Minimisation problems in continuum mechanics and generalised convex hulls

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
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Bonn 2022

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen
Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 18.11.2022
Erscheinungsjahr: 2022

Dedicated to my beloved family
and
to the memory of Helga Röhr

Acknowledgements

This thesis would never have been possible without the support and help from many people.

First of all, I thank Stefan Müller for very helpful discussions, suggestions and the great mathematical insights he gave me. I would also like to thank Sergio Conti, Martin Rumpf and Sebastian Neubert for being members of my thesis committee and Margherita Disertori for being my PhD mentor. I also appreciate the financial support from the Bonn International Graduate school.

From a mathematical viewpoint, this thesis would be far worse without the mathematical discussions with other people and the ideas they gave me. I especially want to highlight Jonas Jansen for insightful discussions and proofreading parts of this thesis and Christina Lienstromberg & Richard Schubert, who introduced me to the quite mysterious world of fluid mechanics.

However, a path to a PhD thesis contains more than doing mathematics and working all-day long. A special thanks goes out to my family, my parents and my sister, for constantly supporting me despite not understanding anything of what I am actually doing. I would not be at this point without their love and their help and I am deeply grateful for this.

Moreover, I thank my colleagues for creating a nice and non-competitive working environment, especially before a world-wide pandemic forced us to work from home. The (sometimes wild) mensa discussions and the Schafkopf/Skat rounds after lunch were an invaluable part of my working day. Especially, I want to thank Jonas Jansen, who had the burden to share an office with me, for our discussions about mathematics, future plans and miscellaneous topics. Moreover, I am very grateful to Christina Lienstromberg, who always helped and encouraged me to carry on when my motivation and my confidence was at a local or even a global minimum.

I also want to thank my friends outside of the mathematical world, especially for very relaxing game/chess evenings, who helped me to forget the (sometimes very frustrating) studies.

Last, I want to thank a few people, who are not directly connected to this thesis, but played a huge role to where I am now. Monika Linden organised the mathematics competitions at my school and is irreplacable at this role. Also, I want to thank Malte & Mareike Mink for establishing the 'Bonner Matheclub' which greatly encourages student's interest in mathematics. Finally, I want to express my gratitude to Dr. med. Anton Rausch and nurse Ewald. Without them, I would not have been able to write this thesis in a very literal sense.

Summary

The goal of this thesis is the analysis of functionals subject to a differential constraint. These functionals appear in minimisation problems which are connected to problems coming from continuum mechanics. In the introductory chapter of this thesis we further explain the connection between the physical problems and their mathematical formulation.

Chapter 2 is concerned with the study of the differential constraint. It mostly features auxiliary results that are needed in later sections. Moreover, we compare two concepts of the constant rank property with respect to two different base fields, $\mathbb{R}$ and $\mathbb{C}$.

The aim of Chapters 3 and 4 is to derive an abstract theory regarding weak continuity and weak lower-semicontinuity of functionals. This is connected to a generalised notion of convexity for functions, so called $\mathcal{A}$-quasiconvexity. Employing the *direct method* of the calculus of variations, these results can directly be applied in the analysis of minimisers for the aforementioned functionals. Chapter 3 studies the significantly stronger notion of $\mathcal{A}$-quasiaffinity and gives and extended version of previously known characterisations of $\mathcal{A}$-quasiaffine function. In contrast, Chapter 4 examines the equivalence between $\mathcal{A}$-quasiconvexity and lower-semicontinuity, with a focus on a weak growth assumption.

The knowledge acquired in Chapters 3 and 4 is applied in Chapter 5. In that chapter, we examine a *data-driven* approach to fluid mechanics in a stationary setting that has previously been employed in the study of solid mechanics. In particular, we show a result that connects convergence of data sets to convergence of corresponding functionals and minimisation problems.

In the second part of this thesis, Chapters 6-8, we consider a notion of convexity for sets that is directly connected to the previously mentioned notion of convexity for functions. This notion of convexity has been analysed subject to the specific constraint of being a gradient, so called *quasiconvexity.* The aim of the second part is to show the validity of some statements that are known for the setting of quasiconvexity to general differential operators. Chapters 7 and 8 are summaries of their respective counterparts, Chapters A and B, which rely on the publications [134] and [20], respectively, and are therefore presented in the appendix.

One of the main statements is that a suitable convex hull of a set does not depend on an exponent whenever the set itself is compact. This result relies on a truncation technique that constructs a cut-off version of a function that still satisfies the differential constraint. The consequences of this truncation theorems in the framework of convex sets is discussed in Chapter 6. The truncation statement itself is shown in two physically relevant settings: in Chapter A for closed differential forms and in Chapter B for the divergence of symmetric 3×3 matrices.

Declarations

I declare that I have authored this thesis independently, that I have not used other than the declared sources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

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1. Introduction

1.1. Outline

This introductory chapter gives an overview of both the physical motivation coming from continuum mechanics and the corresponding mathematical problems and their solution. Hence, it is split into two major parts.

In the first part, Section 1.2, we give a brief introduction to the mathematical formulation of continuum mechanics. In Subsection 1.2.1, we start with the theory of *static* continuum mechanics and the study of equilibria. The treatment then further branches up, dependent on the class of the material considered.

For certain *solids* one observes elastic behaviour. We give a overview of elasticity, hyperelasticity and phase transitions in crystalline structure in the Subsections 1.2.2–1.2.4. In Subsection 1.2.5, we folcus on incompressible *fluids*, where different modelling assumptions are needed.

Different materials, for example water and oil, behave differently under application of forces. Clasically, one models a constitutive equation based on the experimental data and symmetry considerations. This approach is described in Subsection 1.2.6. It leads to a partial differential equation (PDE) for the natural fields (the deformation for solids and the velocity for fluids). A new approach skips the modelling step and directly computes a solution based on the experimental data, cf. Subsection 1.2.7.

The second part of this introduction, Section 1.3, is concerned with an abstract reformulation and solution of the mathematical problems which arise in the theory formulated in Section 1.2. We formulate a generalised version of minimisation problems appearing in the context of continuum mechanics in Subsection 1.3.1. After a slight detour on the underlying PDE constraints in 1.3.2, we focus on the theory of weak lower-semicontinuity and the direct method of the calculus of variations in Subsection 1.3.3. Weak lower-semicontinuity is closely connected to a generalised notion of convexity in presence of a differential constraint, the so called $\mathcal{A}$-quasiconvexity. Here, $\mathcal{A}$-quasiconvexity is a notion for functions $f\colon \mathbb{R}^d \to \mathbb{R}$, which arise as integrands of the minimisation problems considered.

By duality/separation one can also define the notion of $\mathcal{A}$-quasiconvexity for sets, cf. Subsection 1.3.4. This leads to one of the main question of this thesis, Question 1.18, which is concerned with the dependence of a suitable convex hull on the growth exponent p of the underlying class of separating functions. The solution to this question relies on a truncation statement, which is discussed in 1.3.5.

At the end of this chapter, we shall also give a short overview of the structure of this thesis, cf. Section 1.4.

1.2. Modelling and data-driven problems

In this section we briefly outline a family of problems appearing in continuum mechanics. For some problems we are able to derive a variational formulation, which is discussed from a mathematical viewpoint in Section 1.3.

This section is organised as follows: In the first part, we recall some basic notions of continuum mechanics, before going into some detail both for elasticity and fluid mechanics, essentially following [34, 114, 128, 33].

The second part of this section is concerned with two different approaches to obtain *material laws*, which are also referred to as *constitutive equations*. First, we revisit the classical modelling approach and discuss its advantages and disadvantages, cf. Section 1.2.6. Recently, a *data-driven approach* to problems in elasticity and plasticity has been advocated by several authors [87, 41, 40, 131]; we discuss its mathematical formulation and basic consequences in Section 1.2.7.

The mathematical analysis of the ensuing variational problems is the goal of this thesis. This is the focus of the next Section 1.3.

1.2.1. A mathematical formulation of continuum mechanics

We consider a body consisting of a material, which we first assume to be a solid. Mathematically, this body is described to be (the closure of) some open and Lipschitz bounded set $\Omega \subset \mathbb{R}^N$, where usually the space dimension is $N = 2, 3$. This set Ω is often referred to as the *reference configuration*.

If a force is applied to the material, it will deform into a new state. A classical example for this behaviour is a spring, which ideally might be seen as a one-dimensional object, i.e. an interval $\Omega = (0, L) \subset \mathbb{R}$. If we apply some force at its ends, it deforms to occupy a larger domain (and if we stop applying the force it goes back to its initial behaviour).

In this thesis, we are not interested in the behaviour-in-time of this material, but in the new *static equilibrium* after applying the force.

Question 1.1. *After applying a certain (external or internal) force to the material, how does the domain Ω change and where do points in the material move to?*

The change of the domain Ω is modelled as follows. The movement of each particle is described by a map $\varphi\colon \bar{\Omega} \to \mathbb{R}^N$, which is called the *deformation*. The deviation $u = \varphi - \mathrm{Id}$ from the trivial map $\varphi = \mathrm{Id}$ is called the *displacement*. The gradient of φ is called the *deformation gradient* $F = \nabla\varphi$.

The material is subjected to two types of forces. The *body forces* $f\colon \Omega \to \mathbb{R}^N$ are forces acting on each particle of the material. Such body forces usually comprise gravity or electromagnetic forces. *External forces* $g\colon \partial\Omega \to \mathbb{R}^N$ act on the boundary of Ω. Examples for such forces are pressure and centrifugal forces (cf. Figure 1.1).

Euler and Cauchy derived from Newton's principles of mechanics that for any subbody $U \subset \Omega$ the total force F_U exterted by $u(\Omega \setminus U)$ and $u(U)$ can be expressed in terms of

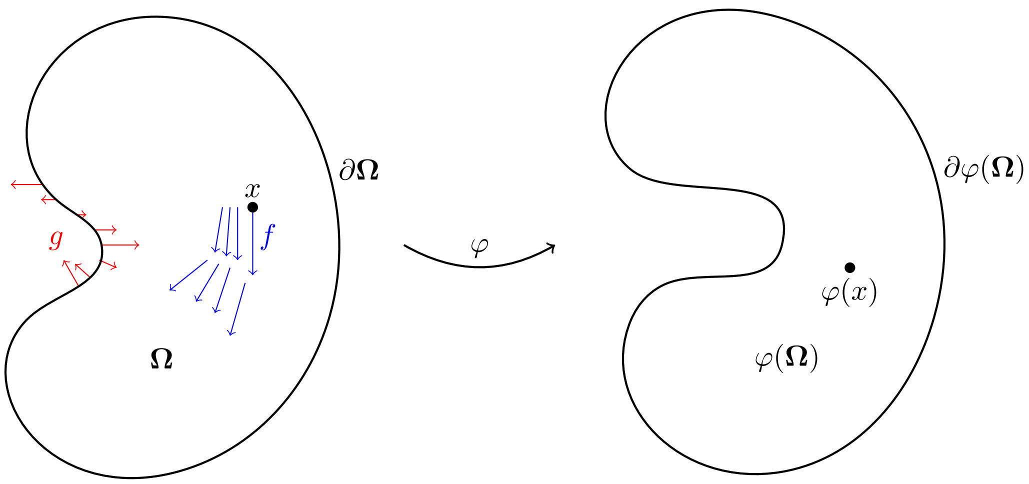

Figure 1.1.: This figure shows a material occupying a bounded Lipschitz set Ω, the forces f and g acting inside Ω and on $\partial\Omega$, respectively. Furthermore, we assume that $\partial\varphi(\Omega) = \varphi(\partial\Omega)$, i.e. particles on the boundary stay on the boundary and particles in the interior still are in the interior.

the Piola-Kirchhoff *stress tensor* σ as $F_U = \int_{\partial U} \sigma \cdot \nu \, \mathrm{d}\mathcal{H}^{N-1}$. Here, ν denotes the outer normal on U and $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. Moreover, for a body in *equilibrium*, the stress tensor satsifies

$$\begin{cases} \operatorname{div}(\sigma(x)) & = & f(x) & \text{in } \Omega, \\ \sigma(x) \cdot \nu & = & g & \text{on } \partial\Omega, \\ \nabla\varphi(x)\sigma(x)^T & = & \sigma(x)\nabla\varphi(x)^T & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, the first equation and second equation guarantee that balance of force inside and on the boundary of the domain. The third equation expresses the balance of angular momentum.

The identities (1.1) hold for *every* elastic body, independent of the material it is made of. To obtain a PDE for the deformation φ for the specific material we need a relation between φ and σ. This relation is referred to as the *material law* or *constitutive law.*

1.2.2. Elasticity

A material is called *elastic* if the stress $\sigma(x)$ only depends on x and on the value of the deformation $\nabla\varphi$. Hence, temporarily disregarding boundary values, we obtain

$$\begin{cases} \operatorname{div}(\sigma(x)) & = & f(x), \\ \sigma(x) & = & \hat{\sigma}(x, \nabla\varphi(x)), \\ \nabla\varphi(x)\sigma(x)^T & = & \sigma(x)\nabla\varphi(x)^T. \end{cases} \tag{1.2}$$

The relation

$$\sigma(x) = \hat{\sigma}(x, F)$$

for a deformation F is often called a *constitutive law* for the material. We further call it homogeneous if $\hat{\sigma}(x, F) = \hat{\sigma}(F)$.

An example of *elastic* behaviour is the previously mentioned spring that deforms under the application of forces. What one can observe (in a reasonable range of forces, cf. Subsection 1.2.4) is that the material follows *Hooke's law*, i.e. that the stress is approximately a linear function of the strain

$$\sigma(F) = CF, \quad C \in \mathbb{R}_+.$$

This linear relation (in higher dimension) is often referred to as *linear elasticity*.

1.2.3. Hyperelasticity

An elastic material is called *hyperelastic* if its constitutive law can be written as a derivative of a *potential* W in the second variable, i.e.

$$\sigma(x, F) = \frac{\partial W}{\partial F}(x, F).$$

The equations

$$\operatorname{div} \sigma = f(x), \quad \sigma(x) = \hat{\sigma}(x, \nabla \varphi(x))$$

can be seen as an Euler–Lagrange equation of a corresponding functional

$$I(\varphi) = \int_\Omega W(x, \varphi(x)) - f\varphi \, \mathrm{d}x. \tag{1.3}$$

In particular, any (sufficiently regular) minimiser of the functional I is a solution to the differential equation (1.2). We call I the *energy* of a deformation φ and W the *stored energy function*.

It is often easier to show that the functional I has a minimiser (see Section 1.3.3 than to solve the PDE (1.2) directly. For existence statements, for minimisation of the functional 1.3 one often assumes that W is convex. Convexity of the energy function W, however, often is incompatible with certain justified physical assumptions, in particular that the stored energy is frame-indifferent (i.e. $W(F) = W(RF)$ for a rotation $R \in \mathrm{SO}(N)$) and diverges if $\det(F)$ tends to zero.

The assumption of *hyperelasticity* is not unreasonable. Indeed, materials following the linear Hooke's law, admit the energy function $W_{\mathrm{H}}(F) = C/2|F|^2$. Other examples of hyperelastic materials are Ogden's, neo-Hookean materials and Mooney-Rivlin materials [34, 120], where, in addition to W_H, the energy functional also depends on the determinant or the cofactor matrix of F.

So far, we only considered a world of *perfect elasticity*. That is, a material deforms after application of force and, if we stop applying the force, it returns into its reference configuration. If the force is not too large, such a behaviour can be observed for many materials, including the example of a spring. However, if the force is too large, *plastic*

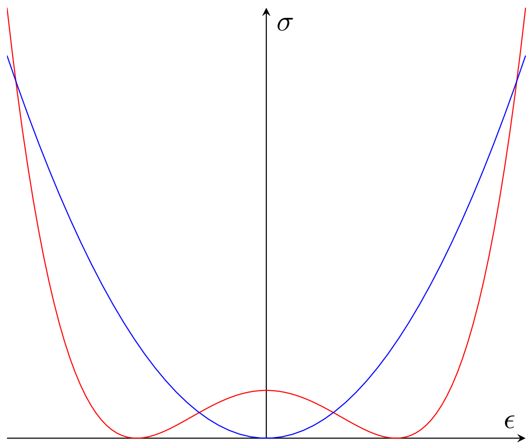

Figure 1.2.: Stored energy functions of a material following Hooke's law and a material with a two-well potential.

behaviour occurs: The material *fails* and we cannot return the reference configuration, even if we stop applying the force; the change is *irreversible*. For example, the atomic structure of the material changes. If even more force is applied to the material, *fracture* might occur and the material breaks up into two pieces.

Disregarding fractures, one might extend the elastic model into two regimes. Let K be a closed subset of $\mathbb{R}^{N\times N}$. If the strain tensor satisfies $(\nabla u(x) + \nabla^T u(x)) \in K$ (the elastic regime) for every $x \in \Omega$, then elastic behaviour occurs, else the deformation is plastic. We revisit a mathematical problem in elasto-plasticity in Section B; for the remainder of this introduction let us assume that the material is perfectly elastic and its constitutive law can be expressed by a stored energy function W.

1.2.4. Phase transitions and microstructures

Energy functionals of the form

$$J(u) = \int_\Omega W(Du(x))\,\mathrm{d}x$$

for a displacement $u = \varphi - \mathrm{id}$ also occur when describing the elastic behaviour of crystalline structures, for example alloys [53, 16]. The local minima of the energy function W can be seen as optimal microscopical states of the lattice. Let us assume that the initial state

$$D\varphi = \mathrm{id}$$

corresponds to a perfectly cubic lattice. Then, for example after a change in temperature, the material might prefer a different energy-optimising configuration, for example a rectangular lattice. This behaviour corresponds to the energy W having local minima on some diagonal matrices.

From a mathematical standpoint, we are interested in the following questions regarding the functional J:

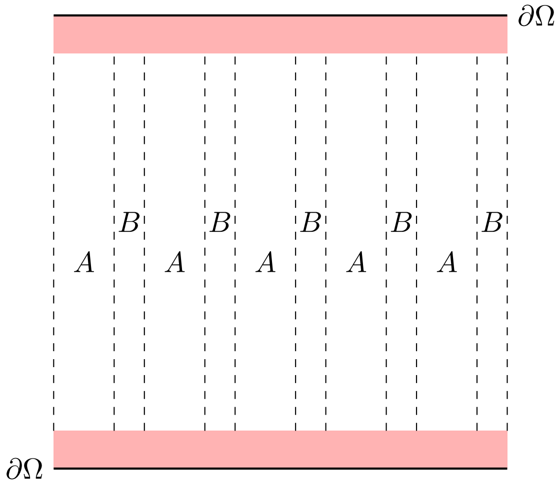

Figure 1.3.: Mathematical formation of a microstructure. In the red region close the boundary, the deformation gradient ∇u is fitting to the boundary condition (e.g. $u(x) = (2/3A + 1/3B)x$), whereas in the interior a microstrucuture consisting of several layers, where either $\nabla u = A$ or $\nabla u = B$, forms.

Question 1.2. *(a) Do minimisers to J exists?*

(b) Do energy-optimal minimisers to J exist? More explicitly: Renormalise the energy W, such that local minima F of W satisfy $W(F) = 0$. Depending on the prescribed boundary values, does there exist a deformation u, such that

$$J(u) = 0?$$

(c) If not, are there at least sequences such that $J(u_n) \to 0$?

Question (a) can be answered by employing the direct method in the calculus of variations, cf. Section 1.3.3 below. The answer to questions (b) and (c) often heavily depend on the prescribed boundary *Dirichlet* boundary to the material. It is worthwile mentioning, that the answer to (c) is of high releveance, both mathematically and physically. To see this in a brief argument, suppose that two matrices A and B are energy-minimising, i.e. $W(A) = W(B) = 0$ and $W > 0$ else (the so called two gradient problem, cf. [16]). For simplicity suppose that $A = -B \in \mathbb{R}^{2\times 2}$, $\Omega = (-1, 1)^2$ and $A = e_{11}$.

A short calculation gives that for zero boundary values $J(u) = 0$ *cannot* be attained, but that there is a sequence of functions u_n with $\lim_{n\to\infty} J(u_n) = 0$ (cf. Figure 1.3).

We see that there is an oscillation pattern between two different phases for u_n. Mathematically, the frequence of the oscillations diverges as $J(u_n) \to 0$. Physically, such an oscillation between two phases also can be observed (for example in Indium-Thallium or Copper-Aluminium-Nickel alloys), with various thickness of the layers (ranging from a thickness of atomic scale to several nanometers or even larger). The reason for this behaviour is that on a microscopic scale further effects come in, in particular the 'energy' of a configuration might not only depend on the first derivative of φ.

1.2.5. Fluid Mechanics

To study fluid mechanics, we need to employ an approach different to the one for solids. If we apply a constant force f to a solid, after some time we reach a new static equilibrium described by the deformation φ. In particular, a particle at a point $x \in \Omega$ in the reference configuration is moved to $\varphi(x)$ and the location of this specific particle does not change in time after reaching the new equilibrium.

For fluids and gases, this is not true. For example, if we start rotating a cylinder containing water, there will be no equilibrium for the displacement u and particles always move around.

Instead of the displacement u, we hence consider the velocity $v\colon \Omega \times [0,\infty) \to \mathbb{R}^N$, that describes the velocity at a point in Ω, i.e. a particle that is at $x_0 \in \Omega$ at time $t = t_0$ has velocity $v(x_0, t_0)$. In this description we might encounter steady states, meaning that the velocity at a point x is constant over time.

We model a fluid as a body occupying a domain $\Omega \subset \mathbb{R}^N$ (which, in this setting, we assume to be time-independent). We describe the behaviour of the fluid by

- the *velocity* field $v\colon \Omega \times [0,\infty) \to \mathbb{R}^N$;
- the *pressure* $\pi\colon \Omega \times [0,\infty) \to \mathbb{R}^N$;
- the *mass density* $\rho\colon \Omega \times [0,\infty) \to \mathbb{R}^N$.

Let us assume that the fluid has a *constant* density. Furthermore suppose that the fluid has a *linear relation* between the stress σ and the rate of strain ϵ. By the usual conservation laws for mass and momentum, and after a suitable non-dimensionalisation, one obtains the *incompressible Navier–Stokes equation* for Newtonian fluids

$$\begin{cases} \partial_t v + (v \cdot \nabla) v &= -\nabla \pi + \mu \operatorname{div} \sigma + f, \\ \operatorname{div} v &= 0, \\ \sigma &= -\pi \operatorname{id} + \mu \dfrac{\nabla v + \nabla^T}{2}, \end{cases} \tag{1.4}$$

where $\mu \in \mathbb{R}_+$ is the *viscosity*[1] of the fluid. Note that in this setting the force term $f\colon \Omega \times [0,\infty) \to \mathbb{R}^N$ may be time dependent. Furthermore, one needs to impose boundary conditions at the spatial boundary $\partial\Omega \times (0,T)$ and an initial condition at $t = 0$, which we shall omit here (cf. Chapter 5).

A *steady state* of the Navier-Stokes equation is a function $u\colon \Omega \to \mathbb{R}^N$, such that

$$v(t,x) = u(x)$$

is a solution to (1.4) with some time-independent force f. Note that for such a steady state, single particles still are in motion given by the velocity $u\colon \Omega \to \mathbb{R}^N$. But, fixing a location $x \in \Omega$ (and not focusing on a fixed particle that moves around), velocity and pressure are always constant in time.

[1] If we assume that the viscosity is 0, we recover the so called Euler equation.

Mathematically, for such u one obtains the *stationary Navier–Stokes equation* for a time-independent velocity field $u\colon \Omega \to \mathbb{R}^N$

$$\begin{cases} (u\cdot\nabla)u &= -\nabla\pi + \mu \operatorname{div}\sigma + f, \\ \operatorname{div} u &= 0, \\ \sigma &= -\pi \operatorname{id} + \mu \dfrac{\nabla u + \nabla^T}{2}. \end{cases} \tag{1.5}$$

Up to now, we assumed that the viscosity ν of the fluid does not depend on the velocity or the gradient of the velocity. That is, the *stress* depends linearly on the symmetric part of ∇v, i.e.

$$\sigma = -\pi \operatorname{id} + \nu\epsilon \text{ for a strain } \epsilon \in \mathbb{R}^{N\times N}_{\text{sym}} \quad \Longrightarrow \quad \sigma(x) = -\pi \operatorname{id} + \nu \frac{\nabla u(x) + \nabla^T u(x)}{2}.$$

Such fluids are called *Newtonian fluids* and, in reality, one can observe that this assumption for the viscosity is almost satisfied by water. Although it is reasonable in many practical applications to assume a fluid being Newtonian, real fluids are in fact non-Newtonian, i.e. they feature a non-linear relation between the stress σ and the rate of strain $\epsilon = \dfrac{\nabla u + \nabla^T u}{2}$. In mathematical terms,

$$\sigma = -\pi \operatorname{id} + \mu(|\epsilon|)\epsilon.$$

This μ is then called the *constitutive law* of the underlying fluid. The suitable version of the stationary Navier Stokes equation for *Non-Newtonian* fluids then reads

$$\begin{cases} (u\cdot\nabla)u &= -\nabla\pi + \operatorname{div}(\sigma) + f, \\ \sigma &= \mu(|\epsilon|)\epsilon, \\ \operatorname{div} u &= 0. \end{cases} \tag{1.6}$$

We hence are interested in the study of solutions to (1.6). This equation and its dynamic counterpart is well-studied in the Newtonian case where the function $\mu(\cdot)$ is constant. A widely-used Non-Newtonian constitutive relation is given by

$$\mu(|\epsilon|) = \mu_0 |\epsilon|^{\alpha-1}, \quad \alpha > 0, \tag{1.7}$$

and the corresponding fluid's are called *power-law fluids* or *Ostwald–de Waele fluids*. The exponent $\alpha > 0$ denotes the so-called *flow-behaviour exponent* and $\mu_0 > 0$ is the *flow consistency index*. In the case $0 < \alpha < 1$ the fluid exhibits a *shear-thinning* behaviour as its viscosity decreases with increasing shear-rate, while the fluid is called *shear-thickening* in the case $\alpha > 1$. In this case the viscosity is an increasing function of the shear rate.

To summarise, the behaviour heavily depends on the fluid's viscosity. Therefore, it is necessary to determine the correct *constitutive law*. Two approaches, namely to either deduce such a law from experimental data or to circumvent the use of a constitutive law and calculate solutions directly, are discussed in the following two sections.

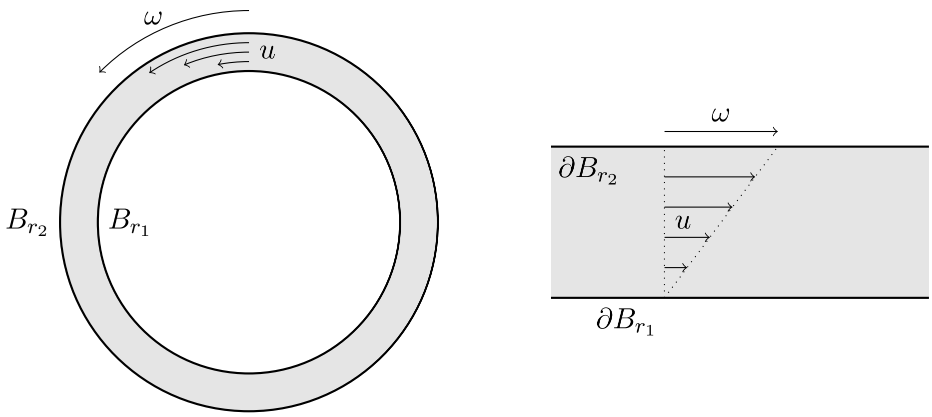

Figure 1.4.: Couette's experimental setup (on the left) and a zoomed-in 'flat' picture (on the right). The experiment consists of two cylinders with a fluid in between. The inner cylinder is at rest and the outer cylinder is moving at angular velocity ω. If this velocity is not too large, the fluid's velocity changes linearly between the two cylinders (right picture). The *viscosity* is calculated by measuring the force needed to rotate the cylinder, as the difference in the velocities u introduces a *shear force*. The higher the viscosity of the fluid, the more force needs to be applied to obtain an angular velocity ω. Furthermore, let us note that if ω is too large, the flow is *not* nicely circular as depicted on the right image, but turbulences occur. Therefore, the experiment only works within a range, where the velocity u has the form as in the right picture, cf. [143]. This range is connected to the thickness $(r_2 - r_1)/r_1$ of the fluid.

1.2.6. Classical Mathematical modelling

For problems in elasticity and for the stationary Navier–Stokes equation, we have so far assumed that at a point $x \in \Omega$ the *stress* σ can be written as a function of the *strain* ϵ (or the deformation gradient $\nabla\varphi$) and the point x. This lead, in the case of hyperelasticity, to a stored energy function W. For fluids, we may determine the viscosity ν dependent on ϵ.

The dependence of σ on the deformation heavily depends on the material. For example, for water a reasonable assumption is that it is Newtonian, i.e. the stress depends linearily on the strain. For other fluids, this is not true, in reality there are many shear-thinning and shear-thickening fluids (i.e. the viscosity decreases when applying shear force, or increases, respectively).

An example for an experimental setup determining the fluid's viscosity is the so called Couette-flow [43]. The experiment features two cylinders of radii r_1 and r_2 with the same center, where the inner cylinder is at rest and the outer cylinder is moving (cf. Figure 1.4).

So, for a given material, the behaviour of the stress has to be determined by experiments. The first step to get a material law $\hat{\sigma}$ is to do as many measurements as possible. As experiments and the equipment tend to be imperfect, a *measuring error* occurs and we cannot expect our final model to be more accurate than the experimental data.

After gathering enough experimental data, we model a function $\epsilon \mapsto \sigma$ that satisfies certain reasonable assumptions and is as close to the experimental data as we can get

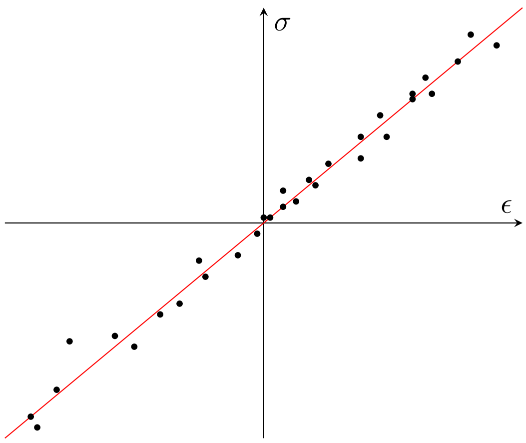

Figure 1.5.: A constitutive law (in red) derived from data points (ϵ, σ). Under a suitable assumption (e.g. that σ is linear in ϵ), the displayed law is closest to the experimental data in is then used in the PDE.

(cf. Figure 1.5). We then take this obtained *material law* and use it for our differential equation.

Such an approach obviously has both advantages and disadvantages. On the one hand, we obtain an explicit partial differential equation, which we might be able to solve (numerically). Moreover, with sensible assumptions on the function $\epsilon \mapsto \sigma$, one might be able to state results about existence and uniqueness, as well as stability of solutions. On the other hand, a drawback of this classical modelling ansatz is that we are exposed to two procedures, where an error might occur. First, the experimental equipment is imperfect. Second, also modelling errors may occur by prescribed assumptions on the map $\hat{\sigma}$.

1.2.7. Data-driven problems

The ability to process huge amounts of data has lead to another approach to tackle the problem of obtaining solutions to problems in continuum mechanics. In the following, we stick to the description of elastic materials.

The essential idea is to directly compute a solution that satisfies the physical laws, i.e. to find a displacement $u\colon \Omega \to \mathbb{R}^N$ and a stress σ that obeys $\operatorname{div} \sigma = f$. For such a displacement u, its strain ϵ and stress tensor σ we now determine how far it is away from the experimental data. We then take a triple of functions (u, ϵ, σ) which is closest to the experimental data.

The advantage of this approach is that we directly get solutions from raw experimental data, in particular, no modelling error occurs. So, in principle, the solution obtained by the data-driven approach should be more accurate than the solution that is calculated from the PDE after a further modelling step.

A mathematical analysis of such a procedure has been done mainly for problems in solid mechanics in [87, 41, 42, 40]. Chapter 5 is concerned with a mathematical analysis of ensuing problems for steady states in fluid mechanics.

Below we formulate the data-driven problem and pose some questions, that are then

answered in Chapter 5 in the context of fluid mechanics.
The experimental data is a set $\mathcal{D}$ of strain-stress pairs $(\epsilon, \sigma) \in \mathbb{R}^{N\times N}_{\text{sym}} \times \mathbb{R}^{N\times N}_{\text{sym}}$. The difficulty is to find a suitable distance function between the pair of functions $\epsilon, \sigma \colon \Omega \to \mathbb{R}^{N\times N}$ and the experimental data. Let us, as a suitable physical law, take ϵ, σ from elasticity [2] obeying

$$\begin{cases} \epsilon(x) &= \dfrac{\nabla + \nabla^T}{2} u(x), \\ \operatorname{div} \sigma(x) &= f(x). \end{cases} \tag{1.8}$$

The simplest way is to just measure the pointwise distance between the solution and the data and integrate over Ω, i.e. we aim to minimise

$$I(\epsilon, \sigma) = \begin{cases} \displaystyle\int_\Omega \operatorname{dist}((\epsilon(x), \sigma(x)), \mathcal{D}) \, \mathrm{d}x & \text{if } (\epsilon, \sigma) \text{ satisfies (1.8)}, \\ \infty & \text{else}, \end{cases} \tag{1.9}$$

for a suitable pointwise distance function dist on $\mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N}$. Before outlining (dis)advantages of this approach, let us shortly pose some mathematical questions arising from the formulation (1.9).

Question 1.3. *(a) Do minimisers to the data-driven problem exist?*

(b) Are minimisers unique for certain data?

(c) Is the data-driven approach consistent with the classic PDE approach? In other words, if the data set $\mathcal{D}$ is prescribed by a constitutive law, $\mathcal{D} = \{(\epsilon, \hat{\sigma}(\epsilon)\}$, is a solution a data-driven solution a solution to the PDE approach, and vice versa?

(d) Is there some form of convergence for data, such that the minimisation problems converge in a suitable sense?

In Chapter 5, which is based on joint work with C. Lienstromberg and R. Schubert [95], we discuss answers to most of these questions in the framework of fluid mechanics. In particular, Section 5.4 is concerned with defining a suitable notion of convergence of data in a pointwise manner. Essentially, two data sets converge to each other if the relative error in measurement goes to zero. In Section 5.5 we use this notion of data convergence to obtain results for the corresponding functional. Finally, consistency with the PDE approach is discussed in Section 5.6. For now, let us note that the data-driven approach is consistent for several constitutive laws like Newtonian or power-law fluids.

Before continuing with an abstract mathematical reformulation of these problems in Section 1.3, let us mention some advantages and disadvantages of the rather simple approach (1.9). On the one hand, given experimental data $\mathcal{D}$ and (ϵ, σ), it is very easy to write down and calculate the functional I. Such a functional I fits into a rather general abstract setting (cf. Section 1.3.1), that is further discussed in Section 1.3.3.

[2] For fluid mechanics one needs to add the inertia term and, in addition, assumes incompressibility of u.

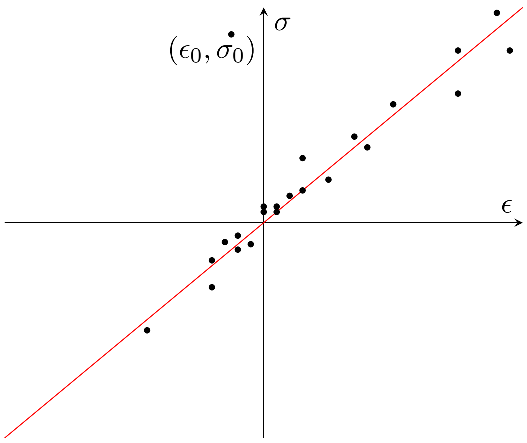

Figure 1.6.: A data set with a single outlier (ϵ_0, σ_0). A sensible modeling approach would esentially ignore this faulty measurement and still give a linear constitutive law. The simple data driven approach in 1.9 has problems with such points.

On the other hand, the approach is simplified and, in reality, often not adjusted to real experimental data. Essentially, we assume that our experimental equipment is very accurate. Moreover, for the notion of data convergence, we need that the relative measurement error tends to zero. Such assumptions are not justified in reality. For example consider experimental data as in Figure 1.6.

From a practical standpoint, one might assume that the point (ϵ_0, σ_0) is just a erroneous measurement and, in the modelling approach to obtain a PDE, the faulty measurement is compensated by the fact that there are many accurate measurements for a similar strain. However, if we do not want to artificially throw out data, the data-driven functional does not distinguish the single data point (ϵ_0, σ_0) and the cluster of data points contradicting the single outliner. Even worse, in the simple model (1.9), being close to the *single* outlier (ϵ_0, σ_0) is equally good as being close to many points.

To circumvent this problem, there are approaches, where either single outliers are ignored (cf. [131]), or a more probabilistic approach to a functional is undertaken (cf. [40]). For the remainder of this thesis, especially in Chapter 5, we however stick to the simplified setting (1.9).

1.3. Mathematical Methods

We now discuss some mathematical techniques to tackle the problems presented in Section 1.2. First, we formulate an abstract version of constrained minimisation problems in Section 1.3.1. Section 1.3.2 focuses on describing the underlying *differential constraint* and discussing some elementary result.

Section 1.3.3 returns to the minimisation problem introduced in 1.3.1. In particular, we recall the *direct method* of the calculus of variations which guarantees existence of minimisers and the crucial requirements to apply this method, namely *weak lower-semicontinuity* and *coercivity*. Applying these abstract results to the setting of distance functions introduced for data-driven problems, we also get a generalised version of convexity for sets, which is discussed in Section 1.3.4. Finally, in Section 1.3.5, we present a *truncation* technique, which is designed to cope with the notion of convexity for sets; these truncation results are major results of the whole thesis.

1.3.1. Constrained minimisation problems

Let us consider a time-independent physical quantity $u\colon \Omega \to \mathbb{R}^d$. For such u we define the functional I as

$$I(u) = \begin{cases} \displaystyle\int_\Omega f(x, u(x))\,\mathrm{d}x & \text{if } u \in \mathcal{C} \\ \infty & \text{else,} \end{cases} \tag{1.10}$$

where

- $I\colon L^p(\Omega, \mathbb{R}^d) \to \mathbb{R}$; we often call $L^p(\Omega, \mathbb{R}^d)$ the *phase space*;
- $\mathcal{C}$ is a set consisting of functions u in the phase space satisfying a certain physical *constraint*, for example (1.8) with $u(x) = (\epsilon(x), \sigma(x))$, or $u = \nabla U$ for a displacement U.
- $f\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ (or $\to [0, \infty)$) is a function locally describing the 'energy' of the physical state u.

To summarise, a minimiser of I is a function satisfying the constraint $u \in \mathcal{C}$ and satisfies

$$I(u) = \inf_{v \in \mathcal{C}} I(v).$$

The main focus is to solve problems of the type described in Section 1.2, i.e. we are interested in the following questions

Question 1.4. *(Q1) Do minimisers exist? Can we further characterise certain properties of minimisers?*

(Q2) How do ***approximate minimisers*** *(i.e. sequences u_n with $I(u_n) \to \inf_{u \in \mathcal{C}} I(u)$) look like? Can we say something about their weak limits?*

(Q3) Can we rewrite ('relax') the functional I, such that it is clearly visible, which functions are weak limits of approximate minimisers?

Before answering questions (Q1)–(Q3), we need to specify the constraint set $\mathcal{C}$. In general, we distinguish between two types of constraints

(a) u needs to satisfy certain boundary conditions;

(b) u satisfies a *differential constraint*. Usually, this comprises

  (b1) $u = \mathcal{B}U$ for some differential operator $\mathcal{B}$ (e.g. $u = \nabla U$)– a *potential constraint*;

  (b2) $\mathcal{A}u = 0$ for a differential operator $\mathcal{A}$ – an *annihilating constraint*.

In the following Section 1.3.2 we see that (b1) and (b2) are essentially equivalent and can be treated in parallel fashion, provided that $\mathcal{A}$ and $\mathcal{B}$ satisfy a technical condition (see also Chapter 2).

1.3.2. Constant rank operators

Let $\mathcal{A}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d) \to C^\infty(\mathbb{R}^N,\mathbb{R}^l)$ be a differential operator that is homogeneous of order $k_\mathcal{A} = k$ and has constant coefficients, i.e.

$$\mathcal{A}u = \sum_{|\alpha|=k} A_\alpha \partial_\alpha u, \tag{1.11}$$

for linear operators $A_\alpha \in \mathrm{Lin}(\mathbb{R}^d,\mathbb{R}^l)$. Murat introduced the following condition on the differential operator $\mathcal{A}$ [119, 137].

Definition 1.5. *For an operator $\mathcal{A}$ as in (1.11) we define the Fourier symbol $\mathbb{A}(\xi) \in \mathrm{Lin}(\mathbb{R}^d,\mathbb{R}^l)$ for $\xi \in \mathbb{R}^N$ as*

$$\mathbb{A}(\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha.$$

*The operator $\mathcal{A}$ is said to satisfy the **constant rank property** if for all $\xi \in \mathbb{R}^N \setminus \{0\}$ the rank of $\mathbb{A}(\xi)$ is constant, i.e. there is $r \in \mathbb{N}$, such that*

$$\dim\ker \mathbb{A}(\xi) = r \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

The Fourier symbol reduces the partial differential operator to an operator acting on functions of one variable; i.e. if $u(x) = v_0\varphi(\xi x)$ for some direction $\xi \in \mathbb{R}^N \setminus \{0\}$, $v_0 \in \mathbb{R}^d$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$, then $u \in \ker \mathcal{A}$ if and only if $v_0 \in \ker \mathbb{A}(\xi)$.

The 'strength' of the constant rank condition for the constrained minimisation problems comes from the fact that it implies several properties. Indeed, the constant rank property for an operator $\mathcal{A}$ is equivalent to the following conditions (cf. [123, 80]):

(a) The existence of a *potential* $\mathcal{B}$ to the differential operator $\mathcal{A}$ (i.e. an operator, such that for functions with average 0 defined on the torus $\mathcal{A}u = 0 \Leftrightarrow u = \mathcal{B}U$), cf. Theorem 2.6;

(b) The existence of an *annihilator* $\mathcal{A}'$ to the differential operator $\mathcal{A}$ (i.e. an operator, such that for functions with average 0 defined on the torus $\mathcal{A}u = v \Leftrightarrow \mathcal{A}'v = 0$), cf. Theorem 2.6;

(c) The existence of a nice projection operator onto the kernel of the differential operator for functions on the torus, cf. Theroem 2.9.

The study of constrained minimisation problems of the type (1.10) has a long history. It has been mainly explored for the differential constraint $u = \nabla U$ (equivalent to $\operatorname{curl} u = 0$) (e.g. [110, 112, 86, 85, 45, 159]). A guiding question for this thesis may be formulated informally as follows.

Question 1.6. *If we extend the minimisation problem from $\mathcal{A} = \operatorname{curl}$ to general differential operators, which properties remain true?*

In the context of weak lower-semicontinuity (cf. the following Subsection 1.3.4) the constant rank property is enough [65]. However, it is not clear whether the constant rank property is enough for other questions.

Question 1.7. *(a) Is there a Poincaré lemma for (topologically trivial) open domains, i.e. a statement á la $\mathcal{A}u = 0 \Rightarrow u = \mathcal{B}U$ not only on the torus, but also for open domains in $\mathbb{R}^N$?*

(b) Is the constant rank property sufficient for truncation statements in the style of Subsection 1.3.5 below?

We return to the second question in Subsection 1.3.5 and focus on the first one for the moment. It is well known that for any curl-free function $u \in C^1(\Omega, \mathbb{R}^N)$ on a simply connected and bounded set Ω one may find a potential $U \colon \Omega \to \mathbb{R}$, such that $u = \nabla U$; i.e. any closed 1-form is also exact. If Ω fails to be simply connected, the question, whether some function u can be written as $u = \nabla U$, also depends on a *topological* feature of Ω, namely the fundamental group of Ω. In this thesis, for simplicity we only consider question (a) in the setting, where Ω is a cube.

Even restricting to this setting, the constant rank property does not suffice to give such a Poincaré lemma. Indeed, there is a simple degenerate counterexample: The operator $\mathcal{B} = 0$ is a potential of any elliptic differential operator, e.g. for the Laplacian $\mathcal{A} = \Delta u$. On the one hand, $\operatorname{Im} \mathcal{B} = \{0\}$ only consists of one function and hence is 0-dimensional. On the other hand, $\ker \mathcal{A}$ is infinite-dimensional.

The situation is still not perfect if, instead of taking $\mathcal{A} = \Delta$, we take $\mathcal{A} = \nabla^k$ to be the k-th gradient. Still $\operatorname{Im} \mathcal{B} = \{0\}$, but now $\ker \mathcal{A}$ is finite-dimensional as it consists of polynomials of degree $\leq k-1$.

The main difference between $\mathcal{A} = \Delta$ and $\mathcal{A} = \nabla^k$ is that the Laplacian is only $\mathbb{R}$-elliptic, but the k-th gradient is $\mathbb{C}$-elliptic (cf. Definition 1.8 below). Applying this knowledge for elliptic operators to general constant rank operators, we come to the following definition. Let us define the complex Fourier symbol $\mathbb{A}(\xi) \in \operatorname{Lin}(\mathbb{C}^d, \mathbb{C}^l)$ as

$$\mathbb{A}(\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha, \quad \xi \in \mathbb{C}^N.$$

To be precise, $A_\alpha \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l)$ can be written as

$$A_\alpha(w) = \sum_{i,j} a_\alpha^{ij} w_j e_i, \quad a_\alpha^{ij} \in \mathbb{R},$$

which can be naturally extended to an operator in $\operatorname{Lin}(\mathbb{C}^d, \mathbb{C}^l)$, using that $a_\alpha^{ij} \in \mathbb{R} \subset \mathbb{C}$.

Definition 1.8 (Constant rank in $\mathbb{C}$)**.** *The operator $\mathcal{A}$ is said to satisfy the **complex constant rank property** if for all $\xi \in \mathbb{C}^N \setminus \{0\}$ the rank of $\mathcal{A}(\xi)$ is constant, i.e. there is*

$r \in \mathbb{N}$ *such that*

$$\dim_{\mathbb{C}} \ker_{\mathbb{C}} \mathbb{A}(\xi) = r \quad \forall \xi \in \mathbb{C}^N \setminus \{0\}.$$

An operator is called $\mathbb{C}$*-elliptic, if it has constant rank in* $\mathbb{C}$ *with* $r = 0$.

A Poincaré lemma is known whenever the potential $\mathcal{B}$ is $\mathbb{C}$-elliptic [82]. Indeed, one of the main observations is the following [138, 76]:

Proposition 1.9. *Let* $\mathcal{B}$ *be a differential operator of constant rank in* $\mathbb{R}$*. Then the following are equivalent*

1. $\mathcal{B}$ *is* $\mathbb{C}$*-elliptic;*
2. *The kernel of* $\mathcal{B}$ *on an open connected set is finite-dimensional and consists of polynomials;*
3. *There exists a differential operator* $\tilde{\mathcal{B}}$ *and* $k \in \mathbb{N}$*, such that* $\nabla^k = \tilde{\mathcal{B}} \circ \mathcal{B}$*.*

In Section 2.6, which is based on joint work with F. Gmeineder, [77] we extend this result to the setting of constant rank operators.

Theorem 1.10 (Section 2.6, [77])**.** *Let* $\mathcal{B}_1, \mathcal{B}_2$ *be two differential operators with constant rank in* $\mathbb{C}$ *and* $\Omega \subset \mathbb{R}^N$ *be open, bounded and connected. Then the following are equivalent*

1. $\ker \mathbb{B}_1(\xi) = \ker \mathbb{B}_2(\xi)$ *for all* $\xi \in \mathbb{C}^N \setminus \{0\}$*;*
2. *The kernels of* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *for functions in* $L^2(\Omega, \mathbb{R}^m)$ *only differ by finite-dimensional spaces, i.e. there are finite-dimensional* X_1 *and* X_2*, such that*
$$\ker \mathcal{B}_1 \cap L^2(\Omega, \mathbb{R}^m) + X_1 = \ker \mathcal{B}_1 \cap L^2(\Omega, \mathbb{R}^m) + X_1;$$
3. *There exist differential operators* $\tilde{B}_1$ *and* $\tilde{B}_2$ *and* $k_1, k_2 \in \mathbb{N}$*, such that*
$$\nabla^{k_1} \circ \mathcal{B}_1 = \tilde{B}_2 \circ \mathcal{B}_2, \quad \nabla^{k_2} \circ \mathcal{B}_2 = \tilde{B}_1 \circ \mathcal{B}_1.$$

For a Poincaré lemma this means the following. If the sequence

$$\mathbb{C}^m \xrightarrow{\mathbb{B}(\xi)} \mathbb{C}^d \xrightarrow{\mathbb{A}(\xi)} \mathbb{C}^l$$

is exact, then possibly we may find a 'natural' annihilator $\tilde{A}$ with the same kernel in Fourier space, but minimal kernel as an operator acting on $L^2(\Omega, \mathbb{R}^d)$; i.e.

$$\ker \tilde{A} \cap L^2(\Omega, \mathbb{R}^d) = \bigcap \big\{ \ker \bar{\mathcal{A}} \cap L^2(\Omega, \mathbb{R}^d) \colon \ker(\bar{\mathbb{A}}(\xi) = \ker \mathbb{A}(\xi) \ \forall \xi \in \mathbb{C}^N \setminus \{0\} \big\}$$

Still one needs to prove a Poincaré lemma for $\tilde{A}$ specifically. This is done in Section 2.6.5 in the special case of space dimension $N = 2$ (also see [8]). In the general setting this is still an open question.

1.3.3. The direct method and weak lower-semicontinuity

The direct method

After the study of the constant-rank property, let us come back to the minimisation problem (1.10). A well-known and powerful technique is to apply the direct method, which was developed in the beginning of the 20^{th} century and, most notably, studied by HILBERT and TONELLI. Abstractly, this method is described as follows:

Proposition 1.11 (The direct method on Banach spaces). *Let X be a reflexive Banach space and $I\colon [-\infty,\infty]$ be a functional on X. Suppose that*

(DM1) $\inf_{x\in X} I(x) < \infty$ *(**the infimum is not** $+\infty$);*

(DM2) $\inf_{x\in X} I(x) > -\infty$ *(**bound from below**);*

(DM3) $\lim_{\|x\|\to\infty} I(x) = \infty$ *(**coercivity**);*

*(DM4) I is **sequentially weakly lower-semicontinuous**, i.e. if $x_n \rightharpoonup x$ in X, then*

$$I(x) \leq \liminf_{n\to\infty} I(x_n).$$

Then I has a minimiser x^*, i.e. $I(x^*) = \inf_{x\in X} I(x^*)$.

A short argument, why the direct method works, is as follows. Take a sequence x_n, such that $I(x_n) \to \inf I(x) \in \mathbb{R}$ as $n \to \infty$; the existence of such a sequence is ensured by (DM1) and (DM2). Coercivity (DM3) ensures that this sequence is a bounded sequence. Reflexivity of the Banach space yields that there is a subsequence x_{n_k} that converges weakly to some x^*. Consequently, due to weak lower-semicontinuity $I(x^*) \leq \liminf_{k\to\infty} I(x_{n_k}) = \inf I$, and we can conclude that x^* is a minimiser.

In this thesis we work on functionals defined on L^p (or sometimes $L^p \times L^q$, or on Sobolev spaces $W^{k,p}$), which are reflexive as long as $1 < p < \infty$ (and $1 < q < \infty$). Using weak-$*$ convergence, one may extend certain results to $p = \infty$. The case $p = 1$ is very different, as just boundedness coming from coercivity does not yield weak compactness of minimising sequences. One needs to redefine the functional on the space of measures (or on BV, BD etc.), cf. [7, 64, 14, 10].

Let us focus on $1 < p < \infty$. For the functional 1.10 properties (DM1) and (DM2) are usually fairly easy to check. Coercivity (DM3) and weak lower-semicontinuity (DM4) are non-trivial properties. If the functional is given by $I(u) = \int_\Omega f(x, u(x))\,\mathrm{d}x$ it is usually assumed that f satisfies p-growth from above and from below, i.e. there are constants $C_1, C_2 > 0$ such that for almost every $x \in \Omega$ and every $v \in \mathbb{R}^d$

$$\frac{1}{C_1}|v|^p - C_1 \leq f(x,v) \leq C_2(1 + |v|^p). \tag{1.12}$$

In this work, we weaken the pointwise coercivity condition (the bound from below in (1.12)); for now let us just refer to the lower-bound in (1.12) as **strong pointwise coercivity** and focus on weak lower-semicontinuity first.

The easiest case for a functional like (1.10) is provided by setting the operator $\mathcal{A}$ to 0; i.e. one arrives at the *unconstrained* functional $J\colon L^p(\Omega, \mathbb{R}^d) \to \mathbb{R}$ given by

$$J(u) = \int_\Omega f(x, u(x))\,\mathrm{d}x.$$

If we consider oscillating sequences $u_n = u(nx)$ for some $\mathbb{Z}^N$-periodic $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ (which converge weakly to the mean of u on $(0,1)^n$), a short calculation gives that the map $u \mapsto f(x,u)$ must be convex for almost every $x \in \Omega$. Indeed, convexity of $f(x,\cdot)$ for almost every x is equivalent to weak lower-semicontinuity [144, 63].

Quasiconvexity

Historically, the next step was to consider the differential constraint $\mathcal{A} = \mathrm{curl}$, i.e. one studies the functional

$$I_\nabla(U) = \int_\Omega f(x, \nabla U(x))\,\mathrm{d}x, \quad U \in W^{1,p}(\Omega, \mathbb{R}^m)$$

and rewrites it in terms of $u = \nabla U$. Convexity of $f\colon \Omega \times \mathbb{R}^{N\times m} \to \mathbb{R}$ in the second variable is sufficient for weak lower-semicontinuity, but indeed not neccessary if $N, m \geq 2$. Morrey [110] introduced the notion of *quasiconvexity* and showed that this is indeed equivalent to weak lower-semicontinuity as long as the integrand f has both p-growth from above and below, (1.12).

Definition 1.12. *A measurable and locally bounded function $f\colon \mathbb{R}^{N\times m} \to \mathbb{R}$ is called* ***quasiconvex****, if for all $\Psi \in C^\infty(T_N, \mathbb{R}^m)$ and all $A \in \mathbb{R}^{N\times m}$*

$$f(A) \leq \int_{T_N} f(A + \nabla\Psi(x))\,\mathrm{d}x. \tag{1.13}$$

f is called ***quasiaffine*** *or a* ***Null-Lagrangian*** *if both f and $-f$ are quasiconvex, i.e.* (1.13) *is satisfied with equality.*

The inequality (1.13) may be seen as a generalised form of Jensen's inequality for convex function; any convex function is automatically quasiconvex. If either $N = 1$ or $m = 1$, also any quasiconvex function is convex. This is not true if both dimensions are larger than 2.

In fact, given a function f, it is not easy to check that it is quasiconvex. For convex functions $g \in C^2(\mathbb{R}^{N\times m})$ there is a simple local condition to check whether g is positive, namely that $D^2 g$ is positive semidefinite; such a condition does not exist for quasiconvexity [145, 92]. For applications we often rely on the following two notions of convexity:

1. **Rank-one convexity** of f as a necessary condition: This means that for each rank-one matrix B and any $A \in \mathbb{R}^{N\times m}$ the function $t \mapsto f(A + tB)$ is convex. This is a

necessary, but in general not sufficient condition for quasiconvexity [145]. In 2×2 dimensions it is still an open question whether rank-one convexity is equivalent to quasiconvexity.

2. **Polyconvexity** of f as a sufficient condition: One considers functions of the form $f(x) = h(M(x))$, where $h\colon \mathbb{R} \to \mathbb{R}$ is a convex function and $M\colon \mathbb{R}^d \to \mathbb{R}$ is quasiaffine.

These different conditions are highlighted by a very instructive example of DACOROGNA and MARCELLINI [6, 48, 45]. Taking $N = m = 2$ and

$$f(A) = |A|^4 - \gamma |A|^2 \det(A)$$

one obtains that

- f is convex, iff $|\gamma| \leq \frac{4}{3}\sqrt{2}$;
- f is polyconvex, iff $|\gamma| \leq 2$;
- f is quasiconvex, iff $|\gamma| \leq 2 + \varepsilon$ for some $\varepsilon > 0$;
- f is rank-one convex, iff $|\gamma| \leq \frac{4\sqrt{3}}{3}$.

$\mathcal{A}$-quasiconvexity

We may replace the differential condition $u = \nabla v$, which is locally equivalent to $\operatorname{curl} u = 0$, by any differential constraint $\mathcal{A}u = 0$ for a constant rank operator $\mathcal{A}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$. Corresponding to quasiconvexity we get the notion of $\mathcal{A}$-quasiconvexity [65].

Definition 1.13 ($\mathcal{A}$-quasiconvexity)**.** *Let $\mathcal{A}$ be a constant rank operator and $f\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable and locally bounded function. f is called **$\mathcal{A}$-quasiconvex** if for all $\psi \in C^\infty(T_N, \mathbb{R}^d)$ with average 0 satisfying the constraint $\mathcal{A}\psi = 0$ and all $v \in \mathbb{R}^d$*

$$f(v) \leq \int_{T_N} f(v + \psi(x))\,\mathrm{d}x. \tag{1.14}$$

*If both f and $-f$ are $\mathcal{A}$-quasiconvex, f is called **$\mathcal{A}$-quasiaffine**. For $f \in C(\mathbb{R}^d)$ we define the **$\mathcal{A}$-quasiconvex hull/envelope** of f as*

$$\mathcal{Q}_{\mathcal{A}} f(v) = \inf\left\{ \int_{T_N} f(v + \psi(y))\,\mathrm{d}y \colon \psi \in C^\infty(T_N, \mathbb{R}^d),\ \psi \in \ker \mathcal{A},\ \int \psi = 0 \right\}.$$

Let us note that $\mathcal{Q}_{\mathcal{A}} f$ is the largest $\mathcal{A}$-quasiconvex that is below f (cf. [65]).

FONSECA and MÜLLER [65] established that indeed this notion of $\mathcal{A}$-quasiconvexity is sufficient and necessary for weak lower-semicontinuity of the functional I, provided that the operator has order one and the function f has p-growth.

In Chapter 4, we give a proof of this equivalence in a setting of higher order operators. This is an extension of the results of the author's master's thesis [133]. In contrast to earlier works [65, 85, 86], the proof is not based on abstract result on Young-measures, but rather on the construction of explicit sequences.

Theorem 1.14. *Let $\mathcal{A}$ be a constant rank operator, $1 < p < \infty$ and $f \in C(\mathbb{R}^d)$ satisfying*

$$0 \leq f(v) \leq C(1 + |v|^p).$$

Then the functional I is weakly lower-semicontinuous if and only if f is $\mathcal{A}$-quasiconvex.

Relaxation and integral-coercivity

The direct method fails, if the functional I is not weakly lower-semicontinuous, i.e. whenever f is not $\mathcal{A}$-quasiconvex. Hence, a minimiser does not need to exist; we are however still interested in the behaviour of *approximate* minimisers, i.e. sequence u_n satisfying

$$\lim_{n\to\infty} I(u_n) = \inf_{u \in L^p} I(u).$$

If I satisfies the coercivity conditon (DM3), we may conclude that a subsequence of u_n converges to some u^*. This map u^* does not need to be a minimiser of I, but is still of interest in some physical application (cf. Section 1.2.5).

The *relaxation* (or sequentially weakly continuous envelope) of the functional I is designed to characterise such u^*. It is abstractly defined via

$$I^*(u) = \inf_{u_n \rightharpoonup u} \liminf_{n\to\infty} I(u_n). \tag{1.15}$$

Then a function u^* is a minimiser of I^* if and only if it there is a sequence u_n with $u_n \rightharpoonup u^*$ and $I(u_n) \to \inf_{u \in L^p} I(u) = \inf_{u \in L^p} I^*(u)$.

While (1.15) gives a formula of the relaxation for any I and any u, one may ask for a condition that guarantees that the infimum in (1.15) is a minimum, i.e. it is attained by some sequence u_n, which we also call *recovery sequence.* A standard technique is to first show that the relaxation in (1.15) exists and then impose the coercivity condition

$$f(v) \geq C_1 |v|^p - C_2$$

which ensures that any sequence $u_{n,\varepsilon}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, satisfying

$$\liminf_{n\to\infty} I(u_{n,\varepsilon}) < I^*(u) + \varepsilon$$

is uniformly bounded in L^p [25]. Hence, taking a suitable diagonal sequence (which is possible, as the weak topology is metrisable on bounded sets), one finds $u_{n,\varepsilon(n)}$ still converging weakly to u^* and satisfying

$$\lim_{n\to\infty} I(u_{n,\varepsilon(n)}) = I^*(u^*).$$

With a careful construction of the recovery sequence, it is possible to weaken the coercivity statement to the following notion of *$\mathcal{A}$-integral coercivity.* That is, for all

$\psi \in C^\infty(T_N, \mathbb{R}^d)$ with average 0 satisfying $\mathcal{A}\psi = 0$ and all $v \in \mathbb{R}^d$ and almost every $x \in \Omega$, we have

$$\int_{T_N} f(x, v + \psi(y))\,\mathrm{d}y \geq C_1 \int_{T_N} |\psi(y)|^p\,\mathrm{d}y - C_2(1 + |v|^p). \tag{1.16}$$

In Section 4.4 we prove that this is enough to ensure the existence of a recovery sequence, i.e.

Theorem 1.15. *Let $1 < p < \infty$, $\mathcal{A}$ be a constant rank operator and let f be a Carathéodory function satisfying p-growth from above and the coercivity condition* (1.16)*. Then for any $u \in L^p(\Omega, \mathbb{R}^d)$ there is a recovery sequence u_n weakly converging to u in L^p and satisfying*

$$\liminf_{n\to\infty} I(u_n) = I^*(u).$$

Moreover, we have the following formula for the relaxed functional I^*:

$$I^*(u) = \begin{cases} \int_\Omega \mathcal{Q}_\mathcal{A} f(x, u(x))\,\mathrm{d}x & \text{if } \mathcal{A}u = 0 \\ \infty & \text{else.} \end{cases}$$

This extends relaxation results obtained in [25]. In particular, we construct explicit recovery sequence that satisfies the statement of Theorem 1.15. This construction allows us to cover the weaker coercivity condition (1.16).

$\mathcal{A}$-quasiconvexity and coercivity in minimisation problems

So far, we discussed $\mathcal{A}$-quasiconvexity and suitable coercivity condition as a tool to apply the direct method in an abstract setting. We want to apply this rather abstract knowledge to the physical setting, introduced in Section 1.2.

First of all, let us note that the $\mathcal{A}$-quasiconvexity condition (1.14) is *not* trivial to verify for a given function f for the same reasons as given for (curl)-quasiconvexity. Once again we introduce a necessary and a different sufficient condition for $\mathcal{A}$-quasiconvexity:

1. **$\Lambda_\mathcal{A}$-convexity** of f as a necessary condition: This means that for any w in the characteristic cone $\Lambda_\mathcal{A} = \bigcup_{\xi\in\mathbb{R}^N\setminus\{0\}} \ker \mathbb{A}(\xi) \subset \mathbb{R}^d$ and any $v \in \mathbb{R}^d$ we have

$$t \mapsto f(v + tw)$$

is convex. Note that for $\mathcal{A}$=curl the characteristic cone comprises only rank-one matrices, so $\Lambda_\mathcal{A}$-convexity is the generalisation corresponding to rank-one-convexity;

2. **$\mathcal{A}$-polyconvexity** of f as a sufficient condition. That is, that f can be written as $f(v) = g(h(v))$ for a convex function $g \in C(\mathbb{R})$ and an $\mathcal{A}$-quasiaffine function $h \in C(\mathbb{R}^d)$.

The notion of $\mathcal{A}$-*polyconvexity* necessitates a closer study of $\mathcal{A}$-quasiaffine functions. In the setting $\mathcal{A} =$ curl it was established that all curl-quasiaffine functions are linear combination of minors, cf. [112, 126, 38, 46]. Following [15, 79, 119], we establish some necessary and sufficient condition for $\mathcal{A}$-quasiaffinity in Chapter 3.

Apart from verifying $\mathcal{A}$-quasiconvexity, it is also quite hard to show the integrated coercivity condition (1.16) for some given f. It is easy to see that 'classical coercivity'

$$f(v) \geq C_1|v|^p - C_2,$$

is stronger, i.e. sufficient for (1.16); but it is also too restrictive for some of the settings we would like to study. However, it is possible to modify the classical coercivity condition by an $\mathcal{A}$-quasiaffine function M, i.e.

$$f(v) \geq C_1|v|^p - C_2 - M(v) \tag{1.17}$$

for an $\mathcal{A}$-quasiaffine function M. Such a condition still implies integral coercivity (if M has at most p-growth). It is however much easier to check the pointwise condition (1.17) in contrast to the integral coercivity (1.16) and therefore, we wil mainly work with a condition similar to (1.17) in Section 5.

1.3.4. $\mathcal{A}$-quasiconvex sets

In this subsection, we introduce the notion of $\mathcal{A}$-quasiconvexity and $\mathcal{A}$-quasiconvex hull for sets. First, we state the definition and justify the name $\mathcal{A}$-quasi*convex* set. After this, we shortly motivate this notion in terms of two physical problems already discussed in Section 1.2. Finally, we raise an interesting question regarding these sets, which provides the motivation for the second part of this thesis, consisting of Chapters 6, A and B.

Definition and relation to convex sets

Definition 1.16. *Let $1 \leq p < \infty$ and $K \subset \mathbb{R}^d$ be a closed set. The $\mathcal{A}$-p-quasiconvex hull of K is defined as*

$$K^{(p)} = \{\mathcal{Q}_{\mathcal{A}} \operatorname{dist}^p(\cdot, K) = 0\}.$$

If $p = \infty$, we define the $\mathcal{A}$-∞-quasiconvex hull of K via

$$K^{(\infty)} = \{x \in \mathbb{R}^d \colon \forall f \in C(\mathbb{R}^d) \text{ that are } \mathcal{A}\text{-quasiconvex and } f_{|K} \leq 0 \text{ also } f(x) \leq 0\}.$$

First of all, let us mention that the definition of $K^{(p)}$ does not depend on the exact definition of the distance function, but only depends on the set K and the behaviour of $\operatorname{dist}^p(y, K)$ if the Euclidean distance between y and K tends to ∞. Moreover, the set $K^{(\infty)}$ may be seen as a natural limit object of $K^{(p)}$, provided that the set K satisfies some reasonable growth conditions, e.g. is compact.

The name 'convex hull' is justified by Minkowski's/Hahn-Banach's separation theorem.

On the one hand, for a closed set K, one can define the convex hull by considering convex combinations of points. On the other hand, these separation theorems allow us to characterise convex hulls by separating hyperplanes. That is, every point which is not in the convex hull of $K \subset \mathbb{R}^d$ can be separated by a $(d-1)$ hyperplane from K, which is between K and the point.

This geometric statement can be restated in terms of functions as follows. There exists an *affine* map, which is ≤ 0 on K and strictly positive in the point we aim to separate. Weakening the condition of affinity to *convexity* does not change the shape of the set. Hence, a characterisation of the *complement* of the convex hull reads as

$$y \notin K^{**} \text{ if } \exists f \text{ convex with } f_{|K} \leq 0 \text{ and } f(y) > 0.$$

Adjusting this characterisation to fit to the convex hull and replacing the property of convexity by $\mathcal{A}$-*quasiconvexity* restores the definition of $K^{(\infty)}$. Moreover, one can show that for compact sets that the *convex* hull K^{**} is also characterised by

$$K^{**} = \left\{(\operatorname{dist}^p(\cdot, K))^{**} = 0\right\}.$$

So, in terms of convexity (that represents the constraint $\mathcal{A} = 0$), both definitions of convex hulls $K^{(p)}$ and $K^{(\infty)}$ coincide.

$\mathcal{A}$-quasiconvex sets in data-driven problems

In the deterministic data-driven approach, cf. Section 1.2.7, we consider an integrand of the form

$$f(x, v) = \operatorname{dist}^p(v, K_x)$$

for a suitable closed set $K_x \subset \mathbb{R}^d$. Such an integrand may of course also appear in the classical formulation, as seen in the context of hyperelasticity and microstructures. For the treatment in this section, let us also assume that $K = K_x$, i.e. that f is not dependent on the first coordinate $x \in \Omega$.

Ideally, a data set coincides with a set given by a reasonable material law. Hence, a minimiser of the corresponding functional I,

$$I(u) = \begin{cases} \int_\Omega \operatorname{dist}^p(u(x), K)\, \mathrm{d}x & \text{if } \mathcal{A}u = 0, \\ \infty & \text{else,} \end{cases}$$

is a classical solution for the PDE with underlying material law, and, vice versa, any solution to the PDE is a minimiser.

We observed in Section 1.2.7, however, that it might be more natural to consider the relaxed functional I^* for the macroscopic behaviour of minimisers. Minimisers of I^* a priori do not need to be minimisers of I and hence no solution to the underlying PDE.

If we want to compare minimisers of I^* to classic PDE solutions, we need to consider the set

$$\{\mathcal{Q}_{\mathcal{A}} \operatorname{dist}^p(\cdot, K) = 0\}$$

instead.

$\mathcal{A}$-quasiconvex sets in microstructures

A prominent example, where curl-quasiconvex sets appear, is in the theory of microstructures for crystals, cf. Section 1.2.4. We have raised the question, which boundary conditions allow for appearances of microstructures with energy converging to 0. Indeed, for affine boundary conditions $u(x) = Fx$, $F \in \mathbb{R}^{N \times N}$, this question can be answered using the notion of $\mathcal{A}$-quasiconvex sets.

Proposition 1.17 (cf. Lemma 6.3)**.** *Let $K \subset \mathbb{R}^{N \times N}$ be compact, $K = \{W = 0\}$ for $W \in C(\mathbb{R}^d, [0, \infty))$. Suppose that W approximately grows like a squared distance function, i.e.*

$$C_1 \operatorname{dist}^2(y, K) - C_2 \leq W(y) \leq C_3 \operatorname{dist}^2(y, K).$$

Then $\inf J(\varphi) = 0$ *for prescribed boundary conditions $u(x) = Fx$ if and only if the matrix F is in the* curl-2-*quasiconvex hull of u, $F \in K^{(2)}$.*

This proposition foreshadows one of the essential questions of this thesis. One might ask, how this set of nice affine boundary conditions changes, if the growth behaviour of the stored energy W varies. This question will be discussed in more detail after the presentation of some examples.

Examples of $\mathcal{A}$-quasiconvex sets and hulls

For certain specific examples, the $\mathcal{A}$-quasiconvex hull can be explicitly computed. As for the notion of $\mathcal{A}$-quasiconvexity for functions, let us remark that for an arbitrary set it is highly non-trivial to find its $\mathcal{A}$-quasiconvex hull and one mainly reduces to an upper- and a lower bound for the hull (as it was the case for functions). This will be discussed extensively in Chapter 6. Let us now shortly outline certain sets, for which at least partial results on the hulls are known.

(a) The so called 'two gradient problem': $K = \{A, B\}$. In this case the behaviour of the hull depends on $A - B$. If $A - B$ is in the characteristic cone of $\mathcal{A}$, then $K^{(\infty)} = \{\lambda A + (1 - \lambda)B, \lambda \in [0, 1]\}$ is just the convex hull, else $K^{(\infty)} = K$ (cf. [16, 50]).

(b) The three gradient problem ($\mathcal{A} = \text{curl}$): $K = \{A, B, C\}$. This has been studied by Šverák [148]. If *no* rank-one connections occur, then the hull $K^{(\infty)}$ coincides with K.

(c) The four gradient problem (K consists of four matrices and $\mathcal{A} = \text{curl}$), where other effects than in the two previous cases may occur. The absence of rank-one connections does in general not imply that $K^{(\infty)} = K$, see ([13, 30, 142, 22] for specific counterexamples and [32, 122, 62] for a more general analysis.

(d) The one-well problem: $K = \mathrm{SO}(N)$, $\mathcal{A} = \mathrm{curl}$. Then $K^{(\infty)} = K$ and moreover, a stronger *rigidity result* hold, which is a statement of the following form: If a function is almost a minimiser to $I(u) = \int_\Omega \mathrm{dist}^2(u, K)$, then it is already close to a minimiser (a constant function) in L^2, [84, 125, 71].

(e) The two-well problem $K = A\mathrm{SO}(N) \cup B\mathrm{SO}(N)$ for $\mathcal{A} = \mathrm{curl}$ has been studied in some special cases in $N = 2, 3$ by [147, 146, 104, 54, 31] and multi-well problems, e.g. [37].

(f) The set of conformal matrices $K = \mathbb{R}_+\mathrm{SO}(N)$ (for $\mathcal{A} = \mathrm{curl}$) is an example for a very interesting behaviour of hulls for non-compact sets. The basic observation is that the hull $K^{(p)}$ coincides with K whenever p is large enough, but $K^{(p)} = \mathbb{R}^{N\times N}$ for p small enough, [152, 116]. Such behaviour will be further examined in Section 6.3 in a geometrically linear setting.

(g) In [41], the authors studied a non-compact set K, which corresponds to a counterpart of the two-well problem in problem for geometrically linear elasticity in the data-driven setting, and its $\mathcal{A}$-quasiconvex hull.

(h) In Section 5.6, we will see some quasiconvex sets in a non-linear setting (with more than one exponent p) arising from common consitutive laws in fluid mechancis.

Main question: Dependence on p

One of the main question of this thesis is the following.

Question 1.18. *Given $K \subset \mathbb{R}^d$ closed, how does the set $K^{(p)}$ depend on the exponent p?*

The aim of Chapters 6, A, B is to give an answer to this question at least in some special cases. The analysis of this question further bifurcates into the treatment of compact and non-compact sets K.

If K is compact, then we obtain the following results:

- For any constant rank operator we have that $K^{(p)} = K^{(q)}$ for for any $1 < p, q < \infty$, cf. [42] for a special case, [20] and Section 6.2.1 for general constant rank operators $\mathcal{A}$.
- If $\mathcal{A} = \mathrm{curl}$, then $K^{(p)} = K^{(q)}$ for any $1 \leq p, q \leq \infty$, which goes back to Zhang [157, 159, 158], based on results in [96, 1].
- In this thesis, we show that $K^{(p)} = K^{(q)}$ for $1 \leq p \leq \infty$ whenever the operator $\mathcal{A}$ satisfy a certain truncation property, which is further elucidated in Subsection 1.3.5 immediately below. In Chapter 7/A, based on the publication [134], we show

that this truncation property holds for closed differential forms, including divergence-free fields. With a similar technique, the same result is obtained for divergence-free symmetric matrices in dimension 3×3 in Chapter B, which closely follows [20] and is summarised in Chapter 8. This solves a question raised in [42] in the context of a model for stress relaxation in amorphous silicia glasses.

The example of conformal matrices shows that the situation for unbounded sets is different. One important observation is that in the case of compact sets any distance function satisfies the classical coercivity condition

$$\operatorname{dist}^p(v, K) \geq C_1 |v|^p - C_2,$$

which cannot be true for unbounded sets. Instead, one has to rely on other notions of coercivity for the compact set, e.g. $\mathcal{A}$-integral coercivity.

As an example, we deal with a geometrically linear example, which includes the data-driven two-well problem from [41] previously outlined (g). In particular, we are able to show that if K is close to a special linear subspace L in a suitable sense, then the $\mathcal{A}$-quasiconvex hulls coincide whenever $1 < p, q < \infty$. For more details, we refer to Section 6.3.

1.3.5. Constrained truncation results

Truncations and $\mathcal{A}$-quasiconvex hulls

Let us summarise the basic idea behind proving that $K^{(1)} = K^{(\infty)}$ in order to motivate the following truncation statement. The inclusion $K^{(\infty)} \subset K^{(1)}$ is trivial, as the $\mathcal{A}$-quasiconvex hull $\mathcal{Q}_{\mathcal{A}} \operatorname{dist}(\cdot, K)$ satisfies all the assertions needed for functions in the definition of $K^{(\infty)}$. The other inclusion turns out to need a truncation statement. Indeed, if $y \in K^{(1)} = \{\mathcal{Q}_{\mathcal{A}} \operatorname{dist}^1(\cdot, K) = 0\}$, we know by Definition 1.13 that there is a sequence of test functions, such that

$$\lim_{n \to \infty} \int_{T_N} \operatorname{dist}^1(y + u_n(x)) \,\mathrm{d}x = 0. \tag{1.18}$$

However, for this sequence we *cannot* infer that for any continuous, $\mathcal{A}$-quasiconvex function $f \in C(\mathbb{R}^d, [0, \infty))$ vanishing on K

$$\lim_{n \to \infty} \int_{T_N} f(y + u_n(x)) \,\mathrm{d}x = 0. \tag{1.19}$$

Indeed, if we can find such a sequence obeying both (1.18) and (1.19), it follows that $y \in K^{(\infty)}$. By employing Lebesgue's dominated convergence theorem, one observes that it would be enough if a sequence satisfies (1.18) and $|f(y + u_n)| \leq C$. The latter is in particular satisfied whenever u_n is uniformly bounded in L^∞. Hence, we are able to formulate a problem whose solution yields a positive answer to Question 1.18.

Question 1.19. *For a constant rank operator $\mathcal{A}$, is it possible to find a truncation as follows: Given $u \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ and some $R > 0$, can we modify u, such that its modification v obeys*

- $\|v\|_{L^\infty} \leq CR$*;*
- $\mathcal{A}v = 0$*;*
- $\|u - v\|_{L^1}$ *is small, whenever* $\operatorname{dist}(u, K)$ *is small for some compact set* K*?*

If the answer to the question is positive, then we can establish that $K^{(1)} = K^{(\infty)}$ for any compact set K.

Lipschitz truncactions, curl-free truncations and beyond

Question 1.19 is answered in the setting $\mathcal{A} = \operatorname{curl}$ in the works [96, 1, 157] via a slightly different viewpoint. Instead of truncating a sequence u_n obeying $\operatorname{curl} u_n = 0$, one might also view u_n as a sequence of gradients $u_n = \nabla U_n$. The condition $u_n \subset L^\infty$ then is equivalent to $U_n \subset W^{1,\infty}$, which is equivalent to U_n being Lipschitz (continuous).

Lipschitz truncation or Lipschitz extension theorems are well-known, the most famous are the Kirszbraun/McShane extension theorem [90, 107]. On a subset of a metric space any Lipschitz function with values in $\mathbb{R}$ or $\mathbb{R}^d$ can be extended in such a way, that it is Lipschitz with the same Lipschitz constant on the whole metric space. Such an extension result can be modified to obtain a truncation as follows: Given some $U \in W^{1,1}(T_N, \mathbb{R})$ divide T_N into a good set X, where U is nicely Lipschitz, and a small bad set X^C. Then replace U on the bad set by an extension of $U_{|X}$.

For general truncations subject to differential constraints, McShane's extension theorem is not suitable[3]. Instead, we employ a *Whitney type extension*, which is far more geometric and allows us to adjust the truncation to differential operators.

With such a Whitney-type construction, which might also be useful to tackle other problems (e.g. [26, 28]), we then are able to prove a truncation theorem answering Question 1.19. In the setting $\mathcal{A} = \operatorname{div}$, such a truncation is stated as follows.

Theorem 1.20. *Let $u \in L^1(T_N, \mathbb{R}^N)$ satisfy $\operatorname{div} u = 0$ in the sense of distributions and let $R > 0$ be fixed. Then there exists $v \in L^1(T_N, \mathbb{R}^N)$ and a purely dimensional constant $C = C(N)$, such that*

(a) $\|\mathcal{A}v\|_{L^\infty} \leq CR$*;*

(b) $\operatorname{div} v = 0$*;*

(c) $\|v - u\|_{L^1} \leq C \int_{\{|u| \geq R\}} |u| \,\mathrm{d}x$*;*

[3] McShane's extension theorem works on metric spaces. For general differential operators (for example higher gradients) one at least needs a geometric structure, e.g. a Riemannian manifold. Such a structure is only used by Whitney's extension theorem

(d) $\mathcal{L}^N(\{u \neq v\}) \leq CR^{-1} \int_{\{|u| \geq R\}} |u| \, \mathrm{d}x.$

Following the work [134] we show the validity of Theorem 1.20 in Chapter A and outline the essential ideas in its summary, chapter 7. It appears as a byproduct of a generalised version stated for closed differential forms. A similar statement for the truncation of symmetric divergence-free matrices is then shown in Chapter B, following [20]. This is summarised in Chapter 8.

1.4. Overview

Let us finish the introduction with a concise overview of this thesis. It is based on the research works [134, 135, 20, 77, 95] and builds on some results of the author's master's thesis [133]. These works have already been mentioned in the introductory sections 1.2 and 1.3. We point to the suitable source at the beginning of each chapter.

First of all, in Chapter 2, we gather information about constant rank differential operators. Sections 2.1–2.4 focus on the constant rank property in $\mathbb{R}$, whereas 2.5 and 2.6 are concerned with the constant rank property in $\mathbb{C}$.

In order to consider $\mathcal{A}$-quasiconvex functions, it is very useful to first study the easier notion of $\mathcal{A}$-quasiaffine functions. We derive several equivalent conditions for a function to be $\mathcal{A}$-quasiaffine in Chapter 3. Properties of $\mathcal{A}$-quasiconvex functions and their relevance for weak-lower semicontinuity results are studied in Chapter 4.

The abstract knowledge that is obtained in Chapters 2, 3 and 4 is used to study a data-driven problems in fluid dynamics, cf. Chapter 5.

The second part of this thesis focuses on the notion of $\mathcal{A}$-quasiconvex sets and hulls. Chapter 6 gives an overview of results in the regime of compact sets. Moreover, we further examine an example of non-compact sets in Section 6.3. As it is outlined in Section 1.3.5, the results for compact sets are shown via a truncation result. As these are quite technical, the proofs are split between the last two chapters. In Chapter A contained in the appendix, we show the validity of the truncation statement for closed differential forms. This chapter is summarised in Chapter 7.

Chapter B is concerned with the truncation for divergence-free symmetric matrices which is summarised in Chapter 8.

Notation

Throughout this thesis, we use the following notation.

Linear Algebra

- $\mathrm{Lin}(V, W)$ is the space of linear maps from a vector space V to W;
- For $L \in \mathrm{Lin}(V, W)$, $\ker L$ is the kernel of the linear map and $\mathrm{Im}\, L$ is the image;
- For $X \subset V$, $\mathrm{span}\, X$ is the span of all vectors in X;
- For a normed vector space V we denote by V^* the dual space of V.

Derivatives and multiindices

- We call $\alpha \in \mathbb{N}^N$, $\alpha = (\alpha_1, ..., \alpha_N)$ a **multiindex**;
- For a multiindex α we define $|\alpha| = \sum_{i=1}^{N} \alpha_i$;
- For $\xi \in \mathbb{R}^N$ and a multiindex α we have $\xi^\alpha = \prod_{i=1}^{N} \xi_i^\alpha \alpha_i$;
- For $k \in \mathbb{N}$ write $[k] = \{1, ..., k\}$.

Function spaces

- $L^p(\Omega, \mathbb{R}^l)$ is the space of all functions $u \colon \Omega \to \mathbb{R}^l$, such that $|u|^p$ is integrable;
- $W^{k,p}(\Omega, \mathbb{R}^l)$ is the space of all functions, such that the first k weak derivatives are in L^p;
- $\mathcal{D}(\Omega, \mathbb{R}^l) = C_c^\infty(\Omega, \mathbb{R}^l)$;
- $\mathcal{D}'(\Omega, \mathbb{R}^l)$ is the space of all distributions;
- For a function space X of integrable functions on a finite-measured set Ω, we denote by $X_\#$ the subspace of $u \in X$, such that $\int_\Omega u = 0$,
- For a function $f \in C(\mathbb{R}^N)$ we denote $\mathrm{spt}(f)$ as the closure of $\{f \neq 0\}$. If f is not continuous, we use $\mathrm{spt}(f) \subset A$ to indicate that $f = 0$ almost everywhere in A^C.

Other notation

- For a sequence x_n consisting of elements in some set X we shortly write $x_n \subset X$;
- $\Omega \subset \mathbb{R}^N$ denotes, if not stated otherwise, an open and bounded set in $\mathbb{R}^N$ (in many chapters it is also assumed to have Lipschitz boundary);
- $A \subset\subset B$ for $A, B \subset \mathbb{R}^N$ denotes that A is compactly contained in B, i.e. $\bar{A} \subset B^\circ$;
- T_N denotes the N-torus;
- $B_\rho(x)$ denotes the open ball around a point x with radius ρ;
- $\mathcal{L}^N$ denotes Lebesgue measure and, for a set $X \subset \mathbb{R}^N$,

$$|X| := \mathcal{L}^N(X);$$

- For a measure μ on $\mathbb{R}^N$ and a μ-measurable set $A \subset \mathbb{R}^N$ with $0 < \mu(A) < \infty$ define the average integral of a μ-measurable function f via

$$⨍_A f \,\mathrm{d}\mu = \frac{1}{\mu(A)} \int_A f \,\mathrm{d}\mu.$$

2. Constant rank operators

This chapter is split into two different parts.
First of all, we summarise some important facts about differential operators satisfying the constant rank property. We point to the exact reference, when it is suitable. Mainly, we follow the preliminary sections of

- [133]: Schiffer, S., *Data-driven problems and generalised convex hulls in elasticity* Master's thesis,
- [95]: Lienstromberg, C., Schiffer, S. and Schubert, R. *A data-driven approach to incompressible viscous fluid mechanics – the stationary case.*

In the second part, we argue that the constant rank condition in $\mathbb{R}$ is enough for minimisation problems, but is too weak to guarantee other properties, for example a Poincaré lemma. Therefore, we introduce the notion of constant rank in $\mathbb{C}$ and discuss some important properties of those operators in Section 2.6.1. Up to minor changes, the remaining part of Section 2.6 coincides with the publication

- [77] Gmeineder, F. and Schiffer, S.: *Natural annihilators and operators of constant rank over* $\mathbb{C}$.

2.1. Introduction

In this chapter, we gather results about the differential constraints that are discussed in the introduction to this thesis. We consider a homogeneous differential operator $\mathcal{A}$ with constant coefficients. That is a differential operator $\mathcal{A}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d) \to C^\infty(\mathbb{R}^N,\mathbb{R}^l)$ of order $k = k_{\mathcal{A}}$ given by

$$\mathcal{A}u = \sum_{|\alpha|=k} A_\alpha \partial_\alpha u, \tag{2.1}$$

where $A_\alpha \in \mathrm{Lin}(\mathbb{R}^d,\mathbb{R}^l)$ are linear maps. Murat and Wilcox [119, 137] advocated the constant rank property as a useful condition to classify these operators. Recall that for $\xi \in \mathbb{R}^N \setminus \{0\}$ we define the Fourier symbol of $\mathcal{A}$ by

$$\mathbb{A}[\xi] = \sum_{|\alpha|=k} A_\alpha \xi^\alpha \in \mathrm{Lin}(\mathbb{R}^d,\mathbb{R}^l). \tag{2.2}$$

Definition 2.1. *(a) We say that the operator $\mathcal{A}$ satisfies the* ***constant rank property*** *if the Fourier symbol has constant rank in $\xi \in \mathbb{R}^N \setminus \{0\}$, i.e. there is $r \in \mathbb{N}$, such* *that*

$$\dim \ker \mathbb{A}[\xi] = r \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{CRP}$$

(b) We call the set

$$\Lambda = \Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker \mathbb{A}[\xi]$$

the ***characteristic cone*** *of $\mathcal{A}$.*

(c) We say that $\mathcal{A}$ satisfies the ***spanning property*** *if the characteristic cone of $\mathcal{A}$ spans up $\mathbb{R}^d$, i.e.*

$$\operatorname{span} \Lambda_{\mathcal{A}} = \mathbb{R}^d. \tag{SP}$$

Example 2.2 (Examples of constant rank operators). (a) The null-operator $\mathcal{A}\colon u \mapsto 0$ has constant rank, as $\ker \mathbb{A}[\xi] = \mathbb{R}^d$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$.

(b) Elliptic operators (in the sense of second order equations, cf. [59, 74, 5]) have constant rank. In particular, $\ker \mathbb{A}[\xi] = \{0\}$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$ if $\mathcal{A}$ is elliptic.

(c) Likewise, the operator $\mathcal{A} = \nabla^k$ (the k-th gradient) also satisfies $\ker \mathbb{A}[\xi] = \{0\}$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$.

(d) The rotation is the differential operator $\operatorname{curl}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^{m\times N}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^m \otimes \mathbb{R}^{N\times N}_{\text{skew}})$ defined by

$$(\operatorname{curl} u)_{ij} = \partial_i u_j - \partial_j u_i, \quad i,j \in \{1, ..., N\}$$

is a constant rank operator. Given $\xi \in \mathbb{R}^N \setminus \{0\}$, note that

$$\ker \mathbb{A}[\xi] = \{a \otimes \xi \colon a \in \mathbb{R}^m\}$$

and therefore the characteristic cone of $\mathcal{A}$ consists entirely of rank-one matrices.

(e) The so called Saint-Venant compability condition (see also Chapter 5)

$$\operatorname{curl}\operatorname{curl}^T \colon C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N}_{\text{sym}}) \to C^\infty(\mathbb{R}^N, (\mathbb{R}^N)^4)$$

defined by

$$\left(\operatorname{curl}\operatorname{curl}^T u\right)_{ijkl} = \partial_{ij} u_{kl} + \partial_{kl} u_{ij} - \partial_{il} u_{kj} - \partial_{kj} \partial_{il}$$

has constant rank. For $\xi \in \mathbb{R}^N \setminus \{0\}$ we have

$$\ker \mathbb{A}[\xi] = \{a \odot \xi \colon a \in \mathbb{R}^N\}$$

and therefore only symmetrised rank-one matrices are the characteristic cone.

(f) The divergence operator $\operatorname{div}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^{m\times N} \to \mathbb{R}^m)$ given by

$$\operatorname{div} u = \sum_{i=1}^{N} \partial_i u_i$$

satisfies the constant rank property and

$$\ker \mathbb{A}[\xi] = \{A \in \mathbb{R}^{m\times N} \colon A \cdot \xi = 0\},$$

i.e. the space of matrices with rank $\leq N-1$ is the characteristic cone. Likewise, the divergence operator applied to symmetric matrices also is a constant rank operator (cf. Chapter B).

2.2. Constant rank operators on the torus

In this section, we gather results about constant rank operators on the torus. This relies on classical Fourier analysis for periodic functions. Note that the constant rank property is formulated as a condition on the Fourier transform of the operator. The constant rank property is therefore the reason for the following results on the torus.

Definition 2.3 (Potentials). *A constant rank operator $\mathcal{B}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)$ is called the* ***potential*** *of $\mathcal{A}$ if for all $\xi \in \mathbb{R}^N \setminus \{0\}$ we have*

$$\operatorname{Im} \mathbb{B}[\xi] = \ker \mathbb{A}[\xi]. \tag{2.3}$$

Likewise, if (2.3) *is satisfied, then $\mathcal{A}$ is called an* ***annihilator*** *of $\mathcal{B}$.*

The definition of potentials can be rewritten as follows. $\mathcal{B}$ is a potential of $\mathcal{A}$ if for all $\xi \in \mathbb{R}^N \setminus \{0\}$

$$\mathbb{R}^m \xrightarrow{\mathbb{B}[\xi]} \mathbb{R}^d \xrightarrow{\mathbb{A}[\xi]} \mathbb{R}^l$$

is an exact sequence.

Example 2.4 (Potential-annihilator pairs).

(a) If $\mathcal{B} = \nabla$, then $\mathcal{A} = \operatorname{curl}$ is the annihilator of $\mathcal{B}$.

(b) Likewise, for the k-th gradient $\mathcal{B} = \nabla^k$, there exists a first-order annihilator (which we shall call $\operatorname{curl}^{(k)}$, cf. [109].

(c) If $\mathcal{A}$ is an $\mathbb{R}$-elliptic operator, i.e. $\ker \mathbb{A}[\xi] = \{0\}$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$, then $\mathcal{B} = 0$ is a potential of $\mathcal{A}$.

(d) For the symmetric gradient $\mathcal{B} = \dfrac{\nabla + \nabla^T}{2}$, the Saint-Venant condition $\mathcal{A} = \operatorname{curl}\operatorname{curl}^T$ is a annihilator.

(e) In dimension $N = 3$, the rotation curl is (after a suitable identification of $\mathbb{R}^{3\times 3}_{\text{skew}}$ to $\mathbb{R}^3$) a potential to $\mathcal{A} = \operatorname{div}$.

(f) In general, the exact sequence of exterior derivatives

$$0 \longrightarrow \mathbb{R}^N \xrightarrow{d[\xi]} \mathbb{R}^N \wedge \mathbb{R}^N \xrightarrow{d[\xi]} \mathbb{R}^N \wedge \mathbb{R}^N \wedge \mathbb{R}^N \xrightarrow{d[\xi]} \dots$$

provides several potential-annihilator pairs.

On the torus, the algebraic condition (2.3) provides us with a nice characterisation of potentials in terms of functions on the torus.

Theorem 2.5 (Potentials on the torus [123, 80]). *Let $\mathcal{A}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d) \to C^\infty(\mathbb{R}^N,\mathbb{R}^l)$ and $\mathcal{B}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^m) \to C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ be two differential operators of order $k_\mathcal{A}$ and $k_\mathcal{B}$, respectively, that obey the constant rank property. The following are equivalent:*

(a) $\mathcal{B}$ is a potential of $\mathcal{A}$.

(b) The sequence

$$W^{k_\mathcal{B},p}_\#(T_N,\mathbb{R}^m) \xrightarrow{\mathcal{B}} L^p_\#(T_N,\mathbb{R}^d) \xrightarrow{\mathcal{A}} W^{-k_\mathcal{A},p}_\#(T_N,\mathbb{R}^l)$$

is exact ***for some*** $1 < p < \infty$.

(c) The sequence

$$W^{k_\mathcal{B},p}_\#(T_N,\mathbb{R}^m) \xrightarrow{\mathcal{B}} L^p_\#(T_N,\mathbb{R}^d) \xrightarrow{\mathcal{A}} W^{-k_\mathcal{A},p}_\#(T_N,\mathbb{R}^l)$$

is exact ***for all*** $1 < p < \infty$.

In particular, Theorem 2.5 means that for $1 < p < \infty$ if $v \in L^p_\#(T_N,\mathbb{R}^d)$ satisfying $\mathcal{A}v = 0$ there is $u \in W^{k_\mathcal{B},p}_\#(T_N,\mathbb{R}^m)$ with $\mathcal{B}u = v$ and

$$\|u\|_{W^{k_\mathcal{B},p}} \le C\|v\|_{L^p}. \tag{2.4}$$

Let us remark that due to Ornstein's non-inequality [121] such a bound is not possible in general for $p = 1$ and $p = \infty$.

Quite recently, Raiţă proved that having a potential is equivalent to the constant rank property.

Theorem 2.6 (Potentials and constant rank properties). *Let $\mathcal{A}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d) \to C^\infty(\mathbb{R}^N,\mathbb{R}^l)$ be a differential operator with constant coefficients of order k. The following are equivalent:*

(a) $\mathcal{A}$ satisfies the constant rank property.

(b) There is a differential operator $\mathcal{B}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^m) \to C^\infty(\mathbb{R}^N,\mathbb{R}^d)$ that is a potential of $\mathcal{A}$.

(c) There is a differential operator $\mathcal{A}'\colon C^\infty(\mathbb{R}^N,\mathbb{R}^l) \to C^\infty(\mathbb{R}^N,\mathbb{R}^{l'})$ that is an annihilator of $\mathcal{A}$.

Proof. For the proof we refer to [123], other methods to prove Theorem 2.6 have been employed in [12, 124]. □

The following definition is of importance for weak lower-semicontinuity results in Chapter 4. Weak convergence of sequences on bounded domains is due to two effects: oscillations and concentrations. The notion of *equi-integrability* allows us to classify sequences, where *no* concentrations occur.

Definition 2.7 (p-equi integrablity)**.** *Let $\Omega \subset \mathbb{R}^N$ (or $\Omega = T_N$) be a bounded and open set, and $X \subset L^p(\Omega, \mathbb{R}^d)$ for $1 \leq p < \infty$. X is called p-equi-integrable if*

$$\lim_{\varepsilon \to 0} \sup_{u \in X} \sup_{|E|<\varepsilon} \int_E |u|^p \, \mathrm{d}x = 0.$$

If $p = 1$, we call 1-equi-integrable sequences just equi-integrable.

Remark 2.8. (a) The notion of p-equi-integrability essentially means that there cannot be a concentration of the L^p mass, e.g. for fixed $u \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^d)$ the bounded sequence

$$u_n(x) = n^{N/p} u(x)$$

is not p-equi-integrable, as mass concentrates around 0.

(b) For a bounded and open set Ω, a subset of $L^1(\Omega, \mathbb{R}^d)$ is weakly compact if and only if it is bounded and equi-integrable (cf. [23, Thm. 4.7.18]).

In the context of minimisation and weak convergence on L^p spaces ($1 < p < \infty$), we want to avoid concentrations and focus on oscillations; i.e. we aim to consider p-equi-integrable sequences only. Lemma 4.11 below justifies ignoring concentrations. Hence, in the following, we want to modify sequences u_n satisfying the present differential constraint $\mathcal{A}u_n = 0$ to some $\tilde{u}_n$ still obeying the differential constraint which is close to u_n in some norm, but now p-equi-integrable.

We first recall the projection theorem on the torus [65, 79].

Theorem 2.9 (Projections on the torus)**.** *Let $\mathcal{A}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ be a constant rank operators. Then there exists a projection operator P with the following properties:*

(a) P is a bounded, linear map, $P\colon L^p(T_N, \mathbb{R}^d) \to L^p(T_N, \mathbb{R}^d)$ for any $1 < p < \infty$;

(b) $P \circ P = P$;

(c) $\mathcal{A} \circ P = 0$;

(d) There is $C = C(p)$, such that for any $u \in L^p(T_N, \mathbb{R}^d)$ we have

$$\|u - Pu\|_{L^p} \leq \|\mathcal{A}u\|_{W^{-k,p}};$$

(e) P maps p-equi-integrable sets into p-equi-integrable sets.

This projection operator is defined as follows: For $\xi \in \mathbb{R}^N \setminus \{0\}$ let us define $\mathbb{P}(\xi)$ to be the orthogonal projection onto $\ker \mathbb{A}[\xi]$. P is then defined as a Fourier multiplier, i.e. if $u(x) = \sum_{\lambda \in \mathbb{Z}^N} \hat{u}(\lambda) e^{-2\pi i x}$, then

$$Pu = \hat{u}(0) + \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \mathbb{P}(\xi) \hat{u}(\lambda) e^{-2\pi i x}.$$

The properties of this projection theorem then classically follow, using smoothness of $\mathbb{P}(\cdot)$, by employing the Hörmander-Mikhlin multiplier theorem (e.g. [78, Theorem 6.2.7]). Equi-integrability just follows from the fact that any smooth, 0-homogeneous Fourier multiplier maps p-equi-integrable sets onto p-equi-integrable sets, c.f. Lemma 2.13.

Indeed, the validity of Theorem 2.9 for a given differential operator $\mathcal{A}$ is even equivalent to the constant rank condition, cf. [80].

2.3. Constant rank operators on open domains

Let us now see how the theory on the torus, which is directly connected to the Fourier transform, generalises to open domains $\Omega \subset \mathbb{R}^N$. For the remainder of this chapter, Ω is an open and bounded domain. By scaling, we moreover may assume that $\Omega \subset\subset (0,1)^N$ is compactly contained in the unit cube and hence might be seen as subset of the N-torus.

We aim to formulate a projection theorem in the spirit of Theorem 2.9 for open domains. The following lemma is concerned with showing an important statement about using cut-offs at the boundary.

Lemma 2.10. *Let $\mathcal{A}$ be a constant rank operator of order k and $\Omega \subset\subset (0,1)^N$, such that Ω can be viewed as an open subset of T_N. Let $\varphi \in C_c^\infty((0,1)^N, \mathbb{R}^d)$ and $1 < p < \infty$.*

(a) For all $u \in L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$ we can identify φu with a function in $L^p(T_N, \mathbb{R}^d)$ (by setting $\varphi u = 0$ on $T_N \setminus \Omega$) and bound

$$\|\mathcal{A}(u\varphi)\|_{W^{-k,p}(T_N,\mathbb{R}^l)} \leq C \|u\|_{W^{-1,p}(T_N,\mathbb{R}^d)} \|\varphi\|_{W^{k+1,\infty}(T_N,\mathbb{R}^d)}. \tag{2.5}$$

(b) If $u_n \rightharpoonup 0$ in $L^p(\Omega, \mathbb{R}^d)$ and $\mathcal{A}u_n = 0$, then $\mathcal{A}(\varphi u_n) \to 0$ in $W^{-k,p}(T_N, \mathbb{R}^l)$.

Proof. Note first that (b) is a direct consequence of (a), as $u_n \rightharpoonup 0$ in $L^p(\Omega, \mathbb{R}^d)$ implies that $u_n \to 0$ in $W^{-1,p}(T_N, \mathbb{R}^d)$, due to the compact Sobolev embedding.

Towards (2.5): If $u \in W^{k,p}(\Omega, \mathbb{R}^d)$, then

$$\mathcal{A}(u\varphi) = (\mathcal{A}u)\varphi + \sum_{|\alpha|=k} \sum_{\beta < \alpha} \binom{\alpha}{\beta} A_\alpha \partial_\beta u \partial_{\alpha-\beta} \varphi.$$

Hence, if $\psi \in W^{k,q}(T_N, \mathbb{R}^l)$,we have

$$\int_{T_N} \mathcal{A}(u_n\varphi)\psi \,\mathrm{d}x = \int_\Omega \mathcal{A}u(\varphi\psi) + \sum_{|\alpha|=k} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \int_{T_N} A_\alpha \partial_\beta u \partial_{\alpha-\beta} \varphi\psi \,\mathrm{d}x$$

$$
\begin{aligned}
&= 0 + \sum_{|\alpha|=k} \sum_{\beta<\alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} \int_{T_N} A_\alpha u \partial_\beta (\partial_{\alpha-\beta} \varphi \psi) \,\mathrm{d}x \\
&\leq C \|u\|_{W^{-1,p}(T_N,\mathbb{R}^d)} \sum_{j=0}^{k-1} \|D^j (D^{k-j} \varphi \psi)\|_{W^{1,q}(T_N,\mathbb{R}^l)} \\
&\leq C \|u\|_{W^{-1,p}(T_N,\mathbb{R}^d)} \|\varphi\|_{W^{k+1,\infty}(T_N)} \|\psi\|_{W^{k,q}(T_N,\mathbb{R}^l)}.
\end{aligned}
$$

Hence, we get (2.5) for $u \in W^{k,p}(\Omega, \mathbb{R}^d)$. By using a density argument, (2.5) also holds for $u \in L^p(\Omega, \mathbb{R}^d)$. □

Based on the statement of the previous Lemma 2.10, the following projection theorem was shown and employed by Fonseca & Müller [65].

Theorem 2.11 (A projection theorem on open domains)**.** *Let* $1 < p < \infty$. *Suppose that* $u_n \rightharpoonup u$ *in* $L^p(\Omega, \mathbb{R}^d)$ *and* $\mathcal{A} u_n \to \mathcal{A} u$ *in* $W^{-k,p}(\Omega, \mathbb{R}^l)$. *Then*

(a) If, in addition, u_n *is p-equi-integrable, there is a sequence* v_n *such that*

  (i) v_n *is still p-equi-integrable;*

  (ii) $\|v_n - u_n\|_{L^p} \to 0$;

  (iii) $\mathcal{A} v_n = 0$.

(b) Fix $1 < q < p$. *Then there exists* $v_n \in L^p(\Omega)$ *such that*

  (i) v_n *is p-equi-integrable;*

  (ii) $\|v_n - u_n\|_{L^q} \to 0$;

  (iii) $\mathcal{A} v_n = 0$.

The proof can be found in [65], but it is also contained in the following Theorem 2.12. Note that Theorem 2.11 does not respect boundary values (e.g. Neumann or Dirichlet boundary data, cf. Corollary 2.16).

To attain a version of Theorem 2.11, which conserves boundary values, we closely follow [95, Section 3.2] until the end of this section (cf. Chapter 5 for the summary of that work).

Theorem 2.12 (Preserving the boundary condition)**.** *Let* $\Omega \subset \mathbb{R}^N$ *have Lipschitz boundary. Suppose that* $\mathcal{A}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ *is a homogeneous differential operator of order* $k_\mathcal{A}$ *satisfying the constant rank property and* $\mathcal{B}$ *is a potential of* $\mathcal{A}$ *in the sense of Definition 2.3. Let* $v_n \rightharpoonup 0$ *in* $L^p(\Omega, \mathbb{R}^d)$, $\mathcal{A} v_n \to 0$ *in* $W^{-k_\mathcal{A},p}(\Omega, \mathbb{R}^l)$. *Then there exists a sequence* $w_n \subset W^{k_\mathcal{B},p}(\Omega, \mathbb{R}^m)$ *such that*

(a) The sequence $\sum_{j=0}^{k_\mathcal{B}} |\nabla^j w_n|$ *is p-equi-integrable;*

(b) $\|\mathcal{B} w_n - v_n\|_{L^q} \to 0$ *as* $n \to \infty$ *for any* $q < p$;

(c) w_n *is compactly supported in* Ω.

To prove this theorem, we need the following three auxiliary results. First of all, we recall the earlier mentioned result about 0-homogeneous Fourier multipliers, cf. [95, Lemma 2.8]. To fix a suitable setting, let $\mathbb{W} : C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^d)$ be a 0-homogeneous Fourier multiplier, i.e. $\mathbb{W}(\lambda\xi) = \mathbb{W}(\xi)$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Define the map

$$Wu(x) = \sum_{\xi \in \mathbb{Z}^N} \mathbb{W}(\xi)(\hat{u}(\xi))e^{-2\pi i x\cdot\xi}, \quad \text{if } u \text{ is given by} \quad u(x) = \sum_{\xi \in \mathbb{Z}^N} (\hat{u}(\xi))e^{-2\pi i x\cdot\xi},$$

and otherwise by density (that this density argument is possible, is shown implicitly by (a) in the following lemma).

Lemma 2.13. *Let $\mathbb{W} : C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^d)$ as above. Then for any $1 < p < \infty$:*

(a) $W : L^p(T_N, \mathbb{R}^d) \to L^p(T_N, \mathbb{R}^d)$ is bounded;

(b) W is continuous from L^p to L^p with respect to the weak topology of L^p;

(c) If $X \subset L^p(T_N, \mathbb{R}^d)$ is a p-equi-integrable and bounded set, then $W(X)$ is also p-equi-integrable.

Proof. (a) follows by the Mikhlin-Hörmander-multiplier theorem (e.g.[65, 78]).

(b) follows from the fact that the adjoint W^* is bounded from $L^{p'}$ to $L^{p'}$.

For (c) we refer to [65, Lemma 2.14 (iv)], where the proof is given in a special case. The proof for the general setting is exactly the same.

□

The second auxiliary result allows us to pick suitable diagonal sequences with respect to the weak topology (which is metrisable on bounded subsets of L^p!).

Lemma 2.14. *Let (X, d_X) be a complete metric space. Suppose that x_n is a sequence in X such that $x_n \to x$ and that, for $m \in \mathbb{N}$, we have $x_{n,m}$ with*

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} d_X(x_{n,m}, x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d_X(x_{n,m}, x) = 0 \quad \text{for all } m \in \mathbb{N}.$$

Then $x_{n,m} \to x$ uniformly in m as $n \to \infty$.

Proof. Let $\varepsilon > 0$. Then there exists m_ε, such that for all $m \geq m_\varepsilon$

$$d_X(x_{n,m}, x_n) < \varepsilon/2$$

and an N_ε, such that for all $n > N_\varepsilon$

$$d_X(x_n, x) < \varepsilon/2.$$

Moreover, there are $N^1, ..., N^{m_\varepsilon}$ such that for all $m = 1, ..., m_\varepsilon$

$$n > N^{m_\varepsilon} \quad \Longrightarrow \quad d_X(x_{n,m}, x) < \varepsilon.$$

Choosing $N = \max\{N_\varepsilon, N^1, ..., N^{m_\varepsilon}\}$ yields that for any $n > N$ and $m \in \mathbb{N}$

$$d(x_{n,m}, x) < \varepsilon$$

which is the required uniform convergence. □

The following result is due to [65, Lemma 2.15]. It allows to construct (p,q)-equi-integrable modified sequences. However, in general these modified sequences fail to conserve the constraints.

Proposition 2.15. *Let v_n be a bounded sequence in $L^p(\Omega, \mathbb{R}^m)$. Then there exists a p-equi-integrable sequence $\tilde{v}_n$, such that*

1. *For almost every $x \in \Omega$ we have $|\tilde{v}_n(x)| \leq |v_n(x)|$;*
2. *For every $q < p$ we have $\lim_{n\to\infty} \|v_n - \tilde{v}_n\|_{L^q} = 0$.*

Finally, we are ready to prove Theorem 2.12.

Proof of Theorem 2.12. **Step 1:** *Construction of the sequence.*
Let us assume by scaling, that $\Omega \subset\subset (0,1)^N$, which can be identified with the N-dimensional torus T_N and extend v_n by 0 outside Ω. Let $m \in \mathbb{N}$. We define open sets V_m and U_m, such that $V_m \subset\subset U_m \subset\subset \Omega$ and such that

$$\begin{aligned}
\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > 2/m\} \subset V_m \subset \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > 1/m\},\\
\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > 4/m\} \subset U_m \subset \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > 3/m\}.
\end{aligned}$$

Then there exist $\varphi_m \subset C_c^\infty(V_m)$ with $\varphi_m \equiv 1$ on U_m and $\psi_m \subset C_c^\infty(\Omega)$ with $\psi_m \equiv 1$ on V_m, such that for all $k, m \in \mathbb{N}$

$$\|\nabla^k \psi_m\|_{L_\infty}, \|\nabla^k \varphi_m\|_{L_\infty} \leq C(k) m^k.$$

By Proposition 2.15 there exists a p-equi-integrable sequence $\tilde{v}_n$, such that $\|\tilde{v}_n - v_n\|_{L^q} \to 0$ for $q < p$. Therefore, as v_n converges weakly to 0, so does $\tilde{v}_n$. Let us now define

$$\begin{aligned}
\bar{v}_{n,m} &= \varphi_m \tilde{v}_n;\\
\bar{w}_{n,m} &= \mathcal{B}^{-1} \bar{v}_{n,m};\\
w_{n,m} &= \psi_m \bar{w}_{n,m}.
\end{aligned}$$

We claim that we can take an appropriate diagonal sequence $w_{n,m(n)}$ with $m(n) \to \infty$ as $n \to \infty$ such that $w_{n,m(n)}$ satisfies the requirements of Theorem 2.12. The purpose of the following steps is to construct such a sequence $m(n)$.

Step 2: *Estimates on $\bar{v}_{n,m}$.*
First of all let us show that

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\|\tilde{v}_n-\bar{v}_{n,m}\|_{L^p}=0. \tag{2.6}$$

To this end, we use that Ω has Lipschitz boundary to get a constant $C>0$ such that

$$|\Omega\setminus V_m|\le|\Omega\setminus U_m|\le Cm^{-1}. \tag{2.7}$$

Then we have

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|\tilde{v}_n-\bar{v}_{n,m}\|_{L^p}&\le\sup_{n\in\mathbb{N}}\|\tilde{v}_n\|_{L^p(\Omega\setminus U_m)}\\&\le\sup_{n\in\mathbb{N}}\sup_{|E|\le|(\Omega\setminus U_m)|}\|\tilde{v}_n\|_{L^p(E)}\\&\le\sup_{n\in\mathbb{N}}\sup_{|E|\le Cm^{-1}}\|\tilde{v}_n\|_{L^p(E)}.\end{aligned}$$

As $\tilde{v}_n$ is p-equi-integrable, this expression converges to 0 as $m\to\infty$, and so (2.6) is established.

Secondly, we want to bound the $W^{-k_{\mathcal{A}},q}$-norm of $\mathcal{A}\bar{v}_{n,m}$. We claim that there exists a sequence $M_1(n)$ with $M_1(n)\to\infty$ as $n\to\infty$, such that for all $m(n)$ with $m(n)\le M_1(n)$ and $m(n)\to\infty$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\|\mathcal{A}\bar{v}_{n,m(n)}\|_{W^{-k_{\mathcal{A}},q}(T_N,\mathbb{R}^l)}=0,\quad\text{for some } 1<q<p. \tag{2.8}$$

Note that if $\tilde{v}_n$ is in $C^k(\Omega,\mathbb{R}^d)$, then we may write

$$\mathcal{A}\bar{v}_{n,m}=\mathcal{A}(\varphi_m\tilde{v}_n)=(\mathcal{A}\tilde{v}_n)\varphi_m+\sum_{|\alpha|=k_{\mathcal{A}}}\sum_{\beta<\alpha}\binom{\alpha}{\beta}A_\alpha\partial_\beta\tilde{v}_n\partial_{\alpha-\beta}\varphi_m.$$

Therefore, we may estimate

$$\|\mathcal{A}\bar{v}_{n,m}\|_{W^{-k_{\mathcal{A}},q}(T_N,\mathbb{R}^l)}\le\|\mathcal{A}\tilde{v}_n\|_{W^{-k_{\mathcal{A}},q}(\Omega;\mathbb{R}^l)}\|\varphi_m\|_{W^{k_{\mathcal{A}},\infty}(\Omega)}+C\|\tilde{v}_n\|_{W^{-1,q}(\Omega)}\|\varphi_m\|_{W^{k_{\mathcal{A}}+1,\infty}(\Omega)}. \tag{2.9}$$

Due to density of $C^k(\Omega;\mathbb{R}^m)$ in $L^p(\Omega;\mathbb{R}^m)$, (2.9) still is valid even if $\tilde{v}_n$ only is in L^p. With the estimates for the derivatives of φ we get

$$\|\mathcal{A}\bar{v}_{n,m}\|_{W^{-k_{\mathcal{A}},q}(T_N,\mathbb{R}^l)}\le C\left(m^{k_{\mathcal{A}}}\|\mathcal{A}\tilde{v}_n\|_{W^{-k_{\mathcal{A}},q}}+m^{k_{\mathcal{A}}+1}\|\tilde{v}_n\|_{W^{-1,q}}\right)$$

Note that, on the one hand, $\mathcal{A}\tilde{v}_n\to0$ in $W^{-k_{\mathcal{A}},q}$, as $\mathcal{A}v_n\to0$ in $W^{-k_{\mathcal{A}},p}$ and $\tilde{v}_n-v_n\to0$ in L^q for $q<p$. On the other hand, as $\tilde{v}_n$ is bounded in L^p and weakly converging to 0, $\tilde{v}_n\to0$ in W_q^{-1} strongly due to the compact embedding of L^q into $W^{-1,q}$. Therefore,

choosing

$$M_1(n) := (\min\{\{\|\mathcal{A}\tilde{v}_n\|_{W^{-k,q}},\ \|\tilde{v}_n\|_{W^{-1,q}}\})^{\frac{-1}{3k_{\mathcal{A}}}} \longrightarrow \infty \quad \text{as } n \to \infty, \tag{2.10}$$

we get

$$\lim_{n\to\infty} \sup_{m \le M_1(n)} \|\mathcal{A}\bar{v}_{n,m}\|_{W^{-k_{\mathcal{A}},p}(T_N,\mathbb{R}^l)} = 0. \tag{2.11}$$

Last, let us note that due to equi-integrability of $\tilde{v}_n$, also the set $\{\bar{v}_{n,m}\}_{n,m\in\mathbb{N}}$ is equi-integrable.

Step 3: *Upper Bound on* $\|\mathcal{B}w_{n,m} - v_n\|_{L^q}$.
First of all, let us note that by definition $w_{n,m}$ is compactly supported in Ω for any $m \in \mathbb{N}$, as ψ_m is compactly supported in Ω. Moreover, observe that

$$\begin{aligned}\|\mathcal{B}w_{n,m} - v_n\|_{L^q} &\le \|\mathcal{B}w_{n,m} - \mathcal{B}\bar{w}_{n,m}\|_{L^q} + \|\mathcal{B}\bar{w}_{n,m} - \bar{v}_{n,m}\|_{L^q} + \|\bar{v}_{n,m} - \tilde{v}_n\|_{L^q} + \|\tilde{v}_n - v_n\|_{L^q}\\ &= \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}.\end{aligned}$$

We already established by the choice of $\tilde{v}_n$ (c.f. Proposition 2.15), that (IV) $\to 0$ as $n \to \infty$. (III) $\to 0$ as $n \to \infty$, whenever $m = m(n)$ goes to ∞, cf. (2.6). Proposition 2.5 and (2.4) yield that

$$\text{(II)} \le |\mathcal{A}\bar{v}_{n,m(n)}| + \int_{T_N} \bar{v}_{n,m(n)}.$$

The first term goes to 0 by (2.11), whenever $m(n) \le M_1(n)$ is a sequence diverging to ∞ as $n \to \infty$, while the mean of $\tilde{v}_{n,m(n)}$ goes to zero since $\tilde{v_n} \rightharpoonup 0$ and because of (2.6). It remains to bound (I). To this end, note that

$$\begin{aligned}\text{(I)} &\le \|(1-\psi_m)\mathcal{B}\bar{w}_{n,m}\|_{L^q} + \sum_{|\alpha|=k_{\mathcal{B}}} \sum_{\beta<\alpha} \|B_\alpha \partial_\beta \bar{w}_{n,m} \partial_{\alpha-\beta}\psi_m\|_{L^q}\\ &\le Cm^{-1}\|\mathcal{B}\bar{w}_{n,m}\|_{L^q} + m^{k_{\mathcal{B}}}\|\bar{w}_{n,m}\|_{W^{k_{\mathcal{B}}-1,q}}.\end{aligned}$$

The first term vanishes as $m \to \infty$. Note that the operator $W = \nabla^{k_{\mathcal{B}}} \circ \mathcal{B}^{-1}$ is a 0-homogeneous, smooth Fourier multiplier. Due to Lemma 2.13 (b), it is continuous from L^q to L^q in the weak topology. Recall, that $\tilde{v}_n \rightharpoonup 0$ as $n \to \infty$ in L^p, that $\bar{v}_{n,m}$ is uniformly bounded in L^p and for fixed $m \in \mathbb{N}$, $\bar{v}_{n,m} = \varphi_m\tilde{v}_n \rightharpoonup 0$. The weak topology of L^p is metrisable on bounded sets, hence we may apply Lemma 2.14 to get that the convergence

$$\bar{v}_{n,m} \rightharpoonup 0 \text{ weakly in } L^p \text{ as } n \to \infty$$

is *uniform* in $m \in \mathbb{N}$. The map $W = \nabla^{k_{\mathcal{B}}} \circ \mathcal{B}^{-1}$ is a smooth 0-homogeneous Fourier multiplier, hence also

$$W\bar{v}_{n,m} \rightharpoonup 0 \text{ weakly in } L^p \text{ uniformly in } m. \tag{2.12}$$

For $s < p^* = \dfrac{Np}{N-p}$ (or $s < \infty$ if $p > N$), the embedding $W^{k_{\mathcal{B}},p} \hookrightarrow W^{k_{\mathcal{B}}-1,s}$ is compact, hence uniform weak convergence of $\nabla^{k_\mathcal{B}}\bar{w}_{n,m}$ (together with the Poincaré inequality) implies that

$$\lim_{n\to\infty}\sup_{m\in\mathbb{N}}\|\bar{w}_{n,m}\|_{W^{k_\mathcal{B}-1,s}} = 0.$$

This holds in particular for $s = p < p^*$. Therefore, choosing $M_2(n)$ as

$$M_2(n) := \left(\sup_{m\in\mathbb{N}}\|\bar{w}_{n,m}\|_{W^{k_\mathcal{B}-1,p}}\right)^{\frac{-1}{2k_\mathcal{B}}}$$

yields that for any sequence $m(n)$ with $m(n) \le \min\{M_1(n), M_2(n)\}$

$$\|\mathcal{B}w_{n,m(n)} - v_n\|_{L^q} \longrightarrow 0 \quad \text{as } n\to\infty.$$

Step 4: *Equi-integrability of* $w_{n,m}$

It remains to show that we may choose the diagonal sequence $w_{n,m(n)}$ in such a fashion, that $\nabla^j w_{n,m(n)}$ is still p-equi-integrable for all $1 \le j \le k_\mathcal{B}$. Note that

$$\nabla^j w_{n,m} = \psi_m\nabla^j\bar{w}_{n,m} + \sum_{i=0}^{j-1}\nabla^i\bar{w}_{n,m}\otimes\nabla^{j-i}\psi_m.$$

$\bar{w}_{n,m}$ is uniformly bounded in m and n in $W^{k_\mathcal{B},p}$, as $\bar{v}_{n,m}$ is uniformly bounded in L^p and $\mathcal{B}^{-1}$ maps L^p to $W^{k_\mathcal{B},p}$. Hence, for $j < k_\mathcal{B}$, $\nabla^j\bar{w}_{n,m}$ is bounded in $L^{\tilde{p}}$ for some $\tilde{p} > p$ and thus $|\psi_m\nabla^j\bar{w}_{n,m}| \le |\nabla^j\bar{w}_{n,m}|$ is p-equi-integrable. Furthermore, observe that we have the pointwise estimate

$$\left|\nabla^i\bar{w}_{n,m}\otimes\nabla^{j-i}\psi_m\right| \le m^{k_\mathcal{B}}\left|\nabla^i\bar{w}_{n,m}\right|1_{\Omega\setminus V_m}.$$

Hence, for p-equi-integrability it suffices to show that there is $M_3(n)\to\infty$ as $n\to\infty$, such that for $i < k_\mathcal{B}$

$$\left\{\nabla^{k_\mathcal{B}}\bar{w}_{n,m}\colon m \le M_3(n)\right\} \quad \text{is } p\text{-equi-integrable,} \tag{2.13}$$

$$\left\{m^{k_\mathcal{B}}\nabla^i\bar{w}_{n,m}1_{\Omega\setminus U_m}\colon m \le M_3(n)\right\} \quad \text{is } p\text{-equi-integrable.} \tag{2.14}$$

Indeed, (2.13) is clear, even for $m\in\mathbb{N}$, as $W = \nabla^{k_\mathcal{B}}\circ\mathcal{B}^{-1}$ is a smooth 0-homogeneous Fourier multiplier. Moreover, $\nabla^{k_\mathcal{B}}\bar{w}_{n,m} = W(\tilde{v}_{n,m})$, which is p-equi-integrable for $m,n\in\mathbb{N}$ due to Step 1.

Note that we already established in (2.3), that

$$\lim_{n\to\infty}\sup_{m\in\mathbb{N}}\|\bar{w}_{n,m}\|_{W^{k_\mathcal{B}-1,s}} = 0$$

for all $s < p^*$. Let now $s\in(p,p^*)$ be fixed. Then for all measurable sets E

$$\int_E|\nabla^i\bar{w}_{n,m}m^{k_\mathcal{B}}1_{\Omega\setminus V_m}|^p \le m^{k_\mathcal{B}p}\int_{E\cap(\Omega\setminus V_m)}|\nabla^i\bar{w}_{n,m}|^p$$

$$\le m^{k_\mathcal{B}p}|E\cap(\Omega\setminus V_m)| \fint_{E\cap(\Omega\setminus V_m)} |\nabla^i \bar{w}_{n,m}|^p$$
$$\le m^{k_\mathcal{B}p}|E\cap(\Omega\setminus V_m)| \left(\fint_{E\cap(\Omega\setminus V_m)} |\nabla^i \bar{w}_{n,m}|^s\right)^{p/s}$$
$$\le m^{k_\mathcal{B}p}|E\cap(\Omega\setminus V_m)|^{\frac{s-p}{s}} \|w_{n,m}\|^p_{W^{k_\mathcal{B}-1,s}}$$
$$\le |E|^{\frac{s-p}{p}} m^{k_\mathcal{B}p} \sup_{\tilde{m}\in\mathbb{N}} \|\bar{w}_{n,\tilde{m}}\|^p_{W^{k_\mathcal{B}-1,s}}.$$

Note that $|E|^{\frac{s-p}{p}} \to 0$ as $|E|\to 0$, hence we assume that $m \le M_3(n)$ defined as

$$M_3(n) := \left(\sup_{m\in\mathbb{N}} \|\bar{w}_{n,m}\|_{W^{k_\mathcal{B}-1,s}}\right)^{\frac{-1}{2k_\mathcal{B}}} \longrightarrow \infty \quad \text{as } n\to\infty. \tag{2.15}$$

We conclude that for any $0\le j\le k_\mathcal{B}$ the set

$$\left\{\nabla^j w_{n,m} \colon n\in\mathbb{N}, m\le M_3(n)\right\}$$

is p-equi-integrable.

Now choosing a sequence $m(n)\to\infty$ as $n\to\infty$ with $m(n)\le \min\{M_1(n),M_2(n),M_3(n)\}\to\infty$ finishes the proof.

□

We can reformulate the statement of Theorem 2.12 if the weak limit is non-zero as follows (both in terms of boundary conditions for the potential and the annihilator, respectively). Note that if the sequence is p-equi-integrable, we can omit the very first step of the proof of Theorem 2.12 and get convergence in L^p.

Corollary 2.16 (Preserving boundary conditions)**.** *Let $v\in L^p(\Omega,\mathbb{R}^d)$ and let $v_n\subset L^p(\Omega,\mathbb{R}^d)$, such that $v_n \rightharpoonup v$ in L^p and $\mathcal{A}v_n\to\mathcal{A}v$ in $W^{-k_\mathcal{A},p}(\Omega,\mathbb{R}^l)$. Let $\mathcal{B}$ be a potential of $\mathcal{A}$.*

(a) Suppose that v can be written as $v=\mathcal{B}u$. There exists a sequence $u_n\subset W^{k_\mathcal{B},p}(\Omega,\mathbb{R}^m)$, such that

(i) u_n-u is compactly supported in Ω;

(ii) $\mathcal{B}u_n$ is p-equi-integrable;

(iii) $\|\mathcal{B}u_n-v_n\|_{L^{\tilde{p}}(\Omega)}\to 0$ for some $1<\tilde{p}<p$.

(b) There is a sequence $\bar{v}_n\subset L^p(\Omega,\mathbb{R}^d)$, such that

(i) $\mathcal{A}\bar{v}_n=\mathcal{A}v$;

(ii) $\bar{v}_n-v$ is compactly supported in Ω;

(iii) $\bar{v}_n$ is p-equi-integrable;

(iv) $\|\bar{v}_n-v_n\|_{L^{\tilde{p}}(\Omega)}\to 0$ for some $1<\tilde{p}<p$.

(c) If v_n is already p-equi-integrable, then we can choose $r=p$ in (a) and (b).

2.4. Non-homogeneous operators and separate constraints

In this section, we shortly look at operators, which are not of the form (2.1), but still can be treated in the same fashion. Previous theorems mostly relied on Fourier analysis and the constant rank property as a suitable condition (cf. Theorem 2.9). In general, non-homogeneous operators do not satisfy such conditions in Fourier space. We therefore specify two situations, in which we can apply the previous theory.

2.4.1. Homogeneous components

We consider differential operators $\mathcal{A}_1$, ..., $\mathcal{A}_k$ which are given by

$$\mathcal{A}_i u = \sum_{|\alpha|=i} A^i_\alpha \partial_\alpha u.$$

In this setting, $\mathcal{A}_i \colon C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{l_i})$ are homogeneous differential operators of order i, i.e. $A^i_\alpha \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^{l_i})$. We then define

$$\mathcal{A}u = (\mathcal{A}_1 u, \ldots, \mathcal{A}_k u),$$

such that, for an open and bounded domain Ω,

$$\mathcal{A} \colon L^p(\Omega, \mathbb{R}^d) \longrightarrow W^{-1,p}(\Omega, \mathbb{R}^{l_1}) \times \ldots \times W^{-k,p}(\Omega, \mathbb{R}^{l_k}).$$

Definition 2.17. *For such a componentwise homogeneous differential operator $\mathcal{A}$ we define the Fourier symbol for $\xi \in \mathbb{R}^N$, $\mathbb{A}[\xi] \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^{l_1} \times \ldots \mathbb{R}^{l_k})$, as follows:*

$$\mathbb{A}[\xi] = (\mathbb{A}_1[\xi], \ldots, \mathbb{A}_k[\xi]).$$

We say that $\mathcal{A}$ satisfies the constant rank property if there is $r \geq 0$, such that for all $\xi \in \mathbb{R}^N \setminus \{0\}$

$$\dim \ker \mathbb{A}[\xi] = r,$$

and likewise, that $\mathcal{A}$ satisfies the spanning property if the characteristic cone

$$\Lambda_{\mathcal{A}} = \bigcup_{\xi \in \mathbb{R}^N \setminus \{0\}} \ker \mathbb{A}[\xi]$$

spans up $\mathbb{R}^d$.

Let us mention that in this framework, only the operator $\mathcal{A}$ needs to satisfy the constant rank property and not $\mathcal{A}_i$. Indeed, taking $\mathcal{A} \colon C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2, \mathbb{R} \times \mathbb{R})$ defined via

$$\mathcal{A}u = (\partial_1 u, \partial_2^2 u)$$

we see that $\mathcal{A}$ satisfies the constant rank property, but its homogeneous parts $\mathcal{A}_1 u = \partial_1 u$ and $\mathcal{A}_2 u = \partial_2^2 u$ do not. On the other hand, even if the homogeneous components all satisfy the constant rank property, validity of the constant rank property is not guaranteed. For example, consider $\mathcal{A}\colon C^\infty(\mathbb{R}^2,\mathbb{R}^2)\to C^\infty(\mathbb{R}^2,\mathbb{R}\times\mathbb{R}^2)$ given by

$$\mathcal{A}_1 u = (\partial_1 u_1 + \partial_2 u_2) = \operatorname{div} u, \quad \mathcal{A}_2 u = \begin{pmatrix} \partial_1(\partial_2 u_1 + \partial_1 u_2) \\ \partial_2(\partial_2 u_1 + \partial_1 u_2) \end{pmatrix}.$$

Then both operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the constant rank property, as

$$\ker \mathbb{A}_1[\xi] = \{\lambda(\xi_2,-\xi_1)\colon \lambda\in\mathbb{R}\}, \quad \ker \mathbb{A}_2[\xi] = \{\lambda(\xi_1,-\xi_2)\colon \lambda\in\mathbb{R}\}.$$

However $\mathcal{A}$ does not enjoy the constant rank property. The dimension for $\ker\mathbb{A}[\xi]$ is zero, except for $\xi=(\pm1,\pm1)$ (where $\dim\ker\mathbb{A}[\xi]=1$).

However, for the problems we consider in this thesis, we can reduce the setting of component-wise homogeneous differential operators $\mathcal{A}$ to the setting of homogeneous differential operators. Note that if $\mathcal{A}_i\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d)\to C^\infty(\mathbb{R}^N,\mathbb{R}^{l_i})$ is a homogeneous differential operator of order i, then $\mathcal{A}_i^k = \nabla^{k-i}\circ\mathcal{A}_1\colon C^\infty(\mathbb{R}^N,\mathbb{R}^{l_i})\to C^\infty(\mathbb{R}^N,(\mathbb{R}^N)^{k-i}\otimes\mathbb{R}^{l_i})$ is homogeneous of order k. Moreover,

$$\ker\mathbb{A}_i^k[\xi] = \ker\mathbb{A}_i[\xi]$$

for any $\xi\in\mathbb{R}^N\setminus\{0\}$. So in terms of the projection operator P defined by Theorem 2.9, we cannot distinguish between $\mathbb{A}_i$ and $\mathbb{A}_i^k$. In particular, if we define

$$\tilde{\mathcal{A}} = (\mathcal{A}_1^k, \mathcal{A}_2^k, \dots, \mathcal{A}_k^k),$$

then $\tilde{\mathcal{A}}$ is a homogeneous differential operator of order k and $\ker\tilde{\mathbb{A}}[\xi]=\ker\mathbb{A}[\xi]$ for any ξ. Hence, we can reformulate all theorems obtained for fully homogeneous $\tilde{\mathcal{A}}$ in Sections 2.2 & 2.3 also for $\mathcal{A}$ instead. For example, the projection on the torus reads as follows.

Corollary 2.18 (Projections on the torus for constant rank, non-homogeneous operators). *Let $\mathcal{A}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d)\to C^\infty(\mathbb{R}^N,\mathbb{R}^{l_1}\times\ldots\times\mathbb{R}^{l_k})$ be a constant rank operator. Then there exists a projection operator P with the following properties*

(a) P maps $L^p(T_N,\mathbb{R}^d)\to L^p(T_N,\mathbb{R}^d)$ boundedly for any $1<p<\infty$;

(b) $P\circ P = P$;

(c) $\mathcal{A}\circ P = 0$;

(d) There is $C=C(p)$, such that for any $u\in L^p(T_N,\mathbb{R}^d)$ we have

$$\|u-Pu\|_{L^p}\le\sum_{i=1}^k\|\mathcal{A}_i u\|_{W^{-i,p}};$$

(e) P maps p-equi-integrable sets into p-equi-integrable sets.

2.4.2. Separate constraints

An easier setting than the previous subsection is the following. We consider $u = (u_1, u_2)$, $u_i : \mathbb{R}^N \to \mathbb{R}^{d_i}$, $i = 1, 2$, i.e. u consists of two different quantities, and differential operators $\mathcal{A}_i$ for $i = 1, 2$ acting on $C^\infty(\mathbb{R}^N, \mathbb{R}^{d_i})$, i.e.

$$\mathcal{A}_i u_i = \sum_{|\alpha|=k_i} A^i_\alpha \partial_\alpha u_i$$

for $A^i_\alpha \in \mathrm{Lin}(\mathbb{R}^{d_i}, \mathbb{R}^{l_i})$. In particular, $\mathcal{A}_i$ is a homogeneous differential operator of order k_i that maps $C^\infty(\mathbb{R}^N, \mathbb{R}^{d_i})$ into $C^\infty(\mathbb{R}^N, \mathbb{R}^{l_i})$. We may summarise the two constraints $\mathcal{A}_1 u_1 = 0$ and $\mathcal{A}_2 u_2 = 0$ as

$$\mathcal{A}u := (\mathcal{A}_1 u_1, \mathcal{A}_2 u_2) = 0.$$

The advantage of splitting up an operator in this fashion, is that we can consider u_1 and u_2 to be in L^p spaces with different exponents, i.e.

$$u = (u_1, u_2) \in L^p(\Omega, \mathbb{R}^{d_1}) \times L^q(\Omega, \mathbb{R}^{d_2}).$$

In such a setting $\mathcal{A}$ maps such functions u into $\mathcal{A}u \in W^{-k_1,p}(\Omega, \mathbb{R}^{d_1}) \times W^{-k_2,q}(\Omega, \mathbb{R}^{d_2})$.

From a standpoint of Fourier analysis, reducing this setting to the fully homogeneous L^p case, is rather simple; we can just treat u_1 and u_2 and the projections etc. separately, i.e. one also gets the following statement.

Corollary 2.19 (Projections on the torus for separate constraints)**.**
Let $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) : C^\infty(\mathbb{R}^N, \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{l_1} \times \mathbb{R}^{l_2})$ be a constant rank operators. Then there exists a projection operator P with the following properties:

(a) P maps $L^p(T_N, \mathbb{R}^{d_1}) \times L^q(T_N, \mathbb{R}^{d_2}) \to L^p(T_N, \mathbb{R}^{d_1}) \times L^q(T_N, \mathbb{R}^{d_2})$ boundedly for any $1 < p, q < \infty$;

(b) $P \circ P = P$;

(c) $\mathcal{A} \circ P = 0$;

(d) There is $C = C(p)$, such that for any $u \in L^p(T_N, \mathbb{R}^d)$ we have

$$\|u - Pu\|_{L^p} \leq \|\mathcal{A}_1 u_1\|_{W^{-k_1,p}} + \|\mathcal{A}_2 u_2\|_{W^{-k_2,q}};$$

(e) P maps (p, q)-equi-integrable sets into (p, q)-equi-integrable sets, i.e. if a set X obeys

$$\lim_{\varepsilon \to 0} \sup_{u \in X} \sup_{|E| < \varepsilon} \int_E |u_1|^p + |u_2|^q \,\mathrm{d}x = 0,$$

then this is also true for $P(X)$:

$$\lim_{\varepsilon \to 0} \sup_{u \in X} \sup_{|E| < \varepsilon} \int_E |(Pu)_1|^p + |(Pu)_2|^q \,\mathrm{d}x = 0.$$

Likewise, projection statements similar to Theorem 2.9 and 2.11 follow.

2.5. What the constant rank condition on $\mathbb{R}$ cannot guarantee

So far, we considered the constant rank condition with respect to $\xi \in \mathbb{R}^N \setminus \{0\}$. We have seen that this condition is sufficient to get

(a) a potential on the torus, i.e. a differential operator $\mathcal{B}$, such that (for functions with average 0) $\mathcal{A}u = 0$ is equivalent to $u = \mathcal{B}v$;

(b) projection theorems on the torus;

(c) projection theorems on open domains (in terms of equi-integrability and the $W^{-k,p}$-norm of $\mathcal{A}u$).

In particular, the sufficiency of the constant rank property in $\mathbb{R}$ for weak lower-semicontinuity problems can be explained by the following heuristic argument: We see in Section 4, that it suffices to consider p-equi-integrable sequences to tackle weakly convergent sequences in the context of lower-semicontinuity for non-negative integrands. Hence, apart from strong convergence, the only effect accounting for weak convergence are fast oscillations. But these are handled by the constant rank property.

If we ask for stronger results, the constant rank property in $\mathbb{R}$ is *not* enough. The result

$$\mathcal{A}u = 0 \Longrightarrow u = \mathcal{B}v$$

only holds on the torus. Consider the example

$$\mathcal{B} = 0 \colon C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N) \text{ and } \mathcal{A} = \Delta = \sum_{i=1}^N \partial_i^2 \colon C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N).$$

In Fourier sense (on $\mathbb{R}$) $\mathcal{B}$ is a potential of $\mathcal{A}$, and indeed for $u \in L^p_\#(T_N)$ with average 0

$$\Delta u = 0 \Longleftrightarrow u = 0.$$

This is obviously not true, if T_N is replaced by any open domain Ω; then the space $\Delta u = 0$ is infinite dimensional, but the image of $\mathcal{B}$ still only is $\{0\}$. This behaviour is expressed in the following statement.

Lemma 2.20 (Potentials on $\mathbb{R}$ on open domains)**.** *Let $\mathcal{A}$ be an operator of constant rank and $\mathcal{B}$ be its potential. Let $\Omega \subset \mathbb{R}^N$ be open and bounded with $C^{\max\{k_\mathcal{A},k_\mathcal{B}\}}$-boundary. Let $1 < p < \infty$. Then there is a vector space $X \subset L^p(\Omega,\mathbb{R}^d)$, such that*

(a) $L^p(\Omega,\mathbb{R}^d) \cap \ker \mathcal{A} = \mathcal{B}(W^{k,p}(\Omega,\mathbb{R}^m)) + X$;

*(b) If $Y \subset X$ is p-equi-integrable and bounded, then Y is compact with respect to the **strong** topology of $L^p(\Omega,\mathbb{R}^d)$.*

Proof (Sketch). We highlight the main ideas of the proof. Write $\operatorname{Im}\mathcal{B} = \mathcal{B}(W^{k,p}(\Omega, \mathbb{R}^m))$. Define X as follows:

$$X = \{u \in L^p(\Omega, \mathbb{R}^d) \colon \mathcal{A}u = 0 \text{ in } \mathcal{D}'(\Omega, \mathbb{R}^l) \text{ and } \mathcal{B}^* u = 0 \text{ in } \mathcal{D}'(\Omega, \mathbb{R}^m)\}. \tag{2.16}$$

To prove Theorem 2.20, it suffices to show both of the following steps.

Step 1: Show that any $u \in L^p(\Omega, \mathbb{R}^d)$ satisfying $\mathcal{A}u = 0$ can be written as $u = \mathcal{B}v_1 + u_2$ for $u_2 \in X$.

Step 2: Any p-equi-integrable sequence u_n in X that converges weakly to some u^* already converges strongly.

A key observation is that we can reduce to the case, where $\mathcal{A}$ and $\mathcal{B}$ have the same order k. If $k_{\mathcal{A}} < k_{\mathcal{B}}$, one may replace $\mathcal{A}$ by $\tilde{A} = \nabla^{k_{\mathcal{B}} - k_{\mathcal{A}}} \circ \mathcal{A}$, which has the same nullspace up to a finite-dimensional space of polynomials, cf. Lemma 2.30. Likewise, if $k_{\mathcal{B}} < k_{\mathcal{A}}$, we may replace $\mathcal{B}$ by $\tilde{B} = \mathcal{B} \circ \operatorname{div}^{k_{\mathcal{A}} - k_{\mathcal{B}}}$.

The basic idea is to solve an elliptic equation $LU = u$ for the elliptic (cf. [5]) operator

$$L = \mathcal{A}^* \circ \mathcal{A} + \mathcal{B} \circ \mathcal{B}^*.$$

Then, due to [4, 5], a solution $U \in W^{2k,p}(\Omega, \mathbb{R}^d) \cap W^{k,p}_0(\Omega, \mathbb{R}^d)$ to $LU = u$ exists. Moreover, $\mathcal{B}(B^*U)$ is $\mathcal{A}$-free (and in $\operatorname{Im}\mathcal{B}$). On the other hand, $\mathcal{A}^* \circ \mathcal{A}U$ is in X, as

$$\mathcal{A}(\mathcal{A}^* \circ \mathcal{A}U) = \mathcal{A}u - \mathcal{A}(\mathcal{B} \circ \mathcal{B}^* u) = 0 \quad \text{and} \quad \mathcal{B}^*(\mathcal{A}^* \circ \mathcal{A}U) = (\mathcal{B}^* \circ \mathcal{A}^*)(\mathcal{A}U) = 0.$$

It suffices to show that X obeys (b). This instantly follows from applying Lemma 2.16 (c) to the constant rank operator L and a sequence $u_n \subset X$ weakly converging to u. Indeed, recall that $\operatorname{spt} u_n - u \subset\subset \Omega$ and $L(u_n - u) = 0$ implies $u_n - u = 0$ for an elliptic operator L.

□

In the following Section we want to establish a condition that further improves the statement of Theorem 2.20. Indeed, it turns out that a more natural condition is *constant rank in* $\mathbb{C}$.

2.6. On Operators with constant rank in $\mathbb{C}$

Summary

After a short introductory text, Section 2.6 coincides, up to minor changes, with the preprint

- [77] Gmeineder, F. and Schiffer, S.: *Natural annihilators and operators of constant rank over* $\mathbb{C}$, https://arxiv.org/abs/2203.10355, 2022.

The research undertaken in the paper in question is a collaboration with F. Gmeineder. Both authors, which, in particular, includes the author of this thesis, have contributed significant parts to each section of the work.

The goal of this section is to give an answer to some questions that have been previously discussed in this chapter in the framework of the constant rank property in $\mathbb{R}$. In particular, the previous Section 2.5 outlined some results that the constant rank property in $\mathbb{R}$ cannot guarantee. The aim of this section is to fill this gap.

To this end, we define the concept of the constant rank property also for the field $\mathbb{C}$ instead of $\mathbb{R}$, i.e.

$$\dim \ker \mathbb{A}[\xi] = r \quad \text{for all } \xi \in \mathbb{C}^N \setminus \{0\}.$$

We shortly discuss, that all the notions we have in $\mathbb{R}$ also apply for $\mathbb{C}$. In particular, we rise the question whether it is possible to derive a Poincaré lemma if the stronger notion of constant rank property in $\mathbb{C}$ holds. This discussion is not part of the preprint, but important to outline in the context of this thesis.

We then follow the lines of [77]. We motivate that, before coming to an answer for the existence of a Poincaré lemma, we need to study some properties of constant rank operators in $\mathbb{C}$. In particular, we observe that a Poincaré lemma cannot hold if operators with coinciding kernels in Fourier space have kernels that differ by infinite-dimensional vector spaces with respect to L^1_{loc} or the space of distributions. So this is the question we need to answer first, which is formulated by Theorem 2.26.

In order to prove Theorem 2.26, we need a suitable version of Hilbert's Nullstellensatz that applies to the present framework. This version is elucidated in Section 2.6.3. We revisit the classical Nullstellensatz which is formulated for (scalar) polynomials acting on an algebraically closed field. Moreover, we define the constant rank property for systems of polynomials that coincides with the notion of constant rank property for differential operators when identifying polynomials to differential operators via the Fourier transform. Then we formulate a vectorial version for the Nullstellensatz that is valid for constant rank systems. This result is a major extension of previous applications of Hilbert's Nullstellensatz in the context of $\mathbb{C}$-elliptic operators, which reduces to a special case in our setting.

Theorem 2.a (=Theorem 2.28)**.** *Let $d, k, l \in \mathbb{N}$ and, for $i \in \{1, ..., d\}$ and $j \in \{1, ..., l\}$, $p_{ij} \in \mathbb{C}[\xi_1, ..., \xi_n]$ be homogeneous polynomials of degree k such that* (2.22) *satisfies the*

constant rank property over $\mathbb{C}$. *Let* $b_1, ..., b_d \in \mathbb{C}[\xi_1, ..., \xi_N]$, $v = (v_1, ..., v_d) \in \mathbb{C}^d$ *and define*

$$B(\xi)(v) := \sum_{i=1}^{d} v_i b_i(\xi).$$

Suppose that for any $\xi = (\xi_1, ..., \xi_N) \in \mathbb{C}^N \backslash \{0\}$ *and* $v = (v_1, ..., v_d) \in \mathbb{C}^d$ *we have that*

$$\Big(\sum_{i=1}^{d} p_{ij}(\xi) v_i = 0 \ \text{ for all } j \in \{1, .., l\} \Big) \quad \Longrightarrow \quad B(\xi)(v) = 0, \tag{2.17}$$

and let $q \in \mathbb{C}[\xi_1, ..., \xi_N]$ *be a homogeneous polynomial of degree* ≥ 1. *Then there exist polynomials* $h_j \in \mathbb{C}[\xi_1, ..., \xi_N]$, $j \in \{1, ..., l\}$, *and an* $s \in \mathbb{N}$ *such that for all* $\xi \in \mathbb{C}^N$ *and all* $v \in \mathbb{C}^d$ *there holds*

$$q^s(\xi) B(\xi)(v) = \sum_{j=1}^{l} h_j(\xi) \sum_{i=1}^{d} v_i p_{ij}(\xi). \tag{2.18}$$

In Section 2.6.4 we return to the statement of differential operators. In particular, we prove that $\ker \mathbb{A}_1[\xi] = \ker \mathbb{A}_2[\xi]$ for all $\xi \in \mathbb{C}^N \setminus \{0\}$ for two differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$ if and only if the nullspaces of the operators differ by a finite-dimensional vector space with respect to L^1:

Theorem 2.b (= Theorem 2.26 + Corollary 2.31)**.** *Let* $\mathcal{A}^{(1)}$ *and* $\mathcal{A}^{(2)}$ *be two homogeneous differential operators of order* $k^{(1)}$ *and* $k^{(2)}$, *which have constant rank over* $\mathbb{C}$ *and both act on* $\mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^d)$. *Moreover, suppose that their Fourier symbols satisfy*

$$\ker(\mathbb{A}^{(1)}[\xi]) = \ker(\mathbb{A}^{(2)}[\xi]).$$

(a) There exists $\tilde{k} \in \mathbb{N}$ *and a differential operator* $\mathcal{D}$ *such that*

$$\nabla^{\tilde{k}} \circ \mathcal{A}^{(2)} = \mathcal{D} \circ \mathcal{A}^{(1)},$$

and, vice versa, $\bar{k} \in \mathbb{N}$ *and* $\bar{\mathcal{D}}$ *such that*

$$\nabla^{\bar{k}} \circ \mathcal{A}^{(1)} = \bar{\mathcal{D}} \circ \mathcal{A}^{(2)}.$$

(b) We may write

$$\{u \in L^1_{\mathrm{loc}} \colon \mathcal{A}^{(1)} u = 0\} + V = \{u \in L^1_{\mathrm{loc}} \colon \mathcal{A}^{(2)} u = 0\} + W$$

for finite-dimensional vector spaces V *and* W *consisting of polynomials.*

It is worthwile mentioning that the second result may be extended to the space of distributions.

This theorem and further theory of ideals over algebraically closed fields lead to the definition of a ‘natural’ annihilator to a constant rank operator $\mathcal{B}$, which is optimal in the sense of inclusion of the corresponding nullspaces.

Finally, in Section 2.6.5 we return to the motivation: the Poincaré lemma for operators with constant rank in $\mathbb{C}$. We show the validity of such a Poincaré lemma for operators acting on a cube in space dimension $N = 2$. The proof relies on adding measures on the boundary of the cube such that we can apply the Poincaré lemma on the torus, cf. Theorem 2.5. We restrict ourselves to the case of $N = 2$, as there we can fully classify any constant rank operator and easily describe the boundary.

2.6.1. Introduction

The Fourier symbol $\mathbb{A}[\xi]$ is initially defined only for $\xi \in \mathbb{R}^N$. Recall that in this case

$$\mathbb{A}[\xi] = \sum_{|\alpha|=k} A_\alpha \partial_\alpha u.$$

The linear maps $A_\alpha \colon \mathbb{R}^d \to \mathbb{R}^l$ can be naturally extended to maps $A_\alpha \colon \mathbb{C}^d \to \mathbb{C}^l$ and hence we may also define a complex-valued Fourier symbol $\mathbb{A}[\xi] \in \mathrm{Lin}(\mathbb{C}^d, \mathbb{C}^l)$ for any $\xi \in \mathbb{C}^N$.

Definition 2.21 (The constant rank property in $\mathbb{C}$)**.** *(a) We say that the operator $\mathcal{A}$ satisfies the **complex constant rank property** if the Fourier symbol has constant rank in $\xi \in \mathbb{C}^N \setminus \{0\}$, i.e. there is $r \in \mathbb{N}$, such that*

$$\dim_{\mathbb{C}} \ker_{\mathbb{C}} \mathbb{A}[\xi] = r \quad \forall \xi \in \mathbb{C}^N \setminus \{0\}.$$

(b) We call the set

$$\Lambda^{\mathbb{C}} = \Lambda^{\mathbb{C}}_{\mathcal{A}} = \bigcup_{\xi \in \mathbb{C}^N \setminus \{0\}} \ker_{\mathbb{C}} \mathbb{A}[\xi] \subset \mathbb{C}^d$$

*the **complex characteristic cone** of $\mathcal{A}$.*

*(c) We say that $\mathcal{A}$ satisfies the **complex spanning property** if the characteristic cone of $\mathcal{A}$ spans up $\mathbb{C}^d$, i.e. $\operatorname{span} \Lambda^{\mathbb{C}}_{\mathcal{A}} = \mathbb{C}^d$.*

Obviously, the constant rank property in $\mathbb{C}$ is a stronger condition than constant rank with respect to the field $\mathbb{R}$; for example, the Laplace operator Δ is $\mathbb{R}$-elliptic ($\ker \Delta(\xi) = \{0\}$ for any $\xi \in \mathbb{R}^N \setminus \{0\}$), but whenever $\xi_1^2 + ... + \xi_N^2 = 0$, the kernel of $\mathbb{A}[\xi]$ is $\mathbb{C}$.

As a consequence, any property which was directly following from the constant rank property in $\mathbb{R}$ also holds for the constant rank property in $\mathbb{C}$. Recall that $\mathbb{C}$ is an algebraically closed field, whereas $\mathbb{R}$ is not; so we may show even more algebraic properties of such operators.

First of all, let us note that using the argumentation of [123, 12], one may obtain the analogue of Theorem 2.6.

Proposition 2.22 (Potentials with respect to the complex constant rank property)**.** *Let $\mathcal{A}$ be a homogeneous differential operator of order k with constant coefficients. The following are equivalent.*

(a) $\mathcal{A}$ satisfies the complex constant rank property;

(b) $\mathcal{A}$ has a complex potential $\mathcal{B}$, i.e. a differential operator $\mathcal{B}$, such that

$$\operatorname{Im}_{\mathbb{C}} \mathbb{B}[\xi] = \ker_{\mathbb{C}} \mathbb{A}[\xi], \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

(c) $\mathcal{A}$ has a complex annihilator $\mathcal{A}'$, i.e a differential operator $\mathcal{A}'$, such that

$$\operatorname{Im}_{\mathbb{C}} \mathbb{B}[\xi] = \ker_{\mathbb{C}} \mathbb{A}[\xi], \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Let us mention that all examples from 2.4 are also potential–annihilator pairs in $\mathbb{C}$.

Example 2.23 ((Non-)Examples of operators of constant rank over $\mathbb{C}$).

(a) $\mathbb{C}$-ellipticity of an operator $\mathcal{B}$ means that $\mathbb{B}[\xi]\colon \mathbb{C}^m \to \mathbb{C}^d$ is injective for any $\xi \in \mathbb{C}^N \setminus \{0\}$. Examples for such operators are the gradient, the higher-order gradient and the symmetric gradient $\epsilon(u) = 1/2(\nabla + \nabla^T)$.

(b) Given $N \geq 2$, the operators curl and $\operatorname{curl}\operatorname{curl}^T$ satisfy the complex constant rank property (cf. the calculation in Example 2.2 (d) and (e)).

(c) The divergence operator (cf. Example 2.2) also has constant rank over $\mathbb{C}$. Likewise, the divergence of symmetric matrices (cf. Chapter B) has constant rank over $\mathbb{C}$.

(d) The Laplacian $\mathcal{B} = \Delta$ does not satisfy the constant rank condition over $\mathbb{C}$. For instance, let $N = 2$. Then

$$\ker(\mathcal{B}[\xi]) = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda(1,i) \text{ or } \xi = \lambda(1,-i),\ \lambda \in \mathbb{R}, \\ \{0\} & \text{otherwise,} \end{cases}$$

and so the constant rank condition is violated over $\mathbb{C}$; still, over the base field $\mathbb{R}$ the Laplacian is elliptic and hence of constant rank over $\mathbb{R}$.

Up to minor changes in notation, the remaining part of this Section 2.6 coincides with the preprint [77].

2.6.2. A Poincaré lemma for $\mathbb{C}$-elliptic operators and the main result

Hitherto, the constant rank property has been mainly studied in the framework of $\mathbb{C}$-*elliptic operators* (cf. [138, 27, 82, 75], which means that $\ker_{\mathbb{C}} \mathbb{A}[\xi] = \{0\}$ for any $\xi \in \mathbb{C}^N \setminus \{0\}$. Indeed, one of the main results for $\mathbb{C}$-elliptic operators is the validity of a Poincaré lemma, i.e. that if $\mathcal{B}$ is $\mathbb{C}$-elliptic and $\mathcal{A}$ is an annihilator, then up to a finite dimensional vector space $X \subset L^2(\Omega, \mathbb{R}^d)$

$$\mathcal{A}u = 0 \quad \Longrightarrow \quad \exists v \in H^{k_{\mathcal{B}}}(\Omega, \mathbb{R}^m) \text{ such that } u - \mathcal{B}v \in X. \tag{2.19}$$

In particular, if the annihilator $\mathcal{A}$ is chosen wisely, one may even take $X = \{0\}$, obtaining a strong Poincaré lemma (cf. [82]).

Let us shortly outline, how one can prove this result. First of all, note that in the case $\mathcal{A} = \operatorname{curl}$ and $\mathcal{B} = \nabla$, the fundamental theorem of calculus provides a suitable operation to obtain a function v as in (2.19). If Ω is star-shaped, i.e. for every $x \in \Omega$ we have $[0, x] \subset \Omega$, and u is continuous, we might define

$$\nabla^{-1} u = v := \int_0^1 u(tx) \cdot x \, \mathrm{d}t. \tag{2.20}$$

Let us mention that for non-regular u and general (still simply connected) domains Ω one has to alter the definition, but the idea stays the same. For higher gradients, one just applies the fundamental theorem multiple times.

Therefore, we know how to obtain a Poincaré lemma for gradients. The second ingredient is to show that one may reduce the treatment of any $\mathbb{C}$-elliptic differential operator to higher order gradients, which is expressed by the following proposition [76].

Proposition 2.24 (Equivalences for $\mathbb{C}$-elliptic operators)**.** *Let $\mathcal{B}$ be a differential operator with constant rank with respect to $\mathbb{R}$. The following statements are equivalent:*

(a) $\mathcal{B}$ is $\mathbb{C}$-elliptic;

(b) There is a differential operator $\tilde{B}$ and $\tilde{k} \in \mathbb{N}$, such that

$$\tilde{B} \circ \mathcal{B} = \nabla^{\tilde{k}};$$

(c) The nullspace of $\mathcal{B}$ (as a subset of $L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^m)$) is finite-dimensional.

The equivalence (a) $\Leftrightarrow$ (b) helps us to come up with a suitable operation obtaining a Poincaré lemma. If u satisfies $\mathcal{A}u = 0$, one may apply the differential operator $\tilde{B}$ and then the inverse $(\nabla^{\tilde{k}})^{-1}$. As one loses a polynomial information when applying $\tilde{B}$, one then obtains a Poincaré lemma up to a finite dimensional vector space.

As a motivation for the remainder of the chapter, let us formulate the following questions arising from the case of $\mathbb{C}$-elliptic operators.

Question 2.25. *(a) Is there a generalised version of Proposition 2.24 in the framework of the constant rank condition?*

(b) If yes, how does this help us to formulate and prove a Poincaré lemma?

(c) Finally, given a differential operator $\mathcal{B}$, can we find an annihilator $\mathcal{A}$, such that the finite dimensional vector space X, for which a Poincaré lemma does not hold, is small (or is this set even empty)?

Let us focus on (a), the question (b) is adressed in the special case $N = 2$, $\Omega = (0,1)^2$ in Subsection 2.6.5. A problem closely related to (c) is answered by Remark 2.34.

The main result of this section is the following version of Proposition 2.24.

Theorem 2.26. *Let $\mathcal{A}$, $\widetilde{\mathcal{A}}$ be two differential operators with constant rank over $\mathbb{C}$. Then the following are equivalent:*

(a) For all $\xi \in \mathbb{C}^n \setminus \{0\}$ *we have*

$$\ker(\mathbb{A}[\xi]) = \ker(\widetilde{\mathbb{A}}[\xi]).$$

(b) There exist two finite dimensional vector subspaces $\mathcal{X}_1, \mathcal{X}_2$ *of the* $\mathbb{R}^d$*-valued polynomials on* $\mathbb{R}^N$ *such that*

$$\ker(\mathcal{A}) + \mathcal{X}_1 = \ker(\widetilde{\mathcal{A}}) + \mathcal{X}_2, \tag{2.21}$$

where ker *is understood as the nullspace in* $\mathcal{D}'(\mathbb{R}^N, \mathbb{R}^d)$*, so e.g.*

$$\ker(\mathcal{A}) = \{T \in \mathcal{D}'(\mathbb{R}^N, W) \colon \ \mathcal{A}T = 0\}.$$

Observe that if we chose one of the operators to be $\mathbb{C}$-elliptic, e.g. $\tilde{A} = \nabla$, we recover the statement (a) $\Leftrightarrow$ (c) from Proposition 2.24. A suitable version of Proposition 2.24 (b) is pointed out in Corollary 2.31. Let us further note that if the Fourier symbols $\mathbb{A}[\xi]$ and $\widetilde{\mathbb{A}}[\xi]$ have the same nullspace for any ξ, then they are both annihilators of some differential operator $\mathcal{B}$ with constant rank in $\mathbb{C}$. Also note that the statement of Theorem 2.26 is false if we drop the assumption that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ satisfy the constant rank property over $\mathbb{C}$ (cf. Example 2.32).

In the language of algebraic geometry, the proof of Theorem 2.26 relies on a vectorial Nullstellensatz to be stated and established in Section 2.6.3 below. Nullstellensatz techniques have been employed in slightly different contexts (see [138, 82, 76]). However, these by now routine applications to differential operators (to be revisited in detail in Section 2.6.3) do not prove sufficient to establish Theorem 2.26.

If a differential operator $\mathcal{B}$ has an annihilator $\mathcal{A}$ of constant complex rank, this annihilator is in some sense minimal when being compared with other annihilators (so e.g. $D \circ \mathcal{A}$ for (real) elliptic operators D on $\mathbb{R}^N$ from X to some finite dimensional real vector space Y). Thus, annihilators of constant complex rank – provided existent – are *natural*. Even though the condition of constant rank over $\mathbb{C}$ appears quite restrictive, it is satisfied for a wealth of operators to be gathered below. As an interesting byproduct, such annihilators can be utilised to derive a Poincaré-type lemma in $N = 2$ dimensions; see Section 2.6.5 for this matter and related open questions in this context.

Organisation of this Section

Apart from this introductory subsection, the section is organised as follows: Subsection 2.6.3 is devoted to a suitable variant of a vectorial Nullstellensatz, that displays the pivotal step in the proof of Theorem 2.26 in Subsection 2.6.4. The section then is concluded by a sample application on a two-dimensional Poincaré-type lemma in Subsection 2.6.5.

Notation

For $k \in \mathbb{N}$, we denote $\mathcal{P}_k(\mathbb{R}^N, \mathbb{R}^d)$ the $\mathbb{R}^d$-valued polynomials on $\mathbb{R}^N$ of degree at most k; the space of $\mathbb{R}^d$-valued polynomials p on $\mathbb{R}^N$ which are homogeneous of degree k, so satisfy $p(\lambda x) = \lambda^k p(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^N$, is denoted as $\mathcal{P}_k^h(\mathbb{R}^N, \mathbb{R}^d)$.

2.6.3. A Nullstellensatz for operators of constant complex rank

The proof of Theorem 2.26 hinges on a variant of the Hilbert Nullstellensatz from algebraic geometry stated in Theorem 2.28 below. For the reader's convenience, let us first display a classical version of the Hilbert Nullstellensatz as a background tool, which may e.g. be found in [81, 117]

Lemma 2.27 (HNS)**.** *Let* $\mathbb{K}$ *be an algebraically closed field,* $p_i \in \mathbb{K}[\xi_1, \ldots, \xi_N]$*,* $i = 1, ..., I$ *be polynomials and* $q \in \mathbb{K}[\xi_1, \ldots, \xi_N]$*, such that*

$$f_i(\xi) = 0 \, \forall \xi \in \mathbb{K}^N \quad \Longrightarrow \quad q(\xi) = 0.$$

Then there is $s \in \mathbb{N}$ *and polynomials* $r_i \in \mathbb{K}[\xi_1, \ldots, \xi_N]$*, such that*

$$q^s = \sum_{i=1}^{I} r_i p_i.$$

The standard use of this result in the context of differential operators (see Remark 2.29 below) does not prove sufficient for Theorem 2.26. Hence let $d, k, l \in \mathbb{N}$. For $i \in \{1, ..., d\}$ and $j \in \{1, ..., l\}$ we consider homogeneous polynomials $p_{ij} \in \mathbb{C}[\xi_1, ..., \xi_N]$ of order k and the system of equations

$$\sum_{i=1}^{d} p_{ij}(\xi) v_i = 0, \quad \xi = (\xi_1, ..., \xi_N) \in \mathbb{C}^N, \; j \in \{1, ..., l\}, \tag{2.22}$$

where $v = (v_1, ..., v_d) \in \mathbb{C}^d$. In accordance with Definition 2.21, we say that the system (2.22) satisfies the *constant rank property over* $\mathbb{C}$ if there exists an $r \in \{0, ..., d\}$ such that for every $\xi \in \mathbb{C}^n \setminus \{0\}$ the vector space

$$\mathcal{X}_\xi((p_{ij})_{ij}) := \left\{ v = (v_1, ..., v_d) \in \mathbb{C}^d \colon \sum_{i=1}^{d} p_{ij}(\xi) v_i = 0 \text{ for all } j \in \{1, ..., l\} \right\}$$

has dimension $(d - r)$ over $\mathbb{C}$. We may now state the main ingredient for the proof of Theorem 2.26, which arises as a generalisation of the usual Hilbert Nullstellensatz:

Theorem 2.28 (Vectorial Nullstellensatz for constant rank operators)**.** *Let* $d, k, l \in \mathbb{N}$ *and, for* $i \in \{1, ..., d\}$ *and* $j \in \{1, ..., l\}$*,* $p_{ij} \in \mathbb{C}[\xi_1, ..., \xi_n]$ *be homogeneous polynomials of degree* k *such that* (2.22) *satisfies the* constant rank property over $\mathbb{C}$*. Let* $b_1, ..., b_d \in \mathbb{C}[\xi_1, ..., \xi_N]$*,*

$v = (v_1, ..., v_d) \in \mathbb{C}^d$ and define

$$B(\xi)(v) := \sum_{i=1}^{d} v_i b_i(\xi).$$

Suppose that for any $\xi = (\xi_1, ..., \xi_N) \in \mathbb{C}^N \backslash \{0\}$ and $v = (v_1, ..., v_d) \in \mathbb{C}^d$ we have that

$$\Big(\sum_{i=1}^{d} p_{ij}(\xi) v_i = 0 \ \textit{ for all } j \in \{1, .., l\} \Big) \quad \Longrightarrow \quad B(\xi)(v) = 0, \tag{2.23}$$

and let $q \in \mathbb{C}[\xi_1, ..., \xi_N]$ be a homogeneous polynomial of degree ≥ 1. Then there exist polynomials $h_j \in \mathbb{C}[\xi_1, ..., \xi_N]$, $j \in \{1, ..., l\}$, and an $s \in \mathbb{N}$, such that for all $\xi \in \mathbb{C}^N$ and all $v \in \mathbb{C}^d$ there holds

$$q^s(\xi) B(\xi)(v) = \sum_{j=1}^{l} h_j(\xi) \sum_{i=1}^{d} v_i p_{ij}(\xi). \tag{2.24}$$

Proof. Let the polynomials p_{ij} satisfy the constant rank property for some fixed $r \in \{0, ..., d\}$. We define sets

$$\mathcal{J} = \{J \subset \{1, ..., l\} \colon |J| = r\}, \quad \mathcal{I} = \{I \subset \{1, ..., d\} \colon |I| = r\}.$$

For a subset $J \in \mathcal{J}$ we write $J = \{j(1), ..., j(r)\}$ for $j(1) < ... < j(r)$ and likewise for $I \in \mathcal{I}$, $I = \{i(1), ..., i(r)\}$ for $i(1) < ... < i(r)$. Define the matrix $M_{IJ} \in \mathbb{C}^{r \times r}$ by its entries via

$$(M_{IJ})_{\beta\gamma} := p_{i(\beta), j(\gamma)}.$$

Now consider an arbitrary $(r \times r)$-minor of $P(\xi) = (p_{ij}(\xi))_{ij}$; any such minor arises as $\det(M_{IJ}(\xi))$ for some $I \in \mathcal{I}, J \in \mathcal{J}$. If $\xi \in \mathbb{C}^N \backslash \{0\}$ is a common zero of all $q_{IJ} := \det(M_{IJ})$, then $\dim_{\mathbb{C}}(\mathcal{X}_\xi((p_{ij})_{ij})) \neq d - r$ by virtue of the constant rank property over $\mathbb{C}$. On the other hand, by homogeneity of the p_{ij}'s, $\xi = 0$ is a common zero of the q_{IJ}'s, and so is the only common zero of the q_{IJ}'s.

On the other hand, $\xi = 0$ is a zero of any homogeneous polynomial $q \in \mathbb{C}[\xi_1, ..., \xi_N]$ of degree ≥ 1. Thus, the Hilbert Nullstellensatz from Lemma 2.27 implies the existence of an $s \in \mathbb{N}$ and polynomials $g_{IJ} \in \mathbb{C}[\xi_1, ..., \xi_N]$ $(I \in \mathcal{I}, J \in \mathcal{J})$ such that

$$q^s = \sum_{J \in \mathcal{J}} \sum_{I \in \mathcal{I}} g_{IJ} \det(M_{IJ}). \tag{2.25}$$

We now come to the definition of h_j as appearing in (2.24). For the matrix M_{IJ} and $\gamma \in \{1, ..., r\}$, we define the matrix M_{IJ}^{γ} as the matrix where the γ-th column vector is replaced by $(b_{i(\beta)})_{\beta=1,...,r}$, i.e.,

$$M_{IJ}^{\gamma} = \begin{pmatrix} p_{i(1)j(1)} & \cdots & p_{i(1)j(\gamma-1)} & b_{i(1)} & p_{i(1)j(\gamma+1)} & \cdots & p_{i(1)j(r)} \\ \cdots & & \cdots & \cdots & \cdots & & \cdots \\ p_{i(r)j(1)} & \cdots & p_{i(r)j(\gamma-1)} & b_{i(r)} & p_{i(r)j(\gamma+1)} & \cdots & p_{i(r)j(r)} \end{pmatrix}.$$

We then define for $j \in \{1, ..., l\}$

$$h_j := \sum_{\gamma=1}^{r} \sum_{I \in \mathcal{I}} \sum_{J \in \mathcal{J}\,:\, j(\gamma)=j} g_{IJ} \det(M_{IJ}^{\gamma}) \tag{2.26}$$

and claim that

$$\sum_{\gamma=1}^{r} p_{ij(\gamma)} \det(M_{IJ}^{\gamma}) = b_i \det M_{IJ} \qquad \text{for all } i \in \{1, ..., d\}, \tag{2.27}$$

$$\sum_{j=1}^{l} h_j \left(\sum_{i=1}^{d} p_{ij} v_i \right) = q^s \sum_{i=1}^{d} b_i v_i, \tag{2.28}$$

so that the h_j's will satisfy (2.24). Let us see how (2.28) follows from (2.27): In fact,

$$\begin{aligned} \sum_{j=1}^{l} h_j \left(\sum_{i=1}^{d} p_{ij} v_i \right) &\overset{(2.26)}{=} \sum_{j=1}^{l} \sum_{i=1}^{d} \sum_{\gamma=1}^{r} \sum_{I \in \mathcal{I}} \sum_{J \in \mathcal{J}\,:\, j(\gamma)=j} g_{IJ} \det(M_{IJ}^{\gamma}) p_{ij} v_i \\ &= \sum_{J \in \mathcal{J}} \sum_{I \in \mathcal{I}} g_{IJ} \left(\sum_{i=1}^{d} \sum_{\gamma=1}^{r} p_{ij(\gamma)} \det(M_{IJ}^{\gamma}) v_i \right) \\ &\overset{(2.27)}{=} \sum_{J \in \mathcal{J}} \sum_{I \in \mathcal{I}} g_{IJ} \det(M_{IJ}) \cdot \left(\sum_{i=1}^{d} b_i v_i \right) \\ &\overset{(2.25)}{=} q^s \sum_{i=1}^{d} b_i v_i. \end{aligned}$$

Hence it remains to show (2.27). To this end, for $\beta, \gamma \in \{1, ..., r\}$ let us define the matrix $M_{I(\beta)J(\gamma)}$ as the $(r-1) \times (r-1)$ matrix, where the γ-th column of M_{IJ} and the β-th row have been removed. By the Laplace expansion formula and the definition of M_{IJ}^{γ}, we then obtain

$$\det(M_{IJ}^{\gamma}) = \sum_{\beta=1}^{r} (-1)^{\beta+\gamma} b_{i(\beta)} \det(M_{I(\beta)J(\gamma)}).$$

Hence,

$$\sum_{\gamma=1}^{r} p_{ij(\gamma)} \det(M_{IJ}^{\gamma}) = \sum_{\beta,\gamma=1}^{r} (-1)^{\beta+\gamma} b_{i(\beta)} \det(M_{I(\beta)J(\gamma)}) p_{ij(\gamma)}. \tag{2.29}$$

Now consider the $(r+1)\times(r+1)$-matrix M defined by

$$M := \begin{pmatrix} p_{i(1)j(1)} & \cdots & p_{i(1)j(r)} & b_{i(1)} \\ \vdots & \ddots & \vdots & \vdots \\ p_{i(r)j(1)} & \cdots & p_{i(r)j(r)} & b_{i(r)} \\ p_{ij(1)} & \cdots & p_{ij(r)} & b_i \end{pmatrix}.$$

By (2.23), for each $\xi \in \mathbb{C}^N \setminus \{0\}$ the subspace of $v \in \mathbb{C}^d$ such that

$$\sum_{i=1}^{d} p_{ij}(\xi)v_i = 0 \text{ for all } j \in \{1, ..., l\}, \qquad \sum_{i=1}^{d} v_i b_i(\xi) = 0$$

is $\mathcal{X}_\xi((p_{ij})_{ij})$ and thus has dimension $(d-r)$. Therefore, all $(r+1)\times(r+1)$ minors of the matrix corresponding to these linear equations vanish. In particular, the determinant of the matrix M is 0. Denote by M^β the $(r \times r)$-submatrix of M, where the last column and the β-th row of M are eliminated. We apply the Laplace expansion formula twice to M (in the last column and then in the last row), to see that

$$\begin{aligned} 0 &= \det(M) \\ &= \left(\sum_{\beta=1}^{r} b_{i(\beta)}(-1)^{r+1+\beta}\det(M^\beta)\right) + b_i \det(M_{IJ}) \\ &= \left(\sum_{\gamma=1}^{r}\sum_{\beta=1}^{r}(-1)^{r+1+\beta}(-1)^{r+\gamma} b_{i(\beta)} p_{ij(\gamma)} \det(M_{I(\beta)J(\gamma)})\right) + b_i \det(M_{IJ}). \end{aligned}$$

Therefore,

$$b_i \det(M_{IJ}) = \sum_{\gamma=1}^{r}\sum_{\beta=1}^{r}(-1)^{\beta+\gamma} b_{i(\beta)} p_{ij(\gamma)} \det(M_{I(\beta)J(\gamma)}),$$

which establishes (2.27). The proof is complete. □

Remark 2.29. In the context of differential operators, the Hilbert Nullstellensatz is typically applied to $\mathbb{C}$-elliptic differential operators $\mathcal{A}$ as follows (cf. [138], [82, Lem. 4, Thm. 5], [76, Prop. 3.2]): Let $\mathcal{A}$ be a first order differential operator on $\mathbb{R}^N$ from $\mathbb{R}^d$ to $\mathbb{R}^l$. Then $\mathbb{C}$-ellipticity of $\mathcal{A}$ implies by virtue of the Hilbert Nullstellensatz that there exists $k \in \mathbb{N}$ with the following property: There exists a linear, homogeneous differential operator $\mathcal{L}$ on $\mathbb{R}^N$ from $\mathbb{R}^l$ to $\mathbb{R}^d \odot^k \mathbb{R}^N$ of order $(k-1)$ such that $D^k = \mathbb{L}\mathbb{A}$. Inserting this relation into the usual Sobolev integral representation of $u \in \mathrm{C}^\infty(\overline{B_1(0)}; V)$ (cf. [3, §4] or [106, Thm. 1.1.10.1]) and integrating by parts then yields a polynomial P of order $(k-1)$ such that

$$u(x) = P(x) + \int_{B_1(0)} K(x,y)\mathbb{A}u(y)\,\mathrm{d}y$$

for all $x \in B_1(0)$ and all $u \in \mathrm{C}^\infty(\overline{B_1(0)}, V)$; here, the function $K\colon B_1(0) \times B_1(0) \to$

$\mathrm{Lin}(\mathbb{R}^l, \mathbb{R}^d)$ is a suitable integral kernel. This, in particular, implies that $\dim(\ker(\mathbb{A})) < \infty$. In our situation, a similar approach does not work. This is so because the operators $\mathcal{A}, \widetilde{\mathcal{A}}$ from Theorem 2.26 do not have finite dimensional nullspaces themselves; we may only assert that the nullspaces differ by finite dimensional vector spaces, and this is why we require the refinement provided by Theorem 2.28.

2.6.4. Proof of Theorem 2.26

Based on Theorem 2.28, the proof of Theorem 2.26 requires two additional ingredients that we record next:

Lemma 2.30. *Let $\mathcal{A}\colon \mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^d) \to \mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^l)$ be a homogeneous differential operator of order k. Define the differential operator*

$$\nabla \circ \mathcal{A}\colon \mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^d) \to \mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^l \times \mathbb{R}^N)$$

componentwisely by

$$((\nabla \circ \mathcal{A})u)_i = \partial_i \mathcal{A} u, \qquad i \in \{1, ..., N\}.$$

Then we have

$$\ker(\nabla \circ \mathcal{A}) = \ker(\mathcal{A}) + \mathcal{P}_k(\mathbb{R}^N, \mathbb{R}^d). \tag{2.30}$$

Observe that this result *does not* require the constant rank property.

Proof. Suppose that $u \in \ker(\nabla \circ \mathcal{A})$. Then $\mathcal{A}u$ is a constant function. Consider the space $W \subset \mathbb{R}^l$ defined by $W := \mathrm{span}\{\mathbb{A}[\xi](\mathbb{R}^d)\colon \xi \in \mathbb{R}^N\}$. Note that, on the one hand, $\mathcal{A}u \in W$ pointwisely, and, on the other hand,

$$W = \mathcal{A}\mathcal{P}_k^h(\mathbb{R}^N, \mathbb{R}^d) = \mathcal{A}\mathcal{P}_k(\mathbb{R}^N, \mathbb{R}^d). \tag{2.31}$$

The last line can be seen by considering, for $|\beta| = k$ and $v \in \mathbb{R}^d$, the polynomials $p_\beta(x) := \frac{x^\beta}{\beta!} v$. Then, for any $\xi \in \mathbb{R}^N$,

$$\mathcal{A}\Big(\sum_{|\beta|=k} \xi^\beta p_\beta\Big) = \sum_{|\alpha|=k} \sum_{|\beta|=k} \xi^\beta \mathcal{A}_\alpha \partial^\alpha p_\beta = \sum_{|\alpha|=k} \xi^\alpha \mathcal{A}_\alpha v$$

and so (2.31) follows by the homogeneity of $\mathcal{A}$ of degree k. In particular, for every $u \in \ker(\nabla \circ \mathcal{A})$, we can find a polynomial p of degree k with $\mathcal{A}(u - p) = 0$. Hence $\ker(\nabla \circ \mathcal{A}) \subset \ker(\mathcal{A}) + \mathcal{P}_k(\mathbb{R}^N, \mathbb{R}^d)$. On the other hand, since $\mathcal{A}$ is homogeneous and of order k, every element of $\ker(\mathcal{A}) + \mathcal{P}_k(\mathbb{R}^N, \mathbb{R}^d)$ belongs to the nullspace of $\nabla \circ \mathcal{A}$. Thus (2.30) follows and the proof is complete. □

Corollary 2.31 (Kernels of annihilators)**.** *Let $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ be two homogeneous differential operators of order $k^{(1)}$ and $k^{(2)}$, which have constant rank over $\mathbb{C}$ and both act on*

$C^\infty(\mathbb{R}^N, \mathbb{R}^d)$. *Moreover, suppose that their Fourier symbols satisfy*

$$\ker(\mathbb{A}^{(1)}[\xi]) \subset \ker(\mathbb{A}^{(2)}[\xi]) \qquad \textit{for all}\, \xi \in \mathbb{C}^n. \tag{2.32}$$

Then the following hold:

(a) There exists $\tilde{k} \in \mathbb{N}$ *and a differential operator* $\mathcal{D}$*, such that*

$$\nabla^{\tilde{k}} \circ \mathcal{A}^{(2)} = \mathcal{D} \circ \mathcal{A}^{(1)}.$$

(b) For the nullspace of $\mathcal{A}^{(1)}$ *we have*

$$\{u \in L^1_{\text{loc}} \colon \mathcal{A}^{(1)} u = 0\} \subset \{u \in L^1_{\text{loc}} \colon \mathcal{A}^{(2)} u = 0\} + V,$$

where V *is a finite dimensional vector space (consisting of polynomials).*

(c) If, in addition,

$$\ker(\mathbb{A}^{(1)}[\xi]) = \ker(\mathbb{A}^{(2)}[\xi]),$$

then we may write

$$\{u \in L^1_{\text{loc}} \colon \mathcal{A}^{(1)} u = 0\} + V = \{u \in L^1_{\text{loc}} \colon \mathcal{A}^{(2)} u = 0\} + W$$

for finite dimensional vector spaces V *and* W *consisting of polynomials.*

Proof. Ad (a). We aim to apply Theorem 2.28, and we explain the setting first. Assuming that $\mathcal{A}^{(1)}$ is $\mathbb{R}^{l_1}$-valued and $\mathcal{A}^{(2)}$ is $\mathbb{R}^{l_2}$-valued, we may write for $v = (v_1, ..., v_d) \in \mathbb{C}^d$

$$\mathbb{A}^{(1)}[\xi]v = \Big(\sum_{i=1}^{d} A^{(1)}_{ij}(\xi) v_i\Big)_{j=1,\dots,l_1} \quad \text{and} \quad \mathbb{A}^{(2)}[\xi] = \Big(\mathbb{A}^{(2)}_m[\xi]v\Big)_{m=1,\dots,l_2},$$

where every $\mathbb{A}^{(2)}_m(\xi)v$ can be written as

$$\mathbb{A}^{(2)}_m[\xi]v = \sum_{i=1}^{d} v_i b_{im}(\xi).$$

For each $m \in \{1, ..., l_2\}$, we apply Theorem 2.28 to $p_{ij}(\xi) = A^{(1)}_{ij}(\xi)$ and $B(\xi) = \mathbb{A}^{(2)}_m[\xi]$; note that its applicability is ensured by (2.32).

In consequence, for every component $\mathbb{A}^{(2)}_m$ with $m \in \{1, ..., l_2\}$ and $a \in \{1, ..., n\}$, we may find $K(a, m) \in \mathbb{N}$ and polynomials $h_{j,a} \in \mathbb{C}[\xi_1, ..., \xi_n]$, such that

$$\xi_a^{N(a,m)} \mathbb{A}^{(2)}_m[\xi] = \sum_{j=1}^{l_1} h_{j,a}(\xi) \sum_{i=1}^{d} A^{(1)}_{ij}[\xi] v_i.$$

Therefore, choosing $\widetilde{k} := N \max_{m \in \{1,\dots,l_2\}, a \in \{1,\dots,n\}} K(a, m)$, we obtain that for every $\alpha \in \mathbb{N}^n$

with $|\alpha| = \tilde{k}$ and $m \in \{1, ..., l_2\}$, there exists $h_{j\alpha}$ such that

$$\xi^\alpha \mathbb{A}^{(2)}_m[\xi] = \sum_{j=1}^{l_1} h_{j\alpha}(\xi) \sum_{i=1}^{d} A^{(1)}_{ij}(\xi) v_i.$$

Defining the differential operator $\mathcal{D}$ according to this Fourier symbol, (a) follows, i.e.,

$$\mathbb{D}[\xi]_{m,\alpha}(w) = \sum_{j=1}^{l_1} h_{j\alpha}(\xi) w_j, \quad m \in \{1, \dots, l_2\}.$$

Ad (b). This directly follows from Lemma 2.30. Indeed, applying Lemma 2.30 $\widetilde{k}$-times, there exists a finite dimensional space $\widetilde{V}$ of polynomials such that

$$\{u \in L^1_{\mathrm{loc}} \colon \nabla^{\tilde{k}} \mathcal{A}^{(2)} u = 0\} = \{u \in L^1_{\mathrm{loc}} \colon \mathcal{A}^{(2)} u = 0\} + \widetilde{V}.$$

As $\ker \mathcal{A}^{(1)} \subset \ker \mathcal{B} \circ \mathcal{A}^{(1)} = \ker \nabla^{\tilde{k}} \circ \mathcal{A}^{(2)}$, the result directly follows. Finally, (c) is immediate by applying (b) in both directions. The proof is complete. ◻

We may now turn to the proof of the main theorem.

Proof of Theorem 2.26. Direction (a)$\Rightarrow$ (b) of Theorem 2.26 is just Corollary 2.31; using convolution one may first observe this for L^1_{loc} functions and then generalise it to $\mathcal{D}'$. On the other hand, direction (b)$\Rightarrow$(a) follows from a routine construction (see e.g. [138, 65, 79]) which we outline for the reader's convenience. Suppose towards a contradiction that there exists $\xi \in \mathbb{C}^N \setminus \{0\}$ such that $\ker(\mathbb{A}[\xi]) \neq \ker(\widetilde{\mathbb{A}}[\xi])$. Without loss of generality, we may then assume there exists $v \in \mathbb{C}^l \setminus \{0\}$ such that $v \in \ker(\mathbb{A}[\xi]) \setminus \ker(\widetilde{\mathbb{A}}[\xi])$. The proof is then concluded by considering the plane waves $u_h(x) := e^{\mathrm{i}x\cdot h\xi} v$ for $h \in \mathbb{Z}$ and sorting by real and imaginary parts; passing to the span of u_h, $h \in \mathbb{Z}$, we obtain an infinite dimensional vector space which, up to the zero function, belongs to $\ker(\mathcal{A}) \setminus \ker(\widetilde{\mathcal{A}})$. ◻

Example 2.32. In general, Theorem 2.26 will fail if $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ *do not* satisfy the complex constant rank property. As one readily verifies, if we take $\mathcal{A} = \Delta$ and $\widetilde{\mathcal{A}} = \Delta^2$ to be the Laplacian and the Bi-Laplacian (and so both violate the constant rank condition over $\mathbb{C}$ by Example (d)) in $n = 2$ dimensions,

$$\ker_{\mathbb{C}}(\mathbb{A}[\xi]) = \ker_{\mathbb{C}}(\widetilde{\mathbb{A}}[\xi]) = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda(1, i)^\top \text{ or } \xi = \lambda(1, -i)^\top, \ \lambda \in \mathbb{C} \\ \{0\} & \text{otherwise.} \end{cases}$$

Denote $\ker(\Delta)$ and $\ker(\Delta^2)$ the nullspaces of Δ or Δ^2, respectively, in $\mathcal{D}'(\mathbb{R}^N)$. Denoting the homogeneous harmonic polynomials on $\mathbb{R}^N$ by $\mathcal{P}_{\mathrm{ho}}(\mathbb{R}^N)$, we have

$$\ker(\Delta) + \widetilde{\mathcal{P}} \subset \ker(\Delta^2),$$

where $\widetilde{\mathcal{P}} = \{v \colon\ \Delta v = p \text{ for some } p \in \mathcal{P}_{\mathrm{ho}}(\mathbb{R}^N)\}$, and from here one sees that the nullspaces of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ differ by an infinite dimensional vector space.

Remark 2.33. Up to now, we assumed that the polynomials p_{ij} are homogeneous polynomials of order k. This assumption is motivated by the fact that we deal with homogeneous differential operators. However, we can also define the *constant rank property* when not all polynomials have the same order. In particular, for polynomials p_{ij} as in (2.22) we may weaken the assumption to p_{ij} having order $k_j \in \mathbb{N}$, and the statement of the vectorial Nullstellensatz still holds true.

For the corresponding differential operator, this includes the following setting. The operator $\mathcal{A} = (\mathcal{A}_0, ..., \mathcal{A}_k)$ is componentwisely defined via homogeneous differential operators $\mathcal{A}_i \colon \mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^d) \to \mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^{l_i})$ of order i (for $i = 0$ the operator $\mathcal{A}_0$ is similarly understood to be a linear map). In particular, $\mathcal{A} \colon \mathrm{C}^\infty(\mathbb{R}^N, V) \to \mathrm{C}^\infty(\mathbb{R}^N, \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_k})$. The constant rank property in this setting means that there exists $r \in \mathbb{N}$ such that

$$\bigcap_{i=0}^{k} \ker(\mathbb{A}_i(\xi)) = r, \quad \text{for all } \xi \in \mathbb{C}^N \setminus \{0\}.$$

Observe that it is not required at all, that each homogeneous component satisfies the constant rank property itself, e.g. $\mathcal{B}u = (\partial_1 u, \partial_2^2 u)$.

In view of Lemma 2.30 we can however also transform this setting into a fully homogeneous one, while only allowing an additional finite-dimensional nullspace. Indeed, the operator $\widetilde{\mathcal{A}}$ given by

$$\widetilde{\mathcal{A}} = (\nabla^k \circ \mathcal{A}_0, \nabla^{k-1} \circ \mathcal{A}_1, \ldots, \mathcal{A}_k)$$

is homogeneous of order k and its nullspace only differs by a finite dimensional space from the nullspace of $\mathcal{A}$.

Remark 2.34. For now, we have seen that if $\mathbb{A}[\xi]$ and $\widetilde{\mathbb{A}}[\xi]$ have the same nullspace for all $\xi \in \mathbb{C}^n \setminus \{0\}$, then their nullspaces as differential operators only differ by finite dimensional spaces. Given the nullspaces $V(\xi) = \ker(\mathbb{A}[\xi])$ for some differential operator $\mathcal{A}$, it is thus natural to ask for a *minimal* differential operator in the sense of nullspaces, i.e., such that if $\ker(\mathcal{B}_0(\xi)) = V(\xi)$ and $\ker(\widetilde{\mathbb{A}}[\xi]) = V(\xi)$ for each $\xi \neq 0$, then $\ker(\mathbb{A}_0) \subset \ker(\widetilde{\mathbb{A}})$.

To this end, let us recall some algebraic facts about ideals. Let $w_1, ..., w_d$ be a basis of W. For a constant coefficient differential operator $\mathcal{B}$ with complex Fourier symbol $\mathbb{B}[\xi]$ we define the set of annihilator polynomials $\mathcal{P}_\mathcal{B}$ as all vector valued polynomials vanishing on $\mathbb{A}[\xi]$, i.e.

$$\mathcal{P}_\mathcal{B} = \{P(\xi_1, ..., \xi_N) = \sum_{i=1}^{d} p_i(\xi) w_i \colon P(\xi_1, ..., \xi_N) \circ \mathbb{A}[\xi] = 0\}$$

This $\mathcal{P}_\mathcal{B}$ generates an ideal $\mathcal{I}_\mathcal{B}$ in $\mathbb{C}[\xi_1, ..., \xi_N, w_1, ..., w_d]$. As every ideal in the ring of polynomials is finitely generated, so is $\mathcal{I}_\mathcal{B}$. In particular, there exists a finite generator $\mathcal{A}_0$ consisting of polynomials in $\mathcal{P}_\mathcal{B}$; these are linear in $w_1, ..., w_d$. As a consequence, every

$P \in \mathcal{P}_{\mathcal{B}}$ can be written as

$$P(\xi) = \sum_{P_j \in \mathcal{A}_0} \alpha_j(\xi) P_j \tag{2.33}$$

for some polynomials α_j. In particular, this set $\mathcal{A}_0$ can be identified with a differential operator $\mathcal{A}_0$, which is component-wise homogeneous (where we view differential operators of degree zero as homogeneous of degree zero). Due to (2.33) every differential operator $\mathcal{A}$ which is an annihilator of $\mathcal{B}$ can be written as

$$\mathcal{A} = \mathcal{B}' \circ \mathcal{A}_0,$$

hence $\ker(\mathcal{A}_0) \subset \ker(\mathcal{A})$. Thus we might consider $\mathcal{A}_0$ as the *natural* annihilator of $\mathcal{B}$.

2.6.5. A Poincaré-type lemma in $N = 2$ dimensions

In this concluding section we give a sample application of the results provided so far by proving a Poincaré lemma in two dimensions. For simplicity, we focus on first order operators and functions defined on a cube $Q = (0,1)^N$. For $\mathcal{A}$-free functions on the torus T_N, it is well-known that $\mathcal{B}$, if $\mathcal{B}$ is a potential in the algebraic sense, it is also a potential in the sense that (cf. Theorem 2.5

$$u \in L^2(T_N, \mathbb{R}^d),\ \mathcal{A}u = 0,\ (u)_{T_N} = 0 \quad \Longrightarrow \quad u = \mathcal{B}v \text{ for some } v \in W^{1,2}(T_N, \mathbb{R}^l).$$

This is shown by use of Fourier methods. We cannot apply such a technique directly for functions on the cubes, as here boundary values cannot assumed to be periodic. Our strategy thus is to add a measure μ supported on ∂Q such that for a function u satisfying $\mathcal{A}u = 0$ in $H^{-1}(Q, W)$, the measure $u + \mu$ satisfies $\mathcal{A}(u + \mu) = 0$ in $H^{-2}(T_N, \mathbb{R}^l)$. We then can apply the theory on the torus to get some $v \in L^2(T_N, \mathbb{R}^m)$ with $\mathcal{B}v = (u + \mu)$, i.e. $\mathcal{B}v = u$ in Q. In dimension $N = 2$, we show that this strategy works for any differential operator of constant rank in $\mathbb{C}$ by adding measures on the one-dimensional faces of Q. In higher dimensions, there might be further restrictions on the operators, but e.g. for $\mathcal{B} = \text{curl}$, $\mathcal{A} = \text{div}$ one may show such a result by adding measures on one- and two-dimensional faces.

For the remainder of this section let $\mathcal{A}$ and $\mathcal{B}$ differential operators of first order given by

$$\mathcal{B}u = \sum_{k=1}^{N} B_k \partial_k u, \quad \mathcal{A}u = \sum_{k=1}^{N} A_k \partial_k u,$$

where $B_k \in \text{Lin}(\mathbb{R}^m, \mathbb{R}^d)$, $A_k \in \text{Lin}(\mathbb{R}^d, \mathbb{R}^l)$. Let $Q = (0,1)^N$ and define

$$L^2_{\mathcal{A}}(Q) = \{u \in L^2(Q, \mathbb{R}^d) \colon \mathcal{A}u = 0 \text{ in } H^{-1}(Q, \mathbb{R}^l)\}$$

and likewise

$$H^1_{\mathcal{A}}(Q) = \{u \in H^1(Q, \mathbb{R}^d) \colon \mathcal{A}u = 0 \text{ in } L^2(Q, \mathbb{R}^l)\},$$

both being equipped with the usual norms on these spaces. For the following, we tacitly assume that $\mathcal{A}$ is an annihilator of $\mathcal{B}$ and that $\mathcal{A}$ has constant rank over $\mathbb{C}$. Our objective of this section is to establish the following result:

Theorem 2.35. *Let $N = 2$. Then there exists a finite dimensional space $X \subset H^1_{\mathcal{A}}(Q)$ consisting of polynomials and a linear, bounded map $\mathcal{B}^{-1} \colon H^1_{\mathcal{A}}(Q) \to L^2(Q, \mathbb{R}^m)$, such that $\mathcal{B} \circ \mathcal{B}^{-1}u - u \in X$. If, in addition, the operator $\mathcal{A}$ satisfies the spanning property, then $X = \{0\}$.*

In consequence, in the situation of Theorem 2.35 we may write $u = \mathcal{B}(\mathcal{B}^{-1}u) + \pi$ for some polynomial $\pi \in X$. We split the proof of Theorem 2.35 into several steps.

Lemma 2.36. *Let $N = 2$. We can decompose*

$$\mathbb{R}^d = V_0 + V_1 + V_2, \tag{2.34}$$

such that $V_i \cap V_j = \{0\}$, $V_i \perp V_j$ for $i, j \in \{0, 1, 2\}$ with $i \neq j$ and

$$V_0 = \left(\operatorname{span}_{\xi \in \mathbb{R}^2 \setminus \{0\}} \ker(\mathbb{A}[\xi])\right)^{\perp} = \left(\operatorname{span}(\ker(\mathbb{A}[e_1]) \cup (\ker \mathbb{A}[e_2]))\right)^{\perp},$$
$$V_2 = \bigcap_{\xi \in \mathbb{R}^2 \setminus \{0\}} \ker(\mathbb{A}[\xi]) = \ker(\mathbb{A}[e_1]) \cap \ker(\mathbb{A}[e_2]).$$

Proof. Clearly, $V_0 \perp V_2$, so V_1 may be just chosen accordingly. It remains to show that V_0 and V_2 can be represented in terms of the behaviour of $\mathbb{A}[e_1]$ and $\mathbb{A}[e_2]$. As $\mathbb{A}$ is of order one, then $v \in \ker \mathbb{A}[e_1] \cap \ker \mathbb{A}[e_2]$ implies by linearity that $v \in \ker \mathbb{A}[\lambda e_1 + \mu e_2]$ for all $\lambda, \mu \in \mathbb{R}$, showing the characterisation of V_2. On the other hand, if $\mathcal{A}$ is of order one, then for all $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{R}^2$

$$\operatorname{Im}(\mathbb{A}[\xi_1 e_1 + \xi_2 e_2]) \subset \operatorname{Im}(\mathbb{A}[e_1]) + \operatorname{Im}(\mathbb{A}[e_2]).$$

As $\operatorname{Im}(\mathbb{B}[\xi]) = \ker(\mathbb{A}[\xi])$, we get the desired result for V_0. □

For the following, observe that we may define another differential operator

$$\widetilde{\mathcal{A}} \colon \mathrm{C}^{\infty}(\mathbb{R}^2; \mathbb{R}^d) \to \mathrm{C}^{\infty}(\mathbb{R}^2; \mathbb{R}^l \times V_0)$$

by defining $\widetilde{\mathcal{A}}(u) = (\mathcal{A}u, P_{V_0}(u))$, where P_{V_0} denotes the orthogonal projection onto V_0. Then $\ker(\widetilde{\mathbb{A}}[\xi]) = \ker(\mathcal{A}[\xi])$ for all $\xi \in \mathbb{C}^2 \setminus \{0\}$. In view of Remark 2.33, we have $\ker(\mathcal{A}) = \ker(\widetilde{\mathcal{A}}) + X$ for some finite dimensional subspace $X \subset L^2_{\mathcal{A}}(Q)$. Note that $\widetilde{\mathcal{A}}$ is not homogeneous in total but in its single components; this will suffice for the following. As a consequence, we may assume from now on that $V_0 = 0$ by considering $\widetilde{\mathcal{A}}$ instead of $\mathcal{A}$. This is why we have the finite dimensional space X in the formulation of Theorem 2.35.

Lemma 2.37. *Suppose that* $\mathcal{A}$ *is* spanning, *i.e.,* $V_0 = \{0\}$ *in* (2.34) *and that the union* $\bigcup_{\xi\in\mathbb{R}^2} \operatorname{Im}(\mathbb{A}[\xi])$ *spans* $\mathbb{R}^l$*. Then we have*

$$\mathbb{R}^l = \operatorname{span}_{\xi\in\mathbb{R}^2\setminus\{0\}} \operatorname{Im}(\mathbb{A}[\xi]) = \operatorname{Im}(\mathbb{A}[e_1]) = \operatorname{Im}(\mathbb{A}[e_2]) = \operatorname{Im}(\mathbb{A}[\xi])$$

for all $\xi \in \mathbb{R}^2 \setminus \{0\}$*.*

Let us shortly remark that for the kernel of the differential operator $\mathcal{A}$, we might restrict our study to operators, such that

$$\mathbb{R}^l = \operatorname{span}_{\xi\in\mathbb{R}^2\setminus\{0\}} \operatorname{Im}(\mathbb{A}[\xi]).$$

If this is not satisfied, we might define the vector space Y as above span and consider $\mathcal{A}' = P_Y \circ \mathcal{A}$, where P_Y is the orthogonal projection onto Y. Then $\ker \mathcal{A}' = \ker \mathcal{A}$ and $\mathcal{A}'$ satisfies

$$\operatorname{span}_{\xi\in\mathbb{R}^2\setminus\{0\}} \operatorname{Im}(\mathbb{A}'[\xi]) = Y,$$

i.e. satisfies the assertions of Lemma 2.37.

Proof of Lemma 2.37. Suppose there exist $\xi_1, \xi_2 \in \mathbb{R}^2\setminus\{0\}$ such that $\operatorname{Im}(\mathbb{A}[\xi_1]) \neq \operatorname{Im}(\mathbb{A}[\xi_2])$. In particular, ξ_1 and ξ_2 are linearly independent. Moreover, $\ker(\mathbb{A}^*[\xi_1]) \neq \ker(\mathbb{A}^*[\xi_2])$ and so there exists some $w \in \mathbb{R}^l$ such that $w \in \ker \mathbb{A}^*[\xi_2]$ but $w \notin \ker \mathbb{A}^*[\xi_1]$. Therefore $0 \neq v := \mathbb{A}^*[\xi_1 + \lambda\xi_2]w \in \operatorname{Im}(\mathbb{A}^*[\xi_1 + \lambda\xi_2])$ for any $\lambda \in \mathbb{R}$. As $\operatorname{Im}(\mathbb{A}^*[\xi]) = (\ker \mathbb{A}[\xi])^\perp$, $P_{\ker(\mathbb{A}[\xi_1+\lambda\xi_2])}(v) = 0$, where again P_V denotes the orthogonal projection onto the subspace $V \subset \mathbb{R}^d$. The map

$$\xi \mapsto P_{\ker(\mathbb{A}[\xi])}(\cdot) \tag{2.35}$$

is homogeneous of degree zero and continuous for $\mathbb{A}$ satisfying the constant rank property [65, Prop. 2.7]. Every $\xi \in \mathbb{R}^2 \setminus \mathbb{R}\xi_2$ can be written as $\xi = \mu(\xi_1 + \lambda\xi_2)$ for suitable $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{0\}$. For such ξ, the zero homogeneity of (2.35) yields $P_{\ker(\mathbb{A}[\xi])}(v) = 0$. On the other hand, choosing $\mu = \lambda^{-1}$ and letting $\lambda \to \infty$, the continuity of (2.35) we conclude that $P_{\ker(\mathbb{A}[\xi])}(v) = 0$ for any $\xi \in \mathbb{R}^2 \setminus \{0\}$. Combining this with the zero homogeneity of (2.35), we also obtain $v \in (\ker(\mathbb{A}(\theta\xi_2)))^\perp$ for all $\theta \in \mathbb{R}\setminus\{0\}$. Hence, $v \in (\ker(\mathbb{A}[\xi]))^\perp$ for all $\xi \in \mathbb{R}^2 \setminus \{0\}$, and this contradicts our assumption $V_0 = \{0\}$. The proof is complete. □

Lemma 2.38. *Let* $\xi_1, \xi_2 \in \mathbb{R}^2$ *be linearly independent and* $\mathcal{A}$ *be spanning in the sense of Lemma 2.37. Then there is a linear map* $L_{\xi_1,\xi_2} \colon \mathbb{R}^l \to \ker(\mathbb{A}[\xi_1])$ *with*

$$\mathbb{A}[\xi_2] \circ L_{\xi_1,\xi_2} = \operatorname{id}_{\mathbb{R}^l}.$$

Proof. For two finite dimensional real vector spaces X_1, X_2, we first recall that a linear map $T\colon X_1 \to X_2$ has a right inverse $S\colon X_2 \to X_1$ if and only if T is surjective. In view of the lemma, we thus have to establish that $\mathbb{A}[\xi_2]|_{\ker(\mathbb{A}[\xi_1])}$ is surjective, and this follows

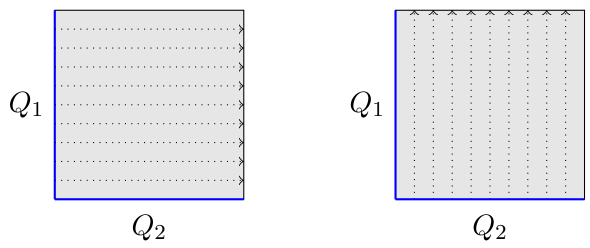

Figure 2.1.: Cube notation and the idea in the proof of Lemma 2.39. We *periodify* the given functions to access the theory on the two-dimensional torus T_2 in Proposition 2.40. To enforce periodicity, the non-periodic contributions of some u are handled by adding suitable correctors defined in terms of horizontal or vertical line integrals, respectively.

from a dimensional argument as follows: Let $r = \dim(V_2)$ and $s = \dim(\ker(\mathbb{A}[\xi]))$, which does not depend on $\xi \in \mathbb{R}^2 \setminus \{0\}$ due to the constant rank property. As $\mathcal{A}$ is spanning,

$$d = \dim(\ker(\mathbb{A}[\xi_1])) + \dim(\ker(\mathbb{A}[\xi_2])) - \dim(\ker(\mathbb{A}[\xi_1]) \cap \ker(\mathbb{A}[\xi_2])) = 2s - r.$$

By Lemma 2.37, $\mathbb{R}^l = \operatorname{Im}(\mathbb{A}[\xi_1])$, and thus the rank-nullity theorem yields

$$l = \dim(\operatorname{Im}(\mathbb{A}[\xi_1]) = d - \dim(\ker(\mathbb{A}[\xi_1]) = (2s - r) - s = s - r.$$

On the other hand, restricting $\mathbb{A}[\xi_2]$ to $\ker \mathbb{A}[\xi_1]$, the nullspace of $\mathbb{A}[\xi_2]|_{\ker(\mathcal{A}[\xi_1])}$ is V_0, hence its dimension is r, and the dimension of its image is $s - r$. Hence, $\mathbb{A}[\xi_2]$ restricted to $\ker \mathbb{A}[\xi_1]$ is still surjective onto $\mathbb{R}^l$, and therefore such a map L_{ξ_1,ξ_2} exists. □

The second key ingredient to establish Theorem 2.35 is the adding of measures on the boundary. In particular, we aim to add a measure μ such that $u + \mu$ is $\mathcal{A}$-free as a measure *on the torus* T_2:

Lemma 2.39 (Adding measures on the boundary)**.** *There are linear maps S_1, S_2 with the following properties:*

1. $S_1 \colon H^1_{\mathcal{A}}(Q) \to \mathcal{P}_2(\mathbb{R}^2; \mathbb{R}^d) \cap \ker(\mathcal{A})$,
2. $S_2 \colon H^1_{\mathcal{A}}(Q) \to L^2(\partial Q, \mathbb{R}^d)(\hookrightarrow H^{-1}(Q, \mathbb{R}^d))$,
3. $\mathcal{A}(u + S_1 u + S_2 u) = 0$ *in* $H^{-2}(T_2, \mathbb{R}^l)$ *for all* $u \in H^1_{\mathcal{A}}(Q)$.

Proof. Recall that the trace operator is bounded from $H^1(Q, \mathbb{R}^d)$ to $L^2(\partial Q, \mathbb{R}^d)$. Define $Q_1 = \{0\} \times [0,1]$ and $Q_2 = [0,1] \times \{0\}$, which may both be seen as subsets of Q and the torus T_2. Define for $u \in H^1_{\mathcal{A}}(Q, \mathbb{R}^d)$

$$w_1(y) := \mathbb{A}[e_1](u(0,y) - u(1,y)) \quad \text{and} \quad w_2(x) := \mathbb{A}[e_2](u(x,0) - u(x,1))$$

for $\mathcal{L}^1$-a.e. $x, y \in [0,1]$. Then $u \mapsto w_j$ is linear and bounded from $H^1_{\mathcal{A}}(Q) \to L^2([0,1], \mathbb{R}^l)$. We then put

$$c_1 := c_1(u) := \int_0^1 w_1(y)\, \mathrm{d}y \quad \text{and} \quad c_2 := c_2(u) := \int_0^1 w_2(x)\, \mathrm{d}x$$

and observe that, because of $u \in H^1_{\mathcal{A}}(Q)$ and a subsequent integration by parts,

$$0 = \int_Q \mathcal{A}u\, \mathrm{d}x = \int_{\partial Q} \mathbb{A}(\nu_{\partial Q}) u\, \mathrm{d}\mathcal{H}^1 \tag{2.36}$$

with the outer unit normal $\nu_{\partial Q}$ to ∂Q. Decomposing ∂Q into its single faces and using the definition of c_1, c_2, we find that $c_1 = -c_2$.

Now define the polynomial $S_1 u$ as follows:

$$S_1 u(x_1, x_2) := a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2 \tag{2.37}$$

for $a_{11}, a_{12}, a_{22} \in \mathbb{R}^d$ defined in terms of the maps L from Lemma 2.38 via

$$a_{12} := -L_{e_1, e_2 - e_1}(c_1), \quad a_{11} := L_{e_2, e_1}(-\mathbb{A}[e_2] a_{12}), \quad a_{22} := L_{e_1, e_2}(-\mathbb{A}[e_1] a_{12}). \tag{2.38}$$

By the properties of the maps L as displayed in Lemma 2.38, we have

$$\begin{aligned} \mathbb{A}[e_1]a_{11} + \mathbb{A}[e_2]a_{12} &= \mathbb{A}[e_1](L_{e_2,e_1}(-\mathbb{A}[e_2]a_{12})) + \mathbb{A}[e_2]a_{12} = 0, \\ \mathbb{A}[e_2]a_{22} + \mathbb{A}[e_1]a_{12} &= \mathbb{A}[e_2](L_{e_1,e_2}(-\mathbb{A}[e_1]a_{12})) + \mathbb{A}[e_1]a_{12} = 0. \end{aligned} \tag{2.39}$$

This particularly implies that

$$\begin{aligned} \mathbb{A} S_1 u &= \mathbb{A}[e_1]\partial_1 S_1 u + \mathbb{A}[e_2]\partial_2 S_1 u \\ &= \mathbb{A}[e_1](2a_{11}x_1 + 2a_{12}x_2) + \mathbb{A}[e_2](2a_{12}x_1 + 2a_{22}x_2) \overset{(2.39)}{=} 0. \end{aligned} \tag{2.40}$$

For future reference, we now record that

$$S_1(S_1 u + u) = 0, \quad \text{for all}\, u \in H^1_{\mathcal{A}}(Q), \tag{2.41}$$

which can be seen as follows: With the obvious definition of $\widetilde{c}_1$,

$$\begin{aligned} \widetilde{c}_1 := \int_0^1 \widetilde{w}_1(y)\, \mathrm{d}y &:= \int_0^1 \mathbb{A}[e_1](S_1 u(0,y) - S_1 u(1,y))\, \mathrm{d}y \\ &= \int_0^1 \mathbb{A}[e_1](2a_{22}y^2 - a_{11} - 2a_{12}y)\, \mathrm{d}y \\ &= \int_0^1 \mathbb{A}[e_1](-a_{11} - 2a_{12}y)\, \mathrm{d}y \qquad \text{(by (2.38) and Lemma 2.38)} \\ &= -\mathbb{A}[e_1]a_{11} - \mathbb{A}[e_1]a_{12} \\ &= \mathbb{A}[e_2 - e_1]a_{12} = -c_1, \end{aligned}$$

the ultimate two equalities being valid by (2.38) and Lemma 2.38 as well. Using that $\mathcal{A}S_1u = 0$, we may argue as in (2.36)ff. to find that

$$\widetilde{c}_2 := \int_0^1 \widetilde{w}_2(x)\,\mathrm{d}y := \int_0^1 \mathbb{A}[e_2](S_1u(x,0) - S_1u(x,1))\,\mathrm{d}x = c_1 = -c_2.$$

This implies that $S_1(S_1u) = -S_1u$ and hereafter (2.41).

We now come to the definition of $S_2u\colon Q_1 \cup Q_2 \to \mathbb{R}^d$. If $S_1u \equiv 0$, we then define

$$S_2u(0,y) := -\int_0^y L_{e_1,e_2}w_1(t)\,\mathrm{d}t, \quad S_2u(x,0) := -\int_0^x L_{e_2,e_1}w_2(t)\,\mathrm{d}t. \tag{2.42}$$

In general, we recall (2.41) and define for general $u \in H^1_{\mathcal{A}}(Q)$

$$S_2u := S_2(u + S_1u).$$

Then S_2u defined on $Q_1 \cup Q_2$ has the following properties:

1. $S_2u(0,0) = S_2u(0,1) = S_2u(1,0) = 0$ due to $c_1 = c_2 = 0$. Indeed, since $u \in H^1_{\mathcal{A}}(Q)$ satisfies $S_1u \equiv 0$, we conclude $a_{12} = 0$. On the other hand, L_{e_1,e_2-e_1} is injective by Lemma 2.38 and so $c_1 = 0$ in light of (2.38); but then $c_2 = -c_1 = 0$ as well.
2. $S_2u \in L^2(Q_1 \cup Q_2; \mathbb{R}^d)$.
3. $S_2u(0,\cdot) \in \ker(\mathbb{A}[e_1])$, $S_2u(\cdot,0) \in \ker(\mathbb{A}[e_2])$ by Lemma 2.38.
4. $S_2u(0,\cdot), S_2u(\cdot,0) \in H^1_0((0,1))$ and, again by Lemma 2.38,

$$\mathbb{A}[e_2]\tfrac{\mathrm{d}}{\mathrm{d}t}S_2u(0,t) = -w_1(t), \quad \mathbb{A}[e_1]\tfrac{\mathrm{d}}{\mathrm{d}t}S_2u(t,0) = -w_2(t).$$

By periodicity, we may view $S_2u \in L^2(\partial Q, \mathbb{R}^d)$, and this can be seen as an element of $H^{-1}(T_2, \mathbb{R}^d)$ by identifying it with the bounded linear functional

$$H^1(T_2,\mathbb{R}^d) \ni \psi \longmapsto \int_{Q_1} S_2u \cdot \mathrm{tr}(\psi)\,\mathrm{d}\mathcal{H}^1 + \int_{Q_2} S_2u \cdot \mathrm{tr}(\psi)\,\mathrm{d}\mathcal{H}^1.$$

Thus, for all $\varphi \in H^2(T_2, \mathbb{R}^l)$ we have

$$\begin{aligned}
\langle \mathcal{A}S_2u, \varphi\rangle_{H^{-2}(T_2)\times H^2(T_2)} &= -\int_{Q_1} S_2u \cdot \mathrm{tr}(\mathcal{A}^*\varphi)\,\mathrm{d}\mathcal{H}^1 - \int_{Q_2} S_2u \cdot \mathrm{tr}(\mathcal{A}^*\varphi)\,\mathrm{d}\mathcal{H}^1\\
&= -\int_{Q_1} (\mathbb{A}[e_1]S_2u)\cdot \mathrm{tr}(\partial_1\varphi) + (\mathbb{A}[e_2]S_2u)\cdot\mathrm{tr}(\partial_2\varphi)\,\mathrm{d}\mathcal{H}^1\\
&\quad - \int_{Q_2} (\mathbb{B}(e_1)S_2u)\cdot \mathrm{tr}(\partial_1\varphi) + (\mathbb{B}(e_2)S_2u)\cdot\mathrm{tr}(\partial_2\varphi)\,\mathrm{d}\mathcal{H}^1\\
&\overset{3}{=} -\int_{Q_1} (\mathbb{A}[e_2]S_2u)\cdot\mathrm{tr}(\partial_2\varphi)\,\mathrm{d}\mathcal{H}^1 - \int_{Q_2} (\mathbb{A}[e_1]S_2u)\cdot\mathrm{tr}(\partial_1\varphi)\,\mathrm{d}\mathcal{H}^1\\
&= \int_{Q_1} (\mathbb{A}[e_2]\partial_2S_2u)\cdot\mathrm{tr}(\varphi)\,\mathrm{d}\mathcal{H}^1 + \int_{Q_2} (\mathbb{A}[e_1]\partial_1S_2u)\cdot\mathrm{tr}(\varphi)\,\mathrm{d}\mathcal{H}^1
\end{aligned}$$

$$\overset{4}{=} -\int_{Q_1} w_1 \cdot \operatorname{tr}(\varphi)\,\mathrm{d}\mathcal{H}^1 - \int_{Q_2} w_2 \cdot \operatorname{tr}(\varphi)\,\mathrm{d}\mathcal{H}^1.$$

On the other hand, for any $\varphi \in H^2(T_2,\mathbb{R}^l)$

$$\begin{aligned}\langle \mathcal{A}u,\varphi\rangle_{H^{-2}(T_2)\times H^2(T_2)} &= -\int_{T_2} u\cdot\mathcal{A}^*\varphi\,\mathrm{d}x = \int_Q \mathcal{A}u\cdot\varphi\,\mathrm{d}x - \int_{\partial Q}(\mathbb{A}[\nu_{\partial\Omega}]u)\cdot\operatorname{tr}(\varphi)\,\mathrm{d}\mathcal{H}^1\\ &= \int_{Q_1} w_1\cdot\operatorname{tr}(\varphi)\,\mathrm{d}\mathcal{H}^1 + \int_{Q_2} w_2\cdot\operatorname{tr}(\varphi)\,\mathrm{d}\mathcal{H}^1.\end{aligned}$$

Hence, $\mathcal{A}(u+S_2u)=0$ in $H^{-1}(T_2,\mathbb{R}^l)$ whenever $u\in H^1_{\mathcal{A}}(Q)\cap\{S_1u\equiv 0\}$. In the general case, we apply the foregoing result to $u+S_1u$ and hence obtain

$$\mathcal{A}((u+S_1u)+S_2(u+S_1u))=0.$$

To conclude, as $S_2u=S_2(u+S_1u)$, we have $\mathcal{A}(u+S_1u+S_2u)=0$ as an element of $H^{-2}(T_2,\mathbb{R}^l)$, and the proof is complete. □

Proposition 2.40. *Suppose that* $\mathcal{A}$ *satisfies the spanning condition. There is a linear and bounded map* $\mathcal{B}^{-1}\colon H^1_{\mathcal{A}}(Q)\to L^2(Q,\mathbb{R}^m)$*, such that* $\mathcal{B}\circ\mathcal{B}^{-1}=\mathrm{id}$*, meaning that for all* $u\in H^1_{\mathcal{A}}(Q)$ *and all* $\varphi\in H^1_0(Q,\mathbb{R}^d)$

$$\int_Q \mathcal{B}^{-1}u\cdot\mathcal{B}^*\varphi = \int_Q u\varphi.$$

Proof. Given $u\in H^1_{\mathcal{A}}(Q)$, we write $u=(u+S_1u)+(-S_1u)=:u_1+u_2$ with S_1 as in the preceding lemma. We treat u_1 and u_2 separately.

Recall that $S_2u_1=S_2u$ for S_2 as in the previous lemma. We write

$$u_1+S_2u_1=u_0+\bar{u}$$

for some $u_0\in\mathbb{R}^d$ and $\bar{u}\in H^{-1}(T_2,\mathbb{R}^d)$, where $\bar{u}$ has zero average over Q, i.e.

$$\langle v,1\rangle_{H^{-1}\times H^1}=0.$$

Note that $\mathcal{A}\bar{u}=0$ in $H^{-2}(T_2,\mathbb{R}^l)$. By the same argument as in Lemma 2.30 we can write $u_0=\mathcal{B}P_1$ for a suitable polynomial P_1 of order one with mean value zero; moreover, the map $u_0\mapsto P_1$ can be arranged to be linear.

For $\bar{u}$, we can apply the theory for constant rank operators on the torus. In particular, by the observation made in (2.4) there exists a linear and bounded operator $\mathcal{B}_T^{-1}\colon H^{-1}(T_2,\mathbb{R}^d)\to L^2(T_2,\mathbb{R}^m)$ that satisfies

$$\mathcal{B}\circ\mathcal{B}_T^{-1}v=v \quad \text{for all } v\in H^{-1}(T_2,\mathbb{R}^d) \text{ with } \mathcal{A}v=0 \text{ and } \langle v,1\rangle_{H^{-1}\times H^1}=0. \tag{2.43}$$

Thus, defining $w:=P_1+\mathcal{B}_T^{-1}(\bar{u}+S_2u)$, we conclude that

1. w depends linearly on u;

2. $\|w\|_{L^2} \le c(\|u_0\|_{L^2} + \|\bar{u} + S_2 u\|_{H^{-1}}) \le C\|u\|_{H^1}$.

3. $\mathcal{B}w = u_1 + S_2 u (= u + S_1 u + S_2 u)$.

We now establish that u_2 can be written as $u_2 = \mathcal{B}P_2$ for a third order polynomial P_2. Recall that

$$S_1 u(x_1, x_2) = a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2$$

with a_{ij} defined as in (2.38). We now define polynomials P_3 and P_4, such that $\mathcal{B}(P_3 + P_4) = -S_1 u$.

Definition of P_3: By the definition of the map L from Lemma 2.38 and by (2.38), the coefficients a_{ij} obey the following:

$$a_{11} \in \ker(\mathbb{A}[e_2]), \quad a_{22} \in \ker(\mathbb{A}[e_1]).$$

The differential operator $\mathcal{B}$ is a potential of $\mathcal{A}$. Therefore, for any $\xi \in \mathbb{R}^2 \setminus \{0\}$, there is a linear map

$$\mathcal{B}^{-1}[\xi] \colon \ker(\mathbb{A}[\xi]) \to (\ker \mathbb{B}[\xi])^{\perp}$$

with $\mathbb{B}[\xi] \circ \mathbb{B}^{-1}(\xi) = \mathrm{Id}_{\ker(\mathcal{B}(\xi))}$ (seen as a Fourier multiplier, this map exactly defines the operator in (2.43)). For future reference, we note that expanding $\mathbb{A}[\xi]\mathbb{B}[\xi] = 0$ for $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{R}^2$ particularly yields

$$\xi_1 \xi_2 (\mathbb{A}[e_1]\mathbb{B}(e_2) + \mathbb{A}[e_2]\mathbb{B}(e_1)) = 0. \tag{2.44}$$

Let us define

$$P_3(x_1, x_2) := -\mathbb{B}^{-1}[e_2](a_{11}) x_1^2 x_2 - \mathbb{B}^{-1}[e_1](a_{22}) x_1 x_2^2.$$

Observe that $(-S_1 u - \mathcal{B}P_3)$ still satisfies $\mathcal{A}(-S_1 u - \mathcal{B}P_3) = 0$ by virtue of $\mathbb{A}[\xi]\mathbb{B}[\xi] = 0$ and (2.40), and has the form

$$\begin{aligned} &(-S_1 u - \mathcal{B}P_3) = a' x_1 x_2, \\ &\quad a' = -2a_{12} + 2\mathbb{A}[e_1]\left(\mathbb{B}^{-1}[e_2](a_{11})\right) + 2\mathbb{B}(e_2)\left(\mathbb{B}^{-1}[e_1](a_{22})\right). \end{aligned} \tag{2.45}$$

Definition of P_4: We define P_4 dependent on a' in (2.45). Note that $\mathcal{A}(a' x_1 x_2) = 0$ and therefore $a' \in \ker(\mathbb{A}[e_1]) \cap \ker(\mathbb{A}[e_2])$. Then define

$$b_2 := \tfrac{1}{2}\mathbb{B}^{-1}[e_1]a', \quad b_1 := \mathbb{B}^{-1}[e_1](-\mathbb{B}[e_2]b_2). \tag{2.46}$$

Note that b_2 is well-defined as $a' \in \ker(\mathbb{A}[e_1])$. Further, note that

$$\mathbb{A}[e_1](-\mathbb{B}[e_2]b_2) \overset{(2.44)}{=} \mathbb{A}[e_2](\mathbb{B}[e_1]b_2) = \tfrac{1}{2}\mathbb{A}[e_2]a' = 0. \tag{2.47}$$

Consequently $\mathbb{B}[e_2]b_2 \in \ker(\mathbb{A}[e_1])$ and so b_1 is well-defined. Let us set $P_4(x_1, x_2) :=$

$(\frac{1}{3}b_1x_1^3 + b_2x_1^2x_2)$. Then

$$\begin{aligned}\mathcal{B}P_4(x_1,x_2) = \mathcal{B}(\tfrac{1}{3}b_1x_1^3 + b_2x_1^2x_2) &= (\mathbb{B}[e_1]b_1 + \mathbb{B}[e_2]b_2)\,x_1^2 + 2\mathbb{B}[e_1]b_2x_1x_2 \\ &\overset{(2.46)}{=} (-\mathbb{B}[e_2]b_2 + \mathbb{B}[e_2]b_2)\,x_1^2 + a'x_1x_2 \\ &\overset{(2.45)_1}{=} (-S_1u - \mathcal{B}P_3).\end{aligned}$$

We conclude that $\mathcal{B}P_3 + \mathcal{B}P_4 = -S_1u$, which is what we wanted to show.

To summarise, we found $w \in L^2(T_2,\mathbb{R}^d)$, such that $\mathcal{B}w = (u+S_1u)+S_2u$ in $H^{-1}(T_2,\mathbb{R}^d)$ and P such that $\mathcal{A}P = -S_1u$. Both w and P depend linearly on u. Let us now define

$$\mathcal{B}^{-1}u := w + P.$$

Then $\mathcal{B}(\mathcal{B}^{-1}u) = u + S_2u$ in $H^{-1}(T_2,\mathbb{R}^d)$. As S_2u is supported on ∂Q, we conclude that $\mathcal{B}(\mathcal{B}^{-1}u) = u$ in $H^{-1}(Q,\mathbb{R}^d)$. □

Using the result for first order operators, we are also able to formulate a version of Theorem 2.35 for higher order operators.

Corollary 2.41. *Let $n = 2$ and let $\mathcal{A}$ be a differential operator of order k. Then there exists a finite dimensional space $X \subset H^k(Q,\mathbb{R}^d) \cap \ker(\mathcal{A})$ consisting of polynomials and a linear, bounded map $\mathcal{B}^{-1}\colon H^1(Q,\mathbb{R}^d) \cap \ker(\mathcal{A}) \to L^2(Q,\mathbb{R}^m)$ such that $u - \mathcal{B}\circ\mathcal{B}^{-1}u \in X$.*

Essentially, the argument is that we can reduce this case to the case of first order operators. First of all, let us reduce to a first-order $\mathcal{A}$. Let $\mathcal{A}$ be of order $l \in \mathbb{N}$. Then $\mathcal{A}u = 0$ if and only if $u^{l-1} = \nabla^{l-1}u$ satisfies

$$\mathcal{A}^{l-1}u^{l-1} = 0 \quad \text{and} \quad \operatorname{curl}^{l-1} u^{l-1} = 0, \tag{2.48}$$

where $\mathcal{A}^{l-1}$ is a suitable reformulation of the differential constraint $\mathcal{A}$ as a first order operator dependent on the $(l-1)$-derivatives; the condition $\operatorname{curl}^{l-1} u^{l-1}$ encodes that u^{l-1} is a $(l-1)$-gradient. Observe that $\mathcal{B}_{l-1} := \nabla^{l-1} \circ \mathcal{B}$ is a potential for the differential operator described in (2.48). For $\mathcal{B}$ of order k observe that $\mathcal{B}v = u$ if and only if for $v^{k-1} = \nabla^{k-1}v$

$$\mathcal{B}^{k-1}v^{k-1} = u \text{ and } \quad \operatorname{curl}^{k-1} v^{k-1} = 0, \tag{2.49}$$

where again, $\mathcal{B}^{k-1}$ is a suitable reformulation of $\mathcal{B}$ in terms of derivatives of order $(k-1)$. Taking (2.48) and (2.49) together and applying Theorem 2.35, up to a finite dimensional vector space, for each u^{l-1} satisfying $\mathcal{A}^{l-1}u^{l-1} = 0$ we might find $\tilde{v}$, such that

$$(\mathcal{B}_{l-1})^{k+l-2}\tilde{v} = u, \quad \operatorname{curl}^{k+l-2}\tilde{v} = 0.$$

and, therefore, v, such that

$$\nabla^{l-1} \circ \mathcal{B}v = u.$$

As a consequence, up to a finite dimensional vector space $\mathcal{X}$, $\mathcal{B}v - u \in \mathcal{X}$.

Remark 2.42. To conclude, let us remark that another approach to the problem described in this section is discussed in [12, Lem. 14] for operators of maximal rank. Whereas we believe that our approach might also apply to other, slightly more general scenarios and since our focus here is more on displaying consequences of the constant rank conditions in the exemplary case of $N = 2$, we shall defer the discussion to higher dimensions to future work.

3. $\mathcal{A}$-quasiaffine functions

Summary

This chapter is loosely based on the preprint

- [135]: Schiffer, S., *A sufficient and necessary condition for $\mathcal{A}$-quasiaffinity.*

In order to fit into this thesis, the results and proofs have been heavily rearranged. This is a single-author manuscript. Hence a detailed description of the doctoral candidate's contribution is not needed.

The goal of this chapter is to derive a characterisation of $\mathcal{A}$-quasiaffine functions. Here, $\mathcal{A}$-quasiaffine functions are functions $f\colon \mathbb{R}^d \to \mathbb{R}$, such that both f and $-f$ are $\mathcal{A}$-quasiconvex, that is for all $\mathcal{A}$-free test functions ψ on the torus we have

$$f(v) \leq \int_{T_N} f(v + \psi(x))\,\mathrm{d}x.$$

This notion is substantially stronger than the notion of $\mathcal{A}$-quasiconvexity. In particular, for operators satisfying the spanning property, the vector space of $\mathcal{A}$-quasiaffine functions is finite-dimensional and consists of polynomials (cf. Theorem 3.8. Indeed, the following characterisation is well-known (cf. [80, 118]):

Theorem 3.a. *[=Proposition 3.2]*

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ and let $\mathcal{A}$ satisfy the constant rank property and the spanning property and let $\mathcal{B}$ be a potential of $\mathcal{A}$. Then the following statements are equivalent.

(a) *f is $\mathcal{A}$-quasiaffine;*

(b) *f is a polynomial and $\forall x \in \mathbb{R}^d$, $\forall r \geq 2$, $\forall \xi_1, ..., \xi_r \in \mathbb{R}^d$ which are linearly dependent and $\forall v_1, ..., v_r \in \mathbb{R}^d$ with $v_i \in \ker \mathbb{A}[\xi_i]$ we have*

$$D^r f(x)[v_1, ..., v_r] = 0; \tag{3.1}$$

(c) *f is C^1 and the Euler-Lagrange equation*

$$\mathcal{B}^T(\nabla f(\mathcal{B}u)) = 0 \tag{3.2}$$

is satisfied in the sense of distributions $\forall u \in C^{k_{\mathcal{B}}}(\bar{\Omega})$, i.e. for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$ we have

$$\int_\Omega \nabla f(\mathcal{B}u) \cdot \mathcal{B}\varphi = 0;$$

(d) *The map $u \mapsto f(u)$ is sequentially weak∗ continuous from $L^\infty(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$ to $L^\infty(\Omega, \mathbb{R}^d)$, i.e. if $u_n \in L^\infty(\Omega, \mathbb{R}^d)$ with $\mathcal{A}u_n = 0$ and $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega, \mathbb{R}^d)$, then also $f(u_n) \overset{*}{\rightharpoonup} f(u)$ in $L^\infty(\Omega, \mathbb{R}^d)$;*

(e) *f is a polynomial of degree $s \leq d$, $p > d$ and the map $u \mapsto f(u)$ is sequentially weakly continuous from $L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$ to $L^{(p/s)}(\Omega)$, i.e. if $u_n \in L^p(\Omega, \mathbb{R}^d)$ with $\mathcal{A}u_n = 0$ and $u_n \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$ then*

$$\lim_{n\to\infty} \int_\Omega f(u_n)\varphi = \int_\Omega f(u)\varphi \quad \forall \varphi \in L^{(p/s)'}(\Omega);$$

(f) *f is a polynomial of degree $s \leq d$ and the map $u \mapsto f(u)$ is sequentially weakly continuous from $L^s(\Omega, \mathbb{R}^d)$ to $\mathcal{D}'(\Omega)$ (the space of distributions on Ω), i.e. if $u_n \in L^s(\Omega, \mathbb{R}^d)$ with $\mathcal{A}u_n = 0$ and $u_n \rightharpoonup u$ in $L^s(\Omega, \mathbb{R}^d)$, then*

$$\lim_{n\to\infty} \int_\Omega f(u_n)\varphi = \int_\Omega f(u)\varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

In Section 3.3, we give a proof of this theorem, which is different to the proofs displayed in [80, 118]. In particular, the proof of the equivalence (a) $\Leftrightarrow$ (d), (a) $\Leftrightarrow$ (e) and (a) $\Leftrightarrow$ (f) does not rely on weak lower-semicontinuity results of Fonseca & Müller [65, 80]. Instead, we just use the definition of $\mathcal{A}$-quasiaffinity and the observation, that any $\mathcal{A}$-quasiaffine function must already be a polynomial map.

In more detail, the directions (d) $\Rightarrow$ (a) etc. rely directly on the definition of $\mathcal{A}$-quasaffinity. Indeed, if (a) is not valid, then using oscillating functions and a test function that is (close to) a characteristic function on a small cube yields the implication.

The other direction is more involved. The following two observations are crucial. First, we see in Theorem 3.8 that any $\mathcal{A}$-quasiaffine function is a polynomial, and moreover, a polynomial is $\mathcal{A}$-quasiaffine if and only if its homogeneous components are $\mathcal{A}$-quasiaffine. Therefore, it suffices to consider homogeneous polynomials of some order $s \in \mathbb{N}$. Second, instead of taking test functions φ and considering

$$\int_\Omega f(u_n)\varphi \,\mathrm{d}x,$$

it suffices to look at test functions of the form $\varphi = \psi^s$. Then we can write

$$\int_\Omega f(u_n)\varphi \,\mathrm{d}x = \int_\Omega f(\psi u_n) \,\mathrm{d}x$$

and we can handle the second integral via the definition of $\mathcal{A}$-quasiaffinity. For the equivalence (a) $\Leftrightarrow$ (b) we use an easier argument than the one used in [119]. This argument is a generalisation of the proof of above statement, which was done in a special case in [15]. The proof is done by induction. The induction hypotheses holds for $r = 2$ due to Plancherel's theorem. To show (a) $\Rightarrow$ (b) we construct an explicit function such that $\mathcal{A}$-quasiaffinity checked for this function implies (b). For the converse direction we use a generalised version of Plancherel's theorem for r terms.

As a mathematical extension to the above results, we then further strengthen condition (b):

Theorem 3.b. *Let $\mathcal{A}$ be a constant rank operator and $\mathcal{B}$ be a potential of $\mathcal{A}$ of order $k_{\mathcal{B}}$. Then (b) from Proposition 3.a is equivalent to*

(b2) f is a polynomial and for all $2 \leq r \leq \min\{k_{\mathcal{B}}, N\} + 1$ and all $\xi_1, ..., \xi_r \in \mathbb{R}^N$ linearly dependent and for all $v_1, ..., v_r \in \mathbb{R}^m$ with $v_i \in \ker \mathbb{A}[\xi_i]$ we have

$$D^r f(x)[v_1, ..., v_r] = 0 \tag{3.3}$$

In particular (b2) is equivalent to $\mathcal{A}$-quasiaffinity of f.

The validity of this theorem is also proven in Section 3.3. It is derived by using basic observations on polynomials.

As a consequence we are able to derive a condition, such that affinity along the characteristic cone $\Lambda_{\mathcal{A}}$ of the differential operator $\mathcal{A}$ guarantees $\mathcal{A}$-quasiaffinity. This is true whenever $\mathcal{A}$ admits a potential $\mathcal{B}$ of first order. It is important to mention that the order of such a potential cannot be directly seen by considering $\mathcal{A}$ alone and in particular, the order of $\mathcal{B}$ is not bounded only in terms of the order of $\mathcal{A}$.

The last Section 3.4 of this chapter is not part in the aforementioned preprint. We give a connection between the notions of $\mathcal{A}$-quasiaffinity and $\mathcal{A}$-quasiconvexity.

3.1. Introduction

3.1.1. Motivation

In this chapter, as a first step towards $\mathcal{A}$-quasiconvexity, we consider a stronger notion first. As discussed in the introduction of this thesis (and also in Chapter 4), a sufficient and necessary condition to weak lower-semicontinuity of a functional $I\colon L^\infty(\Omega, \mathbb{R}^d) \to [0, \infty)$ defined as

$$I(u) = \begin{cases} \displaystyle\int_\Omega f(x, u(x))\,\mathrm{d}x & \text{if } \mathcal{A}u = 0 \\ \infty & \text{else,} \end{cases} \tag{3.4}$$

is $\mathcal{A}$-quasiconvexity of $f(x, \cdot)$ (for $f \in C(\mathbb{R}^d, [0, \infty))$. That is, for almost every $x \in \Omega$, any $v \in \mathbb{R}^d$ and any $\mathcal{A}$-free test function on the torus, cf. Definition 3.1, we have

$$f(x, v) \leq \int_{T_N} f(x, v + \psi(x))\,\mathrm{d}x, \quad \forall \psi \in \mathcal{T}_{\mathcal{A}}. \tag{3.5}$$

This condition is in fact very hard to verify explicitly for given $f \in C(\mathbb{R}^d)$. In this chapter, we study $\mathcal{A}$-quasiaffine functions first. That is, inequality (3.5) is satisfied with equality. From $\mathcal{A}$-quasiaffinity we can infer very strong properties for the function f (c.f. Proposition 3.2). The first part of this chapter is concerned with proving these properties. In

Section 3.4, we discuss how the notion of $\mathcal{A}$-quasiaffinity can be employed for minimisation problems in the context of $\mathcal{A}$-quasiconvexity (3.5).

3.1.2. Definition and Main results

Let us start with the definition of $\mathcal{A}$-quasiaffine functions.

Definition 3.1. *A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is called **$\mathcal{A}$-quasiaffine** if both f and $-f$ are $\mathcal{A}$-quasiconvex, i.e. for all test functions $\psi \in \mathcal{T}_{\mathcal{A}}$,*

$$\mathcal{T}_{\mathcal{A}} = \{\psi \in C^\infty(T_N, \mathbb{R}^d)\colon \mathcal{A}\psi = 0, \int_{T_N} \psi \,\mathrm{d}x = 0\},$$

and all $v \in \mathbb{R}^d$ we have

$$f(v) = \int_{T_N} f(v + \psi(x)) \,\mathrm{d}x. \tag{3.6}$$

Let $\mathcal{B}$ be a potential of the differential operator $\mathcal{A}$. The following characterisation theorem is well-known and shown by Murat [119] ((a) $\Leftrightarrow$ (b)) and Guerra & Raiţă [80] ((a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (f)). For completeness, in Section 3.3 we give a proof of all equivalences, i.e. a modification of Murat's proof based on the proof in the special case $\mathcal{B} = \nabla^k$ of [15] and a proof of the equivalences (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (f), which is not based on the weak lower semi-continuity result of Fonseca & Müller [65].

Proposition 3.2. *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ and let $\mathcal{A}$ satisfy the constant rank property and the spanning property and let $\mathcal{B}$ be a potential of $\mathcal{A}$. Then the following statements are equivalent.*

(a) f is $\mathcal{A}$-quasiaffine;

(b) f is a polynomial and $\forall x \in \mathbb{R}^d$, $\forall r \geq 2$, $\forall \xi_1, ..., \xi_r \in \mathbb{R}^d$ which are linearly dependent and $\forall v_1, ..., v_r \in \mathbb{R}^d$ with $v_i \in \ker \mathbb{A}[\xi_i]$ we have

$$D^r f(x)[v_1, ..., v_r] = 0; \tag{3.7}$$

(c) f is C^1 and the Euler-Lagrange equation

$$\mathcal{B}^T(\nabla f(\mathcal{B}u)) = 0 \tag{3.8}$$

is satisfied in the sense of distributions $\forall u \in C^{k_{\mathcal{B}}}(\bar{\Omega})$, i.e. for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$ we have

$$\int_\Omega \nabla f(\mathcal{B}u) \cdot \mathcal{B}\varphi = 0;$$

(d) The map $u \mapsto f(u)$ is sequentially weak$*$ continuous from $L^\infty(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$ to $L^\infty(\Omega, \mathbb{R}^d)$, i.e. if $u_n \in L^\infty(\Omega, \mathbb{R}^d)$ with $\mathcal{A}u_n = 0$ and $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega, \mathbb{R}^d)$, then also $f(u_n) \overset{*}{\rightharpoonup} f(u)$ in $L^\infty(\Omega, \mathbb{R}^d)$;

(e) *f is a polynomial of degree $s \leq d$, $p > d$ and the map $u \mapsto f(u)$ is sequentially weakly continuous from $L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$ to $L^{(p/s)}(\Omega)$, i.e. if $u_n \in L^p(\Omega, \mathbb{R}^d)$ with $\mathcal{A}u_n = 0$ and $u_n \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$ then*

$$\lim_{n\to\infty} \int_\Omega f(u_n)\varphi = \int_\Omega f(u)\varphi \quad \forall \varphi \in L^{(p/s)'}(\Omega);$$

(f) *f is a polynomial of degree $s \leq d$ and the map $u \mapsto f(u)$ is sequentially weakly continuous from $L^s(\Omega, \mathbb{R}^d)$ to $\mathcal{D}'(\Omega)$ (the space of distributions on Ω), i.e. if $u_n \in L^s(\Omega, \mathbb{R}^d)$ with $\mathcal{A}u_n = 0$ and $u_n \rightharpoonup u$ in $L^s(\Omega, \mathbb{R}^d)$, then*

$$\lim_{n\to\infty} \int_\Omega f(u_n)\varphi = \int_\Omega f(u)\varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

We know that if f is $\mathcal{A}$-quasiaffine, it is a polynomial of order $s \leq d$. Hence, we need to check the validity of (3.7) only for $s \leq d$. However, we can show that this bound can be improved further.

Theorem 3.3. *Let $\mathcal{A}$ be a constant rank operator and $\mathcal{B}$ be a potential of $\mathcal{A}$ of order $k_\mathcal{B}$. Then (b) from Proposition 3.2 is equivalent to*

(b2) *f is a polynomial and for all $2 \leq r \leq \min\{k_\mathcal{B}, N\} + 1$ and all $\xi_1, ..., \xi_r \in \mathbb{R}^N$ linearly dependent and for all $v_1, ..., v_r \in \mathbb{R}^m$ with $v_i \in \ker \mathbb{A}[\xi_i]$ we have*

$$D^r f(x)[v_1, ..., v_r] = 0 \tag{3.9}$$

In particular (b2) is equivalent to $\mathcal{A}$-quasiaffinity of f.

Hence, if the order of the potential $\mathcal{B}$ is one, we can conclude the following statement.

Corollary 3.4 ($\Lambda_\mathcal{A}$-affinity is equivalent to $\mathcal{A}$-quasiaffinity)**.** *Let $\mathcal{A}$ be a constant rank operator and $\mathcal{B}$ be a first-order potential of $\mathcal{A}$. Then $f \colon \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{A}$-quasiaffine if and only if f is $\Lambda_\mathcal{A}$-affine, i.e. for all $v_0 \in \mathbb{R}^d$ and $v \in \Lambda_\mathcal{A}$*

$$t \mapsto f(v_0 + tv)$$

is affine.

The remainder of this chapter is organised as follows. In 3.2 we gather some basic properties and definitions for $\mathcal{A}$-quasiaffine functions. Section 3.3 is devoted to the proofs of the main characterisation theorems. Finally, in Section 3.4, we discuss the connection of $\mathcal{A}$-quasiaffine functions to $\mathcal{A}$-quasiconvex functions, which are examined in Chapter 4.

3.2. Basic properties of $\mathcal{A}$-quasiaffine functions and $\Lambda_{\mathcal{A}}$-affinity

We consider a differential operator $\mathcal{A}$, both satisfying the constant rank and the spanning property and a potential $\mathcal{B}$ of $\mathcal{A}$ of some order $k_{\mathcal{B}}$.

Definition 3.5 ($\mathcal{A}$-quasiaffinity). *(a) We define the space of test functions $\mathcal{T}_{\mathcal{A}}$ as*

$$\mathcal{T}_{\mathcal{A}} = \{\varphi \in C^{\infty}_{\#}(T_N, \mathbb{R}^d) \colon \mathcal{A}\varphi = 0\}.$$

(b) We call a measurable, locally bounded function $f \colon \mathbb{R}^d \to \mathbb{R}$ ***$\mathcal{A}$-quasiaffine*** *if for all $v \in \mathbb{R}^d$ and all $\varphi \in \mathcal{T}_{\mathcal{A}}$*

$$f(v) = \int_{T_N} f(v + \varphi(x))\,\mathrm{d}x. \tag{3.10}$$

(c) We call a measurable, locally bounded function $f \colon \mathbb{R}^d \to \mathbb{R}$ ***$\mathcal{B}$-potential-quasiaffine*** *if for any open and bounded set $\Omega \subset \mathbb{R}^d$, all $v \in \mathbb{R}^d$ and any $\psi \in C_c^{\infty}(\Omega, \mathbb{R}^m)$*

$$f(v) \leq \frac{1}{|\Omega|} \int_{\Omega} f(v + \mathcal{B}\psi(x))\,\mathrm{d}x. \tag{3.11}$$

(d) We say that $f \colon \mathbb{R}^d \to \mathbb{R}$ is ***$\Lambda_{\mathcal{A}}$-affine*** *if for all $v_0 \in \mathbb{R}^d$ and $v \in \Lambda_{\mathcal{A}}$ the function*

$$t \mapsto f(v_0 + tv)$$

is affine.

Proposition 3.6. *Let $\mathcal{A}$ be a homogeneous, constant rank operator, $\mathcal{B}$ be a potential of $\mathcal{A}$ and $f \colon \mathbb{R}^d \to \mathbb{R}$ continuous. Then the following are equivalent.*

(a) f is $\mathcal{A}$-quasiaffine;

(b) f is $\mathcal{B}$-potential-quasiaffine;

(c) For all $\psi \in C_c^{\infty}((0,1)^N, \mathbb{R}^m)$ and for all $v \in \mathbb{R}^d$ we have

$$f(v) = \int_{\Omega} f(v + \mathcal{B}\psi(x))\,\mathrm{d}x;$$

(d) For all $\psi \in C^{\infty}(T_N, \mathbb{R}^m)$ and for all $v \in \mathbb{R}^d$

$$f(v) = \int_{T_N} f(v + \mathcal{B}\psi(x))\,\mathrm{d}x = 0.$$

A proof of this statement (in the setting $\mathcal{B} = \nabla$ and for quasiconvexity instead for quasi-affinity) can for example be found in [115]. For completeness, let us give short arguments.

Proof. The equivalence (a) $\Leftrightarrow$ (d) is clear by the definition of a potential, cf. Theorem 2.5. Furthermore, it is clear that (b) implies its special case (c). Note that (b) is also a special case of (d); by scaling we may assume that $\Omega \subset\subset (0,1)^N$. Then any function $\psi \in C_c^\infty(\Omega, \mathbb{R}^m)$ may be extended by 0 to a function in $C_c^\infty((0,1)^N, \mathbb{R}^m)$, which in turn is in $C^\infty(T_N, \mathbb{R}^m)$ after identifying faces.

The step (c) $\Rightarrow$ (d) requires a different argumentation. If $\psi \in C^\infty(T_N, \mathbb{R}^m)$, let us write

$$\psi_n = n^{-k_{\mathcal{B}}} \psi(nx).$$

We consider a cut-off sequence $\varphi_n \subset C_c^\infty((0,1)^N)$ that is supported in $(n^{-1/2}, 1 - n^{-1/2})^N$ and satisfies

$$\|\nabla^i \varphi_n\|_{L^\infty} \leq C_i n^{i/2} \quad \forall 1 \leq i \leq k_{\mathcal{B}}.$$

A short calculation using continuity of f gives that

$$\int_{T_N} f(v + \mathcal{B}\psi(x))\,\mathrm{d}x = \int_{T_N} f(v + \mathcal{B}\psi_n(x))\,\mathrm{d}x,$$
$$\lim_{n\to\infty} \int_{T_N} \big|f(v + \mathcal{B}\psi_n(x)) - f(v + \mathcal{B}(\varphi_n(x)\psi_n(x)))\big|\,\mathrm{d}x = 0.$$

This implies the validity of (d). □

Before showing crucial properties, let us see a few examples of $\mathcal{A}$-quasiaffine functions.

Example 3.7 ($\mathcal{A}$-quasiaffine functions for selected operators). (a) Consider the operator $\mathcal{B} = \nabla \colon C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times m})$. It is well-known (e.g. [112, 126, 38, 46]), that all $\mathcal{B}$-potential quasiaffine functions are linear combinations of $r \times r$ minors ($1 \leq r \leq \min\{m, N\}$). Likewise, for higher order gradients a characterisation is given by [15]. Essentially, ∇^k-potential-quasiaffine function are already ∇-potential-quasiaffine for the gradient acting on $C^\infty(\mathbb{R}^N, \mathbb{R}^N \odot \ldots \odot \mathbb{R}^N)$.

(b) For the operator $\mathcal{A} = \operatorname{div} \colon C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times l}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^L)$ there are two cases. If $N = 2$, the operator div is a rotation of curl (i.e. 2×2 minors are div-quasiaffine). If $N > 2$, then only affine functions are div-quasiaffine. This can be seen by the fact that these are affine along matrices with rank ≤ 2 (cf. Theorem 3.8 (a) below). This in turn already implies that the map is affine.

(c) An example that is relevant in the context of compensated compactness (e.g. [118, 119, 140, 51, 127, 79]) is the following: Consider an operator $\mathcal{A} \colon C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ of constant rank and a potential $\mathcal{B} \colon C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d)$. Then we may consider the operator $(\mathcal{A}, \mathcal{B}^*) \colon C^\infty(\mathbb{R}^N, \mathbb{R}^d \times \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l \times \mathbb{R}^m)$ defined by

$$(\mathcal{A}, \mathcal{B}^*)(u, v) = (\mathcal{A}u, \mathcal{B}^* v).$$

Note that we have

$$(\ker \mathbb{A}[\xi])^\perp = \ker \mathbb{B}^*[\xi] \quad \forall \xi \in \mathbb{R}^N \backslash \{0\}.$$

Therefore, the map $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(a,b) = a \cdot b$$

is $(\mathcal{A}, \mathcal{B}^*)$-quasiaffine. Prominent examples are the pairs (curl, div) and (curl curl^T, div$_{\text{sym}}$), cf. Chapter 5 and Chapter B, [41].

A key point in proving the characterisation theorem 3.2 is to show that any $\mathcal{A}$-quasiaffine function is $\Lambda_{\mathcal{A}}$-affine.

Theorem 3.8. *(a) Let $M : \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{A}$-quasiaffine. Then M is also $\Lambda_{\mathcal{A}}$-affine.*

(b) Let $M \in C^2(\mathbb{R}^d)$. Then f is Λ_A-affine if and only if for all $x \in \mathbb{R}^d$ and $v \in \Lambda_A$

$$D^2 f(x)[v,v] = \frac{\partial^2}{\partial t^2} f(x+tv)_{|t=0} = 0.$$

(c) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a polynomial of degree 2. Then M is $\mathcal{A}$-quasiaffine if and only if f is $\Lambda_{\mathcal{A}}$-affine.

(d) Any $\Lambda_{\mathcal{A}}$-affine map is a polynomial of degree $\leq d$.

(e) Any partial derivative of a $\Lambda_{\mathcal{A}}$-affine map is also Λ_A-affine.

(f) A homogeneous polynomial $M\colon \mathbb{R}^d \to \mathbb{R}$ of degree ≥ 3 is $\Lambda_{\mathcal{A}}$-affine if all its partial derivatives $\partial_i M$, $i \in \{1,...,d\}$, are $\Lambda_{\mathcal{A}}$-affine.

(g) There exists a basis consisting of homogeneous polynomials of the space of $\Lambda_{\mathcal{A}}$-affine maps.

Proof. (a) follows if we consider test functions $\varphi \in \mathcal{T}_{\mathcal{A}}$ of the form $\Phi(\xi x)v$ for some one-periodic $\Phi \in C^\infty(\mathbb{R})$ and $v \in \ker \mathbb{A}[\xi]$. For (b) one uses that a function $g \in C(\mathbb{R})$ is affine if and only if $g'' = 0$. (c) relies on Plancherel's identity which is valid for quadratic forms. In particular, as all affine functions are automatically $\mathcal{A}$-quasiaffine, we may consider M to be 2-homogeneous. Then, using Plancherel's identity, we find that

$$\int_{T_N} M(u(y))\,\mathrm{d}y = \sum_{\lambda \in \mathbb{Z}^N} f(\hat{u}(\lambda)).$$

As f is homogeneous of degree 2 and $\hat{u}(\lambda) \in \Lambda_{\mathcal{A}}$, it follows that $f(\hat{u}(\lambda)) = 0$ for $\lambda \neq 0$.

Ad (d): Let now $v_1,...,v_d$ be a basis of $\mathbb{R}^d$, which is contained in $\Lambda_{\mathcal{A}}$ and denote by $\lambda_1(y),...,\lambda_n(y)$ the coordinates with respect to this basis. We may write a Λ_A-affine function f as

$$f(y) = \tilde{f}(\lambda_1,...,\lambda_d).$$

Due to $\Lambda_{\mathcal{A}}$-affinity, we know that the map

$$\lambda_i \mapsto \tilde{f}(\lambda_1,...,\lambda_d)$$

is affine for fixed $i \in \{1, ..., d\}$ and fixed λ_j, $j \neq i$. Hence, $\tilde{f}$ must be a polynomial in λ_i. In particular, as $\tilde{f}$ is affine in each λ_i, it has at most degree d.

The property (e) follows from (b). In order to see (f), note that

$$D^2 f(x)[v,v] = \int_0^1 D^3 f(tx)[v,v,x]\,\mathrm{d}t + D^2 f(0)[v,v]$$
$$= \int_0^1 D^2 \left(\frac{\partial}{\partial x} f\right)(tx)[v,v]\,\mathrm{d}t + D^2 f(0)[v,v].$$

As M is homogeneous of degree strictly larger than two, $D^2 M(0) = 0$ and therefore M is $\mathcal{A}$-quasiaffine.

For (g) we use (f). Write $f = \sum_{i=1}^{d} f_i$ for i-homogeneous polynomials f_i. We may consider $\tilde{f} = f - f_0 - f_1$, as f_0 and f_1 are affine and hence $\Lambda_{\mathcal{A}}$-affine. Observe that then $\Lambda_{\mathcal{A}}$-affinity yields $f(x) = 0$ for all $x \in \Lambda_{\mathcal{A}}$. In particular, $f_i(x) = 0$ for all $i = 2, ..., d$ and $x \in \Lambda_{\mathcal{A}}$. But this implies $\Lambda_{\mathcal{A}}$-affinity for f_2. Considering $\bar{f} = \nabla(f - f_0 - f_1 - f_2)$, the statement (f) and an inductive argument, we get that $f_0, ..., f_d$ are all already $\Lambda_{\mathcal{A}}$-affine. Therefore, there must be a basis of homogeneous polynomials for $\Lambda_{\mathcal{A}}$-affine maps. □

Remark 3.9. a) Due to Theorem 3.8 (f), if there is Λ_A-affine polynomial f of degree k, then there is also a $\mathcal{A}$-quasiaffine polynomial of degree $k-1$. In particular, the question of existence of non-affine $\Lambda_{\mathcal{A}}$-affine functions reduces to the existence of quadratic $\Lambda_{\mathcal{A}}$-affine functions. Recall that $\mathcal{A}$-quasiaffine functions are $\Lambda_{\mathcal{A}}$-affine functions and the converse holds true for quadratic functions. Hence, the existence of non-trivial $\mathcal{A}$-quasiaffine functions reduces to the existence of a quadratic function vanishing on $\Lambda_{\mathcal{A}}$.

b) The converse implication in 3.8 (a) is false, i.e. $\Lambda_{\mathcal{A}}$-affinity does not imply $\mathcal{A}$-quasiaffinity (c.f. Lemma 3.10, [15]).

3.3. Proof of the characterisation theorem

3.3.1. Proof of Proposition 3.2

We prove that (a) is equivalent to any other property, i.e. (a) ⇔ (b), (a) ⇔ (c) etc.. We start with the weak continuity statements (d)-(f). Essentially, one could redo the proof of what follows in Chapter 4 for weak lower-semicontinuity (cf. [79]). Instead, we sketch a short argument not relying on the weak lower-semicontinuity result.

For (a) ⇔ (b) note that, as $\mathcal{B}$ is a potential of $\mathcal{A}$, the following condition is equivalent to (b):

(b') M is a polynomial and $\forall x \in \mathbb{R}^d$, $\forall r \geq 2$, $\forall \xi_1, ..., \xi_r \in \mathbb{R}^d$ which are linearly dependent and $\forall w_1, ..., w_r \in \mathbb{R}^m$ we have

$$D^r M(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_r](w_r)] = 0. \tag{3.12}$$

Proof of Proposition 3.2. **(d) ⇒ (a) and (e) ⇒ (a):** We prove this direction by contradiction, so assume that f is not $\mathcal{A}$-quasiaffine and there exists $\psi \in \mathcal{T}_\mathcal{A}$ and $v \in \mathbb{R}^d$, such that

$$f(v) \neq \int_\Omega f(v + \psi(x))\,\mathrm{d}x.$$

We may identify ψ with a $\mathbb{Z}^N$-periodic, $\mathcal{A}$-free function $\bar{\psi} \in C^\infty(\mathbb{R}^N, \mathbb{R}^d)$ and v with a constant function on the torus.

Let now $Q_0 = x_0 + (0,a)^N \subset\subset \Omega$ for some $x_0 \in \Omega$, $a > 0$. Let $\varphi = 1_{Q_0}$ be the characteristic function of Q_0, which is in L^1 (for (d)) and in $L^{p/(p-s)}$ (for (f)). Let us define the sequence function v_n

$$v_n(x) = v + \bar{\psi}(na^{-1}(x - x_0)).$$

Then $\mathcal{A}v_n = 0$, $v_n \rightharpoonup v$ in L^p ($v_n \overset{*}{\rightharpoonup} v$ in L^∞) and

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \varphi f(v_n) = \lim_{n\to\infty} \int_{Q_0} f(v_n) = \lim_{n\to\infty} |Q_0| \int_{T_N} f(v + \psi(nx)) \\ \neq |Q_0| f(v) = \int_\Omega \varphi f(v)\,\mathrm{d}x.\end{aligned}$$

We conclude that $v \mapsto f(v)$ is not weakly continuous from L^p to $L^{p/s}$ (weakly$*$ from L^∞ to L^∞).

(f) ⇒ (a): This direction is quite similar to '(d) ⇒ (a)'. Indeed, the only thing that changes is the test function φ. As $\varphi = 1_{Q_0}$ is not eligible ($\varphi \notin C^\infty$) we instead take $\varphi_\varepsilon \in C_c^\infty(B_\varepsilon(Q_0))$ that converge to 1_{Q_0} in measure. Taking the same test functions v_n and letting $\epsilon \to 0$ leads to a contradiction.

(a) ⇒ (d), (e): We already know (cf. Proposition 3.8) that if f is $\mathcal{A}$-quasiaffine, then f is a polynomial of order $s \leq d$ and that its homogeneous components are $\mathcal{A}$-quasiaffine, i.e. if

$$f(v) = f_0(v) + f_1(v) + ... + f_s(v)$$

for i-homogenous polynomials f_i, then all f_i are $\mathcal{A}$-quasiaffine. Hence, it suffices to prove the statement for homogeneous polynomials.

Let us assume that $\mathcal{A}u_n = \mathcal{A}u = 0$ and that $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega, \mathbb{R}^d)$ (or $u_n \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$ for $s < p < \infty$). Furthermore, let f be a homogeneous polynomial of degree s. We need to show that for all $\varphi \in L^1$ (or $\varphi \in L^{p/(p-s)}$, respectively)

$$\int_\Omega \varphi f(u_n)\,\mathrm{d}x \longrightarrow \int_\Omega \varphi f(u)\,\mathrm{d}x \quad \text{as } n \to \infty. \tag{3.13}$$

It is possible to make the following three reductions:

(R1) $\Omega \subset\subset (0,1)^N$;

(R2) The limit u equals 0;

(R3) It suffices to show (3.13) for all $\varphi \in Y$, where $Y \subset L^1(\Omega,\mathbb{R}^d)$ (or $Y \subset L^{p/(p-s)}$) is a dense subset.

Indeed, the first reduction follows by scaling, and the third reduction is a functional-analytical fact. The second reduction is shown by using an inductive argument over the degree of the polynomials. Indeed, we can write

$$f(u_n) = f(u_n - u) + \sum_{0<|\alpha|\leq s} f_\alpha(u_n - u)u^\alpha$$

for suitable polynomials $f_\alpha = \alpha!\partial_\alpha f$ of order $s - |\alpha|$ (Taylor series). These polynomials are already $\mathcal{A}$-quasiaffine and by the inductive argument

$$\int_\Omega f_\alpha(u_n - u)\,(u^\alpha \varphi)\,\mathrm{d}x \longrightarrow \int_\Omega f_\alpha(0)\,(u^\alpha\varphi)\,\mathrm{d}x = 0 \quad \text{as } n \to \infty,$$

as $u\varphi$ is an admissible test function.

Having made these reductions, take $Y = \{\varphi_+^s - \varphi_-^s \colon \varphi_+, \varphi_- \in C_c^\infty(\Omega)\}$, which is dense in L^1 and $L^{p/(p-s)}$:

- If s is odd, we may take $\varphi_- = 0$ and approximate the $\sqrt[s]{u}$ by C_c^∞ functions and then take this to the power s;
- If s is even, we split u into a positive and a negative part $u = u_+ - u_-$ and approximate $\sqrt[s]{u_+}$ and $\sqrt[s]{u_-}$ by C_c^∞ functions.

Hence, we just show that for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\lim_{n\to\infty} \int_\Omega \varphi^s f(u_n)\,\mathrm{d}x = 0$$

for sequences $u_n \overset{*}{\rightharpoonup} 0$ in L^∞ (or $u_n \rightharpoonup 0$ in L^p) and s-homogeneous $\mathcal{A}$-quasiaffine polynomials f. Note that

$$\int_\Omega \lim_{n\to\infty} f(u_n)\varphi^s\,\mathrm{d}x = \lim_{n\to\infty} \int_{T_N} f(\varphi u_n)\,\mathrm{d}x.$$

The test function φ is fixed and φu_n can be viewed as a function on T_N by extending it by 0 outside Ω (Reduction (R1)). Due to Lemma 2.10, $\mathcal{A}(u_n\varphi) \to 0$ in $W^{-k,q}(T_N,\mathbb{R}^d)$ for all $q < \infty$ (for showing (d)) or in $W^{-k,p}(T_N,\mathbb{R}^d)$, respectively. For this, recall that $u_n \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega,\mathbb{R}^d)$ implies $u_n \to 0$ in $W^{-1,\infty}(\Omega,\mathbb{R}^d)$ (and weak convergence to 0 in $L^p(\Omega,\mathbb{R}^d)$ also implies $u_n \to 0$ in $W^{-1,p}(\Omega,\mathbb{R}^d)$). Also note that still $\varphi u_n \overset{*}{\rightharpoonup} 0$ in L^∞ ($\varphi u_n \rightharpoonup 0$, respectively). Applying projection theorem 2.9, we may find a sequence $\bar{u}_n$, such that

1. $\displaystyle\int_{T_N} \bar{u}_n\,\mathrm{d}x = \int_{T_N} \varphi u_n\,\mathrm{d}x$;
2. $\|\bar{u}_n - \varphi u_n\|_{L^p} \leq \|\mathcal{A}(\varphi u_n)\|_{W^{-k,p}} \to 0$ for $s < p < \infty$ (for (e)) and $\|\bar{u}_n - \varphi u_n\|_{L^q} \leq \|\mathcal{A}(\varphi u_n)\|_{W^{-k,q}} \to 0$ for all $q < \infty$ (for (d));
3. $\mathcal{A}\bar{u}_n = 0$ (as an element of $\mathcal{D}'(T_N,\mathbb{R}^d)$).

By convolution and substracting $\int_{T_N} \varphi u_n \, \mathrm{d}x$ (which tends to 0 as $n \to \infty$), we can find $\tilde{u}_n \in C^\infty(T_N, \mathbb{R}^d)$ satisfying $\mathcal{A}\tilde{u}_n \to 0$, $\int_{T_N} \tilde{u}_n \, \mathrm{d}x = 0$ and

$$\|\tilde{u}_n - \varphi u_n\|_{L^s(T_N,\mathbb{R}^d)} \longrightarrow 0.$$

as $p > s$. As f is a homogeneous polynomial, for $z_1, z_2 \in \mathbb{R}^d$ we have

$$\begin{aligned} |f(z_1) - f(z_2)| &= \left| \sum_{j=1}^{s} \frac{1}{j!} D^j f(z_2) \cdot (z_1 - z_2)^j \right| \le C \sum_{j=1}^{s} |z_2|^{s-j} |z_1 - z_2|^j \\ &\le C(|z_1|^{s-1} + |z_2|^{s-1})|z_1 - z_2|. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \int_{T_N} |f(\varphi u_n) - f(\tilde{u}_n)| \, \mathrm{d}x &\le C \lim_{n\to\infty} \int_{T_N} \left(|\varphi u_n|^{s-1} + |\tilde{u}_n|^{s-1}\right) |\varphi u_n - \tilde{u}_n| \, \mathrm{d}x \\ &\le C \lim_{n\to\infty} (\|\varphi u_n\|_{L^s}^{s-1} + \|\tilde{u}_n\|_{L^s}^{s-1}) \|\varphi u_n - \tilde{u}_n\|_{L^s} \\ &\le C \lim_{n\to\infty} (1 + \|u_n\|_{L^s}) \|\varphi u_n - \tilde{u}_n\|_{L^s} = 0. \end{aligned}$$

By definition of $\mathcal{A}$-quasiaffinity, for all $n \in \mathbb{N}$,

$$f(0) = \int_{T_N} f(\tilde{u}_n) \, \mathrm{d}x$$

and therefore we conclude

$$\lim_{n\to\infty} \int_\Omega \varphi^s f(u_n) \, \mathrm{d}x = \lim_{n\to\infty} \int_{T_N} f(\varphi u_n) \, \mathrm{d}x = \lim_{n\to\infty} \int_{T_N} f(\tilde{u}_n) \, \mathrm{d}x = 0.$$

(a) ⇒ (f): The argument is similar to the previous step, let us shortly outline the differences. Let f be a homogeneous, $\mathcal{A}$-quasiaffine polynomial of degree s (assume that $s \ge 2$, otherwise there is nothing to show). Again we can make the reductions

(R1') $\Omega \subset\subset T_N$;

(R2') $u_n \rightharpoonup 0$ in L^s;

(R3') We show $\int_\Omega \varphi f(u_n) \, \mathrm{d}x \to \int_\Omega \varphi f(u) \, \mathrm{d}x$ for $\varphi \in Y$, where Y is 'dense' in $C_c^\infty(\Omega)$, with respect to the L^∞-norm, i.e. for all $\varphi \in C_c^\infty(\Omega)$, there is $\varphi_h \to \varphi$ in L^∞ with $\varphi_h \in Y$.

The validity of these reduction is established as in the direction '(a) ⇒ (f)'.

Again, we take the subset

$$Y = \{\varphi_+^s - \varphi_-^s : \varphi_+, \varphi_- \in C_c^\infty(\Omega)\}.$$

The argument for density in the L^∞-norm is the same as in '(a) $\Rightarrow$ (d)' [1]. Hence, it suffices to show that for $u_n \rightharpoonup 0$ in $L^s(\Omega, \mathbb{R}^d)$ with $\mathcal{A}u_n = 0$ in $W^{-k,s}(\Omega, \mathbb{R}^l)$ and all $\varphi \in C_c^\infty(\Omega)$

$$\lim_{n\to\infty} \int_{T_N} \varphi^s f(u_n)\,\mathrm{d}x = 0.$$

Again, employing Fourier methods ($s \geq 2$!), we may find $\tilde{u}_n \in C^\infty(T_N, \mathbb{R}^d)$ with average 0, such that $\mathcal{A}\tilde{u}_n = 0$ and

$$\lim_{n\to\infty} \|\tilde{u}_n - \varphi u_n\|_{L^p(T_N,\mathbb{R}^d)} = 0.$$

Hence, by using that f is a polynomial

$$\int_{T_N} |f(\tilde{u}_n) - f(\varphi u_n)|\,\mathrm{d}x \leq (\|\tilde{u}_n\|_{L^s}^{s-1} + \|\varphi u_n\|_{L^s}^{s-1})\|\tilde{u}_n - \varphi u_n\|_{L^s} \longrightarrow 0 \quad \text{as } n \to \infty$$

and so, by definition of a $\mathcal{A}$-quasiaffinity,

$$\lim_{n\to\infty} \int_\Omega \varphi^s f(u_n)\,\mathrm{d}x = \lim_{n\to\infty} \int_{T_N} f(\varphi u_n)\,\mathrm{d}x = \lim_{n\to\infty} \int_{T_N} f(\tilde{u}_n)\,\mathrm{d}x = 0,$$

establishing (a) $\Rightarrow$ (f).

(a) $\Leftrightarrow$ (c):. If M is $\mathcal{A}$-quasiaffine, then by Theorem 3.8, it is a polynomial and hence it is even C^∞. Moreover, for all $u \in C^k(\bar{\Omega})$ and all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\int_\Omega M(\mathcal{B}u(y) + t\mathcal{B}\varphi(y))\,\mathrm{d}y\right)_{|t=0} \\ &= \int_\Omega \frac{d}{dt}\left(M(\mathcal{B}u(y) + t\mathcal{B}\varphi(y))\right)_{|t=0}\,\mathrm{d}y \\ &= \int_\Omega DM(\mathcal{B}u(y)) \cdot \mathcal{B}\varphi(y)\,\mathrm{d}y. \end{aligned} \tag{3.14}$$

Thus, (3.8) holds in the sense of distributions if M is $\mathcal{B}$-potential-quasiaffine. The same calculation as in (3.14) also shows that if (3.8) holds, then M will be $\mathcal{B}$-potential-quasiaffine.

(a) $\Rightarrow$ (b): If $r = 2$, note that $\mathbb{B}[\lambda\xi] = \lambda^{k_\mathcal{B}}\mathbb{B}[\xi]$ for $\xi \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}\backslash\{0\}$. Hence, if ξ_1 and ξ_2 are linearly dependent and nonzero, we may write $\xi_2 = \lambda\xi_1$ and

$$\mathbb{B}[\xi_2](w_2) = \mathbb{B}[\xi_1](\lambda^{k_\mathcal{B}} w_2).$$

[1] In particular, here we only get convergence in $\mathcal{D}'(\Omega)$ and *not* with respect to the weak topology of L^1, as $C_c^\infty(\Omega)$ is not dense in $L^\infty(\Omega)$.

Therefore, we may consider $\xi_1 = \xi_2 = \xi$. Thus,

$$\begin{aligned} D^2M(x)[v_1, v_2] &= D^2M(x)[\mathbb{B}[\xi](w_1), \mathbb{B}[\xi](w_2)] \\ &= \frac{1}{2}D^2M(x)[\mathbb{B}[\xi](w_1 + w_2), \mathbb{B}[\xi](w_1 + w_2)] \\ &- \frac{1}{2}D^2M(x)[\mathbb{B}[\xi](w_1), \mathbb{B}[\xi](w_1)] - \frac{1}{2}D^2M(x)[\mathbb{B}[\xi](w_2), \mathbb{B}[\xi](w_2)] = 0. \end{aligned}$$

We prove the statement for $r > 2$ by induction. Let (3.7) hold for some $r \in \mathbb{N}$. We consider linearly dependent $\xi_1, ..., \xi_{r+1} \in \mathbb{R}^N$ and $w_1, ..., w_{r+1} \in \mathbb{R}^m$. First, suppose that $\xi_1, ..., \xi_r$ are already linearly dependent. Then by the induction hypothesis,

$$D^rM(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_r](w_r)] = 0 \quad \forall x \in \mathbb{R}^d.$$

Taking the derivative in direction $\mathbb{B}(\xi_{r+1})(w_{r+1})$, the result is also 0. Hence,

$$D^{r+1}M(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{r+1}](w_{r+1})] = 0.$$

We may suppose that ξ_{r+1} can be written as a linear combination of linearly independent $\xi_1, ..., \xi_r \in \mathbb{R}^N \backslash \{0\}$. Due to the homogeneity of $\mathbb{B}[\cdot](w)$, we may also assume that

$$\xi_{r+1} = \xi_1 + ... + \xi_r.$$

Let $t_1, ..., t_r \in \mathbb{R}$ be real parameters. Define the function $\varphi \in C^\infty(T_N, \mathbb{R}^m)$ by

$$\varphi(y) := \begin{cases} \sum_{i=1}^{r+1} t_i w_i \cos(2\pi\xi_i \cdot y) & \text{if } k_{\mathcal{B}} \text{ is even,} \\ \sum_{i=1}^{r+1} t_i w_i \sin(2\pi\xi_i \cdot y) & \text{if } k_{\mathcal{B}} \text{ is odd.} \end{cases}$$

For the sake of simplicity we shall consider the case $k_{\mathcal{B}} = 2k$, the other case is rather similar.

Then, $\mathcal{B}\varphi$ is given by

$$\mathcal{B}\varphi(y) = (-4\pi^2)^k \sum_{i=1}^{r+1} t_i \mathbb{B}[\xi_i](w_i) \cos(2\pi\xi_i \cdot y).$$

Now, $\mathcal{B}$-potential-quasiafffinity means that

$$\int_{T_N} M(x + \mathcal{B}\varphi) \, \mathrm{d}y = M(x) \quad \forall x \in \mathbb{R}^d. \tag{3.15}$$

The left-hand side of (3.15) is a polynomial in t_i. The coefficient of $t_1 \cdot ... \cdot t_{r+1}$ is the constant $(-4\pi^2)^k$ times

$$\int_{T_N} D^{r+1}M(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{r+1}](w_{r+1})] \cdot \cos(2\pi\xi_1 \cdot y) \cdot ... \cdot \cos(2\pi\xi_{r+1} \cdot y) \, \mathrm{d}y$$

$$= D^{r+1}M(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{r+1}](w_{r+1})]$$
$$\cdot \int_{[0,1]^N} \cos(2\pi\xi_1 \cdot y) \cdot ... \cdot \cos(2\pi\xi_r \cdot y) \cos(2\pi \sum_{i=1}^{r} \xi_i \cdot y)\, \mathrm{d}y$$
$$= 2^{-r} D^{r+1}M(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{r+1}](w_{r+1})].$$

To calculate the integral in this equation, we just use the addition theorem for cos and Fubini. As the coefficient of $t_1 \cdot ... \cdot t_{r+1}$ on the right-hand side of (3.15) is 0, we get the desired result.

(b) ⇒ (a): We first claim that it suffices to show that $\forall x \in \mathbb{R}^d$, $\forall \varphi \in C^\infty(T_N, \mathbb{R}^m)$ and for all $r \geq 2$

$$\int_{T_N} D^r M(x)[\mathcal{B}\varphi(y), ..., \mathcal{B}\varphi(y)] = 0. \tag{3.16}$$

Suppose that (3.16) holds. We want to show (a). Take arbitrary $x \in \mathbb{R}^d$ and $\varphi \in C^\infty(T_N, \mathbb{R}^m)$. Consider the Taylor series of M at the point x in the direction of $\mathcal{B}\varphi(y) \in \mathbb{R}^d$. As M is a polynomial of some degree s, M equals its Taylor polynomial in x of degree s, i.e.

$$M(x + \mathcal{B}\varphi(y)) = \sum_{r=0}^{s} \frac{1}{r!} D^r M(x)[\mathcal{B}\varphi(y), ..., \mathcal{B}\varphi(y)].$$

Integrating over $y \in T_N$, using (b) and the fact that $\mathcal{B}\varphi$ has average 0, yields

$$\begin{aligned}\int_{T_N} M(x + \mathcal{B}\varphi(y))\, \mathrm{d}y &= \sum_{r=0}^{s} \int_{T_N} \frac{1}{r!} D^r M(x)[\mathcal{B}\varphi(y), ..., \mathcal{B}\varphi(y)]\, \mathrm{d}y \\ &= \int_{T_N} M(x)\, \mathrm{d}y + \int_{T_N} DM(x) \cdot \mathcal{B}\varphi(y)\, \mathrm{d}y \\ &\quad + \sum_{r=2}^{s} \int_{T_N} \frac{1}{r!} D^r M(x)[\mathcal{B}\varphi(y), ..., \mathcal{B}\varphi(y)]\, \mathrm{d}y \\ &= \int_{T_N} M(x)\, \mathrm{d}y = M(x).\end{aligned}$$

It suffices to prove (3.16). To this end, we use the following formula:
If $f_1, ...f_r \in C^0(T_N, \mathbb{R})$, then

$$\int_{T_N} f_1(y) \cdot ... \cdot f_r(y)\, \mathrm{d}y = \sum_{\xi_1, ..., \xi_{r-1} \in \mathbb{Z}^N} \overline{\hat{f}_1(\xi_1)} \cdot \hat{f}_2(\xi_2) \cdot ... \cdot \hat{f}_{r-1}(\xi_{r-1}) \cdot \hat{f}_r\left(\xi_1 - \sum_{i=2}^{r-1} \xi_i\right). \tag{3.17}$$

This equation can be derived using Plancherel's theorem once for f_1 and $f_2 \cdot ... \cdot f_r$ and then using a discrete version of the convolution formula, i.e.

$$(\widehat{f(\cdot)g(\cdot)})(\xi_1) = \sum_{\xi_2 \in \mathbb{Z}^n} \hat{f}(\xi_2) \cdot \hat{g}(\xi_1 - \xi_2).$$

Recall that $D^r M(x)[\cdot, ..., \cdot]$ is a multilinear form (i.e. a homogenenous polynomial in the entries). Therefore, we can use the identity (3.17). Hence

$$\begin{aligned}&\int_{T_N} D^r M(x)[\mathcal{B}\varphi(y), ..., \mathcal{B}\varphi(y)]\\&= \sum_{i=1}^{r-1}\sum_{\xi_i\in\mathbb{Z}} D^r M(x)\left[\mathbb{B}[\xi_1](\hat{\varphi}(\xi_1)), ..., \mathbb{B}[\xi_{r-1}](\hat{\varphi}(\xi_{r-1})), \mathbb{B}[\xi_1 - \sum_{i=2}^{r-1}\xi_i](\hat{\varphi}(\xi_1 - \sum_{i=2}^{r-1}\xi_i))\right]\\&= 0,\end{aligned}$$

as the vectors

$$\xi_1,\ ...,\ \xi_{r-1},\ \xi_1 - \sum_{i=2}^{r-1}\xi_i$$

are linearly dependent. Each summand equals 0 due to condition (3.7) in (b). We have shown the claim and therefore that (b) implies (a). □

3.3.2. Proof of Theorem 3.3

In this section, we prove the improvement of Theorem 3.2.

Proof of Theorem 3.3. We just need to prove that if equation (3.9) is true for $2 \le r \le \min\{k_{\mathcal{B}}, N\} + 1$, then it also holds for $r \in \mathbb{N}$. Let us first deal with the case $\min\{k_{\mathcal{B}}, N\} = N$. Note that then for $j > 2$ and $r = N + j$, there are $N + 1$ vectors ξ_i, which are already linearly dependent, say $\xi_1, ..., \xi_{N+1}$ are linearly dependent. Then,

$$D^{N+1}(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{N+1}](w_{N+1})] = 0.$$

Therefore, also

$$D^{N+j}(x)[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{N+j}](w_{N+j})] = 0.$$

Suppose now that $k_{\mathcal{B}} \le N$. If $k_{\mathcal{B}} = 1$, then for all $\xi_1, \xi_2 \in \mathbb{R}^N\backslash\{0\}$ and $w \in \mathbb{R}^m$

$$\mathbb{B}[\xi_1 + \xi_2](w) = \mathbb{B}[\xi_1](w) + \mathbb{B}[\xi_2](w) \in \operatorname{span}\{\mathbb{B}[\xi_1](w), \mathbb{B}[\xi_2](w)\}.$$

We prove an analogue of this statement for $k_{\mathcal{B}} > 1$. Again, make the reductions from the proof of Theorem 3.2. We just need to show that, for $r > k_{\mathcal{B}} + 1$, $\xi_1, ..., \xi_{r-1} \in \mathbb{R}^N\backslash\{0\}$ linearly independent and $w_1, ..., w_r \in \mathbb{R}^m$, we have

$$D^r M(x)\left[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{r-1}](w_{r-1}), \mathbb{B}[\xi_1 + ... + \xi_{r-1}](w_r)\right] = 0.$$

We claim that

$$\mathbb{B}\left[\sum_{i=1}^{r-1}\xi_i\right](w) \in \operatorname{span}_{\lambda\in I}\left\{\mathbb{B}\left[\sum_{i=1}^{r-1}\lambda_i\xi_i\right](w)\right\}, \tag{3.18}$$

where $r > k_{\mathcal{B}} + 1$ and the set I of coefficients is given by

$$I = \big\{\lambda \in \mathbb{R}^{r-1} \colon \lambda_i = 0 \text{ for some } i \in \{1, ..., r-1\}\big\}.$$

Suppose that (3.18) is proven. Then, for a finite index set $J \subset I$, we can write,

$$\mathbb{B}\left[\sum_{i=1}^{r-1} \xi_i\right](w) = \sum_{\lambda \in J} \mathbb{B}\left[\sum_{i=1}^{r-1} \lambda_i \xi_i\right](w)$$

and use that, for each $\lambda \in J$, there is $i \in \{1, ..., r-1\}$ such that $\lambda_i = 0$. W.l.o.g. $i = 1$ for some fixed $\lambda \in J$. Then

$$\begin{aligned} &D^r M(x)\left[\mathbb{B}[\xi_1](w_1), ..., \mathbb{B}[\xi_{r-1}](w_{r-1}), \mathbb{B}[\sum_{i=2}^{r-1} \lambda_i \xi_{r-1}](w_r)\right] \\ &= \frac{\partial}{\partial \mathbb{B}[\xi_1](w_1)} D^{r-1} M(x)\left[\mathbb{B}[\xi_2](w_2), ..., \mathbb{B}[\xi_{r-1}](w_{r-1}), \mathbb{B}[\sum_{i=2}^{r-1} \lambda_i \xi_{r-1}](w_r)\right]. \end{aligned}$$

Note that we assume that the left-hand side is 0 for $r \leq k_{\mathcal{B}} + 1$. Assuming that (3.18) holds, we can prove this for all $r \in \mathbb{N}$ by an inductive argument.

It remains to prove the validity of (3.18). Consider the polynomial

$$P(t_1, ..., t_{r-1}) = \mathbb{B}\left[\sum_{i=1}^{r-1} t_i \xi_i\right](w_r).$$

This polynomial has degree $k_{\mathcal{B}} < r - 1$. Hence, in every monomial of P of the form $\prod_{i=1}^{r-1} t_i^{\alpha_i}$ there is at least one $j \in \{1, ..., r-1\}$, such that $\alpha_j = 0$. But we can recover the coeffients of these monomials by considering

$$\mathbb{B}\left[\sum_{i=1, i \neq j}^{r-1} t_i \xi_i\right](w_r).$$

In particular, we can recover these coefficients by taking linear combinations of $P(\lambda)$ for $\lambda \in I$. Therefore, (3.18) holds. This concludes the proof of Theorem 3.3. □

Corollary 3.4 is a special case of Theorem 3.3. In this setting, $k_{\mathcal{B}} = 1$, i.e. $\mathcal{A}$-quasiaffinity of M is equivalent to the fact that

$$D^2 M(x)[\mathbb{B}[\xi](w_1), \mathbb{B}[\xi](w_2)] = 0.$$

As it was already established in the proof of Theorem 3.2, this is indeed equivalent to $\Lambda_{\mathcal{A}}$-affinity of M.

Let us recall the Ball-Currie-Olver example showing, $\mathcal{A}$-quasiaffinity does *not* follow if (3.9) does not hold for all $2 \leq r \leq \min\{k_{\mathcal{B}}, N\} + 1$ [15]. Let us consider the setting $k_{\mathcal{B}} = 2$.

Lemma 3.10 (Ball, Currie, Olver)**.** *There is a first-order differential operator $\mathcal{A}$ and a map $L : \mathbb{R}^d \to \mathbb{R}$ which is $\Lambda_{\mathcal{A}}$-affine, but not $\mathcal{A}$-quasiaffine.*

Proof. Consider the differential operator $\mathcal{B} = \nabla^2$, i.e.

$$(\nabla^2 u)_{ijk} = \partial_i \partial_j u_k (i, j = 1, ..., N; k = 1, ..., m)$$

and $\mathcal{A}$ the corresponding first order operator, such that $\mathcal{B}$ is a potential of $\mathcal{A}$ [109]. The characteristic cone of $\mathcal{A}$ is the space of tensors of the form

$$\lambda \otimes \lambda \otimes b \colon \lambda \in \mathbb{S}^{N-1},\ b \in \mathbb{R}^m.$$

Now choose $N = 2$ and $m = 3$ and consider the map L defined via

$$L(\nabla^2 u) = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \partial_x^2 u_{\sigma(1)} \partial_x \partial_y u_{\sigma(2)} \partial_y^2 u_{\sigma(3)}. \tag{3.19}$$

One can check that this is affine in $\Lambda_{\mathcal{A}}$. On the other hand, one can check that, for

$$u(x_1, x_2) = \begin{pmatrix} \cos(2\pi x_1) \\ \cos(2\pi x_2) \\ \cos(2\pi(x_1 + x_2)), \end{pmatrix}$$

we have

$$\int_{T_N} L(u(x_1, x_2))\,\mathrm{d}x = -\frac{1}{4}.$$

□

We have seen in Theorem 3.3 that the answer to the question whether

$$f\ \Lambda_A\text{-convex} \Longrightarrow f\ \mathcal{A}\text{-quasiaffine}$$

depends on the order of the operator $k_{\mathcal{B}}$. We note that the minimal order of $k_{\mathcal{B}}$ of the potential $\mathcal{B}$ cannot be bounded in terms of the order of $\mathcal{A}$. In view of Theorem 3.3, the differential condition on M for being $\mathcal{A}$-quasiaffine therefore depends much more on the order of $\mathcal{B}$ than on the order of $\mathcal{A}$.

Lemma 3.11. *Let $\mathcal{B} \colon C^\infty(\mathbb{R}^2, \mathbb{R}^m) \to C^\infty(\mathbb{R}^2, (\mathbb{R}^2)^k)$ be a differential operator such that*

$$\operatorname{Im} \mathbb{B}[\xi] = \operatorname{Im} \nabla^k[\xi] \quad \forall \xi \in \mathbb{R}^N \backslash \{0\},$$

where $\nabla^k \colon C^\infty(\mathbb{R}^2, \mathbb{R}) \to C^\infty(\mathbb{R}^2, (\mathbb{R}^2)^k)$. Then the operator $\mathcal{B}$ is of order $k_{\mathcal{B}} \geq k$.

Proof. We note that

$$\dim(\operatorname{Im} \nabla^k[\xi]) = 1.$$

Consider $\xi_0 = e_1 + e_2$ and the coordinates of

$$v_{11...1} = \partial_1^k u, \quad v_{22...2} = \partial_2^k u.$$

There exists $v \in \mathbb{R}^m$ such that

$$\mathbb{B}[\xi_0](v) \neq 0, (\mathbb{B}[\xi_0](v))_{1^k} = 1 = (\mathbb{B}[\xi_0](v))_{2^k} = 1.$$

Due to continuity of $\mathbb{B}[\cdot](v)$, there exists an open ball $B_r(\xi_0)$, such that, for all $\xi \in B_r(\xi_0)$,

$$\mathbb{B}[\xi](v) \neq 0.$$

In particular, as the dimension of the image of $\nabla^k[\xi]$ (and therefore also of the image of $\mathbb{B}[\xi]$) is one, we then have, for all $\xi \in B_r(\xi_0)$,

$$\xi_2^k (\mathbb{B}[\xi](v))_{1^k} = \xi_1^k (\mathbb{B}[\xi](v))_{2^k}.$$

Hence, $(\mathbb{B}[\xi](v))_{1^k}$ and $(\mathbb{B}[\xi](v))_{2^k}$ are polynomials of degree larger than k in ξ. Therefore, $\mathcal{B}$ has at least order k. □

Corollary 3.12. *Let $N > 2$.*

(a) For any $k \in \mathbb{N}$, there exists a first-order operator $\mathcal{A}$ such that any potential $\mathcal{B}$ of $\mathcal{A}$ has order $k_{\mathcal{B}} \geq k$.

(b) For any $k \in \mathbb{N}$, there exists a first-order operator $\mathcal{B}$ such that any annihilator $\mathcal{A}$ of $\mathcal{B}$ (i.e. an operator $\mathcal{A}$ such that $\mathcal{B}$ is a potential of $\mathcal{A}$) has order $k_{\mathcal{A}} \geq k$.

Note that (a) follows directly from Lemma 3.11 and the result by Meyers, that ∇^k admits a first-order annihilator $\mathcal{A}^k$ [109]. (b) then follows from the fact that if $\mathcal{B}$ is a potential of $\mathcal{A}$, then $\mathcal{A}^*$ is a potential of $\mathcal{B}^*$. In particular, $\mathcal{B} = (\mathcal{A}^k)^*$ is of first order and only admits annihilators of order $\geq k$.

3.4. $\mathcal{A}$-quasiaffine functions in minimisation problems

Let us shortly see two applications of $\mathcal{A}$-quasiaffine functionals in minimisation problems. To be precise, let us consider the functional $I\colon L^p(\Omega, \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$

$$I(u) = \begin{cases} \displaystyle\int_\Omega f(x, u(x))\,\mathrm{d}x & \text{if } u \in \mathcal{C}, \\ \infty & \text{else.} \end{cases} \tag{3.20}$$

The set $\mathcal{C} \subset L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$ is assumed to be weakly closed in $L^p(\Omega, \mathbb{R}^d)$. To apply the *Direct Method* and get existence of minimisers, the following two properties are crucial:

(I) I needs to be weakly lower-semicontinuous;

(II) I needs to be coercive.

In this chapter, we have shown that if

$$f(x, v) = \varphi(x) \cdot \tilde{f}(v)$$

for some $\varphi \in C_c^\infty(\Omega)$ and an $\mathcal{A}$-quasiaffine function $\tilde{f}$, then I is even weakly continuous. The use of $\mathcal{A}$-quasiaffine functions however goes beyond this observation. Let us outline two different problems, where $\mathcal{A}$-quasiaffine functions might be useful.

3.4.1. Polyconvex functions

We define a further notion of convexity that is easier to handle than $\mathcal{A}$-quasiconvexity.

Definition 3.13 (Polyconvexity)**.** *Let $\mathcal{A}$ be a constant rank operator and $f\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{A}$-quasiaffine. A function g is called **$\mathcal{A}$-polyconvex** if*

$$g(x) = h(f(x))$$

for some convex function $h \in C(\mathbb{R})$.

First of all, observe that polyconvexity is a stronger notion than $\mathcal{A}$-quasiconvexity.

Lemma 3.14. *Any $\mathcal{A}$-polyconvex function g is also $\mathcal{A}$-quasiconvex, i.e. for all $v \in \mathbb{R}^d$ and all $\psi \in \mathcal{T}_\mathcal{A}$ we have*

$$g(v) \leq \int_{T_N} g(v + \psi(x))\,\mathrm{d}x.$$

Proof. Let $v \in \mathbb{R}^d$ and $\psi \in \mathcal{T}_\mathcal{A}$. We use $\mathcal{A}$-quasaffinity of f and then convexity of h:

$$\begin{aligned} g(v) = h(f(v)) &= h\left(\int_{T_N} f(v + \psi(x))\,\mathrm{d}x\right) \\ &\leq \int_{T_N} h(f(v + \psi(x))\,\mathrm{d}x \leq \int_{T_N} g(v + \psi(x))\,\mathrm{d}x. \end{aligned}$$

□

The idea behind introducing this concept of $\mathcal{A}$-polyconvexity is that is is easier to verify polyconvexity of a given function g than verifying $\mathcal{A}$-quasiconvexity. In view of Proposition 3.2, there is an easy pointwise condition to check $\mathcal{A}$-quasiaffinity. Moreover, if $h \in C^2$, convexity is equivalent to $D^2 h$ being positive semidefinite. Hence, checking $\mathcal{A}$-polyconvexity can be done rather explicitly. In contrast to this, $\mathcal{A}$-quasiconvexity is an integrated condition with infinitely many test functions $\psi \in \mathcal{T}_\mathcal{A}$, and thus is much harder to verify. As a consequence, most examples of $\mathcal{A}$-quasiconvex functions are already $\mathcal{A}$-polyconvex. Hence, $\mathcal{A}$-polyconvexity is a sufficient condition for $\mathcal{A}$-quasiconvexity and We see in Chapter 4, that $\mathcal{A}$-quasiconvexity of $f(x,\cdot)$ is, under certain additional growth conditions, equivalent to weak lower-semicontinuity of the functional I. Therefore, $\mathcal{A}$-polyconvexity is a sufficient condition for $\mathcal{A}$-quasiconvexity.

3.4.2. Growth conditions

Coercivity of I means that

$$I(u) \to \infty, \text{ whenever } \|u\| \to \infty. \tag{3.21}$$

Usually, this is ensured by a pointwise constraint on f, which reads

$$f(x, v) \geq C_1|v|^p - C_2. \tag{3.22}$$

Such a coercivity condition is also necessary if the constraint set is $L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$. However, if further conditions are imposed via the constraint set $\mathcal{C}$, one can weaken the pointwise condition (3.22). For example, if

$$\mathcal{C} = \{u \in L^p(T_N, \mathbb{R}^d) \colon \int_{T_N} u(x)\,\mathrm{d}x = a\}, \quad a \in \mathbb{R}^d$$

and $M \in C(\mathbb{R}^d)$ is an $\mathcal{A}$-quasiaffine polynomial, then the growth condition

$$f(x, v) \geq C(1 + |v|^p) - M(v) \tag{3.23}$$

ensures coercivity of the functional (3.21). Pointwise coercivity conditions of the form (3.23) are also useful for boundary conditions. This is further elucidated in Section 4.5 and Section 5.5.

4. Weak lower-semicontinuity and $\mathcal{A}$-quasiconvexity

Sections 4.1–4.4 is a significant extension of the author's master's thesis

- [133]: Schiffer, S., *Data-driven problems and generalised convex hulls in elasticity,* Master's thesis,

with generalised statements and proofs. The last two sections 4.6 and 4.7 are independent of the master's thesis, the latter presents some results, which are also given in the preliminary section of [95]

- [95]: Lienstromberg, C., Schiffer, S. and Schubert, S. *A data-driven approach to incompressible viscous fluid mechanics – the stationary case.*

4.1. Introduction

4.1.1. Overview

In this chapter, we study weak lower-semicontinuity section, relaxation, and existence of minimisers for integral functionals of the form $I(u) = \int_\Omega f(x, u(x))\ \mathrm{d}x$ in L^p, $1 < p < \infty$, subject to a differential constraint $\mathcal{A}u = 0$ in Ω. The main assumptions are that

- $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_l)$ where $\mathcal{A}_i$ are constant coefficient differential operators, cf. Section 2.4.
- $\mathcal{A}$ satisfies the constant rank property,
- $f(x, \cdot)$ satisfies an an integrated coercivity condition for periodic $\mathcal{A}$-free functions.

The main difference to earlier works is a new construction of a recovery sequence, which allows to get a uniform bound on its L^p-norm and to deal with some typical boundary condition. In particular, the results in this section apply to recent examples in the theory of data-driven problems, cf. [41, 42] and Chapter 5.

4.1.2. Functionals with differential constraints

The study of minimisation problem

$$\text{argmin } I_f(u) := \int_\Omega f(x, u(x))\, \mathrm{d}x$$

subject to a differential constraint

$$\mathcal{A}u = 0 \tag{4.1}$$

has a long and distinguished history. For example, for simply connected domain $\Omega \subset \mathbb{R}^N$ the differential constraint $\operatorname{curl} u = 0$ corresponds to the minimisation of integral functionals $J(v) = \int_\Omega f(x, \nabla v(x))\,\mathrm{d}x$.

FONSECA and MÜLLER [65] (see also [111, 46, 25]) have developed a general theory for the lower-semicontinuity the functional I subect to the constraint (4.1) with respect to weak convergence in L^p for $1 < p < \infty$. They assumed that $\mathcal{A}$ is a first-order differential operator with constant coefficients, whose Fourier symbol $\mathbb{A}[\xi]$ satisfies the constant rank condition (cf. [119, 137])

$$\dim\ker \mathbb{A}[\xi] \text{ is constant } \forall \xi \in \mathbb{R}^N \setminus \{0\}$$

and that f is of at most p-growth, i.e.

$$0 \leq f(x,v) \leq C_1(1 + |v|^p) \qquad (p\text{-growth}).$$

For relaxation results and existence of minimisers, one often further assumes that f satisfies the coercivity estimate

$$f(x,v) \geq C_2|v|^p - C_3 \qquad (\text{coercivity}). \tag{4.2}$$

Through the *direct method* of the calculus of variations, the combination of coercivity estimates and lower-semicontinuity immediately implies the existence of minimisers.

Recent works on data-driven elasticity (e.g. [41, 42]) show that the setting of [65] is too restrictive for various interesting applications. First, one may encounter differential constraints which involve operators of different order and order large than one. Second, the coercivity condition on f is too strong; the zero level set of f may not be bounded, ruling out pointwise coercivity estimate of the form $f(x,v) \geq C_2|v|^p - C_3$.

4.1.3. Operators of higher order and another coercivity condition

In this chapter, we address both difficulties outlined before simultaneously. First, we cover differential operators of the form

$$\mathcal{A}u = (\mathcal{A}_1 u, \ldots, \mathcal{A}_k u) \tag{4.3}$$

where $\mathcal{A}_i$ are homogeneous linear, constant coefficients differential operators of order $i \in \mathbb{N}$, i.e.

$$\mathcal{A}_i u = \sum_{|\alpha| = i} A^i_\alpha \partial_\alpha u$$

for linear maps A^i_α. Recall that the Fourier symbol for those operators may be defined as the linear map

$$\mathbb{A}_i[\xi] := \sum_{|\alpha|=i} A^i_\alpha \xi^\alpha$$

We assume that these operators satisfy the following constant rank condition

$$\ker \mathbb{A}[\xi] = \bigcap_{i=1}^{k} \ker \mathbb{A}_i[\xi] = r, \quad \text{for some fixed } r \in \mathbb{N} \text{ for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$

In fact, as we have already observed in Section 2.4, the analysis for operators of this form can be easily reduced to the analysis of operators which are homogeneous of some order $k \in \mathbb{N}$.

Moreover, we replace the pointwise coercivity conditon by the following integral coercivity. Denote by

$$\mathcal{T}_\mathcal{A} = \{\varphi \in C^\infty(T_N, \mathbb{R}^d) \colon \int \varphi = 0 \text{ and } \mathcal{A}\varphi = 0\}$$

the space of all $\mathcal{A}$-free test functions with mean zero. We call f $\mathcal{A}$-integral coercive, if there are constants $C_1, C_2 > 0$, such that for every $x \in \Omega$, $v \in \mathbb{R}^d$ and $\psi \in \mathcal{T}_\mathcal{A}$

$$\int_{T_N} f(x, v + \psi(y)) \, \mathrm{d}y \geq C_1 \int_{T_N} |\psi|^p \, \mathrm{d}y - C_2(1 + |v|^p). \tag{4.4}$$

4.1.4. Main results

The main results of this paper are Theorem 4.1 and 4.2 below. To state them concisely, we focus on homogeneous differential operators of order k. Results for operators of the form (4.3) can be easily deduced, see Section 2.4 and Corollary 4.13 below. We focus on the case $1 < p < \infty$, as L^1 is not reflexive, there are additional effects for $p = 1$ (cf. [14, 11, 9]). Moreover, we only study Carathéodory functions $f \colon \Omega \times \mathbb{R}^d \to [0, \infty)$, i.e. functions which are measurable in the first, and continuous in the second variable.

Following [65], we say that an integrand $g \colon \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{A}$-quasiconvex if for all test functions $\varphi \in \mathcal{T}_\mathcal{A}$ and all $v \in \mathbb{R}^d$ we have the following version of Jensen's inequality:

$$g(v) \leq \int_{T_N} g(v + \varphi(x)) \, \mathrm{d}x. \tag{4.5}$$

For a function $g \in C(\mathbb{R}^d)$ we define the $\mathcal{A}$-quasiconvex envelope of g as follows

$$\mathcal{Q}_\mathcal{A} g(v) = \inf_{\varphi \in \mathcal{T}_\mathcal{A}} \int_{T_N} g(v + \varphi(x)) \, \mathrm{d}x, \tag{4.6}$$

which indeed is the largest $\mathcal{A}$-quasiconvex function, which is pointwise smaller than g (cf. [65], Proposition 4.6). The main theorems study weak-lower semicontinuity of the function

$I_f \colon L^p(\Omega,\mathbb{R}^d) \to \mathbb{R}$ defined via

$$I_f(u) = \begin{cases} \int_\Omega f(x,u(x))\,\mathrm{d}x & \text{if } \mathcal{A}u = 0, \\ \infty & \text{else.} \end{cases}$$

Theorem 4.1 ($\mathcal{A}$-quasiconvexity is sufficient for weak lower-semicontinuity)**.** *Let* $1 < p < \infty$ *and* $\mathcal{A}$ *be a homogeneous differential operator of order* $k \in \mathbb{N}$ *satisfying the constant rank property. Let* $f \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ *be a Carathéodory function satisfying*

$$0 \le f(x,v) \le C(1+|v|^p), \tag{4.7}$$

such that $f(x,\cdot)$ *is* $\mathcal{A}$*-quasiconvex for almost every* $x \in \Omega$*. Then* I_f *is weakly lower-semicontinuous. Moreover, if* f *satisfies the growth condition*

$$f(x,v) \ge \frac{1}{C}(|v|^p - 1)$$

I_f *admits a minimiser in* L^p*.*

Regarding the proof of Theorem 4.1, the key observation is, as in [65], that due to the positivity of f it suffices to show that result for p-equi-integrable sequences rather than general weakly converging sequences. For equi-integrable sequences one can apply a localisation argument. For variety, instead of using rather abstract results about Young measures (cf. [65]), we use a rather explicit argument by restricting to small cubes.

Theorem 4.2 (Relaxation)**.** *Let* $1 < p < \infty$ *and* $\mathcal{A}$ *be a differential operator satisfying the constant rank property,* $f \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ *be a Carathéodory function satisfying* (4.7)*. Then*

$$I_f^*(u) := \inf_{u_n \rightharpoonup u \text{ in } L^p} \liminf_{n\to\infty} I_f(u) = \int_\Omega \mathcal{Q}_\mathcal{A} f(x,u(x))\,\mathrm{d}x \tag{4.8}$$

where $\mathcal{Q}_\mathcal{A} f$ *is defined as in* (4.6)*. Moreover, if* (4.4) *is satisfied, there exists a recovery sequence realising the infimum, i.e.* $u_n \in L^p(\Omega,\mathbb{R}^d)$*, such that* $u_n \rightharpoonup u$ *and*

$$I_f^*(u) = \liminf_{n\to\infty} I_f(u_n). \tag{4.9}$$

A suitable version for first-order operators was shown by Braides, Fonseca and Leoni [25] and, in the setting $p = 1$ for operators of order $k \in \mathbb{N}$ by Arroyo-Rabasa [9]. For this relaxation result, it is mainly assumed that u satisfies the growth condition (4.7). This suffices to show (4.8). If we are given a *global* coercivity condition, i.e.

$$\lim_{\|u\|_{L^p}\to\infty} I(u) = \infty,$$

then naturally we get that sequences almost realising the infimum in (4.8) are uniformly bounded and by choosing an appropriate diagonal sequence we may get a recovery sequence in the sense of (4.9). The classical *pointwise* coercivity condition $f(x,v) \ge C_1|v|^p - C_2$

guarantees coercivity of the functional.
The integrated coercivity condition (4.4) however *does not* imply coercivity of the functional; hence (4.9) does not directly follow from (4.8). Indeed, we need to do a careful construction of the recovery sequence, which guarantees L^p-boundedness (c.f proof of Theorem 4.16).

4.1.5. Outline

We finish the introduction with a short outline of this chapter. In Section 4.2 we introduce some useful notation and recall some fundamental results. Theorem 4.1 is proven in Section 4.3 and Theorem 4.2 in Section 4.4. In Section 4.5 we consider a few examples of functions satisfying the coercivity condition (4.4) and consider a short application to various settings. A detailed application of the results of this chapter can be seen in the following Chapter 5

Sections 4.6 and 4.7 focus on extending the results from this chapter to related settings. Regarding the theory for potentials, there are only minor adjustments needed (cf. Section 4.6). In Section 4.7 we then extend the results to an (p, q)-seeting.

4.2. Basic properties of $\mathcal{A}$-quasiconvex functions

Recall the definition of the set of admissible test functions $\mathcal{T}_\mathcal{A}$:

$$\mathcal{T}_\mathcal{A} = \left\{ w \in C^\infty(T_N, \mathbb{R}^d) \colon \mathcal{A}w = 0, \int_{T_N} w = 0 \right\}.$$

Definition 4.3. *A Borel-function $f \colon \mathbb{R}^d \to \mathbb{R}$ is said to be* ***$\mathcal{A}$-quasiconvex****, if for all $v \in \mathbb{R}^d$, $w \in \mathcal{T}_\mathcal{A}$*

$$f(v) \leq \int_{T_N} f(v + w(x))\,\mathrm{d}x.$$

We define the ***$\mathcal{A}$-quasiconvex envelope*** *of a Borel-function $g \colon \mathbb{R}^d \to \mathbb{R}$ as*

$$\mathcal{Q}_\mathcal{A} g(v) = \inf_{w \in \mathcal{T}_\mathcal{A}} \int_{T_N} g(v + w(x))\,\mathrm{d}x. \tag{4.10}$$

We call $\bigcup_{w \in \mathbb{S}^{N-1}} \ker \mathbb{A}(w) =: \Lambda$ ***the characteristic cone of $\mathcal{A}$****. We say that $f \colon \mathbb{R}^d \to \mathbb{R}$ is Λ-convex, if for all $v \in \Lambda$ and all $x \in \mathbb{R}^d$ the function*

$$f_x(t) = f(x + tv)$$

is convex.

Remark 4.4. (i) $f = \mathcal{Q}_\mathcal{A} f$ if and only if f is $\mathcal{A}$-quasiconvex.

(ii) If f is upper semicontinuous and locally bounded from above then C^∞ in the space of test functions may be replaced by L^∞.

(iii) Convex functions are $\mathcal{A}$-quasiconvex.

(iv) In contrast to convexity, $\mathcal{A}$-quasiconvexity is not a local property in the sense that we only need to look at a small neighbourhood. Kristensen indeed showed in [92] that there exists a non-quasiconvex function f (which is a special case of $\mathcal{A}$-quasiconvexity, see below), but for every point x we can find a quasiconvex function g_x s.t. $g_x = f$ in a neighbourhood of x.

(v) We mainly study operators satisfying the constant rank property (CRP). There is little known about operators with non-constant rank and only a few examples have been studied.

(vi) Due to the result about potentials for $\mathcal{A}$ (c.f [123], Proposition 2.6), in the setting of a homogeneous operator $\mathcal{A}$, $\mathcal{A}$-quasiconvexity equals the notion of $\mathcal{A}$-$\mathcal{B}$-quasiconvexity discussed in [47].

Example 4.5. (i) If the characteristic cone is $\mathbb{R}^d$, then $\mathcal{A}$-quasiconvexity equals convexity. This is for example true for the differential operator div on functions in $C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and the component-wise divergence div acting on functions in $C^\infty(\mathbb{R}^N, \mathbb{R}^N \otimes \mathbb{R}^m)$ as long as $m < N$.

(ii) If $\ker \mathbb{A}(\omega) = \{0\}$ for all $\omega \in \mathbb{S}^{N-1}$, then no functions but constant ones will be in $\ker \mathcal{A}$. In this case every function $f\colon \mathbb{R}^d \to \mathbb{R}$ will be $\mathcal{A}$-quasiconvex.

(iii) A well-studied case is the differential operator $\mathcal{A} = \operatorname{curl}$. This is equivalent to considering functions $u = Dv$ for some $v \in W^{1,1}(T_N, \mathbb{R}^m)$, if $u \in L^1(T_N, \mathbb{R}^{N\times m})$. This special type of $\mathcal{A}$-quasicovexity is simply called *quasiconvexity.*

Proposition 4.6. *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be upper-semicontinuous, $\mathcal{A}$ satisfy the constant rank property* (CRP)*. Then $\mathcal{Q}_\mathcal{A} f$ is $\mathcal{A}$-quasiconvex and upper-semicontinuous. In particular, $\mathcal{Q}_\mathcal{A} f$ is the largest $\mathcal{A}$-quasiconvex function smaller than f.*

The proof can be found in [65, Proposition 3.4].

Proposition 4.7. *Let $\mathcal{A}$ satisfy the constant rank and the spanning property* (CRP) *and* (SP)*. Then*

(i) If f is $\mathcal{A}$-quasiconvex and locally bounded, it is Λ-convex.

(ii) Every locally bounded $\mathcal{A}$-quasiconvex function is continuous.

The proof of (i) is standard (cf. [65]). The statement (ii) then follows from (i) and the spanning property. In particular, the spanning property is necessary and sufficient for continuity of f [79]. Sverak showed, that the converse of (i) is not true in general [145] (for the case $\mathcal{A} = \operatorname{curl}$).

If f satisfies an additional growth condition, then due to Λ-convexity, we can even infer a nice local Lipschitz estimate [105, 88, 79].

Proposition 4.8. *Let $\mathcal{A}$ satisfy* (CRP) *and* (SP)*. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{A}$-quasiconvex and satisfy the growth condition*

$$0 \leq f(v) \leq C_f(1+|v|^p).$$

Then f is locally Lipschitz continuous and there is $C = C(C_f, \Lambda, p)$, such that for all $y, x \in \mathbb{R}^d$

$$|f(y) - f(x)| \leq C(1 + |x|^{p-1} + |y|^{p-1})|y-x|.$$

4.3. Lower-Semicontinuity and Existence of Minimisers

The aim of this section is to prove Theorem 4.1 in our setting. Let $f\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$. We often assume the following hypotheses.

(H1) f is Carathéodory, i.e. measurable in the first and continuous in the second variable. There is $C_0 > 0$ such that for all $x \in \Omega$ and $v \in \mathbb{R}^d$

$$0 \leq f(x,v) \leq C_0(1+|v|^p);$$

(H2) The function f is (uniformly) $\mathcal{A}$-integral coercive, i.e. there are $C_1, C_2 > 0$, such that for all $x \in \Omega$, $v \in \mathbb{R}^d$ and $\psi \in \mathcal{T}_{\mathcal{A}}$

$$\int_{T_N} f(x, v+\psi(y))\,\mathrm{d}y \geq C_1 \int_{T_N} |\psi(y)|^p\,\mathrm{d}y - C_2(1+|v|^p); \tag{4.11}$$

(H3) The function $v \mapsto f(x,v)$ is $\mathcal{A}$-quasiconvex for almost every $x \in \mathbb{R}^d$.

For f satisfying (H1) we consider the following functional $J_f\colon L^p(T_N, \mathbb{R}^d) \to [0,\infty)$ defined by

$$J_f(u) = \int_\Omega f(x, u(x))\,\mathrm{d}x$$

and define I_f as the restriction of J_f onto the kernel of $\mathcal{A}$, i.e.

$$I_f(u) = \begin{cases} J_f(u) & \mathcal{A}u = 0, \\ \infty & \text{else.} \end{cases}$$

The first thing we want to highlight is that J_f is locally Lipschitz continuous in L^p, which directly follows from Proposition 4.8.

Lemma 4.9. *Let f satisfy (H1) and (H3). Then J_f is locally Lipschitz continuous in $L^p(\Omega, \mathbb{R}^d)$ and there is $C_3 > 0$, such that for $u, v \in L^p(\Omega, \mathbb{R}^d)$*

$$|J_f(u) - J_f(v)| \leq C_3\bigl(1 + \|u\|_{L^p}^{p-1} + \|v\|_{L^p}^{p-1}\bigr)\|u-v\|_{L^p}.$$

We first show the weak lower-semicontinuity property.

Theorem 4.10 (Theorem 4.1 part I)**.** *Let $1 < p < \infty$ and let f satisfy (H1) and (H3). Let $u_n \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$ and $\mathcal{A}u_n \to \mathcal{A}u$ in $W^{-k,p}(\Omega, \mathbb{R}^l)$. Then*

$$J_f(u) \leq \liminf_{n\to\infty} J_f(u_n). \tag{4.12}$$

In particular, I_f is weakly lower-semicontinuous.

The following two observations are essential: First of all, in the following Lemma 4.11 we see that it suffices to consider p-equi-integrable sequences. We then subdivide Ω into small cubes Q_a^b and approximate f by functions of the form

$$f_a(x, v) = f_a^b(v) \quad \text{if } x \in Q_a^b$$

and use a 'local' statement.

Lemma 4.11. *Suppose that f satisfies (H1) and (H3). Let $u_n, v_n \subset L^p(\Omega, \mathbb{R}^d)$ be bounded, $\{v_n\}_{n\in\mathbb{N}}$ be p-equi-integrable and let, for some $q < p$, $\|u_n - v_n\|_{L^q} \to 0$. Then*

$$\liminf_{n\to\infty} J_f(v_n) \leq \liminf_{n\to\infty} J_f(u_n). \tag{4.13}$$

Proof of Lemma 4.11. Fix some $\varepsilon > 0$ and choose $0 < \delta < \varepsilon$ such that for all $n \in \mathbb{N}$ and all E measurable with $|E| < \delta$

$$\int_E |v_n(x)|^p \leq \varepsilon.$$

As u_n and v_n are uniformly bounded in L^p by Chebychev's inequality there exists an $R > 0$ such that for all $n \in \mathbb{N}$ $|\{|u_n| \geq R\}| < \delta/2$ and $|\{|v_n| \geq R\}| < \delta/2$. Denote by $X_n = \{x \in \Omega \colon |u_n(x)| < R, |v_n(x)| < R\}$. Note that $v_n \cdot 1_{X_n} - u_n \cdot 1_{X_n} \to 0$ in L^p. Hence,

$$\begin{aligned}
\limsup_{n\to\infty}(I(v_n) - I(u_n)) &= \limsup_{n\to\infty} \int_\Omega f(x, v_n(x)) - f(x, u_n(x))\,\mathrm{d}x \\
&\leq \limsup_{n\to\infty} \int_{X_n} f(x, v_n(x)) - f(x, u_n(x))\,\mathrm{d}x + \int_{X_n^C} f(x, v_n(x)) - f(x, u_n(x))\,\mathrm{d}x \\
&\leq 0 + \sup_{n\in\mathbb{N}} \int_{X_n^C} f(x, v_n(x))\,\mathrm{d}x \\
&\leq \sup_{n\in\mathbb{N}} \sup_{E\subset\Omega\colon |E|<\delta} \int_E f(x, v_n(x))\,\mathrm{d}x \leq \sup_{n\in\mathbb{N}} \sup_{E\subset\Omega\colon |E|<\delta} \int_E C_0(1 + |v_n(x)|^p)\,\mathrm{d}x \\
&\leq C_0(\delta + \varepsilon) \leq 2C_0\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields the equation (4.13). □

Proof of Theorem 4.10. Let $u_n \subset L^p(\Omega, \mathbb{R}^d)$ with $u_n \rightharpoonup u$ and $\mathcal{A}u_n \to \mathcal{A}u$ in $W^{-k,p}(\Omega, \mathbb{R}^d)$. We have seen, that one may reduce to u_n equi-integrable and $\mathcal{A}u_n = \mathcal{A}u$ (cf. Lemma 4.11 or, alternatively, Theorem 2.12).

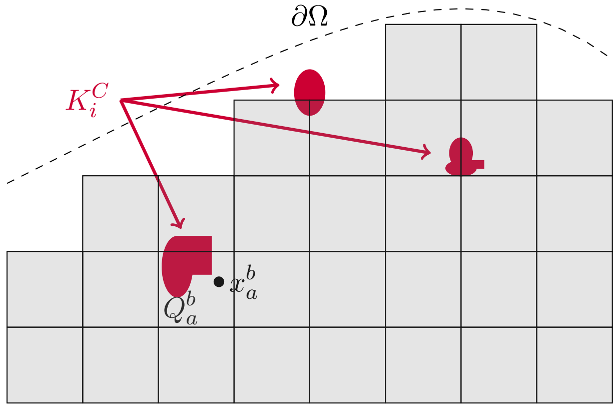

Figure 4.1.: The construction in the proof of Theorem 4.10. The gray cubes are cubes Q_a^b contained in $\mathcal{F}_a$. Consequently, their union (the whole gray area) is F_a. We assume that the measure of the "bad" set K_i^C is smaller than the measure of one cube. For a cube Q_a^b we choose $x_a^b \in Q_a^b \setminus K_i^C$, such that $f(x_a^b)$ is $\mathcal{A}$-quasiconvex.

We now make a few reductions for u. First, approximate u by u^R defined by

$$u^R = \begin{cases} u(x) & |u(x)| \leq R, \\ 0 & |u(x)| > R. \end{cases}$$

and consider $u_n^R = u_n + (u^R - u)$. As $u_n^R \to u_n$ uniformly in n as $R \to \infty$, also

$$\lim_{R\to\infty} \liminf_{n\to\infty} J_f(u_n^R) = \liminf_{n\to\infty} J_f(u_n).$$

Hence, we may assume, $u \in L^\infty(\Omega, \mathbb{R}^d)$. Moreover, by defining

$$\tilde{f}(x,v) := f(x, v - u(x)),$$

which also satisfies (H1) and (H3) (with a larger constant in (H1)), we may assume $u = 0$.

So assume that we are given a p-equi integrable sequence u_n with $\mathcal{A}u_n = 0$, $u_n \rightharpoonup 0$ in $L^p(\Omega, \mathbb{R}^d)$. Let $\varepsilon > 0$, $i \in \mathbb{N}$ and fix some $R > 0$ such that

$$\sup_{n\in\mathbb{N}} |\{x \in \Omega \colon |u_n(x)| \geq R\}| \leq 1/i, \qquad \sup_{n\in\mathbb{N}} \sup_{E\subset\Omega\colon |E|<1/i} \int_E |u_n(x)|^p \,\mathrm{d}x < \varepsilon.$$

By Scorza-Dragoni theorem (cf. [57], p. 235) for any $i \in \mathbb{N}$ there exists $K_i \in \Omega$ compact such that $f_{|K_i\times\mathbb{R}^d}$ is continuous and $|\Omega \backslash K_i| \leq 1/i$. Consider a disjointed family of (semi-open) dyadic cubes $\mathcal{F}_a = \{Q \text{ dyadic cube} \colon l(Q) = 2^{-a},\, Q \subset \Omega\}$ and $F_a = \cup_{Q\in\mathcal{F}_a} Q$.

For a cube $Q_a^b \in \mathcal{F}_a$ pick $x_a^b \in Q_b^a \cap K_i$ such that $f(x,\cdot)$ is $\mathcal{A}$-quasiconvex and define the function

$$f_a^b(v) = \begin{cases} f(x_a^b, v) & \text{if } \mathcal{L}^N(Q_a^b \cap K_i) > 0, \\ 0 & \text{else.} \end{cases}$$

Note that $f_{|K_i \times B(0,R)}$ is uniformly continuous, thus

$$f_a(x,v) := \begin{cases} f_a^b(v) & \text{if } x \in Q_a^b, \\ 0 & \text{else,} \end{cases} \tag{4.14}$$

converges uniformly to f on $K_i \times B(0,R)$ as $a \to \infty$.

Choose a large enough such that $\|f_a - f\|_{L^\infty(K_i \times \mathbb{R}^d)} \leq 1/i$ and $|\Omega \backslash F_a| \leq 1/i$.

$$\begin{aligned}
\liminf_{n\to\infty} \int_\Omega f(x,u_n(x))\,\mathrm{d}x &\geq \liminf_{n\to\infty} \int_{K_i \cap F_a \cap \{|u_n(x)| \leq R\}} f(x,u_n(x))\,\mathrm{d}x \\
&\geq \liminf_{n\to\infty} \int_{K_i \cap F_a \cap \{|u_n(x)| \leq R\}} f_a(x,u_n(x))\,\mathrm{d}x - i^{-1}|\Omega| \\
&\geq \liminf_{n\to\infty} \int_{F_a} f_a(x,u_n(x))\,\mathrm{d}x - (2C_0+1)i^{-1}|\Omega| - 2\varepsilon \\
&= \liminf_{n\to\infty} \sum_{Q_a^b \in \mathcal{F}_a} \int_{Q_a^b} f_a^b(u_n(x))\,\mathrm{d}x - (2C_0+1)i^{-1}|\Omega| - 2\varepsilon \\
&\geq \sum_{Q_a^b \in \mathcal{F}_a} \int_{Q_a^b} f_a^b(0)\,\mathrm{d}x - (2C_0+1)i^{-1}|\Omega| - 2\varepsilon \\
&\geq \int_{K_i \cap F_a \cap \{|u_n(x)| \leq R\}} f_a(x,0)\,\mathrm{d}x - (2C_0+1)i^{-1}|\Omega| - 2\varepsilon \\
&\geq \int_{K_i \cap F_a \cap \{|u_n(x)| \leq R\}} f(x,0)\,\mathrm{d}x - (2C_0+2)i^{-1}|\Omega| - 2\varepsilon \\
&\geq \int_\Omega f(x,0)\,\mathrm{d}x - (5C+2)i^{-1}|\Omega| - 5\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ and $i \to \infty$ yields the weak lower-semicontinuity result. □

Remark 4.12. With very similar methods we can show a lower-semicontinuity result in the setting $p = \infty$ with no growth condition imposed on f (cf. [65]).

One can easily extend Theorem 4.10 to non-homogeneous operators. Using the setting of Section 2.4, we get the following result.

Corollary 4.13 (Reformulation of Theorem 4.10 for non-homogeneous operators)**.** *Let f satisfy the hypotheses (H1) and (H3), and let $\mathcal{A}$ be a differential operator*

$$\mathcal{A}u = (\mathcal{A}_1 u, ..., \mathcal{A}_k u)$$

for homogeneous differential operators $\mathcal{A}_i \colon C^\infty(\mathbb{R}^N,\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R}^N,\mathbb{R}^{l_i})$). Let $\mathcal{A}$ satisfy the constant rank property. Then if $u_n \rightharpoonup u$ and $\mathcal{A}_i u_n \to \mathcal{A}u_n$ in $W^{-i,p}(\Omega,\mathbb{R}^{l_i})$ for all $1 \leq i \leq k$, we have

$$J(u) \leq \liminf_{n\to\infty} J_f(u_n).$$

If, in addition to $\mathcal{A}$-quasiconvexity, we are given a strong coercivity condition, then by the direct method, Theorem 4.10 gives the second part of Theorem 4.1.

Corollary 4.14 (Existence of mininimisers - Theorem 4.1 part II). *Let f satisfy hypotheses (H1) and (H3) and in addition the coercivity condition*

$$f(v) \geq C_1|v|^p - C_2.$$

Let $\mathcal{A}$ satisfy (CRP) *and* (SP) *and X be a weakly closed subset of $L^p(\Omega, \mathbb{R}^d)$. Then I_f has a minimiser in X, i.e. there exists $u \in X$ such that*

$$I_f(u) = \inf_{v \in X} I_f(v).$$

This follows from applying the Direct Method and Theorem 4.10. We see in Section 4.5 that if we restrict to certain space $X \subset L^p(\Omega, \mathbb{R}^d)$, then we can chose a weaker coercivity condition. Let us also remark that pointwise coercivity is neccessary for such a result, if one *does not* impose further conditions on X, for example if $X = L^p(\Omega, \mathbb{R}^d)$.

Example 4.15. Let $\Omega \subset \mathbb{R}^2$ be the unit ball and let $u = (u_1, u_2) \in L^2(\Omega, \mathbb{R}^2 \times \mathbb{R}^2)$ where u_1 and u_2 satisfy the differential constraints

$$\mathcal{A}u = \begin{pmatrix} \mathcal{A}^1 u_1 \\ \mathcal{A}^2 u_2 \end{pmatrix} = \begin{pmatrix} \operatorname{div} u_1 \\ \operatorname{curl} u_2 \end{pmatrix}.$$

Consider the integrand $f(x, u) = |u_1 - u_2|^2$ and the corresponding functional

$$I(u) = \begin{cases} |u_1 - u_2|^2 \, \mathrm{d}x & \text{if } \mathcal{A}u = 0 \\ \infty & \text{else.} \end{cases}$$

The integral coercivity condition (H2) is satisfied, as the function $g(u_1, u_2) = u_1 \cdot u_2$ is $\mathcal{A}$-quasiaffine (i.e. g and $-g$ are $\mathcal{A}$-quasiconvex). In particular, if $u \in L^2(T_N, \mathbb{R}^2 \times \mathbb{R}^2)$ is a function with average 0 satisfying $\mathcal{A}u = 0$, then for all $(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2$

$$\begin{aligned} \int_{T_2} f(v_1 + u_1(y), v_2 + u_2(y)) \, \mathrm{d}y = \int_{T_2} & |(v_1 + u_1(y)|^2 + |v_2 + u_2(y)|^2 \, \mathrm{d}x \\ & - \int_{T_2} 2g(v_1 + u_1(y), v_2 + u_2(y)) \, \mathrm{d}y \\ = \int_{T_2} & |v_1 + u_1(y)|^2 + |v_2 + u_2(y)|^2 \, \mathrm{d}x - 2v_1 v_2 \\ \geq \int_{T_2} & |u_1|^2 + |u_2|^2 \, \mathrm{d}x + |v_1|^2 + |v_2|^2 - 2v_1 v_2 \\ \geq \int_{T_2} & |u_1|^2 + |u_2|^2 \, \mathrm{d}x + |v_1 - v_2|^2 \end{aligned}$$

Therefore (H2) is satisfied. On the other hand, for any harmonic function $U \in W^{1,2}(\Omega)$, the function $u = (\nabla U, \nabla U)$ satisfies the differential constraint $\mathcal{A}u = 0$, $I(u) = 0$, but $\|u\|_{L^2} = 2\|\nabla U\|_{L^2}$ can be chosen arbitrarily large.

The situation improves, if one imposes suitable boundary conditions. Let for example, Γ

be an open subset of $\partial\Omega$ with enough regularity. Consider the boundary conditions

$$\begin{cases} u_1 \cdot \nu = 0 & \text{on } \Gamma \\ u_2 \cdot \tau = g & \text{on } \partial\Omega \setminus \Gamma \end{cases}$$

where $\nu(x), \tau(x) \in \mathbb{R}^2$ are the normal and the tangent vector at some $x \in \partial\Omega$. Using this boundary data, we get that f satisfies a coercivity condition in $L^2(\Omega, \mathbb{R}^2 \times \mathbb{R}^2)$ (cf. [41], Section 4.5 for an argument in a slightly more general situation and Chapter 5 for a treatement in the (p,q)-setting).

4.4. Relaxation and necessity

In this section, we first prove a relaxation result for functionals satisfying both the growth conditions (H1) and (H2). In Theorem 4.16 we first disregard boundary values. Later, in Subsection 4.4.2, we further elaborate on how boundary values can be preserved when relaxing and present a few examples.

4.4.1. Relaxation

In this section we prove Theorem 4.2 about relaxation of functionals. For simplicity, let us write

$$(\mathcal{Q}_{\mathcal{A}} f(x,\cdot))(v) =: \mathcal{Q}_{\mathcal{A}} f(x,v)$$

and denote by J_f^* the candidate for the relaxed functional, i.e.

$$J_f^*(u) := \int_\Omega \mathcal{Q}_{\mathcal{A}} f(x, u(x))\, \mathrm{d}x.$$

Theorem 4.16 (Relaxation and existence of recovery sequences)**.** *Let $\mathcal{A}$ satisfy the constant rank property* (CRP) *and the spanning property* (SP)*. Furthermore, let f satisfy the hypotheses (H1) and (H2). For every $u \in L^p(\Omega, \mathbb{R}^d)$ and there exists a bounded sequence $u_n \subset L^p(\Omega, \mathbb{R}^d)$, such that $u_n \rightharpoonup u$, $\mathcal{A}u_n = \mathcal{A}u$ (as an element in $W^{-k,p}(\Omega, \mathbb{R}^l)$) and*

$$\liminf_{n\to\infty} J_f(u_n) = J_f^*(u).$$

The strategy of the proof will be similar to the proofs for necessity of $\mathcal{A}$-quasiconvexity in Section 4.3. Instead of estimating along a small cube from above, we now take almost optimal functions (in the sense of definition of $\mathcal{A}$-quasiconvexity on small cubes) and try to estimate the error we make from above. To get nice functions on cubes note that we have the following lemma.

Lemma 4.17. *Let $1 < p < \infty$ and let $u \in C^\infty(T_N, \mathbb{R}^d)$ with $\mathcal{A}u = 0$ and $\int_{T_N} u = 0$. Define $u_n(x) = u(nx)$. Then there exists a sequence $v_n \subset C^\infty(T_N, \mathbb{R}^d)$ with $\operatorname{spt} v_n \subset\subset Q = [0,1]^N$, $\|v_n - u_n\|_{L^\infty(T_N, \mathbb{R}^d)} \to 0$ and $\mathcal{A}u_n = 0$.*

We remark that in this lemma it is key that g belongs to $W^{k,p'}(Q,\mathbb{R}^l)$ without zero boundary values, so the result is really non-trivial (otherwise we can use cut-offs at the boundary and are instantly finished).

Proof. We take a standard mollifier η supported in the unit ball $B(0,1)$ with mass 1 and define for $j \in \mathbb{N}$ $\eta_j = \eta(jx) \in C_c^\infty\left(B\left(0,\frac{1}{j}\right)\right)$, $j \in \mathbb{N}$. Define the cube

$$Q_j = \left(\frac{2}{j}, 1-\frac{2}{j}\right)^N$$

and cut-off function $\varphi_j = 1_{Q_j} * \eta_j$. Then

$$\varphi_j \in C_c^\infty\left(\left(\frac{1}{j}, \frac{j-1}{j}\right)\right), \quad \|\partial_\alpha \varphi_j\|_{L^\infty} \leq C_{|\alpha|} j^{|\alpha|},$$

for all $\alpha \in \mathbb{N}^N$. Let $u \in C^\infty(T_N,\mathbb{R}^d)$. Due to Theorem 2.6 (cf. [123]) there is a potential operator $\mathcal{B}$ of some order $k_{\mathcal{B}}$, i.e. there is $U \in W^{k_{\mathcal{B}},p}(T_N,\mathbb{R}^d)$ for any $p \in (1,\infty)$ with

$$\mathcal{B}U = u, \quad \|U\|_{W^{k_{\mathcal{B}},p}} \leq C_p \|u\|_{L^p}$$

for $1 < p < \infty$ (Recall that, in general, $C_p \to \infty$ as $p \to \infty$). In particular,

$$\|U\|_{W^{k_{\mathcal{B}}-1,\infty}} \leq C_p \|u\|_{L^\infty}$$

Define $U_n(x) = n^{-k_{\mathcal{B}}} U(nx)$, such that $u_n = \mathcal{B}U_n$. Define

$$v_{n,j}(x) = \mathcal{B}U_n \eta_j$$

Then, as $\mathcal{B}$ is a potential of $\mathcal{A}$, $\mathcal{A}v_{n,j} = 0$, $v_{n,j}$ is compactly supported in Q and choosing $j(n) = n^{\frac{1}{k_{\mathcal{B}}+1}}$ yields

$$\begin{aligned}
\|v_{n,j(n)} - u_n\|_{L^\infty} &\leq \|(1-\eta_{j(n)})u_n\|_{L^\infty} + C\sum_{i=1}^{k_{\mathcal{B}}} \|\nabla^i \eta_{j(n)}\|_{L^\infty} \|\nabla^{k_{\mathcal{B}}-i} U_n\|_{L^\infty} \\
&\leq \|1-\eta_{j(n)}\|_{L^1} \|u\|_{L^\infty} + C\sum_{i=1}^{k_{\mathcal{B}}} j(n)^i \cdot n^i \\
&\leq \|1-\eta_{j(n)}\|_{L^1} \|u\|_{L^\infty} + Cn^{-\frac{1}{k_{\mathcal{B}}+1}}
\end{aligned}$$

Therefore, $v_{n,j(n)}$ satisfies the requirements of the lemma. □

Lemma 4.17 gives us nice recovery sequences on cubes with zero boundary data. Using this construction we can now prove Theorem 4.16.

Proof of Theorem 4.16. We start by making two reductions. First of all, we show that we can uniformly the L^p-norm of the recovery sequence u_n in terms of the L^p norm of

u. Hence, by local Lipschitz continuity of J_f and density of C_c^∞, it suffices to consider $u \in C_c^\infty(\Omega, \mathbb{R}^d)$.

Moreover, we just show that for any $R > 0$ there is a sequence u_n^R converging weakly to u in L^p with $\mathcal{A}u_n^R = \mathcal{A}u$, such that

$$\liminf_{n\to\infty} J_f(u_n^R) = \int_\Omega \mathcal{Q}_\mathcal{A}^R f(x, u_n(x))\,\mathrm{d}x,$$

where the hull $\mathcal{Q}_\mathcal{A}^R(x,v)$ is defined as

$$\mathcal{Q}_\mathcal{A}^R f(x,v) = \inf_{\psi\in\mathcal{T}_\mathcal{A}\colon \|\psi\|_{L^\infty}\le R} f(x, v+\psi(y))\,\mathrm{d}y. \tag{4.15}$$

As $\mathcal{Q}_\mathcal{A}^R f \le f$ and the convergence $\mathcal{Q}_\mathcal{A}^R f(x,v) \to \mathcal{Q}_\mathcal{A} f(x,v)$ is monotone, again taking an appropriate diagonal sequence $u_n^{R(n)}$ (provided an uniform L^p bound), yields the result.

So consider $u \in C_c^\infty(\Omega, \mathbb{R}^d)$.

Step 1: Construction of a recovery sequence:
Let $R > 0$ and fix $1 > \varepsilon > 0$ and $i \in \mathbb{N}$. We repeat the approximation of f as in Theorem 4.10 (cf. Figure 4.1) For this let K_i be a compact set, such that $\mathcal{L}^N(\Omega\backslash K_i) \le i^{-1}$, such that f is uniformly continuous on $K_i \times B(0,3R)$ and consider a collection $\mathcal{F}_a$ of semi-open dyadic cubes of side length 2^{-a} and $F_a = \cap_{Q\in\mathcal{F}_a} Q$.

Let us assume that that a is large enough, such that $|\Omega\backslash F_a| \le 2i^{-1}$ and that there is no cube Q_a^b in $\mathcal{F}_a$ such that $Q_a^b \cap K_i = \emptyset$ (else set $f_a \equiv 0$ on this cube). For every Q_a^b pick some $x_a^b \in Q_a^b \cap K_i$. Let us define for $v \in \mathbb{R}^d$

$$f_a^b(v) = f_a^b(x_a^b, v), \quad f_a(x,v) = \sum_{Q_a^b\in\mathcal{F}_a} f_a^b(v) 1_{Q_a^b}(x).$$

Due to uniform continuity of f, it is possible to chose a large enough such that for all $(x,v) \in (K_i \cap F_a) \times B_{3R}(0)$

$$|f(x,v) - f_a(x,v)| \le \varepsilon/2$$

and also, as $u \in C_c^\infty$, for all $x, y \in Q_a^b$

$$|u(x) - u(y)| < \varepsilon/2.$$

Let now $\tilde{v}_a^b \in C^\infty(T_N, \mathbb{R}^d)$ with L^∞-norm less than R, such that

$$\int_{T_N} f_a^b(\tilde{v}_a^b(y) + u(x_a^b))\,\mathrm{d}y \le \mathcal{Q}_\mathcal{A}^R f_a^b(u(x_a^b)) + \varepsilon.$$

By Lemma 4.17, scaling T_N down to the cube Q_a^b and by picking a suitable subsequence we may find $v_{a,n}^b \subset C_c^\infty(Q_a^b, \mathbb{R}^d)$ with the following properties:

(v1) $v_{a,n}^b \rightharpoonup 0$ in $L^p(Q_a^b, \mathbb{R}^d)$;

(v2) $\|v_{a,n}^b\|_{L^\infty} \le 2\|\tilde{v}_a^b\|_{L^\infty} \le 2R$;

(v3) $\|v^b_{a,n}\|_{L^p(Q^b_a),\mathbb{R}^d)} \leq 2\|\tilde{v}^b_a\|_{L^p(T_N,\mathbb{R}^d)}\mathcal{L}^N(Q^b_a)$

(v4) $\mathcal{A}v^b_{a,n} = \mathcal{A}u_a$;

(v5) $\liminf_{n\to\infty} \int_{Q^b_a} f^b_a(v^b_{a,n}) \leq \mathcal{L}^N(Q^b_a)(f^b_a(u(x^b_a)) + \varepsilon)$.

The property (v5) follows from the Lipschitz continuity of f^b_a (cf. Lemma 4.9) and the fact that $\tilde{v}^b_a$ almost attains the definition for $\mathcal{Q}^R_{\mathcal{A}} f^b_a$.

Define the recovery sequence $v_{a,n}$ by

$$v_{a,n}(x) = \sum_{Q^b_a \in \mathcal{F}_a} (v^b_{a,n}(x) + u(x^b_a))$$

We now want to show that letting $a, n \to \infty$ and defining a suitable diagonal sequence yields the result.

Step 2: Letting $n \to \infty$:

We define the simple function u_a as follows:

$$u_a(x) = \sum_{Q^b_a \in \mathcal{F}_a} u(x^b_a) 1_{Q^b_a}(x)$$

Then $u_a \to u$ in $L^p(\Omega, \mathbb{R}^d)$ as $a \to \infty$ and in particular, we shall assume that a is large enough, such that

$$\|u_a\|_{L^p} \leq 1 + 2\|u\|_{L^p}.$$

Claim: *Let $a \in \mathbb{N}$ be fixed and large enough according to Step 1. Then*

(1) There is a constant C, only dependent on C_0, C_1, C_2 from (H1) and (H2), such that

$$\|v_{a,n}\|_{L^p} \leq C\|u\|_{L^p};$$

(2) $v_{a,n} \rightharpoonup u_a$ in $L^p(\Omega, \mathbb{R}^d)$ as $n \to \infty$;

(3) $\mathcal{A}v_{a,n} = \mathcal{A}u_a$;

(4) There is $C_R > 0$, such that

$$\liminf_{n\to\infty} \int_\Omega f(x, v_{n,a}(x)) \leq \int_\Omega \mathcal{Q}^R_{\mathcal{A}}(f(x, u(x))\,\mathrm{d}x + C_R(\varepsilon + i^{-1}).$$

Let us start with the uniform bound (1). By $\mathcal{A}$-integral-coercivity we have for $\tilde{v}^b_a$.

$$\begin{aligned} C_1\|\tilde{v}^b_a\|^p - C_2(1 + |u(x^b_a)|^p) &\leq \int_{T_N} f^b_a(\tilde{v}^b_a(y) + u(x^b_a))\,\mathrm{d}y \leq \mathcal{Q}^R_{\mathcal{A}} f^b_a(u(x^b_a)) + \varepsilon \\ &\leq f^b_a(u(x^b_a)) + \varepsilon \leq C_0(1 + |u(x^b_a)^p) + \varepsilon. \end{aligned}$$

Therefore,

$$\|\tilde{v}^b_a\|^p_{L^p(T_N,\mathbb{R}^d)} \leq C(|u(x^b_a)|^p + 1). \tag{4.16}$$

Thus, by construction, we may estimate

$$\|v_{a,n}^b\|^p_{L^p(Q_a^b,\mathbb{R}^d)} \leq \tilde{C}\mathcal{L}^N(Q_a^b)(1+|u(x_a^b)|^p). \tag{4.17}$$

Recalling the definition of $v_{a,n}$ we get

$$\|v_{a,n}\|^p_{L^p(\Omega,\mathbb{R}^d)} \leq \tilde{C}(1+\|u_a\|^p_{L^p}) \leq 2\tilde{C}(1+1\|u\|^p_{L^p})$$

For the weak convergence (2), note that due to (v1) $v_{a,n} \rightharpoonup u(x_a^b)$ in $L^p(Q_a^b,\mathbb{R}^d)$ for each cube Q_a^b, the sequence $v_{a,n}$ is bounded in L^p and that $v_{a,n} = 0$ outside of $F_a = \bigcup_b Q_a^b$. Therefore, $v_{a,n} \rightharpoonup u_a$ in $L^p(\Omega,\mathbb{R}^d)$ as $n \to \infty$.

Property (3) follows directly by the fact that $v_{a,n}^b$ are compactly supported on their cubes and $\mathcal{A}v_{a,n}^b = 0$.

It remains to show that $v_{a,n}$ satisfies (4). First of, all note that we have the L^∞ bound

$$\|v_{a,n}\|_{L^\infty(\Omega,\mathbb{R}^d)} \leq 2R.$$

Hence, we may estimate

$$J_f(v_{a,n}) \leq \int_{K_i\cap F_a} f(x,v_{a,n}(x))\,\mathrm{d}x + 2C_0 i^{-1}(1+R^p) \tag{4.18}$$

On the set $K_i \cap F_a$ we now may replace f by f_a. Thus,

$$\begin{aligned}\int_{K_i\cap F_a} f(x,v_n(x))\,\mathrm{d}x &\leq \sum_{Q_a^b\in\mathcal{F}_a}\int_{Q_a^b} f_a^b(v_{n,a}(x))\,\mathrm{d}x \leq \sum_{Q_a^b\in\mathcal{F}_a}\int_{Q_a^b}\mathcal{Q}_{\mathcal{A}}^R f_a^b(u(x_a^b))\,\mathrm{d}x \\ &\leq \int_{F_a\cap K_i}\mathcal{Q}_{\mathcal{A}}^R f_a(u_a(x))\,\mathrm{d}x + i^{-1}\mathcal{L}^N(\Omega)\end{aligned}$$

u_a is close to $u(x)$ for $x \in Q_a^b$ and $\mathcal{Q}_{\mathcal{A}}^R f$ is uniformly continuous in $K_i \times B(0,R)$, as f is uniformly continuous in $K_i \times B(0,2R)$. Thus,

$$\int_{F_a\cap K_i}\mathcal{Q}_{\mathcal{A}}^R f_a(u_a(x))\,\mathrm{d}x \leq \int_{F_a\cap K_i}\mathcal{Q}_{\mathcal{A}}^R f(u(x))\,\mathrm{d}x + 2\varepsilon\mathcal{L}^N(\Omega). \tag{4.19}$$

Combining (4.18) and (4.19), we get

$$\liminf_{n\to\infty} J_f(v_{a,n}) \leq \int_\Omega \mathcal{Q}_{\mathcal{A}}^R f(x,u(x))\,\mathrm{d}x + 2\mathcal{L}^N(\Omega)(i^{-1}+\varepsilon) + 2C_0 i^{-1}(1+R^p) \tag{4.20}$$

Step 3: Diagonal sequence as $i \to \infty$, $\varepsilon \to 0$:

We have seen that for each $i \in \mathbb{N}$, $\varepsilon > 0$, there is $a_0 = a_0(\varepsilon,i)$, such that for all $a > a_0$ the properties (1)-(4) hold. We now let $i \to \infty$, $\varepsilon(i) = i^{-1} \to 0$ and $a(i) = \max(a_0(\varepsilon(i),i)),i) \to \infty$. Note that we have a uniform L^p bound on $v_{n,a(i)}$. Thus, by appropriately chosing a diagonal sequence $v_i = v_{n(i),a(i)}$ we get

- $v_i \rightharpoonup \lim_{a\to\infty} u_a = u$ as $i \to \infty$;

- $\mathcal{A}v_i \to \lim_{a\to\infty} \mathcal{A}u_a = \mathcal{A}u$ as $i \to \infty$;
- v_i is uniformly bounded in L^p by $C(1 + \|u\|_{L^p})$;
- v_i is a nice recovery sequence, i.e.

$$\liminf_{n\to\infty} J_f(v_{a,n}) \leq \int_\Omega \mathcal{Q}_\mathcal{A}^R f(x, u(x))\,\mathrm{d}x.$$

This proves Theorem 4.16. □

Remark 4.18 (Boundedness of recovery sequence). Let us note that the relaxation Theorem 4.2 directly follows from this theorem and the sufficiency theorem for $\mathcal{A}$-quasiconvexity 4.10. If we *do not* have the coercivity condition (H2) (as it is assumed e.g. in [25, 7]), then we only get bounded sequences $u_n^R \rightharpoonup u$ in $L^p(T_N, \mathbb{R}^d)$, such that

$$\int_\Omega \mathcal{Q}_\mathcal{A}^R f(x, u(x))\,\mathrm{d}x \geq \liminf_{n\to\infty} J_f(u_n^R).$$

These sequences u_n^R are nicely bounded in L^∞ and hence in L^p by CR, but this bound is not uniform in R. Hence we can, in general, not find a suitable diagonal sequence in L^p realising the infimum in

$$\inf \inf_{u_n \rightharpoonup u, \mathcal{A}u_n = \mathcal{A}u} J_f(u_n).$$

Note that the existence of such a diagonal sequence is trivial if we are given a global coercivity condition like

$$J_f(u) \geq C\|u\|_{L^p}^p - C_2$$

holding for all $u \in L^p$ or for all $u \in X$ for a weakly closed subset $X \subset L^p$ (cf. Section 4.4.2). Our version of integrated coercivity (H2) however does not always imply a global coercivity condition (cf. Example 4.15).

Corollary 4.19 (Relaxation). *Let $\mathcal{A}$ satisfy the constant rank property* (CRP) *and the spanning property* (SP). *Furthermore, let f satisfy the hypotheses (H1). Let $I\colon L^p(\Omega, \mathbb{R}^d) \to \mathbb{R}^d$ be a weakly lower-semicontinuous functional, such that $I(u) \leq I_f(u)$ for every $u \in L^p(\Omega, \mathbb{R}^d)$. Then for all $u \in L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$*

$$I(u) \leq \int_\Omega \mathcal{Q}_\mathcal{A} f(x, u(x))\,\mathrm{d}x.$$

In particular I^* is the largest weakly lower-semicontinuous functional below I.

Remark 4.20. One key part of the assumption (H1) is that the function f is Carathédory, meaning that $f(v, \cdot)$ is continuous. As a consequence, the hull $\mathcal{Q}_\mathcal{A} f(v, \cdot)$ is $\mathcal{A}$-quasiconvex and, due to the spanning property (SP), continuous. Therefore, the functional I^* with integrand $\mathcal{Q}_\mathcal{A} f$ is weakly lower-semicontinuous itself. On the other hand, if f is not upper-semicontinuous in the second variable, $\mathcal{Q}_\mathcal{A} f$ might not be $\mathcal{A}$-quasiconvex, and hence the functional might not be lower-semicontinuous.

In fact, using the same observation, we can show that $\mathcal{A}$-quasiconvexity is also necessary for weak-lower semicontinuity of I_f.

Corollary 4.21 (Necessity of $\mathcal{A}$-quasiconvexity for weak lower-semicontinuity)**.** *Let* $1 < p < \infty$*. Let* f *satisfy (H1). Suppose further that if* $u \in L^p(\Omega, \mathbb{R}^d)$*, then for all sequences* $u_n \in L^p(\Omega, \mathbb{R}^d)$ *with* $u_n \rightharpoonup u$ *in* L^p *and* $\mathcal{A}u_n \to \mathcal{A}u$ *in* $W^{-k,p}(\Omega, \mathbb{R}^d)$

$$\liminf_{n\to\infty} I_f(u_n) \geq I_f(u).$$

Then $f(x, \cdot)$ *is* $\mathcal{A}$*-quasiconvex for a.e.* $x \in \Omega$*.*

Proof. Suppose that $f(x, \cdot)$ is not $\mathcal{A}$-quasiconvex for a.e. $x \in \Omega$. Then there exists an $R > 0$ such that $\mathcal{Q}^R_{\mathcal{A}} f(x, v) < f(x, v)$ for $x \in E$ with $|E| > 0$ and some $v = v(x) \in \mathbb{R}^d$. As f is Carathéodory, considering the definition of $\mathcal{Q}^R_{\mathcal{A}}$ v can be chosen to be measurable (or with Scorza–Dragoni even continuous) on a subset of E. Hence, there exists an L^p function u such that

$$\int_\Omega \mathcal{Q}^R_{\mathcal{A}} f(x, u(x))\,\mathrm{d}x \leq \int_\Omega f(x, u(x))\,\mathrm{d}x.$$

But in the proof of Theorem 4.16 we constructed a sequence bounded in L^∞ realising for some $\varepsilon > 0$

$$\int_\Omega \mathcal{Q}^R_{\mathcal{A}} f(x, u(x)) \geq \liminf_{n\to\infty} \int_\Omega f(x, u_n(x))\,\mathrm{d}x - \varepsilon.$$

Note that we only needed the coercivity condition to pass to subsequences as $R \to \infty$ and $\varepsilon \to 0$ which we do not need to do here. Hence, there is a sequence u_n satisfying $\mathcal{A}u_n \to \mathcal{A}u$ and $u_n \rightharpoonup u$ with

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x))\,\mathrm{d}x < \int_\Omega f(x, u(x))\,\mathrm{d}x$$

contradicting the assumption that I was lower-semicontinuous. □

4.4.2. Boundary Values

The construction in the proof of Theorem 4.16 gives us a recovery sequence in L^p. By construction, if f is $\mathcal{A}$-integral coercive, this sequence is bounded. Therefore we get a sequence u_n, such that

$$I^*_f(u_n) = \liminf_{n\to\infty} I_f(u_n).$$

We might encounter boundary problems as follows. The condition $\mathcal{A}u = 0$ implies that suitable components of u have traces on $\partial\Omega$ in suitable negative Sobolev space. In this way we can impose boundary conditions on $\partial\Omega$ or on a sufficiently regular subset $\Gamma \subset \partial\Omega$. Let X_0 denote the (affine) space functions u which satisfy the constraint $\mathcal{A}u = 0$ and a suitable boundary condition. Then we can consider the problem of minimising the functional

$$\tilde{I}(u) := \begin{cases} \displaystyle\int_\Omega f(x, u(x))\,\mathrm{d}x & \text{if } u \in X_0, \\ \infty & \text{else.} \end{cases}$$

and it is natural to ask whether for $u \in X_0$ we can find a recovery sequence $u_n \in X_0$. This is indeed possible. In fact for the recovery sequence u_n constructed in the proof of Theorem 4.16 there exists a sequence $\Omega_n \subset\subset \Omega$ of sets compactly contained in Ω, such that $u_n = u$ on $\Omega \setminus \Omega_n$. Thus u_n and u satisfy the same boundary conditions. Let us also mention that a similar argument is carried out in Chapter 5.

Example 4.22. Suppose that $\Omega \subset\subset (0,1)^N$. We may consider the closed subset X_0 of $L^p(\Omega, \mathbb{R}^d)$ defined via

$$u \in X_0 \text{ whenever } \tilde{u} := \begin{cases} u & \text{on } \Omega \\ 0 & \text{on } (0,1)^n \setminus \Omega \end{cases}, \text{ satisfies } \mathcal{A}\tilde{u} = 0 \text{ in } W^{-k,p}((0,1)^N, \mathbb{R}^l)$$

and for fixed $u_0 \in L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$

$$X_{u_0} = \{u \in L^p(\Omega, \mathbb{R}^d) \colon u - u_0 \in X_0\}$$

Then for every $u \in X_{u_0}$ we may find a recovery sequence $u_n \in X_{u_0}$ to the functional I_f.

Example 4.23. Recall the setup from Example 4.15, i.e. $\mathcal{A}u = (\operatorname{div} u_1, \operatorname{curl} u_2)$. Take the subset $X \subset L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^2)$ of functions satisfying

$$\begin{cases} \mathcal{A}u &= 0, & \\ u_1 \cdot \nu(x) &= 0 & \text{on } \Gamma, \\ u_2 \cdot \tau(x) &= 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

If $u \in X$, then we even may find a recovery sequence $u_n = ((u_1)_n, (u_2)_n)$ converging weakly to u_n, such that

$$(u_1)_n(x) \cdot \nu(x) = u_1(x) \cdot \nu(x), \quad (u_2)_n(x) \cdot \tau(x) = u_2 \cdot \tau(x)$$

for *all* $x \in \partial\Omega$ (in the correct space for traces, i.e. $H^{-1/2}(\Omega) \times H^{-1/2}(\Omega)$). In particular, $u_n \in X$.

Example 4.24 (Boundary conditions for the potential). Instead of considering functionals defined for $u \in L^p(\Omega, \mathbb{R}^d) \cap \ker \mathcal{A}$, we might view these as a functional on $W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$ defined by

$$I(v) = \int_\Omega f(x, \mathcal{B}v(x)) \,\mathrm{d}x$$

for a potential $\mathcal{B}$ of $\mathcal{A}$ in the sense of Raiţă [123], for example

$$\mathcal{A} = \operatorname{curl},\ \mathcal{B} = \nabla \quad \text{or} \quad \mathcal{A} = \operatorname{curl}\operatorname{curl}^T,\ \mathcal{B} = \frac{\nabla + \nabla^T}{2}.$$

Let us assume that v satisfies some boundary condition that $v - v_0 \subset W_0^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$. Then, by a modification of the proof (basically doing the construction on the level of the potential instead of on the level of $\mathcal{A}$-free functions or by using Theorem 2.12), for each v we may find v_n such that $v_n - v \subset W_0^{k_{\mathcal{B}},p}(\Omega,\mathbb{R}^m)$ and

$$\int_\Omega \mathcal{Q}_{\mathcal{A}} f(x,\mathcal{B}v)\,\mathrm{d}x = \liminf_{n\to\infty}\int_\Omega f(x,\mathcal{B}v_n)\,\mathrm{d}x.$$

4.5. Coercivity conditions

Up to now, for a function $f\colon \mathbb{R}^d \to \mathbb{R}$ we have seen the following two coercivity conditions:

(C1) f is **classically coercive** if there are $C_1, C_2 > 0$, such that for all $v \in \mathbb{R}^d$

$$f(v) \geq C_1|v|^p - c_2.$$

(C2) f is **$\mathcal{A}$-integral coercive** if there are $C_3, C_4 > 0$, such that for all $v \in \mathbb{R}^d$, $\psi \in \mathcal{T}_{\mathcal{A}}$

$$\int_{T_N} f(v+\psi(x))\,\mathrm{d}x \geq C_3\|\psi\|_{L^p}^p - C_4(1+|v|^p).$$

For a given function f, (C1) is very easy to check. But it is very restrictive to assume that f satisfies this coercivity condition. If we have more information about the subset, where we want to minimise I_f, $\mathcal{A}$-integral coercivity might be the more suitable condition. It is however very difficult to verify (C2) for general functions f. Therefore, let us shortly define a third concept of coercivity.

Definition 4.25. *We call a function $M\colon \mathbb{R}^d \to \mathbb{R}$ $\mathcal{A}$-quasiaffine, if M and $-M$ are $\mathcal{A}$-quasiconvex.*

*(C3) For an $\mathcal{A}$-quasiaffine function $M\colon \mathbb{R}^d \to \mathbb{R}$, we say that f is **M-polycoercive**, if there are $C_1, C_2 > 0$ for all $v \in \mathbb{R}^d$*

$$f(v) \geq C_1|v|^p - C_2 - M(v). \tag{4.21}$$

We have the following relation between the different types of coercivity:

$$(C1) \Rightarrow (C3) \Rightarrow (C2).$$

It is quite clear, that (C1) implies (C3), as $M = 0$ is $\mathcal{A}$-quasiaffine. If f satisfies (C3), then

$$\int_{T_N} f(v+\psi(x))\,\mathrm{d}x \geq \int_{T_N} C_1|v+\psi(x)|^p - C_2 - M(v) \geq C\|\psi\|_{L^p} - \tilde{C}(1+|v|^p)$$

M-poly-coercivity is considerably weaker than classical coercivity, and has the advantage that the set $\{v\colon f(v) \leq R\}$ can be non-compact if f is M-poly-coercive. But in contrast to $\mathcal{A}$-integral coercivity, it is relatively easy to verify for a given $\mathcal{A}$-quasiaffine function M that a function is M-polycoercive. So let us shortly look at $\mathcal{A}$-quasiaffine functions and typical examples for $\mathcal{A}$.

First, all $\mathcal{A}$-quasiaffine functions are continuous and, moreover, even polynomials (cf. [79, 15]). The degree of those polynomials is bounded by d, so the space of $\mathcal{A}$-quasiaffine functions is finite-dimensional. In particular, there exists a basis of this space consisting of homogeneous polynomials.

Therefore, effectively (4.21) means that there is a homogeneous $\mathcal{A}$-quasiaffine polynomial M of degree $p \in \mathbb{N}$, $p > 1$, such that

$$f(v) \geq C_3|v|^p - C_4 - M(v).$$

In the following we give two examples for behaviour with boundary values. Another example is discussed in the following Chapter 5 in Section 5.5.

4.5.1. Example 1: Boundary Values

Consider the boundary value problem discussed in Example 4.22, i.e. $\Omega \subset\subset (0,1)^N$ and

$$I(u) := \begin{cases} \int_\Omega f(x, u(x))\,\mathrm{d}x & \text{if } u \in X_{u_0}, \\ \infty & \text{else,} \end{cases}$$

where $u \in X_{u_0}$ whenever $\mathcal{A}(u - u_0) = 0$ as an element of $W^{-k,p}(T_N, \mathbb{R}^l)$. Let $p \in \mathbb{N}$ and assume that f satisfies the growth condition

$$f(x, v) \geq C_3|v|^p - C_4 - M(v)$$

As M is a polynomial of degree p and $\mathcal{A}$-quasiaffine, we may write

$$\begin{aligned}\left|\int_\Omega M(u)\,\mathrm{d}x\right| &\leq \left|\int_\Omega M(u_0)\,\mathrm{d}x + M(u - u_0)\,\mathrm{d}x\right| + C\sum_{i=1}^{k-1}\int_\Omega |u|^i|u_0|^{k-i} \\ &= \left|\int_\Omega M(u_0)\,\mathrm{d}x\right| + \sum_{i=1}^{k-1}\|u\|_{L^p}^i\|u_0\|_{L^p}^{k-i}\end{aligned}$$

Thus, by using Young's inequality

$$\left|\int_\Omega M(u)\,\mathrm{d}x\right| \leq \varepsilon\|u\|_{L^p}^p + C_\varepsilon\|u_0\|_{L^p}^p.$$

Therefore,

$$I(u) \geq (C_3 - \varepsilon)\|u\|_{L^p}^p - C_4 - C_\varepsilon\|u_0\|_{L^p}^p$$

and I as a functional therefore is coercive on X_{u_0}.

4.5.2. Example 2: $\mathcal{A} = \operatorname{curl}$

Let $\Omega \subset \mathbb{R}^N$ be Lipschitz and let $u_0 \in W^{1,p}(\Omega, \mathbb{R}^m)$. We consider functionals of the form $I_f\colon W^{1,p}(\Omega, \mathbb{R}^m) \to \mathbb{R}$ given by

$$I_f(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u(x))\, \mathrm{d}x & u - u_0 \in W_0^{k,p}(\Omega, \mathbb{R}^m), \\ \infty & \text{else.} \end{cases} \tag{4.22}$$

Even if it does not directly resemble the functional considered earlier, this is covered by our theory. Indeed, by setting $v = \nabla u$, we see that $\operatorname{curl} v = 0$, where $\operatorname{curl}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^N) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N}_{\text{skew}})$ is defined as

$$\operatorname{curl}_{ij} u = \partial_i u_j - \partial_j u_i.$$

Moreover, the note that the set

$$X_0 = \{v \in L^p(\Omega, \mathbb{R}^{N\times m})\colon \exists u \in W_0^{1,p}(\Omega, \mathbb{R}^m) \text{ such that } v = \nabla u\}$$

is weakly closed.

It is well-known, that a basis the space of curl-quasiaffine functions (also known as Null-Lagrangians) consists of all $r \times r$ minors of the $N \times m$ matrix (e.g. [15, 38, 46]).

By scaling we might assume that $\Omega \subset T_N$. Thus, if $v \in X_0$ and M is a curl-quasiaffine function

$$\int_\Omega M(v(x))\, \mathrm{d}x = \int_{T_N} M(v(x))\, \mathrm{d}x + \mathcal{L}^N(T_N \backslash \Omega) M(0) = M(0).$$

Combining the results of Section 3 and 4, we get:

Proposition 4.26. *Let $f\colon \Omega \times \mathbb{R}^{m\times N} \to [0,\infty)$ be a Carathéodory function and M a $r \times r$ minor, such that*

$$C_1|v|^r - C_2 - C_3 M(v) \le f(x,v) \le C_0(1 + |v|^r).$$

1. *If, in addition, $f(x,\cdot)$ is* curl*-quasiconvex almost everywhere, then for each $u_0 \in W^{1,r}(\Omega, \mathbb{R}^m)$ the functional I_f has a minimiser in $W^{1,r}(\Omega, \mathbb{R}^m)$.*

2. *For every u_0 and every u there exists a bounded minimising sequence $u_n \rightharpoonup u'$ in $W^{1,r}(\Omega, \mathbb{R}^m)$ and*

$$\inf_{u \in W^{1,p}} I_f(u) = \liminf_{n\to\infty} I_f(u_n) = \int_\Omega \mathcal{Q}_{\operatorname{curl}} f(x, u'(x))\, \mathrm{d}x.$$

4.6. Results regarding the potential

Based on the presented methods, the results can easily be modified to fit into a slightly different setting. Let us shortly outline two instances, that will reappear most prominently in Chapter 5. In this section, we deal with functionals on the potential (i.e. on $\mathcal{B}u$) instead of the annihilator, whereas in Section 4.7 we deal with results on $L^p \times L^q$-spaces.

Let $\mathcal{A}$ be a constant rank operator and let $\mathcal{B}$ be a potential of $\mathcal{A}$ of order $k_{\mathcal{B}}$. Let $f\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathédory function. We consider the functional

$$I_{\mathcal{B}}^V(u) = \int_\Omega f(x, \mathcal{B}u(x))\,\mathrm{d}x \tag{4.23}$$

defined on some weakly closed subset $V \subset W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$. Usually, we consider either

$$V = W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m) \tag{4.24}$$

or, for some fixed $u_0 \in W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$, the subset satisfying a Dirichlet boundary conditions, i.e.

$$V_D = \left\{ u \in W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m) \colon u - u_0 \in W_0^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m) \right\} \tag{4.25}$$

Theorem 4.27 (Weak lower-semicontinuity and relaxation for $I_{\mathcal{B}}$)**.** *Let V be as either* (4.24) *or* (4.25)*. Suppose that the function f satisfies the growth condition*

$$0 \le f(x, v) \le C(1 + |v|^p) \quad \text{for a. e. } x \in \Omega,\ \forall v \in \mathbb{R}^d.$$

(a) If $f(x, \cdot)$ is $\mathcal{A}$-quasiconvex for a.e. $x \in \Omega$, then $I_{\mathcal{B}}^V$ is weakly lower-semicontinuous:

(b) If $I_{\mathcal{B}}^V$ is weakly lower-semicontinuous, then $f(x, \cdot)$ is $\mathcal{A}$-quasiconvex for a.e. $x \in \Omega$;

(c) Suppose further that $f(x, \cdot)$ is $\mathcal{A}$-integral coercive uniformly in x. Then for all $u \in W^{k_{\mathcal{B}}}(\Omega, \mathbb{R}^m)$ there exists a sequence $u_n \in W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$, such that $u_n - u \in W_0^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$, $u_n - u \rightharpoonup 0$ in $W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$ and

$$\int_\Omega \mathcal{Q}_{\mathcal{A}} f(x, \mathcal{B}u(x))\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega f(x, \mathcal{B}u_n(x))\,\mathrm{d}x.$$

Remark 4.28. In the language of the differential operator $\mathcal{B}$ uniform $\mathcal{A}$-integral coercivity means that there are constants $C_1, C_2 > 0$, such that for almost every $x \in \Omega$, for any $v \in \mathbb{R}^d$ and any $\psi \in C^\infty(T_N, \mathbb{R}^m)$

$$\int_{T_N} f(x, v + \mathcal{B}u)\,\mathrm{d}x \ge C_1 \int_{T_N} |\mathcal{B}u|^p\,\mathrm{d}x - C_2(1 + |v|^p).$$

Proof. The first statement (a) directly follows from Theorem 4.10. Indeed, if $u_n \rightharpoonup u$ in $W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$, then $v_n = \mathcal{B}u_n$ and $v = \mathcal{B}u$ are $\mathcal{A}$-free and $v_n \rightharpoonup v$ in $L^p(\Omega, \mathbb{R}^d)$. Therefore, due to $\mathcal{A}$-quasiconvexity of f and Theorem 4.10

$$\int_\Omega f(x, v(x))\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega f(x, v_n(x))\,\mathrm{d}x.$$

The second and third statement follow from Theorem 4.16 and the projection result Theorem 2.12. Let $u \in W^{k_{\mathcal{B}},p}(\Omega, \mathbb{R}^m)$ be given and $v = \mathcal{A}u$. Then for any $\varepsilon > 0$, there is a sequence v_n, such that $v_n \rightharpoonup v$, $\mathcal{A}v_n = 0$ and

$$\int_\Omega \mathcal{Q}_\mathcal{A} f(x, v(x))\,\mathrm{d}x + \varepsilon \geq \lim_{n\to\infty} \int_\Omega f(x, v_n(x))\,\mathrm{d}x; \tag{4.26}$$

if $f(x,\cdot)$ is $\mathcal{A}$-integral coercive, we can improve the bound to $\varepsilon = 0$.

By Theorem 2.12 there is a sequence u_n, such that

1. $\sum_{i=0}^{k_\mathcal{B}} \nabla^i u_n(x)$ is p-equi-integrable;
2. $\|\mathcal{B}u_n - v_n\|_{L^q} \to 0$ as $n \to \infty$ for some $1 < q < p$;
3. $u_n - u \subset W_0^{k_\mathcal{B},p}(\Omega, \mathbb{R}^m)$.

Therefore,

$$\liminf_{n\to\infty} \int_\Omega f(x, \mathcal{B}u_n(x))\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega f(x, v_n(x))\,\mathrm{d}x.$$

Together with (4.26) this establishes the validity of parts (b) and (c) □

4.7. Results regarding separate differential constraints

In this section, we consider two separate differential operators of order k_i $(i = 1, 2)$

$$\mathcal{A}_1 \colon C^\infty(\mathbb{R}^N, \mathbb{R}^{d_1}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{l_1}), \quad \mathcal{A}_2 \colon C^\infty(\mathbb{R}^N, \mathbb{R}^{d_2}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{l_2}).$$

From now on, we suppose that both $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the constant rank and the spanning property.

Let us consider a Carathédory function $f \colon \Omega \times (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to \mathbb{R}$, which is measurable in the spacial variable Ω and continuous in the quantity $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. We consider functions

$$v = (v_1, v_2) \in X = L^p(\Omega, \mathbb{R}^{d_1}) \times L^q(\Omega, \mathbb{R}^{d_2}), \quad 1 < p, q < \infty$$

and say that $v \in \ker \mathcal{A}$ if $v_i \in \ker \mathcal{A}_i$. In particular, the differential operator $\mathcal{A}(v_1, v_2) = (\mathcal{A}v_1, \mathcal{A}v_2)$ maps X into $W^{-k_1,p}(\Omega, \mathbb{R}^{l_1}) \times W^{-k_2,q}(\Omega, \mathbb{R}^{l_2})$.

The functionals $I_f, J_f \colon X \to \mathbb{R}$ defined via

$$J(v) = \int_\Omega f(x, v_1(x), v_2(x))\,\mathrm{d}x, \quad I(v) = \begin{cases} J(v) & \text{if } \mathcal{A}v = 0, \\ \infty & \text{else.} \end{cases} \tag{4.27}$$

The same methods employed in the construction for the fully homogeneous setting then yield the following results.

Theorem 4.29 ($\mathcal{A}$-quasiconvexity implies lower-semicontinuity in the (p,q)-setting)**.** *Let $f \colon \Omega \times (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be a Carathéodory function that satisfies the growth condition*

$$0 \leq f(x, v_1, v_2) \leq C(1 + |v_1|^p + |v_2|^q), \quad x \in \Omega, v = (v_1, v_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.$$

Suppose that $u = (u_1, u_2) \in X$ *and that* $u_n = (u_{1,n}, u_{2,n}) \subset X$ *is a sequence that satisfies*

(a) $u_n \rightharpoonup u$ *in* X*, i.e.* $u_{1,n} \rightharpoonup u_1$ *in* $L^p(\Omega, \mathbb{R}^{d_1})$ *and* $u_{2,n} \rightharpoonup u_2$ *in* $L^q(\Omega, \mathbb{R}^{d_2})$*;*

(b) $\mathcal{A}u_n \to \mathcal{A}u$ *in* $W^{-k_1,p}(\Omega, \mathbb{R}^{l_1}) \times W^{-k_2,q}(\Omega, \mathbb{R}^{l_2})$*.*

Suppose that $f(x, \cdot)$ *is* $\mathcal{A}$*-quasiconvex for almost every* $x \in \Omega$*. Then*

$$\liminf_{n\to\infty} J_f(u_n) \leq J_f(u).$$

As a consequence, the functional I *is weakly lower-semicontinuous.*

For a relaxation result we first have to define a suitable notion of coercivity. In the (p, q)-setting, we say that $f\colon \Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ is (uniformly) $\mathcal{A}$-integral coercive if for all $x \in \Omega$, for all $v = (v_1, v_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and all test functions $\psi = (\psi_1, \psi_2) \in \mathcal{T}_{\mathcal{A}}$ we have

$$\int_{T_N} f(x, v_1 + \psi_1(y), v_2 + \psi_2(y))\,\mathrm{d}y \geq C_1 \int_{T_N} |\psi_1|^p + |\psi_2|^q\,\mathrm{d}y - C_2(1 + |v_1|^p + |v_2|^q). \quad (4.28)$$

Using this notion of coercivity, we are able to prove the following relaxation result.

Theorem 4.30 (Relaxation in the (p, q)-setting)**.** *Suppose that* $f\colon \Omega \times (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to \mathbb{R}$ *is a Carathéodory function satisfying both the growth condition*

$$0 \leq f(x, v) \leq C(1 + |v_1|^p + |v_2|^q)$$

and the coercivity condition (4.28)*. Let* $u = (u_1, u_2) \in X$*. Then there exists a recovery sequence* $u_n = (u_{1,n}, u_{2,n}) \in X$*, such that*

1. $u_n \rightharpoonup u$ *in* X*;*

2. $\mathcal{A}u_n = \mathcal{A}u$ *(as elements in* $W^{-k_1,p}(\Omega, \mathbb{R}^{d_1}) \times W^{-k_2,q}(\Omega, \mathbb{R}^{d_2})$*);*

3. $$\lim_{n\to\infty} J(u_n) = \int_\Omega \mathcal{Q}_{\mathcal{A}} f(x, u(x))\,\mathrm{d}x.$$

As the proof of this follows the proof of Theorem 4.4, we only give the arguments that slightly differ from the L^p-setting.

Sketch of Proof. The construction of a recovery sequence for Theorem 4.30 is the same as for Theorem 4.4. Indeed, it suffices to consider the case where we can subdivide Ω into subcubes Q_a^b and the approximation of f by a function

$$f_a(x, v) = \sum_{Q_a^b \in \mathcal{F}_a} f_a^b(v) 1_{Q_a^b}.$$

Moreover, we can reduce to the case where u is constant on the cubes.

For simplicity, let us now assume that

$$f(x,v)=\sum_{Q\in\mathcal{F}} f_Q(v)1_Q \quad \text{and} \quad u(x)=\sum_{Q\in\mathcal{F}} u_Q$$

for a collection $\mathcal{F}$ of disjointed cubes Q. The recovery sequence is constructed by finding $v_{Q,n}=(v_{1,Q,n},v_{2,Q,n})\subset C_c^\infty(Q,\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ that satisfies $\mathcal{A}v_{Q,n}=0$, $\fint_Q v_{Q,n}=0$ and

$$\mathcal{Q}_\mathcal{A} f_Q(u_Q)=\liminf_{n\to\infty}\fint Qf(u_Q+v_{Q,n}(x))\,\mathrm{d}x.$$

The existence of such a sequence can be justified as in the proof of Theorem 4.4. Moreover, we might assume that $v_{Q,n}\rightharpoonup 0$ in $L^p(Q,\mathbb{R}^{d_1})\times L^q(Q,\mathbb{R}^{d_2})$.

The recovery sequence on Ω is the defined as

$$v_n(x)=\sum_{Q\in\mathcal{F}}(u_Q+v_{Q,n}(x))1_Q(x).$$

The main argument from the proof of Theorem 4.4 that one needs to verify in this setting is the uniform $L^p\times L^q$-bound on v_n. This is handled by the assumption that f is $\mathcal{A}$-integral coercive (4.28). Indeed, for the local recovery function $v_{Q,n}$ we obtain for n large enough

$$\mathcal{Q}_\mathcal{A} f_Q(u_q)+\varepsilon\geq\fint_Q f(u_Q+v_{Q,n})(x)\geq\frac{C_1}{|Q|}\int_Q|v_{1,Q,n}|^p+|v_{2,Q,n}|^q\,\mathrm{d}x-C_2(1+|u_{1,Q}|^p+|u_{2,Q}|^q).$$

The upper growth condition for f is also an upper growth condition for $\mathcal{Q}_\mathcal{A} f$. Hence, we conclude

$$\int_Q|v_{1,Q,n}|^p+|v_{2,Q,n}|^q\,\mathrm{d}x\leq C|Q|(1+|u_{1,Q}|^p+|u_{2,Q}|^q).$$

Combining these estimates for all cubes yields

$$\int_\Omega|v_{1,n}|^p+|v_{2,n}|^q\,\mathrm{d}x\leq C\int_\Omega(1+|u_1|^p+|u_2|^q.$$

Therefore, the recovery sequence is uniformly bounded in X.

So the result holds, whenever u and f are of this special form. For the general case we approximate u and f by these functions and take a diagonal sequence. For this we highlight, that taking such a diagonal sequence is only possible due to the uniform bound we have just proven. □

5. Data-driven problems in fluid mechanics

Summary

This chapter is based on joint work in preparation with C. Lienstromberg and R. Schubert

- [95]: Lienstromberg, C., Schiffer, S. and Schubert, R. *A data-driven approach to incompressible viscous fluid mechanics – the stationary case.*

The chapter closely follows the forthcoming article, apart from the preliminary Sections 2&3. Some results of Section 5.2 have also been stated and proven in Chapter 2 and in Sections 4.6 and 4.7. For reference, we restate them here without proof.

The research undertaken in the paper in question is a collaboration with C. Lienstromberg and R. Schubert. All authors and, in particular the author of this thesis, have contributed significant parts to each section of the work.

Our goal is to introduce a data-driven approach to the modelling and analysis of viscous fluids. Instead of including constitutive laws for the fluid's viscosity in the mathematical model, we suggest to directly use experimental data. Only a set of differential constraints derived from first principles and boundary conditions are kept of the classical PDE model and are combined with a data sets. The mathematical framework builds on the recently introduced data-driven approach to solid-mechanics [87, 41].

This chapter is split into six sections. In the introductory Section 5.1, we revisit the PDE approach to the static Navier–Stokes equations and compare it to the new data–driven approach. We furthermore give an overview over the results and the structure of the remaining chapter.

In Section 5.2 we revisit some important notions relevant for the context of minimisation problems, most prominently Γ-convergence and Korn's inequality. Moreover, we see how we can reformulate the problem at hand in terms of constant rank operators, which has been the topic of Chapter 2 of this thesis. Consequently, parts of the corresponding section of the work [95] have been moved to Chapter 2.

Section 5.3 revisits weak lower-semicontinuity results that have been discussed in Chapter 4 of this thesis. In particular, in Theorem 5.11 we justify the important observation that we can reduce our study to so called (p, q)-equi-integrable sequences. That are sequences, where no concentration of mass occurs. It has already been discussed in Chapter 4 that this notion is very helpful for weak lower-semicontinuity statements. Moreover, in Section

5.3.2, we discuss an extension to the linear relaxation result (cf. Theorem 4.16) to a semilinear setting. In particular, we allow the differential constraint, which is $\mathcal{A}v = 0$ in the purely linear setting, to instead feature a specific non-linear right-hand side instead. Later on, this result is applied to the semi-linear inertia term $\operatorname{div}(u \otimes u)$ that appears in the Navier–Stokes equations and its data-driven formulation.

Section 5.4 focuses on a notion of convergence of data sets. Data sets are meant as abstract sets of strain-stress pairs (ϵ, σ), which in applications are derived from experiments. We introduce two different notions of pointwise convergence of data sets. The first concept of convergence expresses, roughly speaking, that the relative error in measurement tends to zero. We then show that this convergence is equivalent to uniform convergence of corresponding data-driven functionals on bounded domains. In the second concept, we weaken the convergence by introducing an increasing range, where the measurements have to be exact. This notion of convergence is in turn equivalent to convergence of certain functionals on (p, q)-equi-integrable sets. As we have seen in the previous Section 5.3, it often suffices to consider (p, q)-equi-integrable sequences and, hence, we work with the second notion of convergence in the remainder of this work.

The knowledge acquired in Sections 5.2, 5.3 and 5.4 is combined to tackle the data-driven problem in fluid mechanics in Section 5.5. First of all, we discuss the different types of boundary values that may apply to this problem. The analysis of the data-driven problem is then split up according to whether the fluid has inertia or not. In both cases we show that the functional is coercive under the prescribed boundary conditions. As a consequence, we are able to derive a Γ-convergence result for the data-driven functionals. It is worthwhile mentioning that in the case of fluids with inertia we need to check that the nonlinearity fits into the setting of the semilinear relaxation results from Section 5.3.

In the last Section 5.6, we check that the data-driven approach is consistent with the classical PDE approach. That is, whenever the data set coincides with some prescribed constitutive law, then the data-driven solution can be associated with a PDE solution and vice versa. We check that this is true for Newtonian fluids, for power-law fluids and for fluids with a monotone strain-stress-relation, which comprises the previous examples, as well as Ellis-law and Herschel–Bulkley fluids.

5.1. Introduction

In this chapter a new approach to the modelling and analysis of viscous fluids is introduced. The hydrostatic behaviour of an incompressible fluid at any instant t in time may be described by its velocity field u which induces a strain

$$\epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right), \tag{5.1}$$

the symmetric gradient of the velocity field. Moreover, the fluid generates a stress field σ which, in the case of an inertialess fluid, satisfies

$$-\operatorname{div}\sigma = f, \tag{5.2}$$

with an external force density f. Both (5.1) and (5.2) are prescribed *differential constraints* and are also called *compatibility conditions.* Both ϵ and σ cannot be *any* field – they have to be a symmetric gradient of another field in the first, and admit a predefined divergence in the second case. For fluids with finite Reynolds number this force balance has to be complemented by the inertial forces proportional to $\partial_t u + (u \cdot \nabla)u$. This results (after suitable non-dimensionalisation) in the equation

$$\partial_t u + (u \cdot \nabla)u - \operatorname{div}\sigma = f.$$

However, in this paper we restrict our analysis to the stationary case $\partial_t u = 0$, i.e.

$$(u \cdot \nabla)u - \operatorname{div}\sigma = f.$$

Since our analysis is mainly based on variational arguments suited for stationary problems, we postpone the time-dependent case to a separate work.

5.1.1. The PDE-based Approach – Constitutive Laws for Viscous Fluids.

Hitherto, the modelling and analysis of the rich set of phenomena in viscous fluid mechanics relies on *constitutive laws* describing the relation between the strain field ϵ and the stress field σ. A commonly used relation is

$$\sigma = -\pi \operatorname{id} + 2\mu(|\epsilon|)\epsilon,$$

which relies on the assumption that the stress comprises two components – the hydrostatic pressure $\pi \operatorname{id}$ and the viscous stress $2\mu(|\epsilon|)\epsilon$. Here, μ denotes the fluid's viscosity. It depends on the strain rate and measures the fluid's resistance to it. Mathematically, the hydrostatic pressure π is the Lagrange multiplier corresponding to the incompressibility condition $\operatorname{div} u = 0$. In the simplest model of a viscous fluid, the viscosity μ is assumed to be constant $\mu = \text{const}$ and the corresponding fluid is called *Newtonian.* In other words, the relation between the viscous forces and the local strain rate is perfectly linear, the constant viscosity being the factor of proportionality. In the case of a *inertialess* incompressible Newtonian fluid one obtains the well-known Stokes' equations

$$\begin{cases} -\mu\Delta u + \nabla\pi = f \\ \operatorname{div} u = 0. \end{cases} \tag{5.3}$$

For incompressible Newtonian fluids *with inertia* one obtains the stationary Navier–Stokes equations

$$\begin{cases} (u\cdot\nabla)u - \mu\Delta u + \nabla\pi = f \\ \operatorname{div} u = 0. \end{cases} \tag{5.4}$$

Although it is reasonable in many practical applications to assume a fluid being Newtonian, real fluids that account for viscosity are in fact non-Newtonian, i.e. they feature a non-linear relation between the stresses σ and the rate of strain ϵ. A widely-used constitutive relation is given by

$$\mu(|\epsilon|) = \mu_0|\epsilon|^{\alpha-1}, \quad \alpha > 0, \tag{5.5}$$

and the corresponding fluid's are called *power-law fluids* or *Ostwald–de Waele fluids*. The exponent $\alpha > 0$ denotes the so-called *flow-behaviour exponent* and $\mu_0 > 0$ is the *flow consistency index*. In the case $0 < \alpha < 1$ the fluid exhibits a *shear-thinning* behaviour as its viscosity decreases with increasing shear-rate, while the fluid is called *shear-thickening* in the case $\alpha > 1$. In this case the viscosity is an increasing function of the shear rate. The corresponding stationary non-Newtonian Navier–Stokes system reads

$$\begin{cases} (u\cdot\nabla)u - \operatorname{div}\big(\mu(|\epsilon(u)|)\epsilon(u)\big) + \nabla\pi = f \\ \operatorname{div} u = 0. \end{cases} \tag{5.6}$$

For $\alpha = 1$ we recover a Newtonian behaviour. In practice, constitutive laws for the fluid's viscosity are derived from experimental measurements, fitting a law belonging to a prescribed class to best approximate the measured data. A large part of the mathematical knowledge in the mechanics of viscous fluids comes from the theoretical and numerical analysis of partial differential equations (Stokes and Navier–Stokes equations), that are derived using constitutive laws. Here, a lot of progress has been made by allowing for increasingly general classes of (nonlinear) viscosity laws [94, 99, 21, 98, 100, 83, 69, 101].

5.1.2. A data-driven Approach.

Nowadays, the availability of big data and the possibility to mine them is increasing drastically. In the present work, instead of including constitutive laws in the mathematical models, we suggest to directly use experimental data in order to find the strain rate ϵ and the stress σ that satisfy the respective differential constraints and, at the same time, approximate the experimental data best. In order to realise this mathematically, we follow the articles [87, 41], where this approach has first been introduced in the context of solid

		Measuring range	
		Constant (unbounded)	**Increasing**
Error	**Constant**	Need to 'throw' out bad data	Need to 'throw' out bad data
	Decreasing	Section 5.4.1	Section 5.4.2

Table 5.1.: Overview of different notions of convergence.

mechanics.

The motivation for replacing the classical PDE-based approach by the data-driven approach is the following. The classical PDE-based approach generates two errors with respect to modelling the real world: First of all, the experimental equipment is imperfect, leading to *measurement errors*. Secondly, the fitting of a material law to the experimental data introduces a *modelling error*. The *data-driven approach* entirely skips this second step.
Turning to the remaining source of errors, with perfect equipment and infinitely many measurements, we expect that it is possible to recover the viscosity law of the fluid (if it exists). In reality, measurements are however restricted by

- the accuracy of the equipment leading to a measurement error;
- a limited number of measurements. This comprises both 'density of measurements' (i.e. given a strain ϵ, how many measurements are taken in a neighbourhood of ϵ?), as well as 'measuring range' (how large are the values of ϵ, where measurements are still taken?).

Nevertheless, if over the course of several consecutive measurement series the measurement error decreases or the number of measurements increases, we expect the experimental data to converge to the material law. Mathematically, we give consideration to this behaviour by introducing different notions of *data convergence*. In this paper we restrict ourselves to the study of the following two settings:

- data with increasing quality and an unbounded range of measurements;
- data with increasing quality and a bounded but increasing range of measurements.

An overview of the possible settings and where they are discussed in this paper is given in Table 5.1.

In the case of non-increasing accuracy, measurements for a given strain rate ϵ might be located in a neighbourhood of the exact value with a certain *likelihood*. In this case the set of data converges in a weak sense to some distribution, see [40]. See also [131] for the analysis of single outliers in measurements.

5.1.3. Mathematical Approach for the data-driven Problem and Main Results.

As mentioned above, we follow the mathematical approach proposed in [41] in a solid mechanical context. To this end, we first split the stress $\sigma = -\pi \operatorname{id} + \tilde{\sigma}$ into its hydrostatic part $\pi \operatorname{id} = -\frac{1}{d} \operatorname{Tr} \sigma \operatorname{id}$ and its viscous part $\tilde{\sigma}$.

Throughout the paper we assume that the *data set* $\mathcal{D}_n$ comprises pairs $(\epsilon, \tilde{\sigma})$ of strain and viscous stress only. The hydrostatic pressure π (i.e. the trace of σ) is not included in the data set, since we allow π to attain arbitrary values. This is due to the fact that the pressure does not play a role in the constitutive law for the viscosity but arises as a Lagrange multiplier corresponding to the incompressibility constraint.

Given a *data set* $\mathcal{D}_n = \{(\epsilon_\beta, \tilde{\sigma}_\beta)\}_{\beta \in B_n}$, consisting of pairs $(\epsilon_\beta, \tilde{\sigma}_\beta)$ of symmetric and trace-free matrices in $\mathbb{R}^{N \times N}$, we consider the functional

$$I_n(\epsilon, \tilde{\sigma}) = \begin{cases} \int_\Omega \operatorname{dist}\left(\left(\epsilon(x), \tilde{\sigma}(x)\right), \mathcal{D}_n\right) \, \mathrm{d}x, & (\epsilon, \tilde{\sigma}) \in \mathcal{C} \\ \infty, & \text{else,} \end{cases} \tag{5.7}$$

as a measure for the distance of functions $(\epsilon, \tilde{\sigma})$, defined on Ω, to the data set. Here, $\mathcal{C}$ is the *constraint set* of fields $\epsilon, \tilde{\sigma}$ satisfying the prescribed differential constraints and suitable boundary conditions, and $\operatorname{dist}(\cdot, \cdot)$ is a suitable distance function.

In the present paper the set of differential constraints is given by (5.1) in combination with either the inertialess or the stationary Navier–Stokes relation. That is, we study both the *linear constraint set*

$$\begin{cases} \epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \\ \operatorname{div} u = 0 \\ -\operatorname{div} \tilde{\sigma} = f - \nabla \pi, \end{cases} \tag{5.8}$$

as well as the *nonlinear constraint set*

$$\begin{cases} \epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \\ \operatorname{div} u = 0 \\ -\operatorname{div} \tilde{\sigma} = f - (u \cdot \nabla) u - \nabla \pi. \end{cases} \tag{5.9}$$

The set of constraints is complemented by suitable boundary conditions. Typical boundary conditions in fluid mechanics are the no-slip condition

$$u = 0 \quad \text{on } \partial\Omega \tag{5.10}$$

and the Navier-slip condition

$$\begin{cases} \tau \cdot (\sigma \nu + \lambda u) = 0, & \tau \in T\partial\Omega \\ u \cdot \nu = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

Here, $\lambda \geq 0$ is the inverse of the so-called slip length and ν denotes the outer normal to $\partial\Omega$. Moreover, $T\partial\Omega$ denotes the tangential bundle of $\partial\Omega$. The case of free slip $\tau \cdot \sigma\nu = 0$ for $\tau \in T\partial\Omega$ is included by $\lambda = 0$. The second condition in (5.11) expresses the non-permeability of the boundary.

Less natural is the Neumann type condition

$$\sigma\nu = 0 \quad \text{on } \partial\Omega. \tag{5.12}$$

We are able to handle all types of boundary conditions (5.10), (5.11), and (5.12) in the linear case (5.8) and the physical conditions (5.10) and (5.11) in the nonlinear case (5.9). In some cases we allow for *inhomogeneous* boundary conditions, i.e. non-zero right-hand sides.

Coming back to (5.7), a minimiser (or a minimising sequence) of the functional I_n always satisfies the compatibility conditions for ϵ and $\tilde{\sigma}$ and is as close to the experimental data $\mathcal{D}_n$ as possible.

In the case in which a sequence $\mathcal{D}_n$ of data sets approximates a limiting set $\mathcal{D}$, corresponding to a material law (as for instance (5.5)), it is expected that the minimisers $v_n = (\epsilon_n, \tilde{\sigma}_n)$ of the functional I_n converge to a solution v of the PDE corresponding to the material law. The main contribution of the present article is to specify conditions under which this assertion is true. We use the following notion for convergence of data sets.

Definition 5.1. *We say that a sequence of closed sets $\mathcal{D}_n$ converges to $\mathcal{D}$, $\mathcal{D}_n \to \mathcal{D}$, if there are sequences $a_n, b_n \to 0$ and $R_n, S_n \to \infty$, such that*

(i) ***Fine approximation on bounded sets:*** *For all $z \in \mathcal{D}$ with $|z| < R_n$ we have*

$$\operatorname{dist}(z, \mathcal{D}_n) \leq a_n(1 + |z|).$$

(ii) ***Uniform approximation on bounded sets:*** *For all $z_n \in \mathcal{D}_n$ with $|z_n| < S_n$ we have*

$$\operatorname{dist}(z_n, \mathcal{D}) \leq b_n(1 + |z_n|).$$

Here, $|\cdot| = \operatorname{dist}(\cdot, 0)$ defines a pseudo-norm.

The sequences a_n and b_n represent the relative error, while S_n and R_n describe the measurement range. Note that condition (i) ensures that every point in the limiting set is approximated by data points in $\mathcal{D}_n$ while condition (ii) ensures that the $\mathcal{D}_n$ approximates $\mathcal{D}$ uniformly.

Moreover, the notion of convergence introduced in Definition 5.1 (ii) is justified from an experimental point of view. Indeed, for a given experimental setup we expect the measurements to be precise only within a certain range, $|z| \leq S_n$. For instance, in the experiment conducted by Couette [44], the aim of which is to measure a fluid's viscosity, the range S_n is linked to the aspect ratio of the rotating cylinders.

It is worthwhile to mention that in our setting we allow the absolute error to grow with the measurement range, which extends the setting studied in [41], where the absolute errors are required to converge to zero.

The main results of this article are the following.

- **Γ-convergence** (Theorem 5.32 and Theorem 5.36): If $\mathcal{D}_n \to \mathcal{D}$ and $\mathcal{D}_n$ satisfies a certain growth condition, then I_n Γ-converges to

$$I^*(\epsilon, \tilde{\sigma}) = \begin{cases} \int_\Omega \mathcal{Q}_\mathcal{A} \operatorname{dist}\big((\epsilon(x), \tilde{\sigma}(x)), \mathcal{D}\big) \,\mathrm{d}x, & (\epsilon, \tilde{\sigma}) \in \mathcal{C} \\ \infty, & \text{else,} \end{cases}$$

  where $\mathcal{Q}_\mathcal{A}$ is a suitable convex envelope of the distance function corresponding to the differential operators defining the compatibility conditions (5.1) and (5.2).

- **Consistency** (Section 5.6): If the data set $\mathcal{D}$ corresponds to a constitutive law, e.g. $\mathcal{D} = (\epsilon, |\epsilon|^{\alpha-1}\epsilon)$ in the case of power-law fluids, then for a function $v = (\epsilon, \tilde{\sigma})$ the following three statements are equivalent:

  (i) v is a minimiser of I^*, i.e. a solution to the *relaxed data-driven problem*;

  (ii) $I^*(v) = 0$, i.e. there exists a sequence $v_n \rightharpoonup v$ with $I(v_n) \to 0$;

  (iii) v is a solution to the corresponding differential equation (i.e. to (5.6) in the nonlinear case) in the classical weak sense.

One of the main difficulties in the proof of the first result is the suitable modification of sequences of functions while preserving differential constraints and given boundary conditions. This is settled by Theorem 2.12, which, for reference, we repeat in its application, Corollary 5.12. One can use this modification result to prove a relaxation statement with a semilinear differential constraint (Theorem 5.15), which, together with the data convergence, leads to the previously mentioned main result about Γ-convergence (Theorem 5.36).

5.1.4. Outline of the Chapter

Let us outline how the rest of this chapter is organised. Section 5.2 aims to contextualise, how the fluid-mechanical problems fit into the general theory of constant rank operators. We introduce some relevant notation and recall the notion of Γ-convergence with respect to the weak topology of L^p-spaces. In Section 5.2.2 we see how the fluid-mechanical setting is translated into the abstract formulation that was introduced in Subsection 2.4.2.

An abstract theory for lower-semicontinuity has been developed by Fonseca & Müller (see also [25] and Chapter 4) and we recall these results at the beginning of Section 5.3. Together with results from Chapter 2 we extend relaxation results, previously attained in [25] and in Chapter 4, to the situation of a semi-linear differential constraint, Theorem 5.15.

For Sections 5.4–5.6 we return to the fluid mechanical setting and apply the abstract results of Section 5.3.

In Section 5.4 we discuss two different notions of *data convergence* purely on set-theoretic level; in particular this convergence is not directly connected to the differential constraints. First, in Subsection 5.4.1 we introduce a form of data convergence which corresponds to the lower-left entry of table 5.1 and show that this is equivalent to a suitable notion of convergence for the corresponding functionals (5.7).

For results about Γ-convergence, however, we can further weaken the notion of convergence to Definition 5.1. This type of convergence is examined in Section 5.4.2. The reason, why this convergence is of interest for Γ-convergence, is discussed earlier at the beginning of Section 5.3 by Theorem 5.11.

The abstract results from Section 5.3 and results about distance functions to data sets $\mathcal{D}_n$ from Section 5.4 are combined in Sections 5.5. In Subsection 5.5.1 and Subsection 5.5.2 we introduce the data-driven problem for inertialess fluids and for fluids with inertia, respectively. We show that, given boundary values and a suitable pointwise coercivity condition, the functionals I_n from (5.7) are coercive on the phase space V. Therefore, we can apply results from Section 5.3 to get the respective Γ-convergence result (Theorem 5.32 and Theorem 5.36).

Finally, Section 5.6 links the (relaxed) data-driven problem $I^*(v) = 0$ to the partial differential equations obtained by including a material law in the modeling. We show that if the data set $\mathcal{D}$ coincides with certain given material laws, i.e. $\mathcal{D} = \{(\epsilon, \tilde{\sigma}) \colon \tilde{\sigma} = \mu(|\epsilon|)\epsilon\}$, then solutions of the relaxed data-driven problem are weak solutions of the classical PDE problem and vice versa.

Comments on the notation

The notation compared to the manuscript and the rest of this thesis has been altered as follows. The dimension of the underlying space still is N. To avoid confusion with the dimension d, which is often used in the context of fluid dynamics instead of N, the differential operator $\mathcal{A}$ now maps from $C^\infty(\mathbb{R}^N, \mathbb{R}^{\mathbf{m}})$ to $C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ and, likewise, $\mathcal{B}$ maps $C^\infty(\mathbb{R}^N, \mathbb{R}^{\mathbf{h}})$ to $C^\infty(\mathbb{R}^N, \mathbb{R}^{\mathbf{m}})$. Moreover, the function f is used as a force term in this section (as it is classically used in the context of fluid dynamics). Apart from distance functions, integrands therefore are denoted by $\mathcal{F}$. Finally, we consider functionals depending on functions $\mathbf{v} = (\epsilon, \tilde{\sigma})$. The function $\mathbf{u}$ takes over the role of a potential (namely the fluid's velocity).

5.2. Functional Analytic Setting of the Fluid Mechanical Problem

In this section, we introduce an abstract functional analytic setting to deal with the differential constraints. First, in Subsection 5.2.1, we recall the notion of Γ-convergence and the notion of constant rank operators. The latter requires a short reminder on some results from Fourier analysis. In Subsection 5.2.2 we see how the differential operators appearing in the fluid mechanical applications fit into the framework of constant rank operators introduced in Subsection 2.4.2.

5.2.1. Γ-Convergence and Constant-Rank Operators

Underlying Function Spaces

Let $\Omega \subset \mathbb{R}^N$ be a bounded, simply connected set with C^1-boundary and let

$$Y = \mathbb{R}^{N\times N}_{\mathrm{sym},0} =: \left\{A \in \mathbb{R}^{N\times N} \colon A = A^T, \mathrm{tr}(A) = 0\right\}$$

be the set of symmetric trace-free matrices in $\mathbb{R}^{N\times N}$. We mainly study functions $v\colon \Omega \to Y \times Y$ and we shall write $v = (\epsilon, \tilde{\sigma})$ to denote their components and $\sigma = -\pi\,\mathrm{id} + \tilde{\sigma}$. One might think of ϵ as the strain and $\tilde{\sigma}$ the viscous part of the stress. For $1 < p, q < \infty$ with $1/p + 1/q = 1$, we consider the *phase space*

$$V = L^p(\Omega, Y) \times L^q(\Omega, Y),$$

equipped with the norm

$$\|v\|_V = \|\epsilon\|_{L^p} + \|\tilde{\sigma}\|_{L^q}.$$

We call $Y \times Y$ the *local phase space*, such that functions $v \in V$ map Ω into $Y \times Y$. Recall that we assume throughout the paper that the pressure π (i.e. the trace of σ) is not considered as part of the data. Consequently, each data set $\mathcal{D}_n$ is a subset of $Y \times Y$. In order to introduce a distance on $Y \times Y$, for pairs $(\epsilon_i, \tilde{\sigma}_i) \in Y \times Y$, $i = 1, 2$, we define

$$\mathrm{dist}((\epsilon_1, \tilde{\sigma}_1), (\epsilon_2, \tilde{\sigma}_2)) = \tfrac{1}{p}|\epsilon_1 - \epsilon_2|^p + \tfrac{1}{q}|\tilde{\sigma}_1 - \tilde{\sigma}_2|^q \quad \text{and} \quad |(\epsilon_1, \tilde{\sigma}_1)|_{p,q} := \mathrm{dist}\big((\epsilon_1, \tilde{\sigma}_1), (0, 0)\big),$$

and therewith

$$d((\epsilon_1, \tilde{\sigma}_1), (\epsilon_2, \tilde{\sigma}_2)) = (\mathrm{dist}\,((\epsilon_1, \tilde{\sigma}_1), (\epsilon_2, \tilde{\sigma}_2)))^{\frac{1}{\max\{p,q\}}}. \tag{5.13}$$

The distance function $d(\cdot,\cdot)$ is defined by taking the p-th, respectively the q-th root of $\mathrm{dist}(\cdot,\cdot)$, in order to guarantee that the triangle inequality is satisfied . Thus, $d(\cdot,\cdot)$ defines a metric on $Y \times Y$. Accordingly, we define the distance on the phase space V by

$$\mathrm{dist}(v_1, v_2) = \int_\Omega \mathrm{dist}\,(v_1(x), v_2(x))\,\mathrm{d}x, \quad v_1, v_2 \in V.$$

We start by proving that the distance function $d(\cdot,\cdot)$, introduced in (5.13), defines a metric.

Lemma 5.2. *The map $d\colon (Y \times Y) \times (Y \times Y) \to \mathbb{R}$ is a metric.*

Proof. Positivity, definiteness and symmetry are clear. The triangle inequality follows from the elementary inequality

$$((a_1 + a_2)^p + (b_1 + b_2)^q)^{\frac{1}{\max\{p,q\}}} \le (a_1^p + b_1^q)^{\frac{1}{\max\{p,q\}}} + (a_2^p + b_2^q)^{\frac{1}{\max\{p,q\}}}, \tag{5.14}$$

being valid for all $a_i, b_i \in [0,\infty)$, $i = 1,2$, and $p \geq q$. Indeed, assume withput loss of generality that $p \geq q$. Then, since the function $s \mapsto s^{q/p}$, $s \in \mathbb{R}$, is concave, we obtain

$$\begin{aligned}
[(a_1+a_2)^p + (b_1+b_2)^q]^{1/p} &\leq \left[(a_1+a_2)^p + \left(b_1^{q/p} + b_2^{q/p}\right)^p\right]^{1/p} \\
&\leq \left[a_1^p + \left(b_1^{q/p}\right)^p\right]^{1/p} + \left[a_2^p + \left(b_2^{q/p}\right)^p\right]^{1/p} \\
&= \left(a_1^p + b_1^q\right)^{1/p} + \left(a_2^p + b_2^q\right)^{1/p}.
\end{aligned}$$

□

In the following we embed Ω into the N-dimensional torus T_N when it is convenient. Without loss of generality we therefore assume that Ω is compactly contained in $(0,1)^N$.

Γ-convergence

In this subsection we recall some well-known results on Γ-convergence that are frequently used throughout the chapter. We use this notion of convergence to consider the behaviour of functionals of the type (5.7) under convergence of the data.

Definition 5.3. *Let (X,d) be a metric space. A sequence of functionals $I_n \colon X \to [-\infty,\infty]$, Γ-converges to $I \colon X \to [-\infty,\infty]$, in symbols $I = \Gamma - \lim_{n\to\infty} I_n$, whenever the following is satisfied.*

(i) ***liminf-inequality:*** *For all $x \in X$ and for all sequences $x_n \to x$ we have*

$$I(x) \leq \liminf_{n\to\infty} I_n(x_n).$$

(ii) ***limsup-inequality:*** *For all $x \in X$ there exists a sequence $x_n \to x$ (called the recovery sequence) such that*

$$I(x) \geq \limsup_{n\to\infty} I_n(x_n).$$

Remark 5.4. (i) In metric spaces the constant sequence $I_n = I$ possesses a Γ-limit I^*, namely the lower-semicontinuous hull of I, given by

$$I^*(x) = \inf_{x_n\to x} \liminf_{n\to\infty} I(x_n). \tag{5.15}$$

I^* is called the *relaxation* of I.

(ii) If each x_n is a minimiser of I_n and $x_n \to x$, then x is a minimiser of I.

(iii) We may define Γ-convergence on topological spaces, cf. [49]. This coincides with the definition on metric spaces when equipped with the standard topology. Weak convergence is not metrisable on Banach spaces. However, it is metrisable on reflexive, separable Banach spaces on bounded sets. Hence, if a functional I satisfies a certain growth condition; i.e.

$$\alpha(\|x\|) \leq I(x) \tag{5.16}$$

for a function $\alpha\colon [0,\infty) \to \mathbb{R}$ with $\alpha(t) \to \infty$ as $t \to \infty$, we just use the metric for weak convergence defined on bounded sets of the Banach space.

(iv) In topological spaces, especially in Banach spaces equipped with the weak topology, the constant sequence $I_n = I$ does in general not possess a sequential Γ-limit, as the infimum in (5.15) does not need to be a minimum.

(v) If I does not satisfy the growth condition (5.16), it is possible to consider the sequential Γ-limit, given as in Definition 5.3. However, this might not exist, even if the topological Γ-limit of a sequence of functionals exists. In particular, the constant sequence might not have a sequential Γ-limit.

In the following we only consider the sequential Γ-limit of sequences in the weak topology of some Banach space (usually $L^p \times L^q$). If the functional I is coercive in the sense of (5.16), then the sequential Γ-limit coincides with the topological Γ-limit. More precisely, the following result holds true.

Lemma 5.5 (Uniform convergence and Γ-convergence)**.** *Let V be a reflexive, separable Banach space equipped with the weak topology. Suppose that $I_n, I\colon V \to [-\infty,\infty]$, such that $I_n(v) \to I(v)$ uniformly on bounded sets of V. If the sequential Γ-limit of the constant sequence I exists, then also I_n possesses a Γ-limit and*

$$\Gamma - \lim_{n\to\infty} I_n = \Gamma - \lim_{n\to\infty} I = I^*.$$

Note that the sequential Γ-limit of the constant sequence I exists if the functional is coercive.

Proof. If $v_n \rightharpoonup v$ is a bounded sequence in V, we have

$$\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} |I_m(v_n) - I(v_n)| = 0.$$

Therefore,

$$\limsup_{n\to\infty} I_n(v_n) = \limsup_{n\to\infty} I(v_n) \le I^*(v) \quad \text{and} \quad \liminf_{n\to\infty} I_n(v_n) = \liminf_{n\to\infty} I(v_n) \ge I^*(v),$$

which establishes both the lim sup-inequality and the lim inf-inequality. □

Korn–Poincaré inequality

In this subsection, we revisit a combination of Korn's inequality (i.e. the full gradient is controlled by its symmetric part) and Poincare's inequality to obtain an estimate of the form

$$\|u\|_{W^{1,p}} \le C\|\epsilon\|_{L^p}, \quad \text{where} \quad 1 < p < \infty \quad \text{and} \quad \epsilon = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right).$$

This estimate is a straightforward consequence of the p-Korn inequality and the Poincaré inequality, cf. for instance [35].

Lemma 5.6 (Abstract Korn–Poincaré inequality). *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$ be open, connected, and bounded with C^1-boundary. Then the following is true.*

(i) There is a constant $C = C(p, \Omega)$, such that for any $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ we have

$$\|u - (A_u x + b_u)\|_{W^{1,p}} \leq C \|\nabla u + \nabla u^T\|_{L^p},$$

where $A_u = \frac{1}{2} \fint_\Omega \nabla u - \nabla u^T \,\mathrm{d}x$ and $b_u = \fint_\Omega u \,\mathrm{d}x$.

(ii) Let $X \subset W^{1,p}(\Omega, \mathbb{R}^N)$ be a closed subspace, such that

$$X \cap \left\{ Ax + b \colon A \in \mathbb{R}^{N \times N}_{\text{skew}}, b \in \mathbb{R}^N \right\} = \{0\}.$$

Then there is a constant $C = C(p, \Omega, X)$, such that for any $u \in X$ we have

$$\|u\|_{W^{1,p}} \leq C \|\nabla u + \nabla u^T\|_{L^p}.$$

5.2.2. The differential operator $\mathcal{A}$ for problems in Fluid mechanics

In this subsection, we discuss how the fluid mechanical constraints (5.8) and (5.9) fit into the abstract setting outlined in Subsection 2.4.2 and in Section 4.7. We consider the two differential operators

$$\begin{cases} \mathcal{A}_1 = \operatorname{curl}\operatorname{curl}^T \colon C^\infty(T_N, Y) \to C^\infty(T_N, (\mathbb{R}^N)^{\otimes 4}) \\ \mathcal{A}_2 = \operatorname{div} \colon C^\infty(T_N, Y) \times C^\infty(T_N, \mathbb{R}) \to C^\infty(T_N, \mathbb{R}^N) \end{cases}$$

as follows

$$\begin{cases} \left(\operatorname{curl}\operatorname{curl}^T(\epsilon)\right)_{ijkl} = \partial_{ij}\epsilon_{kl} + \partial_{kl}\epsilon_{ij} - \partial_{il}\epsilon_{kj} - \partial_{kj}\epsilon_{il}, \quad i,j,k,l = 1, ..., N \\ \left(\operatorname{div}(\tilde{\sigma}, \pi)\right)_i = (\operatorname{div}(\tilde{\sigma} - \pi \,\mathrm{id}))_i = \sum_{j=1}^N \partial_j (\tilde{\sigma} - \pi \,\mathrm{id})_{ij}, \quad i = 1, ..., N. \end{cases}$$

The Fourier symbol of the differential operator $\mathcal{A}_1$ is given by

$$(\mathbb{A}_1[\xi](\epsilon))_{ijkl} = \xi_i\xi_j\epsilon_{kl} + \xi_k\xi_l\epsilon_{ij} - \xi_i\xi_l\epsilon_{kj} - \xi_k\xi_j\epsilon_{il}, \quad \xi \in \mathbb{R}^N \setminus \{0\},\ \epsilon \in Y,\ i,j,k,l = 1, \dots, N.$$

For $\mathcal{A}_2$ the Fourier symbol reads

$$(\mathbb{A}_2[\xi](\tilde{\sigma}, \pi))_i = \sum_{j=1}^N \xi_j \tilde{\sigma}_{ij} - \xi_i \pi, \quad \xi \in \mathbb{R}^N \setminus \{0\},\ (\tilde{\sigma}, \pi) \in Y \times \mathbb{R},\ i = 1, \dots, N.$$

For a fixed $\xi \in \mathbb{R}^N \setminus \{0\}$, the set $\ker \mathbb{A}_1[\xi] \times \ker \mathbb{A}_2[\xi]$ is given as follows. Let $Y_\xi \subset Y$ be defined as

$$Y_\xi = \left\{ a \odot \xi \colon a \in \mathbb{R}^N,\ a \perp \xi \right\},$$

where $a \odot \xi = \frac{1}{2}(a \otimes \xi + \xi \otimes a)$ is the symmetric tensor product. Note that Y_ξ is a $(N-1)$-dimensional subspace of Y. Then we have

$$\ker \mathbb{A}_1[\xi] = Y_\xi \quad \text{and} \quad \ker \mathbb{A}_2[\xi] = \left\{(\tilde{\sigma}, \pi_{\tilde{\sigma}}) \colon \tilde{\sigma} \in Y_\xi^\perp\right\},$$

where $\pi_{\tilde{\sigma}}$ is defined as the unique $\pi \in \mathbb{R}$, such that $\mathbb{A}_2[\xi](\tilde{\sigma}, \pi) = 0$, i.e.

$$\pi_{\tilde{\sigma}} = \frac{\xi^T \tilde{\sigma} \xi}{|\xi|^2}.$$

The differential condition $\operatorname{curl}\operatorname{curl}^T \epsilon = 0$ for $\epsilon \in L^p_\#(T_N, Y)$ encodes that ϵ is a symmetric gradient, i.e. there is $u \in W^{1,p}(T_N, \mathbb{R}^N)$ satisfying

$$\|u\|_{W^{1,p}} \leq C\|\epsilon\|_{L^p}, \quad \epsilon = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right) \quad \text{and} \quad \operatorname{div} u = 0.$$

The differential operator

$$\mathcal{B}_1 \colon C^\infty(T_N, \mathbb{R}^N) \cap \ker \operatorname{div} \longrightarrow C^\infty(T_N, Y) \colon u \longmapsto \tfrac{1}{2}\left(\nabla u + \nabla u^T\right)$$

can be treated as if it was a potential of $\mathcal{A}_1$.

Remark 5.7. Due to the additional constraint $\operatorname{div} u = 0$, $\mathcal{B}_1$ is *not* a potential to $\mathcal{A}_1$ in the sense of Definition (2.3). Note, however that a function $u \in W^{1,p}(T_N, \mathbb{R}^N)$ with zero average satisfies the differential constraint $\operatorname{div} u = 0$ if and only if

$$u = \operatorname{curl}^* U$$

for a suitable function $U \in W^{2,p}\left(T_N, \mathbb{R}^{N\times N}_{\text{skew}}\right)$, where curl^* is the adjoint of curl; in other words curl^* is a potential of div. In particular, this also means that if $\epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, then there exists $U \in W^2_p\left(T_N, \mathbb{R}^{N\times N}_{\text{skew}}\right)$ such that

$$\epsilon = \tfrac{1}{2}\left(\nabla + \nabla^T\right) \circ \operatorname{curl}^* U.$$

Consequently, $\tilde{\mathcal{B}}_1 = \frac{1}{2}\left(\nabla + \nabla^T\right) \circ \operatorname{curl}^*$ is a potential of $\mathcal{A}_1$.

For the purpose of applying Fourier methods, we can use the symmetric gradient $\mathcal{B}_1$ on divergence-free matrices instead. The suitable inverse of $\mathcal{B}_1$ in the Fourier space is

$$\mathcal{B}_1^{-1} = \operatorname{curl}^* \circ \tilde{\mathcal{B}}_1,$$

which is a Fourier multiplier of order $1 + (-2) = -1$.

The potential to the differential operator $\mathcal{A}_2$ is not relevant in this setting. Let us remark that the condition

$$-\operatorname{div} \tilde{\sigma} + \nabla \pi = f,$$

for $(\tilde{\sigma}, \pi) \in L^q(T_N, Y \times \mathbb{R})$ and $f \in W^{-1,p}(T_N, \mathbb{R}^N)$, can be rewritten in terms of $\tilde{\sigma}$ only, as

$$-\operatorname{curl} \circ \operatorname{div} \tilde{\sigma} = \operatorname{curl} f.$$

Another strategy to tackle the linear problem from a "purely' Fourier analytic perspective would be to "forget" about the pressure π by using the operator $\tilde{\mathcal{A}}_2(\tilde{\sigma}) = \operatorname{curl} \circ \operatorname{div} \tilde{\sigma}$. Note that in this approach the operator $\operatorname{curl} \circ \operatorname{div}$ acting on $\tilde{\sigma}$ is the adjoint operator of $\frac{1}{2}\left(\nabla + \nabla^T\right) \circ \operatorname{curl}^*$ which acts on U. For the non-linear problem, cf. Subsection 5.5.2, this approach yields the equation

$$-\operatorname{curl} \operatorname{div} \tilde{\sigma} = \operatorname{curl} f - \operatorname{curl}(u \cdot \nabla) u. \tag{5.17}$$

We believe however, that from the fluid dynamical point of view it is more instructive to include the pressure $\pi \in L^q(\Omega)$ by sticking to the more physical equation

$$-\operatorname{div} \tilde{\sigma} = f - (u \cdot \nabla) u - \nabla \pi.$$

5.3. Existence of minimisers – Weak Lower-Semicontinuity and Coercivity

5.3.1. Weak lower-semicontinuity under differential constraints.

Throughout this paragraph we consider $1 < p, q < \infty$, a Carathéodory function $\mathcal{F} \colon \Omega \times (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}) \to \mathbb{R}$ and functionals $I, J \colon L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2}) \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega \mathcal{F}(x, v(x)) \, \mathrm{d}x \quad \text{and} \quad I(v) = \begin{cases} J(v), & \mathcal{A}v = 0 \\ \infty, & \text{else,} \end{cases} \tag{5.18}$$

The following proposition is a straight-forward adaption of the semi lower-continuity result [65, Theorem 3.6] to the (p, q)-setting (also cf. Proposition 4.29). Recall the notion of $\mathcal{A}$-quasiconvexity as considered in Section 4.7.

Proposition 5.8. *Let $1 < p, q < \infty$ and let $\mathcal{F} \colon \Omega \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition*

$$0 \leq \mathcal{F}(x, z_1, z_2) \leq C(1 + |z_1|^p + |z_2|^q), \quad z_1, z_2 \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}. \tag{5.19}$$

Moreover, let $\mathcal{F}(x, \cdot)$ be $\mathcal{A}$-quasiconvex for a.e. $x \in \Omega$. Then the following holds true:

(i) along all sequences $v_n \rightharpoonup v$ in $L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$ with $\mathcal{A}v_n \to \mathcal{A}v$ strongly in $W^{-k_1,p}(\Omega, \mathbb{R}^{m_1}) \times W^{-k_2,q}(\Omega, \mathbb{R}^{m_2})$ the functional J is sequentially weakly lower-semicontinuous, i.e.

$$J(v) \leq \liminf_{n \to \infty} J(v_n);$$

(ii) the functional I is sequentially weakly lower-semicontinuous on $L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$.

The proof of [65, Theorem 3.6] is based on a suitable notion of equi-integrable sequences.

Definition 5.9. *A set $X \subset L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$ is called $(p,q)-$equi-integrable, if for all $\varepsilon > 0$ there exists a $\delta > 0$, such that*

$$E \text{ measureable }, \ |E| < \delta \quad \Longrightarrow \quad \sup_{v \in X} \int_E |v_1|^p + |v_2|^q \, \mathrm{d}x < \varepsilon,$$

that is $\{v_1\}_{v \in X}$ and $\{v_2\}_{v \in X}$ are p-equi-integrable and q-equi-integrable, respectively.

The key insight for Proposition 5.8 is that it suffices to consider (p,q)-equi-integrable sequences. This is the content of the following proposition which is again a straightforward adaption of the p-setting (cf. Lemma 4.11).

Proposition 5.10. *Let $1 < p, q < \infty$ and let $\mathcal{F} \colon \Omega \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition (5.19). Let $v_n \rightharpoonup v$ in $L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$ and suppose that we are given a (p,q)-equi-integrable sequence $w_n \in L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$ such that for some $\max(1/p, 1/q) < \theta < 1$*

$$\|v_n - w_n\|_{L^{\theta p} \times L^{\theta q}} \longrightarrow 0.$$

Then we have

$$\liminf_{n \to \infty} J(w_n) \leq \liminf_{n \to \infty} J(v_n).$$

The proof of Proposition 5.10 is contained in the proof of the following theorem.

Theorem 5.11. *Let $1 < p, q < \infty$ and let $X \subset L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_1})$ be weakly closed. Moreover, let $\mathcal{F}, \mathcal{F}_n \colon \Omega \times (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}) \to \mathbb{R}$ be Carathéodory functions. We define the functionals $I_n^X, I^X \colon X \to \mathbb{R}$ as*

$$I_n^X(v) = \begin{cases} \displaystyle\int_\Omega \mathcal{F}_n(x, v) \, \mathrm{d}x, & v \in X \\ \infty, & \text{else,} \end{cases} \quad \text{and} \quad I^X(v) = \begin{cases} \displaystyle\int_\Omega \mathcal{F}(x, v) \, \mathrm{d}x, & v \in X \\ \infty, & \text{else,} \end{cases}$$

Suppose that X satisfies the following condition:

(H1) For all bounded sequences $v_n \subset X$ there exists a (p,q)-equi-integrable sequence $w_n \subset X$, such that $w_n - v_n \to 0$ in measure.

Suppose further that $\mathcal{F}_n, \mathcal{F}$ satisfy:

(H2) there exists a constant $C > 0$, such that for all $(z_1, z_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and almost every $x \in \Omega$ we have

$$0 \leq \mathcal{F}_n(x, z_1, z_2), \mathcal{F}(x, z_1, z_2) \leq C(1 + |z_1|^p + |z_2|^q);$$

(H3) $\mathcal{F}$ and $\mathcal{F}_n$ are uniformly continuous on bounded sets of $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, i.e. there exists a function $\nu_R : [0, \infty) \to \mathbb{R}$, such that for all $z_1, z_2 \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ with $|z_1|, |z_2| \leq R$ and for almost every $x \in \Omega$:

$$|\mathcal{F}_n(x, z_1) - \mathcal{F}_n(x, z_2)| + |\mathcal{F}(x, z_1) - \mathcal{F}(x, z_2)| < \nu_R(|z_1 - z_2|);$$

(H4) the functionals with integrands $\mathcal{F}_n$ converge uniformly on equi-integrable subsets, i.e. for all equi-integrable sets $B \subset L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_1})$ and for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$, such that for all $v \in B$ it holds

$$\left| \int_\Omega \mathcal{F}_n(x, v(x)) - \mathcal{F}(x, v(x)) \, \mathrm{d}x \right| \leq \varepsilon, \quad n \geq n_\varepsilon.$$

Then the functionals I_n^X and I^X enjoy the following properties:

(i) for all sequences $v_n \rightharpoonup v$ in X, there is a sequence $w_n \rightharpoonup v$ in X such that

$$\liminf_{n\to\infty} I_n^X(w_n) \leq \liminf_{n\to\infty} I^X(v_n);$$

(ii) for all sequences $v_n \rightharpoonup v$ in X, there is a sequence $\bar{w}_n \rightharpoonup v$ in X such that

$$\liminf_{n\to\infty} I^X(\bar{w}_n) \leq \liminf_{n\to\infty} I_n^X(v_n);$$

(iii) if the sequential Γ-limit of the constant sequence I^X exists, then the sequential Γ-limit of I_n^X exists and

$$\Gamma - \lim_{n\to\infty} I_n^X = \Gamma - \lim_{n\to\infty} I^X.$$

Note that the constraint set $\mathcal{C}$ in the fluid mechanical application is weakly closed and may thus play the role of the set X in the abstract setting.

Proof. **(i)** The main idea of the proof is to show that a suitable version of Proposition 5.10 holds, namely that sequences $w_n \subset X$ as in (H1) already satisfy (i). To this end, let $v_n \subset X$ be bounded, and let $w_n \subset X$ be a (p, q)-equi-integrable sequence, such that $w_n - v_n \to 0$ in measure. Then we have

$$\begin{aligned}\limsup_{n\to\infty} I_n^X(w_n) - I^X(v_n) &= \int_\Omega \mathcal{F}_n(x, w_n) - \mathcal{F}(x, v_n) \, \mathrm{d}x \\ &\leq \limsup_{n\to\infty} \int_\Omega \mathcal{F}_n(x, w_n) - \mathcal{F}(x, w_n) \, \mathrm{d}x + \limsup_{n\to\infty} \int_\Omega \mathcal{F}(x, w_n) - \mathcal{F}(x, v_n) \, \mathrm{d}x.\end{aligned}$$

Due to (H4) and the (p, q)-equi-integrablility of w_n the first term tends to 0. In order to estimate the second term, let $L > 0$ be a constant such that $\|v_n\|_{L^p}, \|w_n\|_{L^p} \leq L$. Then, using (H2), for any $R > 0$ we obtain

$$\begin{aligned}&\int_\Omega \mathcal{F}(x, w_n) - \mathcal{F}(x, v_n) \, \mathrm{d}x \\ &= \int_{\{|w_n|, |v_n| \leq R\}} \mathcal{F}(x, w_n) - \mathcal{F}(x, v_n) \, \mathrm{d}x + \int_{\{|w_n| \geq R\} \cup \{|v_n| \geq R\}} \mathcal{F}(x, w_n) - \mathcal{F}(x, v_n) \, \mathrm{d}x \\ &\leq \int_{\{|w_n|, |v_n| \leq R\}} \mathcal{F}(x, w_n) - \mathcal{F}(x, v_n) \, \mathrm{d}x + \sup_{E \colon |E| < 2(L/R)^{\min(p,q)}} \int_E C(1 + |w_{n,1}|^p + |w_{n,2}|^q) \, \mathrm{d}x.\end{aligned}$$

The first integral on the right-hand side of this inequality converges to 0 as $n \to \infty$, by (H3) and the fact that $w_n - v_n \to 0$ in measure. Moreover, since the sequence w_n is

(p,q)-equi-integrable, the second integral can be bounded by a constant c_R with $c_R \to 0$ as $R \to \infty$. Consequently,

$$\limsup_{n\to\infty} \int \mathcal{F}(x,w_n) - \mathcal{F}(x,v_n)\,\mathrm{d}x \leq 0$$

and we conclude that

$$\limsup_{n\to\infty} I_n^X(w_n) \leq \liminf_{n\to\infty} I^X(v_n). \tag{5.20}$$

(ii) The second statement is obtained in the same way by swapping the roles of $\mathcal{F}_n$ and $\mathcal{F}$. Note that we can uniformly estimate

$$\int_{\{|w_n|,|v_n|\leq R\}} \mathcal{F}_n(x,w_n) - \mathcal{F}_n(x,v_n)\,\mathrm{d}x,$$

as all $\mathcal{F}_n$ have the same modulus of continuity on bounded sets, cf. (H3).

(iii) If the sequential Γ-limit of I^X exists (we denote it by I^{X*}), then for all $v \in X$ the following holds true.

(a) Every sequence $v_n \subset X$ with $v_n \rightharpoonup v$ in X satisfies $I^{X*}(v) \leq \liminf_{n\to\infty} I^X(v_n)$.

(b) There exists a sequence $v_n \subset X$ with $v_n \rightharpoonup v$ in X, such that $I^{X*}(v) \geq \limsup_{n\to\infty} I^X(v_n)$.

The lim inf-inequality for I_n^X is ensured by (ii), i.e. if $v_n \rightharpoonup v$ in X, then

$$\liminf_{n\to\infty} I_n^X(v_n) \geq \liminf_{n\to\infty} I^X(w_n) \geq I^{X*}(v),$$

as $w_n \rightharpoonup v$ in X. On the other hand, the lim sup-inequality follows from (i): we can modify a recovery sequence v_n (or at least a suitable subsequence) to an equi-integrable recovery sequence w_n. By (i), we find that

$$I^{X*}(v) \geq \liminf_{n\to\infty} I^X(v_n) \geq \limsup_{n\to\infty} I^X(w_n).$$

This completes the proof. □

The main challenge in applying Theorem 5.11 to the case in which X is a set given by differential constraints and boundary conditions is to verify Hypothesis (H1). In Section 5.4 we deal with the conditions (H2)–(H4) on the integrand $\mathcal{F}$. Thus, for a given sequence v_n we need to construct a suitable (p,q)-equi-integrable modification w_n that conserves both the differential constraints *and* the boundary conditions. We have already proven this result in Theorem 2.12. For reference, let us give this result without a proof again.

Corollary 5.12 (Preserving boundary conditions)**.** *Let $v \in L^p(\Omega,\mathbb{R}^N)$ and let $v_n \subset L^p(\Omega,\mathbb{R}^N)$, such that $v_n \rightharpoonup v$ in L^p and $\mathcal{A}v_n \to \mathcal{A}v$ in $W^{-k_{\mathcal{A}},p}(\Omega,\mathbb{R}^l)$. Let $\mathcal{B}$ be a potential of $\mathcal{A}$.*

(a) Suppose that v can be written as $v = \mathcal{B}u$. There exists a sequence $u_n \subset W^{k_{\mathcal{B}},p}(\Omega,\mathbb{R}^m)$, such that

(i) $u_n - u$ *is compactly supported in* Ω*;*

(ii) $\mathcal{B}u_n$ *is p-equi-integrable;*

(iii) $\|\mathcal{B}u_n - v_n\|_{L^r(\Omega)} \to 0$ *for some* $1 < r < p$*.*

(b) There is a sequence $\bar{v}_n$*, such that*

(i) $\mathcal{A}\bar{v}_n = \mathcal{A}v$*;*

(ii) $\bar{v}_n - v$ *is compactly supported in* Ω*;*

(iii) $\bar{v}_n$ *is p-equi-integrable;*

(iv) $\|\bar{v}_n - v_n\|_{L^r(\Omega)} \to 0$ *for some* $1 < r < p$*.*

Corollary 5.12 is used to modify sequences of functions in the constraint set $\mathcal{C}$ to obtain equi-integrable sequences while at the same time preserving differential constraints and boundary conditions. Note that in problems of fluid mechanics the boundary conditions are typically given for u, the potential of ϵ, therefore part (a) is suitable for this problem. On the other hand boundary conditions for σ are directly given in terms of the stress. Hence part (b) is suitable there.

5.3.2. Relaxation

If the function $\mathcal{F}$ is not $\mathcal{A}$-quasiconvex, the functional I fails to be weakly lower-semicontinuous. Hence, we cannot ensure existence of minimisers just by using the Direct Method.

However, when studying the Data–Driven problem, it is still sensible to consider approximate minimisers, i.e. sequences v_n with $I(v_n)$ converging to the infimum of I, and their weak limits v^*. In the following we will define a suitable *relaxation* I^* of I, such that any such weak limit v^* is a minimiser to I^* and, vice versa, any minimiser of I^* is a weak limit of approximate minimisers.

Relaxation under a linear differential constraint.

We recall the definition of I from (5.18). For simplicity, we write for the quasiconvex envelope of a function $\mathcal{F}\colon \Omega \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ as

$$\mathcal{Q}_{\mathcal{A}}\mathcal{F}(x, v) = \mathcal{Q}_{\mathcal{A}}(\mathcal{F}(x, \cdot))(v).$$

Note that by Proposition 5.8 the functional I^* given by

$$I^*(v) := \begin{cases} \displaystyle\int_\Omega \mathcal{Q}_{\mathcal{A}}\mathcal{F}(x, v(x))\,\mathrm{d}x, & \mathcal{A}v = 0 \\ \infty, & \text{else}, \end{cases}$$

is weakly lower-semicontinuous in $L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$. That this is indeed the relaxation of I follows from the following (linear) result [25] and also Theorem 4.16.

Proposition 5.13. *Let $\mathcal{F}$ satisfy the following hypotheses*

(A1) $\mathcal{F}\colon \Omega \times (\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}) \to \mathbb{R}$ *is a Carathéodory function;*

(A2) for all $x \in \Omega$ *and* $(v_1, v_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ *we have*

$$0 \leq \mathcal{F}(x, v_1, v_2) \leq C(1 + |v_1|^p + |v_2|^q).$$

Let $(v_1, v_2) \in L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$*. For any* $\varepsilon > 0$*, there exists a bounded sequence* $v^n = (v_1^{n,\varepsilon}, v_2^{n,\varepsilon})$ *in* $L^p(\Omega, \mathbb{R}^{m_1}) \times L^q(\Omega, \mathbb{R}^{m_2})$*, such that*

(i) $v_1^{n,\varepsilon} \rightharpoonup v_1$ *in* $L^p(\Omega, \mathbb{R}^{m_1})$ *and* $v_2^{n,\varepsilon} \rightharpoonup v_2$ *in* $L^q(\Omega, \mathbb{R}^{m_2})$ *as* $n \to \infty$*;*

(ii) $\mathcal{A}_1 v_1^{n,\varepsilon} = \mathcal{A}_1 v_1$ *and* $\mathcal{A}_2 v_2^{n,\varepsilon} = \mathcal{A}_2 v_2$*.*

(iii) v^n *is almost a recovery sequence, i.e.*

$$\int_\Omega \mathcal{Q}_{\mathcal{A}}\mathcal{F}(x, v)\,\mathrm{d}x \geq \lim_{n\to\infty} \int_\Omega \mathcal{F}(x, v^{n,\varepsilon})\,\mathrm{d}x + \varepsilon.$$

Remark 5.14. Recall that (cf. Remark 4.18) the $L^p \times L^q$ bound on the sequence depends on ε, so a priori we might not be able to take a diagonal sequence $v^{n,\varepsilon(n)}$, such that

$$\int_\Omega \mathcal{Q}_{\mathcal{A}}\mathcal{F}(x, v)\,\mathrm{d}x \geq \lim_{n\to\infty} \int_\Omega \mathcal{F}(x, v^{n,\varepsilon(n)})\,\mathrm{d}x.$$

Relaxation under a semi-linear differential constraint

As above, let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz boundary. Instead of considering a linear differential constraint, e.g.

$$\begin{cases} \mathcal{A}_1 v_1 = 0 \\ \mathcal{A}_2 v_2 = f, \end{cases}$$

we include a semilinear term. In the fluid mechanical setting this semilinear term is given by

$$\epsilon \longmapsto (u \cdot \nabla)u,$$

where u is uniquely determined by ϵ due to boundary conditions and the constraint $\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)$.

We fix a suitable setting. Let, as before $\mathcal{A}_1\colon L^p(\Omega, \mathbb{R}^{m_1}) \to W^{-k_1,p}(\Omega, \mathbb{R}^{l_1})$ be a constant rank operator with a potential $\mathcal{B}_1\colon W^{k_{\mathcal{B}_1},p}(\Omega, \mathbb{R}^{h_1}) \to L^p(\Omega, \mathbb{R}^{m_1})$ and $\mathcal{A}_2\colon L^q(\Omega, \mathbb{R}^{m_2}) \to W^{-k_2,p}(\Omega, \mathbb{R}^{l_2})$ be a constant rank operator. In addition, we require the semilinear term to satisfy the following:

(A3) $\theta\colon \Omega \times \mathbb{R}^{h_1} \times (\mathbb{R}^{h_1} \otimes \mathbb{R}^N) \ldots \times (\mathbb{R}^{h_1} \times \mathbb{R}^{h_1} \otimes (\mathbb{R}^N)^{\otimes k_{\mathcal{B}_1}} \to \mathbb{R}^{m_1}$ is a continuous map;

(A4) The map Θ defined on $W^{k_{\mathcal{B}_1},p}(\Omega,\mathbb{R}^{h_1})$ via

$$(\Theta u)(x) = \theta\big(x, u(x), \nabla u(x), \ldots, \nabla^{k_{\mathcal{B}_1}} u(x)\big)$$

is continuous from the *weak* topology of $W^{k_{\mathcal{B}_1},p}(\Omega,\mathbb{R}^{h_1})$ to the strong topology of $W^{-1,r}(\Omega,\mathbb{R}^{l_2})$ for some $r > q$.

We study the following set of constraints:

$$\begin{cases} \mathcal{A}_1 v_1 = 0 \\ v_1 = \mathcal{B}_1 u_1 \\ \mathcal{A}_2 v_2 = \mathcal{A}_2 \Theta(u_1). \end{cases} \tag{5.21}$$

Theorem 5.15. *Let $\mathcal{F}\colon \Omega \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfy the assumptions (A1)–(A2) from Proposition 5.13 and let $\Theta\colon W^{k_{\mathcal{B}_1},p}(\Omega,\mathbb{R}^{h_1}) \to L^r(\Omega,\mathbb{R}^{m_2})$ and $\mathcal{A}_1$, $\mathcal{A}_2$ satisfy the aforementioned hypotheses (A3)–(A4). Suppose that $u_1 \in W^{k_1,p}(\Omega,\mathbb{R}^{h_1})$ and $v = (v_1, v_2) \in L^p(\Omega,\mathbb{R}^{m_1}) \times L^q(\Omega,\mathbb{R}^{m_2})$, such that $u_1 = \mathcal{B}_1 v_1$ and $\mathcal{A}_2 v_2 = \Theta(v_1)$. Then, for all $\varepsilon > 0$, there exist bounded sequences $u^\varepsilon_{1,n} \in W^{k_1,p}(\Omega,\mathbb{R}^{h_1})$ and $v^\varepsilon_n \in L^p(\Omega,\mathbb{R}^{m_1}) \times L^q(\Omega\mathbb{R}^{m_2})$ such that*

(i) $\mathcal{B}_1 u^\varepsilon_{1,n} = v^\varepsilon_{1,n}$*;*

(ii) $u^\varepsilon_{1,n} - u_1$ *is supported in* $\Omega_n \subset\subset \Omega$*;*

(iii) $\mathcal{A}_2 v^\varepsilon_{2,n} = \mathcal{A}_2 \Theta(u^\varepsilon_{1,n})$*;*

(iv) $v^\varepsilon_{2,n} - v_2$ *is supported in* $\Omega_n \subset\subset \Omega$*;*

(v) v^ε_n *is almost a recovery sequence, i.e. it satisfies*

$$\int_\Omega \mathcal{Q}_\mathcal{A}\mathcal{F}(x, v)\,\mathrm{d}x \geq \lim_{n\to\infty} \int_\Omega \mathcal{F}(x, v^\varepsilon_n)\,\mathrm{d}x - \varepsilon.$$

Remark 5.16. (i) The statement of Theorem 5.15 is quite strong concerning boundary conditions. Indeed, the recovery sequence consisting of $u^\varepsilon_{1,n}$ and $v^\varepsilon_{2,n}$ satisfies both Dirichlet boundary conditions for $u^\varepsilon_{1,n}$ *and* a Neumann boundary conditions for $v^\varepsilon_{2,n}$. In the minimisation problem in Section 5.5 below we only require weaker boundary conditions.

(ii) Remark 5.14 is still valid in the setting of Theorem 5.15. More precisely, if we have a coercivity condition on the functional restricted to functions obeying 5.21 and some boundary conditions, then we may find a recovery sequence satisfying (i)–(iv) and

$$\int_\Omega \mathcal{Q}_\mathcal{A}\mathcal{F}(x, v)\,\mathrm{d}x \geq \lim_{n\to\infty} \int_\Omega \mathcal{F}(x, v_n)\,\mathrm{d}x.$$

(iii) In the specific setting of Theorem 5.15 one only needs coercivity in v_1 and in the viscous part $\tilde{\sigma}$. Given $v = (v_1, v_2) = (v_1, (\epsilon, \pi))$, the $L^q(\Omega)$-norm of the pressure π^ϵ_n

may be bounded by

$$\begin{aligned}\|\pi_n^\varepsilon\|_{L^q} &\lesssim \|\nabla\pi_n^\varepsilon\|_{W^{-1,q}} + \|(\tilde\sigma_n^\varepsilon - \pi_n^\varepsilon \operatorname{id}\nu)\|_{W^{-1/q,q}(\partial\Omega)} + \|\operatorname{div}\tilde\sigma_n^\varepsilon\|_{W^{-1,q}}\\ &\lesssim \|\tilde\sigma_n^\varepsilon\|_{L^q} + \|\Theta(u_{1,n}^\varepsilon)\|_{W^{-1,q}} + \|(\tilde\sigma_n^\varepsilon - \pi_n^\varepsilon \operatorname{id})\nu\|_{W^{-1/q,q}(\partial\Omega)}.\end{aligned}$$

In particular, if $\Theta(v_1)$ can be bounded in terms of $\|v_1\|_{L^p}$, then it suffices to consider a coercivity condition of the form

$$(v_1, v_2) \in \mathcal{C},\ (\|v_1\|_{L^p} + \|\sigma\|_{L^q} \to \infty) \quad \Longrightarrow \quad \int_\Omega \mathcal{F}(x, v)\,\mathrm{d}x \to \infty.$$

Proof of Theorem 5.15. By the linear relaxation result Proposition 5.13 there exists a sequence $(\bar v_{1,n}, \bar v_{2,n}) \subset L^p(\Omega,\mathbb{R}^{m_1})\times L^q(\Omega,\mathbb{R}^{m_2})$ weakly converging to $v = (v_1, v_2)$ satisfying

$$\begin{cases} \mathcal{A}_1\bar v_{1,n}^\varepsilon = 0\\ \mathcal{A}_2\bar v_{2,n}^\varepsilon = \mathcal{A}_2 v_2 = \mathcal{A}_2\Theta(u_1)\\ \displaystyle\int_\Omega \mathcal{Q}_{\mathcal{A}}\mathcal{F}(x, v)\,\mathrm{d}x \geq \lim_{n\to\infty}\int_\Omega \mathcal{F}(x, v_n)\,\mathrm{d}x - \varepsilon\end{cases}$$

By Proposition 5.10 and Corollary 5.12 we may take $\tilde u_{1,n}^\varepsilon \subset W^{k_1,p}(\Omega,\mathbb{R}^h)$, and $\tilde v_n \subset L^p(\Omega,\mathbb{R}^{m_1}) \times L^q(\Omega,\mathbb{R}^{m_2})$, such that

(i) $\tilde v_{1,n}^\varepsilon = \mathcal{B}_1\tilde u_{1,n}^\varepsilon$;

(ii) the first k_1-derivatives of $\tilde u_{1,n}^\varepsilon$ are p-equi-integrable;

(iii) $\tilde v_{2,n}^\varepsilon$ is q-equi-integrable;

(iv) $\mathcal{A}_2\tilde v_{2,n}^\varepsilon = \mathcal{A}_2\Theta(u_1)$;

(v) the functions $\tilde u_{1,n}^\varepsilon$ and $\tilde v_{2,n}^\varepsilon$ satisfy the boundary conditions

$$\begin{cases}\operatorname{spt}(\tilde u_{1,n}^\varepsilon - u_1) \subset \Omega_n\\ \operatorname{spt}(\tilde v_{2,n}^\varepsilon - v_2) \subset \Omega_n\end{cases}$$

for some $\Omega_n \subset\subset \Omega$;

(vi) $\displaystyle\int_\Omega \mathcal{Q}_{\mathcal{A}}\mathcal{F}(x, v)\,\mathrm{d}x \geq \lim_{n\to\infty}\int_\Omega \mathcal{F}(x, \tilde v_n^\varepsilon)\,\mathrm{d}x - \varepsilon.$

We set $v_1^n = \tilde v_{1,n}^\varepsilon$ and $u_{1,n}^\varepsilon = \tilde u_{1,n}^\varepsilon$ and modify $\tilde v_{2,n}^\varepsilon$ by

$$v_{2,n}^\varepsilon = \tilde v_{2,n}^\varepsilon + w_{2,n}^\varepsilon$$

such that $\mathcal{A}_2 v_{2,n}^\varepsilon = \Theta(u_{1,n}^\varepsilon)$. In particular, we solve the following equation:

$$\begin{cases}\mathcal{A}_2 w_{2,n}^\varepsilon = \mathcal{A}_2(\Theta(v_{1,n}^\varepsilon) - \Theta(v_1)), \quad x \in \Omega\\ \operatorname{spt}(\tilde w_{2,n}^\varepsilon - v_2) \subset\subset \Omega\end{cases} \tag{5.22}$$

But we know that $w^\varepsilon_{2,n} = \Theta(u^\varepsilon_{1,n}) - \Theta(u_1)$ already is a solution to this system. As $u^\varepsilon_{1,n} - u_1$ is supported inside $\Omega_n \subset\subset \Omega$, so is $u^\varepsilon_{1,n}$ due to the definition of the map Θ, cf. (A3) and (A4). Due to weak-strong continuity we have

$$\|w^\varepsilon_{2,n}\|_{L^r} = \|\Theta(u^\varepsilon_{1,n}) - \Theta(v_1)\|_{L^r} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Then $v^\varepsilon_{2,n} := \tilde{v}^\varepsilon_{2,n} + w^\varepsilon_{2,n}$ still is q-equi-integrable, as $\tilde{v}^\varepsilon_{2,n}$ is q-equi-integrable and $w^\varepsilon_{2,n}$ bounded in $L^r(\Omega, \mathbb{R}^{m_2})$ for some $r > q$; hence also p-equi-integrable. Moreover, as $v^\varepsilon_{1,n} \rightharpoonup v_1$ in $L^p(\Omega, \mathbb{R}^{m_2})$ and Θ is weak-strong continuous,

$$\|\tilde{v}^\varepsilon_{2,n} - v^\varepsilon_{2,n}\|_{L^r} = \|w^\varepsilon_{2,n}\|_{L^r} \longrightarrow 0 \quad \text{as } n \to \infty,$$

and we conclude by Proposition 5.10 that

$$\limsup_{n\to\infty} \int_\Omega \mathcal{F}(x, v^\varepsilon_{1,n}, v^\varepsilon_{2,n})\, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega \mathcal{F}(x, \tilde{v}^\varepsilon_{1,n}, \tilde{v}^\varepsilon_{2,n})\, \mathrm{d}x.$$

As $\tilde{v}^\varepsilon_{2,n} - v_2$ is compactly supported in Ω, $v^\varepsilon_{2,n} - v_2$ satisfies the demanded boundary conditions and $\mathcal{A}v^\varepsilon_{2,n} = \mathcal{A}_2\Theta(v^\varepsilon_{1,n})$. Hence, v^ϵ_n is almost a recovery sequence.

□

Remark 5.17. The statement of Theorem 5.15 is formulated towards its application for fluid dynamics, cf. Subsection 5.5.2. Observe that in the proof of Theorem 5.15, a main step was to solve the differential equation

$$\mathcal{A}_2 w = \mathcal{A}_2\big(\Theta(u^\varepsilon_{1,n}) - \Theta(u_1)\big) \tag{5.23}$$

together with suitable boundary conditions. This equation is solved by the observation, that $(\Theta(u^\varepsilon_{1,n}) - \Theta(u_1))$ already satisfies the boundary conditions.

If we generalise the setting to other non-linearities, we need more assumptions on the non-linearity. For example, consider a constraint like

$$\begin{cases} \mathcal{A}_1 v_1 = 0 \\ v_1 = \mathcal{B}_1 u_1 \\ \mathcal{A}_2 v_2 = \zeta(u_1). \end{cases}$$

for some map $\zeta\colon W^{k_{\mathcal{B}_1},p}(\Omega, \mathbb{R}^{h_1}) \to W^{-k_{\mathcal{A}_2},q}(\Omega, \mathbb{R}^{h_2})$. Then weak-strong continuity is not enough, as one also needs to solve the analogue of (5.22) with suitable boundary conditions. If for example, $\mathcal{A}_2 = \operatorname{div}$, then a further condition is as follows: Whenever u_1 and u_1' satisfy $\operatorname{spt}(u_1 - u_1') \subset\subset \Omega$, then $\displaystyle\int \zeta(u_1) - \zeta(u_1')\, \mathrm{d}x = 0$ (such that the divergence-equation is solvable, cf. [24]).

5.4. Convergence of data sets

In this section, we define two different notions of *data convergence*, i.e. we define a suitable topology on closed subsets of $Y \times Y$. We show that these notions are equivalent to convergence of the unconstrained functionals J. In particular, these notions of data convergence are independent of the underlying differential constraint. Moreover, recall that we assume that the data consist of pairs of strain ϵ and the viscous part $\tilde{\sigma}$ of the stress; the pressure π is not part of the data.

5.4.1. Data convergence on bounded sets

Definition 5.18. *Let $Y \times Y$ be equipped with the metric $d\colon Y \times Y \to \mathbb{R}$, the distance function* dist *and let $(\mathcal{D}_n), \mathcal{D}$ be closed, nonempty subsets of $Y \times Y$. We say that $\mathcal{D}_n$ converges to $\mathcal{D}$ strongly in the topology $\mathcal{T}_{\mathrm{bd}}$, $\mathcal{D}_n \xrightarrow{bd} \mathcal{D}$, if the following is satisfied:*

(i) ***Uniform approximation:*** *There exists a sequence $a_n \to 0$ such that for all $z = (\epsilon, \tilde{\sigma}) \in \mathcal{D}$ it holds*

$$\operatorname{dist}(z, \mathcal{D}_n) \le a_n(1 + |\epsilon|^p + |\tilde{\sigma}|^q).$$

(ii) ***Fine approximation:*** *There exists a sequence $b_n \to 0$ such that for all $z_n = (\epsilon_n, \tilde{\sigma}_n) \in \mathcal{D}_n$ it holds*

$$\operatorname{dist}(z_n, \mathcal{D}) \le b_n(1 + |\epsilon_n|^p + |\tilde{\sigma}_n|^q).$$

Let us consider the functionals defined on V by

$$J(v) = \int_\Omega \operatorname{dist}(v, \mathcal{D}) \,\mathrm{d}x \quad \text{and} \quad J_n(v) = \int_\Omega \operatorname{dist}(v, \mathcal{D}_n) \,\mathrm{d}x.$$

Theorem 5.19. *Let $\mathcal{D}_n, \mathcal{D}$ be closed, nonempty subsets of $Y \times Y$. The following statements are equivalent:*

(i) $\mathcal{D}_n \xrightarrow{bd} \mathcal{D}$;

(ii) For all $v \in V$ it holds that

$$\lim_{n\to\infty} J_n(v) = J(v)$$

and this convergence is uniform on bounded subsets of V.

Proof. **'(i) ⇒ (ii)'.** Suppose without loss of generality that $0 \in \mathcal{D}$. Otherwise we translate the underlying space which at most changes a_n, b_n by a bounded factor. Let $v \in V$, with $\int_\Omega \operatorname{dist}(v, 0) \,\mathrm{d}x \le R$. Then for $n \in \mathbb{N}$ we may estimate

$$\int_\Omega \operatorname{dist}(v, \mathcal{D}) \,\mathrm{d}x = \int_\Omega d(v, \mathcal{D})^p \,\mathrm{d}x \le \int_\Omega (d(v, w_n) + d(w_n, \mathcal{D}))^p \,\mathrm{d}x,$$

where $w_n(x) \in \mathcal{D}_n$ is a point in $\mathcal{D}_n$ such that $d(v(x), w_n(x)) = d(v(x), \mathcal{D}_n)$. Note that, as $0 \in \mathcal{D}$ and due to the uniform approximation property, we obtain a pointwise bound on

w_n, i.e. $d(w_n(x),0) \le 2d(v(x),0)$ for n large enough. Therefore, for some $\varepsilon > 0$ we get

$$\begin{aligned}\int_\Omega \operatorname{dist}(v,\mathcal{D})\,\mathrm{d}x &\le \int_\Omega \Big(d(v,\mathcal{D}_n) + b_n\big(1+\operatorname{dist}(w_n,0)\big)^{1/p}\big)\Big)^p\,\mathrm{d}x \\ &\le \int_\Omega \Big(d(v,\mathcal{D}_n) + 2b_n\big(1+\operatorname{dist}(v,0)\big)^{1/p}\Big)^p\,\mathrm{d}x \\ &\le (1+\varepsilon)\int_\Omega d(v,\mathcal{D}_n)^p + C(\varepsilon,p)b_n^p\big(1+\operatorname{dist}(v,0)\big)\,\mathrm{d}x \\ &\le \int_\Omega \operatorname{dist}(v,\mathcal{D}_n)\,\mathrm{d}x + \Big(\varepsilon\int_\Omega \operatorname{dist}(v,\mathcal{D}_n)\,\mathrm{d}x + C(\varepsilon,p)b_n^p(1+R)\Big).\end{aligned}$$

Note that $\int_\Omega d(v,\mathcal{D}_n)^p\,\mathrm{d}x$ is bounded from above (for n large enough) by $2\int_\Omega d(v,0)^p\,\mathrm{d}x \le 2R$ as $0 \in \mathcal{D}$ and 0 is approximated uniformly by elements of $\mathcal{D}_n$. Therefore, for any $\delta > 0$ we may choose ε and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$\varepsilon\int_\Omega \operatorname{dist}(v,\mathcal{D}_n)\,\mathrm{d}x < \frac{\delta}{2} \quad \text{and} \quad C(\varepsilon,p)b_n^p(1+R) < \frac{\delta}{2}.$$

Consequently, there exists $\delta(R,n) \to 0$, such that for all $v \in V$ with $\int_\Omega \operatorname{dist}(v,0)\,\mathrm{d}x \le R$ it holds that

$$J(u) \le J_n(v) + \delta(R,n). \tag{5.24}$$

For the lower bound on $J(v)$ we can do the same calculation using fine instead of uniform approximation and find that for any $v \in V$ with $\int_\Omega \operatorname{dist}(v,0)\,\mathrm{d}x \le R$ we have

$$\int_\Omega \operatorname{dist}(v,\mathcal{D}_n)\,\mathrm{d}x \le \int_\Omega \operatorname{dist}(v,\mathcal{D})\,\mathrm{d}x + \Big(\varepsilon\int_\Omega \operatorname{dist}(v,\mathcal{D})\,\mathrm{d}x + C(\varepsilon,p)a_n^p(1+R)\Big).$$

We argue as for the lower bound, to obtain $\tilde{\delta}(R,n) \to 0$, such that for all $v \in V$ with $\int \operatorname{dist}(v,0)\,\mathrm{d}x \le R$

$$J_n(v) \le J(v) + \tilde{\delta}(R,h). \tag{5.25}$$

Therefore, the convergence $J_n(v) \to J(v)$ is uniform on bounded subsets of V.

'(ii)⇒ (i)'. We prove the statement by contradiction. Suppose first, that $\mathcal{D}$ is *not* uniformly approximated, i.e. there exists $a > 0$ and a subsequence $z_{n_k} = (\epsilon_{n_k}, \tilde{\sigma}_{n_k}) \subset \mathcal{D}$, such that

$$\operatorname{dist}(z_{n_k},\mathcal{D}_{n_k}) > a\big(1+|\epsilon_{n_k}|^p + |\tilde{\sigma}_{n_k}|^q\big) = a\big(1+\operatorname{dist}(z_{n_k},0)\big).$$

We assume without loss of generality that $0 \in \mathcal{D}$. Let Σ_{n_k} be a subset of Ω with measure $|\Omega|(1+\operatorname{dist}(z_{n_k},0))^{-1}$. We define

$$v_{n_k}(x) := \begin{cases} 0, & x \notin \Sigma_{n_k} \\ z_{n_k}, & x \in \Sigma_{n_k}. \end{cases}$$

Then $\int_\Omega \operatorname{dist}(v_{n_k},0)$ is bounded uniformly from above by $|\Omega|$. Furthermore,

$$\int_\Omega \operatorname{dist}(v_{n_k},\mathcal{D}) = 0, \quad k\in\mathbb{N}.$$

On the other hand,

$$\int_\Omega \operatorname{dist}(v_{n_k},\mathcal{D}_{n_k}) \ge \int_{\Sigma_{n_k}} \operatorname{dist}(z_{n_k},\mathcal{D}_{n_k}) \ge |\Sigma_{n_k}|\cdot a\big(1+\operatorname{dist}(z_{n_k},0)\big) \ge |\Omega| a.$$

Therefore, $J_n(v)$ does not converge to $J(v)$ uniformly on bounded sets of V.

If $\mathcal{D}_n$ is *not* a fine approximation of $\mathcal{D}$, the argumentation is similar. Then there exists $b>0$ and a subsequence $z_{n_k}\in\mathcal{D}_{n_k}$, such that,

$$\operatorname{dist}(z_{n_k},\mathcal{D}) > b\big(1+\operatorname{dist}(z_{n_k},0)\big).$$

Again, assume that $0\in\mathcal{D}$. We may assume that there exists a sequence $z'_n\to 0$ with $z'_n\in\mathcal{D}_n$, otherwise for $v\equiv 0$, it holds that

$$\limsup_{h\to\infty}\int_\Omega \operatorname{dist}(v,\mathcal{D}_n)\,\mathrm{d}x > 0 = \int_\Omega \operatorname{dist}(v,\mathcal{D})\,\mathrm{d}x.$$

Let Σ_{n_k} be a subset of Ω with measure $|\Omega|(1+\operatorname{dist}(z_{n_k},0))^{-1}$ and define

$$v_{n_k}(x) := \begin{cases} 0, & x\notin\Sigma_{n_k} \\ z_{n_k}, & x\in\Sigma_{n_k}. \end{cases}$$

As argued before, $\int_\Omega \operatorname{dist}(v_{n_k},\mathcal{D})\,\mathrm{d}x$ is bounded uniformly by $|\Omega|$ and for $k\in\mathbb{N}$ we find that

$$\int_\Omega \operatorname{dist}(v_{n_k},\mathcal{D}_{n_k})\,\mathrm{d}x = \int_{\Omega\setminus\Sigma_{n_k}} \operatorname{dist}(0,\mathcal{D}_{n_k})\,\mathrm{d}x \longrightarrow 0 \quad \text{as } k\to\infty.$$

But, for the distance to $\mathcal{D}$ we have

$$\int_\Omega \operatorname{dist}(v_{n_k},\mathcal{D}) = \int_{\Sigma_{n_k}} \operatorname{dist}(z_{n_k},\mathcal{D}) \ge |\Sigma_{n_k}|\cdot b\big(1+\operatorname{dist}(z_{n_k},0)\big) = b|\Omega|.$$

Therefore, the convergence $J_n(v)\to J(v)$ cannot be uniform on bounded subsets of V. □

The definition of this type of convergence is motivated by Lemma 5.5. In particular, we have as a consequence that if $\mathcal{D}_n \xrightarrow{bd} \mathcal{D}$, then the sequential Γ-limit of J_n and the constant sequence J coincide, i.e

$$\Gamma - \lim_{n\to\infty} J_n = \Gamma - \lim_{n\to\infty} J.$$

5.4.2. Data convergence on equi-integrable sets

Definition 5.20. *We say that a sequence of closed sets $\mathcal{D}_n \subset Y \times Y$ converges to $\mathcal{D}$ in the $\mathcal{T}_{\text{eq}}$-topology, $\mathcal{D}_n \xrightarrow{eq} \mathcal{D}$, if there are sequences $a_n, b_n \to 0$ and $R_n, S_n \to \infty$ such that the following is satisfied:*

(i) ***Uniform approximation on bounded sets:*** *For all $z \in \mathcal{D}$ with $\operatorname{dist}(z, 0) < R_n$ we have*

$$\operatorname{dist}(z, \mathcal{D}_n) \leq a_n(1 + |\epsilon|^p + |\tilde{\sigma}|^q).$$

(ii) ***Fine approximation on bounded sets:*** *For all $z_n \in \mathcal{D}_n$ with $\operatorname{dist}(z_n, 0) < S_n$ we have*

$$\operatorname{dist}(z, \mathcal{D}_n) \leq b_n(1 + |\epsilon_n|^p + |\tilde{\sigma}_n|^q).$$

Remark 5.21. The following statements are equivalent to the *uniform approximation on bounded sets:*

- For all $R > 0$ there is a sequence $a_n^R \to 0$ such that for all $z \in \mathcal{D}$ with $\operatorname{dist}(z, 0) < R$ we have
$$\operatorname{dist}(z, D_n) \leq a_n^R(1 + |\epsilon|^p + |\tilde{\sigma}|^q).$$
- For all $a > 0$ and $R > 0$, there is an $n(a, R)$ such that for all $z \in \mathcal{D}$ with $\operatorname{dist}(z, 0) < R$ and $n > n(a, R)$ we have
$$\operatorname{dist}(z, D_n) \leq a(1 + |\epsilon|^p + |\tilde{\sigma}|^q).$$

Similar equivalent statements hold for the *fine approximation on bounded sets.*

Theorem 5.22. *Let $\mathcal{D}_n, \mathcal{D}$ be closed, nonempty subsets of $Y \times Y$. The following statements are equivalent:*

(i) $\mathcal{D}_n \xrightarrow{eq} \mathcal{D}$ in the $\mathcal{T}_{\text{eq}}$-topology;

(ii) the functionals J_n converge uniformly to J on (p, q)-equi-integrable subsets of V. That is, if $X \subset V$ is (p, q)-equi-integrable, then

$$\lim_{n \to \infty} \sup_{v \in X} |J_n(v) - J(v)| = 0.$$

Proof. **'(i) ⇒ (ii)':** The proof is similar to the proof of Theorem 5.19. We only prove that fine and uniform approximation imply that, for a (p, q)-equi-integrable subset $X \subset V$, we have

$$\liminf_{n \to \infty} \inf_{v \in X} J_n(u) - J(u) \geq 0. \tag{5.26}$$

The converse inequality follows similarly. For simplicity assume that $0 \in \mathcal{D}$ and that $p \geq q$.

For some fixed $R > 0$ we estimate

$$\begin{aligned} I_n(v) - I(v) &= \int_\Omega \mathrm{dist}(v, \mathcal{D}_n) - \mathrm{dist}(v, \mathcal{D}) \, \mathrm{d}x \\ &= \int_{\{\mathrm{dist}(v,0) \le R\}} \mathrm{dist}(v, \mathcal{D}_n) - \mathrm{dist}(v, \mathcal{D}) \, \mathrm{d}x + \int_{\{\mathrm{dist}(v,0) > R\}} \mathrm{dist}(v, \mathcal{D}_n) - \mathrm{dist}(v, \mathcal{D}) \, \mathrm{d}x \\ &\ge \int_{\{\mathrm{dist}(v,0) \le R\}} \mathrm{dist}(v, \mathcal{D}_n) - \mathrm{dist}(v, \mathcal{D}) \, \mathrm{d}x - C \int_{\{\mathrm{dist}(v,0) > R\}} (1 + |\epsilon|^p + |\tilde{\sigma}|^q) \, \mathrm{d}x. \end{aligned} \tag{5.27}$$

We now estimate both integrals on the right-hand side from below and start with the second term. The set $X \subset V$ is (p, q)-equi-integrable. Hence, there is an increasing function $\omega \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_E (1 + |\varepsilon|^p + |\tilde{\sigma}|^q) \, \mathrm{d}x \le \omega(|E|).$$

The set X is bounded. Thus, defining

$$M := \sup_{v \in X} \int_\Omega 1 + |\varepsilon|^p + |\tilde{\sigma}|^q \, \mathrm{d}x,$$

we find that the measure of $\{\mathrm{dist}(v, 0) > R\}$ is bounded by MR^{-1}. Consequently, we obtain

$$-C \int_{\{\mathrm{dist}(v,0) > R\}} 1 + |\epsilon|^p + |\tilde{\sigma}|^q \, \mathrm{d}x \ge -C\omega(MR^{-1}). \tag{5.28}$$

We turn to the first term in (5.27). If $\mathrm{dist}(v(x), 0) \le R$, we may find some $w(x) \in \mathcal{D}$ with $\mathrm{dist}(w(x), 0) \le (2^p + 2^q)R$, and

$$\mathrm{dist}(v(x), \mathcal{D}) = \mathrm{dist}(v(x), w(x)).$$

Due to uniform approximation for all $w(x)$, we can estimate for n large enough

$$\begin{aligned} \int_{\{\mathrm{dist}(v,0) \le R\}} \mathrm{dist}(v, \mathcal{D}_n) - \mathrm{dist}(v, \mathcal{D}) \, \mathrm{d}x &= \int_{\{\mathrm{dist}(v,0) \le R\}} d(v, \mathcal{D}_n)^p - d(v, \mathcal{D})^p \, \mathrm{d}x \\ &= \int_{\{\mathrm{dist}(v,0) \le R\}} d(v, \mathcal{D}_n)^p - (d(v, w)^p \, \mathrm{d}x \\ &\ge \int_{\{\mathrm{dist}(v,0) \le R\}} d(v, \mathcal{D}_n)^p - \big(d(v, \mathcal{D}_n) + d(w, \mathcal{D}_n)\big)^p \, \mathrm{d}x \\ &\ge \int_{\{\mathrm{dist}(v,0) \le R\}} -\varepsilon d(v, \mathcal{D}_n)^p - C_\varepsilon d(w, \mathcal{D}_n) \, \mathrm{d}x \\ &\ge -\varepsilon M - C_\varepsilon a_n M. \end{aligned}$$

Together with (5.28) this implies

$$J_n(v) - J(v) \ge -C\omega(M/R) - \varepsilon M - C_\varepsilon a_n M.$$

Choosing $R(\varepsilon)$ and n large enough, then for any ε there is n_ε, such that

$$J_n(v) - J(v) \geq -2M\varepsilon, \quad v \in X,\ n \geq n_\varepsilon,$$

which establishes (5.26).

'(ii) ⇒ (i)': This implication is a consequence of the same counterexamples as in Theorem 5.19. Indeed, suppose that the sets $\mathcal{D}_n$ do not uniformly approximate $\mathcal{D}$ on bounded sets. Then there exist $R > 0$, $a > 0$ and a sequence $z_{n_k} \subset \mathcal{D}$, such that $\operatorname{dist}(z_n, 0) \leq R$ and

$$\operatorname{dist}(z_{n_k}, \mathcal{D}_{n_k}) \geq a(1 + |\epsilon_{n_k}|^p + |\tilde{\sigma}_{n_k}|^q).$$

By the same construction as in the proof of Theorem 5.19, that is

$$v_{n_k} := \begin{cases} 0, & x \notin \Sigma_{n_k} \\ z_{n_k}, & x \in \Sigma_{n_k}, \end{cases}$$

we obtain a sequence, such that $J(v_{n_k}) = 0$ and $J_n(v_{n_k}) \geq a|\Omega|$ with v_{n_k} uniformly bounded in $L^\infty(\Omega, Y \times Y)$ and hence v_{n_k} is also (p,q)-equi-integrable. For fine approximation the argument is again very similar. □

5.5. The data-driven problem in fluid mechanics

In this section we apply the theory developed in the previous sections to the setting of fluid mechanics. We thus specialise to an explicit set of constraints $\mathcal{C}$ consisting of differential constraints and boundary conditions. In Subsection 5.5.1 we consider the case of inertialess fluids, leading to a set of linear differential constraints. In Subsection 5.5.2 we consider nonlinear differential constraints. In both cases we work with the following boundary conditions defined on three mutually disjoint and relatively open parts of the boundary $\Gamma_D, \Gamma_R, \Gamma_N \subset \partial\Omega$ that satisfy

$$\overline{\Gamma_D \cup \Gamma_R \cup \Gamma_N} = \partial\Omega \quad \text{and} \quad \mathcal{H}^{N-1}(\bar{\Gamma}_D \setminus \Gamma_D) = \mathcal{H}^{N-1}(\bar{\Gamma}_R \setminus \Gamma_R) = \mathcal{H}^{N-1}(\bar{\Gamma}_N \setminus \Gamma_N) = 0$$

and have C^1-boundary as subsets of the manifold $\partial\Omega$. We consider $(\epsilon, \tilde{\sigma}) \in L^p(\Omega, Y) \times L^q(\Omega, Y)$ with an associated velocity field $u : \Omega \to \mathbb{R}^N$, where $\epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ and a pressure field $\pi : \Omega \to \mathbb{R}$, such that u and σ satisfy the following boundary conditions.

(D) No-slip/Dirichlet boundary conditions:

$$u = g \quad \text{on } \Gamma_D \quad \text{for } g \in W^{1-1/p,p}(\Gamma_D, \mathbb{R}^N).$$

(R) Navier-slip/Robin boundary conditions:

$$\begin{cases} u \cdot \nu = g_\nu \\ P_{T\partial\Omega}\left((\tilde{\sigma} + \pi \operatorname{id})\nu + \lambda u\right) = h_\tau \end{cases} \quad \text{on } \Gamma_R$$

for $g_\nu \in W_p^{1-1/p}(\Gamma_R)$ and $h_\tau \in W^{-1/q,q}(\Gamma_R, \mathbb{R}^N)$. Here, $\lambda \geq 0$ is the inverse slip-length and $P_{T\partial\Omega}$ is the orthogonal projection to the tangent space. Note that the second equation can equivalently be cast as

$$P_{T\partial\Omega}\left(\tilde{\sigma}\nu + \lambda u\right) = h_\tau \quad \text{on } \Gamma_R. \tag{5.29}$$

(N) Neumann boundary conditions:

$$(\tilde{\sigma} + \pi\,\mathrm{id})\nu = h \quad \text{on } \Gamma_N \quad \text{for } h \in W^{-1/q,q}(\Gamma_N, \mathbb{R}^N).$$

Remark 5.23. (i) The boundary conditions for u can be understood as conditions for ϵ in a suitable weak formulation. For instance, if $\Gamma_D = \partial\Omega$, then (D) is equivalent to the following condition on ϵ. For any $\varphi \in W^{1,q}(\Omega, Y)$ with $\operatorname{div}\varphi = 0$ we have

$$\int_\Omega \epsilon \cdot \varphi \,\mathrm{d}x = \int_{\partial\Omega} g(\varphi \cdot \nu)\,\mathrm{d}\mathcal{H}^{N-1}.$$

However, since an ϵ that is contained in the constraint set $\mathcal{C}$ automatically admits a corresponding u, we write the conditions directly for u. A similar remark applies to the appearance of π.

(ii) The Navier-slip boundary condition (R) requires $P_{T\partial\Omega}u \in W^{-1/q,q}(\Gamma_R, \mathbb{R}^N)$ since the other two terms in (5.29) are contained in this space. Since $\epsilon \in L^p(\Omega, Y)$, and by Lemma 5.6, we have $u \in W^{1-1/p,p}(\Gamma_R, \mathbb{R}^N)$. The space $W^{1-1/p,p}(\Gamma_R)$ embeds into $W^{-1/q,q}(\Gamma_R)$, whenever either $p \geq q$ or

$$1 - \tfrac{1}{p} - \tfrac{N-1}{p} \geq -\tfrac{1}{q} - \tfrac{N-1}{q}.$$

Thus, since $q = \frac{p}{p-1}$, we require

$$p \geq \tfrac{2N}{N+1}. \tag{5.30}$$

We can therefore treat the Navier-slip boundary condition in the physically relevant dimensions $N = 2$ and $N = 3$ for $p \geq 4/3$ and for $p \geq 3/2$, respectively.

(iii) The Navier boundary condition (R) includes the so called free-slip boundary condition for $\lambda = 0$.

(iv) For simplicity we assume in the following that either $\Gamma_N = \partial\Omega$ or $\Gamma_D \neq \emptyset$. This allows us to control $\|u\|_{W^{1,p}}$ in terms of $\|\epsilon\|_{L^p}$ and the boundary data via the Korn–Poincaré inequality, cf. Lemma 5.6. If $\Gamma_R \neq \emptyset$, while $\Gamma_D = \emptyset$, it becomes tedious to specify under which conditions this control can still be obtained. See Lemma 5.24 and Remark 5.25 below.

In order to obtain a Korn–Poincaré type inequality, u has to be uniquely determined by the above boundary conditions

$$\begin{cases} u = g, & x \in \Gamma_D \\ u \cdot \nu = g_\nu, & x \in \Gamma_R \end{cases} \tag{5.31}$$

and the constraint

$$\epsilon = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right),$$

or the conditions must be invariant under renormalisation by rigid body motions.

Lemma 5.24 (Validity of the Korn-Poincaré ineqaulity under boundary conditions)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded with C^1-boundary and let $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_R \cup \bar{\Gamma}_N$ be as above. Moreover, suppose that $g \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^N)$, $g_\nu \in W^{1-1/p,p}(\partial\Omega)$ and that for all $A \in \mathbb{R}^{N\times N}_{\text{skew}}$, $b \in \mathbb{R}^N$ we have*

$$\begin{cases} Ax + b = 0, & x \in \Gamma_D \\ (Ax+b)\cdot \nu(x) = 0, & x \in \Gamma_R \end{cases} \implies \quad A = 0,\ b = 0. \tag{5.32}$$

Then the following statements hold true:

(i) If u_1 and u_2 satisfy (5.31) *and*

$$\nabla u_1 + \nabla u_1^T = \nabla u_2 + \nabla u_2^T,$$

then $u_1 = u_2$.

(ii) For all $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ obeying (5.31)*, the Korn–Poincaré inequality*

$$\|u\|_{W^{1,p}} \le C(1 + \|\nabla u + \nabla u^T\|_{L^p}) \tag{5.33}$$

holds for a constant $C = C(\Omega, \Gamma_D, \Gamma_R, g, g_\nu, p)$.

Proof. **(i):** The assertion follows from the fact that if $\nabla u_1 + \nabla u_1^T = \nabla u_2 + \nabla u_2^T$, then $u_1 - u_2 = Ax + b$ for some $A \in \mathbb{R}^{N\times N}_{\text{skew}}$ and $b \in \mathbb{R}^N$. Condition (5.32) then implies that $A = 0$ and $b = 0$.
(ii): The vector space $X \subset W^{1,p}(\Omega, \mathbb{R}^N)$ of functions satisfying the homogeneous boundary conditions in (5.31) satisfies, due to (5.32),

$$X \cap \{Ax + b\colon A \in \mathbb{R}^{N\times N}_{\text{skew}}, b \in \mathbb{R}^N\} = \{0\}.$$

By transposition we get the inhomogeneous version (5.33) for the affine space of functions satisfying (5.31). □

Remark 5.25. Indeed, (5.32) is a rather weak condition on the set Ω. For example, in dimension $N = 2$, the weakest boundary condition in the case $\Gamma_D = \emptyset$ would be

$$(Ax + b) \cdot \nu(x) = 0 \quad \text{on } \Gamma_R.$$

Since $\mathbb{R}^{2\times 2}_{\text{skew}}$ is one-dimensional, we can explicitly set

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It follows that the only sets not satisfying (5.32) are such that Γ_R is a subset of concentric circles. Moreover, if $\Gamma_D \neq \emptyset$, then (5.32) is automatically satisfied.

In dimension $N = 3$, the situation is similar. Indeed, if $\Gamma_D \neq \emptyset$, then (5.32) is satisfied. If $\Gamma_D = \emptyset$, then, if Γ_R is a subset of the boundary of a domain that is rotationally symmetric around a certain axis, (5.32) is not satisfied.

Remark 5.26. Uniqueness of u is only important for fluids with inertia. For inertialess fluids, u only appears in the constraints through boundary conditions. Therefore, even if $\epsilon = \frac{1}{2}(\nabla u_1 + \nabla u_1^T) = \frac{1}{2}(\nabla u_2 + \nabla u_2^T)$ for $u_1 \neq u_2$ enjoying the same boundary conditions, it *does not* matter for the system of equations whether we take u_1 or u_2. In contrast, for fluids with inertia, the contribution $(u \cdot \nabla u)$ in the differential constraints causes the choice of u to be important. Therefore, in the linear setting, even if the prescribed boundary conditions (D), (R) and (N) allow to choose different $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, for example if $\Gamma_N = \partial\Omega$, we may project onto a subspace that does not allow multiple solutions to

$$\epsilon = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right).$$

Consequently, we can apply Lemma 5.6 in this situation.

5.5.1. Inertialess fluids

In this section we study inertialess fluids leading to the set of *linear* differential constraints from (5.8). That is, we consider

$$\begin{cases} \epsilon = \dfrac{1}{2}\left(\nabla u + \nabla u^T\right) \\ \operatorname{div} u = 0 \\ -\operatorname{div} \tilde{\sigma} = f - \nabla \pi, \end{cases} \tag{linD}$$

where $f \in W^{-1,q}(\Omega, \mathbb{R}^N)$ is given. Combining this with the boundary conditions, the constraint set is given by

$$\mathcal{C}_{\text{lin}} := \{(\epsilon, \tilde{\sigma}) \in V : (\text{linD}), (D), (R), \text{ and } (N) \text{ are satisfied}\}. \tag{linC}$$

Note that the statement '$(\epsilon, \tilde{\sigma})$ satisfies (linD)' means that there are $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and $\pi \in L^q(\Omega)$ such that (linD) is satisfied. For data sets $\mathcal{D}_n, \mathcal{D} \subset Y \times Y$ we consider the functionals I_n and I as in (5.7).

Coercivity

In this subsection we verify coercivity of the functionals I_n and I.

Definition 5.27. *We call a function* $\mathcal{F}\colon Y\times Y\to\mathbb{R}$ $(\mathbf{p},\mathbf{q})$*-**coercive**, if there exist* $C_1,C_2>0$ *and* $\gamma\in\mathbb{R}$ *such that*

$$\mathcal{F}(\epsilon,\tilde{\sigma})\geq C_1(|\epsilon|^p+|\tilde{\sigma}|^q)-C_2-\gamma\epsilon\cdot\tilde{\sigma}. \tag{5.34}$$

We say that $\mathcal{F}$ *has* $(\mathbf{p},\mathbf{q})$*-**growth**, if there is* $C_0>0$ *such that*

$$\mathcal{F}(\epsilon,\tilde{\sigma})\leq C_0(1+|\epsilon|^p+|\tilde{\sigma}|^q).$$

For $v\in V$ we define the functional

$$I(v):=\begin{cases}\displaystyle\int_\Omega\mathcal{F}(v)\,\mathrm{d}x, & v\in\mathcal{C}\\ \infty, & \text{else},\end{cases} \tag{5.35}$$

in analogy to (5.7).

Remark 5.28. In Section 5.4 we examine data convergence without the differential constraints, in particular we studied the unconstrained functional J. In general, we do not expect a coercivity statement of the type

$$\|v\|_V\to\infty \quad\Longrightarrow\quad J(v)\to\infty.$$

In the following we prove that coercivity follows in the presence of the differential constraints together with suitable boundary conditions, i.e. it holds that

$$\|v\|_V\to\infty,\ v\in\mathcal{C}_{\text{lin}} \quad\Longrightarrow\quad I(v)=J(v)\to\infty.$$

We can include the term $\epsilon\cdot\tilde{\sigma}$ on the right-hand side of (5.34) because it is a Null-Lagrangian. This becomes clear in Remark 5.29 and in the proof of Lemma 5.30 below. In some sense we only require coercivity away from the collinearity set $\{(\epsilon,\tilde{\sigma}):\epsilon=\beta\tilde{\sigma},\beta\in\mathbb{R}\}$. Because we expect ϵ and $\tilde{\sigma}$ to be colinear for classical fluids, this kind of transversal coercivity is a natural condition for the distance to the data sets which takes the role of $\mathcal{F}$ later on.

Remark 5.29. For the purpose of exposition, we prove a coercivity result for functions on the torus (i.e. we show $\mathcal{A}$-integral coercivity, cf. Chapter 4). Here, *averages* of the functions $(\epsilon,\tilde{\sigma})$ take over the role of *boundary values* and the role of the differential constraints can be isolated more clearly.

Let $\mathcal{F}$ be (p,q)-coercive. We claim that there are constants $C_1,C_2>0$, such that for any $(\epsilon_0,\tilde{\sigma}_0)\in Y\times Y$ and all $(\epsilon,\tilde{\sigma})\in L^p(T_N,Y)\times L^q(T_N,Y)$ satisfying

$$\begin{cases}\displaystyle\int_{T_N}(\epsilon,\tilde{\sigma})\,\mathrm{d}x=0\\ \epsilon=\frac{1}{2}\left(\nabla u+\nabla u^T\right)\\ \operatorname{div}\tilde{\sigma}=\nabla\pi,\end{cases} \tag{5.36}$$

for some $\pi \in L^q(T_N)$, we have the following coercivity:

$$\int \mathcal{F}(\epsilon_0 + \epsilon, \tilde{\sigma}_0 + \tilde{\sigma})\,\mathrm{d}x \geq c_1 \int_{T_N} |\epsilon|^p + |\tilde{\sigma}|^q \,\mathrm{d}x - c_2(1 + |\epsilon_0|^p + |\tilde{\sigma}_0|^q). \tag{5.37}$$

We compute

$$\begin{aligned}
&\int_{T_N} (\epsilon_0 + \epsilon) \cdot (\tilde{\sigma}_0 + \tilde{\sigma})\,\mathrm{d}x \\
&= \int_{T_N} \epsilon \cdot ((\tilde{\sigma}_0 + \tilde{\sigma}) + (\pi_0 + \pi)\,\mathrm{id})\ \mathrm{d}x + \varepsilon_0 \cdot \int_{T_N} ((\tilde{\sigma}_0 + \tilde{\sigma}) + (\pi_0 + \pi)\,\mathrm{id})\ \mathrm{d}x \\
&= \int_{T_N} \frac{1}{2}\left(\nabla u + \nabla u^T\right)((\tilde{\sigma}_0 + \tilde{\sigma}) + (\pi_0 + \pi)\,\mathrm{id})\ \mathrm{d}x + \varepsilon_0 \cdot \int_{T_N} (\tilde{\sigma}_0 + \pi_0\,\mathrm{id})\ \mathrm{d}x \\
&= \int_{T_N} \nabla u\,((\tilde{\sigma}_0 + \tilde{\sigma}) + (\pi_0 + \pi)\,\mathrm{id})\ \mathrm{d}x + \varepsilon_0 \cdot \tilde{\sigma}_0 \\
&= -\int_{T_N} u \cdot \operatorname{div}(\tilde{\sigma} + \pi\,\mathrm{id})\,\mathrm{d}x + \varepsilon_0 \cdot \tilde{\sigma}_0 = \varepsilon_0 \cdot \tilde{\sigma}_0.
\end{aligned}$$

Therefore,

$$\left|\int_{T_N} (\epsilon_0 + \epsilon) \cdot (\tilde{\sigma}_0 + \tilde{\sigma})\,\mathrm{d}x\right| \leq |\epsilon_0|^p + |\tilde{\sigma}_0|^q.$$

We conclude that

$$\begin{aligned}
\int \mathcal{F}(\epsilon_0 + \epsilon, \tilde{\sigma}_0 + \tilde{\sigma}) &\geq C_1 \int_{T_N} |\varepsilon_0 + \varepsilon|^p + |\tilde{\sigma}_0 + \tilde{\sigma}|^q \,\mathrm{d}x - C_2 - \gamma \int_{T_N} \epsilon \cdot \tilde{\sigma}\,\mathrm{d}x \\
&\geq C_1 \int_{T_N} |\epsilon|^p + |\tilde{\sigma}|^q \,\mathrm{d}x - C_2'(1 + |\epsilon_0|^p + |\tilde{\sigma}_0|^q).
\end{aligned}$$

Using the boundary conditions instead of averages, we obtain coercivity of the functional also on bounded domains, as long as the integrand is (p,q)-coercive.

Lemma 5.30 (Coercivity in Ω with boundary values)**.** *Suppose that f, g, g_ν, h_τ, and h are given as in* (linD)*, (D), (R), and (N). We assume that either $\Gamma_N = \partial\Omega$ or $\Gamma_D \neq \emptyset$. If $\Gamma_R \neq \emptyset$, then we additionally assume $p \geq 2d/(d+1)$. Suppose that $\mathcal{F}\colon Y \times Y \to \mathbb{R}$ is (p,q)-coercive and has (p,q)-growth. Then there are $C_3, C_4 > 0$,such that for I from* (5.35) *and for all $v \in V$*

$$I(v) \geq C_3 \int_\Omega (|\epsilon|^p + |\tilde{\sigma}|^q)\,\mathrm{d}x - C_4.$$

Proof. We may assume that $v \in \mathcal{C}_{\mathrm{lin}}$, otherwise there is nothing to show. By the coercivity of $\mathcal{F}$ we have

$$I(v) = \int_\Omega \mathcal{F}(\epsilon, \tilde{\sigma})\,\mathrm{d}x \geq \int_\Omega C_1(|\epsilon|^p + |\tilde{\sigma}|^q) - C_2 - \gamma\varepsilon \cdot \tilde{\sigma}\,\mathrm{d}x. \tag{5.38}$$

Since $v \in \mathcal{C}_{\mathrm{lin}}$,

$$\epsilon = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right),$$

for some u with

$$\|u\|_{W^{1,p}} \leq C\left(\|\epsilon\|_{L^p} + \|g\|_{W^{1-1/p,p}(\Gamma_D)}\right),$$

due to the Korn-Poincaré inequality from Lemma 5.24. Furthermore, we have the following estimate

$$\|(\tilde{\sigma} - \pi \, \mathrm{id})\nu\|_{W^{-1/q,q}(\partial\Omega)} \leq C \left(\|\tilde{\sigma}\|_{L^q} + \|f\|_{W^{-1,q}}\right), \tag{5.39}$$

which is due to $-\operatorname{div}\tilde{\sigma} + \nabla\pi = f$. Let us now estimate the last term in (5.38). The following computations will be done under the assumption that all functions are smooth. The statement follows by density. Observe that

$$\begin{aligned}\int_\Omega \epsilon \cdot \tilde{\sigma} \, \mathrm{d}x &= \int_\Omega \tfrac{1}{2}\left(\nabla u + \nabla u^T\right) \cdot (\tilde{\sigma} - \pi \, \mathrm{id}) \, \mathrm{d}x = \int_\Omega \nabla u \cdot (\tilde{\sigma} - \pi \, \mathrm{id}) \, \mathrm{d}x \\ &= -\int_\Omega u \cdot (\operatorname{div}\tilde{\sigma} - \nabla\pi) \, \mathrm{d}x + \int_{\partial\Omega} u \cdot (\tilde{\sigma} - \pi \, \mathrm{id})\nu \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \int_\Omega u \cdot f \, \mathrm{d}x + \int_{\partial\Omega} u \cdot (\tilde{\sigma} - \pi \, \mathrm{id})\nu \, \mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{5.40}$$

On the one hand, we have the following estimate for the bulk term

$$\left|\int_\Omega u \cdot f \, \mathrm{d}x\right| \leq \|u\|_{L^p}\|f\|_{L^q} \leq C\left(\|\epsilon\|_{L^p} + \|g\|_{W^{1-1/p,p}(\Gamma_D)}\right)\|f\|_{L^q}. \tag{5.41}$$

On the other hand, the boundary contribution can be estimated on the Dirichlet part by

$$\begin{aligned}\left|\int_{\Gamma_D} u \cdot (\tilde{\sigma} - \pi \, \mathrm{id})\nu \, \mathrm{d}\mathcal{H}^{N-1}\right| &= \left|\int_{\Gamma_D} g \cdot (\tilde{\sigma} - \pi \, \mathrm{id})\nu \, \mathrm{d}\mathcal{H}^{N-1}\right| \\ &\leq \|g\|_{W^{1-1/p,p}(\Gamma_D)}\left(\|(\tilde{\sigma} - \pi \, \mathrm{id})\nu\|_{W^{-1/q,q}(\Gamma_D)}\right) \\ &\leq \|g\|_{W^{1-1/p,p}(\Gamma_D)}\left(\|\tilde{\sigma} - \pi \, \mathrm{id}\,\nu\|_{W^{-1/q,q}(\Gamma_D)}\right) \\ &\leq C\left(\|\epsilon\|_{L^p} + \|\tilde{\sigma}\|_{L^q} + \|f\|_{W^{-1,q}}\right), \end{aligned} \tag{5.42}$$

on the Navier part by first isolating the term with sign

$$\int_{\Gamma_R} u \cdot (\tilde{\sigma} - \pi \, \mathrm{id})\nu \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\Gamma_R} g_\nu \nu \cdot (\tilde{\sigma} - \pi \, \mathrm{id})\nu - \lambda|P_{T_x\partial\Omega}u|^2 + P_{T_x\partial\Omega}u \cdot h_\tau \, \mathrm{d}\mathcal{H}^{N-1}, \tag{5.43}$$

and then estimating

$$\left|\int_{\Gamma_R} g_\nu \nu \cdot (\tilde{\sigma} - \pi \, \mathrm{id})\nu + P_{T_x\partial\Omega}u \cdot h_\tau \, \mathrm{d}\mathcal{H}^{N-1}\right| \tag{5.44}$$

$$\begin{aligned}&\leq \|g_\nu\|_{W^{1-1/p,p}(\Gamma_R)}\|(\tilde{\sigma} - \pi \, \mathrm{id})\nu\|_{W^{-1/q,q}(\Gamma_R)} + \|u\|_{W^{1-1/p,p}(\Gamma_R)}\|h_\tau\|_{W^{-1/q,q}(\Gamma_R)} \\ &\leq C_{g_\nu,h_\tau}\left(\|\epsilon\|_{L^p} + \|g\|_{W^{1-1/p,p}(\Gamma_D)} + \|\tilde{\sigma}\|_{L^q} + \|f\|_{W^{-1,q}}\right), \end{aligned} \tag{5.45}$$

and on the Neumann part by

$$\left|\int_{\Gamma_N} u\cdot(\tilde\sigma-\pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1}\right| = \left|\int_{\Gamma_N} u\cdot h\,\mathrm{d}\mathcal{H}^{N-1}\right| \le \|u\|_{W^{1-1/p,p}(\Gamma_N)}\|h\|_{W^{-1/q,q}(\Gamma_N)} \le C_h\|\epsilon\|_{L^p}. \tag{5.46}$$

Inserting (5.43) into (5.40) and using the result together with (5.41), (5.42), (5.45), and (5.46) in (5.38) yields

$$\begin{aligned} I(v) &\ge C_1\left(\|\epsilon\|_{L^p}^p+\|\tilde\sigma\|_{L^q}^q\right)-C_2-\gamma\int_\Omega \epsilon\cdot\tilde\sigma\,\mathrm{d}x\\ &\ge C_1\left(\|\epsilon\|_{L^p}^p+\|\tilde\sigma\|_{L^q}^q\right)-C\left(\|\epsilon\|_{L^p}+\|\tilde\sigma\|_{L^q}+1\right)\\ &\ge \frac{C_1}{2}\left(\|\epsilon\|_{L^p}^p+\|\tilde\sigma\|_{L^q}^q\right)-C, \end{aligned} \tag{5.47}$$

where we used Young's inequality in the last step and the constants depend on the space dimension N, the domain Ω and f, g, g_ν, h, h_τ. □

Lastly we check, that indeed the function $\mathrm{dist}(\cdot,\mathcal{D})$ is (p,q)-coercive if $\mathcal{D}$ contains data for which 'ϵ and $\tilde\sigma$ are aligned well enough'.

Lemma 5.31. *The distance function* $\mathrm{dist}(\cdot,\mathcal{D})$ *to a set* $\mathcal{D}\subset Y\times Y$ *is* (p,q)*-coercive if and only if there are* $c_1\in\mathbb{R}$ *and* $c_2>0$*, such that*

$$\mathcal{D}\subset\{(\epsilon,\tilde\sigma)\in Y\times Y\colon c_1\epsilon\cdot\tilde\sigma+c_2>|\epsilon|^p+|\tilde\sigma|^q\}.$$

Proof. **'⇒':** Suppose first that the distance function to $\mathcal{D}$ is (p,q)-coercive, i.e.

$$\mathrm{dist}((\epsilon,\tilde\sigma),\mathcal{D})\ge C_1(|\epsilon|^p+|\tilde\sigma|^q)-C_2-\gamma\epsilon\cdot\tilde\sigma.$$

Then, for all $(\epsilon,\tilde\sigma)\in\mathcal{D}$ we have

$$0\ge C_1(|\epsilon|^p+|\tilde\sigma|^q)-C_2-\gamma\epsilon\cdot\tilde\sigma$$

and therefore,

$$(\epsilon,\tilde\sigma)\in\mathcal{D}\quad\Longrightarrow\quad|\epsilon|^p+|\tilde\sigma|^q<c_2+c_1\epsilon\cdot\tilde\sigma.$$

'⇐': For the converse direction we need to prove that the distance function to the set

$$\mathcal{D}=\{(\epsilon,\tilde\sigma)\in Y\times Y\colon c_1\epsilon\cdot\tilde\sigma+c_2>|\epsilon|^p+|\tilde\sigma|^q\}$$

is (p,q)-coercive. The constant c_2 only makes $\mathcal{D}$ thicker by a finite amount. To see this, for $(\epsilon,\tilde\sigma)\in\mathcal{D}$, write $\tilde\sigma=\alpha\epsilon+\tilde\sigma^\perp$ with $\epsilon\cdot\tilde\sigma^\perp=0$ and define $\tilde\sigma_\beta=\alpha\epsilon+\beta\tilde\sigma^\perp$. Since $\epsilon\cdot\tilde\sigma=\alpha|\epsilon|^2$ we must have $|\tilde\sigma^\perp|^q\le c_2+c_\alpha|\epsilon|$ because of $(\epsilon,\tilde\sigma)\in\mathcal{D}$. Then $|\tilde\sigma_\beta|^q\le c_q|\alpha\epsilon|^q+\beta^q|\tilde\sigma^\perp|^q$ while $\epsilon\cdot\tilde\sigma=\epsilon\cdot\tilde\sigma_\beta$. Decreasing β, we find a $\tilde\sigma_\beta$ such that $c_1\epsilon\cdot\tilde\sigma>|\epsilon|^p+|\tilde\sigma|^q$and such that $\mathrm{dist}((\epsilon,\tilde\sigma),(\epsilon,\tilde\sigma_\beta))$ is bounded independently of $(\epsilon,\tilde\sigma)$.

Thus, we may assume that $c_2 = 0$ since this only shifts C_2 in (5.34). Then $\mathcal{D}$ is (p,q)-homogeneous, i.e. $(\epsilon, \tilde{\sigma}) \in \mathcal{D} \Rightarrow (\lambda\epsilon, \lambda^{p/q}\tilde{\sigma}) \in \mathcal{D}$ for all $\lambda > 0$. This in turn implies that the distance function is (p,q)-homogeneous, i.e.

$$\operatorname{dist}\Big((\lambda\epsilon, \lambda^{p/q}\tilde{\sigma}), \mathcal{D}\Big) = \lambda^p \operatorname{dist}\left((\epsilon, \tilde{\sigma}), \mathcal{D}\right). \tag{5.48}$$

for all $\lambda > 0$. Let $S = \{|\epsilon|^p + |\tilde{\sigma}|^q = 1\}$ be the unit sphere. Then the set

$$E := S \cap \{2c_1\epsilon \cdot \tilde{\sigma} \leq |\epsilon|^p + |\tilde{\sigma}|^q\}$$

is compact and has positive distance to $\mathcal{D}$, i.e. there exists $a > 0$ such that

$$(\epsilon, \tilde{\sigma}) \in E \quad \Longrightarrow \quad \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}) > a.$$

Hence, setting

$$c = \max_{(\epsilon,\tilde{\sigma})\in E} (|\epsilon|^p + |\tilde{\sigma}|^q - 2c_1\epsilon \cdot \tilde{\sigma}),$$

we have

$$(\epsilon, \tilde{\sigma}) \in S \quad \Longrightarrow \quad \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}) \geq \frac{a}{c}(|\epsilon|^p + |\tilde{\sigma}|^q - 2c_1\epsilon \cdot \tilde{\sigma}),$$

where we use that the right-hand side is smaller than 0 on in the complement of E, while it is smaller than a in E. This and (5.48) show that the distance function dist is (p,q)-coercive.

□

Γ-convergence

Theorem 5.32 (Γ-convergence in the linear setting)**.** *Let $\mathcal{D}_n, \mathcal{D} \subset Y \times Y$ be closed, nonempty sets, and let $\mathfrak{C}_{\text{lin}}$ be given by* (linC)*. Moreover, suppose that*

(i) The distance functions to $\mathcal{D}_n$ and $\mathcal{D}$ are uniformly *(p,q)-coercive, i.e. there are c_1, c_2, such that*

$$\mathcal{D}_n, \mathcal{D} \subset \{(\epsilon, \tilde{\sigma}) \in V \times V : c_1\epsilon \cdot \tilde{\sigma} + c_2 > |\epsilon|^p + |\tilde{\sigma}|^q\};$$

(ii) $\mathcal{D}_n \xrightarrow{eq} \mathcal{D}$;

(iii) if $\Gamma_R \neq \emptyset$, let $p \geq \dfrac{2N}{N+1}$.

Then the functional I_n Γ-converges to I^*, where

$$I^*(v) = \begin{cases} \displaystyle\int_\Omega \mathcal{Q}_\mathcal{A} \operatorname{dist}(v, \mathcal{D})\,\mathrm{d}x, & v \in \mathfrak{C}_{\text{lin}} \\ \infty, & \textit{else.} \end{cases}$$

Proof. The hypotheses of Theorem 5.11 are all satisfied with $\mathcal{F}_n = \operatorname{dist}(\cdot, \mathcal{D}_n)$, $\mathcal{F} = \operatorname{dist}(\cdot, \mathcal{D})$ and $X = \mathfrak{C}_{\text{lin}}$. Indeed, (H1) is Corollary 5.12, (H4) is the assumption $\mathcal{D}_n \xrightarrow{eq} \mathcal{D}$ and (H2) is satisfied by distance functions of sets, such that $\mathcal{D}, \mathcal{D}_n \cap B(0,R) \neq \emptyset$ for some

$R > 0$. This in turn follows from nonemptyness and $\mathcal{D}_n \xrightarrow{eq} \mathcal{D}$. Condition (H3) follows from the fact that the functions $\mathcal{F}$ in our setting are distance functions, hence even locally Lipschitz continuous. Finally, the set $X = \mathcal{C}_{\text{lin}}$ is weakly closed because for a bounded sequence $z_n = (\epsilon_n, \tilde{\sigma}_n) \subset V$ the pressure π_n satisfies, after suitable renormalisation,

$$\|\pi_n\|_{L^q} \leq C \left(\|\tilde{\sigma}_n\|_{L^q} + \|f\|_{W^{-1,q}}\right)$$

and is thus also bounded. Since the differential constraints linD are linear, it is possible to take the limit for a subsequence. Therefore, Theorem 5.11 implies that I_n Γ-converges to the Γ-limit of I, which is given by I^* due to Proposition 5.13.

□

Remark 5.33. Theorem 5.22 establishes equivalence between data convergence and uniform convergence of J_n towards J if there is *no* differential constraint $\mathcal{A}v = 0$. It is not clear whether such an equivalence holds for the constrained functionals I_n and I. Indeed, in an abstract degenerate setting, e.g. $\ker \mathcal{A}[\xi] = \{0\}$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$, so that only constant functions are in $\ker \mathcal{A}$, it is easy to see that the equivalence does not hold. Indeed, uniform approximation for bounded/equi-integrable functions in the constraint set $\mathcal{C}$ is equivalent to *pointwise* uniform approximation on bounded sets. That is, there are $R_n \to \infty$ and $\tilde{a}_n \to 0$, such that for all $z \in \mathcal{D}$ with $\text{dist}(z, 0) \leq R_n$

$$\text{dist}(z, \mathcal{D}_n) \leq \tilde{a}_n.$$

This is considerably weaker than the notions of convergence introduced in Definition 5.18 and Definition 5.20. A similar notion holds for fine approximation. Nevertheless, from a physical viewpoint, the pointwise data convergence $\mathcal{D}_n \xrightarrow{eq} \mathcal{D}$ is a reasonable assumption and we are thus not interested in a complete characterisation of convergence for the constrained functionals.

5.5.2. Fluids with Inertia

In this subsection we consider the system of differential constraints, corresponding to a fluid with inertia

$$\begin{cases} \epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \\ \operatorname{div} u = 0 \\ -\operatorname{div} \tilde{\sigma} = f - \nabla \pi - (u \cdot \nabla) u. \end{cases} \tag{nD}$$

Regarding the boundary conditions, we make the following assumptions throughout this subsection:

(B1) $\Gamma_N = \emptyset$, i.e. there are only no-slip and Navier-type boundary conditions;

(B2) $\Gamma_D \neq \emptyset$;

(B3) One of the following two statements is true

(B3a) $p > 2$;

(B3b) $g = 0$ and $g_\nu = 0$.

Note that assumption (B3b) represents the important case of a non-permeable boundary. In comparison to the linear problem (linD), the set (nD) of differential constraints admits a *direct* coupling between ϵ and $\tilde{\sigma}$ through the inertial term $(u\cdot\nabla)u$. For this set of constraints to still be meaningful, the inertial term $(u\cdot\nabla)u$ needs to be in the same space as f, $\operatorname{div}\tilde{\sigma}$, and $\nabla\pi$. Since $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, for $p < N$ (otherwise we use $u \in W^1_r(\Omega, \mathbb{R}^N)$ for all $r < N$), we have by embedding $u \in L^{Np/(N-p)}(\Omega, \mathbb{R}^N)$ and thus $u \otimes u \in L^{Np/(2N-2p)}(\Omega, \mathbb{R}^{N\times N})$, which implies $(u \cdot \nabla)u = \operatorname{div}(u \otimes u) \in W^{-1,Np/(2N-2p)}(\Omega, \mathbb{R}^N)$. In order for this space to be contained in $W^{-1,q}(\Omega, \mathbb{R}^N)$, we must have

$$q = \frac{p}{p-1} \leq \frac{Np}{2N-2p}, \tag{5.49}$$

which implies

$$p \geq \frac{3N}{N+2}. \tag{5.50}$$

Throughout this section we assume that (5.50) holds. This includes the Newtonian case $p = 2$ in the physical dimensions $N = 2, 3$. Since we have

$$p \geq \frac{3N}{N+2} \geq \frac{2N}{N+1},$$

condition (5.30) is always satisfied. Hence, the Navier boundary condition (R) is well-defined.

In this subsection we consider the constraint set

$$\mathcal{C} := \{(\epsilon, \tilde{\sigma}) \in V \colon (\text{nD}), (D), \text{ and } (R) \text{ are satisfied.}\} \tag{nlC}$$

Coercivity in the semilinear case

In this subsection we check that functionals of the form (5.35), with $\mathcal{C}$ given by (nlC), are still coercive.

Lemma 5.34 (Coercivity in the semi-linear setting)**.** *Let $p \geq 3N/(N+2)$ and assume that the assumptions (B1)–(B3) hold. Let $\mathcal{F}$ be (p,q)-coercive and let $\mathcal{C}$ be given by* (nlC)*. Then there are constants $C_3, C_4 > 0$, such that*

$$I(v) = \int_\Omega \mathcal{F}(\epsilon, \tilde{\sigma})\,\mathrm{d}x \geq C_3\left(\|\epsilon\|^p_{L^p} + \|\tilde{\sigma}\|^q_{L^q}\right) - C_4. \tag{5.51}$$

Proof. Similarly to the proof of Lemma 5.30, we need to estimate $\int \epsilon \cdot \tilde{\sigma}\,\mathrm{d}x$, as for any $(\epsilon, \tilde{\sigma}) \in Y \times Y$

$$\mathcal{F}(\epsilon, \tilde{\sigma}) \geq C_1(|\epsilon|^p + |\tilde{\sigma}|^q) - C_2 - \gamma\epsilon \cdot \tilde{\sigma}. \tag{5.52}$$

Since $v \in \mathcal{C}$, there is a u such that

$$\epsilon = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right),$$

for some u, where

$$\|u\|_{W^{1,p}} \leq C\left(\|\epsilon\|_{L^p} + 1\right) \tag{5.53}$$

due to the Korn–Poincaré inequality, Lemma 5.6 and Lemma 5.24. Furthermore, we have the estimate

$$\|(\tilde{\sigma} - \pi\,\mathrm{id})\nu\|_{W^{-1/q,q}(\partial\Omega)} \leq C\left(\|\tilde{\sigma}\|_{L^q} + \|f\|_{W^{-1,q}} + \|u\|^2_{W^{1,p}}\right), \tag{5.54}$$

which is due to $-\operatorname{div}\tilde{\sigma} + \nabla\pi = f - (u\cdot\nabla)\,u$.

Indeed, repeating the calculation from the proof of Lemma 5.30 and then using the nonlinear force balance, we obtain

$$\begin{aligned}\int_\Omega \epsilon\cdot\tilde{\sigma}\,\mathrm{d}x &= -\int_\Omega u\cdot(\operatorname{div}\tilde{\sigma} - \nabla\pi)\,\mathrm{d}x + \int_{\partial\Omega} u\cdot(\tilde{\sigma} - \pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_\Omega u\cdot(u\cdot\nabla)u + u\cdot f\,\mathrm{d}x + \int_{\partial\Omega} u\cdot(\tilde{\sigma} - \pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_\Omega \operatorname{div}\left(\frac{1}{2}u|u|^2\right) + u\cdot f\,\mathrm{d}x + \int_{\partial\Omega} u\cdot(\tilde{\sigma} - \pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_\Omega u\cdot f\,\mathrm{d}x + \int_{\partial\Omega}\frac{1}{2}(u\cdot\nu)|u|^2 + u\cdot(\tilde{\sigma} - \pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned} \tag{5.55}$$

For the first term we use (5.53) to bound

$$\left|\int_\Omega u\cdot f\,\mathrm{d}x\right| \leq \|u\|_{W^{1,p}}\|f\|_{W^{-1,q}} \leq C\left(\|\epsilon\|_{L^p} + 1\right)\|f\|_{W^{-1,q}}. \tag{5.56}$$

For the boundary term we consider the cases (B3a) and(B3b) separately.
Case (B3a): We split $\partial\Omega = \overline{\Gamma_D \cup \Gamma_R}$ and start with

$$\begin{aligned}\int_{\Gamma_D}\frac{1}{2}(u\cdot\nu)|u|^2 - u\cdot(\tilde{\sigma} - \pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1} &= \int_{\Gamma_D}\frac{1}{2}(g\cdot\nu)|g|^2 - g\cdot(\tilde{\sigma} - \pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1}\\ &\leq \|g\|^3_{L^3(\Gamma_D)} + \|g\|_{W^{1-1/p,p}(\Gamma_D)}\|(\tilde{\sigma} - \pi\,\mathrm{id})\nu\|_{W^{-1/q,q}(\Gamma_D)}\\ &\leq C\left(\|u\|^2_{W^{1,p}} + \|\tilde{\sigma}\|_{L^q} + 1\right)\\ &\leq C\left(\|\epsilon\|^2_{L^p} + \|\tilde{\sigma}\|_{L^q} + 1\right).\end{aligned} \tag{5.57}$$

Note that $W^{1-1/p,p}(\Gamma_D)$ embeds into $L^3(\partial\Omega)$, whenever

$$\frac{1}{3} \geq \frac{1}{p} + \frac{1 - 1/p}{d-1}.$$

This holds in view of assumption (5.50). For the other part of the boundary we estimate

$$\begin{aligned}&\int_{\Gamma_R} \frac{1}{2}(u\cdot\nu)|u|^2 - u\cdot(\tilde\sigma - \pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad= \int_{\Gamma_R} \frac{1}{2}g_\nu|u|^2 - g_\nu\nu\cdot(\tilde\sigma-\pi\,\mathrm{id})\nu + \lambda|P_{T_x\partial\Omega}u|^2 - P_{T_x\partial\Omega}u\cdot h_\tau\,\mathrm{d}\mathcal{H}^{N-1}. \qquad (5.58)\end{aligned}$$

For the terms without sign we obtain

$$\begin{aligned}&\left|\int_{\Gamma_R} \frac{1}{2}g_\nu|u|^2 - g_\nu\nu\cdot(\tilde\sigma-\pi\,\mathrm{id})\nu - P_{T_x\partial\Omega}u\cdot h_\tau\,\mathrm{d}\mathcal{H}^{N-1}\right|\\ &\quad\le \|g_\nu\|_{L^3(\Gamma_R)}\|u\|^2_{L^3(\Gamma_R)} + \|g_\nu\|_{W^{1-1/p,p}(\Gamma_R)}\|(\tilde\sigma-\pi\,\mathrm{id})\nu\|_{W^{-1/q,q}(\Gamma_R)}\\ &\qquad+\|h_\tau\|_{W^{-1/q,q}(\Gamma_R)}\|u\|_{W^{1-1/p,p}(\Gamma_R)}\\ &\quad\le C\left(\|u\|^2_{W^{1,p}} + \|\tilde\sigma\|_{L^q}+1\right)\\ &\quad\le C\left(\|\epsilon\|^2_{L^p} + \|\tilde\sigma\|_{L^q}+1\right). \qquad (5.59)\end{aligned}$$

Inserting (5.58) into (5.55) and using the result together with (5.56), (5.57), (5.59), and the (p,q)-coercivity of $\mathcal{F}$, yields

$$\begin{aligned}I(v) &\ge C_1\left(\|\epsilon\|^p_{L^p} + \|\tilde\sigma\|^q_{L^q}\right) - C_2 - \gamma\int_\Omega \epsilon\cdot\tilde\sigma\,\mathrm{d}x\\ &\ge C_1\left(\|\epsilon\|^p_{L^p} + \|\tilde\sigma\|^q_{L^q}\right) - C\left(\|\epsilon\|^2_{L^p} + \|\tilde\sigma\|_{L^q}+1\right)\\ &\ge \frac{C_1}{2}\left(\|\epsilon\|^p_{L^p} + \|\tilde\sigma\|^q_{L^q}\right) - C,\end{aligned}$$

where we use Young's inequality and the fact that $p > 2$.

Case (B3b): Since $g = 0$ and $g_\nu = 0$, the boundary term simplifies to

$$\begin{aligned}\int_{\partial\Omega} \frac{1}{2}(u\cdot\nu)|u|^2 - u\cdot(\tilde\sigma-\pi\,\mathrm{id})\nu\,\mathrm{d}\mathcal{H}^{N-1} &= -\int_{\Gamma_R} P_{T_x\partial\Omega}u\cdot P_{T_x\partial\Omega}(\tilde\sigma\nu)\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_{\Gamma_R} \lambda|P_{T_x\partial\Omega}u|^2 - P_{T_x\partial\Omega}u\cdot h_\tau\,\mathrm{d}\mathcal{H}^{N-1}. \qquad (5.60)\end{aligned}$$

For the without sign we obtain

$$\left|\int_{\Gamma_R} P_{T_x\partial\Omega}u\cdot h_\tau\,\mathrm{d}\mathcal{H}^{N-1}\right| \le \|u\|_{W^{1-1/p,p}(\Gamma_R)}\|h_\tau\|_{W^{-1/q,q}(\Gamma_R)} \le C\left(\|\epsilon\|_{L^p}+1\right) \qquad (5.61)$$

By inserting (5.60) into (5.55) and using (5.56), (5.61) and the (p,q)-coercivity of $\mathcal{F}$, we obtain

$$\begin{aligned}I(v) &\ge C_1\left(\|\epsilon\|^p_{L^p} + \|\tilde\sigma\|^q_{L^q}\right) - C_2 - \gamma\int_\Omega \epsilon\cdot\tilde\sigma\,\mathrm{d}x\\ &\ge C_1\left(\|\epsilon\|^p_{L^p} + \|\tilde\sigma\|^q_{L^q}\right) - C\left(\|\epsilon\|_{L^p}+1\right)\\ &\ge \frac{C_1}{2}\left(\|\epsilon\|^p_{L^p} + \|\tilde\sigma\|^q_{L^q}\right) - C,\end{aligned}$$

where we use again Young's inequality.

□

Continuity of $\Theta(u) = u \otimes u$

To verify the assumptions of Theorem 5.15, in particular the weak closedness of $\mathcal{C}_{\text{ln}}$, we show that the map

$$u \longmapsto u \otimes u$$

is continuous from the weak topology of $W^{1,p}(\Omega, \mathbb{R}^N)$ to the strong topology of $L^r(\Omega, Y)$ for some $r > q$.

Lemma 5.35. *Let $p > 3N/(N+2)$. Then there is an $r > q = p/(p-1)$, such that Θ is continuous from $W^{1,p}(\Omega, \mathbb{R}^N) \cap \ker \operatorname{div}$, equipped with the weak topology, into to $L^r(\Omega, Y)$.*

In view of Korn's inequality (Lemma 5.6) bounded sets in $L^p(\Omega, Y)$ are mapped to bounded sets in $W^{1,p}(\Omega, \mathbb{R}^N)$ by the map $\epsilon \mapsto u$. Hence, the map Θ might also be seen as a map $\epsilon \mapsto u \otimes u$.

Proof. For $p \geq N$ the result immediately follows from the case $p < N$ by first embedding into $W^{1,\tau}(\Omega, \mathbb{R}^N)$ for some $\tau < N$. Thus, let $p < d$. Then $W^{1,p}(\Omega, \mathbb{R}^N)$ embeds compactly into $L^s(\Omega, \mathbb{R}^N)$ for all $s < Np/(N-p)$. Consequently, for every weakly convergent sequence $u_n \subset W^{1,p}(\Omega, \mathbb{R}^N)$ obeying $\operatorname{div} u_n = 0$, the sequence

$$\Theta(u_n) = u_n \otimes u_n$$

still converges weakly to $\Theta(u)$ in $W^{1,t}(\Omega, Y)$. The exponent $t \in (1, \infty)$ is given in terms of s and p via

$$\frac{1}{t} = \frac{1}{s} + \frac{1}{p}.$$

Consequently, $u_n \otimes u_n \rightharpoonup u \otimes u$ in $W^{1,t}(\Omega, Y)$, whenever

$$t < \frac{Np}{2N - p}.$$

Due to the compact Sobolev embedding, we have $W^{1,t}(\Omega, Y) \hookrightarrow\hookrightarrow L^r(\Omega, Y)$ for $r < Nt/(N-t)$. Therefore, Θ maps $W^{1,p}(\Omega, \mathbb{R}^N)$, equipped with the weak topology, continuously to $L^r(\Omega, Y)$ in the strong topology, whenever

$$r < \frac{N \frac{Np}{2N-p}}{N - \frac{Np}{2N-p}}.$$

This and the condition $q = \dfrac{p}{p-1} < r$ can be satisfies at the same time if

$$p > \frac{3N}{N+2},$$

which is assumption (5.50). □

Γ-**convergence with semilinear constraint.**

Theorem 5.36 (Γ-convergence in the semilinear setting)**.** *Let $\mathcal{D}_n, \mathcal{D} \subset Y \times Y$ be closed, nonempty sets and let $\mathcal{C}$ be given by* (nlC)*. Moreover, suppose that:*

(i) The distance functions to $\mathcal{D}_n$ and $\mathcal{D}$ are uniformly (p,q)-coercive, i.e. there are c_1, c_2, such that

$$\mathcal{D}_n, \mathcal{D} \subset \{(\epsilon, \tilde{\sigma}) \in V \times V \colon c_1 \epsilon \cdot \tilde{\sigma} + c_2 > |\epsilon|^p + |\tilde{\sigma}|^q\};$$

(ii) $\mathcal{D}_n \xrightarrow{eq} \mathcal{D}$;

(iii) $p > \dfrac{3N}{N+2}$;

(iv) assumptions (B1)–(B3) hold.

Then the functional I_n Γ-converges to I^*, where

$$I^*(v) = \begin{cases} \int_\Omega \mathcal{Q}_\mathcal{A} \operatorname{dist}(v, \mathcal{D}) \,\mathrm{d}x, & v \in \mathcal{C} \\ \infty, & \text{else.} \end{cases}$$

Proof. The proof is very similar to the proof of Theorem 5.32. Indeed, as the constraint set $\mathcal{C}$ is weakly closed by Lemma 5.35, the only difficulty, given $v \in \mathcal{C}$, is to find a recovery sequence lying in $\mathcal{C}$. This is achieved in Theorem 5.15. □

5.6. Consistency of data-driven solutions and PDE solutions for material law data

In this section we consider data that are given by a *constitutive law*, i.e.

$$\tilde{\sigma} = \mu(|\epsilon|)\epsilon, \quad \epsilon \in Y,$$

for a viscosity $\mu \colon \mathbb{R} \to \mathbb{R}$. We compare the solutions obtained by the *classical PDE* approach to minimisers of the *data-driven* functional. As before, we assume $\Gamma_N = \emptyset$ and call a pair $(\epsilon, \tilde{\sigma}) \in L^p(\Omega, Y) \times L^q(\Omega, Y)$ a *weak solution* to the stationary Navier–Stokes equation, if there is $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and a pressure $\pi \in L^q(\Omega)$, such that

$$\begin{cases} \epsilon = \frac{1}{2}\big(\nabla u + \nabla u^T\big), & x \in \Omega \\ \operatorname{div} u = 0, & x \in \Omega \\ (u \cdot \nabla)u - \operatorname{div}(\mu(|\epsilon|)\epsilon) + \nabla \pi = f, & x \in \Omega \\ (D), (R), & x \in \partial\Omega, \end{cases} \tag{5.62}$$

where $(5.62)_3$ has to be satisfied in $W^{-1,q}(\Omega, \mathbb{R}^N)$. Note that the system (5.62) is equivalent to

$$\begin{cases} \epsilon = \frac{1}{2}\big(\nabla u + \nabla u^T\big), & x \in \Omega \\ \operatorname{div} u = 0, & x \in \Omega \\ -\operatorname{div} \tilde{\sigma} = f - \nabla \pi - (u \cdot \nabla) u, & x \in \Omega \\ \tilde{\sigma} = \mu(|\epsilon|)\epsilon, & x \in \Omega \\ (D), (R), & x \in \partial\Omega. \end{cases} \tag{5.63}$$

We may interpret the convergence of data sets discussed in Section 5.4 as an increase of the accuracy of measurement. If a constitutive law exists, then the limit $\mathcal{D}$ of data sets $\mathcal{D}_n$ should represent this law. Since we assume that the set $\mathcal{D}$ is given by a constitutive law $\epsilon \mapsto \tilde{\sigma}_c(\epsilon)$, we consider data sets

$$\mathcal{D} = \{(\epsilon, \tilde{\sigma}) \colon \tilde{\sigma} = \tilde{\sigma}_c(\epsilon)\}. \tag{5.64}$$

For typical constitutive laws, a solution to the induced partial differential equation (5.63) exists and it is natural to ask whether (approximate) solutions to the data-driven problem with $\mathcal{D}_n$ converge to a solution of (5.63). It turns out that this is true if the constitutive relation is monotone. Indeed, assume that $(\epsilon, \tilde{\sigma}) \in \mathcal{C}$, i.e. that the differential constraints

$$\begin{cases} \epsilon = \frac{1}{2}\big(\nabla u + \nabla u^T\big), & x \in \Omega \\ \operatorname{div} u = 0, & x \in \Omega \\ -\operatorname{div} \tilde{\sigma} = f - \nabla \pi - (u \cdot \nabla) u, & x \in \Omega \end{cases}$$

are satisfied. If in addition $I(u) = 0$, and thus u is a minimiser, then we have

$$(\epsilon, \tilde{\sigma}) \in \mathcal{D} = \{(\epsilon, \tilde{\sigma}) \colon \tilde{\sigma} = \tilde{\sigma}_c(\epsilon)\} \quad \text{almost everywhere.}$$

Consequently, a minimiser of I satisfying $I(u) = 0$ is a solution to the partial differential equation. Conversely, given a constitutive law $\tilde{\sigma}_c$ and a weak solution to the partial differential equation (5.63), we may construct the set $\mathcal{D}$ as in (5.64) and observe that any solution to the partial differential equation (5.63) is also a minimiser of I.

If the data set $\mathcal{D}$ is a limit of measurement data sets $\mathcal{D}_n$, it s not clear whether a sequence of (approximate) minimisers u_n of I_n converges weakly to a solution u to the partial differential equation because we can only infer $I^*(u) = 0$ and *not* $I(u) = 0$. This is addressed in the following proposition, which directly follows from the relaxation statement Theorem 5.36.

Proposition 5.37. *Let $p > 3N/(N+2)$ and let $\epsilon \mapsto \tilde{\sigma}_c(\epsilon)$ be a given constitutive law. Moreover, assume that the corresponding data set $\mathcal{D}$ is given by* (5.64)*, such that the distance function* $\operatorname{dist}(\cdot,\cdot)$ *is (p, q)-coercive. If the partial differential equation* (5.63) *admits* *a weak solution v, i.e.* $\min_{v\in\mathcal{C}} I(v) = 0$, *then a function v^* is a minimiser of I^* if and only if*

$$v^* \in \{\mathcal{Q}_{\mathcal{A}} \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}) = 0\}$$

almost everywhere. Moreover, if

$$\{\mathcal{Q}_{\mathcal{A}} \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}) = 0\} = \mathcal{D}, \tag{5.65}$$

then any such approximate solution v^* is already a solution to the partial differential equation (5.63)*.*

In the following we characterise some constitutive laws satisfying (5.65). To this end, we study the set

$$\{\mathcal{Q}_{\mathcal{A}} \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}\} = 0\}.$$

Definition 5.38. *Let $1 < p < \infty$ and $q = p/(p-1)$. For a set $\mathcal{D} \subset Y \times Y$ we define the* $\mathcal{A}$-(p,q)***-quasiconvex hull*** *of $\mathcal{D}$ as*

$$\mathcal{D}^{(p,q)} = \{(\epsilon, \tilde{\sigma}) \in Y \times Y \colon \mathcal{Q}_{\mathcal{A}} \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}) = 0\}.$$

We call a set $\mathcal{D} \subset Y \times Y$ $\mathcal{A}$-(p,q)***-quasiconvex*** *if $\mathcal{D} = \mathcal{D}^{(p,q)}$.*

5.6.1. Newtonian fluids

In the Newtonian setting the fluid's viscosity is constant, i.e. $\mu(|\epsilon|) \equiv \mu_0 > 0$ and hence the relation between the local strain ϵ and the viscous stress $\tilde{\sigma}$ is linear with $\tilde{\sigma} = 2\mu_0\epsilon$. In the following, we assume without loss of generality that $\mu_0 = 1/2$. That is, we have $p = q = 2$ and the constitutive law is given by the data set

$$\mathcal{D}_{\mathcal{N}} = \{(\epsilon, \epsilon) \colon \epsilon \in Y\} \subset Y \times Y.$$

Note that, in terms of ϵ and $\tilde{\sigma}$, the Newtonian data set $\mathcal{D}_{\mathcal{N}}$ and the distance function $\operatorname{dist}(\cdot,\cdot)$ can be written as

$$\mathcal{D}_{\mathcal{N}} = \left\{(\epsilon, \tilde{\sigma}) \colon \epsilon \cdot \tilde{\sigma} = \tfrac{1}{2}\left(|\epsilon|^2 + |\tilde{\sigma}|^2\right)\right\} \quad \text{and} \quad \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}_{\mathcal{N}}) = \tfrac{1}{2}|\epsilon - \tilde{\sigma}|^2.$$

Since in this case $\operatorname{dist}((\cdot,\cdot), \mathcal{D}_{\mathcal{N}})$ is already a convex function, it is also $\mathcal{A}$-quasiconvex and we have

$$\mathcal{Q}_{\mathcal{A}} \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}_{\mathcal{N}}) = \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}_{\mathcal{N}}).$$

Consequently, we observe that the $\mathcal{A}$-(p,q)-quasiconvex hull $\mathcal{D}_{\mathcal{N}}^{(p,q)}$ of $\mathcal{D}_{\mathcal{N}}$ is given by

$$\mathcal{D}_{\mathcal{N}}^{(p,q)} = \{(\epsilon, \tilde{\sigma}) \colon \operatorname{dist}((\epsilon, \tilde{\sigma}), \mathcal{D}_N) = 0\} = \mathcal{D}_{\mathcal{N}}.$$

Therefore, any solution to the data-driven problem for Newtonian fluids is also a weak solution to the partial differential equation, in the sense that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ satisfies

$$\begin{cases} (u \cdot \nabla)u = -\nabla \pi + \Delta u, & x \in \Omega \\ \operatorname{div} u = 0, & x \in \Omega \end{cases}$$

and the boundary conditions (D), (R).

5.6.2. Power-law fluids

In the case of power-law fluids, the constitutive law for the fluid's viscosity is $\mu(|\epsilon|) = \mu_0 |\epsilon|^{\alpha-1}\epsilon$ with given flow-consistency index $\mu_0 > 0$ and flow-behaviour exponent $\alpha > 0$. Consequently, we have $\tilde{\sigma} = 2\mu_0|\epsilon|^{\alpha-1}$. As above, we set without loss of generality $\mu_0 = 1/2$. In the previously used notation, we thus consider $1 < p < \infty$, $q = p/(p-1)$ and $\alpha = p/q = 1/(p-1)$ and suppose that the material law is given by the data set

$$\mathcal{D}_{\mathcal{P}} = \big\{(\epsilon, |\epsilon|^{\alpha-1}\epsilon) \colon \epsilon \in Y\big\} \subset Y \times Y.$$

Observe that, for $\alpha \neq 1$, the set $\mathcal{D}_{\mathcal{P}}$ is *not* convex. Consequently, also the corresponding distance function is not convex. However,

$$(\epsilon, \tilde{\sigma}) \in \mathcal{D}_{\mathcal{P}} \Longleftrightarrow \epsilon \cdot \tilde{\sigma} = \tfrac{1}{p}|\epsilon|^p + \tfrac{1}{q}|\tilde{\sigma}|^q.$$

It turns out that the $\mathcal{A}$-(p,q)-quasiconvex hull $\mathcal{D}_{\mathcal{P}}^{(p,q)}$ of $\mathcal{D}_{\mathcal{P}}$ in fact coincides with the data set $\mathcal{D}_{\mathcal{P}}$. In order to verify this, we rely on the following observation (see also [153]).

Lemma 5.39. *Let* $\operatorname{dist}(\cdot, \mathcal{D})$ *be* (p,q)*-coercive. Then*

$$\mathcal{D}^{(p,q)} = \bigcap_{\mathcal{F} \in T_{p,q}} \{\mathcal{F}(z) \leq 0\},$$

where $T_{p,q}$ *is the set of all continuous functions* $\mathcal{F} \in C(Y \times Y)$ *satisfying*

- $\mathcal{F}$ *is* $\mathcal{A}$*-quasiconvex;*
- $\mathcal{F}(z) \leq 0$ *for all* $z \in \mathcal{D}$*;*
- $|\mathcal{F}(\epsilon, \tilde{\sigma})| \leq C(1 + |\epsilon|^p + |\tilde{\sigma}|^q)$.

Proof. **'$\supseteq$':** Since $\mathcal{Q}_{\mathcal{A}} \operatorname{dist}(\cdot, \mathcal{D})$ is contained in $T_{p,q}$, it is clear that $\bigcap_{\mathcal{F} \in T_{p,q}} \{\mathcal{F}(z) \leq 0\}$ is a subset of $\mathcal{D}^{(p,q)}$.
'$\subseteq$': Suppose now that $(\epsilon_0, \tilde{\sigma}_0) \in \mathcal{D}^{(p,q)}$. Then there exists a sequence $(\epsilon_n, \tilde{\sigma}_n) \in L^p(T_N, Y) \times L^q(T_N, Y)$ with zero average, satisfying the differential constraint such that

$$\int_{T_N} \operatorname{dist}\big((\epsilon_0 + \epsilon_n(x), \tilde{\sigma}_0 + \tilde{\sigma}_n(x)), \mathcal{D}\big) \,\mathrm{d}x < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{5.66}$$

Due to the coercivity of the distance function we can bound

$$\|\epsilon_n\|_{L^p} + \|\tilde{\sigma}_n\|_{L^q} \le C(1 + |\epsilon_0|^p + |\tilde{\sigma}_0|^q), \quad n \in \mathbb{N}.$$

Take now $\mathcal{F} \in T_{p,q}$. Then $\mathcal{F}$ is locally Lipschitz continuous thanks to Proposition 4.8 (or, more precisely, a suitable version in a (p,q)-setting). Define $w_n = (\epsilon_n', \tilde{\sigma}_n')$ as the projection of $(\epsilon_0 + \epsilon_n, \tilde{\sigma}_0 + \tilde{\sigma}_n)$ onto $\mathcal{D}$. Then, in view of (5.66) we find that,

$$\|\epsilon_0 + \epsilon_n - \epsilon_n'\|_{L^p} \longrightarrow 0 \quad \text{and} \quad \|\tilde{\sigma}_0 + \tilde{\sigma}_n - \tilde{\sigma}_n'\|_{L^q} \longrightarrow 0.$$

The local Lipschitz continuity of $\mathcal{F}$ and the boundedness of $(\epsilon_n, \tilde{\sigma}_n)$ now imply

$$\left| \int_{T_N} \mathcal{F}(\epsilon_0 + \epsilon_n, \tilde{\sigma}_0 + \tilde{\sigma}_n) - \mathcal{F}(\epsilon_n', \tilde{\sigma}_n') \, \mathrm{d}x \right| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{5.67}$$

Using $\mathcal{A}$-quasiconvexity of $\mathcal{F}$, (5.67), and the non-positivity of $\mathcal{F}$ this implies

$$\mathcal{F}(\epsilon_0, \tilde{\sigma}_0) \le \liminf_{n\to\infty} \int_{T_N} \mathcal{F}(\epsilon_0 + \epsilon_n, \tilde{\sigma}_0 + \tilde{\sigma}_n) \, dx \le \liminf_{n\to\infty} \int_{T_N} \mathcal{F}(\epsilon_n', \tilde{\sigma}_n') \, dx \le 0.$$

Eventually, we find that $(\epsilon_0, \tilde{\sigma}_0) \in \bigcap_{\mathcal{F} \in T_{p,q}} \{\mathcal{F}(z) \le 0\}$ and the proof is complete. □

Corollary 5.40. *Let p, q, α and $\mathcal{D}_{\mathcal{P}}$ be as before. Then*

$$\mathcal{D}_{\mathcal{P}}^{(p,q)} = \mathcal{D}_{\mathcal{P}}.$$

Proof. Lemma 5.39 implies that we only need to find a function $\mathcal{F}$, which is $\mathcal{A}$-quasiconvex, is non-positive in $(\epsilon, \tilde{\sigma})$ if and only if $(\epsilon, \tilde{\sigma}) \in \mathcal{D}_{\mathcal{P}}$ and has (p,q)-growth. The function

$$\mathcal{F}(\epsilon, \tilde{\sigma}) := \tfrac{1}{p}|\epsilon|^p + \tfrac{1}{q}|\tilde{\sigma}|^q - \epsilon \cdot \tilde{\sigma}$$

exactly satisfies these assertions. Therefore, $\mathcal{D}_{\mathcal{P}}^{(p,q)} = \mathcal{D}_{\mathcal{P}}$. □

5.6.3. Monotone material laws

Again, consider $1 < p < \infty$, $q = p/(p-1)$ and $\alpha = p/q$. We consider a constitutive law

$$\tilde{\sigma}(\epsilon) = 2\mu(|\epsilon|)\epsilon \tag{5.68}$$

for a viscosity $\mu \in C(\mathbb{R}_+, \mathbb{R}_+)$. For better readability we omit the factor 2 in (5.68) in the following calculations. Furthermore, throughout this subsection we assume the following:

(i) the material law $\tilde{\sigma}(\cdot)$ is *monotone*, i.e. for all $\epsilon_1, \epsilon_2 \in Y$ we have

$$(\epsilon_1 - \epsilon_2) \cdot (\tilde{\sigma}(\epsilon_1) - \tilde{\sigma}(\epsilon_2)) \ge 0;$$

(ii) $a := \lim_{s\to 0} \mu(s)s$.

The *data set* $\mathcal{D}_{\mathcal{M}}$ corresponding to the constitutive law $\epsilon \mapsto \tilde{\sigma}(\epsilon)$ is given as follows (cf. Figure 5.1):

$$\mathcal{D}_{\mathcal{M}} = \overline{\mathcal{D}}_\epsilon \cup \mathcal{D}_0, \quad \mathcal{D}_\epsilon = \big\{(\epsilon, \tilde{\sigma}(\epsilon)) \colon \epsilon \in Y \setminus \{0\}\big\}, \quad \mathcal{D}_0 = \big\{(0, \tilde{\sigma}) \colon |\tilde{\sigma}| \le a\big\}. \tag{5.69}$$

Remark 5.41. (i) Monotonicity of such a radial-symmetric function $\tilde{\sigma}(\epsilon)$ is equivalent to monotonicity of its one-dimensional counterpart

$$s \longmapsto \mu(s)s.$$

Therefore, the limit $a = \lim_{s\to 0} \mu(s)s$ is well-defined.

(ii) The setting includes the previously discussed cases of Newtonian and power-law fluids, as well as Ellis-law fluids [150]. Furthermore, it allows the strain-stress graph to have a discontinuity at zero, so-called Herschel-Bulkley fluids, cf. [102].

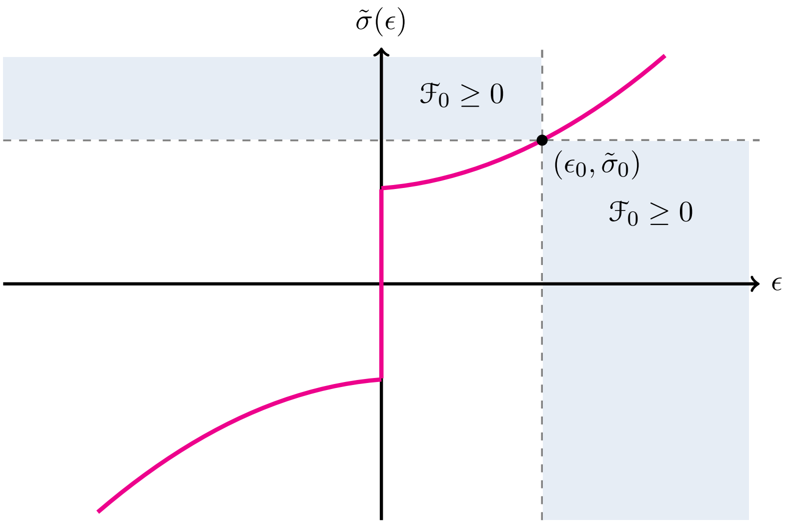

Figure 5.1.: A monotone material set $\mathcal{D}_{\mathcal{M}}$ and the separating function $\mathcal{F}_0$ for a given $(\epsilon_0, \tilde{\sigma}_0) \in \mathcal{D}_{\mathcal{M}}$.

Theorem 5.42. *Let p, q, α and $\mathcal{D}_{\mathcal{M}}$ be as above. Then we have*

$$\mathcal{D}_{\mathcal{M}}^{(p,q)} = \mathcal{D}_{\mathcal{M}}.$$

Proof. As for the proof of Corollary 5.40 for the power-law case, it suffices to find $\mathcal{A}$-quasiconvex separating functions (Lemma 5.39). For $(\epsilon_0, \tilde{\sigma}_0) \in \mathcal{D}_{\mathcal{M}}$ we define the function (cf. Figure 5.1).

$$\mathcal{F}_0(\epsilon, \tilde{\sigma}) = -(\epsilon - \epsilon_0) \cdot (\tilde{\sigma} - \tilde{\sigma}_0).$$

This function is $\mathcal{A}$-quasiconvex (even $\mathcal{A}$-quasiaffine, i.e. $\mathcal{F}$ and $-\mathcal{F}$ are $\mathcal{A}$-quasiconvex) and has (p,q)-growth, as

$$|\mathcal{F}_0(\epsilon,\tilde{\sigma})| \leq \tfrac{1}{p}|\epsilon-\epsilon_0|^p + \tfrac{1}{q}|\tilde{\sigma}-\tilde{\sigma}_0|^q.$$

To conclude that $\mathcal{D}_{\mathcal{M}}^{(p,q)} = \mathcal{D}_{\mathcal{M}}$ we still need to show that

(i) $\mathcal{F}_0$ is non-positive on $\mathcal{D}_{\mathcal{M}}$;

(ii) for all $(\epsilon,\tilde{\sigma}) \notin \mathcal{D}_{\mathcal{M}}$ there is $(\epsilon_0,\tilde{\sigma}_0) \in \mathcal{D}_{\mathcal{M}}$, such that $\mathcal{F}_0(\epsilon,\tilde{\sigma}) > 0$.

(i): Take $(\varepsilon,\tilde{\sigma}) \in \mathcal{D}$. Suppose that $|\varepsilon| \geq |\varepsilon_0|$ (the other case is rather similar). Then

$$\begin{aligned} -\mathcal{F}_0(\epsilon,\tilde{\sigma}) &= (\epsilon-\epsilon_0)\cdot(\tilde{\sigma}-\tilde{\sigma}_0) \\ &= (\epsilon-\epsilon_0)\cdot(\mu(|\epsilon|)\epsilon - \mu(|\epsilon_0|)\epsilon_0) \\ &= \mu(|\epsilon_0|)(\epsilon-\epsilon_0)\cdot(\epsilon-\epsilon_0) + (\epsilon-\epsilon_0)\cdot\big((\mu(|\epsilon_0|)-\mu(|\epsilon_0|))\epsilon\big) \\ &\geq 0 + \big(\mu(|\epsilon_0|)-\mu(|\epsilon_0|)\big)\big(|\epsilon|^2 - |\epsilon||\epsilon_0|\big) \geq 0 \end{aligned}$$

(ii): Suppose that $(\epsilon,\tilde{\sigma}) \notin \mathcal{D}_{\mathcal{M}}$. If $\epsilon \neq 0$, this means that $\tilde{\sigma} \neq \mu(|\varepsilon|)\varepsilon$. In that case, consider

$$\epsilon_t = \epsilon + t(\tilde{\sigma} - \mu(|\epsilon|)\epsilon)$$

and $\tilde{\sigma}_t = \mu(|\epsilon_t|)\epsilon_t$. If $\varepsilon = 0$, simply take $\epsilon_t = te_{11}$. For now, take $\epsilon \neq 0$, the other case is quite similar. Then for $t < 0$ small enough

$$-\mathcal{F}_t(\epsilon,\tilde{\sigma}) = (\epsilon-\epsilon_0)\cdot(\tilde{\sigma}-\tilde{\sigma}_t) = t(\tilde{\sigma}-\mu(|\epsilon|)\epsilon)\cdot(\tilde{\sigma}-\mu(|\epsilon_t|)\epsilon_t) < 0$$

as the map

$$t \mapsto (\tilde{\sigma} - \mu(|\epsilon_t|)\epsilon_t)$$

is continuous. Hence, there is $t < 0$, such that

$$(\tilde{\sigma}-\mu(|\epsilon|)\epsilon)\cdot(\tilde{\sigma}-\mu(|\epsilon_t|)\epsilon_t) > 0.$$

To summarise, there is a function $\mathcal{F}_t \in T_{p,q}$, such that $\mathcal{F}_t(\epsilon,\tilde{\sigma}) > 0$, whenever $(\epsilon,\tilde{\sigma}) \notin \mathcal{D}_{\mathcal{M}}$. □

Remark 5.43. Starting from the constitutive law $\epsilon \mapsto \tilde{\sigma}_c(\varepsilon)$, there are two choices for $\mathcal{D}_{\mathcal{M}}$. We may define $\mathcal{D}_{\mathcal{M}}$ as in (5.69) or only take the set $\overline{\mathcal{D}}_\varepsilon$ introduced in (5.69). For the $\mathcal{A}$-quasiconvex hull this does not make a difference, i.e.

$$\overline{\mathcal{D}}_\varepsilon^{(p,q)} = \mathcal{D}_{\mathcal{M}}^{(p,q)} = \mathcal{D}_{\mathcal{M}}. \tag{5.70}$$

Indeed, (5.70) can be verified by calculating the $\Lambda_{\mathcal{A}}$-*convex hull* of the set $\overline{\mathcal{D}}_\varepsilon$ (that is, we successively take convex combinations along $\Lambda_{\mathcal{A}}$). The $\Lambda_{\mathcal{A}}$-convex hull is a subset of the $\mathcal{A}$-quasiconvex hull. Therefore, it suffices to show that the $\Lambda_{\mathcal{A}}$-convex hull of $\overline{\mathcal{D}}_\varepsilon$ contains

$\mathcal{D}_{\mathcal{M}}$. This in turn follows from the fact that

$$\ker \mathbb{A}_2[\xi] = \{\tilde{\sigma} \in Y : \tilde{\sigma}\xi = 0\} + \mathbb{R}(\xi \otimes \xi) \quad \Longrightarrow \quad \Lambda_{\mathcal{A}_2} = Y.$$

Using this observation, the $\Lambda_{\mathcal{A}}$-convex hull of $\{(0, \tilde{\sigma}) : |\tilde{\sigma}| = a\} \subset \overline{\mathcal{D}}_\varepsilon$ is the convex hull $\mathcal{D}_0$. Consequently, the $\Lambda_{\mathcal{A}}$-convex hull and therefore also the $\mathcal{A}$-quasiconvex hull of $\overline{\mathcal{D}}_\varepsilon$ contain $\mathcal{D}_{\mathcal{M}}$.

6. $\mathcal{A}$-quasiconvex sets and hulls

This chapter discusses results regarding $\mathcal{A}$-quasiconvex sets and is a summary of what we show in remainder of this thesis, Chapters A and B, which are summarised by Chapters 7 and 8, respectively. Consequently, parts of this chapter are based on the two research works

- [134]: Schiffer, S., *L^∞-truncation of closed differential forms*;
- [20]: Behn, L., Gmeineder, F. and Schiffer, S. *On symmetric div-quasiconvex hulls and divsym-free L^∞truncations.*

It is clearly indicated, whenever we refer to these research articles. The remaining part of this chapter (Section 6.3) is based on some unpublished notes.
This chapter is organised as follows. First of all, in Section 6.1 we give a short introduction to $\mathcal{A}$-quasiconvex sets and hulls. We summarise the results obtained in [134] and [20] in Section 6.2.
In Section 6.3 we prove some result regarding non-compact sets which is independent of [134, 20]. The main part of the proofs (i.e. the technique of L^∞-truncations) is then discussed in Chapters A and B.

6.1. Introduction

In this chapter, we give an introduction to the notion of $\mathcal{A}$-quasiconvexity for *sets*. First, we deal with $\mathcal{A}$-quasiconvex hulls of compact sets in Section 6.2. Results in that section rely on rather involved truncation results which are the topic of the last two Chapters A and B. Section 6.3 focuses on an example of $\mathcal{A}$-quasiconvex hulls for non-compact sets.

Towards a definition of $\mathcal{A}$-quasiconvex hull, let $K \subset \mathbb{R}^d$ be a closed set. Motivated by Data-Driven problems in Section 5.6 and Minkowski's and Banach's separation theorem for convex sets, we call a set $\mathcal{A}$-*quasiconvex* if for all $\mathcal{A}$-quasiconvex $f \in C(\mathbb{R}^d)$ we have

$$f_{|K} \leq 0 \text{ and } f(x) \leq 0 \quad \Longrightarrow \quad x \in K.$$

Note that this definition coincides with the standard definition of convex sets, whenever $\mathcal{A} = 0$. For a further motivation we point to the introductory chapter of this thesis, see Section 1.3.4.

This definition may be seen as an L^∞-version of the $\mathcal{A}$-quasiconvex hull discussed in Section 5.6, i.e. a set is $\mathcal{A}$-quasiconvex if it coincides with its hull. In particular, we may derive the following differing concepts.

Definition 6.1 ($\mathcal{A}$-quasiconvex hulls). *Let $K \subset \mathbb{R}^d$ be a closed set and $1 \leq p < \infty$. We define*

(i) The space $S_{\mathcal{A}}(K)$ of ***separating functions*** *as*

$$S_{\mathcal{A}}(K) := \Big\{ f \in C(\mathbb{R}^d)\colon\ f\ \mathcal{A}\text{-quasiconvex},\ f \leq 0 \text{ on } K \Big\}; \tag{6.1}$$

(ii) The ***$\mathcal{A}$-quasiconvex hull*** *$K^{(\infty)}$ as*

$$K^{(\infty)} := \Big\{ x \in \mathbb{R}^d\colon f \in S_{\mathcal{A}}(K) \Rightarrow f(x) \leq 0 \Big\}; \tag{6.2}$$

(iii) The (alternative) ***$\mathcal{A}$-p-quasiconvex hull*** *$K^{(p)*}$ as the set*

$$K^{(p)*} := \Big\{ x \in \mathbb{R}^d\colon f \in S_{\mathcal{A}}(K) \text{ and } f(v) \leq C(1+|v|^p)\ \forall v \in \mathbb{R}^d \Rightarrow f(x) \leq 0 \Big\}; \tag{6.3}$$

(iv) The ***$\mathcal{A}$-p-quasiconvex hull*** *$K^{(p)}$ as*

$$K^{(p)} := \Big\{ x \in \mathbb{R}^d\colon \mathcal{Q}_{\mathcal{A}} \operatorname{dist}^p(x,K) = 0 \Big\}. \tag{6.4}$$

The space $K^{(\infty)}$ can be seen as the natural limiting space of $K^{(p)*}$ as $p \to \infty$. One crucial observation is that we do not need to distinguish between $K^{(p)*}$ and $K^{(p)}$ due to the following result (cf. [133] for the case $p = 2$ and [154] and Lemma 5.39).

Lemma 6.2. *Suppose that the distance function* $\operatorname{dist}^p(\cdot, K)$ *is $\mathcal{A}$-integral coercive* (4.11), *i.e.*

$$\int_{T_N} \operatorname{dist}^p(v + \psi(y), K)\,\mathrm{d}y \geq C_1 \int_{T_N} |\psi(y)|^p\,\mathrm{d}y - C_2(1+|v|^p).$$

Then the sets $K^{(p)}$ and $K^{(p)*}$ coincide.

The assumption that $\operatorname{dist}^p(\cdot, K)$ is coercive plays a huge role in the further analysis of $\mathcal{A}$-quasiconvex hulls. It is clearly satisfied whenever K is a compact set. For unbounded sets we have seen examples in Section 1.3.4 and Section 5.6 in a (p,q)-setting. A further treatment of this case follows in Section 6.3.

Moreover, let us show that the choice of the distance function plays absolutely no role in the $\mathcal{A}$-quasiconvex set, even if dist^p is not $\mathcal{A}$-integral coercive (cf. Proposition 1.17).

Lemma 6.3 (Non-dependence on the distance function). *Suppose that $K \subset \mathbb{R}^d$ is a nonempty, closed set and let $f\colon \mathbb{R}^d \to [0,\infty)$. Let $\omega_1, \omega_2 \in C([0,\infty))$ be two monotonically increasing moduli of continuity for f that satisfy*

(a) $\omega_1(0) = \omega_2(0) = 0$;

(b) $\omega_1(t), \omega_2(t) > 0$, whenever $t > 0$;

(c) $c_1 t^p \leq \omega_1(t) \leq \omega_2(t) \leq c_2 t^p$ for $t > 1$.

Suppose now that $f(v) = 0$ *if and only if* $v \in K$ *and*

$$\omega_1(\mathrm{dist}(v, K)) \leq f(v) \leq \omega_2(\mathrm{dist}(v, K)). \tag{6.5}$$

Then

$$\{\mathcal{Q}_{\mathcal{A}} f = 0\} = \{\mathcal{Q}_{\mathcal{A}} \,\mathrm{dist}^p(\cdot, K) = 0\} = K^{(p)}. \tag{6.6}$$

Observe that if K is a compact set, then we may reduce (6.5) to

$$c_1|v|^p - c_0 \leq f(v) \leq c_2(1 + |v|^p), \quad f(v) = 0 \Leftrightarrow v \in K.$$

In particular, this shows that $K^{(p)}$ does *not* depend on the distance function $\mathrm{dist}^p(\cdot, K)$ and the underlying metric $|\cdot|$.

Proof. Suppose that $v \in K^{(p)}$. Then there is a sequence $u_n \subset \mathcal{T}_{\mathcal{A}}$ such that

$$\lim_{n\to\infty} \int_{T_N} \mathrm{dist}^p(v + u_n(x), K) \,\mathrm{d}x = 0.$$

Subdivide K into two regions:

$$E_n = \{x \in T_N \colon \mathrm{dist}^p(v, u_n(x)) \leq 1\} \text{ and } E_n^C = \{x \in T_N \colon \mathrm{dist}^p(v, u_n(x)) > 1\}.$$

Then, using the moduli of continuity, we get that $1_{E_N} f(v + u_n(x)) \to 0$ in measure and $1_{E_N}|f(v + u_n(x))| \leq \omega_2(1)$. Therefore,

$$\lim_{n\to\infty} \int_{E_n} f(v + u_n(x)) \,\mathrm{d}x = 0. \tag{6.7}$$

On the other hand, on E_n^C we have $f(x, u_n(x)) \leq c_2 \,\mathrm{dist}^p(v, K)$, hence

$$\lim_{n\to\infty} \int_{E_n^C} f(v + u_n(x)) \,\mathrm{d}x = 0. \tag{6.8}$$

Summarising (6.7) and (6.8), we have $v \in \{\mathcal{Q}_{\mathcal{A}} f = 0\}$ and, thus, $K^{(p)} \subset \{\mathcal{Q}_{\mathcal{A}} f = 0\}$.

The same argumentation with the roles of dist^p and f exchanged, shows $\{\mathcal{Q}_{\mathcal{A}} f = 0\} \subset K^{(p)}$.

□

Before continuing with the analysis of hulls of compact sets, let us shortly give a nesting result, which is very helpful for computations of $\mathcal{A}$-quasiconvex sets in specific settings, but is irrelevant for the general approach outlined in Section 6.2, cf. [115, 159]).

Definition 6.4 (Various convex hulls)**.** *Let* $K \subset \mathbb{R}^d$ *be a closed set and let* $\Lambda = \Lambda_{\mathcal{A}}$ *be the characteristic cone of* $\mathcal{A}$*. We define the following hulls:*

(a) *The* ***convex hull*** K^{**} *is given by*

$$K^{**} = \left\{ \sum_{i=0}^{I} \lambda_i x_i \colon x_i \in K,\ I \in \mathbb{N},\ \lambda_i \in [0,1],\ \sum_{i=1}^{I} \lambda_i = 1 \right\}.$$

(b) *The* ***set theoretic*** Λ***-convex*** *hull is defined as*

$$K_\Lambda^{\mathrm{set}} = \overline{\bigcup_{i \in \mathbb{N}} K_\Lambda^i},$$

where K_Λ^i *is inductively defined by* $K_\Lambda^0 = K$ *and*

$$K_\Lambda^{i+1} = \left\{ \lambda x + (1-\lambda) y \colon x, y \in K_\Lambda^i,\ \lambda \in [0,1],\ x - y \in \Lambda \right\}.$$

(c) *The* ***function theoretic*** Λ***-convex hull*** *is defined as*

$$K_\Lambda^{\mathrm{funct}} = \{x \in \mathbb{R}^d \colon f\Lambda\text{-convex and } f_{|K} \leq 0 \Rightarrow f(x) \leq 0\}.$$

(d) *The* $\mathcal{A}$***-polyconvex hull*** *is defined via*

$$K_{\mathcal{A}}^{\mathrm{pc}} = \{x \in \mathbb{R}^d \colon f\mathcal{A}\text{-polyconvex and } f_{|K} \leq 0 \Rightarrow f(x) \leq 0\}.$$

Proposition 6.5 (Relation between the convex hulls)**.** *Let* $K \subset \mathbb{R}^d$ *be a closed set and* $1 \leq p \leq q \leq \infty$*. Then*

$$K \subset K_\Lambda^{\mathrm{set}} \subset K_\Lambda^{\mathrm{funct}} \subset K^{(q)} \subset K^{(p)} \subset K_{\mathcal{A}}^{\mathrm{pc}} \subset K^{**}. \tag{6.9}$$

The same nesting holds if we replace $K^{(p)}$ *and* $K^{(q)}$ *by* $K^{(p)*}$ *and* $K^{(q)*}$*, respectively.*

Most of the nestings follow from the fact that the spaces of separating functions gets smaller. In particular, there are more Λ-convex function than $\mathcal{A}$-quasiconvex functions, more $\mathcal{A}$-quasiconvex functions than polyconvex functions and more polyconvex than convex functions. To show that the inclusions are strict, i.e. the hulls are not the same, we refer to [115]. It is worthwile mentioning that $K_\Lambda^{\mathrm{set}} \neq K_\Lambda^{\mathrm{funct}}$ relies on the four-gradient example, i.e. K consists of four points and $\mathcal{A} = \mathrm{curl}$ (cf. [13, 30, 142, 22]).

6.2. $\mathcal{A}$-quasiconvex hulls of compact sets

First, assume that $K \subset \mathbb{R}^d$ is a compact set. Note that for any $1 \leq p < \infty$ the distance function is classically coercive, i.e.

$$\mathrm{dist}^p(v, K) \geq |v|^p - C$$

for some appropriate $C > 0$. Furthermore, it is important to mention that in such a setting, from a viewpoint of applying the *Direct Method*, including $p = 1$ and $p = \infty$ is reasonable. First of all, the set K and also K^{**} is compact and therefore, any function satisfying $u \in K^{(\infty)}$ automatically is in L^∞.

For the setting $p = 1$ note that if a sequence u_n satisfies $\int_{T_N} \operatorname{dist}(u_n, K)\,\mathrm{d}x \to 0$, then the sequence u_n is equi-integrable, i.e.

$$\lim_{\varepsilon \to 0} \sup_{n \in \mathbb{N}} \sup_{|E| < \varepsilon} \int_E |u_n|\,\mathrm{d}x = 0.$$

Consequently, due to the Dunford-Pettis theorem [23, Thm. 4.7.18], the sequence u_n has a weakly convergent subsequence and we can use the Direct Method, even though L^1 is not reflexive. We conclude that there is a subsequence with $u_{n_k} \rightharpoonup u^*$ and that $u^* \in \{\mathcal{Q}_\mathcal{A} \operatorname{dist}(\cdot, K) = 0\}$ almost everywhere. In the following, we try to answer the following question:

Question 6.6. *How does $K^{(p)}$ depend on p?*

6.2.1. The regime $1 < p < \infty$

Up to minor changes, this subsection coincides with Lemma 5.2 and its proof in [20].

In $1 < p < \infty$ we can use results about Fourier multipliers (as previously obtained in [42]). A modification of their argument and a detailed proof of the following result is as follows, cf. [20].

Theorem 6.7. *Let $\mathcal{A}$ be a constant rank operator, $K \subset \mathbb{R}^d$ be a compact set. Then for all $1 < p < q < \infty$*

$$K^{(p)} = K^{(q)}.$$

As mentioned, the proof relies on the Fourier multiplier result Theorem 2.9 and therefore it shall not work in the setting $p = 1$ and $q = \infty$. We need a more subtle method for this case.

Proof of Theorem 6.7. Let $K \subset B_R(0) \subset \mathbb{R}^d$ and $y \in B_R(0)$.
$\mathbf{K^{(q)}} \subset \mathbf{K^{(p)}}$**:** Write $f_p = \operatorname{dist}^p(\cdot, K)$ and, likewise, $f_q = \operatorname{dist}^q(\cdot, K)$. Let $y \in K^{(q)}$ and let $u_n \subset \mathcal{T}_\mathcal{A}$ be a sequence of test functions such that

$$0 = \mathcal{Q}_\mathcal{A} f_q(y) = \lim_{n \to \infty} \int_{T_N} f_q(y + u_n(x))\,\mathrm{d}x.$$

As K is compact, u_n is bounded in $L^q(T_N, \mathbb{R}^d)$ and, as $q > p$, also bounded in $L^p(T_N, \mathbb{R}^d)$. Also note that for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that $f_p \le \varepsilon + C_\varepsilon f_q$. Therefore,

$$\mathcal{Q}_\mathcal{A} f_p(y) \le \lim_{n \to \infty} \int_{T_N} f_p(y + u_n(x))\,\mathrm{d}x \le \lim_{n \to \infty} \int_{T_N} \varepsilon + C_\varepsilon f_q(y + u_n(x))\,\mathrm{d}x \le \varepsilon.$$

Thus, $y \in K^{(p)}$.

$\mathbf{K^{(p)}} \subset \mathbf{K^{(q)}}$**:** This direction uses a similar, yet easier truncation statement than Theorem 6.14 below. Let $y \in K^{(p)}$ and let $u_n \subset \mathcal{T}_{\mathcal{A}}$ be a test sequence, such that

$$0 = \mathcal{Q}_{\mathcal{A}} f_p(y) = \lim_{n \to \infty} \int_{T_N} f_p(y + u_n(x)) \, \mathrm{d}x.$$

Note that u_n is uniformly bounded in $L^p(T_N, \mathbb{R}^d)$ and that

$$\lim_{n \to \infty} \int_{T_N} \mathrm{dist}^p(u_n(x), B_{2R}(0)) \, \mathrm{d}x = 0.$$

Write

$$\widetilde{u}_n = 1_{\{|u_n| \leq 2R\}} u_n - \fint_{T_N} 1_{\{|u_n| \leq 2R\}}(x) u_n(x) \, \mathrm{d}x$$

and define $v_n := P_{\mathcal{A}} \widetilde{u}_n$ with the projection operator $P_{\mathcal{A}}$ onto the kernel of $\mathcal{A}$ from Theorem 2.9. Observe that

1. $\mathcal{A} v_n = 0$;

2. $(\widetilde{u}_n)$ is bounded in $L^\infty(T_N, \mathbb{R}^d)$ and q-equi-integrable. Since $1 < q < \infty$, the projection $P_{\mathcal{A}} \colon L^q(T_N, \mathbb{R}^d) \to L^q(T_N, \mathbb{R}^d)$ is bounded, v_n is bounded in $L^q(T_N, \mathbb{R}^d)$, q-equi-integrable by Theorem 2.9. Moreover, by Theorem 2.9 and $1 < p < \infty$,

$$\begin{aligned} \|u_n - v_n\|_{L^p(T_N} &\leq \|u_n - \widetilde{u}_n\|_{L^p(T_N)} + \|\widetilde{u}_n - v_n\|_{L^p(T_N)} \\ &\leq \|u_n - \widetilde{u}_n\|_{L^p(T_N)} + C_{\mathcal{A},p} \|\mathcal{A}(\widetilde{u}_n - u_n)\|_{W^{-k,p}(T_N)} \\ &\leq C_{\mathcal{A},p} \|u_n - \widetilde{u}_n\|_{L^p(T_N)} \to 0. \end{aligned}$$

Hence, also

$$\lim_{n \to \infty} \int_{T_N} f_p(y + v_n(x)) \, \mathrm{d}x = 0.$$

We conclude that $f_q(y + v_n) \to 0$ in measure. Combining this with the L^q-boundedness and q-equi-integrability, we obtain

$$\lim_{n \to \infty} \int_{T_N} f_q(y + v_n(x)) \, \mathrm{d}x = 0.$$

Therefore, $y \in K^{(q)}$, concluding the proof.

□

6.2.2. The case $p = 1$ and $q = \infty$: Overview

Our goal is to prove that if $\mathcal{A}$ is a constant rank operator (in $\mathbb{R}$ or in $\mathbb{C}$), that then $K^{(1)} = K^{(\infty)}$. Zhang showed in the 90's that this is true in the setting $\mathcal{A} = \mathrm{curl}$:

Proposition 6.8 ([158])**.** *Let $K \subset \mathbb{R}^{N\times m}$ be a compact set and $\mathcal{A} = \operatorname{curl}$. Then $K^{(1)} = K^{(\infty)}$.*

The goal of Section A is to show that the statement of Proposition 6.8 is true for a wider class of operators, namely differential forms:

Proposition 6.9 ([134])**.** *Let $K \subset \mathbb{R}^m \times (\mathbb{R}^N \wedge ... \wedge \mathbb{R}^N)$ be compact and $\mathcal{A} = d$ be the componentwisely taken outer/Cartan derivative of a k-form. Then $K^{(1)} = K^{(\infty)}$.*

This result in particular applies to $\mathcal{A} = \operatorname{div}$ on $\mathbb{R}^{N\times m}$ matrices. Another variant of this statement is shown in Section B, which is based on [20].

Proposition 6.10 ([20])**.** *Let $K \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ be compact and let $\mathcal{A}$ be the componentwisely taken divergence (the symmetric divergence). Then $K^{(1)} = K^{(\infty)}$.*

6.2.3. L^∞-truncations

Let us shortly discuss the main technique to prove all these theorems. Zhang's proof of Proposition 6.8 is based on the following truncation theorem [1, 2].

Proposition 6.11 (Lipschitz truncation)**.** *Let $u \in W^{1,1}(T_N, \mathbb{R}^m)$ and $L > 0$. Then there is $\bar{u} \in W^{1,\infty}(T_N, \mathbb{R}^m)$ with*

(a) $\|\bar{u}\|_{W^{1,\infty}} \leq CL$;

(b) $\|u - \bar{u}\|_{W^{1,1}} \leq C \int_{\{|u|+|Du|>L\}} |u| + |Du| \,\mathrm{d}x.$

Let us rewrite Proposition 6.11 in terms of $v = \operatorname{curl} u$ to get an appropriate version we try to prove for $\mathcal{A} = d$ and $\mathcal{A} = \operatorname{div}$:

Proposition 6.12 (Lipschitz truncation rewritten as curl-free truncation)**.**
Let $v \in L^1(T_N, \mathbb{R}^{m\times N})$ and $L > 0$. Suppose that $\operatorname{curl} v = 0$ in the sense of distributions. Then there is $\bar{v} \in L^\infty(T_N, \mathbb{R}^{m\times N})$, such that

(a) $\|\bar{v}\|_{L^\infty} \leq CL$,

(b) $\|\bar{v} - v\|_{L^1} \leq C \int_{\{|v|\geq L\}} |v| \,\mathrm{d}x$,

(c) $\operatorname{curl} \bar{v} = 0$.

6.2.4. L^∞-truncation implies $K^{(1)} = K^{(\infty)}$

This subsection is taken from [134], Section 6.1.

In fact, we can show that a truncation theorem à la Proposition 6.12 implies the validity of Propositions 6.9 and 6.10 . Hence, the main task of Sections A and B is to derive a truncation theorem in the style of Proposition 6.12 with curl replaced by the operators $\mathcal{A} = d$ and $\mathcal{A} = \operatorname{div}$, respectively. That motivates the following definition.

Definition 6.13. *We say that $\mathcal{A}$ satisfies the property (ZL) if for all sequences $u_n \subset L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that there exists an $L > 0$ with*

$$\int_{\{y \in T_N :\, |u_n(y)| > L\}} |u_n(y)| \,\mathrm{d}y \longrightarrow 0 \quad \text{as } n \to \infty,$$

there exists a $C = C(\mathcal{A})$ and a sequence $v_n \subset L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

i) $\|v_n\|_{L^\infty(T_N, \mathbb{R}^d)} \leq C_1 L$;

ii) $\|v_n - u_n\|_{L^1(T_N, \mathbb{R}^d)} \to 0$ *as* $n \to \infty$.

For a compact set K we define the set $K^{\mathcal{A}app}$ (cf. [115]) as the set of all $x \in \mathbb{R}^d$ such that there exists a bounded sequence $u_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with

$$\operatorname{dist}(x + u_n, K) \longrightarrow 0 \quad \text{in measure, as } n \to \infty.$$

Theorem 6.14. *Suppose that K is compact and $\mathcal{A}$ is an operator satisfying (ZL). Then*

$$K^{\mathcal{A}app} = K^{(\infty)} = \left\{ x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = 0 \right\}. \tag{6.10}$$

Proof. We first prove $K^{\mathcal{A}app} \subset K^{(\infty)}$. Let $x \in K^{\mathcal{A}app}$ and take an arbitrary $\mathcal{A}$-quasiconvex function $f : \mathbb{R}^d \to [0, \infty)$ with $f_{|K} = 0$. We claim that then $f(x) = 0$.

Take a sequence u_n from the definition of $K^{\mathcal{A}app}$. As f is continuous and hence locally bounded, $f(x + u_n) \to 0$ in measure and $0 \leq f(x + u_n) \leq C$. Quasiconvexity and dominated convergence yield

$$f(x) \leq \liminf_{n \to \infty} \int_{T_N} f(x + u_n(y)) \,\mathrm{d}y = 0.$$

$K^{(\infty)} \subset \left\{ x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = 0 \right\}$ is clear by definition, as $\mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))$ is an admissible separating function.

The proof of the inclusion $\{x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = 0\} \subset K^{\mathcal{A}app}$ uses (ZL). If $\mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K)) = 0$, then there exists a sequence $\varphi_n \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\int_{T_N} \varphi_n = 0$ such that

$$0 = \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = \lim_{n \to \infty} \int_{T_N} \operatorname{dist}(x + \varphi_n(y), K) \,\mathrm{d}y.$$

As K is compact, there exists $R > 0$ such that $K \subset B(0, R)$. Moreover, as $x \in K^{(\infty)}$, also $x \in B(0, R)$. This implies that

$$\lim_{n \to \infty} \int_{T_N \cap \{|\varphi_n| \geq 6R\}} |\varphi_n| \,\mathrm{d}y = 0.$$

We may apply (ZL) and find a sequence $\psi_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$\|\varphi_n - \psi_n\|_{L^1(T_N, \mathbb{R}^d)} \longrightarrow 0 \quad \text{as } n \to \infty$$

and

$$\|\psi_n\|_{L^\infty(T_N,\mathbb{R}^d)} \leq CR.$$

Hence, $x \in K^{\mathcal{A}app}$. □

Remark 6.15. Theorem 6.14 shows that for all $1 \leq p < \infty$

$$K^{\mathcal{A}app} = K^{(\infty)} = \Big\{x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\mathrm{dist}(\cdot, K)^p)(x) = 0\Big\} = K^{(p)}.$$

This follows directly, as all the sets $K^{(p)}$ are nested and, conversely, all the hulls of the distance functions are admissible f in the definition of $K^{(\infty)}$.

Another application of the property (ZL) in the context of Young measures is pointed out in Section A.6.

6.3. $\mathcal{A}$-quasiconvex hulls of non-compact sets

If K is non-compact the situation, may change drastically. Recall that one of the main motivations to study $\mathcal{A}$-quasiconvex hulls was to study the minimisation problem

$$\text{minimise} \quad I(u) = \begin{cases} \displaystyle\int_\Omega \mathrm{dist}^p(u(x), K)\,\mathrm{d}x & \text{if } \mathcal{A}u = 0, \\ \infty & \text{else.} \end{cases}$$

To guarantee existence of minimisers, we need to have some coercivity condition on the distance function. This coercivity is clear in the case of compact sets. For unbounded K we need to assume that for all $v \in \mathbb{R}^d$ and all $\psi \in \mathcal{T}_{\mathcal{A}}$ we have

$$\int_{T_N} \mathrm{dist}^p(v + \psi(x), K)\,\mathrm{d}x \geq c_1 \int_{T_N} |\psi|^p\,\mathrm{d}x - c_2(|v|^p + 1) \tag{6.11}$$

The distance function might be integral coercive for a certain range of $p \in (1, \infty)$, but not for all p. Consequently, we can only expect that $K^{(p)} = K^{(q)}$ for some, but not all pairs $(p, q) \in (1, \infty)^2$. This intuition is highlighted by the following statement [116, 152].

Proposition 6.16. *Let $N \in \mathbb{N}$ be even, $\mathcal{A} = \mathrm{curl}$ acting on $N \times N$ matrices. Let K be the the set of conformal matrices, i.e.*

$$K = \mathbb{R}_+\mathrm{SO}(N) = \{\lambda A \colon \lambda \in [0, \infty),\ A \in \mathrm{SO}(N)\},.$$

Then:

$$K^{(p)} = \begin{cases} K & \text{if } p \geq N/2, \\ \mathbb{R}^{N \times N} & \text{if } p < N/2. \end{cases} \tag{6.12}$$

If N is odd, then $K^{(p)} = K$ for some $p \in (N - \varepsilon, \infty)$, cf. [155, 153], the optimal value for ϵ is still not known. In the following, we show that, under certain circumstances, similar statements are possible, i.e. that $K^{(p)} = K^{(q)}$ for a certain range of p, q.

6.3.1. A geometrically linear example

Consider two differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$ acting both on $C^\infty(\mathbb{R}^N, \mathbb{R}^d)$.

Lemma 6.17. *Let $\mathcal{A}_1, \mathcal{A}_2$ be two such differential operators. The following are equivalent:*

1. *For all $\xi \in \mathbb{R}^N \setminus \{0\}$ we have $\ker \mathbb{A}_1(\xi) = (\ker \mathbb{A}_2(\xi))^\perp$;*

2. *$\mathcal{A}_1^*$ is a potential of $\mathcal{A}_2$;*

3. *$\mathcal{A}_2^*$ is a potential of $\mathcal{A}_1$.*

These statements follow from the algebraic identity $\ker \mathbb{A}_1(\xi) = (\operatorname{Im} \mathbb{A}_1^*(\xi))^\perp$.

Therefore, if $\mathcal{B}$ is a potential of $\mathcal{A} = \mathcal{A}_1$, let us write such a pair of operators as $(\mathcal{A}_1, \mathcal{A}_2) = (\mathcal{A}, \mathcal{B}^*)$. We write $u \in L^p(\mathbb{R}^N, \mathbb{R}^d \times \mathbb{R}^d)$ as $u = (u_1, u_2)$ with $u_i \in L^p(\mathbb{R}^N, \mathbb{R}^d)$. In this work, we have already seen multiple examples of operators, which are exactly of the form $(\mathcal{A}, \mathcal{B}^*)$

Example 6.18. (a) Let $u_1, u_2 \in L^p(\mathbb{R}^N, \mathbb{R}^{m\times N})$ and $\mathcal{A} = \operatorname{curl}$, $\mathcal{B}^* = \operatorname{div}$ taken column-wise. Note that $\mathcal{B} = \nabla$, which is the potential of curl.

(b) Consider $u_1, u_2 \in L^p(\mathbb{R}^N, \mathbb{R}^{N\times N}_{\mathrm{sym}})$, $\mathcal{A} = \operatorname{curl}\operatorname{curl}^T$ and $\mathcal{B}^* = \operatorname{div}$ acting column-wise. Then $\mathcal{B}$ is the symmetric gradient $\left(\dfrac{\nabla + \nabla^T}{2}\right)$, the potential of $\operatorname{curl}\operatorname{curl}^T$.

(c) Recall the operators from Chapter 5, i.e. $u_1, u_2 \in L^p(\mathbb{R}^N, Y)$ for $Y = \{A \in \mathbb{R}^{N\times N}_{\mathrm{sym}} \colon \operatorname{tr}(A) = 0\}$. Let $\mathcal{A}_1 = \operatorname{curl}\operatorname{curl}^T$ and $\mathcal{A}_2(u_2, \nabla\pi) = \operatorname{div} u_2 + \nabla\pi$ for $\pi \in L^p(\mathbb{R}^N, \mathbb{R})$. This setting can be treated like (b). The additional condition that u_1 has trace 0 is 'compensated' by the fact the condition $\mathcal{A}_2 = \operatorname{div} u_2 + \nabla\pi$ is weaker (cf. Remark 5.7).

(d) Let $u \in L^p(\mathbb{R}^2, \mathbb{R}^{2\times 2})$ and $\mathcal{A} = \operatorname{curl}$. We may identify a matrix A via the map $A \mapsto T(A)$, where

$$T\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ -d & c \end{pmatrix} = (T_1(A)\ T_2(A)).$$

Then $\operatorname{curl}(u) = \operatorname{curl}(T_1(A)), \operatorname{div} T_2(A))$ and we recover (a).

Recall that if $\mathcal{B}$ is a potential of $\mathcal{A}$, so is $\mathcal{B} \circ \operatorname{div}$; likewise if $\mathcal{A}$ is an annihilator of $\mathcal{B}$, then also $\nabla \circ \mathcal{A}$ is an annihilator. Hence, we may suppose that the order of $\mathcal{A}$ and the order of $\mathcal{B}$ coincide.

Note that for such operators the map

$$(u_1, u_2) \mapsto u_1 \cdot u_2$$

is $(\mathcal{A}, \mathcal{B}^*)$-quasiaffine. In the following, we consider sets obeying a growth condition of the form

$$\operatorname{dist}^2(u, K) \geq C(|u_1|^2 + |u_2|^2) - C(1 + u_1 \cdot u_2).$$

We define the set L to be the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$, i.e.

$$L = \{(u, u) \colon u \in \mathbb{R}^d\} \subset \mathbb{R}^d \times \mathbb{R}^d. \tag{6.13}$$

Note that this subset L strictly obeys the growth condition

$$\operatorname{dist}^2(u, L) = 1/2|u_1|^2 + 1/2|u_2|^2 - u_1 \cdot u_2.$$

We now study sets K which are close to L such that their distance functions satisfy a similar growth condition.

6.3.2. Sets in a ball around L

Let us assume that $\mathcal{A}$ and $\mathcal{B}$ are two differential operators of order k. Furthermore, let $\mathcal{B}$ be a potential of $\mathcal{A}$. For this subsection, we suppose that the set K obeys the following two hypotheses:

(H1) the set K is close to L, i.e. there is $R_1 > 0$, such that for all $z \in K$ we have

$$\operatorname{dist}(z, L) \leq R_1;$$

(H2) the set L is close to K, i.e. there is $R_2 > 0$, such that for all $y \in L$

$$\operatorname{dist}(z, K) \leq R_2.$$

In other words, (H1) and (H2) ensure that $K \subset B_R(L)$ and $L \subset B_R(K)$.

We use results from Fourier analysis, hence the following argument is crucial. Let us rewrite

$$u = (u_1, u_2) = (v + w, v - w), \quad v, w \in L^p(T_N, \mathbb{R}^d). \tag{6.14}$$

Note that, up to constants, v uniquely determines w, and vice versa, and the following holds:

Lemma 6.19. *Let $(\mathcal{A}, \mathcal{B}^*)$ be a differential operator of order k. Then:*

(a) There are constants $c, C > 0$ such that, for all $v \in L^p(T_N, \mathbb{R}^d)$ satisfying $\int_{T_N} v = 0$, we have

$$c\|(\mathcal{A}, \mathcal{B}^*)(v, v)\|_{W^{-k,p}} \leq \|v\|_{L^p} \leq C\|(\mathcal{A}, \mathcal{B}^*)(v, v)\|_{W^{-k,p}}; \tag{6.15}$$

(b) There are constants $c, C > 0$ such that, for all $w \in L^p(T_N, \mathbb{R}^d)$ satisfying $\int_{T_N} w = 0$, we have

$$c\|(\mathcal{A}, \mathcal{B}^*)(w, -w)\|_{W^{-k,p}} \leq \|w\|_{L^p} \leq C\|(\mathcal{A}, \mathcal{B}^*)(w, -w)\|_{W^{-k,p}}; \tag{6.16}$$

(c) There is a linear, continuous map $\mathbb{M} \colon L^p(T_N, \mathbb{R}^d) \to L^p(T_N, \mathbb{R}^d)$, for all $1 < p < \infty$, such that

(i) $\displaystyle\int_{T_N} \mathbb{M}w = 0$;

(ii) $(\mathcal{A}, \mathcal{B}^*)(\mathbb{M}w + w, \mathbb{M}w - w) = 0$.

Proof. The key insight is that we can write

$$v(x) = \sum_{\lambda \in \mathbb{Z}^N} \hat{v}(\lambda) e^{-2\pi i \lambda \cdot x} = \hat{v}(0) + \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \left(P_{\ker \mathbb{A}(\lambda)} \hat{v}(\lambda) + P_{\ker \mathbb{B}^*(\lambda)} \hat{v}(\lambda) \right) e^{-2\pi i \lambda \cdot x}.$$

where P_V is the orthogonal projection onto a vector space $V \subset \mathbb{R}^d$. Recall (cf. Theorem 2.9) that both

$$\mathbb{P}_1 \colon v \mapsto \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \left(P_{\ker \mathbb{A}(\lambda)} \hat{v}(\lambda) \right) e^{-2\pi i \lambda \cdot x},$$
$$\mathbb{P}_2 \colon v \mapsto \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \left(P_{\ker \mathbb{B}^*(\lambda)} \hat{v}(\lambda) \right) e^{-2\pi i \lambda \cdot x}$$

are Fourier multipliers. Note that $\mathcal{B}^* v = \mathcal{B}^* \mathbb{P}_1 v$ and $\mathcal{A}^* \mathbb{P}_1 v = 0$. As the operator $(A, \mathcal{B}^*)(v, v)$ is elliptic, we get

$$c \|\mathbb{P}_1 v\|_{L^p} \leq \|\mathcal{B}^* \mathbb{P}_1 v\|_{W^{-k,p}} \leq C \|\mathbb{P}_1 v\|_{L^p}.$$

A similar estimate for $\mathbb{P}_2$ establishes (6.15). The same argument for $(w, -w)$ instead of (v, v) gives (6.16). For (c) just use the map

$$\mathbb{M} \colon w \mapsto \mathbb{P}_1 w - \mathbb{P}_2 w$$

which is a L^p-Fourier multiplier for all $1 < p < \infty$ and satisfies the assertions of (c). □

Corollary 6.20. *The distance function* $\operatorname{dist}^p(\cdot, L)$ *and* $\operatorname{dist}^p(\cdot, B_R(L))$ *are* $\mathcal{A}$*-integral coercive.*

Proof. Let $u = (v_0 + v + w_0 + w, v_0 + v - w_0 - w)$ for $v_0, w_0 \in \mathbb{R}^d$ and $v, w \in L^p(T_N, \mathbb{R}^d)$ with average 0 satisfy $(A, \mathcal{B}^*)u = 0$. Note that $\operatorname{dist}^2(u, L) = 2|w_0 + w|^2$ pointwisely. Therefore, using $(\mathcal{A}, \mathcal{B}^*)u = 0$ and the estimates (6.15) and (6.16), we obtain

$$\begin{aligned}
\int_{T_N} \operatorname{dist}^p(u, L) \,\mathrm{d}x &= C \int_{T_N} |w_0 + w|^p \,\mathrm{d}x \\
&\geq C_1 \left(\int_{T_N} |w|^p \,\mathrm{d}x - |w_0|^p \right) \\
&\geq C_2 \left(\|(A, \mathcal{B}^*)(w, -w)\|^p_{W^{-k,p}} + \int_{T_N} |w|^p \,\mathrm{d}x - |w_0|^p \right) \\
&= C_2 \left(\|(A, \mathcal{B}^*)(v, v)\|^p_{W^{-k,p}} + \int_{T_N} |w|^p \,\mathrm{d}x - |w_0|^p \right) \\
&\geq C_3 \left(\|v\|^p_{L^p} + \|w\|^p_{L^p} - |w_0|^p \right).
\end{aligned}$$

This shows coercivity of the distance function to L. The result for $\operatorname{dist}^p(\cdot, B_R(L))$ follows by this and the triangle inequality

$$\operatorname{dist}^p(u, B_R(L)) \geq 2^{-p} \operatorname{dist}^p(u, L) - R^p.$$

□

Using these Fourier arguments, we are now able two prove the following weak truncation argument.[1]

Lemma 6.21. *Suppose that $u_n \in L^p(T_N, \mathbb{R}^d)$ satisfies the differential constraint $\mathcal{A}u_n = 0$ and $\int_{T_N} \operatorname{dist}^p(u_n, B_R(L))\,\mathrm{d}x \to 0$ as $n \to \infty$. Then there exists $\tilde{u}_n \in L^p(T_N, \mathbb{R}^d)$ with the following properties:*

(a) $\|u_n - \tilde{u}_n\|_{L^p} \to 0$;

(b) $(\mathcal{A}, \mathcal{B}^*)\tilde{u}_n = 0$;

(c) $\tilde{u}_n(x) \in B_{2R}(L)$ *almost everywhere.*

Proof. We again use the splitting $u_n = (v_n, v_n) + (w_n, -w_n)$ and that $\operatorname{dist}^2(u_n, L) = 2\|w_n\|^2$. Now note that $\int_{|w_n \geq 2R|} |w_n|^p \to 0$. We define

$$\tilde{w}_n = 1_{|w_n| \leq 2R} w_n.$$

Let $v_n^0 = \int_{T_N} v_n \,\mathrm{d}x$. Define

$$\tilde{u}_n = (v_n^0, v_n^0) + (\mathbb{M}\tilde{w}_n + \tilde{w}_n, \mathbb{M}\tilde{w}_n - \tilde{w}_n).$$

By definition of $\tilde{w}_n$, (c) is satisfied and due to the properties of the map $\mathbb{M}$, $\tilde{u}_n$ obeys (b). It is left to show (a). First of all, note that $w_n - \tilde{w}_n \to 0$ in L^p. We can estimate the remaining difference of $u_n - \tilde{u}_n$ by

$$\begin{aligned}
\|v_n - (v_n^0 + \mathbb{M}\tilde{w}_n)\|_{L^p} &\leq C\|(A, \mathcal{B}^*)((v_n, v_n) - (\mathbb{M}\tilde{w}_n, \mathbb{M}\tilde{w}_n))\|_{W^{-k,p}} \\
&\leq C\| - (\mathcal{A}, \mathcal{B}^*)((w_n, -w_n) + (\tilde{w}_n, -\tilde{w}_n))\|_{W^{-k,p}} \\
&\leq C\|w_n - \tilde{w}_n\|_{L^p} \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, (a) is satisfied. □

Theorem 6.22. *Let K satisfy the hypotheses (H1) and (H2). Then, for all $1 < p, q < \infty$,*

$$K^{(p)} = K^{(q)}$$

with respect to the operator $(\mathcal{A}, \mathcal{B}^*)$.

[1] This is quite similar to the argument we use for the case $1 < p < \infty$ in Theorem 6.7, [20] and is also related to the stament we prove in the compact setting for $p = 1, \infty$ [134, 20].

The validity of this theorem follows directly from the following lemma:

Lemma 6.23. *Let us define* $K_{3R_1}^{(\infty)}$ *as*

$$K_{3R_1}^{(\infty)} = \Big\{x \in \mathbb{R}^d \colon \forall f \in C(\mathbb{R}) \text{ with } f_{|B_{3R_1}(L)} \text{ uniformly continous and } f_{|K} \leq 0, \text{ we have } \mathcal{Q}_{\mathcal{A}} f(x) \leq 0\Big\}.$$

If $K \subset B_{R_1}(L)$ *and* $L \subset B_{R_2}(K)$*, then, for any* $1 < p < \infty$*,*

$$K^{(p)} = K_{3R_1}^{(\infty)}.$$

Proof. First, we prove that $K_{3R_1}^{(\infty)} \subset K^{(p)}$. For this we only need to verify that $\operatorname{dist}^p(\cdot, K)$ is uniformly continuous on $B_{3R_1}(L)$. But a distance function is uniformly continuous on a set whenever it is bounded; by the triangle inequality and (H1) and (H2) we indeed have

$$\operatorname{dist}(z, K) \leq 3R_1 + R_2$$

for all $z \in B_{3R_1}(L)$. Hence, $\operatorname{dist}^p(\cdot, K)$ is bounded and therefore uniformly continuous.

For $K^{(p)} \subset K_{3R_1}^{(\infty)}$ let $(v_0 + w_0, v_0 - w_0) \in K^{(p)}$. As $K \subset B_{R_1}(L)$ and the latter set is convex, $K^{(p)} \subset B_{R_1}(L)$, therefore $|w_0|^2 \leq 2R_1^2$. Take a sequence (v_n, w_n) in L^p with zero average satisfying $(\mathcal{A}, \mathcal{B}^*)(v_n + w_n, v_n - w_n) = 0$ and

$$\lim_{n\to\infty} \int_{T_N} \operatorname{dist}^p\big((v_0 + v_n + w_0 + w_n, v_0 + v_n - w_0 - w_n), K\big)\,\mathrm{d}x = 0.$$

By the previous lemma 6.21, we can find $\tilde{v}_n, \tilde{w}_n$ with average 0 still satisfying the differential constraint, such that $\|\tilde{w}_n\|_{L^\infty} \leq 2R_1$ and $\|\tilde{v}_n - v_n\|_{L^p} + \|\tilde{w}_n - w_n\|_{L^p} \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \int_{T_N} \operatorname{dist}^p\big((v_0 + \tilde{v}_n + w_0 + \tilde{w}_n, v_0 + \tilde{v}_n - w_0 - \tilde{w}_n), K\big)\,\mathrm{d}x = 0.$$

Defining $\tilde{u}_n = (v_0 + \tilde{v}_n + w_0 + \tilde{w}_n, v_0 + \tilde{v}_n - w_0 - \tilde{w}_n)$ we get that $\operatorname{dist}(\tilde{u}_n, K) \to 0$ in measure and that $\tilde{u}_n \in B_{3R_1}(L)$ almost everywhere. If f is uniformly continuous in $B_{3R_1}(L)$, we conclude by applying the dominated convergence theorem

$$\limsup_{n\to\infty} \int_{T_N} f(\tilde{u}_n)\,\mathrm{d}x = \int_{T_N} \limsup_{n\to\infty} f(\tilde{u}_n)\,\mathrm{d}x \leq 0,$$

as $f_{|K} \leq 0$. This means that $(v_0 + w_0, v_0 - w_0) \in K_{3R_1}^{(\infty)}$. □

6.3.3. A sublinear bound on the distance function

In this section, we suppose that K obeys the following two modifications of (H1) and (H2). Let $0 < \beta < 1$ be fixed. We assume:

(H1') there is $R_1 > 0$ such that for all $z \in K$ we have

$$\operatorname{dist}(z, L) \le R_1(1 + |z|^\beta);$$

(H2') there is $R_2 > 0$ such that for all $z \in L$ we have

$$\operatorname{dist}(z, L) \le R_2(1 + |z|^\beta).$$

Note that the degenerate case $\beta = 0$ coincides with the setting of the previous subsection. In this chapter we prove the following theorem.

Theorem 6.24. *Suppose that K satisfies (H1') and (H2'). Then, for all $1 < p < q < \infty$, the $(\mathcal{A}, \mathcal{B}^*)$-quasiconvex hulls coincide, i.e.*

$$K^{(p)} = K^{(q)}.$$

The proof is split up into the following lemmas. First, we see which sets satisfy the hypotheses (H1'). Then we prove that the distance function to such a set is $\mathcal{A}$-integral coercive. After that, we prove a truncation statement in the spirit of Lemma 6.21. As a first step, we show that this truncation statement is valid for $p < q < p/\beta$ (Lemma 6.27) and then conclude its validity for all q in Corollary 6.28. Finally, the statement of Theorem 6.24 can easily be deduced.

Similar to $B_R(L)$, let us define the set

$$L_{\beta,R} = \Big\{(v + w, v - w) \colon v \in \mathbb{R}^d,\ |w| \le R(1 + |v|^\beta)\Big\}$$

Lemma 6.25 (Which sets satisfy (H1')?)**.** *The set K satisfies the assumption (H1') if and only if $K \subset L_{\beta,R}$ for some appropriate $R \in \mathbb{R}$.*

Proof. If $z \in L_{\beta,R}$, we may write $z = (v + w, v - w)$. Then

$$\operatorname{dist}(z, L) = \sqrt{2}|w| \le R(1 + |v|^\beta) \le R(1 + |z|^\beta).$$

This shows the 'only if' direction. On the other hand, if $z \notin L_{\beta,R}$

$$\operatorname{dist}(z, L) = \sqrt{2}|w|| > CR(1 + |v|^\beta) + |w| > C(R)(1 + (|v| + |w|)^\beta) > C(R)(1 + |z|^\beta)$$

and we conclude that if K satisfies (H1'), it must be in some $L_{\beta,R}$. □

Lemma 6.26 (Coercivity of the distance function)**.** *Suppose that K satisfies (H1'). Let $u_0 \in \mathbb{R}^d \times \mathbb{R}^d$ and $u \in L^p(T_N, \mathbb{R}^d \times \mathbb{R}^d)$ with zero average satisfying $(\mathcal{A}, \mathcal{B}^*)u = 0$. Then*

$$\int_{T_N} \operatorname{dist}^p(u_0 + u, K) \,\mathrm{d}x \ge c \int_{T_N} |u|^p \,\mathrm{d}x - C(1 + |u_0|^p), \tag{6.17}$$

where c, C are constants depending on $(\mathcal{A}, \mathcal{B}^*)$, β and R.

Proof. Given such $u_0 + u \in L^p(T_N, \mathbb{R}^d)$, we can find $\tilde{u} \in L^p(T_N, \mathbb{R}^d)$ with average 0 and $\tilde{u}_0 \in \mathbb{R}^d \times \mathbb{R}^d$ such that

(i) $\|(\tilde{u} + \tilde{u}_0) - (u + u_0)\|_{L^p}^p = \int_{T_N} \operatorname{dist}^p(u_0 + u, K)\, \mathrm{d}x$;

(ii) $\tilde{u} + \tilde{u}_0 \in K$ almost everywhere.

Again, let us write $u = (v, v) + (w, -w)$, $u_0 = (v_0, v_0) + (w_0 - w_0)$ and $\tilde{u} = (\tilde{v}, \tilde{v}) + (\tilde{w}, -\tilde{w})$, $\tilde{u}_0 = (\tilde{v}_0, \tilde{v}_0) + (\tilde{w}_0, -\tilde{w}_0)$. The inequality (6.17) can be viewed as an upper bound on the L^p norm of u depending on u_0 and the distance to K. First of all, note that

$$\|u - \tilde{u}\|_{L^p}^p \leq \int_\Omega \operatorname{dist}^p((u_0 + u(x), K)\, \mathrm{d}x. \tag{6.18}$$

Hence, we continue to estimate the L^p norm of $\tilde{u}$ instead. We bound the L^p-norms of $\tilde{w}$ and $\tilde{v}$ separately. First of all, we can estimate $\tilde{w}$ in terms of $\tilde{w}_0$, $\tilde{v}_0$ and $\tilde{v}$ by using that $K \subset L_{\beta, \tilde{R}}$ for some sufficiently large $\tilde{R}$:

$$\begin{aligned}
\|\tilde{w}\|_{L^p}^p &\leq \frac{1}{C_p} \|\tilde{w} + \tilde{w}_0\|_{L^p}^p - C_p |w_0|^p \\
&\leq \frac{1}{C_p} \|\tilde{R}(1 + |v + v_0|^\beta)\|_{L^p}^p - C_p |w_0|^p \\
&\leq \frac{1}{C(R,p)} \|\tilde{v}\|_{L^p}^{\beta p} - C(R,p)(|\tilde{v}_0|^p + |w_0|^p + 1).
\end{aligned}$$

So it suffices to give a bound for $\tilde{v}$. We use Lemma 6.19, i.e. (6.15) and (6.16), and the estimate on $\|\tilde{w}\|_{L^p}$ in order to obtain

$$\begin{aligned}
\|\tilde{v}\|_{L^p}^p &\leq C_p \|(\mathcal{A}, \mathcal{B}^*(\tilde{v}, \tilde{v})\|_{W^{-k,p}} \\
&\leq C_p \left(\|(\mathcal{A}, \mathcal{B}^*)\tilde{u}\|_{W^{-k,p}}^p + \|(\mathcal{A}, \mathcal{B}^*)(\tilde{w}, -\tilde{w})\|_{W^{-k,p}} \right) \\
&= C_p' \left(\|(\mathcal{A}, \mathcal{B}^*)(u - \tilde{u})\|_{W^{-k,p}}^p + \|(\mathcal{A}, \mathcal{B}^*)(\tilde{w}, -\tilde{w})\|_{W^{-k,p}} \right) \\
&\leq C_p'' \left(\|u - \tilde{u}\|_{L^p}^p + \|\tilde{w}\|_{L^p}^p \right) \\
&\leq C_p'' \left(\int_\Omega \operatorname{dist}^p(u, K)\, \mathrm{d}x + \frac{1}{C(R,p)} \|\tilde{v}\|_{L^p}^\beta - C(R,p)\big(|\tilde{v}_0|^p + |\tilde{w}_0|^p + 1\big) \right).
\end{aligned}$$

Using Bernoulli's inequality for $(\|\tilde{v}\|^p)^\beta$ and substracting this term we get

$$\|\tilde{v}\|_{L^p}^p \leq C_1 \int_\Omega \operatorname{dist}^p(u, K)\, \mathrm{d}x - C_2(1 + |\tilde{v}_0|^p + |\tilde{w}_0|^p).$$

Then employing (6.18), the estimate for $\tilde{w}$ and that $|u_0 - \tilde{u}_0|^p \leq C_p \int \operatorname{dist}(u, K)\, \mathrm{d}x$, we conclude

$$\|u\|_{L^p}^p \leq C_1 \int_\Omega \operatorname{dist}^p(u, K)\, \mathrm{d}x - C_2(1 + |v_0|^p + |w_0|^p).$$

This yields (6.17). □

Lemma 6.27. *Let $u_n' = (u_0 + u_n)$ be a bounded sequence in $L^p(T_N, \mathbb{R}^d \times \mathbb{R}^d)$, such that*

(i) $\displaystyle\int_{T_N} \operatorname{dist}^p(u_n', K)\,\mathrm{d}x \to 0$;

(ii) $u_0 \in \mathbb{R}^d \times \mathbb{R}^d$;

(iii) u_n *has zero average and satisfies the differential condition* $(\mathcal{A}, \mathcal{B}^*)u_n = 0$.

Suppose that K satisfies (H1') and (H2'). Let $p < q < \dfrac{p}{\beta}$. Then there is a sequence $\bar{u}_n \in L^p(T_N, \mathbb{R}^d \times \mathbb{R}^d)$ with zero average satisfying

(a) $\|\bar{u}_n - u_n\|_{L^p} \to 0$ *as* $n \to \infty$;

(b) $\displaystyle\int_{T_N} \operatorname{dist}^q(\bar{u}_n + u_0)\,\mathrm{d}x \to 0$ *as* $n \to \infty$;

(c) $(\mathcal{A}, \mathcal{B}^*)\bar{u}_n = 0$.

Proof. Let $u_n' = u_0 + u_n$. As in the previous proofs, we can find a modified sequence $\tilde{u}_n' = \tilde{u}_{0,n} + \tilde{u}_n$ such that $\tilde{u}_n' \in K$ almost everywhere and $\|u_n' - \tilde{u}_n'\|_{L^p} \to 0$. In particular, both

$$|u_0 - \tilde{u}_{0,n}| \longrightarrow 0, \quad \|u_n - \tilde{u}_n\|_{L^p} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Let us write

$$\tilde{u}_n = (\tilde{v}_n + \tilde{w}_n, \tilde{v}_n - \tilde{w}_n).$$

As in the proof of Lemma 6.21 take the Fourier multiplier $\mathbb{M}$ and define

$$\bar{u}_n := (\mathbb{M}\tilde{w}_n + \tilde{w}_n, \mathbb{M}\tilde{w}_n - \tilde{w}_n). \tag{6.19}$$

By Lemma 6.19 (c), we have $(\mathcal{A}, \mathcal{B}^*)(\bar{u}_n) = 0$ and by the estimate on $\|u_n - \tilde{u}_n\|_{L^p}$ and the fact that $\mathbb{M}$ is a Fourier-multiplier, it follows that

$$\|\bar{u}_n - u_n\|_{L^p} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Hence, the only concern is to prove the estimate on the q-distance function, (b). To this end, we can employ the following two pointwise bounds. First, the distance can be estimated in terms of $\bar{u}_n$ and $\tilde{u}_n'$:

$$\operatorname{dist}^q(u_0 + \bar{u}_n(x), K) \le 2^q |u_0 + \bar{u}_n(x) - \tilde{u}_n'(x)|^q, \tag{6.20}$$

as $\tilde{u}_n'(x) \in K$. Moreover, we have a bound on the distance of points in K to L, and vice versa

$$\operatorname{dist}^q(u_0 + \bar{u}_n(x), K) \le C_p(1 + |w_0 + \tilde{w}_n(x)|^{\beta q}) + C_p(1 + |v_0 + \mathbb{M}\tilde{w}_n|^{\beta q}). \tag{6.21}$$

Indeed, (6.21) can be verified by the following argument. The closest point of $u_0 + \bar{u}_n(x)$

to L is $\big(v_0 + \mathbb{M}\tilde{w}_n(x), v_0 + \mathbb{M}\tilde{w}_n(x)\big) \in L$, hence

$$\operatorname{dist}^q(u_0 + \bar{u}_n(x), K) = \operatorname{dist}^q\Big(u_0 + \bar{u}_n(x), \big(v_0 + \mathbb{M}\tilde{w}_n(x), v_0 + \mathbb{M}\tilde{w}_n(x)\big)\Big) \overset{\text{(H1')}}{\leq} R_1^p(1 + |w_0 + \tilde{w}_n(x)|^\beta)^q.$$

However, the distance of this projection point to K can be bounded using (H2')

$$\operatorname{dist}^q(v_0 + \mathbb{M}\tilde{w}_n(x), K) \leq R_2^p(1 + |v_0 + \mathbb{M}\tilde{w}_n(x)|^\beta)^q.$$

Using the triangle inequality and rearranging the terms yields (6.21). We combine estimates (6.20) and (6.21) to get an estimate for dist^q as follows

$$\operatorname{dist}^q(u_0+\bar{u}_n(x), K) \leq C(|u_0+\bar{u}_n(x)-\tilde{u}_n'(x)|^q)^\alpha \left(1 + |w_0 + \tilde{w}_n(x)|^{\beta q} + |v_0 + \tilde{v}_n(x)|^{\beta q}\right)^{1-\alpha} \tag{6.22}$$

for an appropriately chosen $\alpha \in (0,1)$. Using Hölder's inequality with exponents r and r', yields for the integrated identity

$$\begin{aligned}\int_{T_N} &\operatorname{dist}^q(u_0 + \bar{u}_n(x), K)\,\mathrm{d}x \\ &\leq C \int_{T_N} (|u_0 + \bar{u}_n(x) - \tilde{u}_n'(x)|^q)^\alpha \left(1 + |w_0 + \tilde{w}_n(x)|^{\beta q} + |v_0 + \tilde{v}_n(x)|^{\beta q}\right)^{1-\alpha} \mathrm{d}x \\ &\leq \tilde{C} \left(\int_{T_N} |u_0 + \bar{u}_n(x) - \tilde{u}_n'(x)|^{q\alpha r}\,\mathrm{d}x\right)^{1/r} \\ &\qquad \cdot \left(\int_{T_N} \left(1 + |w_0 + \tilde{w}_n(x)|^{\beta q(1-\alpha)r'} + |v_0 + \tilde{v}_n(x)|^{\beta q(1-\alpha)r'}\right) \mathrm{d}x\right)^{1/r'}.\end{aligned}$$

Choose $\alpha = \dfrac{p/q - \beta}{1-\beta} \in (0,1)$ (as $1 > p/q > \beta$) and $r = \dfrac{\beta + \alpha - \beta\alpha}{\alpha}$ (which is larger than 1 as $\alpha < 1$). Then we have

$$q\alpha r = q(\beta + \alpha - \beta\alpha) = q\frac{(\beta - \beta^2) + ((p/q) - \beta) - (\beta(p/q) - \beta^2)}{1-\beta} = q\frac{p}{q} = p \tag{6.23}$$

and

$$\beta q(1-\alpha)r' = q\beta(1-\alpha)\frac{r}{r-1} = q\beta(1-\alpha)\frac{\beta + \alpha - \beta\alpha}{\beta - \beta\alpha} = q(\beta + \alpha - \beta\alpha) = qr\alpha \overset{(6.23)}{=} p. \tag{6.24}$$

This yields

$$\begin{aligned}\int_{T_N} \operatorname{dist}^q(u_0 + \bar{u}_n(x), K)\,\mathrm{d}x &\leq C\|u_0 + \bar{u}_n(x) - \tilde{u}_n'(x)\|_{L^p}^{p/r} \\ &\quad \cdot \left(1 + \|w_0 + \tilde{w}_n(x)\|_{L^p}^{p/r'} + \|v_0 + \tilde{v}_n(x)\|_{L^p}^{p/r'}\right).\end{aligned}$$

The second term is uniformly bounded in n and the first one tends to 0. Therefore,

$$\lim_{n\to\infty}\int_{T_N} \operatorname{dist}^q(u_0+\bar{u}_n(x),K)\,\mathrm{d}x = 0.$$

□

Corollary 6.28. *Let* $1 < p < \infty$ *and* $u_n' = (u_0+u_n)$ *be a bounded sequence in* $L^p(T_N,\mathbb{R}^d\times\mathbb{R}^d)$*, such that*

(i) $\displaystyle\int_{T_N} \operatorname{dist}^p(u_n',K)\,\mathrm{d}x \to 0$*;*

(ii) $u_0 \in \mathbb{R}^d\times\mathbb{R}^d$*;*

(iii) u_n *has zero average and satisfies the differential condition* $(\mathcal{A},\mathcal{B}^*)u_n = 0$*.*

Suppose that K satisfies (H1') and (H2'). Let $1 < p < q < \infty$*. Then there is a sequence* $\bar{u}_n \in L^q(T_N,\mathbb{R}^d\times\mathbb{R}^d)$ *with zero average satisfying*

(a) $\|\bar{u}_n - u_n\|_{L^p} \to 0$ *as* $n\to\infty$*;*

(b) $\displaystyle\int_{T_N} \operatorname{dist}^q(\bar{u}_n+u_0)\,\mathrm{d}x \to 0$ *as* $n\to\infty$*;*

(c) $(\mathcal{A},\mathcal{B}^*)\bar{u}_n = 0$*.*

Proof. This follows by induction and Lemma 6.27. In particular, boundedness in L^q follows from the coercivity Lemma 6.26. □

Using this truncation statement we are ready to prove that the hulls $K^{(p)}$ and $K^{(q)}$ coincide whenever $1 < p,q < \infty$.

Proof of Theorem 6.24. First of all, note that by the integral coercity we have for all $p \in (1,\infty)$ that $K^{(p*)} = K^{(p)}$. Therefore, one gets $K^{(q)} = K^{(q*)} \subset K^{(p*)} = K^{(p)}$. The difficulty adressed in previous lemmas is to show $K^{(p)} \subset K^{(q)}$.

This is shown by Corollary 6.28. If $z \in K^{(p)}$, there is a bounded sequence $u_n \in L^p(T_N,\mathbb{R}^d\times\mathbb{R}^d)$ with zero average satisfying $(\mathcal{A},\mathcal{B}^*)u_n = 0$ and

$$\int_{T_N} \operatorname{dist}^p(z+u_n,K)\,\mathrm{d}x \longrightarrow 0 \quad \text{as } n\to\infty.$$

By Corollary 6.28, the modified sequence $z+\bar{u}_n(x)$ even satisfies

$$\int_{T_N} \operatorname{dist}^q(z+\bar{u}_n,K)\,\mathrm{d}x \longrightarrow 0 \quad \text{as } n\to\infty.$$

Therefore, $z \in K^{(q)}$. □

7. L^∞-truncation: Closed differential forms

This chapter summarises the results obtained in the publication

- [134]: Schiffer, S., *L^∞-truncation of closed differential forms*, https://arxiv.org/abs/2102.07568, 2021.

In particular, only the treatment of $\mathcal{A}$-quasiconvex sets (Section 6.1 in the paper) has already been mentioned in Chapter 6. The paper is given in the first part of the appendix, Chapter A. It is accepted in the peer-review journal 'Calculus of Variations and Partial Differential Equations' published by Springer.

This is a single-author manuscript. Hence a detailed description of the doctoral candidate's contribution is not needed.

7.1. Motivation

The motivation to this chapter comes from the treatment of $\mathcal{A}$-quasiconvex sets. We have seen in Theorem 6.14 that an L^∞-truncation result yields that the hulls $K^{(1)}$ and $K^{(\infty)}$ coincide whenever K is a compact set and (ZL) holds true for the differential operator $\mathcal{A}$. In particular, the main question reads as folllows.

Consider a linear differential operator $\mathcal{A}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d) \to C^\infty(\mathbb{R}^N,\mathbb{R}^l)$ of first order with constant coefficients, and a bounded sequence of functions $u_n \subset L^1(\mathbb{R}^N,\mathbb{R}^d)$ which satisfy $\mathcal{A}u_n = 0$ in the sense of distributions and are close to a bounded set in L^∞, i.e.

$$\lim_{n\to\infty} \int_{\{x\in\mathbb{R}^N\colon |u_n(x)|\geq L\}} |u_n| \,\mathrm{d}x = 0 \tag{7.1}$$

for some $L > 0$. Does there exist a sequence of functions v_n such that $\mathcal{A}v_n = 0$, $\|v_n\|_{L^\infty} \leq CL$ and $(u_n - v_n) \to 0$ in measure (in L^1)?

This question was answered first by Zhang in [157] for sequences of gradients, i.e. for the operator $\mathcal{A} = \operatorname{curl}$. In Chapter A, we give a major extension to this result by showing that it is true for closed differential forms. That is, the result is true whenever $\mathcal{A}$ is an exterior derivative. Moreover, we discuss its applications for $\mathcal{A}$-quasiconvex sets and the additional framework of $\mathcal{A}$-∞-Young measures.

7.2. Main results

Now, we summarise the main results obtained in Chapter A. Indeed, we answer the previously raised question positively, which is expressed via the following theorem:

Theorem 7.a (=Theorem A.1). *Suppose that we have a sequence $u_n \subset L^1(\mathbb{R}^N, \Lambda^r)$ with $du_n = 0$ (in the sense of distributions), and that there exists an $L > 0$ such that*

$$\int_{\{y \in \mathbb{R}^N : \, |u_n(y)| > L\}} |u_n(y)| \, \mathrm{d}y \longrightarrow 0 \quad \text{as } n \to \infty. \tag{7.2}$$

There exists a constant $C_1 = C_1(N, r)$ and a sequence $v_n \subset L^\infty(\mathbb{R}^N, \Lambda^r)$ with $dv_n = 0$ and

i) $\|v_n\|_{L^\infty(\mathbb{R}^N, \Lambda^r)} \leq C_1 L$;

ii) $\|v_n - u_n\|_{L^1(\mathbb{R}^N, \Lambda^r)} \to 0$ *as* $n \to \infty$;

iii) $|\{y \in \mathbb{R}^N : v_n(y) \neq u_n(y)\}| \to 0$.

We outline the idea behind proving this theorem in the following Section 7.3. Further, in a more abstract setting, we show two consequences of the abstract property shown by the theorem above (which already appeared in Chapter 6 as property (ZL)). First, we show that we indeed have equality of the $\mathcal{A}$-quasiconvex hulls. This statement and its proof have also been mentioned in Chapter 6 via Theorem 6.14. Moreover, as a byproduct of the L^∞-truncation, we are able to derive a characterisation of $\mathcal{A}$-∞-Young measures:

Theorem 7.b (=Theorem A.2). *Let $\mathcal{A}$ satisfy the L^∞-trucnation property (ZL). A weak$*$ measurable map $\nu : T_N \to \mathcal{M}(\mathbb{R}^d)$ is an $\mathcal{A}$-∞-Young measure if and only if $\nu_x \geq 0$ a.e. and there exists $K \subset \mathbb{R}^d$ compact and $u \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with*

i) $\operatorname{spt} \nu_x \subset K$ *for a.e.* $x \in T_N$;

ii) $\langle \nu_x, id \rangle = u(x)$ *for a.e.* $x \in T_N$;

iii) $\langle \nu_x, f \rangle \geq f(\langle \nu_x, id \rangle)$ *for a.e.* $x \in T_N$ *and all continuous and $\mathcal{A}$-quasiconvex $f : \mathbb{R}^d \to \mathbb{R}$, i.e. $f \in C(\mathbb{R}^d)$ such that for all $\psi \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$*

$$f\left(\int_{T_N} \psi(x) \, \mathrm{d}x \right) \leq \int f(\psi(x)) \, \mathrm{d}x.$$

7.3. Ideas of proofs

We now shortly outline the ideas behind the proofs of the two previously mentioned theorems. The L^∞-truncation result relies on a generalised version of Whitney's extension theorem. Clasically, Whitney's extension (cf. [151, 139]) extends Lipschitz functions on a closed set $X \subset \mathbb{R}^N$ to be Lipschitz on the whole of $\mathbb{R}^N$ with a Lipschitz constant that is only worse by a multiplicative constant. The main part of the proof is to show that such a Whitney extension is also possible for closed differential forms.

The main ingredients towards this technique are the following:

(a) We need a suitable formulation that is parallel to the notion of being Lipschitz. In more detail, being Lipschitz on a convex set can be expressed by two means:

 (i) the function u is weakly differentiable and satisfies $Du \in L^\infty$ almost everywhere;

 (ii) the function u is continuous and satisfies

$$\frac{|u(x) - u(y)|}{|x - y|} \leq L|x - y|$$

 for all x, y in the set.

A valuable observation by Acerbi & Fusco [1], is that a function is Lipschitz continuous on the set, where the maximal function is small. The counterpart of this observation for closed differential forms is the following. Let M denote the Hardy–Littlewood maximal function.

Theorem 7.c (= Lemma A.7)**.** *There exists a constant $C = C(N, r)$ such that for all $\omega \in C^1(\mathbb{R}^N, \Lambda^r)$, $\lambda > 0$ with $d\omega = 0$ and $x_1, ..., x_{r+1} \in \{M\omega \leq \lambda\}$ we have*

$$\left| \fint_{\mathrm{Sim}(x_1,...,x_{r+1})} \omega(\nu^r(x_1, ..., x_{r+1})) \right| \leq C\lambda \max_{1\leq i,j\leq r+1} |x_i - x_j|^r. \tag{7.3}$$

(b) We need to construct a Whitney extension theorem for sets that satisfy a property in the style of (7.3). This features the classical approach of covering the complement of the sets with Calderón–Zygmund cubes.

(c) The proof itself then can be roughly summarised by the following steps. We do not change the function on the 'good set', where the maximal function is small, and redefine the function on its complement, the 'bad set'. We then show the validity of the extension theorem outlined in the previous item. First, we show a L^∞ bound on the function. Then, we prove that the differential constraint (i.e. that the differential form is closed) is satisfied in a pointwise fashion almost everywhere. Finally, we verify that the exterior derivative as a distribution is actually an L^1-function . This yields that the exterior derivative is zero, i.e. the constructed extension is a closed differential form. The rest of the proof relies on an argument that the complement of the bad set, to which we extend the function, is small in measure.

The proofs of the consequences of this theorem follow the arguments that have been given in the special case $\mathcal{A} = \mathrm{curl}$. In particular, the proof of $K^{(1)} = K^{(\infty)}$ has been seen in Theorem 6.14. The proof of the characterisation result for Young-measures follows its counterpart in the setting $\mathcal{A} = \mathrm{curl}$ from [85, 114].

8. L^∞-truncation: divsym free matrices in dimension three

This chapter summarises the results obtained in the publication

- [20]: Behn, L., Gmeineder, F. and Schiffer, S. *On symmetric div-quasiconvex hulls and divsym-free L^∞ truncations*, https://arxiv.org/abs/2108.05757, 2021.

The treatment of $\mathcal{A}$-quasiconvex sets (Section 5) has been already been mentioned in Chapter 6. The paper is given in the second part of the appendix, Chapter B. It is accepted in the peer-review journal 'Annales de l'Institut Henri Poincaré C: Analyse non linéaire' published by EMS Press.

The research undertaken in the paper in question is a collaboration with L. Behn and F. Gmeineder. All authors and, in particular the author of this thesis, have contributed significant parts to each section of the work.

8.1. Motivation

As for the previous Chapter 7, the motivation for this chapter is the treatment of $\mathcal{A}$-quasiconvex sets. Theorem 6.14 shows that the validity of an L^∞-truncation result yields that the set $K^{(1)} = K^{(\infty)}$.

The goal of this section is to extend the result of Chapter 7 to another differential operator. We show that (ZL) holds for the divergence of symmetric 3×3 matrices. This operator is of relevance in the framework of linear elasticity, which is further outlined in Section B.1.1.

In addition, we derive a slightly weaker result than $K^{(1)} = K^{(\infty)}$ independent of the validity of (ZL). This statement, $K^{(p)} = K^{(q)}$ for $1 < p, q < \infty$, is already featured in Chapter 6 via Theorem 6.7.

8.2. Main result

We now give the main result obtained in the paper and summarise its consequences. The main theorem reads as follows.

Theorem 8.a (= Theorem B.2). *There exists a constant $C > 0$ solely depending on the underlying space dimension $n = 3$ with the following property: For all $u \in L^1(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ with $\operatorname{div}(u) = 0$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ and all $\lambda > 0$ there exists $u_\lambda \in L^1(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfying the* *(a)* L^∞-bound: $\|u_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq C\lambda$;

(b) strong stability: $\|u - u_\lambda\|_{L^1(\mathbb{R}^3)} \leq C \int_{\{|u|>\lambda\}} |u| \,\mathrm{d}x$;

(c) small change: $\mathcal{L}^3(\{u \neq u_\lambda\}) \leq C\lambda^{-1} \int_{\{|u|>\lambda\}} |u| \,\mathrm{d}x$;

(d) differential constraint: $\mathrm{div}(u_\lambda) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$.

The same remains valid when replacing the underlying domain $\mathbb{R}^3$ *by the torus* T_3.

We then show that this truncation theorem in turn implies $K^{(1)} = K^{(\infty)}$ (also see Theorem 6.14 and the treatment in Chapter A).

8.3. Idea of proof

We summarise the main ideas behind the proof to Theorem 8.a. This features an extension of the ideas that are featured in Chapter 7.

8.3.1. $\mathbb{C}$-ellipticity and exact sequences of differential forms

The main insight behind the treatment of differential forms (cf. Chapter A) is that we already know a Lipschitz truncation, i.e. a $W^{1,1}$-$W^{1,\infty}$-truncation of functions. This truncation can be used to derive a curl-free truncation.

For differential forms one can observe the following: The formula for curl-free truncation plays nicely with the geometry of $\mathbb{R}^N$ and is not only suitable as a curl-free truncation, but also for a $W^{1,\mathrm{curl}}$-$W^{\infty,\mathrm{curl}}$-truncation, where

$$W^{p,\mathrm{curl}} := \{u \in L^p \colon \operatorname{curl} u \in L^p\}.$$

This powerful observation then in turn yields an $\mathcal{A}$-free truncation for the annihilator of curl which is the divergence operator div in space dimension three.

Summarised, we can construct L^1-L^∞-truncations along the exact sequence of differential operators, that are exterior derivates.

In Chapter B we show that this technique also holds when the start of the exact sequence is replaced. The result for differential forms relies on a truncation for gradients, which we replace by the result for symmetric gradients (which may be extended, in general, to $\mathbb{C}$-elliptic operators, cf. [19]). In particular, we start with a truncation of the symmetric gradient and then derive a truncation of divergence-free symmetric matrices by following the exact sequence featuring the operators $1/2(\nabla + \nabla^T)$, $\operatorname{curl}\operatorname{curl}^T$ and $\mathrm{div}_{\mathrm{sym}}$. This procedure is further elucidated in Section B.3.

8.3.2. The construction of truncation via Whitney's extension theorem

The technique involved in the proof of Theorem 8.a is very similar to its counterpart in Chapter 7. We first need a suitable pointwise condition for $\mathrm{div}_{\mathrm{sym}}$-free fields that is parallel to Theorem 7.c. We then show Theorem 8.a by the same technique. We use the extension theorem by maintaining the function on a certain good set and changing it on the bad set. First, we show that the extension is $\mathrm{div}_{\mathrm{sym}}$-free which is done in two steps. We prove that the differential condition is satisfied pointwisely almost everywhere. Then, we show that the symmetric divergence of a function, seen as a distribution, is already a L^1-function. The theorem is then established by estimating the measure of the bad set.

The result of Theorem B.1 that $K^{(1)} = K^\infty$ follows by the same means employed in Section A, see also Theorem 6.14.

The validity of $K^{(p)} = K^{(q)}$ for $1 < p, q < \infty$ is independent of the property (ZL) and only relies on the constant rank property. We show this by a significantly weaker version of the truncation statement on the torus, which uses Fourier analysis and the results from Chapter 2, see also Theorem 6.7.

The last section of the work focuses on a slightly weaker truncation statement whose proof also only relies on the constant rank property. This truncation statement is, however, not of relevance for the treatment of $\mathcal{A}$-quasiconvex sets.

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A. L^∞-truncation: Closed differential forms

Up to minor changes, this chapter coincides with the publication.

- [134]: Schiffer, S., *L^∞-truncation of closed differential forms*

In particular, only the treatment of $\mathcal{A}$-quasiconvex sets (Section 6.1 in the paper) already appeared in Chapter 6.

A.1. Introduction

A.1.1. $\mathcal{A}$-free truncations

An interesting question in the calculus of variations and real analysis is the following: Consider a linear differential operator $\mathcal{A}\colon C^\infty(\mathbb{R}^N,\mathbb{R}^d) \to C^\infty(\mathbb{R}^N,\mathbb{R}^l)$ of first order with constant coefficients, and a bounded sequence of functions $u_n \subset L^1(\mathbb{R}^N,\mathbb{R}^d)$ which satisfy $\mathcal{A}u_n = 0$ in the sense of distributions and are close to a bounded set in L^∞, i.e.

$$\lim_{n\to\infty} \int_{\{x\in\mathbb{R}^N\colon |u_n(x)|\geq L\}} |u_n| \,\mathrm{d}x = 0 \tag{A.1}$$

for some $L > 0$. Does there exist a sequence of functions v_n, such that $\mathcal{A}v_n = 0$, $\|v_n\|_{L^\infty} \leq CL$ and $(u_n - v_n) \to 0$ in measure (in L^1)?

This question was answered first by Zhang in [157] for sequences of gradients ($u_n = \nabla w_n$), i.e. for the operator $\mathcal{A} = \mathrm{curl}$, which assigns to a function $u\colon \mathbb{R}^N \to \mathbb{R}^N$ the skew-symmetric $(N \times N)$-matrix with entries $\partial_i u_j - \partial_j u_i$. Zhang's proof, which builds on the works of Liu [96] and Acerbi-Fusco [1], proceeds as follows. Denote by Mf the Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^d)$ and let $u_n = \nabla w_n$. The estimate (A.1) implies that the sets $X^n = \{M(\nabla w_n) \geq L'\}$ have small measure for large n. One then uses (cf. [1]) that

$$|w_n(x) - w_n(y)| \leq CL'|x-y|', \quad x,y \in \mathbb{R}^N \backslash X^n, \tag{A.2}$$

i.e. w_n is Lipschitz continuous on $\mathbb{R}^N\backslash X^n$. The fact that Lipschitz continuous functions on closed subsets of $\mathbb{R}^N$ can be extended to Lipschitz continuous functions on $\mathbb{R}^N$ with the same Lipschitz constant [90] yields the result.

In this chapter, we show that the answer to the previously formulated question is also positive for sequences of differential forms and $\mathcal{A} = d$, the operator of exterior differentiation.

Let us denote by Λ^r the r-fold wedge product of the dual space $(\mathbb{R}^N)^*$ of $\mathbb{R}^N$ and by $d\colon C^\infty(\mathbb{R}^N,\Lambda^r) \to C^\infty(\mathbb{R}^N,\Lambda^{r+1})$ the exterior derivative w.r.t. the standard Euclidean geometry on $\mathbb{R}^N$.

Theorem A.1 (L^∞-truncation of differential forms)**.** *Suppose that we have a sequence $u_n \subset L^1(\mathbb{R}^N,\Lambda^r)$ with $du_n = 0$ (in the sense of distributions), and that there exists an $L > 0$ such that*

$$\int_{\{y\in\mathbb{R}^N\,:\,|u_n(y)|>L\}} |u_n(y)|\,\mathrm{d}y \longrightarrow 0 \quad \text{as } n\to\infty. \tag{A.3}$$

There exists a constant $C_1 = C_1(N,r)$ and a sequence $v_n \subset L^\infty(\mathbb{R}^N,\Lambda^r)$ with $dv_n = 0$ and

i) $\|v_n\|_{L^\infty(\mathbb{R}^N,\Lambda^r)} \le C_1 L$;

ii) $\|v_n - u_n\|_{L^1(\mathbb{R}^N,\Lambda^r)} \to 0$ *as* $n\to\infty$;

iii) $|\{y\in\mathbb{R}^N : v_n(y)\neq u_n(y)\}| \to 0$.

An analogous version of Theorem A.1 holds if $\mathbb{R}^N$ is replaced by the N-torus T_N (cf. Theorem A.17) or by an open Lipschitz set Ω and functions u with zero boundary data (cf. Propostion A.20). Moreover, the result immediately extends to $\mathbb{R}^m$-valued forms by taking truncations coordinatewise (cf. Proposition A.21).

In particular, the result of Theorem A.1 includes a positive answer to the question previously raised for the differential operator $\mathcal{A} = \operatorname{div}$ after suitable identifications of Λ^{N-1} and Λ^N with $\mathbb{R}^N$ and $\mathbb{R}$, respectively.

One key ingredient in the proofs is a version of the Acerbi-Fusco estimate (A.2) for simplices rather than pairs of points in Lemma A.7. For the estimate, let us consider $\omega \in C_c^2(\mathbb{R}^N,\Lambda^r)$ with $d\omega = 0$ and let D be a simplex with vertices $x_1,...,x_{r+1}$ and a normal vector $\nu^r \in \mathbb{R}^N\wedge...\wedge\mathbb{R}^N$ (cf. Section A.2.2 for the precise definition). Assume that $M\omega(x_i)\le L$ for $i=1,...,r+1$. Then

$$\left|\int_D \omega(\nu^r)\right| \le C(N)L \sup_{1\le i,j\le r+1} |x_i - x_j|^r = C(N)L\operatorname{diam}(D)^r. \tag{A.4}$$

The second ingredient is a geometric version of the Whitney extension theorem, which may be of independent interest, cf. Section A.4.

Combining (A.4) and the extension theorem, one easily obtains the assertion for smooth closed forms. The general case follows by a standard approximation argument.

Before turning to an application of the truncation result, let us also mention that in Theorem A.1 the hard part is to get the convergence in ii) just from the rather weak assumption (A.3). A version of Theorem A.1 has been seen for a stronger assumption on the smallness of the sequence in [73]. Regarding solenoidal Lipschitz truncations [26, 28], meaning $W^{1,1}$-$W^{1,\infty}$-truncations instead of L^1-L^∞, the smallness corresponding to (A.3) is also assumed to be slightly different from the present setting.

Moreover, in the setting $\mathcal{A} = \operatorname{curl}$, the statement of Theorem A.1 can be further improved as follows. If K is a compact, convex set and $u_n \to K$ in L^1, we can even get a sequence v_n, such that the L^∞-norm of $\operatorname{dist}(v_n, K)$ converges to 0, cf. [113]. In contrast, Theorem A.1 only implies an L^∞-bound on v_n and convergence in measure to K. MÜLLER'S technique does not rely directly to a curl-free truncation, but on a Lipschitz truncation. It then uses suitable cut-offs and mollifications. There does not seem to be an obvious obstruction, why this technique should not work, if we replace the Lipschitz truncation by a general truncation statement on any potential instead of ∇ (also cf. [73]).

A.1.2. $\mathcal{A}$-∞ Young measures

Truncation results like the result by ZHANG or Theorem A.1 have immediate applications in the calculus of variations. In particular, they provide characterisations of the $\mathcal{A}$-quasiconvex hulls of sets, cf. Section A.6.1 and its discussion in Chapter 6, and the set of Young-measures generated by sequences satisfying $\mathcal{A}u_n = 0$. For a precise definition of $\mathcal{A}$-Young measures we refer to Section A.6 and [65].

The classical result for Young measures generated by sequences of gradients (i.e. sequences of functions u_n satisfying $\operatorname{curl} u_n = 0$) goes back to KINDERLEHRER and PEDREGAL [85, 86]. Here, we show the natural counterpart of their characterisation result, whenever the operator $\mathcal{A}$ admits the following L^∞-truncation result:

We say that $\mathcal{A}$ satisfies the property (ZL) if for all sequences $u_n \subset L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$, such that there exists an $L > 0$ with

$$\int_{\{y \in T_N \colon |u_n(y)| > L\}} |u_n(y)| \,\mathrm{d}y \longrightarrow 0 \quad \text{as } n \to \infty,$$

there exists a $C = C(\mathcal{A})$ and a sequence $v_n \subset L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

i) $\|v_n\|_{L^\infty(T_N,\mathbb{R}^d)} \leq CL$;

ii) $\|v_n - u_n\|_{L^1(T_N,\mathbb{R}^d)} \to 0$ as $n \to \infty$.

By ZHANG [157], the property (ZL) holds for $\mathcal{A} = \operatorname{curl}$ and a version of Theorem A.1 shows this for $\mathcal{A} = d$ (Corollary A.18). Further examples are shortly discussed in Example A.23.

For the characterisation of Young measures, recall that $\operatorname{spt} \nu$ denotes the support of a (signed) Radon measure $\nu \in \mathcal{M}(\mathbb{R}^d)$, and for $f \in C_c(\mathbb{R}^d)$

$$\langle \nu, f \rangle := \int_{\mathbb{R}^d} f \mathrm{d}\mu.$$

If the property (ZL) holds for some differential operator $\mathcal{A}$, then one is able to prove the following statement.

Theorem A.2. *[Classification of $\mathcal{A}$-∞-Young measures] Let $\mathcal{A}$ satisfy (ZL). A weak* measurable map $\nu : T_N \to \mathcal{M}(\mathbb{R}^d)$ is an $\mathcal{A}$-∞-Young measure if and only if $\nu_x \geq 0$ a.e. and there exists $K \subset \mathbb{R}^d$ compact and $u \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with*

i) $\operatorname{spt} \nu_x \subset K$ *for a.e.* $x \in T_N$;

ii) $\langle \nu_x, id\rangle = u(x)$ *for a.e.* $x \in T_N$*;*

iii) $\langle \nu_x, f\rangle \geq f(\langle \nu_x, id\rangle)$ *for a.e.* $x \in T_N$ *and all continuous and* $\mathcal{A}$*-quasiconvex* $f : \mathbb{R}^d \to \mathbb{R}$ *i.e.* $f \in C(\mathbb{R}^d)$*, such that for all* $\psi \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$

$$f\left(\int_{T_N} \psi(x)\,\mathrm{d}x\right) \leq \int f(\psi(x))\,\mathrm{d}x.$$

For further reference to classification of $\mathcal{A}$-p-Young measures for $p < \infty$, let us shortly refer to [65, 66, 130, 93, 9].

A.1.3. Outline

We close the introduction with a brief outline of the paper. In Section A.2, we introduce some notation, recall some basic facts from multilinear algebra and the theory of differential forms. We prove the key estimate (A.4) in Section A.3. Section A.4 is devoted to the proof of the geometric Whitney extension theorem. In Section A.5, the proof of the truncation result (and its local and periodic variant) is given. Section A.6 discusses the applications to $\mathcal{A}$-quasiconvex hulls and $\mathcal{A}$-Young measures. The proofs of the theorems closely follow the arguments in [85] and are discussed in the last subsection A.6.3

A.2. Preliminary results

Define the space Λ^r as the r-fold wedge product of $(\mathbb{R}^N)^*$, i.e.

$$\Lambda^r = \underbrace{(\mathbb{R}^N)^* \wedge \ldots \wedge (\mathbb{R}^N)^*}_{r \text{ copies}}$$

and similarly the space Λ_r as the r-fold wedge product of $\mathbb{R}^N$. Then Λ^r and Λ_r are finite-dimensional vector spaces. For $\mathbb{R}^N$ denote by $\{e_i\}_{i\in[N]}$ the standard basis and by $\cdot$ the standard scalar product. For $(\mathbb{R}^N)^*$ denote by $\theta_1, \ldots, \theta_N$ the corresponding dual basis of $(\mathbb{R}^N)^*$, i.e. θ_i is the map $y \mapsto y \cdot e_i$.

For $k \in I_r := \{l \in [N]^r : l_1 < l_2 < \ldots < l_r\}$ the vectors

$$e^{k,r} = e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_r} \tag{A.5}$$

form a basis of Λ_r. Denote by $\cdot^r$ the scalar product with respect to this basis, i.e. for $k, l \in I_r$

$$e^{k,r} \cdot^r e^{l,r} = \begin{cases} 1 & k = l, \\ 0 & k \neq l. \end{cases}$$

This also provides us with a suitable norm on Λ_r, which we denote by $\|\cdot\|_{\Lambda_r}$. Similarly, using the standard basis of $(\mathbb{R}^n)^*$, we define a basis $\theta^{k,r}$ and a norm $\|\cdot\|_{\Lambda^r}$. Also note that for $0 \leq s \leq r$ there exists (up to sign) a natural map $\Lambda^r \times \Lambda_s \mapsto \Lambda^{r-s}$ (the interior product), as Λ^s is the dual space of Λ_s and $\Lambda^r = \Lambda^s \wedge \Lambda^{r-s}$. In particular, in the special case $s = 1$ for $h_1, ..., h_r \in \mathbb{R}^{N*}$ and $y \in \mathbb{R}^N$

$$(h_1 \wedge .... \wedge h_r)(y) = \sum_{i=1}^{r} (-1)^{i-1} h_i(y) h_1 \wedge ... \wedge h_{i-1} \wedge h_{i+1} ... \wedge h_r. \tag{A.6}$$

In the case $s = r$ and for $h_1, ..., h_r \in (\mathbb{R}^N)^*$ and $y_1, ..., y_r \in \mathbb{R}^N$

$$(h_1 \wedge .... \wedge h_r)(y_1 \wedge ... \wedge y_r) = \sum_{\sigma \in S_r} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{r} h_i(y_{\sigma(i)}) \right), \tag{A.7}$$

where S_r denotes the group of permutations of $[r] = \{1, ..., r\}$. (A.7) also gives us a representation of the map $\Lambda^r \times \Lambda_s \mapsto \Lambda^{r-s}$ as for $h \in \Lambda^r$, $x \in \Lambda_s$ we may consider the element of $\Lambda^{r-s} = (\Lambda_{r-s})^*$ defined by

$$z \longmapsto h(x \wedge z), \quad z \in \Lambda_{r-s}.$$

Let us shortly remark that this notation is slightly different to the usual notation for interior products.

Moreover, note that the space Λ^N is isomorphic to $\mathbb{R}$ via the map I^N defined by

$$a\ \theta_1 \wedge ... \wedge \theta_N \longmapsto a \in \mathbb{R}.$$

A.2.1. Differential forms

In the following, we will define all objects for an open set $\Omega \subset \mathbb{R}^N$, but these definitions are also valid for $\mathbb{R}^N$ and T_N respectively.

We call a map $f \in L^1_{\text{loc}}(\Omega, \Lambda^r)$ an **r-differential form** on Ω. We define the space

$$\Gamma = \bigcup_{r \in \mathbb{N}} C^\infty(\Omega, \Lambda^r).$$

It is well-known (c.f [29, 36]) that there exists a linear map $d \colon \Gamma \mapsto \Gamma$, called the **exterior derivative** with the following properties

i) $d^2 = d \circ d = 0$,

ii) d maps $C^\infty(\Omega, \Lambda^r)$ into $C^\infty(\Omega, \Lambda^{r+1})$,

iii) We have the **Leibniz rule**: If $\alpha \in C^\infty(\Omega, \Lambda^r)$ and $\beta \in C^\infty(\Omega, \Lambda^s)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta, \tag{A.8}$$

iv) $d : C^\infty(\Omega, \Lambda^0) \to C^\infty(\Omega, \Lambda^1)$ is the gradient via the identification $\Lambda^0 = \mathbb{R}$, $\Lambda^1 = (\mathbb{R}^N)^* \cong \mathbb{R}^N$.

We sometimes write d_x to indicate that this derivative is taken in terms of a space variable $x \in \mathbb{R}^N$. This map d has the following representation in terms of the standard coordinates (cf. [36]). Let $\omega \in C^\infty(\Omega, \Lambda^r)$, which, for some $a_k \in C^\infty(\Omega, \mathbb{R})$, can be written as

$$\omega(y) = \sum_{k \in I_r} a_k(y) \theta^{k,r}.$$

Then

$$d\omega(y) = \sum_{k \in I_r} \sum_{l \in [N]} \partial_l a_k(y) \theta_l \wedge \theta^{k,r}. \tag{A.9}$$

Remark A.3. For a fixed $r \in \{0, ..., N-1\}$ we can identify $d \colon C^\infty(\Omega, \Lambda^r) \mapsto C^\infty(\Omega, \Lambda^{r+1})$ with some well-known differential operator $\mathcal{A}$. By definition, for $r = 0$, d can be identified with the gradient. For $r = 1$, after a suitable identification of Λ^2 with $\mathbb{R}^{N \times N}_{skew}$, $d = \text{curl}$, which is the differential operator mapping $u \in C^\infty(\Omega, \mathbb{R}^N)$ to $\operatorname{curl} u \in C^\infty(\Omega, \mathbb{R}^{N\times N}_{skew})$ defined by

$$(\operatorname{curl} u)_{lk} = \partial_l u_k - \partial_k u_l.$$

If $r = N-1$, after identifying Λ^{N-1} with $\mathbb{R}^N$ and Λ^N with $\mathbb{R}$, the differential operator d becomes the divergence of a vector field which is defined for $u \in C^\infty(\Omega, \mathbb{R}^N)$ by

$$\operatorname{div} u = \sum_{k=1}^N \partial_k u_k.$$

Lemma A.4. *We have the following product rules for d:*

i) Let $\omega \in C^1(\Omega, \Lambda^1)$, $z \in \mathbb{R}^N = \Lambda_1$. Then

$$d_y\big(\omega(y)(y-z)\big) = \nabla_y \omega(y) \cdot (y-z) + \omega(y), \tag{A.10}$$

where we define $\nabla_y \omega(y) \cdot (y-z) \in C(\Omega, \Lambda^1)$ as follows:
If $\omega = \sum_{i=1}^N \omega_i \theta_i$ and $(y-z) = \sum_{i=1}^N (y-z)_i e_i$, then

$$\nabla_y \omega(y) \cdot (y-z) := \sum_{l=1}^N \sum_{i=1}^N \partial_l \omega_i(y)(y-z)_i \theta_l.$$

ii) There is a linear bounded map $D^{1,r} \in \operatorname{Lin}((\Lambda^r \times \mathbb{R}^N) \times \mathbb{R}^N, \Lambda^r)$ such that for $\omega \in C^1(\Omega, \Lambda^r)$, $z \in \mathbb{R}^N$ we have

$$d_y\Big(\omega(y)(y-z)\Big) = D^{1,r}\Big(\nabla\omega(y), (y-z)\Big) + \omega(y). \tag{A.11}$$

iii) There is a linear and bounded map $D^{s,r} \in \operatorname{Lin}((\Lambda^r \times \mathbb{R}^N) \times \Lambda_s, \Lambda^{r-s})$ such that for $\omega \in C^1(\Omega, \Lambda^r)$, $z \in \mathbb{R}^N$, $z_2 \in \Lambda_{s-1}$

$$d_y\Big(\omega(y)((y-z) \wedge z_2)\Big) = D^{s,r}\Big(\nabla_y \omega(y), (y-z) \wedge z_2\Big) + (-1)^{s-1}\omega(y)(z_2). \tag{A.12}$$

Proof. i) simply follows from a calculation, i.e., if as mentioned

$$\omega(y) = \sum_{i=1}^{N} \omega_i(y)\theta_i \quad \text{and } (y-z) = \sum_{i=1}^{N} (y-z)_i e_i,$$

then

$$\begin{aligned} d(\omega(y)(y-z)) &= \sum_{l=1}^{N} \partial_l(\omega(y)(y-z))\theta_l \\ &= \sum_{i,l=1}^{N} \partial_l \omega_i(y)(y-z)_i \theta_l + \sum_{l=1}^{N} \omega_l(y)\theta_l, \end{aligned}$$

which is what we claimed. Statement ii) then follows from i) and using (A.6). Likewise, iii) then follows from ii). □

Definition A.5. *For $\omega \in L^1_{\text{loc}}(\Omega, \Lambda^r)$ and $u \in L^1_{\text{loc}}(\Omega, \Lambda^{r+1})$ we say that $d\omega = u$ in the sense of distributions if for all $\varphi \in C_c^\infty(\Omega, \Lambda^{N-r-1})$ we have*

$$\int_\Omega d\varphi \wedge \omega = (-1)^{N-r} \int_\Omega \varphi \wedge u.$$

Note that this definition is equivalent to the following formula: For all $\varphi \in C_c^\infty(\Omega, \Lambda^s)$ with $0 \leq s \leq N-r-1$ and all $\theta \in \Lambda^{N-r-s-1}$ we have

$$(-1)^{r+1} \int_\Omega \omega \wedge d\varphi \wedge \theta = -\int_\Omega u \wedge \varphi \wedge \theta.$$

A.2.2. Stokes' theorem on simplices

We want to establish a suitable notion of Stokes' theorem for differential forms on simplices. Let $1 \leq r \leq N$ and $x_1, ..., x_{r+1} \in \mathbb{R}^N$. Define the simplex $\operatorname{Sim}(x_1, ..., x_{r+1})$ as the convex hull of $x_1, ..., x_{r+1}$. We call this simplex degenerate, if its dimension is strictly less than r.

For $i \in \{1, ..., r+1\}$ consider $\operatorname{Sim}(x_1, ...x_{i-1}, x_{i+1}, ..., x_{r+1}) =: \operatorname{Sim}^i(x_1, ...x_{r+1})$. This is an $(r-1)$ dimensional face of $\operatorname{Sim}(x_1, ..., x_{r+1})$ and a subset of the boundary of the manifold $\operatorname{Sim}(x_1, ..., x_{r+1})$, which, for simplicity, is denoted by $\partial \operatorname{Sim}(x_1, ..., x_{r+1})$. Suppose first that we are given the simplex

$$\{\lambda \in [0,1]^r \, : \, \sum_{i=1}^{r} \lambda_i \leq 1\} \times \{0\}^{N-r} = \operatorname{Sim}(0, e_1, ..., e_r) \subset \mathbb{R}^r \times \{0\}^{N-r} \subset \mathbb{R}^N.$$

Then the classical version of Stokes' theorem on oriented manifolds reads that for every differential form $\tilde{\omega} \in C^1(\mathbb{R}^r \times \{0\}^{N-r}, \mathbb{R}^r \wedge ... \wedge \mathbb{R}^r)$ -$\mathbb{R}^r$ is the corresponding tangential space of the manifold $\operatorname{Sim}(0, e_1, ..., e_r)$- we have

$$\int_{\operatorname{Sim}(0,e_1,...,e_r)} d\tilde{\omega}(y) \, \mathrm{d}\mathcal{H}^r(y) = \int_{\partial^* \operatorname{Sim}(0,e_1,...,e_r)} \tilde{\omega}(y) \wedge \nu(y) \, \mathrm{d}\mathcal{H}^{r-1}(y). \tag{A.13}$$

In (A.13), $\nu(y)$ denotes the outer normal unit vector at $y \in \partial^* \operatorname{Sim}(0, e_1, ...e_r)$ and ∂^* is the reduced boundary of the simplex, where this outer normal exists (the interior of all $(r-1)$-dimensional faces). In our case, we are given a differential form with the underlying space being $\mathbb{R}^N$ and not $\mathbb{R}^r$ (the tangential space of the manifold/simplex), hence we can modify (A.13) to get for $\omega \in C^1(\mathbb{R}^N, \Lambda^{r-1})$

$$\begin{aligned}\int_{\operatorname{Sim}(0,e_1,...,e_r)} & d\omega(y)(e_1 \wedge ... \wedge e_r)\, \mathrm{d}\mathcal{H}^r(y) \\ &= \sum_{i=1}^{r} (-1)^i \int_{\operatorname{Sim}(0,...,e_{i-1},e_{i+1},...,e_r)} \omega(y)(e_1 \wedge ... \wedge e_{i-1} \wedge e_{i+1} \wedge ... \wedge e_r) \\ &+ \int_{\operatorname{Sim}(e_1,...,e_r)} 2^{-r/2} \omega(y)((e_2 - e_1) \wedge (e_3 - e_2) \wedge ... \wedge (e_r - e_{r-1})).\end{aligned} \tag{A.14}$$

Let us write for simplicity that for $x_1, ..., x_{r+1} \in \mathbb{R}^N$

$$\nu^r(x_1, ..., x_{r+1}) = ((x_2 - x_1) \wedge (x_3 - x_2) \wedge ... \wedge (x_{r+1} - x_r)) \in \Lambda_r.$$

The map ν^r has the following properties:

i) ν^r is alternating, i.e. for a permutation $\sigma \in S_r$:

$$\nu^r(y_1, ..., y_{r+1}) = \operatorname{sgn}(\sigma) \nu^r(y_{\sigma(1)}, ..., y_{\sigma(r+1)}).$$

ii) We have the relation

$$\|\nu^r(y_1, ..., y_{r+1})\|_{\Lambda_r} = r\mathcal{H}^r(\operatorname{Sim}(y_1, ..., y_{r+1})).$$

A linear change of coordinates from $\operatorname{Sim}(0, e_1, .., e_r)$ to $\operatorname{Sim}(x_1, ..., x_{r+1})$ leads from (A.14) to the following: For $\omega \in C^\infty(\mathbb{R}^N, \Lambda^{r-1})$ and $x_1, ...x_{r+1} \in \mathbb{R}^N$

$$\begin{aligned}&\frac{1}{r} ⨍_{\operatorname{Sim}(x_1,...,x_{r+1})} d\omega(y)(\nu^r(x_1, ..., x_{r+1}))\, \mathrm{d}\mathcal{H}^r(y) \\ &= \sum_{i=1}^{r+1} \frac{(-1)^i}{r-1} ⨍_{\operatorname{Sim}^i(x_1,...x_{r+1})} \omega(y)(\nu^{r-1}(x_1, ..., x_{i-1}, x_{i+1}, ...x_{r+1}))\, \mathrm{d}\mathcal{H}^{r-1}(y),\end{aligned} \tag{A.15}$$

A.2.3. The maximal function

The Hardy-Littlewood maximal function for $u \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ is defined by

$$Mu(x) = \sup_{R>0} ⨍_{B_R(x)} |u(y)|\, \mathrm{d}y.$$

Again, we can also define the maximal function for functions on the torus using the identification with periodic functions.

Proposition A.6 (Properties of the maximal function (cf. [139]))**.** *M is sublinear, i.e. $M(u+v)(y) \leq Mu(y) + Mv(y)$ for all $u, v \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ and $y \in \mathbb{R}^N$. Moreover, $M : L^p(\mathbb{R}^N, \mathbb{R}^d) \to L^p(\mathbb{R}^N, \mathbb{R})$ is bounded for $1 < p \leq \infty$ and bounded from L^1 to $L^{1,\infty}$. In particular, this means that for $1 \leq p < \infty$*

$$|\{Mu > \lambda\}| \leq C_p \lambda^{-p} \|u\|^p_{L^p(\mathbb{R}^N, \mathbb{R}^d)}.$$

If $u \in L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ is a $\mathbb{Z}^N$-periodic function, i.e. $u \in L^p(T_N, \mathbb{R}^d)$, then

$$|\{Mu > \lambda\} \cap [0,1]^N| \leq C_p \lambda^{-p} \|u\|^p_{L^p([0,1]^N, \mathbb{R}^d)}.$$

A.3. A geometric estimate for closed differential forms

In this section we prove a key lemma for our main theorem.

Lemma A.7. *There exists a constant $C = C(N, r)$ such that for all $\omega \in C^1(\mathbb{R}^N, \Lambda^r)$, $\lambda > 0$ with $d\omega = 0$ and $x_1, ..., x_{r+1} \in \{M\omega \leq \lambda\}$ we have*

$$\left| \fint_{\text{Sim}(x_1,...,x_{r+1})} \omega(\nu^r(x_1, ..., x_{r+1})) \right| \leq C\lambda \max_{1 \leq i,j \leq r+1} |x_i - x_j|^r.$$

This lemma can be seen as a natural analogue of Lipschitz continuity on the set where the maximal function is small. In particular, it has been proven (for example in [1]) that for $u \in W^{1,1}_{loc}(\mathbb{R}^N, \mathbb{R}^m)$ and for $y_1, y_2 \in \{M\nabla u(x) \leq L\}$

$$\left| \int_0^1 \nabla u(ty_1 + (1-t)y_2) \cdot (y_1 - y_2) \, \mathrm{d}t \right| = |u(y_1) - u(y_2)| \leq CL|y_1 - y_2|.$$

Hence, one should view Lemma A.7 as a generalisation of this result.

Proof. For simplicity write $|\omega| := \|\omega\|_{\Lambda^r}$. Recall that

$$\|\nu^r(x_1, ..., x_{r+1})\|_{\Lambda_r} = r\mathcal{H}^r(\text{Sim}(x_1, ..., x_{r+1})) \leq C \max_{1 \leq i,j \leq r+1} |x_i - x_j|^r.$$

It suffices to show that there exists $z \in \mathbb{R}^N$ such that

$$\sum_{i=1}^{r+1} \int_{\text{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} |\omega| \, \mathrm{d}\mathcal{H}^r(y) \leq C\lambda \max_{1 \leq i,j \leq r+1} |x_i - x_j|^r. \tag{A.16}$$

Indeed, to see that (A.16) is enough, note that

$$\begin{aligned} \sum_{i=1}^{r+1} \fint_{\text{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} & \omega(\nu^r(x_1, ...x_{i-1}, z, x_{i+1}, ...)) \, \mathrm{d}\mathcal{H}^r(y) \\ &= \fint_{\text{Sim}(x_1,...,x_{r+1})} \omega(\nu^r(x_1, ..., x_{r+1})) \, \mathrm{d}\mathcal{H}^r(y). \end{aligned} \tag{A.17}$$

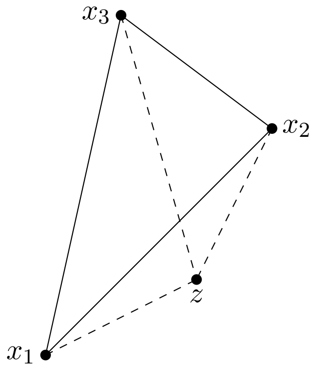

Figure A.1.: Illustration of (A.17) for $r = 2$. The integrals on the dashed 1-dimensional faces cancel out in (A.17) after applying Stokes' theorem.

and

$$\fint_{\mathrm{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} \omega(\nu^r(x_1,...x_{i-1},z,x_{i+1},...))\,\mathrm{d}\mathcal{H}^r(y) \leq \frac{1}{r}\int_{\mathrm{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} |\omega|\,\mathrm{d}\mathcal{H}^r(y).$$

The equation (A.17) can be verified by Stokes' theorem (A.15), using that boundary terms with a simplex with vertex z cancel out on the left-hand side of (A.17).

We now prove (A.16). W.l.o.g. $R = \max_{i,j\in[r+1]} |x_i - x_j| = |x_1 - x_2|$. Note that there exists a dimensional constant C_1 such that

$$|B_R(x_1) \cap B_R(x_2)| \geq C_1 R^N.$$

First, consider $x_1, ..., x_r \in B_R(x_1)$. For $z \in B_R(x_1)$ define $E(z)$ to be the r-dimensional hyperplane going through $x_1, ..., x_r$ and z. This is well-defined if z is not in the $(r-1)$ dimensional hyperplane F going through $x_1, ..., x_r$. Note that for $z, \tilde{z} \notin F$

$$z \in E(\tilde{z}) \Leftrightarrow \tilde{z} \in E(z). \tag{A.18}$$

As $M\omega(x_1) \leq \lambda$, we know that

$$\int_{B_R(x_1)} |\omega|(z)\,\mathrm{d}z \leq \lambda b_N R^N,$$

where b_N is the volume of the N-dimensional unit ball $B_1(0)$. As $\mathcal{H}^r(E(z) \cap B_R(x_1)) = b_r R^r$, it also follows by Fubini and (A.18)

$$\int_{B_R(x_1)} \int_{E(z)\cap B_R(x_1)} |\omega|(y)\,\mathrm{d}\mathcal{H}^r(y)\,\mathrm{d}z \leq \lambda b_N b_r R^{N+r}.$$

Using that $\mathrm{Sim}(x_1,...,x_r,z) \subset E(z) \cap B_R(x_1)$, we conclude that for $\mu > 0$

$$\left|\left\{z \in B_R(x_1)\colon \left|\int_{\mathrm{Sim}(x_1,\dots,x_r,z)} |\omega|(y)\,\mathrm{d}y\right| \geq \mu\right\}\right| \leq \frac{\lambda b_r b_N R^{N+r}}{\mu}. \tag{A.19}$$

Choose now $\mu^* = 2(r+1)b_r b_N R^r \lambda C_1^{-1}$. Plugging this into (A.19), we see that the measure of this set is smaller than $R^N(2(r+1))^{-1}$. Repeating this procedure for all $(r-1)$-dimensional faces of $\mathrm{Sim}(x_1,...,x_{r+1})$, we get that for $i > 1$

$$\left|\left\{z \in B_R(x_1)\colon \left|\int_{\mathrm{Sim}(x_1,\dots,x_{i-1},z,x_{i+1},\dots)} |\omega|(y)\,\mathrm{d}\mathcal{H}^r(y)\right| \geq \mu^*\right\}\right| \leq \frac{C_1 R^N}{2(r+1)},$$

and for $i = 1$

$$\left|\left\{z \in B_R(x_2)\colon \left|\int_{\mathrm{Sim}(z,x_2,\dots x_{r+1})} |\omega|(y)\,\mathrm{d}\mathcal{H}^r(y)\right| \geq \mu^*\right\}\right| \leq \frac{C_1 R^N}{2(r+1)}.$$

Hence, there exists $z \in B_R(x_1) \cap B_R(x_2)$ such that all the summands of (A.16) are smaller than $\mu^* = ((2(r+1))b_r b_N C_1^{-1})R^r\lambda$, i.e.

$$\sum_{i=1}^{r+1} \int_{\mathrm{Sim}(x_1,\dots x_{i-1},z,x_{i+1},\dots)} |\omega|\,\mathrm{d}\mathcal{H}^r(y) \leq \left(2(r+1)^2 b_r b_N C_1^{-1}\right)\lambda \max_{1\leq i,j\leq r+1} |x_i - x_j|^r.$$

This is what we wanted to prove. □

A.4. A Whitney-type extension theorem

First, let us recall the following Lipschitz extension theorem.

Theorem A.8 (Lipschitz extension theorem)**.** *Let $X \subset \mathbb{R}^N$ be a closed set and $u \in C(X,\mathbb{R}^d)$ such that*

$$|u(x) - u(y)| \leq L|x-y|. \tag{A.20}$$

Then there exists a function $v \in C(\mathbb{R}^N,\mathbb{R}^d)$ with $v_{|X} = u$ and such that v is Lipschitz on $\mathbb{R}^N$ with Lipschitz constant at most $C(N)L$ (i.e. the Lipschitz constant does not depend on X).

Of course, there are several ways to prove such a theorem, even with $C(N) = 1$ [90]. However, Whitney's proof [151] plays with the geometry of $\mathbb{R}^N$ quite nicely. Similar geometric ideas lies behind our proof for closed differential forms. First, let us define an analogue of (A.20).

Suppose that X is a closed subset of $\mathbb{R}^N$, such that $X^C = \mathbb{R}^N \backslash X$ is bounded and $|\partial X| = 0$.

Let $u \in C_c^\infty(\mathbb{R}^N,\Lambda^r)$ with $du = 0$. Let $L > 0$ be such that $\|u\|_{L^\infty(X)} \leq L$ and that for all

$x_1, ..., x_{r+1} \in X$ we have

$$\left| \fint_{\mathrm{Sim}(x_1,...,x_{r+1})} u(y)(\nu^r(x_1, ..., x_{r+1}))\,\mathrm{d}y \right| \leq L \max |x_i - x_j|^r. \tag{A.21}$$

Lemma A.9 (Whitney-type extension theorem)**.** *There exists a constant $C = C(N, r)$ such that for all $u \in C_c^\infty(\mathbb{R}^N, \Lambda^r)$ and X meeting the requirements above there exists $v \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \Lambda^r)$ with*

i) $dv = 0$ in the sense of distributions;

ii) $v(y) = u(y)$ for all $y \in X$;

iii) $\|v\|_{L^\infty} \leq CL$.

Remark A.10. The constant C does not depend on the choice of u or X, it is only important that the pair (u, X) satisfies (A.21). The assumption that X^C is bounded makes the proof easier, but may be dropped. It is not clear, whether the assumption that $|\partial X| = 0$ is necessary for the statement to hold or not.

Remark A.11. As one can see in the proof, the assumption $u \in C_c^\infty(\mathbb{R}^N, \Lambda^r)$ can be weakened to $u \in C_c^2(\mathbb{R}^N, \Lambda^r)$, as we only need the first two derivatives of u. However, it is important to remember that we cannot prove Lemma A.9 for the even weaker assumption $u \in L^1_{\mathrm{loc}}$, as (A.21) is not well-defined.

For the proof we follow the classical approach by Whitney with a few little twists. First, we will define the extension in (A.23). Then we prove that v satisfies properties i)-iii). ii) and iii) are quite easy to see from the definition of v, however it is hard to verify that i) holds. On the one hand, we show that the strong derivative of v exists almost everywhere, namely in $\mathbb{R}^N \backslash \partial X$ and that $dv = 0$ almost everywhere, where we use the assumption that the boundary of X is a null-set. On the other hand, we then prove that the distributional derivative dv is in fact also an L^1 function, yielding that $dv = 0$ in the sense of distributions.

We now start with the definition of the extension. Let us recall (cf. [139]) that for $X \subset \mathbb{R}^N$ closed we can find a collection of pairwise disjoint open cubes $\{Q_i^*\}_{i \in \mathbb{N}}$ such that

- Q_i^* are open dyadic cubes;
- $\cup_{i \in \mathbb{N}} \bar{Q}_i^* = X^C$;
- $\mathrm{dist}(Q_i^*, X) \leq l(Q_i^*) \leq 4\,\mathrm{dist}(Q_i^*, X)$, where $l(Q_i^*)$ denotes the sidelength of the cube.

Choose $0 < \varepsilon < 1/4$ and define another collection of cubes by $Q_i = (1+\varepsilon)Q_i^*$ (cube with the same center and sidelength $(1+\varepsilon)l(Q_i^*)$). Then

- $\cup_{i \in \mathbb{N}} Q_i = X^C$;
- For all $i \in \mathbb{N}$, the number of cubes Q_j such that $Q_i \cap Q_j \neq \emptyset$ is bounded by a dimensional constant $C(N)$;

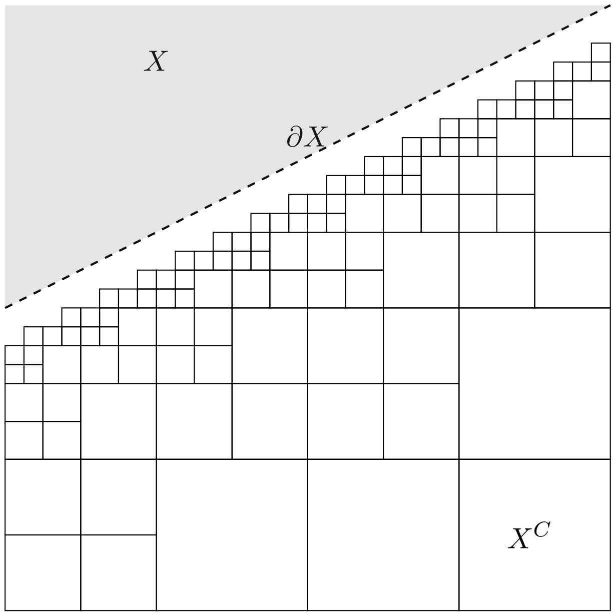

Figure A.2.: A collection of cubes Q_j^* near the boundary (up to a certain size).

- In particular, all $x \in \mathbb{R}^N$ are only contained in at most $C(N)$ cubes Q_i;
- The distance to the boundary is again comparable to the sidelength, i.e.

$$1/2 \operatorname{dist}(Q_i, X) \leq l(Q_i) \leq 8 \operatorname{dist}(Q_i, X).$$

Note that if X is $\mathbb{Z}^N$-periodic, then also Q_i can be chosen to be $\mathbb{Z}^N$ periodic (initially, we have a collection of dyadic cubes). Now consider $\varphi \in C_c^\infty((-1-\varepsilon, 1+\varepsilon)^N, [0,\infty))$ with $\varphi = 1$ on $(-1,1)^N$. We can rescale φ such that we obtain functions $\varphi_j^* \subset C_c^\infty(Q_j)$ with $\varphi_j^* = 1$ on Q_j^*. Define the partition of unity on X^C by

$$\varphi_j = \frac{\varphi_j^*}{\sum_{i \in \mathbb{N}} \varphi_i^*}.$$

Note that $0 \leq \varphi_j \leq 1$ and that there exists a constant $C > 0$ such that for all $j \in \mathbb{N}$

$$|\nabla \varphi_j| \leq C/8 \; l(Q_j)^{-1} \leq C \operatorname{dist}(Q_j, X)^{-1}.$$

For each cube Q_i, we may find an $x \in X$ such that $\operatorname{dist}(Q_i, x) = \operatorname{dist}(Q_i, X)$. Denote this x by x_i. For a multiindex $I = (i_1, ..., i_{r+1}) \in \mathbb{N}^{r+1}$, define

$$G(x_{i_1}, ..., x_{i_{r+1}}) = G(I) := \fint_{\operatorname{Sim}(x_{i_1}, ..., x_{i_{r+1}})} u(y) \, dy.$$

We now define the differential form $\alpha \in L^1(\mathbb{R}^N, \Lambda^r)$ by

$$\alpha(y) := \sum_{I \in \mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))). \tag{A.22}$$

Note that in this setting $G(I)(\nu^r(...)) \in \mathbb{R} = \Lambda^0$.

We claim that the function $v \in L^1_{\text{loc}}(\mathbb{R}^N, \Lambda^r)$ given by

$$v(y) := \begin{cases} u(y) & y \in X, \\ (-1)^r \alpha(y) & y \in X^C \end{cases} \tag{A.23}$$

is the function satisfying all the properties of Lemma A.9.

Lemma A.12. *The differential form α defined in* (A.22) *satisfies $\alpha \in L^1(X^C, \Lambda^r)$ and the sum in* (A.22) *converges pointwise and in L^1.*

Proof. Pointwise convergence is clear, as for fixed $y \in X^C$ only finitely many summands are nonzero in a neighbourhood of y (φ_i is only nonzero in Q_i and any point is only covered by at most $C(N)$ cubes). For L^1 convergence fix some $i_1 \in \mathbb{N}$. Note that there are at most $C(N)^r$ summands in $i_2, ..., i_{r+1}$, which are nonzero, as Q_{i_1} only intersects with $C(N)$ other cubes. Furthermore, note that for all i_l with $Q_{i_l} \cap Q_{i_1} \neq \emptyset$

$$\|d\varphi_{i_l}(y)\|_{\Lambda^1} \leq C \operatorname{dist}(y, X)^{-1} \leq Cl(Q_{i_1})^{-1}.$$

Moreover, we can bound ν^r by

$$\|\nu^r(x_{i_1}, ..., x_{i_{r+1}})\|_{\Lambda_r} \leq \max_{a,b \in \{i_1,..,i_{r+1}\}} |x_a - x_b|^r \leq Cl(Q_{i_1})^r.$$

Hence, we can bound the L^∞-norm of a nonzero summand of (A.22) by $C\|u\|_{L^\infty}$, as $|G(I)| \leq \|u\|_{L^\infty}$. As the support of the summand is contained in Q_{i_1}, we have that its L^1 norm is bounded by

$$C\|u\|_{L^\infty}|Q_{i_1}|.$$

Remember that any point in X^C is covered by only $C(N)$ cubes, such that the sum of $|Q_i|$ is bounded by $C(N)|X^C|$. Hence, the sum in (A.22) converges absolutely in L^1 and its L^1 norm is bounded by $C(N)^{r+1}C\|u\|_{L^\infty}|X^C|$. □

Lemma A.13. *The function v is strongly differentiable almost everywhere and satisfies $dv(y) = 0$ for all $y \in \mathbb{R}^N \backslash \partial X$.*

Proof. Note that $u \in C_c^\infty(\mathbb{R}^N, \Lambda^r)$ and hence v is strongly differentiable in $X \backslash \partial X$. Furthermore, the sum in (A.22) is a finite sum in a neighbourhood of y for all $y \in X^C$. As the summands are also C^∞, the sum is C^∞ in the interior of X^C.

By assumption $du = 0$, hence it remains to prove that $d\alpha(y) = 0$ for all $y \in X^C$. Note that in a neighbourhood of $y \in X^C$ again only finitely many summands are nonzero. Using that $d^2 = 0$ and the Leibniz rule, we get

$$d\alpha(y) = \sum_{I \in \mathbb{N}^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y)(G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))). \tag{A.24}$$

Observe that this term does not converge in L^1 and hence this identity is only valid pointwise.

Pick some $j \in \mathbb{N}$ such that $y \in Q_j$. As all φ_i sum up to 1 in X^C, we have

$$d\varphi_j(y) = -\sum_{I \in \mathbb{N}\setminus\{j\}} d\varphi_i(y).$$

Replace $d\varphi_j$ in the sum in (A.24) by $-\sum_{I \in \mathbb{N}\setminus\{j\}} d\varphi_i(y)$. Recall that $\nu^r(x_1, ..., x_{r+1}) = 0$ if $x_l = x_{l'}$ for some $l \neq l'$. Hence,

$$\begin{aligned}
d\alpha(y) &= \sum_{I \in \mathbb{N}^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&= \sum_{I \in (\mathbb{N}\setminus\{j\})^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&\quad + \sum_{l=1}^{r+1} \sum_{I \in \mathbb{N}^{r+1}:\, i_l = j} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&= \sum_{I \in (\mathbb{N}\setminus\{j\})^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&\quad - \sum_{l=1}^{r+1} \sum_{I \in (\mathbb{N}\setminus\{j\})^{r+1}} d\varphi_{i_{1(y)}} \wedge ... \wedge d\varphi_{i_{r+1}}(y) \\
&\qquad \wedge (G(x_{i_1}, ... x_{i_{l-1}}, x_j, x_{i_{l+1}}, ...)(\nu^r(x_{i_1}, ... x_{i_{l-1}}, x_j, x_{i_{l+1}}, ...))).
\end{aligned}$$

We apply Stokes' theorem (A.15) to the r-form u and the simplex with vertices $x_j, x_{i_1}, ..., x_{i_{r+1}}$, use that $du = 0$ and conclude that this term is 0, i.e.

$$\begin{aligned}
&G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}})) - \sum_{l=1}^{r+1} G(x_{i_1}, ..., x_{i_{l-1}}, x_j, x_{i_{l+1}}, ...)(\nu^r(x_{i_1}, ..., x_j, x_{i_{l+1}}, ...)) \\
&= -\frac{r-1}{r} \fint_{\mathrm{Sim}(x_j, x_{i_1}, ..., x_{i_{r+1}})} du(y)(\nu^{r+1}(x_j, x_{i_1}, ..., x_{i_{r+1}}))\, \mathrm{d}\mathcal{H}^r(y) = 0.
\end{aligned}$$

Hence, the pointwise derivative equals 0 almost everywhere. □

It is important to note that the sum (A.22) in the definition of α converges in L^1, but in general does not converge in $W^{1,1}$, and thus we have no information on the behaviour at the boundary of X^C. However, it suffices to show that the distribution dv for v given by (A.23) is actually an L^1 function. If $dv \in L^1$, we can conclude with Lemma A.13 that $dv = 0$ in the sense of distributions.

Lemma A.14. *The distributional exterior derivative of v defined in* (A.23) *satisfies $dv \in L^1(\mathbb{R}^N, \Lambda^{r+1})$, i.e. there exists an L^1 function $h \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that for all $\psi \in C_c^\infty(\mathbb{R}^N, \Lambda^{N-r-1})$*

$$(-1)^r \int_{X^C} \alpha \wedge d\psi + \int_X u \wedge d\psi = \int_{\mathbb{R}^N} h \wedge \psi.$$

Proof. Consider

$$\int_{X^C} \alpha(y) \wedge d\psi(y)\, \mathrm{d}y.$$

In view of the definition of α, this expression is given by:

$$\int_{\mathbb{R}^N} \sum_{I\in\mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \wedge d\psi \ \mathrm{d}y = (*).$$

We use the splitting $G(I) = (G(I) - u(\cdot)) + u(\cdot)$ and write $(*)$ as

$$\begin{aligned}(*) = &\int_{\mathbb{R}^N} \sum_{I\in\mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge ((G(I) - u(\cdot))(\nu^r(x_{i_1}, ..., x_{i_{r+1}})) \wedge d\psi \\ &+ \int_{\mathbb{R}^N} \sum_{I\in\mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \wedge d\psi \\ &= (\mathrm{I}) + (\mathrm{II}).\end{aligned} \tag{A.25}$$

Note that (I) defines a distribution given by an L^1 function. Indeed, the sum

$$\varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge ((G(I) - u(y))(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))$$

converges in $W^{1,1}(\mathbb{R}^N, \Lambda^{r+1})$. To see this, one can repeat the proof of Lemma A.12 and use that there are additional factors in the estimate of the norms. For this, note that if $z \in Q_{i_1}$

$$\|G(I) - u(z)\|_{\Lambda^r} \le Cl(Q_i)\|\nabla u\|_{L^\infty}$$

and

$$\|\nabla(G(I) - u(\cdot))(z)\|_{\Lambda^r} \le C\|\nabla u\|_{L^\infty}.$$

One gets improved regularity and may integrate by parts to eliminate the derivative of ψ.

Term (II) is not so easy to handle. We prove the following claims:

Claim 1: Let $1 \le s \le r$ and $I' = (i_s, ..., i_{r+1}) \in \mathbb{N}^{r-s+2}$. There exists $h_s \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that

$$\begin{aligned}&\int_{X^C} \sum_{I'\in\mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_s}, ..., x_{i_{r+1}}))) \wedge d\psi \\ &\quad = \int_{X^C} h_s \wedge \psi \\ &\quad\quad - \int_{X^C} \sum_{I'\in\mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s}(x_{i_{s+1}}, ..., x_{i_{r+1}})) \wedge d\psi.\end{aligned} \tag{A.26}$$

Here we use the notation that $\nu^0(x_{i_{r+1}}) = 1 \in \Lambda_0 = \mathbb{R}$.

Claim 2: There is $\tilde{h} \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N} \sum_{I'\in\mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \wedge d\psi \\ &\quad = \int_{X^C} \tilde{h} \wedge \psi + (-1)^r \int_{X^C} u \wedge d\psi.\end{aligned} \tag{A.27}$$

Note that Claim 2 follows from Claim 1 by an inductive argument. The domain of integration in (A.27) can be replaced by X^C as well, as all φ_{i_j} are supported in X^C.

First, let us conclude the proof under the assumption that Claim 1 holds true. Using (A.25) and Claim 2 we see that there is an $h \in L^1(\mathbb{R}^N, \mathbb{R}^d)$ such that

$$\int_{X^C} \alpha \wedge d\psi = \int_{\mathbb{R}^N} h \wedge \psi + (-1)^r \int_{X^C} u \wedge d\psi.$$

Recall that $du = 0$ in the sense of distributions and therefore

$$-\int_{X^C} u \wedge d\psi = \int_X u \wedge d\psi.$$

We conclude that there exists an L^1 function $h \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that

$$\int_{X^C} \alpha \wedge d\psi + (-1)^r \int_X u \wedge d\psi = \int_{\mathbb{R}^N} h \wedge \psi.$$

Thus, dv is an L^1 function.

It remains to prove Claim 1. Note that

$$\nu^{r-s+1}(x_{i_s}, ..., x_{i_{r+1}}) = \sum_{j=s}^{r+1} \nu^{r-s+1}(x_{i_s}, ..., x_{i_{j-1}}, y, x_{i_{j+1}}, ..., x_{i_{r+1}}). \tag{A.28}$$

This can be verified using that the wedge product is alternating and explicitly writing the right-hand side of (A.28).

Using this identity, we may split the right-hand side of (A.26) (denoted by (III)), i.e.

$$\begin{aligned}
\text{(III)} = &\sum_{j=s+1}^{r+1} \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \\
&\qquad \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_s}, ..., x_{i_{j-1}}, y, x_{i_{j+1}}, ..., x_{i_{r+1}}))) \wedge d\psi \\
&+ \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(y, x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi \\
&= \text{(IIIa)} + \text{(IIIb)}.
\end{aligned}$$

Arguing as in Lemma A.12, we see that the sum

$$\sum_{I \in \mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_s}, ..., x_{i_{j-1}}, y, x_{i_{j+1}}, ..., x_{i_{r+1}})))$$

is in fact convergent in L^1. Moreover, the index i_j only appears once in this sum. Recall that for $y \in X^C$

$$\sum_{i_s \in \mathbb{N}} d\varphi_{i_s}(y) = 0.$$

Thus,

$$\text{(IIIa)} = 0.$$

For (IIIb) note that $\sum_{i_1\in\mathbb{N}} \varphi_{i_s} = 1_{X^C}$ and, by the same argument as for (IIIa), we can write

$$(\text{IIIb}) = \int_{X^C} \sum_{I\in\mathbb{N}^{r-s+1}} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(y, x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi.$$

We can now integrate by parts to eliminate the exterior derivative in front of $\varphi_{i_{s+1}}$. Applying Lemma A.4, using $d^2 = 0$, the Leibniz rule and the fact that $\varphi_{i_j} \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$

$$\begin{aligned}
&(-1)^{r-s+1}(\text{IIIb})\\
&= \int_{X^C} \sum_{I\in\mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge d(u(\cdot)(\nu^{r-s+1}(y, x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi\\
&= \int_{X^C} \sum_{I\in\mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge ... \wedge d\varphi_{i_{r+1}}\\
&\qquad\qquad \wedge D^{r-s+1,r}(\nabla u(\cdot), (\nu^{r-s+1}(y, x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi\\
&\quad + (-1)^{(r-s)} \int_{X^C} \sum_{I\in\mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge ... \wedge d\varphi_{i_{r+1}}\\
&\qquad\qquad \wedge u(\cdot)(\nu^{r-s}(x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi
\end{aligned}$$

Arguing similarly to Lemma A.12 and as for term (I), we can show that

$$\sum_{I\in\mathbb{N}^r} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge ... \wedge \varphi_{i_{r+1}} \wedge D^{r-s+1,r}(\nabla u(\cdot), (\nu^{r-s+1}(y, x_{i_{s+1}}, ..., x_{i_{r+1}}))) \in W^{1,1}(\mathbb{R}^N, \Lambda^r),$$

and that this sum is convergent in $W^{1,1}$. Hence, we have shown that there exists $h_s \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that

$$\begin{aligned}
(\text{III}) &= \int_{\mathbb{R}^N} \sum_{I\in\mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_1}, ..., x_{i_{r+1}}))) \wedge d\psi\\
&= \int_{\mathbb{R}^N} h_s \wedge \psi\\
&\quad - \int_{\mathbb{R}^N} \sum_{I\in\mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s}(x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi
\end{aligned} \tag{A.29}$$

Hence, Claim 1 holds, completing the proof of Lemma A.14 □

This proves Lemma A.9. The property that

$$dv = 0 \quad \text{in the sense of distributions}$$

follows from Lemma A.13 and Lemma A.14. By definition, $v = u$ on X. Finally, we can bound the L^∞-norm of v by CL, as in the definition of α

$$\sum_{I\in\mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}})))$$

every summand can be bounded by CL due to (A.21) and the estimate $|d\varphi_j| \le C\,\mathrm{dist}(Q_j, X)^{-1}$. Again, we get the L^∞ bound, as only finitely many summands are nonzero for every $y \in X^C$.

With slight modifications one is able to prove the following variants.

Corollary A.15. *Let $u \in C^\infty(\mathbb{R}^N, \Lambda^r)$ with $du = 0$, let $L > 0$, and let $X \subset \mathbb{R}^N$ be a nonempty closed set such that $\|u\|_{L^\infty(X)} \le L$ and for all $x_1, ..., x_{r+1} \in X$ we have*

$$\left| \fint_{\mathrm{Sim}(x_1,...,x_{r+1})} u(y)(\nu^r(x_1, ..., x_{r+1}))\,\mathrm{d}y \right| \le L \max |x_i - x_j|^r.$$

Suppose further that $|\partial X| = 0$.

There exists a constant $C = C(N, r)$ such that for all $u \in C^\infty(\mathbb{R}^N, \Lambda^r)$ and X meeting these requirements there exists $v \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \Lambda^r)$ with

i) $dv = 0$ in the sense of distributions;

ii) $v(y) = u(y)$ for all $y \in X$;

iii) $\|v\|_{L^\infty} \le CL$.

This statement is proven in the same way as Lemma A.9, but all the statements are only true locally (e.g. the L^1 bounds on α are replaced by bounds in $L^1_{\mathrm{loc}}(X^C, \Lambda^r)$).

If we choose u and X to be $\mathbb{Z}^N$ periodic we get a suitable statement for the torus.

Corollary A.16. *Let $u \in C^\infty(T_N, \Lambda^r)$ with $du = 0$, let $L > 0$, and let $X \subset \mathbb{R}^N$ be a nonempty, closed, $\mathbb{Z}^N$-periodic set (which can be viewed as a subset of T_N) such that $\|u\|_{L^\infty(X)} \le L$ and for all $x_1, ..., x_{r+1} \in X$ we have*

$$\left| \fint_{\mathrm{Sim}(x_1,...,x_{r+1})} \tilde{u}(y)(\nu^r(x_1, ..., x_{r+1}))\,\mathrm{d}y \right| \le L \max |x_i - x_j|^r,$$

where $\tilde{u} \in C^\infty(\mathbb{R}^N, \Lambda^r)$ is the $\mathbb{Z}^N$-periodic representative of u. Suppose further that $|\partial X| = 0$.

There exists a constant $C = C(N, r)$ such that for all $u \in C^\infty(T_N, \Lambda^r)$ and X meeting these requirements there exists $v \in L^1(T_N, \Lambda^r)$ with

i) $dv = 0$ in the sense of distributions;

ii) $v(y) = u(y)$ for all $y \in X \subset T_N$;

iii) $\|v\|_{L^\infty} \le CL$.

As mentioned before, we can choose the cubes Q_j to be rescaled dyadic cubes. As the set X is periodic, the set of cubes (and hence also the partition of unity) and their projection points may also be chosen to be $\mathbb{Z}^N$-periodic. By definition then also the extension will be $\mathbb{Z}^N$-periodic.

A.5. L^∞-truncation

Now we prove the main result of this chapter on the L^∞-truncation of closed forms.

Theorem A.17 (L^∞-truncation of differential forms)**.** *There exist constants $C_1, C_2 > 0$ such that for all $u \in L^1(T_N, \Lambda^r)$ with $du = 0$ and all $L > 0$ there exists $v \in L^\infty(T_N, \Lambda^r)$ with $dv = 0$ and*

i) $\|v\|_{L^\infty(T_N,\Lambda^r)} \le C_1 L$;

ii) $|\{y \in T_N \colon v(y) \neq u(y)|\} \le \dfrac{C_2}{L} \displaystyle\int_{\{y\in T_N\colon |u(y)|>L\}} |u(y)|\,\mathrm{d}y$;

iii) $\|v - u\|_{L^1(T_N,\Lambda^r)} \le C_2 \displaystyle\int_{\{y\in T_N\colon |u(y)|>L\}} |u(y)|\,\mathrm{d}y$.

Given the Whitney-type extension obtained in Lemma A.16 and Lemma A.9 combined with Lemma A.7, the proof now roughly follows ZHANG's proof for Lipschitz truncation in [157]. First, we prove the statement in the case that v is smooth directly using our extension theorem for the set $X = \{Mu \le L\}$. After calculations similar to [157] we are able to show that this extension satisfies the properties of Theorem A.17. Afterwards, we prove the statement for $u \in L^1(T_N, \Lambda^r)$ by a standard density argument.

Proof. First, suppose that $u \in C^\infty(T_N, \Lambda^r)$. For $\lambda > 0$ define the set

$$X_\lambda = \{y \in T_N \colon Mu(y) \le \lambda\}.$$

Choose $2L \le \lambda \le 3L$ such that $|\partial X_\lambda| = 0$. Then, by Lemma A.7 and the extension Lemma A.16, there exists a $v \in L^1(T_N, \Lambda^r)$ with

1. $\{y \in T_N \colon v(y) \neq u(y)\} \subset X_\lambda^C$.
2. $\|v\|_{L^\infty} \le C\lambda$.
3. $dv = 0$ in the sense of distributions.

We need to show that

$$\|v - u\|_{L^1(T_N,\Lambda^r)} \le C_2 \int_{\{y\colon |u(y)|>L\}} |u(y)|\,\mathrm{d}y \tag{A.30}$$

and that

$$|\{y \in T_N \colon v(y) \neq u(y)\}| \le \frac{C_2}{L} \int_{\{y\colon |u(y)|>L\}} |u(y)|\,\mathrm{d}y. \tag{A.31}$$

Indeed, (A.30) follows from (A.31), as $\{v \neq u\} \subset X_\lambda^C$ and thus

$$\begin{aligned}\int_{T_N} |v(y) - u(y)|\,\mathrm{d}y &= \int_{X_\lambda^C} |v(y) - u(y)|\,\mathrm{d}y \\ &\le \int_{\{Mu\ge\lambda\}} |u(y)|\,\mathrm{d}y + \int_{\{Mu\ge\lambda\}} |v(y)|\,\mathrm{d}y\end{aligned}$$

$$\leq \int_{\{|u|\geq\lambda\}} |u(y)|\,\mathrm{d}y + 2CL|\{Mu \geq \lambda\}|.$$

Thus, it suffices to prove (A.31).

To this end, define the function $h\colon \Lambda^r \to \mathbb{R}$ by

$$h(z) = \begin{cases} 0 & \text{if } |z| < L, \\ |z| - L & \text{if } |z| \geq L. \end{cases}$$

Let $y \in \{Mu > \mu\}$ for $\mu \in \mathbb{R}$. Then there exists an $R > 0$ such that

$$\fint_{B_R(y)} |u(z)|\,\mathrm{d}z > \mu.$$

Thus,

$$\begin{aligned} M(h(u))(y) &\geq \fint_{B_R(y)} |h(u)(z)|\,\mathrm{d}z \\ &= \frac{1}{|B_R(y)|}\int_{B_R(y)\cap\{u\geq L\}} |u(z)| - L\,\mathrm{d}z \\ &\geq \fint_{B_R(y)} |u(z)|\,\mathrm{d}z - \frac{1}{|B_R(y)|}\int_{B_R(y)\cap\{u\leq L\}} |u(z)|\,\mathrm{d}z \\ &\qquad - \frac{1}{|B_R(y)|}\int_{B_R(y)\cap\{|u|\geq L\}} L\,\mathrm{d}z \\ &\geq \mu - L. \end{aligned}$$

Thus, $\{y \in T_N\colon Mu > \mu\} \subset \{y \in T_N\colon Mh(u)(y) > \mu - L\}$.

Using the weak-L^1 estimate for the maximal function (Proposition A.6), we get

$$\begin{aligned} |\{y \in T_N\colon Mu(y) \geq \lambda\}| &\leq |\{y \in T_N\colon Mh(u) \geq \lambda - L\}| \\ &\leq \frac{1}{\lambda - L} C \int_{T_N} |h(u)(z)|\,\mathrm{d}z \\ &\leq \frac{C}{L}\int_{T_N\cap\{|u|\geq L\}} |u(z)|\,\mathrm{d}z. \end{aligned} \tag{A.32}$$

This is what we wanted to show. Note that the proof only uses $u \in C^\infty(T_N, \Lambda^r)$ to define v and nowhere else, hence estimate (A.32) is valid for all $u \in L^1(T_N, \Lambda^r)$.

For general $u \in L^1(T_N, \Lambda^r)$, one may consider a sequence $u_n \subset C^\infty(T_N, \Lambda^r)$ with $du_n = 0$ and $u_n \to u$ in L^1 and pointwise almost everywhere. This sequence can be easily constructed by convolving with standard mollifiers.

Observe that for $\lambda > 0$

$$\begin{aligned} \int_{\{|u_n|\geq 2\lambda\}} |u_n|\,\mathrm{d}y &\leq \int_{\{|u_n-u|\geq|u|\}\cap\{|u_n|\geq 2\lambda\}} |u_n|\,\mathrm{d}y + \int_{\{|u_n-u|\leq|u|\}\cap\{|u_n|\geq 2\lambda\}} |u_n|\,\mathrm{d}y \\ &\leq 2\int_{\{|u|\geq\lambda\}} |u|\,\mathrm{d}y + 2\|u_n - u\|_{L^1}. \end{aligned} \tag{A.33}$$

Furthermore, we use the subadditivity of the maximal function and see that for all $y \in T_N$

$$Mu_n(y) \leq Mu(y) + M(u - u_n)(y).$$

Thus,

$$\{y \in T_N \colon Mu_n(y) \geq 2\lambda\} \subset \{y \in T_N \colon Mu(y) \geq \lambda\} \cup \{y \in T_N \colon M(u - u_n)(y) \geq \lambda\}.$$

Using the weak-L^1estimate for the maximal function (Proposition A.6) we see that

$$|\{y \in T_N \colon Mu(y) \leq \lambda\} \cap \{y \in T_N \colon Mu_n(y) \geq 2\lambda\}| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{A.34}$$

Choose some $\lambda \in (4L, 6L)$ such that for all $n \in \mathbb{N}$ $|\partial\{y \in T_N \colon Mu_n(y) \geq 2\lambda\}| = 0$. Then extend like in the first part of the proof to get a sequence v_n with $dv_n = 0$ and

a) $\|v_n\|_{L^\infty(T_N,\Lambda^r)} \leq 2C_1\lambda$;

b) $|\{y \in T_N \colon v_n(y) \neq u_n(y)|\} \leq \dfrac{C_2}{2\lambda} \displaystyle\int_{\{y\colon |u_n(y)|>2\lambda\}} |u_n(y)| \, \mathrm{d}y$;

c) $\|v_n - u_n\|_{L^1(T_N,\Lambda^r)} \leq C_2 \displaystyle\int_{\{y\colon |u_n(y)|>2\lambda\}} |u_n(y)| \, \mathrm{d}y$.

Letting $n \to \infty$, by a) this sequence converges, up to extraction of a subsequence, weakly$*$ to some $v \in L^\infty(T_N, \Lambda^r)$. The weak$*$-convergence implies $dv = 0$. Moreover, by construction, the set $\{y \in T_N \colon v_n \neq u_n\}$ is contained in the set $\{y \in T_N \colon Mu_n(y) \geq 2\lambda\}$. As $u_n \to u$ pointwise a.e. and in L^1, we get using (A.34) that $v = u$ on the set $\{y \in T_N \colon Mu(y) \leq \lambda\}$. (If v_n converges to u in measure on a set A and v_n weakly to some v, then $v = u$ on A.)

Hence, v defined as the weak$*$ limit of v_n satisifies

i) $\|v\|_{L^\infty(T_N,\Lambda^r)} \leq C_1\lambda \leq 6C_1L$;

ii) using (A.32) and $v = u$ on $\{y \in T_N \colon Mu(y) \leq \lambda\}$

$$|\{y \in T_N \colon u(y) \neq v(y)\}| \leq \frac{C_2}{L} \int_{\{y\in T_N\colon |u(y)|>L\}} |u(y)| \, \mathrm{d}y;$$

iii) using triangle inequality and $v_n - u_n \to 0$ in L^1, one obtains

$$\|v - u\|_{L^1(T_N,\Lambda^r)} \leq C_2 \int_{\{y\in T_N\colon |u(y)|>L\}} |u(y)| \, \mathrm{d}y.$$

Hence, v meets the requirements of Theorem A.17. □

Corollary A.18 (L^∞-truncation for sequences)**.** *Suppose that we have a sequence $u_n \subset L^1(\mathbb{R}^N, \Lambda^r)$ with $du_n = 0$, and that there exists $L > 0$ such that*

$$\int_{\{y\in T_N\colon |u_n(y)|>L\}} |u_n(y)| \, \mathrm{d}y \longrightarrow 0 \quad \textit{as } n \to \infty.$$

There exists a $C_1 = C_1(N, r)$ and a sequence $v_n \subset L^1(T_N, \Lambda^r)$ with $dv_n = 0$ and

i) $\|v_n\|_{L^\infty(T_N,\Lambda^r)} \leq C_1 L$;

ii) $\|v_n - u_n\|_{L^1(T_N,\Lambda^r)} \to 0$ *as* $n \to \infty$;

iii) $|\{y \in T_N : v_n(y) \neq u_n(y)\}| \to 0$ *as* $n \to \infty$.

This directly follows by applying Theorem A.17.

The proof of Theorem A.17 also works if L^1 is replaced by L^p for $1 < p < \infty$. Furthermore, we do not need to restrict us to periodic functions on $\mathbb{R}^N$, the statement is also valid for non-periodic functions.

Proposition A.19. *Let $1 \leq p < \infty$. There exist constants $C_1, C_2 > 0$, such that, for all $u \in L^p(\mathbb{R}^N, \Lambda^r)$ with $du = 0$ and all $L > 0$, there exists $v \in L^p(\mathbb{R}^N, \Lambda^r)$ with $dv = 0$ and*

i) $\|v\|_{L^\infty(\mathbb{R}^N,\Lambda^r)} \leq C_1 L$;

ii) $|\{y \in \mathbb{R}^N : v(y) \neq u(y)|\} \leq \frac{C_2}{L^p} \int_{\{y\in\mathbb{R}^N : |u(y)|>L\}} |u(y)|^p \, \mathrm{d}y$;

iii) $\|v - u\|^p_{L^p(\mathbb{R}^N,\Lambda^r)} \leq C_2 \int_{\{y\in\mathbb{R}^N : |u(y)|>L\}} |u(y)|^p \, \mathrm{d}y$.

As described, the proof is pretty much the same as for Theorem A.17. We may also want to truncate closed forms supported on an open bounded subset $\Omega \subset \mathbb{R}^N$ (cf. [28, 26]). This is possible, but we may lose the property, that they are supported in this subset. Let us, for simplicity, consider balls $\Omega = B_\rho(0)$ and, after rescaling, $\rho = 1$.

Proposition A.20. *Let $1 \leq p < \infty$. There exist constants $C_1, C_2 > 0$ such that, for all $u \in L^p(\mathbb{R}^N, \Lambda^r)$ with $du = 0$ and $\mathrm{spt}(u) \subset B_1(0)$ and all $L > 0$, there exists $v \in L^p(\mathbb{R}^N, \Lambda^r)$ with $dv = 0$ and*

i) $\|v\|_{L^\infty(\mathbb{R}^N,\Lambda^r)} \leq C_1 L$;

ii) $|\{y \in \mathbb{R}^N : v(y) \neq u(y)|\} \leq \frac{C_2}{L^p} \int_{\{y\in\mathbb{R}^N : |u(y)|>L\}} |u(y)|^p \, \mathrm{d}y$;

iii) $\|v - u\|^p_{L^p(\mathbb{R}^N,\Lambda^r)} \leq C_2 \int_{\{y\in\mathbb{R}^N : |u(y)|>L\}} |u(y)|^p \, \mathrm{d}y$;

iv) $\mathrm{spt}(v) \subset B_R(0)$, *where R only depends on the L^p-norm of u and on L.*

Again, this proof is very similar to the proof of Theorem A.17. Property iv) comes from the fact that if a function u is supported in $B_1(0)$, then its maximal function $Mu(y)$ decays fast as $y \to \infty$. Regarding the construction made in Section A.4 and Lemma A.7, it is not clear, how to avoid the rather weak statement iv), i.e. we cannot directly deal with arbitrary boundary values and need to modify the truncation.

Let us mention that this result also holds for vector-valued differential forms, i.e. $u \in L^p(\mathbb{R}^N, \Lambda^r \times \mathbb{R}^m)$, where the exterior derivative is taken componentwise.

Proposition A.21 (Vector-valued forms on the torus). *There exist constants $C_1, C_2 > 0$ such that, for all $u \in L^1(T_N, \Lambda^r \times \mathbb{R}^m)$ with $du = 0$ and all $L > 0$, there exists $v \in L^1(T_N, \Lambda^r \times \mathbb{R}^m)$ with $dv = 0$ and*

i) $\|v\|_{L^\infty(T_N,\Lambda^r\times\mathbb{R}^m)} \le C_1 L$;

ii) $|\{y \in T_N : v(y) \neq u(y)|\} \le \dfrac{C_2}{L} \displaystyle\int_{\{y\in T_N:\, |u(y)|>L\}} |u(y)|\,\mathrm{d}y$;

iii) $\|v - u\|_{L^1(T_N,\Lambda^r\times\mathbb{R}^m)} \le C_2 \displaystyle\int_{\{y\in T_N:\, |u(y)|>L\}} |u(y)|\,\mathrm{d}y$.

This statement follows directly from the proof of Theorem A.17 by simply truncating every component of u. Likewise, similar statements as in Propositions A.18, A.19 and A.19 follow for vector-valued differential forms.

A.6. Applications to $\mathcal{A}$-quasiconvex hulls and Young measures

In the following, we consider a linear and homogeneous differential operator of first order, i.e. we are given $\mathcal{A} : C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ of the form

$$\mathcal{A}u = \sum_{k=1}^{N} A_k \partial_k u,$$

where $A_k : \mathbb{R}^d \to \mathbb{R}^l$ are linear maps. We call a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ $\mathcal{A}$-quasiconvex if for all $\varphi \in C^\infty(T_N, \mathbb{R}^d)$ with $\int_{T_N} \varphi(y)\,\mathrm{d}y = 0$ and $\mathcal{A}\varphi = 0$, and for all $x \in \mathbb{R}^d$ then the following version of Jensen's inequality

$$f(x) \le \int_{T_N} f(x + \varphi(y))\,\mathrm{d}y \tag{A.35}$$

holds true. Fonseca and Müller showed that [65][1], if the constant rank condition seen below holds, then $\mathcal{A}$-quasiconvexity is a necessary and sufficient condition for weak$*$ lower-semicontinuity of the functional $I : L^\infty(\Omega, \mathbb{R}^d) \to [0, \infty)$ defined by

$$I(u) = \begin{cases} \displaystyle\int_\Omega f(u(y))\,\mathrm{d}y & \mathcal{A}u = 0 \\ \infty & \text{else.} \end{cases}$$

Definition A.22. *We say that $\mathcal{A}$ satisfies the property (ZL) if for all sequences $u_n \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that there exists an $L > 0$ with*

$$\int_{\{y\in T_N:\, |u_n(y)|>L\}} |u_n(y)|\,\mathrm{d}y \longrightarrow 0 \quad \text{as } n \to \infty,$$

there exists a $C = C(\mathcal{A})$ and a sequence $v_n \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

[1] Also see Chapter 4.

i) $\|v_n\|_{L^\infty(T_N,\mathbb{R}^d)} \le C_1 L$;

ii) $\|v_n - u_n\|_{L^1(T_N,\mathbb{R}^d)} \to 0$ *as* $n \to \infty$.

Our goal now is to show that (ZL) implies further properties for the operator $\mathcal{A}$. We first look at a few examples.

Example A.23.

a) As shown by Zhang [157], the operator $\mathcal{A} = \operatorname{curl}$ has the property (ZL). This is shown by using that its potential is the operator $\mathcal{B} = \nabla$. In fact, most of the applications here have been shown for $\mathcal{B} = \nabla$ relying on (ZL), but can be reformulated for $\mathcal{A}$ satisfying (ZL).

b) Let $W^k = (\mathbb{R}^N \otimes ... \otimes \mathbb{R}^N)_{\text{sym}} \subset (\mathbb{R}^N)^k$. We may identify $u \in C^\infty(T_N, W^k)$ with $\tilde{u} \in C^\infty(T_N, (\mathbb{R}^N)^k)$ and define the operator

$$\operatorname{curl}^{(k)} \colon C^\infty(T_N, W^k) \to C^\infty(T_N, (\mathbb{R}^N)^{k-1} \times \Lambda^2)$$

as taking the curl on the last component of $\tilde{u}$, i.e. for $I \in [N]^{k-1}$

$$(\operatorname{curl}^{(k)} u)_I = 1/2 \sum_{i,j \in \mathbb{N}} \partial_i \tilde{u}_{Ij} - \partial_j \tilde{u}_{Ii} e_i \wedge e_j$$

Note that this operator has the potential $\nabla^k : C^\infty(\mathbb{R}^N, \mathbb{R}) \to C^\infty(\mathbb{R}^N, W^k)$ (cf. [109]). To the best of the author's knowledge the proof of the property (ZL) is in this setting not written down anywhere explicitly, but basically combining the works [1, 67, 139, 157] yields the result.

c) In this work, it has been shown that the exterior derivative d satisfies the property (ZL). The most prominent example is $\mathcal{A} = \operatorname{div}$.

d) The result is also true, if we consider matrix-valued functions instead (cf. Proposition A.20). For example, (ZL) also holds if we consider $\operatorname{div} : C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times M}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^M)$, where

$$\operatorname{div}_i u(x) = \sum_{j=1}^N \partial_j u_{ji}(x).$$

e) Likewise, let $\mathcal{A}_1 \colon C^\infty(T_N, \mathbb{R}^{d_1}) \to C^\infty(T_N, \mathbb{R}^{l_1})$ and $\mathcal{A}_2 \colon C^\infty(T_N, \mathbb{R}^{d_2}) \to C^\infty(T_N, \mathbb{R}^{l_2})$ be two differential operators satisfying (ZL). Then also the operator

$$\mathcal{A} : C^\infty(T_N, \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to C^\infty(T_N, \mathbb{R}^{l_1} \times \mathbb{R}^{l_2})$$

defined componentwise for $u = (u_1, u_2)$ by

$$\mathcal{A}(u_1, u_2) = (\mathcal{A}_1 u_1, \mathcal{A}_2 u_2)$$

satisfies the property (ZL). The truncation is again done separately in the two components. The most prominent example, which is also covered by the result of this paper, is $\mathcal{A}_1 = \text{curl}$ and $\mathcal{A}_2 = \text{div}$, which is highly significant in elasticity and in the framework of compensated compactness.

An overview of the results one is able to prove using property (ZL) can be found in the lecture notes [115, Sec. 4] and in the book [128, Sec. 4,7], where they are formulated for the case of (curl)-quasiconvexity.

A.6.1. $\mathcal{A}$-quasiconvex hulls of compact sets

For $f \in C(\mathbb{R}^d, \mathbb{R})$ we can define the quasiconvex hull of f by (cf. [65, 25])

$$\mathcal{Q}_{\mathcal{A}} f(x) := \inf \left\{ \int_{T_N} f(x + \psi(y))\, \mathrm{d}y \colon \psi \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A},\ \int_{T_N} \psi = 0 \right\}. \quad \text{(A.36)}$$

$\mathcal{Q}_{\mathcal{A}} f$ is the largest $\mathcal{A}$-quasiconvex function below f [65].

In view of the separation theorem for convex sets in Banach spaces we define (cf. [42, 146, 147]) the $\mathcal{A}$-quasiconvex hull of a set $K \subset \mathbb{R}^d$ by

$$K_\infty^{\mathcal{A}qc} := \left\{ x \in \mathbb{R}^d \colon\ \forall f \colon \mathbb{R}^d \to \mathbb{R}\ \mathcal{A}\text{-quasiconvex with } f_{|K} \leq 0 \text{ we have } f(x) \leq 0 \right\},$$

and the $\mathcal{A}$-p-quasiconvex hull for $1 \leq p < \infty$ by

$$K_p^{\mathcal{A}qc} := \Big\{ x \in \mathbb{R}^d \colon\ \forall f \colon \mathbb{R}^d \to \mathbb{R}\ \mathcal{A}\text{-quasiconvex with } f_{|K} \leq 0 \ \text{ and } \\ |f(v)| \leq C(1 + |v|^p) \text{ we have } f(x) \leq 0 \Big\}.$$

The $\mathcal{A}$-p-quasiconvex hull for $1 \leq p < \infty$ can be alternatively defined via

$$K_p^{\mathcal{A}qc*} := \left\{ x \in \mathbb{R}^d \colon (\mathcal{Q}_{\mathcal{A}} \operatorname{dist}^p(\cdot, K))(x) = 0 \right\}.$$

If K is compact, then $K_p^{\mathcal{A}qc} = K_p^{\mathcal{A}qc*}$. Moreover, the spaces $K_p^{\mathcal{A}qc}$ are nested, i.e. $K_q^{\mathcal{A}qc} \subset K_{q'}^{\mathcal{A}qc}$ if $q \leq q'$. In [42] it is shown that equality holds for $\mathcal{A}$ being the symmetric divergence of a matrix, K compact and $1 < q, q' < \infty$. The proof can be adapted for different $\mathcal{A}$, but uses the Fourier transform and is not suitable for the cases $p = 1$ and $p = \infty$. Here, the property (ZL) comes into play.

For a compact set K we define the set $K^{\mathcal{A}app}$ (cf. [115]) as the set of all $x \in \mathbb{R}^d$ such that there exists a bounded sequence $u_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with

$$\operatorname{dist}(x + u_n, K) \longrightarrow 0 \quad \text{in measure, as } n \to \infty.$$

Theorem A.24. *Suppose that K is compact and $\mathcal{A}$ is an operator satisfying (ZL). Then*

$$K^{\mathcal{A}app} = K_\infty^{\mathcal{A}qc} = \left\{ x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = 0 \right\}. \quad \text{(A.37)}$$

Proof. We first prove $K^{\mathcal{A}app} \subset K_\infty^{\mathcal{A}qc}$. Let $x \in K^{\mathcal{A}app}$ and take an arbitrary $\mathcal{A}$-quasiconvex function $f : \mathbb{R}^d \to [0, \infty)$ with $f_{|K} = 0$. We claim that then $f(x) = 0$.

Take a sequence u_n from the definition of $K^{\mathcal{A}app}$. As f is continuous and hence locally bounded, $f(x+u_n) \to 0$ in measure and $0 \le f(x+u_n) \le C$. Quasiconvexity and dominated convergence yield

$$f(x) \le \liminf_{n\to\infty} \int_{T_N} f(x + u_n(y))\,\mathrm{d}y = 0.$$

$K_\infty^{\mathcal{A}qc} \subset \Big\{x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = 0\Big\}$ is clear by definition, as $\mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))$ is an admissible separating function.

The proof of the inclusion $\{x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = 0\} \subset K^{\mathcal{A}app}$ uses (ZL). If $\mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K)) = 0$, then there exists a sequence $\varphi_n \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\int_{T_N} \varphi_n = 0$ such that

$$0 = \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = \lim_{n\to\infty} \int_{T_N} \operatorname{dist}(x + \varphi_n(y), K)\,\mathrm{d}y.$$

As K is compact, there exists $R > 0$ such that $K \subset B(0, R)$. Moreover, as $x \in K_\infty^{\mathcal{A}qc}$, also $x \in B(0, R)$. This implies that

$$\lim_{n\to\infty} \int_{T_N \cap \{|\varphi_n| \ge 6R\}} |\varphi_n|\,\mathrm{d}y = 0.$$

We may apply (ZL) and find a sequence $\psi_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$\|\varphi_n - \psi_n\|_{L^1(T_N, \mathbb{R}^d)} \longrightarrow 0 \quad \text{as } n \to \infty$$

and

$$\|\psi_n\|_{L^\infty(T_N, \mathbb{R}^d)} \le CR.$$

Hence, $x \in K^{\mathcal{A}app}$. □

Remark A.25. Theorem A.24 shows that for all $1 \le p < \infty$

$$K^{\mathcal{A}app} = K_\infty^{\mathcal{A}qc} = \Big\{x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K)^p)(x) = 0\Big\} = K_p^{\mathcal{A}qc}.$$

This follows directly, as all the sets $K_p^{\mathcal{A}qc}$ are nested and, conversely, all the hulls of the distance functions are admissible f in the definition of $K_\infty^{\mathcal{A}qc}$.

Remark A.26. Such a kind of theorem is not true for general unbounded closed sets K. As a counterexample one may consider $\mathcal{A} = \operatorname{curl}$ (i.e. usual quasiconvexity) and look at the set of conformal matrices $K = \{\lambda Q : \lambda \in \mathbb{R}^+, Q \in SO(n)\} \subset \mathbb{R}^{n\times n}$. If $n \ge 2$ is even, by [116], there exists a quasiconvex function $F \colon \mathbb{R}^{n\times n} \to \mathbb{R}$ with $F(x) = 0 \Leftrightarrow x \in K$ and

$$0 \le F(A) \le C(1 + |A|^{n/2}).$$

On the other hand, let $n \ge 4$ be even and $F \colon \mathbb{R}^{n\times n} \to \mathbb{R}$ be a rank-one convex function with $F_{|K} = 0$ and for some $p < n/2$

$$0 \le F(A) \le C(1 + |A|^p).$$

Then $F = 0$ by [152].

A reason for the nice behaviour of compact sets is that for such sets all distance functions are coercive, i.e.

$$\operatorname{dist}(v, K)^p \geq |v|^p - C,$$

which is obviously not true for unbounded sets. Coercivity of a function is often needed for relaxation results (c.f [25]).

A.6.2. $\mathcal{A}$-∞ Young measures

We consider $\mathcal{M}(\mathbb{R}^d)$ the set of signed Radon measures with finite mass. Note that this is the dual space of $C_c(\mathbb{R}^d)$ with the dual pairing

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(y) \, \mathrm{d}\mu(y).$$

For a measurable set $E \subset \mathbb{R}^N$ we call $\mu : E \to \mathcal{M}(\mathbb{R}^d)$ weak$*$ measurable if the map

$$x \longmapsto \langle \mu_x, f \rangle$$

is measurable for all $f \in C_c(\mathbb{R}^d)$. Later, we may consider the space $L^\infty_w(E, \mathcal{M}(\mathbb{R}^d))$, which is the space of all weakly measurable maps such that $\operatorname{spt} \mu_x \subset B(0, R)$ for some $R > 0$ and for a.e. $x \in E$. This space is equipped with the topology $\nu^n \overset{*}{\rightharpoonup} \nu$ iff $\forall f \in C_0(\mathbb{R}^d)$

$$\langle \nu^n_x, f \rangle \overset{*}{\rightharpoonup} \langle \nu_x, f \rangle \text{ in } L^\infty(E).$$

Remark A.27. The topology of $L^\infty_w(E, \mathcal{M}(\mathbb{R}^d))$ is metrisable on bounded sets. In this setting, we call a set $X \subset L^\infty_w(E, \mathcal{M}(\mathbb{R}^d))$ bounded, if

1. There is $R > 0$, such that for all $\mu \in X$ the measure μ_x is supported in $B(0, R)$ for almost every $x \in E$;

2. There is $C > 0$, such that for all $\mu \in X$ the mass $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} \leq C$ for almost every $x \in E$.

Note that ν^n supported on $B(0, R)$ converges to ν if and only if for all $f \in C(\bar{B}(0, R))$ and all $g \in L^1(E)$

$$\int_E \langle \nu^n_x, f \rangle g(x) \, \mathrm{d}x \longrightarrow \int_E \langle \nu_x, f \rangle g(x) \, \mathrm{d}x.$$

If ν^n is bounded, then this equation holds for all f, g if and only if it holds for dense subsets of $C(\bar{B}(0, R))$ and $L^1(E)$. As these spaces are separable, we may consider a countable dense subset $(f_k, g_k)_{k \in \mathbb{N}}$ of $C(\bar{B}(0, R)) \times L^1(E)$ and the pseudo-metric

$$d_k(\nu, \mu) = \left| \int_E \langle \nu_x - \mu_x, f_k \rangle g_k(x) \, \mathrm{d}x \right|,$$

and then define the metric

$$d(\nu,\mu) = \sum_{k\in\mathbb{N}} 2^{-k} \frac{d_k(\nu,\mu)}{1+d_k(\nu,\mu)}.$$

Let us now recall the Fundamental Theorem of Young measures(cf. [18, 140]).

Proposition A.28 (Fundamental Theorem of Young measures)**.** *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and $u_j : E \to \mathbb{R}^d$ a sequence of measurable functions. There exists a subsequence u_{j_k} and a weak∗ measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ such that the following properties hold:*

i) $\nu_x \geq 0$ *and* $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} 1 \, \mathrm{d}\nu_x \leq 1$*;*

ii) $\forall f \in C_0(\mathbb{R}^d)$ *define* $\bar{f}(x) = \langle \nu_x, f\rangle$*. Then* $f(u_{j_k}) \overset{*}{\rightharpoonup} \bar{f}$ *in* $L^\infty(E)$*;*

iii) If $K \subset \mathbb{R}^d$ *is compact, then* $\operatorname{spt}\nu_x \subset K$ *if* $\operatorname{dist}(u_{j_k}, K) \to 0$ *in measure;*

iv) It holds

$$\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1 \text{ for a.e. } x \in E \tag{A.38}$$

if and only if

$$\lim_{M\to\infty} \sup_{k\in\mathbb{N}} |\{|u_{j_k}| \geq M\}| = 0;$$

v) If (A.38) *holds, then for all* $A \subset E$ *measurable and for all* $f \in C(\mathbb{R}^d)$ *such that* $f(u_{j_k})$ *is relatively weakly compact in* $L^1(A)$*, also*

$$f(u_{j_k}) \rightharpoonup \bar{f} \text{ in } L^1(A);$$

vi) If (A.38) *holds, then (iii) holds with equivalence.*

We call such a map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ the Young measure generated by the sequence u_{j_k}. One may show that every weak∗ measurable map $E \to \mathcal{M}(\mathbb{R}^d)$ satisfying (i) is generated by some sequence u_{j_k}.

Remark A.29. If u_k generates a Young measure ν and $v_k \to 0$ in measure (in particular, if $v_k \to 0$ in L^1), then the sequence $(u_k + v_k)$ still generates ν.

If $u : T_N \to \mathbb{R}^d$ is a function, we may consider the oscillating sequence $u_n(x) := u(nx)$. This sequence generates the homogeneous (i.e. $\nu_x = \nu$ a.e.) Young measure ν defined by

$$\langle \nu, f\rangle = \int_{T_N} f(u_n(y))\, \mathrm{d}y.$$

Question A.30. *What happens to the Young measure generated by a sequence u_{j_k} if we impose further conditions on it, for instance $\mathcal{A}u_{j_k} = 0$?*

For $1 \leq p < \infty$ we call a sequence $v_j \subset L^p(\Omega,\mathbb{R}^d)$ p-equi-integrable if

$$\lim_{\varepsilon\to 0} \sup_{j\in\mathbb{N}} \sup_{E\subset\Omega:\, |E|<\varepsilon} \int_E |v_j(y)|^p \, \mathrm{d}y = 0.$$

Definition A.31. *Let $1 \leq p \leq \infty$. We call a map $\nu\colon \Omega \to \mathbb{R}^d$ an $\mathcal{A}$-p-Young measure if there exists a p-equi-integrable sequence $\{v_j\} \subset L^p(\Omega, \mathbb{R}^d)$ (for $p = \infty$ a bounded sequence), such that v_j generates ν and satisfies $\mathcal{A}v_j = 0$.*

For $1 \leq p < \infty$ the set of $\mathcal{A}$-p Young measures was classified by Fonseca and Müller in [65] and for the special case $\mathcal{A} = \operatorname{curl}$ already in [86].

Proposition A.32. *Let $1 \leq p < \infty$ and $\mathcal{A}$ be a constant rank operator. A Young-measure $\nu : T_N \to \mathcal{M}(\mathbb{R}^d)$ is an $\mathcal{A}$-p-Young measure if and only if*

i) $\exists v \in L^p(T_N, \mathbb{R}^d)$ *such that* $\mathcal{A}v = 0$ *and*

$$v(x) = \langle \nu_x, \mathrm{id}\rangle = \int_{\mathbb{R}^d} y \,\mathrm{d}\nu_x(y) \text{ for a.e. } x \in T_N;$$

ii) $\displaystyle\int_{T_N}\int_{\mathbb{R}^d} |z|^p \,\mathrm{d}\nu_x(z)\,\mathrm{d}x < \infty;$

iii) for a.e. $x \in T_N$ *and all continuous* g *with* $|g(v)| \leq C(1 + |v|^p)$ *we have*

$$\langle \nu_x, g\rangle \geq \mathcal{Q}_{\mathcal{A}}g(\langle \nu_x, \mathrm{id}\rangle).$$

Recently, there has also been progress for so-called generalized Young measures ($p = 1$ is a special case), cf. [89, 129, 130, 93, 9].

Proposition A.32 only uses the constant rank property, the property (ZL) is not needed. However, for $p = \infty$ the situation changes. Let us recall the result of Kinderlehrer and Pedregal for $W^{1,\infty}$-Gradient Young measures (cf. [85, 91]), whose proof relies on the validity of (ZL) for curl.

Proposition A.33. *A weak* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^{N\times m})$ is a* curl-∞-*Young measure if and only if $\nu_x \geq 0$ a.e. and there exists $K \subset \mathbb{R}^{N\times m}$ compact, $v \in L^\infty(\Omega, \mathbb{R}^{N\times m})$ such that*

a) $\operatorname{spt}\nu_x \subset K$ *for a.e.* $x \in \Omega$;

b) $\langle \nu_x, \mathrm{id}\rangle = v(x)$ *for a.e.* $x \in \Omega$;

c) for a.e. $x \in \Omega$ *and all continuous* $g\colon \mathbb{R}^{N\times m} \to \mathbb{R}$ *we have*

$$\langle \nu_x, g\rangle \geq \mathcal{Q}_{\mathrm{curl}}g(\langle \nu_x, \mathrm{id}\rangle).$$

It is possible to state such a theorem in the general setting that $\mathcal{A}$ satisfies (ZL). The proofs from [85] mostly rely on this fact and this general case can be treated in the same fashion with few modifications. We do not give all the details of the proofs, but only the crucial steps where we use (ZL).

Let us first state the classification theorem for so called homogeneous $\mathcal{A}$-∞-Young measures, i.e. $\mathcal{A}$-∞-Young measures $\nu\colon T_N \to \mathcal{M}(\mathbb{R}^d)$ with the following properties:

i) $\operatorname{spt}\nu_x \subset K$ for a.e. $x \in T_N$ where $K \subset \mathbb{R}^d$ is compact;

ii) ν is a homogeneous Young measure, i.e. there exists $\nu_0 \in \mathcal{M}(\mathbb{R}^d)$ such that $\nu_x = \nu_0$ for a.e. $x \in T_N$.

Define the set $\mathcal{M}^{\mathcal{A}qc}(K)$ by (cf. [149])

$$\mathcal{M}^{\mathcal{A}qc}(K) = \left\{\nu \in \mathcal{M}(\mathbb{R}^d) \colon \nu \geq 0,\ \operatorname{spt}\nu \subset K,\ \langle \nu, f\rangle \geq f(\langle \nu, \mathrm{id}\rangle)\ \forall f \colon \mathbb{R}^d \to \mathbb{R}\ \mathcal{A}\text{-qc}\right\}. \tag{A.39}$$

Denote by $H_{\mathcal{A}}(K)$ the set of homogeneous $\mathcal{A}$-∞-Young measures supported on K. We are now able to formulate the classification of these measures (cf.[85, Theorem 5.1.]).

Proposition A.34 (Characterisation of homogeneous $\mathcal{A}$-∞-Young measures)**.** *Let $\mathcal{A}$ satisfy the property (ZL) and K be a compact set. Then*

$$H_{\mathcal{A}}(K) = \mathcal{M}^{\mathcal{A}qc}(K).$$

Using this result, one may prove the Characterisation of $\mathcal{A}$-∞-Young measures (c.f [85, Theorem 6.1]).

Proposition A.35 (Characterisation of $\mathcal{A}$-∞-Young measures)**.** *Suppose that $\mathcal{A}$ satisfies the property (ZL). A weak$*$ measurable map $\nu : T_N \to \mathcal{M}(\mathbb{R}^d)$ is an $\mathcal{A}$-∞-Young measure if and only if $\nu_x \geq 0$ a.e. and there exists $K \subset \mathbb{R}^d$ compact and $u \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with*

i) $\operatorname{spt}\nu_x \subset K$ *for a.e.* $x \in T_N$.

ii) $\langle \nu_x, \mathrm{id}\rangle = u$ *for a.e.* $x \in T_N$,

iii) $\langle \nu_x, f\rangle \geq f(\langle \nu_x, \mathrm{id}\rangle)$ *for a.e.* $x \in T_N$ *and all continuous and $\mathcal{A}$-quasiconvex* $f : \mathbb{R}^d \to \mathbb{R}$.

As mentioned, the proofs in the case $\mathcal{A} = \operatorname{curl}$ can be found in [85, 115, 128]. Let us shortly describe the strategy of the proofs. For Proposition A.34 one may prove that $H_{\mathcal{A}}(K)$ is weakly compact, that averages of (non-homogeneous) $\mathcal{A}$-infty Young measures are in $H_{\mathcal{A}}(K)$ and that the set $H^x_{\mathcal{A}}(K) = \{\nu \in H_{\mathcal{A}} \colon \langle \nu, \mathrm{id}\rangle = x\}$ is weak* closed and convex. The characterisation theorem then follows by using Hahn-Banachs separation theorem and showing that any $\mu \in M^{\mathcal{A}qc}$ cannot be separated from $H_{\mathcal{A}}(K)$, i.e. for all $f \in C(K)$ and for all $\mu \in M^{\mathcal{A}qc}(K)$ with $\langle \mu, \mathrm{id}\rangle = 0$

$$\langle \nu, f\rangle \geq 0 \text{ for all } \nu \in H^0_{\mathcal{A}}(K) \Rightarrow \langle \mu, f\rangle \geq 0.$$

Proposition A.35 then can be shown using Proposition A.34 and a localisation argument.

A.6.3. On the proofs of Propositions A.34 and A.35

In this section, we present the proof of Proposition A.34, basing on its counterpart for gradient Young measures in [115]. After that we shortly sketch the proof of A.35, which is then done by a standard technique of approximation on small cubes [2].

The property (ZL) is helpful due to the following two observations:

1. If $\nu \in H_{\mathcal{A}}(K)$ is a homogeneous $\mathcal{A}$-∞-Young measure, then by using (ZL) we can find a sequence generating ν with an L^∞-bound only depending on $|K|_\infty := \sup_{y\in K} |y|$ (cf. Lemma A.37)

2. A Young measure ν is an $\mathcal{A}$-∞-Young measure if there is $v_n \subset L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ and $L > 0$ such that

$$\int_{|u_n|\geq L} |u_n| \,\mathrm{d}x \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Remark A.36. Moreover, note that, if a sequence $u_n \subset L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ generates a homogeneous Young measure ν, we can find $v_n \subset C_c^\infty((0,1)^N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\|v_n\|_{L^\infty} \leq C\|u_n\|_{L^\infty}$ and $\|u_n - v_n\|_{L^1} \to 0$. In particular, v_n still generates the same homogeneous Young measure.

To find such a sequence, recall that there is a potential $\mathcal{B}$ of order $k_{\mathcal{B}}$ to the differential operator $\mathcal{A}$. Let us, for simplicity, assume that all u_n have zero average. Then we can write

$$u_n = \mathcal{B}U_n$$

with $\|U_n\|_{W^{k_{\mathcal{B}},q}} \leq C_q\|u_n\|_{L^q} \leq C_q\|u_n\|_{L^\infty}$ for all $1 < q < \infty$ and a constant $C_q > 0$. Let us define

$$U_{n,i,j}(x) = \varphi_j(x) i^{-k_{\mathcal{B}}} U_n(ix), \quad u_{n,i,j}(x) = \mathcal{B}U_{n,i,j}(x),$$

for a suitable sequence of cut-offs $\varphi_j \to 1$ in $L^1((0,1)^N, \mathbb{R})$. Picking suitable subsequences $i(n)$ and $j(n)$ we obtain a sequence $u_{n,i(n),j(n)}$ bounded in L^∞, still generating ν, but with compact support in $(0,1)^N$. Convolution with a standard mollifier gives a sequence v_n that is also in $C_c^\infty((0,1)^N, \mathbb{R}^d)$

Lemma A.37. *(Properties of $H_{\mathcal{A}}(K)$)*

i) If $\nu \in H_{\mathcal{A}}(K)$ with $\langle \nu, \mathrm{id}\rangle = 0$, then there exists a sequence $u_j \subset L^\infty(T_N, \mathbb{R}^d)$ such that $\mathcal{A}u_j = 0$, u_j generates ν and $\|u_j\|_{L^\infty(T_N,\mathbb{R}^d)} \leq C \sup_{z\in K} |z| = C|K|_\infty$.

ii) $H_{\mathcal{A}}(K)$ is weakly$*$ compact in $\mathcal{M}(\mathbb{R}^d)$.

Proof. i) follows from the definition of $H_{\mathcal{A}}(K)$. The uniform bound on the L^∞ norm of u_j can be guaranteed by (ZL) and vi) in Theorem A.28.

For the weak$*$ compactness note that $H_{\mathcal{A}}(K)$ is contained in the weak$*$ compact set $\mathcal{P}(K)$ of probability measures on K. As the weak$*$ topology is metrisable on $\mathcal{P}(K)$ it suffices to show that $H_{\mathcal{A}}(K)$ is sequentially closed. Hence, we consider a sequence $\nu_k \subset H_{\mathcal{A}}(K)$ with $\nu_k \overset{*}{\rightharpoonup} \nu$ and show that $\nu \in H_{\mathcal{A}}(K)$.

[2] which is quite similar to the argumentation in Chapter 4 in the proofs of Theorem 4.10 and Theorem 4.16.

Due to the definition of Young measures, we may find sequences $u_{j,k} \subset L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $u_{j,k}$ generates ν_k for $j \to \infty$. Recall that the topology of generating Young measures is metrisable on bounded set of $L^\infty(T_N, \mathbb{R}^d)$ (c.f. Remark A.27). We may find a subsequence $u_{j_k,k}$ which generates ν. As we know that $\|u_{j_k,k}\|_{L^\infty} \leq C|K|_\infty$, $\nu \in H_{\mathcal{A}}(K)$ and hence $H_{\mathcal{A}}(K)$ is closed. □

Lemma A.38. *Let ν be an $\mathcal{A}$-∞-Young measure generated by a bounded sequence $u_k \subset L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$. Then the measure $\bar{\nu}$ defined via duality for all $f \in C_0(\mathbb{R}^d)$ by*

$$\langle \bar{\nu}, f \rangle = \int_{T_N} \langle \nu_x, f \rangle \, \mathrm{d}x$$

is in $H_{\mathcal{A}}(K)$.

Proof. For $n \in \mathbb{N}$ define $u_k^n \subset L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ by $u_k^n(x) = u_k(nx)$. Then for all $f \in C_0(\mathbb{R}^d)$

$$f(u_k^n) \overset{*}{\rightharpoonup} \int_{T_N} f(u_k) \text{ in } L^\infty(T_N, \mathbb{R}^d) \quad \text{as } n \to \infty.$$

Note that by Theorem A.28 ii) we also have

$$\int_{T_N} f(u_k(x)) \, \mathrm{d}x \longrightarrow \int_{T_N} \langle \nu_x, f \rangle \, \mathrm{d}x \quad \text{as } k \to \infty.$$

Due to metrisability on bounded sets (Remark A.27), we can find a subsequence $u_k^{k(n)}$ in $L^\infty(T_N, \mathbb{R}^d)$ such that

$$f(u_k^{n(k)}) \overset{*}{\rightharpoonup} \int_{T_N} \langle \nu_x, f \rangle \, \mathrm{d}x \quad \text{as } k \to \infty.$$

Thus, $\bar{\nu} \in H_{\mathcal{A}}(K)$. □

Lemma A.39. *Define the set $H_{\mathcal{A}}^x(K) := \{\nu \in H_{\mathcal{A}} \colon \langle \nu, \mathrm{id} \rangle = x\}$. Then $H_{\mathcal{A}}^x(K)$ is weak$*$ closed and convex.*

Proof. Weak$*$-closedness is clear by Lemma A.37. For convexity, let ν_1, ν_2 be $\mathcal{A}$-∞-Young measures. By an argumentation following Remark A.36 (and Lemma 3.6), we can find sequences $v_n \subset C_c^\infty((0,\lambda) \times (0,1)^{N-1}, \mathbb{R}^d)$ and $w_n \subset C_c^\infty((\lambda,1) \times (0,1)^{N-1}, \mathbb{R}^d)$ that generate ν_1 and ν_2, respectively. Define

$$u_n = \begin{cases} v_n & \text{in } (0,\lambda) \times (0,1)^{N-1}, \\ w_n & \text{in } (\lambda,1) \times (0,1)^{N-1}. \end{cases}$$

and $u_{n,i}$ via $u_{n,i}(x) = u_n(ix)$. Then proceeding as in Lemma A.38, picking a suitable subsequence $i(n)$ yields that $u_{n,i(n)}$ generates $\lambda\nu_1 + (1-\lambda)\nu_2$. □

We proceed with the proof of the characterisation of homogeneous $\mathcal{A}$-∞-Young measures.

Proof of Theorem A.34: We have that $H_{\mathcal{A}}(K) \subset \mathcal{M}^{\mathcal{A}qc}$ due to the fundamental theorem of Young measures: $\nu \geq 0$ and $\operatorname{spt} \nu \subset K$ are clear by i) and iii) of Theorem A.28. The corresponding inequality follows by $\mathcal{A}$-quasiconvexity, i.e. if $u_n \subset L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ generates the Young measure ν, then

$$\langle \nu, f \rangle = \lim_{n\to\infty} \int_{T_N} f(u_n(y))\,\mathrm{d}y \geq \liminf_{n\to\infty} f\left(\int_{T_N} u_n(y)\,\mathrm{d}y\right) = f(\langle \nu, \mathrm{id}\rangle).$$

To prove $M^{\mathcal{A}qc}(K) \subset H_{\mathcal{A}}(K)$, w.l.o.g. consider a measure such that $\langle \nu, \mathrm{id}\rangle = 0$. We just proved that $H^0_{\mathcal{A}}(K)$ is weak$*$ closed and convex. Remember that $C(K)$ is the dual space of the space of signed Radon measures $\mathcal{M}(K)$ with the weak$*$ topology (see e.g. [132]). Hence, by Hahn-Banach separation theorem, it suffices to show that for all $f \in C(K)$ and all $\mu \in M^{\mathcal{A}qc}(K)$ with $\langle \mu, \mathrm{id}\rangle = 0$

$$\langle \nu, f \rangle \geq 0 \text{ for all } \nu \in H^0_{\mathcal{A}}(K) \quad \Longrightarrow \quad \langle \mu, f \rangle \geq 0.$$

To this end, fix some $f \in C(K)$, consider a continuous extension to $C_0(\mathbb{R}^d)$ and let

$$f_k(x) = f(x) + k\,\mathrm{dist}^2(x, K).$$

We claim that

$$\lim_{k\to\infty} \mathcal{Q}_{\mathcal{A}} f_k(0) \geq 0. \tag{A.40}$$

If we show (A.40), μ satisfies

$$\langle \mu, f \rangle = \langle \mu, f_k \rangle \geq \langle \mu, \mathcal{Q}_{\mathcal{A}} f_k \rangle \geq \mathcal{Q}_{\mathcal{A}} f_k(0),$$

finishing the proof. For the identity $\langle \mu, f \rangle = \langle \mu, f_k \rangle$ recall that μ is supported in K and $\mathrm{dist}(x, K) = 0$ for $x \in K$.

Hence, suppose that (A.40) is wrong. As f_k is strictly increasing, there exists $\delta > 0$ such that

$$\mathcal{Q}_{\mathcal{A}} f_k(0) \leq -2\delta, \quad k \in \mathbb{N}.$$

Using the definition of the $\mathcal{A}$-quasiconvex envelope for functions(4.10), we get $u_k \subset L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\int_{T_N} u_k(y)\,\mathrm{d}y = 0$ and

$$\int_{T_N} f_k(u_k(y))\,\mathrm{d}y \leq -\delta. \tag{A.41}$$

We may assume that $u_k \rightharpoonup 0$ in $L^2(T_N, \mathbb{R}^d)$ and also that $\mathrm{dist}^2(u_k, K) \to 0$ in $L^1(T_N)$. By property (ZL), there exists a sequence $v_k \in \ker \mathcal{A}$ bounded in $L^\infty(T_N, \mathbb{R}^d)$ with $\|u_k - v_k\|_{L^1} \to 0$. v_k generates (up to taking subsequences) a Young measure ν with $\operatorname{spt} \nu_x \subset K$.

Then for fixed $j \in \mathbb{N}$, using Lemma A.38 and that $\bar{\nu} \in H_{\mathcal{A}}(K) \subset M^{\mathcal{A}qc}(K)$,

$$\liminf_{k\to\infty} \int_{T_N} f_j(u_k(y))\,\mathrm{d}y \geq \liminf_{k\to\infty} \int_{T_N} f_j(v_k(y))\,\mathrm{d}y = \int_{T_N}\int_{\mathbb{R}^d} f_j \;\mathrm{d}\nu_x\,\mathrm{d}x = \langle \bar{\nu}, f\rangle \geq 0.$$

But this is a contradiction to (A.41), as $f_k \geq f_j$ if $k \geq j$. □

Let us finally outline the strategy of the proof for Proposition A.35. For details we refer to [85, 115].

Proof of Propostion A.35(Sketch). Necessity of condition i)-iii) is established by the following argument. i) and ii) follow directly from the fact that the Young-measure μ is generated by an $\mathcal{A}$-free sequence that, up to a subsequence, has a weak-$*$-limit u. iii) follows from the lower-semicontinuity statement of Fonseca and Müller [65].

To prove sufficiency of these conditions, one needs to construct a sequence generating the Young-measure ν. Let us suppose that $u = 0$, otherwise we define the Young-measure $\tilde{\nu} = \nu - u$. Then we find a sequence v_n generating $\tilde{\nu}$ and, consequently, $v_n + u$ generates ν.

To find such a sequence one divides T_N into subcubes and approximates ν by maps $\nu_n \colon T_N \to \mathcal{M}(\mathbb{R}^d)$, which are constant on the subcubes. For each subcube Q one then constructs a sequence $v^Q_{n,m} \subset L^\infty(Q,\mathbb{R}^d) \cap \ker\mathcal{A}$, $m \in \mathbb{N}$, that generates ν_n and satisfies $v^Q_{n,m} \in C^\infty_c(Q,\mathbb{R}^d)$. These $v^Q_{n,m}$ then give a sequence $v_{n,m}$ generating ν_n and taking a suitable diagonal sequence one may find a sequence generating ν (cf. [115, Proof of Theorem 4.7]). □

B. L^∞-truncation: divsym free matrices in dimension three

Up to minor changes, this chapter coincides with the publication.

- [20]: Behn, L., Gmeineder, F. and Schiffer, S. *On symmetric div-quasiconvex hulls and divsym-free L^∞ truncations*

The treatment of $\mathcal{A}$-quasiconvex sets (Section B.6) is also part of Chapter 6.

B.1. Introduction

B.1.1. Aim and scope

One of the key problems in continuum mechanics is the mathematical description of the plasticity behaviour of solids. Such solids are usually modelled by reference configurations $\Omega \subset \mathbb{R}^3$ subject to loads or forces and corresponding *velocity fields* $v\colon \Omega \to \mathbb{R}^3$. The (elasto)plastic behaviour of the material is mathematically described in terms of the stress tensor $\sigma\colon \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and is dictated by the precise target $K \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ where it takes values; K is usually referred to as the *elastic domain*. When ideal plasticity is assumed and potential hardening effects are excluded, K is a compact set in $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ with non-empty interior. As prototypical examples, in the Von Mises or Tresca models used for the description of metals or alloys, we have $K = \{\sigma \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\colon\ \mathbf{f}(\sigma^D) \leq \theta\}$ with a threshold $\theta > 0$, the deviatoric stress $\sigma^D := \sigma - \dfrac{1}{3}\mathrm{tr}(\sigma)E_{3\times 3}$ and *convex* $\mathbf{f}\colon \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$. Generalising this to $K = \{\sigma \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\colon\ \mathbf{f}(\sigma^D) + \vartheta\mathrm{tr}(\sigma) \leq \theta\}$ for $\vartheta > 0$ as in the Drucker-Prager or Mohr-Coulomb models for concrete or sand (cf. [55, 97]), such models take into account persisting volumetric changes induced by the hydrostatic pressure as plasticity effects. In all of these models, K is a *convex* set. This opens the gateway to the techniques from convex analysis, and we refer to [72, 97] for more detail.

As the main motivation for the present chapter, the convexity assumption on the elastic domain K is *not satisfied* by all materials. A prominent example where the non-convexity of K can be observed explicitely is fused silica glass (cf. Meade & Jeanloz [108]). Slightly more generally, for amorphous solids being deformed subject to shear, experiments on the molecular dynamics (cf. Maloney & Robbins [103]) exhibit the formation of characteristic patterns in the underlying deformation fields. As a possible explanation of this phenomenon, the emergence of such patterns on the *microscopic* level displays the effort of the material to cope with the enduring *macroscopic* deformations. Within the framework of limit analysis [97], Schill et al. [136] offer a link between the non-convexity of K and the appearance of such fine microstructure. Working from plastic dissipation principles, the corresponding static problem is identified in [136] as

$$\sup_\sigma \inf_v \left\{ \int_\Omega \sigma \cdot \nabla v \, \mathrm{d}x \colon \; \sigma \in L^\infty_{\mathrm{div}}(\Omega, K), \;\; v \in W^{1,1}(\Omega, \mathbb{R}^N), \quad v = g \text{ on } \partial\Omega \right\} \tag{B.1}$$

for given boundary data $g \colon \partial\Omega \to \mathbb{R}^3$. Here, $L^\infty_{\mathrm{div}}(\Omega, K)$ is the space of all $L^\infty(\Omega, K)$-maps which are row-wise divergence-free (or solenoidal) in the sense of distributions; note that, if even we admitted general $\sigma \in L^\infty(\Omega, K)$ in (B.1), the variational principle would be non-trivial only for $\sigma \in L^\infty_{\mathrm{div}}(\Omega, K)$. Stability under microstructure formation, in turn, is linked to the existence of solutions of (B.1); cf. Müller [115] for a discussion of the underlying principles. Towards the existence of solutions, the direct method of the Calculus of Variations requires semicontinuity, and it is here where the set K must be relaxed. By the constraints on σ, this motivates the passage to the *symmetric div-quasiconvex hull* of K as studied by Conti, Müller & Ortiz [42]. In the present paper, we complete the characterisation of such hulls (cf. Theorem B.1 below) and thereby answer a conjecture posed in [42] in the affirmative. To state our result, we pause and remind the reader of the requisite terminology first.

B.1.2. Divsym-quasiconvexity and the main result

Following [42], we call a Borel measurable, locally bounded function $F \colon \mathbb{R}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}$ *symmetric div-quasiconvex* if

$$F(\xi) \leq \int_{T_N} F(\xi + \varphi(x)) \, \mathrm{d}x \tag{B.2}$$

holds for all $\xi \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$ and all admissible test maps

$$\varphi \in \mathcal{T}_\mathcal{A} := \left\{ \varphi \in \mathrm{C}^\infty(T_N, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \quad \mathrm{div}(\varphi) = 0, \int_{T_N} \varphi \, \mathrm{d}x = 0 \right\}, \tag{B.3}$$

where T_N denotes the N-dimensional torus. Here, the divergence is understood in the row- (or equivalently, column-)wise manner. Accordingly, the *symmetric div-quasiconvex (or divsym-quasiconvex) envelope* of a Borel measurable, locally bounded function $F \colon \mathbb{R}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}$ is defined as the largest symmetric div-quasiconvex function below F; more explicitely,

$$\mathcal{Q}_\mathcal{A} F(\xi) := \inf \left\{ \int_{T_N} F(\xi + \varphi(x)) \, \mathrm{d}x \colon \; \varphi \in \mathcal{T}_\mathcal{A} \right\}. \tag{B.4}$$

Divsym-quasiconvexity is a strictly weaker notion than convexity, which can be seen by Tartar's example [141] $f \colon \mathbb{R}^{N\times N}_{\mathrm{sym}} \ni \xi \mapsto (N-1)|\xi|^2 - \mathrm{tr}(\xi)^2$. The discussion in Section B.1.1 necessitates a notion of divsym-quasiconvexity *for sets*. Inspired by the separation theory from convex analysis, we call a compact set $K \subset \mathbb{R}^{N\times N}_{\mathrm{sym}}$ *symmetric div-quasiconvex* provided for each $\xi \in \mathbb{R}^{N\times N}_{\mathrm{sym}} \setminus K$ there exists a symmetric div-quasiconvex $g \in \mathrm{C}(\mathbb{R}^{N\times N}_{\mathrm{sym}}; [0, \infty))$ such that $g(\xi) > \max_K g$. The relaxation of the elastic domains $K \subset$

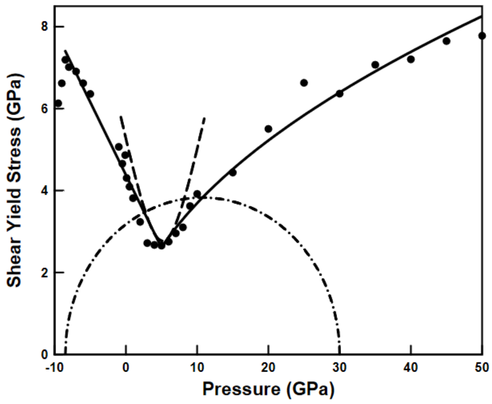

Figure B.1.: Molecular dynamics computations for fused silica glass linking pressure and shear yield stress, taken from SCHILL et al. [136, Fig. 17(b)]. Within the framework of limit analysis [97], the non-convexity of the critical state line (thick line) is linked to the instability for microstructure formation (cf. [136, Sec. 4]) and so a suitable relaxation is required.

$\mathbb{R}^{N\times N}_{\mathrm{sym}}$ in turn is defined in terms of the symmetric div-quasiconvex envelopes of distance functions. For a compact subset $K\subset\mathbb{R}^{N\times N}_{\mathrm{sym}}$ and $1\leq p<\infty$, put $f_p(\xi):=\mathrm{dist}^p(\xi,K)$. The *p-symmetric div-quasiconvex hull* of K then is defined by

$$K^{(p)}:=\{\xi\in\mathbb{R}^{N\times N}_{\mathrm{sym}}\colon\ \mathscr{Q}_{\mathcal{A}}f_p(\xi)=0\},\tag{B.5}$$

whereas we set for $p=\infty$:

$$K^{(\infty)}:=\left\{\xi\in\mathbb{R}^{N\times N}_{\mathrm{sym}}\colon\ \begin{array}{c}g(\xi)\leq\max_K g \text{ for all symmetric}\\ \text{div-quasiconvex } g\in\mathrm{C}(\mathbb{R}^{N\times N}_{\mathrm{sym}};[0,\infty))\end{array}\right\}.\tag{B.6}$$

Both (B.5) and (B.6) are the natural generalisations of the usual convex hulls to the symmetric div-quasiconvex context, and one easily sees that $K^{(\infty)}$ is the smallest symmetric div-quasiconvex, compact set containing K. If the distance function to K is nicely coercive, which is in particular satisfied for compact sets, then the definition of $K^{(\infty)}$ can be viewed as the limiting object of $K^{(p)}$, since in this case (cf. Lemma 6.2)

$$K^{(p)}=\left\{\xi\in\mathbb{R}^{N\times N}_{\mathrm{sym}}\colon\ \begin{array}{c}g(\xi)\leq\max_K g\text{ for all symmetric div-quasiconvex}\\ g\in\mathrm{C}(\mathbb{R}^{N\times N}_{\mathrm{sym}};[0,\infty))\text{ with } g(z)\leq C(1+|z|^p)\text{ for all } z\in\mathbb{R}^{N\times N}_{\mathrm{sym}}\end{array}\right\}.$$

By our discussion in Section B.1.1, it is particularly important to understand the properties of the symmetric div-quasiconvex hulls. In [42], CONTI, MÜLLER & ORTIZ established that $K^{(p)}$ is independent of $1<p<\infty$. Specifically, they conjectured in [42, Rem. 3.9] that $K^{(1)}=K^{(\infty)}$ in analogy with the usual quasiconvex envelopes (see ZHANG [158] or MÜLLER [115, Thm. 4.10]). The truncation result presented in this chapter answers this question in the affirmative , leading to the main result.

Theorem B.1 (Main result)**.** *Let $K \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ be compact. Then $K^{(1)} = K^{(\infty)}$ and so*

$$K^{(p)} = K^{(1)} = K^{(\infty)} \qquad \text{for all} \quad 1 \leq p \leq \infty. \tag{B.7}$$

Let us note that the p-symmetric div-quasiconvex hulls satisfy the antimonotonicity property with respect to inclusions, i.e., if $1 \leq p \leq q \leq \infty$, then $K^{(q)} \subset K^{(p)}$. For Theorem B.1, it thus suffices to establish $K^{(1)} \subset K^{(\infty)}$ and this is exactly what shall be achieved in Section B.6 [1]. We wish to point out that for the present chapter, our focus is on compact sets K and not on potentially unbounded ones, for which even in the usual quasiconvex case only a few contributions are available; see, e.g., [60, 116, 152, 155, 160].

From a proof perspective, any underlying argument relies, as in Chapter A, on a L^∞-truncation of suitable recovery sequences, simultaneously keeping track of the differential constraint. Contrary to routine mollification, truncations leave the input functions unchanged on a large set and display an important tool in the study of nonlinear problems [1, 17, 68, 70, 113, 156]. It is here where Theorem B.1 cannot be established by analogous means as in [42, Sec. 3], where a higher order truncation argument in the spirit of Acerbi & Fusco [2] and Zhang [157] is employed. More precisely, for $1 < p < q < \infty$, the critical inclusion $K^{(p)} \subset K^{(q)}$ is established in [42] by passing to the corresponding potentials of divsym-free fields, and as these potentials are of second order, performing a $W^{2,\infty}$-truncation on the potentials; this shall be referred to as *potential truncation*. The underlying potential operators are obtained as suitable Fourier multiplier operators, which is why they only satisfy strong L^p-L^p-bounds for $1 < p < \infty$. It is well-known that such Fourier multiplier operators do not map $L^1 \to L^1$ boundedly (cf. Ornstein [121] and, more recently, [39, 61, 88]), and so this approach is bound to fail in view of Theorem B.1. In the regime $1 < p < \infty$, this strategy can readily be employed in the general context of $\mathcal{A}$-quasiconvex hulls in the sense of Fonseca & Müller [65], but is not even required for the inclusion $K^{(p)} \subset K^{(q)}$, $p < q$, and can be established by more elementary means, cf. Theorem 6.7 in Section 6.2.1 and Lemma B.17 in this chapter. The key tool in establishing Theorem B.1 therefore consists in the following truncation result, allowing us to truncate a div-free L^1-map $u \colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ while still preserving the constraint $\operatorname{div}(u) = 0$:

Theorem B.2 (Main truncation theorem)**.** *There exists a constant $C > 0$ solely depending on the underlying space dimension $n = 3$ with the following property: For all $u \in L^1(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ with $\operatorname{div}(u) = 0$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ and all $\lambda > 0$ there exists $u_\lambda \in L^1(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ satisfying the*

(a) L^∞-bound: $\|u_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq C\lambda$.

(b) strong stability: $\|u - u_\lambda\|_{L^1(\mathbb{R}^3)} \leq C \int_{\{|u|>\lambda\}} |u| \, \mathrm{d}x$.

(c) small change: $\mathcal{L}^3(\{u \neq u_\lambda\}) \leq C\lambda^{-1} \int_{\{|u|>\lambda\}} |u| \, \mathrm{d}x$.

[1] see also Theorem 6.14

(d) differential constraint: $\operatorname{div}(u_\lambda) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$.

The same remains valid when replacing the underlying domain $\mathbb{R}^3$ *by the torus* T_3.

We have seen that Theorem B.2 implies the validity of Theorem B.1 via Theorem 6.14. Therefore, the rest of this chapter is devoted to proving Theorem B.2.

Here we heavily rely on the *strong stability property* from item (b), without which the proof of Theorem B.1 is not clear to us. The detailed construction that underlies the proof of Theorem B.2, reminiscent of a geometric version of the WHITNEY smoothing or extension procedure [151], is explained in Section B.3 and carried out in detail in Section B.4. Here we understand by *geometric* that the construction is directly taylored to the problem at our disposal, meaning that the solenoidality constraint $\operatorname{div}(u) = 0$ is visible in our construction in terms of the Gauß-Green theorem on certain simplices. The line of argument employed in the proof can also be applied to higher dimensions, but to focus on the essentials for the physically relevant case we here stick to $n = 3$ dimensions for expository reasons.

To conclude, let us note that by MÜLLER's improvement [113, Thm. 2] of the aforementioned ZHANG truncation lemma [157, Lem. 3.1] for convex sets, one might wonder whether an analogous result can be achieved in the framework discussed in the present paper. Even though the underlying mollification strategy in [113] should be compatible with our approach, the precise technical implementation needs some refinement and shall be deferred to future work. Still, such a result will only concern convex (and not symmetric div-quasiconvex) sets, as even MÜLLER's original result for convex sets seems to be open for quasiconvex sets.

B.1.3. Organisation of the chapter

Apart from this introductory section, the chapter is organised as follows: In Section B.2, we fix notation and gather auxiliary material on maximal operators and basic facts from harmonic analysis. Section B.3 then explains the idea underlying the construction employed in the proof of Theorem B.2, and is then carried out in detail in Section B.4.

How Theorem B.1 follows from Theorem B.2 has already been discussed in Chapter 6, Theorem 6.14. For completeness, we give the proof again in Section B.6.

B.2. Preliminaries

B.2.1. Notation

We denote $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ the N-dimensional Lebesgue or $(N-1)$-dimensional Hausdorff measures, respectively. For notational brevity, we shall also write $\mathrm{d}^{N-1} = \mathrm{d}\mathcal{H}^{N-1}$. Given N or $(N-1)$-dimensional measurable subsets Ω and Σ of $\mathbb{R}^N$ with $\mathcal{L}^N(\Omega), \mathcal{H}^{N-1}(\Sigma) \in (0, \infty)$, respectively, we use the shorthand

$$\fint_\Omega u \,\mathrm{d}x := \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u \,\mathrm{d}x \quad \text{and} \quad \fint_\Sigma v \,\mathrm{d}^{N-1}x := \frac{1}{\mathcal{H}^{N-1}(\Sigma)} \int_\Sigma v \,\mathrm{d}^{n-1}x$$

for $\mathcal{L}^N$- or $\mathcal{H}^{N-1}$-measurable maps $u\colon \Omega \to \mathbb{R}^m$ and $v\colon \Sigma \to \mathbb{R}^m$. As we shall mostly assume $n = 3$, we denote $B_r(z)$ the open ball of radius r centered at $z \in \mathbb{R}^3$, whereas we reserve the notation $\mathbb{B}_r(z)$ to denote the corresponding open balls in the symmetric (3×3)-matrices $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$; moreover, we put $\omega_3 := \mathcal{L}^3(B_1(0))$. By *cubes* Q we understand non-degenerate cubes throughout, and use $\ell(Q)$ to denote their sidelength. Lastly, for $x_1, ..., x_j \in \mathbb{R}^3$, we denote $\langle x_1, ..., x_j\rangle$ the convex hull of the vectors $x_1, ..., x_j$, and if x_1, x_2, x_3 do not lie on a joint line, $\mathrm{aff}(x_1, x_2, x_3)$ the affine hyperplane containing x_1, x_2, x_3.

B.2.2. Maximal operator, bad sets and Whitney covers

For a finite dimensional real vector space V, $w \in L^1(\mathbb{R}^N, V)$ and $R > 0$, we recall the (restricted) *centered Hardy-Littlewood maximal operators* to be defined by

$$\begin{aligned} \mathcal{M}_R w(x) &:= \sup_{0<r<R} \fint_{B_r(x)} |w|\,\mathrm{d}y, \qquad x \in \mathbb{R}^N,\\ \mathcal{M} w(x) &:= \sup_{r>0} \fint_{B_r(x)} |w|\,\mathrm{d}y, \qquad x \in \mathbb{R}^N. \end{aligned} \tag{B.8}$$

Note that, by lower semicontinuity of $\mathcal{M}_R w$, the superlevel sets $\{\mathcal{M}_R w > \lambda\}$ are open for all $\lambda > 0$. Moreover, we record that $\mathcal{M}$ is of weak-$(1,1)$-type, meaning that there exists $c = c(n) > 0$ such that

$$\mathcal{L}^n(\{\mathcal{M}w > \lambda\}) \leq \frac{c}{\lambda}\|w\|_{L^1(\mathbb{R}^N)} \qquad \text{for all } w \in L^1(\mathbb{R}^N, V). \tag{B.9}$$

See [78, 139] for more background information. Now let $\Omega \subset \mathbb{R}^N$ be open. Then there exists a *Whitney cover* $\mathcal{W} = (Q_j)$ for Ω. By this we understand a sequence of open cubes Q_j with the following properties:

(W1) $\Omega = \bigcup_{j\in\mathbb{N}} Q_j$.

(W2) $\frac{1}{5}\ell(Q_j) \leq \mathrm{dist}(Q_j, \Omega^\complement) \leq 5\ell(Q_j)$ for all $j \in \mathbb{N}$.

(W3) *Finite overlap:* There exists a number $M = M(n) > 0$ such that at most M elements of $\mathcal{W}$ overlap; i.e., for each $i \in \mathbb{N}$,

$$|\{j \in \mathbb{N}\colon\; Q_j \in \mathcal{W} \text{ and } Q_i \cap Q_j \neq \emptyset\}| \leq M.$$

(W4) *Comparability for touching cubes:* There exists a constant $c(N) > 0$ such that if $Q_i, Q_j \in \mathcal{W}$ satisfy $Q_i \cap Q_j \neq \emptyset$, then

$$\frac{1}{c(N)}\ell(Q_i) \leq \ell(Q_j) \leq c(N)\ell(Q_i).$$

Whenever such a Whitney cover is considered, we tacitly understand x_j to be the *centre* of the corresponding cube Q_j. Based on the Whitney cover $\mathcal{W}$ from above, we choose a partition of unity (φ_j) subject to $\mathcal{W}$ with the following properties:

(P1) For any $j \in \mathbb{N}$, $\varphi_j \in \mathrm{C}_c^\infty(Q_j;[0,1])$.

(P2) $\sum_{j\in\mathbb{N}} \varphi_j = 1$ in Ω.

(P3) For each $l \in \mathbb{N}$, there exists a constant $c = c(n,l) > 0$ such that

$$|\nabla^l \varphi_j| \leq \frac{c}{\ell(Q_j)^l} \qquad \text{for all } j \in \mathbb{N}.$$

B.2.3. Differential operators and projection maps

For the following sections, we require some terminology for differential operators and a suitable projection property to be gathered in the sequel. Let $\mathcal{A}$ be a constant coefficient, linear and homogeneous differential operator of order $k \in \mathbb{N}$ on $\mathbb{R}^N$ (or T_N) between $\mathbb{R}^d$ and $\mathbb{R}^N$, so $\mathcal{A}$ has a representation

$$\mathcal{A}u = \sum_{|\alpha|=k} \mathcal{A}_\alpha \partial^\alpha u, \qquad u\colon \mathbb{R}^N \to \mathbb{R}^d, \tag{B.10}$$

with fixed $\mathcal{A}_\alpha \in \mathcal{L}\backslash(\mathbb{R}^d;\mathbb{R}^N)$ for $|\alpha| = k$. Following [119, 137] we say that $\mathcal{A}$ has *constant rank* (in $\mathbb{R}$) provided the rank of the Fourier symbol $\mathcal{A}[\xi] = \sum_{|\alpha|=k} \mathcal{A}_\alpha \xi^\alpha \colon \mathbb{R}^d \to \mathbb{R}^N$ is independent of $\xi \in \mathbb{R}^N \setminus \{0\}$. A constant coefficient differential operator $\mathcal{B}$ of order $j \in \mathbb{N}$ on $\mathbb{R}^N$ (or T_N) between $\mathbb{R}^\ell$ and $\mathbb{R}^d$ consequently is called a *potential* of $\mathcal{A}$ provided for each $\xi \in \mathbb{R}^N \setminus \{0\}$ the Fourier symbol sequence

$$\mathbb{R}^\ell \xrightarrow{\mathcal{B}[\xi]} \mathbb{R}^d \xrightarrow{\mathcal{A}[\xi]} \mathbb{R}^N$$

is exact at every $\xi \in \mathbb{R}^n \setminus \{0\}$, i.e., $\mathbb{A}[\xi](\mathbb{R}^\ell) = \ker(\mathbb{A}[\xi])$ for each such ξ. We moreover say that $\mathcal{A}$ has *constant rank* (in $\mathbb{C}$) provided $\mathbb{A}[\xi]\colon \mathbb{C}^d \to \mathbb{C}^l$ has rank independent of $\xi \in \mathbb{C}^N \setminus \{0\}$. If we only speak of *constant rank*, then we tacitly understand constant rank in $\mathbb{R}$. In Section B.7, we require the following two auxiliary results, ensuring both the existence of potentials and suitable projection operators (cf. Theorem 2.5 and Theorem 2.9, respectively).

Lemma B.3 (Existence of potentials, [123, Thm. 1, Lem. 5])**.** *Let $\mathcal{A}$ be a differential operator with constant rank over $\mathbb{R}$. Then $\mathcal{A}$ possesses a potential $\mathcal{B}$. Moreover, if $u \in \mathrm{C}^\infty(T_N,\mathbb{R}^d)$ satisfies $\int_{T_N} u \,\mathrm{d}x = 0$ and $\mathcal{A}u = 0$, there exists $v \in \mathrm{C}^\infty(T_N,\mathbb{R}^\ell)$ with $\mathcal{B}v = u$. Equally, for each $u \in \mathcal{S}(\mathbb{R}^N,\mathbb{R}^d)$ with $\mathcal{A}u = 0$ there exists $v \in \mathcal{S}(\mathbb{R}^N,\mathbb{R}^\ell)$ with $\mathcal{B}v = u$.*

Lemma B.4 (Projection maps on the torus, [65, Lem. 2.14])**.** *Let $1 < p < \infty$ and let $\mathcal{A}$ be a differential operator of order k with constant rank in $\mathbb{R}$. Then there is a bounded, linear projection map $P_\mathcal{A}\colon L^p(T_N,\mathbb{R}^d) \to L^p(T_N,\mathbb{R}^d)$ with the following properties:*

1. *$P_\mathcal{A} u \in \ker \mathcal{A}$ and $P_\mathcal{A} \circ P_\mathcal{A} = P_\mathcal{A}$.*

2. $\|u - P_{\mathcal{A}}u\|_{L^p(T_N)} \le C_{\mathcal{A},p}\|\mathcal{A}u\|_{W^{-k,p}(T_N)}$ *whenever* $\fint_{T_N} u \,\mathrm{d}x = 0$.

3. If $(u_j) \subset L^p(T_N, \mathbb{R}^d)$ *is bounded and p-equiintegrable, i.e.,*

$$\lim_{\varepsilon \searrow 0}\left(\sup_{j\in\mathbb{N}}\sup_{E\colon \mathcal{L}^n(E)<\varepsilon}\int_E |u_j|^p \,\mathrm{d}x\right) = 0,$$

then also $(P_{\mathcal{A}}u_j)$ *is p-equiintegrable.*

As alluded to in the introduction, Lemma B.4 does not extend to $p = 1$ in general, the reason being Ornstein's Non-Inequality [121]; also see [39, 61, 88] for more recent approaches to the matter and Grafakos [78, Thm. 4.3.4] for a full characterisation of L^1-multipliers.

B.3. On the construction of divsym-free truncations

Before embarking on the proof of Theorem B.2 in Section B.4, we comment on the underlying idea and how it is implemented in conceptually easier settings (see Sections B.3.2 and B.3.3 below). To elaborate on the connections to divsym-truncations, we premise a discussion of the general framework first.

B.3.1. Potential truncations versus $\mathcal{A}$-free truncations

We start by streamlining terminology as follows: Let Ω either be T_N or $\mathbb{R}^N$. Given a constant rank differential operator $\mathcal{B}$ on Ω between $\mathbb{R}^m$ and $\mathbb{R}^d$ and $1 \le p \le \infty$, we define Sobolev-type spaces $W^{\mathcal{B},p}(\Omega) := \{u \in L^p(\Omega, \mathbb{R}^m)\colon\ \mathcal{B}u \in L^p(\Omega, \mathbb{R}^d)\}$. A family of operators $(S_\lambda)_{\lambda>0}$ with $S_\lambda\colon W^{\mathcal{B},p}(\Omega) \to W^{\mathcal{B},\infty}(\Omega)$ is called a $W^{\mathcal{B},p}$-$W^{\mathcal{B},\infty}$*-truncation* provided there exists a constant $c = c(\mathcal{B}, p) > 0$ such that, for all $u \in W^{\mathcal{B},p}(\Omega)$ and $\lambda > 0$,

1. $\|S_\lambda u\|_{L^\infty(\Omega)} + \|\mathcal{B}S_\lambda u\|_{L^\infty(\Omega)} \le c\lambda$.
2. $\|u - S_\lambda u\|_{L^p(\Omega)} + \|\mathcal{B}u - \mathcal{B}S_\lambda u\|_{L^p(\Omega)} \le c\int_{\{|u|+|\mathcal{B}u|>\lambda\}} |u|^p + |\mathcal{B}u|^p \,\mathrm{d}x$.
3. $\mathcal{L}^n(\{u \ne S_\lambda u\}) \le \dfrac{c}{\lambda^p}\int_{\{|u|+|\mathcal{B}u|>\lambda\}} |u|^p + |\mathcal{B}u|^p \,\mathrm{d}x$.

If $\mathcal{B} = \nabla^k$, then we simply speak of a $W^{k,p}$-$W^{k,\infty}$*-truncation*. Conversely, if $\mathcal{B}$ is a potential of the differential operator $\mathcal{A}$ and $1 \le p \le \infty$, we define $L^p_{\mathcal{A}}(\Omega) := \{u \in L^p(\Omega, \mathbb{R}^d)\colon\ \mathcal{A}u = 0\}$. A family of operators $(T_\lambda)_{\lambda>0}$ with $T_\lambda\colon L^p_{\mathcal{A}}(\Omega) \to L^\infty_{\mathcal{A}}(\Omega)$ is called an $\mathcal{A}$*-free* L^p-L^∞*-truncation* (or simply $\mathcal{A}$-free L^∞-truncation) provided there exists $c = c(\mathcal{A}, p) > 0$ such that the following hold for all $u \in L^\infty_{\mathcal{A}}(\Omega)$ and $\lambda > 0$:

1. $\|T_\lambda u\|_{L^\infty(\Omega)} \le c\lambda$.
2. $\|u - T_\lambda u\|_{L^p(\Omega)} \le c\int_{\{|u|>\lambda\}} |u|^p \,\mathrm{d}x$.

3. $\mathcal{L}^n(\{u \neq T_\lambda u\}) \leq \dfrac{c}{\lambda^p} \displaystyle\int_{\{|u|>\lambda\}} |u|^p \,\mathrm{d}x.$

Originally, $W^{1,p}$-$W^{1,\infty}$-truncations as in ACERBI & FUSCO [2] leave $u \in W^{1,p}(\Omega)$ unchanged on $\{\mathcal{M}u \leq \lambda\} \cap \{\mathcal{M}(\nabla u) \leq \lambda\}$. Here, the functions satisfy the Lipschitz estimate

$$|u(x) - u(y)| \lesssim |x-y|(\mathcal{M}(\nabla u)(x) + \mathcal{M}(\nabla u)(y)) \lesssim \lambda|x-y|$$

for $\mathcal{L}^n$-a.e. $x, y \in \{\mathcal{M}(\nabla u) \leq \lambda\}$ and thus can be extended to a $c\lambda$-Lipschitz function $S_\lambda u$ by virtue of MC SHANE's extension theorem [58, Chpt. 3.1.1., Thm. 1]. Note that, if u is divergence-free, then $S_\lambda u$ is not in general. In view of preserving differential constraints, this necessitates a more flexible approach that allows to geometrically handle the action of differential operators. Instead of appealing to the MC SHANE extension, one may directly perform a WHITNEY-type extension [151] and truncate $u \in W^{1,1}(\Omega)$ on the bad set $\mathcal{O}_\lambda = \{\mathcal{M}u > \lambda\} \cup \{\mathcal{M}(\nabla u) > \lambda\}$ via

$$\widetilde{\mathbf{S}}_\lambda u(x) = \begin{cases} \displaystyle\sum_{j\in\mathbb{N}} \varphi_j (u)_{Q_j}, & x \in \mathcal{O}_\lambda, \\ u(x), & x \in \mathcal{O}_\lambda^\complement, \end{cases} \quad \text{or} \quad \mathbf{S}_\lambda u(x) = \begin{cases} \displaystyle\sum_{j\in\mathbb{N}} \varphi_j u(y_j), & x \in \mathcal{O}_\lambda, \\ u(x), & x \in \mathcal{O}_\lambda^\complement, \end{cases} \tag{B.11}$$

where $y_j \in \mathcal{O}_\lambda^\complement$ are chosen suitably and (φ_j) is a partition of unity subordinate to the Whitney covering of $\mathcal{O}_\lambda$ (cf. Section B.2.2). Then $\widetilde{\mathbf{S}}_\lambda$ and $\mathbf{S}_\lambda$ define $W^{1,1}$-$W^{1,\infty}$-truncations; cf. [52, 139]. Setting $v = \nabla u$, this formula gives a curl-free L^1-L^∞-truncation, as $\operatorname{curl}(v) = 0 \Leftrightarrow v = \nabla u$ for some function u. Using (P1)–(P3), we can, however, rewrite $\widetilde{v} := \nabla \mathbf{S}_\lambda u$ purely in terms of v, i.e.

$$\widetilde{v}(x) = \begin{cases} \displaystyle\sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t & x \in \mathcal{O}_\lambda, \\ v(x) & x \in \mathcal{O}_\lambda^\complement. \end{cases} \tag{B.12}$$

To see the validity of (B.12), we first note that (φ_i) is a partition of unity on $\mathcal{O}_\lambda$, i.e., $\sum_{i\in\mathbb{N}} \varphi_i(y) = 1$ for $y \in \mathcal{O}_\lambda$ and also that, due to the same fact, $\sum_{j\in\mathbb{N}} \nabla\varphi_j(y) = 0$ for any $y \in \mathcal{O}_\lambda$. Using this fact at $(*)$, we conclude

$$\begin{aligned} \widetilde{v}(x) = \nabla \mathbf{S}_\lambda u(x) &= \sum_{j\in\mathbb{N}} \nabla\varphi_j u(y_j) \\ &\overset{(*)}{=} \sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j (u(y_j) - u(y_i)) \\ &= \sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 \nabla u(ty_j + (1-t)y_i)\cdot(y_j - y_i)\ \mathrm{d}t \\ &\overset{\nabla u = v}{=} \sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\ \mathrm{d}t, \end{aligned} \tag{B.13}$$

which is (B.12). The previous calculation yields that we may skip the step of going to the potential u of v, as the truncation $\widetilde{v}$ *does not* depend on the choice of u.

B.3.2. The construction of divergence-free truncations

In an intermediate step, we explain how (B.12) gives rise to divergence-free L^1-L^∞-truncations [2]. Here, given a divergence-free map $w \in (L^1 \cap \mathrm{C}^\infty)(\Omega, \mathbb{R}^3)$, we may write $w = \mathrm{curl}(v)$ for some $v \in W^{\mathrm{curl},1}(\Omega)$.

The key observation is that the truncation formula (B.12) does not only give a curl-free L^1-L^∞-truncation, but is stronger and gives a $W^{\mathrm{curl},1}$-$W^{\mathrm{curl},\infty}$-truncation, if we redefine the bad set to be $\widetilde{\mathcal{O}}_\lambda := \{\mathcal{M}v > \lambda\} \cup \{\mathcal{M}\,\mathrm{curl}(v) > \lambda\}$. Temporarily accepting this fact and hereafter that

$$S_\lambda^{\mathrm{curl}} v = \begin{cases} \sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t, & x \in \widetilde{\mathcal{O}}_\lambda, \\ v(x), & x \in \widetilde{\mathcal{O}}_\lambda^{\complement} \end{cases} \tag{B.14}$$

defines a $W^{\mathrm{curl},1}$-$W^{\mathrm{curl},\infty}$-truncation of $v \in W^{\mathrm{curl},1}(\Omega,\mathbb{R}^3)$, we may then apply $S_\lambda^{\mathrm{curl}}$ to v. Most importantly, we here *directly truncate the curl instead of the full gradients*, and so are in position to use that $w = \mathrm{curl}(v) \in L^1$. Returning to $\widetilde{w} := \mathrm{curl}(S_\lambda^{\mathrm{curl}} v)$, we then arrive at the requisite truncation. For $n = 3$, this can be written explicitely for $y \in \mathcal{O}_\lambda$ via

$$\begin{aligned} \widetilde{w}(y) &= (\widetilde{w}_1(y), \widetilde{w}_2(y), \widetilde{w}_3(y)) \\ &= \mathrm{curl}(S_\lambda^{\mathrm{curl}} v)(y) = \sum_{i,j\in\mathbb{N}} \mathrm{curl}(\varphi\nabla\varphi_j) \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\ \mathrm{d}t, \end{aligned} \tag{B.15}$$

and for future comparison with divsym-free truncations, we carry out the computation for $\widetilde{w}_1$. For brevity, we put $A(i,j) := \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\ \mathrm{d}t$. Then, artificially introducing a third variable k, we obtain

$$\begin{aligned} \widetilde{w}_1(y) &= \sum_{i,j\in\mathbb{N}} \big(\partial_2(\varphi_i\partial_3\varphi_j) - \partial_3(\varphi_i\partial_2\varphi_j)\big) A(i,j) \\ &= 2\sum_{i,j\in\mathbb{N}} \partial_2\varphi_i\partial_3\varphi_j A(i,j) \qquad \text{(permuting } i \leftrightarrow j \text{ and using } A(i,j) = -A(j,i)) \\ &= 2\sum_{i,j,k\in\mathbb{N}} \varphi_k\partial_2\varphi_i\partial_3\varphi_j (A(i,j) + A(j,k) + A(k,i)) \quad \Big(\text{by } \sum_l \nabla\varphi_l = 0,\ l \in \{i,j,k\}\Big). \end{aligned}$$

Instead of using the fundamental theorem of calculus, we use Stokes' theorem to write

$$(A(i,j) + A(j,k) + A(k,i)) = \fint_{\langle x_i,x_j,x_k\rangle} \mathrm{curl}\,v \cdot ((y_i - y_j)\times(y_j - y_k))\ \mathrm{d}\mathcal{H}^2,$$

[2] This is the procedure that is carried out in Chapter A

for the triangle $\langle x_i, x_j, x_k\rangle$ with vertices x_i, x_j and x_k. Since $\operatorname{curl} v = w$, we then arrive at

$$\widetilde{w}_1(y) = \sum_{i,j,k\in\mathbb{N}} \varphi_k \partial_2\varphi_i \partial_3\varphi_j \fint_{\langle x_i,x_j,x_k\rangle} w\cdot((y_i-y_j)\times(y_j-y_k))\ \mathrm{d}\mathcal{H}^2 \tag{B.16}$$

Using formula (B.16), instead of going to the potential of div, we may directly construct truncations of div-free functions.

Pursuing the strategy explained above, the reader might notice that the effective difficulty for div-free fields is to verify that (B.14) defines a $W^{\operatorname{curl},1}$-$W^{\operatorname{curl},\infty}$-truncation. For divsym-free L^1-fields, the main argument (to be explained in Section B.3.3 and carried out in detail in Section B.4) will be centered around constructing the more involved $W^{\operatorname{curl}\operatorname{curl}^\top,1}$-$W^{\operatorname{curl}\operatorname{curl}^\top,\infty}$-truncations rather than $W^{\operatorname{curl},1}$-$W^{\operatorname{curl},\infty}$-truncations. To motivate the need of such truncations, a quick homological discussion in the div-free case is in order. By the construction in (B.14)ff., we are able to formulate an $\mathcal{A}$-free L^1-L^∞-truncation of the annihilator $\mathcal{A}$ of curl, which is div in three dimensions. As discussed in Chapter A, [134], this approach works for all potential-annihilator pairs along the exact sequence of exterior derivatives. This is the exact sequence of differential operators starting with ∇, that is

$$\begin{aligned}0 \longrightarrow C^\infty_\#(T_N,\mathbb{R}) &\xrightarrow{\nabla} C^\infty_\#(T_N,\mathbb{R}^N) \xrightarrow{\operatorname{curl}} C^\infty_\#(T_N,\mathbb{R}^{N\times N}_{\mathrm{skew}}) \longrightarrow \ldots\\ &\longrightarrow C^\infty_\#(T_N,\mathbb{R}^N) \xrightarrow{\operatorname{div}} C^\infty_\#(T_N,\mathbb{R}) \longrightarrow 0.\end{aligned}$$

To summarise the above procedure for div-free fields, one

(D1) *first* picks a suitable $W^{\nabla,1}$-$W^{\nabla,\infty}$-truncation as in (B.11),

(D2) *second* rewrites it by considering gradients only as in (B.12)

(D3) *third* shows that the resulting operator as in (B.14) defines a $W^{\operatorname{curl},1}$-$W^{\operatorname{curl},\infty}$-truncation.

This consequently gives rise to a div-free L^1-L^∞-truncation.

B.3.3. Truncations involving the symmetric gradient

Let $N = 3$. Towards divsym-free L^1-L^∞-truncations, we now aim to modify the procedure (D1)–(D3) from above. Here we work from the exact sequence

$$\begin{aligned}0 \longrightarrow C^\infty_\#(T_3,\mathbb{R}^3) &\xrightarrow{\varepsilon} C^\infty_\#(T_3,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \xrightarrow{\operatorname{curl}\operatorname{curl}^\top} C^\infty_\#(T_3,\mathbb{R}^{3\times3}_{\mathrm{sym}})\\ &\xrightarrow{\operatorname{div}} C^\infty_\#(T_3,\mathbb{R}^3) \longrightarrow 0,\end{aligned} \tag{B.17}$$

where $\operatorname{curl}\operatorname{curl}^\top$ is the potential of the divergence of symmetric matrices, defined in $n = 3$ dimensions by

$$\operatorname{curl}\operatorname{curl}^\top v = \begin{pmatrix} w_{2323} & w_{2331} & w_{2312}\\ w_{3123} & w_{3131} & w_{3112}\\ w_{1223} & w_{1231} & w_{1212}\end{pmatrix} \quad \text{for } v\in \mathrm{C}^2(\mathbb{R}^3,\mathbb{R}^{3\times3}_{\mathrm{sym}}),\ \text{where}$$

$$w_{abcd} := \partial_a\partial_c v_{bd} + \partial_b\partial_d v_{ac} - \partial_a\partial_d v_{bc} - \partial_b\partial_c v_{ad}.$$

Note that in dimension $N = 3$, the exact sequence starting with symmetric gradients has three non-zero elements (ε, $\operatorname{curl}\operatorname{curl}^\top$ and the symmetric divergence); in higher dimension it is longer, for simplicity we therefore restrict ourselves to $N = 3$. We then proceed by analogy with (D1)–(D3), namely

(DS1) *first* pick a suitable $W^{\varepsilon,1}$-$W^{\varepsilon,\infty}$-truncation,

(DS2) *second* rewrite it by considering symmetric gradients only

(DS3) *third* show that the resulting operator defines a $W^{\operatorname{curl}\operatorname{curl}^\top,1}$-$W^{\operatorname{curl}\operatorname{curl}^\top,\infty}$-truncation.

Towards (DS1), we note that $W^{\mathcal{B},1}$-$W^{\mathcal{B},\infty}$-truncations are also known in settings where $\mathcal{B} \neq \nabla$. In this work, we use that such a truncation exists for the symmetric gradient, i.e. $\mathcal{B} = \varepsilon = \dfrac{1}{2}(\nabla + \nabla^\top)$ (cf. [56, 19]). As an analogue of formula (B.11), we now use

$$\mathbf{S}^\varepsilon_\lambda u(x) = \begin{cases} \sum_{j\in\mathbb{N}} \varphi_j(x) P_j u(x), & x \in \mathcal{O}_\lambda, \\ u(x), & x \in \mathcal{O}_\lambda^\complement, \end{cases} \tag{B.18}$$

with suitable projections P_j onto the rigid deformations, so the nullspace of the symmetric gradient ε. Such projections can be obtained via

$$P_j u(x) = \fint_{Q_j} u(\xi) + \tfrac{1}{2}(\nabla - \nabla^\top)u(\xi)(x-\xi)\ \mathrm{d}\mu_j(\xi)$$

for suitable measures μ_j, so that $(\nabla - \nabla^\top)$ becomes invisible after integrating by parts. As an adaptation of (B.13) and hereafter (B.14), one may then follow (DS2) to obtain

$$S^{\operatorname{curl}\operatorname{curl}^\top}_\lambda v(x)_{ab} = \begin{cases} \frac{1}{2}\sum_{i,j\in\mathbb{N}} \varphi_i\partial_a\varphi_j(G_b(i,j)+H_b(i,j)) + \varphi_i\partial_b\varphi_j(G_a(i,j)+H_a(i,j)), \\ \qquad\qquad x\in\mathcal{O}_\lambda, \\ v_{ab}(x), \qquad\qquad x \in \mathcal{O}_\lambda^\complement \end{cases}$$

for $a, b \in \{1, 2, 3\}$ as a substitute for (B.14), where G_a, G_b and H_a, H_b are defined in terms of v and the previously mentioned measures μ_j. In view of (DS3), we then need to establish that the resulting operator in fact yields a $W^{\operatorname{curl}\operatorname{curl}^\top,1}$-$W^{\operatorname{curl}\operatorname{curl}^\top,\infty}$-truncation, and this is *in essence* what we establish in Section B.4. More precisely, we directly prove that when applying $\operatorname{curl}\operatorname{curl}^\top$ to $S^{\operatorname{curl}\operatorname{curl}^\top}_\lambda v$ and rewriting the result purely in terms of $w = \operatorname{curl}\operatorname{curl}^\top(v)$ (just as (B.16) rewrites $\operatorname{curl}(S^{\operatorname{curl}}_\lambda v)$ purely in terms of w), we obtain the requisite truncation operator. Omitting the details of the derivation, the truncation operator is written down explicitely in (B.23), and the entire Section B.4 is centered around establishing that it features the desired properties.

Indeed, the treatment in Chapter A and in the current chapter (together with the previously outline strategy) lead to the following conjecture:

Conjecture B.5 (Theorem B.2 for operators with constant rank in $\mathbb{C}$). Let

$$0 \to C^\infty_\#(T_N, \mathbb{R}^{d_0}) \xrightarrow{\mathcal{A}_1} C^\infty_\#(T_N, \mathbb{R}^{d_1}) \xrightarrow{\mathcal{A}_2} ... \xrightarrow{\mathcal{A}_k} C^\infty_\#(T_N, \mathbb{R}^{d_k}) \xrightarrow{\mathcal{A}_{k+1}} ...$$

be an exact sequence of differential operators with constant rank in $\mathbb{C}$, in particular, $\mathcal{A}_1$ being $\mathbb{C}$-elliptic. This is equivalent to

$$0 \to \mathbb{C}^{d_0} \xrightarrow{\mathbb{A}_1[\xi]} \mathbb{C}^{d_1} \xrightarrow{\mathbb{A}_2[\xi]} \mathbb{C}^{d_2} \xrightarrow{\mathbb{A}_3[\xi]} ... \xrightarrow{\mathbb{A}_k[\xi]} \mathbb{C}^{d_k} \xrightarrow{\mathbb{A}_{k+1}[\xi]} ...$$

being exact for all $\xi \in \mathbb{C}^N \setminus \{0\}$. Then for any differential operator $\mathcal{A}_k$ contained in this exact sequence there is $C_k > 0$, such that for $u \in L^1(T_N, \mathbb{R}^{d_k})$ with $\mathcal{A}_k u = 0$ in $\mathcal{D}'(T_N, \mathbb{R}^{d_{k+1}})$ and $\lambda > 0$, there is $u_\lambda \in L^1(\mathbb{R}^N, \mathbb{R}^{d_k})$ satisfying

1. $\|u_\lambda\|_{L^\infty} \leq C\lambda$. ($L^\infty$-bound)
2. $\|u - u_\lambda\|_{L^1} \leq C \int_{\{|u|>\lambda\}} |u| \, \mathrm{d}x$. (Strong stability)
3. $\mathcal{L}^n(\{u \neq u_\lambda\} \leq C\lambda^{-1} \int_{\{|u|>\lambda\}} |u| \, \mathrm{d}x$. (Small change)
4. $\mathcal{A}_k u_\lambda = 0$, i.e. the differential constraint is still satisfied.

If any differential operator $\mathcal{A}$ with constant rank over $\mathbb{C}$ is a part of such an exact sequence, this means that the $\mathcal{A}$-free truncation is possible for every such operator.

B.4. Construction of the truncation and the proof of Theorem B.2

In this section, we establish Theorem B.2. As a main ingredient, we shall prove the following variant for smooth maps that will be shown to imply Theorem B.2 in Section B.4.7:

Proposition B.6. *Let* $w \in (\mathrm{C}^\infty \cap L^1)(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfy* $\operatorname{div}(w) = 0$. *Then there exists a constant* $c > 0$ *such that for all* $\lambda > 0$ *there exists an open set* $\mathcal{U}_\lambda \subset \mathbb{R}^3$ *and a function* $w_\lambda \in (L^1 \cap L^\infty)(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *with the following properties:*

1. $w = w_\lambda$ *on* $\mathcal{U}_\lambda^\complement$ *and* $\mathcal{L}^3(\{w \neq w_\lambda\}) < \dfrac{c}{\lambda} \displaystyle\int_{\{|w|>\frac{\lambda}{2}\}} |w| \, \mathrm{d}x$.
2. $\operatorname{div}(w_\lambda) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$.
3. $\|w_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq c\lambda$.

B.4.1. A short outline of the proof of Proposition B.6

As the proof of Proposition B.6 involves several rather technical steps, let us briefly outline its strategy:

1. In Section B.4.2 we define the truncation pointwisely (which is derived by following the steps explained in Section B.3.2 and B.3.3) and collect auxiliary properties of the terms involved in Lemma B.7.

2. Lemma B.8 is designed to bound single terms appearing as a summand when proving in Lemma B.9 that our truncation actually maps into L^∞.

3. We then show that the truncation actually is a smooth function on the bad set $\mathcal{O}_\lambda$. Therefore, we can check the constraint $\operatorname{div}(T_\lambda w) = 0$ pointwisely in $\mathcal{O}_\lambda$ (cf. Lemma B.10), which involves a technical computation given in the Section B.5.

4. Consequently, the truncation is div-free both in the interior of $\mathcal{O}_\lambda$ and its complement. To show global solenoidality, we verify that the distributional divergence actually is an L^1-function, cf. Lemma B.11. We then conclude $\operatorname{div}(T_\lambda w) \in L^1$ and $\operatorname{div}(T_\lambda w) = 0$ almost everywhere, hence $\operatorname{div}(T_\lambda w) = 0$.

5. Finally, we conclude by estimating the measure of the bad set to get a bound on the measure of the set $\{w \neq T_\lambda w\}$, cf. Lemma B.13.

B.4.2. Definition of T_λ

Let $w = (w_1, w_2, w_3) \in (\mathrm{C}^\infty \cap L^1)(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ satisfy $\operatorname{div}(w) = 0$. In view of locally redefining our given map w on $\mathcal{O}_\lambda = \{\mathcal{M}w > \lambda\}$, we put

$$\begin{aligned} \mathfrak{A}_{\alpha,\beta}(i,j,k)(y) &:= \fint_{\langle x_i,x_j,x_k\rangle} ((y-\xi)_\beta w_\alpha(\xi) - (y-\xi)_\alpha w_\beta(\xi))\nu_{ijk} \,\mathrm{d}^2\xi, \\ \mathfrak{B}_\alpha(i,j,k) &:= \fint_{\langle x_i,x_j,x_k\rangle} w_\alpha(\xi)\cdot \nu_{ijk} \,\mathrm{d}^2\xi \end{aligned} \tag{B.19}$$

provided the simplex $\langle x_i, x_j, x_k\rangle$ (i.e., the convex hull of x_i, x_j, x_k) is non-degenerate; if it is degenerate, we then define $\mathfrak{A}_{\alpha,\beta}(i,j,k) := 0$ and $\mathfrak{B}_\alpha(i,j,k) := 0$. Here and in what follows, we use

$$\nu_{x_i,x_j,x_k} := \nu_{ijk} := \frac{1}{2}(x_i - x_j)\times(x_k - x_j), \tag{B.20}$$

provided the simplex $\langle x_i, x_j, x_k\rangle$ is non-degenerate. Consider a three-tuple

$$(\alpha,\beta,\gamma) \in \{(1,2,3),(2,3,1),(3,1,2)\}.$$

For $(i,j,k)\in\mathbb{N}^3$ and centre points $x_l\in Q_l$ for $l\in\{i,j,k\}$, we then define

$$\begin{aligned}\widetilde{w}^{(k)}_{\alpha\beta} &= 3\sum_{i,j\in\mathbb{N}}(\partial_\gamma\varphi_j\partial_\alpha\varphi_i\mathfrak{B}_\alpha(i,j,k)+\partial_\beta\varphi_j\partial_\gamma\varphi_i\mathfrak{B}_\beta(i,j,k))\\ &\quad+\sum_{i,j\in\mathbb{N}}(\partial_{\beta\gamma}\varphi_j\partial_\gamma\varphi_i-\partial_{\gamma\gamma}\varphi_j\partial_\beta\varphi_i)\mathfrak{A}_{\beta,\gamma}(i,j,k)\\ &\quad+\sum_{i,j\in\mathbb{N}}(\partial_{\alpha\gamma}\varphi_j\partial_\gamma\varphi_i-\partial_{\gamma\gamma}\varphi_j\partial_\alpha\varphi_i)\mathfrak{A}_{\gamma,\alpha}(i,j,k)\\ &\quad+\sum_{i,j\in\mathbb{N}}(\partial_{\alpha\gamma}\varphi_j\partial_\beta\varphi_i+\partial_{\beta\gamma}\varphi_j\partial_\alpha\varphi_i-2\partial_{\alpha\beta}\varphi_j\partial_\gamma\varphi_i)\mathfrak{A}_{\alpha,\beta}(i,j,k).\end{aligned}\tag{B.21}$$

We define $\widetilde{w}^{(k)}_{\beta\alpha}=\widetilde{w}^{(k)}_{\alpha\beta}$ by symmetry. For the diagonal terms, we put

$$\begin{aligned}\widetilde{w}^{(k)}_{\alpha\alpha} &= 6\sum_{i,j\in\mathbb{N}}\partial_\beta\varphi_j\partial_\gamma\varphi_i\mathfrak{B}_\alpha(i,j,k)\\ &\quad+2\sum_{i,j\in\mathbb{N}}(\partial_{\gamma\gamma}\varphi_j\partial_\beta\varphi_i-\partial_{\beta\gamma}\varphi_j\partial_\gamma\varphi_i)\mathfrak{A}_{\gamma,\alpha}(i,j,k)\\ &\quad+2\sum_{i,j\in\mathbb{N}}(\partial_{\beta\beta}\varphi_j\partial_\gamma\varphi_i-\partial_{\beta\gamma}\varphi_j\partial_\beta\varphi_i)\mathfrak{A}_{\alpha,\beta}(i,j,k).\end{aligned}\tag{B.22}$$

Note that, since at most M cubes Q_j overlap by (W3), each of the sums in (B.21) and (B.22) are, in a neighbourhood of each point $x\in\mathcal{O}_\lambda$, actually *finite* sums and hence $\widetilde{w}^{(k)}:=(w^{(k)}_{\alpha\beta})_{\alpha\beta}$ is well-defined. Based on (B.21), we define the truncation operator T_λ by

$$T_\lambda w := w-\sum_k\varphi_k(w-\widetilde{w}^{(k)})=\begin{cases} w & \text{in } \mathcal{O}_\lambda^{\complement},\\ \sum_k\varphi_k\widetilde{w}^{(k)} & \text{in } \mathcal{O}_\lambda.\end{cases}\tag{B.23}$$

Note that on $\mathcal{O}_\lambda$, $T_\lambda w$ is a locally finite sum of C^∞-maps and thus is equally of class $\mathrm{C}^\infty(\mathcal{O}_\lambda;\mathbb{R}^{3\times3}_{\mathrm{sym}})$.

B.4.3. Auxiliary properties of $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$

In this section, we record some useful properties and auxiliary bounds on the maps $\mathfrak{A}_{\alpha,\beta}(i,j,k)$ and the (constant) maps $\mathfrak{B}_\alpha(i,j,k)$ that will play an instrumental role in the proof of Proposition B.6. We begin by gathering elementary properties of $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ to be utilised crucially when performing index permutations for the sums appearing in (B.23):

Lemma B.7. *Let* $w\in\mathrm{C}^1(\mathbb{R}^3,\mathbb{R}^{3\times3}_{\mathrm{sym}})$ *satisfy* $\mathrm{div}(w)=0$, $i,j,k,l\in\mathbb{N}$ *and define* $\mathfrak{A}_{\alpha\beta},\mathfrak{B}_\alpha$ *for* $\alpha,\beta\in\{1,2,3\}$ *by* (B.19). *Then the following hold:*

(a) $\partial_\alpha\mathfrak{A}_{\alpha,\beta}(i,j,k)=-\mathfrak{B}_\beta(i,j,k)$;

(b) $\partial_\beta\mathfrak{A}_{\alpha,\beta}(i,j,k)=\mathfrak{B}_\alpha(i,j,k)$;

(c) Antisymmetry of $\mathfrak{A}_{\alpha,\beta}$: $\mathfrak{A}_{\alpha,\beta}(i,j,k)=-\mathfrak{A}_{\alpha,\beta}(j,i,k)=\mathfrak{A}_{\alpha,\beta}(j,k,i)$;

(d) Antisymmetry of $\mathfrak{B}_\alpha$: $\mathfrak{B}_\alpha(i,j,k) = -\mathfrak{B}_\alpha(j,i,k) = \mathfrak{B}_\alpha(j,k,i)$;

(e) $\operatorname{div}_\xi((y-\xi)_\beta w_\alpha(\xi) - (y-\xi)_\alpha w_\beta(\xi)) = 0$;

(f) $\mathfrak{B}_\alpha(i,j,k) - \mathfrak{B}_\alpha(l,j,k) - \mathfrak{B}_\alpha(i,l,k) - \mathfrak{B}_\alpha(i,j,l) = 0$;

(g) $\mathfrak{A}_{\alpha,\beta}(i,j,k) - \mathfrak{A}_{\alpha,\beta}(l,j,k) - \mathfrak{A}_{\alpha,\beta}(i,l,k) - \mathfrak{A}_{\alpha,\beta}(i,j,l) = 0$.

Proof. Properties (a)–(d) are immediate consequences of the definitions. Property (e) holds, since

$$\operatorname{div}_\xi((y-\xi)_\beta w_\alpha(\xi) - (y-\xi)_\alpha w_\beta(\xi)) = -w_{\alpha\beta}(\xi) - \xi_\beta \operatorname{div}(w_\alpha) + w_{\beta\alpha}(\xi) + \xi_\alpha \operatorname{div}(w_\beta) = 0.$$

To prove (f) we use that by the definition of $\mathfrak{B}_\alpha$ and the Gauß-Green theorem we have

$$\mathfrak{B}_\alpha(i,j,k) - \mathfrak{B}_\alpha(l,j,k) - \mathfrak{B}_\alpha(i,l,k) - \mathfrak{B}_\alpha(i,j,l) = \int_{\langle x_i,x_j,x_k,x_m\rangle} \operatorname{div}(w_\alpha)\ \mathrm{d}x = 0.$$

Note that this calculation also holds in the case that one or multiple of the simplices are degenerate. Analogously, we can prove (g) by applying the Gauß-Green theorem as well as (e) to get

$$\begin{aligned}\mathfrak{A}_{\alpha,\beta}(i,j,k) &- \mathfrak{A}_{\alpha,\beta}(l,j,k) - \mathfrak{A}_{\alpha,\beta}(i,l,k) - \mathfrak{A}_{\alpha,\beta}(i,j,l)\\ &= \int_{\langle x_i,x_j,x_k,x_m\rangle} \operatorname{div}_\xi((y-\xi)_\beta w_\alpha(\xi) - (y-\xi)_\alpha w_\beta(\xi))\ \mathrm{d}x = 0.\end{aligned}$$

The proof is complete. □

Lemma B.8. [3] *Let $u \in (L^1 \cap \mathrm{C}^1)(\mathbb{R}^3, \mathbb{R}^3)$ satisfy $\operatorname{div}(u) = 0$ and $z_0 \in \{\mathcal{M}_{2R} u \leq \lambda\}$, where $R > 0$. Let, in addition, $x_1, x_2, x_3 \in B_R(z_0)$. Then*

$$\left| \fint_{\langle x_1,x_2,x_3\rangle} u(\xi)\cdot \nu_{123}\ \mathrm{d}^2\xi \right| \leq C\lambda R^2. \tag{B.24}$$

Moreover, if $w \in (L^1 \cap \mathrm{C}^1)(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ satisfies $\operatorname{div}(w) = 0$ and the cubes Q_i, Q_j, Q_k have non-empty intersection, $y \in Q_i \cap Q_j \cap Q_k$, we have for $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ as defined in (B.19)

(a) $|\mathfrak{A}_{\alpha,\beta}(i,j,k)(y)| \leq C\lambda \ell(Q_i)^3$;

(b) $|\mathfrak{B}_\alpha(i,j,k)| \leq C\lambda\ell(Q_i)^2$.

The constant $C = C(3)$ is a dimensional constant, that does not depend on u, i,j,k and the shape of $\mathcal{O}_\lambda$.

Proof. Let $x_1, x_2, x_3, z_0 \in \mathbb{R}^3$ be according to the assumption, $z_0 = (z_0^1, z_0^2, z_0^3)$. Then, using that $\operatorname{div} u = 0$, we find by Gauß' theorem for an arbitrary $\eta \in \mathbb{R}^3$

$$\left| \fint_{\langle x_1,x_2,x_3\rangle} u\cdot\nu_{123}\ \mathrm{d}^2\xi \right| \leq \Big(\int_{\langle \eta,x_2,x_3\rangle} + \int_{\langle x_1,\eta,x_3\rangle} + \int_{\langle x_1,x_2,\eta\rangle} \Big) |u|\ \mathrm{d}^2\xi. \tag{B.25}$$

[3] This corresponds to Lemma A.7 in the simple divergence-free setting.

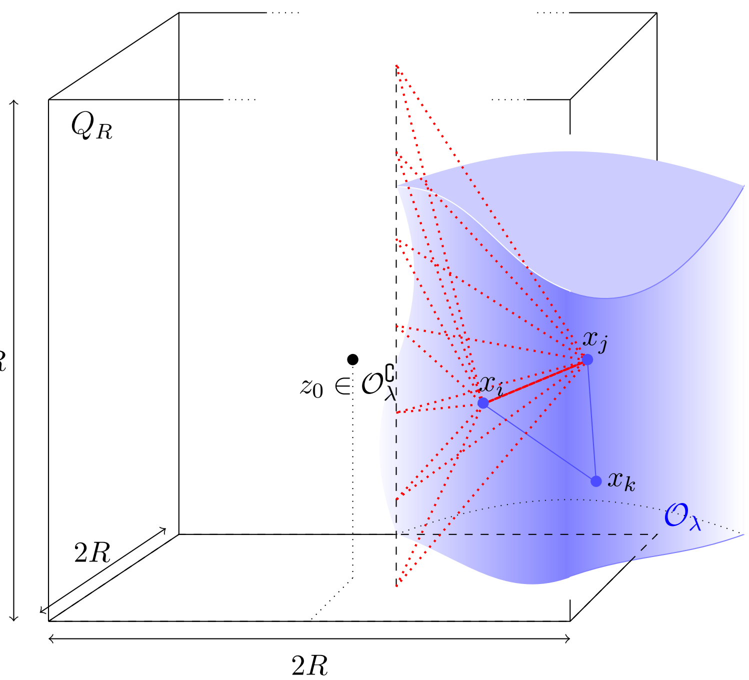

Figure B.2.: The construction in the proof of Lemma B.8. The point $z_0 \in \mathcal{O}_\lambda^\complement$ is chosen such that it is close to x_i, x_j and x_k respectively. Instead of estimating the integral on the triangle with vertices x_i, x_j and x_k directly, we estimate integrals along triangles with vertices x_i, x_j and $z \in Q_R(z_0)$ (the triangles with red dashed lines) and use Gauß' theorem.

Recalling from Section B.2.1 that $\operatorname{aff}(x_i, x_j, x_k)$ denotes the affine hyperplane containing x_i, x_j, x_k, we now establish the existence of some $\eta \in \mathbb{R}^3 \setminus \operatorname{aff}(x_i, x_j, x_k)$ such that the right-hand side of (B.25) is bounded by $CR^2\lambda$ for some $C > 0$ solely depending on the underlying space dimension $N = 3$. Denote $Q_R(z_0)$ the cube centered at z_0 with faces parallel to the coordinate planes and sidelength $2R$ so that $B_R(z_0) \subset Q_R(z_0) \subset B_{\sqrt{3}R}(z_0)$. Then, with the maximal operator $\mathcal{M}_{2R}$ from (B.8),

$$\begin{aligned}
\int_{B_R(z_0)} \int_{\langle x_1, x_2, z\rangle} |u(\xi)| \, \mathrm{d}^2\xi \, \mathrm{d}z &\leq \int_{Q_R(z_0)} \int_{\langle x_1, x_2, z\rangle} |u(\xi)| \, \mathrm{d}^2\xi \, \mathrm{d}z \\
&= \int_{z_0^1 - R}^{z_0^1 + R} \int_{z_0^2 - R}^{z_0^2 + R} \int_{z_0^3 - R}^{z_0^3 + R} \int_{\langle x_1, x_2, (z^1, z^2, z^3)\rangle} |u(\xi)| \, \mathrm{d}^2\xi \, \mathrm{d}z^3 \, \mathrm{d}z^2 \, \mathrm{d}z^1 \\
&\leq \int_{z_0^1 - R}^{z_0^1 + R} \int_{z_0^2 - R}^{z_0^2 + R} \int_{Q_R(z_0)} |u| \, \mathrm{d}x \, \mathrm{d}z^2 \, \mathrm{d}z^1 \\
&\leq \omega_3 (\sqrt{3}R)^3 \int_{z_0^1 - R}^{z_0^1 + R} \int_{z_0^2 - R}^{z_0^2 + R} \fint_{B_{\sqrt{3}R}(z_0)} |u| \, \mathrm{d}x \, \mathrm{d}z^2 \, \mathrm{d}z^1 \\
&\leq \omega_3 (2R)^3 (2R)^2 \mathcal{M}_{2R} u(z_0) \\
&\leq c\lambda R^5.
\end{aligned} \tag{B.26}$$

Here $c > 0$ is a constant solely depending on the space dimension $n = 3$. In consequence, by Markov's inequality,

$$\mathcal{L}^3(\mathcal{U}_{x_1,x_2,\cdot}[u,\lambda';B_R(z_0)]) := \mathcal{L}^3\Big(\Big\{z\in B_R(z_0)\colon \int_{\langle x_1,x_2,z\rangle}|u(\xi)|\,\mathrm{d}^2\xi>\lambda'\Big\}\Big) \overset{(B.26)}{\leq} c\frac{\lambda}{\lambda'}R^5 \qquad \text{for any } \lambda'>0,$$

where $\mathcal{U}_{x_1,x_2,\cdot}[u,\lambda';B_R(z_0)]$ is defined in the obvious manner. The same argument equally works for the remaining simplices that appear in (B.25), and therefore, setting

$$\mathcal{U} := \mathcal{U}_{x_1,x_2,\cdot}[u,\lambda';B_R(z_0)]\cup\mathcal{U}_{\cdot,x_2,x_3}[u,\lambda';B_R(z_0)]\cup\mathcal{U}_{x_1,\cdot,x_3}[u,\lambda';B_R(z_0)]$$

with an obvious definition of the sets appearing on the right-hand side, we obtain

$$\mathcal{L}^3(\mathcal{U})\leq\frac{4c\lambda}{\lambda'}R^5.$$

We still have the freedom to choose $\lambda'>0$ and consequently put $\lambda' := \frac{16}{\omega_3}c\lambda R^2$ so that $\mathcal{L}^3(\mathcal{U}^\complement)\geq\frac{3}{4}\mathcal{L}^3(B_R(z_0))$. We may thus pick $\eta\in B_R(z_0)\setminus\operatorname{aff}(x_i,x_j,x_k)$ such that $\eta\in\mathcal{U}^\complement$, and by definition of $\mathcal{U}$, this choice of η gives

$$\left|\fint_{\langle x_1,x_2,x_3\rangle}u\cdot\nu_{123}\,\mathrm{d}^2\xi\right|\leq c\lambda R^2$$

with some purely dimension dependent constant $c>0$. This completes the proof of (B.24).

The estimates in (a) and (b) are consequences of (B.24). For (a) note that there is $z_0\in\mathcal{O}_\lambda^\complement$ with $\operatorname{dist}(z_0,Q_i)\leq C\ell(Q_i)$ and $Q_i\cap Q_j\cap Q_k\subset B_{C\ell(Q_i)}(z_0)$ by (W2) and (W4). Moreover, $\mathcal{M}w(z_0)\leq\lambda$ by definition of $\mathcal{O}_\lambda$ and therefore, for fixed $y\in Q_i$

$$\mathcal{M}_{2R}((y-\cdot)_\beta w_\alpha(\cdot)-(y-\cdot)_\alpha w_\beta)(z_0)\leq 2\sup_{z\in B_{2R}(z_0)}|y-z|\cdot\mathcal{M}w(z_0).$$

Setting $R=C\ell(Q_i)$ and using Lemma B.7 (e) yields the estimate (a). The estimate for $\mathfrak{B}_\alpha$ directly uses the existence of a point $z_0\in\mathcal{O}_\lambda^\complement$, such that $Q_i,Q_j,Q_k\subset B_{C\ell(Q_i)}(z_0)$ and that w_α is divergence-free. Applying (B.24) in this setting yields (b). □

B.4.4. Elementary properties of T_λ

We now record various properties of T_λ that play an instrumental role in the proof of Theorem B.2. Throughout this section, we tacitly suppose that $w\in(\mathrm{C}^\infty\cap L^1)(\mathbb{R}^3,\mathbb{R}^{3\times3}_{\mathrm{sym}})$, and begin with providing the corresponding L^∞-bounds:

Lemma B.9. *There exists a purely dimensional constant $c>0$ such that*

$$\|T_\lambda w\|_{L^\infty(\mathbb{R}^3)}\leq c\lambda \qquad \textit{holds for all } \lambda>0. \tag{B.27}$$

Proof. Since $|w|\leq\lambda$ on $\mathcal{O}_\lambda^\complement$, it suffices to prove $\|T_\lambda w\|_{L^\infty(\mathcal{O}_\lambda)}\leq c\lambda$ for some suitable $c>0$.

Hence let $x \in \mathcal{O}_\lambda$. Then, by (W1) and (W3), $x \in Q_k$ for some $k \in \mathbb{N}$, and there are only finitely many cubes Q_i, Q_j such that $Q_i \cap Q_j \cap Q_k \neq \emptyset$; note that the number of such cubes solely depends on the underlying space dimension $n = 3$. For any choice of $\alpha', \beta', \gamma' \in \{1, 2, 3\}$ and $\ell_1 + \ell_2 = 2$ we have

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j| \leq c \frac{\mathbb{1}_{Q_i \cap Q_j \cap Q_k}}{\ell(Q_k)^2} \tag{B.28}$$

and similarly, if $\ell_1 + \ell_2 = 3$,

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j| \leq c \frac{\mathbb{1}_{Q_i \cap Q_j \cap Q_k}}{\ell(Q_k)^3}, \tag{B.29}$$

which is seen by combining (W4) and (P3). Again, $c > 0$ is a purely dimensional constant. By definition of $\widetilde{w}^{(k)}$, cf. (B.21) and (B.22), on $\mathcal{O}_\lambda$ every summand in (B.23) containing some $\mathfrak{B}_\delta(i, j, k)$, $\delta \in \{\alpha, \beta, \gamma\}$, is of the form $\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{B}_\delta(i, j, k)$ with $\ell_1 + \ell_2 = 2$. Here we may invoke Lemma B.8 (b) in conjunction with (B.28) to find

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{B}_\delta(i, j, k)| \leq c\lambda.$$

Conversely, every summand in (B.23) on $\mathcal{O}_\lambda$ that contains some $\mathfrak{A}_{\delta,\kappa}(i, j, k)$, $\delta, \kappa \in \{\alpha, \beta, \gamma\}$, is of the form $\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{A}_{\delta,\kappa}(i, j, k)$ with $\ell_1 + \ell_2 = 3$, and in this case Lemma B.8 (a) in conjunction with (B.29) yields

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{A}_{\delta,\kappa}(i, j, k)| \leq c\lambda.$$

By the uniformly finite overlap of the cubes, cf. (W3), this completes the proof. □

Lemma B.10. *For every* $\alpha \in \{1, 2, 3\}$*,* $T_\lambda(w_{\alpha 1}, w_{\alpha 2}, w_{\alpha 3})$ is solenoidal *on* $\mathcal{O}_\lambda$*.*

The proof of this lemma relies on a slightly elaborate computation, mutually hinging on index permutations and the properties of the maps $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ as gathered in Lemma B.7. For expository purposes, we thus accept Lemma B.10 for the time being and refer the reader to the computational section B.5.1 for its proof.

B.4.5. Global divsym-freeness

As the last ingredient towards Proposition B.6, we next address the regularity of $\mathrm{div}(T_\lambda w)$. Here, we do not assert that $T_\lambda w$ belongs to the Sobolev space $W^{1,1}(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$; this is so because $T_\lambda w$ is precisely constructed in a way such that handling of the divergence is possible (cf. Lemma B.11 below), whereas the control of the full gradients does not come up as a consequence of Lemma B.8; in particular, there seems to be no reason for the series in (B.23) to converge in $W_0^{1,1}(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Note that, if it did, we could directly infer from Lemma B.10 that $\mathrm{div}(T_\lambda w) = 0$.

Lemma B.11. *Let* $w \in (\mathrm{C}^\infty \cap L^1)(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfy* $\mathrm{div}(w) = 0$ *and define* $T_\lambda w$ *for* $\lambda > 0$ *by* (B.23)*. Then the distributional divergence of* $T_\lambda w$ *is an* $\mathbb{R}^3$*-valued regular distribution,* *that is,* $\operatorname{div}(T_\lambda w) \in L^1(\mathbb{R}^3, \mathbb{R}^3)$.

Proof. We focus on the first column $(T_\lambda w)_1$ of $T_\lambda w$; the other columns are treated by analogous means. Let $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^3)$. By a technical, yet elementary computation to be explained in detail below (cf. Section B.5.2), we have [4].

$$\begin{aligned}\int_{\mathcal{O}_\lambda} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x &= 2\sum_{i,j,k} \int_{\mathcal{O}_\lambda} \varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{B}_1(i,j,k)\partial_1\psi \,\mathrm{d}x \\ &\quad + 2\sum_{i,j,k} \int_{\mathcal{O}_\lambda} \varphi_k(\partial_3\varphi_j)(\partial_1\varphi_i)\mathfrak{B}_1(i,j,k)\partial_2\psi \,\mathrm{d}x \\ &\quad + 2\sum_{i,j,k} \int_{\mathcal{O}_\lambda} \varphi_k(\partial_1\varphi_j)(\partial_2\varphi_i)\mathfrak{B}_1(i,j,k)\partial_3\psi \,\mathrm{d}x \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{B.30}$$

We focus on term I first and consider the functions

$$\begin{aligned} v_{\mathrm{I},(1)}(y) &:= \sum_{i,j,k} v_{\mathrm{I}}^{ijk}(y) := \sum_{i,j,k} \varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(\mathfrak{B}_1(i,j,k) - w_1(y)\cdot\nu_{ijk}), \\ w_{\mathrm{I}}(y) &:= \sum_{i,j,k} w_{\mathrm{I}}^{ijk}(y) := \sum_{i,j,k} \varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{ijk}). \end{aligned} \tag{B.31}$$

We claim that $v_{\mathrm{I},(1)} \in W_0^{1,1}(\mathcal{O}_\lambda)$. Note that each summand belongs to $\mathrm{C}_c^\infty(\mathcal{O}_\lambda)$, and so it suffices to establish that the overall sum in (B.31) converges absolutely in $W^{1,1}(\mathcal{O}_\lambda)$. We give bounds on the single summands: For $i,j,k \in \mathbb{N}$, note that whenever $y \in Q_i \cap Q_j \cap Q_k$, then

$$\begin{aligned} |\mathfrak{B}_1(i,j,k) - w_1(y)\cdot\nu_{ijk}| &\le \fint_{\langle x_i,x_j,x_k\rangle} |w_1(\xi) - w_1(y)|\,|\nu_{ijk}| \,\mathrm{d}^2\xi \\ &\le c\|\nabla w_1\|_{L^\infty(\mathbb{R}^3)}\ell(Q_k)^3 \end{aligned} \tag{B.32}$$

as a consequence of the usual Lipschitz estimate, $\operatorname{dist}(y, \langle x_i,x_j,x_k\rangle) \le c\ell(Q_k)$ and $|\nu_{ijk}| \le c\ell(Q_k)^2$ by (W4). Now, by (W4) and (P3), we consequently obtain by (B.32)

$$\begin{aligned} \|v_{\mathrm{I}}^{ijk}\|_{L^1(Q_k)} &\le c\ell(Q_k)^4\|\nabla w_1\|_{L^\infty(\mathbb{R}^3)}, \\ \|\nabla v_{\mathrm{I}}^{ijk}\|_{L^1(Q_k)} &\le c\ell(Q_k)^3\|\nabla w_1\|_{L^\infty(\mathbb{R}^3)}, \end{aligned}$$

so that, by the uniformly finite overlap of the cubes,

$$\begin{aligned} \sum_{i,j,k} \|v_{\mathrm{I}}^{ijk}\|_{W^{1,1}(\mathcal{O}_\lambda)} &\le c\sum_k (\ell(Q_k)^4 + \ell(Q_k)^3)\|\nabla w_1\|_{L^\infty(\mathbb{R}^3)} \\ &\le c(1 + \mathcal{L}^3(\mathcal{O}_\lambda)^{\frac{1}{3}})\sum_k \ell(Q_k)^3\|\nabla w_1\|_{L^\infty(\mathbb{R}^3)} \\ &\le c(1 + \mathcal{L}^3(\mathcal{O}_\lambda)^{\frac{1}{3}})\mathcal{L}^3(\mathcal{O}_\lambda)\|\nabla w_1\|_{L^\infty(\mathbb{R}^3)} < \infty. \end{aligned}$$

[4] We already verified, that the term below is a divergence-free truncation in Chapter A, cf. Lemma A.14

Hence, $v_{\mathrm{I},(1)} \in W_0^{1,1}(\mathcal{O}_\lambda)$. Extend $v_{\mathrm{I},(1)}$ by zero to the entire $\mathbb{R}^3$ to obtain $v_{\mathrm{I},(2)} \in W_0^{1,1}(\mathbb{R}^3)$. Then an integration by parts yields

$$\begin{aligned} \mathrm{I} &= 2\int_{\mathcal{O}_\lambda} v_{\mathrm{I},(1)}\partial_1\psi \;\mathrm{d}y + 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}\partial_1\psi \;\mathrm{d}y \\ &= 2\int_{\mathbb{R}^3} v_{\mathrm{I},(2)}\partial_1\psi \;\mathrm{d}y + 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}\partial_1\psi \;\mathrm{d}y \\ &\overset{v_{\mathrm{I},(2)}\in W_0^{1,1}(\mathbb{R}^3)}{=} -2\int_{\mathbb{R}^3} (\partial_1 v_{\mathrm{I},(2)})\psi \;\mathrm{d}y + 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}\partial_1\psi \;\mathrm{d}y =: \mathrm{I}_1 + \mathrm{I}_2, \end{aligned} \tag{B.33}$$

and $\partial_1 v_{\mathrm{I},(2)} \in L^1(\mathbb{R}^3)$. Towards term I_2, observe that for all $y \in \mathbb{R}^3$,

$$\begin{aligned} -2\nu_{ijk} &= -(x_i - x_j) \times (x_k - x_j) \\ &= (y - x_j) \times (x_j - x_k) + (x_i - y) \times (y - x_k) + (x_i - x_j) \times (x_j - y), \end{aligned} \tag{B.34}$$

which follows by direct computation using that $(x_j - y) \times (y - x_j) = 0$. Working from the definition of w_{I} as in (B.31), we consequently find by (B.34)

$$\begin{aligned} \mathrm{I}_2 = 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}(y)\partial_1\psi \;\mathrm{d}y &= 2\int_{\mathcal{O}_\lambda} \sum_{i,j,k} \varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{y,x_j,x_k})\partial_1\psi \;\mathrm{d}y && (=0) \\ &+ 2\int_{\mathcal{O}_\lambda} \sum_{i,j,k} \varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,y,x_k})\partial_1\psi \;\mathrm{d}y && (=0) \\ &+ 2\int_{\mathcal{O}_\lambda} \sum_{i,j} (\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi \;\mathrm{d}y =: \mathrm{I}_3, \end{aligned}$$

where we have used that $\sum_i \partial_3\varphi_i = 0$ on $\mathcal{O}_\lambda$ for the first, $\sum_j \partial_2\varphi_j = 0$ on $\mathcal{O}_\lambda$ for the second and $\sum_k \varphi_k = 1$ on $\mathcal{O}_\lambda$ for the ultimate term. By a similar argument as above, the sum in the integrand of I_3 has an integrable majorant, whereby we may change the sum and the integral. Hence, integrating by parts with respect to ∂_2,

$$\begin{aligned} \mathrm{I}_3 = \mathrm{I}_3^1 := 2\sum_{ij}\int_{\mathcal{O}_\lambda} &\partial_2(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\;\mathrm{d}y && (=T_1) \\ - 2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_j(\partial_{23}\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\;\mathrm{d}y && (=T_2) \\ - 2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_j(\partial_3\varphi_i)(\partial_2 w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\;\mathrm{d}y && (=T_3) \\ - 2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\partial_2\nu_{x_i,x_j,y})\partial_1\psi)\;\mathrm{d}y && (=T_4) \\ - 2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_{12}\psi)\;\mathrm{d}y && (=T_5), \end{aligned}$$

but on the other hand, now integrating by parts with respect to ∂_3,

$$
\begin{aligned}
\mathrm{I}_3 = \mathrm{I}_3^2 := 2\sum_{ij}\int_{\mathcal{O}_\lambda} &\partial_3(\varphi_i(\partial_2\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_6)\\
-2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_i(\partial_{23}\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_7)\\
-2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_i(\partial_2\varphi_j)(\partial_3 w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_8)\\
-2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_i(\partial_2\varphi_j)(w_1(y)\cdot\partial_3\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_9)\\
-2\sum_{ij}\int_{\mathcal{O}_\lambda} &(\varphi_i(\partial_2\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_{13}\psi)\,\mathrm{d}y && (=T_{10}).
\end{aligned}
$$

We then have $\mathrm{I}_3 = \frac{1}{2}(\mathrm{I}_3^1+\mathrm{I}_3^2)$. To proceed further, note that $T_1 = T_6 = 0$ by the fundamental theorem of calculus. Moreover, $\frac{1}{2}(T_2+T_7) = 0$, which follows from permuting indices $i \leftrightarrow j$ in T_2 and using the antisymmetry property $\nu_{x_i,x_j,y} = -\nu_{x_j,x_i,y}$:

$$
\begin{aligned}
T_2 &= -2\sum_{ji}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_{23}\varphi_j)(w_1(y)\cdot\nu_{x_j,x_i,y})\partial_1\psi)\,\mathrm{d}y\\
&= 2\sum_{ji}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_{23}\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y = -T_7.
\end{aligned}
$$

For treating terms T_3 and T_8, define the smooth function $v_{\mathrm{I},(3)}\colon \mathcal{O}_\lambda \to \mathbb{R}$ by

$$
v_{\mathrm{I},(3)} := \sum_{ij}(\varphi_j(\partial_3\varphi_i)(\partial_2 w_1(y)\cdot\nu_{x_i,x_j,y})) + (\varphi_i(\partial_2\varphi_j)(\partial_3 w_1(y)\cdot\nu_{x_i,x_j,y})). \tag{B.35}
$$

By an argument similar to the one employed in (B.31)ff., we have $v_{\mathrm{I},(3)} \in W_0^{1,1}(\mathcal{O}_\lambda)$. More precisely, for all finite index sets $\mathcal{I},\mathcal{J} \subset \mathbb{N}$ the functions

$$
\begin{aligned}
z_{\mathcal{I},\mathcal{J}} &:= \sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} z_{ij}\\
&:= \sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}}(\varphi_j(\partial_3\varphi_i)(\partial_2 w_1(y)\cdot\nu_{x_i,x_j,y})) + (\varphi_i(\partial_2\varphi_j)(\partial_3 w_1(y)\cdot\nu_{x_i,x_j,y}))
\end{aligned}
$$

are finite sums of $\mathrm{C}_c^\infty(\mathcal{O}_\lambda)$-functions. By the Leibniz rule in conjunction with (W2)–(W4) and (P3), we obtain

$$
\begin{aligned}
\sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}}\|z_{ij}\|_{W^{1,1}(\mathcal{O}_\lambda)} &= \sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}}\|z_{ij}\|_{L^1(\mathcal{O}_\lambda)} + \|\nabla z_{ij}\|_{L^1(\mathcal{O}_\lambda)}\\
&\le c\sum_{i\in\mathcal{I}}\ell(Q_i)^4\|\nabla w_1\|_{L^\infty(\mathbb{R}^3)}
\end{aligned}
$$

$$+ c \sum_{i \in \mathcal{I}} \big(\ell(Q_i)^3 \|\nabla w_1\|_{L^\infty(\mathbb{R}^3)} + \ell(Q_i)^4 \|\nabla^2 w_1\|_{L^\infty(\mathbb{R}^3)} \big)$$
$$\leq (1 + \mathcal{L}^3(\mathcal{O}_\lambda)^{\frac{1}{3}}) \|w_1\|_{W^{2,\infty}(\mathbb{R}^3)},$$

where c is a purely dimensional constant. Since the ultimate term in the previous estimation is independent of $\mathcal{I}$ and $\mathcal{J}$, we conclude that the sum in (B.35) converges absolutely in the Banach space $W_0^{1,1}(\mathcal{O}_\lambda)$. Hence, in particular, it converges in $W_0^{1,1}(\mathcal{O}_\lambda)$ and so $v_{\mathrm{I},(3)} \in W_0^{1,1}(\mathcal{O}_\lambda)$.

Extending $v_{\mathrm{I},(3)}$ by zero to $v_{\mathrm{I},(4)} \in W_0^{1,1}(\mathbb{R}^3)$, then obtain

$$\tfrac{1}{2}(T_3 + T_8) = \int_{\mathbb{R}^3} (\partial_1 v_{\mathrm{I},(4)}) \psi \, \mathrm{d}y. \tag{B.36}$$

Since $\mathrm{I}_3 = \dfrac{1}{2}(\mathrm{I}_3^1 + \mathrm{I}_3^2)$, the above arguments, permuting $i \leftrightarrow j$ in I_3^2 and (B.36) combine to

$$\begin{aligned}
\mathrm{I}_3 = &- \frac{1}{2} \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j (\partial_3 \varphi_i)(w_1(y) \cdot ((x_i - x_j) \times e_2)) \partial_1 \psi) \, \mathrm{d}y && (= \tfrac{1}{2} T_4) \\
&+ \frac{1}{2} \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j (\partial_2 \varphi_i)(w_1(y) \cdot ((x_i - x_j) \times e_3)) \partial_1 \psi) \, \mathrm{d}y && (= \tfrac{1}{2} T_9) \\
&- \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j (\partial_3 \varphi_i)(w_1(y) \cdot \nu_{x_i, x_j, y}) \partial_{12} \psi) \, \mathrm{d}y && (= \tfrac{1}{2} T_5) \\
&+ \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j (\partial_2 \varphi_i)(w_1(y) \cdot \nu_{x_i, x_j, y}) \partial_{13} \psi) \, \mathrm{d}y && (= \tfrac{1}{2} T_{10}) \\
&+ \int_{\mathbb{R}^3} (\partial_1 v_{\mathrm{I},(4)}) \psi \, \mathrm{d}y.
\end{aligned}$$

Next note that, expanding and using $\sum_i \varphi_i = 1$ as well as $\sum_i \partial_3 \varphi_i = 0$ on $\mathcal{O}_\lambda$,

$$\begin{aligned}
\tfrac{1}{2} T_4 = &- \frac{1}{2} \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j (\partial_3 \varphi_i)(w_1(y) \cdot ((x_i - y) \times e_2)) \partial_1 \psi) \, \mathrm{d}y \\
&- \frac{1}{2} \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j (\partial_3 \varphi_i)(w_1(y) \cdot ((y - x_j) \times e_2)) \partial_1 \psi) \, \mathrm{d}y \quad (= 0) \\
= &- \frac{1}{2} \sum_{i} \int_{\mathcal{O}_\lambda} ((\partial_3 \varphi_i)(w_1(y) \cdot ((x_i - y) \times e_2)) \partial_1 \psi) \, \mathrm{d}y \\
= &\ \frac{1}{2} \sum_{i} \int_{\mathcal{O}_\lambda} (\varphi_i \partial_3 w_1(y) \cdot ((x_i - y) \times e_2)) \partial_1 \psi) \, \mathrm{d}y \\
&+ \frac{1}{2} \int_{\mathcal{O}_\lambda} (w_1(y) \cdot (-e_3 \times e_2) \partial_1 \psi) \, \mathrm{d}y \\
&+ \frac{1}{2} \sum_{i} \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y) \cdot ((x_i - y) \times e_2) \partial_{13} \psi) \, \mathrm{d}y.
\end{aligned} \tag{B.37}$$

By a similar argument as for (B.35)ff., we use $w \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ to see that the function

$$v_{\mathrm{I},(5)}(y) := -\frac{1}{2}\sum_i \varphi_i \partial_3 w_1(y) \cdot ((x_i - y) \times e_2)) \tag{B.38}$$

belongs to $W_0^{1,1}(\mathcal{O}_\lambda)$, and hence, again denoting its trivial extension to $\mathbb{R}^3$ by $v_{\mathrm{I},(6)}$ and recalling that $e_2 \times e_3 = e_1$,

$$\begin{aligned}\tfrac{1}{2}T_4 = \int_{\mathbb{R}^3} (\partial_1 v_{\mathrm{I},(6)})\psi \,\mathrm{d}x &+ \frac{1}{2}\int_{\mathcal{O}_\lambda} (w_{11}(y)\partial_1\psi)\,\mathrm{d}y \\ &+ \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y) \cdot ((x_i - y) \times e_2)\partial_{13}\psi)\,\mathrm{d}y\end{aligned} \tag{B.39}$$

We handle the term $\frac{1}{2}T_9$ in the same fashion (swapping the roles of the indices 2 and 3): Introducing $v_{\mathrm{I},(7)} \in W_0^{1,1}(\mathcal{O}_\lambda)$ by

$$v_{\mathrm{I},(7)}(y) := \frac{1}{2}\sum_i \varphi_i \partial_2 w_1(y) \cdot ((x_i - y) \times e_3)$$

as a substitute for (B.38) and denoting its trivial extension to $\mathbb{R}^3$ by $v_{\mathrm{I},(8)}$, we arrive at

$$\begin{aligned}\tfrac{1}{2}T_9 = \int_{\mathbb{R}^3} (\partial_1 v_{\mathrm{I},(8)})\psi \,\mathrm{d}x &+ \frac{1}{2}\int_{\mathcal{O}_\lambda} (w_{11}(y)\partial_1\psi)\,\mathrm{d}y \\ &- \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y) \cdot ((x_i - y) \times e_3)\partial_{12}\psi)\,\mathrm{d}y.\end{aligned} \tag{B.40}$$

Working from (B.39) and (B.40), we then arrive at

$$\begin{aligned}\tfrac{1}{2}(T_4 + T_9) &= \int_{\mathbb{R}^3} (\partial_1 (v_{\mathrm{I},(6)} + v_{\mathrm{I},(8)})\psi \,\mathrm{d}y + \int_{\mathcal{O}_\lambda} (w_{11}(y)\partial_1\psi)\,\mathrm{d}y \\ &+ \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y) \cdot ((x_i - y) \times e_2)\partial_{13}\psi)\,\mathrm{d}y \\ &- \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y) \cdot ((x_i - y) \times e_3)\partial_{12}\psi)\,\mathrm{d}y.\end{aligned} \tag{B.41}$$

To summarise, by (B.30), (B.33) and (B.41), there exists $v_{\mathrm{I}} \in W_0^{1,1}(\mathbb{R}^3)$, such that

$$\begin{aligned}\mathrm{I} &= \int_{\mathbb{R}^3} (\partial_1 v_{\mathrm{I}})\psi \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} (w_{11}(y)\partial_1\psi)\,\mathrm{d}y \\ &+ \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y) \cdot ((x_i - y) \times e_2)\partial_{13}\psi)\,\mathrm{d}y \\ &- \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y) \cdot ((x_i - y) \times e_3)\partial_{12}\psi)\,\mathrm{d}y \\ &- \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j (\partial_3\varphi_i)(w_1(y) \cdot \nu_{x_i x_j y})\partial_{12}\psi)\,\mathrm{d}y\end{aligned}$$

$$+\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_2\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{13}\psi)\,\mathrm{d}y \tag{B.42}$$

The same calculations with the coordinates $1\to 2\to 3\to 1$ permuted imply that there exist $v_{\mathrm{II}}, v_{\mathrm{III}}\in W_0^{1,1}(\mathbb{R}^3)$, such that

$$\begin{aligned}\mathrm{II} = &\int_{\mathbb{R}^3}(\partial_2 v_{\mathrm{II}})\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}(w_{12}(y)\partial_2\psi)\,\mathrm{d}y\\
&+\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_3)\partial_{21}\psi)\,\mathrm{d}y\\
&-\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_1)\partial_{23}\psi)\,\mathrm{d}y\\
&-\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_1\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{23}\psi)\,\mathrm{d}y\\
&+\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{21}\psi)\,\mathrm{d}y\end{aligned} \tag{B.43}$$

and

$$\begin{aligned}\mathrm{III} = &\int_{\mathbb{R}^3}(\partial_3 v_{\mathrm{III}})\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}(w_{13}(y)\partial_3\psi)\,\mathrm{d}y\\
&+\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_1)\partial_{32}\psi)\,\mathrm{d}y\\
&-\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_2)\partial_{31}\psi)\,\mathrm{d}y\\
&-\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_2\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{31}\psi)\,\mathrm{d}y\\
&+\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_1\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{32}\psi)\,\mathrm{d}y,\end{aligned} \tag{B.44}$$

and $\partial_1 v_{\mathrm{I}}, \partial_2 v_{\mathrm{II}}, \partial_3 v_{\mathrm{III}}$ all vanish outside $\mathcal{O}_\lambda$. Combining (B.42), (B.43) and (B.44), we get that there is $h\in L^1(\mathcal{O}_\lambda)$, $h=\partial_1 v_{\mathrm{I}}+\partial_2 v_{\mathrm{II}}+\partial_3 v_{\mathrm{III}}$, such that

$$\int_{\mathcal{O}_\lambda}(T_\lambda w)_1\cdot\nabla\psi\,\mathrm{d}x = \int_{\mathcal{O}_\lambda}h\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}w_1\cdot\nabla\psi\,\mathrm{d}x. \tag{B.45}$$

Recall that w satisfies $\mathrm{div}(w)=0$ and that $T_\lambda w = w$ on $\mathcal{O}_\lambda^{\complement}$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^3}(T_\lambda w)_1\cdot\nabla\psi\,\mathrm{d}x &= \int_{\mathcal{O}_\lambda^{\complement}}(T_\lambda w)_1\cdot\nabla\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}(T_\lambda w)_1\cdot\nabla\psi\,\mathrm{d}x\\
&= \int_{\mathcal{O}_\lambda^{\complement}}w_1\cdot\nabla\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}w_1\cdot\nabla\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}h\psi\,\mathrm{d}x\\
&= \int_{\mathcal{O}_\lambda}h\psi\,\mathrm{d}x.\end{aligned}$$

Therefore, $\mathrm{div}((T_\lambda w)_1)\in L^1(\mathbb{R}^3)$. Arguing in the exactly same way for the other columns,

$\operatorname{div}(T_\lambda w) \in L^1(\mathbb{R}^3, \mathbb{R}^3)$, and the proof is complete. □

As an immediate consequence of Lemmas B.10 and B.11, we obtain the following

Corollary B.12. *Let* $w \in (\mathrm{C}^\infty \cap L^1)(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfy* $\operatorname{div}(w) = 0$ *and define* $T_\lambda w$ *for* $\lambda > 0$ *by* (B.23). *Then for* $\mathcal{L}^1$*-almost every* $\lambda > 0$, $\operatorname{div}(T_\lambda w) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$.

Proof. Observe that on $\mathbb{R}^3 \setminus \partial\mathcal{O}_\lambda$ the function $T_\lambda w$ is strongly differentiable and, as w is (row-wise) solenoidal on $\mathbb{R}^3$ and $\operatorname{div}(T_\lambda w) = 0$ on $\mathcal{O}_\lambda$ (Lemma B.10), $\operatorname{div}(T_\lambda w) = 0$ on the open set $\mathbb{R}^3 \setminus \partial\mathcal{O}_\lambda$. As $w \in \mathrm{C}^\infty$, $\mathcal{M}w \in \mathrm{C}(\mathbb{R}^3)$ and the set

$$\{\lambda > 0 \colon \mathcal{L}^3(\partial\mathcal{O}_\lambda) \neq 0\} \subset \{\lambda > 0 \colon \mathcal{L}^3(\{\mathcal{M}w = \lambda\}) \neq 0\}$$

is an $\mathcal{L}^1$-null set. Hence, for all λ not contained in this set, $\operatorname{div}(T_\lambda w) \in L^1(\mathbb{R}^3, \mathbb{R}^3)$ and $\operatorname{div}(T_\lambda w) = 0$ $\mathcal{L}^3$-a.e.. Thus, for $\mathcal{L}^1$-almost every λ, $\operatorname{div}(T_\lambda w) = 0$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$. □

B.4.6. Strong stability and the proof of Proposition B.6

In view of Lemma B.9 and Corollary B.12, Proposition B.6 will follow provided we can prove the strong stability (cf. Proposition B.6 1). Towards this aim, we begin with

Lemma B.13. *Then there exists a purely dimensional constant* $C > 0$ *such that, for each* $w \in L^1(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *and each* $\lambda > 0$, *we have*

$$\mathcal{L}^3(\{\mathcal{M}w > \lambda\}) \leq \frac{C}{\lambda} \int_{\{|w| > \lambda/2\}} |w(x)| \,\mathrm{d}x$$

The rough idea of the proof of this statement is to use the weak-$(1,1)$-estimate for the Hardy-Littlewood maximal operator $\mathcal{M}$ (cf. (B.8)) for the function h defined via

$$h(x) = \max\{0, |w(x)| - \lambda/2\}, \tag{B.46}$$

see ZHANG [157] for the details (also see Lemma A.17). As an important consequence of Lemma B.13 and the L^∞-bound of w_λ is the following:

Corollary B.14. *Let* $w \in L^1(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfy* $\operatorname{div}(w) = 0$. *Moreover, for* $\lambda > 0$, *let* $w_\lambda := T_\lambda w$ *be as in* (B.23). *Then we have with a purely dimensional constant* $C > 0$

$$\|w - w_\lambda\|_{L^1(\mathbb{R}^3)} \leq C \int_{\{|w| > \lambda/2\}} |w| \,\mathrm{d}x. \tag{B.47}$$

Proof. Recall that $\mathcal{O}_\lambda := \{\mathcal{M}w > \lambda\}$. By construction, $w = w_\lambda$ on $\mathcal{O}_\lambda^\complement$. Therefore,

$$\|w - w_\lambda\|_{L^1(\mathbb{R}^3)} \leq \int_{\mathcal{O}_\lambda} |w - w_\lambda| \,\mathrm{d}x \leq \int_{\mathcal{O}_\lambda} |w| \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} |w_\lambda| \,\mathrm{d}x. \tag{B.48}$$

On the one hand, Lemma B.13 gives us

$$\int_{\mathcal{O}_\lambda} |w| \,\mathrm{d}x \leq \lambda\mathcal{L}^3(\mathcal{O}_\lambda) + \int_{\{|w| > \lambda\}} |w| \,\mathrm{d}x \leq C \int_{\{|w| > \lambda/2\}} |w| \,\mathrm{d}x, \tag{B.49}$$

and, on the other hand, using Lemma B.9 and Lemma B.13,

$$\int_{\mathcal{O}_\lambda} |w_\lambda| \,\mathrm{d}x \le \|w_\lambda\|_{L^\infty(\mathbb{R}^3)} \mathcal{L}^3(\mathcal{O}_\lambda) \le C \int_{\{|w|>\lambda/2\}} |w| \,\mathrm{d}x, \tag{B.50}$$

$C > 0$ still being a purely dimensional constant. In view of (B.48), (B.49) and (B.50), we obtain (B.47), and this completes the proof. □

Proof of Proposition B.6. Let $w \in (\mathrm{C}^\infty \cap L^1)(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfy $\mathrm{div}(w) = 0$ and let $\lambda > 0$. Pick some $\widetilde{\lambda} \in (\lambda, 2\lambda)$ such that $\mathcal{L}^3(\partial\mathcal{O}_{\widetilde{\lambda}}) = 0$ and define $w_\lambda := T_{\widetilde{\lambda}} w$ and $\mathcal{U}_\lambda := \mathcal{O}_{\widetilde{\lambda}}$. Then

1. $w = w_\lambda$ on $\mathcal{U}_\lambda^{\complement}$ by construction.
2. Lemma B.13 implies that
$$\mathcal{L}^3(\{w \neq w_\lambda\}) \le \frac{c}{\widetilde{\lambda}} \int_{\{|w|>\widetilde{\lambda}/2\}} |w| \,\mathrm{d}x \le \frac{c}{\lambda} \int_{\{|w|>\lambda/2\}} |w| \,\mathrm{d}x.$$
3. $\mathrm{div}(w_\lambda) = 0$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ by Corollary B.12.
4. $\|w_\lambda\|_{L^\infty(\mathbb{R}^3)} \le c\tilde{\lambda} \le 2c\lambda$ by Lemma B.9.

To summarise, w_λ satisfies all the required properties, and the proof is complete. □

B.4.7. Proof of Theorem B.2

We now establish Theorem B.2, and hence let $\lambda > 0$ be given. Let $u \in L^1(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfy $\mathrm{div}(u) = 0$ and pick a sequence $(w^j) \subset (\mathrm{C}^\infty \cap L^1)(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ such that $w^j \to u$ strongly in $L^1(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ as $j \to \infty$, still satisfying $\mathrm{div}(w^j) = 0$ for each $j \in \mathbb{N}$. Such a sequence can be constructed by convolution with smooth bumps.

For $\lambda > 0$ consider the truncation $w^j_{4\lambda}$ of w^j according to Proposition B.6. Note that this sequence is uniformly bounded in L^∞ by $4c\lambda$. Therefore, a suitable, non-relabeled subsequence converges in the weak*-sense to some u^λ in $L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$. First of all,

$$\|u^\lambda\|_{L^\infty(\mathbb{R}^3)} \le \sup_{j\in\mathbb{N}} \|w^j_{4\lambda}\|_{L^\infty(\mathbb{R}^3)} \le 4c\lambda, \quad \mathrm{div}(u^\lambda) = 0.$$

We claim that $w^j_{4\lambda} \to u$ strongly in L^1 on the set $\{\mathcal{M}u \le 2\lambda\}$ as $j \to \infty$, and hence $u^\lambda = u$ on $\{\mathcal{M}u \le 2\lambda\}$. If this claim is proven, then Lemma B.13 and Corollary B.14 imply the small change and strong stability properties (b), (c) of Theorem B.2. Therefore u^λ will satisfy all properties displayed in Theorem B.2 and thus finish the proof.

It remains to show the claim. Recall that the maximal function $\mathcal{M}$ is sublinear. Thus,

$$\{\mathcal{M}w^j > 4\lambda\} \setminus \{\mathcal{M}(w^j - u) > 2\lambda\} \subset \{\mathcal{M}u > 2\lambda\}. \tag{B.51}$$

Note that $\mathcal{L}^3(\{\mathcal{M}(w^j - u) > 2\lambda\})$ converges to zero as $j \to \infty$ since $w^j - u \to 0$ in L^1 and $\mathcal{M}$ is weak-$(1,1)$. After picking a suitable, non-relabeled subsequence of (w^j) we may suppose that $\|w^j - u\|_{L^1(\mathbb{R}^3)} \le 2^{-j}\lambda$ for all $j \in \mathbb{N}$ and hence

$$\mathcal{L}^3\{\mathcal{M}(w^j - u) > 2\lambda\} \le C2^{-j} \qquad \text{for all } j \in \mathbb{N}.$$

Therefore, for each $J \in \mathbb{N}$, the $\mathcal{L}^3$-measure of the set

$$E_J := \bigcup_{j>J} \{\mathcal{M}(w^j - u) > 2\lambda\}$$

can be bounded by $C2^{-J}$. Due to (B.51), we have $\{\mathcal{M}u \le 2\lambda\} \setminus E_J \subset \{\mathcal{M}w^j \le 4\lambda\}$ for $j > J$. Let us fix $J \in \mathbb{N}$ and bound the L^1-norm of $w^j_{4\lambda} - u$ on $\{\mathcal{M}u \le 2\lambda\}$ for $j > J$:

$$\begin{aligned}
\int_{\{\mathcal{M}u \le 2\lambda\}} |w^j_{4\lambda} - u| \, \mathrm{d}x &\le \int_{E_J} |w^j_{4\lambda} - u| \, \mathrm{d}x + \int_{\{\mathcal{M}u \le 2\lambda\}\setminus E_J} |w^j_{4\lambda} - u| \, \mathrm{d}x \\
&\le \int_{E_J} |w^j_{4\lambda}| + |u| \, \mathrm{d}x + \int_{\{\mathcal{M}w^j \le 4\lambda\}} |w^j_{4\lambda} - u| \, \mathrm{d}x \\
&\le C2^{-J}\lambda + \int_{E_J} |u| \, \mathrm{d}x + \int_{\{\mathcal{M}w^j \le 4\lambda\}} |w^j - u| \, \mathrm{d}x \\
&\le C2^{-J}\lambda + \int_{E_J} |u| \, \mathrm{d}x + \|w^j - u\|_{L^1(\mathbb{R}^3)}.
\end{aligned}$$

Letting $J \to \infty$ yields $w^j_{4\lambda} - u \to 0$ in $L^1(\{\mathcal{M}u \le 2\lambda\})$. As $(w^j_{4\lambda})$ weakly*-converges to u^λ in $L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, we conclude that $u = u^\lambda$ on $\{\mathcal{M}u \le 2\lambda\}$, proving the claim. □

B.5. Computational details for proofs

In this section, we give the computational details for some of the identities used in the main part of the paper. We will need the following lemma.

Lemma B.15. *Let $a, b, c \in \mathbb{N}^3$ be multi-indices with $|a|, |b|, |c| \ge 1$ and $\alpha, \beta \in \{1, 2, 3\}$. Then on the set $\mathcal{O}_\lambda$ have*

$$\sum_{ijk} \partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \mathfrak{B}_\alpha(i, j, k) = 0, \tag{B.52}$$

and

$$\sum_{ijk} \partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \mathfrak{A}_{\alpha,\beta}(i, j, k) = 0. \tag{B.53}$$

Proof. Recall from the definition of the φ_l that $\sum \varphi_l \equiv 1$ on $\mathcal{O}_\lambda$. We therefore have $\sum \partial_a \varphi_l = \sum \partial_b \varphi_l = \sum \partial_c \varphi_l = 0$. We can use this to get

$$\begin{aligned}
&\sum_{ijk} \partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \mathfrak{B}_\alpha(i, j, k) \\
= &\sum_{ijkm} \partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \Big(\mathfrak{B}_\alpha(i, j, k) - \mathfrak{B}_\alpha(m, j, k) - \mathfrak{B}_\alpha(i, m, k) - \mathfrak{B}_\alpha(i, j, m) \Big)
\end{aligned}$$

Now (B.52) follows from Lemma B.7 (f); (B.53) can be shown completely analogously. □

B.5.1. Proof of Lemma B.10

We focus on the case $\alpha = 1$. Let thus $D := \operatorname{div}(T_\lambda w)_1$. To avoid notational overload we omit the arguments i, j and k of $\mathfrak{A}_{\alpha,\beta}(i,j,k)$ and $\mathfrak{B}_\alpha(i,j,k)$ in the following equation. Thus, all $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ implicitly depend on the summation indices. By the definition of $T_\lambda w$ on $\mathcal{O}_\lambda$, (B.23), we have

$$
\begin{aligned}
D =& 6\sum_{ijk} \partial_1(\varphi_k \partial_2\varphi_j \partial_3\varphi_i)\mathfrak{B}_1 && (= T_1)\\
&+ 2\sum_{ijk} \partial_1(\varphi_k(\partial_{33}\varphi_j\partial_2\varphi_i - \partial_{23}\varphi_j\partial_3\varphi_i))\mathfrak{A}_{3,1} && (= T_2)\\
&+ 2\sum_{ijk} \varphi_k(\partial_{33}\varphi_j\partial_2\varphi_i - \partial_{23}\varphi_j\partial_3\varphi_i)\partial_1\mathfrak{A}_{3,1} && (= T_3)\\
&+ 2\sum_{ijk} \partial_1(\varphi_k(\partial_{22}\varphi_j\partial_3\varphi_i - \partial_{23}\varphi_j\partial_2\varphi_i))\mathfrak{A}_{1,2} && (= T_4)\\
&+ 2\sum_{ijk} \varphi_k(\partial_{22}\varphi_j\partial_3\varphi_i - \partial_{32}\varphi_j\partial_2\varphi_i)\partial_1\mathfrak{A}_{1,2} && (= T_5)\\
&+ 3\sum_{ijk} \partial_2(\varphi_k \partial_3\varphi_j \partial_1\varphi_i)\mathfrak{B}_1 && (= T_6)\\
&+ 3\sum_{ijk} \partial_2(\varphi_k \partial_2\varphi_j \partial_3\varphi_i)\mathfrak{B}_2 && (= T_7)\\
&+ \sum_{ijk} \partial_2(\varphi_k(\partial_{23}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_2\varphi_i))\mathfrak{A}_{2,3} && (= T_8)\\
&+ \sum_{ijk} \varphi_k(\partial_{23}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_2\varphi_i)\partial_2\mathfrak{A}_{2,3} && (= T_9)\\
&+ \sum_{ijk} \partial_2(\varphi_k(\partial_{13}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_1\varphi_i))\mathfrak{A}_{3,1} && (= T_{10})\\
&+ \sum_{ijk} \varphi_k(\partial_{13}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_1\varphi_i)\partial_2\mathfrak{A}_{3,1} && (= T_{11})\\
&+ \sum_{ijk} \partial_2(\varphi_k(\partial_{13}\varphi_j\partial_2\varphi_i + \partial_{23}\varphi_j\partial_1\varphi_i - 2\partial_{12}\varphi_j\partial_3\varphi_i))\mathfrak{A}_{1,2} && (= T_{12})\\
&+ \sum_{ijk} (\varphi_k(\partial_{13}\varphi_j\partial_2\varphi_i + \partial_{23}\varphi_j\partial_1\varphi_i - 2\partial_{12}\varphi_j\partial_3\varphi_i))\partial_2\mathfrak{A}_{1,2} && (= T_{13})\\
&+ 3\sum_{ijk} \partial_3(\varphi_k \partial_2\varphi_j \partial_3\varphi_i)\mathfrak{B}_3 && (= T_{14})\\
&+ 3\sum_{ijk} \partial_3(\varphi_k \partial_1\varphi_j \partial_2\varphi_i)\mathfrak{B}_1 && (= T_{15})\\
&+ \sum_{ijk} \partial_3(\varphi_k(\partial_{12}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_1\varphi_i))\mathfrak{A}_{1,2} && (= T_{16})\\
&+ \sum_{ijk} (\varphi_k(\partial_{12}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_1\varphi_i))\partial_3\mathfrak{A}_{1,2} && (= T_{17})
\end{aligned}
$$

$$\begin{aligned}
&+\sum_{ijk}\partial_3(\varphi_k(\partial_{23}\varphi_j\partial_2\varphi_i-\partial_{22}\varphi_j\partial_3\varphi_i))\mathfrak{A}_{2,3} && (=T_{18})\\
&+\sum_{ijk}(\varphi_k(\partial_{23}\varphi_j\partial_2\varphi_i-\partial_{22}\varphi_j\partial_3\varphi_i))\partial_3\mathfrak{A}_{2,3} && (=T_{19})\\
&+\sum_{ijk}\partial_3(\varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i+\partial_{12}\varphi_j\partial_3\varphi_i-2\partial_{13}\varphi_j\partial_2\varphi_i))\mathfrak{A}_{3,1} && (=T_{20})\\
&+\sum_{ijk}(\varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i+\partial_{12}\varphi_j\partial_3\varphi_i-2\partial_{13}\varphi_j\partial_2\varphi_i))\partial_3\mathfrak{A}_{3,1} && (=T_{21})\\
&=\sum_{ijk} f^{(1)}_{ijk}\mathfrak{B}_1+f^{(2)}_{ijk}\mathfrak{B}_2+f^{(3)}_{ijk}\mathfrak{B}_3+f^{(1,2)}_{ijk}\mathfrak{A}_{1,2}+f^{(2,3)}_{ijk}\mathfrak{A}_{2,3}+f^{(3,1)}_{ijk}\mathfrak{A}_{3,1} && =:(*)
\end{aligned}$$

for suitable coefficient maps $f^{(\cdot)}_{ijk}$ or $f^{(\cdot,\cdot)}_{ijk}$, respectively. To achieve this grouping we use Lemma B.7 (a) and (b) as well as the fact that $T_{11}=T_{17}=0$. In the following we will show that each of the six sums in $(*)$ vanishes individually. This is done by a very similar calculation every time.

Ad $f^{(1)}_{ijk}$. Here the coefficients are determined by terms T_1, T_6, T_{13}, T_{15} and T_{21}. Therefore,

$$\begin{aligned}
f^{(1)}_{ijk}=\ &6\partial_1\varphi_k\partial_2\varphi_j\partial_3\varphi_i+6\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i+6\varphi_k\partial_2\varphi_j\partial_{13}\varphi_i+3\partial_2\varphi_k\partial_3\varphi_j\partial_1\varphi_i\\
&+3\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i+3\varphi_k\partial_3\varphi_j\partial_{12}\varphi_i+\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i+\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i\\
&+(-2)\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i+3\partial_3\varphi_k\partial_1\varphi_j\partial_2\varphi_i+3\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i+3\varphi_k\partial_1\varphi_j\partial_{23}\varphi_i\\
&+(-1)\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i+(-1)\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i+2\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i=:P^{ijk}_1+...+P^{ijk}_{15}.
\end{aligned}$$

In the next step we group those of the P^{ijk}_l together, that have the same structure apart from a permutation of the indices i,j and k. For example, we have

$$P^{ijk}_1=2P^{jki}_4=2P^{kij}_{10}.$$

We now group all the terms and then perform the corresponding index permutations:

$$\begin{aligned}
\sum_{ijk} f^{(1)}_{ijk}\mathfrak{B}_1(i,j,k)&=\sum_{ijk}\Big[(P^{ijk}_1+P^{ijk}_4+P^{ijk}_{10})+(P^{ijk}_2+P^{ijk}_6+P^{ijk}_9+P^{ijk}_{14})\\
&\quad+(P^{ijk}_3+P^{ijk}_7+P^{ijk}_{11}+P^{ijk}_{15})+(P^{ijk}_5+P^{ijk}_8+P^{ijk}_{12}+P^{ijk}_{13})\Big]\mathfrak{B}_1(i,j,k)\\
&=\sum_{ijk}P^{ijk}_1\big(\mathfrak{B}_1(i,j,k)+\tfrac12\mathfrak{B}_1(j,k,i)+\tfrac12\mathfrak{B}_1(k,i,j)\big)\\
&\quad+P^{ijk}_2\big(\mathfrak{B}_1(i,j,k)+\tfrac12\mathfrak{B}_1(j,i,k)-\tfrac13\mathfrak{B}_1(i,j,k)-\tfrac16\mathfrak{B}_1(i,j,k)\big)\\
&\quad+P^{ijk}_3\big(\mathfrak{B}_1(i,j,k)+\tfrac16\mathfrak{B}_1(j,i,k)+\tfrac12\mathfrak{B}_1(j,i,k)+\tfrac13\mathfrak{B}_1(j,i,k)\big)\\
&\quad+P^{ijk}_5\big(\mathfrak{B}_1(i,j,k)+\tfrac13\mathfrak{B}_1(i,j,k)+\mathfrak{B}_1(j,i,k)-\tfrac13\mathfrak{B}_1(i,j,k)\big)\\
&=2\sum_{ijk}P^{ijk}_1\mathfrak{B}_1(i,j,k)=:(**),
\end{aligned}$$

where we used Lemma B.7 (d) to get the last equality. Finally, Lemma B.15 implies that $(**)$ vanishes identically.

Ad $f^{(2)}_{ijk}$. For the corresponding coefficients, only terms T_5, T_7 and T_{19} matter here. Therefore,

$$\begin{aligned} f^{(2)}_{ijk} &= -2\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{23}\varphi_j\partial_2\varphi_i + 3\partial_2\varphi_k\partial_2\varphi_j\partial_3\varphi_i + 3\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i \\ &\quad + 3\varphi_k\partial_2\varphi_j\partial_{23}\varphi_i + \varphi_k\partial_{23}\varphi_j\partial_2\varphi_i + (-1)\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i \quad =: Q^{ijk}_1 + ... + Q^{ijk}_7. \end{aligned}$$

Grouping similar terms and permuting indices as above we get

$$\begin{aligned} \sum_{ijk} f^{(1)}_{ijk}\mathfrak{B}_2(i,j,k) &= \sum_{ijk}\Big[(Q^{ijk}_1 + Q^{ijk}_4 + Q^{ijk}_7) + (Q^{ijk}_2 + Q^{ijk}_5 + Q^{ijk}_6) + Q^{ijk}_3\Big]\mathfrak{B}_2(i,j,k) \\ &= \sum_{ijk} Q^{ijk}_1\big(\mathfrak{B}_2(i,j,k) - \tfrac{3}{2}\mathfrak{B}_2(i,j,k) + \tfrac{1}{2}\mathfrak{B}_2(i,j,k)\big) \\ &\qquad + Q^{ijk}_2\big(\mathfrak{B}_2(i,j,k) + \tfrac{3}{2}\mathfrak{B}_2(j,i,k) + \tfrac{1}{2}\mathfrak{B}_2(i,j,k)\big) + Q^{ijk}_3\mathfrak{B}_2(i,j,k) \\ &= \sum_{ijk} Q^{ijk}_3\mathfrak{B}_2(i,j,k) = 0, \end{aligned}$$

where we again used Lemma B.7 (d) and in the last step Lemma B.15.

Ad $f^{(3)}_{ijk}$. Here, only terms T_3, T_9, T_{14} contribute to the corresponding coefficients. Thus,

$$\begin{aligned} f^{(3)}_{ijk} = \;& 2\varphi_k\partial_{33}\varphi_j\partial_2\varphi_i + (-2)\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + (-1)\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + \varphi_k\partial_{33}\varphi_j\partial_2\varphi_i \\ & + 3\partial_3\varphi_k\partial_2\varphi_j\partial_3\varphi_i + 3\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + 3\varphi_k\partial_2\varphi_j\partial_{33}\varphi_i \quad =: S^{ijk}_1 + ... + S^{ijk}_7. \end{aligned}$$

We thus get

$$\begin{aligned} \sum_{ijk} f^{(3)}_{ijk}\mathfrak{B}_3(i,j,k) &= \sum_{ijk}\Big[(S^{ijk}_1 + S^{ijk}_4 + S^{ijk}_7) + (S^{ijk}_2 + S^{ijk}_3 + S^{ijk}_6) + S^{ijk}_5\Big]\mathfrak{B}_3(i,j,k) \\ &= \sum_{ijk} S^{ijk}_1(\mathfrak{B}_3(i,j,k) + \tfrac{1}{2}\mathfrak{B}_3(i,j,k) + \tfrac{3}{2}\mathfrak{B}_3(j,i,k)) \\ &\qquad + S^{ijk}_2(\mathfrak{B}_3(i,j,k) + \tfrac{1}{2}\mathfrak{B}_3(i,j,k) - \tfrac{3}{2}\mathfrak{B}_3(i,j,k)) + S^{ijk}_5\mathfrak{B}_3(i,j,k) \\ &= \sum_{ijk} S^{ijk}_5\mathfrak{B}_3(i,j,k) = 0. \end{aligned}$$

Ad $f^{(1,2)}_{ijk}$. These coefficients are determined by T_4, T_{12} and T_{16}. In consequence,

$$\begin{aligned} f^{(1,2)}_{ijk} = \;& 2\partial_1\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{122}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{22}\varphi_j\partial_{13}\varphi_i + (-2)\partial_1\varphi_k\partial_{23}\varphi_j\partial_2\varphi_i \\ & + (-2)\varphi_k\partial_{123}\varphi_j\partial_2\varphi_i + (-2)\varphi_k\partial_{23}\varphi_j\partial_{12}\varphi_i + \partial_2\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{123}\varphi_j\partial_2\varphi_i \\ & + \varphi_k\partial_{13}\varphi_j\partial_{22}\varphi_i + \partial_2\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i + \varphi_k\partial_{223}\varphi_j\partial_1\varphi_i + \varphi_k\partial_{23}\varphi_j\partial_{12}\varphi_i \\ & + (-2)\partial_2\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i + (-2)\varphi_k\partial_{122}\varphi_j\partial_3\varphi_i + (-2)\varphi_k\partial_{12}\varphi_j\partial_{23}\varphi_i + \partial_3\varphi_k\partial_{12}\varphi_j\partial_2\varphi_i \\ & + \varphi_k\partial_{123}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{12}\varphi_j\partial_{23}\varphi_i + (-1)\partial_3\varphi_k\partial_{22}\varphi_j\partial_1\varphi_i + (-1)\varphi_k\partial_{223}\varphi_j\partial_1\varphi_i \\ & + (-1)\varphi_k\partial_{22}\varphi_j\partial_{13}\varphi_i \quad =: U^{ijk}_1 + ... + U^{ijk}_{21}. \end{aligned}$$

Here we can first note that by Lemma B.15 for each $l \in \{1,4,7,10,13,16,19\}$ the terms $U_l^{ijk}\mathfrak{A}_{1,2}(i,j,k)$ sum up to zero. We thus have

$$\begin{aligned}\sum_{ijk} f_{ijk}^{(1,2)}\mathfrak{A}_{1,2}(i,j,k) &= \sum_{ijk}\Big[(U_2^{ijk}+U_{14}^{ijk})+(U_3^{ijk}+U_9^{ijk}+U_{21}^{ijk})+(U_5^{ijk}+U_8^{ijk}+U_{17}^{ijk})\\ &\qquad +(U_6^{ijk}+U_{12}^{ijk}+U_{15}^{ijk}+U_{18}^{ijk})+(U_{11}^{ijk}+U_{20}^{ijk})\Big]\mathfrak{A}_{1,2}(i,j,k)\\ &= \sum_{ijk} U_2^{ijk}(\mathfrak{A}_{1,2}(i,j,k)-\mathfrak{A}_{1,2}(i,j,k))\\ &\qquad + U_3^{ijk}(\mathfrak{A}_{1,2}(i,j,k)+\tfrac12\mathfrak{A}_{1,2}(j,i,k)-\tfrac12\mathfrak{A}_{1,2}(i,j,k))\\ &\qquad + U_5^{ijk}(\mathfrak{A}_{1,2}(i,j,k)-\tfrac12\mathfrak{A}_{1,2}(i,j,k)-\tfrac12\mathfrak{A}_{1,2}(i,j,k))\\ &\qquad + U_6^{ijk}(\mathfrak{A}_{1,2}(i,j,k)-\tfrac12\mathfrak{A}_{1,2}(i,j,k)+\mathfrak{A}_{1,2}(j,i,k)-\tfrac12\mathfrak{A}_{1,2}(j,i,k))\\ &\qquad + U_{11}^{ijk}(\mathfrak{A}_{1,2}(i,j,k)-\mathfrak{A}_{1,2}(i,j,k)) \quad = 0.\end{aligned}$$

Ad $f_{ijk}^{(2,3)}$. Only the terms T_8 and T_{18} matter here. In particular,

$$\begin{aligned}f_{ijk}^{(2,3)} = \;& \partial_2\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + (-1)\partial_2\varphi_k\partial_{33}\varphi_j\partial_2\varphi_i + \partial_3\varphi_k\partial_{23}\varphi_j\partial_2\varphi_i + (-1)\partial_3\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i\\ &+ 2\varphi_k\partial_{23}\varphi_j\partial_{23}\varphi_i + (-1)\varphi_k\partial_{33}\varphi_j\partial_{22}\varphi_i + (-1)\varphi_k\partial_{22}\varphi_j\partial_{33}\varphi_i =: V_1^{ijk}+...+V_7^{ijk}\end{aligned}$$

We first note that the terms $V_l^{ijk}\mathfrak{A}_{2,3}(i,j,k)$ for $l\in\{1,2,3,4\}$ all sum up to zero (Lemma B.15). Consequently,

$$\begin{aligned}\sum_{ijk} f_{ijk}^{(2,3)}\mathfrak{A}_{2,3}(i,j,k) &= \sum_{ijk}\Big[(V_6^{ijk}+V_7^{ijk})+V_5^{ijk}\Big]\mathfrak{A}_{2,3}(i,j,k)\\ &= \sum_{ijk} V_6^{ijk}(\mathfrak{A}_{2,3}(i,j,k)+\mathfrak{A}_{2,3}(j,i,k)) + V_5^{ijk}\mathfrak{A}_{2,3}(i,j,k)\\ &= \sum_{ijk} V_5^{ijk}\mathfrak{A}_{2,3}(i,j,k).\end{aligned}$$

To see that the final term vanishes, we notice $V_5^{ijk}=V_5^{jik}$ and thus

$$\sum_{ijk} V_5^{ijk}\mathfrak{A}_{2,3}(i,j,k) = \sum_{ijk} V_5^{ijk}(\tfrac12\mathfrak{A}_{2,3}(i,j,k)+\tfrac12\mathfrak{A}_{2,3}(j,i,k)) = 0.$$

Ad $f_{ijk}^{(3,1)}$. Here, only the terms T_2, T_{10} and T_{20} are relevant and therefore

$$\begin{aligned}f_{ijk}^{(3,1)} = \;& 2\partial_1\varphi_k\partial_{33}\varphi_j\partial_2\varphi_1 + (-2)2\partial_1\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{133}\varphi_j\partial_2\varphi_i + (-2)\varphi_k\partial_{123}\varphi_j\partial_3\varphi_i\\ &+ 2\varphi_k\partial_{33}\varphi_j\partial_{12}\varphi_i + (-2)\varphi_k\partial_{23}\varphi_j\partial_{13}\varphi_i + \partial_2\varphi_k\partial_{13}\varphi_j\partial_3\varphi_i + (-1)\partial_2\varphi_k\partial_{33}\varphi_j\partial_1\varphi_i\\ &+ \varphi_k\partial_{123}\varphi_j\partial_3\varphi_i + (-1)\varphi_k\partial_{233}\varphi_j\partial_1\varphi_i + \varphi_k\partial_{13}\varphi_j\partial_{23}\varphi_i + (-1)\varphi_k\partial_{33}\varphi_j\partial_{12}\varphi_i\\ &+ \partial_3\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i + \partial_3\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i + (-2)\partial_3\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{233}\varphi_j\partial_1\varphi_i\\ &+ \varphi_k\partial_{123}\varphi_j\partial_3\varphi_i + (-2)\varphi_k\partial_{133}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{23}\varphi_j\partial_{13}\varphi_i + \varphi_k\partial_{12}\varphi_j\partial_{33}\varphi_i\\ &+ (-2)\varphi_k\partial_{13}\varphi_j\partial_{23}\varphi_i \qquad =: W_1^{ijk}+...+W_{21}^{ijk}\end{aligned}$$

We first apply Lemma $B.15$ to see that we can ignore the terms corresponding to W_l^{ijk} for $l \in \{1, 2, 7, 8, 13, 14, 15\}$. For the remaining terms we calculate

$$\begin{aligned}
\sum_{ijk} f_{ijk}^{(3,1)} \mathfrak{A}_{3,1}(i,j,k) &= \sum_{ijk} \Big[(W_3^{ijk} + W_{18}^{ijk}) + (W_4^{ijk} + W_9^{ijk} + W_{17}^{ijk}) + (W_5^{ijk} + W_{12}^{ijk} + W_{20}^{ijk}) \\
&\quad + (W_6^{ijk} + W_{11}^{ijk} + W_{19}^{ijk} + W_{21}^{ijk}) + (W_{10}^{ijk} + W_{16}^{ijk}) \Big] \mathfrak{A}_{3,1}(i,j,k) \\
&= \sum_{ijk} W_3^{ijk} (\mathfrak{A}_{3,1}(i,j,k) - \mathfrak{A}_{3,1}(i,j,k)) \\
&\quad + W_4^{ijk} (\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k)) \\
&\quad + W_5^{ijk} (\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k) + \tfrac{1}{2}\mathfrak{A}_{3,1}(j,i,k)) \\
&\quad + W_6^{ijk} (\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(j,i,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k) + \mathfrak{A}_{3,1}(j,i,k)) \\
&\quad + W_{10}^{ijk} (\mathfrak{A}_{3,1}(i,j,k) - \mathfrak{A}_{3,1}(i,j,k)) \quad = 0.
\end{aligned}$$

We thus have shown that $D = (*) = 0$, yielding that the truncation is solenoidal on $\mathcal{O}_\lambda$.

B.5.2. Proof of the identity (B.30)

Let $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^3)$ be arbitrary. In order to obtain formula (B.30), we write

$$\begin{aligned}
\int_{\mathcal{O}_\lambda} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x &= \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{A}_{1,2}, \nabla\psi) \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{A}_{2,3}, \nabla\psi) \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{A}_{3,1}, \nabla\psi) \,\mathrm{d}x \\
&\quad + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{B}_1, \nabla\psi) \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{B}_2, \nabla\psi) \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{B}_3, \nabla\psi) \,\mathrm{d}x \\
&=: \sum_{\ell=1}^{6} S_\ell,
\end{aligned}$$

where we indicate e.g. by $\mathbf{T}(\mathfrak{A}_{1,2}, \nabla\psi)$ that, when writing out $w_1 \cdot \nabla\psi$ directly by means of (B.21) and (B.22), $\mathbf{T}(\mathfrak{A}_{1,2}, \nabla\psi)$ contains all appearances of $\mathfrak{A}_{1,2}(i,j,k)$ and analogously for the remaining terms. The underlying procedure of dealing with the different terms is analogous for the remaining columns w_2 and w_3, which is why we exclusively focus on w_1 but give all the details in this case.

In the following, we will frequently interchange the triple sum $\sum_{ijk}$ and the integral over $\mathcal{O}_\lambda$, which allows us treat the single terms via integration by parts. This interchanging of sums and integrals is allowed since every sum $\sum_{ijk}(...)$ has an integrable majorant, in turn being seen similarly to the reasoning that underlies the proof of Lemma B.9.

We begin with S_1. This term is constituted by three parts S_1^1, S_1^2, S_1^3 given below, which stem from $w_{11}\partial_1\psi$, $w_{12}\partial_2\psi$ and $w_{13}\partial_3\psi$ (in this order). Here we have

$$\begin{aligned}
S_1^1 &= 2\sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k (\partial_{22}\varphi_j \partial_3\varphi_i - \partial_{23}\varphi_j \partial_2\varphi_i) \mathfrak{A}_{1,2}(i,j,k) \partial_1\psi \,\mathrm{d}x \\
&= -2\sum_{ijk} \int_{\mathcal{O}_\lambda} (\partial_2\varphi_j)(\partial_2\varphi_k \partial_3\varphi_i \mathfrak{A}_{1,2}(i,j,k) \partial_1\psi) \,\mathrm{d}x \qquad (= T_1^1)
\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_{23}\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi)\,\mathrm{d}x && (=T_2^1)\\
&-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\partial_2\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi)\,\mathrm{d}x && (=T_3^1)\\
&-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_{12}\psi)\,\mathrm{d}x && (=T_4^1)\\
&-2\sum_{i,j,k}\int_{\mathcal{O}_\lambda}\varphi_k\partial_{23}\varphi_j\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi\,\mathrm{d}x && (=T_5^1).
\end{aligned}
$$

Permuting indices $j\leftrightarrow k$ and using the antisymmetry from Lemma B.7 (c), we obtain

$$
\begin{aligned}
T_1^1 &= -2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\partial_2\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi)\,\mathrm{d}x\\
&= 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\partial_2\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,k,j)\partial_1\psi)\,\mathrm{d}x\\
&= 2\sum_{ikj}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\partial_2\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,k,j)\partial_1\psi)\,\mathrm{d}x = -T_1^1,
\end{aligned}
\tag{B.54}
$$

and hence $T_1^1=0$. Equally, permuting $i\leftrightarrow j$, we find that $T_2^1+T_5^1=0$. Therefore, using Lemma B.7 (b) for T_3^1 and integrating by parts in term T_4^1 with respect to ∂_1,

$$
\begin{aligned}
S_1^1 = T_3^1+T_4^1 &= -2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi)\,\mathrm{d}x && (=T_6^1)\\
&\quad+2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{12}\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_7^1)\\
&\quad+2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\partial_1\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_8^1)\\
&\quad+2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_{13}\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_9^1)\\
&\overset{\text{Lem. B.7 (a)}}{=}\; 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_{10}^1).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
S_1^2 &= \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{13}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_1^2)\\
&\quad+\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_2^2)\\
&\quad-\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(2\partial_{12}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_3^2)
\end{aligned}
$$

We finally turn to S_1^3. Here we have

$$\begin{aligned}
S_1^3 &= \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k(\partial_{12}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi \;\mathrm{d}x \\
&= -\sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\partial_2\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi \;\mathrm{d}x && (= T_1^3) \\
&\quad - \sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_{22}\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi \;\mathrm{d}x && (= T_2^3) \\
&\quad - \sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\partial_2\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi \;\mathrm{d}x && (= T_3^3) \\
&\quad - \sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_{23}\psi \;\mathrm{d}x && (= T_4^3) \\
&\quad - \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k(\partial_{22}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi \;\mathrm{d}x && (= T_5^3)
\end{aligned}$$

Again, T_1^3 vanishes by the same argument as for (B.54), $T_2^3 + T_5^3 = 0$ by permuting indices $i \leftrightarrow j$, and so we obtain analogously to above

$$\begin{aligned}
S_1^3 = &-\sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi \;\mathrm{d}x && (= T_6^3) \\
&+ \sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_{13}\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi \;\mathrm{d}x && (= T_7^3) \\
&+ \sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\partial_3\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi \;\mathrm{d}x && (= T_8^3) \\
&+ \sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_{23}\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi \;\mathrm{d}x && (= T_9^3) \\
&+ \sum_{ijk} \int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i \underbrace{\partial_3\mathfrak{A}_{1,2}(i,j,k)}_{=0} \partial_2\psi \;\mathrm{d}x.
\end{aligned}$$

Permuting indices $i \leftrightarrow j$ in T_1^2 and T_7^3 yields by virtue of the antisymmetry property of $\mathfrak{A}_{1,2}$ that $T_9^1 + T_1^2 + T_7^3 = 0$, and we directly find that $T_7^1 + T_3^2 = 0$. For terms T_8^1 and T_8^3, we permute indices $i \leftrightarrow j$ and $j \leftrightarrow k$ in term T_8^3 to obtain

$$T_8^1 + T_8^3 = 3\sum_{ijk} \int_{\mathcal{O}_\lambda} (\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi \;\mathrm{d}x \tag{B.55}$$

For terms T_2^2 and T_9^3, we permute indices $i \leftrightarrow j$ in T_9^3 to obtain $T_2^2 + T_9^3 = 0$. Having left

T_6^1 and T_6^3 untouched, we thus obtain

$$
\begin{aligned}
S_1 = &-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi)\;\mathrm{d}x \qquad &&(=T_6^1)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}\partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi\;\mathrm{d}x &&(=T_6^3)\\
&-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi\;\mathrm{d}x &&(=T_{10}^1)\\
&+3\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\;\mathrm{d}x &&(=T_8^1+T_8^3)\\
&=: \mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3+\mathbf{S}_4'.
\end{aligned}
\tag{B.56}
$$

We now claim that $\mathbf{S}_4'=0$. Let us first note that the overall sum in the definition of $\mathbf{S}_4'$ converges absolutely in $L^1(\mathcal{O}_\lambda)$. This can be seen similarly to the proof of Lemma B.9, and is a consequence of (P3), Lemma B.8 (b) and $\mathcal{L}^3(\mathcal{O}_\lambda)<\infty$, together with the bound

$$
\sum_{ijk}\int_{\mathcal{O}_\lambda}|(\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi|\;\mathrm{d}x\leq c\lambda\|\nabla w_1\|_{L^1(\mathbb{R}^3)}\mathcal{L}^3(\mathcal{O}_\lambda),
$$

where $c=c(3)>0$ is a constant only depending on the underlying space dimension $n=3$. By Lemma B.15, we have

$$
\sum_{ijk}(\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\equiv 0 \qquad \text{pointwisely in } \mathcal{O}_\lambda, \tag{B.57}
$$

to be understood as the limit of the corresponding partial sums. Therefore,

$$
\begin{aligned}
S_1 = &-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi)\;\mathrm{d}x \qquad &&(=T_6^1)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}\partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi\;\mathrm{d}x &&(=T_6^3)\\
&-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi\;\mathrm{d}x &&(=T_{10}^1)\\
&=: \mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3.
\end{aligned}
\tag{B.58}
$$

We now turn to S_2. Our line of action is similar to that for dealing with S_1 and so, integrating by parts twice, we successively obtain

$$
\begin{aligned}
S_2 = &\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{23}\varphi_j\partial_3\varphi_i-\partial_{33}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi\;\mathrm{d}x\\
&+\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{23}\varphi_j\partial_2\varphi_i-\partial_{22}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi\;\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{ijk}(-1)\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\partial_3\varphi_k\partial_3\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi)\,\mathrm{d}x && (=T_1)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_{33}\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi)\,\mathrm{d}x && (=T_2)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\partial_3\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi)\,\mathrm{d}x && (=T_3)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_{23}\psi)\,\mathrm{d}x && (=T_4)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\varphi_k\partial_{33}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi\,\mathrm{d}x && (=T_5)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\partial_2\varphi_k\partial_2\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi)\,\mathrm{d}x && (=T_6)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_{22}\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi)\,\mathrm{d}x && (=T_7)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_2\varphi_i\partial_2\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi)\,\mathrm{d}x && (=T_8)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_2\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_{23}\psi)\,\mathrm{d}x && (=T_9)\\
&-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi\,\mathrm{d}x && (=T_{10}).
\end{aligned}
$$

Terms T_1 and T_6 vanish by the same argument as in (B.54). Permuting indices $i \leftrightarrow j$, we then obtain $T_2+T_5=0$, and in a similar manner we see that $T_7+T_{10}=0$ and $T_4+T_9=0$. To conclude, we use Lemma B.7 to obtain

$$
\begin{aligned}
S_2 = T_3 + T_8 = &-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi)\,\mathrm{d}x\\
&+\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_2\varphi_i\mathfrak{B}_3(i,j,k)\partial_3\psi)\,\mathrm{d}x =: \mathbf{S}_4+\mathbf{S}_5.
\end{aligned}
\tag{B.59}
$$

Term S_3 is given by

$$
\begin{aligned}
S_3 := 2&\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{33}\varphi_j\partial_2\varphi_i-\partial_{23}\varphi_i\partial_3\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_1\psi\\
&+\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{13}\varphi_j\partial_3\varphi_i-\partial_{33}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_2\psi\\
&+\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{23}\varphi_j\partial_1\varphi_i+\partial_{12}\varphi_j\partial_3\varphi_i-2\partial_{13}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi\\
&=: S_3^1+S_3^2+S_3^3
\end{aligned}
$$

Terms S_3^1 and S_3^2 are treated as as term S_1^1, where we now integrate by parts with respect to ∂_3 in S_3^1 or with respect to ∂_1 in S_3^2, respectively. Similary to the computation underlying

S_1, this gives us

$$\begin{aligned}
S_3 = 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_3\varphi_j)\varphi_k(\partial_2\varphi_i)\mathfrak{B}_1(i,j,k)\partial_1\psi && (= T_1') \\
+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_{13}\varphi_j)\varphi_k\partial_2\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_2') \\
+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_3\varphi_j)\partial_1\varphi_k\partial_2\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_3') \\
+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_3\varphi_j)\varphi_k\partial_{12}\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_4') \\
+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_3\varphi_j)\varphi_k\partial_2\varphi_i\mathfrak{B}_3(i,j,k)\partial_3\psi && (= T_5') \\
+ \sum_{ijk}\int_{\mathcal{O}_\lambda} & \partial_1\varphi_j\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_2\psi \,\mathrm{d}x && (= T_6') \\
+ \sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_2\partial_1\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_7') \\
+ \sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_1\varphi_j)\partial_2\varphi_k\partial_3\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_8') \\
+ \sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_1\varphi_j)\varphi_k\partial_{23}\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_9') \\
+ \sum_{ijk}\int_{\mathcal{O}_\lambda} & \varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_{10}') \\
+ \sum_{ijk}\int_{\mathcal{O}_\lambda} & \varphi_k(\partial_{12}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_{11}') \\
- 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & \varphi_k\partial_{13}\varphi_j\partial_2\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_{12}').
\end{aligned}$$

By an argument analogous to (B.56)ff., $T_3' = T_8' = 0$. Moreover, permuting indices yields as above $T_4' + T_7' + T_{11}' = 0$ and $T_9' + T_{10}' = 0$, whereas $T_2' + T_{12}' = 0$ follows directly. Therefore,

$$\begin{aligned}
S_3 = 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_3\varphi_j)\varphi_k(\partial_2\varphi_i)\mathfrak{B}_1(i,j,k)\partial_1\psi \\
+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda} & (\partial_3\varphi_j)\varphi_k\partial_2\varphi_i\mathfrak{B}_3(i,j,k)\partial_3\psi \\
+ \sum_{ijk}\int_{\mathcal{O}_\lambda} & \partial_1\varphi_j\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_2\psi \,\mathrm{d}x =: \mathbf{S}_6 + \mathbf{S}_7 + \mathbf{S}_8
\end{aligned} \tag{B.60}$$

Until now, we have only considered the contributions from $\mathfrak{A}_{1,2}$, $\mathfrak{A}_{3,1}$ and $\mathfrak{A}_{2,3}$. The contributions containing $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ then read as

$$\begin{aligned} S_4 + S_5 + S_6 = 6 &\sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_2 \varphi_j \partial_3 \varphi_i \mathfrak{B}_1(i,j,k) \partial_1 \psi \\ &+ 3 \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_3 \varphi_j \partial_1 \varphi_i \mathfrak{B}_1(i,j,k) \partial_2 \psi \\ &+ 3 \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_1 \varphi_j \partial_2 \varphi_i \mathfrak{B}_1(i,j,k) \partial_3 \psi \\ &+ 3 \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_2 \varphi_j \partial_3 \varphi_i \mathfrak{B}_2(i,j,k) \partial_2 \psi \\ &+ 3 \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_2 \varphi_j \partial_3 \varphi_i \mathfrak{B}_3(i,j,k) \partial_3 \psi \\ &= \mathbf{S}_9 + \mathbf{S}_{10} + \mathbf{S}_{11} + \mathbf{S}_{12} + \mathbf{S}_{13}. \end{aligned}$$

Combining this with (B.58), (B.59) and (B.60), we may then build the overall sum $S_1 + ... + S_6 = \mathbf{S}_1 + ... + \mathbf{S}_{13}$. Summing up all terms, we note by an analogous permutation argument that $\mathbf{S}_3 + \mathbf{S}_4 + \mathbf{S}_{12} = 0$, $\mathbf{S}_5 + \mathbf{S}_7 + \mathbf{S}_{13} = 0$, and so

$$\begin{aligned} \int_{\mathcal{O}_\lambda} (T_\lambda w)_1 \cdot \nabla \psi \, \mathrm{d}x = 2 &\sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_2 \varphi_j \partial_3 \varphi_i \mathfrak{B}_1(i,j,k) \partial_1 \psi \, \mathrm{d}x \qquad (\sim \mathbf{S}_1 + \mathbf{S}_6 + \mathbf{S}_9) \\ &+ 2 \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_1 \varphi_i \partial_3 \varphi_j \mathfrak{B}_1(i,j,k) \partial_2 \psi \, \mathrm{d}x \qquad (\sim \mathbf{S}_8 + \mathbf{S}_{10}) \\ &+ 2 \sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k \partial_1 \varphi_j \partial_2 \varphi_i \mathfrak{B}_1(i,j,k) \partial_3 \psi \, \mathrm{d}x \qquad (\sim \mathbf{S}_2 + \mathbf{S}_{11}), \end{aligned}$$

where we use the symbol '$\sim$' to indicate where the single terms stem from. This is precisely (B.30), and so the proof is complete.

B.6. Proof of Theorem B.1

The proof of Theorem B.1 heavily depends on the validity of the truncation theorem B.2. In fact, Theorem B.1 has been proven in a different setting, where the divergence is replaced by some other differential operator (e.g. [157, 134]). For convenience of the reader, let us shortly present the argument here. First of all, note that the statement of Theorem B.2 also holds if we consider functions $u \in L^1(T_3, \mathbb{R}^{3\times3}_{\text{sym}})$ instead of functions defined on $\mathbb{R}^3$.

Proposition B.16. *There exists $C > 0$ with the following property: For all $u \in L^1(T_3\mathbb{R}^{3\times3}_{\text{sym}})$ with $\operatorname{div}(u) = 0$ in $\mathcal{D}'(T_3, \mathbb{R}^3)$ and $\lambda > 0$, there is $u_\lambda \in L^1(T_3, \mathbb{R}^{3\times3}_{\text{sym}})$ satisfying*

(a) $\|u_\lambda\|_{L^\infty} \le C\lambda$. ($L^\infty$-bound)

(b) $\|u - u_\lambda\|_{L^1} \le C \int_{\{|u|>\lambda\}} |u| \, \mathrm{d}x$. (Strong stability)

(c) $\mathcal{L}^3(\{u \neq u_\lambda\}) \leq C\lambda^{-1} \int_{\{|u|>\lambda\}} |u| \, \mathrm{d}x$. (Small change)

(d) $\operatorname{div}(u_\lambda) = 0$, *i.e., the differential constraint is still satisfied.*

To see this, one can either repeat the proof presented in Section B.4 or write $u \in L^1(T_3.\mathbb{R}^{3\times3}_{\mathrm{sym}})$ as a $\mathbb{Z}^3$-periodic function on $\mathbb{R}^3$ and apply the obvious L^1_{loc}-version of Theorem B.2.

Proof of Theorem B.1. As $\mathcal{Q}_\mathcal{A} f_1$ is a continuous symmetric div-quasiconvex function vanishing on K, all $y \in K^{(\infty)}$ are by definition also in $K^{(1)}$. It remains to show the other direction. Suppose that $\xi \in K^{(1)}$ and $(u_m) \subset L^1(T_3, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \cap \mathcal{T}$ is a test sequence with

$$0 = \mathcal{Q}_\mathcal{A} f_1(\xi) = \lim_{m\to\infty} \int_{\mathbb{T}_3} f_1(\xi + u_m(x)) \, \mathrm{d}x. \tag{B.61}$$

As K is a compact set, we find $R > 0$ with $K \subset \mathbb{B}_R(0)$ and $\xi \in \mathbb{B}_R(0)$. Thus, by (B.61),

$$\lim_{m\to\infty} \int_{\{|u_m|>3R\}} |u_m| \, \mathrm{d}x = 0. \tag{B.62}$$

Applying Proposition B.16 gives a sequence $\widetilde{v}_m \in L^\infty(T_3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$, such that

(i) $\operatorname{div}(\widetilde{v}_m) = 0$.

(ii) $\|\widetilde{v}_m - u_m\|_{L^1(\mathbb{T}_3)} \to 0$ as $m \to \infty$.

(iii) $\|\widetilde{v}_m\|_{L^\infty(\mathbb{T}_3)} \leq CR$.

Mollification and subtracting the average gives a sequence $(v_m) \subset L^\infty(T_3, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \cap \mathcal{T}_\mathcal{A}$ also satisfying properties (i)–(iii). Hence,

$$0 = \mathcal{Q}_\mathcal{A} f_1(\xi) = \lim_{m\to\infty} \int_{T_3} f_1(\xi + v_m(x)) \, \mathrm{d}x. \tag{B.63}$$

Take now a symmetric div-quasiconvex function $g \in \mathrm{C}(\mathbb{R}^{3\times3}_{\mathrm{sym}})$. We may suppose that $\max g(K) = 0$ and, as $\max\{0, g\}$ is again symmetric div-quasiconvex, that $g \equiv 0$ on K. Using uniform boundedness of v_m we may estimate with $C > 0$ as in (iii)

$$|g(\xi + v_m(x))| \leq \sup_{\eta\in\mathbb{B}_{(2C+1)R}(0)} |g(\eta)| < \infty. \tag{B.64}$$

Due to (B.63), $\operatorname{dist}(\xi + v_m, K) \to 0$ in measure, and by passing to a non-relabeled subsequence, we may assume that $\operatorname{dist}(\xi + v_m, K) \to 0$ $\mathcal{L}^3$-a.e.. As g is uniformly continuous on $\mathbb{B}_{(2C+1)R}(0)$, we get by (B.64) and dominated convergence

$$g(\xi) \leq \lim_{m\to\infty} \int_{\mathbb{T}_3} g(\xi + v_m(x)) \, \mathrm{d}x \leq \int_{\mathbb{T}_3} \lim_{m\to\infty} g(\xi + v_m(x)) \, \mathrm{d}x = 0. \tag{B.65}$$

Therefore, $\xi \in K^{(\infty)}$. The proof is complete. □

Let us, for the sake of completeness, also discuss a proof of the statement $K^{(p)} = K^{(q)}$, $1 < p, q < \infty$, which can be easily adapted to general constant rank operators $\mathcal{A}$ of the form (B.10). To this end, recall that a Borel measurable function $F\colon \mathbb{R}^d \to \mathbb{R}$ is called $\mathcal{A}$-quasiconvex provided it satisfies (B.2) for all $\xi \in \mathbb{R}^d$ and $\varphi \in \mathcal{T}$, where $\mathcal{T}_{\mathcal{A}}$ is now the set of all $\varphi \in \mathrm{C}^\infty(T_N, \mathbb{R}^d)$ with zero mean and $\mathcal{A}\varphi = 0$. The $\mathcal{A}$-quasiconvexifications $\mathcal{Q}_{\mathcal{A}} f$ of functions f and, for non-empty, compact sets $K \subset \mathbb{R}^d$, the corresponding sets $K^{(p)}$ for $1 \leq p \leq \infty$ are defined as in (B.4), now systematically replacing the divsym-quasiconvexity by $\mathcal{A}$-quasiconvexity. In contrast to [42], we even do not need to use potentials, but can directly appeal to Lemma B.4. Note that the construction of the projection $P_{\mathcal{A}}$ from Lemma B.4 crucially relies on Fourier multipliers and hence is not applicable for $p = 1$ and $p = \infty$. Using this projection operator $P_{\mathcal{A}}$, we can prove the following statement.

Lemma B.17. *Let $\mathcal{A}$ be a constant rank operator of the form* (B.10) *and let $K \subset \mathbb{R}^d$ be compact. Then, for $1 < p < q < \infty$, $K^{(p)} = K^{(q)}$.*

Proof. With slight abuse of notation, let $K \subset \mathbb{B}_R(0) := \{\eta \in \mathbb{R}^d\colon\ |\eta| < R\}$ and $y \in \mathbb{B}_R(0)$. Ad '$K^{(q)} \subset K^{(p)}$'. Let $y \in K^{(q)}$ and let $(u_m) \subset \mathcal{T}_{\mathcal{A}}$ be a test sequence such that

$$0 = \mathcal{Q}_{\mathcal{A}} f_q(y) = \lim_{m\to\infty} \int_{T_N} f_q(y + u_m(x))\ \mathrm{d}x.$$

As K is compact, (u_m) is bounded in $L^q(T_N, \mathbb{R}^d)$ and, as $q > p$, also bounded in $L^p(T_N, \mathbb{R}^d)$. Also note that for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that $f_p \leq \varepsilon + C_\varepsilon f_q$. Therefore,

$$\mathcal{Q}_{\mathcal{A}} f_p(y) \leq \lim_{m\to\infty} \int_{T_N} f_p(y + u_m(x))\ \mathrm{d}x \leq \lim_{m\to\infty} \int_{T_N} \varepsilon + C_\varepsilon f_q(y + u_m(x))\ \mathrm{d}x \leq \varepsilon.$$

Thus, $y \in K^{(p)}$. The direction $K^{(p)} \subset K^{(q)}$ uses a similar, yet easier truncation statement than Theorem B.1. Let $y \in K^{(p)}$ and let $(u_m) \subset \mathcal{T}_{\mathcal{A}}$ be a test sequence, such that

$$0 = \mathcal{Q}_{\mathcal{A}} f_p(y) = \lim_{m\to\infty} \int_{T_N} f_p(y + u_m(x))\ \mathrm{d}x.$$

Note that (u_m) is uniformly bounded in $L^p(T_N, \mathbb{R}^d)$ and that

$$\lim_{m\to\infty} \int_{T_N} \mathrm{dist}^p(u_m(x), \mathbb{B}_{2R}(0))\ \mathrm{d}x = 0.$$

Write

$$\widetilde{u}_m = \mathbb{1}_{\{|u_m| \leq 2R\}} u_m - \fint_{T_N} \mathbb{1}_{\{|u_m| \leq 2R\}}(x) u_m(x)\ \mathrm{d}x$$

and define $v_m := P_{\mathcal{A}} \widetilde{u}_m$ with the projection operator $P_{\mathcal{A}}$ from Lemma B.4. Observe that

1. $\mathcal{A} v_m = 0$ by Lemma B.4 1.
2. $(\widetilde{u}_m)$ is bounded in $L^\infty(T_N, \mathbb{R}^d)$ and q-equi-integrable. Since $1 < q < \infty$, the projection $P_{\mathcal{A}}\colon L^q(T_N, \mathbb{R}^d) \to L^q(T_N, \mathbb{R}^d)$ is bounded, (v_m) is bounded in $L^q(T_N, \mathbb{R}^d)$,

q-equi-integrable by Lemma B.4 3, Moreover, by Lemma B.4 2 and $1 < p < \infty$,

$$\begin{aligned}\|u_m - v_m\|_{L^p(T_N)} &\le \|u_m - \widetilde{u}_m\|_{L^p(T_N)} + \|\widetilde{u}_m - v_m\|_{L^p(T_N)} \\ &\le \|u_m - \widetilde{u}_m\|_{L^p(T_N)} + C_{\mathcal{A},p}\|\mathcal{A}(\widetilde{u}_m - u_m)\|_{W^{-k,p}(T_N)} \\ &\le C_{\mathcal{A},p}\|u_m - \widetilde{u}_m\|_{L^p(T_N)} \to 0.\end{aligned}$$

Hence, also

$$\lim_{m\to\infty} \int_{T_N} f_p(y + v_m(x))\,\mathrm{d}x = 0.$$

We conclude that $f_q(y + v_m) \to 0$ in measure. Combining this with the L^q-boundedness and q-equiintegrability, we obtain

$$\lim_{m\to\infty} \int_{T_N} f_q(y + v_m(x))\,\mathrm{d}x = 0.$$

Therefore, $y \in K^{(q)}$, concluding the proof. □

B.7. Potential truncations

In this concluding section, we come back to the potential truncations alluded to in the introduction and discuss the limitations of this strategy in view of Theorems B.1 and B.2. Let $\mathcal{A}$ be a constant rank operator. Recall that the potential truncation strategy, originally pursued in [28] for $\mathcal{A} = \operatorname{div}$, is to represent $u \in L^p(T_N, \mathbb{R}^d)$ with $\mathcal{A}u = 0$ and $\fint_{T_N} u\,\mathrm{d}x = 0$ as $u = \mathcal{B}v$ for some potential $\mathcal{B}$ of order $l \in \mathbb{N}$ (cf. Proposition 2.5) and then performing a $W^{l,p}$-$W^{l,\infty}$-truncation on the potential v. We then write with slight abuse of notation[5] $v = \mathcal{B}^{-1}u$. Since it is of independent interest but also motivates the need for a different strategy for Theorem B.2 for $p = 1$, we record the following.

Proposition B.18. *Let $\mathcal{A}$ be a constant rank differential operator of order $k \in \mathbb{N}$ and $\mathcal{B}$ be a potential of $\mathcal{A}$ of order $l \in \mathbb{N}$. Let $1 < p < \infty$. Then there exists a constant $C > 0$ such that the following hold: If $u \in L^p(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ and $\lambda > 0$ then there exists $u_\lambda \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ satisfying the*

1. L^∞-bound: $\|u_\lambda\|_{L^\infty(T_N)} \le C\lambda$.

2. weak stability:

$$\|u_\lambda - u\|^p_{L^p(T_N)} \le C \int_{\{\sum_{j=0}^l |\nabla^j \circ \mathcal{B}^{-1}u| > \lambda\}} \sum_{j=0}^{l} |\nabla^j \circ \mathcal{B}^{-1}u|^p\,\mathrm{d}x.$$

[5]The notation $\mathcal{B}^{-1}$ is only symbolic as $\mathcal{B}$ might be non-invertible.

3. small change:

$$\mathcal{L}^n(\{u_\lambda \neq u\}) \leq \frac{C}{\lambda^p} \int_{\{\sum_{j=0}^{l} |\nabla^j \circ \mathcal{B}^{-1} u| > \lambda\}} \sum_{j=0}^{l} |\nabla^j \circ \mathcal{B}^{-1} u|^p \, \mathrm{d}x$$

For simplicity, we state this result on T_N; a version on $\mathbb{R}^N$ follows by analogous means.

Proof. We start by outlining the $W^{m,p}$-$W^{m,\infty}$-truncation that seems hard to be traced in the literature; here, we choose a direct approach instead of appealing to McShane-type extensions. Let $m \in \mathbb{N}$. For $v \in W^{m,p}(T_N, \mathbb{R}^d)$, let $\mathcal{O}_\lambda := \{\sum_{j=0}^{m} \mathcal{M}(\nabla^j v) > \lambda\}$. Since the sum of lower semicontinuous functions is lower semicontinuous, $\mathcal{O}_\lambda$ is open. We choose a Whitney decomposition $\mathcal{W} = (Q_j)$ of $\mathcal{O}_\lambda$ satisfying (W1)–(W4), and a partition of unity (φ_j) subject to $\mathcal{W}$ with (P1)–(P3). We note that the Whitney cover can be arranged in a way such that $\mathcal{L}^N(Q_j \cap Q_{j'}) \geq c \max\{\mathcal{L}^N(Q_j), \mathcal{L}^N(Q_{j'})\}$ holds for some $c = c(N) > 0$ and all $j, j' \in \mathbb{N}$ such that $Q_j \cap Q_{j'} \neq \emptyset$. For each $j \in \mathbb{N}$, we then denote $\pi_j[v]$ the $(m-1)$-th order averaged Taylor polynomial of v over Q_j; cf. [106, Chpt. 1.1.10]. In particular, we have the scaled version of Poincaré's inequality

$$\fint_{Q_j} |\partial^\alpha (w - \pi_j[w])|^q \, \mathrm{d}x \leq c(q,m,N) \ell(Q_j)^{q(m-|\alpha|)} \fint_{Q_j} |\nabla^m w|^q \, \mathrm{d}x \tag{B.66}$$

for all $1 \leq q < \infty$, $w \in W^{m,q}(T_N, \mathbb{R}^d)$ and $|\alpha| \leq m$. We then put

$$v_\lambda := v - \sum_j \varphi_j (v - \pi_j[v]) = \begin{cases} v & \text{in } \mathcal{O}_\lambda^{\complement}, \\ \sum_j \varphi_j \pi_j[v] & \text{in } \mathcal{O}_\lambda. \end{cases} \tag{B.67}$$

Then $v_\lambda \in W^{m,p}(T_N, \mathbb{R}^d)$, which can be seen as follows: On $\mathcal{O}_\lambda$, v_λ is a locally finite sum of C^∞-maps and hence of class C^∞ too. For an arbitrary $|\alpha| \leq m$, (B.66) yields

$$\begin{aligned} \sum_j \|\partial^\alpha(\varphi_j(v - \pi_j[v])\|_{L^q(\mathcal{O}_\lambda)}^q &\overset{\text{(P3)}}{\leq} \sum_j \sum_{\beta+\gamma=\alpha} \frac{c(N,q)}{\ell(Q_j)^{q(|\beta|+|\gamma|)}} \ell(Q_j)^{q|\gamma|} \|\partial^\gamma (v - \pi_j[v])\|_{L^q(Q_j)}^q \\ &\leq c(N,m,q) \sum_j \ell(Q_j)^{q(m-|\alpha|)} \|\nabla^m v\|_{L^q(Q_j)}^q \\ &\overset{\text{(W3)}}{\leq} c(N,m,q) \mathcal{L}^n(\mathcal{O}_\lambda)^{\frac{q(m-|\alpha|)}{n}} \|\nabla^m v\|_{L^q(\mathcal{O}_\lambda)}^q. \end{aligned}$$

In conclusion, applying the previous inequality with $q = 1$, on $(0,1)^N$ the series in (B.67) converges absolutely in $W_0^{m,1}((0,1)^N; \mathbb{R}^d)$ and hence $v_\lambda \in W^{m,1}(T_N, \mathbb{R}^d)$; then applying the previous inequality with $q = p$ yields $v_\lambda \in W^{m,p}(T_N, \mathbb{R}^d)$. Whenever $x \in Q_{j_0}$ for some $j_0 \in \mathbb{N}$, (W2) implies that we may blow up Q_{j_0} by a fixed factor $c > 0$ so that $cQ_{j_0} \cap \mathcal{O}_\lambda^{\complement} \neq \emptyset$. Fix some $z \in cQ_{j_0} \cap \mathcal{O}_\lambda^{\complement}$. Then, for some $c' = c'(N) > 0$, $Q_{j_0} \subset B_{c'\ell(Q_{j_0})}(z)$

and so

$$\fint_{Q_{j_0}} |\partial^\alpha v|\,\mathrm{d}x \leq c(N)\fint_{B_{c'(N)\ell(Q_{j_0})}(z)} |\partial^\alpha v|\,\mathrm{d}x \leq c(N)\mathcal{M}(\nabla^{|\alpha|}v)(z) \leq c(N)\lambda \tag{B.68}$$

for all $|\alpha| \leq m$. Now let $Q_j \in \mathcal{W}$ be another cube with $Q_j \cap Q_{j_0} \neq \emptyset$; by (W3), there are only $M = M(n) < \infty$ many such cubes. Since $\nabla^m \pi_{j_0}[v] = 0$ and $\sum_j \varphi_j = 1$ on $\mathcal{O}_\lambda$,

$$\begin{aligned}
|\nabla^m v_\lambda(x)| &\leq \Bigg| \sum_{j:\ Q_j\cap Q_{j_0}\neq\emptyset} \nabla^m(\varphi_j(\pi_j[v]-\pi_{j_0}[v]))(x)\Bigg| \\
&\overset{\text{(P3)}}{\leq} c \sum_{\substack{j:\ Q_j\cap Q_{j_0}\neq\emptyset\\ |\alpha|+|\beta|=m}} \frac{1}{\ell(Q_j)^{|\alpha|}} \|\nabla^{|\beta|}(\pi_j[v]-\pi_{j_0}[v])\|_{L^\infty(Q_j\cap Q_{j_0})} \\
&\overset{(*)}{\leq} c \sum_{\substack{j:\ Q_j\cap Q_{j_0}\neq\emptyset\\ |\alpha|+|\beta|=m}} \frac{1}{\ell(Q_j)^{|\alpha|}} \Big(\fint_{Q_j} |\nabla^{|\beta|}(\pi_j[v]-v)|\,\mathrm{d}x + \fint_{Q_{j_0}} |\nabla^{|\beta|}(v-\pi_{j_0}[v])|\,\mathrm{d}x \Big) \\
&\leq c \sum_{j:\ Q_j\cap Q_{j_0}\neq\emptyset} \fint_{Q_j} |\nabla^m v|\,\mathrm{d}x \quad \text{(by (B.66))} \\
&\leq c\lambda \qquad\qquad \text{(by (B.68) and (W3))},
\end{aligned}$$

where have used at $(*)$ that on the polynomials of degree at most $(m-1)$ on cubes, all norms are equivalent (in particular, the L^1- and L^∞-norms), and scaling (recall that $\mathcal{L}^n(Q_j \cap Q_{j_0}) \geq c\max\{\mathcal{L}^n(Q_j), \mathcal{L}^n(Q_{j_0})\}$) whenever $Q_j \cap Q_{j_0} \neq \emptyset$, and (W3)). Hence,

(i) $\|\nabla^m v\|_{L^\infty(T_N)} \leq c(m,N)\lambda$,

(ii) $\mathcal{L}^N(\{u \neq u_\lambda\}) \leq \dfrac{c(m,N,p)}{\lambda^p} \displaystyle\sum_{j=0}^m \|\nabla^j v\|^p_{L^p(T_N)}$.

We now let $u \in L^p(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ satisfy $\int_{(0,1)} u\,\mathrm{d}x = 0$. Since $\mathcal{B}^{-1}$ has a Fourier symbol of class C^∞ off zero and homogeneous of degree $(-l)$, $\nabla^l \circ \mathcal{B}^{-1}$ has a Fourier symbol of class C^∞ off zero and homogeneous of degree zero. By Mihlin's theorem (cf. [139]), applicable because of $1 < p < \infty$ and by Poincaré's inequality, we thus find that $\mathcal{B}^{-1}u \in W^{l,p}(T_N)$ together with $\|\mathcal{B}^{-1}u\|_{W^{l,p}(T_N)} \leq c\|u\|_{L^p(T_N)}$. We then perform a $W^{l,p}$-$W^{l,\infty}$-truncation on $v = \mathcal{B}^{-1}u$ as in the first part of the proof, yielding v_λ, and define $u_\lambda := \mathcal{B}v_\lambda$. By the properties gathered in the first part of the proof, we may employ Zhang's trick (see (B.46) ff.) to conclude 2 and 3 as well. The proof is complete. □

Remark B.19 (Strong stability and $1 < p < \infty$ versus $p = 1$). It is clear from the above proof that the potential truncation only works fruitfully in the case $1 < p < \infty$ by the entering of Mihlin's theorem; indeed, the operator $\mathcal{B}^{-1}$ is defined via Fourier multipliers and by Ornstein's Non-Inequality, we cannot conclude that $\mathcal{B}^{-1}u \in W^{l,1}$ provided $u \in L^1$. However, the potential truncations from Proposition B.18 do *not satisfy the strong stability* *property* $\|u-u_\lambda\|^p_{L^p(T_N)} \leq C\int_{\{|u|>\lambda\}} |u|^p \,\mathrm{d}x$. The underlying reason is that $\nabla^l \circ \mathcal{B}^{-1}$ is a Fourier multiplication operator with symbol smooth off zero and homogeneous of degree zero; by Ornstein's Non-Inequality, we only have that $\nabla^l \circ \mathcal{B}^{-1}\colon L^\infty \to \mathrm{BMO}$ in general, and here BMO *cannot* be replaced by L^∞. The potential truncation is performed on the sets where $\sum_{j=0}^{l} \mathcal{M}(\nabla^j \circ \mathcal{B}^{-1}u) > \lambda$. Thus, even if $u \in L^\infty(T_N, \mathbb{R}^d)$ is $\mathcal{A}$-free with $\|u\|_{L^\infty(T_N)} \leq \lambda$, the potential truncation might modify u regardless of $\lambda > 0$ and hence strong stability cannot be achieved. As established by Conti, Müller and Ortiz [42], in the case $1 < p < \infty$ this issue still can be circumvented to arrive at Lemma B.17, but in the context of $p = 1$ the underlying techniques break down. In essence, this was the original motivation for the different proof displayed in Sections B.3 and B.4.

We conclude the paper with possible other approaches and extensions of Theorem B.2.

Remark B.20. As mentioned in the introduction, [26] constructs a divergence-free $W^{1,p}$-$W^{1,\infty}$-truncation. Here a Whitney-type truncation is performed first, leading to a non-divergence-free truncation. To arrive at a divergence-free truncation, the local divergence overshoots are then corrected by subtracting special solutions of suitable divergence equations. This is achieved by invoking the *Bogovskiĭ* operator [24], which selects specific solutions of the (heavily underdetermined) divergence equation $\mathrm{div}(Y) = f$ with $Y|_{\partial\Omega} = 0$ by

$$Y(x) = \mathrm{Bog}(f)(x) := \int_\Omega f(y)\frac{x-y}{|x-y|^N}\int_{|x-y|}^{\infty} \omega_R\Big(y + s\frac{x-y}{|x-y|}\Big)s^{N-1}\,\mathrm{d}s\,\mathrm{d}y, \quad x \in \Omega,$$

provided $\Omega \subset \mathbb{R}^N$ is star-shaped with respect to a ball $B_R(x_0) \subset\subset \Omega$, f has integral zero over Ω and ω_R is a scaled cut-off relative to $B_R(x_0)$.

In our situation, the main drawback of the Bogovskiĭ operator is that if equations $\mathrm{div}(Y) = f$ for $f\colon (0,1)^N \to \mathbb{R}^N$ are considered, then the solution Y obtained by the row-wise application of the Bogovskiĭ operator does not necessarily take values in $\mathbb{R}^{N\times N}_{\mathrm{sym}}$; note that passing to the symmetric part Y^{sym} destroys the validity of the divergence equation. While this potentially could be repaired by passing to different solution operators, the method requires tools that are not fully clear to us in the present lower regularity context of Theorem B.2. With our proof in Section B.4 being taylored to divergence constraints, in principle it can be modified to yield divergence-free $W^{1,p}$-$W^{1,\infty}$-truncations as well. We shall pursue this together with possible extensions of the approach in [26] elsewhere.