begin{aligned} \partial_t f + v \cdot \nabla_x f + \operatorname{div}((u-v)f) = 0, \\ -\Delta u + \nabla p + \rho u - j = h, \quad \operatorname{div} u = 0, \\ \rho = \textstyle\int f \,\mathrm{d}v, \quad j = \textstyle\int v f \,\mathrm{d}v, \end{aligned} \right. \tag{A.1.1}$$

where $f(t,x,v)$ is the particle density and h is some external force acting on the fluid. For questions regarding modeling and applications of this system, we refer the reader to [Bou+15] and the references therein.

The rigorous derivation of these equations from a microscopic system is a wide open problem. The main difficulty lies in the nature of the interaction of the particles which is only implicitly given through the fluid. Moreover it is singular and long range. A natural preliminary step towards the rigorous derivation of the Vlasov(-Navier)-Stokes equations consists in the derivation of the limit fluid equations in (A.1.1) without taking into account the particle evolution. These are the so-called Brinkman equations. The additional term $\rho u - j$ describes the effective drag force that the particles exert on the fluid: The drag force of a single particle in a Stokes flow is given by

$$F_i = 6\pi R(V_i - u_i),$$

where R is the particle radius, V_i its velocity and u_i is the unperturbed fluid velocity at the position of the particle. Therefore, the total drag will be of order one if the number of particles m (in a finite volume) times their radius R_m is of order one. By making the convenient choice

$$R_m = \frac{1}{6\pi m}, \tag{A.1.2}$$

the Brinkman equations in the form above arise based on a superposition principle for the drag forces.

The rigorous derivation of the Brinkman equations has attracted increasing attention over the last years, with results both in the cases of zero and non-zero particle velocities, see e.g. [All90b; GH19a; Gér22] and [DGR08; HMS19; CH20], respectively. The most recent results focus on the derivation under very mild assumptions for (random) particle configurations. Such investigations seem compulsory in order to eventually accomplish the rigorous derivation of the Vlasov(-Navier)-Stokes equations. In this regard, it is also desirable to develop very accurate explicit approximations for the microscopic solution u_m and to characterize its convergence rate to the limit u as well as the associated fluctuations. In our paper, we focus on these aspects.

Statement of the main result

We consider the perforated domain

$$\Omega_m = \mathbb{R}^3 \setminus \bigcup_{i=1}^m \overline{B}_i,$$

where the particles are given by $B_i = B_{R_m}(X_i)$ with R_m as in (A.1.2). The particle positions $X_1, \dots, X_m$ as well as their velocities $V_1, \dots, V_m$ are random variables in $\mathbb{R}^3$. For $h \in \dot{H}^{-1}(\mathbb{R}^3; \mathbb{R}^3)$, we study the solution u_m to the Stokes equations

$$\begin{cases} -\Delta u_m + \nabla p_m = h, \quad \operatorname{div} u_m = 0 & \text{in } \Omega_m, \\ \qquad\qquad\qquad\qquad\quad u_m = V_i & \text{in } B_i, \ i = 1, \dots, m. \end{cases} \tag{A.1.3}$$

We consider the case when $Z_i = (X_i, V_i)$ are i.i.d. according to $f \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$. We impose the following hypotheses on f:

(H1) $\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(\mathrm{d}x, \mathrm{d}v) < \infty$;

(H2) the distribution of the centers $\rho(\cdot) := \int_{\mathbb{R}^3} f(\cdot, \mathrm{d}v) \in W^{1,\infty}(\mathbb{R}^3)$ is compactly supported;

(H3) $j(\cdot) := \int_{\mathbb{R}^3} v f(\cdot, \mathrm{d}v) \in H^1(\mathbb{R}^3)$.

We remark that we in particular allow to choose $f(\mathrm{d}x, \mathrm{d}v) = \rho(x)\,\mathrm{d}x\delta_{v=0}$ which means that all particle velocities are zero.

We note that the Stokes equations (A.1.3) are only well-posed if the particles do not overlap. However, in our setting, overlapping does not occur with probability approaching 1 as $m \to \infty$. This follows from the following standard result that can for example be found in [Hau09, Proposition A.3].

Lemma A.1.1. *For $\nu \geq 0$, $L > 0$ let*

$$\mathcal{O}_{m,\nu,L} = \left\{ (Z_i)_{i=1}^m = ((X_i, V_i))_{i=1}^m : \min_{i \neq j} |X_i - X_j| > Lm^\nu R_m \right\}.$$

Then, for all $0 \leq \nu < 1/3$ and all $L > 0$, there exists $m_0 > 0$ such that for all $m \geq m_0$

$$\mathbb{P}(\mathcal{O}_{m,\nu,L}) \leq CLm^{\nu - 1/3},$$

where C depends only on ρ.

For overcoming the problem of the ill-posedness of (A.1.3) for overlapping particles, we could restrict ourselves to configurations of non-overlapping particles. However, this results in the loss of the independence of the particle positions. Thus, for technical reasons, we prefer to define u_m to be the solution to (A.1.3) for $(Z_i)_{i=1}^m \in \mathcal{O}_{m,0,2}$ and $u_m = u$ for $(Z_i)_{i=1}^m \notin \mathcal{O}_{m,0,2}$.

For the statement of our main result, we introduce $u \in \dot{H}^1(\mathbb{R}^3)$ as the unique weak solution to the Brinkman equations

$$-\Delta u + (\rho u - j) + \nabla p = h, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3. \tag{A.1.4}$$

Moreover, we introduce the solution operator A for the Brinkman equations with vanishing flux j. More precisely, A, which depends on ρ, maps g to to the solution w of the equation

$$-\Delta w + \rho w + \nabla p = g, \quad \operatorname{div} w = 0 \quad \text{in } \mathbb{R}^3. \tag{A.1.5}$$

Theorem A.1.2. *Let $h \in \dot{H}^{-1}(\mathbb{R}^3; \mathbb{R}^3)$ and let u_m and u be defined as in* (A.1.3) *and* (A.1.4).

(i) For any $\beta < 1/2$ and any compact set $K \subset \mathbb{R}^3$

$$m^\beta \|u_m - u\|_{L^2(K)} \longrightarrow 0 \quad \text{in probability.}$$

(ii) For every $g \in L^2(\mathbb{R}^3)$ with compact support,

$$\xi_m[g] := m^{1/2}(g, u_m - u) \longrightarrow \xi[g]$$

in distribution, where ξ is a Gaussian field with mean zero and covariance

$$\begin{aligned} \mathbb{E}[\xi[g_1]\xi[g_2]] = &\int_{\mathbb{R}^3 \times \mathbb{R}^3} \big((u(x) - v) \cdot (Ag_1)(x)\big)\big((u(x) - v) \cdot (Ag_2)(x)\big) f(\mathrm{d}x, \mathrm{d}v) \\ &- (\rho u - j, Ag_1)_{L^2} (\rho u - j, Ag_2)_{L^2} \end{aligned} \tag{A.1.6}$$

for all $g_1, g_2 \in L^2(\mathbb{R}^3)$ with compact support.

Remark A.1.3. (i) The analogous result holds when the Stokes equations are replaced by the Poisson equation. In this case, the quantities V_i are scalars as well as u_m, u, h, j, etc. Moreover, reflecting that the capacity of a ball of radius R is $4\pi R$, one should replace (A.1.2) by

$$R_m = \frac{1}{4\pi m}. \tag{A.1.7}$$

(ii) Formally, we can write $\xi = A\zeta$, where ζ accounts for the fluctuations of the drag force $j - \rho u$. The appearance of the second term on the right-hand side of (A.1.6) is classical for the fluctuations in m-particle systems, see e.g. [BH77], and is supposed to disappear if we modeled the particles by a Poisson Point Process instead. In particular, one can expect in this case, at least formally, $\xi = A\zeta$ with

$$\zeta = \left(\int (v-u)^2 f(\cdot, \mathrm{d}v) \right)^{\frac{1}{2}} W,$$

where W is space white noise.

(iii) The rate of convergence in part (i) of Theorem A.1.2 is optimal in view of part (ii). By interpolating the estimate in part (i) with the energy bound, one obtains a convergence in H^s_{loc} for any $s < 1$ with rate $m^{-\beta+s/2}$ for any $\beta < 1/2$, though. This might not be optimal, though. Indeed, we will show that the fluctuations ξ_m are bounded in H^s_{loc}, $s < 1/2$ (cf. Proposition A.3.3).

Possible generalizations

We briefly comment on three aspects of possible generalizations and improvements of our main result. The first aspect addresses random radii of the particles and the second space dimensions different from $d = 3$. Finally, we comment in better notions of probabilistic convergence in part (i) of the theorem.

Indeed, it is not difficult to extend the above result to the case where the radii of the particles are also random. More precisely, assume that the radius of each particle is $R_i^m = r_i R_m$ with R_m as in (A.1.2), respectively. Assume that the radii r_i are independent bounded random variables, also independent of the positions, with expectation $\mathbb{E}r = 1$. Then, the assertions of Theorem A.1.2 still hold with an additional factor $\mathbb{E}r^2$ in front of the first term on the right-hand side of the covariance. In order not to further burden the presentation, we restrict our attention to the case of identical radii.

Regarding the space dimension, our analysis is restricted to the physically most relevant three-dimensional case. Applying the same techniques in dimension $d = 2$ seems possible with additional technicalities due the usual issues regarding the capacity of a set in $d = 2$.

We emphasize though that, for $d \geq 4$, we do not expect Theorem A.1.2 to continue to hold. One reason for this is that the volume occupied by the particles becomes too big. Indeed, the critical scaling of the radius of m spherical particles in dimension $d \geq 3$ is $R_m \sim m^{-1/(d-2)}$. The results cited above ensure that under this scaling we still have $u_m \rightharpoonup u$ weakly in $\dot{H}^1(\mathbb{R}^d)$. However, in the case when the particle velocities are all zero, i.e. $f = \rho \otimes \delta_0$, we obtain as a trivial upper bound for the rate of convergence in L^2_{loc}

$$\|u_m - u\|_{L^2_{\mathrm{loc}}(\mathbb{R}^d)} \geq \|u_m - u\|_{L^2(\cup_{i=1}^m B_i)} = \|u\|_{L^2(\cup_{i=1}^m B_i)} \sim \left(\mathcal{L}^d\left(\bigcup_{i=1}^m B_i\right)\right)^{\frac{1}{2}} \sim m^{-\frac{1}{d-2}}.$$

This shows that Theorem A.1.2 cannot hold in this form for $d \geq 5$. Moreover, in dimension $d = 4$, this error is of critical order, which suggests that the analysis of the fluctuations is much more delicate.

Our techniques are restricted to dimension $d = 3$ for another reason. Namely, we will several times use the fact that the fundamental solution to the Stokes equations is in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ which is no longer true in higher dimensions.

Instead of convergence in probability, one could aim for convergence in L^p. Following the proof of the theorem reveals that we actually prove

$$\mathbb{E}_m[\mathbf{1}_{\mathcal{O}^c_{m,0,5}} \|u_m - u\|^2_{L^2_{\mathrm{loc}}}] \leq Cm^{-1}.$$

This implies $\mathbb{E}_m[\|u_m - u\|_{L^2_{\mathrm{loc}}}] \leq Cm^{-1/6}$ by Lemma A.1.1, provided an a priori bound $\mathbb{E}_m[\|u_m - u\|^2_{L^2_{\mathrm{loc}}}] \leq C$. Such a bound has been obtained in [CH20]. Although different particle distributions are considerd in [CH20], one readily checks that [CH20, Lemma 3.4] also implies such an a priori estimate in our setting. Again, the power $m^{-1/6}$ is presumably not optimal and one could aim for an estimate $\mathbb{E}_m[\|u_m - u\|^2_{L^2_{\mathrm{loc}}}] \leq Cm^{-1}$. Following our present approach, one would need to adapt the approximation that we use for u_m in the set $\mathcal{O}_{m,0,5}$. The adaptation needs to take into account in a more precise way the geometry of the particle configuration and one could take inspiration from the proof of [CH20, Lemma 3.4]. However, it seems unavoidable that this approach would drastically increase the technical part of our proof.

Comments on assumption (H1)–(H3)

The second moment bound in the first assumption, (H1), is very natural. It ensures that the solution u_m is bounded in $L^2(\Omega; \dot{H}^1(\mathbb{R}^3))$, where Ω denotes the probability space. Moreover, the covariance of the fluctuations provided in Theorem A.1.2 involve this second moment.

The regularity assumptions on ρ and j, (H2)–(H3), are of more technical nature: they ensure that both j and ρu, which appear in the Brinkman equations (A.1.4), lie in $\dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$. The $\dot{H}^{-1}$ property will be very useful to treat those terms as source terms of the Stokes equations. On the other hand, the H^1-regularity allows us to quantify the differences of those terms to some discrete and averaged versions involved in the setup of appropriate approximations for u_m that we detail in Section A.2.

Discussion of related results

Previous results on the derivation of the Brinkman equations

As indicated at the beginning of this introduction, there is a huge literature on the derivation of the Brinkman equations and corresponding results for the Poisson equation where one could mention for instance [MK74; CM82a; PV80; Oza83; DG94; GHV18]. For a more complete list and discussion of this literature, we refer the reader to [GHV18; GH19a].

In [GH19a; CH20], the Brinkman equations have been derived under very mild assumptions on the particle configurations. In [GH19a], the authors considered zero particle veolcities. The particle positions can be distributed to rather general stationary processes, and the radii are i.i.d. with only a $(1+\beta)$ moment bound. This allows for many clusters of overlapping particles. A corresponding result for the Poisson equation has been obtained in [GHV18].

On the other hand, in [CH20], the particle radii are identical but their velocities are not necessarily zero. The authors consider more general particle distributions than i.i.d. configurations. The Brinkman equations are derived in this setting under assumptions including a 5th moment bound of the velocities. The result in [CH20] comes with an estimate of the convergence rate $u_m \to u$ in L^2_{loc}. However, this does not allow to deduce convergence faster than $m^{-\beta}$ with $\beta < 1/95$.

Results about explicit approximations for u_m

A widespread approach to homogenization of the Poisson and Stokes equations in perforated domains with homogeneous Dirichlet boundary conditions is the so-called method of oscillating testfunctions which is used for instance in [CM82a; All90b]. An oscillating testfunction w_m is constructed in such a way that it vanishes in the particles and converges to 1 weakly in H^1_{loc}. This function w_m carries the information of the capacity (or resistance) of the particles. A natural question is then, how well $w_m u$ approximates u_m. Since the function w_m is usually constructed explicitly, this allows for an explicit approximation for u_m. In [KM89; All90b] it is shown that for periodic configurations $\|u_m - w_m u\|_{\dot H^1} \leq Cm^{-1/3}$. This error is of the order of the particle distance and thus the optimal error that one can expect due to the discretization. Similar results have been obtained in [Giu21b] for the random configurations studied in [GHV18], with a larger error due to particle clusters.

In the recent papers [Fep22; FJ21], higher order approximations for the Poisson and the Stokes equations in periodically perforated domains are analyzed.

In the present paper, we do not work with oscillating test functions. However, we derive equally explicit approximations for u_m which we will denote by $\tilde u_m$ (see Section A.2). As we will show in Theorem A.3.1 , we have $\|u_m - \tilde u_m\|_{\dot H^1} \leq Cm^{-\beta}$ for all $\beta < 1$. This error is much smaller than the one obtained in [KM89; All90b]. The reason for that is twofold. First, we take into account the leading order discretization error in terms of fluctuations. Second, we benefit from the randomness which reduces the higher order dicretization errors on average. We believe that Theorem A.3.1 could be of independent interest. In particular concerning the rigorous derivation of the Vlasov-Stokes equations (A.1.1), such explicit accurate approximations of u_m in good norms seem essential. Indeed, for the related derivation of the transport-Stokes system for inertialess suspensions in [Höf18], corresponding approximations have been crucial.

Related results concerning fluctuations and preliminary comments on our proof

In the classical theory of stochastic homogenization of elliptic equations with oscillating coefficients, the study of fluctuations has been a very active research field in recent years. Of the vast literature, one could mention for example [AKM17; DGO20].

Regarding the homogenization in perforated domains, the literature is much more sparse. In the recent paper [DG21], the authors were able to adapt some of the techniques of quantitative stochastic homogenization of elliptic equations with oscillating coefficients to the Stokes equations in perforated domains with sedimentation boundary conditions which are different from the ones considered here.

Related results to Theorem A.1.2 have been obtained in [FOT85] for the Poisson equation and in [Rub86] for the Stokes equations. However, in these papers, the authors were only able to treat the Poisson and the Stokes equations corresponding to (A.1.3) with an additional large massive term λu_m: they obtained a result corresponding to Theorem A.1.2 provided that λ is sufficiently large (depending on ρ).

The approach in [FOT85; Rub86] follows the approximation of the solution u_m by the so-called method of reflections. The idea behind this method is to express the solution operator of the problem in the perforated domain in terms of the solutions operators when only one of the particles is present. More precisely, let v_0 be the solution of the problem in the whole space without any particles. Then, define $v_1 = v_0 + \sum_i v_{1,i}$ in such a way that $v_0 + v_{1,i}$ solves the problem if i was the only particle. Since $v_{1,i}$ induces an error in B_j for $j \neq i$, one adds further functions $v_{2,i}$, this time starting from v_1. Iterating this procedure yields a sequence v_k. In general, v_k is not convergent. With the additional massive term though, one can show that the method of reflections does converge, provided that λ is sufficiently large.

In [HV18], the first author and Velázquez showed how the method of reflections can be modified to ensure convergence without a massive term and how this modified method can be used to obtain convergence results for the homogenization of the Poisson and Stokes equations. In order to study the fluctuations, a high accuracy of the approximation of u_m is needed. This would make it necessary to analyze many of the terms arising from the modified method of reflections which we were allowed to disregard for the qualitative convergence result of u_m in [HV18]. It seems very hard to control sufficiently well these additional terms which either do not arise or are of higher order for the (unmodified) method of reflections used in [FOT85; Rub86].

Thus, in the present paper, we do not use the method of reflections but follow an alternative approach to obtain an approximation for u_m. Again, we approximate u_m by $\tilde{u}_m = w_0 + \sum_i w_i$, where w_i solves the homogeneous Stokes equations outside of $\overline{B}_i$. However, we do not take w_i as in the method of reflections, where it is expressed in terms of w_0. Instead w_i will depend on u, exploiting that we already know that u_m converges to u. In contrast to the approximation obtained from the method of reflections, we will be able to choose w_i in such a way that the approximation $\tilde{u}_m = w_0 + \sum_i w_i$ is sufficient to capture the fluctuations.

A related approach has recently been used in a parallel work by Gérard-Varet in [Gér22] to give a very short proof of the homogenization result $u_m \rightharpoonup u$ weakly in $\dot{H}^1$ under rather mild assumptions on the positions of the particles. However, since we study the fluctuations in this paper, we need a more refined approximation than the one used in [Gér22]. More precisely, to leading order, the function w_i will only depend on V_i and the value of u at B_i. However, w_i will also include a lower-order term which is still relevant for the fluctuations. As we will see, this lower-order term will depend in some way on the fluctuations of the positions of all the other particles.

Organization of the paper

The rest of the paper is devoted to the proof of the main result, Theorem A.1.2.

In Section A.2, we give a precise definition of the approximation $\tilde{u}_m = w_0 + \sum_i w_i$, outlined in the paragraph above, as well as a heuristic explanation for this choice.

In Section A.3, we state three key estimates regarding this approximation and show how the proof of Theorem A.1.2 follows from these estimates.

The proof of these key estimates contains a purely analytic part as well as a stochastic part which are given in Sections A.4 and A.5, respectively.

A.2 The approximation for the microscopic solution u_m

NOTATION

We introduce the following notation that is used throughout the paper.

We denote by $G\colon \dot H^{-1}(\mathbb{R}^3) \to \dot H^1(\mathbb{R}^3)$ the solution operator for the Stokes equations. This operator is explicitly given as a convolution operator with kernel g, the fundamental solution to the Stokes equations, i.e.,

$$g(x) = \frac{1}{8\pi}\left(\frac{\mathrm{Id}}{|x|} + \frac{x \otimes x}{|x|^3}\right). \tag{A.2.1}$$

We recall from Theorem A.1.2 that $A\colon \dot H^{-1}(\mathbb{R}^3) \to \dot H^1(\mathbb{R}^3)$ is the solution operator for the limit problem (A.1.5). We observe the identities

$$(1 + G\rho)A = G, \qquad A(1 + \rho G) = G, \qquad A = G - A\rho G. \tag{A.2.2}$$

We remark that multiplication by ρ maps from $\dot H^1(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3) \cap \dot H^{-1}(\mathbb{R}^3)$. Indeed, this follows from $\rho \in W^{1,\infty}(\mathbb{R}^3)$ with compact support and the fact that $\dot H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ which implies $L^{6/5}(\mathbb{R}^3) \subset \dot H^{-1}(\mathbb{R}^3)$. Furthermore, observe that A and G are bounded operators from $L^2(\mathbb{R}^3) \cap H^{-1}(\mathbb{R}^3)$ to $C^{0,\alpha}(\mathbb{R}^3)$, $\alpha \leq 1/2$, and from $H^1(\mathbb{R}^3) \cap H^{-1}(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$. In particular, $A\rho$ and $G\rho$ are bounded operators from $L^2(\operatorname{supp}\rho)$ (and in particular from $\dot H^1(\mathbb{R}^3)$) to $L^\infty(\mathbb{R}^3)$ and from $\dot H^1(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$.

We denote $G^{-1} = -\Delta$. Then we have $GG^{-1} = G^{-1}G = P_\sigma$, where P_σ is the projection to the divergence free functions. In fact, we will use G^{-1} in the expression AG^{-1} only. We observe that $A = AP_\sigma$ and thus

$$AG^{-1}G = A.$$

We denote by $B^m(x) = B_{R_m}(x)$ and the normalized Hausdorff measure on the sphere $\partial B^m(x)$ by

$$\delta_x^m := \frac{\mathcal{H}^2|_{\partial B^m(x)}}{\mathcal{H}^2(\partial B^m(x))},$$

and write $\delta_i^m := \delta_{X_i}^m$.

Moreover, we denote for any function $\varphi \in L^1(B^m(x))$ the average on $B^m(x)$ by $(\varphi)_x$, i.e.

$$(\varphi)_x := \fint_{B^m(x)} \varphi(y)\,\mathrm{d}y := \frac{1}{|B^m(x)|}\int_{B^m(x)} \varphi(y)\,\mathrm{d}y,$$

and we abbreviate $(\varphi)_i := (\varphi)_{X_i}$.

We will need a cut-off version of the fundamental solution. To this end, let $\eta \in C_c^\infty(B_3(0))$ with $\mathbf{1}_{B_2(0)} \leq \eta \leq \mathbf{1}_{B_3(0)}$ and $\eta_m(x) := \eta(x/R_m)$. Now consider $\tilde g_m =$

$(1-\eta_m)g$. We need an additional term in order to make $\tilde g^m$ divergence free. This is obtained through the classical Bogovski operator (see e.g. [Gal11, Theorem 3.1]) which provides the existence of a sequence $\psi_m \in C_c^\infty(B_{3R_m} \setminus B_{2R_m})$ such that $\operatorname{div} \psi_m = \operatorname{div}(\eta_m g)$ and

$$\|\nabla^k \psi_m\|_{L^p(\mathbb{R}^3)} \leq C(p,k) \|\nabla^{k-1} \operatorname{div}(\eta_m g)\|_{L^p(\mathbb{R}^3)} \tag{A.2.3}$$

for all $1 < p < \infty$ and all $k \geq 1$. By scaling considerations, the constant C is independent of m. Then, we define G^m as the convolution operator with kernel

$$g^m = (1-\eta_m)g + \psi_m. \tag{A.2.4}$$

Approximation of u_m using monopoles induced by u

To find a good approximation for u_m, we observe that u_m satisfies

$$-\Delta u_m + \nabla p = h \mathbf{1}_{\Omega_m} + \sum_i h_i, \quad \text{in } \mathbb{R}^3 \tag{A.2.5}$$

for some functions $h_i \in \dot H^{-1}(\mathbb{R}^3)$, each supported in $\overline{B}_i$, which are the charge distributions induced in the particles due to the Dirichlet boundary conditions.

We begin by observing that for most of the configurations of particles, the particles are sufficiently separated which allows us to determine good approximations for h_i by ignoring its direct interaction with another particle. As we will see, our approximation for h_i will only incorporate the effect of the other particles through the limit u.

To be more precise, let $0 < \nu < 1/3$. Then, by Lemma A.1.1, we know that, for most of the particles, $B_{m^\nu R_m}(X_i)$ only contains the particle B_i. In this case, h_i is uniquely determined by the problem

$$\begin{cases} -\Delta v_i + \nabla p = h & \text{in } B_{m^\nu R_m}(X_i) \setminus \overline{B_i}, \\ v_i = V_i & \text{in } \overline{B}_i, \\ v_i = u_m & \text{on } \partial B_{m^\nu R_m}(X_i). \end{cases} \tag{A.2.6}$$

We simplify this problem to derive an approximation for h_i. First, we drop the right-hand side h in (A.2.6). Its contribution is expected to be negligible, since the volume of $B_{m^\nu R_m}(X_i) \setminus \overline{B_i}$ is small compared to the difference of the boundary data at ∂B_i and $\partial B_{m^\nu R_m}(X_i)$ which is typically of order 1. Next, we know that typically $\partial B_{m^\nu R_m}(X_i)$ is very far from any particle. Since $u_m \rightharpoonup u$ in $\dot H^1(\mathbb{R}^3)$, we therefore replace (A.2.6) by

$$\begin{cases} -\Delta v_i + \nabla p = 0 & \mathbb{R}^3 \setminus \overline{B_i}, \\ v_i = V_i & \text{in } \overline{B}_i, \\ v_i(x) \to (u)_i & \text{as } |x - X_i| \to \infty. \end{cases} \tag{A.2.7}$$

Here, we could also have chosen $u(X_i)$ instead of $(u)_i$. The precise choice that we make will turn out to be convenient later. By our choice of R_m in (A.1.2), the explicit solution of (A.2.7) is given by v_i which solves $-\Delta v_i + \nabla p = h_i$ in $\mathbb{R}^3$ with

$$h_i = \frac{V_i - (u)_i}{m} \delta_i^m.$$

Therefore, resorting to (A.2.5), we are led to approximate u_m by

$$\tilde u_m := G\left[h - \frac{1}{m} \sum_{i=1}^m ((u)_i - V_i)\delta_i^m\right]. \tag{A.2.8}$$

We emphasize that for this approximation it is not important to know the function u. We only used that $u_m \rightharpoonup u$ in $\dot{H}^1(\mathbb{R}^3)$ which is always true for a subsequence by standard energy estimates. On the contrary, we can now identify the limit u. Indeed, if we believe that $\tilde{u}_m$ approximates u_m sufficiently well,

$$u \leftharpoonup u_m \approx \tilde{u}_m = G\left[h - \frac{1}{m}\sum_{i=1}^{m}((u)_i - V_i)\delta_i^m\right] \rightharpoonup G[h + j - \rho u] \qquad \text{(A.2.9)}$$

which shows that u indeed solves (A.1.4).

This approximation $\tilde{u}_m$ cannot fully capture the fluctuations, though. In the next subsection we thus show how to refine this approximation.

We end this subsection by comparing this approximation to the one used in [FOT85; Rub86] through the method of reflections. The first order approximation of the method of reflections is given by $\tilde{u}_m$ as defined in (A.2.8) but with Gh instead of u on the right-hand side. Since this is a much cruder approximation, one needs to iterate the approximation scheme. This only yields a convergent series in [FOT85; Rub86] due to the additional large massive term. On the other hand, this series then approximates u_m sufficiently well without the refinement that we introduce in the next subsection.

Refined approximation to capture the fluctuations

We make the ansatz that, macroscopically,

$$u_m = u + m^{-\frac{1}{2}}\xi_m + o(m^{-\frac{1}{2}}), \qquad \text{(A.2.10)}$$

where ξ_m is a random function which needs to be determined. We assume that the fluctuations ξ_m are in some sense macroscopic, just as u, such that we can follow the same approximation scheme as in the previous subsection.

More precisely, we adjust the Dirichlet problem (A.2.7) by adding $m^{-\frac{1}{2}}(\xi_m)_i$ on the right-hand side of the third line. This leads to the definition

$$\tilde{u}_m := G\left[h - \frac{1}{m}\sum_{i=1}^{m}(u - V_i + m^{-\frac{1}{2}}\xi_m)_i\delta_i^m\right]. \qquad \text{(A.2.11)}$$

We have not defined ξ_m yet. To make a good choice for ξ_m, the idea is to use a similar argument as in (A.2.9) but only to take the limit $m \to \infty$ in terms which are of lower order. More precisely, we observe, again taking for granted that $\tilde{u}_m$ approximates u_m sufficiently well and using $u = G(h + j - \rho u)$,

$$\begin{aligned} u + m^{-1/2}\xi_m \approx u_m \approx \tilde{u}_m &= G\left[h - \frac{1}{m}\sum_{i=1}^{m}(u - V_i + m^{-\frac{1}{2}}\xi_m)_i\delta_i^m\right] \\ &= u + G\left[\rho u - j - \frac{1}{m}\sum_{i=1}^{m}((u)_i - V_i)\delta_i^m\right] - G\left[\frac{1}{m}\sum_{i=1}^{m}(m^{-\frac{1}{2}}\xi_m)_i\delta_i^m\right]. \end{aligned} \qquad \text{(A.2.12)}$$

We expect

$$G\left[\sum_{j\neq i}\frac{m^{-\frac{1}{2}}(\xi_m)_j}{m}\delta_j^m\right] = G(\rho m^{-\frac{1}{2}}\xi_m) + O(m^{-1}). \qquad \text{(A.2.13)}$$

Inserting this into (A.2.12), leads to

$$m^{-1/2}\xi_m + G(\rho m^{-\frac{1}{2}}\xi_m) \approx G\left[\rho u - j - \frac{1}{m}\sum_{i=1}^{m}((u)_i - V_i)\delta_i^m\right]. \tag{A.2.14}$$

This equation could be used as a definition of ξ_m. Although this turns out to be a good approximation on the level of equation (A.2.10), we will now argue that this is not the case for the definition of $\tilde{u}_m$ in (A.2.11). Indeed, the right-hand side of (A.2.14) is equal to $(u)_i - V_i$ in B_i to leading order. Hence, $(m^{-1/2}\xi_m)_i$ would be of the same order which would yield a contribution to $\tilde{u}_m$ through ξ_m of order 1 instead of order $m^{-1/2}$.

Therefore, we need to be more careful and go back to microscopic considerations: Since $u_m = V_i$ in B_i and $\tilde{u}_m \approx u_m$, we want to define ξ_m in such a way that $\tilde{u}_m \approx V_i$ in B_i. Thus we want to compute $\tilde{u}_m$ in B_i in order to find a good definition of ξ_m. Since we expect $\tilde{u}_m = \tilde{u}_m(X_i) + O(m^{-1})$ in B_i (at least on average), we only compute $\tilde{u}_m(X_i)$, and by the same reasoning, we replace any average $(\xi_m)_i$ by $\xi_m(X_i)$ at will. Then, we find, using again $u = G(h + j - \rho u)$,

$$\begin{aligned}\tilde{u}_m(X_i) &\approx u(X_i) + (G(\rho u - j))(X_i) - u(X_i) + V_i - m^{-\frac{1}{2}}\xi_m(X_i)\\ &\quad - G\left[\frac{1}{m}\sum_{j\neq i}(u - V_j + m^{-\frac{1}{2}}\xi_m)_j\delta_j^m\right](X_i)\\ &= V_i - m^{-\frac{1}{2}}\xi_m(X_i) + G\left[\rho u - j - \frac{1}{m}\sum_{j\neq i}((u)_j - V_j + m^{-\frac{1}{2}}\xi_m)_j)\delta_j^m\right](X_i).\end{aligned} \tag{A.2.15}$$

Requiring $\tilde{u}_m(X_i) = V_i$ yields

$$m^{-\frac{1}{2}}\xi_m(X_i) + G\left[\frac{1}{m}\sum_{j\neq i}m^{-\frac{1}{2}}(\xi_m)_j\delta_j^m\right](X_i) = G\left[\rho u - j - \frac{1}{m}\sum_{j\neq i}((u)_j - V_j)\delta_j^m\right](X_i). \tag{A.2.16}$$

In order to define ξ_m from this equation, we want the sum on the right-hand side to include i such that the function is the same for every i. To this end, we notice that by Lemma A.1.1, with high probability, we have for all i and all $W \in \mathbb{R}^3$

$$G^m\delta_i^m W = 0 \quad \text{in } B_i, \qquad G\delta_j^m W = G^m\delta_j^m W \quad \text{in } B_i \quad \text{for all } j \neq i, \tag{A.2.17}$$

where G^m is the operator introduced at the end of Section A.2. Hence, we replace the right-hand side of (A.2.16) by

$$m^{-\frac{1}{2}}\Theta_m := G(\rho u - j) - \frac{1}{m}\sum_{i=1}^{m}G^m\left(((u)_i - V_i)\delta_i^m\right). \tag{A.2.18}$$

We expect $\Theta_m \sim 1$ since the right-hand side of (A.2.18) represents the fluctuations of the discrete approximation of $G(\rho u - j)$. As before, we replace the sum on the left-hand side of (A.2.16) by $\rho\xi_m$. Combining these approximations leads to

$$m^{-\frac{1}{2}}(1 + G\rho)\xi_m = m^{-\frac{1}{2}}\Theta_m. \tag{A.2.19}$$

In view of (A.2.2), it holds $(1+G\rho)AG^{-1} = P_\sigma$. Since, Θ_m is divergence free, (A.2.19) leads to define ξ_m to be the solution of

$$\xi_m = AG^{-1}\Theta_m. \tag{A.2.20}$$

Note that the only difference between this definition of ξ_m and (A.2.14) is the replacement of G by G^m. As mentioned above, we expect that, on a macroscopic scale, the operators G and G^m are almost the same (we will make this argument rigorous in Lemma A.5.4). Therefore, in equation (A.2.10), we expect, that it does not play a role (in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$) whether we take G or G^m. Consequently, as an approximation for ξ_m, we introduce

$$\begin{aligned} \tau_m &:= AG^{-1}\tilde{\Theta}_m, \\ m^{-1/2}\tilde{\Theta}_m &:= G(\rho u - j) - \frac{1}{m}\sum_{i=1}^m G((u(X_i) - V_i)\delta_{X_i}). \end{aligned} \tag{A.2.21}$$

This function bears the advantage that it is the sum of i.i.d. random variables. Hence, it is straightforward to study the limit properties of $\tau_m[g] := (g, \tau_m)$. Notice that we both replaced the average $(u)_i$ by the value in the center of the ball $u(X_i)$ and δ_i^m by δ_{X_i}. Since $u \in \dot{H}^1(\mathbb{R}^3)$, τ_m is not defined for every realization of particles. However, as we will see, it is well-defined as an L^2-function on the probability space with values in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

A.3 Proof of the main result

The first step of the proof is to rigorously justify the approximation of u_m by $\tilde{u}_m$, defined in (A.2.11) with ξ_m and Θ_m as in (A.2.20) and (A.2.18).

Theorem A.3.1. *For all $\varepsilon > 0$ and all $\beta < 1$*

$$\lim_{m\to\infty} \mathbb{P}_m \left[m^\beta \|u_m - \tilde{u}_m\|_{\dot{H}^1(\mathbb{R}^3)} > \varepsilon \right] \to 0.$$

The next step is to show that we actually have

$$\tilde{u}_m = u + m^{-1/2}\xi_m + o(m^{-1/2})$$

which was the starting point of our heuristics, i.e. ξ_m indeed describes the fluctuations of $\tilde{u}_m$ around u. In contrast to Theorem A.3.1, we can only expect local L^2-estimates since not even $u_m - u$ is small in the strong topology of $\dot{H}^1(\mathbb{R}^3)$.

Proposition A.3.2. *For all $\varepsilon > 0$, all bounded sets $K' \subset \mathbb{R}^3$ and all $\beta < 1$*

$$\lim_{m\to\infty} \mathbb{P}_m \left[m^\beta \|\tilde{u}_m - u - m^{-1/2}\xi_m\|_{L^2(K')} > \varepsilon \right] \to 0.$$

Combining Proposition A.3.1 and A.3.2, we observe that we only have to prove the statements of Theorem A.1.2 with $u_m - u$ replaced by $m^{-1/2}\xi_m$. We postpone the proofs of Theorem A.3.1 and Proposition A.3.2 to Section A.4.

The next proposition shows that, instead of ξ_m, we can actually consider τ_m introduced in the previous section.

Proposition A.3.3. *For any bounded set $K' \subset \mathbb{R}^3$ and every $0 \le s < \frac{1}{2}$ there is a constant $C_s(K') > 0$ independent of m such that*

$$\mathbb{E}_m[\|\xi_m\|^2_{H^s(K')}] \le C_s(K').$$

Let τ_m be defined by (A.2.21)*. Then,*

$$\limsup_{m\to\infty} m^{1-2s}\mathbb{E}_m\left[\|\xi_m-\tau_m\|^2_{H^s(K')}\right]\le C_s(K').$$

We postpone the proof of Proposition A.3.3 to Section A.5.

Note that for $s=0$, these estimates include the case $L^2(K')$ which we will use now in order to prove Theorem A.1.2. Indeed, Theorem A.1.2 is a direct consequence of the above results together with the classical Central Limit Theorem.

Proof of Theorem A.1.2. Due to the uniform bound on $\mathbb{E}_m[\|\xi_m\|^2_{L^2(K)}]$ from Proposition A.3.3, assertion (i) of the main theorem follows immediately from Theorem A.3.1 and Proposition A.3.2 since $\dot H^1(\mathbb{R}^3)$ embeds into $L^2_{loc}(\mathbb{R}^3)$.

Since convergence in probability implies convergence in distribution, Theorem A.3.1 and Propositions A.3.2 and A.3.3 imply that it suffices to prove assertion (ii) of Theorem A.1.2 with $\xi_m[g]$ replaced by $\tau_m[g]:=(g,\tau_m)_{L^2(\mathbb{R}^3)}$, i.e we need to prove that

$$\tau_m[g]\to\xi[g]$$

in distribution for any $g\in L^2(\mathbb{R}^3)$ with compact support. Since $\tau_m[g]$ is a sum of independent random variables, this is a direct consequence of the Central Limit Theorem and the following computation for covariances: let $g_1,g_2\in L^2(\mathbb{R}^3)$ with compact support, then

$$\begin{aligned}
&\mathbb{E}_m\left[\tau_m[g_1]\tau_m[g_2]\right]\\
&=m^{-1}\mathbb{E}_m\Bigg[\left(g_1,\sum_{i=1}^m A\left(\rho u-j-(u(X_i)-V_i)\delta_{X_i}\right)\right)_{L^2(\mathbb{R}^3)}\\
&\qquad\qquad\left(g_2,\sum_{j=1}^m A\left(\rho u-j-(u(X_j)-V_j)\delta_{X_j}\right)\right)_{L^2(\mathbb{R}^3)}\Bigg]\\
&=\int_{\mathbb{R}^3\times\mathbb{R}^3}(g_1,A(\rho u-j-(u(x)-v)\delta_x))_{L^2(\mathbb{R}^3)}\cdot\\
&\quad\cdot(g_2,A(\rho u-j-(u(x)-v)\delta_x))_{L^2(\mathbb{R}^3)}\,f(\mathrm{d}x,\mathrm{d}v)\\
&=\int_{\mathbb{R}^3\times\mathbb{R}^3}(g_1,A((u(x)-v)\delta_x))_{L^2(\mathbb{R}^3)}\,(g_2,A((u(x)-v)\delta_x))_{L^2(\mathbb{R}^3)}\,f(\mathrm{d}x,\mathrm{d}v)\\
&\qquad-(Ag_1,\rho u-j)_{L^2(\mathbb{R}^3)}(Ag_2,\rho u-j)_{L^2(\mathbb{R}^3)}\\
&=\int_{\mathbb{R}^3\times\mathbb{R}^3}((u(x)-v)\cdot(Ag_1)(x))((u(x)-v)\cdot(Ag_2)(x))f(\mathrm{d}x,\mathrm{d}v)\\
&\qquad-(\rho u-j,Ag_1)_{L^2(\mathbb{R}^3)}\,(\rho u-j,Ag_2)_{L^2(\mathbb{R}^3)}\,.
\end{aligned}$$

Here we used that $A\delta_x\in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ (see Lemma A.5.3) and that A is a symmetric operator on $L^2(\mathbb{R}^3)$. This finishes the proof. □

A.4 Proofs of Theorem A.3.1 and Proposition A.3.2

In this section, we will reduce the proof of Theorem A.3.1 and Proposition A.3.2 to proving the following single probabilistic lemma. The proof of this lemma, which is given in Section A.5, is the main technical part of this paper. It makes rigorous the heuristic equation (A.2.13).

As we discussed in the heuristic arguments, we will exploit in the following that the probability of having very close particles is vanishing as stated in Lemma A.1.1. In the notation of this lemma, we abbreviate

$$\mathcal{O}_m = \mathcal{O}_{m,0,5}.$$

Lemma A.4.1. *Let* Λ_m,Γ_m, Ξ_m *and* $\tilde{\tilde{\Xi}}_m$ *be defined by*

$$\Lambda_m := (G^m - G)\left(\frac{1}{m}\sum_i ((u)_i - V_i)\delta_i^m\right),$$
$$\Gamma_m := G^m\left[\sum_i \frac{(u)_i - V_i}{m}\delta_i^m\right] + G(\rho m^{-\frac{1}{2}}\xi_m),$$
$$\Xi_m := G(\rho m^{-\frac{1}{2}}\xi_m) - G^m\left[\sum_i \frac{m^{-\frac{1}{2}}(\xi_m)_i}{m}\delta_i^m\right],$$
$$\tilde{\tilde{\Xi}}_m := G(\rho m^{-\frac{1}{2}}\xi_m) - G\left[\sum_i \frac{m^{-\frac{1}{2}}(\xi_m)_i}{m}\delta_i^m\right].$$

Then,

$$\limsup_{m\to\infty} m^2\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\nabla\left(Gh + \Gamma_m + \Xi_m\right)\|^2_{L^2(\cup_i B_i)}\right] < \infty,$$
$$\limsup_{m\to\infty} m^4\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\Xi_m\|^2_{L^2(\cup_i B_i)}\right] < \infty,$$
$$\limsup_{m\to\infty} m^2\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\tilde{\tilde{\Xi}}_m + \Lambda_m\|^2_{L^2_{\mathrm{loc}}(\mathbb{R}^3)}\right] < \infty.$$

The proof of this lemma is the main technical work of the present paper. We postpone it to Section A.5.

Proof of Proposition A.3.2. Recall the definition of $\tilde{u}_m$ from (A.2.11). We compute using $u = G(h - \rho u + j)$ and $\xi_m = AG^{-1}\Theta_m = \Theta_m - G\rho\xi_m$ (cf. (A.2.2)) and the definition of Θ_m from (A.2.18)

$$\begin{aligned}
&\tilde{u}_m - u - m^{-1/2}\xi_m \\
&= G\left(h - \frac{1}{m}\sum_i (u - V_i + m^{-1/2}\xi_m)_i\delta_i^m\right) - u - m^{-1/2}\xi_m \\
&= G\left(\rho u - j - \frac{1}{m}\sum_i (u - V_i + m^{-1/2}\xi_m)_i\delta_i^m\right) - m^{-1/2}\Theta_m + m^{-1/2}G\rho\xi_m \\
&= m^{-1/2}G\left(\rho\xi_m - \frac{1}{m}\sum_i (\xi_m)_i\delta_i^m\right) + (G^m - G)\left(\frac{1}{m}\sum_i ((u)_i - V_i)\delta_i^m\right) \\
&= \tilde{\tilde{\Xi}}_m + \Lambda_m.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}_m\left[m^\beta\|\tilde{u}_m - u - m^{-1/2}\xi_m\|_{L^2(K')} > \varepsilon\right] \\
&\leq \mathbb{P}_m[\mathcal{O}_m^c] + C\varepsilon^{-2}m^{2\beta}\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\tilde{\tilde{\Xi}}_m + \Lambda_m\|^2_{L^2(K')}\right]
\end{aligned}$$

and we now conclude by Lemmas A.1.1 and A.4.1. □

Proof of Theorem A.3.1. We observe that the assertion follows from the following claim: There exists a universal constant C such that for all $(X_1, \dots, X_m) \in \mathcal{O}_m$ and all m sufficiently large

$$\begin{aligned}\|\tilde{u}_m - u_m\|^2_{\dot{H}^1(\mathbb{R}^3)} \leq C\|\nabla(u + G(\rho u - j)) + \nabla\Gamma_m\|^2_{L^2(\cup_i B_i)} + \|\nabla\Xi_m\|^2_{L^2(\cup_i B_i)} \\ + Cm^2\|\Xi_m\|^2_{L^2(\cup_i B_i)}.\end{aligned} \tag{A.4.1}$$

Indeed, accepting the claim for the moment, let $\beta < 1$ and $\varepsilon > 0$. Then, using again $u = G(h - \rho u + j)$

$$\begin{aligned}&\mathbb{P}_m\left[m^\beta\|\tilde{u}_m - u_m\|_{\dot{H}^1(\mathbb{R}^3)} > \varepsilon\right] \\ &\leq \mathbb{P}_m[\mathcal{O}_m^c] + C\varepsilon^{-2}m^{2\beta}\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\left(\|\nabla\left(Gh + \Gamma_m + \Xi_m\right)\|^2_{L^2(\cup_i B_i)} + m^2\|\Xi_m\|^2_{L^2(\cup_i B_i)}\right)\right].\end{aligned}$$

Thus, the assertion follows again from Lemmas A.1.1 and A.4.1.

It remains to prove the claim above. It follows from the fact that $u_m - \tilde{u}_m$ solves the homogeneous Stokes equations outside of the particles.

Let $(X_1, \dots X_m) \in \mathcal{O}_m$. Then, by definition of this set, the balls $B_{2R_m}(X_i)$ are disjoint for m sufficiently large and we may assume in the following that this is satisfied.

By definition of u_m and $\tilde{u}_m$, we have $-\Delta(\tilde{u}_m - u_m) + \nabla p = 0$ in $\mathbb{R}^3 \setminus \cup_i \overline{B_i}$. By classical arguments which we include for convenience, this implies

$$\|\tilde{u}_m - u_m\|^2_{\dot{H}^1(\mathbb{R}^3)} \leq C\left(\|\nabla\tilde{u}_m\|^2_{L^2(\cup_i B_i)} + \frac{1}{m}\sum_i(\tilde{u}_m - V_i)_i^2\right). \tag{A.4.2}$$

Indeed, $\tilde{u}_m - u_m$ minimizes the $\dot{H}^1(\mathbb{R}^3)$-norm among all divergence free functions w with $w = \tilde{u}_m - u_m = \tilde{u}_m - V_i$ in $\cup_i B_i$. Thus, to show (A.4.2), it suffices to construct a divergence free function w with $w = \tilde{u}_m - V_i$ in $\cup_i B_i$ such that $\|w\|_{\dot{H}^1(\mathbb{R}^3)}$ is bounded by the right-hand side of (A.4.2). Since the balls $B_{2R_m}(X_i)$ are disjoint as $(X_1, \dots, X_m) \in \mathcal{O}_m$, we only need to construct divergence free functions w_i such that $w_i \in H^1_0(B_{2R_m}(X_i))$, $w_i = \tilde{u}_m - V_i$ in B_i and

$$\|w_i\|^2_{\dot{H}^1(\mathbb{R}^3)} \leq C\left(\|\nabla\tilde{u}_m\|^2_{L^2(B_i)} + \frac{1}{m}(\tilde{u}_m - V_i)_i^2\right).$$

It is not difficult to see that such functions w_i exist. For the convenience of the reader, we state this result in Lemma A.4.2 below. Thus, the estimate (A.4.2) holds.

It remains to prove that the right-hand side of (A.4.2) is bounded by the right-hand side of (A.4.1). To this end, let $x \in B_i$ for some $1 \leq i \leq m$. We resort to the definition of $\tilde{u}_m$ in (A.2.11) to deduce, analogously as in (A.2.15), that

$$\begin{aligned}\tilde{u}_m(x) &= u(x) - (u)_i + V_i - m^{-\frac{1}{2}}(\xi_m)_i + G(\rho u - j)(x) \\ &\quad - G\left[\sum_{j\neq i}\frac{(u)_j - V_j}{m}\delta_j^m\right](x) - G\left[\sum_{j\neq i}\frac{m^{-\frac{1}{2}}(\xi_m)_j}{m}\delta_j^m\right](x).\end{aligned}$$

The definitions of ξ_m and Θ_m from (A.2.20) and (A.2.18), the identity $\xi_m = \Theta_m - G\rho\xi_m$ implies that for all $y \in B_i$

$$m^{-\frac{1}{2}}\xi_m(y) = G(\rho u - j)(y) - G\left[\sum_{j\neq i}\frac{(u)_j - V_j}{m}\delta_j^m\right](y) - G(\rho m^{-\frac{1}{2}}\xi_m)(y),$$

where we used that $(X_1, \dots, X_m) \in \mathcal{O}_m$ to replace G^m by G. Thus,

$$\begin{aligned}
&\tilde{u}_m(x) - V_i \\
&= u(x) - (u)_i + G(\rho u - j)(x) - (G(\rho u - j))_i + G\left[\sum_{j\neq i} \frac{(u)_j - V_j}{m}\delta_j^m\right]_i \\
&\quad - G\left[\sum_{j\neq i} \frac{(u)_j - V_j}{m}\delta_j^m\right](x) + (G(\rho m^{-\frac{1}{2}}\xi_m))_i - G\left[\sum_{j\neq i} \frac{m^{-\frac{1}{2}}(\xi_m)_j}{m}\delta_j^m\right](x) \\
&= (u + G(\rho u - j))(x) - (u + G(\rho u - j))_i + \Gamma_m(x) - (\Gamma_m)_i + \Xi_m(x).
\end{aligned}$$

To conclude the proof, we again use $(X_1, \dots, X_m) \in \mathcal{O}_m$ to replace G by G^m appropriately. Finally, we combine this identity with (A.4.2) and the estimate $(\Xi_m)_i^2 \leq Cm^3\|\Xi_m\|^2_{L^2(B_i)}$. □

Lemma A.4.2. *Let $x \in \mathbb{R}^3$, $R > 0$ and $w \in H^1(B_R(x))$ be divergence free. Then, there exists a divergence free function $\varphi \in H_0^1(B_{2R}(x))$ with $\varphi = w$ in $B_R(x)$ and*

$$\|\varphi\|^2_{\dot{H}^1(\mathbb{R}^3)} \leq C\left(\|\nabla w\|^2_{L^2(B_R(x))} + R(w)^2_{x,R}\right),$$

where $(w)_{x,R} = \fint_{B_R(x)} w \,\mathrm{d}x$ and C is a universal constant.

Proof. We write $w = w - (w)_{x,R} + (w)_{x,R}$. By a classical extension result for Sobolev function, there exists $\varphi_1 \in H_0^1(B_{2R}(x))$ such that $\varphi_1 = w - (w)_{x,R}$ in $B_R(x)$ and

$$\|\nabla\varphi_1\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla w\|_{L^2(B_R(x))}.$$

By scaling, the constant C does not depend on R.

Furthermore, we take $\varphi_2 = (w)_{x,R}\theta_R$ where $\theta_R \in C_c^\infty(B_{2R}(x))$ is a cut-off function with $\theta_R = 1$ in $B_R(x)$ and $\|\nabla\theta_R\|_\infty \leq CR^{-1}$. Then,

$$\|\nabla\varphi_2\|^2_{L^2(\mathbb{R}^3)} \leq CR(w)^2_{x,R}.$$

Finally, applying a standard Bogovski operator, there exists a function $\varphi_3 \in H_0^1(B_{2r}(x)\setminus B_R(x))$ such that $\operatorname{div}\varphi_3 = -\operatorname{div}(\varphi_1 + \varphi_2)$ and

$$\|\nabla\varphi_3\|_{L^2(\mathbb{R}^3)} \leq C\|\operatorname{div}(\varphi_1 + \varphi_2)\|_{L^2(\mathbb{R}^3)}.$$

Again, the constant C is independent of R by scaling considerations.

Choosing $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ finishes the proof. □

A.5 Proof of probabilistic statements

This section contains the main technical part of the proof of our main result, the probabilistic estimates stated in Proposition A.3.3 and Lemma A.4.1. The strategy that we will use to estimate all these terms is to expand the square of sums over the particles and then to use independence of the positions of the particles to calculate the expectations, distinguishing between terms where different particles appear and where one or more particles appear more than once. Then, it will remain to observe that combinatorially relevant terms cancel and that the remaining terms can be bounded sufficiently well, uniformly in m. This proof is quite lengthy. Indeed, expanding the square will lead to terms with up to 5 indices, thus giving rise to a huge number of cases that need to be distinguished.

However, there are only relatively few and basic analytic tools that we will rely on to obtain these cancellations and estimates. These are collected in the following subsection. Their proofs are postponed to the appendix.

Some of those estimates concern expressions that will recurrently appear when we take expectations. Indeed, since many of the terms in Lemma A.4.1 contain L^2-norms in the particles B_i, we will often deal with terms of the form

$$\mathbb{E}_m\left[\mathbf{1}_{B_i^m}(x)\right] = \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{B_y^m}(x) f(\mathrm{d}y, \mathrm{d}v) = \int_{\mathbb{R}^3} \mathbf{1}_{B_y^m}(x)\rho(y)\,\mathrm{d}y = m^{-3}(\rho)_x.$$

Another term that recurrently appears due to the definitions of $\tilde{u}_m$ and ξ_m is

$$(\mathcal{R}w)(x) := \mathbb{E}_m\left[(w)_i\delta_i^m\right](x) = \int_{\mathbb{R}^3} \rho(y)(w)_y\delta_y^m(x)\,\mathrm{d}y = \fint_{\partial B_x^m} \rho(y)(w)_y\,\mathrm{d}y. \tag{A.5.1}$$

To justify this formal computation one tests the expression with a function $\varphi \in C_c^\infty(\mathbb{R}^3)$ and performs some changes of variables.

For the sake of a more compact notation, we introduce

$$W_i := (u)_i - V_i, \tag{A.5.2}$$

$$F := \rho u - j, \tag{A.5.3}$$

$$\begin{aligned}\mathcal{F}(x) := \mathbb{E}_m\left[W_i\delta_i^m\right](x) &= \int_{\mathbb{R}^3\times\mathbb{R}^3} ((u)_y - v)\delta_y^m(x) f(\mathrm{d}y, \mathrm{d}v) \\ &= \fint_{\partial B_x^m} \rho(y)(u)_y - j(y)\,\mathrm{d}\mathcal{H}^2(y).\end{aligned} \tag{A.5.4}$$

Some analytic estimates

In this subsection, we collect some auxiliary observations and estimates for future reference. All the proofs of the results in this subsection can be found in subsection A.6 of the appendix.

In the following, we denote by K the bounded set defined by

$$K := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \operatorname{supp}\rho) \le 1\}. \tag{A.5.5}$$

Note that $B_i \subset K$ almost surely for all $1 \le i \le m$ and all $m \ge 1$.

Lemma A.5.1. *(i) For all $1 \le p \le \infty$ and all $w \in L^p(\mathbb{R}^3)$*

$$\|(w)_\cdot\|_{L^p(\mathbb{R}^3)} \le \|w\|_{L^p(\mathbb{R}^3)}. \tag{A.5.6}$$

(ii) For all $\alpha > 0$, all $1 \le p \le \infty$, and all $w \in L^p(K)$, we have

$$\|\rho^\alpha(w)_\cdot\|_{L^p(\mathbb{R}^3)} \le C\|w\|_{L^p(K)}, \tag{A.5.7}$$

where the constant C depends only on ρ, p and α.

(iii) For all $w \in \dot{H}^1(\mathbb{R}^3)$

$$\|w - (w)_\cdot\|_{L^2(\mathbb{R}^3)} \le m^{-1}\|w\|_{\dot{H}^1(\mathbb{R}^3)}. \tag{A.5.8}$$

(iv) The operator $\mathcal{R}$ defined in (A.5.1) *is a bounded operator from $L^2(K)$ to $L^2(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$ and from $H^1(K)$ to $H^1(\mathbb{R}^3)$. Moreover, there is a constant C depending only on ρ such that*

$$\|(\mathcal{R} - \rho)w\|_{L^2(\mathbb{R}^3)} \leq Cm^{-1}\|w\|_{H^1(K)}, \tag{A.5.9}$$

$$\|(\mathcal{R} - \rho)w\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq Cm^{-1}\|w\|_{L^2(K)}. \tag{A.5.10}$$

(v) We have

$$\sup_m \Big[\|F\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \|\mathcal{F}\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \|F\|_{L^2(\mathbb{R}^3)} + \|\mathcal{F}\|_{L^2(\mathbb{R}^3)} + \mathbb{E}_m[W_1^2]\Big] < \infty, \tag{A.5.11}$$

and there is a constant C depending only on ρ and j such that

$$\|F - \mathcal{F}\|_{L^2(\mathbb{R}^3)} + \|F - \mathcal{F}\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq Cm^{-1}\left(\|u\|_{H^1(K)} + \|j\|_{H^1(\mathbb{R}^3)}\right) \tag{A.5.12}$$

Lemma A.5.2. *There exists a constant C such that for all $x, y \in \mathbb{R}^3$ and all $m \geq 1$, we have*

$$|G\delta_y^m|(x) \leq C\frac{1}{|x-y| + m^{-1}}, \tag{A.5.13}$$

$$|A\delta_y^m|(x) \leq C\left(1 + \frac{1}{|x-y| + m^{-1}}\right), \tag{A.5.14}$$

$$|\nabla G\delta_y^m|(x) \leq C\frac{1}{|x-y|^2 + m^{-2}}. \tag{A.5.15}$$

In particular, for any bounded set K'

$$\sup_{y\in\mathbb{R}^3} \left(\|G\delta_y^m\|_{L^2(K')} + \|A\delta_y^m\|_{L^2(K')}\right) \leq C(K'). \tag{A.5.16}$$

Moreover, for all $m \geq 1$ and $y \in \mathbb{R}^3$, it holds

$$\|\delta_y^m\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq Cm^{1/2}, \tag{A.5.17}$$

with a constant independent of y and m.

Lemma A.5.3. *For every $0 \leq s < \frac{1}{2}$ and every bounded set K'*

$$\sup_{y\in\mathbb{R}^3} \|A\delta_y\|_{H^s(K')} + \|G\delta_y\|_{H^s(K')} \leq C_s(K'). \tag{A.5.18}$$

Furthermore, for every $0 < \varepsilon \leq \frac{1}{2}$

$$\|\delta_y^m - \delta_y\|_{H^{-3/2-\varepsilon}(K')} \leq C(K')m^{-\varepsilon}. \tag{A.5.19}$$

Lemma A.5.4. *For any $k \in \mathbb{N}$, G^m is a bounded operator from $\dot{H}^k(\mathbb{R}^3)$ to $\dot{H}^{k+2}(\mathbb{R}^3)$. Moreover, there is a constant C that depends only on k such that*

$$\|G - G^m\|_{\dot{H}^k(\mathbb{R}^3)\to\dot{H}^k(\mathbb{R}^3)} \leq Cm^{-2}, \tag{A.5.20}$$

$$\|G - G^m\|_{\dot{H}^k(\mathbb{R}^3)\to\dot{H}^{k+1}(\mathbb{R}^3)} \leq Cm^{-1}. \tag{A.5.21}$$

PROOF OF PROPOSITION A.3.3

For the proof of Proposition A.3.3, we first introduce another function, σ_m, intermediate between τ_m and ξ_m. We first show that ξ_m is close to σ_m in the following lemma, which we will also use in the proof of Lemma A.4.1.

From now on, we will use the notation $A \lesssim B$ for scalar quantities A and B whenever there is a constant $C > 0$ such that $A \leq CB$ and where C depends neither directly nor indirectly on m.

Lemma A.5.5. *Using the notation from* (A.5.2) *and* (A.5.3)*, let σ_m be defined by*

$$\sigma_m := AG^{-1}\hat{\Theta}_m, \tag{A.5.22}$$
$$m^{-1/2}\hat{\Theta}_m := GF - \frac{1}{m}\sum_{i=1}^{m} G(W_i\delta_i^m).$$

Then, for every bounded $K' \subset \mathbb{R}^3$

$$\mathbb{E}_m\left[\|\xi_m - \sigma_m\|^2_{L^2(K')}\right] \leq Cm^{-1}$$

and

$$\mathbb{E}_m\left[\|\nabla\xi_m - \nabla\sigma_m\|^2_{L^2(\mathbb{R}^3)}\right] \leq Cm.$$

Proof. Let K be the set defined in (A.5.5). We argue that AG^{-1} satisfies

$$\|AG^{-1}w\|_{L^2(K')} \lesssim \|w\|_{L^2(K')} \tag{A.5.23}$$

for any $K' \supset K$ and any (divergence free) $w \in L^2(K')$. Indeed, by (A.2.2), we observe that

$$AG^{-1} = (1 - A\rho)P_\sigma,$$

and therefore (A.5.23) follows from the regularity of $A\rho$ discussed after (A.2.2).

We recall that both G and G^m (cf. (A.2.4)) map to divergence free functions. Thus, by (A.5.23), we have for any bounded set $K' \supset K$

$$\begin{aligned}
&\mathbb{E}_m\left[\|\xi_m - \sigma_m\|^2_{L^2(K')}\right] \\
&= \frac{1}{m}\mathbb{E}_m\left[\left\|\sum_i AG^{-1}(G - G^m)(W_i\delta_i^m)\right\|^2_{L^2(K')}\right] \\
&\lesssim \frac{1}{m}\mathbb{E}_m\left[\sum_i\sum_{j\neq i}\int_{K'}\Big(AG^{-1}(G - G^m)(W_i\delta_i^m)\Big)\Big(AG^{-1}(G - G^m)(W_j\delta_j^m)\Big)\right] \\
&+ \frac{1}{m}\mathbb{E}_m\left[\sum_i\int_{K'}|(G - G^m)(W_i\delta_i^m)|^2\right] \\
&=: I_1 + I_2.
\end{aligned}$$

Recalling the notation (A.5.4) and using (A.5.20), we deduce

$$\begin{aligned}
I_1 &= (m-1)\|AG^{-1}(G - G^m)\mathcal{F}\|^2_{L^2(K')} \lesssim (m-1)\|(G - G^m)\mathcal{F}\|^2_{L^2(K')} \\
&\lesssim m^{-3}\|\mathcal{F}\|^2_{L^2(\mathbb{R}^3)} \lesssim m^{-3}
\end{aligned}$$

due to (A.5.11). It remains to bound I_2. By combining (A.5.21) with (A.5.17), we obtain

$$\|(G-G^m)(\delta_y^m)\|^2_{L^2(\mathbb{R}^3)} \lesssim m^{-2}\|\delta_y^m\|^2_{\dot H^{-1}(\mathbb{R}^3)} \lesssim m^{-1}.$$

Thus, by (A.5.11)

$$I_2 \lesssim m^{-1}\mathbb{E}_m[W_1]^2 \lesssim m^{-1}.$$

For the gradient estimate, we can argue similarly: Since AG^{-1} is bounded from $\dot H^1(\mathbb{R}^3)$ to $\dot H^1(\mathbb{R}^3)$

$$\begin{aligned}
&\mathbb{E}_m\left[\|\nabla(\xi_m-\sigma_m)\|^2_{L^2(\mathbb{R}^3)}\right]\\
&=\frac{1}{m}\mathbb{E}_m\left[\left\|\sum_{i=1}^m \nabla AG^{-1}(G-G^m)(W_i\delta_i^m)\right\|^2_{L^2(\mathbb{R}^3)}\right]\\
&\lesssim \frac{1}{m}\mathbb{E}_m\left[\sum_{i=1}^m\sum_{j\neq i}\int_{\mathbb{R}^3}\Big(\nabla AG^{-1}(G-G^m)(W_i\delta_i^m)\Big)\Big(\nabla AG^{-1}(G-G^m)(W_j\delta_j^m)\Big)\right]\\
&+\frac{1}{m}\mathbb{E}_m\left[\sum_{i=1}^m\int_{\mathbb{R}^3}|\nabla(G-G^m)(W_i\delta_i^m)|^2\right]\\
&=: I_1+I_2.
\end{aligned}$$

Using (A.5.21), we deduce

$$\begin{aligned}
I_1 &= (m-1)\|\nabla AG^{-1}(G-G^m)\mathcal{F}\|^2_{L^2(\mathbb{R}^3)} \lesssim (m-1)\|\nabla(G-G^m)\mathcal{F}\|^2_{L^2(\mathbb{R}^3)}\\
&\lesssim m^{-1}\|\mathcal{F}\|^2_{L^2(\mathbb{R}^3)} \lesssim m^{-1}.
\end{aligned}$$

It remains to bound I_2. Using that both G^m and G are bounded operators from H^{-1} to $\dot H^1$, we find with (A.5.17)

$$\|\nabla(G-G^m)(\delta_y^m)\|^2_{L^2(\mathbb{R}^3)} \lesssim \|\delta_y^m\|^2_{\dot H^{-1}(\mathbb{R}^3)} \lesssim m.$$

Thus,

$$I_2 \lesssim m\mathbb{E}_m[W_1^2] \lesssim m.$$

This finishes the proof. □

Corollary A.5.6. *For every $0 \le s < \frac{1}{2}$ and every $K' \subset \mathbb{R}^3$ bounded, there is a constant $C_s(K') > 0$ independent of m such that*

$$\mathbb{E}_m\left[\|\xi_m-\sigma_m\|^2_{H^s(K')}\right] \le C_s(K')m^{-1+2s}.$$

Proof. This follows from Lemma A.5.5 and the interpolation inequality

$$\begin{aligned}
\mathbb{E}_m\left[\|\xi_m-\sigma_m\|^2_{H^s(K')}\right] &\lesssim \mathbb{E}_m\left[\|\xi_m-\sigma_m\|^{2(1-s)}_{L^2(K')}\|\nabla\xi_m-\nabla\sigma_m\|^{2s}_{L^2(K')}\right]\\
&\le \mathbb{E}_m\left[\|\xi_m-\sigma_m\|^2_{L^2(K')}\right]^{1-s}\mathbb{E}_m\left[\|\nabla\xi_m-\nabla\sigma_m\|^2_{L^2(K')}\right]^s\\
&\lesssim m^{-1+2s}.
\end{aligned}$$

This finishes the proof. □

Proof of Proposition A.3.3. By Lemma A.5.5, it suffices to prove

$$\mathbb{E}_m\left[\|\sigma_m-\tau_m\|^2_{H^s(K')}\right]\le Cm^{-1+2s} \tag{A.5.24}$$
$$\mathbb{E}_m[\|\tau_m\|^2_{\dot{H}^s(K')}]\le C_s(K')$$

for every $0\le s<\frac{1}{2}$. We introduce $\tilde{W}_i:=u(X_i)-V_i$. It is easily seen that $\mathbb{E}_m[\tilde{W}_1^2]\le C$ and $\mathbb{E}_m[|W_1-\tilde{W}_1|]\lesssim\frac{1}{m}$ uniformly in m. Since $\tilde{W}_i\delta_{X_i}$ are independent identically distributed random variables, we obtain

$$\begin{aligned}\mathbb{E}_m\left[\|\tau_m\|_{\dot{H}^s(K')}\right]&=\frac{1}{m}\mathbb{E}_m\left[\left\|\sum_{i=1}^m AF-A\tilde{W}_i\delta_{X_i}\right\|^2_{\dot{H}^s(K')}\right]\\&=\mathbb{E}_m\left[\left\|AF-A\tilde{W}_1\delta_{X_1}\right\|^2_{\dot{H}^s(K')}\right]\\&\le C_s(K')\end{aligned}$$

by (A.5.18).

Finally, have to estimate $\sigma_m-\tau_m$:

$$\begin{aligned}&\mathbb{E}_m\left[\|\sigma_m-\tau_m\|^2_{\dot{H}^s(K')}\right]\\&=\frac{1}{m}\mathbb{E}_m\left[\left\|\sum_{i=1}^m A\big(W_i\delta_i^m-\tilde{W}_i\delta_{X_i}\big)\right\|^2_{\dot{H}^s(K')}\right]\\&\le\frac{1}{m}\sum_{i,j=1}^m\mathbb{E}_m\left[\left\|A\big(W_i\delta_i^m-\tilde{W}_i\delta_{X_i}\big)\right\|_{\dot{H}^s(K')}\left\|A\big(W_j\delta_j^m-\tilde{W}_j\delta_{X_j}\big)\right\|_{\dot{H}^s(K')}\right]\\&=\frac{1}{m}\sum_{j\ne i=1}^m\mathbb{E}_m\left[\left\|A\big(W_i\delta_i^m-\tilde{W}_i\delta_{X_i}\big)\right\|_{\dot{H}^s(K')}\left\|A\big(W_j\delta_j^m-\tilde{W}_j\delta_{X_j}\big)\right\|_{\dot{H}^s(K')}\right]\\&\quad+\frac{1}{m}\sum_{i=1}^m\mathbb{E}_m\left[\left\|A\big(W_i\delta_i^m-\tilde{W}_i\delta_{X_i}\big)\right\|^2_{\dot{H}^s(K')}\right]\\&=I_1+I_2.\end{aligned}$$

For I_1, notice that by (A.5.12)

$$\begin{aligned}I_1&=(m-1)\|A(F-\tilde{F})\|^2_{H^s(K')}\\&\le(m-1)\|A(F-\tilde{F})\|^2_{\dot{H}^1(K')}\\&\le m^{-1}.\end{aligned}$$

For I_2, we estimate

$$\begin{aligned}\|A(W_i\delta_i^m-\tilde{W}_i\delta_{X_i})\|_{\dot{H}^s(K')}&\le\left\|A(W_i-\tilde{W}_i)\delta_i^m\right\|_{\dot{H}^s(K')}+\left\|A\tilde{W}_i(\delta_i^m-\delta_{X_i})\right\|_{\dot{H}^s(K')}\\&\le|W_i-\tilde{W}_i|\|A\delta_i^m\|_{\dot{H}^s(K')}+|\tilde{W}_i|\|A(\delta_i^m-\delta_{X_i})\|_{\dot{H}^s(K')}\\&\lesssim|W_i-\tilde{W}_i|+m^{s-\frac{1}{2}}|\tilde{W}_i|\end{aligned}$$

by (A.5.18) and by combining (A.5.19) with the fact that A is a bounded operator from $\dot H^{s-2}(K')$ to $\dot H^s(K')$. Inserting this above, we find that

$$\begin{aligned} I_2 &\lesssim \frac{1}{m}\sum_{i=1}^m \mathbb{E}_m\left[\left(W_i - \tilde W_i| + m^{s-\frac12}|\tilde W_i|\right)^2)\right] \\ &\lesssim \mathbb{E}_m\left[|W_i - \tilde W_i|^2\right] + m^{-1+2s}\mathbb{E}_m\left[|\tilde W_i|^2)\right] \\ &\lesssim m^{-1+2s}. \end{aligned}$$

Combining the estimates for I_1 and I_2 yields (A.5.24) which finishes the proof. □

Proof of Lemma A.4.1

We begin the proof of Lemma A.4.1 by observing that we have actually already proved the required estimate for Λ^m. Indeed, $\Lambda^m = m^{-1/2}(\Theta^m - \hat\Theta^m)$ with $\hat\Theta_m$ as in Lemma A.5.5. Moreover, in the proof of Lemma A.5.5, we showed $\|\Theta^m - \hat\Theta^m\|^2_{L^2_{\text{loc}}(\mathbb{R}^3)} \lesssim m^{-1}$.

We divide the rest of proof of Lemma A.4.1 into three steps corresponding to the three terms

$$\begin{aligned} I_1 &:= \mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\nabla(u + G(\rho u - j))\|^2_{L^2(\cup_i B_i)}\right], \\ I_2 &:= \mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\nabla\Gamma_m\|^2_{L^2(\cup_i B_i)}\right], \\ I_3 &:= m^2\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\Xi_m\|^2_{L^2(\cup_i B_i)}\right] + \mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\tilde\Xi_m\|^2_{L^2(K')}\right] + \mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\nabla\Xi_m\|^2_{L^2(\cup_i B_i)}\right], \end{aligned} \tag{A.5.25}$$

where K' is a bounded set. We need to prove $I_k \le Cm^{-2}$ for $k = 1, 2, 3$, uniformly in m with a constant depending only on h, ρ and K'.

Step 1: Estimate of $\mathbf{I_1}$.

Since $\nabla Gh \in L^2(\mathbb{R}^3)$ is deterministic, and the positions of the particles B_i are independent, we estimate

$$\begin{aligned} I_1 = \mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\nabla Gh\|^2_{L^2(\cup_i B_i)}\right] \le \mathbb{E}_m\left[\|\nabla Gh\|^2_{L^2(\cup_i B_i)}\right] &= m^{-2}\int_{\mathbb{R}^3}(\rho)_x|\nabla Gh|^2\,\mathrm{d}x \\ &\lesssim m^{-2}\|\nabla Gh\|^2_{L^2(\mathbb{R}^3)} \lesssim m^{-2}. \end{aligned}$$

Here we used (A.5.6) together with $\rho \in L^\infty(\mathbb{R}^3)$.

Step 2: Estimate of $\mathbf{I_2}$.

Since Γ_m depends on m, the computation is more involved. According to the definition of Γ, we split I_2 again. More precisely, it suffices to estimate

$$\begin{aligned} I_{2,1} &:= \mathbb{E}_m\left[\left\|\nabla G\left[\sum_{j\neq i}\frac{(u)_j - V_j}{m}\delta_j^m\right]\right\|^2_{L^2(\cup_i B_i)}\right], \\ I_{2,2} &:= \mathbb{E}_m\left[\|\nabla G(\rho m^{-\frac12}\xi_m)\|^2_{L^2(\cup_i B_i)}\right]. \end{aligned}$$

In the first term, we used that for $(Z_1, \dots, Z_m) \in \mathcal{O}_m$ we can replace G^m by G according to (A.2.17).

We first consider $I_{2,1}$. We expand the square to obtain for any fixed i

$$I_{2,1} = m\mathbb{E}_m \left[\int_{B_i} \left(\nabla G \left[\frac{1}{m} \sum_{j \neq i} \frac{(u)_j - V_j}{m} \delta_j^m \right] \right) (x) \left(\nabla G \left[\frac{1}{m} \sum_{k \neq i} \frac{(u)_k - V_k}{m} \delta_k^m \right] \right) (x) \right]$$

$$=: \frac{1}{m} \sum_{j \neq i} \sum_{k \neq i} I_{2,1}^{j,k}.$$

We distinguish the cases $j \neq k$ and $j = k$. In the case $j \neq k$, we apply a similar reasoning as for I_1: due to the independence of Z_i, Z_j, Z_k, we have with $\mathcal{F}$ as in (A.5.4)

$$\begin{aligned} I_{2,1}^{jk} &= m^{-4} \int_{\mathbb{R}^3} (\rho)_x \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla G \left[((u)_y - v)\, \delta_y^m \right] (x) f(\mathrm{d}y, \mathrm{d}v) \right)^2 \mathrm{d}x \\ &= m^{-4} \int_{\mathbb{R}^3} (\rho)_x \left(\nabla G[\mathcal{F}](x) \right)^2 \mathrm{d}x \lesssim m^{-4} \| \nabla G[\mathcal{F}] \|^2_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where we used again (A.5.6). Since by (A.5.11), $\mathcal{F}$ is bounded in $\dot{H}^{-1}(\mathbb{R}^3)$, we therefore conclude that

$$\sum_{j \neq i} \sum_{k \notin \{i,k\}} I_{2,1}^{jk} \lesssim m^{-2}.$$

It remains to estimate $I_{2,1}^{jj}$. We compute

$$\begin{aligned} I_{2,1}^{jj} &= m^{-4} \int_{\mathbb{R}^3} (\rho)_x \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left(\nabla G \left[((u)_y - v)\, \delta_y^m \right] (x) \right)^2 f(\mathrm{d}y, \mathrm{d}v)\, \mathrm{d}x \\ &\lesssim m^{-4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} ((u)_y - v)^2 \, \| \nabla G \delta_y^m \|^2_{L^2(\mathbb{R}^3)} f(\mathrm{d}y, \mathrm{d}v). \end{aligned}$$

By (A.5.17)

$$\| \nabla G \delta_y^m \|^2_{L^2(\mathbb{R}^3)} \lesssim m.$$

Combining this with (A.5.7), we conclude

$$\begin{aligned} \sum_{j \neq i} I_{2,1}^{jj} &\lesssim m^{-2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} ((u)_y - v)^2 f(\,\mathrm{d}y,\, \mathrm{d}v) \\ &\lesssim m^{-2} \left(\| \rho^{1/2} (u)_\cdot \|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(\mathrm{d}y, \mathrm{d}v) \right) \\ &\lesssim m^{-2} \left(\| u \|^2_{L^2(K)} + \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(\mathrm{d}y, \mathrm{d}v) \right) \lesssim m^{-2}, \end{aligned}$$

by assumption (H1).

We now turn to $I_{2,2}$. We estimate

$$I_{2,2} \leq \mathbb{E}_m \left[\| \nabla G(\rho m^{-\frac{1}{2}} \sigma_m) \|^2_{L^2(\cup_i B_i)} \right] + \mathbb{E}_m \left[\| \nabla G(\rho m^{-\frac{1}{2}} (\xi_m - \sigma_m) \|^2_{L^2(\cup_i B_i)} \right],$$

with σ_m from Lemma A.5.5. Using this lemma and the fact that $G\rho$ is a bounded operator from $\dot{H}^1(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$, we find

$$\mathbb{E}_m\left[\|\nabla G(\rho m^{-\frac{1}{2}}(\xi_m-\sigma_m))\|^2_{L^2(\cup_i B_i)}\right] \lesssim m^{-2}\|m^{-\frac{1}{2}}(\xi_m-\sigma_m)\|^2_{\dot{H}^1(\mathbb{R}^3)} \lesssim m^{-2}.$$

Recalling the definition of σ_m from Lemma A.5.5, we have

$$\begin{aligned}\mathbb{E}_m\left[\|\nabla G(\rho m^{-\frac{1}{2}}\sigma_m)\|^2_{L^2(\cup_i B_i)}\right] &\leq \sum_{i=1}^m \mathbb{E}_m\left[\left\|\nabla G\left(\rho A\left[F-\frac{1}{m}\sum_{j=1}^m W_j\delta_j^m\right]\right)\right\|^2_{L^2(B_i)}\right]\\ &\lesssim \sum_{i=1}^m \mathbb{E}_m\left[\|\nabla G\left(\rho A F\right)\|^2_{L^2(B_i)}\right]\\ &\quad+\sum_{i=1}^m \mathbb{E}_m\left[\left\|\nabla G\left(\rho A\left[\frac{1}{m}\sum_{j=1}^m [W_j\delta_j^m]\right]\right)\right\|^2_{L^2(B_i)}\right]\\ &=: I_{2,2,1}+I_{2,2,2}.\end{aligned}$$

This is a very rough estimate, since we actually expect cancellations from the difference. However, these cancellations are not needed here for the desired bound. Indeed, since $G\rho A$ is a bounded operators from $\dot{H}^{-1}(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$, $I_{2,2,1}$ is controlled analogously as I_1.

It remains to estimate $I_{2,2,2}$. We expand the square again and write

$$\begin{aligned}&I_{2,2,2}\\ &=\sum_{i=1}^m \mathbb{E}_m\left[\int_{B_i}\left(\nabla G\left(\rho A\left[\frac{1}{m}\sum_{j=1}^m W_j\delta_j^m\right]\right)\right)\cdot\left(\nabla G\left(\rho A\left[\frac{1}{m}\sum_{k=1}^m W_k\delta_k^m\right]\right)\right)\mathrm{d}x\right]\\ &=: \sum_{i=1}^m\sum_{j=1}^m\sum_{k=1}^m I^{i,j,k}_{2,2,2}.\end{aligned}$$

We have to distinguish the cases where all i,j,k are distinct, the case where $j=k$ but $j\neq i$, the case where $i=j$ or $i=k$ but $j\neq k$, and, finally, the case where $i=j=k$.

In the first case, we can proceed analogously as for $I^{j,k}_{2,1}$. In particular, we use the definition of $\mathcal{F}$ to deduce

$$\begin{aligned}\sum_{i=1}^m\sum_{j\neq i}\sum_{k\notin\{i,j\}} I^{i,j,k}_{2,2,2} &= m^{-3}\frac{m(m-1)(m-2)}{m^2}\int_{\mathbb{R}^3}(\rho)_x(\nabla G\rho A\mathcal{F})^2\,\mathrm{d}x\\ &\lesssim m^{-2}\|\nabla G\rho A\mathcal{F}\|^2_{L^2(\mathbb{R}^3)} \lesssim m^{-2}\|\mathcal{F}\|^2_{\dot{H}^{-1}(\mathbb{R}^3)} \lesssim m^{-2},\end{aligned}$$

since $G\rho A$ is also bounded from $\dot{H}^{-1}(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$.

Next, we estimate $I^{i,j,j}_{2,2,2}$. Analogously as for $I^{j,j}_{2,1}$, we obtain

$$\begin{aligned}\sum_{i=1}^m\sum_{j\neq i} I^{i,j,j}_{2,2,2} &= m^{-3}\frac{m(m-1)}{m^2}\int_{\mathbb{R}^3}(\rho)_x\int_{\mathbb{R}^3}\left(\nabla G\rho A((u)_y-v)\delta^m_y(x)\right)^2 f(\mathrm{d}y,\mathrm{d}v)\,\mathrm{d}x\\ &\lesssim m^{-3}\int_{\mathbb{R}^3}((u)_y-v)^2\left\|\nabla G\rho A\delta^m_y(x)\right\|^2_{L^2_x(\mathbb{R}^3)} f(\mathrm{d}y,\mathrm{d}v).\end{aligned}$$

Since ∇GV is a bounded operator from $\dot{H}^1(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$, we obtain by (A.5.17) combined with (A.5.7) and using (H1)

$$\sum_{i=1}^{m} \sum_{j\neq i} I_{2,2,2}^{i,j,j} \lesssim m^{-2} \left(\|\rho^{1/2}(u)\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(\mathrm{d}y, \mathrm{d}v) \right) \lesssim m^{-2}.$$

The third estimate concerns $I_{2,2,2}^{i,i,k}$. By symmetry, $I_{2,2,2}^{i,j,i}$ is dealt with analogously. We have, using (A.5.17), (A.5.11), and (A.5.7) together with (A.5.6),

$$\begin{aligned}
&\sum_{i=1}^{m} \sum_{k\neq i} I_{2,2,2}^{i,i,k} \\
&= \frac{m(m-1)}{m^2} \int_{\mathbb{R}^3} \mathbb{E}_m \left[\mathbf{1}_{B_i} \nabla G \left(\rho A \left[W_i \delta_i^m \right] \right) \right] \nabla G \left(\rho A \left[\mathcal{F} \right] \right) \, \mathrm{d}x \\
&\lesssim \|\nabla G \rho A \mathcal{F}\|_{L^2(\mathbb{R}^3)} \left\| \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{B^m(y)} \nabla G \left(\rho A \left[((u)_y - v)\delta_y^m \right] \right) f(\mathrm{d}y, \mathrm{d}v) \right\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \sup_{y\in\mathbb{R}^3} \|\nabla G \rho A \delta_y^m\|_{L^\infty} \left\| \int_{\mathbb{R}^3\times\mathbb{R}^3} ((u)_y - v) \mathbf{1}_{B^m(y)} f(\mathrm{d}y, \mathrm{d}v) \right\|_{L^2(\mathbb{R}^3)} \\
&\lesssim m^{1/2} m^{-3} \|(\rho(u)_\cdot - j)_\cdot\|_{L^2(\mathbb{R}^3)} \lesssim m^{-5/2}.
\end{aligned}$$

We also used that the operator $\nabla G \rho A$ maps $\dot{H}^{-1}(\mathbb{R}^3)$ into $L^\infty(\mathbb{R}^3)$, as well as $j \in L^2(\mathbb{R}^3)$ by assumption (H3).

Finally, we estimate $I_{2,2,2}^{i,i,i}$. Using (A.5.17) and (A.5.7), we obtain

$$\begin{aligned}
\sum_{i=1}^{m} I_{2,2,2}^{i,i,i} &= \frac{m}{m^2} \int_{\mathbb{R}^3} \mathbb{E}_m \left[\mathbf{1}_{B_i} |\nabla G \left(\rho A \left[W_i \delta_i^m \right] \right)|^2) \right] \, \mathrm{d}x \\
&= \frac{1}{m} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{B^m(y)} \left| \nabla G \left(\rho A \left[((u)_y - v)\delta_y^m \right] \right) \right|^2 f(\mathrm{d}y, \mathrm{d}v) \, \mathrm{d}x \\
&\lesssim \frac{1}{m} \sup_{y\in\mathbb{R}^3} \|\nabla G \rho A \delta_y^m\|_{L^\infty(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{B^m(y)} \left((u)_y - v \right)^2 f(\mathrm{d}y, \mathrm{d}v) \, \mathrm{d}x \\
&\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{1}_{B^m(y)} \left(|(u)_y|^2 + |v|^2 \right) f(\mathrm{d}y, \mathrm{d}v) \, \mathrm{d}x \\
&\lesssim m^{-3} \left(\int_{\mathbb{R}^3} \rho(y) |(u)_y|^2 \, \mathrm{d}y \, \mathrm{d}x + \int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(\mathrm{d}y, \mathrm{d}v) \right) \\
&\lesssim m^{-3}.
\end{aligned}$$

This finishes the estimate of $I_{2,2,2}$. Therefore, the estimate of $I_{2,2}$ is complete, which also finishes the estimate of I_2.

Step 3: Estimate of $\mathbf{I_3}$.

We recall from (A.5.25) that I_3 consists of three terms, which we denote by J_1, J_2 and J_3. We will focus on the proof on J_1 as this is the most difficult term. We will comment on the adjustments needed to treat J_2 and J_3 along the estimates for J_1. Roughly speaking, the main difference between J_1 and J_2 is that one considers $L^2(\cup_i B_i)$ for J_1 and $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ for J_2. Naively, J_1 should therefore be better by a factor $|\cup_i B_i| \sim m^{-2}$, which is exactly the estimate we obtain. Moreover, J_3 concerns the gradient of the terms in J_1. Since we may loose a factor m^{-2} going from J_1 to J_3, it will not be difficult to adapt the estimates for J_1 to J_3 using the gradient estimates in Section A.5. For the sake of completeness we detail the estimates for J_3 in the appendix.

Step 3.1: Expansion of the terms

As in the previous step, we first want to replace all occurrences of G^m by G. Note that G^m is present both explicitly in the definition of Ξ^m and also implicitly through ξ_m. By (A.2.17) and independence of the position of the particles, it holds

$$\begin{aligned}
&m^2\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\|\Xi_m\|^2_{L^2(\cup_i B_i)}\right]\\
&\leq m^2\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\sum_{i=1}^m\int_{B_i}\left|G(\rho m^{-\frac12}\xi_m)-G\left[\sum_{j\neq i}\frac{m^{-\frac12}(\xi_m)_j}{m}\delta_j^m\right]\right|^2\mathrm{d}x\right]\\
&= m^3\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\int_{B_i}\left|G(\rho m^{-\frac12}\xi_m)-G\left[\sum_{j\neq i}\frac{m^{-\frac12}(\xi_m)_j}{m}\delta_j^m\right]\right|^2\mathrm{d}x\right]\\
&\lesssim m^3\mathbb{E}_m\left[\int_{B_i}\left|G(\rho m^{-1/2}(\xi_m-\sigma_m))\right|^2\mathrm{d}x\right]\\
&+m^3\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\int_{B_i}\left|G(Vm^{-\frac12}\sigma_m)-G\left[\sum_{j\neq i}\frac{m^{-\frac12}(\xi_m)_j}{m}\delta_j^m\right]\right|^2\mathrm{d}x\right],
\end{aligned}$$

where on the right-hand side, i is any of the m identically distributed particles. We use that $G\rho$ is a bounded operator from $L^2(K)$ to $L^\infty(B_i)$ and Lemma A.5.5 to deduce

$$\begin{aligned}
m^3\mathbb{E}_m\left[\int_{B_i}\left|G(\rho m^{-1/2}(\xi_m-\sigma_m))\right|^2\right]&\lesssim \mathbb{E}_m\left[\left\|G(\rho m^{-1/2}(\xi_m-\sigma_m))\right\|^2_{L^\infty(B_i)}\right]\\
&\lesssim m\mathbb{E}_m\left[\left\|m^{-1/2}(\xi_m-\sigma_m)\right\|^2_{L^2(K)}\right]\\
&\lesssim m^{-2}.
\end{aligned}$$

This implies, that for the estimate of J_1, it suffices to show that

$$\mathfrak{J}_1:=\mathbb{E}_m\left[\mathbf{1}_{\mathcal{O}_m}\int_{B_i}\left|G(\rho m^{-\frac12}\sigma_m)-G\left[\sum_{j\neq i}\frac{m^{-\frac12}(\xi_m)_j}{m}\delta_j^m\right]\right|^2\mathrm{d}x\right]\lesssim m^{-5}.$$

By the definitions of $m^{-\frac12}\xi_m$ and $m^{-\frac12}\rho_m$ (cf. (A.2.20) and (A.5.22)) together with (A.2.17), we have in $\mathcal{O}_m$

$$\begin{aligned}
G(\rho m^{-\frac12}\sigma_m)-G\left[\sum_{j\neq i}\frac{m^{-\frac12}(\xi_m)_j}{m}\delta_j^m\right]&=\frac1m\sum_{k=1}^m\sum_{j=1}^m\Psi_{jk},\\
\Psi_{j,k}(x):=G\left[\rho A\left(F-W_k\delta_k^m\right)\right]-(1-\delta_{ij})G&\left[\left(A\left(F-(1-\delta_{jk})W_k\delta_k^m\right)\right)_j\delta_j^m\right] \quad \text{(A.5.26)}
\end{aligned}$$

(Strictly speaking $\Psi_{j,k}$ depends on i, but we omit this dependence for the ease of notation.)

Thus,

$$\mathfrak{J}_1 \le m^{-1} \sum_{j=1}^m \sum_{k=1}^m \sum_{n=1}^m \sum_{\ell=1}^m I_3^{i,j,k,n,\ell},$$
$$I_3^{i,j,k,n,\ell} := \mathbb{E}_m \left[\int_{B_i} \Psi_{j,k}(x) \Psi_{n,\ell}(x) \, \mathrm{d}x \right].$$

Similarly, we have the estimate

$$J_3 \lesssim \mathbb{E}_m \left[\int_{\cup_i B_i} \left| \nabla G(\rho m^{-\frac{1}{2}} (\xi_m - \sigma_m)) \right|^2 \right] + \mathfrak{J}_3 \lesssim m^{-2} + \mathfrak{J}_3,$$
$$\mathfrak{J}_3 := m^{-3} \sum_{j=1}^m \sum_{k=1}^m \sum_{n=1}^m \sum_{\ell=1}^m \mathbb{E}_m \left[\int_{B_i} \nabla \Psi_{j,k}(x) \nabla \Psi_{n,\ell}(x) \, \mathrm{d}x \right], \tag{A.5.27}$$

with the same proof as before using that $\nabla G \rho$ is a bounded operator from $\dot{H}^1(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$ and the second part of Lemma A.5.5.

Furthermore,

$$J_2 \lesssim \mathbb{E}_m \left[\left\| G(\rho m^{-\frac{1}{2}} (\xi_m - \sigma_m)) \right\|_{L^2(K')}^2 \right] + \mathfrak{J}_2 \lesssim m^{-2} + \mathfrak{J}_2,$$
$$\mathfrak{J}_2 := m^{-4} \sum_{j=1}^m \sum_{k=1}^m \sum_{n=1}^m \sum_{\ell=1}^m \int_{K'} \mathbb{E}_m \left[\tilde{\Psi}_{j,k}(x) \tilde{\Psi}_{n,\ell}(x) \right] \, \mathrm{d}x,$$

where $\tilde{\Psi}_{j,k}$ denotes the function that is obtained by omitting the factor $(1-\delta_{ij})$ in (A.5.26).

Relying on this structure enables us to make more precise the argument why the estimate for $\mathfrak{J}_1$ is most difficult compared to $\mathfrak{J}_2$ and $\mathfrak{J}_3$. Indeed, for the estimate for $\mathfrak{J}_3$, one just follows the same argument as for $\mathfrak{J}_1$. The relevant estimates in Section A.5 show that whenever ∇G instead of G appears, we loose (at most) a factor m^{-1}. For completeness, we provide the proof of the estimates regarding $\mathfrak{J}_3$ in the appendix.

On the other hand, for $\mathfrak{J}_2$, we can use the estimates that we will prove for the terms of $\mathfrak{J}_1$ in the case when the index i is different from all the other indices. Indeed, in those cases, $\Psi_{j,k} = \tilde{\Psi}_{j,k}$, and we will always estimate

$$|I_3^{i,j,k,n,\ell}| = \left| m^{-3} \int_{\mathbb{R}^3} (\rho)_x \mathbb{E}_m \left[\Psi_{j,k} \Psi_{n,\ell} \right] \, \mathrm{d}x \right| \lesssim m^{-3} \left\| \mathbb{E}_m \left[\Psi_{j,k}(x) \Psi_{n,\ell}(x) \right] \right\|_{L^1_{\mathrm{loc}}(\mathbb{R}^3)}.$$

Thus, the bound for $\mathfrak{J}_2$ is a direct consequence of the estimates we will derive to bound $\mathfrak{J}_1$.

Recall that we need to prove $|\mathfrak{J}_1| \lesssim m^{-2}$. We will split the sum into the cases $\#\{i,j,k,n,\ell\} = \alpha$, $\alpha = 1, \dots, 5$. Then, since i is fixed, there will be $m^{\alpha-1}$ summands for the case $\#\{i,j,k,n,\ell\} = \alpha$. Thus, it is enough to show that in each of these cases

$$|I_3^{i,j,k,n,\ell}| \lesssim m^{-\alpha}, \quad \alpha = \#\{i,j,k,n,\ell\}.$$

To prove this estimate, we have to rely on cancellations between the terms that $\Psi_{j,k}$ is composed of. To this end, we denote the first part of $\Psi_{j,k}$ by

$$\Psi_k^{(1)} := \Psi^{(1,1)} + \Psi_k^{(1,2)} := G \left[\rho A F - \rho A [W_k \delta_k^m] \right],$$

and the second part by

$$\Psi^{(2)}_{j,k} := \Psi^{(2,1)}_{j} + \Psi^{(2,2)}_{j,k} := (1-\delta_{ij}) G\left[\left(A\left(F - (1-\delta_{jk}) W_k \delta^m_k\right)\right)_j \delta^m_j \right].$$

We observe that

$$\begin{aligned} \mathbb{E}_m[\Psi^{(1,1)}] &= G\rho AF, \\ \mathbb{E}_m[\Psi^{(1,2)}_k] &= G\rho A\mathcal{F}, \\ \mathbb{E}_m[\Psi^{(2,1)}_j] &= (1-\delta_{ij}) G\mathcal{R}AF, \\ \mathbb{E}_m[\Psi^{(2,2)}_{j,k}] &= (1-\delta_{ij})(1-\delta_{jk}) G\mathcal{R}A\mathcal{F}. \end{aligned} \tag{A.5.28}$$

Step 3.2: The cases in which at most 2 indices are equal

In many cases, we can rely on cancellations within $\Psi^{(1)}_k$ and $\Psi^{(2)}_{j,k}$. Indeed, we will prove the following lemma:

Lemma A.5.7. *Let $K' \subset \mathbb{R}^3$ be bounded. Then,*

$$\left\| \mathbb{E}_m[\Psi^{(1)}_k] \right\|_{L^2(K')} \lesssim m^{-1}, \tag{A.5.29}$$

$$\left\| \mathbb{E}_m[\Psi^{(2)}_{j,k}] \right\|_{L^2(K')} \lesssim m^{-1} \quad \textit{if } j \neq k. \tag{A.5.30}$$

There are only three cases (up to symmetry), where we have to rely on cancellations between $\Psi^{(1)}_k$ and $\Psi^{(2)}_{j,k}$ to estimate $I_3^{i,j,k,n,\ell}$. These are the cross terms, when either $j = n$, or $k = \ell$, or $j = \ell$, and all the other indices are different. In these cases, we will rely on the following lemma:

Lemma A.5.8. *Let $K' \subset \mathbb{R}^3$ be bounded. Then,*

$$\|\mathbb{E}_m\left[\Psi_{j,k}\Psi_{j,\ell}\right]\|_{L^1(K')} \lesssim m^{-2} \quad \textit{if } \#\{i,j,k,\ell\} = 4, \tag{A.5.31}$$

$$\|\mathbb{E}_m\left[\Psi_{j,k}\Psi_{n,k}\right]\|_{L^1(K')} \lesssim m^{-2} \quad \textit{if } \#\{i,j,k,n\} = 4, \tag{A.5.32}$$

$$\|\mathbb{E}_m\left[\Psi_{j,k}\Psi_{n,j}\right]\|_{L^1(K')} \lesssim m^{-2} \quad \textit{if } \#\{i,j,k,n\} = 4. \tag{A.5.33}$$

Finally, we obtain the following estimates, useful in particular for the cases in which $i = k$.

Lemma A.5.9. *Let $K' \subset \mathbb{R}^3$ be bounded. Then, for any i, j, k,*

$$\begin{aligned} &\left\| \mathbb{E}_m[\Psi^{(1,1)}] \right\|_{L^2(K')} + \left\| \mathbb{E}_m[\Psi^{(1,2)}_k] \right\|_{L^2(K')} + \left\| \mathbb{E}_m[\Psi^{(2,1)}_j] \right\|_{L^2(K')} \\ &\quad + \left\| \mathbb{E}_m[\Psi^{(2,2)}_{j,k}] \right\|_{L^2(K')} \lesssim 1. \end{aligned} \tag{A.5.34}$$

$$\begin{aligned} &\left\| \mathbb{E}_m[\mathbf{1}_{B^m_i} \Psi^{(1,1)}] \right\|_{L^2(\mathbb{R}^3)} + \left\| \mathbb{E}_m[\mathbf{1}_{B^m_i} \Psi^{(1,2)}_k] \right\|_{L^2(\mathbb{R}^3)} + \left\| \mathbb{E}_m[\mathbf{1}_{B^m_i} \Psi^{(2,1)}_j] \right\|_{L^2(\mathbb{R}^3)} \\ &\quad + \left\| \mathbb{E}_m[\mathbf{1}_{B^m_i} \Psi^{(2,2)}_{k,j}] \right\|_{L^2(\mathbb{R}^3)} \lesssim m^{-3}. \end{aligned} \tag{A.5.35}$$

Combining these lemmas allows us to estimate $I_3^{i,j,k,n,\ell}$ in all the cases when $\alpha = \#\{i,j,k,n,\ell\} \geq 4$.

Corollary A.5.10. *The following estimates hold true where the implicit constants are independent of m:*

1. *If $\#\{i,j,k,n,\ell\} = 5$, then*
$$|I_3^{i,j,k,n,\ell}| \lesssim m^{-5}.$$
2. *If $\#\{i,j,k,n,\ell\} = 4$, then*
$$|I_3^{i,j,k,n,\ell}| \lesssim m^{-4}.$$

Proof. If $\#\{i,j,k,n,\ell\} = 5$, then by independence, the Hölder inequality and Lemma A.5.7
$$\begin{aligned}|I_3^{i,j,k,n,\ell}| &\leq \left\| \mathbb{E}_m \left[\mathbf{1}_{B_{\frac{1}{m}}(w_i)} \right] \right\|_{L^\infty(\mathbb{R}^3)} \|\mathbb{E}_m [\Psi_{j,k}]\|_{L^2(K)} \|\mathbb{E}_m [\Psi_{n,\ell}]\|_{L^2(K)} \\ &\lesssim m^{-3} m^{-1} m^{-1} = m^{-5}.\end{aligned}$$

If $\#\{i,j,k,n,\ell\} = 4$, we need to distinguish all the possible combinations of two indices being equal. Depending on which indices coincide, we split the product by independence of the other indices. If $j = n$, $k = \ell$ or $j = \ell$ (or $k = n$ which is the same), we rely on Lemma A.5.8 and gain an additional factor m^{-3} from the expectation of $\mathbf{1}_{B_i^m}$.

If $j = k$ (or analogously $n = \ell$), the expectation completely factorizes into $\mathbb{E}_m[\mathbf{1}_{B_i^m}]\mathbb{E}_m[\Psi_{jj}]\mathbb{E}_m[\Psi_{n\ell}]$ and we can apply (A.5.34) for the second factor and Lemma A.5.7 for the third factor.

Finally, in all the other cases we can, without loss of generality, split the expectation into $\mathbb{E}_m[\mathbf{1}_{B_i^m}\Psi_{jk}]\mathbb{E}_m[\Psi_{n\ell}]$ and apply (A.5.35) for the first factor and Lemma A.5.7 for the second factor. □

We finish this step by giving the proofs of Lemmas A.5.7, A.5.8 and A.5.9.

Proof of Lemma A.5.7. By (A.5.28), we have
$$\mathbb{E}_m[\Psi_k^{(1)}] = G\rho A(F - \mathcal{F}),$$
and using (A.5.12) yields (A.5.29). Similarly, for $j \neq k$, $i \neq j$,
$$\mathbb{E}_m[\Psi_{j,k}^{(2)}] = G\mathcal{R}A(F - \mathcal{F}).$$
Using again (A.5.12) and recalling from Lemma A.5.1 that $\mathcal{R}$ is a bounded operator from $L^2(K)$ to $\dot{H}^{-1}(\mathbb{R}^3)$ yields (A.5.30). For $i = j$, $\Psi_{j,k}^{(2)} = 0$ and there is nothing to prove. □

Proof of Lemma A.5.8. Regarding (A.5.31), we have
$$\begin{aligned}&\mathbb{E}_m [\Psi_{j,k}\Psi_{j,\ell}] \\ &= \iiint \left(G\left[\rho A\left(F - ((u)_{y_2} - v_2)\delta_{y_2}^m\right)\right] - G\left[\left(A\left(F - ((u)_{y_2} - v_2)\delta_{y_2}^m\right)\right)_{y_1} \delta_{y_1}^m\right]\right) \\ &\qquad \left(G\left[\rho A\left(F - ((u)_{y_3} - v_3)\delta_{y_3}^m\right)\right] - G\left[\left(A\left(F - ((u)_{y_3} - v_3)\delta_{y_3}^m\right)\right)_{y_1} \delta_{y_1}^m\right]\right) \\ &\qquad\qquad f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2) f(\mathrm{d}y_3, \mathrm{d}v_3) \\ &= \int \rho(y_1) \left(G\rho A(F - \mathcal{F}) - (A(F - \mathcal{F}))_{y_1} G\delta_{y_1}^m\right)^2 \,\mathrm{d}y_1.\end{aligned}$$

We obtain

$$\|\mathbb{E}_m\left[\Psi_{j,k}\Psi_{j,\ell}\right]\|_{L^1(K')} \lesssim \|G\rho A(F-\mathcal{F})\|^2_{L^2(K')} + \int \rho(y)(A(F-\mathcal{F}))_y^2 \|G\delta_y^m\|^2_{L^2(K)}\,\mathrm{d}y$$
$$\lesssim m^{-2} + \|A(F-\mathcal{F})\|^2_{L^2(K')} \lesssim m^{-2},$$

where we used (A.5.12) for both terms and (A.5.16) and (A.5.7) for the second term.

Regarding (A.5.32), we compute

$$\begin{aligned}
&\mathbb{E}_m\left[\Psi_{j,k}\Psi_{n,k}\right] \\
&= \iiint \left(G\left[\rho A\left(F - \left((u)_{y_2} - v_2\right)\delta_{y_2}^m\right)\right] - G\left[\left(A\left(F - \left((u)_{y_2} - v_2\right)\delta_{y_2}^m\right)\right)_{y_1} \delta_{y_1}^m\right]\right) \\
&\qquad \left(G\left[\rho A\left(F - \left((u)_{y_2} - v_2\right)\delta_{y_2}^m\right)\right] - G\left[\left(A\left(F - \left((u)_{y_2} - v_2\right)\delta_{y_2}^m\right)\right)_{y_3} \delta_{y_3}^m\right]\right) \\
&\qquad\qquad f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2) f(\mathrm{d}y_3, \mathrm{d}v_3) \\
&= \int \rho(y_2)\left(G(\rho - \mathcal{R})AF - \left((u)_{y_2} - v_2\right)G(\rho-\mathcal{R})A\delta_{y_2}^m\right)^2 f(\mathrm{d}y_2, \mathrm{d}v_2).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\|\mathbb{E}_m\left[\Psi_{j,k}\Psi_{n,k}\right]\|_{L^1(K')} &\lesssim \|G(\rho-\mathcal{R})AF\|^2_{L^2(K')} \\
&\quad + \sup_z \|G(\rho-\mathcal{R})A\delta_z^m\|^2_{L^2(K')} \int ((u)_z - v)^2 f(\mathrm{d}z, \mathrm{d}v) \\
&\lesssim m^{-2},
\end{aligned}$$

where we used (A.5.16) for both terms and (A.5.7) and (H1) for the second term.

Finally, to prove (A.5.33), we just apply Young's inequality to reduce to the previous two estimates. Indeed,

$$\begin{aligned}
\mathbb{E}_m\left[\Psi_{j,k}\Psi_{n,j}\right] &= \int \left(G\rho A(F-\mathcal{F}) - (A(F-\mathcal{F})u)_y\, G\delta_y^m\right) \\
&\qquad \left(G(\rho-\mathcal{R})AF - \left((u)_y - v\right)G(\rho-\mathcal{R})A\delta_y^m\right) f(\mathrm{d}y, \mathrm{d}v) \\
&\le \int \rho(y)\left(G\rho A(F-\mathcal{F}) - (A(F-\mathcal{F})u)_y\, G\delta_y^m\right)^2 \mathrm{d}y \\
&\quad + \int \left(G(\rho-\mathcal{R})AF - \left((u)_y - v\right)G(\rho-\mathcal{R})A\delta^m\right)^2 f(\mathrm{d}y, \mathrm{d}v).
\end{aligned}$$

These two terms are exactly the ones we have estimated in the previous two steps. □

Proof of Lemma A.5.9. The first estimate (A.5.34) follows directly from (A.5.28) and (A.5.11) together with the fact that the operators $G\rho A$, $G\rho A$, $G\mathcal{R}A$ and $G\mathcal{R}A$ are all bounded from $\dot{H}^1(\mathbb{R}^3)$ to $L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

Regarding (A.5.35), we first observe that these estimates follow directly from (A.5.34) in the cases, when $i \neq k$. Indeed, if i is different from both j and k, the expectation factorizes. Moreover, the case $i = j$ is trivial, since the terms with index j vanish for $i = j$.

If $i = k$, we only need to consider those terms, where k appears, i.e. $\Psi_k^{(1,2)}$ and $\Psi_{j,k}^{(2,2)}$. Again, we only need to consider the case $j \neq k = i$.

We have for $\Psi_k^{(1,2)}$

$$\begin{aligned}\|\mathbb{E}_m[\mathbf{1}_{B_i^m}\Psi_i^{(1,2)}]\|_{L^2(\mathbb{R}^3)} &= \left\|\int \mathbf{1}_{B^m(y)} G\rho A\left[\big((u)_y - v\big)\delta_y^m\right] f(\mathrm{d}y,\mathrm{d}v)\right\|_{L^2(\mathbb{R}^3)} \\ &\le \sup_{y\in\mathbb{R}^3}\|G\rho A\delta_y^m\|_{L^\infty(\mathbb{R}^3)}\left\|\int\big((u)_y - v\big)\mathbf{1}_{B^m(y)} f(\mathrm{d}y,\mathrm{d}v)\right\|_{L^2(\mathbb{R}^3)} \\ &\lesssim m^{-3}\|(\rho(u)_\cdot - j)_\cdot\|_{L^2(\mathbb{R}^3)} \lesssim m^{-3},\end{aligned}$$

where we used (A.5.16), (A.5.6) and (A.5.7). Since for $j \neq i$,

$$\mathbb{E}_m[\mathbf{1}_{B_i^m}\Psi_{j,i}^{(2,2)}] = \int \mathbf{1}_{B^m(y)} G\mathcal{R}A\left[\big((u)_y - v\big)\delta_y^m\right] f(\mathrm{d}y,\mathrm{d}v),$$

the estimate of this term is analogous. □

Step 3.3: The cases in which the number of different indices is 3 or less.

It remains to estimate $|I_3^{i,j,k,n,\ell}|$, when $\#\{i,j,k,n,\ell\} \le 3$. We will show that $|I_3^{i,j,k,n,\ell}| \lesssim m^{-3}$ for $\#\{i,j,k,n,\ell\} = 3$, and $|I_3^{i,j,k,n,l}| \lesssim m^{-2}$ for $\#\{i,j,k,n,\ell\} \le 2$. Formally, a factor m^{-3} can be expected to come from the term $\mathbf{1}_{B_i^m}$, so that cancellations are not needed for the estimates of those term. We will see that this strategy works for all the terms except for $I_3^{i,j,i,j,\ell}$ with i, j, ℓ mutually distinct.

Thus, in all cases except $I_3^{i,j,i,j,\ell}$ with i, j, ℓ mutually distinct, we just brutally estimate the product $\Psi_{j,k}\Psi_{n,\ell}$ via the triangle inequality

$$\left|I_3^{i,j,k,n,\ell}\right| \le \sum_{\alpha,\beta,\gamma,\delta=1}^{2}\int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\Psi_{j,k}^{(\alpha,\beta)}\Psi_{n,\ell}^{(\gamma,\delta)}\right]\right|,$$

with the convention that $\Psi_{j,k}^{(1,1)} = \Psi^{(1,1)}$, and similarly for $\Psi_{j,k}^{(1,2)}$ and $\Psi_{j,k}^{(2,1)}$.

We now consider all possible cases of $(\alpha,\beta,\gamma,\delta) \in \{1,2\}^4$ and $\#\{i,j,k,n,\ell\} \le 3$. Since $\Psi^{(1,1)}$ does not depend on any index and both $\Psi_k^{(1,2)}$ and $\Psi_j^{(2,1)}$ only depend on one index (not taking into account the dependence of i since $\Psi_i^{(2,1)} = 0$ anyway), the number of cases to be considered considerably reduces for these terms.

In order to exploit this in the sequel, we introduce the following slightly abusive notation. When considering the term $\mathbb{E}_m[\mathbf{1}_{B_i^m}\Psi_{j,k}^{(\alpha,\beta)}\Psi_{n,\ell}^{(\gamma,\delta)}]$ for fixed $\alpha,\beta,\gamma,\delta$, we define the notion of *relevant indices* to be the subset of indices $\{i,j,k,n,\ell\}$ appearing in this product after replacing $\Psi_{j,k}^{(1,1)}$ by $\Psi^{(1,1)}$ and similarly for $\Psi_{j,k}^{(1,2)}$, $\Psi_{j,k}^{(2,1)}$ and for the indices n,ℓ.

To further reduce the number of cases that we have to consider, we next argue that we do not have to consider the cases $\{j,k,n,\ell\}$ with $J \cap \{j,k\} \cap \{n,\ell\} = \emptyset$, where J is the set of relevant indices. Indeed, in all these cases, the expectation factorizes, and we conclude by the bounds provided by Lemma A.5.9. In particular, we do not have to consider any case where $\Psi^{(1,1)}$ appears.

Moreover, if j is a relevant index and $i = j$, then $\Psi_{j,k}^{(2,2)} = \Psi_j^{(2,1)} = 0$, and therefore, there is nothing to estimate. If j and k are both relevant indices and $j = k$, then $\Psi_{j,j}^{(2,2)} = 0$, and therefore, there is nothing to estimate either. The same reasoning applies to the cases where $i = n$ and $n = \ell$, respectively.

We now list all the cases that are left to consider. Cases that are equivalent by symmetry we list only once. We use the convention here, that we only specify which relevant indices coincide; relevant indices which are not explicitly denoted as equal are assumed to be different. The indices which are not relevant may take any number, in particular coinciding with each other or with relevant indices.

1. $(\alpha,\beta,\gamma,\delta) = (2,2,2,2)$: Relevant indices: $\{i,j,k,n,\ell\}$. Since all the indices are relevant, we only have to consider cases where at least two pairs or three indices coincide. All the other cases are already covered when we have estimated $I^{i,j,k,n,\ell}$ with $\#\{i,j,k,n,\ell\} \geq 4$. The cases left to consider are
   a) $i = k, j = n$,
   b) $i = k, j = \ell$,
   c) $i = k = \ell$,
   d) $j = n, k = \ell$,
   e) $j = \ell, k = n$,
   f) $i = k = \ell, j = n$.
2. $(\alpha,\beta,\gamma,\delta) = (2,1,2,2)$: Relevant indices: $\{i,j,n,\ell\}$. Cases to consider:
   a) $j = n$,
   b) $j = \ell$,
   c) $i = \ell, j = n$.
3. $(\alpha,\beta,\gamma,\delta) = (2,1,2,1)$: Relevant indices: $\{i,j,n\}$. Only case to consider: $j = n$.
4. $(\alpha,\beta,\gamma,\delta) = (1,2,2,2)$: Relevant indices: $\{i,k,n,\ell\}$. Cases to consider:
   a) $i = k = \ell$,
   b) $i = \ell, k = n$,
   c) $k = n$.
5. $(\alpha,\beta,\gamma,\delta) = (1,2,2,1)$: Relevant indices: $\{i,k,n\}$. Only case to consider: $k = n$.
6. $(\alpha,\beta,\gamma,\delta) = (1,2,1,2)$: Relevant indices: $\{i,k,\ell\}$. Cases to consider:
   a) $k = \ell$,
   b) $i = k = \ell$.

In order to conclude the proof of the lemma, it now remains to give estimates for the cases listed above.

The case (1a): As mentioned at the beginning of Step 3.3, this is the case, where we rely on cancellations with $\Psi^{(2,1)}$ coming from case (2c). We estimate

$$
\begin{aligned}
&\mathbb{E}_m \left[\mathbf{1}_{B_i^m}(x) \Psi^{(2,2)}_{j,i}(x) (\Psi^{(2,1)}_{j\ell} - \Psi^{(2,2)}_{j\ell})(x) \right] \\
&= \iint \mathbf{1}_{B^m(y_1)}(x) G \left[\left(A \left[((u)_{y_1} - v_1) \delta^m_{y_1} \right] \right)_{y_2} \delta^m_{y_2} \right](x) \cdot \\
&\qquad \cdot G \left[\left(A \left(F - \mathcal{F} \right) \right)_{y_2} \delta^m_{y_2} \right](x) f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2) \\
&= \iint \rho(y_2) \mathbf{1}_{B^m(y_1)}(x) \left(A \left[((u)_{y_1} - v_1) \delta^m_{y_1} \right] \right)_{y_2} (G \delta^m_{y_2})^2(x) \left(A(F - \mathcal{F}) \right)_{y_2} f(\mathrm{d}y_1, \mathrm{d}v_1) \, \mathrm{d}y_2.
\end{aligned}
$$

Hence, since A maps $L^2(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ and by (A.5.12)

$$\begin{aligned}
&\int \left| \mathbb{E}_m \left[\mathbf{1}_{B_i^m} \Psi_{ji}^{(2,2)} (\Psi_{j\ell}^{(2,1)} - \Psi_{j\ell}^{(2,2)}) \right] \right| \, \mathrm{d}x \\
&\lesssim m^{-1} \iiint \rho(y_2) \mathbf{1}_{B^m(y_1)}(x) \left| \left(A \left[\left((u)_{y_1} - v_1 \right) \delta_{y_1}^m \right] \right)_{y_2} \right| (G\delta_{y_2}^m)^2(x) f(\mathrm{d}y_1, \mathrm{d}v_1) \, \mathrm{d}y_2 \, \mathrm{d}x.
\end{aligned}$$

By (A.5.13)

$$\int \mathbf{1}_{B^m(y_1)}(x) (G\delta_{y_2}^m)^2(x) \, \mathrm{d}x \lesssim m^{-3} \frac{1}{|y_2 - y_1|^2 + m^{-2}}. \tag{A.5.36}$$

Combining this with the pointwise estimate (A.5.14) yields

$$\begin{aligned}
&\int \left| \mathbb{E}_m \left[\mathbf{1}_{B_i^m} \Psi_{ji}^{(2,2)} (\Psi_{j\ell}^{(2,1)} - \Psi_{j\ell}^{(2,2)}) \right] \right| \, \mathrm{d}x \\
&\lesssim m^{-4} \iint \rho(y_2) |(u)_{y_1} - v_1| \frac{1}{|y_2 - y_1|^2 + m^{-2}} \left(1 + \frac{1}{|y_2 - y_1| + m^{-1}} \right) f(\mathrm{d}y_1, \mathrm{d}v_1) \, \mathrm{d}y_2 \\
&\lesssim m^{-4} \log m \int |(u)_{y_1} - v_1| f(\mathrm{d}y_1, \mathrm{d}v_1) \lesssim m^{-4} \log m,
\end{aligned}$$

where we used (A.5.7) and (H1).

The case (1b) is similar. However, it turns out to be easier, since the singularity is subcritical, so we do not need to take into account cancellations. Indeed,

$$\begin{aligned}
&\mathbb{E}_m \left[\mathbf{1}_{B_i^m}(x) \Psi_{ji}^{(2,2)}(x) \Psi_{nj}^{(2,2)}(x) \right] \\
&= \iint \mathbf{1}_{B^m(y_1)}(x) G \left[\left(A \left[\left((u)_{y_1} - v_1 \right) \delta_{y_1}^m \right] \right)_{y_2} \delta_{y_2}^m \right] (x) \\
&\cdot G \left[\int \left(A \left[\left((u)_{y_2} - v_2 \right) \delta_{y_2}^m \right] \right)_{y_3} \delta_{y_3}^m f(\mathrm{d}y_3, \mathrm{d}v_3)] \right] (x) f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2) \\
&= \iint \left((u)_{y_1} - v_1 \right) \left((u)_{y_2} - v_2 \right) \mathbf{1}_{B^m(y_1)}(x) \left(A\delta_{y_1}^m \right)_{y_2} (G\delta_{y_2}^m)(x) \cdot \\
&\quad \cdot \left(G\mathcal{R}A\delta_{y_2}^m \right)(x) f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2).
\end{aligned}$$

Thus, since $G\mathcal{R}$ maps $L^2(K)$ to $L^\infty(\mathbb{R}^3)$ and by (A.5.16)

$$\begin{aligned}
&\int \left| \mathbb{E}_m \left[\mathbf{1}_{B_i^m} \Psi_{ji}^{(2,2)} \Psi_{nj}^{(2,2)} \right] \right| \, \mathrm{d}x \\
&\lesssim \iint \left((u)_{y_1} - v_1 \right) \left((u)_{y_2} - v_2 \right) \mathbf{1}_{B^m(y_1)}(x) \left| \left(A\delta_{y_1}^m \right)_{y_2} \right| \left| (G\delta_{y_2}^m) \right| (x) f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2).
\end{aligned} \tag{A.5.37}$$

Now we proceed as in the previous case to estimate

$$\begin{aligned}
&\int \left| \mathbb{E}_m \left[\mathbf{1}_{B_i^m} \Psi_{ji}^{(2,2)} \Psi_{nj}^{(2,2)} \right] \right| \, \mathrm{d}x \\
&\lesssim m^{-3} \iint \left(\left((u)_{y_1} - v_1 \right)^2 + \left((u)_{y_2} - v_2 \right)^2 \right) \frac{1 + \frac{1}{|y_2 - y_1| + m^{-1}}}{|y_2 - y_1| + m^{-1}} f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2) \\
&\lesssim m^{-3}.
\end{aligned}$$

The case (1c): We have

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{B_i^m}(x)\Psi^{(2,2)}_{ji}(x)\Psi^{(2,2)}_{ni}(x)\right]\\
&=\int \mathbf{1}_{B^m(y_1)}(x)\left(G\left[\int \rho(y_2)\left(A\left[((u)_{y_1}-v_1)\delta^m_{y_1}\right]\right)_{y_2}\delta^m_{y_2}\,\mathrm{d}z\right]\right)(x)^2 f(\mathrm{d}y_1,\mathrm{d}v_1)\\
&=\int \left((u)_{y_1}-v_1\right)^2\mathbf{1}_{B^m(y_1)}(x)\left(G\mathcal{R}A\delta^m_{y_1}\right)(x)^2 f(\mathrm{d}y_1,\mathrm{d}v_1).
\end{aligned}$$

Thus, using first that $\|G\mathcal{R}A\delta^m_{y_1}\|_{L^\infty(\mathbb{R}^3)}\lesssim 1$ as above, (H1) and (A.5.6) together with (A.5.7).

$$\int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\Psi^{(2,2)}_{ji}\Psi^{(2,2)}_{ni}\right]\right|\,\mathrm{d}x\lesssim\int\int\left((u)_{y_1}-v_1\right)^2\mathbf{1}_{B^m(y_1)}(x)f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}x\lesssim m^{-3}.$$

The case (1d): We compute

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{B_i^m}(x)\Psi^{(2,2)}_{jk}(x)\Psi^{(2,2)}_{jk}(x)\right]\\
&=m^{-3}\iint(\rho)_x\rho(y_2)\left(G\left[\left(A\left[((u)_{y_1}-v_1)\delta^m_{y_1}\right]\right)_{y_2}\delta^m_{y_2}\right](x)\right)^2 f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2\\
&=m^{-3}\iint(\rho)_x\rho(y_2)\left((u)_{y_1}-v_1\right)^2\left(A\delta^m_{y_1}\right)^2_{y_2}(G\delta^m_{y_2})^2(x)f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2.
\end{aligned}$$

Using (A.5.16) twice, (A.5.7) together with (H2) and (H1), we can successively estimate the integral in x, y_2 and (y_1,v_1) to deduce

$$\begin{aligned}
\int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\Psi^{(2,2)}_{jk}\Psi^{(2,2)}_{jk}\right]\right|\,\mathrm{d}x&\lesssim m^{-3}\int\rho(y_2)\left((u)_{y_1}-v_1\right)^2\left(A\left[\delta^m_{y_1}\right]\right)^2_{y_2}f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2\\
&\lesssim m^{-3}\int\left((u)_{y_1}-v_1\right)^2 f(\mathrm{d}y_1,\mathrm{d}v_1)\lesssim m^{-3}.
\end{aligned}$$

The case (1e): We just observe that by Young's inequality

$$\int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\Psi^{(2,2)}_{jk}\Psi^{(2,2)}_{kj}\right]\right|\,\mathrm{d}x\le\int\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\left(\left(\Psi^{(2,2)}_{jk}\right)^2+\left(\Psi^{(2,2)}_{kj}\right)^2\right)\right]\,\mathrm{d}x.$$

Thus, this case is reduced to case (1d).

The case (1f). Note that $\#\{i,j,k,n,\ell\}=2$. Hence, we only need a bound m^{-2}. We have

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{B_i^m}(x)\Psi^{(2,2)}_{ji}(x)\Psi^{(2,2)}_{ji}(x)\right]\\
&=\iint\rho(y_2)\mathbf{1}_{B^m(y_1)}(x)\left(G\left[\left(A\left[((u)_{y_1}-v_1)\delta^m_{y_1}\right]\right)_{y_2}\delta^m_{y_2}\right](x)\right)^2 f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2\\
&=\iint\rho(y_2)\left((u)_{y_1}-v_1\right)^2\mathbf{1}_{B^m(y_1)}(x)\left(A\delta^m_{y_1}\right)^2_{y_2}(G\delta^m_{y_2})^2(x)f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2.
\end{aligned}$$

We can estimate the integral in x using again (A.5.36)

$$\begin{aligned}
&\int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\Psi^{(2,2)}_{ji}\Psi^{(2,2)}_{ji}\right]\right|\,\mathrm{d}x\\
&\le\int\rho(y_2)\left((u)_{y_1}-v_1\right)^2\mathbf{1}_{B^m(y_1)}(x)\left(A\delta^m_{y_1}\right)^2_{y_2}(G\delta^m_{y_2})^2(x)f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2\,\mathrm{d}x\\
&\lesssim m^{-3}\int\rho(y_2)\left((u)_{y_1}-v_1\right)^2\left(A\delta^m_{y_1}\right)^2_{y_2}\frac{1}{|y_2-y_1|^2+m^{-2}}f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2.
\end{aligned}$$

Moreover, using (A.5.14), we find

$$\int \left| \mathbb{E}_m \left[\mathbf{1}_{B_i^m} \Psi_{ji}^{(2,2)} \Psi_{ji}^{(2,2)} \right] \right| \,\mathrm{d}x$$

$$\lesssim m^{-3} \int \rho(y_2) \big((u)_{y_1} - v_1 \big)^2 \left(\frac{1}{|y_2 - y_1|^2 + m^{-2}} + \frac{1}{|y_2 - y_1|^4 + m^{-4}} \right) f(\mathrm{d}y_1, \mathrm{d}v_1)\,\mathrm{d}y_2$$

$$\lesssim m^{-2} \int \big((u)_{y_1} - v_1 \big)^2 f(\mathrm{d}y_1, \mathrm{d}v_1) \lesssim m^{-2},$$

where we used (A.5.7) and (H1) in the last estimate. Note that this estimate is sufficient, since the number of different indices in this case is only 2.

The cases (2a) and (2b) are reduced to the cases (3) and (1d) by Young's inequality, analogously as in the case (1e).

The case (2c) was estimated together with the case (1a) if k is different from the other indices.

If k coincides with one of the other indices, the number of different indices is 2 and we can reduce the case to the cases (3) and (1f) by Young's inequality.

The case (3): In this case we get a factor m^{-3} from $\mathbf{1}_{B_i^m}$ and thus the desired estimate follows from

$$\|\mathbb{E}_m[|\Psi_j^{(2,1)}|^2]\|_{L^1(K)} \lesssim \int \rho(y_1) |(AF)_{y_1}|^2 \|G\delta_{y_1}^m\|_{L^2(K)}^2 \,\mathrm{d}y_1 \lesssim 1,$$

where we used (A.5.16) and (A.5.7).

The case (4a) is estimated by an analogous computation as the one at the end of the proof of Lemma A.5.9, relying on the fact that

$$\|\Psi_k^{(1,2)}\|_{L^\infty(\mathbb{R}^3)} \lesssim |(u)_k - V_k|, \tag{A.5.38}$$

which is a direct consequence of (A.5.16) and the fact that $G\rho$ is bounded from $L^2(K)$ to $L^\infty(\mathbb{R}^3)$. Since the index n is free, a similar bound can be used for $\Psi_{n,\ell}^{(2,2)}$. More precisely,

$$|\mathbb{E}_m[\mathbf{1}_{B_i^m} \Psi_i^{(1,2)} \Psi_{n,i}^{(2,2)}]|$$

$$\leq \int \mathbf{1}_{B^m(y_1)} \left| G\mathcal{R}A \left[\big((u)_{y_1} - v_1 \big) \delta_{y_1}^m \right] \right| \left| G\rho A \left[\big((u)_{y_1} - v_1 \big) \delta_{y_1}^m \right] \right| f(\mathrm{d}y_1, \mathrm{d}v_1)$$

$$\lesssim \int \mathbf{1}_{B^m(y_1)} |(u)_{y_1} - v_1|^2 f(\mathrm{d}y_1, \mathrm{d}v_1),$$

since $G\mathcal{R}$ and $G\rho$ map $L^2(K)$ to $L^\infty(\mathbb{R}^3)$ and using again (A.5.16). As before, integrating in x yields a factor m^{-3}.

The case (4b): Using (A.5.38) yields

$$|\mathbb{E}_m[\mathbf{1}_{B_i^m} \Psi_k^{(1,2)} \Psi_{k,i}^{(2,2)}]|$$

$$\lesssim \int \mathbf{1}_{B^m(y_1)} |(u)_{y_1} - v_1| |(u)_{y_2} - v_2| G[\delta_{y_2}^m] |(A\delta_{y_1}^m)_{y_2}| f(\mathrm{d}y_1, \mathrm{d}v_1) f(\mathrm{d}y_2, \mathrm{d}v_2),$$

which is the same as (A.5.37) which we have already estimated.

The case (4c) is reduced to the cases (6a) and (1d) by Young's inequality.

The case (5) is reduced to the cases (6a) and (3) by Young's inequality.

The cases (6a) and (6b) are estimated by an analogous computation as the one at the end of the proof of Lemma A.5.9, relying on (A.5.38) again.

A.6 Appendix

PROOFS OF THE AUXILIARY ESTIMATES FROM SECTION A.5

Proof of Lemma A.5.1. **(i)** Define

$$[w](x) = \fint_{\partial B_x^m} w(y)\,\mathrm{d}\mathcal{H}^2(y).$$

We observe that for $w \in W^{1,p}(\mathbb{R}^3)$, $1 \leq p < \infty$

$$\begin{aligned}
\|[w]\|_{L^p(\mathbb{R}^3)}^p &= \int_{\mathbb{R}^3} \left| \fint_{\partial B^m(x)} w(y)\,\mathrm{d}\mathcal{H}^2(y) \right|^p \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \fint_{\mathbb{R}^3} \mathbf{1}_{|x-y|=m^{-1}} |w(y)|^p \,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \fint_{\mathbb{R}^3} \mathbf{1}_{|y'|=m^{-1}} |w(y'+x)|^p \,\mathrm{d}\mathcal{H}^2(y')\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \fint_{\mathbb{R}^3} \mathbf{1}_{|y'|=m^{-1}} |w(x')|^p \,\mathrm{d}\mathcal{H}^2(y')\,\mathrm{d}x' \\
&= \|w\|_{L^p(\mathbb{R}^3)}^p.
\end{aligned}$$

By density, the operator $[\cdot]$ is defined on $L^p(\mathbb{R}^3)$. Using an analogous argument also for the average $(\cdot)$ over the full ball yields (A.5.6).

(ii) If $w \in L^p(K)$, the fact that $\rho \in L^\infty$ has compact support in K implies (A.5.7).

(iii) To prove (A.5.8), we first establish the following inequality:

Let $R > 0$ and $\varphi \in L^1(\mathbb{R}^3)$ with $\varphi \geq 0$, $\operatorname{supp}\varphi \subset B_R(0)$ and $\|\varphi\|_{L^1} = 1$. Let $w \in \dot{H}^1(\mathbb{R}^3)$, then

$$\|\varphi * w - w\|_{L^2(\mathbb{R}^3)} \lesssim R\|\nabla w\|_{L^2(\mathbb{R}^3)}. \tag{A.6.1}$$

There are several ways to prove this. By scaling, it is enough to consider the case $R = 1$. We can use the Fourier transform: observe that $\hat{\varphi} \in C^\infty(\mathbb{R}^3)$ with

$$|\nabla\hat{\varphi}| = |\mathcal{F}(x\varphi)| \in L^\infty(\mathbb{R}^3).$$

Since $\hat{\varphi}(0) = 1$, this shows that there is a constant $C > 0$ such that $|(1-\hat{\varphi})(k)| \leq C|k|$. Hence,

$$\|\varphi * w - w\|_{L^2(\mathbb{R}^3)}^2 = \|(1-\hat{\varphi})\hat{w}\|_{L^2(\mathbb{R}^3)}^2 \leq C\|k\hat{w}\|_{L^2(\mathbb{R}^3)}^2 \leq C\|\nabla w\|_{L^2(\mathbb{R}^3)}^2.$$

Now, (A.5.8) follows by choosing $\varphi(x) = \mathbf{1}_{B^m(0)}(x)$.

(iv) We note that $\mathcal{R}w = [\rho(w).]$. Thus, $\mathcal{R}$ is a bounded operator from $L^2(K)$ to $L^2(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$ and from $H^1(K)$ to $H^1(\mathbb{R}^3)$ by the previous estimates, together with the assumption that $\rho \in W^{1,\infty}$ with compact support and $L^{6/5}(\mathbb{R}^3) \subset \dot{H}^{-1}(\mathbb{R}^3)$.

For the estimate (A.5.9), we compute, for $w \in \dot{H}^1(\mathbb{R}^3)$,

$$\begin{aligned}
&\|\mathcal{R}w - \rho w\|_{L^2(\mathbb{R}^3)} \\
&= \left\| \fint_{\partial B^m(x)} \rho(y)(w)_y \,\mathrm{d}\mathcal{H}^2(y) - \rho(x)w(x) \right\|_{L^2(\mathbb{R}^3)} \\
&\le \left\| \fint_{\partial B^m(x)} (\rho(y) - \rho(x))\,(w)_y \,\mathrm{d}\mathcal{H}^2(y) \right\|_{L^2(\mathbb{R}^3)} \\
&\quad + \left\| \fint_{\partial B^m(x)} \rho(x)\,((w)_y - w(x))\ \mathrm{d}\mathcal{H}^2(y) \right\|_{L^2(\mathbb{R}^3)} \\
&=: J_1 + J_2.
\end{aligned}$$

Further, it is by Jensen's inequality

$$\begin{aligned}
J_1^2 &= \int_{\mathbb{R}^3} \left| \fint_{\partial B^m(x)} (\rho(y) - \rho(x))\,(w)_y \,\mathrm{d}\mathcal{H}^2(y) \right|^2 \mathrm{d}x \\
&\le \int_{\mathbb{R}^3} \fint_{\partial B^m(x)} |\rho(y) - \rho(x)|^2\, |(w)_y|^2 \,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x \\
&\le m^{-2} \|\nabla \rho\|^2_{L^\infty(\mathbb{R}^3)} \|w\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

where we used (A.5.6). Moreover,

$$\begin{aligned}
J_2^2 &= \int_{\mathbb{R}^3} \left| \fint_{\partial B^m(x)} \rho(x) \fint_{B^m(y)} w(z) - w(x)\,\mathrm{d}z\,\mathrm{d}y \right|^2 \mathrm{d}x \\
&\le \|\rho\|^2_{L^\infty} \int_{\mathbb{R}^3} \left| \fint_{\partial B^m(x)} \fint_{B^m(y)} w(z)\,\mathrm{d}z\,\mathrm{d}y - w(x) \right|^2 \mathrm{d}x \\
&= \|\rho\|^2_{L^\infty} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \left(\fint_{\partial B^m(x)} |B^m|^{-1} \mathbf{1}_{|y-z| \le R_m}\,\mathrm{d}y \right) (w(z))\,\mathrm{d}z - w(x) \right|^2 \mathrm{d}x \\
&= \|\rho\|^2_{L^\infty} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \varphi(x-z) w(z)\,\mathrm{d}z - w(x) \right|^2 \mathrm{d}x,
\end{aligned}$$

with the choice

$$\varphi(x) = \fint_{\partial B^m(x)} |B^m|^{-1} \mathbf{1}_{|y| \le R_m}\,\mathrm{d}y.$$

Using Fubini, we easily see that φ satisfies the assumptions to apply (A.6.1). Hence

$$J_2^2 \le C m^{-2} \|\rho\|^2_{L^\infty(\mathbb{R}^3)} \|\nabla w\|^2_{L^2(\mathbb{R}^3)}.$$

This proves (A.5.9). Finally, estimate (A.5.10) follows from testing with $\psi \in \dot{H}^1(\mathbb{R}^3)$

$$\langle \rho w - \mathcal{R}w, \psi \rangle = \langle w, \rho\psi - \mathcal{R}\psi \rangle \le m^{-1} \|w\|_{L^2(\mathbb{R}^3)} \|\rho\|_{W^{1,\infty}(\mathbb{R}^3)} \|\psi\|_{\dot{H}^1(\mathbb{R}^3)}.$$

To justify the first line, observe that

$$\begin{aligned}
\int_{\mathbb{R}^3} (\mathcal{R}w)(x)\psi(x)\,\mathrm{d}x &= \int \rho(x)(w)_x \fint_{\partial B^m(x)} \psi(y)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x \\
&= \int \rho(x) \left(\fint_{\mathbb{R}^3} \mathbf{1}_{|x-z|\le 1/m} w(z)\,\mathrm{d}z \right) \fint_{\partial B^m(x)} \psi(y)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} w(z) \left(\fint_{\mathbb{R}^3} \mathbf{1}_{|x-z|\le 1/m} \rho(x) \fint_{\partial B^m(x)} \psi(y)\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x \right) \mathrm{d}z \\
&= \int_{\mathbb{R}^3} w(z)(\mathcal{R}\psi)(z)\,\mathrm{d}z.
\end{aligned}$$

(v) Recall that $F = \rho u - j$. Since $\rho \in L^\infty$ has compact support and $u \in \dot{H}^1(\mathbb{R}^3)$, we have $\rho u \in L^2(\mathbb{R}^3)$. Furthermore, from hypotheses (H3) we have $j \in L^2(\mathbb{R}^3)$. Since $\mathcal{F} = \mathcal{R}u - [j]$ and $u \in L^2(K)$, we have $\mathcal{F} \in L^2(\mathbb{R}^3)$. Finally, we have with $W_1 = (u)_1 - V_1$

$$\begin{aligned}
\mathbb{E}_m[W_1^2] = \int_{\mathbb{R}^3\times\mathbb{R}^3} |(u)_x - v|^2 f(\mathrm{d}x, \mathrm{d}v) &\le 2\int_{\mathbb{R}^3} \rho(x)|(u)_x|^2\,\mathrm{d}x + 2\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(\mathrm{d}x, \mathrm{d}v) \\
&\le C\|u\|_{L^2(K)} + 2\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(\mathrm{d}x, \mathrm{d}v)
\end{aligned}$$

which is uniformly bounded by (A.5.7) and (H1).

To prove (A.5.12), we first focus on estimating the L^2-norm. Note that

$$F - \mathcal{F} = \rho u - j - (\mathcal{R}u - [j])\,.$$

Hence, it is

$$\|F - \mathcal{F}\|_{L^2(\mathbb{R}^3)} \le \|\rho u - \mathcal{R}u\|_{L^2(\mathbb{R}^3)} + \|j - [j]\|_{L^2(\mathbb{R}^3)}\,.$$

Using (A.5.9), it is enough to see

$$\|w - [w]\|_{L^2(\mathbb{R}^3)} \lesssim m^{-1}\,\|w\|_{\dot{H}^1(\mathbb{R}^3)} \qquad \text{for all } w \in \dot{H}^1(\mathbb{R}^3).$$

First, let $w \in \mathcal{S}(\mathbb{R}^3)$. Then

$$\begin{aligned}
\|w - [w]\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} \left| \fint_{\partial B^m(x)} w(x) - w(y)\,\mathrm{d}\mathcal{H}^2(y) \right|^2 \mathrm{d}x \\
&\le \int_{\mathbb{R}^3} \fint_{\partial B^m(x)} |w(x) - w(y)|^2\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^3} \fint_{\partial B^m(x)} \int_0^1 |\nabla w(x + t(y-x))|^2 |x-y|^2\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^2(y)\,\mathrm{d}x \\
&\lesssim m^{-2} \fint_{\partial B^m(x)} \int_0^1 \|\nabla w\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^2(y) \\
&= m^{-2}\|w\|^2_{\dot{H}^1(\mathbb{R}^3)},
\end{aligned}$$

where we used Jensen's inequality twice and the fundamental theorem of calculus. Now by density of $\mathcal{S}(\mathbb{R}^3)$ in $\dot{H}^1(\mathbb{R}^3)$, we obtain the estimate of the L^2-norm in (A.5.12). To estimate the $\dot H^{-1}$-norm in (A.5.12), we again argue by testing with $\psi \in \dot H^1(\mathbb{R}^3)$. By (A.5.10), it is enough to see

$$|\langle j-[j],\psi\rangle| = |\langle j,\psi-[\psi]\rangle| \le \|j\|_{L^2(\mathbb{R}^3)}\|\psi-[\psi]\|_{L^2(\mathbb{R}^3)} \le m^{-1}\|\psi\|_{\dot H^1(\mathbb{R}^3)}.$$

This finishes the proof. □

Proof of Lemma A.5.2. Recalling the definition of $B^m(y) = B_{R_m}(y)$ and (A.1.7), it is well-known that

$$G\delta_y^m(x) = \begin{cases} m\mathrm{Id} & x \in B^m(y) \\ g(x-y) - \frac{R_m^2}{6}\Delta g(x-y) & x \in \mathbb{R}^3 \setminus B^m(y), \end{cases}$$

with g as in (A.2.1). Then (A.5.13), (A.5.15) and (A.5.16) follow immediately. (A.5.15) implies that $\|G\delta\|_{\dot H^1(\mathbb{R}^3)} \lesssim m^{1/2}$ and, since G is an isometry from $\dot H^{-1}(\mathbb{R}^3)$ to $\dot H^1(\mathbb{R}^3)$, this proves (A.5.17). The bounds for A follow by using the identity $A = G - A\rho G$ and that $A\rho$ maps $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ □

Proof of Lemma A.5.3. To deduce the bound for $G\delta_y$ in $H^s_{\mathrm{loc}}(\mathbb{R}^3)$, note for example that $e^{-|x-y|}G\delta_y = e^{-|x-y|}/(4\pi|x-y|) \in H^s(\mathbb{R}^3)$ (e.g. by Fourier). The corresponding estimate for A follows from the identity $A = G - A\rho G$ (cf. (A.2.2)) and the fact that $A\rho$ maps H^s_{loc} to H^s_{loc}.

For the second estimate, observe that $H^{3/2+\varepsilon}(K')$ embeds into the space of ε-Hölder continuous functions on K'. Hence, we may estimate, for every $w \in H^{3/2+\varepsilon}(K')$

$$\langle \delta_y^m - \delta_y, w\rangle \le \fint_{B^m(y)} |w(x) - w(y)| \,\mathrm{d}H^2(x) \le m^{-\varepsilon}\|w\|_{C^\varepsilon(K')} \le Cm^{-\varepsilon}\|w\|_{H^{3/2+\varepsilon}(K')}.$$

This concludes the proof. □

Proof of Lemma A.5.4. By (A.2.4), $G - G^m$ is a convolution operator with convolution kernel

$$\bar g_m := \eta_m g - \psi_m.$$

Thus, to prove (A.5.20) and (A.5.21) it suffices to show

$$\|\nabla^l \bar g_m\|_{L^1(\mathbb{R}^3)} \le m^{-2+l} \tag{A.6.2}$$

for $l = 0, 1$. Moreover, (A.6.2) for $l = 2$ implies that G^m is a bounded operator from $\dot H^l(\mathbb{R}^3)$ to $\dot H^{l+2}(\mathbb{R}^3)$ since we know that G is a bounded operator from $\dot H^l(\mathbb{R}^3)$ to $\dot H^{l+2}(\mathbb{R}^3)$.

By definition of η_m, we have for all $l \in \mathbb{N}$

$$|\nabla^l(\eta_m g)| \lesssim m^{1+l}\mathbf{1}_{B_{3R_m}(0)\setminus B_{2R_m}(0)}.$$

In particular, for all $1 \le p \le \infty$ and all $l \in \mathbb{N}$

$$\|\nabla^l(\eta_m g)\|_{L^p(\mathbb{R}^3)} \lesssim m^{1+l-3/p}. \tag{A.6.3}$$

In view of (A.2.3), this implies

$$\|\nabla^l(\eta_m g)\|_{L^p(\mathbb{R}^3)} \lesssim m^{1+l-3/p}, \tag{A.6.4}$$

for all $l \ge 1$ and all $1 < p < \infty$. By the Hölder inequality, this bound also holds for $p = 1$ and by the Poicaré inequality also for $l = 0$. Combining (A.6.3) and (A.6.4) yields (A.6.2). □

ESTIMATES FOR $\mathfrak{J}_3$

In this part of the appendix, we detail the estimates of $\mathfrak{J}_3$ from (A.5.27) We follow the same strategy as for $\mathfrak{J}_1$ described in Steps 3.2 and 3.3 of the proof of Lemma A.4.1. Therefore, we just name and prove the relevant lemmas. Observe that we need weaker bounds. If we want to show $|\mathfrak{J}_3| \lesssim m^{-2}$, this requires

$$I_{3,\nabla}^{i,j,k,l} = \mathbb{E}_m \left[\int_{B_i} \nabla\Psi_{j,k}(x) \nabla\Psi_{n,\ell}(x) \,\mathrm{d}x \right] \lesssim m^{-\alpha+2}, \quad \alpha = \#\{i,j,k,n,\ell\}.$$

As before, we write $\nabla\Psi_{j,l} = \nabla\Psi_k^{(1)} + \nabla\Psi_{j,l}^{(2)}$, where

$$\nabla\Psi_k^{(1)} := \nabla\Psi^{(1,1)} + \nabla\Psi_k^{(1,2)} := \nabla G \left[\rho A\left(F - W_k \delta_k^m\right)\right],$$

and

$$\nabla\Psi_{j,k}^{(2)} := \nabla\Psi_j^{(2,1)} + \nabla\Psi_{j,k}^{(2,2)} := (1-\delta_{ij}) \nabla G \left[\left(A\left(F - W_k \delta_k^m\right) \right)_j \delta_j^m \right].$$

Recall that $W_k = (u)_k - V_k$ and $F = \rho u - j$.

We observe that

$$\begin{aligned} \mathbb{E}_m[\nabla\Psi^{(1,1)}] &= \nabla G \rho A F, \\ \mathbb{E}_m[\nabla\Psi_k^{(1,2)}] &= \nabla G \rho A \mathcal{F}, \\ \mathbb{E}_m[\nabla\Psi_j^{(2,1)}] &= (1-\delta_{ij}) \nabla G \mathcal{R} A F, \\ \mathbb{E}_m[\nabla\Psi_{j,k}^{(2,2)}] &= (1-\delta_{ij})(1-\delta_{jk}) \nabla G \mathcal{R} A \mathcal{F}. \end{aligned} \tag{A.6.5}$$

Furthermore, we observe that the only difference to the discussion of $\mathfrak{J}_1$ is that the outmost G is replaced by ∇G. Hence, we we will apply the same strategy as before using the analogous auxiliary estimates for the gradient.

We start by giving the corresponding lemmas in the case $\#\{i,j,k,n,\ell\} \geq 4$.

Lemma A.6.1.

$$\left\| \mathbb{E}_m[\nabla\Psi_k^{(1)}] \right\|_{L^2(\mathbb{R}^3)} \lesssim m^{-1}, \tag{A.6.6}$$

$$\left\| \mathbb{E}_m[\nabla\Psi_{j,k}^{(2)}] \right\|_{L^2(\mathbb{R}^3)} \lesssim m^{-1} \quad \textit{if } j \neq k. \tag{A.6.7}$$

Lemma A.6.2.

$$\|\mathbb{E}_m\left[\nabla\Psi_{j,k} \nabla\Psi_{j,\ell}\right]\|_{L^1(\mathbb{R}^3)} \lesssim m^{-1} \quad \textit{if } \#\{i,j,k,\ell\} = 4, \tag{A.6.8}$$

$$\|\mathbb{E}_m\left[\nabla\Psi_{j,k} \nabla\Psi_{n,k}\right]\|_{L^1(\mathbb{R}^3)} \lesssim m^{-1} \quad \textit{if } \#\{i,j,k,n\} = 4, \tag{A.6.9}$$

$$\|\mathbb{E}_m\left[\nabla\Psi_{j,k} \nabla\Psi_{n,j}\right]\|_{L^1(\mathbb{R}^3)} \lesssim m^{-1} \quad \textit{if } \#\{i,j,k,n\} = 4. \tag{A.6.10}$$

Lemma A.6.3. *We have for any* i,j,k

$$\begin{aligned} &\left\| \mathbb{E}_m[\nabla\Psi^{(1,1)}] \right\|_{L^2(\mathbb{R}^3)} + \left\| \mathbb{E}_m[\nabla\Psi_k^{(1,2)}] \right\|_{L^2(\mathbb{R}^3)} \\ &\quad + \left\| \mathbb{E}_m[\nabla\Psi_j^{(2,1)}] \right\|_{L^2(\mathbb{R}^3)} + \left\| \mathbb{E}_m[\nabla\Psi_{j,k}^{(2,2)}] \right\|_{L^2(\mathbb{R}^3)} \lesssim m. \end{aligned} \tag{A.6.11}$$

$$\begin{aligned} &\left\| \mathbb{E}_m[\mathbf{1}_{B_i^m} \nabla\Psi^{(1,1)}] \right\|_{L^2(\mathbb{R}^3)} + \left\| \mathbb{E}_m[\mathbf{1}_{B_i^m} \nabla\Psi_k^{(1,2)}] \right\|_{L^2(\mathbb{R}^3)} \\ &\quad + \left\| \mathbb{E}_m[\mathbf{1}_{B_i^m} \nabla\Psi_j^{(2,1)}] \right\|_{L^2(\mathbb{R}^3)} + \left\| \mathbb{E}_m[\mathbf{1}_{B_i^m} \nabla\Psi_{j,k}^{(2,2)}] \right\|_{L^2(\mathbb{R}^3)} \lesssim m^{-5/2}. \end{aligned} \tag{A.6.12}$$

Proof of Lemma A.6.1. By (A.6.5), we have

$$\mathbb{E}_m[\Psi_k^{(1)}] = \nabla G\rho A(F - \mathcal{F}).$$

Using (A.5.12) yields (A.6.6).

Similarly, for $j \neq k$, $i \neq j$,

$$\mathbb{E}_m[\Psi_{j,k}^{(2)}] = \nabla G\mathcal{R}A(F - \mathcal{F}).$$

Using again (A.5.12) yields (A.6.7). □

Proof of Lemma A.6.2. Regarding (A.6.8), we have

$$\mathbb{E}_m\left[\nabla\Psi_{j,k}\nabla\Psi_{j,\ell}\right] = \int \rho(y_1)\left(\nabla G\rho A(F-\mathcal{F}) - \left(A(F-\mathcal{F})\right)_{y_1}\nabla G\delta_{y_1}^m\right)^2 \,\mathrm{d}y_1,$$

and hence

$$\begin{aligned}
&\|\mathbb{E}_m\left[\nabla\Psi_{j,k}\nabla\Psi_{j,\ell}\right]\|_{L^1(\mathbb{R}^3)} \\
&\lesssim \|\nabla G\rho A(F-\mathcal{F})\|_{L^2(\mathbb{R}^3)}^2 + \int \rho(y_1)\left(A(F-\mathcal{F})\right)_{y_1}\|\nabla G\delta_{y_1}^m\|_{L^2(K)}^2 \,\mathrm{d}y_1 \\
&\lesssim m^{-2} + m^{-1} \\
&\lesssim m^{-1},
\end{aligned}$$

where we used (A.5.12) for both terms and (A.5.17) for the second term.

Regarding (A.6.9), we compute

$$\begin{aligned}
&\mathbb{E}_m\left[\nabla\Psi_{j,k}\nabla\Psi_{n,k}\right] \\
&= \int \rho(y_1)\left(\nabla G(\rho-\mathcal{R})AF - \left((u)_{y_2} - v_2\right)\nabla G(\rho-\mathcal{R})A\delta_{y_2}^m\right)^2 f(\mathrm{d}y_2, \mathrm{d}v_2).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|\mathbb{E}_m\left[\nabla\Psi_{j,k}\nabla\Psi_{n,k}\right]\|_{L^1(\mathbb{R}^3)} \\
&\lesssim \|\nabla G(\rho-\mathcal{R})A\rho u\|_{L^2(\mathbb{R}^3)}^2 + \sup_{y_1}\|\nabla G(\rho-\mathcal{R})A\delta_{y_1}^m\|_{L^2(\mathbb{R}^3)}^2 \int \left((u)_{y_2} - v_2\right)^2 f(\mathrm{d}y_2, \mathrm{d}v_2) \\
&\lesssim m^{-1},
\end{aligned}$$

where we used (A.5.10) for both terms and (A.5.16) and (H1) for the second term. Finally, (A.6.10) follows from (A.6.8) and (A.6.9) via Young's inequality. □

Proof of Lemma A.6.3. The first estimate, (A.6.11), follows directly from (A.6.5) and (A.5.11) together with the fact that the operators $\nabla G\rho A$, $\nabla G\rho A$, $\nabla G\mathcal{R}A$ and $\nabla G\mathcal{R}A$ are all bounded operators from $\dot{H}^{-1}(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$.

Regarding (A.6.12), these estimates follow from (A.6.11) if $i \neq k$. If $i = k$, we only need to consider those terms, in which k appears, i.e. $\nabla\Psi_k^{(1,2)}$ and $\nabla\Psi_{j,k}^{(2,2)}$. Again, we only need to consider the case $j \neq k = i$.

Then

$$\left\|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_{j,i}^{(1,2)}\right]\right\| = \left\|\int \mathbf{1}_{B_{y_1}^m}\nabla G\rho A[\big((u)_{y_1}-v_1\big)\delta_{y_1}^m]f(\mathrm{d}y_1,\mathrm{d}v_1)\right\|$$
$$\leq \sup_{y_1\in\mathbb{R}^3}\|\nabla G\rho A\delta_{y_1}^m\|_{L^\infty(\mathbb{R}^3)}\left\|\int\big((u)_{y_1}-v_1\big)\mathbf{1}_{B_{y_1}^m}f(\mathrm{d}y_1,\mathrm{d}v_1)\right\|_{L^2(\mathbb{R}^3)}$$
$$\lesssim m^{-5/2}.$$

Here, we used (A.5.17) and that $G\rho$ maps $\dot{H}^1(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$ for the first term, and (H1) as well as (A.5.6) followed by (A.5.7) for the second. Since for $j\neq i$,

$$\mathbb{E}_m[\mathbf{1}_{B_i^m}\nabla\Psi_{j,i}^{(2,2)}] = \int \mathbf{1}_{B^m(y_1)}\nabla G\mathcal{R}A\left[\big((u)_{y_1}-v_1\big)\delta_{y_1}^m\right]f(\mathrm{d}y_1,\mathrm{d}v_1),$$

the estimate of this term is analogous. □

This finishes the cases in which at most 2 indices are equal. For the remaining cases, we can again follow the same strategy as for $\mathfrak{J}_1$. We provide here only the necessary estimates. All the other estimates follow by applying Young's inequality and reducing the proofs to the estimates given here, just as in the proof for $\mathfrak{J}_1$.

Lemma A.6.4. *The corresponding estimates in the case* $(\alpha,\beta,\gamma,\delta)=(2,2,2,2)$ *are:*

$$i=k, j=n: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_{j,i}^{(2,2)}(\nabla\Psi_j^{(2,1)}-\nabla\Psi_{j,\ell}^{(2,2)})\right]\right|\,\mathrm{d}x\lesssim m^{-2}. \tag{A.6.13}$$
$$i=k, j=\ell: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_{j,i}^{(2,2)}\nabla\Psi_{n,j}^{(2,2)}\right]\right|\,\mathrm{d}x\lesssim m^{-2}. \tag{A.6.14}$$
$$i=k=\ell: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_{j,i}^{(2,2)}\nabla\Psi_{n,i}^{(2,2)}\right]\right|\,\mathrm{d}x\lesssim m^{-2}. \tag{A.6.15}$$
$$j=n, k=\ell: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\left|\nabla\Psi_{j,k}^{(2,2)}\right|^2\right]\right|\,\mathrm{d}x\lesssim m^{-2}. \tag{A.6.16}$$
$$i=k=\ell, j=n: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\left|\nabla\Psi_{j,i}^{(2,2)}\right|^2\right]\right|\,\mathrm{d}x\lesssim 1. \tag{A.6.17}$$

The corresponding estimate in the case $(\alpha,\beta,\gamma,\delta)=(2,1,2,1)$ *is:*

$$j=n: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\left|\nabla\Psi_j^{(2,1)}\right|^2\right]\right|\,\mathrm{d}x\lesssim m^{-2}. \tag{A.6.18}$$

The corresponding estimates in the case $(\alpha,\beta,\gamma,\delta)=(1,2,2,2)$ *are:*

$$i=k=\ell: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_i^{(1,2)}\nabla\Psi_{n,i}^{(2,2)}\right]\right|\,\mathrm{d}x\lesssim m^{-2}. \tag{A.6.19}$$
$$i=\ell, k=n: \quad \int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_k^{(1,2)}\nabla\Psi_{k,i}^{(2,2)}\right]\right|\,\mathrm{d}x\lesssim m^{-1}. \tag{A.6.20}$$

Proof of Lemma A.6.4. For (A.6.13), it is

$$\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_{j,i}^{(2,2)}(\nabla\Psi_{j,\ell}^{(2,1)}-\nabla\Psi_{j,\ell}^{(2,2)})\right]$$
$$=\iint\rho(y_2)\mathbf{1}_{B_{y_1}^m}(x)\left(A\left[\big((u)_{y_1}-v_1\big)\delta_{y_1}^m\right]\right)_{y_2}(\nabla G\delta_{y_2}^m)^2(x)\left(A(F-\mathcal{F})\right)_{y_2}f(\mathrm{d}y_1,\mathrm{d}v_1)\mathrm{d}y_2.$$

By (A.5.15), it holds

$$\int \mathbf{1}_{B^m_{y_1}}(x)(\nabla G\delta^m_{y_2})^2(x)\,\mathrm{d}x \lesssim m^{-3}\frac{1}{|y_2-y_1|^4+m^{-4}}, \tag{A.6.21}$$

and thus analogously as in the corresponding term for $\mathfrak{J}_1$ using (A.5.14)

$$\begin{aligned}
&\int \left|\mathbb{E}_m\left[\mathbf{1}_{B^m_i}\nabla\Psi^{(2,2)}_{j,i}(\nabla\Psi^{(2,1)}_{j,\ell}-\nabla\Psi^{(2,2)}_{j,\ell})\right]\right|\,\mathrm{d}x\\
&\lesssim m^{-4}\iint \rho(y_2)|(u)_{y_1}-v_1|\frac{1}{|y_2-y_1|^4+m^{-4}}\left(1+\frac{1}{|y_2-y_1|+m^{-1}}\right)f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2\\
&\lesssim m^{-2}.
\end{aligned}$$

Regarding (A.6.14), we compute

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{B^m_i}(x)\nabla\Psi^{(2,2)}_{ji}(x)\nabla\Psi^{(2,2)}_{nj}(x)\right]\\
&=\iint \big((u)_{y_1}-v_1\big)\big((u)_{y_2}-v_2\big)\mathbf{1}_{B^m(y_1)}(x)\left(A\delta^m_{y_1}\right)_{y_2}(\nabla G\delta^m_{y_2})(x)\cdot\\
&\quad\cdot\left(\nabla G\mathcal{R}A\delta^m_{y_2}\right)(x)f(\mathrm{d}y_1,\mathrm{d}v_1)f(\mathrm{d}y_2,\mathrm{d}v_2).
\end{aligned}$$

Now we use that $G\mathcal{R}$ maps $\dot{H}^1(\mathbb{R}^3)$ to $W^{1,\infty}(\mathbb{R}^3)$ to deduce as in the previous case

$$\begin{aligned}
&\int \left|\mathbb{E}_m\left[\mathbf{1}_{B^m_i}(x)\nabla\Psi^{(2,2)}_{ji}(x)\nabla\Psi^{(2,2)}_{nj}(x)\right]\right|\,\mathrm{d}x\\
&\lesssim m^{1/2}m^{-3}\int\left(\big((u)_{y_1}-v_1\big)^2+\big((u)_{y_1}-v_1\big)^2\right)\frac{1+\frac{1}{|y_2-y_1|+m^{-1}}}{|y_2-y_1|^2+m^{-2}}f(\mathrm{d}y_1,\mathrm{d}v_1)f(\mathrm{d}y_2,\mathrm{d}v_2)\\
&\lesssim m^{-5/2}\log m.
\end{aligned}$$

For (A.6.15), we get

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{B^m_i}(x)\nabla\Psi^{(2,2)}_{ji}(x)\nabla\Psi^{(2,2)}_{ni}(x)\right]\\
&=\int\big((u)_{y_1}-v_1\big)^2\mathbf{1}_{B^m_{y_1}}(x)\left(\nabla G\mathcal{R}A\delta^m_{y_1}\right)(x)^2f(\mathrm{d}y_1,\mathrm{d}v_1).
\end{aligned}$$

Thus by (A.5.16), (A.5.7) and (H1), it is

$$\int\left|\mathbb{E}_m\left[\mathbf{1}_{B^m_i}(x)\nabla\Psi^{(2,2)}_{ji}(x)\nabla\Psi^{(2,2)}_{ni}(x)\right]\right|\,\mathrm{d}x\lesssim m^{-2}.$$

The case (A.6.16):

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{B^m_i}(x)\nabla\Psi^{(2,2)}_{jk}(x)\nabla\Psi^{(2,2)}_{jk}(x)\right]\\
&=m^{-3}\iint(\rho)_x\rho(y_2)\big((u)_{y_1}-v_1\big)^2\left(A\delta^m_{y_1}\right)^2_{y_2}(\nabla G\delta^m_{y_2})^2(x)f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2.
\end{aligned}$$

Using (A.5.17),(A.5.16), (A.5.7) and (H1), we get

$$\begin{aligned}
&\int\left|\mathbb{E}_m\left[\mathbf{1}_{B^m_i}\Psi^{(2,2)}_{jk}\Psi^{(2,2)}_{jk}\right]\right|\,\mathrm{d}x\\
&\lesssim m^{-2}\int\rho(y_2)\big((u)_{y_1}-v_1\big)^2\left(A\delta^m_{y_1}\right)^2_{y_2}f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2\lesssim m^{-2}.
\end{aligned}$$

For the next estimate (A.6.17), we get

$$\begin{aligned}
&\mathbb{E}_m\left[\mathbf{1}_{B_i^m}(x)\nabla\Psi_{ji}^{(2,2)}(x)\nabla\Psi_{ji}^{(2,2)}(x)\right] \\
&= \iint \rho(y_2)\big((u)_{y_1}-v_1\big)^2 \mathbf{1}_{B_{y_1}^m}(x)\left(A\delta_{y_1}^m\right)_{y_2}^2 (\nabla G\delta_{y_2}^m)^2(x) f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2.
\end{aligned}$$

By using again (A.6.21) and (A.5.14), we get

$$\begin{aligned}
&\int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_{ji}^{(2,2)}\nabla\Psi_{ji}^{(2,2)}\right]\right|\,\mathrm{d}x \\
&\lesssim m^{-3}\int\rho(y_2)\big((u)_{y_1}-v_1\big)^2\left(\frac{1}{|y_2-y_1|^4+m^{-4}}+\frac{1}{|y_2-y_1|^6+m^{-6}}\right) f(\mathrm{d}y_1,\mathrm{d}v_1)\,\mathrm{d}y_2 \\
&\lesssim \int\big((u)_{y_1}-v_1\big)^2 f(\mathrm{d}y_1,\mathrm{d}v_1)\lesssim 1.
\end{aligned}$$

To estimate (A.6.18), observe

$$\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\left|\nabla\Psi_j^{(2,1)}\right|^2\right]\lesssim m^{-3}\int(\rho)_x\rho(y_1)(AF)_{y_1}^2|\nabla G\delta_{y_1}^m|(x)^2\,\mathrm{d}y_1,$$

and hence by (A.5.17), it holds

$$\int\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\left|\nabla\Psi_j^{(2,1)}\right|^2\right]\,\mathrm{d}x\lesssim m^{-2}\int\rho(y_1)(AVu)_{y_1}^2\,\mathrm{d}y_1\lesssim m^{-2}.$$

For (A.6.19), it holds

$$\begin{aligned}
&\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_i^{(1,2)}\nabla\Psi_{n,i}^{(2,2)}\right]\right| \\
&\le \int \mathbf{1}_{B_{y_1}^m}\left|\nabla G\mathcal{R}A\left[\big((u)_{y_1}-v_1\big)\delta_{y_1}^m\right]\right|\left|\nabla G\rho A\left[\big((u)_{y_1}-v_1\big)\delta_{y_1}^m\right]\right| f(\mathrm{d}y_1,\mathrm{d}v_1) \\
&\lesssim m\int\mathbf{1}_{B_{y_1}^m}\big((u)_{y_1}-v_1\big)^2 f(\mathrm{d}y_1,\mathrm{d}v_1),
\end{aligned}$$

where we used (A.5.17). Thus

$$\int|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_i^{(1,2)}\nabla\Psi_{n,i}^{(2,2)}\right]|\,\mathrm{d}x\lesssim m^{-2}.$$

Finally for (A.6.20), it is

$$\begin{aligned}
&\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_k^{(1,2)}\nabla\Psi_{k,i}^{(2,2)}\right]\right| \\
&\le\int\mathbf{1}_{B_{y_1}^m}\rho(y_1)\left|(u)_{y_2}-v_2\right|^2\left|\nabla G\rho A\delta_{y_2}^m\right|\left|(A\delta_{y_1}^m)_{y_2}\right|\left|\nabla G\delta_{y_2}^m\right|\,\mathrm{d}y_1 f(\mathrm{d}y_2,\mathrm{d}v_2) \\
&\lesssim m^{1/2}\int\mathbf{1}_{B_{y_1}^m}\rho(y_1)\left|(u)_{y_1}-v_1\right|^2\left|(A\delta_{y_1}^m)_{y_2}\right|\left|\nabla G\delta_{y_2}^m\right|\,\mathrm{d}y_1 f(\mathrm{d}y_2,\mathrm{d}v_2),
\end{aligned}$$

where we used (A.5.16). This is estimated as in (A.6.14) to get

$$\int\left|\mathbb{E}_m\left[\mathbf{1}_{B_i^m}\nabla\Psi_k^{(1,2)}\nabla\Psi_{k,i}^{(2,2)}\right]\right|\,\mathrm{d}x\lesssim m^{-1}.$$

This finishes the proof. □

B Long-Time Behaviour and Stability for Quasilinear Doubly Degenerate-Parabolic Equations of Higher Order

Abstract

We study the long-time behaviour of solutions to quasilinear doubly degenerate parabolic problems of fourth order. The equations model for instance the dynamic behaviour of a non-Newtonian thin-film flow on a flat impermeable bottom and with zero contact angle. We consider a shear-rate dependent fluid the rheology of which is described by a constitutive power-law or Ellis-law for the fluid viscosity. In all three cases, positive constants (i.e. positive flat films) are the only positive steady-state solutions. Moreover, we can give a detailed picture of the long-time behaviour of solutions with respect to the $H^1(\Omega)$-norm. In the case of shear-thickening power-law fluids, one observes that solutions which are initially close to a steady state, converge to equilibrium in finite time. In the shear-thinning power-law case, we find that steady states are polynomially stable in the sense that, as time tends to infinity, solutions which are initially close to a steady state, converge to equilibrium at rate $1/t^{1/\beta}$ for some $\beta > 0$. Finally, in the case of an Ellis-fluid, steady states are exponentially stable in $H^1(\Omega)$.

B.1 Introduction

Aim of the paper

The present paper is concerned with the asymptotic behaviour of positive weak solutions to fourth-order quasilinear (doubly) degenerate parabolic problems as they arise in the modelling of non-Newtonian thin-film flows. It turns out that, for large times, fluids with a shear-rate dependent viscosity exhibit a specific asymptotic behaviour, depending on their shear-thickening or shear-thinning nature, respectively.

We consider a thin layer of a viscous, non-Newtonian and incompressible fluid on an impermeable flat bottom, as sketched in Figure B.1.

In addition to the non-Newtonian fluid rheology, the following modelling assumptions are crucial for the analysis of the resulting partial differential equations. First, the fluid flow is assumed to be uniform in one horizontal direction (in y-direction in Figure B.1), such that we obtain a (spatially) one-dimensional problem. Moreover, we assume

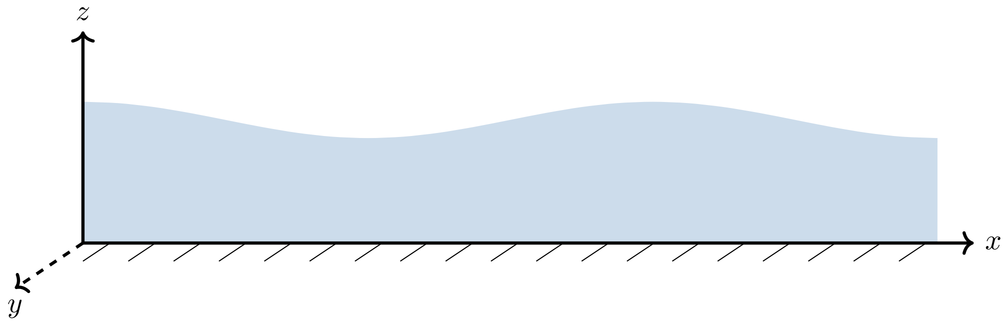

Figure B.1: Cross section of fluid film on impermeable solid bottom.

that the characteristic height of the fluid layer is rather thin compared to its characteristic length and consider the asymptotic limit of a vanishing aspect ratio. Based on a non-Newtonian Navier–Stokes system, we use the so-called lubrication approximation [GO03; GP08; OO95] in order to derive an evolution equation for the height $u = u(t,x) \geq 0$ of the fluid film at time $t > 0$ and spatial position $x \in \Omega$, where $\Omega \subset \mathbb{R}$ is a bounded interval. We neglect gravitational effects and assume that the dynamics of the flow is driven by capillarity only. Finally, we prescribe a no-slip condition on the lower boundary of the fluid film. However, the mathematical analysis of the present paper does also apply to the case of Navier-slip conditions.

As constitutive laws for the non-Newtonian shear-dependent fluid we consider so-called **power-law fluids**, also called **Ostwald-de Waele fluids**, and so-called **Ellis-fluids**; see below for more details on these material laws. In the case of power-law fluids, when prescribing a no-slip condition on the lower boundary, the resulting evolution problem reads

$$\begin{cases} u_t + \left(u^{\alpha+2}|u_{xxx}|^{\alpha-1}u_{xxx}\right)_x = 0, & t > 0,\ x \in \Omega, \\ u_x(t,x) = u_{xxx}(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{B.1.1}$$

Note that (B.1.1)$_1$ is a fourth-order quasilinear parabolic equation that is doubly degenerate in the sense that the degeneracy occurs both with respect to the unknown u and with respect to its third spatial derivative u_{xxx}. The Neumann-type boundary conditions $u_x = u_{xxx} = 0$ on $\partial\Omega$ reflect the zero-contact angle condition and the no-flux condition at the lateral boundary, respectively. Finally, $u_0 > 0$ denotes the given positive initial film height. Note that for $0 < \alpha < 1$, the coefficients of the highest-order term depend only Hölder continuously on the unknown and lower-order derivatives.

In the case of Ellis-fluids, we obtain the evolution equation

$$\begin{cases} u_t + a\left(u^3\left[1 + b|uu_{xxx}|^{\alpha-1}\right]u_{xxx}\right)_x = 0, & t > 0,\ x \in \Omega, \\ u_x(t,x) = u_{xxx}(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{B.1.2}$$

Here, $a, b > 0$ are positive physical parameters, depending on the constant surface tension, the flow-behaviour exponent α and the characteristic viscosity of the fluid. However, for clarity of presentation, we drop these parameters in our analysis since they do not affect our arguments. This equation has for instance been studied in [AG02; LM20] in the context of self-similar solutions and local strong solutions, respectively.

The main difference in the classification of (B.1.1) and (B.1.2) is that (B.1.1) is doubly degenerate in the sense that we loose parabolicity if either the unknown u or its third spatial derivative u_{xxx} become zero. In contrast, (B.1.2) is degenerate only in the unknown u itself.

For $\alpha = 1$ we recover in both equations (B.1.1) and (B.1.2) the well-known Newtonian thin-film equation

$$u_t + \big(u^3 u_{xxx}\big)_x = 0, \quad t > 0,\ x \in \Omega. \tag{B.1.3}$$

This equation is studied extensively in the mathematical literature. For results concerning existence, uniqueness and stability of weak solutions to (B.1.3) we refer the reader for instance to the works [BF90; BBD95; BP96].

Main results of the paper – Stability of steady states and long-time behaviour of positive weak solutions

In the present paper we study the behaviour of positive weak solutions to (B.1.1) and (B.1.2), respectively, for large times. Note that we consider only the case of strictly positive initial values $u_0 > 0$ since these allow us to find a positive time up to which solutions remain strictly positive.

The main results of the paper are the following: We prove local existence of positive weak solutions to the power-law thin-film equation (B.1.1) for all flow-behaviour exponents $\alpha > 0$, see Theorem B.3.2 below. In the case $\alpha > 1$ of shear-thinning power-law fluids, even global existence of non-negative weak solutions has been established in [AG04], using a two-step regularisation scheme, Galerkin approximation and energy/entropy methods. Since the present paper is concerned with stability of positive steady states, we are only interested in positive weak solutions. Therefore, we use a simpler regularisation method that allows us (only) to construct local positive weak solutions, but for all flow-behaviour exponents $\alpha > 0$. These solutions can then be extended to global weak solutions as long as they are close to steady states.

Moreover, again for all $\alpha > 0$, we can characterise positive steady states of the power-law thin-film equation by positive constants, cf. Theorem B.3.5 below. As already mentioned, the long-time behaviour of solutions that are initially close to a steady state $\bar{u}_0 = ⨍_\Omega u_0\, dx$ depends strongly on the choice of the flow-behaviour exponent α, i.e., on the shear-thinning, respectively shear-thickening nature of the fluid. The main result concerning global existence and stability properties of steady states is the following:

Theorem.
Fix $\alpha > 0$. Then there exists an $\varepsilon > 0$ such that, for all positive initial values $u_0 \in H^1(\Omega)$ with $\|u_0 - \bar{u}_0\|_{H^1(\Omega)} \leq \varepsilon$, problem (B.1.1) *possesses at least one global positive weak solution*

$$u \in C\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1,loc}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha},loc}\big((0,\infty); (W^1_{\alpha+1,B}(\Omega))'\big),$$

satisfying the boundary condition $u_x = 0$ on $\partial\Omega$ pointwise for almost every $t \geq 0$. Moreover, this global solution has the following asymptotic behaviour:

(i) In the shear-thickening case $0 < \alpha < 1$, there exists a positive but finite time $0 < t^* < \infty$ such that

$$u(t,\cdot) \longrightarrow \bar{u}_0 \text{ in } H^1(\Omega), \text{ as } t \to t^*, \quad \text{and} \quad u(t,x) = \bar{u}_0, \quad t \geq t^*,\ x \in \Omega.$$

(ii) In the shear-thinning case $1 < \alpha < \infty$*, there exists a constant* $C > 0$ *such that*

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \leq \frac{C\varepsilon}{\left(1 + C\varepsilon^{\alpha-1}t\right)^{\frac{1}{\alpha-1}}}, \quad 0 \leq t < \infty.$$

(iii) In the Newtonian case $\alpha = 1$*, there exist positive constants* $C, \gamma > 0$ *such that*

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \leq Ce^{-\gamma t}, \quad 0 \leq t < \infty.$$

Note that statement (iii) of this theorem is already well-known [BBD95; BP96] and can even be proved in 'better' function spaces with standard theory, see for instance the text books [HI11; Lun12]. Moreover, in the shear-thinning case (ii), convergence to steady states has already been proved in [AG04] but without rate of convergence. In the cylindrical Taylor–Couette setting, statement (iii) has first been shown in [PV20] in the framework of stable center manifolds. Similarly, the results in (i) and (ii) have been obtained in [LPV22] and [LV22], also in the cylindrical Taylor–Couette geometry.

Finally, we prove global existence of positive weak solutions to the Ellis-law thin-film equation (B.1.2) and provide a description of their asymptotic behaviour. For $\alpha \geq 2$, stability and exponential decay to equilibrium can again be obtained by standard techniques [Lun12; HI11]. However, for $1 < \alpha < 2$, these techniques are not applicable since the coefficients of the differential operator are merely Hölder continuous. For this range of flow-behaviour exponents we use energy methods to prove exponential asymptotic stability of steady states in $H^1(\Omega)$.

Shear-Dependent non-Newtonian Fluids

Many common liquids and gases, such as water and air, may reasonably be considered Newtonian. However, there is still a multitude of real fluids which are in fact non-Newtonian. Newtonian fluids are characterised by a perfectly linear dependence of the shear stress $\sigma(\epsilon)$ on the local strain rate ϵ, the constant fluid viscosity $\mu > 0$ being the factor of proportionality. In contrast to that, shear-dependent non-Newtonian fluids feature a non-linear relation between the shear-rate and the viscous stress, $\sigma(\epsilon) = \mu(|\epsilon|)\epsilon$, where $\mu(|\epsilon|)$ is the shear-dependent viscosity. That is, these fluids become more solid or more liquid under shear force. In the case in which the fluid viscosity increases with increasing shear rate, the corresponding fluids are called **shear-thickening**. On the contrary, fluids are called **shear-thinning** if their viscosity decreases with increasing shear-rate. In this paper, we are concerned with two classes of non-Newtonian fluids, so-called **power-law fluids** or **Ostwald–de Waele fluids** and **Ellis-fluids**.

Power-Law Fluids. For **power-law fluids** or **Ostwald–de Waele fluids** the constitutive law for the effective fluid viscosity reads

$$\mu(|\epsilon|) = \mu_0|\epsilon|^{\frac{1}{\alpha}-1}, \tag{B.1.4}$$

with a characteristic viscosity $\mu_0 > 0$ and a flow-behaviour exponent $\alpha > 0$. For these fluids, the relation between the local strain and the viscous stress is

$$\sigma(\epsilon) = \mu_0|\epsilon|^{\frac{1}{\alpha}-1}\epsilon.$$

Note that the corresponding fluid is shear-thickening for flow-behaviour exponents $0 < \alpha < 1$, while it is shear-thinning for $\alpha > 1$. In the case $\alpha = 1$, we recover the Newtonian regime $\mu(|\epsilon|) \equiv \mu_0 > 0$ of a constant viscosity.

However, it is observed in real-world applications (e.g. in polymeric systems) that, at 'intermediate' shear rates, fluids behave according to (B.1.4), while the at rather low and/or rather high shear rates, the viscosity approaches a Newtonian plateau. This is obviously not reflected by (B.1.4).

Ellis fluids. As a second class of shear-dependent non-Newtonian fluids we consider fluids the rheology of which is described by the so-called **Ellis constitutive law** [WS94]

$$\frac{1}{\mu(|\epsilon|)} = \frac{1}{\mu_0}\left(1 + \left|\frac{\sigma(\epsilon)}{\sigma_{1/2}}\right|^{\alpha-1}\right), \quad \alpha \geq 1,\ 0 < \sigma_{1/2} < \infty, \tag{B.1.5}$$

where $\sigma(\epsilon) = \mu(|\epsilon|)\epsilon$ is the viscous shear stress. Here, $\mu_0 > 0$ denotes the viscosity at zero shear stress and $\sigma_{1/2} > 0$ is the viscous shear stress at which the viscosity is reduced to $\mu_0/2$. Thus, for $\alpha > 1$ and $0 < \sigma_{1/2} < \infty$ the Ellis constitutive law describes a shear-thinning behaviour, i.e., the fluid viscosity decreases with increasing shear rate. For $\alpha = 1$ or for $\sigma_{1/2}^{\alpha-1} \to \infty$, we recover a Newtonian behaviour. As an advantage over (B.1.4), the Ellis law (B.1.5) has the ability to describe a shear-thinning behaviour for 'moderate' shear rates and a Newtonian plateau for rather low shear stresses, since for all $\sigma \in \mathbb{R}$,

$$\frac{1}{\mu(|\epsilon|)} = \frac{1}{\mu_0}\left(1 + \left|\frac{\sigma(\epsilon)}{\sigma_{1/2}}\right|^{\alpha-1}\right) \longrightarrow \frac{1}{\mu_0}, \quad \text{as} \quad \sigma_{1/2}^{\alpha-1} \to \infty.$$

For the majority of polymers and polymer solutions the flow-behaviour exponent α in (B.1.5) varies in a range between 1 and 2, see e.g. [BAH87; MB65].

A plot of the different constitutive laws for the fluid viscosity (Newtonian fluids, shear-thickening and shear-thinning power-law fluids and Ellis fluids) is offered in Figure B.2.

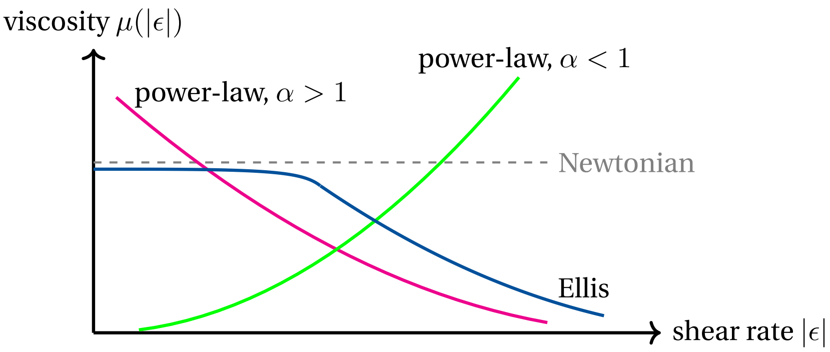

Figure B.2: Constitutive viscosity laws: Newtonian fluid (dashed), shear-thinning power-law fluid (pink), shear-thickening power-law fluid (green) and Ellis-fluid (blue).

Outline of the paper

The structure of the paper is as follows: In Section B.2 we introduce the functional setting we will work in.

In Section B.3 we prove local existence of positive weak solutions to the power-law thin-film equation and characterise positive steady states by positive constants.

In Section B.4 we derive regularity estimates for weak solutions that are valid as long as the solution stays bounded away from zero. More precisely, we prove a Łojasiewicz–Simon-type inequality that estimates the dissipation functional in terms of powers of the energy functional. Moreover, we provide a local L_1-in-time estimate for the dissipation functional in terms of the energy at a slightly earlier time.

Section B.5 is concerned with the dynamic behaviour of solutions to the shear-thickening power-law problem. First, we prove global existence of positive weak solutions for initial film heights that are initially close to a constant in $H^1(\Omega)$. Moreover, we show that these solutions converge to a positive constant in finite time and stay constant for all later times.

Section B.6 is concerned with the stability properties of solutions to the shear-thinning power-law thin-film equation. As in the shear-thickening case, it is shown that weak solutions exist globally time and stay positive if they are initially close to a steady state. Moreover, these positive global weak solutions are polynomially stable in $H^1(\Omega)$ in the sense that they converge to a steady state (positive constant) at rate $1/t^{1/(\alpha-1)}$, as time tends to infinity.

In Section B.7 we study the non-Newtonian thin-film equation that arises when the constitutive law for the fluid viscosity is the Ellis-law. The corresponding Ellis fluids have a Newtonian plateau for small shear rates and behave like a shear-thinning power-law fluid for high shear rates. We observe exponential asymptotic stability of steady states in the $H^1(\Omega)$-norm.

B.2 Functional framework

In this section we provide the functional setting that will be needed for the study of both the power-law (B.1.1) and Ellis-law (B.1.2) thin-film equations.

Throughout this paper, we assume that $\Omega \subset \mathbb{R}$ is a bounded interval. For $k \in \mathbb{N}$ and $p \in [1,\infty)$ we denote by $W_p^k(\Omega)$ the usual Sobolev spaces with norm

$$\|v\|_{W_p^k(\Omega)} = \left(\sum_{j=0}^{k} \|\partial^j v\|_{L_p(\Omega)}^p \right)^{1/p}.$$

We then define the seminorm

$$[v]_{W_p^s(\Omega)} = \int_\Omega \int_\Omega \frac{|v(x)-v(z)|^p}{|x-z|^{1+sp}}\, dx\, dz, \quad 1 \le p < \infty,\ 0 < s < 1,$$

and introduce the **fractional Sobolev spaces** by

$$W_p^s(\Omega) = \left\{ v \in W_p^{[s]}(\Omega);\, \|v\|_{W_p^s(\Omega)} < \infty \right\}, \quad 1 \le p < \infty,\ s \in \mathbb{R}_+ \setminus \mathbb{N},$$

where

$$\|v\|_{W_p^s(\Omega)} = \left(\|v\|_{W_p^{[s]}(\Omega)}^p + [\partial^{[s]} v]_{W_p^{s-[s]}(\Omega)}^p \right)^{1/p}, \quad 1 \le p < \infty,\ s \in \mathbb{R}_+ \setminus \mathbb{N},$$

with $[s]$ denoting the largest integer such that $[s] \le s$.

We now recall some important properties of these spaces. It is well-known (see, for instance, [Tri78]) that, for $0 \le s_0 < s_1 < \infty$, $1 < p < \infty$, and $0 < \rho < 1$, the space $W_p^s(\Omega)$ with $s = (1-\rho)s_0 + \rho s_1$, is the complex interpolation space between $W_p^{s_1}(\Omega)$ and $W_p^{s_0}(\Omega)$, in symbols

$$W_p^s(\Omega) = [W_p^{s_0}(\Omega), W_p^{s_1}(\Omega)]_\rho.$$

In order to take the Neumann-type boundary conditions into account, we further introduce the Banach spaces

$$W_{p,B}^{4\rho}(\Omega) = \begin{cases} \{v \in W_p^{4\rho}(\Omega); v_x = v_{xxx} = 0 \text{ on } \partial\Omega\}, & 3 + \frac{1}{p} < 4\rho \le 4, \\ \{v \in W_p^{4\rho}(\Omega); v_x = 0 \text{ on } \partial\Omega\}, & 1 + \frac{1}{p} < 4\rho \le 3 + \frac{1}{p}, \\ W_p^{4\rho}(\Omega), & 0 \le 4\rho \le 1 + \frac{1}{p}. \end{cases}$$

For $4\rho \in (0,4) \setminus \{1 + 1/p, 3 + 1/p\}$, the spaces $W_{p,B}^{4\rho}(\Omega)$ are closed linear subspaces of $W_p^{4\rho}(\Omega)$ and satisfy the interpolation property [Tri78, Theorem 4.3.3]

$$W_{p,B}^{4\rho}(\Omega) = \big(L_p(\Omega), W_{p,B}^4(\Omega)\big)_{\rho,p}, \quad 1 < p < \infty.$$

Lastly, we use $W_{p,0}^1(\Omega)$ to denote the space of functions belonging to $W_p^1(\Omega)$ with zero boundary condition.

B.3 Local existence for the power-law thin-film equation

In this section we prove local existence of positive weak solutions to the evolution problem

$$\begin{cases} u_t + \big(u^{\alpha+2}|u_{xxx}|^{\alpha-1}u_{xxx}\big)_x = 0, & t > 0,\ x \in \Omega, \\ u_x(t,x) = u_{xxx}(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{B.3.1}$$

for flow-behaviour exponents $\alpha > 0$, i.e., for both shear-thinning $(\alpha > 1)$ and shear-thickening $(\alpha < 1)$ power-law fluids. Moreover, we characterise the positive steady states of (B.3.1) by positive constants (flat films of positive height).

Our analysis strongly relies on an energy-dissipation estimate for the **energy functional**

$$E[u] = \frac{1}{2}\int_\Omega |u_x|^2\,dx.$$

Formally testing the equation with the second derivative u_{xx}, one finds that $E[u](t)$ decreases along solutions to (B.3.1). More precisely, solutions u to (B.3.1) satisfy

$$\frac{d}{dt}E[u](t) = -D[u](t) = -\int_\Omega u^{\alpha+2}|u_{xxx}|^{\alpha+1}\,dx.$$

We call $D[\cdot]$ the **dissipation functional**.

For the purpose of local existence, we introduce in Section B.3 a regularised version of (B.3.1) that removes the degeneracy in the third derivative u_{xxx}. For the regularised problem we apply standard parabolic theory in order to prove existence of positive strong solutions, emanating from positive initial values. In Section B.3 we provide uniform a-priori bounds for the solutions to the regularised problem and pass to the limit of a vanishing regularisation parameter in order to obtain local existence of positive weak solutions to the original problem (B.3.1).

Note that for $\alpha > 1$ (shear-thinning fluids) existence of global non-negative weak solutions is already proved in [AG04], where the authors use a more involved regularisation scheme. However, in the present paper we are only interested in positive solutions, but for all flow-behaviour exponents $\alpha > 0$.

In order to simplify notation, we introduce, for a fixed $\alpha > 0$, the function

$$\psi\colon \mathbb{R} \to \mathbb{R}, \quad s \mapsto \psi(s) = |s|^{\alpha-1}s,$$

and rewrite the partial differential equation $(\text{B.3.1})_1$ as

$$u_t + \big(u^{\alpha+2}\psi(u_{xxx})\big)_x = 0, \quad t > 0,\ x \in \Omega.$$

Note that if $\alpha \geq 1$, then $\psi \in C^1(\mathbb{R})$ with $\psi'(s) = \alpha|s|^{\alpha-1}$. For $\alpha < 1$ the function ψ is only α-Hölder-continuous.

Definition B.3.1. *For a given $T > 0$ and initial value $u_0 \in H^1(\Omega)$, a weak solution to* (B.3.1) *is defined as a function*

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$$

that has the following properties:

(i) (Weak formulation) u satisfies the differential equation $(\text{B.3.1})_1$ *in the weak sense, i.e.,*

$$\int_0^T \langle u_t, \varphi\rangle_{W^1_{\alpha+1}(\Omega)}\, dt = \int_0^T \int_\Omega u^{\alpha+2}\psi(u_{xxx})\,\varphi_x\, dx\, dt$$

for all test functions $\varphi \in L_{\alpha+1}\big((0,T); W^1_{\alpha+1,B}(\Omega)\big)$.

(ii) (Initial and boundary values) u satisfies the contact angle condition $u_x = 0$ on $\partial\Omega$ and the initial condition $(\text{B.3.1})_3$ *pointwise.*

The following theorem contains the main result of this section.

Theorem B.3.2 (Local existence of positive weak solutions). *Given a positive initial value $u_0 \in W^{4\rho}_{\alpha+1,B}(\Omega)$, $4\rho > 3 + 1/(\alpha+1)$, with $u_0(x) > 0$, $x \in \bar{\Omega}$, there exists a time $T > 0$ such that problem* (B.3.1) *admits at least one positive weak solution*

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$$

on $(0,T)$ in the sense of Definition B.3.1. Moreover, such a solution has the following properties:

(i) (Positivity) u is bounded away from zero

$$0 < C_T \leq u(t,x), \quad 0 \leq t \leq T,\ x \in \bar{\Omega}.$$

(ii) (Conservation of mass) u *conserves its mass in the sense that*

$$\|u(t)\|_{L_1(\Omega)} = \|u_0\|_{L_1(\Omega)}, \quad 0 \le t \le T.$$

(iii) (Energy-dissipation identity) Energy is dissipated along solutions

$$E[u](t) + \int_0^t D[u](s)\,ds = E[u_0] \tag{B.3.2}$$

for almost every $t \in [0,T]$.

Observe that due to the positivity of a solution u to (B.3.1) we have

$$\int_\Omega u(t,x)\,dx = \|u(t)\|_{L_1(\Omega)} = \|u_0\|_{L_1(\Omega)}, \quad 0 \le t \le T.$$

Remark B.3.3. In fact, the above theorem holds true for initial values $u_0 \in H^1(\Omega)$. We choose u_0 in the smaller space $W^{4\rho}_{\alpha+1}(\Omega)$ since the solutions u to the original problem are constructed as accumulation points of **strong** solutions u^σ to a regularised problem, not only as functions satisfying a suitable weak formulation. In order to apply semigroup theory, we require the initial value to satisfy $u_0 \in W^{4\rho}_{\alpha+1}(\Omega)$. That $u_0 \in H^1(\Omega)$ is enough can be seen by replacing u_0 by $u_0^\sigma \in W^{4\rho}_{\alpha+1}(\Omega)$ with

$$u_0^\sigma(x) > 0,\ x \in \bar{\Omega},\ \ \bar{u}_0^\sigma = \bar{u}_0 = ⨍_\Omega u_0\,dx \ \text{ and } \ u_0^\sigma \longrightarrow u_0 \ \text{ strongly in } H^1(\Omega),\ \text{as } \sigma \searrow 0.$$

This can for instance be obtained by a symmetric extension of the initial value $u_0 \in H^1(\Omega)$ at the lateral boundaries and mollification.

Remark B.3.4. Given a positive weak solution

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

to (B.3.1) as obtained in Theorem B.3.2, we may extend it beyond time T by restarting the equation with initial datum $u(T)$ and using that $u(T,x) > 0$ for all $x \in \bar{\Omega}$ and Remark B.3.3. In fact, in this way we can construct a weak solution to (B.3.1) in the sense of Definition B.3.1 up to a time $T_* > 0$ at which $u(T_*,x) = 0$ for some $x \in \bar{\Omega}$. Note though, that the solutions in Theorem B.3.2 are not unique, so that the 'maximal' time T_* of existence of positive solutions is not unique.

Positive Steady states of (B.3.1). We are interested in the stability properties of steady-state solutions to (B.3.1), i.e., in functions $u^* \in W^3_{\alpha+1,B}(\Omega)$ that solve the ordinary differential equation

$$U^{\alpha+2}|U'''|^{\alpha-1}U''' = 0, \quad x \in \Omega. \tag{B.3.3}$$

In physical parlance, (B.3.3) says that there is no flux of the fluid through the boundaries of the interval. Positive steady states of (B.3.1) may be easily characterised by the following theorem.

Theorem B.3.5 (Characterisation of positive steady states). *A function* $u \in W^3_{\alpha+1,B}(\Omega)$ *is a positive steady-state solution of* (B.3.1) *if and only if* $u \equiv u^* \in \mathbb{R}_{>0}$ *is given by a positive constant.*

Proof. **(i)** Let $u \equiv u^* \in \mathbb{R}_{>0}$. Then $u^* \in W^3_{\alpha+1,B}(\Omega)$ clearly satisfies the ODE (B.3.3).
(ii) Let $u = u^* \in W^3_{\alpha+1,B}(\Omega)$ be an arbitrary positive steady-state solution of (B.3.1), i.e., a solution to the ODE (B.3.3). Then u^* satisfies

$$0 = \frac{d}{dt}E[u^*] = -D[u^*] = -\int_\Omega |u^*|^{\alpha+2}|u^*_{xxx}|^{\alpha+1}\,dx.$$

Since the integrand on right-hand side of this equation is non-negative and $u^*(x) > 0,\ x \in \bar{\Omega}$, it follows that $u^*_{xxx} \equiv 0$ on $\bar{\Omega}$. Consequently, u^*_{xx} is constant and this in turn implies that u^*_x is linear. Taking the Neumann boundary conditions into account, we find that u^* must be constant. □

Local Existence of Positive Solutions to the Regularised Problem and Uniform A-Priori Bounds

In order to handle the difficulties caused by the doubly nonlinear and doubly degenerate nature of the evolution problem (B.3.1), we introduce, for a fixed regularisation parameter $\sigma \in (0,1)$ and all $s \in \mathbb{R}$, the smooth function

$$\psi_\sigma(s) = (s^2 + \sigma^2)^{\frac{\alpha-1}{2}}s, \quad s \in \mathbb{R},$$

and substitute the nonlinear term $\psi(u_{xxx})$ in (B.3.1) accordingly. The **regularised problem** corresponding to (B.3.1) then reads

$$\begin{cases} u^\sigma_t + \big((u^\sigma)^{\alpha+2}\psi_\sigma(u^\sigma_{xxx})\big)_x = 0, & t > 0,\ x \in \Omega, \\ u^\sigma_x(t,x) = u^\sigma_{xxx}(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u^\sigma(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{P_σ}$$

It follows from standard parabolic theory [Ama93; Eid69; LM20] that the regularised problem (P_σ) possesses, for each fixed $\sigma \in (0,1)$ and suitable initial data, a unique maximal strong solution u^σ. This is the content of Theorem B.3.7 below. Moreover, in Lemma B.3.8 below, we provide a-priori bounds for the strong solution that are uniform in the regularisation parameter $\sigma > 0$. First, though, we define what we mean by a maximal strong solution to (P_σ).

Definition B.3.6. *Fix $\alpha > 0$ and $\sigma \in (0,1)$. Let $1 < p < \infty$. Given a positive initial value $u_0 \in L_p(\Omega)$, we call a function $u\colon [0,T_u) \to L_p(\Omega)$ with $u(t,x) > 0$ for $t \in [0,T_u)$ and $x \in \bar{\Omega}$ a* ***maximal positive strong solution*** *to (P_σ) on $[0,T_u)$ in $L_p(\Omega)$ if the following conditions are satisfied:*

(i) $u \in C\big([0,T_u); L_p(\Omega)\big) \cap C^1\big((0,T_u); L_p(\Omega)\big)$;

(ii) $u(0) = u_0 \in L_p(\Omega)$ and $u(t) \in W^4_{p,B}(\Omega)$ for all $t \in (0,T_u)$;

(iii) (Positivity) $u(t,x) > 0$ for $t \in [0,T_u)$ and $x \in \bar{\Omega}$;

(iv) u satisfies the differential equation $(P_\sigma)_1$ pointwise;

(v) (Maximality) There is no other solution v on $[0,T_v)$ with $T_u < T_v$.

Clearly, solutions to (P_σ), as obtained in the following theorem, do also dissipate energy. We therefore introduce the notation

$$D_\sigma[u^\sigma](t) = \int_\Omega (u^\sigma)^{\alpha+2} |u^\sigma_{xxx}|^{\alpha+1}\, dx$$

for the dissipation functional corresponding to the energy functional $E[\cdot]$ and the regularised equation (P_σ).

Theorem B.3.7 (Local existence for (P_σ)). *Fix $\alpha > 0$ and $\sigma \in (0,1)$. Let $1/(\alpha+1) < s < r < 1$. Moreover, let $\theta = \frac{3+s}{4}$ and $\rho = \frac{3+r}{4}$. Then, given an initial film height $u_0 \in W^{4\rho}_{\alpha+1,B}(\Omega)$ such that $u_0(x) > 0$ for all $x \in \bar{\Omega}$, problem (P_σ) possesses a unique maximal solution*

$$\begin{aligned} u^\sigma \in\, & C\big([0,T_\sigma); W^{4\rho}_{\alpha+1,B}(\Omega)\big) \cap C^\rho\big([0,T_\sigma); L_{\alpha+1}(\Omega)\big) \cap C\big((0,T_\sigma); W^4_{\alpha+1,B}(\Omega)\big) \\ & \cap C^1\big((0,T_\sigma); L_{\alpha+1}(\Omega)\big). \end{aligned}$$

Moreover, the solution enjoys the following properties.

(i) (Positivity) u^σ is positive

$$u^\sigma(t,x) > 0, \quad 0 \le t < T_\sigma,\ x \in \bar{\Omega}.$$

(ii) (Conservation of mass) u^σ conserves its mass in the sense that

$$\|u^\sigma(t)\|_{L_1(\Omega)} = \|u_0\|_{L_1(\Omega)}, \quad 0 \le t < T_\sigma. \tag{B.3.4}$$

(iii) (Energy-dissipation identity) u^σ satisfies the energy-dissipation identity

$$E[u^\sigma](t) + \int_0^t D_\sigma[u^\sigma](s) = E[u_0], \quad 0 \le t < T_\sigma. \tag{B.3.5}$$

(iv) (Maximal time of existence) Suppose that $T_\sigma < \infty$. Then

$$\liminf_{t \nearrow T_\sigma} \frac{1}{\min_{x\in\bar{\Omega}} u^\sigma(t)} + \|u^\sigma(t)\|_{W^{4\gamma}_{\alpha+1,B}(\Omega)} = \infty$$

for all $\gamma \in (\theta, 1]$.

Proof. **(i) Local existence, uniqueness and positivity.** In order to prove local existence and uniqueness of a strong solution we apply [LM20, Theorem 4.2]. To this end, we verify that (P_σ) fits into the corresponding abstract functional setting. Moreover, after rewriting (P_σ) in non-divergence form, we define for $v(t) \in W^{4\theta}_{\alpha+1,B}(\Omega)$ with $\theta = (3+s)/4$ such that $v(x) > 0,\ x \in \bar{\Omega}$, the linear differential operator $\mathcal{A}(v(t)) \in \mathcal{L}\big(W^4_{\alpha+1,B}(\Omega); L_{\alpha+1}(\Omega)\big)$ of fourth order by

$$\mathcal{A}(v(t))u^\sigma = A(v(t))\partial_x^4 u^\sigma \quad \text{with} \quad A(v(t)) = v^{\alpha+2}\psi'_\sigma(v_{xxx}),$$

where

$$\begin{aligned} \psi'_\sigma(s) &= (\alpha-1)(s^2+\sigma^2)^{\frac{\alpha-3}{2}} s^2 + (s^2+\sigma^2)^{\frac{\alpha-1}{2}} \\ &= \alpha(s^2+\sigma^2)^{\frac{\alpha-1}{2}} - \sigma^2(\alpha-1)(s^2+\sigma^2)^{\frac{\alpha-3}{2}}, \quad s \in \mathbb{R}. \end{aligned}$$

Note that for positive $\sigma \in (0,1)$ we have $\psi'_\sigma(s) > C_{\sigma,\alpha} > 0$ for all $s \in \mathbb{R}$ and all fixed $\alpha > 0$. Moreover, we introduce the right-hand side

$$\mathcal{F}(v(t)) = -(\alpha+2)v^{\alpha+1}v_x\,\psi_\sigma(v_{xxx})$$

and perceive (P_σ) as an abstract quasilinear Cauchy problem

$$\begin{cases} \dot{u^\sigma} + \mathcal{A}(u^\sigma)u^\sigma = \mathcal{F}(u^\sigma), & t > 0, \\ u^\sigma(0) = u_0. \end{cases}$$

Note that the Neumann-type boundary conditions $(P_\sigma)_2$ are incorporated in the domain $W^4_{\alpha+1,B}(\Omega)$ of the operator $\mathcal{A}(v(t))$. Due to the smoothness of ψ_σ the maps

$$\mathcal{A}\colon W^{3+s}_{\alpha+1,B}(\Omega) \longrightarrow \mathcal{L}\big(W^4_{\alpha+1,B}(\Omega); L_{\alpha+1}(\Omega)\big) \quad \text{and} \quad \mathcal{F}\colon W^{3+s}_{\alpha+1,B}(\Omega) \longrightarrow L_{\alpha+1}(\Omega)$$

are, for all $\alpha > 0$, locally Lipschitz continuous. In order to guarantee parabolicity, we extend the differential operator $\mathcal{A}$ to the differential operator

$$\bar{\mathcal{A}}_\varepsilon(v(t)) \in \mathcal{L}\big(W^4_{\alpha+1,B}(\Omega); L_{\alpha+1}(\Omega)\big), \quad \bar{\mathcal{A}}_\varepsilon(v(t))u^\sigma = \bar{A}_\varepsilon(v(t))\partial_x^4 u^\sigma,$$

where

$$\bar{A}_\varepsilon(v(t)) = \max\big\{v_+^{\alpha+2}\psi'_\sigma(v_{xxx}), \varepsilon/2\big\}$$

and $v_+ = \max\{v, 0\}$. Following the lines of [LM20, Chapter 5], we study the extended parabolic problem with $\bar{\mathcal{A}}_\varepsilon$ instead of $\mathcal{A}$ and show that the corresponding local positive solution $u^\sigma = u^\sigma(\varepsilon)$ also solves the non-extended problem (P_σ) for a short but strictly positive time. More precisely, the extended regularised problem is, for each fixed $\sigma, \varepsilon \in (0,1)$, parabolic in the sense that $\bar{\mathcal{A}}_\varepsilon(v(t))$ generates an analytic semigroup on $L_{\alpha+1}(\Omega)$. Indeed, due to the embedding $W^{3+s}_{\alpha+1,B}(\Omega) \hookrightarrow C^3(\bar{\Omega})$ and the positivity of $\sigma, \varepsilon > 0$, we have that $\bar{A}_\varepsilon(v(t,\cdot)) \in C(\bar{\Omega})$. Moreover, the principal symbol $a_\varepsilon(x,\xi)$ satisfies

$$\text{Re}(a_\varepsilon(x,\xi)\eta|\eta) \geq C_{\sigma,\alpha,\varepsilon}(i\xi)^4\eta^2 > 0, \quad (x,\xi) \in \bar{\Omega} \times \{-1,1\},\, \eta \in \mathbb{R}\setminus\{0\},$$

for a positive constant $C_{\sigma,\alpha,\varepsilon} > 0$. Consequently, $\bar{\mathcal{A}}_\varepsilon(v(t))$, together with the Neumann-type boundary conditions, is normally elliptic in the sense of [Ama93, Example 4.3(d)] and we can apply [Ama93, Theorem 4.1 and Remark 4.2(b)] to conclude that $\bar{\mathcal{A}}_\varepsilon(v(t))$ generates an analytic semigroup on $L_{\alpha+1}(\Omega)$. Thus, we are in the abstract setting of [LM20, Theorem 4.2] which yields existence and uniqueness of a local positive strong solution to the extended problem in $L_{\alpha+1}(\Omega)$. On a potentially smaller time interval, this solution $u^\sigma = u^\sigma(\varepsilon)$ is, for ε small enough, also a local positive strong solution to (P_σ), see step (iii) in the proof of [LM20, Theorem 5.1].

(ii) Conservation of mass. This follows by testing the regularised partial differential equation $(P_\sigma)_1$ with the constant function $\varphi \equiv 1$, integration by parts and using the Neumann boundary conditions $(P_\sigma)_2$.

(iii) Energy-dissipation identity. Since the solution obtained in step (i) enjoys the regularity

$$u_x^\sigma \in C\big((0,T); W^1_{\alpha+1,0}(\Omega)\big) \cap C^1\big((0,T); (W^1_{\alpha+1,0}(\Omega))'\big),$$

we may apply [LM20, Proposition 6.1] in order to guarantee that the expression

$$\frac{d}{dt}E[u^\sigma](t) = \int_\Omega u^\sigma_{xt}u^\sigma_x\,dx = -\int_\Omega |u^\sigma|^{\alpha+2}\big(|u^\sigma_{xxx}|^2 + \sigma^2\big)^{\frac{\alpha-1}{2}}|u^\sigma_{xxx}|^2\,dx = -D_\sigma[u^\sigma](t)$$

is well-defined for all $t \in (0,T)$. Integrating with respect to time gives the energy-dissipation identity.
(iv) Maximal time of existence. Using the notation introduced in step (i), this result is a minor adaptation of [LM20, Theorem 7.1].

□

In order to prove the local-existence result for the original problem (Theorem B.3.2), we need suitable uniform (in σ) a-priori estimates for the solution to (P_σ) as given in the following lemma.

Lemma B.3.8 (Uniform bounds). *Let u^σ be the maximal solution to (P_σ) for a fixed $\sigma \in (0,1)$ and an initial value $u_0 \in W^{4\rho}_{\alpha+1,B}(\Omega)$ such that $u_0(x) > 0$ for all $x \in \bar{\Omega}$. Then the following holds true. There is $T > 0$ such that the family $(u^\sigma)_\sigma$ has the following properties:*

(i) $(u^\sigma)_\sigma$ is uniformly bounded in $L_\infty\big((0,T); H^1(\Omega)\big)$;

(ii) $\big(|u^\sigma|^{\alpha+2}\,\psi_\sigma(u^\sigma_{xxx})\big)_\sigma$ is uniformly bounded in $L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big)$;

(iii) $(u^\sigma_t)_\sigma$ is uniformly bounded in $L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$;

(iv) $(u^\sigma_{xxx})_\sigma$ is uniformly bounded in $L_{\alpha+1}\big((0,T)\times\Omega\big)$;

(v) $(u^\sigma)_\sigma$ is uniformly bounded in $L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$;

(vi) $((u^\sigma_x)_t)_\sigma$ is uniformly bounded in $L_{\frac{\alpha+1}{\alpha}}\big((0,T); \big(W^1_{\alpha+1,0}(\Omega)\cap W^2_{\alpha+1}(\Omega)\big)'\big)$.

Proof. Note that once we have proved items (i) and (iii), the Aubin–Lions–Simon lemma [Sim86] implies that the family $(u^\sigma)_\sigma$ is equicontinuous. Hence, we may choose $T > 0$ such that u^σ is bounded away uniformly from zero on the interval $[0,T]$.

Within this proof, $C > 0$ denotes a positive constant, possibly depending on α, Ω, and $\|u_0\|_{W^{4\rho}_{\alpha+1}(\Omega)}$, but independent of σ.
(i) Since

$$D_\sigma[u^\sigma](t) = \int_\Omega (u^\sigma)^{\alpha+2}\psi_\sigma(u^\sigma_{xxx})\,u^\sigma_{xxx}\,dx \geq 0, \quad t\in[0,T_\sigma),$$

we have

$$E[u^\sigma](t) = \frac{1}{2}\|u^\sigma_x(t)\|^2_{L_2(\Omega)} \leq E[u_0], \quad t\in[0,T_\sigma). \tag{B.3.6}$$

Using Poincaré's inequality and (B.3.4), we obtain for $t\in[0,T_\sigma)$

$$\|u^\sigma(t)\|_{L_2(\Omega)} \leq \|u^\sigma(t)-\bar{u}^\sigma(t)\|_{L_2(\Omega} + \|\bar{u}^\sigma(t)\|_{L_2(\Omega)} \leq C\|u^\sigma_x(t)\|_{L_2(\Omega)} + \|\bar{u}_0\|_{L_2(\Omega)},$$

which, together with (B.3.6), yields

$$\sup_{0\leq t\leq T_\sigma}\|u^\sigma(t)\|_{H^1(\Omega)} \leq C\big(\|\bar{u}_0\|_{L_2(\Omega)} + E[u_0]^{1/2}\big).$$

Hence, $(u^\sigma)_\sigma$ is uniformly bounded in $L_\infty\big((0,T_\sigma); H^1(\Omega)\big)$.

(ii) First we consider the case $0<\alpha<1$. Observe that

$$\begin{aligned}&\big\||u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx})\big\|^{\frac{\alpha+1}{\alpha}}_{L_{\frac{\alpha+1}{\alpha}}((0,T_\sigma)\times\Omega)}\\&=\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{(\alpha+2)\frac{\alpha+1}{\alpha}}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{\alpha+1}{\alpha}}\,dx\,dt\\&=\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{(\alpha+2)\frac{\alpha+1}{\alpha}}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{1-\alpha}{\alpha}}|u^\sigma_{xxx}|^2\,dx\,dt.\end{aligned}$$

Using that $\frac{1-\alpha}{\alpha}>0$, we get the pointwise estimate $|u^\sigma_{xxx}|^{\frac{1-\alpha}{\alpha}}\le\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{1-\alpha}{2\alpha}}$. Furthermore, in view of (i) and $H^1(\Omega)\hookrightarrow L_\infty(\Omega)$, we find that $(u^\sigma)_\sigma$ is uniformly bounded in $L_\infty\big((0,T_\sigma);L_\infty(\Omega)\big)$. Combining this, we obtain the estimate

$$\begin{aligned}&\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{(\alpha+2)\frac{\alpha+1}{\alpha}}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{1-\alpha}{\alpha}}|u^\sigma_{xxx}|^2\,dx\,dt\\&\quad\le\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{(\alpha+2)\frac{\alpha+1}{\alpha}}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}}|u^\sigma_{xxx}|^2\,dx\,dt\\&\quad\le C\int_0^{T_\sigma}D_\sigma[u^\sigma](t)\,dt\\&\quad\le CE[u_0],\end{aligned}$$

where the last step is due to (B.3.5). In the case $1<\alpha<\infty$, we have to use a different argument. Note that by (i) and (B.3.5), we have

$$\begin{aligned}&\big\||u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx})\big\|^{\frac{\alpha+1}{\alpha}}_{L_{\frac{\alpha+1}{\alpha}}((0,T_\sigma)\times\Omega)}\\&=\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{(\alpha+2)\frac{\alpha+1}{\alpha}}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{\alpha+1}{\alpha}}\,dx\,dt\\&\le C\int_{\{|u^\sigma_{xxx}|\le\sigma\}}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{1-\alpha}{\alpha}}|u^\sigma_{xxx}|^2\,dx\,dt\\&\quad+\int_{\{|u^\sigma_{xxx}|>\sigma\}}|u^\sigma|^{(\alpha+2)\frac{\alpha+1}{\alpha}}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{\alpha+1}{\alpha}}\,dx\,dt\\&\le CT_\sigma\sigma^{\alpha+1}+C\int_0^{T_\sigma}D_\sigma[u^\sigma](t)\,dt\\&\le C\big(T_\sigma\sigma^{\alpha+1}+E[u_0]\big).\end{aligned}$$

(iii) Since u^σ is a weak solution to (P_σ), we have

$$\int_0^{T_\sigma}\langle u^\sigma_t,\varphi\rangle_{W^1_{\alpha+1}(\Omega)}\,dt=\int_0^{T_\sigma}\int_\Omega(u^\sigma)^{\alpha+2}\,\psi_\sigma(u^\sigma_{xxx})\,\varphi_x\,dx\,dt$$

for all $\varphi \in L_{\alpha+1}\big((0,T_\sigma);W^1_{\alpha+1,B}(\Omega)\big)$. Applying Hölder's inequality and (i), we obtain

$$\left|\int_0^{T_\sigma} \langle u_t^\sigma, \varphi\rangle_{W^1_{\alpha+1}(\Omega)}\,dt\right| \le \int_0^{T_\sigma}\int_\Omega |u^\sigma|^{\alpha+2}\,|\psi_\sigma(u^\sigma_{xxx})|\;|\varphi_x|\,dx\,dt$$
$$\le \left(\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{\alpha+2}|\varphi_x|^{\alpha+1}\,dx\,dt\right)^{\frac{1}{\alpha+1}}\cdot$$
$$\cdot\left(\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{\alpha+2}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{\alpha+1}{\alpha}}\,dx\,dt\right)^{\frac{\alpha}{\alpha+1}}$$
$$\le C\|\varphi\|_{L_{\alpha+1}((0,T_\sigma);W^1_{\alpha+1}(\Omega))}\left(\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{\alpha+2}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{\alpha+1}{\alpha}}\,dx\,dt\right)^{\frac{\alpha}{\alpha+1}}.$$

For $0<\alpha<1$, we obtain similar as in step (ii) that

$$\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{\alpha+2}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{\alpha+1}{\alpha}}\,dx\,dt \le \int_0^{T_\sigma} D_\sigma[u^\sigma](t)\,dt \le E[u_0].$$

For $1<\alpha<\infty$, we get, similarly as in step (ii),

$$\int_0^{T_\sigma}\int_\Omega |u^\sigma|^{\alpha+2}\big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}\frac{\alpha+1}{\alpha}}|u^\sigma_{xxx}|^{\frac{\alpha+1}{\alpha}}\,dx\,dt \le CT_\sigma\sigma^{\alpha+1} + C\int_0^{T_\sigma} D_\sigma[u^\sigma](t)\,dt$$
$$\le C\left(T_\sigma\sigma^{\alpha+1}+E[u_0]\right).$$

(iv) We prove that $(u^\sigma_{xxx})_\sigma$ is uniformly bounded in $L_{\alpha+1,\mathrm{loc}}\big((0,T_\sigma)\times\Omega\big)$. Note that by definition of T_σ and continuity of u^σ, we have $u^\sigma(t,x) > c_\delta > 0$ for all $(t,x)\in[0,T_\sigma-\delta)\times\Omega$, for every $\delta>0$.

In the case $1<\alpha<\infty$, we get

$$\int_0^{T_\sigma-\delta}\int_\Omega |u^\sigma_{xxx}|^{\alpha+1}\,dx\,dt \le \int_0^{T_\sigma-\delta}\int_\Omega \big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}}|u^\sigma_{xxx}|^2\,dx\,dt$$
$$\le C\int_0^{T_\sigma-\delta} D_\sigma[u^\sigma](t)\,dt$$
$$\le CE[u_0],$$

where the constant C depends also on δ, and where in the last step we used (B.3.5).

Now we consider the case $0<\alpha<1$. We have

$$\int_0^{T_\sigma-\delta}\int_\Omega |u^\sigma_{xxx}|^{\alpha+1}\,dx\,dt = \int_{\{|u^\sigma_{xxx}|\le\sigma\}} |u^\sigma_{xxx}|^{\alpha+1}\,dx\,dt + \int_{\{|u^\sigma_{xxx}|>\sigma\}} |u^\sigma_{xxx}|^{\alpha+1}\,dx\,dt$$
$$\le C(T_\sigma-\delta)\sigma^{\alpha+1} + \int_{\{|u^\sigma_{xxx}|>\sigma\}} |u^\sigma_{xxx}|^{\alpha+1}\,dx\,dt.$$

Using the inequality

$$|x|^{\alpha+1} = \left(\tfrac12|x|^2+\tfrac12|x|^2\right)^{\frac{\alpha-1}{2}}|x|^2 \le \left(\tfrac12\right)^{\frac{\alpha-1}{2}}\left(|x|^2+\sigma^2\right)^{\frac{\alpha-1}{2}}|x|^2,\qquad |x|>\sigma,\ x\in\mathbb{R},$$

we obtain

$$\begin{aligned}
&\int_0^{T_\sigma-\delta}\int_\Omega |u^\sigma_{xxx}|^{\alpha+1}\,dx\,dt\\
&\le C(T_\sigma-\delta)\sigma^{\alpha+1}+C\int_0^{T_\sigma-\delta}\int_\Omega \big(|u^\sigma_{xxx}|^2+\sigma^2\big)^{\frac{\alpha-1}{2}}|u^\sigma_{xxx}|^2\,dx\,dt\\
&\le C(T_\sigma-\delta)\sigma^{\alpha+1}+C\int_0^{T_\sigma-\delta} D_\sigma[u^\sigma](t)\,dt\\
&\le C\big((T_\sigma-\delta)\sigma^{\alpha+1}+E[u_0]\big)
\end{aligned}$$

with C depending also on δ. In the last step we used again (B.3.5).
(v) As observed in (i), u^σ is uniformly bounded in $L_\infty\big((0,T_\sigma);L_\infty(\Omega)\big)$, and hence also in $L_{\alpha+1}\big((0,T_\sigma)\times\Omega\big)$. From (iv), we also know that u^σ_{xxx} is uniformly bounded in $L_{\alpha+1,\mathrm{loc}}\big((0,T_\sigma)\times\Omega\big)$.Combining this, we find that u^σ is uniformly bounded in $L_{\alpha+1,\mathrm{loc}}\big((0,T_\sigma);W^3_{\alpha+1,B}(\Omega)\big)$ by interpolation.
(vi) This follows as in (iii) using a duality argument. □

Proof of Theorem B.3.2: Local Existence of Positive Weak Solutions to the Original Problem

In this section we pass to the limit of a vanishing regularisation parameter $\sigma\searrow 0$. Using the uniform bounds provided in Lemma B.3.8, we show that the family $(u^\sigma)_\sigma$ admits an accumulation point that is a positive weak solution to the original problem (B.3.1). As usual, we use Minty's trick in order to identify the (nonlinear) limit flux.

Lemma B.3.9 (Convergence of approximations). *Let u^σ be the maximal solution to (P_σ) for a fixed $\sigma\in(0,1)$ and a positive initial value $u_0\in W^{4\rho}_{p,B}(\Omega)$ such that $u_0(x)>0$ for all $x\in\bar\Omega$. Then the following holds true. There are a positive time $T>0$ and a subsequence $(u^\sigma)_\sigma$ (not relabelled) such that, as $\sigma\searrow 0$, we have convergence in the following sense:*

(i) $u^\sigma\to u$ *strongly in* $C\big([0,T];C^\rho(\bar\Omega)\big)$*;*

(ii) $|u^\sigma|^{\alpha+2}\,\psi_\sigma(u^\sigma_{xxx})\rightharpoonup\chi$ *weakly in* $L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big)$ *for some limit function* χ*;*

(iii) $u^\sigma_t\rightharpoonup u_t$ *weakly in* $L_{\frac{\alpha+1}{\alpha}}\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big)$*;*

(iv) $u^\sigma_{xxx}\rightharpoonup u_{xxx}$ *weakly in* $L_{\alpha+1}\big((0,T)\times\Omega\big)$*;*

(v) $(u^\sigma_x)_t\rightharpoonup u_{xt}$ *weakly in* $L_{\frac{\alpha+1}{\alpha}}\big((0,T);\big(W^1_{\alpha+1,0}(\Omega)\cap W^2_{\alpha+1}(\Omega)\big)'\big)$.

Since the proof of this lemma differs only very slightly from that in [AG04; LPV22; LV22], we shift it to the appendix.

We are left to prove the convergence of the nonlinear flux term $\big(|u^\sigma|^{\alpha+2}\,\psi_\sigma(u^\sigma_{xxx})\big)\rightharpoonup\big(|u|^{\alpha+2}\,\psi(u_{xxx})\big)$ in $L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big)$. This is done in the next lemma the proof of which is based on the monotonicity of the regularisation and Minty's trick.

Lemma B.3.10. *Given $\sigma \in (0,1)$, let u^σ be the maximal solution to (P_σ), corresponding to an initial value $u_0 \in W^{4\rho}_{\alpha+1,B}(\Omega)$. Then there exists a subsequence $(u^\sigma)_\sigma$ (not relabelled) such that*

$$|u^\sigma|^{\alpha+2}\,\psi_\sigma(u^\sigma_{xxx}) \rightharpoonup |u|^{\alpha+2}\psi(u_{xxx}) \quad \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big)$$

as $\sigma \searrow 0$.

The proof of the above stated lemma uses the same arguments as the one in [LPV22]. For the sake of completeness, we include it in the appendix.

Remark B.3.11. Note that the limit u is bounded in $C\big([0,T];H^1(\Omega)\big)$. Indeed, from Lemma B.3.9 (i) we already know that

$$u \in C\big([0,T];C^\rho(\bar\Omega)\big) \hookrightarrow C\big([0,T];L_2(\Omega)\big).$$

Furthermore,

$$u_x \in L_{\alpha+1}\big((0,T);W^1_{\alpha+1,0}(\Omega)\cap W^2_{\alpha+1}(\Omega)\big)$$

and

$$u_{xt} \in L_{\frac{\alpha+1}{\alpha}}\big((0,T);\big(W^1_{\alpha+1,0}(\Omega)\cap W^2_{\alpha+1}(\Omega)\big)'\big)$$

due to Lemma B.3.9 (iv) and (v) and lower semicontinuity of the norm. Using [Ber88, Remark 3.4], this yields that $u_x \in C\big([0,T];L_2(\Omega)\big)$. Therefore, $u \in C\big([0,T];H^1(\Omega)\big)$.

Proof of Theorem B.3.2. **(i)** We first show that the limit u is bounded away from zero on $[0,T]\times\bar\Omega$. This follows immediately from the positivity of u^σ on $[0,T_\sigma)\times\bar\Omega$ and the convergence in Lemma (B.3.9) (i).
(ii) Thanks to Lemma B.3.8 (iii) and (iv) and Remark B.3.11 above, we obtain the regularity properties

$$u \in C\big([0,T];H^1(\Omega)\big)\cap L_{\alpha+1}\big((0,T);W^3_{\alpha+1,B}(\Omega)\big)$$

and

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big).$$

(iii) We now prove that u satisfies the weak integral formulation in Definition B.3.1. To do so, note that for solutions to the regularised problem (P_σ) we have that

$$\int_0^T \langle u^\sigma_t,\varphi\rangle_{W^1_{\alpha+1}(\Omega)}\,dt = \int_0^T\int_\Omega |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx})\,\varphi_x\,dx\,dt$$

for all test functions $\varphi \in L_{\alpha+1}\big((0,T);W^1_{\alpha+1,B}(\Omega)\big)$. On the one hand, since $\varphi_x \in L_{\alpha+1}\big((0,T)\times\Omega\big)$, it follows from Lemma B.3.10 that

$$\int_0^T \langle u^\sigma_t,\varphi\rangle_{W^1_{\alpha+1}(\Omega)}\,dt \longrightarrow \int_0^T\int_\Omega |u|^{\alpha+2}\psi(u_{xxx})\,\varphi_x\,dx\,dt.$$

On the other hand, Lemma B.3.8 (iii) gives

$$\int_0^T \langle u^\sigma_t,\varphi\rangle_{W^1_{\alpha+1}(\Omega)}\,dt \longrightarrow \int_0^T \langle u_t,\varphi\rangle_{W^1_{\alpha+1}(\Omega)}\,dt.$$

Combining both, we then find that u satisfies the desired integral identity

$$\int_0^T \langle u_t,\varphi\rangle_{W^1_{\alpha+1}(\Omega)}\,dt = \int_0^T\int_\Omega u^{\alpha+2}\psi(u^\sigma_{xxx})\,\varphi_x\,dx\,dt$$

for all $\varphi \in L_{\alpha+1}\big((0,T); W^1_{\alpha+1,B}(\Omega)\big)$.
(iv) By Lemma B.3.9 (i) the initial condition is satisfied in the limit. That the first boundary condition in (B.3.1)$_2$ is fulfilled by u follows from Lemma B.3.9 (v).
(v) This follows from the conservation of mass property

$$\int_\Omega u^\sigma(t)\,dx = \int_\Omega u_0\,dx, \quad t \in [0, T_\sigma),$$

for the approximation u^σ (see Theorem B.3.7 (ii)) and the convergence in Lemma B.3.9 (i).
(vi) In Lemma B.3.10 we have already shown that the solution u to the original problem (B.3.1) satisfies the energy-dissipation identity for almost every $t \in [0, T]$. □

B.4 Differential inequality for the energy and regularity estimates

The content of this section is twofold. First, we derive a differential inequality of Łojasiewicz–Simon type for the energy functional E which is valid as long as the weak solution to (B.3.1) remains bounded away from zero. Then, we derive L_1-in-time regularity estimates for the weak solution to (B.3.1). The results are the same as in the cylindrical Taylor–Couette setting in [LPV22; LV22]. However, since the present paper deals with the flat case, the proofs cannot rely on Fourier analysis.

Proposition B.4.1. *Fix $\alpha > 0$ and a positive initial value $u_0 \in H^1(\Omega)$ with $u_0(x) > 0$ for $x \in \bar\Omega$. Let*

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$$

be a weak solution to (B.3.1) *with initial value u_0, as obtained in Theorem B.3.2. Let* $m = \min_{(t,x)\in[0,T]\times\bar\Omega} u(t,x) > 0$. *Then there is a constant $C = C_{\alpha,\Omega,m} > 0$ such that*

$$\frac{d}{dt}E[u](t) = -D[u](t) \le -C\big(E[u](t)\big)^{\frac{\alpha+1}{2}}$$

for almost every $t \in [0,T]$.

The proof of Proposition B.4.1 is based on the following crucial Poincaré estimate. It is worthwhile to emphasise that this estimate is valid in both the shear-thinning case and the shear-thickening case.

Lemma B.4.2. *Fix $\alpha > 0$ and let $v \in H^1(\Omega) \cap W^3_{\alpha+1,B}(\Omega)$ with $\bar v = 0$ and $v_x(x) = 0$ for $x \in \partial\Omega$. Then there exists a constant $C = C_{\alpha,\Omega} > 0$ such that*

$$E[v] \le C\|v_{xxx}\|^2_{L_{\alpha+1}(\Omega)}.$$

Proof. We distinguish the cases $\alpha = 1, \alpha > 1$ and $\alpha < 1$.
The case $\alpha = 1$. This is just a direct application of Poincaré's inequality.

The case $\alpha > 1$**.** Define $w = v_{xx} \in W^1_{\alpha+1}(\Omega) \subset L_2(\Omega)$. Observe that v is a weak solution to the Neumann boundary-value problem given by

$$\begin{cases} v_{xx} = w, & x \in \Omega, \\ v_x = 0, & x \in \partial\Omega. \end{cases}$$

Hence, we obtain the estimate $\|v_x\|_{L_2(\Omega)} \leq C\|w\|_{L_2(\Omega)}$. Furthermore, note that $\bar{w} = 0$. Using this, applying Poincaré's inequality and then Jensen's inequality for the concave function $s \mapsto s^{2/(\alpha+1)}$, $s \in (0,\infty)$, we find that

$$\begin{aligned} E[v] &= \frac{1}{2}\|v_x\|^2_{L_2(\Omega)} \leq C\|w\|^2_{L_2(\Omega)} \leq C\|w_x\|^2_{L_2(\Omega)} = C\int_\Omega |v_{xxx}|^{(\alpha+1)\frac{2}{\alpha+1}}\,dx \\ &\leq C\left(\int_\Omega |v_{xxx}|^{\alpha+1}\,dx\right)^{\frac{2}{\alpha+1}}. \end{aligned}$$

The case $\alpha < 1$**.** In this case we have $2/(\alpha+1) > 1$ and we cannot use Jensen's inequality anymore. Instead, we rely on the Sobolev embedding and a-priori estimates for the Bi-Laplace equation. Define $w = v_{xxx} \in L_{\alpha+1}(\Omega)$. Then v is a weak solution to

$$\begin{cases} v_{xxxx} = w_x, & x \in \Omega, \\ v_x = v_{xxx} = 0, & x \in \partial\Omega, \end{cases}$$

in the sense that

$$\int_\Omega v_{xx}\varphi_{xx}\,dx = -\int_\Omega w\varphi_x\,dx \qquad \text{for all } \varphi \in W^2_{\frac{\alpha+1}{\alpha},B}(\Omega).$$

Since $v \in C^2(\bar{\Omega})$ by the Sobolev embedding, we may use $v \in W^2_{\frac{\alpha+1}{\alpha},B}(\Omega)$ as a test function and find that

$$\|v_{xx}\|^2_{L_2(\Omega)} \leq \int_\Omega |w||v_x|\,dx \leq \|w\|_{L_{\alpha+1}(\Omega)}\|v_x\|_{L_{\frac{\alpha}{\alpha+1}}(\Omega)} \leq C\|w\|_{L_{\alpha+1}(\Omega)}\|v_{xx}\|_{L_2(\Omega)}.$$

Dividing by $\|v_{xx}\|_{L_2(\Omega)}$, we conclude that $\|v_{xx}\|^2_{L_2(\Omega)} \leq C\|w\|^2_{L_{\alpha+1}(\Omega)}$. Finally, the desired estimate

$$E[v] = \frac{1}{2}\|v_x\|^2_{L_2(\Omega)} \leq C\|v_{xx}\|^2_{L_2(\Omega)} \leq C\|w\|^2_{L_{\alpha+1}(\Omega)} = C\|v_{xxx}\|^2_{L_{\alpha+1}(\Omega)}$$

follows by Poincaré's inequality. □

Proof of Proposition B.4.1. From Theorem B.3.2 we know that weak solutions to (B.3.1) satisfy the energy-dissipation identity (B.3.2). Taking the derivative in time, we find that

$$\frac{d}{dt}E[u](t) + D[u](t) = 0$$

for almost every $t \in [0,T]$. Furthermore, since $m = \min_{(t,x)\in[0,T]\times\bar{\Omega}} u(t,x) > 0$ and by Lemma B.4.2, we obtain

$$D[u](t) = \int_\Omega |u|^{\alpha+2}|u_{xxx}|^{\alpha+1}\,dx \geq m^{\alpha+2}\|u_{xxx}(t)\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} \geq Cm^{\alpha+2}\big(E[u](t)\big)^{\frac{\alpha+1}{2}}$$

for almost every $t \in [0,T]$. This concludes the proof. □

Next, we turn to L_1-in-time bounds for the dissipation functional in terms of the energy. The proof is a simplified version of the one in [LV22] for general degenerate parabolic problems of fourth order. In our case, it relies on testing the partial differential equation with a time cut-off of the second spatial derivative.

Theorem B.4.3. *Fix $\alpha > 0$ and a positive initial value $u_0 \in H^1(\Omega)$ with $u_0(x) > 0$ for $x \in \bar{\Omega}$. Let*

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$$

be a positive weak solution to (B.3.1) *on* $(0,T)$*, as obtained in Theorem B.3.2. Then there exists a constant $C > 0$, independent of t, such that the dissipation functional $D[u]$ enjoys the L_1-in-time bound*

$$\int_{t/2}^{t} D[u](s)\,ds \le \frac{C}{t}\int_{t/4}^{t/2} E[u](s)\,ds \le \frac{C}{4}E[u]\big(\tfrac{t}{4}\big).$$

Proof. We choose a cut-off function $\chi \in C^\infty(\mathbb{R})$ in time such that $0 \le \chi \le 1$, $\chi(s) = 1$ for $s \ge t/2$, $\chi(s) = 0$ for $s \le t/4$ and $\chi'(s) \le C/t$ for some constant C, independent of t. Now we define the test function $\varphi(s,x) = \chi(s)u_{xx}(s,x) \in L_{\alpha+1}\big((0,T); W^1_{\alpha+1,B}(\Omega)\big)$. Since u is a weak solution to (B.3.1) on the time interval $[0,t]$, we obtain

$$\begin{aligned}\int_0^t \langle u_t, \chi(s)u_{xx}\rangle_{W^1_{\alpha+1}}\,ds &= \int_0^t\int_\Omega u^{\alpha+2}\psi(u_{xxx})u_{xxx}\chi(s)\,dx\,ds \\ &= \int_0^t \chi(s)D[u](s)\,ds.\end{aligned} \tag{B.4.1}$$

Moreover, since $\chi(0) = 0$ and $\chi(t) = 1$, we have the inequality

$$\begin{aligned}0 \le E[u](t) &= \int_0^t \frac{d}{ds}\big(\chi(s)E[u](s)\big)\,ds \\ &= \int_0^t \chi'(s)E[u](s)\,ds - \int_0^t \chi(s)\langle u_s, u_{xx}\rangle_{W^1_{\alpha+1}}\,ds.\end{aligned} \tag{B.4.2}$$

Combining (B.4.1) and (B.4.2) and using that $\chi \equiv 1$ on $[t/2,t]$ and $D[u](s) \ge 0$ for all $0 \le s \le t$, we conclude that

$$\int_{t/2}^t D[u](s)\,ds \le \int_0^t \chi(s)D[u](s)\,ds \le \int_0^t \chi'(s)E[u](s)\,ds \le \frac{C}{t}\int_{t/4}^{t/2} E[u](s)\,ds.$$

Finally, since $E[u]$ decreases along solutions, we may estimate

$$\frac{C}{t}\int_{t/4}^{t/2} E[u](s)\,ds \le \frac{C}{4}E[u]\big(\tfrac{t}{4}\big).$$

This completes the proof. ∎

B.5 Shear-thickening power-law fluids ($\alpha < 1$) – Global existence and convergence to steady states in finite time

This section deals with the long-time asymptotics of shear-thickening power-law fluids, i.e. we consider flow-behaviour exponents $\alpha < 1$ in (B.3.1). We prove that for positive initial values $u_0 \in H^1(\Omega)$ that are close to a steady state in the sense that

$$\frac{1}{2}\bar{u}_0 < u_0(x) < 2\bar{u}_0, \quad x \in \bar{\Omega}, \qquad \text{where} \quad \bar{u}_0 = \fint_\Omega u_0\, dx,$$

problem (B.3.1) with $\alpha < 1$ possesses a globally-in-time defined positive weak solution that converges to a steady state in finite time. As in the circular Taylor–Couette setting [LPV22], the corresponding proof relies mainly on the differential inequality derived in Proposition B.4.1. This differential inequality guarantees that the energy becomes zero in finite time $0 < t^* < \infty$. We construct a globally-in-time defined positive weak solution by constant extension at time t^*.

By Theorem B.3.2 and Remark B.3.3 there exists a weak solution $u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$ with $u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$ to (B.3.1). We define the time

$$\tau = \sup\Big\{\tilde{T} > 0;\ \exists \text{ a weak solution } u \text{ to (B.3.1) on } [0,\tilde{T}] \tag{B.5.1}$$
$$\text{with } \tfrac{1}{2}\bar{u}_0 \le u(t,x) \le 2\bar{u}_0\ \forall\, 0 \le t \le \tilde{T}\Big\},$$

up to which solutions are bounded away from zero and bounded above. Note that by continuity of weak solutions, we have $0 < \tau$. By Remark B.3.4 we may also assume that $\tau \le T$. In particular, we can apply the results of Section B.4 up to time τ.

Theorem B.5.1 (Global existence and convergence in finite time). *Fix $0 < \alpha < 1$. There exists $\varepsilon > 0$ such that, for all positive initial values $u_0 \in H^1(\Omega)$ with $\|u_0 - \bar{u}_0\|_{H^1(\Omega)} < \varepsilon$, there is a positive global weak solution*

$$u \in C\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1,\mathrm{loc}}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha},loc}\big((0,\infty); (W^1_{\alpha+1,B}(\Omega))'\big).$$

Moreover, there exists a time $0 < t^* < \infty$ such that

$$u(t,\cdot) \longrightarrow \bar{u}_0 \text{ in } H^1(\Omega), \text{ as } t \to t^*, \quad \text{and} \quad u(t,x) = \bar{u}_0, \quad t \ge t^*,\ x \in \bar{\Omega}.$$

Proof. Let $u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$ the solution to (B.3.1) provided by Theorem B.3.2 and Remark B.3.3 with initial datum $u_0 > \bar{u}_0/2$ in $\bar{\Omega}$. Write $u(t,x) = \bar{u}_0 + v(t,x)$ for $(t,x) \in [0,T] \times \Omega$, where due to conservation of mass $\int_\Omega v\, dx = 0$ for all $t \in [0,T] \times \bar{\Omega}$. Then, by continuity and the definition of τ, we have $|v(t,x)| \le \bar{u}_0/2$ for $(t,x) \in [0,\tau] \times \bar{\Omega}$. Thus, there exists a constant $C > 0$ such that for almost every $t \in [0,\tau]$ it holds

$$\int_\Omega |v_{xxx}|^{\alpha+1}\, dx \le C \int_\Omega |u|^{\alpha+2} |v_{xxx}|^{\alpha+1}\, dx.$$

Hence, using the energy-dissipation identity (B.3.2) and Lemma B.4.2, we obtain

$$\begin{aligned}\frac{d}{dt}E[v](t) = \frac{d}{dt}E[u](t) &= -\int_\Omega |u|^{\alpha+2}|v_{xxx}|^{\alpha+1}\, dx \\ &\le -C\|v_{xxx}(t)\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} \le -C\big(E[v](t)\big)^{\frac{\alpha+1}{2}}\end{aligned}$$

for almost every $t \in [0,\tau]$. This inequality implies that the energy $E[v](\cdot) = E[u](\cdot)$ is decreasing and hence $\tau = T$. Furthermore, it follows that

$$\frac{d}{dt}\Big(\big(E[v](t)\big)^{\frac{1-\alpha}{2}}\Big) \le -C_\alpha,$$

as long as $E[v](t) > 0$, and integration from 0 to t yields

$$\big(E[v](t)\big)^{\frac{1-\alpha}{2}} \le \big(E[v_0]\big)^{\frac{1-\alpha}{2}} - C_\alpha t, \quad t \in [0,T], \text{ if } E[v](t) > 0.$$

Thus, we conclude that

$$E[v](t) \le \Big(\big(E[v_0]\big)^{\frac{1-\alpha}{2}} - C_\alpha t\Big)^{\frac{2}{1-\alpha}}, \quad t \in [0,T], \text{ if } E[v](t) > 0,$$

which implies the existence of a finite time $t^* \ge 0$ with $t^* \le (E[v_0])^{\frac{1-\alpha}{2}}/C_\alpha$ such that

$$E[v](t) = 0, \quad t \ge t^*.$$

We may choose $\varepsilon > 0$ small enough so that we obtain $t^* < T$. Finally, note that $E[v](t) = 0$ for $t \ge t^*$ and $\bar{v}(t) = 0$ implies that $v(t,x) = 0$ for all $t \ge t^*$ and $x \in \Omega$. Hence, the solution u may be extended by the constant solution $\bar{u}_0$ for times $t \ge t^*$ to a global-in-time weak solution $u \in C\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1,\mathrm{loc}}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$ and we have

$$u(t,x) \longrightarrow \bar{u}_0 \quad \text{in } H^1(\Omega)$$

and uniformly as $t \to t^*$ in finite time. □

B.6 Shear-thinning power-law fluids ($\alpha > 1$) – Global existence and polynomial stability of steady states

In this section we study the long-time behaviour of solutions to the shear-thinning power-law equation. More precisely, we fix a flow-behaviour exponent $\alpha > 1$ in (B.3.1) and consider positive initial values $u_0 \in H^1(\Omega)$ that are close to a steady state in the sense that

$$\frac{1}{2}\bar{u}_0 < u_0(x) < 2\bar{u}_0, \quad x \in \bar{\Omega},$$

where $\bar{u}_0 = \fint_\Omega u_0\, dx$. We show that there exist global positive weak solutions u to (B.3.1) with $\alpha > 1$ that remain ε-close to the steady state for all times and converge at rate $1/t^{\frac{1}{\alpha-1}}$ to equilibrium, as $t \to \infty$. Note that convergence to equilibrium has already been proved for the global non-negative weak solutions constructed in [AG04], but with no rate of convergence. The result on the rate of convergence is the same as in [LV22] for the cylindrical Taylor–Couette setting. The proof relies again on the differential inequality for the energy, derived in Proposition B.4.1. However, in the shear-thinning case also the L_1-in-time bound of Theorem B.4.3 is crucial.

Theorem B.6.1 (Global existence and polynomial stability). *Fix $1 < \alpha < \infty$. There exists $\varepsilon > 0$ such that for all positive initial values $u_0 \in H^1(\Omega)$ with $\|u_0 - \bar{u}_0\|_{H^1(\Omega)} < \varepsilon$, there is a global positive weak solution*

$$u \in C\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1,\mathrm{loc}}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha},loc}\big((0,\infty);(W^1_{\alpha+1,B}(\Omega))'\big).$$

Moreover, there is a constant $C > 0$ *such that*

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \le \frac{C\varepsilon}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{1}{\alpha-1}}}, \quad 0 \le t < \infty.$$

Furthermore, the dissipation decreases polynomially along the solution in the following L_1*-in-time sense*

$$\int_{t/2}^{t} D[u](s)\,ds \le \frac{C\varepsilon^2}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{2}{\alpha-1}}} \tag{B.6.1}$$

for all $0 \le t < \infty$.

Remark B.6.2. Note that the weak solution

$$u \in C\big([0,\infty);H^1(\Omega)\big) \cap L_{\alpha+1,\mathrm{loc}}\big((0,\infty);W^3_{\alpha+1,B}(\Omega)\big)$$

obtained in Theorem B.6.1 satisfies $u(t,x) \ge \bar{u}_0/2$ for all $(t,x) \in [0,\infty)\times\bar{\Omega}$. Hence, (B.6.1) implies that the $W^3_{\alpha+1}(\Omega)$-norm is also controlled in the L_1-in-time sense by

$$\int_{t/2}^{t}\int_\Omega |u_{xxx}(s)|^{\alpha+1}\,dx\,ds \le \frac{C\varepsilon}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{1}{\alpha-1}}}$$

for all $0 \le t < \infty$.

Proof of Theorem B.6.1. First, we show that there exists an $\varepsilon > 0$ such that for all initial values $u_0 \in H^1(\Omega)$ with $\bar{u}_0 = 0$ and $\|u_0 - \bar{u}_0\| < \varepsilon$, there is a constant $C > 0$ independent of ε such that

$$E[u](t) \le \frac{C\varepsilon^2}{\big(1 + \varepsilon^{\alpha-1}t\big)^{\frac{2}{\alpha-1}}}, \quad 0 \le t < \infty.$$

Let $u \in C\big([0,T];H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T);W^3_{\alpha+1,B}(\Omega)\big)$ the solution to (B.3.1) provided by Theorem B.3.2 and Remark B.3.3 with initial datum $u_0 \in H^1(\Omega)$ satisfying $u_0 > \bar{u}_0/2$ in $\bar{\Omega}$. As in the proof of Theorem B.5.1, we write $u(t,x) = \bar{u}_0 + v(t,x)$ for $(t,x) \in [0,T]\times\bar{\Omega}$, where due to conservation of mass $\int_\Omega v\,dx = 0$ for all $t \in [0,T]$. Then, by continuity and the definition of τ (see (B.5.1)), we have $|v(t,x)| \le \bar{u}_0/2$ for $(t,x) \in [0,\tau]\times\bar{\Omega}$. By Lemma B.4.2 we then conclude that

$$\begin{aligned} E[u](t) = E[v](t) &\le C\left(\int_\Omega |v_{xxx}|^{\alpha+1}\,dx\right)^{\frac{2}{\alpha+1}} \\ &\le C\left(\int_\Omega |u|^{\alpha+2}|v_{xxx}|^{\alpha+1}\,dx\right)^{\frac{2}{\alpha+1}} = C\big(D[u](t)\big)^{\frac{2}{\alpha+1}} \end{aligned}$$

for almost every $t \in [0,\tau]$. Inserting this into the energy-dissipation identity (B.3.2), we find that

$$\frac{d}{dt}E[u](t) = -D[u](t) \le -C\big(E[u](t)\big)^{\frac{\alpha+1}{2}} \tag{B.6.2}$$

for almost every $t \in [0,\tau]$. This implies that the energy $E[u](\cdot)$ is decreasing and hence $\tau = T$. Furthermore, we can rewrite estimate (B.6.2) as

$$\frac{2}{1-\alpha}\frac{d}{dt}\big(E[u](t)\big)^{\frac{1-\alpha}{2}} \le -C, \quad 0 \le t \le \tau,$$

so that, after integration, we obtain

$$\frac{2}{1-\alpha}\big(E[u](t)\big)^{\frac{1-\alpha}{2}} \leq -Ct + \frac{2}{1-\alpha}\big(E[u_0]\big)^{\frac{1-\alpha}{2}}, \quad 0 \leq t \leq \tau.$$

Since $\alpha > 1$, we can rearrange this inequality to

$$E[u](t) \leq \left(\big(E[u_0]\big)^{\frac{1-\alpha}{2}} + \frac{C(\alpha-1)}{2}t\right)^{\frac{2}{1-\alpha}} = \frac{E[u_0]}{\Big(1 + C\big(E[u_0]\big)^{\frac{\alpha-1}{2}}t\Big)^{\frac{2}{\alpha-1}}}, \quad 0 \leq t \leq \tau.$$

Since the function $s \mapsto \frac{s}{\left(1+Cs^{\frac{\alpha-1}{2}}t\right)^{\frac{2}{\alpha-1}}}$ is increasing on $[0,\infty)$ and $E[u_0] \leq \varepsilon^2$ by assumption, we infer that

$$E[u](t) \leq \frac{C\varepsilon^2}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{2}{\alpha-1}}}, \quad 0 \leq t \leq \tau.$$

Now, we choose $\varepsilon > 0$ such that

$$\|u_0 - \bar{u}_0\|_{L_\infty(\Omega)} \leq C\big(E[u_0]\big)^{\frac{1}{2}} \leq \frac{\bar{u}_0}{2},$$

where the first estimate is due to the embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ and the Poincaré inequality. This, together with the fact that $E[u](\cdot)$ is decreasing, guarantees that

$$\|u(t) - \bar{u}_0\|_{L_\infty(\Omega)} \leq C\big(E[u](t)\big)^{\frac{1}{2}} \leq C\big(E[u_0]\big)^{\frac{1}{2}} \leq \frac{\bar{u}_0}{2}, \quad 0 \leq t \leq \tau.$$

Hence, solutions u to (B.3.1) on $[0,\tau]$ remain strictly bounded away from zero and by bootstrapping as in Remark B.3.4, we may extend it beyond time τ to a global-in-time weak solution $u \in C\big([0,\infty); H^1(\Omega)\big)$
$\cap L_{\alpha+1,\text{loc}}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$ that satisfies

$$E[u](t) \leq \frac{C\varepsilon^2}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{2}{\alpha-1}}}, \quad 0 \leq t < \infty.$$

Since by Poincaré's inequality we have

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \leq C\sqrt{E[u](t)} \leq \frac{C\varepsilon}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{1}{\alpha-1}}}, \quad 0 \leq t < \infty,$$

we conclude the polynomial stability in $H^1(\Omega)$. For the L_1-in-time estimate, we apply Theorem B.4.3 and obtain

$$\int_{t/2}^{t} D[u](s)\,ds \leq CE[u]\big(\tfrac{t}{4}\big) \leq \frac{C\varepsilon^2}{\big(1 + C\varepsilon^{\alpha-1}t\big)^{\frac{2}{\alpha-1}}}, \quad 0 \leq t < \infty.$$

This completes the proof. □

B.7 Global existence and exponential stability for the Ellis-law thin-film equations

Now we turn to fluids with Ellis-law rheology. These are fluids whose viscosity approaches a Newtonian plateau for low shear rates, while for big shear rates the viscosity is shear-thinning. The corresponding thin-film equation is given by

$$\begin{cases} u_t + \big(u^3(1+|uu_{xxx}|^{\alpha-1})u_{xxx}\big)_x = 0, & t>0,\ x\in\Omega, \\ u_x(t,x) = u_{xxx}(t,x) = 0, & t>0,\ x\in\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{B.7.1}$$

for flow-behaviour exponents $\alpha \geq 1$. Here $\Omega \subset \mathbb{R}$ denotes, as before, a bounded interval.

Definition B.7.1. *Let $\alpha > 1$. For a given $T > 0$ a weak solution to* (B.7.1) *is defined as a function*

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$$

that has the following properties:

- *(i) (Weak formulation) u satisfies the differential equation* (B.7.1)$_1$ *in the weak sense, i.e.,*

$$\int_0^T \langle u_t, \varphi\rangle_{W^1_{\alpha+1}(\Omega)}\, dt = \int_0^T \int_\Omega u^3\big(1+|uu_{xxx}|^{\alpha-1}\big)u_{xxx}\,\varphi_x\, dx\, dt$$

  for all test functions $\varphi \in L_{\alpha+1}\big((0,T); W^1_{\alpha+1,B}(\Omega)\big)$.

- *(ii) (Initial and boundary values) u satisfies the contact angle condition $u_x = 0$ on $\partial\Omega$ and the initial condition* (B.7.1)$_3$ *pointwise.*

In the case of Ellis fluids we naturally obtain the dissipation functional

$$D[u] = \int_\Omega u^3\big(1+|uu_{xxx}|^{\alpha-1}\big)|u_{xxx}|^2\, dx.$$

For general positive initial data in $H^1(\Omega)$ we can show existence of local-in-time positive weak solutions.

Theorem B.7.2 (Local existence of positive weak solutions). *Let $\alpha > 1$. Given a positive initial value $u_0 \in H^1(\Omega)$ with $u_0(x) > 0,\ x \in \bar\Omega$, there exists a time $T > 0$ such that problem* (B.7.1) *admits at least one positive weak solution*

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T); (W^1_{\alpha+1,B}(\Omega))'\big)$$

on $(0,T)$ in the sense of Definition B.7.1. Moreover, such a solution has the following properties:

- *(i) (Positivity) u is bounded away from zero*

$$0 < C_T \leq u(t,x), \quad 0 \leq t \leq T,\ x \in \bar\Omega.$$

(ii) (Conservation of mass) u conserves its mass in the sense that

$$\|u(t)\|_{L_1(\Omega)} = \|u_0\|_{L_1(\Omega)}, \quad 0 \le t \le T.$$

(iii) (Energy-dissipation identity) Energy is dissipated along solutions

$$E[u](t) + \int_0^t D[u](s)\, ds = E[u_0] \tag{B.7.2}$$

for almost every $t \in [0,T]$.

Remark B.7.3. For positive initial datum $u_0 \in W^{4\rho}_{\alpha+1,B}(\Omega)$, $4\rho > 3 + 1/(\alpha+1)$ with $u_0(x) > 0, x \in \bar{\Omega}$, the problem (B.7.1) actually possesses a unique maximal strong solution [LM20]

$$\begin{aligned} u \in & C\big([0,T_{\max}); W^{4\rho}_{\alpha+1,B}(\Omega)\big) \cap C^{\rho}\big([0,T_{\max}); L_{\alpha+1}(\Omega)\big) \\ & \cap C\big((0,T_{\max}); W^{4}_{\alpha+1,B}(\Omega)\big) \cap C^1\big((0,T_{\max}); L_{\alpha+1}(\Omega)\big). \end{aligned}$$

Moreover, the solution enjoys the following properties:

(i) (Positivity) u is positive

$$u(t,x) > 0, \quad 0 \le t < T_{\max},\ x \in \bar{\Omega}.$$

(ii) (Conservation of mass) u conserves its mass in the sense that

$$\|u(t)\|_{L_1(\Omega)} = \|u_0\|_{L_1(\Omega)}, \quad 0 \le t < T_{\max}.$$

(iii) (Energy-dissipation identity) u satisfies the energy-dissipation identity

$$E[u](t) + \int_0^t D[u](s) = E[u_0], \quad 0 \le t < T_{\max}. \tag{B.7.3}$$

(iv) (Maximal time of existence) Suppose that $T_{\max} < \infty$. Then

$$\liminf_{t \nearrow T_{\max}} \frac{1}{\min_{x\in\bar{\Omega}} u(t)} + \|u(t)\|_{W^{4\gamma}_{\alpha+1,B}(\Omega)} = \infty$$

for all $\gamma \in (\theta, 1]$.

Proof of Theorem B.7.2. For initial data $u_0 \in W^{4\rho}_{\alpha+1,B}(\Omega)$, $4\rho > 3 + 1/(\alpha+1)$ with $u_0(x) > 0$ we obtain local-in-time strong solutions. Choosing a sequence $\big(u_0^{(k)}\big)_{k\in\mathbb{N}}$ with $u_0^{(k)}(x) > 0$, $x \in \bar{\Omega}$, and $\bar{u}_0^{(k)} = \bar{u}_0$ such that $u_0^{(k)} \to u_0$ strongly in $H^1(\Omega)$ guarantees, together with the energy-dissipation identity (B.7.3) and similar a-priori bounds as in Lemma B.3.8 that the corresponding strong solutions $u^{(k)}$ converge weakly in $L_\infty\big((0,T); H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$ to a weak solution $u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$. Positivity, conservation of mass and the energy-dissipation identity for almost every $t \in [0,T]$ are preserved under taking the weak limit. □

Steady states of (B.7.1)**.** We now turn to stability. First, we find that the same characterisation of positive steady states as before holds true. This is the content of the following theorem which has already been proved in [LM20, Corollary 6.3].

Theorem B.7.4 (Characterisation of positive steady states). *A function* $u \in W^3_{\alpha+1,B}(\Omega)$ *is a positive steady-state solution of* (B.7.1) *if and only if* $u \equiv u_* \in \mathbb{R}_{>0}$ *is given by positive constant.*

Global existence and exponential stability for (B.7.1)**.** It is well-known that for the Newtonian thin-film equation

$$\begin{cases} u_t + \big(u^3 u_{xxx}\big)_x = 0, & t > 0,\ x \in \Omega, \\ u_x(t,x) = u_{xxx}(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

solutions close to positive steady states converge exponentially fast to equilibrium [BP96]. We now prove that the same behaviour can be found for Ellis-law thin films.

Theorem B.7.5 (Global existence and exponential stability). *Fix* $1 < \alpha < \infty$. *There exists* $\varepsilon > 0$ *such that, for all positive initial values* $u_0 \in H^1(\Omega)$ *with* $\|u_0 - \bar{u}_0\|_{H^1(\Omega)} < \varepsilon$, *there is a global positive weak solution*

$$u \in C\big([0,\infty); H^1(\Omega)\big) \cap L_{\alpha+1,\mathrm{loc}}\big((0,\infty); W^3_{\alpha+1,B}(\Omega)\big)$$

with

$$u_t \in L_{\frac{\alpha+1}{\alpha},loc}\big((0,\infty); (W^1_{\alpha+1,B}(\Omega))'\big).$$

Moreover, there is $\lambda > 0$ *and a constant* $C > 0$ *such that*

$$\|u(t) - \bar{u}_0\|_{H^1(\Omega)} \le Ce^{-\lambda t}\|u_0\|_{H^1(\Omega)}.$$

Furthermore, we find that the dissipation decreases exponentially along the solution in the following L_1*-in-time sense:*

$$\int_{t/2}^{t} D[u](s)\, ds \le Ce^{-2\lambda t}\|u_0\|^2_{H^1(\Omega)}.$$

Proof. Let $u_0 \in H^1(\Omega)$ with $\bar{u}_0/2 < u_0(x) < 2\bar{u}_0$, $x \in \bar{\Omega}$ and $u \in C\big([0,T]; H^1(\Omega)\big) \cap L_{\alpha+1}\big((0,T); W^3_{\alpha+1,B}(\Omega)\big)$ the solution to (B.7.1) provided by Theorem B.7.2. We also define

$$\tau = \sup\Big\{\tilde{T} > 0;\ \exists \text{ a weak solution } u \text{ to (B.7.1) on } [0,\tilde{T}] \\ \text{with } \tfrac{1}{2}\bar{u}_0 \le u(t,x) \le 2\bar{u}_0\ \forall\, 0 \le t \le \tilde{T}\Big\}.$$

Then $\tau \le T$ because otherwise we can extend weak solutions beyond time τ.

Next, write $u(t,x) = \bar{u}_0 + v(t,x)$ for $(t,x) \in [0,T] \times \Omega$, where due to conservation of mass $\int_\Omega v(t,x)\, dx = 0$ for all $t \in [0,T]$. Then, by continuity and the definition of τ, we have $|v(t,x)| \le \bar{u}_0/2$ for $0 \le t \le \tau$.

We then find, by the energy-dissipation identity (B.7.2) and the definition of τ, that

$$\begin{aligned} \frac{d}{dt}E[u](t) = -D[u](t) &= -\int_\Omega u^3(t,x)\big(1 + |u(t,x)u_{xxx}(t,x)|^{\alpha-1}\big)|u_{xxx}(t,x)|^2\, dx \\ &\le -\int_\Omega u^3(t,x)|u_{xxx}(t,x)|^2\, dx \le -C\int_\Omega |u_{xxx}(t,x)|^2\, dx \le -CE[u](t) \end{aligned}$$

for almost every $t \in [0,\tau]$, where in the last step we have applied Lemma B.4.2. This yields that $E[u](t)$ is decreasing and so $\tau = T$. Applying Gronwall's inequality, we deduce that

$$E[u](t) \leq E[u_0]e^{-Ct}$$

for all $t \in [0,\tau]$. Now choose $\varepsilon > 0$ small enough so that

$$\|u_0 - \bar{u}_0\|_{L^\infty(\Omega)} \leq CE[u_0]^{\frac{1}{2}} \leq \frac{\bar{u}_0}{2},$$

where in the first estimate we have used the embedding $H^1(\Omega) \hookrightarrow L_\infty(\Omega)$ and Poincaré's inequality. Using this and the fact that $E[u](t)$ is decreasing, we get

$$\|u(t) - \bar{u}_0\|_{L^\infty(\Omega)} \leq CE[u](t)^{\frac{1}{2}} \leq CE[u_0]^{\frac{1}{2}} \leq \frac{\bar{u}_0}{2}$$

for all $t \in [0,T]$. We can then extend the solution beyond time T to a global-in-time weak solution $u \in C\big([0,\infty);H^1(\Omega)\big) \cap L_{\alpha+1,\text{loc}}\big((0,\infty);W^3_{\alpha+1,B}(\Omega)\big)$ to (B.7.1) that satisfies

$$E[u](t) \leq E[u_0]e^{-Ct}, \quad 0 \leq t < \infty.$$

By Poincaré's inequality, we then conclude that

$$\|u(t) - u_0\|_{H^1(\Omega)} \leq CE[u(t)]^{1/2} \leq C\|\nabla u_0\|_{L^2(\Omega)}e^{-\lambda t},$$

for some $\lambda > 0$ and all $t \in (0,\infty)$.

The L_1-in-time estimate follows from adapting Theorem B.4.3 to the new dissipation functional. □

B.8 Appendix: Proofs of Lemma B.3.9 and Lemma B.3.10

Here we give precise proofs of the auxiliary results needed to establish local existence of positive weak solutions to the original problem (B.3.1) in Section B.3.

Proof of Lemma B.3.9. **(i)** In Lemma B.3.8 (i) and (iii) we have shown that

$$\begin{cases} (u^\sigma)_\sigma \text{ is uniformly bounded in } L_\infty\big((0,T);H^1(\Omega)\big) \\ (u^\sigma_t)_\sigma \text{ is uniformly bounded in } L_{\frac{\alpha+1}{\alpha}}\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big). \end{cases}$$

Moreover, in view of the Rellich-Kondrachov theorem, see e.g. [AF03, Thm. 6.3], we have

$$H^1(\Omega) \overset{c}{\hookrightarrow} C^\rho(\bar{\Omega}) \hookrightarrow (W^1_{\alpha+1}(\Omega))', \quad \rho \in [0,1/2),$$

where $\overset{c}{\hookrightarrow}$ indicates compactness of the embedding. This enables us to use [Sim86, Cor. 4], which gives that the sequence

$$(u^\sigma)_\sigma \text{ is relatively compact in } C\big([0,T];C^\rho(\bar{\Omega})\big)$$

with $\rho \in [0,1/2)$ as above.
(ii) This is an immediate consequence of Lemma B.3.8 (ii).
(iii) By Lemma B.3.8 (iii), we can extract a subsequence $(u^\sigma_t)_\sigma$ such that

$$u^\sigma_t \rightharpoonup v \quad \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big) \hookrightarrow \mathcal{D}'\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big)$$

for some limit function $v \in L_{\frac{\alpha+1}{\alpha}}\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big)$. Since, in addition,

$$u^\sigma \longrightarrow u \quad \text{in } C\big([0,T];C^\rho(\bar\Omega)\big) \hookrightarrow \mathcal{D}'\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big) \quad \rho \in [0,1/2),$$

we conclude that

$$u^\sigma_t \longrightarrow u_t \quad \text{in } \mathcal{D}'\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big)\big),$$

and thus, $v = u_t \in L_{\frac{\alpha+1}{\alpha}}\big((0,T);(W^1_{\alpha+1,B}(\Omega))'\big)$.

(iv) Note that the strong convergence $u^\sigma \to u$ in $C\big([0,T];C^\rho(\bar\Omega)\big)$, $\rho \in [0,1/2)$, in (i) implies uniform convergence

$$u^\sigma \longrightarrow u \quad \text{in } C\big([0,T]\times\bar\Omega\big). \tag{B.8.1}$$

Moreover, by Lemma B.3.8 (v), there exists some $\hat u \in L_{\alpha+1}\big((0,T);W^3_{\alpha+1,B}(\Omega)\big)$ such that

$$u^\sigma \rightharpoonup \hat u \quad \text{in } L_{\alpha+1}\big((0,T);W^3_{\alpha+1,B}(\Omega)\big). \tag{B.8.2}$$

Because of the uniqueness of the limit function, we infer from (B.8.1) and (B.8.2) that

$$u^\sigma \rightharpoonup u \quad \text{in } L_{\alpha+1}\big((0,T);W^3_{\alpha+1,B}(\Omega)\big).$$

In virtue of the weak lower-semicontinuity of the norm and Lemma B.3.8 (iv) and (v), we finally obtain

$$\begin{cases} \|u_{xxx}\|_{L_{\alpha+1}((0,T)\times\Omega)} \le \liminf_{\sigma\to0} \|u^\sigma_{xxx}\|_{L_{\alpha+1}((0,T)\times\Omega)} \le C \\ \|u\|_{L_{\alpha+1}((0,T);W^3_{\alpha+1,B}(\Omega))} \le \liminf_{\sigma\to0} \|u^\sigma\|_{L_{\alpha+1}((0,T);W^3_{\alpha+1,B}(\Omega))} \le C \end{cases}$$

for some generic constant $C > 0$ that is independent of σ.

(v) This follows by reasoning similarly to (iii) and the proof is complete. □

Proof of Lemma B.3.10. The proof is divided into several steps. Throughout the proof, when there is no fear of ambiguity, we pass to a subsequence without relabelling it.

(i) First, by Lemma B.3.9 (ii), we know that $|u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx})$ is weakly sequentially compact, i.e., there is an element $\chi \in L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big)$ such that

$$|u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx}) \rightharpoonup \chi \quad \text{weakly in } L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big).$$

It remains to identify the limit flux χ.

(ii) Next, in view of Lemma B.3.8 (v) and the lower semicontinuity of the norm,

$$u_x \in L_{\alpha+1}\big((0,T);W^1_{\alpha+1,0}(\Omega)\cap W^2_{\alpha+1}(\Omega)\big).$$

Thus, we can take $\varphi = u_{xx} \in L_{\alpha+1}\big((0,T);W^1_{\alpha+1}(\Omega)\big)$ as a test function in the equation (P_σ) for u^σ. This gives

$$\int_0^T\int_\Omega u^\sigma_t u_{xx}\,dx\,dt + \int_0^T\int_\Omega |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx})u_{xxx}\,dx\,dt = 0.$$

Using Lemma B.3.9 (iii), the first term satisfies

$$\int_0^T\int_\Omega u^\sigma_t\,u_{xx}\,dx\,dt \longrightarrow \int_0^T\int_\Omega u_t\,u_{xx}\,dx\,dt = E[u](T) - E[u](0)$$

as $\sigma \searrow 0$. For the second term, we infer from Lemma B.3.9 (ii) that

$$\int_0^T \int_\Omega |u^\sigma|^{\alpha+2} \psi_\sigma(u^\sigma_{xxx}) u_{xxx}\, dx\, dt \longrightarrow \int_0^T \int_\Omega \chi u_{xxx}\, dx\, dt,$$

as $\sigma \searrow 0$. Consequently, we obtain the identity

$$E[u](t) + \langle \chi | u_{xxx} \rangle_{L_{\alpha+1}} = E[u_0]$$

for almost every $t \in [0,T]$.
(iii) We now use Minty's trick to identify the limit flux χ. Note that the operator

$$\begin{cases} \psi_\sigma \colon L_{\alpha+1}\big((0,T) \times \Omega\big) \longrightarrow L_{\frac{\alpha+1}{\alpha}}\big((0,T) \times \Omega\big), \\ \psi_\sigma(v) = \big(|v|^2 + \sigma^2\big)^{\frac{\alpha-1}{2}} v \end{cases}$$

is monotone, i.e. for all $v, w \in L_{\alpha+1}\big((0,T) \times \Omega\big)$ with $v \neq w$ it holds that

$$\langle \psi_\sigma(v) - \psi_\sigma(w) | v - w \rangle_{L_{\alpha+1}} = \int_0^T \int_\Omega \big(\psi_\sigma(v) - \psi_\sigma(w)\big)(v - w)\, dx\, dt > 0.$$

This follows immediately from the monotonicity of the function

$$\psi_\sigma \colon \mathbb{R} \to \mathbb{R} \;:\; s \mapsto (s^2 + \sigma^2)^{\frac{\alpha-1}{2}} s.$$

From now on, we simply write $\langle v | w \rangle$ for the dual pairing $\langle v|w\rangle_{L_{\alpha+1}((0,T)\times\Omega)}$ between $v \in L_{\frac{\alpha+1}{\alpha}}\big((0,T) \times \Omega\big)$ and $w \in L_{\alpha+1}\big((0,T) \times \Omega\big)$. Let now $\varphi \in W^3_{\alpha+1}\big((0,T) \times \Omega\big)$. In view of the monotonicity of ψ_σ, we have

$$\begin{aligned} 0 &\leq \big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx}) - |u^\sigma|^{\alpha+2}\psi_\sigma(\varphi_{xxx}) | (u^\sigma - \varphi)_{xxx} \big\rangle \\ &= \big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx}) | u^\sigma_{xxx} \big\rangle - \big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx}) | \varphi_{xxx} \big\rangle \\ &\quad - \big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(\varphi_{xxx}) | u^\sigma_{xxx} \big\rangle + \big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(\varphi_{xxx}) | \varphi_{xxx} \big\rangle. \end{aligned}$$

We consider the four dual pairings on the right-hand side separately.

First, we rewrite the energy-dissipation identity for the problem (P_σ) as

$$\big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx}) | u^\sigma_{xxx} \big\rangle = E[u_0] - E[u^\sigma](t) \quad \text{for almost every } t \in [0,T].$$

Thanks to Lemma B.3.9 (i) we know that $u^\sigma(t) \to u(t)$ in $H^1(\Omega)$ for almost every $t \in [0,T]$, and hence, as $\sigma \searrow 0$, we have

$$\big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx}) | u^\sigma_{xxx} \big\rangle \longrightarrow E[u_0] - E[u](t) \quad \text{for almost every } t \in [0,T]. \tag{B.8.3}$$

For the second dual pairing, we get from Lemma B.3.9 (ii) that

$$\big\langle |u^\sigma|^{\alpha+2}\psi_\sigma(u^\sigma_{xxx}) | \varphi_{xxx} \big\rangle \longrightarrow \langle \chi | \varphi_{xxx} \rangle, \quad \text{as } \sigma \searrow 0.$$

For the third pairing, we use Lemma B.3.9 (i) and (iv) to obtain

$$\begin{cases} u^\sigma \longrightarrow u & \text{strongly in } C\big([0,T] \times \bar{\Omega}\big) \\ u^\sigma_{xxx} \rightharpoonup u_{xxx} & \text{weakly in } L_{\alpha+1}\big((0,T) \times \Omega\big), \end{cases}$$

and this implies

$$\left\langle |u^\sigma|^{\alpha+2}\psi_\sigma(\varphi_{xxx})|u^\sigma_{xxx}\right\rangle \longrightarrow \left\langle |u|^{\alpha+2}\psi(\varphi_{xxx})|u_{xxx}\right\rangle, \quad \text{as } \sigma \searrow 0.$$

Clearly, for the fourth pairing, we have

$$\left\langle |u^\sigma|^{\alpha+2}\psi_\sigma(\varphi_{xxx})|\varphi_{xxx}\right\rangle \longrightarrow \left\langle |u|^{\alpha+2}\psi(\varphi_{xxx})|\varphi_{xxx}\right\rangle, \quad \text{as } \sigma \searrow 0. \tag{B.8.4}$$

Combining (B.8.3)–(B.8.4) yields the inequality

$$0 \leq E[u_0] - E[u](t) - \langle \chi|\varphi_{xxx}\rangle - \left\langle |u|^{\alpha+2}\psi(\varphi_{xxx})|(u-\varphi)_{xxx}\right\rangle,$$

and taking into account the identity

$$E[u](t) + \langle \chi|u_{xxx}\rangle = E[u_0]$$

proved in step (ii), for almost every $t \in [0,T]$, we get that

$$0 \leq \left\langle \chi - |u|^{\alpha+2}\psi(\varphi_{xxx})|(u-\varphi)_{xxx}\right\rangle.$$

Choosing $\varphi = u - \lambda v$ for some arbitrary $v \in W^3_{\alpha+1}\big((0,T)\times\Omega\big)$ and $\lambda > 0$, gives the inequality

$$\left\langle \chi - |u|^{\alpha+2}\psi\big((u-\lambda v)_{xxx}\big)|v_{xxx}\right\rangle \geq 0$$

and thus in the limit $\lambda \searrow 0$ we deduce

$$\left\langle \chi - |u|^{\alpha+2}\psi(u_{xxx})|v_{xxx}\right\rangle \geq 0, \quad v \in W^3_{\alpha+1}\big((0,T)\times\Omega\big),$$

for almost every $t \in [0,T]$. Now taking $\varphi = u + \lambda v$, we see that

$$\left\langle \chi - |u|^{\alpha+2}\psi(u_{xxx})|v_{xxx}\right\rangle \leq 0, \quad v \in W^3_{\alpha+1}\big((0,T)\times\Omega\big).$$

Hence, we have shown that

$$\left\langle \chi - |u|^{\alpha+2}\psi(u_{xxx})|v_{xxx}\right\rangle = 0, \quad v \in W^3_{\alpha+1}\big((0,T)\times\Omega\big),$$

from which, since $v \in W^3_{\alpha+1}\big((0,T)\times\Omega\big)$ is arbitrary, we are able to identify

$$\chi = |u|^{\alpha+2}\psi(u_{xxx}) \in L_{\frac{\alpha+1}{\alpha}}\big((0,T)\times\Omega\big).$$

This completes the proof. □