Topics in the L^p theory for outer measure spaces

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Par vardar dentro ai cieli sereni
là su sconti da nuvoli neri
go lasà le me vali e i me orti
par salir su le cime de i monti.

So rivà su le cime de i monti
go vardà dentro ai cieli sereni
vedarò le me vali e i me orti
là zo sconti da nuvoli neri?

Giacomo Noventa

Di quel che udire e che parlar vi piace,
noi udiremo e parleremo a voi,
mentre che ’l vento, come fa, ci tace.

Dante

[...] not because they are easy, but because they are hard.

John F. Kennedy

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Abstract

The theory of L^p spaces for outer measures, or outer L^p spaces, was introduced by Do and Thiele. Their main interest was its application to the study of the boundedness properties of some multilinear forms satisfying certain invariances arising in the context of Calderón-Zygmund theory and time-frequency analysis.

However, the theory can be developed in a broader generality of settings. It requires a set X, an outer measure μ to evaluate the magnitude of subsets of X, and a size S to evaluate the magnitude of functions on X when localized to the elements of a certain collection $\mathcal{A}$ of subsets of X. Then, the outer $L^p_\mu(S)$ quasi-norms are defined by the interplay between μ and S via a layer cake integral. For every $p \in (0,\infty)$ and every function f on X, we define

$$\|f\|_{L^\infty_\mu(S)} = \sup\Big\{S(f)(A) \colon A \in \mathcal{A}\Big\},$$
$$\|f\|_{L^p_\mu(S)} = \Big(\int_0^\infty p\lambda^p \inf\Big\{\mu(B) \colon B \subseteq X, \|f1_{B^c}\|_{L^\infty_\mu(S)} \leqslant \lambda\Big\}\frac{\mathrm{d}\lambda}{\lambda}\Big)^{\frac{1}{p}},$$

and the outer $L^\infty_\mu(S)$ and $L^p_\mu(S)$ spaces to be the sets of functions on X for which $\|f\|_{L^\infty_\mu(S)}$, and $\|f\|_{L^p_\mu(S)}$ are finite respectively. For example, the mixed L^p spaces on the Cartesian product of σ-finite measure spaces can be exhibited as outer L^p spaces for an appropriate choice of (X,μ,S).

Do and Thiele developed the theory of outer L^p spaces in the direction of their real interpolation properties, such as Hölder's inequality and Marcinkiewicz interpolation. This thesis is concerned with further developing the theory of these spaces. The focus is towards the Banach space properties analogous to those of the mixed L^p spaces, such as Köthe duality, triangle inequality for countably many summands, and Minkowski's inequality.

The thesis consists of four chapters.

Chapter 1 is an introduction. We recall definitions and properties of outer L^p spaces from the article of Do and Thiele and we introduce a list of examples. We also comment on the results about the Banach space properties of outer L^p spaces appearing in the following chapters.

In Chapter 2, we study single iterated outer L^p spaces, when the size is a suitably averaged local classical L^r quasi-norm associated with a measure ω on X. For $p, r \in (1,\infty)$ we prove that the outer L^p quasi-norms are equivalent to norms up to a constant uniform in the setting (X,μ,ω). We also focus on the setting on $\mathbb{R}^d \times (0,\infty)$ associated with Calderón-Zygmund theory.

In Chapter 3, we study double iterated outer L^p spaces, when the size is a suitably averaged local single iterated outer L^q quasi-norm on the setting (X,ν,ω). Under additional assumptions on μ and ν, for $p, q, r \in (1,\infty)$ we prove that the outer L^p quasi-norms are equivalent to norms up to a constant uniform in the setting (X,μ,ν,ω). We provide

counterexamples showing the necessity of additional assumptions. We also focus on the setting on $\mathbb{R}^2 \times (0, \infty)$ associated with time-frequency analysis.

In Chapter 4, we address additional questions about outer L^p spaces. For example, we prove a version of Minkowski's inequality for single iterated outer L^p quasi-norms. We conclude with some open conjectures.

Contents

Chapter 1

Introduction

The L^p theory for outer measure spaces, or theory of outer L^p spaces, was introduced by Do an Thiele in [DT15] in the study of the boundedness properties of linear and multilinear operators satisfying certain symmetries. This is the case for many important operators in harmonic analysis. For example, operators with translation and dilation invariances in Calderón-Zygmund theory (paraproducts, singular integral operators, $T(1)$ theorem) and operators with additional modulation invariances in time-frequency analysis (Carleson operator, bilinear Hilbert transform). We refer to the books of Grafakos [Gra08, Gra09], Muscalu and Schlag [MS13a, MS13b], Stein [Ste70, Ste93], and Thiele [Thi06] for a thorough treatment of the study of these operators in harmonic analysis. In this Introduction, we are satisfied with considering two prototypical examples, one in the context of Calderón-Zygmund theory and the other in the context of time-frequency analysis.

The example for the first case is the form associated with the Hilbert transform. For all Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$, we define the form Λ_{H} by

$$\Lambda_{\mathrm{H}}(f,g) = \mathrm{p.\,v.} \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{t} f(x-t) g(x) \,\mathrm{d}t \,\mathrm{d}x.$$

It satisfies translation and dilation invariances.

The example for the second case is the form associated with the bilinear Hilbert transform. For all Schwartz functions $f_1, f_2, f_3 \in \mathcal{S}(\mathbb{R})$, we define the form Λ_{BH} by

$$\Lambda_{\mathrm{BH}}(f,g,h) = \mathrm{p.\,v.} \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{t} f_1(x-t) f_2(x) f_3(x+t) \,\mathrm{d}t \,\mathrm{d}x.$$

It satisfies translation, dilation, and modulation invariances.

We can prove the boundedness of these forms on the Cartesian product of classical L^p spaces, namely

$$\begin{aligned} |\Lambda_{\mathrm{H}}(f,g)| &\leqslant C_{\mathrm{H}}(p,q) \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}, \\ |\Lambda_{\mathrm{BH}}(f_1,f_2,f_3)| &\leqslant C_{\mathrm{BH}}(p_1,p_2,p_3) \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})} \|f_3\|_{L^{p_3}(\mathbb{R})}, \end{aligned} \tag{1.0.1}$$

where the exponents $p, q, p_i \in (1,\infty)$ satisfy the Hölder type condition, namely

$$\frac{1}{p}+\frac{1}{q}=1, \qquad \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1.$$

Moreover, in both cases we can express the forms in terms of an integral over the set of invariances, namely

$$\begin{aligned} \Lambda_{\mathrm{H}}(f,g) &= A_{\mathrm{H}} \int_{\mathbb{R}\times(0,\infty)} F(x,s)G(x,s)\,\mathrm{d}x\frac{\mathrm{d}s}{s}, \\ \Lambda_{\mathrm{BH}}(f,g,h) &= A_{\mathrm{BH}} \int_{\mathbb{R}\times\mathbb{R}\times(0,\infty)} F_1(x,\xi,s)F_2(x,\xi,s)F_3(x,\xi,s)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned} \tag{1.0.2}$$

In the previous display, $F = F(f)$, $G = G(g)$, and $F_i = F_i(f_i)$ are functions obtained sampling f, g, and f_i with appropriately translated, dilated and modulated copies of certain wave-packets. We refer to the article of Do and Thiele [DT15] and the book of Thiele [Thi06] for the details of the equalities in (1.0.2). Informally speaking, the singular kernel is decomposed along the set of invariances and absorbed in the samplings of the functions.

In particular, the left hand sides of the inequalities in (1.0.1) become similar to the left hand side of Hölder's inequality. This suggests a strategy to prove the inequalities in (1.0.1) based on the following two-step programme. First, we would apply a version of Hölder's inequality to the integrals in (1.0.2), namely

$$\begin{aligned} |\Lambda_{\mathrm{H}}(f,g)| &\leqslant B\|F\|_{L^p}\|G\|_{L^q}, \\ |\Lambda_{\mathrm{BH}}(f,g,h)| &\leqslant B\|F_1\|_{L^{p_1}}\|F_2\|_{L^{p_2}}\|F_3\|_{L^{p_3}}, \end{aligned}$$

for appropriate abstract L^p quasi-norms on the sets $\mathbb{R}\times(0,\infty)$ and $\mathbb{R}\times\mathbb{R}\times(0,\infty)$. Next, we would prove the boundedness of the maps from f to F, namely

$$\|F\|_{L^p} \leqslant C\|f\|_{L^p(\mathbb{R})}, \qquad \|F_i\|_{L^{p_i}} \leqslant C\|f_i\|_{L^{p_i}(\mathbb{R})}.$$

In the case of the form Λ_{H}, this proof strategy is not essentially different from the classical one, passing through the maximal and square functions. The major novelty is that the new approach encodes the classical one in the case of forms satisfying translation and dilation invariances, and generalizes it to the case of forms satisfying additional modulation invariances. As a matter of fact, such a proof strategy is a cornerstone in the context of time-frequency analysis tracing back to the seminal works of Lacey and Thiele [LT97, LT99, LT00] and Thiele [Thi00, Thi02] on the bilinear Hilbert transform and the Carleson operator. Among the more recent applications of the two-step programme outlined above, we point out the following works. The articles of Do and Thiele [DT15], Di Plinio and Ou [DPO18b], Amenta and Uraltsev [AU20b], and the Ph.D. thesis of Warchalski [War18] on the bilinear Hilbert transform. The articles of Uraltsev [Ura16], Di Plinio, Do and Uraltsev [DPDU18], Amenta and Uraltsev [AU22] on the Carleson operator. The article of Do, Muscalu and

Thiele [DMT17] on the bilinear iterated Fourier inversion operator. Further references can be found in the introductions of [Fra21] and [Fra22], namely Chapter 2 and Chapter 3 of this thesis, and we direct the interested reader to them. We briefly comment that also the proof of the boundedness of the forms associated with a sparse family of dyadic cubes can be interpreted as a variant of the two-step programme outlined above.

The theory of outer L^p spaces provides a framework to formalize the proof strategy outlined above. In fact, it turns out that the right structure on the sets of invariances is that of an outer measure space, and the abstract L^p spaces to evaluate the magnitude of the embedding function sending f to F are the outer L^p spaces.

An outer measure μ on a set X is a monotone, subadditive function on the collection of subsets of X, attaining value zero on the empty set. It provides a way to evaluate the magnitude of subsets of X. On one hand, the lack of additivity on disjoint subsets prevents the development of a linear theory of integrals. On the other, there is no restriction on the subsets on which the outer measure is defined. We could always restrict to the collection of Carathéodory measurable subsets, hence recovering a measure, and consider the L^p theory associated with the new measure space. This is the classical use of outer measures in the introduction of measures, the main example being the Lebesgue measure, see for example the book of Rudin [Rud74]. However, in the cases of interest, there would be very few Carathéodory measurable subsets, sometimes only the trivial ones, the empty set and the whole set itself.

Nevertheless, there is still hope to develop a quasi-subadditive theory of L^p spaces. For every set X endowed with an outer measure μ, lacking a theory of integrals for outer measure spaces, we use the layer cake integral and we measure the super level sets via μ to define the outer L^p quasi-norms. For example, for every outer measure μ on a set X, for every $p \in (0, \infty]$, we can define the outer L^p spaces associated with the following quasi-norms. For every function f on X, we define

$$\begin{aligned} \|f\|_{L^\infty(X,\mu)} &= \sup \Big\{ \lambda \in [0,\infty) \colon \mu\Big(\Big\{x \in X \colon |f(x)| > \lambda\Big\}\Big) > 0 \Big\}, \\ \|f\|_{L^p(X,\mu)} &= \Big(\int_0^\infty p \lambda^p \mu\Big(\Big\{x \in X \colon |f(x)| > \lambda\Big\}\Big) \frac{\mathrm{d}\lambda}{\lambda} \Big)^{\frac{1}{p}}. \end{aligned} \tag{1.0.3}$$

Actually, these are the definitions of the L^p quasi-norms and spaces appearing in the context of the well-developed theory of capacities and Choquet integrals, see for example the articles of Choquet [Cho54] and Adams [Ada98].

In fact, in the previous display we used implicitly an additional ingredient, the collection of functionals associated with the point evaluation at each point of X. Its role can be played by other collections of functionals, leading to the introduction of sizes. A size S on a set X is a collection of functionals, one for each element of a collection $\mathcal{A}$ of subsets of X. For each element of $\mathcal{A}$, the associated functional is defined on the same vector space $\mathcal{M}$ of functions on X, and it is homogeneous, monotone, and subadditive in the variable in $\mathcal{M}$. The size provides a way to evaluate the magnitude of functions. In (1.0.3), this is achieved by the point evaluations, namely the size is the L^∞ norm with respect to the counting measure. Another example of a size is the functional associated with the property defining the Carleson measures on the upper half space $\mathbb{R}^d \times (0,\infty)$, see for example the book of Stein [Ste93]. For the purpose of this introduction, we restrict to the case of measures on the upper half space defined by densities with respect to the Lebesgue measure. Then, the functional has the form of a suitably averaged local classical L^1 norm of such densities on specific subsets of the upper half space. Such a functional is the prototype of the sizes we are interested in throughout this thesis.

Then, the outer L^p quasi-norms with respect to S are defined by an interplay between the outer measure and the size. This interplay is analogous to that between the measure ω and the classical $L^\infty(X,\omega)$ norm appearing in the layer cake representation of the classical $L^p(X,\omega)$ quasi-norms on the measure space (X,ω).

In view of its application to the two-step programme outlined above, in their article [DT15], Do and Thiele developed the theory of outer L^p spaces mainly in the direction of the real interpolation properties. These include versions of Hölder's inequality and Marcinkiewicz interpolation for the outer L^p spaces.

In this thesis we are mostly concerned with investigating the Banach space properties of the outer L^p spaces. For example, whether or not the outer L^p quasi-norms are equivalent to norms, namely whether or not they satisfy a quasi-triangle inequality with constant uniform in the number of summands. We postpone the description of the other Banach space properties we are interested in to the final section of this Introduction.

1.1 Definition and properties of outer L^p spaces

We proceed with the formal definition of the outer L^p spaces and the statement of relevant properties, following the article of Do and Thiele [DT15]. We postpone the examples to Subsections 1.2.1 – 1.2.13, after the introduction of the whole formal framework.

The first ingredient in the definition of outer L^p spaces is that of an outer measure.

Definition 1.1.1 (Outer measure). *Let X be a set. Let $\mathcal{P}(X)$ be the collection of all the subsets of X. An* outer measure *μ on X is a function*

$$\mu \colon \mathcal{P}(X) \to [0,\infty],$$

satisfying the following properties.

(i) $\mu(\varnothing) = 0$.

(ii) For all subsets $A, B \subseteq X$, $A \subseteq B$, we have $\mu(A) \leqslant \mu(B)$.

(iii) For every collection $\{A_n \colon n \in \mathbb{N}\} \subseteq \mathcal{P}(X)$ of subsets of X, we have

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \leqslant \sum_{n\in\mathbb{N}} \mu(A_n).$$

A standard way to define an outer measure μ on a set X is to start with a function, called pre-measure, defined on a particular collection of subsets of X, and to generate μ via minimal coverings. We briefly recall the construction.

Definition 1.1.2 (Pre-measure)**.** *Let X be a set. Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. A* pre-measure *$(\sigma, \mathcal{E})$ on X is a function*

$$\sigma \colon \mathcal{E} \to [0, \infty].$$

To simplify the notation, we will often avoid to mention $\mathcal{E}$, as it is implicitly determined by σ as its domain.

We define the outer measure $\mu = \mu(\sigma, \mathcal{E})$ on X as follows. For every subset $A \subseteq X$,

$$\mu(A) = \inf \Big\{ \sum_{E \in \mathcal{E}'} \sigma(E) \colon \mathcal{E}' \subseteq \mathcal{E}, A \subseteq \bigcup_{E \in \mathcal{E}'} E \Big\}, \tag{1.1.1}$$

where the infimum is taken over all the countable subcollections of $\mathcal{E}$ covering A. Moreover, the sum over an empty collection is understood to be 0. Furthermore, if there exists no countable subcollection of $\mathcal{E}$ covering A, then the infimum is understood to be ∞. We refer to Proposition 2.1 in [DT15] for a proof that μ is indeed an outer measure. We point out that, for every subset $E \in \mathcal{E}$, we have

$$\mu(E) \leqslant \sigma(E),$$

but in general the inequality in the opposite direction need not hold true.

The second ingredient in the definition of outer L^p spaces is that of a size.

Definition 1.1.3 (Size)**.** *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. Let $\mathcal{M}(X, \mathcal{A})$ be a vector space of functions on X. A* size *$(S, \mathcal{A}, \mathcal{M}(X, \mathcal{A}))$ on X is a function*

$$S \colon \mathcal{M}(X, \mathcal{A}) \to [0, \infty]^{\mathcal{A}},$$

satisfying the following properties.

(i) For every $\lambda \in \mathbb{R}$, for every function $f \in \mathcal{M}(X, \mathcal{A})$, for every subset $A \in \mathcal{A}$, we have

$$S(\lambda f)(A) = |\lambda| S(f)(A).$$

(ii) For all functions $f, g \in \mathcal{M}(X, \mathcal{A})$ such that $|f| \leqslant |g|$, for every subset $A \in \mathcal{A}$, we have

$$S(f)(A) \leqslant S(g)(A).$$

(iii) There exists a constant $C > 0$ such that, for all functions $f, g \in \mathcal{M}(X, \mathcal{A})$, for every subset $A \in \mathcal{A}$, we have

$$S(f + g)(A) \leqslant C(S(f)(A) + S(g)(A)).$$

To simplify the notation, we will often avoid to mention $\mathcal{M}(X, \mathcal{A})$ or both $\mathcal{A}$ and $\mathcal{M}(X, \mathcal{A})$, as they are implicitly determined by S as its domain.

We are now ready to define the outer L^p quasi-norms and spaces with respect to a size on a set endowed with an outer measure. We start with the outer L^∞ quasi-norm of a function. It is the maximal magnitude achieved by the function in terms of the size.

Definition 1.1.4 (Outer $L^\infty_\mu(S)$ quasi-norm and space)**.** *Let X be a set, let μ be an outer measure on X, and let $(S, \mathcal{A}, \mathcal{M}(X, \mathcal{A}))$ be a size. For every function $f \in \mathcal{M}(X, \mathcal{A})$, we define the* outer $L^\infty_\mu(S)$ quasi-norm *of the function f by*

$$\|f\|_{L^\infty_\mu(S)} = \|f\|_{L^{\infty,\infty}_\mu(S)} \coloneqq \sup\Big\{S(f)(A) \colon A \in \mathcal{A}\Big\}, \tag{1.1.2}$$

and the outer $L^\infty_\mu(S)$ space *to be the set of functions in $\mathcal{M}(X, \mathcal{A})$ for which $\|f\|_{L^\infty_\mu(S)}$ is finite.*

The outer L^∞ quasi-norm allows us to introduce the super level measure of a function with respect to the size. It is the magnitude of a minimal set outside of which the outer L^∞ quasi-norm of the function is controlled by λ, minimal in terms of the outer measure.

Definition 1.1.5 (Super level measure)**.** *Let X be a set, let μ be an outer measure on X, and let $(S, \mathcal{A}, \mathcal{M}(X, \mathcal{A}))$ be a size. For every function $f \in \mathcal{M}(X, \mathcal{A})$, for every $\lambda \in (0, \infty)$, we define the* super level measure *of the function f at level λ with respect to the size S by*

$$\mu(S(f) > \lambda) \coloneqq \inf\Big\{\mu(B) \colon B \subseteq X, f 1_{B^c} \in \mathcal{M}(X, \mathcal{A}), \|f 1_{B^c}\|_{L^\infty_\mu(S)} \leqslant \lambda\Big\}.$$

The super level measure allows us to define the outer L^p and $L^{p,\infty}$ quasi-norms and spaces with respect to a size on a set endowed with an outer measure for $p \in (0, \infty)$.

Definition 1.1.6 (Outer $L^p_\mu(S)$ and $L^{p,\infty}_\mu(S)$ quasi-norms and spaces)**.** *Let X be a set, let μ be an outer measure on X, and let $(S, \mathcal{A}, \mathcal{M}(X, \mathcal{A}))$ be a size. For every $p \in (0, \infty)$, for every function $f \in \mathcal{M}(X, \mathcal{A})$, we define the* outer $L^p_\mu(S)$ and $L^{p,\infty}_\mu(S)$ quasi-norms *of the function f by*

$$\begin{aligned} \|f\|_{L^p_\mu(S)} &\coloneqq \Big(\int_0^\infty p \lambda^p \mu(S(f) > \lambda) \frac{\mathrm{d}\lambda}{\lambda} \Big)^{\frac{1}{p}}, \\ \|f\|_{L^{p,\infty}_\mu(S)} &\coloneqq \sup\Big\{ \lambda \mu(S(f) > \lambda)^{\frac{1}{p}} \colon \lambda > 0 \Big\}, \end{aligned} \tag{1.1.3}$$

and the outer $L^p_\mu(S)$ and $L^{p,\infty}_\mu(S)$ spaces *to be the sets of functions in $\mathcal{M}(X, \mathcal{A})$ for which $\|f\|_{L^p_\mu(S)}$ and $\|f\|_{L^{p,\infty}_\mu(S)}$ are finite respectively.*

The outer L^p and $L^{p,\infty}$ quasi-norms and spaces described in Definition 1.1.4 and Definition 1.1.6 satisfy some expected properties for a meaningful L^p theory.

- The quantities defined in (1.1.2) and (1.1.3) are indeed monotone quasi-norms, see Proposition 3.1 in [DT15]. Therefore, they can be used to define recursively new sizes, hence iterated outer L^p quasi-norms and spaces. The recursive definition of outer L^p spaces is described in details in the introduction of [Fra22], namely Chapter 3 of this thesis.
- The outer L^p and $L^{p,\infty}$ quasi-norms and spaces are well-behaved with respect to the pull-back of a map Φ between different settings (X_1, μ_1, S_1) and (X_2, μ_2, S_2), provided Φ is well-behaved with respect to the outer measures and the sizes, see Proposition 3.2 in [DT15].
- The outer L^p and $L^{p,\infty}$ quasi-norms and spaces are well-behaved with respect to the real interpolation properties of L^p spaces, such as logarithmic convexity of the outer L^p quasi-norms, outer Hölder's inequality, and Marcinkiewicz interpolation, see Propositions 3.3 – 3.5 in [DT15].
- The outer L^p quasi-norms satisfy a Radon-Nikodym type property, see Proposition 3.6 in [DT15].

Before moving to the next section, we recall the statements of outer Hölder's inequality and the Radon-Nikodym type property.

Theorem 1.1.7 (Outer Hölder's inequality, Proposition 3.4 in [DT15])**.** *Let X be a set, $\mathcal{A}$ a collection of subsets of X. Let μ, μ_1, μ_2 be three outer measures on X such that, for every subset $A \subseteq X$, we have*

$$\mu(A) \leqslant \mu_1(A), \qquad \mu(A) \leqslant \mu_2(A).$$

Let $(S, \mathcal{A}, \mathcal{M}(X, \mathcal{A}))$, $(S_1, \mathcal{A}, \mathcal{M}(X, \mathcal{A}))$, $(S, \mathcal{A}, \mathcal{M}(X, \mathcal{A}))$ be three sizes satisfying the following three properties.

(i) For all functions $f_1, f_2 \in \mathcal{M}(X, \mathcal{A})$, we have $f_1 f_2 \in \mathcal{M}(X, \mathcal{A})$.

(ii) For every subset $A \in \mathcal{A}$, there exist $A_1 \in \mathcal{A}$ and $A_2 \in \mathcal{A}$ such that, for all functions $f_1, f_2 \in \mathcal{M}(X, \mathcal{A})$, we have

$$S(f_1 f_2)(A) \leqslant S_1(f_1)(A_1) S_2(f_2)(A_2).$$

Then, for all $p, p_1, p_2 \in (0, \infty]$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},$$

there exists a constant $C = C(p, p_1, p_2)$ such that, for all functions $f_1 \in \mathcal{M}(X, \mathcal{A}_1)$ and $f_2 \in \mathcal{M}(X, \mathcal{A}_2)$, we have

$$\|f_1 f_2\|_{L^p_\mu(S)} \leqslant C \|f_1\|_{L^{p_1}_{\mu_1}(S_1)} \|f_2\|_{L^{p_2}_{\mu_2}(S_2)}.$$

Theorem 1.1.8 (Radon-Nikodym, Proposition 3.6 in [DT15]). *Let (X, Σ, ω) be a σ-finite measure space, μ an outer measure, and $(S, \mathcal{A}, \mathcal{M}(X, \Sigma))$ a size with $\mathcal{A} \subseteq \Sigma$ and $\mathcal{M}(X, \Sigma)$ the vector space of measurable functions. If either, for every measurable subset $A \in \mathcal{A}$, we have*

$$\mu(A) = 0 \Rightarrow \omega(A) = 0,$$

or if there exists a constant $\widetilde{C}$ such that, for every measurable subset $A \in \mathcal{A}$, for every measurable function $f \in \mathcal{M}(X, \Sigma)$, we have

$$\mu(A)^{-1} \|f 1_A\|_{L^1(X,\omega)} \leqslant \widetilde{C} \|f\|_{L^\infty_\mu(S)},$$

then, there exists a constant C such that, for every measurable function $f \in L^\infty_\mu(S)$, we have

$$\|f\|_{L^1(X,\omega)} \leqslant C \|f\|_{L^1_\mu(S)}.$$

1.2 Examples

We proceed with the description of some relevant examples of outer measures and sizes. In particular, we point out that throughout this thesis, we will be concerned with sizes of the form of suitably averaged local classical and outer L^r quasi-norms. Therefore, our examples are triples of quadruples made of a set, one or two outer measures, and a measure.

We start by introducing σ-finite settings in Subsection 1.2.1, which satisfy some reasonable additional assumptions. First, an assumption on the absolute continuity between the outer measures and the measure. Next, an assumption on the σ-finiteness of the set with respect to the outer measures and the measure. A particular case of such settings is that of the finite ones we define in Subsection 1.2.2. Next, in Subsection 1.2.3, we have general settings, where we drop the additional assumptions made on σ-finite settings. However, in this case we reduce the collection of sizes to that associated with the classical L^∞ norm. After that, in Subsection 1.2.4, we define the settings on the Cartesian product of σ-finite measure spaces exhibiting the mixed L^p spaces as outer L^p ones. These settings are our point of reference in the analysis of the Banach space properties of the outer L^p spaces. Then, in Subsections 1.2.5 – 1.2.7, we introduce some finite settings where the outer measure satisfies particular subadditivity properties, providing sources for counterexamples. Finally, we conclude with the settings involved in the study of Calderón-Zygmund theory and time-frequency analysis. The former ones, defined on the upper half space $\mathbb{R}^d \times (0, \infty)$ or its discrete model in Subsections 1.2.8 – 1.2.10. The latter ones defined on the upper half 3-space $\mathbb{R}^2 \times (0, \infty)$ or its discrete model in Subsections 1.2.11 – 1.2.13.

Upon first reading, it is enough to focus on the examples described in Subsection 1.2.1, the standard setting of the results presented in this thesis, in Subsection 1.2.4, the point of reference in the analysis of the properties of the outer L^p spaces, and in Subsection 1.2.8 and Subsection 1.2.11.

Before introducing the examples, we briefly comment on the use of the notation ℓ^r throughout this thesis. Usually, this symbol denotes the sequence spaces with r-integrability and the quasi-norms associated with them, namely the classical L^r spaces and quasi-norms on the measure space $\mathbb{N}$ with the counting measure. Instead, throughout this thesis, this symbol will denote certain sizes defined by suitably averaged local classical and outer L^r quasi-norms. In this regard, keeping in mind the symbol L^r, the symbol ℓ^r is chosen to imitate the relation between the symbols $\int$ and $\fint$.

1.2.1 σ-finite setting

Let (X, Σ) be a measurable space, where Σ is the σ-algebra of measurable subsets of X. Let $\mathcal{E}, \mathcal{U} \subseteq \Sigma$ be two collections of measurable subsets of X. Let σ and τ be pre-measures defined on $\mathcal{E}$ and $\mathcal{U}$ respectively, and we assume them to attain only strictly positive finite values. Let μ and ν be the outer measures on X generated via minimal coverings as in (1.1.1) by the pre-measures σ and τ on the collections $\mathcal{E}$ and $\mathcal{U}$ respectively. Let ω be a measure on (X, Σ). To guarantee that the sizes we are interested in are well-defined, we make two additional assumptions. First, we assume a certain absolute continuity between the outer measures and the measure. Namely, in case we consider only μ and ω, we assume that, for every measurable subset $A \in \Sigma$, we have

$$\mu(A) = 0 \Rightarrow \omega(A) = 0. \tag{1.2.1}$$

In case we consider μ, ν and ω, we assume that, for every measurable subset $A \in \Sigma$, we have

$$\mu(A) = 0 \Rightarrow \nu(A) = 0, \qquad \nu(A) = 0 \Rightarrow \omega(A) = 0.$$

Next, we assume that the set is σ-finite with respect to the outer measures and the measure. Namely, there exist three collections $\{A_n : n \in \mathbb{N}\}, \{B_n : n \in \mathbb{N}\}, \{C_n : n \in \mathbb{N}\} \subseteq \Sigma$ of measurable subsets of X such that

$$\begin{gathered} \mu(A_n) < \infty, \qquad \nu(B_n) < \infty, \qquad \omega(C_n) < \infty, \qquad \text{for every } n \in \mathbb{N}, \\ X = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} C_n. \end{gathered} \tag{1.2.2}$$

Under these assumptions, we define (X, μ, ω) and (X, μ, ν, ω) *σ-finite settings*.

Before defining the sizes, we introduce some auxiliary notation. We define $\mathcal{M}(X, \Sigma)$ to be the collection of Σ-measurable functions on X with values in $\mathbb{R}$. We define the collection $\widetilde{\Sigma} \subseteq \Sigma$ of measurable subsets of X by

$$\widetilde{\Sigma} := \Big\{ A \in \Sigma : \mu(A) \neq \infty \Big\},$$

and, for every measurable subset $A \in \Sigma$, we define the collection $\Sigma'_\omega(A) \subseteq \Sigma$ of measurable subsets of A by

$$\Sigma'_\omega(A) := \Big\{ B \in \Sigma : B \subseteq A, \omega(A \backslash B) = 0 \Big\}.$$

First, we define the size $(\ell^\infty_\omega, \Sigma, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every measurable subset $A \in \Sigma$,

$$\ell^\infty_\omega(f)(A) \coloneqq \| f 1_A \|_{L^\infty(X,\omega)}. \tag{1.2.3}$$

Moreover, for every $r \in (0,\infty)$, we define the size $(\ell^r_\omega, \widetilde{\Sigma}, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every subset $A \in \widetilde{\Sigma}$,

$$\ell^r_\omega(f)(A) \coloneqq \begin{cases} 0, & \text{if } \mu(A) = 0, \\ \mu(A)^{-\frac{1}{r}} \| f 1_A \|_{L^r(X,\omega)}, & \text{if } \mu(A) \neq 0. \end{cases} \tag{1.2.4}$$

Then, for all $p, r \in (0,\infty]$, we define the *single iterated outer* $L^p_\mu(\ell^r_\omega)$ *and* $L^{p,\infty}_\mu(\ell^r_\omega)$ *quasi-norms and spaces* on the setting (X,μ,ω) with respect to each of the sizes appearing in the previous two displays as in Definition 1.1.4 and Definition 1.1.6.

In particular, for every measurable function f on X, we have

$$\| f \|_{L^\infty_\mu(\ell^\infty_\omega)} = \| f \|_{L^\infty(X,\omega)}. \tag{1.2.5}$$

Moreover, for every $p \in (0,\infty)$, for every measurable function f on X, we have

$$\begin{aligned} \| f \|_{L^p_\mu(\ell^\infty_\omega)} &= \Big(\int_0^\infty p \lambda^p \inf \Big\{ \mu(B_\lambda) \colon B_\lambda \in \Sigma'_\omega(A_\lambda) \Big\} \frac{\mathrm{d}\lambda}{\lambda} \Big)^{\frac{1}{p}}, \\ \| f \|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} &= \sup \Big\{ \lambda \inf \Big\{ \mu(B_\lambda) \colon B_\lambda \in \Sigma'_\omega(A_\lambda) \Big\}^{\frac{1}{p}} \colon \lambda > 0 \Big\}, \end{aligned} \tag{1.2.6}$$

where, for every $\lambda \in (0,\infty)$, we define the measurable subset $A_\lambda \subseteq X$ by

$$A_\lambda = \Big\{ x \in X \colon |f(x)| > \lambda \Big\}.$$

Therefore, for every measurable subset $A \in \Sigma$, we have

$$\| 1_A \|_{L^p_\mu(\ell^\infty_\omega)} = \| 1_A \|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} = \inf \Big\{ \mu(B) \colon B \in \Sigma'_\omega(A) \Big\}^{\frac{1}{p}}. \tag{1.2.7}$$

Finally, if μ is the outer measure generated via minimal coverings as in (1.1.1) by ω considered as a pre-measure on the collection Σ of measurable subsets, for all $p, r \in (0,\infty]$, for every measurable function f on X, we have

$$\| f \|_{L^p_\omega(\ell^r_\omega)} = \| f \|_{L^p(X,\omega)}, \qquad \| f \|_{L^{p,\infty}_\omega(\ell^r_\omega)} = \| f \|_{L^{p,\infty}(X,\omega)}.$$

This concludes our observations about single iterated outer L^p quasi-norms and spaces.

Next, for every $r \in (0,\infty]$, we define the size $(\ell^\infty_\nu(\ell^r_\omega), \Sigma, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every subset $A \in \Sigma$,

$$\ell^\infty_\nu(\ell^r_\omega)(f)(A) \coloneqq \| f 1_A \|_{L^\infty_\nu(\ell^r_\omega)}.$$

Moreover, for all $q \in (0,\infty)$, $r \in (0,\infty]$, we define the size $(\ell^q_\nu(\ell^r_\omega), \widetilde{\Sigma}, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every subset $A \in \widetilde{\Sigma}$,

$$\ell^q_\nu(\ell^r_\omega)(f)(A) \coloneqq \begin{cases} 0, & \text{if } \mu(A) = 0, \\ \mu(A)^{-\frac{1}{q}} \| f 1_A \|_{L^q_\nu(\ell^r_\omega)}, & \text{if } \mu(A) \neq 0. \end{cases}$$

Then, for every $p,q,r \in (0,\infty]$, we define the *double iterated outer* $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ *and* $L^{p,\infty}_\mu(\ell^q_\nu(\ell^r_\omega))$ *quasi-norms and spaces* on the setting (X,μ,ν,ω) with respect to each of the sizes appearing in the previous two displays as in Definition 1.1.4 and Definition 1.1.6.

Remark 1.2.1. *If the outer measure μ is generated via minimal coverings by the pre-measure σ on the collection $\mathcal{E}$, we can define additional sizes. Before defining them, we introduce some auxiliary notation. We define the collection $\widetilde{\mathcal{E}} \subseteq \mathcal{E}$ by*

$$\widetilde{\mathcal{E}} = \Big\{ E \in \mathcal{E} \colon \sigma(E) \neq \infty \Big\}.$$

First, we define the size $(\ell^\infty_{\omega,\sigma}, \mathcal{E}, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every measurable subset $E \in \mathcal{E}$,

$$\ell^\infty_{\omega,\sigma}(f)(E) \coloneqq \| f 1_E \|_{L^\infty(X,\omega)}.$$

Moreover, for every $r \in (0,\infty)$, we define the size $(\ell^r_{\omega,\sigma}, \widetilde{\mathcal{E}}, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every measurable subset $E \in \widetilde{\mathcal{E}}$,

$$\ell^r_{\omega,\sigma}(f)(E) \coloneqq \begin{cases} 0, & \text{if } \sigma(E) = 0, \\ \sigma(E)^{-\frac{1}{r}} \| f 1_E \|_{L^r(X,\omega)}, & \text{if } \sigma(E) \neq 0. \end{cases}$$

For $r \in (0,\infty)$, the sizes ℓ^r_ω and $\ell^r_{\omega,\sigma}$ need not be equal on the subsets in $\widetilde{\mathcal{E}}$.

However, in Lemma 2.A.3 in Chapter 2, we prove that, for all $p,r \in (0,\infty]$, for every measurable function f on X, we have

$$\| f \|_{L^p_\mu(\ell^r_{\omega,\sigma})} = \| f \|_{L^p_\mu(\ell^r_\omega)},$$

therefore the single iterated outer L^p quasi-norms and spaces with respect to the two sizes are equal.

Analogously, for every $r \in (0,\infty]$, we define the size $(\ell^\infty_{\nu,\sigma}(\ell^r_\omega), \mathcal{E}, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every measurable subset $E \in \mathcal{E}$,

$$\ell^\infty_{\nu,\sigma}(\ell^r_\omega)(f)(E) \coloneqq \| f 1_E \|_{L^\infty_\nu(\ell^r_\omega)}.$$

Moreover, for all $q \in (0,\infty)$, $r \in (0,\infty]$, we define the size $(\ell^q_{\nu,\sigma}(\ell^r_\omega), \widetilde{\mathcal{E}}, \mathcal{M}(X,\Sigma))$ as follows. For every measurable function f on X, for every subset $E \in \widetilde{\mathcal{E}}$,

$$\ell^q_{\nu,\sigma}(\ell^r_\omega)(f)(E) \coloneqq \begin{cases} 0, & \text{if } \sigma(E) = 0, \\ \sigma(E)^{-\frac{1}{q}} \| f 1_E \|_{L^q_\nu(\ell^r_\omega)}, & \text{if } \sigma(E) \neq 0. \end{cases}$$

Once again, for $q \in (0,\infty)$, $r \in (0,\infty]$, the sizes $\ell^q_\nu(\ell^r_\omega)$ and $\ell^q_{\nu,\sigma}(\ell^r_\omega)$ need not be equal on the subsets in $\widetilde{\mathcal{E}}$.

However, for $q \leqslant r$ or $q = \infty$, by an argument analogous to that in Lemma 2.A.3 in Chapter 2 together with Lemma 3.3.1 in Chapter 3, we can prove that the double iterated outer L^p quasi-norms with respect to the two sizes are equivalent, and the double iterated outer L^p spaces are equal.

Instead, for $q > r$, for every $p \in (0,\infty]$, in general we only have that, for every measurable function f on X,

$$\|f\|_{L^p_\mu(\ell^q_{\nu,\sigma}(\ell^r_\omega))} \leqslant \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

However, we can use an argument analogous to that in the case $q \leqslant r$ to recover the equivalence in the case $q > r$ as long as the subsets in $\mathcal{E}$ satisfy good enough compatibility conditions with respect to the outer measure ν. One sufficient compatibility condition is that there exists two constants $\Phi, K \geqslant 1$ such that the following properties hold true. For every subset $A \subseteq X$, there exists a collection $\mathcal{A} \subseteq \mathcal{E}$ of pairwise disjoint elements such that

$$A \subseteq \bigcup_{E \in \mathcal{A}} E, \qquad \sum_{E \in \mathcal{A}} \sigma(E) \leqslant \Phi\mu(A), \tag{1.2.8}$$

and, for every subset $U \subseteq X$, we have

$$\sum_{E \in \mathcal{A}} \nu(E \cap U) \leqslant K\nu(U). \tag{1.2.9}$$

1.2.2 Finite setting

Let X be a finite set, and let ω be a measure on $(X, \mathcal{P}(X))$. Since we assumed that every subset of X is measurable, then all the functions on X are measurable. Moreover, we assume that, for every $x \in X$, we have $\omega(x) = \omega(\{x\}) \in (0,\infty)$. Let μ and ν be outer measures on X. We assume that, for every subset $A \subseteq X$, we have $\mu(A), \nu(A) \in (0,\infty)$. These assumptions are reasonable, as subsets of X of zero or infinite measure or outer measure contribute only trivially to any L^p theory on X. Under these assumptions, we define (X,μ,ω) and (X,μ,ν,ω) *finite settings*. In particular, all finite settings (X,μ,ω) and (X,μ,ν,ω) are σ-finite settings with $\Sigma = \mathcal{E} = \mathcal{U} = \mathcal{P}(X)$, $\sigma = \mu$, and $\tau = \nu$.

In particular, for every $p \in (0,\infty)$, for every function f on X, we have

$$\|f\|_{L^p_\mu(\ell^\infty_\omega)} = \Big(\int_0^\infty p\lambda^p \mu\Big(\Big\{x \in X \colon |f(x)| > \lambda\Big\}\Big) \frac{\mathrm{d}\lambda}{\lambda} \Big)^{\frac{1}{p}},$$
$$\|f\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} = \sup\Big\{\lambda\mu(\{x \in X \colon |f(x)| > \lambda\})^{\frac{1}{p}} \colon \lambda > 0\Big\}.$$

Therefore, for every subset $A \subseteq X$, we have

$$\|1_A\|_{L^p_\mu(\ell^\infty_\omega)} = \|1_A\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} = \mu(A)^{\frac{1}{p}}. \tag{1.2.10}$$

1.2.3 General setting

Let (X, Σ) be a measurable space, where Σ is the σ-algebra of measurable subsets of X. Let μ and ν be outer measures on X, and let ω be a measure on (X, Σ). We define (X, μ, ω) a *general setting*.

Since we dropped any additional assumption on the setting, we only restrict to the ℓ^∞_ω size. We define it as in (1.2.3) and, for every $p \in (0, \infty]$, we define the outer $L^p_\mu(\ell^\infty_\omega)$ and $L^{p,\infty}_\mu(\ell^\infty_\omega)$ quasi-norms and spaces on the setting (X, μ, ω) as in Definition 1.1.4 and Definition 1.1.6. In particular, for every measurable function f on X, we have the same properties described in the equalities in (1.2.5), (1.2.6), and (1.2.7).

1.2.4 Cartesian product of σ-finite measure spaces

For all σ-finite measure spaces (Y, Σ_Y, ω_Y) and (Z, Σ_Z, ω_Z), let

$$\begin{aligned} X &= Y \times Z, \\ \mathcal{E} &= \Big\{ Y' \times Z : Y' \in \Sigma_Y \Big\}, \\ \sigma(Y' \times Z) &= \omega_Y(Y'), && \text{for every } Y' \in \Sigma_Y, \end{aligned}$$

let μ be the outer measure on X generated via minimal coverings as in (1.1.1) by the pre-measure σ on the collection $\mathcal{E}$, and let ω the canonical product measure on X associated with ω_Y and ω_Z, see for example the book of Rudin [Rud74]. In particular, we have $\mu = \omega_Y \circ \pi_Y$, where $\pi_Y : X \to Y$ is the projection onto Y. The setting (X, μ, ω) is σ-finite.

For all $p, r \in (0, \infty]$, for every measurable function f on X, we have

$$\| f \|_{L^p_\mu(\ell^r_\omega)} = \| \| f(\cdot, \cdot) \|_{L^r(Z, \omega_Z)} \|_{L^p(Y, \omega_Y)}.$$

Next, for all σ-finite measure spaces (Y, Σ_Y, ω_Y), (Z, Σ_Z, ω_Z), and (W, Σ_W, ω_W), let

$$\begin{aligned} X &= Y \times Z \times W, \\ \mathcal{E} &= \Big\{ Y' \times Z \times W : Y' \in \Sigma_Y \Big\}, \\ \sigma(Y' \times Z \times W) &= \omega_Y(Y'), && \text{for every } Y' \in \Sigma_Y, \\ \mathcal{U} &= \Big\{ Y' \times Z' \times W : Y' \in \Sigma_Y, Z' \in \Sigma_Z \Big\}, \\ \tau(Y' \times Z' \times W) &= \omega_Y(Y') \omega_Z(Z'), && \text{for all } Y' \in \Sigma_Y,\ Z' \in \Sigma_Z, \end{aligned}$$

let μ and ν be the outer measures on X generated via minimal coverings as in (1.1.1) by the pre-measures σ and τ on the collections $\mathcal{E}$ and $\mathcal{U}$ respectively, and let ω the canonical product measure on X associated with ω_Y, ω_Z, and ω_W. As above, we have $\mu = \omega_Y \circ \pi_Y$ and $\nu = \rho \circ \pi_{Y \times Z}$, where ρ is the canonical product measure on $Y \times Z$ associated with ω_Y and ω_Z. The settings (X, μ, ω), (X, ν, ω), and (X, μ, ν, ω) are σ-finite.

For all $p, q, r \in (0, \infty]$, for every measurable function f on X, we have

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} = \|\|\|f(\cdot,\cdot,\cdot)\|_{L^r(W,\omega_W)}\|_{L^q(Z,\omega_Z)}\|_{L^p(Y,\omega_Y)}.$$

We can iterate the definitions above in the case of the Cartesian product of arbitrarily many σ-finite measure spaces, reproducing the *mixed L^p spaces* as iterated outer L^p ones.

This class of examples exhibits the paradigmatic comparability of the additive behaviours between the different outer measures.

Remark 1.2.2. *In the setting on the Cartesian product of σ-finite measure spaces just described, the elements of $\mathcal{E}$ satisfy the sufficient compatibility condition with respect to the outer measure ν stated in Remark 1.2.1. In fact, for every subset $A \subseteq X$, the collection $\{\pi_Y(A) \times Z \times W\} \subseteq \mathcal{E}$ satisfies the properties in* (1.2.8) *and* (1.2.9)*. Therefore, we can prove that, for all $p, q, r \in (0, \infty]$, for every measurable function f on X, we have*

$$\|f\|_{L^p_\mu(\ell^q_{\nu,\sigma}(\ell^r_\omega))} = \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))},$$

and the double iterated outer L^p quasi-norms and spaces with respect to the two sizes are equal.

1.2.5 Constant outer measure

Let (X, Σ, ω) be a σ-finite measure space. Let μ be the outer measure on X defined as follows. For every subset $A \subseteq X$, $A \neq \varnothing$,

$$\mu(A) = 1.$$

We define such an outer measure to be the *constant outer measure*. The setting (X, μ, ω) is σ-finite, and we observe that there are no Carathéodory measurable subsets of X with respect to μ other than $\{\varnothing, X\}$. Next, let (X, ν, ω) be a σ-finite setting on X. Then the setting (X, μ, ν, ω) is σ-finite as well.

For all $p, r \in (0, \infty]$, for every measurable function f on X, we have

$$\|f\|_{L^p_\mu(\ell^r_\omega)} = \|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} = \|f\|_{L^\infty_\mu(\ell^r_\omega)} = \|f\|_{L^r(X,\omega)}.$$

Moreover, for all $p, q, r \in (0, \infty]$, for every measurable function f on X, we have

$$\begin{aligned}
\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} &= \|f\|_{L^{p,\infty}_\mu(\ell^q_\nu(\ell^r_\omega))} = \|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} = \|f\|_{L^q_\nu(\ell^r_\omega)}, \\
\|f\|_{L^p_\nu(\ell^q_\mu(\ell^r_\omega))} &= \|f\|_{L^p_\nu(S^{q,r}_\omega)}, \\
\|f\|_{L^{p,\infty}_\nu(\ell^q_\mu(\ell^r_\omega))} &= \|f\|_{L^{p,\infty}_\nu(S^{q,r}_\omega)},
\end{aligned}$$

where, for every $r \in (0, \infty]$, the size $(S^{\infty,r}_\omega, \Sigma, \mathcal{M}(X, \Sigma))$ is defined by, for every measurable function f on X, for every measurable subset $A \in \Sigma$,

$$S^{\infty,r}_\omega(f)(A) \coloneqq \|f 1_A\|_{L^r(X,\omega)},$$

and, for all $q \in (0,\infty)$, $r \in (0,\infty]$, the size $(S^{q,r}_{\omega}, \widetilde{\Sigma}, \mathcal{M}(X,\Sigma))$ is defined by, for every measurable function f on X, for every measurable subset $A \in \Sigma$, $\nu(A) \neq \infty$,

$$S^{q,r}_{\omega}(f)(A) \coloneqq \begin{cases} 0, & \text{if } \nu(A) = 0, \\ \nu(A)^{-\frac{1}{q}} \|f 1_A\|_{L^r(X,\omega)}, & \text{if } \nu(A) \neq 0. \end{cases}$$

This class of examples exploit the strong subadditivity properties of the constant outer measure, namely the failure of additivity on any collection of disjoint subsets. In the case of double iterated outer L^p spaces, this class provides a source of counterexamples to the uniformity of the constants in the quasi-triangle inequality for countably many summands, see Subsection 3.3.4 in Chapter 3.

1.2.6 Hypercube packing

For every $m \in \mathbb{N}$, we define the subset $S_m \subseteq (\mathbb{Z}/m\mathbb{Z})^m$ by

$$S_m \coloneqq \{1, \dots, m-1\}^m.$$

Moreover, for every $x \in (\mathbb{Z}/m\mathbb{Z})^m$, we define the subset $E(x) \subseteq (\mathbb{Z}/m\mathbb{Z})^m$ by

$$E(x) \coloneqq (x + S_m)/(m\mathbb{Z})^m.$$

Next, for every $m \in \mathbb{N}$, let

$$\begin{aligned} X_m &= (\mathbb{Z}/m\mathbb{Z})^m = \mathbb{Z}^m/(m\mathbb{Z})^m, \\ \mathcal{E}_m &= \Big\{ E(x) \colon x \in X_m \Big\}, \\ \sigma_m(E(x)) &= 1, && \text{for every } x \in X_m, \\ \omega_m(x) &= 1, && \text{for every } x \in X_m, \end{aligned}$$

and let μ_m be the outer measure on X_m generated via minimal coverings as in (1.1.1) by the pre-measure σ_m on the collection $\mathcal{E}_m$. The setting (X_m, μ_m, ω_m) is finite, and we observe that there are no Carathéodory measurable subsets of X_m with respect to μ_m other than $\{\varnothing, X_m\}$.

This class of examples, suggested by the articles of Herer and Christensen [HC75], and Topsøe [Top76], exploits the weak subadditivity properties of the outer measures, namely the failure of uniform subadditivity with multiplicity, see Remark 4.2.9. In the case of the outer $L^p_{\mu}(\ell^{\infty}_{\omega})$ spaces, this class provides a source of counterexamples to the uniformity of the constant in the quasi-triangle inequality for countably many summands.

The outer measure space (X_m, μ_m) can be understood as follows. Let Y_m be the collection of m-dimensional hypercubes in $\mathbb{R}^m$ with sidelength 1 and vertices in $\mathbb{Z}^m$. Let Z_m be the collection of m-dimensional hypercubes in $\mathbb{R}^m$ with sidelength $m-1$ and vertices in $\mathbb{Z}^m$. Let $\sim_m$ be the equivalence between elements of Y_m defined by the grid $(m\mathbb{Z})^m$, namely, for all $y, y' \in Y_m$, we say $y \sim_m y'$ if and only if there exists $\vec{a} \in \mathbb{Z}^m$ such that

$$y' = \left\{\vec{s} + m\vec{a} \colon \vec{s} \in y \subseteq \mathbb{R}^m\right\}.$$

Then, we have $X_m \equiv Y_m / \sim_m$, and the outer measure μ_m is generated via minimal coverings as in (1.1.1) by the pre-measure σ_m attaining value 1 on each element of the collection $\{z/\sim_m \colon z \in Z_m\} \subseteq \mathcal{P}(X_m)$.

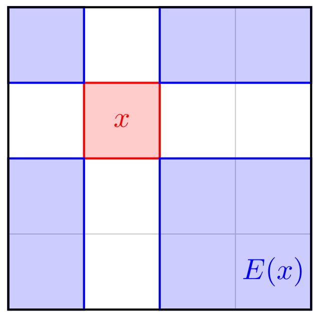

Figure 1.1: Subset $E(x)$ associated with $x \in X_m$ in the representation of the setting as a hypercube.

1.2.7 Dyadic trees of arbitrary depth

For all $n, l \in \mathbb{Z}$, we define the dyadic interval $I(n, l)$ in $\mathbb{R}$ by

$$I(n, l) = (2^l n, 2^l (n+1)],$$

and the collection $\mathcal{I}$ of dyadic intervals in $\mathbb{R}$ by

$$\mathcal{I} = \left\{I(n, l) \colon n, l \in \mathbb{Z}\right\}.$$

Moreover, for every $m \in \{0\} \cup \mathbb{N}$, for every dyadic interval $I \in \mathcal{I}$ such that

$$I \subseteq (0, 1], \qquad |I| = 2^{-m},$$

where $|I|$ is the Lebesgue measure of I, we define the subset $E(I) \subseteq \mathcal{I}$ by

$$E(I) \coloneqq \left\{J \in \mathcal{I} \colon I \subseteq J \subseteq (0, 1]\right\}.$$

Next, for every $m \in \{0\} \cup \mathbb{N}$, let

$$\begin{aligned}
X_m &= \left\{I \in \mathcal{I} \colon I \subseteq (0, 1], |I| \geqslant 2^{-m}\right\}, \\
\mathcal{E}_m &= \left\{E(I) \colon I \in X_m, |I| = 2^{-m}\right\}, \\
\sigma_m(E(I)) &= 1, && \text{for every } I \in X_m,\ |I| = 2^{-m}, \\
\omega_m(J) &= 1, && \text{for every } J \in X_m,
\end{aligned}$$

and let μ_m be the outer measure on X_m generated via minimal coverings as in (1.1.1) by the pre-measure σ_m on the collection $\mathcal{E}_m$. The setting (X_m, μ_m, ω_m) is finite, and we observe that there are no Carathéodory measurable subsets of X_m with respect to μ_m other than $\{\varnothing, X_m\}$.

This class of examples exploits the weak subadditivity properties of the outer measures. In the case of the outer $L^1_\mu(\ell^r_\omega)$ spaces, this class provides a source of counterexamples to the uniformity of the constant in the quasi-triangle inequality for countably many summands.

The outer measure spaces (X_m, μ_m) can be understood as follows. Let Y be a rooted tree with a bifurcation at each level of depth, where the level of depth of the root is assumed to be the 0-th level. Let Z be the collection of branches in Y, namely the subsets of Y obtained by starting from the root of the tree and subsequently choosing one possibility for each bifurcation at every level of depth. Let $\sim_m$ be the equivalence between subsets of Y defined by the identity when restricted to the first m levels of depth. Then, we have $X_m \equiv Y/\sim_m$, and the outer measure μ_m is generated via minimal coverings as in (1.1.1) by the pre-measure σ_m attaining value 1 on each element of the collection $\{z/\sim_m : z \in Z\} \subseteq \mathcal{P}(Y_m)$.

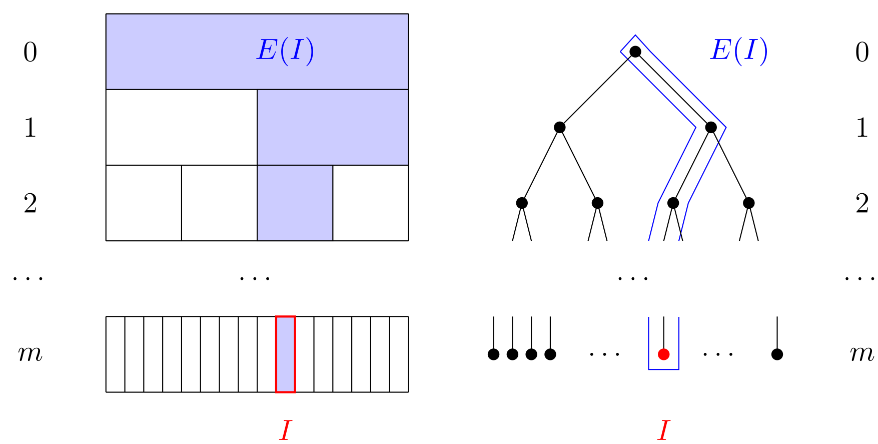

Figure 1.2: Subset $E(I)$ associated with the dyadic interval $I \subseteq (0, 1]$, $|I| = 2^{-m}$ in both of the representations of the setting, as a collection of dyadic intervals and as a tree.

1.2.8 Discrete model of the upper half space: dyadic cubes

For every $d \in \mathbb{N}$, for all $\vec{m} \in \mathbb{Z}^d$, $l \in \mathbb{Z}$, we define the *dyadic cube* $Q(\vec{m}, l)$ in $\mathbb{R}^d$ by

$$Q(\vec{m}, l) \coloneqq \prod_{i=1}^{d} (2^l m_i, 2^l (m_i + 1)],$$

and the collection $\mathcal{Q}_d$ of dyadic cubes in $\mathbb{R}^d$ by

$$\mathcal{Q}_d \coloneqq \Big\{ Q(\vec{m}, l) \colon \vec{m} \in \mathbb{Z}^d, l \in \mathbb{Z} \Big\}.$$

Moreover, for all $\vec{m} \in \mathbb{Z}^d$, $l \in \mathbb{Z}$, we define the *upper half dyadic cubic box* $B(\vec{m}, l)$ in the upper half space $\mathbb{R}^d \times (0, \infty)$ by

$$B(\vec{m}, l) = B(Q(\vec{m}, l)) \coloneqq Q(\vec{m}, l) \times (2^{l-1}, 2^l],$$

and the collection $\mathcal{B}_d$ of upper half dyadic cubic boxes in the upper half space $\mathbb{R}^d \times (0, \infty)$ by

$$\mathcal{B}_d \coloneqq \Big\{ B(\vec{m}, l) \colon \vec{m} \in \mathbb{Z}^d, l \in \mathbb{Z} \Big\}.$$

Furthermore, for all $\vec{m} \in \mathbb{Z}^d$, $l \in \mathbb{Z}$, we define the subset $E(\vec{m}, l) \subseteq \mathcal{B}_d$ by

$$E(\vec{m}, l) = E(Q(\vec{m}, l)) = E(B(\vec{m}, l)) \coloneqq \Big\{ B \in \mathcal{B}_d \colon B \subseteq Q(\vec{m}, l) \times (0, 2^l] \Big\}.$$

Next, for every $d \in \mathbb{N}$, let

$$\begin{aligned} X_d &= \mathcal{B}_d, \\ \mathcal{E}_d &= \Big\{ E(\vec{m}, l) \colon \vec{m} \in \mathbb{Z}^d, l \in \mathbb{Z} \Big\}, \\ \sigma_d(E(\vec{m}, l)) &= 2^{dl}, && \text{for all } \vec{m} \in \mathbb{Z}^d,\ l \in \mathbb{Z}, \\ \omega_d(B(\vec{m}, l)) &= 2^{dl}, && \text{for all } \vec{m} \in \mathbb{Z}^d,\ l \in \mathbb{Z}, \end{aligned}$$

and let μ_d be the outer measure on X_d generated via minimal coverings as in (1.1.1) by the pre-measure σ_d on the collection $\mathcal{E}_d$. The setting (X_d, μ_d, ω_d) is σ-finite, and we observe that the σ-algebra of the Carathéodory measurable subsets of X_d with respect to μ_d is generated by the subsets in the collection

$$\Big\{ X_{\vec{a}} \colon \vec{a} \in \{-1, 1\}^d \Big\}.$$

For every $\vec{a} \in \{-1, 1\}^d$, we define the subset $X_{\vec{a}} \subseteq X_d$ by

$$X_{\vec{a}} = \Big\{ B \in \mathcal{B}_d \colon B \subseteq \Big(\prod_{i=1}^{d} X_{a_i} \Big) \times (0, \infty) \Big\},$$

where we define the subsets $X_{+1}, X_{-1} \subseteq \mathbb{R}$ by

$$X_{+1} = (0, \infty), \qquad X_{-1} = X \backslash X_{+1} = (-\infty, 0]. \tag{1.2.11}$$

1.2.9 Dyadic upper half space

For every $d \in \mathbb{N}$, for all $\vec{m} \in \mathbb{Z}^d$, $l \in \mathbb{Z}$, we define the *dyadic tent* or *dyadic cubic box* $E(\vec{m}, l)$ in the upper half space $\mathbb{R}^d \times (0, \infty)$ by

$$E(\vec{m}, l) = E(Q(\vec{m}, l)) = E(B(\vec{m}, l)) \coloneqq Q(\vec{m}, l) \times (0, 2^l].$$

Next, for every $d \in \mathbb{N}$, let

$$\begin{aligned} X_d &= \mathbb{R}^d \times (0, \infty), \\ \mathcal{E}_d &= \left\{ E(\vec{m}, l) \colon \vec{m} \in \mathbb{Z}^d, l \in \mathbb{Z} \right\}, \\ \sigma_d(E(\vec{m}, l)) &= 2^{dl}, && \text{for all } \vec{m} \in \mathbb{Z}^d,\ l \in \mathbb{Z}, \\ \mathrm{d}\omega_d(\vec{y}, t) &= \mathrm{d}\vec{y} \frac{\mathrm{d}t}{t}, && \text{for all } \vec{y} \in \mathbb{R}^d,\ t \in (0, \infty), \end{aligned}$$

and let μ_d be the outer measure on X_d generated via minimal coverings as in (1.1.1) by the pre-measure σ_d on the collection $\mathcal{E}_d$. The setting (X_d, μ_d, ω_d) is σ-finite, and we observe that the σ-algebra of the Carathéodory measurable subsets of X_d with respect to μ_d is generated by the subsets in the collection

$$\left\{ X_{\vec{a}} \colon \vec{a} \in \{-1, 1\}^d \right\},$$

For every $\vec{a} \in \{-1, 1\}^d$, we define the subset $X_{\vec{a}} \subseteq X_d$ by

$$X_{\vec{a}} = \left(\prod_{i=1}^{d} X_{a_i} \right) \times (0, \infty),$$

where the subsets $X_{+1}, X_{-1} \subseteq \mathbb{R}$ are defined in (1.2.11).

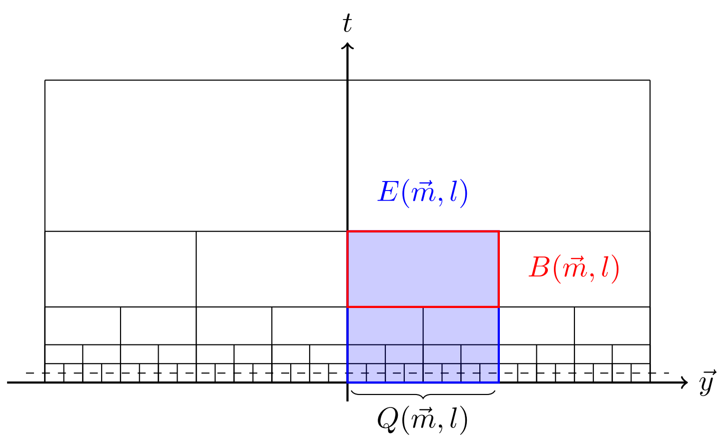

Figure 1.3: Dyadic cube $Q(\vec{m}, l)$, upper half dyadic cubic box $B(\vec{m}, l)$, and dyadic tent $E(\vec{m}, l)$ in the upper half space with coordinates $(\vec{y}, t)$.

1.2.10 Continuous upper half space

For every $d \in \mathbb{N}$, for all $\vec{x} \in \mathbb{R}^d$, $s \in (0,\infty)$, we define the *continuous tent* or *continuous cubic box* $E(\vec{x},s)$ in the upper half space $\mathbb{R}^d \times (0,\infty)$ by

$$E(\vec{x},s) \coloneqq \prod_{i=1}^{d} (x_i, x_i + s] \times (0,s].$$

Next, for every $d \in \mathbb{N}$, let

$$\begin{aligned}
X_d &= \mathbb{R}^d \times (0,\infty), \\
\mathcal{E}_d &= \Big\{ E(\vec{x},s) \colon \vec{x} \in \mathbb{R}^d, s \in (0,\infty) \Big\}, \\
\sigma_d(E(\vec{x},s)) &= s^d, && \text{for all } \vec{x} \in \mathbb{R}^d,\ s \in (0,\infty), \\
\mathrm{d}\omega_d(\vec{y},t) &= \mathrm{d}\vec{y}\frac{\mathrm{d}t}{t}, && \text{for all } \vec{y} \in \mathbb{R}^d,\ t \in (0,\infty),
\end{aligned}$$

and let μ_d be the outer measure on X_d generated via minimal coverings as in (1.1.1) by the pre-measure σ_d on the collection $\mathcal{E}_d$. The setting (X_d, μ_d, ω_d) is σ-finite, and we observe that there are no Carathéodory measurable subsets of X_d with respect to μ_d other than $\{\varnothing, X_d\}$.

1.2.11 Discrete model of the upper half 3-space: Heisenberg dyadic tiles

For all $m, l \in \mathbb{Z}$, we define the *dyadic interval* $I(m,l)$ in $\mathbb{R}$ by

$$I(m,l) \coloneqq (2^l m, 2^l(m+1)],$$

and the collection $\mathcal{I}$ of dyadic intervals in $\mathbb{R}$ by

$$\mathcal{I} \coloneqq \Big\{ I(m,l) \colon m, l \in \mathbb{Z} \Big\}.$$

Moreover, for all $m, n, l \in \mathbb{Z}$, we define the *dyadic rectangle of area* 1 $R(m,n,l)$ in $\mathbb{R}^2$ by

$$R(m,n,l) \coloneqq I(m,l) \times I(n,-l),$$

and the collection $\mathcal{R}$ of dyadic rectangles of area 1 in $\mathbb{R}^2$ by

$$\mathcal{R} \coloneqq \Big\{ R(m,n,l) \colon m, n, l \in \mathbb{Z} \Big\}.$$

Furthermore, for all $m, n, l \in \mathbb{Z}$, we define the *Heisenberg upper half dyadic tile* $H(m,n,l)$ in the upper half 3-space $\mathbb{R}^2 \times (0,\infty)$ by

$$H(m,n,l) \coloneqq H(I(m,l), I(n,-l)) = R(m,n,l) \times (2^{l-1}, 2^l],$$

and the collection $\mathcal{H}$ of Heisenberg upper half dyadic tiles in the upper half 3-space $\mathbb{R}^2 \times (0,\infty)$ by

$$\mathcal{H} \coloneqq \Big\{ H(m,n,l) \colon m,n,l \in \mathbb{Z} \Big\}.$$

Finally, for all $n,l,l' \in \mathbb{Z}$, $l' \leqslant l$, we define $N(n,l') \in \mathbb{Z}$ by the condition

$$I(n,-l) \subseteq I(N(n,l'),-l').$$

Then, for all $m,l \in \mathbb{Z}$, we define the subset $E(m,l) \subseteq \mathcal{H}$ by

$$E(m,l) = E(I(m,l)) \coloneqq \Big\{ H \in \mathcal{H} \colon H \subseteq I(m,l) \times \mathbb{R} \times (0,2^l] \Big\},$$

and, for all $m,n,l \in \mathbb{Z}$, we define the subset $T(m,n,l) \subseteq \mathcal{H}$ by

$$\begin{aligned} T(m,n,l) &= T(I(m,l),I(n,-l)) = T(H(m,n,l)) \\ &\coloneqq \Big\{ H \in \mathcal{H} \colon H \subseteq \bigcup_{l' \in \mathbb{Z}, l' \leqslant l} \Big(I(m,l) \times I(N(n,l'),-l') \times (0,2^{l'}] \Big) \Big\}. \end{aligned}$$

Next, let

$$\begin{aligned} X &= \mathcal{H}, \\ \mathcal{E} &= \Big\{ E(m,l) \colon m,l \in \mathbb{Z} \Big\}, \\ \sigma(E(m,l)) &= 2^l, && \text{for all } m,l \in \mathbb{Z}, \\ \mathcal{T} &= \Big\{ T(m,n,l) \colon m,n,l \in \mathbb{Z} \Big\}, \\ \tau(T(m,n,l)) &= 2^l, && \text{for all } m,n,l \in \mathbb{Z}, \\ \omega(H(m,n,l)) &= 2^l, && \text{for all } m,n,l \in \mathbb{Z}, \end{aligned}$$

and let μ and ν be the outer measures on X generated via minimal coverings as in (1.1.1) by the pre-measures σ and τ on the collections $\mathcal{E}$ and $\mathcal{T}$ respectively. The settings (X,μ,ω), (X,ν,ω), and (X,μ,ν,ω) are σ-finite. Moreover, we observe that there are no Carathéodory measurable subsets of X with respect to μ other than $\{\varnothing, X_+, X_-, X\}$, where the subsets $X_+, X_- \subseteq X$ are defined by

$$X_+ = \Big\{ H \in \mathcal{H} \colon H \subseteq (0,\infty) \times \mathbb{R} \times (0,\infty) \Big\}, \qquad X_- = X \backslash X_+.$$

Furthermore, the σ-algebra of the Carathéodory measurable subsets of X with respect to ν is generated by the subsets $\{X_{+,+}, X_{+,-}, X_{-,+}, X_{-,-}\}$ defined by

$$\begin{aligned} X_{+,+} &= \Big\{ H \in \mathcal{H} \colon H \subseteq (0,\infty) \times (0,\infty) \times (0,\infty) \Big\}, && X_{+,-} = X_+ \backslash X_{+,+}, \\ X_{-,+} &= \Big\{ H \in \mathcal{H} \colon H \subseteq (-\infty,0] \times (0,\infty) \times (0,\infty) \Big\}, && X_{-,-} = X_- \backslash X_{-,+}. \end{aligned}$$

1.2.12 Dyadic upper half 3-space

For all $n, l, l' \in \mathbb{Z}$, $l' \leqslant l$, we define $N(n,l') \in \mathbb{Z}$ by the condition

$$I(n,-l) \subseteq I(N(n,l'),-l').$$

For all $m, l \in \mathbb{Z}$, we define the *dyadic stripe* $E(m,l)$ in the upper half 3-space $\mathbb{R}^2 \times (0,\infty)$ by

$$E(m,l) = E(I(m,l)) \coloneqq I(m,l) \times \mathbb{R} \times (0, 2^l],$$

and, for all $m, n, l \in \mathbb{Z}$, we define the *dyadic tree* $T(m,n,l)$ in the upper half 3-space $\mathbb{R}^2 \times (0,\infty)$ by

$$\begin{aligned} T(m,n,l) &= T(I(m,l), I(n,-l)) = T(H(m,n,l)) \\ &\coloneqq \bigcup_{l' \in \mathbb{Z}, l' \leqslant l} \Big(I(m,l) \times I(N(n,l'),-l') \times (0, 2^{l'}] \Big). \end{aligned}$$

Next, let

$$\begin{aligned} X &= \mathbb{R} \times \mathbb{R} \times (0,\infty), \\ \mathcal{E} &= \Big\{ E(m,l) \colon m, l \in \mathbb{Z} \Big\}, \\ \sigma(E(m,l)) &= 2^l, && \text{for all } m, l \in \mathbb{Z}, \\ \mathcal{T} &= \Big\{ T(m,n,l) \colon m, n, l \in \mathbb{Z} \Big\}, \\ \tau(T(m,n,l)) &= 2^l, && \text{for all } m, n, l \in \mathbb{Z}, \\ \mathrm{d}\omega(y,\eta,t) &= \mathrm{d}y \, \mathrm{d}\eta \, \mathrm{d}t, && \text{for all } y, \eta \in \mathbb{R},\ t \in (0,\infty), \end{aligned}$$

and let μ and ν be the outer measures on X generated via minimal coverings as in (1.1.1) by the pre-measures σ and τ on the collections $\mathcal{E}$ and $\mathcal{T}$ respectively. The settings (X,μ,ω), (X,ν,ω), and (X,μ,ν,ω) are σ-finite. Moreover, we observe that there are no Carathéodory measurable subsets of X with respect to μ other than $\{\varnothing, X_+, X_-, X\}$, where the subsets $X_+, X_- \subseteq X$ are defined by

$$X_+ = (0,\infty) \times \mathbb{R} \times (0,\infty), \qquad X_- = X \backslash X_+.$$

Furthermore, the σ-algebra of the Carathéodory measurable subsets of X with respect to ν is generated by the subsets $\{X_{+,+}, X_{+,-}, X_{-,+}, X_{-,-}\}$ defined by

$$\begin{aligned} X_{+,+} &= (0,\infty) \times (0,\infty) \times (0,\infty), && X_{+,-} = X_+ \backslash X_{+,+}, \\ X_{-,+} &= (-\infty, 0] \times (0,\infty) \times (0,\infty), && X_{-,-} = X_- \backslash X_{-,+}. \end{aligned}$$

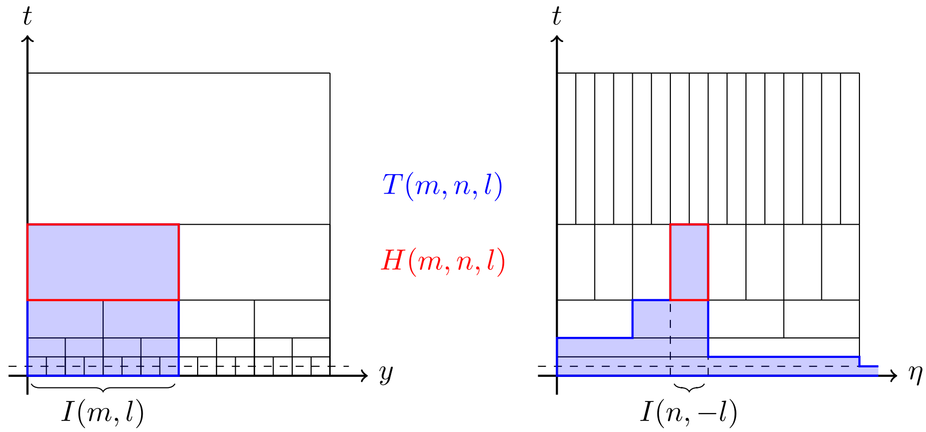

Figure 1.4: Dyadic intervals $I(m,l)$, $I(n,-l)$, Heisenberg upper half dyadic tile $H(m,n,l)$, and dyadic tree $T(m,n,l)$ in the upper half 3-space with coordinates (y,η,t) projected onto the upper half planes with coordinates (y,t) and (η,t) respectively.

For every Heisenberg upper half dyadic tile $H \in \mathcal{H}$, we define the dyadic intervals $I_H, \widetilde{I}_H \in \mathcal{I}$ by

$$I_H \coloneqq \pi(H), \qquad \widetilde{I}_H \coloneqq \widetilde{\pi}(H),$$

where $\pi\colon X \to \mathbb{R}$ is the projection onto the first coordinate and $\widetilde{\pi}\colon X \to \mathbb{R}$ onto the second. Moreover, we define the dyadic tree $T_H \in \mathcal{T}$ by

$$T_H \coloneqq T(I_H, \widetilde{I}_H) = T(H).$$

For every dyadic tree $T \in \mathcal{T}$, we define the dyadic intervals $I_T, \widetilde{I}_T \in \mathcal{I}$ by

$$I_T \coloneqq \pi(T), \qquad |\widetilde{I}_T| \coloneqq |I_T|^{-1}, H(I_T, \widetilde{I}_T) \subseteq T,$$

where $\pi\colon X \to \mathbb{R}$ is the projection onto the first coordinate. Moreover, we define the Heisenberg upper half dyadic tile $H_T \in \mathcal{H}$ by

$$H_T \coloneqq H(I_T, \widetilde{I}_T).$$

1.2.13 Continuous upper half 3-space

For all $x \in \mathbb{R}$, $s \in (0,\infty)$, we define the *continuous stripe* $E(x,s)$ in the upper half 3-space $\mathbb{R}^2 \times (0,\infty)$ by

$$E(x,s) \coloneqq (x, x+s] \times \mathbb{R} \times (0,s],$$

and, for all $x, \xi \in \mathbb{R}$, $s \in (0,\infty)$, we define the *continuous tree* $T(x,\xi,s)$ in the upper half 3-space $\mathbb{R}^2 \times (0,\infty)$ by

$$T(x,\xi,s) \coloneqq \bigcup_{s' \in (0,s]} \Big((x, x+s] \times \Big(\xi - \frac{1}{s'}, \xi + \frac{1}{s'}\Big] \times (0,s'] \Big).$$

Next, let

$$\begin{aligned}
X &= \mathbb{R} \times \mathbb{R} \times (0, \infty), \\
\mathcal{E} &= \Big\{ E(x,s) \colon x \in \mathbb{R}, s \in (0,\infty) \Big\}, \\
\sigma(E(x,s)) &= s, && \text{for all } x \in \mathbb{R},\ s \in (0,\infty), \\
\mathcal{T} &= \Big\{ T(x,\xi,s) \colon x, \xi \in \mathbb{R}, s \in (0,\infty) \Big\}, \\
\tau(T(x,\xi,s)) &= s, && \text{for all } x, \xi \in \mathbb{R},\ s \in (0,\infty), \\
\mathrm{d}\omega(y,\eta,t) &= \mathrm{d}y\, \mathrm{d}\eta\, \mathrm{d}t, && \text{for all } y, \eta \in \mathbb{R},\ t \in (0,\infty),
\end{aligned}$$

and let μ and ν be the outer measures on X generated via minimal coverings as in (1.1.1) by the pre-measures σ and τ on the collections $\mathcal{E}$ and $\mathcal{T}$ respectively. The settings (X, μ, ω), (X, ν, ω), and (X, μ, ν, ω) are σ-finite, and we observe that there are no Carathéodory measurable subsets of X with respect to μ or ν other than $\{\varnothing, X\}$.

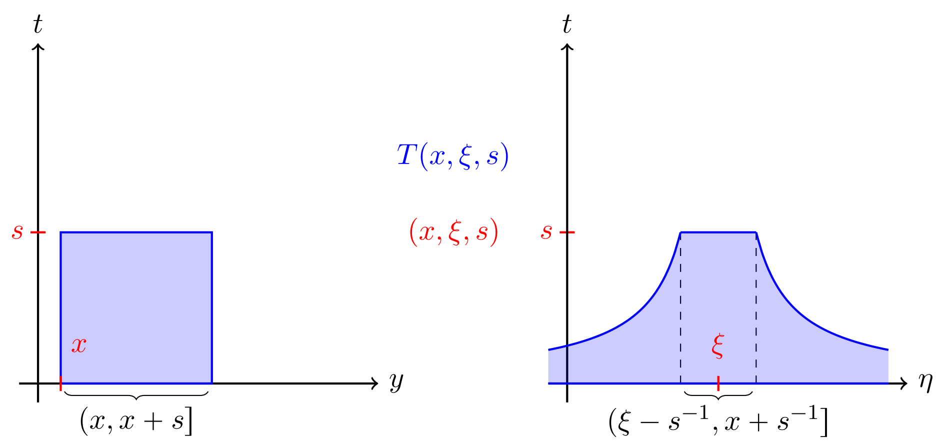

Figure 1.5: Continuous tree $T(x,\xi,s)$ in the upper half 3-space with coordinates (y,η,t) projected onto the two upper half planes with coordinates (y,t) and (η,t) respectively.

Remark 1.2.3. *In the settings on the upper half* 3*-space or its discrete model described in Subsections 1.2.11 – 1.2.13, the elements of* $\mathcal{E}$ *satisfy the sufficient compatibility condition with respect to the outer measure* ν *stated in Remark 1.2.1. Therefore, we can prove that, for all* $p, q, r \in (0, \infty]$*, there exists a constant* $C = C(p,q,r)$ *such that, for every measurable function* f *on* X*, we have*

$$C^{-1} \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \|f\|_{L^p_\mu(\ell^q_{\nu,\sigma}(\ell^r_\omega))} \leqslant C \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))},$$

the double iterated outer L^p *quasi-norms with respect to the two sizes are equivalent, and the double iterated outer* L^p *spaces with respect to the two sizes are equal.*

1.3 Main results: further properties of outer L^p spaces

We turn to the study of the Banach space properties of the outer L^p spaces. In our investigation, we keep as a point of reference the properties of the mixed L^p quasi-norms and spaces on the Cartesian product of σ-finite measure spaces. As we saw in the description of the setting on the Cartesian product of σ-finite measure spaces in Subsection 1.2.4, they can be exhibited as outer L^p quasi-norms and spaces.

The mixed L^p quasi-norms and spaces on the Cartesian product of σ-finite measure spaces are well-studied mathematical objects, see for example the article of Benedek and Panzone [BP61]. They satisfy many properties, other than those already listed in the previous sections, for example the following ones.

(i) **Collapsing of exponents.** For every $p \in (0, \infty]$, for every measurable function f on $Y \times Z$, we have

$$\| \| f \|_{L^p(Z,\omega_Z)} \|_{L^p(Y,\omega_Y)} = \| f \|_{L^p(Y \times Z,\omega)},$$

where ω is the canonical product measure on $Y \times Z$ associated with ω_Y and ω_Z. This property is Fubini's Theorem, more precisely the Fubini-Tonelli Theorem, see for example the book of Rudin [Rud74].

(ii) **Köthe duality.** For all $p, r \in [1, \infty]$, for every measurable function f on $Y \times Z$, we have

$$\| \| f \|_{L^r(Z,\omega_Z)} \|_{L^p(Y,\omega_Y)} = \sup \Big\{ \| fg \|_{L^1(Y \times Z,\omega)} \colon \| \| g \|_{L^{r'}(Z,\omega_Z)} \|_{L^{p'}(Y,\omega_Y)} = 1 \Big\},$$

where ω is the canonical product measure on $Y \times Z$ associated with ω_Y and ω_Z.

(iii) **Triangle inequality.** For all $p, r \in [1, \infty]$, for every collection $\{ f_n \colon n \in \mathbb{N} \}$ of measurable functions on $Y \times Z$, we have

$$\Big\| \Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^r(Z,\omega_Z)} \Big\|_{L^p(Y,\omega_Y)} \leqslant \sum_{n \in \mathbb{N}} \| \| f_n \|_{L^r(Z,\omega_Z)} \|_{L^p(Y,\omega_Y)}.$$

(iv) **Minkowski's inequality.** For all $p, r \in (0, \infty]$, $p \geqslant r$, for every measurable function f on $Y \times Z$, we have

$$\| \| f \|_{L^r(Z,\omega_Z)} \|_{L^p(Y,\omega_Y)} \leqslant \| \| f \|_{L^p(Y,\omega_Y)} \|_{L^r(Z,\omega_Z)}.$$

It is then natural to ask whether these properties hold true also for the outer L^p quasi-norms and spaces on more general settings. In particular, we allow for the equalities and inequalities with constant 1 in the previous four displays to be replaced by equivalences and inequalities up to constants that may depend on the exponents p, r, but not on the setting (X, μ, ω). Namely, we ask whether, for all $p, r \in (0, \infty]$, there exists a constant $C = C(p, r)$ such that, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the following properties hold true.

(i) **Collapsing of exponents.** For every $p \in (0,\infty]$, for every measurable function f on X, we have

$$C^{-1}\|f\|_{L^p_\mu(\ell^p_\omega)} \leqslant \|f\|_{L^p(X,\omega)} \leqslant C\|f\|_{L^p_\mu(\ell^p_\omega)}. \tag{\textbf{CoE}}$$

(ii) **Köthe duality.** For all $p, r \in [1,\infty]$, for every measurable function f on X, we have

$$C^{-1}\|f\|_{L^p_\mu(\ell^r_\omega)} \leqslant \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^{r'}_\omega)} = 1\Big\} \leqslant C\|f\|_{L^p_\mu(\ell^r_\omega)}. \tag{\textbf{KD}}$$

(iii) **Quasi-triangle inequality for countably many summands.** For all $p, r \in [1,\infty]$, for every collection $\{f_n \colon n \in \mathbb{N}\}$ of measurable functions on X, we have

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^p_\mu(\ell^r_\omega)} \leqslant C\sum_{n\in\mathbb{N}}\|f_n\|_{L^p_\mu(\ell^r_\omega)}. \tag{\textbf{qTI}}$$

(iv) **Minkowski's inequality.** There exists an outer measure υ on X such that, for all $p, r \in (0,\infty]$, $p \geqslant r$, for every measurable function f on X, we have

$$\|f\|_{L^p_\mu(\ell^r_\omega)} \leqslant C\|f\|_{L^r_\upsilon(\ell^p_\omega)}, \qquad \|f\|_{L^p_\upsilon(\ell^r_\omega)} \leqslant C\|f\|_{L^r_\mu(\ell^p_\omega)}. \tag{\textbf{MI}}$$

The use of the term Köthe duality in this context is a slight abuse. In general, the Köthe dual space is a notion defined with respect to Banach function spaces on a measure space, see for example the books of Bennett and Sharpley [BS88], Lindenstrauss and Tzafriri [LT79]. A Banach function space $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ on a σ-finite measure space (X,ω) is defined by three conditions. First, it is a normed space. Next, for all measurable functions f and g on X, if g belongs to $\mathcal{L}$ and f is bounded by g in absolute value ω-almost everywhere, then f belongs to $\mathcal{L}$ and $\|f\|_{\mathcal{L}} \leqslant \|g\|_{\mathcal{L}}$. Moreover, for every measurable subset $E \subseteq X$ such that $\omega(E)$ is finite, then its characteristic function 1_E belongs to $\mathcal{L}$. Now, the outer $L^p_\mu(\ell^r_\omega)$ quasi-norms are not norms, a priori. However, the second property defining a Banach function space is satisfied by the outer $L^p_\mu(\ell^r_\omega)$ spaces. Finally, the collection $\{1_E \colon E \subseteq X, \omega(E) < \infty\}$ may not be contained in the outer $L^p_\mu(\ell^r_\omega)$ spaces. Nevertheless, in the case of σ-finite settings we can prove that there exists a countable subcollection of $\{1_E \colon E \subseteq X, \omega(E) < \infty, \mu(E) < \infty\}$ contained in the outer $L^p_\mu(\ell^r_\omega)$ spaces, and whose corresponding subsets cover X.

We point out that the properties listed above are ordered. Collapsing of exponents in (**CoE**) implies Köthe duality in (**KD**) for the exponents $p = r \in [1,\infty]$. Moreover, collapsing of exponents in (**CoE**) for $p = 1$ and Köthe duality in (**KD**) for a couple of exponents $p, r \in [1,\infty]$ imply the quasi-triangle inequality in (**qTI**) for the same exponents. Finally, the quasi-triangle inequality in (**qTI**) for a couple of exponents $p, r \in [1,\infty]$ is a special case of Minkowski's inequality in (**MI**) for certain double iterated outer L^p spaces. Namely, the quasi-triangle inequality for the single iterated outer $L^p_\mu(\ell^r_\omega)$ spaces on the setting (X,μ,ω) is a form of Minkowski's inequality for the double iterated outer $L^p_{\widetilde{\mu}}(\ell^r_{\widetilde{\omega}}(\ell^1_\rho))$ spaces on the setting $(X \times \mathbb{N}, \widetilde{\mu} = \mu \circ \pi_X, \widetilde{\omega} = \omega \circ \pi_X, \rho)$, where ρ is the canonical product measure on $X \times \mathbb{N}$ associated with ω and the counting measure on $\mathbb{N}$.

Properties analogous to those in (**CoE**) – (**MI**) can be investigated in the case of iterated outer L^p quasi-norms and spaces for any arbitrary degree of iteration. We point out that, as the degree of iterations increases, multiple different phenomena of collapsing of exponents and Minkowski's inequality are possible. Given a collection $\{p_n \in (0,\infty] \colon n \in \mathbb{N}\}$ of exponents and a collection $\{\mu_n \colon n \in \{0\} \cup \mathbb{N}\}$ of outer measures on a measure space (X,ω) such that $\mu_0 = \omega$, we list some properties that can be investigated.

(i') **Collapsing of exponents.** Fix $i, j \in \mathbb{N}$, $i > j$. There exist an outer measure υ on X such that, for every $p \in (0,\infty]$, for every measurable function f on X, we have

$$C^{-1}\|f\|_{L^{p_n}_{\mu_n}(\dots\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p}_{\mu_i}(\ell^{p}_{\mu_{i-1}}(\dots\ell^{p}_{\mu_j}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega}))))))} \leqslant \|f\|_{L^{p_n}_{\mu_n}(\dots(\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p}_{\upsilon}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega})))))},$$
$$\|f\|_{L^{p_n}_{\mu_n}(\dots(\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p}_{\upsilon}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega})))))} \leqslant C\|f\|_{L^{p_n}_{\mu_n}(\dots\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p}_{\mu_i}(\ell^{p}_{\mu_{i-1}}(\dots\ell^{p}_{\mu_j}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega}))))))}.$$

(iv') **Minkowski's inequality.** Fix $i, j \in \mathbb{N}$, $i > j$. There exist a collection $\{\upsilon_k \colon j \leqslant k \leqslant i\}$ of outer measures on X such that, for all $p_i, p_{i-1}, \dots, p_j \in (0,\infty]$, $p_i \geqslant p_{i-1}, \dots, p_j$, for every measurable function f on X, we have

$$\begin{aligned}
&\|f\|_{L^{p_n}_{\mu_n}(\dots\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p_i}_{\mu_i}(\dots\ell^{p_j}_{\mu_j}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega})))))} \leqslant \\
&\qquad\qquad \leqslant C\|f\|_{L^{p_n}_{\mu_n}(\dots\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p_{i-1}}_{\upsilon_i}(\dots\ell^{p_j}_{\upsilon_{j+1}}(\ell^{p_i}_{\upsilon_j}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega}))))))}, \\
&\|f\|_{L^{p_n}_{\mu_n}(\dots\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p_i}_{\upsilon_i}(\dots\ell^{p_j}_{\upsilon_j}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega})))))} \leqslant \\
&\qquad\qquad \leqslant C\|f\|_{L^{p_n}_{\mu_n}(\dots\ell^{p_{i+1}}_{\mu_{i+1}}(\ell^{p_{i-1}}_{\mu_i}(\dots\ell^{p_j}_{\mu_{j+1}}(\ell^{p_i}_{\mu_j}(\ell^{p_{j-1}}_{\mu_{j-1}}(\dots\ell^{p_0}_{\omega}))))))}.
\end{aligned}$$

Some partial results about the properties in (**CoE**) – (**MI**) in the case of single iterated outer L^p spaces follow from the propositions proved by Do and Thiele in [DT15]. In particular, the second inequality in (**CoE**) follows from the Radon-Nikodym type result for the outer L^p quasi-norms (Theorem 1.1.8), while the second inequality in (**KD**) follows from the Radon-Nikodym type result for the outer L^1 quasi-norms and outer Hölder's inequality (Theorem 1.1.7).

In the remaining part of the Introduction, we discuss the original results contained in this thesis.

For single iterated outer L^p spaces, we have the following result, whose main point is the uniformity in the setting (X,μ,ω) of the constants associated with collapsing of exponents, Köthe duality, quasi-triangle inequality, and Minkowski's inequality. The first three properties of this result are discussed in the case of a finite setting in Theorem 2.1.1 in Chapter 2, and extended to the case of a σ-finite setting in Theorem 4.2.1 in Chapter 4. The fourth property is discussed in Theorem 4.3.9 in Chapter 4.

Theorem 1.3.1. *For all $p, r \in (0, \infty]$, there exists a constant $C = C(p, r)$ such that, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the following properties hold true.*

(i) For every $p \in (0, \infty]$, for every measurable function f on X, we have (**CoE**)*.*

(ii) For all $p \in (1, \infty]$, $r \in [1, \infty)$ or $p = r \in \{1, \infty\}$, for every measurable function f on X, we have (**KD**)*.*

(iii) For all $p \in (1, \infty]$, $r \in [1, \infty)$ or $p = r \in \{1, \infty\}$, for every collection $\{f_n : n \in \mathbb{N}\}$ of measurable functions on X, we have (**qTI**)*.*

(iv) There exists an outer measure $\upsilon = \upsilon(\mu, \omega)$ on X such that, for all $p, r \in (0, \infty]$, $p \geqslant r$, for every measurable function f on X, we have (**MI**)*.*

It is worth noting that the outer measure υ associated with (**MI**) has a better subadditivity behaviour than general outer measures, as proved in Lemma 4.3.12 in Chapter 4.

The main ingredient in the proof of Theorem 1.3.1 is a recursive greedy selection algorithm providing a sequence of disjoint subsets of X satisfying the following properties. For every level $\lambda \in (0, \infty)$, they exhaust the subsets where the size is bigger than λ, but at the same time they guarantee a lower bound on the super level measure associated with $C\lambda$.

In general, in the remaining cases for the exponents p, r we have counterexamples to the existence of a constant in (**KD**) and (**qTI**) that is uniform in the setting (X, μ, ω). This is showed in Lemma 2.3.4 in Chapter 2 for all $p = 1$, $r \in (1, \infty]$, and in Lemma 4.2.5 in Chapter 4 for all $p \in [1, \infty)$, $r = \infty$. In fact, uniformity in the setting of the constant in a weak version of (**qTI**) for $p = 1$, $r = \infty$ is equivalent to a certain subadditivity condition on the outer measure μ, as proved in Lemma 4.2.7 in Chapter 4.

However, not all the counterexamples can be reproduced on the settings on the upper half space and upper half 3-space or their discrete models described in Subsections 1.2.8 – 1.2.13. Therefore, we recover (**KD**) and (**qTI**) for the single iterated outer L^p spaces in at least some of the endpoint cases for the exponents p, r. These results are discussed in Theorem 2.1.2 in Chapter 2 in the case of the upper half space setting and its discrete model, and in Theorem 4.4.1 in Chapter 4 in the case of the upper half 3-space setting and its discrete model.

Theorem 1.3.2. *Let (X_d, μ_d, ω_d) be any setting on the upper half space or its discrete model described in Subsections 1.2.8 – 1.2.10. Then, for all $p, r \in [1, \infty]$, there exists a constant $C = C(p, r)$ such that we have* (**KD**) *and* (**qTI**)*.*

Let (X, ν, ω) be any setting on the upper half 3-space or its discrete model described in Subsections 1.2.11 – 1.2.13. Then, for all $p \in (1, \infty]$, $r \in [1, \infty]$ or $p = r = 1$, there exists a constant $C = C(p, r)$ such that we have (**KD**) *and* (**qTI**)*. For all $p = 1$, $r \in (1, \infty]$, we have a counterexample to the existence of a constant in* (**KD**) *and* (**qTI**)*.*

The result for the single iterated outer L^p spaces on the upper half space settings allows us to relate them with the more classical notion of tent spaces introduced by Coifman, Meyer, and Stein in [CMS83, CMS85]. The latter are spaces of functions on the upper half space defined by quasi-norms that are akin to mixed L^p, although strictly speaking they are not mixed L^p quasi-norms. We refer to the original papers or the introduction of Chapter 2 for their definition. The Köthe duality result for outer $L^p_\mu(\ell^r_\omega)$ and tent T^p_r spaces, together with easy comparability between the quasi-norms for certain exponents p, r, implies the equivalence between the two notions for all $p, r \in (0, \infty]$. This result is discussed in Theorem 2.1.3 in Chapter 2.

Theorem 1.3.3. *For all $p, r \in (0, \infty]$, there exists a constant $C = C(p, r)$ such that, for every setting (X_d, μ_d, ω_d) on the upper half space described in Subsections 1.2.9 – 1.2.10, for every measurable function f on X_d, we have*

$$C^{-1}\|f\|_{T^p_r} \leqslant \|f\|_{L^p_{\mu_d}(\ell^r_{\omega_d})} \leqslant C\|f\|_{T^p_r},$$

hence $L^p_{\mu_d}(\ell^r_{\omega_d}) = T^p_r$.

The result for the outer L^p spaces on the upper half 3-space settings in the case of a size ℓ^r_ω leads us to consider the case of the size S with variable exponent appearing in the article of Do and Thiele [DT15]. The size S is of the form of a sum of sizes ℓ^∞_ω and ℓ^2_ω restricted to certain subsets of each tree in the upper half 3-space. The proof strategy used for Theorem 1.3.1 cannot be adapted to the case of the outer L^p spaces with respect to S or any of its components. In fact, for the outer $L^p_\nu(S)$ spaces we exhibit counterexamples to the existence of a constant in a version of Köthe duality with an appropriate dual size S', see Lemma 4.5.2 in Chapter 4. For the outer L^p spaces with respect to each of the components of S, we could still prove a version of Köthe duality, but we would have to substitute the classical $L^1(X, \omega)$ norm used to measure the product of functions. In particular, we would substitute it with a quasi-norm that does not satisfy quasi-triangle inequality for countably many summands, see Lemma 4.5.1. Therefore, the outer L^p spaces would not inherit such property.

For arbitrary iterated outer L^p spaces, one would hope to be able to apply the same arguments used to prove Theorem 1.3.1 recursively to obtain the desired uniform results. For example, let $\{\mu_i : i \in \mathbb{N}\}$ be a collection of outer measures on a measure space (X, ω) such that, for every $i \in \mathbb{N}$, the setting (X, μ_i, ω) is σ-finite. Moreover, let $p \in (0, \infty]$. Applying the result in (**CoE**) to the setting (X, μ_1, ω), then, for every $n \in \mathbb{N}$, for every measurable function f on X, we have

$$C(p)^{-1}\|f\|_{L^p_{\mu_n}(\dots\ell^p_{\mu_2}(\ell^p_{\mu_1}(\ell^p_\omega)))} \leqslant \|f\|_{L^p_{\mu_n}(\dots\ell^p_{\mu_2}(\ell^p_\omega))} \leqslant C(p)\|f\|_{L^p_{\mu_n}(\dots\ell^p_{\mu_2}(\ell^p_{\mu_1}(\ell^p_\omega)))},$$

and iterating the application of the same result changing recursively the setting (X, μ_i, ω), we obtain

$$C(p)^{-n}\|f\|_{L^p_{\mu_n}(\dots\ell^p_{\mu_i}(\dots\ell^p_\omega))} \leqslant \|f\|_{L^p(X,\omega)} \leqslant C(p)^n\|f\|_{L^p_{\mu_n}(\dots\ell^p_{\mu_i}(\dots\ell^p_\omega))},$$

where the constant $C(p)$ is independent of f, n, X, ω, and $\{\mu_i : i \in \mathbb{N}\}$. However, in general the iteration of the same arguments is not going to be possible, and we have to be more careful in our analysis.

Already in the case of double iterated outer L^p spaces, to recover uniform results at least in a class of settings, we need to require additional conditions. Trying to replicate the same arguments used to prove Theorem 1.3.1, we face a problem given by the lack of q-orthogonality between the outer $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions with arbitrary disjoint supports. For single iterated outer L^p spaces, the corresponding property is the r-orthogonality between the classical $L^r(X,\omega)$ quasi-norms of functions with arbitrary disjoint supports. This property is easily verified by the additivity of the integral associated with the measure ω.

Instead, for double iterated outer L^p spaces, according to the cases $q > r$ or $q < r$, only one between a sub- and super-q-orthogonality results still holds true. However, to replicate the arguments used to prove Theorem 1.3.1 to a complete extent, we need the full q-orthogonality. Such a property depends on the compatibility between the (sub)additive behaviours of the outer measures μ and ν, and the compatibility is the subject of the additional conditions. In fact, the necessity of some additional conditions, at least in a certain open range of exponents $p, q, r \in (1, \infty)$, is not an artefact of the proof strategy we pursue, as exhibited by a collection of counterexamples in Chapter 3.

To state the additional conditions, we need to introduce some auxiliary definitions. They depend on two parameters $\Phi, K \geqslant 1$. First, given a subset $A \subseteq X$, we say that a subset $B \subseteq X$ is a *μ-parent set of A (with parameter Φ)* if $A \subseteq B$ and we have

$$\mu(B) \leqslant \Phi\mu(A).$$

A *μ-parent function* $\mathbf{B}$ *(with parameter Φ)* is then a monotone function from $\mathcal{P}(X)$ to itself, associating every subset $A \subseteq X$ with a μ-parent set (with parameter Φ) $\mathbf{B}(A)$.

Moreover, given a collection $\mathcal{E}$ of subsets of X, we say that a function $\mathcal{C}$ from $\mathcal{P}(X)$ to the set of subcollections of pairwise disjoint elements in $\mathcal{E}$ is a *μ-covering function (with parameter Φ)* if the function $\mathbf{B}_{\mathcal{C}}$ from $\mathcal{P}(X)$ to itself defined by

$$\mathbf{B}_{\mathcal{C}}(A) = \bigcup_{E \in \mathcal{C}(A)} E,$$

is a μ-parent function (with parameter Φ).

Next, we say that a collection $\mathcal{A}$ of pairwise disjoint subsets of X is *ν-Carathéodory (with parameter K)* if, for every subset $U \subseteq X$, we have

$$\sum_{A \in \mathcal{A}} \nu(U \cap A) \leqslant K \nu\Big(U \cap \bigcup_{A \in \mathcal{A}} A\Big).$$

In particular, we observe that the classical Carathéodory measurability test for a subset E with respect to an outer measure μ corresponds to checking that the couple of disjoint subsets $\{E, E^c\}$ is μ-Carathéodory with parameter $K = 1$.

Finally, we define two conditions for the quadruple $(X, \mu, \nu, \mathcal{C})$.

Condition 1.3.4 (Canopy). *We say that $(X, \mu, \nu, \mathcal{C})$ satisfies the* canopy condition (with parameters Φ, K) *if $\mathcal{C}$ is a μ-covering function with parameter Φ, and, for every collection $\mathcal{A}$ that is a ν-Carathéodory collection with parameter K, for every subset $D \subseteq X$ disjoint from $\mathbf{B}_{\mathcal{C}}(\bigcup_{A \in \mathcal{A}} A)$, the collection $\mathcal{A} \cup \{D\}$ is still ν-Carathéodory with the same parameter K.*

Condition 1.3.5 (Crop). *We say that $(X, \mu, \nu, \mathcal{C})$ satisfies the* crop condition (with parameters Φ, K) *if $\mathcal{C}$ is a μ-covering function with parameter Φ, and, for every collection $\mathcal{A} \subseteq \mathcal{E}$, there exists a subcollection $\mathcal{D} \subseteq \mathcal{A}$ that is a ν-Carathéodory collection with parameter K and such that, for every subset $F \subseteq X$ disjoint from $\bigcup_{D \in \mathcal{D}} D$, we have*

$$\mathbf{B}_{\mathcal{C}}(F) = \mathbf{B}_{\widetilde{\mathcal{C}}}(F),$$

where

$$\widetilde{\mathcal{C}}(F) = \mathcal{C}(F) \backslash \mathcal{A}.$$

In particular, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the quadruple $(X, \mu, \omega, \mathrm{Id})$ satisfies the canopy and crop conditions with parameters $\Phi = K = 1$.

For double iterated outer L^p spaces, we have the following result, whose main point, once again, is the dependence of the constants on the setting and their uniformity up to additional conditions. This result is discussed in Theorem 3.1.3 and Theorem 3.1.4 in Chapter 3, and Theorem 4.6.1 in Chapter 4.

Theorem 1.3.6. *For all $p, q, r \in (0, \infty]$, $\Phi, K \geqslant 1$, there exist constants $C_1 = C_1(q, r, \Phi, K)$, $C_2 = C_2(q, r, \Phi, K)$, $C = C(p, q, r, \Phi, K)$ such that, for every finite setting (X, μ, ν, ω) described in Subsection 1.2.2, for every μ-covering function $\mathcal{C}$, the following properties hold true.*

(i) If $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 1.3.4, then for every function f on X, we have

$$C_1^{-1} \|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C_2 \|f\|_{L^q_\nu(\ell^r_\omega)}.$$

If $q < r$ or $q = \infty$, the constant C_1 does not depend on Φ, K. If $q > r$, the constant C_2 does not depend on Φ, K.

If $q = r \in (0, \infty]$, the constants C_1, C_2 do not depend on Φ, K.

(ii) If $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 1.3.4, then for all $p, q, r \in (1, \infty)$, $q \leqslant r$, for every function f on X, we have

$$C^{-1} \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \sup\left\{ \|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1 \right\} \leqslant C \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

If $(X, \mu, \nu, \mathcal{C})$ satisfies the crop condition 1.3.5, then for all $p, q, r \in (1, \infty)$, $q \geqslant r$, the same inequality holds true.

Moreover, if $p, q, r \in [1, \infty]$ *satisfy one of the following conditions*

$$p = \infty, q \in (1, \infty), r \in (q, \infty),$$
$$p \in (1, \infty], q = r \in [1, \infty),$$
$$p = q = r \in \{1, \infty\},$$

the constant C *does not depend on* Φ*,* K*.*

(iii) If $(X, \mu, \nu, \mathcal{C})$ *satisfies the canopy condition 1.3.4, then for all* $p, q, r \in (1, \infty)$*,* $q \leqslant r$*, for every collection* $\{f_n \colon n \in \mathbb{N}\}$ *of functions on* X*, we have*

$$\| \sum_{n \in \mathbb{N}} f_n \|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \sum_{n \in \mathbb{N}} \| f_n \|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

If $(X, \mu, \nu, \mathcal{C})$ *satisfies the crop condition 1.3.5, then for all* $p, q, r \in (1, \infty)$*,* $q \geqslant r$*, the same inequality holds true.*

Moreover, if $p, q, r \in [1, \infty]$ *satisfy one of the following conditions*

$$p = \infty, q \in (1, \infty], r \in [1, \infty),$$
$$p \in (1, \infty], q = r \in [1, \infty),$$
$$p = q = r \in \{1, \infty\},$$

the constant C *does not depend on* Φ*,* K*.*

We point out the dichotomy between the cases $q > r$ and $q < r$ and its relation with the canopy and crop conditions. For collapsing of exponents, the distinction between the cases is clarified by counterexamples exhibiting the failure of the uniformity in Φ, K of either of the two constants C_1 and C_2. We refer to Subsection 3.3.4 in Chapter 3 for these counterexamples. For Köthe duality, the distinction could be just an artefact of the proof strategy. It would be interesting to understand more clearly the nature of necessary and sufficient conditions to obtain uniformity of the constants in collapsing of exponents and Köthe duality.

In particular, for every σ-finite setting on the upper half 3-space or its discrete model described in Subsections 1.2.11 – 1.2.13, both the canopy and crop conditions are satisfied, and the properties stated in Theorem 1.3.6 still hold true. This result is discussed in Theorem 3.1.5 in Chapter 3.

The remaining part of this thesis is organized into three Chapters.
In Chapter 2, we report the article [Fra21].

Marco Fraccaroli. Duality for outer $L^p_\mu(\ell^r)$ spaces and relation to tent spaces. *J. Fourier Anal. Appl.*, 27(4):Paper No. 67, 48, 2021.

In Chapter3, we report the article [Fra22].

Marco Fraccaroli. Duality for double iterated outer L^p spaces.

in the revised version accepted for forthcoming publication in *Studia Mathematica*.
In Chapter 4, we prove additional properties of outer L^p spaces and we collect some open conjectures.

Notation

For every measure space (X, ω), for every $p \in (0, \infty]$, the notation $\|f\|_{L^p(X,\omega)}$ stands for the classical L^p quasi-norm.

For every $p \in [1, \infty]$, the notation p' stands for the Hölder's conjugate exponent, namely $p' \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Unless explicitly stated otherwise, a constant C is a finite strictly positive real number, namely $C \in (0, \infty)$.

Unless explicitly stated otherwise, the notation $A \sim_p B$ means that there exists a constant $C = C(p)$ such that $A \leqslant CB$ and $B \leqslant CA$.

We denote by $\mathbb{N}$ the set of strictly positive integer numbers, namely

$$\mathbb{N} = \{1, 2, \dots, n, n+1, \dots\}.$$

In particular, the number 0 does not belong to $\mathbb{N}$.

Unless explicitly stated otherwise, the elements of a double sequence are parametrized by pairs (k, n) with $k \in \mathbb{Z}$, $n \in \mathbb{N}_k$, where $\mathbb{N}_k$ is either $\mathbb{N}$ or a finite initial string of it, possibly empty. On the set of couples we consider the lexicographic order as follows: $(l, m) < (k, n)$ if either $l > k$, or $l = k$, $m < n$.

Chapter 2

Single iterated outer L^p space

In this chapter, we report the article [Fra21].

Marco Fraccaroli. Duality for outer $L^p_\mu(\ell^r)$ spaces and relation to tent spaces. *J. Fourier Anal. Appl.*, 27(4):Paper No. 67, 48, 2021.

At the end of the chapter, we collect some typos discovered after the publication.

Abstract

We study the outer L^p spaces introduced by Do and Thiele on sets endowed with a measure and an outer measure. We prove that, in the case of finite sets, for $1 < p \leqslant \infty, 1 \leqslant r < \infty$ or $p = r \in \{1, \infty\}$, the outer $L^p_\mu(\ell^r)$ quasi-norms are equivalent to norms up to multiplicative constants uniformly in the cardinality of the set. This is obtained by showing the expected duality properties between the corresponding outer $L^p_\mu(\ell^r)$ spaces uniformly in the cardinality of the set. Moreover, for $p = 1, 1 < r \leqslant \infty$, we exhibit a counterexample to the uniformity in the cardinality of the finite set. We also show that in the upper half space setting the desired properties hold true in the full range $1 \leqslant p, r \leqslant \infty$. These results are obtained via greedy decompositions of functions in the outer $L^p_\mu(\ell^r)$ spaces. As a consequence, we establish the equivalence between the classical tent spaces T^p_r and the outer $L^p_\mu(\ell^r)$ spaces in the upper half space. Finally, we give a full classification of weak and strong type estimates for a class of embedding maps to the upper half space with a fractional scale factor for functions on $\mathbb{R}^d$.

2.1 Introduction

A classical research topic in harmonic analysis is the study of linear and multilinear operators defined on functions on $\mathbb{R}^d$ and satisfying certain symmetries. It is the case of Calderón-Zygmund theory, when the symmetries are given by translations and dilations, and time-frequency analysis, when additional modulation symmetries are included. The symmetries are parametrized by the upper half space $\mathbb{R}^d \times (0, \infty)$ in the first case, and the upper half 3-space $\mathbb{R} \times (0, \infty) \times \mathbb{R}$ in the second. In fact, in both cases we can use a wave packet decomposition to encode the information of a function on $\mathbb{R}^d$ in the space parametrizing the specific symmetries.

In [DT15], the authors introduced in both the previous settings a new type of function spaces, the so called outer L^p spaces. These spaces were defined via quasi-norms with a structure reminiscent of the iteration of classical Lebesgue norms. The purpose was to formalize a paradigm in proving the boundedness of operators in time-frequency analysis by a two-step program. In particular, the program consisted of a version of Hölder inequality for outer L^p spaces followed by estimates from classical to outer L^p spaces on the embedding maps associated with wave packet decompositions. This is for example the case of the bilinear Hilbert transform in [AU20b, DPO18b, DT15], the variational Carleson operator in [DPDU18, Ura16], the variational bilinear iterated Fourier inversion operator in [DMT17], a family of trilinear multiplier forms with singularity over a one-dimensional subspace in [CDPO18], and the uniform bilinear Hilbert transform in [War18]. Analogous applications of the outer L^p spaces framework in other settings with different geometries can be found in [AU20a], [DPGTZK18], [DPO18a], [DT15], [MT17], [TTV15].

Moreover, in [DT15] the authors pointed out that the two-step program outlined above, when applied to the outer L^p spaces on $\mathbb{R}^d \times (0, \infty)$, recovers some results of classical Calderón-Zygmund theory, as detailed for example in [Ste70, Ste93]. In fact, in this particular setting, the outer L^p spaces are competing with the more classical tent spaces introduced in [CMS83, CMS85]. The tent spaces are defined by iterated Lebesgue norms, and they have been thoroughly studied and used in the literature. Due to the many analogies in their definition and use, the equivalence between the outer L^p spaces and the tent ones has been conjectured since the publication of [DT15] but never formally established. We prove the equivalence in Theorem 2.1.3.

In order to formalize the two-step program described above, in [DT15], the authors developed the framework of the outer L^p spaces focusing on their real interpolation features, such as Marcinkiewicz interpolation and Hölder's inequality, while other aspects of the theory of these spaces remained untouched. For example, whether the outer L^p quasi-norms are equivalent to norms, or whether they can be recovered as a supremum of a pairing with functions in another appropriate outer $L^{p'}$ space.

Already these simple questions turn out to be difficult. We begin their study in this paper from the case of the outer L^p spaces of functions on $\mathbb{R}^d \times (0, \infty)$ described in [DT15]. We provide a positive answer to both of the questions in Theorem 2.1.2. The study of the same questions in the case of the outer L^p spaces on $\mathbb{R} \times (0, \infty) \times \mathbb{R}$ described in [DT15] is beyond the purpose of the paper, and it will be addressed in future work. We briefly comment on the difference with the previous case. The geometry of the outer measure on the upper half 3-space can be addressed substantially analogously to that on the upper half space. The source of difficulty is the so called size, the object corresponding to the inner Lebesgue norm of the iterated L^p nature of the outer L^p spaces. While on $\mathbb{R}^d \times (0, \infty)$ the size is given by a single Lebesgue norm, on $\mathbb{R} \times (0,\infty) \times \mathbb{R}$ the size is given by the sum of different Lebesgue norms instead. As a consequence, it is more complicated to treat and requires further investigation.

We turn now to a more detailed introduction of the outer L^p spaces. Differently from [DT15], we specialize the sizes to be themselves Lebesgue norms, so that we can view the L^p theory for outer measure spaces as a generalization of the classical product, or iteration, of L^p quasi-norms. We first focus on the finite setting. This allows us to introduce meaningful outer L^p spaces while at the same time dealing with the least possible amount of technicalities possible. For a more general setting, we refer the interested reader to Appendix 2.A.

We start recalling that on the Cartesian product X of two finite sets equipped with strictly positive weights $(Y,\mu), (Z,\nu)$, we can define the classical product, or iterated, $L^\infty L^r, L^p L^r$ spaces for $0 < p, r < \infty$ by the quasi-norms

$$\begin{aligned} \|f\|_{L^\infty((Y,\mu),L^r(Z,\nu))} &= \sup_{y \in Y} \Big(\sum_{z \in Z} \nu(z) |f(y,z)|^r \Big)^{\frac{1}{r}} \\ &= \sup_{y \in Y} \Big(\mu(y)^{-1} \sum_{z \in Z} \omega(y,z) |f(y,z)|^r \Big)^{\frac{1}{r}}, \\ \|f\|_{L^p((Y,\mu),L^r(Z,\nu))} &= \Big(\sum_{y \in Y} \mu(y) \Big(\sum_{z \in Z} \nu(z) |f(y,z)|^r \Big)^{\frac{p}{r}} \Big)^{\frac{1}{p}}, \end{aligned} \tag{2.1.1}$$

where we denote by $\omega = \mu \otimes \nu$ the induced weight on X. In both cases, the inner L^r quasi-norm may be replaced by an L^∞ norm as well. For $1 \leqslant p, r \leqslant \infty$, the objects defined in the display are in fact norms.

The L^p spaces associated with an outer measure space (X,μ), or outer L^p spaces, generalize this construction. An *outer measure* μ on X is a monotone, subadditive function from $\mathcal{P}(X)$, the power set of X, to the extended positive half-line, attaining the value 0 on the empty set. In general, an outer measure need not generate an interesting measure by restriction to the Carathéodory measurable sets. For instance, when μ is constantly 1 on every nonempty element of $\mathcal{P}(X)$, the Carathéodory σ-algebra is trivial. A standard way to generate an outer measure is via a pre-measure σ, a function from a collection of subsets $\mathcal{E} \subseteq \mathcal{P}(X)$ to the positive half-line, by means of covering an arbitrary subset of X by elements of $\mathcal{E}$. Namely, for every $A \subseteq X$, we define

$$\mu(A) = \inf \Big\{ \sum_{E \in \mathcal{E}'} \sigma(E) \colon \mathcal{E}' \subseteq \mathcal{E}, A \subseteq \bigcup_{E \in \mathcal{E}'} E \Big\}, \tag{2.1.2}$$

with the understanding that an empty sum is 0 and that if A is not covered by $\mathcal{E}$, then the infimum is ∞. In fact, this is the way the authors introduced the outer measures in the upper half space and in the upper half 3-space in [DT15].

For the purpose of defining the outer L^p spaces in the most streamlined fashion, we make the reasonable assumption on μ to be strictly positive and finite on every singleton in $\mathcal{P}(X)$. Next, for a strictly positive weight ω on X, $0<r<\infty$, let ℓ^∞, ℓ^r be the functions from the set of functions on X to $[0,\infty]^{\mathcal{P}(X)}$ defined by

$$\begin{aligned} \ell^\infty(f)(A) &= \sup_{x\in A}|f(x)|, \\ \ell^r(f)(A) &= \Big(\mu(A)^{-1}\sum_{x\in A}\omega(x)|f(x)|^r\Big)^{\frac{1}{r}}. \end{aligned} \tag{2.1.3}$$

The reader familiar with the theory of outer L^p spaces developed in [DT15] can recognize that ℓ^∞, ℓ^r are *sizes*.

For $0<p<\infty, 0<r\leqslant\infty$, we define the outer $L^\infty_\mu(\ell^r), L^p_\mu(\ell^r), L^{p,\infty}_\mu(\ell^r)$ spaces by the quasi-norms

$$\|f\|_{L^\infty_\mu(\ell^r)} = \|f\|_{L^{\infty,\infty}_\mu(\ell^r)} = \sup_{A\subseteq X}\ell^r(f)(A), \tag{2.1.4}$$

$$\|f\|_{L^p_\mu(\ell^r)} = \Big(\int_0^\infty p\lambda^p \inf\{\mu(A)\colon A\subseteq X, \|f1_{A^c}\|_{L^\infty(\ell^r)}\leqslant\lambda\}\frac{\mathrm{d}\lambda}{\lambda}\Big)^{\frac{1}{p}}, \tag{2.1.5}$$

$$\|f\|_{L^{p,\infty}_\mu(\ell^r)} = \Big(\sup_{\lambda>0}\lambda^p \inf\{\mu(A)\colon A\subseteq X, \|f1_{A^c}\|_{L^\infty(\ell^r)}\leqslant\lambda\}\Big)^{\frac{1}{p}}. \tag{2.1.6}$$

The integral in (2.1.5) is reminiscent of the layer-cake representation for the classical L^p norm on a measure space. The novelty and the subtle point of the theory of outer L^p spaces discussed in [DT15] we want to stress is the different way to evaluate the magnitude of a function to define the level sets. This is done through L^r averages rather than L^∞ norm. As a consequence, due to the L^r averaging interplay between μ and ω, the infima in (2.1.5) and (2.1.6) do not stand for outer measures of super level sets $\{f>\lambda\}$ of the function f. In general, this happens only when $r=\infty$, and the L^p quasi-norm becomes a Choquet integral. To shorten the notation, we drop the subscript μ in $L^p_\mu(\ell^r)$ and we refer to the outer L^p spaces with the symbol $L^p(\ell^r)$. Moreover, we denote the infima in (2.1.5) and (2.1.6) associated with f,λ by

$$\mu(\ell^r(f)>\lambda), \tag{2.1.7}$$

and we refer to it as the *super level measure*.

As a side remark, we comment on the definition of the outer L^p quasi-norms in the case of an outer measure μ generated by a pre-measure. Let σ be a pre-measure attaining only strictly positive values on a collection of sets $\mathcal{E}$ covering X, so that μ is strictly positive and finite on every singleton in $\mathcal{P}(X)$. In this case, in (2.1.4), and hence in (2.1.5) and (2.1.6), we can equivalently take the supremum over the elements of $\mathcal{E}$ of the following quantity

$$\begin{aligned} \ell^\infty_\sigma(f)(E) &= \ell^\infty(f)(E), \\ \ell^r_\sigma(f)(E) &= \Big(\sigma(E)^{-1}\sum_{x\in E}\omega(x)|f(x)|^r\Big)^{\frac{1}{r}}, \end{aligned} \tag{2.1.8}$$

as we will see in Lemma 2.A.3 in Appendix 2.A.

An example of the setting just described is the realisation of the classical iterated $L^\infty L^r, L^p L^r$ spaces discussed above as outer L^p spaces. Let X be the set $Y \times Z$, ω be the strictly positive weight $\mu \otimes \nu$, σ be the pre-measure defined on the collection $\mathcal{E} = \{\{y\} \times Z : y \in Y\}$ of subsets of X by

$$\sigma(\{y\} \times Z) = \mu(y),$$

and consider the outer measure generated by σ as in (2.1.2). Then, the quasi-norms in (2.1.1) are the same of those in (2.1.4) and (2.1.5) in this setting. In particular, the outer L^p quasi-norms are in fact norms, at least in a certain range of exponents.

In the first part of this paper, we develop the theory of outer L^p spaces addressing the question of the equivalence of the corresponding quasi-norms to norms. The first novelty is to provide a positive answer in the case of the outer $L^p(\ell^r)$ spaces on finite sets. It follows by the sharpness of the Hölder's inequality in the sense of the following inequality,

$$\|f\|_{L^p(\ell^r)} \leqslant C \sup_{\|g\|_{L^{p'}(\ell^{r'})}=1} \|fg\|_{L^1(X,\omega)}, \tag{2.1.9}$$

where the constant C is independent of $f \in L^p(\ell^r)$, and $L^1(X,\omega)$ stands for the classical L^1 space on X with the measure associated with the weight ω.

Theorem 2.1.1. *Let $0 < p, r \leqslant \infty$. There exists a constant $C = C(p,r)$ such that, for every finite set X, finite outer measure μ strictly positive on every singleton in $\mathcal{P}(X)$, and strictly positive weight ω, the following properties hold true.*

(i) For $0 < p = r \leqslant \infty$, for every $f \in L^p(\ell^p)$,

$$\frac{1}{C}\|f\|_{L^p(X,\omega)} \leqslant \|f\|_{L^p(\ell^p)} \leqslant C\|f\|_{L^p(X,\omega)}.$$

(ii) For $1 < p \leqslant \infty, 1 \leqslant r < \infty$ or $p = r \in \{1,\infty\}$, for every $f \in L^p(\ell^r)$,

$$\frac{1}{C} \sup_{\|g\|_{L^{p'}(\ell^{r'})}=1} \|fg\|_{L^1(X,\omega)} \leqslant \|f\|_{L^p(\ell^r)} \leqslant C \sup_{\|g\|_{L^{p'}(\ell^{r'})}=1} \|fg\|_{L^1(X,\omega)}.$$

(iii) For $1 < p \leqslant \infty, 1 \leqslant r < \infty$ or $p = r \in \{1,\infty\}$, for every $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\ell^r)$,

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^p(\ell^r)} \leqslant C \sum_{n\in\mathbb{N}} \|f_n\|_{L^p(\ell^r)}.$$

Therefore, for $1 < p \leqslant \infty, 1 \leqslant r < \infty$ or $p = r \in \{1,\infty\}$, the outer $L^p(\ell^r)$ quasi-norm is equivalent to a norm, and the outer $L^p(\ell^r)$ space is the Köthe dual space of the outer $L^{p'}(\ell^{r'})$ space.

The main point of the theorem is the uniformity of the constant in (X, μ, ω). In fact, for every fixed finite setting, both statements in $(ii), (iii)$ are verified by a certain constant also for $p = 1, 1 < r \leqslant \infty$ or $1 < p < \infty, r = \infty$, and hence the final considerations of the theorem hold true as well. However, for $p = 1, 1 < r \leqslant \infty$, the constant is not uniform in (X, μ, ω), and we exhibit a counterexample in Lemma 2.3.4. For $1 < p < \infty, r = \infty$, the question about uniformity remains open. The uniformity of the constant suggests that if an infinite setting is suitably approximated by finite restrictions, the same results could possibly be obtained through a limiting process.

There is a slight abuse in the use of the term Köthe dual space in the statement of Theorem 2.1.1, since this object is in general defined for Banach function spaces. A *Banach function space*, or *Köthe function space*, $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ on a σ-finite measure space $(X, \widetilde{\omega})$ is a Banach space of measurable functions containing all the simple functions and such that if f is a measurable function with absolute value bounded $\widetilde{\omega}$-almost everywhere by $g \in \mathcal{L}$, then $f \in \mathcal{L}$ with norm bounded by that of g. The *Köthe dual space*, or *associate space*, of $\mathcal{L}$ is then defined as the space of measurable functions such that the $L^1(X, \widetilde{\omega})$ pairing with every element of $\mathcal{L}$ is finite, endowed with the norm of the dual space, see for example [BS88, LT79]. In our setting, we have both a measure associated with the weight ω and an outer measure μ on X. Although it is not clear whether a priori the simple functions with respect to ω belong to the outer $L^p(\ell^r)$ space, it is straight-forward to check that the simple functions with respect to μ belong to $L^p(\ell^r)$. Therefore, with a slight abuse of terminology, we extend the definition of the Köthe duality to the outer $L^p(\ell^r)$ spaces with respect to the $L^1(X, \omega)$ pairing.

The first inequalities of both statements in $(i), (ii)$ were already proved as consequences of more general results obtained in [DT15, Ura17], see Proposition 2.A.7 and Proposition 2.A.5 in Appendix 2.A of the present paper. It would be interesting to investigate whether, for example, the outer L^p quasi-norms are equivalent to norms in the generality of sizes discussed in [DT15] and recalled in Appendix 2.A.

We further develop our research in the case of the outer L^p spaces with size defined by an L^r norm on the infinite setting associated with Calderón-Zygmund theory. We address the question of the equivalence to norms of the outer L^p quasi-norms on functions on the upper half space described in [DT15]. In particular, let X be $\mathbb{R}^d \times (0, \infty)$ with the topology inherited from $\mathbb{R}^{d+1}$, $\mathcal{D}$ be the collection of the open dyadic cubic boxes with sides parallel to the axes and base on $\mathbb{R}^d$. Let σ be the function on $\mathcal{D}$ given by the classical volume of the base of the box, μ be the outer measure on X generated by σ on $\mathcal{D}$ as in (2.1.2). Finally, let ω be the measure defined by the density $\omega(y, t) = t^{-1}$ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0, \infty)$, where $y \in \mathbb{R}^d, t \in (0, \infty)$. For $0 < r < \infty$, let $\ell^\infty_\sigma, \ell^r_\sigma$ be the functions from $\mathcal{B}(X)$, the set of Borel measurable functions on X, to $[0, \infty]^{\mathcal{D}}$ defined by

$$
\begin{aligned}
&\ell^\infty_\sigma(f)(D) = \|f 1_D\|_{L^\infty(X, \omega(y,t)\, \mathrm{d}y\, \mathrm{d}t)}, \\
&\ell^r_\sigma(f)(D) = \Big(\sigma(D)^{-1} \int_D |f(y,t)|^r \, \mathrm{d}y \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = \sigma(D)^{-\frac{1}{r}} \|f 1_D\|_{L^r(X, \omega(y,t)\, \mathrm{d}y\, \mathrm{d}t)}.
\end{aligned}
\tag{2.1.10}
$$

For $0 < p, r \leqslant \infty$, let the outer $L^p(\ell^r_\sigma), L^{p,\infty}(\ell^r_\sigma)$ spaces be defined as in (2.1.4), (2.1.5) and (2.1.6), taking the supremum of the quantity in the previous display over the elements of $\mathcal{D}$ in (2.1.4). In analogy with the remark concerning the quantities in (2.1.8), we drop the subscript σ in $L^p(\ell^r_\sigma)$.

In this infinite setting, we prove the analogous statement of Theorem 2.1.1. The properties $(ii), (iii)$ hold true even in the endpoint cases $p = 1, 1 < r \leqslant \infty$ and $1 \leqslant p < \infty, r = \infty$.

Theorem 2.1.2. *Let (X, μ, ω) be the upper half space setting just described, $0 < p, r \leqslant \infty$. There exists a constant $C = C(p, r)$ such that the analogous properties stated in Theorem 2.1.1 hold true in the following ranges, property (i) for $0 < p = r \leqslant \infty$, properties $(ii), (iii)$ for $1 \leqslant p, r \leqslant \infty$.*

Therefore, for $1 \leqslant p, r \leqslant \infty$, the outer $L^p(\ell^r)$ quasi-norm is equivalent to a norm, and the outer $L^p(\ell^r)$ space is the Köthe dual space of the outer $L^{p'}(\ell^{r'})$ space.

As we recalled in the first part of the introduction, in the upper half space setting there are already classical spaces with a different iterated $L^p L^r$ structure, namely the tent spaces. Let $\Gamma(x)$ be the cone with vertex in $x \in \mathbb{R}^d$, $T(x, s)$ be the tent over the ball in $\mathbb{R}^d$ centred in x with radius s,

$$\Gamma(x) = \{(y, t) \in \mathbb{R}^d \times (0, \infty) \colon |x - y| < t\},$$
$$T(x, s) = \{(y, t) \in \mathbb{R}^d \times (0, \infty) \colon |x - y| < s - t\}.$$

For $0 < p < \infty, 0 < r \leqslant \infty$, let

$$\begin{aligned} A_r(f)(x) &= \|f\|_{L^r(\Gamma(x), \mathrm{d}y \frac{\mathrm{d}t}{t^{d+1}})}, \\ \|f\|_{T^p_r} &= \|A_r(f)\|_{L^p(\mathbb{R}^d, \mathrm{d}x)}. \end{aligned} \tag{2.1.11}$$

For $p = \infty, 0 < r \leqslant \infty$, let

$$\begin{aligned} C_r(f)(x) &= \sup_{s \in (0, \infty)} \|f\|_{L^r(T(x,s), \omega)}, \\ \|f\|_{T^\infty_r} &= \|C_r(f)\|_{L^\infty(\mathbb{R}^d, \mathrm{d}x)}. \end{aligned} \tag{2.1.12}$$

For $0 < p, r \leqslant \infty$, the tent space T^p_r is defined by the T^p_r quasi-norm. Sometimes in the literature an additional continuity condition is assumed on functions in T^p_∞, see for example [CMS85], but we do not, in order to preserve a uniformity in the definition of the spaces. For $1 \leqslant p, r \leqslant \infty$, the quasi-norms defined in the last two displays are in fact norms.

The third result of this paper is to establish the equivalence between the outer $L^p(\ell^r)$ spaces and the tent spaces T^p_r.

Theorem 2.1.3. *For $0 < p, r \leqslant \infty$, there exists a constant $C = C(p, r)$ such that, for every $f \in L^p(\ell^r)$,*

$$\frac{1}{C}\|f\|_{T^p_r} \leqslant \|f\|_{L^p(\ell^r)} \leqslant C\|f\|_{T^p_r}.$$

Moreover, we have $L^p(\ell^r) = T^p_r$.

It is worth noting that while the tent spaces require to pass from cones to tents in order to define T_r^∞, the definition of the outer $L^p(\ell^r)$ spaces always relies on the boxes, or equivalently on the tents.

In the second part of the paper, we turn our focus to embedding maps of functions on $\mathbb{R}^d$ to the upper half space $\mathbb{R}^d \times (0,\infty)$. These embeddings are obtained by pairing a function on $\mathbb{R}^d$ with translated and dilated versions of a given test function. More precisely, given a test function ϕ satisfying certain boundedness and decay properties, we define, for every locally integrable function f on $\mathbb{R}^d$, the embedded function $F_\phi(f)$ on $\mathbb{R}^d \times (0,\infty)$ by

$$F_\phi(f)(y,t) = \int_{\mathbb{R}^d} f(x) t^{-d} \phi(t^{-1}(y-x)) \, \mathrm{d}x. \tag{2.1.13}$$

A prominent example of such an embedding is the harmonic extension of a function on $\mathbb{R}^d$ to the upper half space, where ϕ is the Poisson kernel. The interest in embedding maps is part of the aforementioned two-step program to prove the boundedness of operators in Calderón-Zygmund theory.

We study continuous inclusions between outer L^p spaces in the upper half space and continuous embeddings from classical L^p spaces on $\mathbb{R}^d$ to outer L^p spaces in this setting. We start with an improvement over a previous result on Hardy-Littlewood-Sobolev inclusions between tent spaces in [Ame18]. We obtain the boundedness of the map

$$T^p_{r_1} \hookrightarrow T^q_{r_2}, f \mapsto t^{\frac{d}{p}-\frac{d}{q}} f,$$

for $0 < p < q \leqslant \infty, 0 < r_2 \leqslant r_1 \leqslant \infty$, or equivalently the same statement for outer $L^p(\ell^r)$ spaces. The improvement over the result in [Ame18] consists of allowing for r_1 to be strictly greater than r_2.

These inclusions allow to recover strong type (p,q) estimates for the embedding maps with a fractional scale factor

$$L^p(\mathbb{R}^d) \hookrightarrow L^q(\ell^r), f \mapsto t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f),$$

for $0 < p < q \leqslant \infty, 0 < r \leqslant \infty$ from the ones for $p = q, r = \infty$. The fourth result of the paper is then the full classification of all positive and negative results regarding strong and weak type estimates for a family of embedding maps with a fractional scale factor in Theorem 2.6.1. More precisely, for $\varepsilon > 0, f \in \mathcal{S}(\mathbb{R}^d)$, let the embedded function $F_\varepsilon(f) = F(f)$ be defined by

$$F(f)(y,t) = \sup_\phi F_\phi(f)(y,t),$$

where the supremum is taken over the set of functions ϕ such that

$$|\phi(z)| \leqslant (1+|z|)^{-d-\varepsilon}. \tag{2.1.14}$$

With respect to the strong type estimates, we extract the following statement from Theorem 2.6.1.

Theorem 2.1.4. *Let*

$$1 \leqslant p, q \leqslant \infty, 0 < r \leqslant \infty. \tag{2.1.15}$$

Then, for (p, q, r) satisfying one of the following conditions

$$\begin{aligned} 1 < p < q \leqslant \infty, 0 < r \leqslant \infty, \\ 1 < p = q \leqslant \infty, r = \infty, \\ p = 1, q = \infty, 0 < r \leqslant \infty, \end{aligned} \tag{2.1.16}$$

there exists a constant $C = C(p, q, r, d, \varepsilon)$ such that, for every $f \in L^p(\mathbb{R}^d)$,

$$\|t^{\frac{d}{p}-\frac{d}{q}} F(f)\|_{L^q(\ell^r)} \leqslant C\|f\|_{L^p(\mathbb{R}^d)}.$$

For all the triples (p, q, r) satisfying (2.1.15) *but none of the conditions in* (2.1.16)*, no strong type (p, q) estimate holds true.*

It is worth noting that the strong type $(1, \infty)$ estimates hold true for $0 < r \leqslant \infty$, even if for $r = \infty$ only the weak type $(1, 1)$ estimate holds true. Moreover, in the endpoint $p = q = 1, r = \infty$, we prove in Proposition 2.6.2 a substitute of the strong type $(1, 1)$ estimate, namely the boundedness of the embedding map

$$H^1(\mathbb{R}^d) \hookrightarrow L^1(\ell^\infty), f \mapsto F_\varphi(f),$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

We conclude the paper with some applications of these embedding theorems yielding alternative proofs of classical results such as the Hardy-Littlewood-Sobolev inequality, and the Gagliardo-Nirenberg-Sobolev inequality up to the endpoint in the spirit of the aforementioned two-step program.

Guide to the paper

In Section 2 we start with two decomposition results for functions in the outer $L^p(\ell^r)$ spaces in both finite and upper half space settings. We use them to prove Theorem 2.1.2 and Theorem 2.1.1 in Section 3. Moreover, in Lemma 2.3.4, we provide a counterexample to the uniformity of the statements in $(ii), (iii)$ in Theorem 2.1.1 for $p = 1, 1 < r \leqslant \infty$. In Section 4 we prove Theorem 2.1.3. In Section 5, Theorem 2.5.1, we improve over the result of Amenta on Hardy-Littlewood-Sobolev inclusions between tent spaces. In Section 6, Theorem 2.6.1, we prove a full classification of all positive and negative results regarding strong and weak type estimates for a family of embedding maps with a fractional scale factor from classical L^p spaces on $\mathbb{R}^d$ to outer $L^p(\ell^r)$ spaces on $\mathbb{R}^d \times (0, \infty)$. Moreover, in Proposition 2.6.2 we prove the boundedness of the embedding map defined by a test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ from $H^1(\mathbb{R}^d)$ to the outer $L^1(\ell^\infty)$ space. We use the strong type estimates from both results to prove the Hardy-Littlewood-Sobolev inequality, and the Gagliardo-Nirenberg-Sobolev inequality up to the endpoint in the spirit of the aforementioned two-step program in Section 7. Finally, in Appendix 2.A, we review the definitions and recall some results of the theory of outer L^p spaces in the level of generality discussed in [DT15]. In Appendix 2.B, we prove some properties of the outer measure μ on the upper half space described above.

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2.2 Decompositions for outer $L^p(\ell^r)$ spaces

In this section we state and prove two crucial preparatory decomposition results for functions in the outer $L^p(\ell^r)$ spaces in both finite and upper half space settings, used in proving Theorem 2.1.1 and Theorem 2.1.2, respectively. Both consist of a recursive greedy selection algorithm that provides a sequence of maximal disjoint subsets of X exhausting the elements of $\mathcal{P}(X)$ where the quantity defined in (2.1.3) is in the interval $[2^k, 2^{k+1}), k \in \mathbb{Z}$. This property guarantees not only an upper bound but also a lower bound on the super level measure in (2.1.7) at level $\lambda = 2^k, k \in \mathbb{Z}$, in terms of the outer measures of the selected subsets, thus providing a concrete substitute for it. Without loss of generality, we can restrict our attention only to these levels. In fact, we can replace the integral in (2.1.5) with an equivalent discrete version, namely

$$\Big(\sum_{k \in \mathbb{Z}} 2^{kp} \mu(\ell^r(f) > 2^k) \Big)^{\frac{1}{p}},$$

due to the monotonicity in λ of the super level measure of a fixed function. This quantity is no longer homogeneous in f, hence it is not a quasi-norm, but the discrete levels fit better the recursive process we want to describe.

The decompositions in the two cases are analogous. We could state and prove a unified result in the general setting described in Appendix 2.A, at least in the range of exponents $0 < p, r < \infty$. It would require some adjustments to address the technicalities due to the non-finiteness of the selection process and the generation of the outer measure by a pre-measure. In this exposition, we prefer to focus separately on the two specific settings for the following reasons.

The finite setting offers a full view on the mechanism of the recursive selection algorithm and the proof of the decomposition properties. Moreover, we do not have to worry about our selection process being well-defined, since at each step only finitely many choices are available, and we can choose any subset of X. Again, we stress that the main point in this case is the uniformity of constants in (X, μ, ω).

The upper half space setting serves two purposes. On one hand, as a privileged case of the general setting described in Appendix 2.A, it provides an example of addressing the technicalities we referred to above. On the other hand, due to the geometry of the outer measure, it allows for an improved version of the decomposition result. First, we can extend it to the case $r = \infty$, which is not included in the finite setting. Second, the decomposition of a function in the outer $L^1(\ell^r)$ space, for $1 < r \leqslant \infty$, is subtly more efficient for our purpose, as will be clarified in Remark 2.3.2. We could state sufficient conditions on the geometry of the outer measure to ensure this refined decomposition in a broader generality, but these considerations are beyond the purpose of the paper, and they will be developed in future work.

We start with the finite setting. Let X be a finite set, μ an outer measure strictly positive and finite on every singleton in $\mathcal{P}(X)$, ω a strictly positive weight. We have the following uniform decomposition result for functions in the outer $L^p(\ell^r)$ spaces defined by (2.1.5).

Proposition 2.2.1. *Let $0 < p, r < \infty$. There exists a constant $C = C(p, r)$ such that, for every finite set X, finite outer measure μ strictly positive on every singleton in $\mathcal{P}(X)$, and strictly positive weight ω, the following property holds true. For $f \in L^p(\ell^r)$, there exists a sequence of sets $\{E_k : k \in \mathbb{Z}\} \subseteq \mathcal{P}(X)$ such that if*

$$F_k = \bigcup_{l \geqslant k} E_l,$$

then, for every $k \in \mathbb{Z}$,

$$\ell^r(f 1_{F_{k+1}^c})(E_k) > 2^k, \qquad \text{when } E_k \neq \varnothing, \tag{2.2.1}$$

$$\|f 1_{F_k^c}\|_{L^\infty(\ell^r)} \leqslant 2^k, \tag{2.2.2}$$

$$\mu(\ell^r(f) > 2^k) \leqslant \sum_{l \geqslant k} \mu(E_l), \tag{2.2.3}$$

$$\mu(E_k) \leqslant C \mu(\ell^r(f) > 2^{k-1}). \tag{2.2.4}$$

Proof. First, we observe qualitatively that by outer Hölder's inequality, Proposition 2.A.5 in Appendix 2.A, we have $L^p(\ell^r) \subseteq L^\infty(\ell^r)$, because $\mu(X)$ is finite.

We define E_k by backward recursion on $k \in \mathbb{Z}$. For k large enough such that

$$\|f\|_{L^\infty(\ell^r)} \leqslant 2^k,$$

we set E_k to be empty. Now fix k and assume we have selected E_l for $l > k$. In particular, F_{k+1} is already well-defined. If there exists a set $A \subseteq X$ such that

$$\ell^r(f 1_{F_{k+1}^c})(A) > 2^k, \tag{2.2.5}$$

then we choose such a set A to be E_k, making sure that

$$\|f1_{(A\cup F_{k+1})^c}\|_{L^\infty(\ell^r)} \leqslant 2^k. \tag{2.2.6}$$

In fact, if there exists a set $B \subseteq X$ such that

$$\ell^r(f1_{(A\cup F_{k+1})^c})(B) > 2^k,$$

then by the subadditivity of the outer measure, we have

$$\ell^r(f1_{F_{k+1}^c})(A \cup B) > 2^k.$$

Due to the finiteness of X, the condition (2.2.6) can be achieved in finitely many steps. If no A satisfying (2.2.5) exists, we set E_k to be empty, and proceed the recursion with $k-1$.

By construction, we have (2.2.1) for every nonempty selected set E_k, (2.2.2) and (2.2.3) for every $k \in \mathbb{Z}$.

We observe that for every k such that 2^k is greater than the $L^\infty(\ell^r)$ quasi-norm of f, the statement (2.2.4) is true. To prove (2.2.4) for any other k, let A_{k-1} be a set witnessing the super level measure at level 2^{k-1}. In particular,

$$\|f1_{A_{k-1}^c}\|_{L^\infty(\ell^r)} \leqslant 2^{k-1},$$
$$\mu(\ell^r(f) > 2^{k-1}) = \mu(A_{k-1}).$$

By (2.2.2) for $k+1$, we have

$$\mu(A_{k-1}) \geqslant 2^{-r(k+1)} \sum_{x\in A_{k-1}\setminus F_{k+1}} \omega(x)|f(x)|^r. \tag{2.2.7}$$

By the definition of A_{k-1} and E_k, we have

$$\sum_{x\in E_k\setminus A_{k-1}} \omega(x)|f(x)|^r \leqslant 2^{r(k-1)}\mu(E_k),$$
$$\sum_{x\in E_k\setminus F_{k+1}} \omega(x)|f(x)|^r > 2^{rk}\mu(E_k),$$

hence

$$\sum_{x\in(A_{k-1}\cap E_k)\setminus F_{k+1}} \omega(x)|f(x)|^r > C2^{r(k-1)}\mu(E_k).$$

Combining this with (2.2.7) gives

$$\mu(\ell^r(f) > 2^{k-1}) \geqslant C\mu(E_k),$$

concluding the proof of (2.2.4) for the given k. $\square$

Now we move to the upper half space setting. Let X be the upper half space and μ the outer measure generated by the pre-measure σ on $\mathcal{D}$, the collection of the open dyadic cubic boxes in the upper half space, as in (2.1.2). In particular,

$$\begin{aligned} X &= \mathbb{R}^d \times (0,\infty), \\ \mathcal{D} &= \{(x,0)+(0,2^j)^{d+1} \colon x \in 2^j\mathbb{Z}^d, j \in \mathbb{Z}\}, \\ \sigma(E) &= |B(E)|, \qquad \text{for every } E \in \mathcal{D}, \\ \omega(y,t) &= t^{-1}, \end{aligned} \tag{2.2.8}$$

where $B(E)$ is the base in $\mathbb{R}^d$ of the dyadic box $E \in \mathcal{D}$, and $|B(E)|$ its volume. Moreover, for every dyadic box $E = (x,0) + (0,s)^{d+1} \in \mathcal{D}$, we define E^+ by

$$E^+ = (x,0) + ((0,s)^d \times (s/2,s)).$$

Finally, let ω be the measure defined by the density $\omega(y,t)$ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0,\infty)$, where $y \in \mathbb{R}^d, t \in (0,\infty)$, and for every $0 < r \leqslant \infty$ let ℓ^r be the size defined in (2.1.10).

We make the following observations involving the geometry of the elements of $\mathcal{D}$ and the values of σ, μ on them. We postpone the proofs to Appendix 2.B.

Lemma 2.2.2. *For every two dyadic boxes $E_1, E_2 \in \mathcal{D}$ with nonempty intersection, we have either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$.*

Lemma 2.2.3. *Let $\{E_n \colon n \in \mathbb{N}\}$ be a collection of pairwise disjoint dyadic boxes in $\mathcal{D}$, and let $\{D_n \colon n \in \mathbb{N}\}$ be a collection of subsets of X such that, for every $n \in \mathbb{N}$, we have $D_n \subseteq E_n$ and $D_n \cap E_n^+ \neq \varnothing$. Then we have*

$$\mu\Big(\bigcup_{n\in\mathbb{N}} D_n\Big) = \sum_{n\in\mathbb{N}} \sigma(E_n).$$

In the following statement, the elements of a double sequence are parametrized by a pair (k,n), for $k \in \mathbb{Z}, n \in \mathbb{N}_k$, where $\mathbb{N}_k$ is either the set of positive natural numbers or a possibly empty finite initial string of positive natural numbers. We consider the lexicographic order of such pairs as follows: $(l,m) < (k,n)$ if either $l > k$, or $l = k$ and $m < n$.

We have the following decomposition result for functions in the intersection between the outer $L^p(\ell^r)$ and $L^\infty(\ell^r)$ spaces defined by (2.1.5) and (2.1.4), respectively.

Proposition 2.2.4. *Let $0 < p < \infty, 0 < r \leqslant \infty$. There exists a constant $C = C(p,r)$ such that the following property holds true. For $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, there exists a double sequence of dyadic boxes $\{E_{k,n} \colon k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \mathcal{D}$ such that if*

$$\begin{aligned} F_k &= \bigcup_{n\in\mathbb{N}_k} F_{k,n}, \\ F_{k,n} &= F_{k,n-1} \cup E_{k,n}, \\ F_{k,0} &= \bigcup_{i\in I_k} Q_i, \end{aligned}$$

where $\{Q_i : i \in I_k\} \subseteq \mathcal{D}$ *is the collection of maximal dyadic boxes such that*

$$|B(Q_i)| \leqslant 2\Big|B(Q_i) \cap \bigcup_{(l,m):\, l>k} B(E_{l,m})\Big|, \tag{2.2.9}$$

then, for every $k \in \mathbb{Z}, n \in \mathbb{N}_k$,

$$\ell^r(f 1_{F^c_{k,n-1}})(E_{k,n}) > 2^k, \qquad \textit{when } E_{k,n} \neq \varnothing, \tag{2.2.10}$$

$$\|f 1_{F^c_k}\|_{L^\infty(\ell^r)} \leqslant 2^k, \tag{2.2.11}$$

$$\mu(\ell^r(f) > 2^k) \leqslant C \sum_{(l,m):\, l \geqslant k} \sigma(E_{l,m}), \tag{2.2.12}$$

$$\sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}) \leqslant C\mu(\ell^r(f) > 2^{k-1}). \tag{2.2.13}$$

Moreover, the collection $\{B(E_{k,n}) : k \in \mathbb{Z}, n \in \mathbb{N}_k\}$ *of the bases of the chosen boxes is* 2*-Carleson, i.e. for every dyadic box* $E \in \mathcal{D}$

$$\sum_{(k,n):\, E_{k,n} \subseteq E} \sigma(E_{k,n}) \leqslant 2\sigma(E). \tag{2.2.14}$$

For the definition of the Λ-Carleson condition and, later in the proof, of the η-sparse condition for collections of cubes, as well as for their equivalence, we refer for example to [LN19].

Before starting the proof, we briefly comment that a dyadic box satisfies the condition in (2.2.9) for a certain $k \in \mathbb{Z}$ when at least half of its base is covered by the bases of the elements of the double sequence selected up to the level $k+1$.

Proof. **Case I:** $0 < r < \infty$**.** The selection algorithm is analogous to that described in the previous proof. We define $E_{k,n}$ by a double recursion, backward on $k \in \mathbb{Z}$, and, for every fixed k, forward on $n \in \mathbb{N}_k$. In parallel, we prove the properties in (2.2.10) – (2.2.13) by backward induction on $k \in \mathbb{Z}$.

For k large enough such that

$$\|f\|_{L^\infty(\ell^r)} \leqslant 2^k,$$

we set $\mathbb{N}_k$ empty. The properties in (2.2.10) – (2.2.13) are trivially satisfied.

Now fix (k,n) and assume we have selected $E_{l,m}$ for $(l,m) < (k,n)$, and that the properties in (2.2.10) – (2.2.13) are satisfied for every $l > k$. In particular, F_{k+1} is already well-defined and satisfies (2.2.11), and $F_{k,n-1}$ is already well-defined. If there exists a dyadic box $A \in \mathcal{D}$ such that

$$\ell^r(f 1_{F^c_{k,n-1}})(A) > 2^k, \tag{2.2.15}$$

then we choose such a dyadic box A to be $E_{k,n}$, making sure that $\sigma(A)$ is maximal. The maximality of $\sigma(A)$ is achieved because the set of values of σ is discrete and doubling, namely it is $\{2^{id} \colon i \in \mathbb{Z}\}$, and we have an upper bound on $\sigma(A)$ when A satisfies the condition (2.2.15). In fact, we have

$$\sigma(A) \leqslant C\mu(\ell^r(f) > 2^{k-1}) \leqslant C2^{-kp}\|f\|_{L^p(\ell^r)}^p < \infty. \tag{2.2.16}$$

To prove the first inequality, we use an argument analogous to that used to prove (2.2.4) above. For every $\varepsilon > 0$, let $A_{k-1}(\varepsilon)$ be an optimal set witnessing the super level measure at level 2^{k-1} up to the multiplicative constant $(1+\varepsilon)$. Next, let $\mathcal{E}_{k-1}(\varepsilon)$ be an optimal covering of $A_{k-1}(\varepsilon)$ witnessing its outer measure up to the multiplicative constant $(1+\varepsilon)$. In particular,

$$\begin{gathered}\|f1_{A_{k-1}(\varepsilon)^c}\|_{L^\infty(\ell^r)} \leqslant 2^{k-1},\\ A_{k-1}(\varepsilon) \subseteq \bigcup_{E \in \mathcal{E}_{k-1}(\varepsilon)} E,\\ (1+\varepsilon)^2\mu(\ell^r(f) > 2^{k-1}) \geqslant (1+\varepsilon)\mu(A_{k-1}(\varepsilon)) \geqslant \sum_{E \in \mathcal{E}_{k-1}(\varepsilon)} \sigma(E).\end{gathered}$$

By (2.2.11) for $k+1$, we have, for every $E \in \mathcal{E}_{k-1}(\varepsilon)$,

$$\sigma(E) \geqslant 2^{-r(k+1)}\|f1_{E \setminus F_{k+1}}\|_{L^r(X,\omega)}^r,$$

which yields, together with the covering of $A_{k-1}(\varepsilon)$ by the elements of $\mathcal{E}_{k-1}(\varepsilon)$,

$$\begin{aligned}\sum_{E \in \mathcal{E}_{k-1}(\varepsilon)} \sigma(E) &\geqslant \sum_{E \in \mathcal{E}_{k-1}(\varepsilon)} 2^{-r(k+1)}\|f1_{E \setminus F_{k+1}}\|_{L^r(X,\omega)}^r\\ &\geqslant 2^{-r(k+1)}\|f1_{(\bigcup_{E \in \mathcal{E}_{k-1}(\varepsilon)} E)\setminus F_{k+1}}\|_{L^r(X,\omega)}^r\\ &\geqslant 2^{-r(k+1)}\|f1_{A_{k-1}(\varepsilon)\setminus F_{k+1}}\|_{L^r(X,\omega)}^r.\end{aligned} \tag{2.2.17}$$

By the definition of $A_{k-1}(\varepsilon)$ and A, we have

$$\begin{aligned}\|f1_{A \setminus A_{k-1}(\varepsilon)}\|_{L^r(X,\omega)}^r &\leqslant 2^{r(k-1)}\sigma(A),\\ \|f1_{A \setminus F_{k+1}}\|_{L^r(X,\omega)}^r &\geqslant \|f1_{A \setminus F_{k,n-1}}\|_{L^r(X,\omega)}^r > 2^{rk}\sigma(A),\end{aligned}$$

hence

$$\|f1_{(A_{k-1}(\varepsilon)\cap A)\setminus F_{k+1}}\|_{L^r(X,\omega)}^r > C2^{r(k-1)}\sigma(A).$$

Combining this with (2.2.17) and taking ε arbitrarily small give the desired inequality

$$\mu(\ell^r(f) > 2^{k-1}) \geqslant C\sigma(A).$$

If no A satisfying (2.2.15) exists, we set $\mathbb{N}_k = \{1, \ldots, n-1\}$, $\mathbb{N}_k$ empty if $n = 1$. If we are able to choose $E_{k,n}$ for all $n \in \mathbb{N}$, we fix such $E_{k,n}$. Before proceeding the recursion with $(k-1, 1)$, we prove the properties in (2.2.10) – (2.2.13) for k.

By construction, we have (2.2.10) for every nonempty selected dyadic box $E_{k,n}$.

The proof of (2.2.13) for k assuming (2.2.11) for $k+1$, which we have by the induction hypothesis, is analogous to that of the first inequality in (2.2.16). In fact, we have

$$\|f1_{\bigcup_{n\in\mathbb{N}_k} E_{k,n}\setminus A_{k-1}(\varepsilon)}\|^r_{L^r(X,\omega)} \leqslant \sum_{n\in\mathbb{N}_k} \|f1_{E_{k,n}\setminus A_{k-1}(\varepsilon)}\|^r_{L^r(X,\omega)} \leqslant 2^{r(k-1)} \sum_{n\in\mathbb{N}_k} \sigma(E_{k,n}),$$

$$\|f1_{\bigcup_{n\in\mathbb{N}_k} E_{k,n}\setminus F_{k+1}}\|^r_{L^r(X,\omega)} \geqslant \sum_{n\in\mathbb{N}_k} \|f1_{E_{k,n}\setminus F_{k,n-1}}\|^r_{L^r(X,\omega)} > 2^{rk} \sum_{n\in\mathbb{N}_k} \sigma(E_{k,n}),$$

hence

$$\|f1_{(A_{k-1}(\varepsilon)\cap\bigcup_{n\in\mathbb{N}_k} E_{k,n})\setminus F_{k+1}}\|^r_{L^r(X,\omega)} > C2^{r(k-1)} \sum_{n\in\mathbb{N}_k} \sigma(E_{k,n}),$$

where $A_{k-1}(\varepsilon)$ is defined as above. We conclude as above

Now we prove (2.2.11) for k. If $\mathbb{N}_k$ is finite, then by construction there is no dyadic box $A \in \mathcal{D}$ such that

$$\ell^r(f1_{F_k^c})(A) > 2^k.$$

If $\mathbb{N}_k$ is infinite, we observe by (2.2.13) for this k, that

$$\sum_{n\in\mathbb{N}_k} \sigma(E_{k,n}) < \infty,$$

since $f \in L^p(\ell^r)$. Therefore, $\sigma(E_{k,n})$ tends to zero as n tends to ∞. Since each $E_{k,n}$ is chosen to maximize $\sigma(E_{k,n})$, there exists no dyadic box $A \in \mathcal{D}$ which can violate (2.2.11) as such A would contradict the choice of $E_{k,n}$ for sufficiently large n. This concludes the proof of (2.2.11) for the given k.

With (2.2.11), we also have (2.2.12). In fact, we have

$$\begin{aligned}
\mu(F_k) &\leqslant \mu(F_{k-1,0}) \\
&\leqslant \sum_{i\in I_{k-1}} |B(Q_i)| \\
&\leqslant 2\Big|\bigcup_{i\in I_{k-1}} B(Q_i) \cap \bigcup_{(l,m)\colon l\geqslant k} B(E_{l,m})\Big| \\
&\leqslant C \sum_{(l,m)\colon l\geqslant k} \sigma(E_{l,m}),
\end{aligned}$$

where we used (2.2.9) and the disjointness of the elements of $\{Q_i \colon i \in I_{k-1}\}$ in the third inequality.

Case II: $r = \infty$**.** The only difference is in the selection of $E_{k,n}$. Fix (k,n) and assume we have selected $E_{l,m}$ for $(l,m) < (k,n)$, and that the properties in (2.2.10) – (2.2.13) are satisfied for every $l > k$. If there exists a dyadic box $A \in \mathcal{D}$ such that

$$\ell^\infty(f1_{F^c_{k,n-1}}1_{A^+})(A) > 2^k, \tag{2.2.18}$$

then we choose such a dyadic box A to be $E_{k,n}$, making sure that $\sigma(A)$ is maximal.

As in the previous case, the maximality of $\sigma(A)$ is achieved because the set of values of σ is discrete and doubling, and we have an upper bound on $\sigma(A)$ when A satisfies the condition (2.2.18). In fact, we have

$$\sigma(A) \leqslant \mu(\ell^\infty(f) > 2^{k-1}) \leqslant C 2^{-kp} \|f\|^p_{L^p(\ell^\infty)} < \infty. \tag{2.2.19}$$

To prove the first inequality, we observe that for $E = A^+ \cap \{|f| \geqslant 2^k\}$, we have $\omega(E) > 0$, hence

$$\mu(E) \leqslant \mu(\ell^\infty(f) > 2^{k-1}).$$

We conclude by Lemma 2.2.3.

The proof of (2.2.10) – (2.2.13) for k then follows in a straight-forward way. As in the previous case, the proof of (2.2.13) is analogous to that of the first inequality in (2.2.19). In fact, we observe that for $D_{k,n} = E^+_{k,n} \cap \{|f| \geqslant 2^k\}$, we have $\omega(D_{k,n}) > 0$, hence

$$\mu\Big(\bigcup_{n \in \mathbb{N}_k} D_{k,n}\Big) \leqslant \mu(\ell^\infty(f) > 2^{k-1}).$$

We conclude by Lemma 2.2.3 upon observing that for fixed k, the selected dyadic boxes $E_{k,n}$ are pairwise disjoint, by Lemma 2.2.2 and the definition of $E_{k,n}$.

To conclude, for every $0 < r \leqslant \infty$, we observe that the collection $\{B(E_{k,n})\colon k \in \mathbb{Z}, n \in \mathbb{N}_k\}$ is 1/2-sparse, i.e. one can choose pairwise disjoint measurable sets $\widetilde{B}_{k,n} \subseteq B(E_{k,n})$ with $|\widetilde{B}_{k,n}| \geqslant |B(E_{k,n})|/2$. This follows by (2.2.9) and the maximality in the choice of $E_{k,n}$. Therefore, the collection is 2-Carleson. □

2.3 Equivalence with norms

In this section we prove Theorem 2.1.2 and Theorem 2.1.1. We start with the upper half space setting. First, we prove property (i). After that, for every $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, for $1 \leqslant p, r \leqslant \infty$, we provide a candidate function g to realize (2.1.9), up to normalization of its outer $L^{p'}(\ell^{r'})$ quasi-norm. Upon showing an upper bound on the outer $L^{p'}(\ell^{r'})$ quasi-norm of g and a lower bound on the $L^1(X, \omega)$ norm of fg, properties $(ii), (iii)$ follow. Then we turn to the finite setting and when possible we follow analogous arguments to prove properties $(i), (ii)$, and (iii). In almost all the definitions and proofs we make use of the decompositions provided by Proposition 2.2.4 and Proposition 2.2.1. Finally, in Lemma 2.3.4 we exhibit a counterexample to the uniformity in every finite setting (X, μ, ω) of both statements in $(ii), (iii)$ for $p = 1, 1 < r \leqslant \infty$.

We start with the upper half space setting, where (X, μ, ω) is the setting described in (2.2.8).

Proof of Theorem 2.1.2, property (i). The case $p = \infty$ follows by definition.

Therefore, we can assume without loss of generality $p = 1$, since

$$\|f\|^p_{L^p(\ell^p)} = \|f^p\|_{L^1(\ell^1)}.$$

For $f \in L^1(\ell^1) \cap L^\infty(\ell^1)$, let $\{E_{k,n}\}$ be the collection of the dyadic boxes from Proposition 2.2.4. We have

$$\begin{aligned}
\|f\|_{L^1(\ell^1)} &\leqslant C \sum_{k\in\mathbb{Z}} 2^k \mu(\ell^1(f) > 2^k) \\
&\leqslant C \sum_{k\in\mathbb{Z}} 2^k \sum_{(l,m)\colon l \geqslant k} \sigma(E_{l,m}) \\
&\leqslant C \sum_{l\in\mathbb{Z}} \sum_{m\in\mathbb{N}_l} 2^l \sigma(E_{l,m}) \\
&\leqslant C \sum_{l\in\mathbb{Z}} \sum_{m\in\mathbb{N}_l} \|f\|_{L^1(E_{l,m}\setminus F_{l,m-1},\omega)} \\
&\leqslant C\|f\|_{L^1(X,\omega)},
\end{aligned}$$

where we used (2.2.12) in the second inequality, Fubini and the bounds on the geometric series in the third, (2.2.10) in the fourth, and disjointness of the sets in the fifth.

We note that f vanishes ω-almost everywhere outside the union of all the selected dyadic boxes $\{E_{k,n}\}$, since $\mathcal{D}$ covers all of X. We have

$$\begin{aligned}
\|f\|_{L^1(X,\omega)} &= \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}_k} \|f 1_{E_{k,n}\setminus F_{k,n-1}}\|_{L^1(X,\omega)} + \sum_{k\in\mathbb{Z}} \|f 1_{F_{k,0}\setminus F_{k+1}}\|_{L^1(X,\omega)} \\
&\leqslant \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}_k} \|f 1_{E_{k,n}\setminus F_{k+1}}\|_{L^1(X,\omega)} + \sum_{k\in\mathbb{Z}} \sum_{i\in I_k} \|f 1_{Q_i\setminus F_{k+1}}\|_{L^1(X,\omega)} \\
&\leqslant \sum_{k\in\mathbb{Z}} 2^{k+1} \sum_{n\in\mathbb{N}_k} \sigma(E_{k,n}) + \sum_{k\in\mathbb{Z}} 2^{k+1} \sum_{i\in I_k} \sigma(Q_i) \\
&\leqslant \sum_{k\in\mathbb{Z}} 2^{k+1} \sum_{n\in\mathbb{N}_k} \sigma(E_{k,n}) + \sum_{k\in\mathbb{Z}} 2^{k+1} \sum_{(l,m)\colon l>k} \sigma(E_{l,m}) \\
&\leqslant \sum_{k\in\mathbb{Z}} 2^{k+1} \sum_{n\in\mathbb{N}_k} \sigma(E_{k,n}) + C\sum_{l\in\mathbb{Z}} 2^{l+1} \sum_{m\in\mathbb{N}_l} \sigma(E_{l,m}) \\
&\leqslant C \sum_{k\in\mathbb{Z}} 2^{k-1} \mu(\ell^1(f) > 2^{k-1}) \\
&\leqslant C\|f\|_{L^1(\ell^1)},
\end{aligned}$$

where we used (2.2.11) in the second inequality, (2.2.9) and the disjointness of the dyadic boxes $\{Q_i\}$ in the third, Fubini and the bounds on the geometric series in the fourth, and (2.2.13) in the fifth.

A standard approximation argument yields the result for arbitrary $f \in L^1(\ell^1)$. □

Now we provide the candidate function g for $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, for $1 \leqslant p, r \leqslant \infty$. We separate the definition into four cases depending on p and r.

Case 1: $1 \leqslant p, r < \infty$**.** For $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, let $\{E_{k,n}\}$ be the collection from Proposition 2.2.4, and define

$$g(x,s) = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{k(p-r)} 1_{E_{k,n} \setminus F_{k,n-1}}(x,s) |f(x,s)|^{r-1}.$$

Case 2: $1 \leqslant p < \infty$ **and** $r = \infty$**.** For $f \in L^p(\ell^\infty) \cap L^\infty(\ell^\infty)$, let $\{E_{k,n}\}$ be the collection from Proposition 2.2.4, and define

$$g(x,s) = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{k(p-1)} 1_{\widetilde{E}_{k,n}}(x,s) (\ell^1(1_{\widetilde{E}_{k,n}})(E_{k,n}))^{-1},$$

where

$$\widetilde{E}_{k,n} = E^+_{k,n} \cap \{|f| > 2^k\},$$

and $E^+_{k,n}$ is the upper half of $E_{k,n}$.

Case 3: $p = \infty$ **and** $1 \leqslant r < \infty$**.** For $f \in L^\infty(\ell^r)$, let the dyadic box $E \in \mathcal{D}$ witness the outer $L^\infty(\ell^r)$ quasi-norm of f up to a factor 2, and define

$$g(x,s) = 1_E(x,s) |f(x,s)|^{r-1}.$$

Case 4: $p = r = \infty$**.** For $f \in L^\infty(\ell^\infty)$, let the dyadic box $E \in \mathcal{D}$ witness the outer $L^\infty(\ell^\infty)$ quasi-norm of f up to a factor 2 in a subset of strictly positive measure in E^+, and define

$$g(x,s) = 1_{\widetilde{E}}(x,s) (\ell^1(1_{\widetilde{E}})(E))^{-1},$$

where

$$\widetilde{E} = E^+ \cap \{|f| > \|f\|_{L^\infty(\ell^\infty)}/2\}.$$

We have the following upper bounds on the outer $L^{p'}(\ell^{r'})$ quasi-norm of g, where g is defined according to the four (p,r)-dependent cases.

Lemma 2.3.1. ***Case I:*** *$p = 1$ **and** $1 \leqslant r \leqslant \infty$**.** We have*

$$\|g\|_{L^\infty(\ell^{r'})} \leqslant C.$$

Case II: *$1 < p < \infty$ **and** $1 \leqslant r \leqslant \infty$**.** We have*

$$\|g\|^{p'}_{L^{p'}(\ell^{r'})} \leqslant C \|f\|^p_{L^p(\ell^r)}.$$

Case III: *$p = \infty$ **and** $1 \leqslant r < \infty$**.** We have*

$$\|g\|_{L^1(\ell^{r'})} \leqslant \|f\|^{r-1}_{L^\infty(\ell^r)} \sigma(E).$$

Case IV: *$p = r = \infty$**.** We have*

$$\|g\|_{L^1(\ell^1)} \leqslant \sigma(E).$$

Proof. **Case I:** $p = 1$ **and** $1 \leqslant r \leqslant \infty$**.** Let $1 < r < \infty$. For every dyadic box $A \in \mathcal{D}$, we have

$$\begin{aligned}
(\ell^{r'}(g)(A))^{r'} &= \frac{1}{\sigma(A)} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{-kr} \int_{A \cap (E_{k,n} \setminus F_{k,n-1})} |f(y,t)|^r \omega(y,t) \, \mathrm{d}y \, \mathrm{d}t \\
&\leqslant \frac{1}{\sigma(A)} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{-kr} \int_{A \cap (E_{k,n} \setminus F_{k+1})} |f(y,t)|^r \omega(y,t) \, \mathrm{d}y \, \mathrm{d}t \\
&\leqslant C \frac{1}{\sigma(A)} \Big(\sigma(A) + \sum_{(k,n) \colon E_{k,n} \subseteq A} \sigma(E_{k,n}) \Big) \\
&\leqslant C,
\end{aligned} \tag{2.3.1}$$

where we used (2.2.11) and the nested structure of $\mathcal{D}$, namely the fact that for $A, B \in \mathcal{D}, A \cap B \neq \varnothing$, then either $A \subseteq B$ or $B \subseteq A$, in the second inequality, and (2.2.14) in the third.

In an analogous way, for every dyadic box $A \in \mathcal{D}$, for $r = \infty$, we have

$$\ell^1(g)(A) \leqslant C,$$

and it is easy to see that, for $r = 1$, we have

$$\ell^\infty(g)(A) \leqslant 1.$$

Therefore, for $1 \leqslant r \leqslant \infty$, we have

$$\|g\|_{L^\infty(\ell^{r'})} \leqslant C.$$

Case II: $1 < p < \infty$ **and** $1 \leqslant r \leqslant \infty$**.** Let $1 < r < \infty$. For a fixed k and every dyadic box $A \in \mathcal{D}$, we have

$$\begin{aligned}
(\ell^{r'}(g 1_{F_k^c})(A))^{r'} &= \frac{1}{\sigma(A)} \sum_{(l,m) \colon l < k} 2^{l(p-r)r'} \int_{A \cap (E_{l,m} \setminus F_{l,m-1})} |f(y,t)|^r \omega(y,t) \, \mathrm{d}y \, \mathrm{d}t \\
&\leqslant \sum_{l<k} 2^{l(p-r)r'} \frac{1}{\sigma(A)} \int_{A \setminus F_{l+1}} |f(y,t)|^r \omega(y,t) \, \mathrm{d}y \, \mathrm{d}t \\
&\leqslant c \sum_{l<k} 2^{l(p-r+r-1)r'} \\
&\leqslant c 2^{k(p-1)r'},
\end{aligned} \tag{2.3.2}$$

where we used (2.2.11) in the second inequality, and the bounds on the geometric series in the third.

In an analogous way, for every dyadic box $A \in \mathcal{D}$, for $r = \infty$, we have

$$\begin{aligned}\ell^1(g1_{F_k^c})(A) &= \frac{1}{\sigma(A)} \sum_{l<k} \sum_{m \in \mathbb{N}_l} 2^{l(p-1)} \int_{A \cap \widetilde{E}_{l,m}} (\ell^1(1_{\widetilde{E}_{l.m}})(E_{l.m}))^{-1} \omega(y,t) \,\mathrm{d}y \,\mathrm{d}t \\ &\leqslant \sum_{l<k} 2^{l(p-1)} \frac{1}{\sigma(A)} \sum_{m\colon E_{l,m} \subseteq A} \sigma(E_{l,m}) \\ &\leqslant c 2^{k(p-1)},\end{aligned} \tag{2.3.3}$$

where we used the disjointness of the elements of $\{E_{l,m} \colon m \in \mathbb{N}_l\}$ due to the maximality in their choice, and the bounds on the geometric series in the second inequality.

It is easy to see that, for every dyadic box $A \in \mathcal{D}$, for $r = 1$, we have

$$\ell^\infty(g1_{F_k^c})(A) \leqslant 2^{k(p-1)}.$$

As a consequence, for $1 \leqslant r \leqslant \infty$, for every dyadic box $A \in \mathcal{D}$, we have

$$\ell^{r'}(g1_{F_k^c})(A) \leqslant c2^{k(p-1)},$$

hence

$$\mu(\ell^{r'}(g) > c2^{k(p-1)}) \leqslant \mu(F_k) \leqslant C \sum_{(l,m)\colon l \geqslant k} \sigma(E_{l,m}). \tag{2.3.4}$$

Therefore, we have

$$\begin{aligned}\|g\|^{p'}_{L^{p'}(\ell^{r'})} &\leqslant C \sum_{k \in \mathbb{Z}} 2^{kp} \mu(\ell^{r'}(g) > c2^{k(p-1)}) \\ &\leqslant C \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{(l,m)\colon l \geqslant k} \sigma(E_{l,m}) \\ &\leqslant C \sum_{l \in \mathbb{Z}} 2^{lp} \sum_{m \in \mathbb{N}_l} \sigma(E_{l,m}) \\ &\leqslant C \sum_{l \in \mathbb{Z}} 2^{lp} \mu(\ell^r(f) > 2^{l-1}) \\ &\leqslant C \|f\|^p_{L^p(\ell^r)},\end{aligned}$$

where we used (2.3.4) in the second inequality, Fubini and the bounds on the geometric series in the third, and (2.2.13) in the fourth.

Case III: $p = \infty$ **and** $1 \leqslant r < \infty$**.** By construction we have

$$\|g\|_{L^\infty(\ell^{r'})} \leqslant \|f\|^{r-1}_{L^\infty(\ell^r)},$$

therefore, by outer Hölder's inequality, Proposition 2.A.5, we have

$$\|g\|_{L^1(\ell^{r'})} \leqslant \|g\|_{L^\infty(\ell^{r'})} \|1_E\|_{L^1(\ell^\infty)} \leqslant \|f\|^{r-1}_{L^\infty(\ell^r)} \sigma(E).$$

Case IV: $p = r = \infty$**.** In an analogous way, we have

$$\|g\|_{L^1(\ell^1)} \leqslant \sigma(E),$$

since by construction $\|g\|_{L^\infty(\ell^1)} = 1$. □

Remark 2.3.2. *Without the crucial property of the decomposition established by* (2.2.14)*, the argument in* (2.3.1) *above produces the empty upper bound*

$$(\ell^{r'}(g)(E))^{r'} \leqslant \sum_{k\in\mathbb{Z}} 1.$$

Nevertheless, when $1 < p < \infty$*, in* (2.3.2) *and in* (2.3.3) *we can already get a summable decay in* $l < k$ *for the upper bound on the* $\ell^{r'}$ *size of* g *over the sets* $A \cap (F_l \backslash F_{l+1})$*, and it is not necessary to invoke* (2.2.14)*.*

We have the following lower bounds on the $L^1(X,\omega)$ norm of fg, where as above g is defined according to the four (p,r)-dependent cases.

Lemma 2.3.3. ***Case I:*** $1 \leqslant p < \infty$ ***and*** $1 \leqslant r \leqslant \infty$***.*** *We have*

$$\|fg\|_{L^1(X,\omega)} \geqslant C\|f\|^p_{L^p(\ell^r)}.$$

Case II: $p = \infty$ ***and*** $1 \leqslant r < \infty$***.*** *We have*

$$\|fg\|_{L^1(X,\omega)} \geqslant C\|f\|^r_{L^\infty(\ell^r)}\sigma(E).$$

Case III: $p = r = \infty$***.*** *We have*

$$\|fg\|_{L^1(X,\omega)} \geqslant C\|f\|_{L^\infty(\ell^r)}\sigma(E).$$

Proof. **Case I:** $1 \leqslant p < \infty$ **and** $1 \leqslant r \leqslant \infty$**.** Let $1 \leqslant r < \infty$. For every fixed (k,n) such that $E_{k,n}$ is not empty, we have

$$\ell^1(fg1_{F^c_{k,n-1}})(E_{k,n}) = 2^{k(p-r)}(\ell^r(f1_{F^c_{k,n-1}})(E_{k,n}))^r > 2^{kp}, \tag{2.3.5}$$

where we used (2.2.10) in the inequality.

For $r = \infty$, by the definition of g, we have the same inequality.

Therefore, for $1 \leqslant r \leqslant \infty$, we have

$$\begin{aligned}
\|fg\|_{L^1(X,\omega)} &\geqslant \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}_k} \|fg\|_{L^1(E_{k,n}\backslash F_{k,n-1},\omega)} \\
&\geqslant \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{N}_k} 2^{kp}\sigma(E_{k,n}) \\
&\geqslant C\sum_{l\in\mathbb{Z}} 2^{lp} \sum_{(k,n)\colon l\leqslant k} \sigma(E_{k,n}) \\
&\geqslant C\sum_{l\in\mathbb{Z}} 2^{lp}\mu(\ell^r(f) > 2^l) \\
&\geqslant C\|f\|^p_{L^p(\ell^r)},
\end{aligned}$$

where we used (2.3.5) in the second inequality, the bounds on the geometric series and Fubini in the third, and (2.2.12) in the fourth.

Case II: $p = \infty$ **and** $1 \leqslant r < \infty$**.** Let $E \in \mathcal{D}$ be the dyadic box associated with g, in particular

$$\ell^r(f)(E) \geqslant C\|f\|_{L^\infty(\ell^r)}.$$

Therefore, we have

$$\begin{aligned}\|fg\|_{L^1(X,\omega)} &= \|f1_E\|^r_{L^r(X,\omega)} \\ &= \ell^r(f)(E)^r\sigma(E) \\ &\geqslant C\|f\|^r_{L^\infty(\ell^r)}\sigma(E).\end{aligned}$$

Case III: $p = r = \infty$**.** In an analogous way, we have

$$\|fg\|_{L^1(X,\omega)} \geqslant C\|f\|_{L^\infty(\ell^r)}\sigma(E).$$

□

Proof of Theorem 2.1.2, properties $(ii), (iii)$. The first inequality in (ii) is given by outer Hölder's inequality, Proposition 2.A.5.

The second inequality in (ii) is a corollary of the previous Lemmata for $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$. A standard approximation argument yields the case of an arbitrary $f \in L^p(\ell^r)$.

The statement in (iii) is a corollary of the triangle inequality for the $L^1(X, \omega)$ norm and property (ii). □

We conclude the part of the section about the upper half space with the following observation.

Let X be the upper half space and ν the outer measure generated by the pre-measure σ on $\mathcal{E}$, the collection of all the open cubic boxes in the upper half space, as in (2.1.2). In particular,

$$\begin{aligned}X &= \mathbb{R}^d \times (0, \infty), \\ \mathcal{E} &= \{(x, 0) + (0, s)^{d+1} : x \in \mathbb{R}^d, s \in (0, \infty)\}, \\ \sigma(E) &= |B(E)|, \quad \text{for every } E \in \mathcal{E}, \\ \omega(y, t) &= t^{-1},\end{aligned} \tag{2.3.6}$$

where $B(E)$ is the base in $\mathbb{R}^d$ of the box E, and $|B(E)|$ its volume. We observe that $\mathcal{D} \subseteq \mathcal{E}$, and every box in $\mathcal{E}$ can be covered up to a set of measure zero by finitely many dyadic boxes in $\mathcal{D}$ of comparable pre-measure. Therefore, the outer $L^p(\ell^r)$ space quasi-norms in the settings (2.2.8) and (2.3.6) are equivalent by Proposition 2.A.4. As a consequence, all the previous results obtained in the setting (2.2.8) extend to the setting (2.3.6). An analogous argument applies to the outer measure structure generated by triangular tents in place of cubic boxes.

We turn now to the finite setting.

Proof of Theorem 2.1.1. The proof of property (i) and, for $1 < p \leqslant \infty, 1 \leqslant r < \infty$, of property (ii) follows by arguments analogous to those in the previous proofs, using the decomposition in Proposition 2.2.1.

For $p = r \in \{1, \infty\}$, the statement in (ii) follows by the equivalence between $L^p(\ell^p)$ and $L^p(X, \omega)$ by property (i).

The statement in (iii) is again a corollary of the triangle inequality for the $L^1(X, \omega)$ norm and property (ii). □

Lemma 2.3.4. *Let $1 < r \leqslant \infty$. For every $M > 0$, there exist a finite set X, a finite outer measure μ strictly positive on every singleton in $\mathcal{P}(X)$, a strictly positive weight ω, functions $f, f_n \in L^1(\ell^r)$ such that*

$$\|f\|_{L^1(\ell^r)} \geqslant M \sup_{\|g\|_{L^\infty(\ell^{r'})}=1} \|fg\|_{L^1(\ell^1)},$$
$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^1(\ell^r)} \geqslant M \sum_{n \in \mathbb{N}} \|f_n\|_{L^1(\ell^r)}.$$

Proof. Let $\mathcal{D}$ be the set of dyadic intervals. For every $m \in \mathbb{N}$, let

$$\begin{aligned}
X_m &= \{I \in \mathcal{D} \colon I \subseteq [0,1], |I| \geqslant 2^{-m}\}, \\
\mathcal{E}_m &= \{E_I = \{J \in \mathcal{D} \colon I \subseteq J \subseteq [0,1]\} \colon I \in X_m, |I| = 2^{-m}\}, \\
\sigma_m(E_I) &= 1, \quad \text{for every } I \in X_m, |I| = 2^{-m}, \\
\omega_m(J) &= 1, \quad \text{for every } J \in X_m, \\
f_m(J) &= 2^m |J|, \\
f_I(J) &= 1_{E_I}(J), \quad \text{for every } I \in X_m, |I| = 2^{-m}.
\end{aligned}$$

We have

$$\Big\| \sum_{I \in X_m, |I| = 2^{-m}} f_I \Big\|_{L^1(\ell^r)} = \|f_m\|_{L^1(\ell^r)} \geqslant 2^m \frac{m+1}{2},$$
$$\sum_{I \in X_m, |I| = 2^{-m}} \|f_I\|_{L^1(\ell^r)} = \sum_{I \in X_m, |I| = 2^{-m}} (m+1)^{\frac{1}{r}} = 2^m (m+1)^{\frac{1}{r}}.$$

For m big enough, we get the second statement. In particular, this yields a counterexample to the uniformity of the constant in the statement of Theorem 2.1.1, property (iii). Therefore, also the uniformity of the constant in the statement of Theorem 2.1.1, property (ii) does not hold true. □

2.4 Equivalence with tent spaces

In this section we prove the equivalence between the outer $L^p(\ell^r)$ spaces in the upper half space setting (2.3.6) and the tent spaces T^p_r stated in Theorem 2.1.3. First, in Lemma 2.4.1

we prove the equivalence for certain exponents p, r. After that, we extend it to the full range $0 < p, r \leqslant \infty$ via the Köthe duality result for the outer $L^p(\ell^r)$ spaces, equivalent to that stated in Theorem 2.1.2, property (ii), and the analogous result for tent spaces T^p_r, stated in Proposition 2.4.2.

Lemma 2.4.1. *For $p = \infty, 0 < r < \infty$ or $0 < p < \infty, r = \infty$, there exists a constant $C = C(p,r)$ such that, for every $f \in L^p(\ell^r)$,*

$$\frac{1}{C}\|f\|_{T^p_r} \leqslant \|f\|_{L^p(\ell^r)} \leqslant C\|f\|_{T^p_r}.$$

Proof. Without loss of generality, it is enough to consider the cases

$$\begin{aligned} &p = \infty, r = 1, \\ &p = 1, r = \infty. \end{aligned} \tag{2.4.1}$$

In fact, let $q < \infty$ be the minimum of p and r. We have

$$\begin{aligned} \|f\|^q_{T^p_r} &= \|f^q\|_{T^{p/q}_{r/q}}, \\ \|f\|^q_{L^p(\ell^r)} &= \|f^q\|_{L^{p/q}(\ell^{r/q})}, \end{aligned}$$

where $\infty/q = \infty$, thus recovering one of the cases in (2.4.1).

Case I: $p = \infty, r = 1$**.** The quantities associated with the spaces $L^\infty(\ell^1), T^\infty_1$ are equivalent by definition, up to a constant determined by a simple covering argument between boxes and tents.

Case II: $p = 1, r = \infty$**.** Let $f \in L^1(\ell^\infty)$. For every $\lambda > 0$, let $\mathcal{E}_\lambda \subseteq \mathcal{E}$ be a covering witnessing the super level measure at level λ up to a factor 2. In particular, we have

$$2\mu(\ell^\infty(f) > \lambda) \geqslant \sum_{E \in \mathcal{E}_\lambda} \sigma(E).$$

For

$$B_\lambda = \bigcup_{\mathcal{E}_\lambda} 10B(E) \subseteq \mathbb{R}^d,$$

where $10B$ is the cube in $\mathbb{R}^d$ with the same centre of B and 10 times its side length, we have

$$|B_\lambda| \leqslant C \sum_{E \in \mathcal{E}_\lambda} \sigma(E) \leqslant C\mu(\ell^\infty(f) > \lambda).$$

Moreover, for every $x \in B^c_\lambda$, we have

$$A_\infty(f)(x) \leqslant \lambda,$$

otherwise we get a contradiction with the definition of $\mathcal{E}_\lambda$. Therefore, we have

$$|\{x \in \mathbb{R}^d \colon A_\infty(f)(x) > \lambda\}| \leqslant C\mu(\ell^\infty(f) > \lambda). \tag{2.4.2}$$

Now let $f \in T^p_\infty$. For every $\lambda > 0$, let D_λ be

$$D_\lambda = \{x \in \mathbb{R}^d \colon A_\infty(f)(x) > \lambda\},$$

and define

$$E_\lambda = \bigcup_{i \in I_\lambda} 10 Q_i \subseteq X,$$

where $\{B(Q_i)\}$ is a Whitney decomposition of D_λ, and $10Q$ is the box whose base $B(10Q)$ has the same centre of $B(Q)$ and 10 times its side length. In particular, we have

$$\mu(E_\lambda) \leqslant C|D_\lambda|.$$

Moreover, for every $E \in \mathcal{E}$, we have

$$\ell^\infty(f 1_{E^c_\lambda})(E) \leqslant \lambda,$$

otherwise we get a contradiction with the definition of D_λ. Therefore, we have

$$\mu(\ell^\infty(f) > \lambda) \leqslant C|\{x \in \mathbb{R}^d \colon A_\infty(f)(x) > \lambda\}|. \tag{2.4.3}$$

The desired equivalence follows by integrating the inequalities (2.4.2), (2.4.3) over all levels $\lambda > 0$. □

For the tent spaces T^p_r we have the following Köthe duality result, see for example Theorem 5.2 in [Hua16].

Proposition 2.4.2. *For $1 \leqslant p, r \leqslant \infty$, for every $f \in T^p_r$,*

$$\sup_{\|g\|_{T^{p'}_{r'}}=1} \|fg\|_{L^1(X,\omega)} \leqslant \|f\|_{T^p_r} \leqslant \sup_{\|g\|_{T^{p'}_{r'}}=1} \|fg\|_{L^1(X,\omega)}.$$

Proof of Theorem 2.1.3. Without loss of generality, it is enough to consider the cases

$$p = r = \infty,$$
$$1 < p \leqslant \infty, r = 1,$$
$$p = 1, 1 \leqslant r \leqslant \infty,$$

due to an argument analogous to that in the previous proof.

Case I: $p = r = \infty$**.** The equivalence between $L^\infty(\ell^\infty), T^\infty_\infty$ follows by definition.

Case II: $1 < p \leqslant \infty, r = 1$**.** For $p = \infty$ the quantities associated with the spaces $L^\infty(\ell^1), T^\infty_1$ are equivalent by Lemma 2.4.1.

For $1 < p < \infty$, let $f \in L^p(\ell^1)$. By Theorem 2.1.2, property (ii), we have

$$\frac{1}{C} \sup_{\|g\|_{L^{p'}(\ell^\infty)} \leqslant 1} \|fg\|_{L^1(X,\omega)} \leqslant \|f\|_{L^p(\ell^1)} \leqslant C \sup_{\|g\|_{L^{p'}(\ell^\infty)} \leqslant 1} \|fg\|_{L^1(X,\omega)}.$$

Applying Lemma 2.4.1 to g, we have

$$\frac{1}{C} \sup_{\|g\|_{T^{p'}_\infty} \leqslant 1} \|fg\|_{L^1(X,\omega)} \leqslant \|f\|_{L^p(\ell^1)} \leqslant C \sup_{\|g\|_{T^{p'}_\infty} \leqslant 1} \|fg\|_{L^1(X,\omega)}.$$

Finally, by Proposition 2.4.2, we conclude

$$\frac{1}{C}\|f\|_{T^p_1} \leqslant \|f\|_{L^p(\ell^1)} \leqslant C\|f\|_{T^p_1}.$$

Case III: $p = 1, 1 \leqslant r \leqslant \infty$**.** For $p = 1, r = \infty$, the quantities associated with the spaces $L^1(\ell^\infty), T^1_\infty$ are equivalent by Lemma 2.4.1.

For $p = 1, 1 \leqslant r < \infty$, an argument analogous to that used to prove Case II yields the desired equivalence. If $p = r = 1$, we use Case I in place of Lemma 2.4.1.

To conclude, we observe that the set of bounded functions with compact support in X is dense in T^p_r for $1 \leqslant p < \infty, r = 1$ and $p = 1, 1 \leqslant r < \infty$. However, these functions are also in $L^p(\ell^r)$. Therefore, the two spaces coincide. □

2.5 Hardy-Littlewood-Sobolev inclusions for tent spaces

In this section we improve over a result of Amenta on continuous inclusions between tent spaces T^p_r, see Theorem 2.19 and Lemma 2.20 in [Ame18]. In his notation, we have the weighted tent spaces $T^{p,r}_s$ defined, for $0 < p, r \leqslant \infty, s \in \mathbb{R}$, by

$$T^{p,r}_s = \{f : t^{-ds} f \in T^p_r\}, \qquad \|f\|_{T^{p,r}_s} = \|t^{-ds} f\|_{T^p_r},$$

where T^p_r is defined in (2.1.11) and (2.1.12), and the continuous inclusions

$$T^{p,r}_0 \hookrightarrow T^{q,r}_{\frac{1}{q}-\frac{1}{p}}, f \mapsto f,$$

for $0 < p < q \leqslant \infty, 0 < r \leqslant \infty$. The improvement consists of allowing for two different values of r, under certain conditions, in each of the two spaces in the last display.

Due to the equivalence proved in the previous section, we get an analogous result for the outer $L^p(\ell^r)$ spaces in the upper half space setting (2.3.6). This result is auxiliary in proving strong type estimates in the following section.

Theorem 2.5.1. *For $0 < p < q \leqslant \infty, 0 < r_2 \leqslant r_1 \leqslant \infty$, there exists a constant $C = C(p, q, r_1, r_2)$ such that, for every $f \in T^p_{r_1}$,*

$$\|t^{\frac{d}{p}-\frac{d}{q}} f\|_{T^q_{r_2}} \leqslant C\|f\|_{T^p_{r_1}}.$$

Equivalently, for every $f \in L^p(\ell^{r_1})$,

$$\|t^{\frac{d}{p}-\frac{d}{q}} f\|_{L^q(\ell^{r_2})} \leqslant C\|f\|_{L^p(\ell^{r_1})}.$$

The main ingredient is the following. We define a function a to be a T^p_r-atom associated with the ball $B \subseteq \mathbb{R}^d$ if a is essentially supported in $T(B)$ and

$$\|a\|_{T^r_r} \leqslant |B|^{\frac{1}{r}-\frac{1}{p}}. \tag{2.5.1}$$

Lemma 2.5.2. *Let $1 < q \leqslant r_2 \leqslant r_1 \leqslant \infty$. Suppose that a is a $T^1_{r_1}$-atom. Then a is in $T^q_{r_2}$ with norm smaller than 1.*

Proof. For $q < \infty$, let $0 < r, s \leqslant \infty$ be such that

$$\frac{1}{r} + \frac{1}{r_1} = \frac{1}{r_2}, \qquad \frac{1}{s} + \frac{1}{r_1} = \frac{1}{q}.$$

We have

$$\begin{aligned} \|t^{d-\frac{d}{q}} a\|_{T^q_{r_2}} &= \|A_{r_2}(t^{d-\frac{d}{q}} a)\|_{L^q(B)} \\ &\leqslant \|A_r(t^{d-\frac{d}{q}} 1_{T(B)}) A_{r_1}(a)\|_{L^q(B)} \\ &\leqslant \|A_r(t^{d-\frac{d}{q}} 1_{T(B)})\|_{L^s(B)} \|A_{r_1}(a)\|_{L^{r_1}(B)} \\ &\leqslant |B|^{1-\frac{1}{r_1}} \|a\|_{T^{r_1}_{r_1}} \\ &\leqslant 1, \end{aligned}$$

where we used Hölder's inequality in the first and in the second inequality, and (2.5.1) in the fourth.

For $q = r_2 = r_1 = \infty$, the statement follows directly from (2.5.1). □

Proof of Theorem 2.5.1. The proof of the first statement follows along the lines of that of Theorem 2.19 in [Ame18], using Lemma 2.5.2 above in place of Lemma 2.20.

The second statement then follows by Theorem 2.1.3. □

2.6 Embedding into outer $L^p(\ell^r)$ spaces with a fractional scale factor

In this section we state and prove a full classification of all positive and negative results regarding strong and weak type estimates for a family of embedding maps with a fractional scale factor from classical L^p spaces on $\mathbb{R}^d$ to outer $L^p(\ell^r)$ spaces in the upper half space setting.

The positive results for $d = 1, 1 \leqslant p = q \leqslant \infty, r = \infty$ were already proved in [DT15], see Theorem 4.1. Although there ϕ was assumed to be smooth and compactly supported, the same argument can be extended with minor adjustments to the test functions satisfying the boundedness and decay condition (2.1.14) and to all dimensions.

We conclude the section by stating and proving an embedding theorem with a fractional scale factor for functions in the Hardy space $H^1(\mathbb{R}^d)$ into the outer $L^1(\ell^\infty)$ space. The embedded function in this case is that defined in (2.1.13) for a smooth test function $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Theorem 2.6.1. *Let*

$$1 \leqslant p, q \leqslant \infty, 0 < r \leqslant \infty. \tag{2.6.1}$$

Then, for (p, q, r) satisfying one of the following conditions, which are also displayed in Fig. 1 below,

$$\begin{gathered} 1 < p < q \leqslant \infty, 0 < r \leqslant \infty, \\ 1 < p = q \leqslant \infty, r = \infty, \\ p = 1, q = \infty, 0 < r \leqslant \infty, \end{gathered} \tag{2.6.2}$$

there exists a constant $C = C(p, q, r, d, \varepsilon)$ such that, for every $f \in L^p(\mathbb{R}^d)$,

$$\| t^{\frac{d}{p} - \frac{d}{q}} F(f) \|_{L^q(\ell^r)} \leqslant C \| f \|_{L^p(\mathbb{R}^d)}.$$

For all the triples (p, q, r) satisfying (2.6.1) *but none of the conditions in* (2.6.2)*, no strong type (p, q) estimate holds true.*

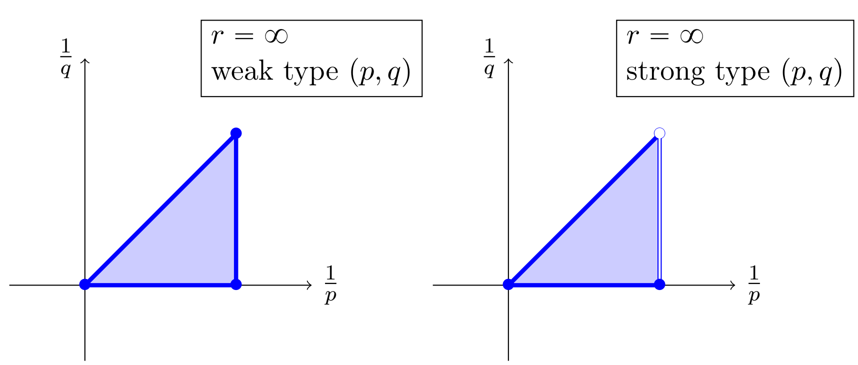
Moreover, for (p, q, r) satisfying one of the following conditions, which are also displayed in Fig. 1 below,

$$\begin{gathered} 1 = p < q < \infty, 0 < r \leqslant \infty, \\ p = q = 1, r = \infty, \end{gathered} \tag{2.6.3}$$

there exists a constant $C = C(q, r, d, \varepsilon)$ such that, for every $f \in L^1(\mathbb{R}^d)$,

$$\| F(f) \|_{L^{q,\infty}(\ell^r)} \leqslant C \| f \|_{L^1(\mathbb{R}^d)}.$$

For all the triples (p, q, r) satisfying (2.6.1) *but none of the conditions in* (2.6.2),(2.6.3)*, no weak type (p, q) estimate holds true.*



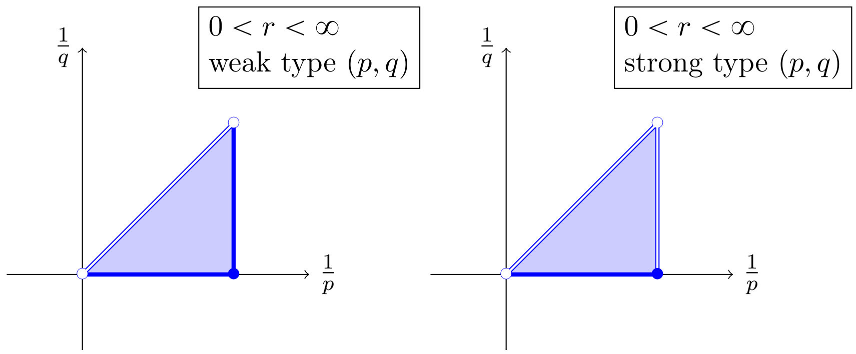


Figure 1: range of exponents p, q, r and weak/strong type estimates.

In the next proof, the constants c, C are allowed to depend on d, ε, p, q, r but not on f.

Proof of Theorem 2.6.1. Without loss of generality, we can assume f to be nonnegative. In fact, by definition (2.1.13), we have the pointwise bound

$$|F_\phi(f)(y,t)| \leqslant F_{|\phi|}(|f|)(y,t) \leqslant F(|f|)(y,t).$$

In particular, we have

$$F(f)(y,t) = \int_{\mathbb{R}^d} f(z) t^{-d} (1 + t^{-1}|y-z|)^{-d-\varepsilon} \,\mathrm{d}z.$$

This expression can be bounded either by means of the centred maximal function

$$F(f)(y,t) \leqslant CMf(y), \tag{2.6.4}$$

or by Young's convolution inequality

$$F(f)(y,t) \leqslant Ct^{-\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.6.5}$$

2.6.1 Strong type (p, q) estimates for $0 < r \leqslant \infty$ in the range for $p \neq 1, q$ displayed in Fig. 1

The strong type (p, q) estimates in the range $1 < p < q \leqslant \infty, 0 < r \leqslant \infty$ follow by the already known strong type (p, p) estimate for $1 < p \leqslant \infty, r = \infty$ and Theorem 2.5.1.

2.6.2 Strong type $(1, \infty)$ estimates for $0 < r \leqslant \infty$

We aim to prove that, for every $E \in \mathcal{E}$,

$$\ell^r(t^d F(f))(E) \leqslant C\|f\|_{L^1(\mathbb{R}^d)}. \tag{2.6.6}$$

If $r = \infty$, the claim follows by (2.6.5).

Now let $0 < r < \infty$. By Theorem 2.1.2, property (*iii*), the decay property of ϕ, and the translation invariance of the $L^\infty(\ell^r)$ quasi-norm, it is enough to prove the inequality assuming that f is supported in $(-1,1)^d$ and $\phi = 1_{(-1,1)^d}$. In this case, we have

$$F_\phi(f)(y,t) \leqslant Ct^{-d}\|f 1_{y+(-t,t)^d}\|_{L^1(\mathbb{R}^d)} 1_{\{(-1-s,1+s)^d\times\{s\},s>0\}}(y,t),$$

and it is enough to prove (2.6.6) for the elements of $\mathcal{E}$ of the form

$$E_{x,u} = (x+(-u,u)^d)\times(0,2u) \in \mathcal{E},$$

for every $u > 0, x \in (-1-u,1+u)^d$. We distinguish two cases, $r \geqslant 1$ and $0 < r < 1$.

Case I: $r \geqslant 1$**.** Let $r = 1$. We have

$$\begin{aligned}
\ell^1(t^d F_\phi(f))(E_{x,u}) &\leqslant \frac{C}{u^d}\int_0^{2u}\int_{x+(-u,u)^d}\int_{(-1,1)^d} f(z)1_{y+(-t,t)^d}(z)\,\mathrm{d}z\,\mathrm{d}y\frac{\mathrm{d}t}{t} \\
&\leqslant \frac{C}{u^d}\int_{(-1,1)^d} f(z)\int_0^{2u}\int_{x+(-u,u)^d} 1_{z+(-t,t)^d}(y)\,\mathrm{d}y\frac{\mathrm{d}t}{t}\,\mathrm{d}z \\
&\leqslant \frac{C}{u^d}\|f\|_{L^1(\mathbb{R}^d)}\int_0^{2u} t^d\frac{\mathrm{d}t}{t} \\
&\leqslant C\|f\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

where we used Fubini in the second inequality.

If $1 < r < \infty$, Proposition 2.A.8 implies the strong type $(1,\infty)$ estimate for $L^\infty(\ell^r)$ from those for $L^\infty(\ell^1), L^\infty(\ell^\infty)$.

Case II: $0 < r < 1$**.** We have

$$\begin{aligned}
\ell^r(t^d F_\phi(f))(E_{x,u}) &\leqslant \ell^1(t^{d-\frac{1}{2}}F_\phi(f))(E_{x,u})\ell^{\frac{r}{1-r}}(t^{\frac{1}{2}})(E_{x,u}) \\
&\leqslant C\Big(\frac{1}{u^d}\int_0^{2u}\int_{x+(-u,u)^d} t^{-\frac{1}{2}}\int_{(-1,1)^d} f(z)1_{y+(-t,t)^d}(z)\,\mathrm{d}z\,\mathrm{d}y\frac{\mathrm{d}t}{t}\Big)\times \\
&\quad\times\Big(\frac{1}{u^d}\int_0^{2u}\int_{x+(-u,u)^d} t^{\frac{r}{2(1-r)}}\,\mathrm{d}y\frac{\mathrm{d}t}{t}\Big)^{\frac{1-r}{r}} \\
&\leqslant C\|f\|_{L^1(\mathbb{R}^d)}\Big(\frac{1}{u^d}\int_0^{2u} t^{d-\frac{1}{2}}\frac{\mathrm{d}t}{t}\Big)\Big(\int_0^{2u} t^{\frac{r}{2(1-r)}}\frac{\mathrm{d}t}{t}\Big)^{\frac{1-r}{r}} \\
&\leqslant C\|f\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

where we used Hölder's inequality with exponents $(1,\frac{r}{1-r})$ in the first inequality, and then we proceeded as in the previous case.

2.6.3 Weak type $(1,q)$ estimates for $0 < r \leqslant \infty$ in the range for $q \neq \infty$ displayed in Fig. 1

We aim to prove that, for every $\lambda > 0$,

$$\lambda^q\mu(\ell^r(t^{d-\frac{d}{q}}F(f)) > \lambda) \leqslant C\|f\|^q_{L^1(\mathbb{R}^d)}.$$

This requires to construct, for every $\lambda > 0$, a set with appropriate outer measure approximating the super level measure at level λ.

For fixed f and $\lambda > 0$, let D_λ be the set

$$D_\lambda = \{x \in \mathbb{R}^d \colon Mf(x) > \lambda^q \|f\|_{L^1(\mathbb{R}^d)}^{1-q}\}.$$

We have

$$|D_\lambda| \leqslant C\lambda^{-q} \|f\|_{L^1(\mathbb{R}^d)}^q,$$

because of the weak type $(1,1)$ estimate for the maximal function operator on $\mathbb{R}^d$.

Let $\{B_i \colon i \in I_\lambda\}$ be a Whitney covering of D_λ up to a set of measure 0 by pairwise disjoint open dyadic cubes in $\mathbb{R}^d$, and denote by x_i and s_i the centre and the side length of B_i, respectively. Let $Q(B_i) = Q_i \in \mathcal{D}$ be the open dyadic box over the cube B_i, and define

$$E_\lambda = \bigcup_{i \in I_\lambda} Q_i \subseteq X.$$

In particular, we have

$$\mu(E_\lambda) \leqslant |D_\lambda| \leqslant C\lambda^{-q} \|f\|_{L^1(\mathbb{R}^d)}^q.$$

We are left with proving that for every $E \in \mathcal{E}$,

$$\ell^r(t^{d-\frac{d}{q}} F(f) 1_{E_\lambda^c})(E) \leqslant C\lambda.$$

If $(x,s) \in E_\lambda^c, x \in D_\lambda$, then $x \in Q_i$ for some $i \in I_\lambda$, $s > s_i$, and there exists $u \in \mathbb{S}^{d-1}$ such that $x + s'u \in D_\lambda^c$, for $cs_i \leqslant s' \leqslant Cs_i$. As a consequence, for $t \geqslant s$, we have

$$t^{d-\frac{d}{q}} F(f)(x,t) \leqslant C(t+s')^{d-\frac{d}{q}} F(f)(x+s'u, t+s').$$

Therefore, we have

$$\ell^r(t^{d-\frac{d}{q}} F(f) 1_{E_\lambda^c})(E) \leqslant C \sup_{x \in D_\lambda^c} \|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\}\times(0,\infty), \frac{\mathrm{d}t}{t})},$$

and it is enough to show that for every $x \in D_\lambda^c$, we have

$$\|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\}\times(0,\infty), \frac{\mathrm{d}t}{t})} \leqslant C\lambda.$$

We split the norm on the left hand side at height $0 < R(x) < \infty$ soon to be fixed

$$\|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\}\times(0,R(x)), \frac{\mathrm{d}t}{t})} + \|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\}\times(R(x),\infty), \frac{\mathrm{d}t}{t})}. \tag{2.6.7}$$

We bound $F(f)$ by (2.6.4) in the first summand obtaining

$$CMf(x)R(x)^{d-\frac{d}{q}},$$

and by (2.6.5) in the second summand obtaining

$$C\|f\|_{L^1(\mathbb{R}^d)}R(x)^{-\frac{d}{q}}.$$

If $0 < r < \infty$, we require the additional hypothesis $q > 1$ to guarantee the L^r-integrability at 0 of the estimate for the first summand.

Optimizing the choice of $R(x)$ with

$$R(x) = CMf(x)^{-\frac{1}{d}}\|f\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}},$$

we get the bound for (2.6.7)

$$CMf(x)^{\frac{1}{q}}\|f\|_{L^1(\mathbb{R}^d)}^{1-\frac{1}{q}}.$$

We conclude by the estimate for every $x \in D_\lambda^c$,

$$Mf(x) \leqslant \lambda^q\|f\|_{L^1(\mathbb{R}^d)}^{1-q}.$$

2.6.4 Counterexample to the strong type $(1,q)$ estimates for $1 \leqslant q < \infty, 0 < r \leqslant \infty$

In the following counterexamples we are going to use test functions ϕ satisfying the condition (2.1.14) with a multiplicative factor different from 1. While it does not effect the nature of the counterexamples, it spares us the definition of other appropriate constants.

For $f = 1_{(-1,1)^d}, \phi = 1_{(-1,1)^d}$, we have

$$F_\phi(f)(y,t) \geqslant t^{-d}1_{\{(-s,s)^d\times\{s\}, s\geqslant 1\}}(y,t).$$

For every $u \geqslant 1$, let

$$E_u = (0,u)^{d+1} \in \mathcal{E}.$$

Then, for $0 < r < \infty$, we have

$$\ell^r(t^{d-\frac{d}{q}}F(f)1_{(\mathbb{R}^d\times(0,u))^c})(E_{2u}) \geqslant \Big(\frac{1}{(2u)^d}\int_u^{2u}\int_{(0,u)^d} t^{-\frac{dr}{q}}\,\mathrm{d}y\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} \geqslant Cu^{-\frac{d}{q}},$$

and it is easy to see that, for $r = \infty$, we have

$$\ell^\infty(t^{d-\frac{d}{q}}F(f)1_{(\mathbb{R}^d\times(0,u))^c})(E_{2u}) = u^{-\frac{d}{q}}.$$

Therefore, for every fixed $u \geqslant 1$, if $A \subseteq X$ is such that

$$\ell^r(t^{d-\frac{d}{q}} F(f) 1_{A^c})(E_{2u}) \leqslant C u^{-\frac{d}{q}},$$

then $A \backslash (\mathbb{R}^d \times (0,u)) \neq \varnothing$, hence we have

$$\mu(\ell^r(t^{d-\frac{d}{q}} F(f)) > C u^{-\frac{d}{q}}) \geqslant u^d.$$

As a consequence, we have

$$\|t^{d-\frac{d}{q}} F_\phi(f)\|^q_{L^q(\ell^r)} \geqslant C \int_0^C u^{-d} \mu(\ell^r(t^{d-\frac{d}{q}} F(f)) > C u^{-\frac{d}{q}}) \frac{\mathrm{d}u}{u} = \infty.$$

2.6.5 Counterexample to the weak type (p,q) estimates for $1 \leqslant q \leqslant p \leqslant \infty, 0 < r < \infty$ and $1 \leqslant q < p \leqslant \infty, r = \infty$

For f, ϕ as above, we have

$$F_\phi(f)(y,t) \geqslant 1_{\{(-1+s,1-s)^d \times \{s\}, s \leqslant 1\}}(y,t).$$

For every $x \in (0,\frac{1}{4})^d, u \leqslant \frac{1}{4}$, let

$$E_{x,u} = (x + (-u,u)^d) \times (0,2u) \in \mathcal{E}.$$

Then, for $1 \leqslant q \leqslant p \leqslant \infty, 0 < r < \infty$, we have

$$\ell^r(t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f))(E_{x,u}) \geqslant \Big(\frac{1}{(2u)^d} \int_0^{2u} \int_{x+(-u,u)^d} t^{\frac{dr}{p}-\frac{dr}{q}} \,\mathrm{d}y \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = \infty,$$

thus exhibiting a counterexample in the case $p = q = \infty$. Moreover, it is easy to see that, for $1 \leqslant q < p \leqslant \infty, r = \infty$, we have

$$\ell^\infty(t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f))(E_{x,u}) = \infty.$$

Let $A \subseteq (-1,1)^d \times (0,\infty)$ be such that, for every $x \in (0,\frac{1}{4})^d, u \leqslant \frac{1}{4}$,

$$\ell^r(t^{\frac{d}{p}-\frac{d}{q}} F(f) 1_{A^c})(E_{x,u}) < \infty. \tag{2.6.8}$$

For every finite collection $\mathcal{E}' \subseteq \mathcal{E}$ covering A, let

$$A_{\mathcal{E}'} = \bigcup_{E \in \mathcal{E}'} \overline{B(E)},$$

where $B(E)$ is the base in $\mathbb{R}^d$ of E, and $\overline{B}$ is the closure of B in $\mathbb{R}^d$. If $A_{\mathcal{E}'} \cap [0, \frac{1}{4}]^d \neq \varnothing$, there would exist x, u such that $E_{x,u} \cap A = \varnothing$, hence contradicting (2.6.8). Therefore, for every $\lambda > 0$, we have

$$\mu(\ell^r(t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f)) > \lambda) \geqslant C,$$

where C does not depend on λ.

As a consequence, for $q \neq \infty$, we have

$$\|t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f)\|^q_{L^{q,\infty}(\ell^r)} \geqslant C \sup_{\lambda>0} \lambda^q = \infty.$$

□

Before stating and proving the embedding result for functions in $H^1(\mathbb{R}^d)$, we recall the definition of H^1-atom. A function f is a H^1-atom associated with the cube $B \subseteq \mathbb{R}^d$ if f is essentially supported in B and

$$\int_B f(x)\,\mathrm{d}x = 0, \qquad \|f\|_{L^\infty(\mathbb{R}^d)} \leqslant |B|^{-1}.$$

Proposition 2.6.2. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then there exists a constant $C = C(d, \varphi)$ such that, for every $f \in H^1(\mathbb{R}^d)$,*

$$\|F_\varphi(f)\|_{L^1(\ell^\infty)} \leqslant C\|f\|_{H^1(\mathbb{R}^d)}.$$

Proof. By Theorem 2.1.2, property (iii), the decay properties of φ and its derivatives, and the definition of the Hardy space $(H^1(\mathbb{R}^d), \|\cdot\|_{H^1(\mathbb{R}^d)})$, it is enough to prove the inequality assuming that φ is a smooth function compactly supported in a cube of side length 2 and f is a H^1-atom associated with a cube B. Moreover, due to the translation invariance of the $L^1(\ell^\infty)$ quasi-norm, we can assume that both f, φ are supported in cubes centred in the origin. Therefore it is enough to show that

$$\|F_\varphi(f)\|_{L^1(\ell^\infty)} \leqslant C.$$

Let $2B$ be the cube with the same centre of B and double the side length. For $0 < t < |B|^{\frac{1}{d}}, y \in 2B$, we have

$$|F_\varphi(f)(y,t)| \leqslant C|B|^{-1},$$

where we used the L^∞ bounds for f.

For $t \geqslant |B|^{\frac{1}{d}}, y \in (-|B|^{\frac{1}{d}} - t, |B|^{\frac{1}{d}} + t)^d$, we have

$$\begin{aligned}
|F_\varphi(f)(y,t)| &= Ct^{-d}\Big|\int_B f(z)\varphi(t^{-1}(y-z))\,\mathrm{d}z\Big| \\
&= Ct^{-d}\Big|\int_B f(z)(\varphi(t^{-1}(y-z)) - \varphi(t^{-1}y))\,\mathrm{d}z\Big| \\
&\leqslant Ct^{-d}\int_B |f(z)|t^{-1}|z|\,\mathrm{d}z \\
&\leqslant C|B|^{\frac{1}{d}}t^{-(d+1)},
\end{aligned}$$

where we used the L^∞ bounds, the localized support and the cancellation property of f together with the smoothness of φ.

For all the others (y,t), we have $F_\varphi(f)$ is 0, since the supports of f and the dilated version of φ are disjoint.

As a consequence, for $\lambda > C|B|^{-1}$, we have

$$\mu(\ell^\infty(F_\varphi(f)) > \lambda) = 0,$$

and for $0 < \lambda \leqslant C|B|^{-1}$, we have

$$\mu(\ell^\infty(F_\varphi(f)) > \lambda) \leqslant C|B|^{\frac{1}{d+1}}\lambda^{-\frac{d}{d+1}}.$$

Therefore, we have

$$\|F_\varphi(f)\|_{L^1(\ell^\infty)} \leqslant C\int_0^{C|B|^{-1}} \mu(\ell^\infty(F_\varphi(f)) > \lambda)\,\mathrm{d}\lambda \leqslant C.$$

□

2.7 Applications

In this section we show some applications of the strong type estimates in Theorem 2.6.1 and Proposition 2.6.2. We use them to give alternative proofs of the Hardy-Littlewood-Sobolev inequality, and the Gagliardo-Nirenberg-Sobolev inequality up to the endpoint in the spirit of the two-step program outlined in the introduction.

Theorem 2.7.1 (HLS inequality)**.** *For $1 < p, q < \infty, 0 < \alpha < d$ such that*

$$\frac{1}{p}+\frac{1}{q}+\frac{\alpha}{d} = 2,$$

there exists a constant $C = C(p,q,d)$ such that, for every $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$,

$$\Big|\int_{\mathbb{R}^{2d}} \frac{f(x)g(y)}{|x-y|^\alpha}\,\mathrm{d}x\,\mathrm{d}y\Big| \leqslant C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

Proof. Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that $\operatorname{supp}\hat{\psi} \subseteq [\frac{1}{2}, 2], \int_0^\infty \hat{\psi}^2(t)\frac{\mathrm{d}t}{t} = 1$, and define $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^d)$ by

$$\hat{\Psi}(\xi) = \hat{\psi}(|\xi|), \hat{\Phi}(\xi) = |\xi|^{\alpha-d}\hat{\psi}(|\xi|).$$

Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. By a frequency localization argument, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^{2d}} \frac{f(x)g(y)}{|x-y|^{\alpha}}\,\mathrm{d}x\,\mathrm{d}y\Big| &\leqslant C\Big|\int_{\mathbb{R}^{2d}} \hat{f}(\xi)\hat{g}(\eta)|\xi-\eta|^{\alpha-d}\delta(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\Big| \\ &\leqslant C\Big|\int_{\mathbb{R}^{2d}\times(0,\infty)} \hat{f}(\xi)\hat{g}(\eta)|\xi-\eta|^{\alpha-d}\delta(\xi+\eta)\hat{\psi}^2(t)\,\mathrm{d}\xi\,\mathrm{d}\eta\frac{\mathrm{d}t}{t}\Big| \\ &\leqslant C\Big|\int_{\mathbb{R}^{d}\times(0,\infty)} t^{d-\alpha}\hat{f}(\xi)\hat{\psi}(|\xi|t)\hat{g}(-\xi)(t|\xi|)^{\alpha-d}\hat{\psi}(|\xi|t)\,\mathrm{d}\xi\frac{\mathrm{d}t}{t}\Big| \\ &\leqslant C\Big|\int_{\mathbb{R}^{d}\times(0,\infty)} t^{d-\alpha}F_{\Psi}(f)(y,t)G_{\Phi}(g)(y,t)\,\mathrm{d}y\frac{\mathrm{d}t}{t}\Big|.\end{aligned}$$

By Theorem 2.1.2, property (i), the integral in the last display is bounded by

$$\|t^{d-\alpha}F_{\Psi}(f)G_{\Phi}(g)\|_{L^1(\ell^1)}.$$

Applying outer Hölder's inequality, Proposition 2.A.5, we estimate it in terms of

$$\|t^{d-\alpha}F_{\Psi}(f)\|_{L^{q'}(\ell^1)}\|G_{\Phi}(g)\|_{L^q(\ell^\infty)},$$

which by the strong type estimates in Theorem 2.6.1 is bounded by

$$\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

A standard approximation argument yields the result for arbitrary $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$. □

Theorem 2.7.2 (GNS inequality)**.** *For $1 \leqslant p < d$, there exists a constant $C = C(p,d)$ such that, for every $f \in W^{1,p}(\mathbb{R}^d)$,*

$$\|f\|_{L^{p*}(\mathbb{R}^d)} \leqslant C\|\nabla f\|_{L^p(\mathbb{R}^d)},$$

where $p_* = \frac{dp}{d-p}$.

Moreover, there exists a constant $C = C(d)$ such that, for every $f \in W^{1,d}(\mathbb{R}^d)$,

$$\|f\|_{BMO(\mathbb{R}^d)} \leqslant C\|\nabla f\|_{L^d(\mathbb{R}^d)}.$$

Proof. Let $\{\varphi_i\}_{i=1}^d$ be a smooth partition of the unity on the set $\{\frac{1}{2} \leqslant |\xi| \leqslant 2\}$ such that $\operatorname{supp}\varphi_i \subseteq \{|\xi_i| > \frac{1}{4d}\} \cap \{\frac{1}{4} \leqslant |\xi| \leqslant 4\}$.

For $\psi \in \mathcal{S}(\mathbb{R})$ as above, let $\Psi_i \in \mathcal{S}(\mathbb{R}^d)$ be defined by

$$\hat{\Psi}_i(\xi) = \frac{\hat{\psi}(|\xi|)}{\xi_i}\varphi_i(\xi).$$

For $1 < p < d$, let $f, g \in \mathcal{S}(\mathbb{R}^d)$. By a frequency localization argument, we have

$$\begin{aligned}
|\langle f, g\rangle| &\leqslant C\Big|\int_{\mathbb{R}^{2d}} \hat{f}(\xi)\hat{g}(\eta)\delta(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\Big| \\
&\leqslant C\Big|\int_{\mathbb{R}^{2d}\times(0,\infty)} \hat{f}(\xi)\hat{g}(\eta)\delta(\xi+\eta)\hat{\psi}^2(t)\,\mathrm{d}\xi\,\mathrm{d}\eta\frac{\mathrm{d}t}{t}\Big| \\
&\leqslant C\sum_{i=1}^{d}\Big|\int_{\mathbb{R}^{d}\times(0,\infty)} t\xi_i\hat{f}(\xi)\frac{\hat{\psi}(|\xi|t)}{t\xi_i}\varphi_i(\xi)\hat{g}(-\xi)\hat{\psi}(|\xi|t)\,\mathrm{d}\xi\frac{\mathrm{d}t}{t}\Big| \\
&\leqslant C\sum_{i=1}^{d}\Big|\int_{\mathbb{R}^{d}\times(0,\infty)} tF_{\Psi_i}(\partial_i f)(y,t)G_{\Psi}(g)(y,t)\,\mathrm{d}y\frac{\mathrm{d}t}{t}\Big|.
\end{aligned}$$

By Theorem 2.1.2, property (i), the integral in the last display is bounded by

$$\sum_{i=1}^{d}\|tF_{\Psi_i}(\partial_i f)G_{\Psi}(g)\|_{L^1(\ell^1)}.$$

Applying outer Hölder's inequality, Proposition 2.A.5, we estimate it in terms of

$$\sum_{i=1}^{d}\|tF_{\Psi_i}(\partial_i f)\|_{L^{p*}(\ell^1)}\|G_{\Psi}(g)\|_{L^{p*'}(\ell^\infty)},$$

which by the strong type estimates in Theorem 2.6.1 is bounded by

$$\sum_{i=1}^{d}\|\partial_i f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p*'}(\mathbb{R}^d)}.$$

The duality between $L^p(\mathbb{R}^d)$ spaces and the density of Schwartz functions in $L^p(\mathbb{R}^d)$ yield the desired inequality. A standard approximation argument yields the result for arbitrary $f \in W^{1,p}(\mathbb{R}^d)$.

For $p = d$, we proceed in the same way with $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in H^1(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$, getting

$$|\langle f, g\rangle| \leqslant \sum_{i=1}^{d}\|tF_{\Psi_i}(\partial_i f)\|_{L^\infty(\ell^1)}\|G_{\Psi}(g)\|_{L^1(\ell^\infty)},$$

which by the strong type estimates in Theorem 2.6.1 and by Proposition 2.6.2 is bounded by

$$\sum_{i=1}^{d}\|\partial_i f\|_{L^d(\mathbb{R}^d)}\|g\|_{H^1(\mathbb{R}^d)}.$$

The duality between the spaces $BMO(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ and the density of Schwartz functions in $H^1(\mathbb{R}^d)$ yield the desired inequality. A standard approximation argument yields the result for arbitrary $f \in W^{1,d}(\mathbb{R}^d)$.

For $p = 1, d > 1$, the statement can be classically proved by the Loomis-Whitney inequality. □

2.A Outer L^p spaces theory

In this Appendix we review the theory of outer L^p spaces in the level of generality discussed in [DT15].

Definition 2.A.1 (Outer measure, pre-measure)**.** *Let X be a set. An outer measure μ on X is a function from $\mathcal{P}(X)$, the power set of X, to $[0,\infty]$ that satisfies the following properties.*

(1) $\mu(\varnothing) = 0$.

(2) If $E \subseteq F$ for two subsets of X, then $\mu(E) \leqslant \mu(F)$.

(3) If $\{E_i\}$ is a countable collection of sets in X, then

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leqslant \sum_{i=1}^{\infty} \mu(E_i).$$

A pre-measure $(\sigma,\mathcal{E})$ on X is defined by a collection $\mathcal{E}$ of subsets of X and a function σ from $\mathcal{E}$ to $[0,\infty)$.

Since $\mathcal{E}$ is implicit in the definition of σ, we drop it in the notation $(\sigma,\mathcal{E})$, and we refer to the pre-measure with the symbol σ.

Definition 2.A.2 (Size)**.** *Let (X,Σ) be a measurable space. A size $(S,\mathcal{A})$ on X is defined by a collection $\mathcal{A}$ of measurable subsets of X and a function S from $\mathcal{M}(X)$, the set of measurable functions on X, to $[0,\infty]^{\mathcal{A}}$, that satisfies, for every $f,g \in \mathcal{M}(X)$, for every $A \in \mathcal{A}$, the following properties.*

(1) If $|f| \leqslant |g|$, then $S(f)(A) \leqslant S(g)(A)$.

(2) $S(\lambda f)(A) = |\lambda| S(f)(A)$ for every $\lambda \in \mathbb{C}$.

(3) There exists a constant C depending only on S but not on f,g,A such that

$$S(f+g)(A) \leqslant C[S(f)(A) + S(g)(A)].$$

Since $\mathcal{A}$ is implicit in the definition of S, we drop it in the notation $(S,\mathcal{A})$, and we refer to the size with the symbol S.

Now, let (X,Σ) be a measurable space, and let $\mathcal{E}$ be a countable collection of measurable subsets of X such that

$$X = \bigcup_{E\in\mathcal{E}} E.$$

Let σ be a pre-measure defined on $\mathcal{E}$ attaining only strictly positive values, and let μ be the outer measure generated by σ as in (2.1.2). Let $(S,\mathcal{A})$ be a size on X.

In particular, let ω be a measure on (X, Σ), and assume that for all $A \in \Sigma$

$$\mu(A) = 0 \Rightarrow \omega(A) = 0.$$

For $0 < r < \infty$, we can define the following sizes. First, let $\ell^\infty_\sigma, \ell^r_\sigma$ be the sizes defined by, for every function $f \in \mathcal{M}(X)$, for every $E \in \mathcal{E}$,

$$\begin{aligned} \ell^\infty_\sigma(f)(E) &= \|f 1_E\|_{L^\infty(X,\omega)}, \\ \ell^r_\sigma(f)(E) &= \sigma(E)^{-\frac{1}{r}} \|f 1_E\|_{L^r(X,\omega)}. \end{aligned} \tag{2.A.1}$$

Next, for $\widetilde{\Sigma}$ defined by

$$\widetilde{\Sigma} = \{A \in \Sigma \colon \mu(A) \in (0, \infty)\},$$

let ℓ^∞, ℓ^r be the sizes defined by, for every function $f \in \mathcal{M}(X)$, for every $A \in \widetilde{\Sigma}$,

$$\begin{aligned} \ell^\infty(f)(A) &= \|f 1_A\|_{L^\infty(X,\omega)}, \\ \ell^r(f)(A) &= \mu(A)^{-\frac{1}{r}} \|f 1_A\|_{L^r(X,\omega)}. \end{aligned} \tag{2.A.2}$$

For every function $f \in \mathcal{M}(X)$, we define

$$\|f\|_{L^\infty(S)} = \sup_{A \in \mathcal{A}} S(f)(A),$$

and the outer $L^\infty(S)$ space to be the set of functions in $\mathcal{M}(X)$ for which this quantity is finite.

For $\lambda > 0$, we define the super level measure

$$\mu(S(f) > \lambda) = \inf\{\mu(A) \colon A \in \Sigma, \|f 1_{A^c}\|_{L^\infty(S)} \leqslant \lambda\}.$$

For $0 < p < \infty$, for every function $f \in \mathcal{M}(X)$, we define

$$\begin{aligned} \|f\|_{L^p(S)} &= \Big(\int_0^\infty p \lambda^p \mu(S(f) > \lambda) \frac{\mathrm{d}\lambda}{\lambda} \Big)^{\frac{1}{p}}, \\ \|f\|_{L^{p,\infty}(S)} &= \Big(\sup_{\lambda > 0} \lambda^p \mu(S(f) > \lambda) \Big)^{\frac{1}{p}}, \end{aligned}$$

and the outer $L^p(S), L^{p,\infty}(S)$ spaces to be the sets of functions in $\mathcal{M}(X)$ for which these quantities are finite, respectively.

We have the following equality between the outer L^p spaces associated with the ℓ^r_σ sizes and the ℓ^r ones under some reasonable assumptions.

Lemma 2.A.3. *Let (X, Σ) be a measurable space, and let $\mathcal{E}$ be a countable collection of measurable subsets of X such that*

$$X = \bigcup_{E \in \mathcal{E}} E.$$

Let σ be a pre-measure defined on $\mathcal{E}$ attaining only strictly positive values, and let μ be the outer measure generated by σ as in (2.1.2)*. Let ω be a measure on (X, Σ), and assume that for all $A \in \Sigma$*

$$\mu(A) = 0 \Rightarrow \omega(A) = 0.$$

Let ℓ^r_σ, ℓ^r be the sizes defined in (2.A.1), (2.A.2)*. Then, for $0 < p, r \leqslant \infty$, for every function $f \in \mathcal{M}(X)$, we have*

$$\|f\|_{L^p(\ell^r_\sigma)} = \|f\|_{L^p(\ell^r)}.$$

Proof. It is enough to prove the equality for $p = \infty$. The case $0 < p < \infty$ follows by this particular case and the definition of the outer L^p quasi-norm.

Case I: $r = \infty$. For every $E \in \mathcal{E}$, we have

$$\mu(E) \leqslant \sigma(E) < \infty. \tag{2.A.3}$$

Now, if $\mu(E) = 0$, then $\omega(E) = 0$, hence

$$\ell^\infty_\sigma(f)(E) = 0 \leqslant \|f\|_{L^\infty(\ell^\infty)}.$$

If $\mu(E) \neq 0$, then $E \in \widetilde{\Sigma}$, hence

$$\ell^\infty_\sigma(f)(E) = \ell^\infty(f)(E) \leqslant \|f\|_{L^\infty(\ell^\infty)}.$$

Next, for every $A \in \widetilde{\Sigma}$, there exists a countable collection $\mathcal{E}_A \subseteq \mathcal{E}$ such that

$$A \subseteq \bigcup_{E \in \mathcal{E}_A} E,$$

hence

$$\ell^\infty(f)(A) \leqslant \sup_{E \in \mathcal{E}_A} \ell^\infty_\sigma(f)(E) \leqslant \|f\|_{L^\infty(\ell^\infty_\sigma)}.$$

Therefore, we have

$$\|f\|_{L^\infty(\ell^\infty_\sigma)} = \sup_{E \in \mathcal{E}} \ell^\infty_\sigma(f)(E) \leqslant \|f\|_{L^\infty(\ell^\infty)} = \sup_{A \in \widetilde{\Sigma}} \ell^\infty(f)(A) \leqslant \|f\|_{L^\infty(\ell^\infty_\sigma)}.$$

Case II: $0 < r < \infty$. Let $E \in \mathcal{E}$. If $\mu(E) = 0$, then $\omega(E) = 0$, hence

$$\ell^r_\sigma(f)(E) = 0 \leqslant \|f\|_{L^\infty(\ell^r)}.$$

If $\mu(E) \neq 0$, then $E \in \widetilde{\Sigma}$, hence we have, by (2.A.3),

$$\ell^r_\sigma(f)(E) = \Big(\sigma(E)^{-1} \sum_{x \in E} \omega(x)|f(x)|^r\Big)^{\frac{1}{r}} \leqslant \Big(\mu(E)^{-1} \sum_{x \in E} \omega(x)|f(x)|^r\Big)^{\frac{1}{r}} \leqslant \ell^r(f)(E).$$

Next, let $A \in \widetilde{\Sigma}$. For every $\varepsilon > 0$, there exists a countable collection $\mathcal{E}_A(\varepsilon) \subseteq \mathcal{E}$ such that

$$A \subseteq \bigcup_{E \in \mathcal{E}_A(\varepsilon)} E,$$
$$\mu(A) \leqslant \sum_{E \in \mathcal{E}_A(\varepsilon)} \sigma(E) \leqslant (1+\varepsilon)\mu(A),$$

hence

$$\begin{aligned}\ell^r(f)(A) &\leqslant \mu(A)^{-\frac{1}{r}} \Big(\sum_{E \in \mathcal{E}_A} \|f 1_E\|^r \Big)^{\frac{1}{r}} \\ &\leqslant \mu(A)^{-\frac{1}{r}} \sup_{E \in \mathcal{E}_A} \ell^r_\sigma(f)(E) \Big(\sum_{E \in \mathcal{E}_A} \sigma(E) \Big)^{\frac{1}{r}} \\ &\leqslant (1+\varepsilon) \|f\|_{L^\infty(\ell^r_\sigma)},\end{aligned}$$

By taking ε arbitrarily small, we obtain the desired inequality.

Therefore, we have

$$\|f\|_{L^\infty(\ell^r_\sigma)} = \sup_{E \in \mathcal{E}} \ell^r_\sigma(f)(E) \leqslant \|f\|_{L^\infty(\ell^r)} = \sup_{A \in \widetilde{\Sigma}} \ell^r(f)(A) \leqslant \|f\|_{L^\infty(\ell^r_\sigma)}.$$

□

Finally, we recall some important results in a setting satisfying the properties stated above with the additional assumption $\mathcal{E} = \mathcal{A}$.

Proposition 2.A.4 (Pull back, Proposition 3.2 in [DT15]). *For $i = 1, 2$, let (X_i, Σ_i) be a measurable space, $(\sigma_i, \mathcal{A}_i)$ be a pre-measure satisfying the properties stated above, and $(S_i, \mathcal{A}_i)$ be a size. Let $\Phi \colon X_1 \to X_2$ be a measurable map. Assume that for every $E_2 \in \mathcal{A}_2$ we have*

$$\mu_1(\Phi^{-1} E_2) \leqslant A \mu_2(E_2).$$

Further assume that for each $E_1 \in \mathcal{A}_1$, there exists $E_2 \in \mathcal{A}_2$ such that for every $f \in \mathcal{M}(X_2)$ we have

$$S_1(f \circ \Phi)(E_1) \leqslant B S_2(f)(E_2).$$

Then we have for every $f \in \mathcal{M}(X_2)$ and $0 < p \leqslant \infty$ and some universal constant C

$$\|f \circ \Phi\|_{L^p(S_1)} \leqslant A^{1/p} B C \|f\|_{L^p(S_2)},$$
$$\|f \circ \Phi\|_{L^{p,\infty}(S_1)} \leqslant A^{1/p} B C \|f\|_{L^{p,\infty}(S_2)}.$$

Proposition 2.A.5 (outer Hölder's inequality, Proposition 3.4 in [DT15])**.** *Let (X, Σ) be a measurable space. Let $(\sigma, \mathcal{A})$, $(\sigma_1, \mathcal{A}_1)$, $(\sigma_2, \mathcal{A}_2)$ be three pre-measures on X satisfying the properties stated above and such that the generated outer measures satisfy $\mu \leqslant \mu_i$, for $i = 1, 2$. Let $(S, \mathcal{A}), (S_1, \mathcal{A}_1), (S_2, \mathcal{A}_2)$ be three sizes on X such that for any $A \in \mathcal{A}$, there exist $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ such that for all $f_1, f_2 \in \mathcal{M}(X)$ we have*

$$S(f_1 f_2)(A) \leqslant S_1(f_1)(A_1) S_2(f_2)(A_2).$$

Let $p, p_1, p_2 \in (0, \infty]$ such that $1/p = 1/p_1 + 1/p_2$. Then, for every $f_1, f_2 \in \mathcal{M}(X)$,

$$\|f_1 f_2\|_{L^p(S)} \leqslant 2 \|f_1\|_{L^{p_1}(S_1)} \|f_2\|_{L^{p_2}(S_2)}.$$

Proposition 2.A.6 (Marcinkiewicz interpolation)**.** *Let (X, Σ) be a measurable space, $(\sigma, \mathcal{A})$ be a pre-measure satisfying the properties stated above, and $(S, \mathcal{A})$ be a size. Let (Y, ν) be a measure space. Let $1 \leqslant p_1 < p_2 \leqslant \infty, 1 \leqslant q_1 \neq q_2 \leqslant \infty$ such that $p_i \leqslant q_i$, for $i = 1, 2$. Let T be a homogeneous quasi-subadditive operator that maps $L^{p_1}(Y, \nu)$ and $L^{p_2}(Y, \nu)$ to the space $\mathcal{M}(X)$ such that*

$$\|T(f)\|_{L^{q_1,\infty}(S)} \leqslant A_1 \|f\|_{L^{p_1}(Y,\nu)},$$
$$\|T(f)\|_{L^{q_2,\infty}(S)} \leqslant A_2 \|f\|_{L^{p_2}(Y,\nu)}.$$

Then we also have

$$\|T(f)\|_{L^q(S)} \leqslant A_1^{\theta} A_2^{1-\theta} C_{p,p_1,p_2} \|f\|_{L^p(Y,\nu)},$$

where $0 < \theta < 1$ is such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{p} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. See Appendix B in [Ste70]. It is enough, for a function h on X, to replace the quantity $\mu(\{h > \lambda\})$ with the super level measure at level λ in the definition of the non-increasing rearrangement h^*. In particular, for a function $h\colon X \to \mathbb{R}$, the function $h^*\colon (0, \infty) \to (0, \infty)$ is defined by

$$h^*(t) = \inf\{\lambda\colon \mu(S(h) > \lambda) \leqslant t\}.$$

□

Proposition 2.A.7 (Radon-Nikodym measure differentiation, Proposition 3.6 in [DT15], Proposition 1.9 in [Ura17])**.** *Let (X, ω) be a measure space, $(\sigma, \mathcal{A})$ be a pre-measure satisfying the properties stated above, and $(S, \mathcal{A})$ be a size. Then, if either for all $A \in \mathcal{A}$*

$$\mu(A) = 0 \Rightarrow \omega(A) = 0,$$

or for all $A \in \mathcal{A}$

$$\mu(A)^{-1} \int_A |f(x)| \, \mathrm{d}\omega(x) \leqslant C \|f\|_{L^\infty(S)},$$

we have, for every $f \in L^\infty(S)$,

$$\Big| \int_X f(x) \, \mathrm{d}\omega(x) \Big| \leqslant C \|f\|_{L^1(S)},$$

where the implicit constant C is independent of $\|f\|_{L^\infty(S)}$.

Proposition 2.A.8. *Let (X, ω) be a measure space, and $(\sigma, \mathcal{A})$ be a pre-measure satisfying the properties stated above. For $0 < r_1 < r_2 \leqslant \infty$, let $\ell_\sigma^{r_1}, \ell_\sigma^{r_2}$ be the sizes defined in* (2.A.1). *Then, for every $0 < p \leqslant \infty, r_1 < r < r_2$, there exists a constant $C = C(p, r, r_1, r_2)$ such that, for every $f \in \mathcal{M}(X)$,*

$$\|f\|_{L^p(\ell_\sigma^r)} \leqslant C(\|f\|_{L^p(\ell_\sigma^{r_1})} + \|f\|_{L^p(\ell_\sigma^{r_2})}),$$
$$\|f\|_{L^{p,\infty}(\ell_\sigma^r)} \leqslant C(\|f\|_{L^{p,\infty}(\ell_\sigma^{r_1})} + \|f\|_{L^{p,\infty}(\ell_\sigma^{r_2})}).$$

Proof. It is enough to prove that there exists a constant $c = c(r, r_1, r_2)$ such that, for every $\lambda > 0$,

$$\mu(\ell_\sigma^r(f) > c\lambda) \leqslant C(\mu(\ell_\sigma^{r_1}(f) > \lambda) + \mu(\ell_\sigma^{r_2}(f) > \lambda)).$$

The desired inequalities follow by multiplying the last display by λ^p and either integrating or taking the supremum over all levels $\lambda > 0$.

Let $E_1, E_2 \subseteq X$ be two sets witnessing the super level measure at λ up to a factor 2 with respect to the sizes $\ell_\sigma^{r_1}$ and $\ell_\sigma^{r_2}$, respectively. In particular, we have

$$2\mu(\ell_\sigma^{r_1}(f) > \lambda) \geqslant \mu(E_1), \qquad 2\mu(\ell_\sigma^{r_2}(f) > \lambda) \geqslant \mu(E_2).$$

Now let $E' = E_1 \cup E_2$. Then, for every $A \subseteq X$, we have

$$\ell_\sigma^r(f 1_{(E')^c})(A) \leqslant c(\ell_\sigma^{r_1}(f 1_{(E_1)^c})(A))^\theta (\ell_\sigma^{r_1}(f 1_{(E_2)^c})(A))^{1-\theta} \leqslant c\lambda,$$

by logarithmic convexity of the L^r spaces, where $0 < \theta < 1$ satisfies

$$\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}.$$

To conclude, we observe that $\mu(E') \leqslant \mu(E_1) + \mu(E_2)$. ☐

2.B Geometry of the dyadic upper half space

In this Appendix we prove Lemma 2.2.2 and Lemma 2.2.3.

Proof of Lemma 2.2.2. For every dyadic box $E \in \mathcal{D}$, we have

$$E = B(E) \times (0, |B(E)|),$$

where $B(E)$ is its base in $\mathbb{R}^d$, and $|B(E)|$ the volume of the base.

Therefore, the desired property follows from the analogous one for the dyadic cubes $B(E_1), B(E_2)$. ☐

We state and prove an auxiliary result.

Lemma 2.B.1. *For every dyadic box $E \in \mathcal{D}$ and for every collection of pairwise disjoint dyadic boxes $\{E_n \colon n \in \mathbb{N}\}$ such that $E_n \subseteq E$ for every $n \in \mathbb{N}$, we have*

$$\sum_{n\in\mathbb{N}} \sigma(E_n) \leqslant \sigma(E).$$

Proof. The dyadic cubes in the collection $\{B(E_n) \colon n \in \mathbb{N}\}$ are pairwise disjoint and such that $B(E_n) \subseteq B(E)$ for every $n \in \mathbb{N}$. Therefore, we have

$$\sum_{n\in\mathbb{N}} \sigma(E_n) = \sum_{n\in\mathbb{N}} |B(E_n)| = \Big| \bigcup_{n\in\mathbb{N}} B(E_n) \Big| \leqslant |B(E)| = \sigma(E).$$

□

Proof of Lemma 2.2.3. The inequality

$$\mu\Big(\bigcup_{n\in\mathbb{N}} D_n \Big) \leqslant \sum_{n\in\mathbb{N}} \sigma(E_n)$$

is trivially satisfied by the definition of μ.

If the left hand side is infinite, there is nothing else to prove. If it is finite, for every $\varepsilon > 0$, let $\mathcal{E}(\varepsilon) \subseteq \mathcal{D}$ be an optimal covering of the union of the elements of $\{D_n \colon n \in \mathbb{N}\}$ witnessing its outer measure up to the multiplicative constant $(1+\varepsilon)$. In particular,

$$\begin{gathered} \bigcup_{n\in\mathbb{N}} D_n \subseteq \bigcup_{E\in\mathcal{E}(\varepsilon)} E, \\ \mu\Big(\bigcup_{n\in\mathbb{N}} D_n \Big) \leqslant \sum_{E\in\mathcal{E}(\varepsilon)} \sigma(E) \leqslant (1+\varepsilon)\mu\Big(\bigcup_{n\in\mathbb{N}} D_n \Big) < \infty. \end{gathered} \tag{2.B.1}$$

Without loss of generality, we can assume the elements of $\mathcal{E}(\varepsilon)$ to be pairwise disjoint. In fact, given two elements of $\mathcal{E}(\varepsilon)$ with nonempty intersection, by Lemma 2.2.2 one is contained in the other, and we can discard the smaller from the collection. The upper bound on $\sigma(E)$ for every $E \in \mathcal{E}(\varepsilon)$ by (2.B.1) guarantees that we still end up with a collection.

Next, we observe that for every E_n, there exists an element of $\mathcal{E}(\varepsilon)$ such that $E_n^+ \cap E \neq \varnothing$, hence $E_n \subseteq E$ by Lemma 2.2.2. Since the elements of $\mathcal{E}(\varepsilon)$ are pairwise disjoint, the element E is unique, hence we can split the collection $\{E_n \colon n \in \mathbb{N}\}$ into pairwise disjoint subcollections $\mathcal{D}(E) = \{E_n \colon n \in \mathbb{N}, E_n \subseteq E\}$, one for each $E \in \mathcal{E}(\varepsilon)$.

By Lemma 2.B.1, we have

$$\sum_{n\in\mathbb{N}} \sigma(E_n) = \sum_{E\in\mathcal{E}(\varepsilon)} \sum_{E_n\in\mathcal{D}(E)} \sigma(E_n) \leqslant \sum_{E\in\mathcal{E}(\varepsilon)} \sigma(E).$$

Combining this with (2.B.1) and taking ε arbitrarily small give the desired inequality. □

Typos

- The fourth display in the proof of Lemma 2.4.1 should be

$$B_\lambda = \bigcup_{E \in \mathcal{E}_\lambda} 10B(E) \subseteq \mathbb{R}^d.$$

- The statement of Lemma 2.5.2 should be

  Lemma. *Let $1 < q \leqslant r_2 \leqslant r_1 \leqslant \infty$. Suppose that a is a $T^1_{r_1}$-atom. Then $t^{d-\frac{d}{q}}a$ is in $T^q_{r_2}$ with norm smaller than* 1.

- The first line of the pre-last display in the proof of Lemma 2.A.3 should be

$$\ell^r(f)(A) \leqslant \mu(A)^{-\frac{1}{r}} \Big(\sum_{E \in \mathcal{E}_A} \|f 1_E\|^r_{L^r(\omega)} \Big)^{\frac{1}{r}}.$$

- The second part of the last display in the statement of Proposition 2.A.6 should be

$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Chapter 3

Double iterated outer L^p space

In this chapter, we report the article [Fra22].

Marco Fraccaroli. Duality for double iterated outer L^p spaces.

in the revised version accepted for forthcoming publication in *Studia Mathematica*. An earlier version is available on arXiv

Marco Fraccaroli. Duality for double iterated outer L^p spaces. *arXiv e-prints*, page arXiv:2104.09472, Apr 2021.

Abstract

We study the double iterated outer L^p spaces, namely the outer L^p spaces associated with three exponents and defined on sets endowed with a measure and two outer measures. We prove that in the case of finite sets, under certain conditions between the outer measures, the double iterated outer L^p spaces are isomorphic to Banach spaces uniformly in the set, the measure, and the outer measures. We achieve this by showing the expected duality properties between them. We also provide counterexamples demonstrating that the uniformity does not hold in arbitrary settings on finite sets without further assumptions, at least in a certain range of exponents. We prove the isomorphism to Banach spaces and the duality properties between the double iterated outer L^p spaces also in the upper half 3-space infinite setting described by Uraltsev, going beyond the case of finite sets.

3.1 Introduction

The theory of L^p spaces for outer measures, or outer L^p spaces, was introduced by Do and Thiele in [DT15] in the context of time-frequency analysis. It provides a framework to encode the boundedness of linear and multilinear operators satisfying certain symmetries in a two-step programme. The programme consists of a version of Hölder's inequality for outer

L^p spaces together with the boundedness of certain embedding maps between classical and outer L^p spaces associated with wave packet decompositions. This scheme of proof turns out to be applicable not only in time-frequency analysis, see for example [AU20a], [AU20b], [AU22], [CDPO18], [DPDU18], [DMT17], [Ura16], [Ura17], [War18], but in other contexts too, see for example [DPGTZK18], [DPO18a], [DT15], [Fra21], [MT17], [TTV15].

Although the theory of outer L^p spaces comes in a broad generality of settings, the outer L^p spaces used in [DT15] are specifically defined by quasi-norms reminiscent in nature of iterated Lebesgue norms. In particular, the two Lebesgue norms involved in the definition of outer L^p quasi-norms are associated with the two structures on a set provided by a measure and an outer measure. We recall that an outer measure μ on a set X is a monotone, subadditive function from $\mathcal{P}(X)$, the power set of X, to the extended positive half line, attaining the value 0 on the empty set. Similarly, in [Ura16] Uraltsev considered outer L^p spaces associated with three structures on a set, namely a measure and two outer measures, once again in the context of time-frequency analysis and in the spirit of the aforementioned two-step programme. Outer L^p spaces associated with three structures where used in [AU20a], [AU20b], [AU22], [DPDU18], [Ura16], [Ura17], [War18].

As a matter of fact, one can define outer L^p spaces associated with arbitrary $(n+1)$ structures on a set, namely a measure and n outer measures. We refer to these spaces as *iterated outer L^p spaces*, and we provide a definition in detail. We start recalling the classical product of L^p spaces on a set with a Cartesian product structure. Given a collection of couples of finite sets with strictly positive weights (X_i, ω_i), we define recursively the product L^p quasi-norms for functions on their Cartesian product as follows. For any $n \in \mathbb{N}$, let

$$Y^n = \prod_{i=1}^{n} X_i,$$

where, for $n = 0$, the empty Cartesian product is intended to be $\{\varnothing\}$. Note that, for any $x \in X_n$, a function f on Y^n defines a function $f(\cdot, x)$ on Y^{n-1}. Given a collection of exponents $p_i \in (0, \infty]$, we define the classical product $\mathbb{L}_n$ quasi-norm of a function f on Y^n, where

$$\mathbb{L}_n = L^{p_n}_{\omega_n}(L^{p_{n-1}}_{\omega_{n-1}}(\dots L^{p_1}_{\omega_1})),$$

by the recursion

$$\|f(x)\|_{\mathbb{L}_0} = |f(x)|, \tag{3.1.1}$$

$$\|f\|_{\mathbb{L}_n} = \|\|f(\cdot, x)\|_{\mathbb{L}_{n-1}}\|_{L^{p_n}(X_n, \omega_n)}. \tag{3.1.2}$$

The theory of outer L^p spaces allows for a generalization of this definition to settings where the underlying set has no Cartesian product structure. For the purpose of this paper, we provide the definition of the iterated outer L^p quasi-norms in the form of a recursion analogous to that in (3.1.1), (3.1.2).

Let X be a finite set together with a collection of outer measures μ_i on it. To avoid cumbersome details, we make the harmless assumption that every μ_i is finite and strictly positive on every nonempty element of $\mathcal{P}(X)$. In fact, it is reasonable that subsets of X on which either of the outer measures is 0 or ∞ should contribute only trivially to the iterated outer L^p spaces on X, and we ignore them altogether. Throughout the paper, we avoid recalling this assumption, but the reader should always consider it implicitly stated whenever we refer to outer measures.

Given a collection of exponents $p_i \in (0,\infty]$, we define the iterated outer $\mathbf{L}_n$ quasi-norm of a function f on X, where

$$\mathbf{L}_n = L^{p_n}_{\mu_n}(\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1})),$$

by the recursion

$$\|f\|_{\mathbf{L}_0} = \sup_{x\in X}|f(x)|, \tag{3.1.3}$$

$$\mathbf{I}_n(f) = \sup_{\varnothing\neq A\subseteq X} \mu_n(A)^{-(p_{n-1})^{-1}}\|f 1_A\|_{\mathbf{L}_{n-1}}, \tag{3.1.4}$$

$$\|f\|_{\mathbf{L}_n} = \begin{cases} \mathbf{I}_n(f), & \text{if } p_n = \infty, \\ \Big[\displaystyle\int_0^\infty p_n \lambda^{p_n} \inf\{\mu_n(B) \colon \mathbf{I}_n(f 1_{B^c}) \leqslant \lambda\} \frac{\mathrm{d}\lambda}{\lambda}\Big]^{\frac{1}{p_n}}, & \text{if } p_n \neq \infty, \end{cases} \tag{3.1.5}$$

where $p_0 = \infty$, and the exponent ∞^{-1} is intended to be 0. We refer to the space defined by the quantity in (3.1.5) as the *iterated outer L^p space* $\mathbf{L}_n$ or $L^{p_n}_{\mu_n}(\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1}))$, where we denote the argument of the supremum in (3.1.4) as

$$\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1})(f)(A) = \mu_n(A)^{-(p_{n-1})^{-1}}\|f 1_A\|_{\mathbf{L}_{n-1}}, \tag{3.1.6}$$

and the infimum in (3.1.5) as

$$\mu_n(\ell^{p_{n-1}}_{\mu_{n-1}}(\dots \ell^{p_1}_{\mu_1})(f) > \lambda) = \inf\{\mu_n(B) \colon \mathbf{I}_n(f 1_{B^c}) \leqslant \lambda\}. \tag{3.1.7}$$

In the language of the L^p theory for outer measure spaces, the quantity in (3.1.6) defines a *size*, and that in (3.1.7) defines the *super level measure* of a function f at level λ with respect to the size.

If the outer measure μ_1 is a measure ω, then we have, for every $p_1 \in (0,\infty]$,

$$\|f\|_{\mathbf{L}_1} = \|f\|_{L^{p_1}(X,\omega)},$$

hence we can begin the recursion in (3.1.3), (3.1.4), (3.1.5) from $\mathbf{L}_1$. In fact, the general case already had this form. The quasi-norm defined by the collections of outer measures μ_i and exponents p_i is the same one defined by the collections of outer measures $\widetilde{\mu}_i$ and exponents $\widetilde{p}_i$, where $\widetilde{\mu}_1$ is the counting measure, $\widetilde{p}_1 = \infty$, and $\widetilde{\mu}_{i+1} = \mu_i, \widetilde{p}_{i+1} = p_i$ for every $i \in \mathbb{N}$.

Therefore, without loss of generality, we always assume that μ_1 is a measure ω associated with a finite and strictly positive weight that we denote by ω as well, with a slight abuse of notation. As before, throughout the paper, we avoid recalling this assumption, but the reader should always consider it implicitly stated whenever we refer to measures.

The classical product $\mathbb{L}_n$ quasi-norms defined in (3.1.2) are a special case of the iterated outer $\mathbf{L}_n$ ones defined in (3.1.5), with the same collection of exponents and the following collection of outer measures μ_i. For any $1 \leqslant j \leqslant n$, we define

$$Y_j^n = \prod_{i=j}^{n} X_i,$$

and we observe that the set Y^n has a canonical partition $\mathcal{Z}_j$, namely

$$\mathcal{Z}_j = \{Y_1^{j-1} \times z \colon z \in Y_j^n\}.$$

where the set $Y_1^0 \times z$ is intended to be the singleton $\{z\}$. For every $A \subseteq Y^n$, let

$$\mu_i(A) = \inf_Z \{\sum_{z \in Z} \prod_{j=i}^{n} \omega_j\big(\pi_j(z)\big)\}, \tag{3.1.8}$$

where π_j is the projection in the coordinate in X_j, and the infimum is taken over all subsets Z of Y_i^n such that A is covered by the elements of $\mathcal{Z}_i$ associated with Z.

The theory of classical product of L^p spaces is well-developed, see for example [BP61]. In the range of exponents $p_i \in [1, \infty]$, the quantities defined in (3.1.2) are norms, and they satisfy the expected duality properties. On the other hand, the theory of outer L^p spaces is a theory of quasi-norms, mainly developed in [DT15] towards the real interpolation features of these quantities like Radon–Nikodym results, Hölder's inequality, and Marcinkiewicz interpolation, due to the aforementioned two-step programme.

However, as showed in [DT15], the iterated outer L^p spaces satisfy some properties analogous to those of the iterated classical ones. In particular, a one-direction "collapsing effect" and a version of Hölder's inequality up to a uniform constant, namely

$$\|f\|_{L^1(X,\omega)} \leqslant C\|f\|_{L^1_{\mu_n}(\ell^1_{\mu_{n-1}}(\dots\ell^1_\omega))}, \tag{3.1.9}$$

$$\sup_g \{\|fg\|_{L^1_{\mu_n}(\dots\ell^1_\omega)} \colon \|g\|_{L^{p_n'}_{\mu_n}(\dots\ell^{p_1'}_\omega)} \leqslant 1\} \leqslant C\|f\|_{L^{p_n}_{\mu_n}(\dots\ell^{p_1}_\omega)}, \tag{3.1.10}$$

where, for every $1 \leqslant i \leqslant n$, the exponent p_i' is the Hölder conjugate of p_i, satisfying

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1.$$

In [Fra21], we studied the opposite inequalities in (3.1.9) and in (3.1.10) in the single iterated case, namely when $n = 2$. We proved the equivalence in both cases up to constants depending on $p_i \in (1, \infty)$ but independent of the measure ω, the outer measure $\mu = \mu_2$, and the set X, as long as X is finite. These in turn imply the equivalence of the outer $L^{p_2}_\mu(\ell^{p_1}_\omega)$ quasi-norms to the norms defined by the supremum in (3.1.10). The endpoint cases $p_1 = \infty$ and $p_2 = 1$ exhibit a different behaviour, and we refer to [Fra21] for more details.

In the present paper, we focus on the analogous opposite inequalities in (3.1.9) and in (3.1.10) in the double iterated case, namely when $n = 3$. Already in this case, the study of the opposite inequalities becomes substantially more difficult due to the interplay between the subadditivity of the two outer measures and the exponents. We start recalling the setting. Let X be a finite set, μ, ν outer measures, and ω a measure. Given three exponents $p, q, r \in (0, \infty]$, we define the double iterated outer L^p space $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ through the quasi-norm in (3.1.5), with $\mu_1 = \omega$, $\mu_2 = \nu$, $\mu_3 = \mu$, and $p_1 = r$, $p_2 = q$, $p_3 = p$.

Before stating our main results, we introduce some auxiliary definitions. They depend on parameters $\Phi, K \geqslant 1$ that we are going to avoid recalling every time.

Given a subset A of X, we say that a subset B of X is a *μ-parent set of A (with parameter Φ)* if $A \subseteq B$ and we have

$$\mu(B) \leqslant \Phi\mu(A). \tag{3.1.11}$$

A *μ-parent function* $\mathbf{B}$ *(with parameter Φ)* is then a monotone function from $\mathcal{P}(X)$ to itself, associating every subset A of X with a μ-parent set (with parameter Φ) $\mathbf{B}(A)$.

Moreover, given a collection $\mathcal{E}$ of subsets of X, we say that a function $\mathcal{C}$ from $\mathcal{P}(X)$ to the set of subcollections of pairwise disjoint elements in $\mathcal{E}$ is a *μ-covering function (with parameter Φ)* if the function $\mathbf{B}_{\mathcal{C}}$ from $\mathcal{P}(X)$ to itself defined by

$$\mathbf{B}_{\mathcal{C}}(A) = \bigcup_{E \in \mathcal{C}(A)} E,$$

is a μ-parent function (with parameter Φ).

Next, we say that a collection $\mathcal{A}$ of pairwise disjoint subsets of X is *ν-Carathéodory (with parameter K)* if, for every subset U of X, we have

$$\sum_{A \in \mathcal{A}} \nu(U \cap A) \leqslant K\nu\big(U \cap \bigcup_{A \in \mathcal{A}} A\big). \tag{3.1.12}$$

Finally, we define two conditions for the quadruple $(X, \mu, \nu, \mathcal{C})$.

Condition 3.1.1 (Canopy). *We say that $(X, \mu, \nu, \mathcal{C})$ satisfies the* canopy condition (with parameters Φ, K) *if $\mathcal{C}$ is a μ-covering function (with parameter Φ), and for every ν-Carathéodory collection (with parameter K) $\mathcal{A}$, for every subset D of X disjoint from $\mathbf{B}_{\mathcal{C}}\big(\bigcup_{A \in \mathcal{A}} A\big)$, the collection $\mathcal{A} \cup \{D\}$ is still ν-Carathéodory (with parameter K).*

Condition 3.1.2 (Crop). *We say that $(X, \mu, \nu, \mathcal{C})$ satisfies the* crop condition (with parameters Φ, K) *if $\mathcal{C}$ is a μ-covering function (with parameter Φ), and for every collection*

$\mathcal{A}$ in $\mathcal{E}$, there exists a ν-Carathéodory subcollection (with parameter K) $\mathcal{D}$ of $\mathcal{A}$ such that, for every subset F of X disjoint from $\bigcup_{D \in \mathcal{D}} D$, we have

$$\mathbf{B}_{\mathcal{C}}(F) = \mathbf{B}_{\widetilde{\mathcal{C}}}(F),$$

where

$$\widetilde{\mathcal{C}}(F) = \mathcal{C}(F) \backslash \mathcal{A}.$$

We are now ready to state our main results.

Theorem 3.1.3. *For all $q, r \in (0, \infty]$, $\Phi, K \geqslant 1$, there exist constants $C_1 = C_1(q, r, \Phi, K)$, $C_2 = C_2(q, r, \Phi, K)$ such that the following property holds true.*

Let X be a finite set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function such that $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 3.1.1. Then, for every function $f \in L^q_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$C_1^{-1} \|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C_2 \|f\|_{L^q_\nu(\ell^r_\omega)}. \tag{3.1.13}$$

If $q \leqslant r$ or $q = \infty$, the constant C_1 does not depend on Φ, K.

If $q \geqslant r$, the constant C_2 does not depend on Φ, K.

Theorem 3.1.4. *For all $p, q \in (1, \infty)$, $r \in [q, \infty)$, $\Phi, K \geqslant 1$, there exists a constant $C = C(p, q, r, \Phi, K)$ such that the following property holds true.*

Let X be a finite set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function such that $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 3.1.1. Then

(i) For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$C^{-1} \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \sup_{\|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1} \|fg\|_{L^1(X,\omega)} \leqslant C \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{3.1.14}$$

(ii) For every collection of functions $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$\|\sum_{n \in \mathbb{N}} f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \sum_{n \in \mathbb{N}} \|f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{3.1.15}$$

For all $p, q \in (1, \infty)$, $r \in (1, q]$, $\Phi, K \geqslant 1$, there exists a constant $C = C(p, q, r, \Phi, K)$ such that the analogous property holds true for every finite set X, outer measures μ, ν, measure ω, and μ-covering function $\mathcal{C}$ such that $(X, \mu, \nu, \mathcal{C})$ satisfies the crop condition 3.1.2.

If $q = r$, the constant C does not depend on Φ, K.

The first result describes one instance of the "collapsing effect". When we have two consecutive outer L^p space structures associated with the same exponent, under certain conditions, the "exterior" one can be disregarded. We recall that, as a consequence of the "collapsing effect" in the single iterated case, property (i) of Theorem 1.1 in [Fra21], for all $p, r \in (0, \infty]$, we have

$$C^{-1}\|f\|_{L^p_\mu(\ell^r_\omega)} \leqslant \|f\|_{L^p_\mu(\ell^r_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^p_\mu(\ell^r_\omega)},$$

where the constant $C = C(p, r)$ does not depend on Φ, K, and it is uniform in X, μ, ν, ω. Hence, the double iterated outer L^p spaces are reduced to single iterated ones. In particular, when $p = q = r \in (0, \infty]$, we have the full "collapsing effect"

$$C^{-1}\|f\|_{L^r(X,\omega)} \leqslant \|f\|_{L^r_\mu(\ell^r_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^r(X,\omega)}, \tag{3.1.16}$$

with constant $C = C(r)$ uniform in X, μ, ν, ω.

The second result yields the sharpness of outer Hölder's inequality. As a consequence, the iterated outer $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ quasi-norm inherits from the $L^1(X, \omega)$-pairing a quasi-triangle inequality up to a constant uniform in the number of the summands, which is stated in the second property. Therefore, in the prescribed range of exponents, the double iterated outer L^p space is uniformly isomorphic to a Banach space with norm defined by the supremum in (3.1.14). Moreover, it is the Köthe dual space of the outer $L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))$ space, and we refer to [Fra21] for an explanation of the use of the term Köthe duality in this context.

The main focus of both of the theorems is on the dependence of the constants in (3.1.13), (3.1.14), and (3.1.15). A priori, for every fixed finite setting (X, μ, ν, ω) the constants are finite, but they depend on (X, μ, ν, ω). The theorems state that the constants depend on (X, μ, ν, ω) only through the parameters Φ, K associated with the canopy condition 3.1.1 or the crop condition 3.1.2. Moreover, we can exhibit counterexamples showing that, at least for the exponents p, q, r in a certain range, the constants cannot be chosen uniformly in Φ, K. We present the counterexamples in Subsection 3.3.4. It might be of interest to provide conditions weaker than the canopy condition 3.1.1 and the crop condition 3.1.2 that would still give a control on the constants. However, this line of research is beyond the scope of the paper. We also comment that the range of exponents p, q, r interested by the aforementioned counterexamples points out a substantial difference between the cases of single iterated and double iterated outer L^p spaces. In the former case, there are no pathological behaviours of the outer L^p spaces in the range of exponents $(1, \infty)^2$. In the latter case, as we describe in Subsection 3.3.4, the range of exponents interested by the counterexamples contains an open subset of $(1, \infty)^3$. Finally, we mention the dichotomy between the cases $q > r$ and $q < r$ in the statement of the two theorems, in particular in view of the reduction to the single iterated outer L^p spaces in the case $q = r$. In Theorem 3.1.3 the dichotomy is in part explained by the counterexamples we exhibit in Subsection 3.3.4. It would be interesting to clarify whether in Theorem 3.1.4 the dichotomy is an intrinsic feature of the problem or it is just an artefact of the argument used in the proof. If the former case were true, it would be interesting to clarify how the dichotomy relates to the conditions guaranteeing a control on the constants.

Before moving on, we comment on the definition of ν-Carathéodory collections and the conditions we stated before the results. We start observing that the Carathéodory measurability test with respect to an outer measure μ^* corresponds to checking that the couple $\{E, E^c\}$ is μ^*-Carathéodory with parameter 1. In particular, when ν is a measure, every collection of pairwise disjoint measurable subsets of X is ν-Carathéodory with parameter $K = 1$. This fact implies that, in the single iterated case, we can always deal with ν-Carathéodory collections, which come with desirable properties. In particular, for every set X, outer measure μ, measure ω, the quadruple $(X, \mu, \omega, \mathrm{Id})$ satisfies both the canopy condition 3.1.1 and the crop condition 3.1.2 with parameters $\Phi = K = 1$.

The extension of the results stated in Theorem 3.1.3 and Theorem 3.1.4 to infinite settings under reasonable assumptions should not be a surprise. However, this level of generality is beyond the scope of the paper. We concern ourselves only with two specific infinite settings, namely the one described by Uraltsev in [Ura16] and a slight variation of it, both of them defined on the upper half 3-space. Although not equivalent, these settings exhibit similar geometric properties. We focus mainly on the latter, which allows for a better exploitation of them.

We briefly recall the setting that we describe in detail in Subsection 3.4.3. Let X be the upper half 3-space $\mathbb{R} \times (0, \infty) \times \mathbb{R}$, and ω the measure induced on it by the Lebesgue measure $\mathrm{d}y \, \mathrm{d}t \, \mathrm{d}\eta$ on $\mathbb{R}^3$. On X, we define two outer measures by means of the following covering construction. Given a collection $\mathcal{S}$ of subsets of X and a pre-measure $\sigma \colon \mathcal{S} \to (0, \infty)$, we define the outer measure $\mu \colon \mathcal{P}(X) \to [0, \infty]$ on an arbitrary subset A of X by

$$\mu(A) = \inf\{ \sum_{S \in \mathcal{S}'} \sigma(S) \colon \mathcal{S}' \subseteq \mathcal{S}, A \subseteq \bigcup_{S \in \mathcal{S}'} S \}. \tag{3.1.17}$$

First, for any dyadic interval $I \subseteq \mathbb{R}$, let $D(I)$ be the dyadic strip given by the Cartesian product between I, the interval $(0, |I|]$ and $\mathbb{R}$. Let $\mathcal{D}$ be the collection of all the dyadic strips, and, for every $D(I) \in \mathcal{D}$, let σ be the length of the base I.

Second, for any couple of dyadic intervals $I, \widetilde{I} \subseteq \mathbb{R}$ with inverse lengths, let $T(I, \widetilde{I})$ be the dyadic tree given by the union of the Cartesian products between a dyadic interval $J \subseteq I$, the interval $(0, |J|]$, and the dyadic interval $\widetilde{J} \supseteq \widetilde{I}$ with inverse length of J. Let $\mathcal{T}$ be the collection of all the dyadic trees, and, for every $T(I, \widetilde{I}) \in \mathcal{T}$, let τ be the length of the base I.

Now, let μ, ν be the outer measures on X associated with $(\mathcal{D}, \sigma), (\mathcal{T}, \tau)$ respectively as in (3.1.17). As we will see in Appendix 3.A, for every dyadic strip D in $\mathcal{D}$ and every dyadic tree T in $\mathcal{T}$, we have

$$\mu(D) = \sigma(D), \qquad \nu(T) = \tau(T).$$

We define the double iterated outer L^p space $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ in the upper half 3-space setting through the quasi-norm in (3.1.5) for ω-measurable functions. We use $\mu_1 = \omega$, $\mu_2 = \nu$, $\mu_3 = \mu$, and we restrict the supremum in $\mathbf{I}_1$ to the ω-measurable sets, that in $\mathbf{I}_2$ to the dyadic trees in $\mathcal{T}$, and that in $\mathbf{I}_3$ to the dyadic strips in $\mathcal{D}$.

In this setting, we have both the "collapsing effect" and the sharpness of outer Hölder's inequality described in the finite setting in the previous theorems.

Theorem 3.1.5. *Let (X, μ, ν, ω) be the dyadic upper half 3-space setting just described, $p, q, r \in (0, \infty]$. There exists a constant $C = C(p, q, r)$ such that the following properties hold true.*

(i) For all $q, r \in (0, \infty)$, for every function $f \in L^q_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$C^{-1}\|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^q_\nu(\ell^r_\omega)}. \tag{3.1.18}$$

(ii) For all $p, q, r \in (1, \infty)$, for every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$C^{-1}\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \sup_{\|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))}=1} \|fg\|_{L^1(X,\omega)} \leqslant C\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{3.1.19}$$

(iii) For all $p, q, r \in (1, \infty)$, for every collection of functions $\{f_n : n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$\|\sum_{n\in\mathbb{N}} f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \sum_{n\in\mathbb{N}} \|f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{3.1.20}$$

The result analogous to Theorem 3.1.5 holds true even in the upper half 3-space setting with arbitrary strips and trees originally considered in [Ura16] that we describe in detail in Subsection 3.5.3.

We conclude pointing out that the outer L^p spaces used by Uraltsev are different from those defined in (3.1.5). In [Ura16], the innermost size, namely the quantity in (3.1.6) for $n = 2$, is not defined by a single Lebesgue norm with respect to the measure ω, but by a sum of an L^2 and an L^∞ norms, making it more complicated to treat. The first step in the study of these spaces would be to extend the results stated in Theorem 3.1.5 to the case $r = \infty$. This is likely to be achieved exploiting the geometric properties of the strips and trees in the upper half 3-space in the same fashion of the boxes in the upper half space in [Fra21]. The second step, the one requiring new considerations, would be to address the issue of the variable exponent Lebesgue norm.

Guide to the paper

In Section 3.2, we review some preliminaries about outer L^p quasi-norms and, more specifically, single iterated outer L^p ones from [Fra21]. In Section 3.3, we prove Theorem 3.1.3 and Theorem 3.1.4. Moreover, at least in a certain range of exponents $p, q, r \in (0, \infty]$, we present the counterexamples showing that the constants appearing in the statements of these theorems are not independent of the setting (X, μ, ν, ω), as discussed in the Introduction. In Section 3.4, we describe some settings in which we define a μ-covering function satisfying the canopy condition 3.1.1 and the crop condition 3.1.2. In Section 3.5, we prove Theorem 3.1.5 in the dyadic upper half 3-space setting reducing the problem to an equivalent one in a finite setting via an approximation argument. The proof relies on the geometric properties of the outer measures and the approximation properties of functions in iterated outer L^p spaces that we will prove in Appendix 3.A and Appendix 3.B, respectively.

3.2 Preliminaries

In this section, we make some observations about the outer L^p quasi-norms. Moreover, we review the decomposition result for functions in a single iterated outer L^p space, which is the main ingredient in proving the results corresponding to Theorem 3.1.3 and Theorem 3.1.4 in [Fra21]. It provides a model for the decomposition in the case of double iterated outer L^p spaces that we perform in Section 3.3.

First, for every $p \in (0, \infty)$, we observe that we can replace the integral defining the outer L^p quasi-norm in (3.1.5) by a discrete version of it. For every $\Psi > 1$, we have

$$\|f\|^p_{L^p_\mu(S)} \sim_{\Psi,p} \sum_{k \in \mathbb{Z}} \Psi^{kp} \mu(S(f) > \Psi^k) \sim_{\Psi,p} \sum_{k \in \mathbb{Z}} \Psi^{kp} \sum_{l \geqslant k} \mu(S(f) > \Psi^l), \tag{3.2.1}$$

where S is a size of the form ℓ^r_ω or $\ell^q_\nu(\ell^r_\omega)$, and more generally an arbitrary size in the definition in [DT15]. The equivalences in (3.2.1) follow by the monotonicity of the super level measure, Fubini and the bounds on the geometric series.

Next, let X be a finite set, μ, ν outer measures, and ω a measure. Since μ, ν are finite and strictly positive on every nonempty subset of X, by outer Hölder's inequality, Proposition 3.4 in [DT15], we have

$$\begin{gathered} L^q_\nu(\ell^r_\omega) \subseteq L^\infty_\nu(\ell^r_\omega), \\ L^p_\mu(\ell^q_\nu(\ell^r_\omega)) \subseteq L^\infty_\mu(\ell^q_\nu(\ell^r_\omega)) \cap L^\infty_\mu(\ell^\infty_\nu(\ell^r_\omega)). \end{gathered} \tag{3.2.2}$$

Finally, we recall two results for single iterated outer L^p spaces already appearing, explicitly or implicitly stated, in Proposition 2.1 in [Fra21], with their proofs.

Lemma 3.2.1. *For all $r \in (0, \infty)$, $N \geqslant 1$, there exist constants $C = C(r, N)$, $c = c(N)$ such that the following property holds true.*

Let X be a set, ν an outer measure, and ω a measure. Let $f \in L^\infty_\nu(\ell^r_\omega)$ be a function on X, let $k \in \mathbb{Z}$ satisfy

$$\|f\|_{L^\infty_\nu(\ell^r_\omega)} \in (2^k, 2^{k+1}], \tag{3.2.3}$$

and let A be a subset of X such that

$$\|f 1_A\|^r_{L^r(X,\omega)} > 2^{(k-N)r} \nu(A). \tag{3.2.4}$$

Then we have

$$\nu(A) \leqslant C \nu(\ell^r_\omega(f) > c 2^k). \tag{3.2.5}$$

Proof. Let $\varepsilon > 0$. Let $V(c2^k, \varepsilon)$ be an optimal set associated with the super level measure $\nu(\ell^r_\omega(f) > c2^k)$ up to the multiplicative constant $(1+\varepsilon)$, namely

$$\|f 1_{V(c2^k,\varepsilon)^c}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant c2^k, \tag{3.2.6}$$

$$(1+\varepsilon)\nu(\ell^r_\omega(f) > c2^k) \geqslant \nu\big(V(c2^k,\varepsilon)\big), \tag{3.2.7}$$

where c will be fixed later. We have

$$\begin{aligned}
\nu\big(V(c2^k,\varepsilon)\big) &\geqslant 2^{-(k+1)r}\|f 1_{V(c2^k,\varepsilon)} 1_A\|^r_{L^r(X,\omega)} \\
&\geqslant 2^{-(k+1)r}\big(\|f 1_A\|^r_{L^r(X,\omega)} - \|f 1_{A\setminus V(c2^k,\varepsilon)}\|^r_{L^r(X,\omega)}\big) \\
&\geqslant 2^{-(k+1)r}\big(2^{(k-N)r} - c^r 2^{kr}\big)\nu(A),
\end{aligned}$$

where we used the monotonicity of ν and (3.2.3) in the first inequality, the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets in the second, (3.2.4) and (3.2.6) in the third. By choosing

$$c = 2^{-N-1},$$

and taking ε arbitrarily small, the previous chain of inequalities together with (3.2.7) yields the desired inequality in (3.2.5). □

Proposition 3.2.2. *For all $q, r \in (0,\infty)$, there exist constants $C = C(q,r)$, $c = c(q,r)$ such that the following decomposition properties hold true.*

Let X be a finite set, ν an outer measure, ω a measure. For every function $f \in L^q_\nu(\ell^r_\omega)$ on X, there exists a collection $\{U_j : j \in \mathbb{Z}\}$ of pairwise disjoint subsets of X such that, if we set

$$V_j = \bigcup_{l \geqslant j} U_l,$$

then, for every $j \in \mathbb{Z}$, we have

$$\ell^r_\omega(f 1_{V^c_{j+1}})(U_j) > 2^j, \qquad \textit{when } U_j \neq \varnothing, \tag{3.2.8}$$

$$\|f 1_{V^c_j}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant 2^j, \tag{3.2.9}$$

$$\nu(\ell^r_\omega(f) > 2^j) \leqslant \nu(V_j), \tag{3.2.10}$$

$$\nu(U_j) \leqslant C\nu(\ell^r_\omega(f) > c2^j). \tag{3.2.11}$$

In particular, we have

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} \sim_{r,q} \sum_{j\in\mathbb{Z}} 2^{jq}\nu(U_j) \sim_{r,q} \sum_{j\in\mathbb{Z}} 2^{jq} \sum_{l\geqslant j} \nu(U_l). \tag{3.2.12}$$

Proof. The first four statements and their proof appeared already in Proposition 2.1 in [Fra21]. The equivalences in (3.2.12) follow by (3.2.1), (3.2.10), the definition of V_j, (3.2.11), Fubini, and the bounds for the geometric series. □

Throughout the paper, we use the observations made in this section without necessarily further referring to them. For example, the reader should always have in mind the equivalences in (3.2.1) whenever we consider an outer L^p quasi-norm, and the list of properties (3.2.8)–(3.2.12) whenever we perform such a decomposition.

3.3 Equivalence with norms

In this section, we study the equivalence of double iterated outer L^p quasi-norms with norms uniformly in the finite setting.

First, for all $q, r \in (0,\infty)$, we study the q-orthogonality behaviour of the outer $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on disjoint sets. Accordingly, we show decomposition results for functions in the double iterated outer L^p space with respect to a size of the form $\ell^q_\nu(\ell^r_\omega)$. We use them to prove Theorem 3.1.3.

After that, for all $p, q, r \in (1,\infty)$, we produce a function g for which we have a good lower bound on the $L^1(X,\omega)$-pairing with f and a good upper bound on the $L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))$ quasi-norm of g. We use it to prove Theorem 3.1.4.

Finally, we conclude the section with the promised class of counterexamples.

3.3.1 q-orthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norm

We start with a result about the sub- and q-superorthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on arbitrary disjoint sets according to the case distinction $q \geqslant r$ or $q \leqslant r$. We present counterexamples to the validity of the inequality in the opposite directions in both cases $q > r$ or $q < r$ in Subsection 3.3.4.

Lemma 3.3.1. *For all $q \in (0,\infty)$, $r \in (0,\infty]$, there exists a constant $C = C(q,r)$ such that the following properties hold true.*

Let X be a finite set, ν an outer measure, ω a measure. Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of X. Then, for every function f on X, we have

$$\sum_{A\in\mathcal{A}} \|f1_A\|^q_{L^q_\nu(\ell^r_\omega)} \leqslant C\|f1_B\|^q_{L^q_\nu(\ell^r_\omega)}, \qquad \text{for } q \geqslant r, \tag{3.3.1}$$

$$\|f1_B\|^q_{L^q_\nu(\ell^r_\omega)} \leqslant C \sum_{A\in\mathcal{A}} \|f1_A\|^q_{L^q_\nu(\ell^r_\omega)}, \qquad \text{for } q \leqslant r, \tag{3.3.2}$$

where $B = \bigcup_{A\in\mathcal{A}} A$.

Proof. Without loss of generality, we assume $q = 1$. In fact, for $\frac{r}{q} \in (0,\infty]$, we have

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} = \|f^q\|_{L^1_\nu(\ell^{r/q}_\omega)}.$$

Case I: $q = 1, r = \infty$**.** We have

$$\nu(\ell^\infty_\omega(f) > \lambda) = \nu(\{x \in X : |f(x)| > \lambda\}). \tag{3.3.3}$$

Together with the subadditivity of ν, this yields

$$\nu(\ell^\infty_\omega(f1_B) > \lambda) \leqslant \sum_{A \in \mathcal{A}} \nu(\ell^\infty_\omega(f1_A) > \lambda).$$

By integrating in $(0,\infty)$ on both sides, we obtain the desired inequality in (3.3.2).

Case II: $q = 1$, $r \in (0,1]$. We start with the following observation. Let $\mathcal{E}$ be a collection of pairwise disjoint nonempty subsets of X such that, for every $E \in \mathcal{E}$, we have

$$\ell^r_\omega(f)(E) \in (2^j, 2^{j+1}]. \tag{3.3.4}$$

Together with the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets and the subadditivity of ν, this yields

$$\ell^r_\omega(f)\big(\bigcup_{E \in \mathcal{E}} E\big) > \Big(\nu\big(\bigcup_{E \in \mathcal{E}} E\big)^{-1} \sum_{E \in \mathcal{E}} 2^{jr}\nu(E)\Big)^{\frac{1}{r}} \geqslant 2^j. \tag{3.3.5}$$

Next, we have

$$\begin{aligned}
\nu\big(\bigcup_{E \in \mathcal{E}} E\big)^{-1} \sum_{E \in \mathcal{E}} \ell^r_\omega(f)(E)\nu(E) &\leqslant 2^{j+1}\nu\big(\bigcup_{E \in \mathcal{E}} E\big)^{-1} \sum_{E \in \mathcal{E}} \nu(E) \\
&\leqslant 2^{j+1}\Big(\nu\big(\bigcup_{E \in \mathcal{E}} E\big)^{-1} \sum_{E \in \mathcal{E}} \nu(E)\Big)^{\frac{1}{r}} \\
&\leqslant 2\Big(\nu\big(\bigcup_{E \in \mathcal{E}} E\big)^{-1} \sum_{E \in \mathcal{E}} \|f1_E\|^r_{L^r(X,\omega)}\Big)^{\frac{1}{r}} \\
&\leqslant 2\ell^r_\omega(f)\big(\bigcup_{E \in \mathcal{E}} E\big),
\end{aligned}$$

where we used the upper bound on $\ell^r_\omega(f)(E)$ in (3.3.4) for every $E \in \mathcal{E}$ in the first inequality, the subadditivity of ν and $r \leqslant 1$ in the second, the lower bound on $\ell^r_\omega(f)(E)$ in (3.3.4) for every $E \in \mathcal{E}$ in the third, and the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets in the fourth. The previous chain of inequalities yields

$$\sum_{E \in \mathcal{E}} \ell^r_\omega(f)(E)\nu(E) \leqslant 2\ell^r_\omega(f)\big(\bigcup_{E \in \mathcal{E}} E\big)\nu\big(\bigcup_{E \in \mathcal{E}} E\big). \tag{3.3.6}$$

Moreover, let $j \in \mathbb{Z}$ and, for every $k \in \mathbb{Z}$, $k \leqslant j$, let $\mathcal{E}_k$ be a collection of pairwise disjoint subsets of X such that, for every nonempty $E \in \mathcal{E}_k$, we have

$$\ell^r_\omega(f)(E) \in (2^k, 2^{k+1}],$$

and, for every nonempty $\bigcup_{E \in \mathcal{E}_k} E$, we have

$$\ell^r_\omega(f)\big(\bigcup_{E \in \mathcal{E}_k} E\big) \in (2^j, 2^{j+1}].$$

By (3.3.6) applied twice, we have

$$\begin{aligned}\sum_{k\leqslant j}\sum_{E\in\mathcal{E}_k}\ell^r_\omega(f)(E)\nu(E) &\leqslant 2\sum_{k\leqslant j}\ell^r_\omega(f)\Big(\bigcup_{E\in\mathcal{E}_k}\Big)\nu\Big(\bigcup_{E\in\mathcal{E}_k}\Big)\\ &\leqslant 4\ell^r_\omega(f)\Big(\bigcup_{k\leqslant j}\bigcup_{E\in\mathcal{E}_k}\Big)\nu\Big(\bigcup_{k\leqslant j}\bigcup_{E\in\mathcal{E}_k}\Big),\end{aligned}\tag{3.3.7}$$

where the sums run only over the nonempty subsets E and $\bigcup_{E\in\mathcal{E}_k}E$.

Now, let $\{A_j\colon j\in\mathbb{Z}\}$, $\{B_j\colon j\in\mathbb{Z}\}$ be the collections associated with the decomposition in Proposition 3.2.2 of the functions $f1_A$, $f1_B$, respectively. By (3.3.5) and (3.3.7), we can pass from the collection $\{A_j\colon A\in\mathcal{A}, j\in\mathbb{Z}\}$ of pairwise disjoint subsets of X to a collection $\mathcal{E}=\{E_l\colon l\in\mathbb{Z}\}$ of pairwise disjoint subsets of X with strictly fewer elements such that

$$\ell^r_\omega(f)(E_l)\in(2^l,2^{l+1}],\qquad \text{when } E_l\neq\varnothing,\tag{3.3.8}$$

$$\sum_{A\in\mathcal{A}}\sum_{j\in\mathbb{Z}}2^j\nu(A_j)\leqslant C\sum_{l\in\mathbb{Z}}2^l\nu(E_l).\tag{3.3.9}$$

By the monotonicity of ν, we have

$$\|f1_{E_l\cap\left(\bigcup_{k\geqslant l-1}B_k\right)^c}\|^r_{L^r(X,\omega)}\leqslant 2^{(l-1)r}\nu\big(E_l\cap\big(\bigcup_{k\geqslant l-1}B_k\big)^c\big)\leqslant 2^{(l-1)r}\nu(E_l).$$

Together with (3.3.8), this yields

$$\begin{aligned}\sum_{k\geqslant l-1}\|f1_{E_l\cap B_k}\|^r_{L^r(X,\omega)} &= \|f1_{E_l\cap\bigcup_{k\geqslant l-1}B_k}\|^r_{L^r(X,\omega)}\\ &= \|f1_{E_l}\|^r_{L^r(X,\omega)}-\|f1_{E_l\cap(\bigcup_{k\geqslant l-1}B_k)^c}\|^r_{L^r(X,\omega)}\\ &\geqslant c2^{lr}\nu(E_l).\end{aligned}\tag{3.3.10}$$

Therefore, we have

$$\begin{aligned}\sum_{l\in\mathbb{Z}}2^l\nu(E_l) &\leqslant C\sum_{l\in\mathbb{Z}}2^{l(1-r)}\sum_{k\geqslant l-1}\|f1_{E_l\cap B_k}\|^r_{L^r(X,\omega)}\\ &\leqslant C\sum_{k\in\mathbb{Z}}2^{k(1-r)}\sum_{l\leqslant k+1}\|f1_{E_l\cap B_k}\|^r_{L^r(X,\omega)}\\ &\leqslant C\sum_{k\in\mathbb{Z}}2^{k(1-r)}\|f1_{B_k}\|^r_{L^r(X,\omega)}\\ &\leqslant C\sum_{k\in\mathbb{Z}}2^k\nu(B_k),\end{aligned}$$

where we used (3.3.10) in the first inequality, $r\leqslant 1$ in the second, and the r-orthogonality of the classical L^r quasi-norms of functions supported on disjoint sets in the third. Together with (3.2.12) for the collections $\{A_j : j \in \mathbb{Z}\}$, $\{B_j : j \in \mathbb{Z}\}$, and (3.3.9), the previous chain of inequalities yields the desired inequality in (3.3.1).

Case III: $q = 1$, $r \in [1, \infty)$. Let A_j, B_j be defined as before. We have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} 2^j \nu(B_j) &\leqslant \sum_{j\in\mathbb{Z}} 2^{j(1-r)} \|f 1_{B_j}\|^r_{L^r(X,\omega)} \\
&\leqslant \sum_{A\in\mathcal{A}} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} 2^{j(1-r)} \|f 1_{A_k\cap B_j}\|^r_{L^r(X,\omega)} \\
&\leqslant \sum_{A\in\mathcal{A}} \sum_{k\in\mathbb{Z}} \big(2^{k(1-r)} \sum_{j\geqslant k} \|f 1_{A_k\cap B_j}\|^r_{L^r(X,\omega)} + \sum_{j<k} 2^{j(1-r)} \|f 1_{A_k\cap B_j}\|^r_{L^r(X,\omega)}\big) \\
&\leqslant C \sum_{A\in\mathcal{A}} \sum_{k\in\mathbb{Z}} \big(2^{k(1-r)} \|f 1_{A_k}\|^r_{L^r(X,\omega)} + \sum_{j<k} 2^{j(1-r)} 2^{jr} \nu(A_k \cap B_j)\big) \\
&\leqslant C \sum_{A\in\mathcal{A}} \sum_{k\in\mathbb{Z}} \big(2^k \nu(A_k) + \sum_{j<k} 2^j \nu(A_k)\big),
\end{aligned}$$

where we used the r-orthogonality of the classical L^r quasi-norms for functions with disjoint supports in the second and in the fourth inequality, and $r \geqslant 1$ in the third. Together with (3.2.12) for the collections $\{A_j : j \in \mathbb{Z}\}$, $\{B_j : j \in \mathbb{Z}\}$, the previous chain of inequalities yields the desired inequality in (3.3.2). □

We continue with a result about the full q-orthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on disjoint sets forming a ν-Carathéodory collection.

Lemma 3.3.2. *For all $q \in (0, \infty)$, $r \in (0, \infty]$, $K \geqslant 1$, there exist constants $C_1 = C_1(q, r, K)$, $C_2 = C_2(q, r, K)$ such that the following property holds true.*

Let X be a set, ν an outer measure, ω a measure. Let $\mathcal{A}$ be a ν-Carathéodory collection of pairwise disjoint subsets of X. Then, for every function f on X, we have

$$C_1^{-1} \|f 1_B\|^q_{L^q_\nu(\ell^r_\omega)} \leqslant \sum_{A\in\mathcal{A}} \|f 1_A\|^q_{L^q_\nu(\ell^r_\omega)} \leqslant C_2 \|f 1_B\|^q_{L^q_\nu(\ell^r_\omega)}, \tag{3.3.11}$$

where $B = \bigcup_{A\in\mathcal{A}} A$.

Proof. As before, without loss of generality, we assume $q = 1$.

Expanding the definition of the outer $L^1_\nu(\ell^r_\omega)$ quasi-norms in (3.3.11), we have

$$\begin{aligned}
\|f 1_B\|_{L^1_\nu(\ell^r_\omega)} &= \int_0^\infty \nu(\ell^r_\omega(f 1_B) > \lambda) \,\mathrm{d}\lambda, \\
\sum_{A\in\mathcal{A}} \|f 1_A\|_{L^1_\nu(\ell^r_\omega)} &= \int_0^\infty \sum_{A\in\mathcal{A}} \nu(\ell^r_\omega(f 1_A) > \lambda) \,\mathrm{d}\lambda.
\end{aligned}$$

To show the desired inequalities, it is enough to prove that there exist constants $c = c(r,K)$, $C = C(r,K)$ such that, for every $\lambda > 0$, we have

$$\nu(\ell^r_\omega(f1_B) > c\lambda) \leqslant \sum_{A\in\mathcal{A}} \nu(\ell^r_\omega(f1_A) > \lambda) \leqslant C\nu(\ell^r_\omega(f1_B) > \lambda). \tag{3.3.12}$$

By integrating in $(0,\infty)$ on both sides, we obtain the desired inequalities in (3.3.11).

Case I: $r = \infty$**.** By the subadditivity of ν and the ν-Carathéodory condition (3.1.12), we have

$$\begin{aligned}\nu(\{x \in B\colon f(x) > \lambda\}) &\leqslant \sum_{A\in\mathcal{A}} \nu(\{x\in A\colon f(x) > \lambda\}) \\ &\leqslant K\nu(\{x\in B\colon f(x)>\lambda\}).\end{aligned}$$

Together with (3.3.3), this yields the desired inequalities in (3.3.12).

Case II: $r \in (0,\infty)$**.** We start with the first inequality in (3.3.12). Let $\varepsilon > 0$. For every $A \in \mathcal{A}$, let $V(A,\lambda,\varepsilon)$ be an optimal set associated with the super level measure $\nu(\ell^r_\omega(f1_A) > \lambda)$ up to the multiplicative constant $(1+\varepsilon)$, namely

$$\|f1_A 1_{V(A,\lambda,\varepsilon)^c}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant \lambda, \tag{3.3.13}$$

$$(1+\varepsilon)\nu(\ell^r_\omega(f1_A) > \lambda) \geqslant \nu\big(V(A,\lambda,\varepsilon)\big), \tag{3.3.14}$$

and set

$$V = \bigcup_{A\in\mathcal{A}} V(A,\lambda,\varepsilon).$$

For every $U \subseteq X$, we have

$$\begin{aligned}\big(\ell^r_\omega(f1_B1_{V^c})(U)\big)^r &\leqslant \nu(U)^{-1}\sum_{A\in\mathcal{A}} \|f1_A1_{V(A,\lambda,\varepsilon)^c}1_U\|^r_{L^r(X,\omega)} \\ &\leqslant \nu(U)^{-1}\sum_{A\in\mathcal{A}} \lambda^r\nu(U\cap A) \\ &\leqslant K\lambda^r,\end{aligned}$$

where we used the r-orthogonality of the classical L^r quasi-norms of functions with disjoint support in the first inequality, (3.3.13) in the second, and the ν-Carathéodory condition (3.1.12) in the third. Together with the subadditivity of ν and (3.3.14), the previous chain of inequalities yields

$$\nu(\ell^r_\omega(f1_B) > K^{1/r}\lambda) \leqslant (1+\varepsilon)\sum_{A\in\mathcal{A}} \nu(\ell^r_\omega(f1_A) > \lambda).$$

By taking ε arbitrarily small, we obtain the desired first inequality in (3.3.12).

We turn to the second inequality in (3.3.12). Let $\varepsilon > 0$. Let $V(\lambda, \varepsilon)$ be an optimal set associated with the super level measure $\nu(\ell^r_\omega(f 1_B) > \lambda)$ up to the multiplicative constant $(1+\varepsilon)$, namely

$$\|f 1_{V(\lambda,\varepsilon)^c}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant \lambda, \tag{3.3.15}$$

$$(1+\varepsilon)\nu(\ell^r_\omega(f 1_B) > \lambda) \geqslant \nu\big(V(\lambda,\varepsilon)\big). \tag{3.3.16}$$

For every $U \subseteq X$, we have

$$\big(\ell^r_\omega(f 1_A 1_{V(\lambda,\varepsilon)^c})(U)\big)^r \leqslant \nu(U)^{-1} \|f 1_B 1_{V(\lambda,\varepsilon)^c} 1_U\|^r_{L^r(X,\omega)} \leqslant \lambda^r,$$

where we used the monotonicity of the classical L^r quasi-norms in the first inequality, and (3.3.15) in the second. Together with the ν-Carathéodory condition (3.1.12) and (3.3.16), the previous chain of inequalities yields

$$\sum_{A \in \mathcal{A}} \nu(\ell^r_\omega(f 1_A) > \lambda) \leqslant \sum_{A \in \mathcal{A}} \nu(V(\lambda,\varepsilon) \cap A) \leqslant (1+\varepsilon) K \nu(\ell^r_\omega(f 1_B) > \lambda).$$

By taking ε arbitrarily small, we obtain the desired second inequality in (3.3.12). □

3.3.2 Decomposition for double iterated outer L^p spaces.

We start with the result corresponding to Lemma 3.2.1 in the case of sizes given by single iterated outer L^p quasi-norms. The proof relies on the q-suborthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions with disjoint supports as stated in (3.3.1) or in the second inequality in (3.3.11). Therefore, according to the relation between the exponents q, r, we allow the constants to depend on the parameter associated with the ν-Carathéodory collection formed by the disjoint sets.

Lemma 3.3.3. *For all $q \in (0,\infty)$, $r \in (0,\infty]$, $K \geqslant 1$, $N \geqslant 1$, there exist constants $C = C(q,r,K,N)$, $c = c(q,r,K,N)$ such that the following property holds true.*

Let X be a set, μ, ν outer measures, and ω a measure. Let $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ be a function on X, let $k \in \mathbb{Z}$ satisfy

$$\|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \in (2^k, 2^{k+1}], \tag{3.3.17}$$

and let $\mathcal{A}$ be a ν-Carathéodory collection of subsets of X such that, for every $A \in \mathcal{A}$,

$$\|f 1_A\|^q_{L^q_\nu(\ell^r_\omega)} > 2^{(k-N)q} \mu(A). \tag{3.3.18}$$

Then we have

$$\sum_{A \in \mathcal{A}} \mu(A) \leqslant C \mu(\ell^q_\nu(\ell^r_\omega)(f) > c 2^k). \tag{3.3.19}$$

If $q \geqslant r$ and X is finite, the constants C, c do not depend on K.

Proof. **Case I: arbitrary q, r.** Let $\varepsilon > 0$. Let $F(c2^k, \varepsilon)$ be an optimal set associated with the super level measure $\mu(\ell^q_\nu(\ell^r_\omega)(f) > c2^k)$ up to the multiplicative constant $(1+\varepsilon)$, namely

$$\|f 1_{F(c2^k,\varepsilon)^c}\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant c2^k, \tag{3.3.20}$$

$$(1+\varepsilon)\mu(\ell^q_\nu(\ell^r_\omega)(f) > c2^k) \geqslant \mu\big(F(c2^k, \varepsilon)\big), \tag{3.3.21}$$

where c will be fixed later. For $B = \bigcup_{A \in \mathcal{A}} A$, we have

$$\begin{aligned}
\mu\big(F(c2^k, \varepsilon)\big) &\geqslant 2^{-(k+1)q} \|f 1_{F(c2^k,\varepsilon)} 1_B\|^q_{L^q_\nu(\ell^r_\omega)} \\
&\geqslant C 2^{-(k+1)q} \sum_{A \in \mathcal{A}} \|f 1_{F(c2^k,\varepsilon)} 1_A\|^q_{L^q_\nu(\ell^r_\omega)} \\
&\geqslant C 2^{-(k+1)q} \sum_{A \in \mathcal{A}} (C_\Delta^{-1} \|f 1_A\|_{L^q_\nu(\ell^r_\omega)} - \|f 1_{A \setminus F(c2^k,\varepsilon)}\|_{L^q_\nu(\ell^r_\omega)})^q \\
&\geqslant C 2^{-(k+1)q} \sum_{A \in \mathcal{A}} (C_\Delta^{-1} 2^{k-N} - c2^k)^q \mu(A),
\end{aligned}$$

where we used the monotonicity of μ and (3.3.17) in the first inequality, Lemma 3.3.2 applied to the ν-Carathéodory collection $\mathcal{A}$ in the second, the quasi-triangle inequality for the outer L^p quasi-norm of two summands in the third, and (3.3.18) and (3.3.20) in the fourth. By choosing

$$c = (2^{N+1} C_\Delta)^{-1},$$

and taking ε arbitrarily small, the previous chain of inequalities together with (3.3.21) yields the desired inequality in (3.3.19).

Case II: $q \geqslant r$. We use (3.3.1) from Lemma 3.3.1 applied to every arbitrary collection $\mathcal{A}$ of pairwise disjoint subsets of X in place of Lemma 3.3.2. □

We are now ready to state and prove a series of decomposition results for functions in the outer L^p space with respect to a size of the form $\ell^q_\nu(\ell^r_\omega)$. Although the statements, as well as the proofs, are similar, we provide them separately in order to highlight the differences. The proofs rely on the selection of disjoint subsets where the size achieves the levels Ψ^k, for a certain $\Psi > 1$. The key ingredient in order to perform such a selection exhaustively at each step is the q-suborthogonality of the $L^q_\nu(\ell^r_\omega)$ quasi-norms of functions supported on certain disjoint sets. Therefore, according to the relation between the exponents q, r, we require the canopy condition 3.1.1, and we allow the constants to depend on the parameters associated with it.

We start with a decomposition result in the full range of exponents under the assumption of the canopy condition 3.1.1 on the setting.

Proposition 3.3.4. *For all $p, q, r \in (0, \infty)$, $\Phi, K \geqslant 1$, there exist constants $C = C(p, q, r, \Phi, K)$, $c = c(p, q, r, \Phi, K)$ such that the following property holds true.*

Let X be a finite set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function such that $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 3.1.1. For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, there exists a collection $\{E_k \colon k \in \mathbb{Z}\}$ of pairwise disjoint subsets of X such that, if we set

$$F_k = \mathbf{B}_{\mathcal{C}}\big(\bigcup_{l \geqslant k} E_l\big),$$

then, for every $k \in \mathbb{Z}$, we have

$$\ell^q_\nu(\ell^r_\omega)(f 1_{F^c_{k+1}})(E_k) > c2^k, \qquad \text{when } E_k \neq \varnothing, \tag{3.3.22}$$

$$\|f 1_{F^c_k}\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant 2^k, \tag{3.3.23}$$

$$\mu(\ell^q_\nu(\ell^r_\omega)(f) > 2^k) \leqslant \mu(F_k), \tag{3.3.24}$$

$$\mu(E_k) \leqslant C\mu(\ell^q_\nu(\ell^r_\omega)(f) > c2^k). \tag{3.3.25}$$

In particular, we have

$$\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \sim_{p,q,r,\Phi,K} \sum_{k\in\mathbb{Z}} 2^{kp}\mu(E_k) \sim_{p,q,r,\Phi,K} \sum_{k\in\mathbb{Z}} 2^{kp} \sum_{l\geqslant k} \mu(E_l). \tag{3.3.26}$$

Proof. By (3.2.2), we have $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$. We define the collection $\{E_k \colon k \in \mathbb{Z}\}$ by a backward recursion on $k \in \mathbb{Z}$. For k large enough such that

$$\|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant 2^k,$$

we set E_k to be empty. Now, we fix k and assume to have selected E_l for every $l > k$. In particular, F_{k+1} is already well-defined. If there exists no subset A of X disjoint from F_{k+1} such that

$$\ell^q_\nu(\ell^r_\omega)(f)(A) > 2^k, \tag{3.3.27}$$

then we set E_k to be empty, and proceed the recursion with $k - 1$.

If there exists a subset A of X disjoint from F_{k+1} satisfying (3.3.27), we define an auxiliary ν-Carathéodory collection $\{E_{k,n} \colon n \in \mathbb{N}_k\}$ of subsets of X by a forward recursion on $n \in \mathbb{N}_k$. The existence of A provides the starting point $E_{k,1}$ for the recursion. Now, we fix n, assume to have selected $E_{k,m}$ for every $m \in \mathbb{N}, m < n$ forming a ν-Carathéodory collection, and set

$$F_{k,n-1} = F_{k+1} \cup \mathbf{B}_{\mathcal{C}}\big(\bigcup_{m<n} E_{k,m}\big).$$

If there exists a subset A of X disjoint from $F_{k,n-1}$ satisfying (3.3.27), then we choose such a set A to be $E_{k,n}$. By the canopy condition 3.1.1, we have that the collection $\{E_{k,m} \colon m \leqslant n\}$ is still ν-Carathéodory. If no A satisfying (3.3.27) exists, we set $\mathbb{N}_k$ to be $\{1, \dots, n-1\}$, stop the forward recursion, set

$$E_k = \bigcup_{n\in\mathbb{N}_k} E_{k,n},$$

and proceed the backward recursion with $k-1$.

By construction, we have (3.3.23) and (3.3.24) for every $k \in \mathbb{Z}$. By construction and Lemma 3.3.2 applied to the ν-Carathéodory collection $\{E_{k,n} : n \in \mathbb{N}_k\}$, we have (3.3.22) for every nonempty E_k. To prove (3.3.25), we observe that for every k such that 2^k is greater than the $L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ quasi-norm of f, the statement is true. For every other k, the proof follows by construction and Lemma 3.3.3.

The equivalences in (3.3.26) follow by (3.3.24), the definition of F_k, (3.3.25), Fubini, and the bounds for the geometric series. □

Under the assumption $q \geqslant r$ on the exponents, we can drop the assumption of the canopy condition 3.1.1 on the setting. Moreover, for every function f, the collection $\{E_k : k \in \mathbb{Z}\}$ produced by the decomposition forms a partition of the support of f.

Proposition 3.3.5. *For all $p, q \in (0, \infty)$, $r \in (0, q]$, there exist constants $C = C(p, q, r)$, $c = c(p, q, r)$ such that the following property holds true.*

Let X be a finite set, μ, ν outer measures, ω a measure. For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, there exists a collection $\{E_k : k \in \mathbb{Z}\}$ of pairwise disjoint subsets of X forming a partition of the support of f such that, if we set

$$F_k = \bigcup_{l \geqslant k} E_l.$$

then we have the same properties stated in (3.3.22)–(3.3.26).

Proof. The argument is analogous to that in the previous proof. The only difference is in the definition of E_k, for which we do not need a second forward recursion.

In fact, we fix k and assume to have selected E_l for every $l > k$. In particular, F_{k+1} is already well-defined. We set $\mathcal{E}_k$ to be the collection of nonempty subsets of X disjoint from F_{k+1} satisfying (3.3.27). If $\mathcal{E}_k$ is empty, we set E_k to be empty, and proceed the recursion with $k-1$. If $\mathcal{E}_k$ is not empty, we choose a subcollection $\mathcal{E}'_k$ of $\mathcal{E}_k$ satisfying the following conditions. First, the elements of $\mathcal{E}'_k$ are pairwise disjoint. Moreover, every element of $\mathcal{E}_k$ intersects at least one element of $\mathcal{E}'_k$. We can fulfil these conditions in finitely many steps, due to the finiteness of X. In fact, if there exists an element of $\mathcal{E}_k$ pairwise disjoint from every element of $\mathcal{E}'_k$, we add it to $\mathcal{E}'_k$. Then, we set E_k to be the union of the subsets of X in $\mathcal{E}'_k$, so that the subset F_k satisfies the property in (3.3.23) by construction. By (3.3.1) in Lemma 3.3.1 and the subadditivity of ν, the subset E_k satisfies the property in (3.3.22).

Due to the definition of F_k, the collection $\{E_k : k \in \mathbb{Z}\}$ forms a partition of the support of f. □

In fact, under the assumption of the canopy condition 3.1.1 on the setting, we can obtain a slightly different decomposition result improving that in Proposition 3.3.4 in the full range of exponents. The refinement we obtain is that we produce a partition of the support of the function f in terms of two ν-Carathéodory collections $\{\widetilde{E}^1_k : k \in \mathbb{Z}\}, \{\widetilde{E}^2_k : k \in \mathbb{Z}\}$. These collections are associated with $\{E_k : k \in \mathbb{Z}\}$, the collection of pairwise disjoint subsets of X we define by backward recursion according to the values of the size $\ell^q_\nu(\ell^r_\omega)$, and the collections are involved in an equivalence analogous to (3.3.26). The improvement over Proposition 3.3.4 is clarified by the following observations. First, the collection $\{E_k : k \in \mathbb{Z}\}$ in Proposition 3.3.4 is a ν-Carathéodory collection, but in general it is not a partition of the support of the function f. Next, the collection $\{F_k \backslash F_{k+1} : k \in \mathbb{Z}\}$ in Proposition 3.3.4 is a partition of the support of the function f, but in general it is not a ν-Carathéodory collection. Obtaining a partition of the support of the function f in terms of ν-Carathéodory collections is important in order to prove Theorem 3.1.3. The minor price we have to pay to obtain the refinement described before is to change the levels from $\{2^k : k \in \mathbb{Z}\}$ to $\{\Psi^k : k \in \mathbb{Z}\}$, for a certain $\Psi > 1$ depending on the exponents and the parameters.

Proposition 3.3.6. *For all $p, q, r \in (0, \infty)$, $\Phi, K \geqslant 1$, there exist constants $C = C(p, q, r, \Phi, K)$, $c = c(p, q, r, \Phi, K)$, $\Psi = \Psi(\Phi, p)$ such that the following property holds true.*

Let X be a set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function such that $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 3.1.1. For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, there exists a collection $\{E_k : k \in \mathbb{Z}\}$ of pairwise disjoint subsets of X such that, if we set

$$F_k = \mathbf{B}_{\mathcal{C}}\big(\mathbf{B}_{\mathcal{C}}(F_{k+1} \cup E_k)\big),$$

then we have the same properties stated in (3.3.22)–(3.3.25) with 2^k replaced by Ψ^k.

In particular, the ν-Carathéodory collections $\{\widetilde{E}^1_k : k \in \mathbb{Z}\}, \{\widetilde{E}^2_k : k \in \mathbb{Z}\}$ defined by

$$\widetilde{E}^1_k = \mathbf{B}_{\mathcal{C}}(F_{k+1} \cup E_k) \backslash F_{k+1}, \qquad \widetilde{E}^2_k = F_k \backslash \mathbf{B}_{\mathcal{C}}(F_{k+1} \cup E_k), \tag{3.3.28}$$

form a partition of the support of f, and we have

$$\begin{aligned} \|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} &\sim_{p,q,r,\Phi,K} \sum_{k \in \mathbb{Z}} \Psi^{kp} \mu(E_k) \\ &\sim_{p,q,r,\Phi,K} \sum_{k \in \mathbb{Z}} \Psi^{kp} \big(\mu(\widetilde{E}^1_k) + \mu(\widetilde{E}^2_k)\big). \end{aligned} \tag{3.3.29}$$

Proof. The argument is analogous to that in the proof of Proposition 3.3.4. The only difference is that we replace the levels 2^k with the levels Ψ^k, where

$$\Psi = \Phi^{\frac{3}{p}}.$$

In fact, we define E_k by a double recursion as before, and $\widetilde{E}^1_k, \widetilde{E}^2_k$ as in (3.3.28). Due to their definition, the collections $\{\widetilde{E}^1_k : k \in \mathbb{Z}\}, \{\widetilde{E}^2_k : k \in \mathbb{Z}\}$ are ν-Carathéodory and they form a partition of the support of f.

We turn now to the proof of the desired equivalences in (3.3.29). By the properties corresponding to (3.3.25) and (3.3.24) in this setting, and the definition of F_k, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(E_k) &\leqslant C \sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(\ell^q_\nu(\ell^r_\omega)(f) > c\Psi^k) \\ &\leqslant C\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \\ &\leqslant C \sum_{k\in\mathbb{Z}} \Psi^{kp}\mu(\ell^q_\nu(\ell^r_\omega)(f) > \Psi^k) \\ &\leqslant C \sum_{k\in\mathbb{Z}} \Psi^{kp} \sum_{l\geqslant k} \big(\mu(\widetilde{E}^1_l) + \mu(\widetilde{E}^2_l)\big).\end{aligned}$$

Moreover, by (3.3.28), $\mathcal{C}$ being a μ-covering function, and the definition of Ψ, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \Psi^{kp} \sum_{l\geqslant k} \big(\mu(\widetilde{E}^1_l) + \mu(\widetilde{E}^2_l)\big) &\leqslant C \sum_{k\in\mathbb{Z}} \Psi^{kp} \sum_{l\geqslant k} \sum_{j\geqslant l} \Phi^{2(j-l)}\mu(E_j) \\ &\leqslant C \sum_{k\in\mathbb{Z}} \Psi^{kp} \sum_{j\geqslant k} \Phi^{2(j-k)}\mu(E_j) \\ &\leqslant C \sum_{k\in\mathbb{Z}} \sum_{j\geqslant k} \Phi^{k-j}\Psi^{jp}\mu(E_j) \\ &\leqslant C \sum_{j\in\mathbb{Z}} \Psi^{jp}\mu(E_j).\end{aligned}$$

□

We are now ready to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. The case $q = \infty$ follows by definition. Therefore, without loss of generality, we assume $q = 1$.

Case I: arbitrary $r \in (0, \infty]$. For a function $f \in L^1_\mu(\ell^1_\nu(\ell^r_\omega))$, let $\{E_k : k \in \mathbb{Z}\}$, $\{\widetilde{E}^1_k : k \in \mathbb{Z}\}$, $\{\widetilde{E}^2_k : k \in \mathbb{Z}\}$ be the collections of pairwise disjoint subsets of X as in Proposition 3.3.6. By (3.3.29), the property corresponding to (3.3.22), and Lemma 3.3.2, we have

$$\begin{aligned}\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))} &\leqslant C \sum_{k\in\mathbb{Z}} \Psi^k\mu(E_k) \leqslant C \sum_{k\in\mathbb{Z}} \|f1_{E_k}\|_{L^1_\nu(\ell^r_\omega)} \leqslant C\|\sum_{k\in\mathbb{Z}} f1_{E_k}\|_{L^1_\nu(\ell^r_\omega)} \\ &\leqslant C\|f\|_{L^1_\nu(\ell^r_\omega)}.\end{aligned}$$

Moreover, by the quasi-triangle inequality for the outer L^p quasi-norm of two summands, Lemma 3.3.2, the property corresponding to (3.3.23), and (3.3.29), we have

$$\begin{aligned}\|f\|_{L^1_\nu(\ell^r_\omega)} &\leqslant C(\|\sum_{k\in\mathbb{Z}} f 1_{\widetilde{E}^1_k}\|_{L^1_\nu(\ell^r_\omega)} + \|\sum_{k\in\mathbb{Z}} f 1_{\widetilde{E}^2_k}\|_{L^1_\nu(\ell^r_\omega)}) \\ &\leqslant C(\sum_{k\in\mathbb{Z}} \|f 1_{\widetilde{E}^1_k}\|_{L^1_\nu(\ell^r_\omega)} + \sum_{k\in\mathbb{Z}} \|f 1_{\widetilde{E}^2_k}\|_{L^1_\nu(\ell^r_\omega)}) \\ &\leqslant C \sum_{k\in\mathbb{Z}} \Psi^k\big(\mu(\widetilde{E}^1_k) + \mu(\widetilde{E}^2_k)\big) \\ &\leqslant C\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))}.\end{aligned}$$

Case II: $q \geqslant r$**.** For a function $f \in L^1_\mu(\ell^1_\nu(\ell^r_\omega))$, let $\{E_k : k \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of X as in Proposition 3.3.5. By the properties corresponding to (3.3.26) and (3.3.22), and (3.3.1) in Lemma 3.3.1, we have

$$\begin{aligned}\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))} &\leqslant C \sum_{k\in\mathbb{Z}} 2^k \mu(E_k) \leqslant C \sum_{k\in\mathbb{Z}} \|f 1_{E_k}\|_{L^1_\nu(\ell^r_\omega)} \leqslant C\|\sum_{k\in\mathbb{Z}} f 1_{E_k}\|_{L^1_\nu(\ell^r_\omega)} \\ &\leqslant C\|f\|_{L^1_\nu(\ell^r_\omega)}.\end{aligned}$$

Case III: $q \leqslant r$**.** For a function $f \in L^1_\mu(\ell^1_\nu(\ell^r_\omega))$, let $\{A_k : k \in \mathbb{Z}\}$ be the collection of optimal sets associated with the super level measures $\mu(\ell^1_\nu(\ell^r_\omega)(f) > 2^k)$, namely

$$\|f 1_{A^c_k}\|_{L^\infty_\mu(\ell^1_\nu(\ell^r_\omega))} \leqslant 2^k, \tag{3.3.30}$$

$$\mu(\ell^1_\nu(\ell^r_\omega)(f) > 2^k) = \mu(A_k). \tag{3.3.31}$$

By (3.3.2) in Lemma 3.3.1, (3.3.30), the monotonicity of μ, and (3.3.31), we have

$$\begin{aligned}\|f\|_{L^1_\nu(\ell^r_\omega)} &\leqslant C \sum_{k\in\mathbb{Z}} \|f 1_{A_k\setminus A_{k+1}}\|_{L^1_\nu(\ell^r_\omega)} \leqslant C \sum_{k\in\mathbb{Z}} 2^{k+1}\mu(A_k\setminus A_{k+1}) \leqslant C \sum_{k\in\mathbb{Z}} 2^k \mu(A_k) \\ &\leqslant C\|f\|_{L^1_\mu(\ell^1_\nu(\ell^r_\omega))}.\end{aligned}$$

□

3.3.3 Dualizing function candidate

We start recalling the setting. Let $p, q, r \in (1, \infty)$, $\Phi, K \geqslant 1$. Let X be a finite set, μ, ν outer measures, ω a measure, and $\mathcal{C}$ a μ-covering function. For $q < r$, we assume $(X, \mu, \nu, \mathcal{C})$ to satisfy the canopy condition 3.1.1. For $q > r$, we assume $(X, \mu, \nu, \mathcal{C})$ to satisfy the crop condition 3.1.2.

When $q = r$, the double iterated outer L^p quasi-norm is isomorphic to a single iterated one, and the results stated in Theorem 3.1.4 correspond to properties (ii), (iii) of Theorem 1.1 in [Fra21].

When $q \neq r$, for a function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we provide the candidate dualizing function g on X. We distinguish two cases.

Case 1: $q > r$**.** Let $\{E_k : k \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of X associated with the function f and the size $\ell^q_\nu(\ell^r_\omega)$ as in Proposition 3.3.5.

Case 2: $q < r$**.** Let $\{E_k : k \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of X associated with the function f and the size $\ell^q_\nu(\ell^r_\omega)$ as in Proposition 3.3.4.

In both cases, let $\{U^k_j : j \in \mathbb{Z}\}$ be the collection of pairwise disjoint subsets of E_k associated with the function $f 1_{E_k}$ and the size ℓ^r_ω as in Proposition 3.2.2. We define

$$\begin{aligned} f_{k,j}(x) =& f(x) 1_{U^k_j}(x), \\ f_k(x) =& \sum_{j \in \mathbb{Z}} f_{k,j}(x) = f(x) \sum_{j \in \mathbb{Z}} 1_{U^k_j}(x). \end{aligned}$$

When $q > r$, let

$$M = 2 + \Big\lfloor \frac{\log_2 K}{r} \Big\rfloor,$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. For

$$\mathcal{F}^k_j = \{F \in \mathcal{E} : \ell^r_\omega(f_{k,j})(F) \leqslant 2^{j-M}\},$$

let $\mathcal{G}^k_j$ be its ν-Carathéodory subcollection as in the crop condition 3.1.2, and set

$$\widetilde{U}^k_j = U^k_j \setminus \bigcup_{G \in \mathcal{G}^k_j} G.$$

We set

$$W^k_j = \begin{cases} \widetilde{U}^k_j, & \text{for } q > r, \\ U^k_j, & \text{for } q < r. \end{cases}$$

and we define

$$\begin{aligned} g_{k,j}(x) =& f(x)^{r-1} 1_{W^k_j}(x), \\ g_k(x) =& \sum_{j \in \mathbb{Z}} 2^{j(q-r)} g_{k,j}(x) = f(x)^{r-1} \sum_{j \in \mathbb{Z}} 2^{j(q-r)} 1_{W^k_j}(x), \\ g(x) =& \sum_{k \in \mathbb{Z}} 2^{k(p-q)} g_k(x) = f(x)^{r-1} \sum_{k \in \mathbb{Z}} 2^{k(p-q)} \sum_{j \in \mathbb{Z}} 2^{j(q-r)} 1_{W^k_j}(x). \end{aligned} \tag{3.3.32}$$

Lemma 3.3.7. *Let $p, q, r \in (1, \infty)$, $q \neq r$, $\Phi, K \geqslant 1$. There exists a constant $c = c(r, K)$ such that, for every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have*

$$\| f^r_{k,j} 1_{W^k_j} \|_{L^1(X,\omega)} \geqslant c 2^{jr} \nu(U^k_j). \tag{3.3.33}$$

Proof. **Case I:** $q > r$**.** We have

$$\begin{aligned}\|f_{k,j}^r 1_{W_j^k}\|_{L^1(X,\omega)} &\geqslant \|f_{k,j}^r\|_{L^1(X,\omega)} - \sum_{G\in\mathcal{G}_j^k} \|f_{k,j}^r 1_G\|_{L^1(X,\omega)} \\ &\geqslant 2^{jr}\nu(U_j^k) - \sum_{G\in\mathcal{G}_j^k} 2^{(j-M)r}\nu(U_j^k \cap G) \\ &\geqslant 2^{jr}\nu(U_j^k) - K2^{(j-M)r}\nu(U_j^k) \\ &\geqslant c2^{jr}\nu(U_j^k),\end{aligned}$$

where we used (3.2.8) and the control on the size ℓ_ω^r defining the elements of $\mathcal{F}_j^k$ in the second inequality, the ν-Carathéodory condition (3.1.12) for the collection $\mathcal{G}_j^k$ in the third, and the definition of M in the fourth.

Case II: $q < r$**.** The desired inequality follows by (3.2.8). □

The definition of g guarantees the following good lower bound on the classical L^1 norm of fg, and good upper bound on the outer $L_\mu^{p'}(\ell_\nu^{q'}(\ell_\omega^{r'}))$ quasi-norm of g.

Lemma 3.3.8. *Let $p, q, r \in (1, \infty)$, $q \neq r$, $\Phi, K \geqslant 1$. There exists a constant $c = c(p, q, r, \Phi, K)$ such that, for every function $f \in L_\mu^p(\ell_\nu^q(\ell_\omega^r))$ on X, for g defined by* (3.3.32)*, then*

$$\|fg\|_{L^1(X,\omega)} \geqslant c\|f\|^p_{L_\mu^p(\ell_\nu^q(\ell_\omega^r))}.$$

Proof. By (3.3.33) and (3.2.12), we have

$$\begin{aligned}\|fg\|_{L^1(X,\omega)} &= \sum_{k\in\mathbb{Z}} 2^{k(p-q)} \sum_{j\in\mathbb{Z}} 2^{j(q-r)} \|f_{k,j}^r 1_{W_j^k}\|_{L^1(X,\omega)} \\ &\geqslant c \sum_{k\in\mathbb{Z}} 2^{k(p-q)} \sum_{j\in\mathbb{Z}} 2^{jq}\nu(U_j^k) \\ &\geqslant c \sum_{k\in\mathbb{Z}} 2^{k(p-q)} \|f_k\|^q_{L_\nu^q(\ell_\omega^r)}.\end{aligned}$$

For $q < r$, by (3.3.22) and (3.3.26), we have

$$\sum_{k\in\mathbb{Z}} 2^{k(p-q)} \|f_k\|^q_{L_\nu^q(\ell_\omega^r)} \geqslant c \sum_{k\in\mathbb{Z}} 2^{kp}\mu(E_k) \geqslant c\|f\|^p_{L_\mu^p(\ell_\nu^q(\ell_\omega^r))}.$$

For $q > r$, the properties in Proposition 3.3.5 corresponding to (3.3.22) and (3.3.26) yield the analogous chain of inequalities. □

Lemma 3.3.9. *Let $p, q, r \in (1, \infty)$, $q \neq r$, $\Phi, K \geqslant 1$. There exists a constant $C = C(p, q, r, \Phi, K)$ such that, for every function $f \in L_\mu^p(\ell_\nu^q(\ell_\omega^r))$ on X, for g defined by* (3.3.32)*, then*

$$\|g\|^{p'}_{L_\mu^{p'}(\ell_\nu^{q'}(\ell_\omega^{r'}))} \leqslant C\|f\|^p_{L_\mu^p(\ell_\nu^q(\ell_\omega^r))}. \tag{3.3.34}$$

Proof. **Case I: q > r.** Let $\widetilde{k}, j$ be fixed. For every subset F of X, for every subset U of F, we have

$$\begin{aligned}\ell_{\omega}^{r'}(g_{\widetilde{k}} 1_F 1_{(V_j^{\widetilde{k}})^c})(U) &\leqslant \sum_{\widetilde{j}<j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \| g_{\widetilde{k},\widetilde{j}} 1_{U \setminus V_{\widetilde{j}+1}^{\widetilde{k}}} \|_{L^{r'}(X,\omega)}^{r'})^{\frac{1}{r'}} \\ &\leqslant \sum_{\widetilde{j}<j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \| f_{\widetilde{k},\widetilde{j}} 1_{U \setminus V_{\widetilde{j}+1}^{\widetilde{k}}} \|_{L^{r}(X,\omega)}^{r})^{\frac{1}{r'}} \\ &\leqslant c 2^{j(q-1)},\end{aligned}$$

where we used the triangle inequality for the classical $L^{r'}$ norm in the first inequality, and (3.2.9) in the third. The previous chain of inequalities yields

$$\nu(\ell_{\omega}^{r'}(g_{\widetilde{k}} 1_F) > c 2^{j(q-1)}) \leqslant \sum_{\widetilde{j} \geqslant j} \nu(W_{\widetilde{j}}^{\widetilde{k}} \cap F). \tag{3.3.35}$$

Moreover, for every fixed $\widetilde{j} \in \mathbb{Z}$, for $E = \mathbf{B}_{\mathcal{C}}(F)$, we have

$$\nu(W_{\widetilde{j}}^{\widetilde{k}} \cap F) \leqslant C \nu(\ell_{\omega}^{r}(f_{\widetilde{k}} 1_E) > \widetilde{c} 2^{\widetilde{j}}). \tag{3.3.36}$$

In fact, we have two cases.

(i) If $W_{\widetilde{j}}^{\widetilde{k}} \cap F = \varnothing$, the left hand side in (3.3.36) is 0, and the inequality holds true.

(ii) If $W_{\widetilde{j}}^{\widetilde{k}} \cap F \neq \varnothing$, by the crop condition 3.1.2, we have that $E' = \mathbf{B}_{\mathcal{C}}(W_{\widetilde{j}}^{\widetilde{k}} \cap F) \subseteq E$ is covered by a collection of disjoint subsets that are not in $\mathcal{F}_{\widetilde{j}}^{\widetilde{k}}$, so that

$$\ell_{\omega}^{r}(f_{\widetilde{k},\widetilde{j}} 1_E)(U_{\widetilde{j}}^{\widetilde{k}} \cap E') \geqslant \widetilde{c} 2^{\widetilde{j}},$$

hence, by Lemma 3.2.1, we obtain (3.3.36).

Therefore, by (3.3.35) and (3.3.36), we have

$$\begin{aligned}\| g_{\widetilde{k}} 1_F \|_{L_{\nu}^{q'}(\ell_{\omega}^{r'})}^{q'} &\leqslant C \sum_{j \in \mathbb{Z}} 2^{jq} \nu(\ell_{\omega}^{r'}(g_{\widetilde{k}} 1_F) > c 2^{j(q-1)}) \\ &\leqslant C \sum_{j \in \mathbb{Z}} 2^{jq} \sum_{\widetilde{j} \geqslant j} \nu(\ell_{\omega}^{r}(f_{\widetilde{k}} 1_E) > \widetilde{c} 2^{\widetilde{j}}) \\ &\leqslant C \| f_{\widetilde{k}} 1_E \|_{L_{\nu}^{q}(\ell_{\omega}^{r})}^{q}.\end{aligned} \tag{3.3.37}$$

Hence, we have

$$\begin{aligned}\ell^{q'}_\nu(\ell^{r'}_\omega)(g1_{F_k^c})(F) &\leqslant C\sum_{\widetilde{k}<k} 2^{\widetilde{k}(p-q)}(\mu(F)^{-1}\|g_{\widetilde{k}}1_F\|^{q'}_{L^{q'}_\nu(\ell^{r'}_\omega)})^{\frac{1}{q'}}\\ &\leqslant C\sum_{\widetilde{k}<k} 2^{\widetilde{k}(p-q)}(\mu(F)^{-1}\|f_{\widetilde{k}}1_E\|_{L^q_\nu(\ell^r_\omega)})^{\frac{1}{q'}}\\ &\leqslant C2^{k(p-1)},\end{aligned}$$

where we used the quasi-triangle inequality for the outer $L^{q'}_\nu(\ell^{r'}_\omega)$ quasi-norm proved in [Fra21] in the first inequality, (3.3.37) in the second, the property in Proposition 3.3.5 corresponding to (3.3.23) and (3.1.11) in the third. The previous chain of inequalities yields

$$\mu(\ell^{q'}_\nu(\ell^{r'}_\omega)(g) > C2^{k(p-1)}) \leqslant \mu(F_k) \leqslant \widetilde{C}\sum_{\widetilde{k}\geqslant k}\mu(E_{\widetilde{k}}).$$

Together with the property in Proposition 3.3.5 corresponding to (3.3.26), this yields

$$\begin{aligned}\|g\|^{p'}_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} &\leqslant \widetilde{C}\sum_{k\in\mathbb{Z}} 2^{kp}\mu(\ell^{q'}_\nu(\ell^{r'}_\omega)(g) > C2^{k(p-1)})\\ &\leqslant \widetilde{C}\sum_{k\in\mathbb{Z}} 2^{kp}\sum_{\widetilde{k}\geqslant k}\mu(E_{\widetilde{k}})\\ &\leqslant \widetilde{C}\|f\|^p_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.\end{aligned}$$

Case II: $q<r$**.** Let $\widetilde{k}$ be fixed. It is enough to prove that, for every subset F of X, we have

$$\|g_{\widetilde{k}}1_F\|^{q'}_{L^{q'}_\nu(\ell^{r'}_\omega)} \leqslant C\|f_{\widetilde{k}}1_F\|^q_{L^q_\nu(\ell^r_\omega)}. \tag{3.3.38}$$

The desired inequality in (3.3.34) then follows as in the previous case.

Let j be fixed. Let $V(2^j)$ be an optimal set associated with the super level measure $\nu(\ell^r_\omega(f_{\widetilde{k}}1_F) > 2^j)$, namely

$$\|f_{\widetilde{k}}1_F1_{V(2^j)^c}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant 2^j, \tag{3.3.39}$$

$$\nu(\ell^r_\omega(f_{\widetilde{k}}1_F) > 2^j) = \nu\big(V(2^j)\big). \tag{3.3.40}$$

For every subset U of F, we have

$$\begin{aligned}\ell^{r'}_{\omega}(g_{\widetilde{k}} 1_F 1_{V(2^j)^c})(U) &\leqslant \sum_{\widetilde{j}<j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \| g_{\widetilde{k},\widetilde{j}} 1_{U \setminus V^{\widetilde{k}}_{\widetilde{j}+1}} \|^{r'}_{L^{r'}(X,\omega)})^{\frac{1}{r'}} + \\ &\quad + (\nu(U)^{-1} \| \sum_{\widetilde{j}\geqslant j} 2^{\widetilde{j}(q-r)} g_{\widetilde{k},\widetilde{j}} 1_F 1_{U\setminus V(2^j)} \|^{r'}_{L^{r'}(X,\omega)})^{\frac{1}{r'}} \\ &\leqslant \sum_{\widetilde{j}<j} 2^{\widetilde{j}(q-r)} (\nu(U)^{-1} \| f_{\widetilde{k},\widetilde{j}} 1_{U \setminus V^{\widetilde{k}}_{\widetilde{j}+1}} \|^{r}_{L^{r}(X,\omega)})^{\frac{1}{r'}} + \\ &\quad + 2^{j(q-r)} (\nu(U)^{-1} \| \sum_{\widetilde{j}\geqslant j} f_{\widetilde{k},\widetilde{j}} 1_F 1_{U\setminus V(2^j)} \|^{r'}_{L^{r'}(X,\omega)})^{\frac{1}{r'}} \\ &\leqslant c 2^{j(q-1)},\end{aligned}$$

where we used the triangle inequality for the classical $L^{r'}$ norm in the first inequality, the condition $q < r$ in the second, (3.3.23) and (3.3.39) in the third. Together with (3.3.40), the previous chain of inequalities yields, for every $j \in \mathbb{Z}$,

$$\nu(\ell^{r'}_{\omega}(g_{\widetilde{k}} 1_F) > c 2^{j(q-1)}) \leqslant \nu(\ell^{r}_{\omega}(f_{\widetilde{k}} 1_F) > 2^j).$$

The inequality in (3.3.38) follows multiplying by 2^{jq} and summing in $j \in \mathbb{Z}$ on both sides. □

We are now ready to prove Theorem 3.1.4.

Proof of Theorem 3.1.4. When $q = r$, the double iterated outer L^p quasi-norm is isomorphic to a single iterated one, and the proof corresponds to the one of properties (ii), (iii) of Theorem 1.1 in [Fra21].

When $q \neq r$, we proceed as follows.

Property (i). By (3.1.16), the $L^1(X,\omega)$-pairing of two functions f, g is equivalent to the outer $L^1_\mu(\ell^1_\nu(\ell^1_\omega))$ quasi-norm of the product fg. The second inequality in (3.1.14) is then given by outer Hölder's inequality, Proposition 3.4 in [DT15]. The first inequality in (3.1.14) is a corollary of Lemma 3.3.8 and Lemma 3.3.9 for $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$.

Property (ii). The inequality in (3.1.15) is a corollary of the triangle inequality for the $L^1(X,\omega)$ norm and property (i). □

3.3.4 Counterexamples

For every $m \in \mathbb{N}$, we introduce the finite setting

$$\begin{aligned} X_m &= \{x_i \colon 1 \leqslant i \leqslant m\}, & \\ \omega_m(A) &= \mu_m(A) = |A|, & \text{for every } A \subseteq X_m, \\ \nu_m(A) &= 1, & \text{for every } \varnothing \neq A \subseteq X_m, \\ f_i &= 1_{x_i}, & \text{for every } 1 \leqslant i \leqslant m, \\ f &= 1_{X_m}. & \end{aligned}$$

In particular, the collection of singletons $\{\{x_i\}: 1 \leqslant i \leqslant m\}$ satisfies the ν_m-Carathéodory condition with parameter $K_m \geqslant m$.

First, we observe that, for every exponent $r \in (0, \infty]$, for every function g, for every nonempty subset A of X_m, we have

$$\ell^r_{\omega_m}(g)(A) = \|g 1_A\|_{L^r(X_m, \omega_m)}.$$

Therefore, for every exponent $r \in (0, \infty]$, for every function g, we have

$$\nu_m(\ell^r_{\omega_m}(g) > \lambda) = \begin{cases} \nu_m(X_m) = 1, & \text{for } \lambda \in [0, \|g\|_{L^\infty_{\nu_m}(\ell^r_{\omega_m})}), \\ \nu_m(\varnothing) = 0, & \text{for } \lambda \in [\|g\|_{L^\infty_{\nu_m}(\ell^r_{\omega_m})}, \infty), \end{cases}$$

where, here and later as well, for every level λ, we provide a subset of X_m realizing the infimum in the definition of the super level measure in (3.1.7).

Hence, for all exponents $q, r \in (0, \infty]$, we have

$$\|g\|_{L^q_{\nu_m}(\ell^r_{\omega_m})} = \|g\|_{L^\infty_{\nu_m}(\ell^r_{\omega_m})} = \|g\|_{L^r(X_m, \omega_m)}.$$

In particular, for every exponent $r \in (0, \infty]$, we have

$$\sum_{i=1}^m \|f_i\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = \sum_{i=1}^m 1 = m,$$
$$\|\sum_{i=1}^m f_i\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = \|f\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = m^{\frac{1}{r}}.$$

When $r \in (0, \infty]$, $r \neq 1$, one of the constants C_1, C_2 of q-super- or suborthogonality in (3.3.11) blows up as m grows to infinity.

Next, we observe that, for all exponents $q, r \in (0, \infty]$, for every function g, for every nonempty subset A of X_m, we have

$$\ell^q_{\nu_m}(\ell^r_{\omega_m})(g)(A) = \mu_m(A)^{-\frac{1}{q}} \|g 1_A\|_{L^q_{\nu_m}(\ell^r_{\omega_m})} = |A|^{-\frac{1}{q}} \|g 1_A\|_{L^r(X_m, \omega_m)},$$

hence, for every exponent $r \in [1, \infty]$, for every strict subset B of X_m, we have

$$\|f 1_{B^c}\|_{L^\infty_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} = 1 = \ell^1_{\nu_m}(\ell^r_{\omega_m})(f 1_{B^c})(\{x_i\}), \qquad \text{for every } x_i \notin B.$$

Therefore, for every exponent $r \in [1, \infty]$, we have

$$\mu_m(\ell^1_{\nu_m}(\ell^r_{\omega_m})(f) > \lambda) = \begin{cases} \mu_m(X_m) = m, & \text{for } \lambda \in [0, 1), \\ \mu_m(\varnothing) = 0, & \text{for } \lambda \in [1, \infty). \end{cases}$$

In particular, for every exponent $r \in [1, \infty]$, we have

$$\|f\|_{L^1_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} = m.$$

When $r \in (1, \infty]$, the constant C_2 of the "collapsing effect" in (3.1.13) blows up as m grows to infinity.

Finally, we observe that, for all exponents $q \in (1, \infty)$, $r \in (1, q]$, for every strict subset B of X_m, we have

$$\|f 1_{B^c}\|_{L^\infty_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} = |X_m \backslash B|^\alpha = \ell^q_{\nu_m}(\ell^r_{\omega_m})(f 1_{B^c})(B^c),$$

where $\alpha = \alpha(r, q) = \frac{1}{r} - \frac{1}{q}$. Therefore, for all exponents $q \in (1, \infty)$, $r \in (1, q]$, we have, for $1 \leqslant i \leqslant m$,

$$\mu_m(\ell^q_{\nu_m}(\ell^r_{\omega_m})(f) > \lambda) = \begin{cases} \mu_m(X_m^{m-i+1}) = m - i + 1, & \text{for } \lambda \in [(i-1)^\alpha, i^\alpha), \\ \mu_m(\varnothing) = 0, & \text{for } \lambda \in [m^\alpha, \infty), \end{cases}$$

where X_m^j is any arbitrary subset of X_m of cardinality j.

In particular, for all exponents $p, q \in (1, \infty)$, $r \in (1, q]$, there exists a constant $c = c(p, q, r)$ such that, for every $m \in \mathbb{N}$ big enough, we have

$$\sum_{i=1}^m \|f_i\|_{L^p_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} = \sum_{i=1}^m 1 = m,$$

$$\|\sum_{i=1}^m f_i\|_{L^p_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} = \|f\|_{L^p_{\mu_m}(\ell^q_{\nu_m}(\ell^r_{\omega_m}))} \geqslant c m^{\frac{1}{p} - \frac{1}{q} + \frac{1}{r}}.$$

Therefore, the constants of the sharpness of outer Hölder's inequality in (3.1.14) and the triangle inequality in (3.1.15) blow up as m grows to infinity when

$$p, q, r \in (1, \infty), \qquad \frac{1}{p} - \frac{1}{q} + \frac{1}{r} > 1.$$

Now, for every $m \in \mathbb{N}$, we slightly modify the previous finite setting

$$\begin{aligned} X_m &= \{x_i \colon 1 \leqslant i \leqslant m\}, & & \\ \omega_m(A) &= |A|, & & \text{for every } A \subseteq X_m, \\ \nu_m(A) &= 1, & & \text{for every } A \subseteq X_m, \\ \sigma_m(\{x_i\}) &= 2^{\beta(i-1)}, & & \text{for every } 1 \leqslant i \leqslant m, \\ f &= 1_{X_m}, & & \end{aligned}$$

where $\beta = \beta(r) = \frac{2}{r}$, and let μ_m be the measure generated via (3.1.17) from σ_m. As in the previous setting, the collection of singletons $\{\{x_i\} \colon 1 \leqslant i \leqslant m\}$ satisfies the ν_m-Carathéodory condition with parameter $K_m \geqslant m$.

As in the previous setting, for all exponents $q, r \in (0, \infty]$, for every function g, for every nonempty subset A of X_m, we have

$$\ell^q_{\nu_m}(\ell^r_{\omega_m})(g)(A) = \mu_m(A)^{-\frac{1}{q}} \|g 1_A\|_{L^q_{\nu_m}(\ell^r_{\omega_m})} = \mu_m(A)^{-\frac{1}{q}} \|g 1_A\|_{L^r(X_m, \omega_m)},$$

hence, for every exponent $r \in (0, 1]$, for every strict subset B of X_m, we have

$$\|f 1_{B^c}\|_{L^\infty_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} = 2^{-\beta(j-1)} = \ell^1_{\nu_m}(\ell^r_{\omega_m})(f 1_{B^c})(\{x_j\}),$$

where $j = \min\{i\colon 1 \leqslant i \leqslant m, x_i \notin B\}$. Therefore, for every exponent $r \in (0, 1]$, we have, for $1 \leqslant j < m$,

$$\mu_m(\ell^1_{\nu_m}(\ell^r_{\omega_m})(f) > \lambda) = \begin{cases} \mu_m(X_m) = \sum_{i=1}^{m} 2^{\beta(i-1)}, & \text{for } \lambda \in [0, 2^{-\beta(m-1)}), \\ \mu_m(X_m^j) = \sum_{i=1}^{j} 2^{\beta(i-1)}, & \text{for } \lambda \in [2^{-\beta j}, 2^{-\beta(j-1)}), \\ \mu_m(\varnothing) = 0, & \text{for } \lambda \in [1, \infty), \end{cases}$$

where $X_m^j = \{x_i\colon 1 \leqslant i \leqslant j\} \subseteq X_m$.

In particular, for every exponent $r \in (0, 1]$, there exists a constant $C = C(r)$ such that we have

$$\|f\|_{L^1_{\nu_m}(\ell^r_{\omega_m})} = m^{\frac{1}{r}},$$
$$\|f\|_{L^1_{\mu_m}(\ell^1_{\nu_m}(\ell^r_{\omega_m}))} \leqslant Cm.$$

When $r \in (0, 1)$, the constant C_1 of the "collapsing effect" in (3.1.13) blows up as m grows to infinity.

3.4 Examples

In this section we present three settings in which we provide a μ-covering function $\mathcal{C}$ satisfying the canopy condition 3.1.1 and the crop condition 3.1.2.

3.4.1 Finite set with three measures

Let X be a finite set, μ, ν, ω be three measures on it. The function $\mathcal{C}$ defined by

$$\mathcal{E} = \{\{x\}\colon x \in X\}, \qquad \mathcal{C}(A) = \{\{x\}\colon x \in A\},$$

is a μ-covering function with parameter $\Phi = 1$. The canopy and the crop conditions with parameters $\Phi = K = 1$ are satisfied because every collection of pairwise disjoint subsets of X is ν-Carathéodory with parameter $K = 1$, since ν is a measure, and the very definition of $\mathcal{C}$. The same conditions are satisfied by

$$\mathcal{E}' = \mathcal{P}(X), \qquad \mathcal{C}'(A) = A.$$

3.4.2 Cartesian product of three finite sets with measures

Let X_1, X_2, X_3 be finite sets with measures $\omega_1, \omega_2, \omega_3$. Let μ, ν, ω be the outer measures μ_1, μ_2, μ_3 defined on X as in (3.1.8). The function $\mathcal{C}$ defined by

$$\mathcal{E} = \{X_1 \times X_2 \times \{z\} : z \in X_3\}, \qquad \mathcal{C}(A) = \{X_1 \times X_2 \times \{z\} : z \in \pi_3(A)\},$$

where π_3 is the projection in X_3, is a μ-covering function with parameter $\Phi = 1$. The canopy and the crop conditions with parameters $\Phi = K = 1$ are satisfied because every collection of disjoint subsets of X of the form $X_1 \times X_2 \times Z$ is ν-Carathéodory with parameter $K = 1$, since on these sets ν behaves like the measure $\omega_2 \otimes \omega_3$, and the very definition of $\mathcal{C}$. The same conditions are satisfied by

$$\mathcal{E}' = \{X_1 \times X_2 \times Z : Z \in \mathcal{P}(X_3)\}, \qquad \mathcal{C}'(A) = X_1 \times X_2 \times \pi_3(A).$$

3.4.3 Upper half 3-space with dyadic strips and trees

Let X be the upper half 3-space, together with the measure induced by the Lebesgue measure on $\mathbb{R}^3$,

$$\begin{aligned} &X = \mathbb{R}^3_+ = \mathbb{R}^2_+ \times \mathbb{R} = \mathbb{R} \times (0, \infty) \times \mathbb{R}, \\ &\mathrm{d}\omega(y, t, \eta) = \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}\eta. \end{aligned} \tag{3.4.1}$$

To define the outer measures, we start recalling the set $\mathcal{I}$ of dyadic intervals in $\mathbb{R}$,

$$\begin{aligned} &I(m, l) = (2^l m, 2^l(m+1)], \\ &\mathcal{I} = \{I(m, l) : m, l \in \mathbb{Z}\}. \end{aligned}$$

Moreover, for all $m, l, n \in \mathbb{Z}$, we define the dyadic upper half tile $H(m, l, n)$ by

$$H(m, l, n) = I(m, l) \times (2^{l-1}, 2^l] \times I(n, -l). \tag{3.4.2}$$

Now, let μ be the outer measure generated by the pre-measure σ on $\mathcal{D}$, the collection of dyadic strips, as in (3.1.17), namely

$$\begin{aligned} &D(m, l) = D\big(I(m, l)\big) = \bigcup_{l' \leqslant l} \bigcup_{m' = 2^{l-l'} m}^{2^{l-l'}(m+1)-1} \bigcup_{n' \in \mathbb{Z}} H(m', l', n'), \\ &\mathcal{D} = \{D(m, l) : m, l \in \mathbb{Z}\} = \{D(I) : I \in \mathcal{I}\}, \\ &\sigma\big(D(m, l)\big) = |I(m, l)| = 2^l, \qquad \text{for all } m, l \in \mathbb{Z}. \end{aligned} \tag{3.4.3}$$

Analogously, let ν be the outer measure generated by the pre-measure τ on $\mathcal{T}$, the collection of dyadic trees, as in (3.1.17), namely

$$\begin{aligned}
&T(m,l,n) = T\big(I(m,l), I(n,-l)\big) = \bigcup_{l' \leqslant l} \bigcup_{m'=2^{l-l'}m}^{2^{l-l'}(m+1)-1} H\big(m', l', N(n,l')\big),\\
&\mathcal{T} = \{T(m,l,n) \colon m,l,n \in \mathbb{Z}\} = \{T(I,\widetilde{I}) \colon I, \widetilde{I} \in \mathcal{I}, |I||\widetilde{I}| = 1\},\\
&\tau\big(T(m,l,n)\big) = |I(m,l)| = 2^l, \qquad \text{for all } m,l,n \in \mathbb{Z},
\end{aligned} \tag{3.4.4}$$

where $N(n,l')$ is defined by the condition

$$I(n,-l) \subseteq I(N(n,l'), -l'). \tag{3.4.5}$$

From now on, we assume all the strips and trees in this subsection to be dyadic, and we avoid repeating it.

Next, for every $L \in \mathbb{Z}$, we define

$$Y_L = \mathbb{R} \times (0, 2^L] \times \mathbb{R}, \tag{3.4.6}$$

On Y_L, we have the measure ω_L and the outer measures μ_L, ν_L induced by ω, μ, ν. In particular, the outer measures μ_L, ν_L are equivalently generated as in (3.1.17) by the pre-measures σ, τ restricting the collections of dyadic strips and trees to those contained in Y_L, namely

$$\begin{aligned}
\mathcal{D}_L &= \{D(m,l) \colon m,l \in \mathbb{Z}, l \leqslant L\},\\
\mathcal{T}_L &= \{T(m,l,n) \colon m,l,n \in \mathbb{Z}, l \leqslant L\}.
\end{aligned}$$

Moreover, we drop the subscript L in all the notation, as the definitions are consistent with the inclusion $Y_{L_1} \subseteq Y_{L_2}$ for $L_1 \leqslant L_2$.

To define the function $\mathcal{C}$ and check that it satisfies the conditions, we recall some properties of the geometry of dyadic strips and trees and introduce some auxiliary functions and state their properties. We postpone the proofs to Appendix 3.A.

To make the notation more compact in the following definitions, we introduce a new symbol for the union of the elements of a collection of subsets of X,

$$\begin{aligned}
\mathcal{L} \colon \mathcal{P}(\mathcal{P}(X)) &\to \mathcal{P}(X),\\
\mathcal{L}(\mathcal{A}) &= \bigcup_{A \in \mathcal{A}} A.
\end{aligned}$$

We start with two observations about the geometry of the intersections between strips, and between a strip and a tree.

Lemma 3.4.1. *Given two strips D_1, D_2 in $\mathcal{D}$, their intersection is again a strip in $\mathcal{D}$, possibly empty. If it is nonempty, we have either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.*

Lemma 3.4.2. *Given a strip D in $\mathcal{D}$ and a tree T in $\mathcal{T}$, their intersection is again a tree T' in $\mathcal{T}$, possibly empty.*

After that, we follow up with some observations about the behaviour of the outer measures μ, ν on strips, trees, their unions and their intersections.

Lemma 3.4.3. *For every strip D in $\mathcal{D}$ and for every tree T in $\mathcal{T}$, we have*

$$\mu(D) = \sigma(D) = |\pi(D)|, \tag{3.4.7}$$
$$\nu(T) = \tau(T) = |\pi(T)|, \tag{3.4.8}$$

where π is the projection in the first coordinate.

Moreover, for every tree T in $\mathcal{T}$, we have

$$\nu(T) = |\pi(T)| = |\pi\big(D(T)\big)| = \mu\big(D(T)\big), \tag{3.4.9}$$

where $D(T)$ is the strip in $\mathcal{D}$ containing T defined by

$$D(T) = \pi(T) \times (0, |\pi(T)|] \times \mathbb{R}.$$

Lemma 3.4.4. *For every collection $\mathcal{D}_1$ of pairwise disjoint strips in $\mathcal{D}$, we have*

$$\mu\big(\mathcal{L}(\mathcal{D}_1)\big) = \sum_{D_1 \in \mathcal{D}_1} \mu(D_1) = \sum_{D_1 \in \mathcal{D}_1} |\pi(D_1)|. \tag{3.4.10}$$

Analogously, for every collection $\mathcal{T}_1$ of pairwise disjoint trees in $\mathcal{T}$, we have

$$\nu\big(\mathcal{L}(\mathcal{T}_1)\big) = \sum_{T_1 \in \mathcal{T}_1} \nu(T_1) = \sum_{T_1 \in \mathcal{T}_1} |\pi(T_1)|. \tag{3.4.11}$$

Moreover, for every collection $\mathcal{D}_1$ of pairwise disjoint strips in $\mathcal{D}$, for every tree T in $\mathcal{T}$, we have

$$\nu\big(T \cap \mathcal{L}(\mathcal{D}_1)\big) = \sum_{D_1 \in \mathcal{D}_1} \nu(T \cap D_1). \tag{3.4.12}$$

Finally, we introduce the auxiliary functions. First, we define the function $\mathcal{Q}$ by

$$\mathcal{Q} \colon \mathcal{P}(X) \to \mathcal{P}(\mathcal{D}),$$
$$\mathcal{Q}(A) = \{E \colon E \in \mathcal{D}, E_+ \cap A \neq \varnothing\},$$

where E_+ is the upper half part of the strip E,

$$E_+ = \{(x,s,\xi) \in E \colon s > \sigma(E)/2\}.$$

It satisfies the following properties

$$A \subseteq \mathcal{L}(\mathcal{Q}(A)), \tag{3.4.13}$$
$$A_1 \subseteq A_2 \Rightarrow \mathcal{L}(\mathcal{Q}(A_1)) \subseteq \mathcal{L}(\mathcal{Q}(A_2)), \tag{3.4.14}$$
$$\mu(\mathcal{L}(\mathcal{Q}(A))) = \mu(A). \tag{3.4.15}$$

After that, we define the function $\mathcal{N}$ by

$$\mathcal{N}: \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D}),$$
$$\mathcal{N}(\mathcal{D}_1) = \{E: E \in \mathcal{D}, |\pi(E) \cap \pi(\mathcal{L}(\mathcal{D}_1))| \geqslant |\pi(E)|/2\}.$$

It associates a collection of strips $\mathcal{D}_1$ to the collection of strips whose associated space interval is at least half covered by the space intervals associated with the elements of $\mathcal{D}_1$. It satisfies the following properties

$$\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{N}(\mathcal{D}_1)), \tag{3.4.16}$$
$$\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{D}_2) \Rightarrow \mathcal{L}(\mathcal{N}(\mathcal{D}_1)) \subseteq \mathcal{L}(\mathcal{N}(\mathcal{D}_2)), \tag{3.4.17}$$
$$\mu(\mathcal{L}(\mathcal{N}(\mathcal{D}_1))) \leqslant 2\mu(\mathcal{L}(\mathcal{D}_1)). \tag{3.4.18}$$

Finally, we define the function $\mathcal{M}$ by

$$\mathcal{M}: \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D}),$$
$$\mathcal{M}(\mathcal{D}_1) = \{E: E \in \mathcal{D}_1, \forall D_1 \in \mathcal{D}_1 \backslash \{E\} \text{ we have } E \nsubseteq D_1\}.$$

It associates a collection of strips $\mathcal{D}_1$ to the subcollection of maximal elements with respect to inclusion. In particular, it is well-defined because, for every $L \in \mathbb{Z}$, the space Y_L is bounded in the second variable. In fact, by Lemma 3.4.1, the function $\mathcal{M}$ maps into the subset of collections of pairwise disjoint strips in $\mathcal{D}$. Moreover, it satisfies the following properties

$$\mathcal{L}(\mathcal{D}_1) = \mathcal{L}(\mathcal{M}(\mathcal{D}_1)), \tag{3.4.19}$$
$$\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{D}_2) \Rightarrow \mathcal{L}(\mathcal{M}(\mathcal{D}_1)) \subseteq \mathcal{L}(\mathcal{M}(\mathcal{D}_2)), \tag{3.4.20}$$
$$\mu(\mathcal{L}(\mathcal{D}_1)) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{D}_1))) = \sum_{E \in \mathcal{M}(\mathcal{D}_1)} \mu(E). \tag{3.4.21}$$

We define the function $\mathcal{C}: \mathcal{P}(X) \to \dot{\mathcal{P}}(\mathcal{E})$ by

$$\mathcal{E} = \mathcal{D}, \qquad \mathcal{C}(A) = \mathcal{M}(\mathcal{N}(\mathcal{Q}(A))),$$

where $\dot{\mathcal{P}}(\mathcal{E})$ stands for the set of subcollections of pairwise disjoint elements in $\mathcal{E}$.

We prove now that the function $\mathcal{C}$ is a μ-covering function and that the setting $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 3.1.1 and the crop condition 3.1.2.

Lemma 3.4.5. *The function $\mathcal{C}$ is a μ-covering function for every choice of the parameter $\Phi \geqslant 2$.*

Proof. We recall that

$$\mathbf{B}_{\mathcal{C}}(A) = \mathcal{L}(\mathcal{M}(\mathcal{N}(\mathcal{Q}(A)))).$$

By (3.4.13), (3.4.16) and (3.4.19), we have

$$A \subseteq \mathbf{B}_{\mathcal{C}}(A).$$

By (3.4.14), (3.4.17) and (3.4.20), we have

$$A_1 \subseteq A_2 \Rightarrow \mathbf{B}_{\mathcal{C}}(A_1) \subseteq \mathbf{B}_{\mathcal{C}}(A_2).$$

Moreover, by (3.4.21), (3.4.18) and (3.4.15), we have

$$\mu\big(\mathbf{B}_{\mathcal{C}}(A)\big) \leqslant 2\mu(A).$$

□

Lemma 3.4.6. *The setting $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 3.1.1 for every choice of parameters $\Phi, K \geqslant 2$.*

Proof. Let $\mathcal{A}$ be a ν-Carathéodory collection of subsets of X with parameter K, and $\widetilde{D}$ a subset of X disjoint from $\mathbf{B}_{\mathcal{C}}\big(\mathcal{L}(\mathcal{A})\big)$. We claim that the collection $\mathcal{A} \cup \{\widetilde{D}\}$ is still ν-Carathéodory with the same parameter K. In particular, we want to prove that for every subset U of X, we have

$$\sum_{A \in \mathcal{A}} \nu(U \cap A) + \nu(U \cap \widetilde{D}) \leqslant K\nu(U). \tag{3.4.22}$$

Without loss of generality, we assume $U \cap \widetilde{D} \neq \varnothing$, otherwise the inequality follows by the ν-Carathéodory property for the collection $\mathcal{A}$. In particular, we have $\widetilde{D} \neq \varnothing$.

First, we prove (3.4.22) under some additional assumptions on $\widetilde{D}$ and U. After that, we obtain the general case in a series of generalization steps.

Step 1. Let $\widetilde{D}$ be a nonempty set of the form

$$D \backslash \mathbf{B}_{\mathcal{C}}\big(\mathcal{L}(\mathcal{A})\big), \tag{3.4.23}$$

where D is a strip in $\mathcal{D}$, and $\mathbf{B}_{\mathcal{C}}\big(\mathcal{L}(\mathcal{A})\big) \subsetneq D$. We claim that, for every tree T in $\mathcal{T}$, we have

$$\sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap D) \leqslant K\nu(T). \tag{3.4.24}$$

The version of (3.4.22) for the particular choices of T and $\widetilde{D}$ follows by the monotonicity of ν.

Without loss of generality, we assume T to be contained in D. The result for an arbitrary tree T follows by that for $T \cap D$, which by Lemma 3.4.2 is a tree as well, and the monotonicity of ν.

For every tree T contained in D with nonempty intersection with $\widetilde{D}$, we have

$$D(T) \notin \mathcal{N}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))).$$

Together with (3.4.9), this yields

$$\nu(T) = |\pi\big(D(T)\big)| \geqslant 2|\pi(D(T) \cap \mathcal{L}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))))|.$$

By (3.4.19) and the disjointness of the elements of a collection $\mathcal{M}(\mathcal{D}_1)$ for every $\mathcal{D}_1 \subseteq \mathcal{D}$, we have

$$\begin{aligned} |\pi(D(T) \cap \mathcal{L}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))))| &= |\pi(D(T) \cap \mathcal{L}(\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))))| \\ &= \sum_{E \in \mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} |\pi(D(T) \cap E)|. \end{aligned}$$

By the monotonicity of the Lebesgue measure, Lemma 3.4.2, and (3.4.9), we have

$$\begin{aligned} \sum_{E \in \mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} |\pi(D(T) \cap E)| &\geqslant \sum_{E \in \mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} |\pi(T \cap E)| \\ &\geqslant \sum_{E \in \mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} \nu(T \cap E). \end{aligned}$$

By (3.4.12) and the monotonicity of ν, we have

$$\sum_{E \in \mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A})))} \nu(T \cap E) \geqslant \nu(T \cap \mathcal{L}(\mathcal{M}(\mathcal{Q}(\mathcal{L}(\mathcal{A}))))) \geqslant \nu(T \cap \mathcal{L}(\mathcal{A})).$$

Together with the condition $K \geqslant 2$ and the ν-Carathéodory property for the collection $\mathcal{A}$, the previous chains of inequalities yield

$$\begin{aligned} K\nu(T) &\geqslant \nu(T \cap D) + 2(K-1)\nu(T \cap \mathcal{L}(\mathcal{A})) \\ &\geqslant \nu(T \cap D) + K\nu(T \cap \mathcal{L}(\mathcal{A})) \\ &\geqslant \nu(T \cap D) + \sum_{A \in \mathcal{A}} \nu(T \cap A). \end{aligned}$$

Step 2. Let $\widetilde{D}$ be a nonempty set of the form

$$\widetilde{D} = \bigcup_{D' \in \mathcal{D}'} \widetilde{D}' = \bigcup_{D' \in \mathcal{D}'} \big(D' \backslash \mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))\big),$$

where $\mathcal{D}'$ is a collection of pairwise disjoint strips. We claim that, for every tree T in $\mathcal{T}$, we have (3.4.22) for the particular choices of T and $\widetilde{D}$.

By definition, for every strip $\mathcal{D}'$, we have

$$D' \nsubseteq \mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A})).$$

Therefore, by Lemma 3.4.1, we have

$$\mathcal{C}(\mathcal{L}(\mathcal{A})) = \mathcal{C}_1 \cup \bigcup_{D' \in \mathcal{D}'} \mathcal{C}_{D'},$$

where the elements of $\mathcal{C}_1$ are disjoint from $\mathcal{L}(\mathcal{D}')$, while, for every D' in $\mathcal{D}'$, the elements of $\mathcal{C}_{D'}$ are contained in D'. In particular, we have

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_1 \cup \bigcup_{D' \in \mathcal{D}'} \mathcal{A}_{D'} \\ &= \{A \colon A \in \mathcal{A}, A \subseteq \mathcal{L}(\mathcal{C}_1)\} \cup \bigcup_{D' \in \mathcal{D}'} \{A \colon A \in \mathcal{A}, A \subseteq \mathcal{L}(\mathcal{C}_{D'})\}. \end{aligned}$$

Then

$$\begin{aligned} K\nu(T) &\geqslant K\nu\big(T \cap (\mathcal{L}(\mathcal{C}(\mathcal{L}(\mathcal{A}))) \cup \bigcup_{D' \in \mathcal{D}'} D')\big) \\ &\geqslant K\nu\big(T \cap \mathcal{L}(\mathcal{C}_1)\big) + K \sum_{D' \in \mathcal{D}'} \nu(T \cap D') \\ &\geqslant \sum_{A \in \mathcal{A}_1} \nu(T \cap A) + \sum_{D' \in \mathcal{D}'} \big(\sum_{A \in \mathcal{A}_{D'}} \nu(T \cap A) + \nu(T \cap D')\big) \\ &\geqslant \sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \mathcal{L}(\mathcal{D}')) \\ &\geqslant \sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \widetilde{D}). \end{aligned} \tag{3.4.25}$$

where we used the monotonicity of ν in the first and in the fifth inequality, (3.4.12) in the second, the ν-Carathéodory property for the collection $\{A \colon A \in \mathcal{A}, A \subseteq \mathcal{L}(\mathcal{C}_1)\}$ and (3.4.24) for each D' in $\mathcal{D}'$ in the third, Fubini and (3.4.12) in the fourth.

Step 3. Let $\widetilde{D}$ be an arbitrary nonempty set disjoint from $\mathbf{B}_{\mathcal{C}}\big(\mathcal{L}(\mathcal{A})\big)$. We claim that, for every tree T in $\mathcal{T}$, we have (3.4.22) for the particular choices of T and $\widetilde{D}$.

For $\mathcal{D}' = \mathcal{M}(\mathcal{Q}(\widetilde{D}))$, we define

$$\widetilde{D}_1 = \bigcup_{D' \in \mathcal{D}'} \big(D' \backslash \mathbf{B}_{\mathcal{C}}(\mathcal{L}(\mathcal{A}))\big).$$

By (3.4.25) and the monotonicity of ν, we have

$$K\nu(T) \geqslant \sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \widetilde{D}_1) \geqslant \sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \widetilde{D}). \tag{3.4.26}$$

Step 4. Let $\widetilde{D}$ be an arbitrary nonempty set disjoint from $\mathbf{B}_{\mathcal{C}}\big(\mathcal{L}(\mathcal{A})\big)$. We claim that, for every subset U of X, we have (3.4.22).

In fact, there exists a collection $\mathcal{T}' \subseteq \mathcal{T}$ covering U ν-optimally, namely

$$U \subseteq \bigcup_{T \in \mathcal{T}'} T, \tag{3.4.27}$$

$$\sum_{T \in \mathcal{T}'} \tau(T) = \nu(U). \tag{3.4.28}$$

By (3.4.26) for every tree T in $\mathcal{T}'$, the subadditivity of ν, and (3.4.27), we have

$$\begin{aligned} K \sum_{T \in \mathcal{T}'} \nu(T) &\geqslant \sum_{T \in \mathcal{T}'} \big(\sum_{A \in \mathcal{A}} \nu(T \cap A) + \nu(T \cap \widetilde{D})\big) \\ &\geqslant \sum_{A \in \mathcal{A}} \sum_{T \in \mathcal{T}'} \nu(T \cap A) + \sum_{T \in \mathcal{T}'} \nu(T \cap \widetilde{D}) \\ &\geqslant \sum_{A \in \mathcal{A}} \nu(U \cap A) + \nu(U \cap \widetilde{D}). \end{aligned}$$

Together with (3.4.28), this yields the desired inequality in (3.4.22). □

Lemma 3.4.7. *The setting $(X, \mu, \nu, \mathcal{C})$ satisfies the crop condition 3.1.2 for every choice of parameters $\Phi \geqslant 2, K \geqslant 1$.*

Proof. For every collection $\mathcal{A}$ of strips in $\mathcal{D}$, let $\mathcal{B} = \mathcal{M}(\mathcal{A})$. The subcollection $\mathcal{B}$ is ν-Carathéodory with parameter $K = 1$. Moreover, for every subset F of X disjoint from $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{A})$, we have

$$\mathcal{C}(F) \cap \mathcal{A} = \mathcal{Q}(F) \cap \mathcal{A} = \varnothing,$$

and this yields

$$\mathbf{B}_{\mathcal{C}}(F) = \mathbf{B}_{\widetilde{\mathcal{C}}}(F).$$

□

3.5 Double iterated outer L^p spaces on the upper half 3-space

In this section we prove Theorem 3.1.5 in the dyadic upper half 3-space setting described in (3.4.1), (3.4.3) and (3.4.4), reducing the problem to an equivalent one in a finite setting via an approximation argument.

We start stating some auxiliary results about the approximation of functions in outer L^p spaces. We use them to prove the approximation of functions in outer L^p spaces on the upper half 3-space X by functions with support in X_J for a certain $J \in \mathbb{N}$, where

$$X_J = (-2^J J, 2^J J] \times (2^{-J}, 2^J] \times (-2^J J, 2^J J]. \tag{3.5.1}$$

On X_J, we have the measure ω_J and the outer measures μ_J, ν_J induced by ω, μ, ν. In particular, this setting inherits the definition of the function $\mathcal{C}$ on Y_J, for Y_J defined in (3.4.6), and its properties (Lemma 3.4.5, Lemma 3.4.6, Lemma 3.4.7).

Next, for any $J \in \mathbb{N}$, we introduce a finite setting X'_J and exhibit a map between functions on X_J and on X'_J preserving the double iterated outer L^p quasi-norms. We use Theorem 3.1.3, Theorem 3.1.4 in the finite settings to prove Theorem 3.1.5.

Finally, we conclude the section with some observations about the result analogous to Theorem 3.1.5 for double iterated outer L^p spaces in the upper half 3-space setting where the outer measures are defined by arbitrary strips and trees originally considered in [Ura16].

3.5.1 Approximation results

First, we state a result about the approximation of functions in $L^p_\mu(S)$ by functions in $L^p_\mu(S) \cap L^\infty_\mu(S)$, for a size S of the form ℓ^r_ω or $\ell^q_\nu(\ell^r_\omega)$, and more generally an arbitrary size in the definition in [DT15].

Lemma 3.5.1. *For every $p \in (0,\infty)$, there exists a constant $C = C(p)$ such that the following property holds true.*

Let X be a set, μ an outer measure, and S a size. For every $f \in L^p_\mu(S)$, there exists a subset A of X such that $f1_A$ is in $L^p_\mu(S) \cap L^\infty_\mu(S)$ and we have

$$\|f\|_{L^p_\mu(S)} \leqslant C\|f1_A\|_{L^p_\mu(S)}.$$

Next, we state a result about the behaviour of the super level measures for single iterated outer L^p spaces for monotonically increasing cut offs of a function in a general setting.

Lemma 3.5.2 (Monotonic convergence I)**.** *For every $r \in (0,\infty)$, there exist constants $C = C(r)$, $c = c(r)$ such that the following property holds true.*

Let X be a set, ν an outer measure, and ω a measure. Let $\{X_J \colon J \in \mathbb{N}\}$ be a monotonically increasing sequence of subsets of X such that

$$X = \bigcup_{J \in \mathbb{N}} X_J,$$

and let $f \in L^\infty_\nu(\ell^r_\omega)$ be a function on X. Then, for every $k \in \mathbb{Z}$, there exists $J = J(r,f,k) \in \mathbb{N}$ such that

$$\nu(\ell^r_\omega(f) > 2^k) \leqslant C \sum_{l \geqslant k} \nu(\ell^r_\omega(f1_{X_J}) > c2^l).$$

Finally, we state a result about the behaviour of the super level measures for double iterated outer L^p spaces for monotonically increasing cut offs of a function in the dyadic upper half 3-space setting.

Lemma 3.5.3 (Monotonic convergence II)**.** *For all $q, r \in (0, \infty)$, there exist constants $C = C(q, r)$, $c = c(q, r)$ such that the following property holds true.*

Let $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ be a function on $X = \mathbb{R} \times (0, \infty) \times \mathbb{R}$, and let $\{X_J : J \in \mathbb{N}\}$ be the monotonically increasing sequence of subsets of X defined in (3.5.1)*. Then, for every $k \in \mathbb{Z}$, there exists $J = J(q, r, f, k) \in \mathbb{Z}$ such that*

$$\mu(\ell^q_\nu(\ell^r_\omega)(f) > 2^k) \leqslant C \sum_{l \geqslant k} \mu(\ell^q_\nu(\ell^r_\omega)(f 1_{X_J}) > c2^l).$$

We postpone the proofs of the previous three results to Appendix 3.B. We use them to prove the following results about the approximation of functions in $L^q_\nu(\ell^r_\omega)$ and $L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ by functions with support in X_j for a certain $j \in \mathbb{N}$.

Lemma 3.5.4. *For all $q, r \in (0, \infty)$, there exists a constant $C = C(q, r)$ such that the following property holds true.*

For every function $f \in L^q_\nu(\ell^r_\omega)$, there exists $J = J(q, r, f) \in \mathbb{N}$ such that

$$\|f 1_{X_J}\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant C \|f 1_{X_J}\|_{L^q_\nu(\ell^r_\omega)}.$$

Proof. The first inequality follows by the monotonicity of the outer L^p quasi-norms.

To prove the second inequality, by Lemma 3.5.1, we assume f to be in $L^q_\nu(\ell^r_\omega) \cap L^\infty_\nu(\ell^r_\omega)$. Next, we observe that there exists $K = K(q, r, f) \in \mathbb{N}$ such that

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} \leqslant C \sum_{k \in \mathbb{Z}} 2^{kq} \nu(\ell^r_\omega(f) > 2^k) \leqslant C \sum_{k \in [-K, K]} 2^{kq} \nu(\ell^r_\omega(f) > 2^k).$$

By Lemma 3.5.2, for every $k \in [-K, K]$, there exists a $\widetilde{J} = \widetilde{J}(r, f, k) \in \mathbb{N}$ such that

$$\nu(\ell^r_\omega(f) > 2^k) \leqslant C \sum_{l \geqslant k} \nu(\ell^r_\omega(f 1_{X_{\widetilde{J}}}) > c2^l).$$

By taking $J = \max_{k \in [-K, K]} \widetilde{J}(k, f, r)$, the previous inequalities yield

$$\|f\|^q_{L^q_\nu(\ell^r_\omega)} \leqslant C \sum_{k \in [-K, K]} 2^{kq} \sum_{l \geqslant k} \nu(\ell^r_\omega(f 1_{X_J}) > c2^l) \leqslant C \|f 1_{X_J}\|^q_{L^q_\nu(\ell^r_\omega)}.$$

□

Lemma 3.5.5. *For all $p, q, r \in (0, \infty)$. There exists a constant $C = C(p, q, r)$ such that the following property holds true.*

For every function $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$, there exists $J = J(p, q, r, f) \in \mathbb{N}$ such that

$$\|f 1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \|f 1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

Proof. The inequalities follow via the same argument used in the previous proof, with Lemma 3.5.2 replaced by Lemma 3.5.3. □

3.5.2 Equivalence with finite settings

We introduce the following finite setting,

$$\begin{aligned}
&X' = \mathbb{Z}^3,\\
&\omega'(m,l,n) = 1,\\
&D'(m,l) = \{(m',l',n') \colon m' \in [2^{l-l'}m, 2^{l-l'}(m+1)), l' \leqslant l, n' \in \mathbb{Z}\},\\
&\mathcal{D}' = \{D'(m,l) \colon m,l \in \mathbb{Z}\},\\
&\sigma'\big(D'(m,l)\big) = 2^l, \qquad \text{for all } m,l \in \mathbb{Z},\\
&T'(m,l,n) = \{(m',l',n') \colon m' \in [2^{l-l'}m, 2^{l-l'}(m+1)), l' \leqslant l, n' = N(n,l')\},\\
&\mathcal{T}' = \{T'(m,l,n) \colon m,l,n \in \mathbb{Z}\},\\
&\tau'\big(T'(m,l,n)\big) = 2^l, \qquad \text{for all } m,l,n \in \mathbb{Z},
\end{aligned}$$

where $N(n,l')$ is defined by the condition (3.4.5), and μ',ν' are defined by σ',τ' as in (3.1.17). Moreover, for every $J \in \mathbb{N}$, we define

$$X'_J = \{(m,l,n) \in X' \colon l \in (-J,J], m \in [-J2^{J-l}, J2^{J-l}), n \in [-J2^{J+l}, J2^{J+l})\},$$

On X_J, we have the measure ω'_J and the outer measures μ'_J, ν'_J induced by ω', μ', ν'. In fact, the outer measure μ'_J is equivalently generated by the pre-measure σ'_J on $\mathcal{D}'_J$ as in (3.1.17), namely

$$\begin{aligned}
&D'_J(m,l) = D'(m,l) \cap X'_J,\\
&\mathcal{D}'_J = \{D'_J(m,l) \colon m,l \in \mathbb{Z}, D'_J(m,l) \neq \varnothing\},\\
&\sigma_J\big(D'_J(m,l)\big) = 2^l, \qquad \text{for all } m,l \in \mathbb{Z}, D'_J(m,l) \neq \varnothing,
\end{aligned}$$

and the outer measure ν'_J by the pre-measure τ'_J on $\mathcal{T}'_J$ as in (3.1.17), namely

$$\begin{aligned}
&T'_J(m,l,n) = T'(m,l,n) \cap X'_J,\\
&\mathcal{T}'_J = \{T'_J(m,l,n) \colon m,l,n \in \mathbb{Z}, T'_J(m,l,n) \neq \varnothing\},\\
&\tau'_J\big(T'_J(m,l,n)\big) = 2^l, \qquad \text{for all } m,l,n \in \mathbb{Z}, T'_J(m,l,n) \neq \varnothing.
\end{aligned}$$

The setting on X'_J inherits the definition of the function $\mathcal{C}$ on X_J and its properties (Lemma 3.4.5, Lemma 3.4.6, Lemma 3.4.7) via the map associating every triple $(m,l,n) \in X'$ to $H(m,l,n)$, the pairwise disjoint subsets of X defined in (3.4.2).

Moreover, every function f on X that is in $L^r_{\text{loc}}(X,\omega)$ for some $r \in (0,\infty]$ defines a function $F(f,r)$ on X' by

$$F(f,r)(m,l,n) = \|f 1_{H(m,l,n)}\|_{L^r(X,\omega)}.$$

For every fixed $r \in (0,\infty]$, the map between functions on X and on X' just described preserves the iterated outer L^p quasi-norms.

Lemma 3.5.6. *Let $p,q,r \in (0,\infty)$. For every f supported in X_J for any $J \in \mathbb{N}$, we have*

$$\|f\|_{L^q_\nu(\ell^r_\omega)} = \|F(f,r)\|_{L^q_{\nu'}(\ell^r_{\omega'})},$$
$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} = \|F(f,r)\|_{L^p_{\mu'}(\ell^q_{\nu'}(\ell^r_{\omega'}))}.$$

Proof. Let $J \in \mathbb{N}$ be fixed, and assume that f is supported in X_J.

We start observing that $F(f,r)$ is supported in X'_J. Moreover, in both cases, we can restrict to consider only the elements of $\mathcal{D}_J, \mathcal{T}_J$ and $\mathcal{D}'_J, \mathcal{T}'_J$, since we have

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} = \|f\|_{L^p_{\mu_J}(\ell^q_{\nu_J}(\ell^r_{\omega_J}))},$$
$$\|F(f,r)\|_{L^p_{\mu'}(\ell^q_{\nu'}(\ell^r_{\omega'}))} = \|F(f,r)\|_{L^p_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))}.$$

In particular, for any $U \in \mathcal{T}_J$, we have $U = T_J(m,l,n)$, and we define $U' \in \mathcal{T}'_J$ by $U' = T'_J(m,l,n)$, hence satisfying

$$\nu_J(U) = \tau_J(U) = \tau'_J(U') = \nu'_J(U'). \tag{3.5.2}$$

Now, for any two collections $\mathcal{U}_1, \mathcal{U}_2$ of elements in $\mathcal{T}_J$, we define, for $i = 1,2$,

$$U_i = \mathcal{L}(\mathcal{U}_i), U'_i = \mathcal{L}(\mathcal{U}'_i),$$

and we have

$$F(f1_{U_1\setminus U_2}, r) = F(f,r)1_{U'_1\setminus U'_2}. \tag{3.5.3}$$

Next, by the definition of $F(f,r)$, we have

$$\|f\|_{L^r(X_J,\omega_J)} = \|F(f,r)\|_{L^r(X'_J,\omega'_J)}. \tag{3.5.4}$$

Therefore, for any element U in $\mathcal{T}_J$, we have

$$\|f1_U\|_{L^r(X_J,\omega_J)} = \|F(f1_U,r)\|_{L^r(X'_J,\omega'_J)} = \|F(f,r)1_{U'}\|_{L^r(X'_J,\omega'_J)}, \tag{3.5.5}$$

where we used (3.5.4) in the first equality, and (3.5.3) in the second. Moreover, for any $A \subseteq X_J$, there exists a finite subcollection $\mathcal{U}$ of $\mathcal{T}_J$ such that $A \subseteq \mathcal{L}(\mathcal{U})$ and

$$\nu_J(A) = \sum_{U\in\mathcal{U}} \tau_J(U) = \sum_{U\in\mathcal{U}} \nu_J(U). \tag{3.5.6}$$

In particular, we have

$$\begin{aligned}\nu_J(A)^{-1}\|f1_A\|^r_{L^r(X_J,\omega_J)} &\leqslant \nu_J(A)^{-1}\sum_{U\in\mathcal{U}}\|f1_U\|^r_{L^r(X_J,\omega_J)} \\ &\leqslant \nu_J(A)^{-1}\max_{V\in\mathcal{U}}\nu_J(V)^{-1}\|f1_V\|^r_{L^r(X_J,\omega_J)}\sum_{U\in\mathcal{U}}\nu_J(U) \\ &\leqslant \max_{V\in\mathcal{U}}\nu_J(V)^{-1}\|f1_V\|^r_{L^r(X_J,\omega_J)},\end{aligned} \tag{3.5.7}$$

where we used the monotonicity and the r-orthogonality of the classical L^r quasi-norm in the first inequality, Hölder's inequality in the second, and (3.5.6) in the third. The analogous properties hold true for any F supported in X'_J.

Therefore, for any $\lambda > 0$, we have, for $F = F(f,r)$,

$$\begin{aligned}
&\nu_J(\ell^r_{\omega_J}(f) > \lambda) = \\
&= \inf\{\nu_J(A) \colon A \subseteq X_J, \sup\{\nu_J(B)^{-1/r}\|f 1_B 1_{A^c}\|_{L^r(X_J,\omega_J)} \colon B \subseteq X_J\} \leqslant \lambda\} \\
&= \inf\{\nu_J(\mathcal{L}(\mathcal{U})) \colon \mathcal{U} \subseteq \mathcal{T}_J, \sup\{\nu_J(V)^{-1/r}\|f 1_V 1_{\mathcal{L}(\mathcal{U})^c}\|_{L^r(X_J,\omega_J)} \colon V \in \mathcal{T}_J\} \leqslant \lambda\} \\
&= \inf\{\nu'_J(\mathcal{L}(\mathcal{U}')) \colon \mathcal{U}' \subseteq \mathcal{T}'_J, \\
&\qquad\qquad \sup\{\nu'_J(V')^{-1/r}\|F 1_{V'} 1_{\mathcal{L}(\mathcal{U}')^c}\|_{L^r(X'_J,\omega'_J)} \colon V' \in \mathcal{T}'_J\} \leqslant \lambda\} \\
&= \inf\{\nu'_J(A') \colon A' \subseteq X'_J, \\
&\qquad\qquad \sup\{\nu'_J(B')^{-1/r}\|F 1_{B'} 1_{(A')^c}\|_{L^r(X'_J,\omega'_J)} \colon B' \subseteq X'_J\} \leqslant \lambda\} \\
&= \nu'_J(\ell^r_{\omega'_J}(F) > \lambda),
\end{aligned}$$

where we used (3.5.6) and (3.5.7) in the second equality, (3.5.2) and (3.5.5) in the third, the analogous of (3.5.6) and (3.5.7) in the fourth. Hence

$$\|f\|_{L^q_{\nu_J}(\ell^r_{\omega_J})} = \|F(f,r)\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})}.$$

Applying an analogous argument to the "exterior" level of definition of the double iterated outer L^p space, we obtain

$$\|f\|_{L^p_{\mu_J}(\ell^q_{\nu_J}(\ell^r_{\omega_J}))} = \|F(f,r)\|_{L^p_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))}.$$

□

We are now ready to prove Theorem 3.1.5.

Proof of Theorem 3.1.5. Let $p,q,r \in (0,\infty]$. By Lemma 3.5.4 and Lemma 3.5.5, for every $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$, there exists $J = J(f,p,q,r) \in \mathbb{N}$ such that

$$\begin{aligned}
\|f 1_{X_J}\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant C\|f 1_{X_J}\|_{L^q_\nu(\ell^r_\omega)}, \\
\|f 1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f 1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))},
\end{aligned} \tag{3.5.8}$$

where C is independent of f and J. By Lemma 3.5.6, we have

$$\begin{aligned}
\|f 1_{X_J}\|_{L^q_\nu(\ell^r_\omega)} = \|F(f 1_{X_J}, r)\|_{L^q_{\nu'}(\ell^r_{\omega'})} = \|F(f,r) 1_{X'_J}\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})}, \\
\|f 1_{X_J}\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} = \|F(f 1_{X_J}, r)\|_{L^p_{\mu'}(\ell^q_{\nu'}(\ell^r_{\omega'}))} = \|F(f,r) 1_{X'_J}\|_{L^p_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))}.
\end{aligned} \tag{3.5.9}$$

Property (i). Let $q, r \in (0, \infty)$. By Theorem 3.1.3, we have

$$\begin{aligned} C^{-1} \|F(f,r)1_{X'_J}\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})} &\leqslant \|F(f,r)1_{X'_J}\|_{L^q_{\mu'_J}(\ell^q_{\nu'_J}(\ell^r_{\omega'_J}))} \\ &\leqslant C \|F(f,r)1_{X'_J}\|_{L^q_{\nu'_J}(\ell^r_{\omega'_J})}, \end{aligned}$$

where C is independent of f and J. Together with (3.5.8) and (3.5.9), the previous chain of inequalities yields the desired equivalence in (3.1.18).

Property (ii). Let $p, q, r \in (1, \infty)$. By Theorem 3.1.4, for every $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$, there exists a function G on X'_J with unitary outer $L^{p'}_{\mu'_J}(\ell^{q'}_{\nu'_J}(\ell^{r'}_{\omega'_J}))$ quasi-norm such that

$$\begin{aligned} C^{-1} \|F(f,r)1_{X'_J}\|_{L^p_{\mu'_J}(\ell^p_{\nu'_J}(\ell^r_{\omega'_J}))} &\leqslant \|F(f,r)1_{X'_J} G\|_{L^1(X'_J, \omega'_J)} \\ &\leqslant C \|F(f,r)1_{X'_J}\|_{L^p_{\mu'_J}(\ell^p_{\nu'_J}(\ell^r_{\omega'_J}))}, \end{aligned} \tag{3.5.10}$$

where C is independent of f and J. We define a function g on X by

$$g(x,s,\xi) = |f(x,s,\xi)|^{r-1} \sum_{m,l,n \in \mathbb{Z}} F(f,r)(m,l,n)^{1-r} G(m,l,n) 1_{H(m,l,n)}(x,s,\xi).$$

By construction, we have

$$F(g, r') = G.$$

Together with Lemma 3.5.6, this yields

$$\|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = \|G\|_{L^{p'}_{\mu'}(\ell^{q'}_{\nu'}(\ell^{r'}_{\omega'}))} = \|G\|_{L^{p'}_{\mu'_J}(\ell^{q'}_{\nu'_J}(\ell^{r'}_{\omega'_J}))} = 1.$$

Moreover, by construction we have

$$\begin{aligned} \|fg\|_{L^1_\omega} &= \|F(f,r)G\|_{L^1(X'_J, \omega'_J)} = \|F(f,r)G\|_{L^1(X'_J, \omega'_J)} \\ &= \|F(f,r)1_{X'_J} G\|_{L^1(X'_J, \omega'_J)}. \end{aligned}$$

Together with (3.5.8), (3.5.9), and (3.5.10), the last two chains of equalities yield the desired equivalence in (3.1.19).

Property (iii). The inequality in (3.1.20) is a corollary of the triangle inequality for the $L^1(X, \omega)$ norm and property (ii). $\square$

3.5.3 Upper half 3-space with arbitrary strips and trees

We turn to the case of double iterated outer L^p spaces on the upper half 3-space setting where the outer measures are defined by arbitrary strips and trees. In particular, let

$$\begin{aligned}
&X = \mathbb{R}^3_+ = \mathbb{R}^2_+ \times \mathbb{R} = \mathbb{R} \times (0,\infty) \times \mathbb{R}, \\
&\mathrm{d}\omega(y,t,\eta) = \mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\eta, \\
&\widetilde{D}(x,s) = \{(y,t,\eta) \colon y \in x + (0,s], t \in (0,s], \eta \in \mathbb{R}\}, \\
&\widetilde{\mathcal{D}} = \{\widetilde{D}(x,s) \colon x \in \mathbb{R}, s \in (0,\infty)\}, \\
&\widetilde{\sigma}\big(\widetilde{D}(x,s)\big) = s, \qquad \text{for all } x \in \mathbb{R}, s \in (0,\infty), \\
&\widetilde{T}(x,s,\xi) = \{(y,t,\eta) \colon y \in x + (0,s], t \in (0,s], \eta \in \xi + (-t^{-1}, t^{-1}]\}, \\
&\widetilde{\mathcal{T}} = \{\widetilde{T}(x,s,\xi) \colon x \in \mathbb{R}, s \in (0,\infty), \xi \in \mathbb{R}\}, \\
&\widetilde{\tau}\big(\widetilde{T}(x,s,\xi)\big) = s, \qquad \text{for all } x \in \mathbb{R}, s \in (0,\infty), \xi \in \mathbb{R},
\end{aligned} \tag{3.5.11}$$

where $\widetilde{\mu}, \widetilde{\nu}$ are defined by $\widetilde{\sigma}, \widetilde{\tau}$ as in (3.1.17).

On one hand, the outer measures generated by dyadic strips and arbitrary ones are equivalent and we can substitute the outer measure $\widetilde{\mu}$ with μ. In particular, we have $\mathcal{D} \subseteq \widetilde{\mathcal{D}}$, and every element of $\widetilde{\mathcal{D}}$ is covered by at most two elements of $\mathcal{D}$ with comparable pre-measure.

On the other hand, the outer measures generated by dyadic trees and arbitrary ones are not equivalent. In fact, while for every dyadic tree T in $\mathcal{T}$ we have

$$\widetilde{\nu}(T) \leqslant \nu(T),$$

instead for every arbitrary tree $\widetilde{T}$ in $\widetilde{\mathcal{T}}$ we have

$$\nu(\widetilde{T}) = \infty, \tag{3.5.12}$$

and we postpone the proof to Appendix 3.A. Therefore, we can not trivially deduce the same result stated in Theorem 3.1.5 in the setting described in (3.5.11) from Theorem 3.1.5 itself.

However, a reduction of the problem to an equivalent one in a finite setting via an approximation argument analogous to that described in the previous subsections still yields the desired result. We briefly comment on some additional observations, providing guidance to the readers interested in a complete proof.

First, we observe that the outer measure $\widetilde{\nu}$ is equivalent to $\widetilde{\nu}_d$, the outer measure defined as in (3.1.17) by the pre-measure $\widetilde{\tau}$ restricting the collection $\widetilde{\mathcal{T}}$ of trees to those associated with dyadic intervals, namely

$$\widetilde{\mathcal{T}}_d = \{\widetilde{T}(2^l m, 2^l, 2^{-l} n) \colon m,l,n \in \mathbb{Z}\} \subseteq \widetilde{\mathcal{T}}.$$

The geometry of the elements of $\mathcal{D}, \widetilde{\mathcal{T}}_d$ and their intersections is analogous to that of the elements of $\mathcal{D}, \mathcal{T}$. Therefore, for every function f in a double iterated outer L^p space in the setting $(X, \mu, \widetilde{\nu}_d, \omega)$, we can pass to a cut off $f 1_{X_J}$ approximating the double iterated outer L^p quasi-norm of f, for X_J defined in (3.5.1).

Next, for every fixed $J \in \mathbb{N}$, we consider the outer measure $\widetilde{\nu}_{d,J}$ induced on Y_J by $\widetilde{\nu}_d$, where Y_J is defined in (3.4.6). We observe that $\widetilde{\nu}_{d,J}$ is equivalent to the outer measure generated as in (3.1.17) by the pre-measure $\widetilde{\tau}$ restricting the collection $\widetilde{\mathcal{T}}_d$ of trees to those contained in Y_J, namely

$$\widetilde{\mathcal{T}}_{d,J} = \{\widetilde{T}(2^l m, 2^l, 2^{-l} n) \colon m, l, n \in \mathbb{Z}, l \leqslant J\} \subseteq \widetilde{\mathcal{T}}_d.$$

In the setting $(Y_J, \mu_J, \widetilde{\mathcal{T}}_{d,J}, \omega_J)$, we can state definitions and prove results based on the geometry of the elements of $\mathcal{D}_J, \widetilde{\mathcal{T}}_{d,J}$ analogous to those in Section 3.4. Therefore, for every $J \in \mathbb{N}$, we can define a μ_J-covering function $\widetilde{\mathcal{C}}$ satisfying the canopy condition 3.1.1 and the crop condition 3.1.2. In particular, this definition is inherited by $X_J \subseteq Y_J$.

After that, for every fixed $J \in \mathbb{N}$, we observe that the elements of $\mathcal{D}_J, \widetilde{\mathcal{T}}_{d,J}$ with nonempty intersection with X_J are finitely many. Therefore, we can introduce a finite setting with a point for every intersection and the induced measure and outer measures. In particular, we conclude the result corresponding to that stated in Theorem 3.1.5 via an argument analogous to that of the previous subsection.

3.A Geometry of the dyadic upper half 3-space setting

In this appendix, we present the postponed proofs of the results involving the geometry of the dyadic strips and trees in the upper half 3-space stated in Section 3.4, and in (3.5.12) in Section 3.5.

We start recalling that every dyadic strip D in $\mathcal{D}$ is determined by a dyadic interval I_D in $\mathcal{I}$, and has the form

$$D = I_D \times (0, |I_D|] \times \mathbb{R} = \pi(D) \times (0, |\pi(D)|] \times \mathbb{R}, \tag{3.A.1}$$

and every dyadic tree T in $\mathcal{T}$ is determined by two dyadic intervals $I_T, \widetilde{I}_T$ in $\mathcal{I}$ such that $|I_T||\widetilde{I}_T| = 1$ and has the form

$$T = \bigcup_{J \in \mathcal{I}, J \subseteq I_T} J \times (0, |J|] \times \widetilde{J}(T, J) = \bigcup_{J \in \mathcal{I}, J \subseteq \pi(T)} J \times (0, |J|] \times \widetilde{J}(T, J), \tag{3.A.2}$$

where the dyadic interval $\widetilde{J}(T, J)$ in $\mathcal{I}$ is defined by the conditions

$$\begin{gathered} |\widetilde{J}(T, J)| = |J|^{-1}, \\ \widetilde{I}_T = \widetilde{J}\big(T, \pi(T)\big) \subseteq \widetilde{J}(T, J). \end{gathered}$$

Proof of Lemma 3.4.1. If $D_1 \cap D_2$ is empty, the statement is trivially verified. Therefore, we assume that the strips D_1, D_2 have a nonempty intersection. Hence the dyadic intervals $\pi(D_1), \pi(D_2)$ have a nonempty intersection as well. Therefore, we have either $\pi(D_1) \subseteq \pi(D_2)$ or $\pi(D_2) \subseteq \pi(D_1)$. Without loss of generality, we can restrict to the first case, the second being analogous. We have $|\pi(D_1)| \leqslant |\pi(D_2)|$, hence by (3.A.1)

$$D_1 \subseteq D_2.$$

□

Proof of Lemma 3.4.2. If $D \cap T$ is empty, the statement is trivially verified. Therefore, we assume that the strip D and the tree T have a nonempty intersection. Hence the dyadic intervals $\pi(D), \pi(T)$ have a nonempty intersection as well. Therefore, we have either $\pi(D) \subseteq \pi(T)$ or $\pi(T) \subseteq \pi(D)$. In the first case, we have $|\pi(D)| \leqslant |\pi(T)|$, hence by (3.A.1) and (3.A.2)

$$D \cap T = T\Big(\pi(D), \widetilde{J}\big(T, \pi(D)\big)\Big).$$

In the second case, we have $|\pi(T)| \leqslant |\pi(D)|$, hence by (3.A.1) and (3.A.2)

$$D \cap T = T.$$

□

Proof of Lemma 3.4.3. Let D be a strip in $\mathcal{D}$. Then

$$\mu(D) = \inf\{\sum_{D_1 \in \mathcal{D}_1} \sigma(D_1) \colon \mathcal{D}_1 \subseteq \mathcal{D}, D \subseteq \mathcal{L}(\mathcal{D}_1)\}.$$

Therefore, the inequality

$$\mu(D) \leqslant \sigma(D)$$

follows trivially. To prove the opposite inequality, we observe that for every covering $\mathcal{D}_1$ of D by means of strips in $\mathcal{D}$, there exists a strip E in $\mathcal{D}_1$ such that

$$(x_D, |\pi(D)|, 0) \in E,$$

where x_D is the middle point of the dyadic interval $\pi(D)$. In particular, this implies

$$\sigma(E) \geqslant |\pi(D)|.$$

Therefore, we have

$$\sum_{D_1 \in \mathcal{D}_1} \sigma(D_1) \geqslant \sigma(D),$$

By taking the infimum among all the possible coverings of D, we obtain the desired equality in (3.4.7).

The statement for a tree T in $\mathcal{T}$ in (3.4.8) follows by an analogous argument considering the point

$$(x_T, |\pi(T)|, \xi_T),$$

where x_T is the middle point of the dyadic interval $\pi(T)$, and ξ_T is the middle point of the dyadic interval $\widetilde{J}\big(T, \pi(T)\big)$.

The statement in (3.4.9) follows by the definition of $D(T)$, (3.4.7), and (3.4.8). □

Proof of Lemma 3.4.4. Let $\mathcal{D}_1$ be a collection of pairwise disjoint strips in $\mathcal{D}$. The inequality

$$\mu(\mathcal{L}(\mathcal{D}_1)) \leqslant \sum_{D_1 \in \mathcal{D}_1} \mu(D_1),$$

follows by the subadditivity of μ. To prove the opposite inequality, we consider a covering $\mathcal{D}_2$ of $\mathcal{L}(\mathcal{D}_1)$. Without loss of generality, we assume that every E in $\mathcal{D}_2$ is not strictly contained in any element of D_1, otherwise it would be useless to the purpose of covering. Therefore, we have $E \not\subset \mathcal{L}(\mathcal{D}_1)$, and, by Lemma 3.4.1, we have

$$\mathcal{D}_1 = \mathcal{D}_{1,E} \cup \widetilde{\mathcal{D}}_1,$$

where every element of $\mathcal{D}_{1,E}$ is contained in E, and every element of the other collection is disjoint from E. In particular,

$$\mathcal{L}(\mathcal{D}_{1,E}) \subseteq E. \tag{3.A.3}$$

As a consequence, we have

$$\sigma(E) = |\pi(E)| \geqslant |\pi\big(\mathcal{L}(\mathcal{D}_{1,E})\big)| = \sum_{D_1 \in \mathcal{D}_{1,E}} |\pi(D_1)| = \sum_{D_1 \in \mathcal{D}_{1,E}} \mu(D_1),$$

where we used (3.4.7) in the first and in the third equality, (3.A.3) and the monotonicity of π and the Lebesgue measure in the inequality, the distributivity of the projection over set union and the additivity of the Lebesgue measure on the disjoint intervals in $\pi(\mathcal{D}_1)$ in the second equality. Together with the observation that for every element D_1 of $\mathcal{D}_1$ there exists at least one E in $\mathcal{D}_2$ such that $D_1 \in \mathcal{D}_{1,E}$, we obtain

$$\sum_{E \in \mathcal{D}_2} \sigma(E) \geqslant \sum_{E \in \mathcal{D}_2} \sum_{D_1 \in \mathcal{D}_{1,E}} \mu(D_1) \geqslant \sum_{D_1 \in \mathcal{D}_1} \mu(D_1).$$

By taking the infimum among all the possible coverings of $\mathcal{L}(\mathcal{D}_1)$, we obtain the desired equality in (3.4.10).

The statement for a collection $\mathcal{T}_1$ of pairwise disjoint trees in (3.4.11) follows by an analogous argument. The additional observation is that the collection of trees $\mathcal{T}$ splits into two families

$$\mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_-,$$

where the elements of $\mathcal{T}_+$ are all contained in $\mathbb{R} \times (0,\infty) \times (0,\infty)$, while the elements of $\mathcal{T}_-$ are all contained in $\mathbb{R} \times (0,\infty) \times (-\infty, 0]$. In particular, every element of the first family is disjoint from every element of the second one.

The statement in (3.4.12) follows by Lemma 3.4.2 and (3.4.11). □

Proof of (3.4.13), (3.4.14). Let A be a subset of X. For every point (x,s,ξ) in A, there exist $l \in \mathbb{Z}$ such that $s \in (2^{l-1}, 2^l]$, and $m \in \mathbb{Z}$ such that $x \in I(m,l)$. Hence, we have

$$(x,s,\xi) \in D(m,l)_+,$$

proving (3.4.13).

Next, let A_1, A_2 be two subsets of X such that $A_1 \subseteq A_2$. By the definition of $\mathcal{Q}$, we have $\mathcal{Q}(A_1) \subseteq \mathcal{Q}(A_2)$. Taking the union of the elements of the collection in both cases, we obtained the desired inclusion, proving (3.4.14). □

Proof of (3.4.16), (3.4.17). Let $\mathcal{D}_1$ be a collection of strips. By the definition of $\mathcal{N}$, we have $\mathcal{D}_1 \subseteq \mathcal{N}(\mathcal{D}_1)$. Taking the union of the elements of the collection in both cases, we obtained the desired inclusion, proving (3.4.16).

Next, let $\mathcal{D}_1, \mathcal{D}_2$ be two collections of strips such that $\mathcal{L}(\mathcal{D}_1) \subseteq \mathcal{L}(\mathcal{D}_2)$. In particular, $\pi(\mathcal{L}(\mathcal{D}_1)) \subseteq \pi(\mathcal{L}(\mathcal{D}_2))$. By the definition of $\mathcal{N}$, we have $\mathcal{N}(\mathcal{D}_1) \subseteq \mathcal{N}(\mathcal{D}_1)$. Taking the union of the elements of the collection in both cases, we obtained the desired inclusion, proving (3.4.17). □

Proof of (3.4.19), (3.4.20). Let $\mathcal{D}_1$ be a collection of strips. Since $\mathcal{M}(\mathcal{D}_1) \subseteq \mathcal{D}_1$, we have the inclusion $\mathcal{L}(\mathcal{M}(\mathcal{D}_1)) \subseteq \mathcal{L}(\mathcal{D}_1)$.

To prove the inclusion in the opposite direction, we observe that for every strip D' in $\mathcal{D}_1 \backslash \mathcal{M}(\mathcal{D}_1)$, there exists a finite collection of strips in $\mathcal{D}$ strictly containing D'. In particular, there exists a maximal one in $\mathcal{D}_1$, which then belongs to $\mathcal{M}(\mathcal{D}_1)$ and is unique by definition. Taking the union of the elements of the collection in both cases, we obtained the desired inclusion, proving (3.4.19).

The monotonicity property in (3.4.20) follows trivially. □

Proof of (3.4.21), (3.4.18), (3.4.15). The equalities in (3.4.21) follow by (3.4.19) and (3.4.10).

Now, we turn to the proof of the inequality in (3.4.18). By (3.4.19), we have

$$\mathcal{N} \circ \mathcal{M} = \mathcal{N},$$

hence

$$\mu(\mathcal{L}(\mathcal{N}(\mathcal{D}_1))) = \mu(\mathcal{L}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))).$$

By (3.4.19) and (3.4.10), we have

$$\mu(\mathcal{L}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1))))) = \sum_{E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))} |\pi(E)|,$$

$$\mu(\mathcal{L}(\mathcal{D}_1)) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{D}_1))) = \sum_{E \in \mathcal{M}(\mathcal{D}_1)} |\pi(E)|.$$

By the disjointness of the elements in $\mathcal{M}(\mathcal{D}_1)$ and Lemma 3.4.1, we can partition the collection $\mathcal{M}(\mathcal{D}_1)$ into pairwise disjoint subcollections $\mathcal{M}(\mathcal{D}_1)_E$, one for each element $E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))$, so that

$$\mathcal{L}(\mathcal{M}(\mathcal{D}_1)_E) \subseteq E.$$

By the definition of $\mathcal{N}$, we have

$$\sum_{E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))} |\pi(E)| \leqslant 2 \sum_{E \in \mathcal{M}(\mathcal{N}(\mathcal{M}(\mathcal{D}_1)))} \sum_{F \in \mathcal{M}(\mathcal{D}_1)_E} |\pi(F)| \leqslant 2 \sum_{F \in \mathcal{M}(\mathcal{D}_1)} |\pi(F)|.$$

Together with the previous chains of equalities, this yields the desired inequality in (3.4.18).

Finally, we turn to the proof of the equality in (3.4.15). The inequality

$$\mu(A) \leqslant \mu(\mathcal{L}(\mathcal{Q}(A))),$$

follows by (3.4.13) and the monotonicity of μ. The inequality

$$\mu(\mathcal{L}(\mathcal{Q}(A))) = \mu(\mathcal{L}(\mathcal{M}(\mathcal{Q}(A)))) \leqslant \mu(A),$$

follows by an argument analogous to the one used to prove (3.4.10) upon observing that for every E in $\mathcal{M}(\mathcal{Q}(A))$, the intersection between E_+ and A is nonempty. □

Proof of (3.5.12). Without loss of generality, we assume the arbitrary tree $\widetilde{T} \in \widetilde{\mathcal{T}}$ to be of the form $\widetilde{T}(0,1,1)$, namely

$$\widetilde{T}(0,1,1) = \{(y,t,\eta) \colon y \in (0,1], t \in (0,1], \eta \in 1 + (-t^{-1}, t^{-1}]\}.$$

Next, let $\widetilde{T}_0$ be the subset of $\widetilde{T}$ defined by

$$\widetilde{T}_0 = \widetilde{T}(0,1,1) \cap (0,1] \times (0,1] \times (0,\infty).$$

Due to the monotonicity of ν, it is enough to show that

$$\nu(\widetilde{T}_0) = \infty.$$

Now, let $\mathcal{U}_0 \subseteq \mathcal{T}$ be a covering of $\widetilde{T}_0$ by dyadic trees. For every $l \in \mathbb{N}$, let V_l be the subset of $\widetilde{T}_0$ defined by

$$V_l = (0,1] \times (2^{-l-1}, 2^{-l}] \times (2^l, 2^l + 1],$$

and let $\mathcal{U}_0(l)$ be the subcollection of $\mathcal{U}_0$ defined by its dyadic tree with nonempty intersection with V_l. In particular, we have

$$V_l \subseteq \mathcal{L}(\mathcal{U}_0(l)),$$

and, for every $l' \in \mathbb{N}, l' \neq l$, for every $U \in \mathcal{U}_0(l)$, we claim that

$$U \cap V_{l'} = \varnothing.$$

In particular, the dyadic tree U has the form $T\big(m, -j, n(l,j)\big)$, where $j \in \mathbb{Z}, j \leqslant l$, $m \in \mathbb{Z}, 0 \leqslant m < 2^j$, and $n(l,j) \in \mathbb{Z}$ is defined by the condition

$$I(n(l,j),j) \subseteq I(1,l).$$

If $j > l'$, we have

$$U \subseteq \mathbb{R} \times (0, 2^{l'-1}] \times \mathbb{R},$$
$$V_{l'} \subseteq \mathbb{R} \times (2^{l'-1}, 2^{l'}] \times \mathbb{R},$$

yielding the desired disjointness.

If $j < l'$, we distinguish two cases.

Case I: $l < l'$**.** We have

$$I(n(l,j),j) \subseteq I(1,l) \subseteq I(0,l'),$$
$$(2^{l'}, 2^{l'}+1] \subseteq I(1,l'),$$

yielding the desired disjointness.

Case II: $l > l'$**.** We have

$$I(n(l,j),j) \subseteq I(1,l),$$
$$(2^{l'}, 2^{l'}+1] \subseteq I(1,l') \subseteq I(0,l),$$

yielding the desired disjointness.

Therefore, the subcollections $\mathcal{U}_0(l)$ are pairwise disjoint, and we have

$$\sum_{T \in \mathcal{U}_0} \tau(T) \geqslant \sum_{l \in \mathbb{N}} \sum_{T \in \mathcal{U}_0(l)} \tau(T) \geqslant \sum_{l \in \mathbb{N}} \nu(V_l).$$

It is enough to observe that, for every $l \in \mathbb{N}$, we have

$$\nu(V_l) = 1.$$

In fact, for every covering $\mathcal{V}_l$ of V_l by dyadic trees in $\mathcal{T}$, we have

$$\pi(V_l) \subseteq \pi\Big(\bigcup_{V \in \mathcal{V}_l} V\Big) \subseteq \bigcup_{V \in \mathcal{V}_l} \pi(V),$$

hence

$$1 = |\pi(V_l)| \leqslant \sum_{V \in \mathcal{V}_l} |\pi(V)| = \sum_{V \in \mathcal{V}_l} \tau(V).$$

□

3.B Approximation for outer L^p spaces

In this appendix, we present the postponed proofs of the approximation results stated in Section 3.5.

Proof of Lemma 3.5.1. We have

$$\|f\|^p_{L^p_\mu(S)} \leqslant C \sum_{k\in\mathbb{Z}} 2^{kp}\mu(S(f) > 2^k).$$

In particular, there exists $k_0 \in \mathbb{N}$ such that, for every $\widetilde{k} \in \mathbb{N}, \widetilde{k} \geqslant k_0$, we have

$$\|f\|^p_{L^p_\mu(S)} \leqslant C \sum_{k\leqslant\widetilde{k}} 2^{kp}\mu(S(f) > 2^k). \tag{3.B.1}$$

If $\mu(S(f) > 2^{k_0}) = 0$, we have that $f \in L^\infty_\mu(S)$, and we can take $A = X$.

Otherwise, we claim that there exists $k_1 \in \mathbb{N}, k_1 > k_0$ such that

$$\mu(S(f) > 2^{k_1-1}) > 2^p\mu(S(f) > 2^{k_1}). \tag{3.B.2}$$

If not, for every $k \in \mathbb{N}, k > k_0$, we would have

$$2^{kp}\mu(S(f) > 2^k) \geqslant 2^{k_0p}\mu(S(f) > 2^{k_0}) > 0,$$

yielding the contradiction

$$\|f\|^p_{L^p_\mu(S)} \geqslant C \sum_{k=k_0+1}^{\infty} 2^{kp}\mu(S(f) > 2^k) \geqslant C \sum_{k=k_0+1}^{\infty} 2^{k_0p}\mu(S(f) > 2^{k_0}) = \infty.$$

Now, let B be an optimal set associated with $\mu(\ell^r(f) > 2^{k_1})$ up to a factor $2^{-1}(1+2^p)$, namely

$$\|f1_{B^c}\|_{L^\infty_\mu(S)} \leqslant 2^{k_1}, \tag{3.B.3}$$

$$\mu(S(f) > 2^{k_1}) \leqslant \mu(B) \leqslant \frac{1+2^p}{2}\mu(S(f) > 2^{k_1}), \tag{3.B.4}$$

and define $A = B^c$, so that $f1_A \in L^\infty_\mu(S)$.

We claim that for every $k \in \mathbb{N}, k < k_1$, we have

$$\mu(S(f1_A) > 2^k) \geqslant \frac{1-2^{-p}}{2}\mu(S(f) > 2^k). \tag{3.B.5}$$

If not, there would exist $\widetilde{k} \in \mathbb{N}, \widetilde{k} < k_1$ such that

$$\mu(S(f1_A) > 2^{\widetilde{k}}) < \frac{1-2^{-p}}{2}\mu(S(f) > 2^{\widetilde{k}}),$$

yielding the contradiction

$$\begin{aligned}\mu(S(f) > 2^{\tilde{k}}) &\leqslant \mu(S(f1_A) > 2^{\tilde{k}}) + \mu(B)\\ &< \frac{1-2^{-p}}{2}\mu(S(f) > 2^{\tilde{k}}) + \frac{1+2^p}{2}2^{-p}\mu(S(f) > 2^{k_1-1})\\ &\leqslant \mu(S(f) > 2^{\tilde{k}}),\end{aligned}$$

where we used (3.B.3) and the subadditivity of μ in the first inequality, (3.B.4) and (3.B.2) in the second, and the monotonicity of the super level measure $\mu(S(f) > \lambda)$ in λ in the third.

Therefore, by (3.B.1) and (3.B.5), we have

$$\begin{aligned}\|f\|^p_{L^p_\mu(S)} &\leqslant C\sum_{k<k_1} 2^{kp}\mu(S(f) > 2^k) \leqslant C\sum_{k<k_1} 2^{kp}\mu(S(f1_A) > 2^k)\\ &\leqslant C\|f1_A\|^p_{L^p_\mu(S)}.\end{aligned}$$

□

Proof of Lemma 3.5.2. Without loss of generality, upon normalization of f, we assume that

$$1 < \|f\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant 2.$$

For every $k \in \mathbb{Z}, k > 0$, the super level measure of f associated with the level 2^k is zero, and the desired inequality is trivially satisfied.

For the remaining $k \in \mathbb{Z}, k \leqslant 0$, we prove the desired inequality by induction. In particular, we prove that there exist constants $C = C(r)$, $c = c(r)$, and a bounded sequence $\{C_k\colon C_k < C, k \in \mathbb{Z}, k \leqslant 0\}$ such that

$$\nu(\ell^r_\omega(f) > 2^k) \leqslant C_k \sum_{l \geqslant k} \nu(\ell^r_\omega(f1_{X_j}) > c2^l).$$

Case I: $k = 0$. By the r-orthogonality of the classical L^r quasi-norm on sets with disjoint supports, there exists a set B_0 such that

$$\ell^r_\omega(f)(B_0) > 1, \tag{3.B.6}$$
$$\nu(\ell^r_\omega(f) > 1) \leqslant \nu(B_0). \tag{3.B.7}$$

By the monotonicity of the classical L^r quasi-norm and (3.B.6), there exists $j \in \mathbb{N}$ such that

$$\ell^r_\omega(f1_{X_j})(B_0) > 1.$$

Since we have

$$\|f1_{X_j}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant 2,$$

we obtain, by Lemma 3.2.1,

$$\nu(B_0) \leqslant C_0 \nu(\ell^r_\omega(f 1_{X_j}) > c).$$

Together with (3.B.7), this yields the desired inequality.

Case II: $k < 0$**.** We assume that there exists $j = j(r, f, k+1) \in \mathbb{N}$ such that

$$\nu(\ell^r_\omega(f) > 2^{k+1}) \leqslant C_{k+1} \sum_{l \geqslant k+1} \nu(\ell^r_\omega(f 1_{X_j}) > c 2^l). \tag{3.B.8}$$

Now, for every $\varepsilon > 0$, there exists a set A_{k+1} such that

$$\|f 1_{A^c_{k+1}}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant 2^{k+1}, \tag{3.B.9}$$

$$\nu(\ell^r_\omega(f) > 2^{k+1}) \leqslant \nu(A_{k+1}) \leqslant (1+\varepsilon)\nu(\ell^r_\omega(f) > 2^{k+1}). \tag{3.B.10}$$

We will fix ε later. In particular, we have

$$\nu(\ell^r_\omega(f) > 2^k) \leqslant \nu(A_{k+1}) + \nu(\ell^r_\omega(f 1_{A^c_{k+1}}) > 2^k). \tag{3.B.11}$$

If we have

$$\|f 1_{A^c_{k+1}}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant 2^k,$$

we obtain

$$\nu(\ell^r_\omega(f) > 2^k) \leqslant \nu(A_{k+1}) \leqslant (1+\varepsilon) C_{k+1} \sum_{l \geqslant k+1} \nu(\ell^r_\omega(f 1_{X_j}) > c 2^l).$$

Otherwise, we have

$$2^k < \|f 1_{A^c_{k+1}}\|_{L^\infty_\nu(\ell^r_\omega)} \leqslant 2^{k+1}.$$

Applying to the function $f 1_{A^c_{k+1}}$ an argument analogous to that of the previous case, we obtain $j = j(r, f, k) \in \mathbb{N}$, without loss of generality greater than $j(r, f, k+1)$, such that

$$\nu(\ell^r_\omega(f 1_{A^c_{k+1}}) > 2^k) \leqslant C_0 \nu(\ell^r_\omega(f 1_{A^c_{k+1}} 1_{X_j}) > c 2^k) \leqslant C_0 \nu(\ell^r_\omega(f 1_{X_j}) > c 2^k).$$

Together with (3.B.11), (3.B.10), and (3.B.8), the previous chain of inequalities yields

$$\nu(\ell^r_\omega(f) > 2^k) \leqslant (1+\varepsilon) C_{k+1} \sum_{l \geqslant k+1} \nu(\ell^r_\omega(f 1_{X_j}) > c 2^l) + C_0 \nu(\ell^r_\omega(f 1_{X_j}) > c 2^k).$$

By choosing $\varepsilon = \varepsilon(k) = 2^{2^k} - 1$ and defining $C_k = 2^{1-2^k} C_0, C = 2C_0$, we obtain the desired inequality. □

Proof of Lemma 3.5.3. The proof is analogous to that of Lemma 3.5.2 upon the following observation. Without loss of generality, it is enough to comment in the case

$$1 < \|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant 2.$$

Therefore, for every dyadic strip $E \in \mathcal{D}$, we have $f 1_E \in L^q_\nu(\ell^r_\omega)$. Moreover, there exists a collection of maximal dyadic strips $\{E_n \colon E_n \in \mathcal{D}, n \in \mathbb{N}\}$ such that

$$\ell^q_\nu(\ell^r_\omega)(f)(E_n) > 1,$$
$$\mu(\ell^q_\nu(\ell^r_\omega)(f) > 1) \leqslant \sum_{n \in \mathbb{N}} \mu(E_n).$$

In particular, there exists a finite subcollection such that

$$\mu(\ell^q_\nu(\ell^r_\omega)(f) > 1) \leqslant 2 \sum_{n=1}^{N} \mu(E_n).$$

Since the dyadic strips are maximal, then they are disjoint, hence, by Lemma 3.4.4, they are ν-Carathéodory with parameter 1.

Now we apply an argument analogous to that used to prove Lemma 3.5.2 with the monotonicity of the classical L^r quasi-norms replaced by Lemma 3.5.4, and Lemma 3.2.1 replaced by Lemma 3.3.3. □

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Chapter 4

Further results

4.1 Introduction

In this chapter, we collect some statements and proofs of additional properties in the context of the L^p theory for outer measures, as well as some open conjectures.

In Section 4.2, we focus on the properties of single iterated outer L^p quasi-norms and spaces on σ-finite and finite settings described in Subsections 1.2.1 – 1.2.2.

Next, in Section 4.3, we study Minkowski's inequality and the relative embeddings in the case of single iterated outer L^p quasi-norms and spaces on σ-finite settings described in Subsection 1.2.1.

In the following two sections, we investigate the Banach space properties of outer L^p quasi-norms and spaces on the settings on the upper half 3-space or its discrete model described in Subsections 1.2.11 – 1.2.13. First, in Section 4.4, we study the case of the sizes ℓ^r_ω. Then, in Section 4.5, we pass to the case of the size with variable exponent appearing in the article of Do and Thiele [DT15].

After that, in Section 4.6, we make an observation about the Banach space properties of the double iterated outer $L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ spaces on finite settings described in Subsections 1.2.2.

In Section 4.7, we consider the embedding maps via cancellative wavelets from classical L^p spaces to single iterated outer L^p spaces on the settings on the upper half space or its discrete model described in Subsections 1.2.8 – 1.2.10. We comment on some positive and negative results about their boundedness.

We conclude by collecting some conjectures about the L^p theory of outer measure spaces in Section 4.8.

4.2 Outer $L^p_\mu(\ell^r_\omega)$ quasi-norms and spaces

In this section, we further investigate the properties of the single iterated outer L^p quasi-norms and spaces.

First, in Theorem 4.2.1 in Subsection 4.2.1, we extend the results about the Banach space properties of the single iterated outer L^p spaces obtained in Chapter 2 in the case of finite settings described in Subsection 1.2.2 to the case of σ-finite settings described in Subsection 1.2.1. In this regard, we recall that all the finite settings are also σ-finite settings.

Next, in Lemma 4.2.4 in Subsection 4.2.2, we prove that, given two outer measures on the same measure space, an inequality between them implies certain embeddings between the respective single iterated outer L^p spaces.

After that, in Lemma 4.2.5 in Subsection 4.2.3, we provide a counterexample to the uniformity in the finite setting (X, μ, ω) of the constant in the weak quasi-triangle inequality for countably many summands for the outer $L^p_\mu(\ell^\infty_\omega)$ spaces, clarifying the remaining case of the analysis begun in Chapter 2.

Finally, in Lemma 4.2.7 and Lemma 4.2.10 in Subsection 4.2.4, we exhibit necessary and sufficient conditions on the outer measure μ to recover the uniformity in the setting (X, μ, ω) of the constant in the weak and strong quasi-triangle inequality for countably many summands for the outer $L^1_\mu(\ell^\infty_\omega)$ spaces.

4.2.1 Banach space properties of the outer $L^p_\mu(\ell^r_\omega)$ spaces on σ-finite settings

We prove the uniformity in the σ-finite setting (X, μ, ω) of the constants in collapsing of exponents, Köthe duality, and quasi-triangle inequality for countably many summands for the single iterated outer L^p spaces.

Theorem 4.2.1. *For all $p, r \in (0, \infty]$, there exists a constant $C = C(p, r)$ such that, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the following properties hold true.*

(i) For every $p \in (0, \infty]$, for every measurable function $f \in L^p_\mu(\ell^p_\omega)$ on X, we have

$$C^{-1} \|f\|_{L^p_\mu(\ell^p_\omega)} \leqslant \|f\|_{L^p(X,\omega)} \leqslant C \|f\|_{L^p_\mu(\ell^p_\omega)}.$$

(ii) For all $p \in (1, \infty]$, $r \in [1, \infty)$ or $p = r \in \{1, \infty\}$, for every measurable function $f \in L^p_\mu(\ell^r_\omega)$ on X, we have

$$C^{-1} \|f\|_{L^p_\mu(\ell^r_\omega)} \leqslant \sup \left\{ \|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^{r'}_\omega)} = 1 \right\} \leqslant C \|f\|_{L^p_\mu(\ell^r_\omega)}.$$

(iii) For all $p \in (1, \infty]$, $r \in [1, \infty)$ or $p = r \in \{1, \infty\}$, for every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^r_\omega)$ of measurable functions on X, we have

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^p_\mu(\ell^r_\omega)} \leqslant C \sum_{n \in \mathbb{N}} \|f_n\|_{L^p_\mu(\ell^r_\omega)}.$$

The proof of the previous statement follows from an argument analogous to that used to prove Theorem 2.1.1 in Chapter 2 in the case of finite settings. The main ingredient in the latter case is a decomposition result stated in Proposition 2.2.1 in Chapter 2. The same role in the former case is played by Proposition 4.2.2 stated below. This is a decomposition result with respect to the size ℓ^r_ω for measurable functions on X in the intersection between the outer $L^p_\mu(\ell^r_\omega)$ and $L^\infty_\mu(\ell^r_\omega)$ spaces. We point out that, in the case of finite settings, the outer $L^p_\mu(\ell^r_\omega)$ space is contained in the outer $L^\infty_\mu(\ell^r_\omega)$ space. However, in the case of σ-finite settings, in general the inclusion does not hold true.

Therefore, we need an additional ingredient to reduce the study of functions in the outer $L^p_\mu(\ell^r_\omega)$ space to that of functions in the outer $L^p_\mu(\ell^r_\omega) \cap L^\infty_\mu(\ell^r_\omega)$ space in the case of σ-finite settings. The approximation results needed are stated in Lemma 3.5.1 and Lemma 3.5.2 in Chapter 3. In particular, in Lemma 3.5.2, we use the monotone convergence theorem for the classical $L^p(X,\omega)$ spaces, see for example the book of Rudin [Rud74].

We refer to the end of Chapter 1 for the notation of a double sequence parametrized by pairs (k,n) with $k \in \mathbb{Z}$, $n \in \mathbb{N}_k$ appearing in the following statement.

Proposition 4.2.2. *For all $p, r \in (0,\infty)$, there exists a constant $C = C(p,r)$ such that, for every σ-finite setting (X,μ,ω) described in Subsection 1.2.1, the following property holds true.*

For every measurable function $f \in L^p_\mu(\ell^r_\omega) \cap L^\infty_\mu(\ell^r_\omega)$ on X, there exist $k_0 \in \mathbb{Z}$ and a double sequence $\{E_{k,n} \colon k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \Sigma$ of measurable subsets of X such that

- *For every $k \in \mathbb{Z}$, $k > k_0$, we have $\mathbb{N}_k = \varnothing$.*
- *If we set*

$$\begin{aligned}
F_k &= \varnothing, && \text{for every } k \in \mathbb{Z},\ k > k_0,\\
F_{k,0} &= F_{k+1}, && \text{for every } k \in \mathbb{Z},\ k \leqslant k_0,\\
F_{k,n} &= F_{k,n-1} \cup E_{k,n}, && \text{for every } k \in \mathbb{Z},\ k \leqslant k_0, \text{ for every } n \in \mathbb{N}_k,\\
F_k &= F_{k+1} \cup \bigcup_{n\in\mathbb{N}_k} F_{k,n}, && \text{for every } k \in \mathbb{Z},\ k \leqslant k_0,
\end{aligned}$$

then, for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_k$, we have

$$\ell^r_\omega(f 1_{F^c_{k,n-1}})(E_{k,n}) > 2^k, \qquad \text{when } E_{k,n} \neq \varnothing, \tag{4.2.1}$$

$$\|f 1_{F^c_k}\|_{L^\infty_\mu(\ell^r_\omega)} \leqslant 2^k, \tag{4.2.2}$$

$$\mu(\ell^r_\omega(f) > 2^k) \leqslant \mu(F_k), \tag{4.2.3}$$

$$\sum_{n\in\mathbb{N}_k} \mu(E_{k,n}) \leqslant C\mu(\ell^r_\omega(f) > 2^{k-1}). \tag{4.2.4}$$

In particular, we have

$$\|f\|^p_{L^p_\mu(\ell^r_\omega)} \sim_{p,r} \sum_{k\in\mathbb{Z}} 2^{kp} \sum_{n\in\mathbb{N}_k} \mu(E_{k,n}) \sim_{p,r} \sum_{k\in\mathbb{Z}} 2^{kp} \sum_{l\in\mathbb{Z}, l\geqslant k} \sum_{m\in\mathbb{N}_l} \mu(E_{l,m}).$$

Proof. The selection algorithm is analogous to that described in **Case I** in the proof of Proposition 2.2.4 in Chapter 2. We define the collection $\{E_{k,n} \colon k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \Sigma$ of measurable subsets of X by a double recursion, backward on $k \in \mathbb{Z}$, and, for every fixed k, forward on $n \in \mathbb{N}_k$. In parallel, we prove the properties in (4.2.1) – (4.2.4) by backward induction on $k \in \mathbb{Z}$.

We briefly comment on the modification needed. For all $\widetilde{k} \in \mathbb{Z}$, $\widetilde{k} \leqslant k_0$, $\widetilde{n} \in \mathbb{N}_{\widetilde{k}}$, we define the collection $\mathcal{E}_{\widetilde{k},\widetilde{n}} \subseteq \Sigma$ of measurable subsets of X by

$$\mathcal{E}_{\widetilde{k},\widetilde{n}} = \Big\{ E \in \Sigma \colon \ell^r_\omega(f 1_{F^c_{\widetilde{k},\widetilde{n}-1}})(E) > 2^{\widetilde{k}} \Big\}.$$

If $\mathcal{E}_{\widetilde{k},\widetilde{n}}$ is empty, we define $\mathbb{N}_{\widetilde{k}} \subseteq \mathbb{N}$ by

$$\mathbb{N}_{\widetilde{k}} = \begin{cases} \varnothing, & \text{if } \widetilde{n} = 1, \\ \{1, \dots, \widetilde{n}-1\}, & \text{if } \widetilde{n} \in \mathbb{N}, \widetilde{n} > 1. \end{cases}$$

If $\mathcal{E}_{\widetilde{k},\widetilde{n}}$ is not empty, by an argument analogous to that used to prove the first inequality in (2.16) in Chapter 2, we can prove that, for every measurable subset $E \in \mathcal{E}_{\widetilde{k},\widetilde{n}}$, we have

$$\mu(E) + \sum_{n \in \mathbb{N}_{\widetilde{k}}, n < \widetilde{n}} \mu(E_{\widetilde{k},n}) \leqslant C \mu(\ell^r_\omega(f) > 2^{\widetilde{k}-1}) \leqslant C 2^{-\widetilde{k}p} \|f\|^p_{L^p_\mu(\ell^r_\omega)} < \infty, \tag{4.2.5}$$

hence there exists $j_0 = j_0(\widetilde{k}, \widetilde{n}) \in \mathbb{Z}$ such that

$$\sup \Big\{ \mu(E) \colon E \in \mathcal{E}_{\widetilde{k},\widetilde{n}} \Big\} \in (2^{j_0}, 2^{j_0+1}].$$

We choose $E_{\widetilde{k},\widetilde{n}} \in \mathcal{E}_{\widetilde{k},\widetilde{n}}$ such that

$$\mu(E_{\widetilde{k},\widetilde{n}}) \in (2^{j_0}, 2^{j_0+1}].$$

By the inequality in (4.2.5), we have

$$j_0(\widetilde{k}, \widetilde{n}+1) \leqslant j_0(\widetilde{k}, \widetilde{n}) - 1,$$

hence, for every $\widetilde{k} \in \mathbb{Z}$, the sequence $\{\mu(E_{\widetilde{k},n}) \colon n \in \mathbb{N}_{\widetilde{k}}\}$ is strictly decreasing. □

4.2.2 Domination between outer measures and embeddings between outer L^p spaces

We start with the definition of domination between outer measures.

Definition 4.2.3. *Let μ and ν be two outer measures on a set X. We say that ν* dominates *μ or equivalently μ* is dominated *by ν if, for every subset $A \subseteq X$, we have*

$$\mu(A) \leqslant \nu(A). \tag{4.2.6}$$

Given two outer measures μ and ν on the same measure space (X,ω) such that ν dominates μ, we obtain certain embeddings between the single iterated outer L^p spaces associated with them.

Lemma 4.2.4. *For all $p, r \in (0,\infty]$, there exists a constant $C = C(p,r)$ such that, for all σ-finite settings (X,μ,ω) and (X,ν,ω) described in Subsection 1.2.1, if ν dominates μ, then, for every measurable function f on X, we have*

$$\text{if } p \geqslant r, \qquad \|f\|_{L^p_\nu(\ell^r_\omega)} \leqslant C\|f\|_{L^p_\mu(\ell^r_\omega)},$$
$$\text{if } p \leqslant r, \qquad \|f\|_{L^p_\mu(\ell^r_\omega)} \leqslant C\|f\|_{L^p_\nu(\ell^r_\omega)}.$$

Proof. We split the proof into four cases according to the values of p and r.

Case I: $p = r \in (0,\infty]$. The desired inequalities follow from collapsing of exponents, property (i) in Theorem 4.2.1, for the σ-finite settings (X,μ,ω) and (X,ν,ω).

Case II: $r < p = \infty$ or $p < r = \infty$. The desired inequalities follow from the definition of the outer L^p quasi-norms in Definition 1.1.4 and Definition 1.1.6, the definition of the sizes ℓ^r_ω in (1.2.3) and (1.2.4), and the domination between the outer measures.

Case III: $r < p < \infty$. Without loss of generality, we assume $r = 1$, since, for every setting (X,μ,ω), we have

$$\|f\|^r_{L^p_\mu(\ell^r_\omega)} = \|f^r\|_{L^{\frac{p}{r}}_\mu(\ell^1_\omega)}.$$

In particular, we have $p > 1$, hence

$$\begin{aligned}
\|f\|_{L^p_\nu(\ell^1_\omega)} &\leqslant C \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\nu(\ell^\infty_\omega)} = 1\Big\} \\
&\leqslant C \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^\infty_\omega)} = 1\Big\} \\
&\leqslant C \sup\Big\{\|fg\|_{L^1_\mu(\ell^1_\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^\infty_\omega)} = 1\Big\} \\
&\leqslant C\|f\|_{L^p_\mu(\ell^1_\omega)},
\end{aligned}$$

where we used Köthe duality, property (ii) in Theorem 4.2.1, for the σ-finite setting (X,ν,ω) in the first inequality, the inequality proved in **Case II** in the second, the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) for the σ-finite setting (X,μ,ω) in the third, and outer Hölder's inequality (Theorem 1.1.7) for the σ-finite setting (X,μ,ω) in the fourth.

Case IV: $p < r < \infty$. Without loss of generality, we assume $p = 2$. In particular, we have $1 < r' < p' = 2$, hence

$$\begin{aligned}
\|f\|_{L^2_\mu(\ell^r_\omega)} &\leqslant C \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^2_\mu(\ell^{r'}_\omega)} = 1\Big\} \\
&\leqslant C \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^2_\nu(\ell^{r'}_\omega)} = 1\Big\} \\
&\leqslant C \sup\Big\{\|fg\|_{L^1_\nu(\ell^1_\omega)} \colon \|g\|_{L^2_\nu(\ell^{r'}_\omega)} = 1\Big\} \\
&\leqslant C\|f\|_{L^2_\nu(\ell^r_\omega)},
\end{aligned}$$

where we used Köthe duality, property (ii) in Theorem 4.2.1, for the σ-finite setting (X,μ,ω) in the first inequality, the inequality proved in **Case III** in the second, the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) for the σ-finite setting (X,ν,ω) in the third, and outer Hölder's inequality (Theorem 1.1.7) for the σ-finite setting (X,ν,ω) in the fourth. □

4.2.3 Counterexample to uniform weak quasi-triangle inequality for the outer $L^p_\mu(\ell^\infty_\omega)$ spaces

For $p \in [1,\infty)$, we consider the dependence on the finite setting (X,μ,ω) of the constant in the inequality

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} \leqslant C(X,\mu,\omega) \sum_{n\in\mathbb{N}} \| f_n \|_{L^p_\mu(\ell^\infty_\omega)},$$

where $\{f_n \colon n \in \mathbb{N}\}$ is any arbitrary collection of functions on X. For every $p \in [1,\infty)$, we exhibit a counterexample to the uniformity of the constant in the finite setting (X,μ,ω). This failure implies the existence of counterexamples to the uniformity in the finite setting (X,μ,ω) of the constant in Köthe duality for the outer $L^p_\mu(\ell^\infty_\omega)$ spaces with $p \in [1,\infty)$ as well.

The counterexample is suggested by the articles of Herer and Christensen [HC75] and Topsøe [Top76], where the authors studied the existence of *pathological submeasures*. A pathological submeasure μ on a set X is a non-zero outer measure such that the only measure on X dominated by μ as in Definition 4.2.3 is the zero measure.

Lemma 4.2.5. *Let $p \in [1,\infty)$. For every $M > 0$, there exist a finite setting (X,μ,ω) and a collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^\infty_\omega)$ of functions on X such that*

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} \geqslant M \sum_{n\in\mathbb{N}} \| f_n \|_{L^p_\mu(\ell^\infty_\omega)}.$$

Proof. For every $m \in \mathbb{N}$, let (X_m,μ_m,ω_m) be the setting on the m-dimensional hypercube of sidelength m described in Subsection 1.2.6. We refer to that subsection for the definition of the subset $E(x)$.

For every $x \in X_m$, we define the function f_x on X_m by

$$f_x = 1_{E(x)}.$$

Summing over all the elements of X_m, we obtain

$$\sum_{x\in X_m} f_x = (m-1)^m 1_{X_m}.$$

Next, for every $n \in \mathbb{N}$, $n \leqslant m$, for every collection $\{x_1,\dots,x_n\} \subseteq X_m$, we define $x \in X_m$ by

$$x = (\pi_1(x_1),\dots,\pi_n(x_n),0,\dots,0),$$

where $\pi_i \colon X_m \to \mathbb{Z}/m\mathbb{Z}$ is the projection onto the i-th coordinate. Hence, for every $n \in \mathbb{N}$, $n \leqslant m$, we have

$$x \notin \bigcup_{i=1}^{n} E(x_i).$$

Therefore, since the outer measure μ_m is generated via minimal coverings as in (1.1.1) by the pre-measure σ_m on the collection $\{E(x) \colon x \in X_m\}$ and $\sigma_m(E(x)) = 1$ for every $x \in X_m$, we have

$$\mu_m(X_m) > m.$$

Moreover, for every $x \in X_m$, we have

$$\mu_m(E(x)) = 1.$$

Together with the equality for the outer L^p and $L^{p,\infty}$ quasi-norms of a characteristic function in finite settings in (1.2.10), the previous two displays yield

$$\Big\| \sum_{x \in X_m} f_x \Big\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} = (m-1)^m \|1_{X_m}\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} > (m-1)^m m^{\frac{1}{p}},$$

$$\sum_{x \in X_m} \|f_x\|_{L^p_\mu(\ell^\infty_\omega)} = \sum_{x \in X_m} 1 = m^m.$$

Taking $m \in \mathbb{N}$ big enough, we obtain the desired inequality. □

4.2.4 Necessary and sufficient conditions for strong and weak quasi-triangle inequalities for the outer $L^1_\mu(\ell^\infty_\omega)$ space

We start by recalling an already known necessary and sufficient condition on the outer measure μ to recover the strong triangle inequality for countably many summands for the outer $L^1_\mu(\ell^\infty_\omega)$ space on general settings. The condition on μ is called *strong subadditivity* in the article of Choquet [Cho54], and *submodularity* in the book of Denneberg [Den94]. We refer to [Cho54] and [Den94] for the proof of the following statement. We point out that the constant in the triangle inequality in (4.2.8) is 1.

Lemma 4.2.6 (Theorem in Subsection 54.2 in [Cho54], Theorem 6.3 in [Den94])**.** *For every general setting (X, μ, ω) described in Subsection 1.2.3 where, for every $x \in X$, we have*

$$\omega(\{x\}) \in (0, \infty), \tag{4.2.7}$$

the following properties are equivalent.

(i) For every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\mu(\ell^\infty_\omega)$ of functions on X, we have

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^1_\mu(\ell^\infty_\omega)} \leqslant \sum_{n \in \mathbb{N}} \|f_n\|_{L^1_\mu(\ell^\infty_\omega)}. \tag{4.2.8}$$

(ii) For all subsets $A, B \subseteq X$, we have

$$\mu(A \cup B) + \mu(A \cap B) \leqslant \mu(A) + \mu(B).$$

Next, we provide new necessary and sufficient conditions on the outer measure μ to recover the uniformity in the general setting (X, μ, ω) of the constant in the weak and strong quasi-triangle inequalities for countably many summands for the outer $L^1_\mu(\ell^\infty_\omega)$ space. In particular, we allow the constants appearing in the quasi-triangle inequalities in (4.2.9) and (4.2.16) to be different from 1. We start with the weak quasi-triangle inequality. We recall that, in the case of general settings, for every measurable function f on X, we have the same properties described in the equalities in (1.2.6) and (1.2.7) in the case of σ-finite settings.

Lemma 4.2.7. *Let $C \geqslant 1$. For every general setting (X, μ, ω) described in Subsection 1.2.3, the following properties are equivalent.*

(i) For every collection $\{f_n : n \in \mathbb{N}\} \subseteq L^1_\mu(\ell^\infty_\omega)$ of measurable functions on X, we have

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)} \leqslant C \sum_{n \in \mathbb{N}} \| f_n \|_{L^1_\mu(\ell^\infty_\omega)}. \tag{4.2.9}$$

(ii) For every collection $\{A_n : n \in \mathbb{N}\}$ of measurable subsets of X, we have

$$\Big\| \sum_{n \in \mathbb{N}} 1_{A_n} \Big\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)} \leqslant C \sum_{n \in \mathbb{N}} \| 1_{A_n} \|_{L^1_\mu(\ell^\infty_\omega)}. \tag{4.2.10}$$

Moreover, if ω satisfies the condition in (4.2.7), the properties are equivalent to the following one.

(iii) For every collection $\{A_n : n \in \mathbb{N}\}$ of subsets of X, we have

$$\min\Big\{ \sum_{n \in \mathbb{N}} 1_{A_n}(x) \colon x \in \bigcup_{n \in \mathbb{N}} A_n \Big\} \mu\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) \leqslant C \sum_{n \in \mathbb{N}} \mu(A_n). \tag{4.2.11}$$

Proof. **Case I:** $(i) \Rightarrow (ii)$**.** Let $\{A_n : n \in \mathbb{N}\}$ be a collection of measurable subsets of X. For every $n \in \mathbb{N}$, we define the measurable function f_n on X by

$$f_n = 1_{A_n}.$$

The inequality in (4.2.9) for the collection $\{f_n : n \in \mathbb{N}\}$ of measurable functions on X yields the inequality in (4.2.10).

Case II: $(ii) \Rightarrow (iii)$**.** We define $j \in \mathbb{N}$ by

$$j = \min\Big\{ \sum_{n \in \mathbb{N}} 1_{A_n}(x) \colon x \in \bigcup_{n \in \mathbb{N}} A_n \Big\}.$$

By the definition of $j \in \mathbb{N}$, the condition on ω in (4.2.7), and the equality for the outer $L^{1,\infty}$ quasi-norm in (1.2.6), we have

$$j\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sup\Big\{\lambda\mu\Big(\ell^\infty_\omega\Big(\sum_{n\in\mathbb{N}} 1_{A_n}\Big) > \lambda\Big) \colon \lambda \in (0,j)\Big\} \leqslant \Big\|\sum_{n\in\mathbb{N}} 1_{A_n}\Big\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)}.$$

Moreover, by the equality for the outer L^1 quasi-norm of a characteristic function in (1.2.7) and the condition on ω in (4.2.7), for every $n \in \mathbb{N}$, we have

$$\|1_{A_n}\|_{L^1_\mu(\ell^\infty_\omega)} = \mu(A_n).$$

Together with the inequality in (4.2.10), the previous three displays yield the desired inequality in (4.2.11).

Case III: $(ii) \Rightarrow (i)$**.** Let $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\mu(\ell^\infty_\omega)$ be a collection of measurable functions on X. Without loss of generality, we assume that all the functions $\{f_n \colon n \in \mathbb{N}\}$ are non-negative. We start by considering functions on X with values in $\mathbb{N} \cup \{0\}$.

For every $j \in \mathbb{N}$, we define the measurable subset $A_j \subseteq X$ by

$$A_j = \Big\{x \in X \colon \sum_{n\in\mathbb{N}} f_n(x) \geqslant j\Big\},$$

and, for all $n, m \in \mathbb{N}$, we define the measurable subset $A_{n,m} \subseteq X$ by

$$A_{n,m} = \Big\{x \in X \colon f_n(x) \geqslant m\Big\}.$$

By the equalities for the outer $L^{1,\infty}$ and L^1 quasi-norms in (1.2.6) and (1.2.7), since the functions in the collection $\{f_n \colon n \in \mathbb{N}\}$ have values in $\mathbb{N} \cup \{0\}$, we have

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)} = \sup\Big\{\|j1_{A_j}\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)} \colon j \in \mathbb{N}\Big\},$$

and, for every $n \in \mathbb{N}$, we have

$$\|f_n\|_{L^1_\mu(\ell^\infty_\omega)} = \sum_{m\in\mathbb{N}} \|1_{A_{n,m}}\|_{L^1_\mu(\ell^\infty_\omega)}. \tag{4.2.12}$$

Moreover, for every $j \in \mathbb{N}$, by the definition of the measurable subsets A_j and $A_{n,m}$, we have

$$j1_{A_j} \leqslant \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} 1_{A_{n,m}}.$$

For every $j \in \mathbb{N}$, together with the inequality in (4.2.10) for the collection $\{A_{n,m} \colon n, m \in \mathbb{N}\}$ of measurable subsets of X, the previous two displays yield the inequality

$$\|j1_{A_j}\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)} \leqslant \sum_{n\in\mathbb{N}} \|f_n\|_{L^1_\mu(\ell^\infty_\omega)}.$$

Taking the supremum over $j \in \mathbb{N}$, together with the equality in (4.2.12), the previous display yields the desired inequality in (4.2.9).

By standard homogeneity and approximation arguments, we extend the result to functions on X with values first in $\mathbb{Q}$ and then in $\mathbb{R}$.

Case IV: *(iii)* $\Rightarrow$ *(ii)*. Let $\{A_n \colon n \in \mathbb{N}\}$ be a collection of subsets of X. For every $j \in \mathbb{N}$, we define the subset $B_j \subseteq X$ by

$$B_j = \Big\{x \in X \colon \sum_{n \in \mathbb{N}} 1_{A_n}(x) \geqslant j\Big\}.$$

Therefore, we have

$$j 1_{B_j} \leqslant \sum_{n \in \mathbb{N}} 1_{B_j \cap A_n}. \tag{4.2.13}$$

By the equality for the outer $L^{1,\infty}$ quasi-norm in (1.2.6) and the condition on ω in (4.2.7), we have

$$\Big\| \sum_{n \in \mathbb{N}} 1_{A_n} \Big\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)} = \sup \Big\{ j\mu(B_j) \colon j \in \mathbb{N} \Big\}, \tag{4.2.14}$$

and, by the equality for the outer L^1 quasi-norm of a characteristic function in (1.2.7) and the condition on ω in (4.2.7), for every $n \in \mathbb{N}$, we have

$$\| 1_{A_n} \|_{L^1_\mu(\ell^\infty_\omega)} = \mu(A_n). \tag{4.2.15}$$

For every $j \in \mathbb{N}$, together with the inequality in (4.2.11) for the collection $\{B_j \cap A_n \colon n \in \mathbb{N}\}$ of subsets of X, the equality in (4.2.15) and the inequality in (4.2.13) yield the inequality

$$j\mu(B_j) \leqslant C \sum_{n \in \mathbb{N}} \| 1_{A_n} \|_{L^1_\mu(\ell^\infty_\omega)}.$$

Taking the supremum over $j \in \mathbb{N}$, together with the equality in (4.2.14), the previous display yields the desired inequality in (4.2.10). □

Remark 4.2.8. *Every finite setting described in Subsection 1.2.2 is an example of a general setting (X, μ, ω) where, for every $x \in X$, we have $\omega(\{x\}) \in (0, \infty)$.*

Remark 4.2.9. *The condition associated with the inequality in* (4.2.11) *is called* quasi-subadditivity of order infinity *in [AL85], and it can be understood as follows.*

For every subset $A \subseteq X$, for every function $\phi \colon X \to \mathbb{N}$, we define the subset $\widetilde{E}(A, \phi) \subseteq X \times \mathbb{N}$ by

$$\widetilde{E}(A, \phi) = \Big\{ (x, \phi(x)) \in X \times \mathbb{N} \colon x \in A \Big\}.$$

Next, let

$$\begin{aligned} \widetilde{X} &= X \times \mathbb{N}, \\ \widetilde{\mathcal{E}} &= \Big\{ \widetilde{E}(A, \phi) \colon A \subseteq X, \phi \colon A \to \mathbb{N} \Big\}, \\ \widetilde{\sigma}(\widetilde{E}(A, \phi)) &= \mu(A), && \text{for all } A \subseteq X,\ \phi \colon A \to \mathbb{N}, \end{aligned}$$

and let $\widetilde{\mu}$ be the outer measure generated via minimal coverings as in (1.1.1) *by the pre-measure $\widetilde{\sigma}$ on the collection $\widetilde{\mathcal{E}}$. The condition associated with the inequality in* (4.2.11) *states that, up to a bounded multiplicative constant C, for every $j \in \mathbb{N}$, for every subset $A \subseteq X$, we have*

$$j\mu(A) \leqslant C\widetilde{\mu}(A \times \{1, 2, \ldots, j\}).$$

Therefore, the collection $\{A \times \{i\} \colon i \in \mathbb{N}, i \leqslant j\}$ of subsets of $\widetilde{X}$ provides a covering of the set $A \times \{1, 2, \ldots, j\}$ defining the value of the outer measure $\widetilde{\mu}$ via $\widetilde{\sigma}$, up to a multiplicative factor C. We point out that, by the definition of $\widetilde{\mu}$, we have

$$\widetilde{\mu}(A \times \{1, 2, \ldots, j\}) \leqslant j\mu(A).$$

We follow up with the strong quasi-triangle inequality.

Lemma 4.2.10. *Let $C \geqslant 1$. For every general setting (X, μ, ω) described in Subsection 1.2.3 where ω satisfies the condition in* (4.2.7)*, the following properties are equivalent.*

(i) For every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\mu(\ell^\infty_\omega)$ of functions on X, we have

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^1_\mu(\ell^\infty_\omega)} \leqslant C \sum_{n \in \mathbb{N}} \| f_n \|_{L^1_\mu(\ell^\infty_\omega)}. \tag{4.2.16}$$

(ii) For every collection $\{A_n \colon n \in \mathbb{N}\}$ of subsets of X such that

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots, \tag{4.2.17}$$

we have, for the outer measure $\widetilde{\mu}$ defined in Remark 4.2.9,

$$\sum_{n \in \mathbb{N}} \mu(A_n) \leqslant C\widetilde{\mu}\Big(\bigcup_{n \in \mathbb{N}} (A_n \times \{n\}) \Big). \tag{4.2.18}$$

Proof. **Case I:** $(i) \Rightarrow (ii)$**.** Let $\{A_n \colon n \in \mathbb{N}\}$ be a collection of subsets of X. For the collection $\widetilde{\mathcal{E}}$ of subsets of $X \times \mathbb{N}$ of the form $\widetilde{E}(A, \phi)$ defined in Remark 4.2.9, let $\{\widetilde{E}(B_i, \phi_i) \colon i \in \mathbb{N}\} \subseteq \widetilde{\mathcal{E}}$ be a collection such that

$$\bigcup_{n \in \mathbb{N}} (A_n \times \{n\}) \subseteq \bigcup_{i \in \mathbb{N}} \widetilde{E}(B_i, \phi_i). \tag{4.2.19}$$

Then, for every $i \in \mathbb{N}$, we define the function f_i on X by

$$f_i = 1_{B_i}.$$

By the equality for the outer L^1 quasi-norm of a characteristic function in (1.2.7) and the condition on ω in (4.2.7), we have

$$\| f_i \|_{L^1_\mu(\ell^\infty_\omega)} = \mu(B_i) = \widetilde{\sigma}(\widetilde{E}(B_i, \phi_i)).$$

Next, by the inclusions in (4.2.17) and (4.2.19), we have

$$\sum_{n\in\mathbb{N}} 1_{A_n} \leqslant \sum_{i\in\mathbb{N}} f_i.$$

Moreover, by the equality for the outer L^1 quasi-norm in (1.2.6) and the condition on ω in (4.2.7), we have

$$\Big\|\sum_{n\in\mathbb{N}} 1_{A_n}\Big\|_{L^1_\mu(\ell^\infty_\omega)} = \sum_{j\in\mathbb{N}} \mu\Big(\Big\{x \in X \colon \sum_{n\in\mathbb{N}} 1_{A_n}(x) \geqslant j\Big\}\Big),$$

and, by the inclusion in (4.2.17), for every $j \in \mathbb{N}$, we have

$$\mu\Big(\Big\{x \in X \colon \sum_{n\in\mathbb{N}} 1_{A_n}(x) \geqslant j\Big\}\Big) = \mu(A_j).$$

Together with the monotonicity of the outer L^p quasi-norms and the inequality in (4.2.16) for the collection $\{f_i \colon i \in \mathbb{N}\}$ of functions on X, the previous four displays yield the inequality

$$\sum_{n\in\mathbb{N}} \mu(A_n) \leqslant C \sum_{i\in\mathbb{N}} \widetilde{\sigma}(\widetilde{E}(B_i, \phi_i)).$$

Taking the infimum over the coverings of $\bigcup_{n\in\mathbb{N}}(A_n \times \{n\})$ by countable subcollections of $\widetilde{\mathcal{E}}$, by the definition of $\widetilde{\mu}$, we obtain the desired inequality in (4.2.18).

Case II: *(ii)* $\Rightarrow$ *(i)***.** Let $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\mu(\ell^\infty_\omega)$ be a collection of functions on X. Without loss of generality, we assume that all the functions $\{f_n \colon n \in \mathbb{N}\}$ are non-negative. We start by considering functions on X with values in $\mathbb{N} \cup \{0\}$.

For every $j \in \mathbb{N}$, we define the subset $A_j \subseteq X$ by

$$A_j = \Big\{x \in X \colon \sum_{n\in\mathbb{N}} f_n(x) \geqslant j\Big\}.$$

Therefore, we have

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_j \supseteq A_{j+1} \supseteq \cdots.$$

Moreover, for all $n, m \in \mathbb{N}$, we define the subset $B_{n,m} \subseteq X$ by

$$B_{n,m} = \Big\{x \in X \colon f_n(x) \geqslant m\Big\},$$

and we define the function $\phi_{n,m}$ by

$$\phi_{n,m} \colon B_{n,m} \to \mathbb{N}, \qquad \phi_{n,m}(x) = 1 + \sum_{l\in\mathbb{N}, l<n} \sum_{k\in\mathbb{N}} 1_{B_{l,k}}(x) + \sum_{k\in\mathbb{N}, k<m} 1_{B_{n,k}}(x).$$

For the subsets of $X \times \mathbb{N}$ of the form $\widetilde{E}(A,\phi)$ defined in Remark 4.2.9, we have

$$\bigcup_{j\in\mathbb{N}}(A_j \times \{j\}) \subseteq \bigcup_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}} \widetilde{E}(B_{n,m},\phi_{n,m}).$$

By the equality for the outer L^1 quasi-norm in (1.2.6) and the condition on ω in (4.2.7), we have

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^1_\mu(\ell^\infty_\omega)} = \sum_{j\in\mathbb{N}}\mu(A_j),$$

and, for every $n \in \mathbb{N}$, we have

$$\|f_n\|_{L^1_\mu(\ell^\infty_\omega)} = \sum_{m\in\mathbb{N}}\mu(B_{n,m}) = \sum_{m\in\mathbb{N}}\widetilde{\sigma}(\widetilde{E}(B_{n,m},\phi_{n,m})).$$

Together with the inequality in (4.2.18) for the collection $\{A_j \colon j \in \mathbb{N}\}$ of subsets of X and the definition of the outer measure $\widetilde{\mu}$, the previous three displays yield the desired inequality in (4.2.16).

By standard homogeneity and approximation arguments, we extend the result to functions on X with values first in $\mathbb{Q}$ and then in $\mathbb{R}$. □

4.3 Minkowski's inequality for the outer $L^p_\mu(\ell^r_\omega)$ quasi-norms

In this section, we extend the classical Minkowski's inequality in the case of mixed L^p spaces on the Cartesian product of σ-finite measure spaces to the case of single iterated outer L^p spaces on σ-finite settings described in Subsection 1.2.1.

This extension requires the definition of an additional outer measure υ associated with every σ-finite setting (X,μ,ω). To define υ, we introduce a canonical construction in Subsection 4.3.1. Iterating the canonical construction updating recursively the setting, we obtain a collection of new outer measures on X which are related between themselves, as we show in Subsection 4.3.2. As an example, in Subsections 4.3.3 – 4.3.4, we study this collection in the finite settings described in Subsections 1.2.6 – 1.2.7. Next, in Theorem 4.3.9, we state and prove the desired Minkowski's inequality between the outer $L^p_\mu(\ell^r_\omega)$ and $L^r_\upsilon(\ell^p_\omega)$ quasi-norms on σ-finite settings, and the embeddings between the respective single iterated outer L^p spaces.

After that, in Subsection 4.3.6, we comment on the outer measures generated by the iterations of the canonical construction in all the remaining settings described in Subsections 1.2.1 – 1.2.13.

In Subsection 4.3.7, we conclude the section by showing that the canonical construction defining the outer measure υ guarantees some additional regularity in terms of the subadditivity behaviour of υ. In particular, the outer measure υ satisfies the condition associated with the inequality in (4.2.10) with constant 1, and it behaves nicely in terms of the Banach space properties of the outer $L^p_\upsilon(\ell^\infty_\omega)$ spaces with $p \in (1,\infty]$.

4.3.1 Recursive construction of canonical outer measures

We start by observing that, for every $r \in [1,\infty]$, for every σ-finite setting (X,μ,ω), the outer $L^\infty_\mu(\ell^r_\omega)$ quasi-norm with $r \in [1,\infty]$ inherits the triangle inequality from the classical $L^r(X,\omega)$ norm.

Lemma 4.3.1. *For every $r \in [1,\infty]$, for every σ-finite setting (X,μ,ω) described in Subsection 1.2.1, the outer $L^\infty_\mu(\ell^r_\omega)$ quasi-norm is a norm.*

Proof. We claim that, for all measurable functions $f,g \in L^\infty_\mu(\ell^r_\omega)$ on X, we have

$$\|f+g\|_{L^\infty_\mu(\ell^r_\omega)} \leqslant \|f\|_{L^\infty_\mu(\ell^r_\omega)} + \|g\|_{L^\infty_\mu(\ell^r_\omega)}. \tag{4.3.1}$$

In fact, for every measurable subset $A \in \Sigma$, $\mu(A) = 0$, by the definition of the size ℓ^r_ω in (1.2.3) and (1.2.4), we have

$$\ell^r_\omega(f+g)(A) = 0 \leqslant \|f\|_{L^\infty_\mu(\ell^r_\omega)} + \|g\|_{L^\infty_\mu(\ell^r_\omega)}.$$

Moreover, for every measurable subset $A \in \Sigma$, $\mu(A) \notin \{0,\infty\}$, we have

$$\begin{aligned} \ell^r_\omega(f+g)(A) &= \mu(A)^{-\frac{1}{r}} \|(f+g)1_A\|_{L^r(X,\omega)} \\ &\leqslant \mu(A)^{-\frac{1}{r}} (\|f1_A\|_{L^r(X,\omega)} + \|g1_A\|_{L^r(X,\omega)}) \\ &\leqslant \|f\|_{L^\infty_\mu(\ell^r_\omega)} + \|g\|_{L^\infty_\mu(\ell^r_\omega)}, \end{aligned}$$

where we used the triangle inequality for the classical $L^r(X,\omega)$ norm in the first inequality. Taking the supremum over the measurable subsets $A \in \Sigma$, $\mu(A) \neq \infty$, the previous two displays yield the desired inequality in (4.3.1). □

For every σ-finite setting (X,μ,ω), we define the pre-measure $\sigma_\upsilon = \sigma_\upsilon(\mu,\omega)$ on the collection Σ of all the measurable subsets of X by

$$\sigma_\upsilon(A) \coloneqq \begin{cases} 0, & \text{if } \mu(A) = 0, \\ \|1_A\|_{L^\infty_\mu(\ell^1_\omega)}, & \text{if } \mu(A) \neq 0. \end{cases} \tag{4.3.2}$$

Next, we define $\upsilon = \upsilon(\mu,\omega)$ to be the outer measure on X generated via minimal coverings as in (1.1.1) by the pre-measure σ_υ on the collection Σ. By Lemma 4.3.1, for every measurable subset $A \in \Sigma$, we have

$$\upsilon(A) = \sigma_\upsilon(A). \tag{4.3.3}$$

In particular, for every measurable subset $A \in \Sigma$, if we define the collection $\Sigma_\mu(A) \subseteq \Sigma$ of measurable subsets by

$$\Sigma_\mu(A) = \Big\{ B \in \Sigma \colon B \subseteq A, \mu(B) \notin \{0,\infty\} \Big\},$$

then, we have

$$\upsilon(A) = \sup \Big\{ \frac{\omega(B)}{\mu(B)} \colon B \in \Sigma_\mu(A) \Big\}, \tag{4.3.4}$$

where the supremum over an empty collection is understood to be 0. Moreover, if the outer measure μ is generated via minimal coverings as in (1.1.1) by a pre-measure σ on a collection $\mathcal{E}$, by Lemma 2.A.3 in Chapter 2, for every measurable subset $A \in \Sigma$, we have

$$\upsilon(A) = \sup \Big\{ \frac{\omega(A \cap E)}{\sigma(E)} \colon E \in \mathcal{E}, \sigma(E) \notin \{0, \infty\} \Big\}. \tag{4.3.5}$$

We observe that, starting with a σ-finite setting (X, μ, ω), the construction described in (4.3.2) defines another σ-finite setting (X, υ, ω).

Lemma 4.3.2. *For every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the setting (X, υ, ω) is σ-finite as well.*

Proof. We split the proof into two parts, the absolute continuity of the measure ω with respect to the outer measure υ, and the σ-finiteness of X with respect to the outer measure υ.

Part I: absolute continuity. We claim that, for every measurable subset $A \in \Sigma$, we have

$$\upsilon(A) = 0 \Rightarrow \omega(A) = 0.$$

Let $A \subseteq X$ be a measurable subset such that $\upsilon(A) = 0$.

We distinguish three cases.

Case I: $\mu(A) = 0$. By the absolute continuity of the measure ω with respect to the outer measure μ assumed for the σ-finite setting (X, μ, ω), we have $\omega(A) = 0$.

Case II: $\mu(A) \notin \{0, \infty\}$. By the equality in (4.3.4) for υ, we have $\omega(A) = 0$.

Case III: $\mu(A) = \infty$. By the σ-finiteness of X with respect to the outer measure μ, there exists a collection $\{A_n \colon n \in \mathbb{N}\}$ of disjoint measurable subsets of A such that

$$\begin{aligned} &\mu(A_n) < \infty, && \text{for every } n \in \mathbb{N}, \\ &A = \bigcup_{n \in \mathbb{N}} A_n, \end{aligned}$$

hence, by **Case I** and **Case II** for every $n \in \mathbb{N}$ and the subadditivity of the measure ω, we have $\omega(A) = 0$.

Part II: σ-finiteness. We claim that there exists a collection $\{A_n \colon n \in \mathbb{N}\} \subseteq \Sigma$ of measurable subsets of X such that

$$\begin{aligned} &\upsilon(A_n) < \infty, && \text{for every } n \in \mathbb{N}, \\ &X = \bigcup_{n \in \mathbb{N}} A_n. \end{aligned}$$

By the σ-finiteness of X with respect to the outer measure μ and the measure ω assumed for the σ-finite setting (X, μ, ω), the set X is covered by a countable collection of subsets with finite outer measure μ and measure ω. Therefore, without loss of generality, we assume that

$$\mu(X) \in [0, \infty), \qquad \omega(X) \in [0, \infty).$$

By collapsing of exponents, property (i) in Theorem 4.2.1, for the σ-finite setting (X, μ, ω), we have

$$C^{-1} \|1_X\|_{L^1_\mu(\ell^1_\omega)} \leqslant \|1_X\|_{L^1(X,\omega)} = \omega(X) \leqslant C \|1_X\|_{L^1_\mu(\ell^1_\omega)}. \tag{4.3.6}$$

We define the collection $\{B_j : j \in \{0\} \cup \mathbb{N}_X\} \subseteq \Sigma$ of measurable subsets of X such that

$$\begin{aligned} B_0 &= X, & & \\ B_j &\subseteq B_{j-1}, & & \text{for every } j \in \mathbb{N}_X, \\ \mu(B_j) &\leqslant C 2^{1-j} \omega(X), & & \text{for every } j \in \mathbb{N}_X, \end{aligned}$$

by a forward recursion on $j \in \{0\} \cup \mathbb{N}_X$, where $\mathbb{N}_X$ is either $\mathbb{N}$ or a finite initial string of it, possibly empty. In particular, we set

$$\begin{aligned} X_1 &= B_0 = X, & & \\ X_j &= B_{j-1}, & & \text{for every } j \in \mathbb{N}. \end{aligned}$$

Fix $j \in \mathbb{N}$ and assume we have selected B_l for every $l \in \{0\} \cup \mathbb{N}$, $l < j$. In particular, B_{j-1} is already well-defined. By the definition of the super level measure in Definition 1.1.5, for every $j \in \mathbb{N}$, there exists a measurable subset $B_j \subseteq X_j$ such that

$$\|1_{B_j^c}\|_{L^\infty_\mu(\ell^1_\omega)} \leqslant 2^j, \qquad \mu(B_j) \leqslant 2\mu(\ell^1_\omega(1_{X_j}) > 2^j).$$

Together with the monotonicity of the size ℓ^1_ω, Chebyshev's inequality for the outer L^1 quasi-norm, and the second inequality in (4.3.6), the previous display yields

$$\mu(B_j) \leqslant 2\mu(\ell^1_\omega(1_{X_j}) > 2^j) \leqslant 2\mu(\ell^1_\omega(1_X) > 2^j) \leqslant 2^{1-j} \|1_X\|_{L^1_\mu(\ell^1_\omega)} \leqslant C 2^{1-j} \omega(X).$$

and proceed the recursion with $j + 1$. In particular, if there exists $j_0 \in \mathbb{N}$ such that

$$\mu(X_{j_0}) = 0, \qquad \text{or} \qquad \omega(X_{j_0}) = 0,$$

then, by the equality in (4.3.4) for υ, we have

$$\upsilon(X_{j_0}) = 0,$$

and we stop the recursion.

Next, we define the measurable subset $B \subseteq X$ by

$$B = \bigcap_{j \in \mathbb{N}} B_j.$$

We have

$$\mu(B) = \lim_{j \to \infty} \mu(B_j) = 0,$$

hence, by the equality in (4.3.4) for υ, we have

$$\upsilon(B) = 0.$$

Finally, for every $j \in \mathbb{N}$, let $\{E_{j,k,n} \colon j \in \mathbb{N}, k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \Sigma$ be the collection of measurable subsets of X produced by the decomposition of $1_{B_{j-1} \setminus B_j} \in L^1_\mu(\ell^1_\omega) \cap L^\infty_\mu(\ell^1_\omega)$ with respect to the size ℓ^1_ω at levels $\{2^k \colon k \in \mathbb{Z}\}$ provided by Proposition 4.2.2. In particular, we have

$$\begin{aligned} &\upsilon(E_{j,k,n}) = \ell^1_\mu(1_X)(E_{j,k,n}) \in (2^k, 2^{k+1}], \\ &\upsilon\Big((B_{j-1} \setminus B_j) \setminus \bigcup_{k \in \mathbb{Z}} \bigcup_{n \in \mathbb{N}_k} E_{j,k,n}\Big) = 0. \end{aligned}$$

The countable collection of measurable subsets of X defined by

$$\{B\} \cup \Big\{E_{j,k,n} \colon j \in \mathbb{N}, k \in \mathbb{Z}, n \in \mathbb{N}_k\Big\} \cup \Big\{(B_{j-1} \setminus B_j) \setminus \bigcup_{k \in \mathbb{Z}} \bigcup_{n \in \mathbb{N}_k} E_{j,k,n} \colon j \in \mathbb{N}\Big\},$$

prove that X is σ-finite with respect to the outer measure υ. □

4.3.2 Iterations of the canonical construction

Iterating the construction described in (4.3.2) starting with the setting (X, υ, ω), we generate another outer measure $\widetilde{\mu} = \widetilde{\mu}(\upsilon, \omega) = \widetilde{\mu}(\mu, \omega)$ on X.

Lemma 4.3.3. *For every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the outer measure μ dominates the outer measure $\widetilde{\mu}$ as in Definition 4.2.3.*

Proof. It is enough to prove that, for every measurable subset $A \in \Sigma$, we have

$$\widetilde{\mu}(A) \leqslant \mu(A).$$

Then, by the equality in (4.3.3) for $\widetilde{\mu}$, for every measurable subset $A \in \Sigma$, we have

$$\sigma_{\widetilde{\mu}}(A) \leqslant \mu(A),$$

and the outer measure $\widetilde{\mu}$ generated via minimal coverings as in (1.1.1) by the pre-measure $\sigma_{\widetilde{\mu}}$ on the collection Σ inherits the domination.

We distinguish three cases.

Case I: $\mu(A) = 0$**.** By the equality in (4.3.4) for υ, we have $\upsilon(A) = 0$. By the equality in (4.3.4) for $\widetilde{\mu}$, we have $\widetilde{\mu}(A) = 0$.

Case II: $\mu(A) = \infty$**.** The desired inequality is trivially satisfied.

Case III: $\mu(A) \notin \{0, \infty\}$**.** By the equality in (4.3.4) for $\widetilde{\mu}$ and υ, we have

$$\begin{aligned}\widetilde{\mu}(A) &= \sup\left\{\frac{\omega(B)}{\upsilon(B)} : B \in \Sigma_\upsilon(A)\right\}\\ &= \sup\left\{\inf\left\{\frac{\omega(B)}{\omega(C)}\mu(C) : C \in \Sigma_\mu(B)\right\} : B \in \Sigma_\upsilon(A)\right\}.\end{aligned}$$

If the collection $\Sigma_\upsilon(A)$ is empty, then we have $\widetilde{\mu}(A) = 0$, and the desired inequality is trivially satisfied. Otherwise, there exists a measurable subset $B \in \Sigma_\upsilon(A)$, and we claim that

$$\mu(B) \notin \{0, \infty\}.$$

In fact, if $\mu(B) = 0$, then $\upsilon(B) = 0$, yielding a contradiction. If $\mu(B) = \infty$, then $\mu(A) = \infty$, yielding a contradiction. Therefore, we have $B \in \Sigma_\mu(B)$, hence

$$\widetilde{\mu}(A) \leqslant \sup\left\{\frac{\omega(B)}{\omega(B)}\mu(B) : B \in \Sigma_\upsilon(A)\right\} \leqslant \mu(A).$$

□

Further iterating the construction described in (4.3.2) starting with the setting $(X, \widetilde{\mu}, \omega)$, we generate another outer measure $\widetilde{\upsilon} = \widetilde{\upsilon}(\widetilde{\mu}, \omega) = \widetilde{\upsilon}(\mu, \omega)$ on X.

Lemma 4.3.4. *For every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, let υ, $\widetilde{\mu}$, and $\widetilde{\upsilon}$ be the outer measures defined via the recursive application of the construction described in* (4.3.2)*. Then, we have $\upsilon = \widetilde{\upsilon}$.*

Proof. The domination of $\widetilde{\upsilon}$ by υ follows from Lemma 4.3.3 for the σ-finite setting (X, υ, ω). We prove the domination of υ by $\widetilde{\upsilon}$. It is enough to prove that, for every measurable subset $A \in \Sigma$, we have

$$\upsilon(A) \leqslant \widetilde{\upsilon}(A).$$

Then, by the equality in (4.3.3) for $\widetilde{\mu}$, for every measurable subset $A \in \Sigma$, we have

$$\sigma_\upsilon(A) \leqslant \sigma_{\widetilde{\upsilon}}(A),$$

and the outer measures υ and $\widetilde{\upsilon}$ generated via minimal coverings as in (1.1.1) by the pre-measures σ_υ and $\sigma_{\widetilde{\upsilon}}$ on the collection Σ respectively inherit the domination.

By the equality in (4.3.4) for $\widetilde{\upsilon}$ and υ, for every measurable subset $A \in \Sigma$, we have

$$\widetilde{\upsilon}(A) = \sup\Big\{\frac{\omega(D)}{\widetilde{\mu}(D)} \colon D \in \Sigma_{\widetilde{\mu}}(A)\Big\},$$
$$\upsilon(A) = \sup\Big\{\frac{\omega(B)}{\mu(B)} \colon B \in \Sigma_{\mu}(A)\Big\} = \sup\Big\{\frac{\omega(B)}{\mu(B)} \colon B \in \Sigma_{\mu}(A), \omega(B) \neq 0\Big\}.$$

We claim that, for every measurable subset $B \in \Sigma$, if $B \subseteq A$, $\mu(B) \notin \{0, \infty\}$, and $\omega(B) \neq 0$, then

$$\widetilde{\mu}(B) \notin \{0, \infty\}.$$

In fact, by Lemma 4.3.3, we have $\widetilde{\mu}(B) < \infty$. Next, since $\omega(B) \neq 0$, if $\widetilde{\mu}(B) = 0$, then $\upsilon(B) = 0$, yielding a contradiction with the absolute continuity of the measure ω with respect to the outer measure μ proved in Lemma 4.3.2. Therefore, we have $B \in \Sigma_{\widetilde{\mu}}(A)$, hence, by the domination of $\widetilde{\mu}$ by μ, we have

$$\upsilon(A) \leqslant \sup\Big\{\frac{\omega(B)}{\mu(B)} \colon B \in \Sigma_{\widetilde{\mu}}(A)\Big\} \leqslant \sup\Big\{\frac{\omega(B)}{\widetilde{\mu}(B)} \colon B \in \Sigma_{\widetilde{\mu}}(A)\Big\} = \widetilde{\upsilon}(A).$$

□

Corollary 4.3.5. *For every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the recursive application of the construction described in* (4.3.2) *produces at most two outer measures on X different from μ, namely υ and $\widetilde{\mu}$.*

No further a priori relation between the outer measures μ and $\widetilde{\mu}$ can be established, as the examples studied in the following two subsections show.

4.3.3 First example: no uniform equivalence

Neither the equality nor the uniform equivalence between μ and $\widetilde{\mu}$ are guaranteed.

Lemma 4.3.6. *For every $M > 0$, there exist a finite setting (X, μ, ω) and a subset $A \subseteq X$ such that*

$$\mu(A) \geqslant M\widetilde{\mu}(A).$$

Proof. For every $m \in \mathbb{N}$, let (X_m, μ_m, ω_m) be the setting on the m-dimensional hypercube of sidelength m described in Subsection 1.2.6. We refer to that subsection for the definition of the subset $E(x)$. Let υ_m and $\widetilde{\mu}_m$ be the outer measures defined via the recursive application of the construction described in (4.3.2). Then, by the equality in (4.3.5) for υ_m, for every subset $A \subseteq X_m$, we have

$$\upsilon_m(A) = \sup\Big\{\omega_m(A \cap E(x)) \colon x \in X_m\Big\}. \tag{4.3.7}$$

Next, from the proof of Lemma 4.2.5, we recall that

$$\mu_m(X_m) > m.$$

Moreover, we observe that every $y \in X_m$ belongs to $(m-1)^m$ many subsets of the form $E(x)$ with $x \in X_m$. Therefore, for every subset $B \subseteq X_m$, $B \neq \varnothing$, we have

$$(m-1)^m \omega_m(B) = \sum_{x \in X_m} \omega_m(B \cap E(x)) \leqslant m^m \sup \left\{ \omega_m(B \cap E(x)) \colon x \in X_m \right\}.$$

Together with the equality in (4.3.7), the previous display yields

$$\widetilde{\mu}_m(X_m) = \sup \left\{ \frac{\omega_m(B)}{\upsilon_m(B)} \colon B \subseteq X_m, B \neq \varnothing \right\} \leqslant \frac{m^m}{(m-1)^m}.$$

Taking $m \in \mathbb{N}$ big enough, for $A = X_m$, we obtain the desired inequality. □

4.3.4 Second example: equality

The equality between μ and $\widetilde{\mu}$ may be achieved in certain setting. For every $m \in \{0\} \cup \mathbb{N}$, let (X_m, μ_m, ω_m) be the setting on the dyadic tree of depth m described in Subsection 1.2.7. We refer to that subsection for the definitions of the subset $E(I)$ and the collection $\mathcal{E}_m$. Let υ_m and $\widetilde{\mu}_m$ be the outer measures defined via the recursive application of the construction described in (4.3.2). Then by the equality in (4.3.5) for υ_m, for every subset $A \subseteq X_m$, we have

$$\upsilon_m(A) = \sup \left\{ \omega_m(A \cap E(I)) \colon I \in X_m, |I| = 2^{-m} \right\}.$$

We claim the equality between μ_m and $\widetilde{\mu}_m$ in this setting.

Lemma 4.3.7. *For every $m \in \{0\} \cup \mathbb{N}$, for every subset $A \subseteq X_m$, we have*

$$\mu_m(A) = \widetilde{\mu}_m(A). \tag{4.3.8}$$

Before proving the lemma, we describe an auxiliary construction, associating every subset $A \subseteq X_m$, $A \neq \varnothing$ to a subset $B = B(A) \subseteq A$ of its own. First, we define a collection $\mathcal{A} \subseteq \mathcal{E}_m$ such that

$$A \subseteq \bigcup_{E \in \mathcal{A}} E,$$

as follows. We set

$$\begin{aligned} A_0 &= A \cap X_0, \\ A_k &= A \cap (X_k \backslash X_{k-1}), \qquad && \text{for every } k \in \{1, \ldots, m\}. \end{aligned}$$

We define the elements of $\mathcal{A}$ by backward recursion on k. For $k = m$, we define the collection $\mathcal{A}_m \subseteq \mathcal{E}_m$ by

$$\mathcal{A}_m = \Big\{E(a) \colon a \in A_m\Big\},$$

allowing $\mathcal{A}_m$ to be empty. Next, fix $k < m$ and suppose we have defined $\mathcal{A}_j \subseteq \mathcal{E}_m$ for every $j \in \{k+1, \dots, m\}$. We define the collection $\widetilde{\mathcal{A}}_k \subseteq \mathcal{E}_k$ by

$$\widetilde{\mathcal{A}}_k = \Big\{\widetilde{E}(a) \colon a \in A_k \setminus \bigcup_{j=k+1}^{m} \bigcup_{E \in \mathcal{A}_j} E\Big\}.$$

If $\widetilde{\mathcal{A}}_k$ is empty, we define $\mathcal{A}_k$ to be empty as well. Otherwise, for every $\widetilde{E}(a) \in \widetilde{\mathcal{A}}_k$, we define $E(a) = E(\widetilde{E}(a)) \in \mathcal{E}_m$ to be any arbitrary element of $\mathcal{E}_m$ such that $E(a) \cap X_k = \widetilde{E}(a)$, and we define the collection $\mathcal{A}_k \subseteq \mathcal{E}_m$ by

$$\mathcal{A}_k = \Big\{E(a) = E(\widetilde{E}(a)) \colon \widetilde{E}(a) \in \widetilde{\mathcal{A}}_k\Big\}.$$

After that, we define the collection $\mathcal{A} \subseteq \mathcal{E}_m$ by

$$\mathcal{A} = \bigcup_{k=0}^{m} \mathcal{A}_k.$$

Finally, we define the subset $B = B(A) \subseteq A$ by

$$B = \bigcup_{k=0}^{m} \Big(A_k \setminus \bigcup_{j=k+1}^{m} \bigcup_{E \in \mathcal{A}_j} E\Big).$$

Lemma 4.3.8. *For every $m \in \{0\} \cup \mathbb{N}$, for every subset $A \subseteq X_m$, $A \neq \varnothing$, let the subset $B \subseteq A$ be defined by the previous construction. Then, we have*

$$\omega_m(B) = \mu_m(A), \qquad \upsilon_m(B) = 1.$$

Proof. By construction, we have

$$\omega_m(B) = |\mathcal{A}| \geqslant \mu_m(A), \qquad \upsilon_m(B) = 1,$$

where $|\mathcal{A}|$ is the cardinality of $\mathcal{A}$. To prove that $\mu_m(A) = |\mathcal{A}|$, we distinguish two cases.

Case I: $\omega_m(B) = 1$**.** The desired equality is trivially satisfied.

Case II $\omega_m(B) > 1$**.** We argue by contradiction and we suppose that $\mu_m(A) < |\mathcal{A}|$, hence $\mu_m(B) < |\mathcal{A}|$. In particular, there exists a collection $\mathcal{B} \subseteq \mathcal{E}_m$ such that $|\mathcal{B}| < |\mathcal{A}|$ and

$$B \subseteq \bigcup_{E \in \mathcal{B}} E.$$

Therefore, there exists $E \in \mathcal{B}$ such that $|E \cap B| \geqslant 2$. As a consequence, there exist $a, b \in E \cap B$, $a \neq b$, such that

$$\mu_m(\{a, b\}) = 1.$$

To prove that this equality yields a contradiction, we distinguish two cases.

Case II.i. There exists $k \in \{1, \ldots, m\}$ such that

$$a, b \in A_k \backslash \bigcup_{j=k+1}^{m} \bigcup_{E \in \mathcal{A}_j} E.$$

The contradiction follows noting that, for every $E \in \mathcal{E}_m$, the set $E \cap (X_k \backslash X_{k-1})$ has only one element.

Case II.ii. There exist $k, k' \in \{1, \ldots, m\}$, $k \neq k'$, such that

$$a \in A_k \backslash \bigcup_{j=k+1}^{m} \bigcup_{E \in \mathcal{A}_j} E, \qquad b \in A_{k'} \backslash \bigcup_{j=k'+1}^{m} \bigcup_{E \in \mathcal{A}_j} E,$$

and, without loss of generality, we assume $k > k'$. We argue by contradiction and we suppose that $\mu_m(\{a, b\}) = 1$. Therefore, there exists $E \in \mathcal{E}_m$ such that $a, b \in E$. In particular, we have $E \cap X_k = \widetilde{E}(a)$, hence $b \in E(\widetilde{E}(a)) \in \mathcal{A}_k$, thus $b \notin B$, yielding a contradiction. □

We are ready to prove the equality between μ_m and $\widetilde{\mu}_m$ in the case of the setting (X_m, μ_m, ω_m) with $m \in \{0\} \cup \mathbb{N}$.

Proof of Lemma 4.3.7. By Lemma 4.3.3, the outer measure μ_m dominates the outer measure $\widetilde{\mu}_m$. By Lemma 4.3.8, for every $m \in \{0\} \cup \mathbb{N}$, for every subset $A \subseteq X_m$, $A \neq \varnothing$, for the subset $B \subseteq A$ defined by the previous construction, we have

$$\widetilde{\mu}_m(A) \geqslant \frac{\omega_m(B)}{\upsilon_m(B)} = \mu_m(A).$$

□

4.3.5 Outer Minkowski's inequality for σ-finite settings

We turn to the proof of Minkowski's inequality between the single iterated outer L^p quasi-norms on σ-finite settings and the embeddings between the respective single iterated outer L^p spaces.

Theorem 4.3.9. *For all $p, r \in (0, \infty]$, $p \geqslant r$, there exists a constant $C = C(p, r)$ such that, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, for the outer measure υ defined via the construction described in* (4.3.2)*, for every measurable function f on X, we have*

$$\|f\|_{L^p_\mu(\ell^r_\omega)} \leqslant C \|f\|_{L^r_\upsilon(\ell^p_\omega)}, \qquad \|f\|_{L^p_\upsilon(\ell^r_\omega)} \leqslant C \|f\|_{L^r_\mu(\ell^p_\omega)}.$$

Proof. We split the proof into three cases according to the values of p and r.

Case I: $p = r \in (0, \infty]$. The desired inequalities follow from collapsing of exponents, property (i) in Theorem 4.2.1, for the σ-finite settings (X, μ, ω) and (X, ν, ω).

Case II: $r < p = \infty$. We define the collections $\Sigma_\mu(X), \Sigma_\upsilon(X) \subseteq \Sigma$ of measurable subsets of X by

$$\Sigma_\mu(X) = \Big\{A \in \Sigma \colon \mu(A) \notin \{0, \infty\}\Big\},$$
$$\Sigma_\upsilon(X) = \Big\{A \in \Sigma \colon \upsilon(A) \notin \{0, \infty\}\Big\}.$$

Moreover, for every measurable subset $A \in \Sigma$, we define the collection $\Sigma'_\omega(A) \subseteq \Sigma$ of measurable subsets of A by

$$\Sigma'_\omega(A) = \Big\{B \in \Sigma \colon B \subseteq A, \omega(A \backslash B) = 0\Big\}.$$

Next, for every measurable subset $A \in \Sigma$, for every $\lambda \in (0, \infty)$, we define the measurable subset $A_\lambda \subseteq A$ by

$$A_\lambda = \Big\{x \in A \colon |f(x)| > \lambda\Big\}.$$

We claim that, for every measurable subset $A \in \Sigma_\mu(X)$, for every $\lambda \in (0, \infty)$, for every subset $B_\lambda \in \Sigma'_\omega(A_\lambda)$, we have

$$\frac{\omega(A_\lambda)}{\mu(A)} \leqslant \upsilon(B_\lambda). \tag{4.3.9}$$

To prove the claim, we distinguish two cases.

If $\mu(B_\lambda) = 0$, then $\omega(B_\lambda) = 0$ by the absolute continuity of the measure ω with respect to the outer measure μ assumed for the σ-finite setting (X, μ, ω), and $\omega(A_\lambda) = 0$ by the assumption $B_\lambda \in \Sigma'_\omega(A_\lambda)$. Then, the inequality in (4.3.9) is trivially satisfied.

If $\mu(B_\lambda) \neq 0$, we have

$$\frac{\omega(A_\lambda)}{\mu(A)} \leqslant \frac{\omega(B_\lambda)}{\mu(B_\lambda)} \leqslant \upsilon(B_\lambda),$$

where we used the assumption $B_\lambda \in \Sigma'_\omega(A_\lambda)$ and the monotonicity of the outer measure μ in the first inequality, the assumptions $B_\lambda \subseteq A_\lambda \subseteq A$, $\mu(A) < \infty$, $\mu(B_\lambda) \neq 0$, and the equality in (4.3.4) for υ in the second.

By the same argument and Lemma 4.3.3, for every measurable subset $A \in \Sigma_\upsilon(X)$, for every $\lambda \in (0, \infty)$, for every subset $B_\lambda \in \Sigma'_\omega(A_\lambda)$, we have

$$\frac{\omega(A_\lambda)}{\upsilon(A)} \leqslant \widetilde{\mu}(B_\lambda) \leqslant \mu(B_\lambda). \tag{4.3.10}$$

By the inequality in (4.3.9), we have

$$\begin{aligned}
\|f\|^r_{L^\infty_\mu(\ell^r_\omega)} &= \sup\Big\{\mu(A)^{-1}\|f1_A\|^r_{L^r(X,\omega)} \colon A \in \Sigma_\mu(X)\Big\} \\
&= \sup\Big\{\mu(A)^{-1}\int_0^\infty r\lambda^r \omega(A_\lambda)\frac{\mathrm{d}\lambda}{\lambda} \colon A \in \Sigma_\mu(X)\Big\} \\
&\leqslant \sup\Big\{\int_0^\infty r\lambda^r \inf\Big\{\upsilon(B_\lambda) \colon B_\lambda \in \Sigma'_\omega(A_\lambda)\Big\}\frac{\mathrm{d}\lambda}{\lambda} \colon A \in \Sigma_\mu(X)\Big\} \\
&= \sup\Big\{\|f1_A\|^r_{L^r_\upsilon(\ell^\infty_\omega)} \colon A \in \Sigma_\mu(X)\Big\} \\
&= \|f\|^r_{L^r_\upsilon(\ell^\infty_\omega)}.
\end{aligned}$$

By the same chain of inequalities, exchanging the roles of μ and υ, replacing the inequality in (4.3.9) by that in (4.3.10), we have

$$\|f\|^r_{L^\infty_\upsilon(\ell^r_\omega)} \leqslant \|f\|^r_{L^r_\mu(\ell^\infty_\omega)}.$$

Case III: $r < p < \infty$**.** Without loss of generality, we assume $r = 1$, since, for every setting (X, μ, ω), we have

$$\|f\|^r_{L^p_\mu(\ell^r_\omega)} = \|f^r\|_{L^{\frac{p}{r}}_\mu(\ell^1_\omega)}.$$

In particular, we have $p > 1$, hence

$$\begin{aligned}
\|f\|_{L^p_\mu(\ell^1_\omega)} &\leqslant C\sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^\infty_\omega)} = 1\Big\} \\
&\leqslant C\sup\Big\{\|fg\|_{L^1_\upsilon(\ell^1_\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^\infty_\omega)} = 1\Big\} \\
&\leqslant C\sup\Big\{\|f\|_{L^1_\upsilon(\ell^p_\omega)}\|g\|_{L^\infty_\upsilon(\ell^{p'}_\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^\infty_\omega)} = 1\Big\} \\
&\leqslant C\|f\|_{L^1_\upsilon(\ell^p_\omega)},
\end{aligned}$$

where we used Köthe duality, property (ii) in Theorem 4.2.1, for the σ-finite setting (X, μ, ω) in the first inequality, the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) for the σ-finite setting (X, υ, ω) in the second, outer Hölder's inequality (Theorem 1.1.7) for the σ-finite setting (X, υ, ω) in the third, and the inequality proved in **Case II** in the fourth. We prove the remaining inequality analogously, exchanging the roles of μ and υ. □

Corollary 4.3.10. *For all $p, r \in (0, \infty]$, $p \geqslant r$, there exists a constant $C = C(p, r)$ such that, for every σ-finite setting (X, μ, ω), for the outer measures υ and $\widetilde{\mu}$ defined via the recursive application of the construction described in* (4.3.2)*, for every measurable function f on X, we have*

$$\begin{aligned}
\|f\|_{L^p_\mu(\ell^r_\omega)} &\leqslant C\|f\|_{L^p_{\widetilde{\mu}}(\ell^r_\omega)} \leqslant C^2\|f\|_{L^r_\upsilon(\ell^p_\omega)}, \\
\|f\|_{L^p_\upsilon(\ell^r_\omega)} &\leqslant C\|f\|_{L^r_{\widetilde{\mu}}(\ell^p_\omega)} \leqslant C^2\|f\|_{L^r_\mu(\ell^p_\omega)}.
\end{aligned}$$

Proof. The desired inequalities follow from Lemma 4.2.4, Lemma 4.3.3, Lemma 4.3.4, and Theorem 4.3.9. □

4.3.6 Examples

We comment on the outer measures υ and $\widetilde{\mu}$ defined via the recursive application of the construction described in (4.3.2) in the remaining settings described in Chapter 1.

- Let (X, μ, ω) be a σ-finite setting described in Subsection 1.2.1 such that μ is the outer measure generated via minimal coverings as in (1.1.1) by ω considered as a pre-measure on the collection Σ of measurable subsets. In particular, μ coincide with ω on the measurable subsets. Then υ is the constant outer measure attaining the value 1 on every non-empty subset of X, and $\widetilde{\mu} = \mu$. Moreover, if (X, μ, ω) is a finite setting described in Subsection 1.2.2, then $\widetilde{\mu} = \mu = \omega$.

- Let (X, μ, ω) be the setting on the Cartesian product of σ-finite measure spaces described in Subsection 1.2.4. Then, for every measurable subset $A \subseteq X$, we have

$$\upsilon(A) \geqslant \sup \Big\{ \omega_Z(Z') \colon Z' \in \Sigma_Z, \exists Y' \in \Sigma_Y, \omega_Y(Y') \neq 0, Y' \times Z' \subseteq A \Big\},$$
$$\widetilde{\mu}(A) \geqslant \sup \Big\{ \omega_Y(Y') \colon Y' \in \Sigma_Y, \exists Z' \in \Sigma_Z, \omega_Z(Z') \neq 0, Y' \times Z' \subseteq A \Big\},$$

  where Σ_Y is the collection of measurable subsets of Y, and Σ_Z of Z. In general the equality between μ and $\widetilde{\mu}$ is not guaranteed. For example, for all measurable subsets $Y' \in \Sigma_Y$, $Z' \in \Sigma_Z$ such that

$$\omega_Y(Y') \neq 0, \qquad \omega_Z(Z') = 0, \qquad Z' \neq \varnothing,$$

  we have

$$\mu(Y' \times Z') = \omega_Y(Y'), \qquad \widetilde{\mu}(Y' \times Z') = \omega(Y' \times Z') = 0.$$

  However, for all measurable subsets $Y' \in \Sigma_Y$, $Z' \in \Sigma_Z$, we have

$$\upsilon(Y \times Z') = \omega_Z(Z'),$$
$$\widetilde{\mu}(Y' \times Z) = \omega_Y(Y') = \mu(Y' \times Z).$$

  In particular, if the measure spaces are finite sets, for every subset $A \subseteq X$, we have

$$\upsilon(A) = \sup \Big\{ \omega_Z(\pi_Z(A \cap \{y\} \times Z)) \colon y \in Y \Big\},$$
$$\widetilde{\mu}(A) \geqslant \sup \Big\{ \omega_Y(\{y\}) \colon y \in \pi_Y(A) \Big\}.$$

- Let (X, μ, ω) be the setting described in Subsection 1.2.5, where the outer measure attains the constant value 1 on every non-empty subset of X. Then υ is the outer measure generated via minimal coverings as in (1.1.1) by ω considered as a pre-measure on the collection Σ of measurable subsets, and $\widetilde{\mu} = \mu$.

- Let (X_d, μ_d, ω_d) be the setting on the collection of dyadic cubes described in Subsection 1.2.8. Then, for every subset $A \subseteq X_d$, we have

$$\upsilon_d(A) = \sup \Big\{ \frac{\omega_d(A \cap E(\vec{m}, l))}{\mu_d(E(\vec{m}, l))} : \vec{m} \in \mathbb{Z}^d, l \in \mathbb{Z} \Big\},$$

namely $\upsilon_d(A)$ is the minimal Carleson constant associated with the collection $\mathcal{Q}_d(A) \subseteq \mathcal{Q}_d$ of dyadic cubes given by the bases of the elements in A, and

$$\widetilde{\mu}_d(A) = \mu_d(A). \tag{4.3.11}$$

In fact, let $B \subseteq A$ be the collection of upper half dyadic boxes associated with the maximal dyadic cubes in $\mathcal{Q}_d(A)$, maximal in terms of set inclusion. Then $\omega_d(B) = \mu_d(A)$ and $\upsilon_d(B) = 1$.

- Let (X_d, μ_d, ω_d) be the setting on the upper half space described in Subsection 1.2.9, where the outer measure μ_d is generated via minimal coverings by the collection of dyadic tents. Then, for every measurable subset $A \subseteq X_d$, we have

$$\upsilon_d(A) = \sup \Big\{ \frac{\omega_d(A \cap E(\vec{m}, l))}{\mu_d(E(\vec{m}, l))} : \vec{m} \in \mathbb{Z}^d, l \in \mathbb{Z} \Big\},$$

namely $\upsilon_d(A)$ is the dyadic Carleson constant associated with the measure $1_A \omega$. Next, we claim that, for every measurable subset $A \subseteq X_d$, we have

$$\widetilde{\mu}_d(A) = \widetilde{\mu}_d(\widetilde{A}) = \mu_d(\widetilde{A}), \tag{4.3.12}$$

where the measurable subset $\widetilde{A} = \widetilde{A}(A) \subseteq A$ is defined by the following auxiliary construction. We refer to Subsection 1.2.8 for the definitions of the dyadic cube $Q(\vec{m}, l)$, the collection of dyadic cubes $\mathcal{Q}_d$, and the upper half dyadic cubic box $B(\vec{m}, l)$.

For all $\vec{m} \in \mathbb{Z}^d$, $l \in \mathbb{Z}$, we define the measurable subset $A(\vec{m}, l) \subseteq A$ by

$$A(\vec{m}, l) = A \cap B(\vec{m}, l).$$

Moreover, for all $\vec{m} \in \mathbb{Z}^d$, $l \in \mathbb{Z}$, for every measurable subset $A \subseteq X_d$, for every collection $\mathcal{Q} \subseteq \mathcal{Q}_d$, we define the subset $B(\vec{m}, l, A, \mathcal{Q}) \subseteq A(\vec{m}, l)$ by

$$B(\vec{m}, l, A, \mathcal{Q}) = \begin{cases} \varnothing, & \text{if } Q(\vec{m}, l) \notin \mathcal{Q}, \\ A(\vec{m}, l), & \text{if } Q(\vec{m}, l) \in \mathcal{Q}, \end{cases}$$

and we define the measurable subset $B(A,\mathcal{Q}) \subseteq A$ by

$$B(A,\mathcal{Q}) = \bigcup_{\vec{m}\in\mathbb{Z}^d}\bigcup_{l\in\mathbb{Z}} B(\vec{m},l,A,\mathcal{Q}).$$

Finally, for every measurable subset $A \subseteq X_d$, we define the collection $\mathcal{Q}(A) \subseteq \mathcal{Q}_d$ by

$$\mathcal{Q}(A) = \Big\{Q(\vec{m},l) \in \mathcal{Q}_d \colon \omega_d(A(\vec{m},l)) > 0\Big\}, \tag{4.3.13}$$

and we define the measurable subset $\widetilde{A} \subseteq A$ by

$$\widetilde{A} = B(A,\mathcal{Q}(A)). \tag{4.3.14}$$

In particular, by construction we have

$$\omega(A \backslash \widetilde{A}) = 0. \tag{4.3.15}$$

We turn to the proof of the inequalities in (4.3.12).

The first equality in (4.3.12) claims

$$\begin{aligned}\widetilde{\mu}_d(A) \geqslant \widetilde{\mu}_d(\widetilde{A}) = \sup\Big\{\frac{\omega(\widetilde{B})}{\upsilon_d(\widetilde{B})} \colon \widetilde{B}\in\Sigma, \widetilde{B}\subseteq\widetilde{A}, \upsilon(\widetilde{B}) \neq \{0,\infty\}\Big\} \geqslant \\ \geqslant \sup\Big\{\frac{\omega(B)}{\upsilon_d(B)} \colon B\in\Sigma, B\subseteq A, \upsilon(B) \neq \{0,\infty\}\Big\} = \widetilde{\mu}_d(A),\end{aligned}$$

where we used the equality in (4.3.4) for $\widetilde{\mu}_d$. The first inequality in the previous display follows from the monotonicity of the outer measure $\widetilde{\mu}_d$. The second inequality follows from the monotonicity of the outer measure υ_d and the equality in (4.3.15).

The second equality in (4.3.12) follows from the domination of $\widetilde{\mu}_d$ by μ_d, and the following observation.

For every subset $A \subseteq X_d$, we define the collection $\mathcal{Q}(A) \subseteq \mathcal{Q}_d$ as in (4.3.13), and we define $\widetilde{\mathcal{Q}}(A)$ to be the collection of maximal dyadic cubes in $\mathcal{Q}(A)$, maximal in terms of set inclusion. By Lemma 2.2.3 in Chapter 2, we have

$$\mu_d(B(A,\widetilde{\mathcal{Q}}(A))) = \sum_{\vec{m}\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}} \mu_d(B(\vec{m},l)) 1_{\widetilde{\mathcal{Q}}(A)}(Q(\vec{m},l)) = \mu_d(\widetilde{A}).$$

Next, we distinguish two cases.

Case I: $\mu_d(\widetilde{A}) < \infty$**.** For every $\varepsilon > 0$, there exists a finite collection $\widetilde{\mathcal{Q}}(A,\varepsilon) \subseteq \widetilde{\mathcal{Q}}(A)$ such that

$$\mu_d(B(A,\widetilde{\mathcal{Q}}(A,\varepsilon))) \geqslant \mu_d(B(A,\widetilde{\mathcal{Q}}(A))) - \varepsilon.$$

We define $\rho(\varepsilon) > 0$ by

$$\rho(\varepsilon) = \min\Big\{\frac{\omega_d(A(\vec{m},l))}{\mu_d(B(\vec{m},l))} \colon Q(\vec{m},l) \in \widetilde{\mathcal{Q}}(A,\varepsilon)\Big\},$$

therefore, for every dyadic cube $Q(\vec{m},l) \in \widetilde{\mathcal{Q}}(A,\varepsilon)$, there exists a subset $B(\vec{m},l,A,\varepsilon) \subseteq A(\vec{m},l)$ such that

$$\omega_d(B(\vec{m},l,A,\varepsilon)) = \rho(\varepsilon)\mu_d(B(\vec{m},l)).$$

We define the subset $B(A,\varepsilon) \subseteq \widetilde{A}$ by

$$B(A,\varepsilon) = \bigcup_{\vec{m}\in\mathbb{Z}^d}\bigcup_{l\in\mathbb{Z}} B(\vec{m},l,A,\varepsilon),$$

hence

$$\omega_d(B(A,\varepsilon)) \geqslant \rho(\varepsilon)(\mu_d(\widetilde{A}) - \varepsilon), \qquad\qquad \upsilon_d(B(A,\varepsilon)) = \rho(\varepsilon).$$

Therefore, we have

$$\widetilde{\mu}_d(\widetilde{A}) \geqslant \frac{\omega_d(B(A,\varepsilon))}{\upsilon_d(B(A,\varepsilon))} \geqslant \mu_d(\widetilde{A}) - \varepsilon.$$

Taking ε arbitrarily small, the previous display yields the desired inequality.

Case II: $\mu_d(\widetilde{A}) = \infty$. For every $M > 0$, there exists a finite collection $\widetilde{\mathcal{Q}}(A,M) \subseteq \widetilde{\mathcal{Q}}(A)$ such that

$$\mu_d(B(A,\widetilde{\mathcal{Q}}(A,M))) \geqslant M.$$

By the same argument used in **Case I**, we have

$$\widetilde{\mu}_d(\widetilde{A}) \geqslant M.$$

Taking $M > 0$ arbitrarily big, the previous display yields the desired inequality.

- Let (X_d, μ_d, ω_d) be the setting on the upper half space described in Subsection 1.2.10, where the outer measure μ_d is generated via minimal coverings by the collection of continuous tents. Then, for every measurable subset $A \subseteq X_d$, we have

$$\upsilon_d(A) = \sup\Big\{\frac{\omega_d(A\cap E(x,s))}{\mu_d(E(x,s))} \colon x \in \mathbb{R}^d, s \in (0,\infty)\Big\},$$

namely $\upsilon_d(A)$ is the minimal Carleson constant associated with the measure $1_A\omega$. Next, we claim that, for every measurable subset $A \subseteq X_d$, we have

$$\widetilde{\mu}_d(A) = \widetilde{\mu}_d(\widetilde{A}) = \mu_d(\widetilde{A}). \tag{4.3.16}$$

where the measurable subset $\widetilde{A} = \widetilde{A}(A) \subseteq A$ is defined by the following auxiliary construction. We refer to Subsection 1.2.10 for the definitions of the continuous cubic box $E(\vec{x},s)$ and the collection of continuous cubic boxes $\mathcal{E}_d$. Next, for every $\vec{x}\in\mathbb{R}^d$, $s\in(0,\infty)$, we define the *upper half cubic box* $B(\vec{x},s)$ in the upper half space $\mathbb{R}^d\times(0,\infty)$ by

$$B(\vec{x},s)=E(\vec{x},s)\cap\Big(\mathbb{R}^d\times\Big(\frac{s}{2},s\Big]\Big),$$

and we define the collection $\mathcal{E}_{d,\mathbb{Q}}\subseteq\mathcal{E}_d$ of continuous tents with rational coordinates by

$$\mathcal{E}_{d,\mathbb{Q}}=\Big\{E(\vec{q},s)\in\mathcal{E}_d\colon \vec{q}\in\mathbb{Q}^d, s\in\mathbb{Q}\cap(0,\infty)\Big\}.$$

For all $\vec{q}\in\mathbb{Q}^d$, $s\in\mathbb{Q}\cap(0,\infty)$, we define the measurable subset $A(\vec{q},s)\subseteq A$ by

$$A(\vec{q},s)=A\cap B(\vec{q},s),$$

we define the collection $\mathcal{A}\subseteq\mathcal{P}(A)$ by

$$\mathcal{A}=\Big\{A(\vec{q},s)\subseteq A\colon \vec{q}\in\mathbb{Q}^d, s\in\mathbb{Q}\cap(0,\infty),\omega_d(A(\vec{q},s))=0\Big\},$$

and we define the measurable subset $\widetilde{A}\subseteq A$ by

$$\widetilde{A}=A\backslash\bigcap_{A'\in\mathcal{A}}A'. \tag{4.3.17}$$

In particular, by construction we have

$$\omega(A\backslash\widetilde{A})=0.$$

The equalities in (4.3.16) follow by arguments analogous to those used to prove the equalities in (4.3.12). We briefly comment on the modifications needed. We can no longer rely on the dyadic structure to define the collection $\widetilde{\mathcal{Q}}(A)$, nor to use Lemma 2.2.3 in Chapter 2 as in the previous setting. We distinguish two cases.

Case I: $\mu_d(\widetilde{A})<\infty$**.** For every $\delta>0$, there exists a collection $\mathcal{E}_d(\widetilde{A},\delta)\subseteq\mathcal{E}_{d,\mathbb{Q}}$ of continuous tents with rational coordinates such that

$$\widetilde{A}\subseteq\bigcup_{E\in\mathcal{E}_d(\widetilde{A},\delta)}E,$$

$$\mu_d(\widetilde{A})\geqslant\sum_{E\in\mathcal{E}_d(\widetilde{A},\delta)}\mu_d(E)-\delta,$$

and, for every $E(\vec{q},s)\in\mathcal{E}_d(\widetilde{A},\delta)$, we have

$$\omega_d\Big((A\cap B(\vec{q},s))\backslash\bigcup_{E\in\mathcal{E}_d(\widetilde{A},\delta),E\neq E(\vec{q},s)}E\Big)\neq 0.$$

The collection $\mathcal{E}_d(\widetilde{A}, \delta)$ replace the collection $\widetilde{\mathcal{Q}}(A)$, and, by the same argument used in **Case I** in the previous setting, we can prove

$$\widetilde{\mu}_d(\widetilde{A}) \geqslant \mu_d(\widetilde{A}) - \delta.$$

Taking δ arbitrarily small, the previous display yields the desired inequality.

Case II: $\mu_d(\widetilde{A}) = \infty$**.** For every $M > 0$, there exists a collection $\mathcal{E}_d(\widetilde{A}, M) \subseteq \mathcal{E}_{d,\mathbb{Q}}$ of continuous tents with rational coordinates such that

$$\sum_{E \in \mathcal{E}_d(\widetilde{A},\delta)} \mu_d(E) \geqslant M,$$

and, for every $E(\vec{q}, s) \in \mathcal{E}_d(\widetilde{A}, \delta)$, we have

$$\omega_d\Big((A \cap B(\vec{q}, s)) \setminus \bigcup_{E \in \mathcal{E}_d(\widetilde{A},\delta), E \neq E(\vec{q},s)} E\Big) \neq 0.$$

By the same argument used in **Case II** in the previous setting, we can prove

$$\widetilde{\mu}_d(\widetilde{A}) \geqslant M.$$

Taking $M > 0$ arbitrarily big, the previous display yields the desired inequality.

- Let (X, ν, ω) be the setting on the collection of Heisenberg upper half dyadic tiles described in Subsection 1.2.11. Then, for every subset $A \subseteq X$, we have

$$\begin{aligned} \upsilon(A) &= \sup\Big\{\frac{\omega(A \cap T(m, n, l))}{\nu(T(m, n, l))} : m, n, l \in \mathbb{Z}\Big\}, \\ \widetilde{\nu}(A) &= \nu(A). \end{aligned} \tag{4.3.18}$$

The second equality follows from combining the arguments used to prove the equalities in (4.3.8) and (4.3.11) for $d = 1$.

- Let (X, ν, ω) be the setting on the upper half 3-space described in Subsection 1.2.12, where the outer measure ν is generated via minimal coverings by the collection of dyadic trees. Then, for every measurable subset $A \subseteq X$, we have

$$\begin{aligned} \upsilon(A) &= \sup\Big\{\frac{\omega(A \cap T(m, n, l))}{\nu(T(m, n, l))} : m, n, l \in \mathbb{Z}\Big\}, \\ \widetilde{\nu}(A) &= \widetilde{\nu}(\widetilde{A}) = \nu(\widetilde{A}), \end{aligned} \tag{4.3.19}$$

where $\widetilde{A} \subseteq A$ is defined by an auxiliary construction analogous to that in (4.3.14) as in the case of the setting described in Subsection 1.2.9, with the collection of upper half dyadic cubic boxes $\mathcal{B}_d$ replaced by the collection of Heisenberg upper half dyadic tiles $\mathcal{H}$. The equalities in the second line in the previous display follow by arguments analogous to those used to prove the equalities in (4.3.8) and (4.3.12) for $d = 1$, with Lemma 2.2.3 in Chapter 2 replaced by Lemma 4.4.5 stated below.

- Let (X, ν, ω) be the setting on the upper half 3-space described in Subsection 1.2.13, where the outer measure ν is generated via minimal coverings by the collection of continuous trees. Then, for every measurable subset $A \subseteq X$, we have

$$\begin{aligned} \upsilon(A) &= \sup\Big\{\frac{\omega(A \cap T(x,\xi,s))}{\nu(T(x,\xi,s))} \colon x, \xi \in \mathbb{R}, s \in (0,\infty)\Big\}, \\ \widetilde{\nu}(A) &= \widetilde{\nu}(\widetilde{A}) = \nu(\widetilde{A}), \end{aligned} \tag{4.3.20}$$

where $\widetilde{A} \subseteq A$ is defined by an auxiliary construction analogous to that in (4.3.17) in the case of the setting described in Subsection 1.2.10, with the continuous tents with rational coordinates replaced by the continuous trees with rational coordinates. The equalities in the second line in the previous display follow by arguments analogous to those used to prove the equalities in (4.3.8) and (4.3.16) for $d = 1$.

We point out that in the case of the settings on the upper half space described in Subsections 1.2.9 – 1.2.10, the outer measures $\widetilde{\mu}_d$ and μ_d are not equivalent. For example, for every $\vec{x} \in \mathbb{R}^d$, we have

$$\mu_d(\{\vec{x}, 1\}) = 1, \qquad \widetilde{\mu}_d(\{\vec{x}, 1\}) = 0.$$

However, the single iterated outer L^p quasi-norms and spaces associated with them are equal. We have the same properties for the outer measures $\widetilde{\nu}$ and ν in the case of the settings on the upper half 3-space described in Subsections 1.2.12 – 1.2.13.

Lemma 4.3.11. *For all $p, r \in (0,\infty]$, $p \geqslant r$, for every setting (X_d, μ_d, ω_d) on the upper half space described in Subsections 1.2.9 – 1.2.10, for the outer measure $\widetilde{\mu}_d = \widetilde{\mu}_d(\mu_d, \omega_d)$ defined via the construction described in* (4.3.2)*, for every measurable function f on X_d, we have*

$$\|f\|_{L^p_{\mu_d}(\ell^r_\omega)} = \|f\|_{L^p_{\widetilde{\mu}_d}(\ell^r_\omega)}, \qquad \|f\|_{L^{p,\infty}_{\mu_d}(\ell^r_\omega)} = \|f\|_{L^{p,\infty}_{\widetilde{\mu}_d}(\ell^r_\omega)}.$$

For every setting (X, ν, ω) on the upper half 3-space described in Subsections 1.2.12 – 1.2.13, for the outer measure $\widetilde{\nu} = \widetilde{\nu}(\nu, \omega)$ defined via the construction described in (4.3.2)*, we have the same statement for the outer $L^p_\nu(\ell^r_\omega)$ and $L^{p,\infty}_\nu(\ell^r_\omega)$ spaces.*

Proof. In the case of the settings (X_d, μ_d, ω_d) on the upper half space, for every measurable subset $A \subseteq X_d$, we have

$$\omega_d(A \backslash \widetilde{A}) = 0,$$

where $\widetilde{A} \subseteq A$ is the subset defined by the auxiliary construction.

Analogously, in the settings (X, ν, ω) on the upper half 3-space, for every measurable subset $A \subseteq X$, we have

$$\omega_d(A \backslash \widetilde{A}) = 0,$$

where $\widetilde{A} \subseteq A$ is the subset defined by the auxiliary construction.

Therefore, the desired equalities follow from the equalities for the outer L^p and $L^{p,\infty}$ quasi-norms in (1.2.6), and the definition of the sizes ℓ^r_ω in (1.2.3) and (1.2.4). □

4.3.7 Banach space properties of the outer $L^p_\upsilon(\ell^\infty_\omega)$ spaces

The outer measures defined via the construction described in (4.3.2) have a better subadditivity behaviour than general outer measures. In particular, they guarantee the uniformity of the constants in Köthe duality and quasi-triangle inequality for countably many summands for the outer $L^p_\upsilon(\ell^\infty_\omega)$ spaces with $p \in (1,\infty)$, as well as in the weak quasi-triangle inequality for countably many summands for the outer $L^1_\upsilon(\ell^\infty_\omega)$ space.

Lemma 4.3.12. *For every $p \in [1,\infty]$, there exists a constant $C = C(p)$ such that, for every σ-finite setting (X,μ,ω) described in Subsection 1.2.1, for the outer measure υ defined via the construction described in* (4.3.2)*, the following properties hold true.*

(i) For every $p \in (1,\infty]$, for every measurable function $f \in L^p_\upsilon(\ell^\infty_\omega)$ on X, we have

$$C^{-1}\|f\|_{L^p_\upsilon(\ell^\infty_\omega)} \leqslant \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\upsilon(\ell^1_\omega)} = 1\Big\} \leqslant C\|f\|_{L^p_\upsilon(\ell^\infty_\omega)}.$$

(ii) For every $p \in (1,\infty]$, for every collection of measurable functions $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\upsilon(\ell^\infty_\omega)$ on X, we have

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^p_\upsilon(\ell^\infty_\omega)} \leqslant C\sum_{n\in\mathbb{N}}\|f_n\|_{L^p_\upsilon(\ell^\infty_\omega)}.$$

(iii) For every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\upsilon(\ell^\infty_\omega)$ of measurable functions on X, we have

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^{1,\infty}_\upsilon(\ell^\infty_\omega)} \leqslant \sum_{n\in\mathbb{N}}\|f_n\|_{L^1_\upsilon(\ell^\infty_\omega)}.$$

Proof. **Proof of property** (i)**.** For $p = \infty$, the inequalities follow from collapsing of exponents, properties (i) in Theorem 4.2.1, for the σ-finite setting (X,υ,ω).

For $p \in (1,\infty)$, let $f \in L^p_\upsilon(\ell^\infty_\omega)$. The second inequality follows from the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) and outer Hölder's inequality (Theorem 1.1.7) for the σ-finite setting (X,υ,ω).

To prove the first inequality, for every $k \in \mathbb{Z}$, we define the measurable subset $A_k \subseteq X$ by

$$A_k = \Big\{x \in X \colon |f(x)| > 2^k\Big\}.$$

By the assumption $f \in L^p_\upsilon(\ell^\infty_\omega)$, we have

$$\inf\Big\{\upsilon(B_k) \colon B_k \in \Sigma, B_k \subseteq A_k, \omega(A_k \backslash B_k) = 0\Big\} \leqslant 2^{-kp}\|f\|_{L^p_\upsilon(\ell^\infty_\omega)} < \infty.$$

We define a collection $\{D_k \colon k \in \mathbb{Z}\}$ of measurable subsets of X as follows.

We distinguish two cases.

If

$$\inf \Big\{ \upsilon(B_k) \colon B_k \in \Sigma, B_k \subseteq A_k, \omega(A_k \backslash B_k) = 0 \Big\} = 0,$$

then we define the subset D_k to be the empty set.

If

$$\inf \Big\{ \upsilon(B_k) \colon B_k \in \Sigma, B_k \subseteq A_k, \omega(A_k \backslash B_k) = 0 \Big\} \in (0, \infty),$$

then, for every $\varepsilon > 0$, we define the subset $\widetilde{B}_k = \widetilde{B}_k(\varepsilon) \in \Sigma$ such that $\widetilde{B}_k \subseteq A_k$, $\omega(A_k \backslash \widetilde{B}_k) = 0$, and

$$\inf \Big\{ \upsilon(B_k) \colon B_k \in \Sigma, B_k \subseteq A_k, \omega(A_k \backslash B_k) = 0 \Big\} \leqslant \upsilon(\widetilde{B}_k),$$
$$\upsilon(\widetilde{B}_k) \leqslant (1+\varepsilon) \inf \Big\{ \upsilon(B_k) \colon B_k \in \Sigma, B_k \subseteq A_k, \omega(A_k \backslash B_k) = 0 \Big\},$$

and the measurable subset $D_k \subseteq \widetilde{B}_k$ such that

$$\frac{\omega(D_k)}{\mu(D_k)} \leqslant \upsilon(\widetilde{B}_k) \leqslant (1+\varepsilon) \frac{\omega(D_k)}{\mu(D_k)}.$$

In particular, we have $\mu(D_k) \neq 0$.

We define the measurable function g on X by

$$g(x) = \sum_{k \in \mathbb{Z}} 2^{k(p-1)} \rho_k 1_{D_k}(x),$$

where, for every $k \in \mathbb{Z}$, we define $\rho_k \in [0, \infty)$ by

$$\rho_k = \begin{cases} 0, & \text{if } D_k = \varnothing, \\ \mu(D_k)^{-1}, & \text{if } D_k \neq \varnothing. \end{cases}$$

There exists a constant $c = c(p, \varepsilon)$ such that

$$\| fg \|_{L^1(X,\omega)} \geqslant c \sum_{k \in \mathbb{Z}} 2^{kp} \upsilon(\widetilde{B}_k) \geqslant c \| f \|^p_{L^p_\upsilon(\ell^\infty_\omega)},$$

where we used the definitions of g, D_k and $\widetilde{B}_k$ in the first inequality, and the definition of the outer $L^p_\upsilon(\ell^\infty_\omega)$ quasi-norm in the second.

Moreover, there exists a constant $C_1 = C_1(p)$ such that, for every $j \in \mathbb{Z}$, for every subset

$E \subseteq X$, we have

$$\begin{aligned} \upsilon(E)^{-1}\|g 1_{(\bigcup_{k\in\mathbb{Z},k\geqslant j}\widetilde{B}_k)^c} 1_E\|_{L^1(X,\omega)} &= \sum_{k\in\mathbb{Z},k<j\colon \omega(D_k\cap E)\neq 0} 2^{k(p-1)}\frac{\omega(D_k\cap E)}{\upsilon(E)\mu(D_k)} \\ &\leqslant \sum_{k\in\mathbb{Z},k<j\colon \omega(D_k\cap E)\neq 0} 2^{k(p-1)}\frac{\omega(D_k\cap E)}{\upsilon(E)\mu(D_k\cap E)} \\ &\leqslant \sum_{k\in\mathbb{Z},k<j} 2^{k(p-1)}\frac{\upsilon(D_k\cap E)}{\upsilon(E)} \\ &\leqslant C_1 2^{j(p-1)}, \end{aligned}$$

where we used the monotonicity of the outer measure υ in the first inequality, the absolute continuity of the measure ω with respect to the outer measure μ thus

$$\omega(D_k\cap E)\neq 0 \Rightarrow \mu(D_k\cap E)\neq 0,$$

and the equality in (4.3.4) for υ in the second, the monotonicity of the outer measure υ and the bounds on the geometric series in the third. Hence, by the subadditivity of the outer measure υ, we have

$$\upsilon(\ell^1_\omega(g) > C_1 2^{j(p-1)}) \leqslant \upsilon\Big(\bigcup_{k\in\mathbb{Z},k\geqslant j}\widetilde{B}_k\Big) \leqslant \sum_{k\in\mathbb{Z},k\geqslant j}\upsilon(\widetilde{B}_k),$$

and, by Fubini and the bounds on the geometric series, there exists a constant $C_2 = C_2(p)$ such that we have

$$\|g\|^{p'}_{L^{p'}_\upsilon(\ell^1_\omega)} \leqslant C_2\sum_{j\in\mathbb{Z}}2^{jp}\sum_{k\in\mathbb{Z},k\geqslant j}\upsilon(\widetilde{B}_k) \leqslant C_2^2\sum_{k\in\mathbb{Z}}2^{kp}\upsilon(\widetilde{B}_k) \leqslant C_2^3\|f\|^p_{L^p_\upsilon(\ell^\infty)}.$$

Proof of property (ii)**.** The desired inequality is a corollary of the triangle inequality for the classical $L^1(X,\omega)$ norm and the previous property.

Proof of property (iii)**.** Let $\{A_n : n\in\mathbb{N}\}$ be a collection of measurable subsets of X. For every $j\in\mathbb{N}$, we define the measurable subset $B_j \subseteq X$ by

$$B_j = \Big\{x\in X\colon \Big|\sum_{n\in\mathbb{N}} 1_{A_n}(x)\Big| \geqslant j\Big\}.$$

Therefore, we have

$$j 1_{B_j} \leqslant \sum_{n\in\mathbb{N}} 1_{A_n}.$$

Next, for every measurable subset $A\in\Sigma$, by Theorem 4.3.9, we have

$$\|1_A\|_{L^\infty_\mu(\ell^1_\omega)} \leqslant \|1_A\|_{L^1_\upsilon(\ell^\infty_\omega)},$$

and, by the equality for the outer L^1 quasi-norm of a characteristic function in (1.2.7), we have

$$\|1_A\|_{L^1_\upsilon(\ell^\infty_\omega)} = \inf\Big\{\upsilon(B)\colon B \in \Sigma, B \subseteq A, \omega(A\backslash B) = 0\Big\} \leqslant \upsilon(A) = \|1_A\|_{L^\infty_\mu(\ell^1_\omega)}.$$

Moreover, by the equalities for the outer L^1 and $L^{1,\infty}$ quasi-norms in (1.2.6), we have

$$\Big\|\sum_{n\in\mathbb{N}} 1_{A_n}\Big\|_{L^{1,\infty}_\upsilon(\ell^\infty_\omega)} = \sup\Big\{\|j1_{B_j}\|_{L^1_\upsilon(\ell^\infty_\omega)}\colon j \in \mathbb{N}\Big\}.$$

Together with Lemma 4.3.1, the previous four displays yield the inequality

$$\Big\|\sum_{n\in\mathbb{N}} 1_{A_n}\Big\|_{L^{1,\infty}_\upsilon(\ell^\infty_\omega)} \leqslant \sum_{n\in\mathbb{N}} \|1_{A_n}\|_{L^1_\upsilon(\ell^\infty_\omega)}.$$

The desired inequality follows from Lemma 4.2.7. □

Remark 4.3.13. *We point out that, a priori, the outer measure υ does not guarantee the uniformity of the constants in Köthe duality and quasi-triangle inequality for countably many summands for the outer $L^1_\upsilon(\ell^r_\omega)$ spaces with $r \in (1,\infty]$. This is clarified by the equality between $\widetilde{\mu}_m$ and μ_m showed for every $m \in \{0\} \cup \mathbb{N}$ in Subsection 4.3.4 in the case of the setting on the dyadic tree of depth m described in Subsection 1.2.7. In fact, in this collection of settings we exhibited counterexamples to the uniformity in the finite setting of the constant in the quasi-triangle inequality for countably many summands for the outer $L^1_\mu(\ell^r_\omega)$ spaces with $r \in (1,\infty]$, see Lemma 2.3.4 in Chapter 2.*

4.4 Outer L^p spaces on the upper half 3-space settings

In this section, we study the Banach space properties of the single iterated outer L^p spaces on the settings on the upper half 3-space or its discrete model, with a particular focus on the case of the outer $L^p_\nu(\ell^\infty_\omega)$ and $L^1_\nu(\ell^r_\omega)$ spaces.

Theorem 4.4.1. *For all $p, r \in (0,\infty]$, there exists a constant $C = C(p,r)$ such that, for every setting (X,ν,ω) on the upper half 3-space or its discrete model described in Subsections 1.2.11 – 1.2.13, the following properties hold true.*

(i) For every $p \in (0,\infty]$, for every measurable function $f \in L^p_\nu(\ell^p_\omega)$ on X, we have

$$C^{-1}\|f\|_{L^p_\nu(\ell^p_\omega)} \leqslant \|f\|_{L^p(X,\omega)} \leqslant C\|f\|_{L^p_\nu(\ell^p_\omega)}.$$

(ii) For all $p \in (1,\infty]$, $r \in [1,\infty]$ or $p = r = 1$, for every measurable function $f \in L^p_\nu(\ell^r_\omega)$ on X, we have

$$C^{-1}\|f\|_{L^p_\nu(\ell^r_\omega)} \leqslant \sup\Big\{\|fg\|_{L^1(X,\omega)}\colon \|g\|_{L^{p'}_\nu(\ell^{r'}_\omega)} = 1\Big\} \leqslant C\|f\|_{L^p_\nu(\ell^r_\omega)}.$$

(iii) *For all* $p \in (1,\infty]$, $r \in [1,\infty]$ *or* $p = r = 1$, *for every collection* $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\nu(\ell^\infty_\omega)$ *of measurable functions on* X, *we have*

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^p_\nu(\ell^r_\omega)} \leqslant C \sum_{n\in\mathbb{N}} \| f_n \|_{L^p_\nu(\ell^r_\omega)}.$$

(iv) *For every collection* $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\nu(\ell^\infty_\omega)$ *of measurable functions on* X, *we have*

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^{1,\infty}_\nu(\ell^\infty_\omega)} \leqslant C \sum_{n\in\mathbb{N}} \| f_n \|_{L^1_\nu(\ell^\infty_\omega)}.$$

(v) *For every* $r \in (1,\infty]$, *for every* $M > 0$, *there exists a collection* $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\nu(\ell^r_\omega)$ *of measurable functions on* X *such that*

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^1_\nu(\ell^r_\omega)} \geqslant M \sum_{n\in\mathbb{N}} \| f_n \|_{L^1_\nu(\ell^r_\omega)}.$$

Property (i) and, for all $p \in (1,\infty]$, $r \in [1,\infty)$ or $p = r \in \{1,\infty\}$, properties (ii) and (iii) are a corollary of Theorem 4.2.1. Moreover, for all $p \in (1,\infty)$, $r = \infty$, properties (ii) and (iii), and property (iv) are a corollary of Lemma 4.3.11 and Lemma 4.3.12, as well as the equalities in (4.3.18), (4.3.19), and (4.3.20). Since we only sketched the proof of those equalities, in this section we provide an alternative proof of properties (ii) and (iii) in the remaining cases, as well as properties (iv) and (v).

In Subsection 4.4.2, we start with the case of the setting (X,ν,ω) on the upper half 3-space described in Subsection 1.2.12, where we denote by ν the outer measure generated via minimal coverings by the collection of dyadic trees. Then the case of the setting on the collection of Heisenberg upper half dyadic tiles described in Subsection 1.2.11 follows straight-forwardly.

After that, in Subsection 4.4.3, we comment on how to adjust the arguments in the case of the setting $(X,\bar{\nu},\omega)$ on the upper half 3-space described in Subsection 1.2.13, where we denote by $\bar{\nu}$ the outer measure generated via minimal coverings by the collection of continuous trees.

The separate analysis of the two cases is due to the fact that the outer measures ν and $\bar{\nu}$ are not equivalent. Moreover, also the single iterated outer L^p quasi-norms associated with the two outer measures are not equivalent. In fact, in Subsection 4.4.4, we prove the following result.

Theorem 4.4.2. *For all* $p, r \in (0,\infty]$, $p > r$, *we have*

$$L^r_\nu(\ell^p_\omega) \hookrightarrow L^r_{\bar{\nu}}(\ell^p_\omega) \qquad L^r_\nu(\ell^p_\omega) \neq L^r_{\bar{\nu}}(\ell^p_\omega),$$
$$L^p_{\bar{\nu}}(\ell^r_\omega) \hookrightarrow L^p_\nu(\ell^r_\omega) \qquad L^p_{\bar{\nu}}(\ell^r_\omega) \neq L^p_\nu(\ell^r_\omega).$$

We recall that, instead, in the case of the settings on the upper half space described in Subsections 1.2.9 – 1.2.10, the outer measures generated via minimal coverings by the collections of dyadic and continuous tents respectively are equivalent. As a consequence, also the single iterated outer L^p quasi-norms associated with the two outer measures are equivalent, and we refer to Section 2.3 in Chapter 2 for the details.

4.4.1 Geometry of the dyadic trees in the upper half 3-space

Let (X, ν, ω) be the setting on the upper half 3-space described in Subsection 1.2.12. We recall that the outer measure ν is generated via minimal coverings by the pre-measure τ on the collection $\mathcal{T}$ of dyadic trees, and each dyadic tree is denoted by $T(m, n, l)$ for certain $m, n, l \in \mathbb{Z}$.

We start with some auxiliary observations about the geometry of dyadic trees and their intersections, and the values of the outer measure ν on them.

We refer to Subsection 1.2.12 for the definitions of the dyadic tree $T_H \in \mathcal{T}$ associated with a Heisenberg upper half dyadic tile $H \in \mathcal{H}$, and the Heisenberg upper half dyadic tile $H_T \in \mathcal{H}$ associated with a dyadic tree $T \in \mathcal{T}$.

Lemma 4.4.3. *Let T be a dyadic tree in $\mathcal{T}$, let H be a Heisenberg upper half dyadic tile in $\mathcal{H}$ such that $T \cap H \neq \varnothing$. Then, we have $T_H \subseteq T$.*

Proof. The statement is a straight-forward consequence of the definition of dyadic trees in $\mathcal{T}$ and the pairwise disjointness between different elements in $\mathcal{H}$. □

Lemma 4.4.4. *Let T be a dyadic tree in $\mathcal{T}$ and let $\{T_n : n \in \mathbb{N}\} \subseteq \mathcal{T}$ be a collection of dyadic trees such that, for every $n \in \mathbb{N}$, we have*

$$T_n \subseteq T, \tag{4.4.1}$$

$$T_n \nsubseteq \bigcup_{m \in \mathbb{N}, m \neq n} T_m. \tag{4.4.2}$$

Then, we have

$$\sum_{n \in \mathbb{N}} \tau(T_n) \leqslant \tau(T).$$

Proof. For every $n \in \mathbb{N}$, we define the Heisenberg upper half dyadic tiles $H, H_n \in \mathcal{H}$ by

$$H = H_T \subseteq T, \qquad H_n = H_{T_n} \subseteq T_n.$$

We claim that the dyadic intervals in the collection $\{\pi(H_n) : n \in \mathbb{N}\}$ are pairwise disjoint, where $\pi \colon X \to \mathbb{R}$ is the projection onto the first coordinate.

In fact, by the definition of dyadic trees in $\mathcal{T}$ and (4.4.1), for every $n \in \mathbb{N}$, we have

$$\widetilde{\pi}(H) \subseteq \widetilde{\pi}(H_n), \tag{4.4.3}$$

where $\widetilde{\pi}\colon X \to \mathbb{R}$ is the projection onto the second coordinate. Next suppose there exist $n, n' \in \mathbb{N}$ such that the dyadic intervals $\pi(H_n)$ and $\pi(H_{n'})$ have non-empty intersection. Therefore, they are contained one in the other and, without loss of generality, we assume $\pi(H_{n'}) \subseteq \pi(H_n)$. Then, we have $|\pi(H_{n'})| \leqslant |\pi(H_n)|$, hence $|\widetilde{\pi}(H_{n'})| \geqslant |\widetilde{\pi}(H_n)|$. Moreover, by the inclusion in (4.4.3), the dyadic intervals $\widetilde{\pi}(H_n)$ and $\widetilde{\pi}(H_{n'})$ have non-empty intersection, hence $\widetilde{\pi}(H_n) \subseteq \widetilde{\pi}(H_{n'})$. As a consequence, we have $H_{n'} \subseteq T_n$. Hence, by Lemma 4.4.3, we have $T_{n'} \subseteq T_n$, yielding a contradiction with the condition in (4.4.2).

By the inclusion in (4.4.1), for every $n \in \mathbb{N}$, the dyadic interval $\pi(H_n)$ is contained in $\pi(T)$. Therefore, we have

$$\sum_{n \in \mathbb{N}} \tau(T_n) = \sum_{n \in \mathbb{N}} |\pi(H_n)| = \Big| \bigcup_{n \in \mathbb{N}} \pi(H_n) \Big| \leqslant |\pi(T)| = \tau(T),$$

where we used the fact that the dyadic intervals in the collection $\{\pi(H_n) \colon n \in \mathbb{N}\}$ are pairwise disjoint in the second equality. ☐

Lemma 4.4.5. *Let $\{H_n \colon n \in \mathbb{N}\} \subseteq \mathcal{H}$ be a collection of Heisenberg upper half dyadic tiles such that, for every $n \in \mathbb{N}$, we have*

$$H_n \nsubseteq \bigcup_{m \in \mathbb{N}, m \neq n} T_{H_m}. \tag{4.4.4}$$

Let $\{W_n \colon n \in \mathbb{N}\}$ be a collection of measurable subsets of X such that, for every $n \in \mathbb{N}$, we have

$$W_n \subseteq T_{H_n}, \qquad W_n \cap H_n \neq \varnothing. \tag{4.4.5}$$

Then, we have

$$\sum_{n \in \mathbb{N}} \tau(T_{H_n}) = \nu\Big(\bigcup_{n \in \mathbb{N}} W_n \Big).$$

In particular, for every dyadic tree T in $\mathcal{T}$, we have

$$\tau(T) = \nu(T).$$

Proof. Since the outer measure ν is generated via minimal coverings by the pre-measure τ on the collection $\mathcal{T}$ of dyadic trees, by the inclusion in (4.4.5), we have

$$\nu\Big(\bigcup_{n \in \mathbb{N}} W_n \Big) \leqslant \sum_{n \in \mathbb{N}} \tau(T_{H_n}).$$

Therefore, if the left hand side in the previous display is infinite, then the right hand side is infinite as well, yielding the desired equality. If the left hand side is finite, for every $\varepsilon > 0$, there exists a collection $\mathcal{U}(\varepsilon) \subseteq \mathcal{T}$ of dyadic trees such that

$$\bigcup_{n \in \mathbb{N}} W_n \subseteq \bigcup_{U \in \mathcal{U}(\varepsilon)} U, \tag{4.4.6}$$

$$\sum_{U \in \mathcal{U}(\varepsilon)} \tau(U) \leqslant (1 + \varepsilon) \nu\Big(\bigcup_{n \in \mathbb{N}} W_n \Big) < \infty. \tag{4.4.7}$$

Without loss of generality, we assume that, for every $U \in \mathcal{U}(\varepsilon)$, we have

$$U \nsubseteq \bigcup_{V \in \mathcal{U}(\varepsilon), V \neq U} V,$$

otherwise we would drop U from the collection $\mathcal{U}(\varepsilon)$, preserving the inclusion in (4.4.6) and decreasing the left hand side in the inequality in (4.4.7). Next, by the inclusion in (4.4.6), for every $n \in \mathbb{N}$, there exists an element $U \in \mathcal{U}(\varepsilon)$ such that $W_n \cap U \neq \varnothing$, hence $H_n \cap U \neq \varnothing$. In particular, by Lemma 4.4.3, we have $T_{H_n} \subseteq U$. Then, for every $U \in \mathcal{U}(\varepsilon)$, we define the subcollection $\mathcal{T}(U) \subseteq \{T_{H_n} : n \in \mathbb{N}\}$ by

$$\mathcal{T}(U) = \Big\{T_{H_n} : n \in \mathbb{N}, T_{H_n} \subseteq U\Big\}.$$

We have

$$\Big\{T_{H_n} : n \in \mathbb{N}\Big\} = \bigcup_{U \in \mathcal{U}(\varepsilon)} \mathcal{T}(U).$$

The condition in (4.4.4) on the collection $\{H_n : n \in \mathbb{N}\}$ implies the condition in (4.4.2) on each of the collections $\mathcal{T}(U)$. By Lemma 4.4.4, we have

$$\sum_{n \in \mathbb{N}} \tau(T_{H_n}) \leqslant \sum_{U \in \mathcal{U}(\varepsilon)} \sum_{T_{H_n} \subseteq \mathcal{T}(U)} \tau(T_{H_n}) \leqslant \sum_{U \in \mathcal{U}(\varepsilon)} \tau(U).$$

Taking ε arbitrarily small, together with the inequality in (4.4.7), the previous display yields the desired inequality. □

4.4.2 Outer $L^p_\nu(\ell^\infty_\omega)$ and $L^1_\nu(\ell^r_\omega)$ spaces on the upper half 3-space setting with dyadic trees

Let (X, ν, ω) be the setting on the upper half 3-space described in Subsection 1.2.12. We recall that the outer measure ν is generated via minimal coverings by the pre-measure τ on the collection $\mathcal{T}$ of dyadic trees, and each dyadic tree is denoted by $T(m, n, l)$ for certain $m, n, l \in \mathbb{Z}$. We have the following decomposition result with respect to the size ℓ^∞_ω for measurable functions on X in the intersection between the outer $L^p_\mu(\ell^\infty_\omega)$ and $L^\infty_\mu(\ell^\infty_\omega)$ spaces.

We refer to the end of Chapter 1 for the notation of a double sequence parametrized by pairs (k, n) with $k \in \mathbb{Z}$, $n \in \mathbb{N}_k$ appearing in the following statement.

Proposition 4.4.6. *For every $p \in (0, \infty)$, there exists a constant $C = C(p)$ such that the following property holds true.*

For every measurable function $f \in L^p_\nu(\ell^\infty_\omega) \cap L^\infty_\nu(\ell^\infty_\omega)$ on X, there exist $k_0 \in \mathbb{Z}$ and a double sequence $\{T_{k,n} : k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \mathcal{T}$ of dyadic trees such that

- *For every $k \in \mathbb{Z}$, $k > k_0$, we have $\mathbb{N}_k = \varnothing$.*

- *If we set*

$$
\begin{aligned}
U_k &= \varnothing, && \textit{for every } k \in \mathbb{Z},\ k > k_0,\\
U_{k,0} &= U_{k+1}, && \textit{for every } k \in \mathbb{Z},\ k \leqslant k_0,\\
U_{k,n} &= U_{k,n-1} \cup T_{k,n}, && \textit{for every } k \in \mathbb{Z},\ k \leqslant k_0,\ \textit{for every } n \in \mathbb{N}_k,\\
U_k &= U_{k+1} \cup \bigcup_{n \in \mathbb{N}_k} U_{k,n}, && \textit{for every } k \in \mathbb{Z},\ k \leqslant k_0,
\end{aligned}
$$

then, for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_k$, *we have*

$$\ell^\infty_\omega(f 1_{U^c_{k,n-1}})(T_{k,n}) > 2^k, \qquad \textit{when } T_{k,n} \neq \varnothing, \tag{4.4.8}$$

$$\|f 1_{U^c_k}\|_{L^\infty(\ell^\infty_\omega)} \leqslant 2^k, \tag{4.4.9}$$

$$\nu(\ell^\infty_\omega(f) > 2^k) \leqslant \nu(U_k), \tag{4.4.10}$$

$$\sum_{n \in \mathbb{N}_k} \nu(T_{k,n}) \leqslant \nu(\ell^\infty_\omega(f) > 2^k). \tag{4.4.11}$$

In particular, we have

$$\|f\|^p_{L^p_\nu(\ell^\infty_\omega)} \sim_p \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{n \in \mathbb{N}_k} \nu(T_{k,n}) \sim_p \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{l \in \mathbb{Z}, l \geqslant k} \sum_{m \in \mathbb{N}_l} \nu(T_{l,m}).$$

Moreover, for every $k \in \mathbb{Z}$, *the Heisenberg upper half dyadic tiles in the collection*

$$\Big\{ H_{k,n} = H_{T_{k,n}} \colon n \in \mathbb{N}_k \Big\} \subseteq \mathcal{H}, \tag{4.4.12}$$

satisfy the geometric property in (4.4.4).

Proof. The selection algorithm is analogous to that described in **Case II** in the proof of Proposition 2.2.4 in Chapter 2. We define the collection $\{T_{k,n} \colon k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \mathcal{T}$ by a double recursion, backward on $k \in \mathbb{Z}$, and, for every fixed k, forward on $n \in \mathbb{N}_k$. In parallel, we prove the properties in (4.4.8) – (4.4.11) by backward induction on $k \in \mathbb{Z}$.

We briefly comment on the modifications needed. Dyadic cubic boxes, namely dyadic tents, are replaced by dyadic trees. The upper half dyadic cubic box E^+ associated with a dyadic cubic box E is replaced by the Heisenberg upper half dyadic tile H_T associated with a dyadic tree T. The geometric properties of dyadic tents observed in Lemma 2.2.2 and Lemma 2.2.3 in Chapter 2 are replaced by those of the dyadic trees observed in Lemma 4.4.3 and Lemma 4.4.5 respectively.

The geometric property of the collection defined in (4.4.12) follows from making the maximal choice for the dyadic tree $T_{k,n}$ at each step of the selection algorithm, maximal in terms of the outer measure ν. ☐

We are ready to prove the remaining properties stated in Theorem 4.4.1 in the case of the setting on the upper half 3-space described in Subsection 1.2.12, where the outer measure is generated via minimal coverings by the collection of dyadic trees.

Proof of Theorem 4.4.1 for the setting described in Subsection 1.2.12. We prove the properties that do not follow as a corollary of Theorem 4.2.1.

Proof of property (ii) **for all** $p \in (1,\infty)$**,** $r = \infty$**.** Let $f \in L^p_\nu(\ell^\infty_\omega)$. The second inequality follows from the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) and outer Hölder's inequality (Theorem 1.1.7) for the setting (X,ν,ω). To prove the first inequality, by the approximation result stated in Lemma 3.5.1 in Chapter 3 for functions in the outer $L^p_\nu(\ell^\infty_\omega)$ spaces, without loss of generality, we assume $f \in L^p_\nu(\ell^\infty_\omega) \cap L^\infty_\nu(\ell^\infty_\omega)$. Let $\{T_{k,n} \colon k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \mathcal{T}$ be the collection of dyadic trees produced by the decomposition of f with respect to the size ℓ^∞_ω at levels $\{2^k \colon k \in \mathbb{Z}\}$ provided by Proposition 4.4.6. In particular, for every fixed $k \in \mathbb{Z}$, the collection $\{T_{k,n} \colon n \in \mathbb{N}_k\} \subseteq \mathcal{T}$ of dyadic trees satisfies the geometric property in (4.4.2), and the Heisenberg upper half dyadic tiles in the collection $\{H_{k,n} = H_{T_{k,n}} \colon n \in \mathbb{N}_k\} \subseteq \mathcal{H}$ satisfy the geometric property in (4.4.4). Moreover, there exists a collection $\{W_{k,n} \subseteq H_{k,n} \colon k \in \mathbb{Z}, n \in \mathbb{N}_k\}$ of pairwise disjoint measurable subsets of X such that, for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_k$, we have $\omega(W_{k,n}) > 0$ and

$$|f(x,\xi,s)| \in (2^k, 2^{k+1}], \qquad \text{for every } (x,\xi,s) \in W_{k,n}.$$

We define the measurable function g on X by

$$g(x,\xi,s) = \sum_{k\in\mathbb{Z}} 2^{k(p-1)} \sum_{n\in\mathbb{N}_k} \frac{\nu(T_{k,n})}{\omega(W_{k,n})} 1_{W_{k,n}}(x,\xi,s).$$

There exist constants $c = c(p)$ and $C = C(p)$ such that

$$\|fg\|_{L^1(X,\omega)} \geqslant c\|f\|^p_{L^p_\nu(\ell^\infty_\omega)}, \qquad \|g\|_{L^{p'}_\nu(\ell^1_\omega)} \leqslant C\|f\|^{p-1}_{L^p_\nu(\ell^\infty_\omega)}.$$

The proof of the first inequality in the previous display is straight-forward. The proof of the second is analogous to that of **Case II** with $r = \infty$ in the proof of Lemma 2.3.1 in Chapter 2. We briefly comment on the modifications needed. Dyadic cubic boxes, namely dyadic tents, are replaced by dyadic trees. The geometric properties of dyadic tents observed in Lemma 2.B.1 and Lemma 2.2.3 in Chapter 2 and implicitly used are replaced by those of the dyadic trees observed in Lemma 4.4.4 for subcollections of $\{T_{k,n} \colon n \in \mathbb{N}_k\}$ and Lemma 4.4.5 for subcollections of $\{H_{k,n} \colon n \in \mathbb{N}_k\}$ respectively.

Proof of property (iii) **for all** $p \in (1,\infty)$**,** $r = \infty$**.** The desired inequality is a corollary of the triangle inequality for the classical $L^1(X,\omega)$ norm and the previous property.

Proof of property (iv)**.** We define the measurable function f on X by

$$f = \sum_{n\in\mathbb{N}} f_n.$$

For every $\lambda \in (0,\infty)$, we define the collection $\mathcal{H}_\lambda \subseteq \mathcal{H}$ of Heisenberg upper half dyadic tiles by

$$\mathcal{H}_\lambda = \Big\{ H \in \mathcal{H} \colon \|f 1_H\|_{L^\infty(X,\omega)} > \lambda \Big\}.$$

First, we assume $f \in L^{1,\infty}_\nu(\ell^\infty_\omega)$. Therefore, for every $\lambda \in (0,\infty)$, we have

$$\sup \Big\{ |\pi(H)| \colon H \in \mathcal{H}_\lambda \Big\} \leqslant \nu\Big(\bigcup_{H \in \mathcal{H}_\lambda} H \Big) < \infty,$$

where $\pi \colon X \to \mathbb{R}$ is the projection onto the first coordinate. Hence, the collection $\mathcal{H}_\lambda$ has maximal elements, namely we define the collection $\widetilde{\mathcal{H}}_\lambda \subseteq \mathcal{H}_\lambda$ by

$$\widetilde{\mathcal{H}}_\lambda = \Big\{ H \in \mathcal{H}_\lambda \colon \nexists H' \in \mathcal{H}_\lambda, H' \neq H, \pi(H) \subseteq \pi(H'), \widetilde{\pi}(H') \subseteq \widetilde{\pi}(H) \Big\},$$

where $\widetilde{\pi} \colon X \to \mathbb{R}$ is the projection onto the second coordinate. In particular, we have

$$\bigcup_{H \in \mathcal{H}_\lambda} H = \bigcup_{H \in \widetilde{\mathcal{H}}_\lambda} H,$$

and the Heisenberg upper half dyadic tiles in the collection $\widetilde{\mathcal{H}}_\lambda$ satisfy the geometric property in (4.4.4). Since $f \in L^{1,\infty}_\nu(\ell^\infty_\omega)$, there exists $\Lambda \in (0,\infty)$ such that

$$\|f\|_{L^{1,\infty}_\nu(\ell^\infty_\omega)} \leqslant 2\Lambda \nu(\ell^\infty_\omega(f) > \Lambda) \leqslant 2 \sum_{H \in \widetilde{\mathcal{H}}_\Lambda} \|f 1_H\|_{L^\infty(X,\omega)} \nu(H). \tag{4.4.13}$$

Next, for every $n \in \mathbb{N}$, we split $\widetilde{\mathcal{H}}_\Lambda$ into the subcollections $\widetilde{\mathcal{H}}_{\Lambda,k} = \widetilde{\mathcal{H}}_{\Lambda,k}(n)$, one for each $k \in \mathbb{Z}$, defined by

$$\widetilde{\mathcal{H}}_{\Lambda,k} = \Big\{ H \in \widetilde{\mathcal{H}}_\Lambda, \|f_n 1_H\|_{L^\infty(X,\omega)} \in (2^k, 2^{k+1}] \Big\},$$

hence, by Lemma 4.4.5 for the collection $\widetilde{\mathcal{H}}_{\Lambda,k}$, we have

$$\sum_{H \in \widetilde{\mathcal{H}}_{\Lambda,k}} \nu(H) = \nu\Big(\bigcup_{H \in \widetilde{\mathcal{H}}_{\Lambda,k}} H \Big) \leqslant \nu(\ell^\infty_\omega(f_n) > 2^k).$$

Therefore, we have

$$\begin{aligned} \sum_{H \in \widetilde{\mathcal{H}}_\Lambda} \|f_n 1_H\|_{L^\infty(X,\omega)} \nu(H) &\leqslant 2 \sum_{k \in \mathbb{Z}} 2^k \sum_{H \in \widetilde{\mathcal{H}}_{\Lambda,k}} \nu(H) \\ &\leqslant 2 \sum_{k \in \mathbb{Z}} 2^k \nu(\ell^\infty_\omega(f_n) > 2^k) \\ &\leqslant C \|f_n\|_{L^1_\nu(\ell^\infty_\omega)}. \end{aligned} \tag{4.4.14}$$

Together with the triangle inequality for the classical $L^\infty(X,\omega)$ norm, the inequalities in (4.4.13) and (4.4.14) yield the desired inequality.

Next, we assume $f \notin L^{1,\infty}_\nu(\ell^\infty_\omega)$. Therefore, for every $M > 0$, there exists $\Lambda \in (0,\infty)$ and $U \subseteq X$ such that

$$M \leqslant \|f 1_U\|_{L^{1,\infty}_\nu(\ell^\infty_\omega)} < \infty.$$

By the same argument used in the previous case, we have

$$M \leqslant C \sum_{n\in\mathbb{N}} \|f_n\|_{L^1_\nu(\ell^\infty_\omega)}.$$

Taking $M > 0$ arbitrarily big, the previous display yields the desired inequality.

Proof of property (v)**.** Fix $j \in \mathbb{N}$. For all $n, l \in \mathbb{Z}$, $0 \leqslant l \leqslant j$, we define $\widetilde{N}(n,l) \in \mathbb{Z}$ by the condition

$$I(n,-j) \subseteq I(\widetilde{N}(n,l),-l).$$

For all $n, l \in \mathbb{Z}$, $0 \leqslant l \leqslant j$, we define the measurable subset $\widetilde{H}(n,l) \subseteq X$ by

$$\widetilde{H}(n,l) = \bigcup_{m=0}^{2^{j-l}-1} H(m,n,l) = I(0,j) \times I(n,-l) \times (2^{l-1}, 2^l], \tag{4.4.15}$$

and, for every $n' \in \mathbb{Z}$, $0 \leqslant n' < 2^j$, we define the measurable subset $E(n') \subseteq X$ by

$$E(n') = \bigcup_{l=0}^{j} \widetilde{H}(\widetilde{N}(n',l),l).$$

In particular, for every $n' \in \mathbb{Z}$, $0 \leqslant n' < 2^j$, we have

$$H(0,n',j) \subseteq E(n') \subseteq T(0,n',j),$$

hence, by Lemma 4.4.5, we have

$$\nu(E(n')) = 2^j.$$

Next, we define the measurable functions $f_{n'}$ and f on X by

$$\begin{aligned} f_{n'}(x,\xi,s) &= 1_{E(n')}(x,\xi,s), \\ f(x,\xi,s) &= \sum_{n'=0}^{2^j-1} f_{n'}(x,\xi,s) = \sum_{l=0}^{j} \sum_{n=0}^{2^l-1} 2^{j-l} 1_{\widetilde{H}(n,l)}(x,\xi,s). \end{aligned}$$

There exist constants $c = c(r)$ and $C = C(r)$ such that

$$\begin{aligned} &\|f\|_{L^1_\nu(\ell^r_\omega)} \geqslant c 2^{2j}(j+1), \\ &\sum_{n'=0}^{2^j-1} \|f_{n'}\|_{L^1_\nu(\ell^r_\omega)} \leqslant C \sum_{n'=0}^{2^j-1} \|f_{n'}\|_{L^1_\nu(\ell^\infty_\omega)} \|f_{n'}\|_{L^\infty_\nu(\ell^r_\omega)} = C 2^j 2^j (j+1)^{\frac{1}{r}}. \end{aligned}$$

Taking $j \in \mathbb{N}$ big enough, we obtain the desired inequality. □

Remark 4.4.7. *The counterexample is analogous to that in Lemma 2.3.4 in Chapter 2. We point out the similarities between the figure on the left in Figure 1.2 and that on the right in Figure 1.4, seen upside down.*

Remark 4.4.8. *The existence of counterexamples to the quasi-triangle inequality for countably many summands for the outer $L^1_\nu(\ell^r_\omega)$ spaces exhibited in property (v) implies the existence of counterexamples to Köthe duality for the same spaces.*

Proof of Theorem 4.4.1 for the setting described in Subsection 1.2.11. Let (X, ν, ω) be the setting on the collection of Heisenberg upper half dyadic tiles described in Subsection 1.2.11, let $(\bar{X}, \bar{\nu}, \bar{\omega})$ be the setting on the upper half 3-space described in Subsection 1.2.12.

We reduce the proof in the case of the setting (X, ν, ω) to that in the case of the setting $(\bar{X}, \bar{\nu}, \bar{\omega})$ upon the following observation. For every function f on X, we define the measurable function $F(f)$ on $\bar{X}$ by

$$F(f)(x, \xi, s) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} f(H(m, n, l)) 1_{H(m,n,l)}(x, \xi, s).$$

Then, for all $p, r \in (0, \infty]$, we have

$$\|f\|_{L^p_\nu(\ell^r_\omega)} = \|F(f)\|_{L^p_{\bar{\nu}}(\ell^r_{\bar{\omega}})}.$$

□

4.4.3 Outer $L^p_\nu(\ell^\infty_\omega)$ and $L^1_\nu(\ell^r_\omega)$ spaces on the upper half 3-space setting with continuous trees

Let $(X, \bar{\nu}, \omega)$ be the setting on the upper half 3-space described in Subsection 1.2.13. We recall that the outer measure $\bar{\nu}$ is generated via minimal coverings by the pre-measure $\bar{\tau}$ on the collection $\bar{\mathcal{T}}$ of continuous trees, and each continuous tree is denoted by $\bar{T}(x, \xi, s)$ for certain $x, \xi \in \mathbb{R}$, $s \in (0, \infty)$. Moreover, let ν be the outer measure on the upper half 3-space described in Subsection 1.2.12. We recall that the outer measure ν is generated via minimal coverings by the pre-measure τ on the collection $\mathcal{T}$ of dyadic trees, and each dyadic tree is denoted by $T(m, n, l)$ for certain $m, n, l \in \mathbb{Z}$.

The outer measure $\bar{\nu}$ is equivalent to $\bar{\nu}_{\text{dya}}$, the outer measure on the upper half 3-space X generated via minimal coverings by the pre-measure $\bar{\tau}_{\text{dya}}$ on the collection $\bar{\mathcal{T}}_{\text{dya}} \subseteq \bar{\mathcal{T}}$ of continuous trees associated with dyadic intervals, namely

$$\bar{\mathcal{T}}_{\text{dya}} = \left\{ \bar{T}(2^l m, 2^{-l} n, 2^l) \colon m, n, l \in \mathbb{Z} \right\},$$
$$\bar{\tau}_{\text{dya}}(\bar{T}(2^l m, 2^{-l} n, 2^l)) = 2^l, \qquad \text{for all } m, n, l \in \mathbb{Z}.$$

In fact, for every $x \in \mathbb{R}$, $\xi \in \mathbb{R}$, $s \in (0, \infty)$, there exist $m, n, l \in \mathbb{Z}$ such that

$$x \in (2^l m, 2^l (m+1)], \qquad \xi \in (2^{-l} n, 2^{-l}(n+1)], \qquad s \in (2^{l-1}, 2^l],$$

hence

$$\bar{T}(x,\xi,s) \subseteq \bar{T}(2^l m, 2^{-l} n, 2^l) \cup \bar{T}(2^l(m+1), 2^{-l} n, 2^l) \cup \\ \cup \bar{T}(2^l m, 2^{-l}(n+1), 2^l) \cup \bar{T}(2^l(m+1), 2^{-l}(n+1), 2^l).$$

Therefore, for every subset $A \subseteq X$, we have

$$\bar{\nu}(A) \leqslant \bar{\nu}_{\mathrm{dya}}(A) \leqslant 8\bar{\nu}(A),$$

hence, for all $p, r \in (0,\infty]$, there exists a constant $C = C(p,r)$ such that, for every measurable function f on X, we have

$$C^{-1}\|f\|_{L^p_{\bar{\nu}}(\ell^r_\omega)} \leqslant \|f\|_{L^p_{\bar{\nu}_{\mathrm{dya}}}(\ell^r_\omega)} \leqslant C\|f\|_{L^p_{\bar{\nu}}(\ell^r_\omega)}. \tag{4.4.16}$$

Next, for all $m, n, l \in \mathbb{Z}$, we define the *Heisenberg upper half continuous tile* $\bar{H}(m,n,l)$ in X by

$$\bar{H}(m,n,l) \coloneqq \bar{T}(2^l m, 2^{-l} n, 2^l) \cap (\mathbb{R}^2 \times (2^{l-1}, 2^l]),$$

and we define the collection $\bar{\mathcal{H}}$ of Heisenberg upper half continuous tiles in X by

$$\bar{\mathcal{H}} \coloneqq \left\{ \bar{H}(m,n,l) \colon m,n,l \in \mathbb{Z} \right\}.$$

We observe that at least two and at most four elements of $\bar{\mathcal{H}}$ can overlap at the same time.

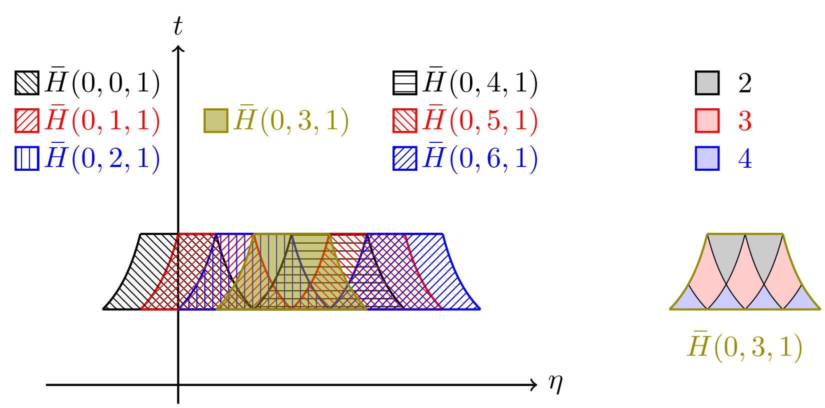

Figure 4.1: Heisenberg upper half continuous tiles in the upper half 3-space with coordinates (y,η,t) projected onto the upper half plane with coordinates (η,t), and multiplicity of their overlapping.

Finally, for every Heisenberg upper half continuous tile $\bar{H} = \bar{H}(m,n,l) \in \bar{\mathcal{H}}$, we define the continuous tree $\bar{T}_{\bar{H}} \in \bar{\mathcal{T}}$ by

$$\bar{T}_{\bar{H}} = \bar{T}_{\bar{H}(m,n,l)} \coloneqq \bar{T}(2^l m, 2^{-l}(2n+1), 2^l).$$

The setting $(X, \bar{\nu}_{\mathrm{dya}}, \omega)$ satisfies geometric properties analogous to those of the setting (X, ν, ω) described in Lemma 4.4.3, Lemma 4.4.4, and Lemma 4.4.5. In fact, we can prove the following results.

Lemma 4.4.9. *Let $\bar{T}$ be a continuous tree in $\bar{\mathcal{T}}_{\mathrm{dya}}$, let $\bar{H}$ be a Heisenberg upper half dyadic tile in $\bar{\mathcal{H}}$ such that $\bar{T} \cap \bar{H} \neq \varnothing$. Then, there exists a continuous tree $\bar{T}' \in \bar{\mathcal{T}}_{\mathrm{dya}}$ such that*

$$\bar{\tau}_{\mathrm{dya}}(\bar{T}') = \bar{\tau}_{\mathrm{dya}}(\bar{T}), \qquad \bar{T}_{\bar{H}} \subseteq \bar{T} \cup \bar{T}'.$$

Lemma 4.4.10. *Let $\bar{T}$ be a continuous tree in $\bar{\mathcal{T}}_{\mathrm{dya}}$ and let $\{\bar{T}_n \colon n \in \mathbb{N}\} \subseteq \bar{\mathcal{T}}_{\mathrm{dya}}$ be a collection of continuous trees such that, for every $n \in \mathbb{N}$, we have*

$$\bar{T}_n \subseteq \bar{T},$$
$$\bar{T}_n \nsubseteq \bigcup_{m \in \mathbb{N}, m \neq n} \bar{T}_m.$$

Then, we have

$$\sum_{n \in \mathbb{N}} \bar{\tau}_{\mathrm{dya}}(\bar{T}_n) \leqslant \bar{\tau}_{\mathrm{dya}}(\bar{T}).$$

Lemma 4.4.11. *Let $\{\bar{H}_n \colon n \in \mathbb{N}\} \subseteq \bar{\mathcal{H}}$ be a collection of Heisenberg upper half dyadic tiles such that, for every $n \in \mathbb{N}$, we have*

$$\bar{H}_n \nsubseteq \bigcup_{m \in \mathbb{N}, m \neq n} \bar{T}_{\bar{H}_m}.$$

Let $\{\overline{W}_n \colon n \in \mathbb{N}\}$ be a collection of measurable subsets of X such that, for every $n \in \mathbb{N}$, we have

$$\overline{W}_n \subseteq \bar{T}_{\bar{H}_n}, \qquad \overline{W}_n \cap \bar{H}_n \neq \varnothing.$$

Then, we have

$$\bar{\nu}_{\mathrm{dya}}\Big(\bigcup_{n \in \mathbb{N}} \overline{W}_n\Big) \leqslant \sum_{n \in \mathbb{N}} \bar{\tau}_{\mathrm{dya}}(\bar{T}_{\bar{H}_n}) \leqslant 4 \bar{\nu}_{\mathrm{dya}}\Big(\bigcup_{n \in \mathbb{N}} \overline{W}_n\Big).$$

In particular, replacing the setting (X, ν, ω) with the setting $(X, \bar{\nu}_{\mathrm{dya}}, \omega)$, we can prove a decomposition result for the outer $L^p_{\bar{\nu}_{\mathrm{dya}}}(\ell^\infty_\omega)$ spaces analogous to that stated in Proposition 4.4.6. We briefly comment on the modifications needed. Dyadic trees in $\mathcal{T}$ are replaced by continuous trees in $\bar{\mathcal{T}}_{\mathrm{dya}}$. The Heisenberg upper half dyadic tile H_T associated with a dyadic tree $T \in \mathcal{T}$ is replaced by the Heisenberg upper half continuous tile $\bar{H}_{\bar{T}}$ associated with a continuous tree $\bar{T} \in \bar{\mathcal{T}}_{\mathrm{dya}}$. The geometric properties of dyadic trees in $\mathcal{T}$ observed in Lemma 4.4.3, Lemma 4.4.4, and Lemma 4.4.5 are replaced by those of the continuous trees in $\bar{\mathcal{T}}_{\mathrm{dya}}$ observed in Lemma 4.4.9, Lemma 4.4.10 and Lemma 4.4.11 respectively.

As a consequence, we can prove Theorem 4.4.1 in the case of the setting on the upper half 3-space described in Subsection 1.2.13, where the outer measure is generated via minimal coverings by the collection of continuous trees.

Proof of Theorem 4.4.1 for the setting described in Subsection 1.2.13. We prove the properties that do not follow as a corollary of Theorem 4.2.1.

Proof of properties (ii) **and** (iii) **for all** $p \in (1,\infty)$**,** $r=\infty$**, and property** (iv)**.** The proof follows from the equivalence in (4.4.16) and the decomposition result for the outer $L^p_{\bar\nu_{\mathrm{dya}}}(\ell^\infty_\omega)$ spaces on the setting $(X,\bar\nu_{\mathrm{dya}},\omega)$ analogous to that stated in Proposition 4.4.6.

Proof of property (v)**.** For fixed $j \in \mathbb{N}$, $\varepsilon \in (0,2^{-j-10}]$, for all $n,l \in \mathbb{Z}$, $0 \leqslant l \leqslant j$, let

$$\begin{aligned}
\widetilde{I}_-(n,l) &= (2^{-l}(2n+1) - 2^{-j-1} - \varepsilon, 2^{-l}(2n+1) - 2^{-j-1} + \varepsilon],\\
\widetilde{I}_+(n,l) &= (2^{-l}(2n+1) + 2^{-j-1} - \varepsilon, 2^{-l}(2n+1) + 2^{-j-1} + \varepsilon],\\
\widetilde{H}(n,l) &= (0,2^j] \times (\widetilde{I}_-(n,l) \cup \widetilde{I}_+(n,l)) \times (2^l - \varepsilon, 2^l].
\end{aligned}$$

The construction of the counterexamples proceeds as in the previous proof in the case of the setting on the upper half 3-space described in Subsection 1.2.12. □

4.4.4 Outer L^p spaces on the upper half 3-space settings with dyadic and continuous trees

Let (X,ν,ω) be the setting on the upper half 3-space described in Subsection 1.2.12. We recall that the outer measure ν is generated via minimal coverings by the pre-measure τ on the collection $\mathcal{T}$ of dyadic trees, and each dyadic tree is denoted by $T(m,n,l)$ for certain $m,n,l \in \mathbb{Z}$. Moreover, let $\bar\nu$ be the outer measure on the upper half 3-space described in Subsection 1.2.13. We recall that the outer measure $\bar\nu$ is generated via minimal coverings by the pre-measure $\bar\tau$ on the collection $\bar{\mathcal{T}}$ of continuous trees, and each continuous tree is denoted by $T(x,\xi,s)$ for certain $x,\xi \in \mathbb{R}$, $s \in (0,\infty)$.

The outer measures ν and $\bar\nu$ are not equivalent. In fact, for every subset $A \subseteq X$, we have

$$\bar\nu(A) \leqslant \nu(A), \tag{4.4.17}$$

but, for every continuous tree $\bar T \in \bar{\mathcal{T}}$, we have

$$\bar\nu(\bar T) < \nu(\bar T) = \infty, \tag{4.4.18}$$

and we refer to Appendix 3.A in Chapter 3 for the details. Moreover, the single iterated outer L^p spaces associated with the outer measures ν and $\bar\nu$ are different, as stated in Theorem 4.4.2.

Before proving the theorem, we state and prove two auxiliary geometric observations about the following setting. We recall the definition of the dyadic tree $T(0,0,0) \in \mathcal{T}$ and the continuous tree $\bar T = \bar T(0,0,1) \in \bar{\mathcal{T}}$

$$\begin{aligned}
T(0,0,0) &= \Big\{(y,\eta,t) \in X \colon y \in (0,1], \eta \in (0,2^{-\lceil \log_2 t \rceil}], t \in (0,1]\Big\},\\
\bar T = \bar T(0,0,1) &= \Big\{(y,\eta,t) \in X \colon y \in (0,1], \eta \in (-t^{-1},t^{-1}], t \in (0,1]\Big\},
\end{aligned}$$

where, for every $x \in \mathbb{R}$, we define $\lceil x \rceil \in \mathbb{Z}$ to be the smallest integer number greater than or equal to x. Next, let

$$\begin{aligned}
\widetilde{U} &= (\bar{T} \cap (\mathbb{R} \times (0,\infty) \times (0,\infty))) \backslash T(0,0,0), \\
\widetilde{H}_j &= I(0,0) \times I(1,-j) \times (2^{j-1}, 2^j], \qquad \text{for every } j \in \mathbb{Z},\ j \leqslant 0, \\
\widetilde{U}_j &= \widetilde{U} \cap \widetilde{H}_j, \qquad \text{for every } j \in \mathbb{Z},\ j \leqslant 0.
\end{aligned}$$

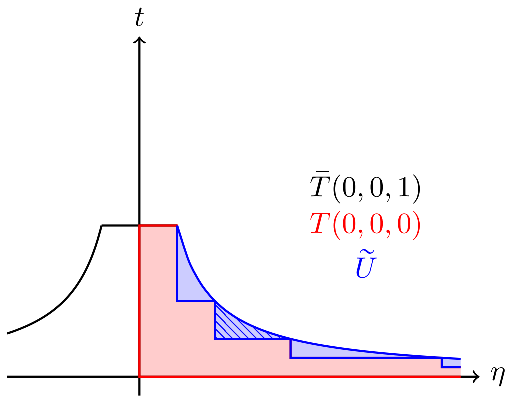

Figure 4.2: Continuous tree $\bar{T}(0,0,1)$, dyadic tree $T(0,0,0)$, and subset $\widetilde{U}$ in the upper half 3-space with coordinates (y,η,t) projected onto the upper half plane with coordinates (η,t). The area covered by the line pattern corresponds to subset $\widetilde{U}_{-1}$.

Lemma 4.4.12. *For every dyadic tree $T \in \mathcal{T}$, its intersection with $\widetilde{U}$ is either empty or contained in at most one set $\widetilde{H}_j$, and we have*

$$\omega(T \cap \widetilde{U}) \leqslant \nu(T). \tag{4.4.19}$$

Moreover, for every $j \in \mathbb{Z}$, $j \leqslant 0$, for the dyadic tree $T(0,1,j) \in \mathcal{T}$, we have $T(0,1,j) \cap \widetilde{U} \subseteq \widetilde{U}_j$ and

$$\omega(T(0,1,j) \cap \widetilde{U}) \geqslant \frac{\nu(T(0,1,j))}{12}. \tag{4.4.20}$$

Proof. For all $l, l' \in \mathbb{Z}$, we have

$$I(1,-l) \cap I(1,-l') = \varnothing,$$

therefore, for all $m, m', l, l' \in \mathbb{Z}$, we have

$$H(m,1,l) \cap T(m',1,l') = \varnothing.$$

Therefore, for all $M, N, L \in \mathbb{Z}$, there exists at most one value $l(L) \in \mathbb{Z}$ such that, for some $m \in \mathbb{Z}$, we have $H(m,1,l(L)) \subseteq T(M,N,L)$.

Next, we observe

$$\widetilde{U} \subseteq \bigcup_{l\in\mathbb{Z}, l\leqslant 0} \Big(\bigcup_{m\in\{0,\ldots,2^{-l}-1\}} H(m,1,l) \Big).$$

Every dyadic tree $T \in \mathcal{T}$ is of the form $T(M,N,L)$ for certain $M,N,L \in \mathbb{Z}$. We distinguish two cases.

Case I: $l(L) > 0$**.** We have $T(M,N,L) \cap \widetilde{U} = \varnothing$, and the desired inequality in (4.4.19) is trivially satisfied.

Case II: $l(L) \leqslant 0$**.** We have

$$\omega(T(M,N,L) \cap \widetilde{U}) \leqslant \omega\Big(\bigcup_{m\in\mathcal{M}} H(m,1,l(L)) \Big) \leqslant 2^L = \nu(T(M,N,L)),$$

where the subset $\mathcal{M} \subseteq \mathbb{Z}$ is defined by

$$\mathcal{M} = \{0, \ldots, 2^{-l(L)} - 1\} \cap \{2^{L-l(L)}M, \ldots, 2^{L-l(L)}(M+1) - 1\},$$

yielding the desired inequality in (4.4.19).

This concludes the proof of the first part of the statement, and we pass to the second. For every $j \in \mathbb{Z}$, $j \leqslant 0$, we have

$$T(0,1,j) \cap \widetilde{U} = H(0,1,j) \cap \widetilde{U} \subseteq \widetilde{U}_j,$$

and we have

$$\begin{aligned} \omega(T(0,1,j) \cap \widetilde{U}) &= \omega(H(0,1,j) \cap \widetilde{U}_j) \\ &\geqslant \omega\Big((0,2^j] \times \Big(2^{-j}, \frac{3}{2}2^{-j}\Big] \times \Big(2^{j-1}, \frac{2}{3}2^j\Big] \Big) \\ &\geqslant \frac{2^j}{12}, \end{aligned}$$

yielding the desired inequality in (4.4.20). $\square$

Lemma 4.4.13. *For every collection $\bar{\mathcal{T}}' \subseteq \bar{\mathcal{T}}$ of continuous trees such that*

$$\sum_{\bar{T}'\in\bar{\mathcal{T}}'} \bar{\tau}(\bar{T}') < 1, \tag{4.4.21}$$

there exists a continuous tree $\bar{V} \in \bar{\mathcal{T}}$ contained in $\bar{T}$ and pairwise disjoint from every element of $\bar{\mathcal{T}}'$.

Proof. By the inequality in (4.4.21), we have

$$\Big| \pi\Big(\bigcup_{\bar{T}'\in\bar{\mathcal{T}}'} \bar{T}' \Big) \Big| \leqslant \sum_{\bar{T}'\in\bar{\mathcal{T}}'} |\pi(\bar{T}')| = \sum_{\bar{T}'\in\bar{\mathcal{T}}'} \bar{\tau}(\bar{T}') < 1 = |(0,1]|,$$

where $\pi\colon X \to \mathbb{R}$ is the projection onto the first coordinate. Therefore, there exists an interval $J = (x_J, x_J + |J|]$ contained in $(0,1]$ and pairwise disjoint from every interval in the collection $\{\pi(\bar{T}')\colon \bar{T}' \subseteq \bar{\mathcal{T}}'\}$. Then $\bar{V} = \bar{T}(x_J, 0, |J|) \in \bar{\mathcal{T}}$ is a continuous tree satisfying the desired properties. □

Proof of Theorem 4.4.2. Without loss of generality, we assume $r = 1$, since, for every setting (X, μ, ω), we have

$$\|f\|^r_{L^p_\mu(\ell^r_\omega)} = \|f^r\|_{L^{\frac{p}{r}}_\mu(\ell^1_\omega)}.$$

In particular, we have $p > 1$.

The embeddings between single iterated outer L^p spaces follow from Lemma 4.2.4 and the domination of outer measures stated in (4.4.17). To prove that the single iterated outer L^p spaces are different, for every $p \in (1, \infty]$, we define the measurable function u on X by

$$u = \sum_{j=0}^{\infty} (j+1)^{-\frac{p+1}{2p}} 1_{\widetilde{U}_{-j}},$$

where the exponent for $p = \infty$ is understood to be $-\frac{1}{2}$. We claim that, for every $p \in (1, \infty]$, we have

$$u \in L^1_{\bar{\nu}}(\ell^p_\omega) \backslash L^1_\nu(\ell^p_\omega), \qquad u \in L^p_\nu(\ell^1_\omega) \backslash L^p_{\bar{\nu}}(\ell^1_\omega).$$

Case I: $u \in L^1_{\bar{\nu}}(\ell^p_\omega) \backslash L^1_\nu(\ell^p_\omega)$. By outer Hölder's inequality (Theorem 1.1.7) and collapsing of exponents, property (i) in Theorem 4.4.1, for the setting $(X, \bar{\nu}, \omega)$, there exists a constant $C = C(p) \geqslant 1$ such that

$$\|u\|_{L^1_{\bar{\nu}}(\ell^p_\omega)} \leqslant C \|1_{\widetilde{U}}\|_{L^{p'}_{\bar{\nu}}(\ell^\infty_\omega)} \|u\|_{L^p_{\bar{\nu}}(\ell^p_\omega)} \leqslant C^2 \bar{\nu}(\bar{T})^{\frac{1}{p'}} \|u\|_{L^p(X,\omega)} < \infty.$$

Moreover, there exists a constant $c = c(p) \leqslant 1$ such that, for every $j \in \mathbb{Z}$, $j \leqslant 0$, we have

$$\begin{aligned} \ell^p_{\omega,\tau}(u)(T(0,1,j)) &= \tau(T(0,1,j))^{-\frac{1}{p}} \|u 1_{T(0,1,j)}\|_{L^p(X,\omega)} \\ &= (j+1)^{-\frac{p+1}{2p}} \tau(T(0,1,j))^{-\frac{1}{p}} \omega(T(0,1,j) \cap \widetilde{U})^{\frac{1}{p}} \\ &\geqslant c (j+1)^{-\frac{p+1}{2p}} \tau(T(0,1,j))^{-1} \omega(T(0,1,j) \cap \widetilde{U}) \\ &= c \ell^1_{\omega,\tau}(u)(T(0,1,j)), \end{aligned} \tag{4.4.22}$$

where we refer to Remark 1.2.1 for the notation $\ell^r_{\omega,\tau}$, and we used the inequality in (4.4.20) and the observation that u is constant on the subset $\widetilde{H}_j$ for every $j \in \mathbb{Z}$, $j \leqslant 0$, in the second.

Therefore, there exists a constant $c = c(p) \leqslant 1$ such that

$$\|u\|_{L^1_\nu(\ell^p_\omega)} = \|u\|_{L^1_\nu(\ell^p_{\omega,\tau})} \geqslant c \|u\|_{L^1_\nu(\ell^1_{\omega,\tau})} = c \|u\|_{L^1_\nu(\ell^1_\omega)} \geqslant c^2 \|u\|_{L^1(X,\omega)} = \infty,$$

where we used Lemma 2.A.3 in Chapter 2 in the first and the second equality, the inequality in (4.4.22) and the definition of the outer L^p quasi-norms in Definition 1.1.6 in the first inequality, and collapsing of exponents, property (i) in Theorem 4.4.1, for the setting (X, ν, ω) in the second inequality.

Case II: $u \in L^p_\nu(\ell^1_\omega) \backslash L^p_{\bar{\nu}}(\ell^1_\omega)$**.** For every subset $A \subseteq X$ such that $\bar{\nu}(A) < 1$, there exists a collection $\bar{\mathcal{A}} \subseteq \bar{\mathcal{T}}$ of continuous trees such that

$$A \subseteq \bigcup_{\bar{T}' \in \bar{\mathcal{A}}} \bar{T}', \qquad \sum_{\bar{T}' \in \bar{\mathcal{A}}} \bar{\tau}(\bar{T}') < 1.$$

By Lemma 4.4.13, there exists a continuous tree $\bar{V} = \bar{V}(\bar{\mathcal{A}}) \in \bar{\mathcal{T}}$ pairwise disjoint from every element of $\bar{\mathcal{A}}$. Then, we have

$$\|u 1_{A^c}\|_{L^\infty_{\bar{\nu}}(\ell^1_\omega)} \geqslant \ell^1_\omega(u)(\bar{V}) = \infty.$$

Therefore, for every $\lambda \in (0, \infty)$, we have

$$\bar{\nu}(\ell^1_\omega(u) > \lambda) = 1,$$

hence

$$\|u\|_{L^p_{\bar{\nu}}(\ell^1_\omega)} = \infty.$$

Moreover, by the same argument used to prove the last chain of inequalities in **Case I**, there exists a constant $C = C(p) \geqslant 1$ such that

$$\|u\|_{L^p_\nu(\ell^1_\omega)} = \|u\|_{L^p_\nu(\ell^1_{\omega,\tau})} \leqslant C \|u\|_{L^p_\nu(\ell^p_{\omega,\tau})} = C \|u\|_{L^p_\nu(\ell^p_\omega)} \leqslant C^2 \|u\|_{L^p(X,\omega)} < \infty,$$

and we refer to Remark 1.2.1 for the notation $\ell^r_{\omega,\tau}$. □

Remark 4.4.14. *Consider the following variant of the setting (X, ν, ω) on the upper half 3-space described in Subsection 1.2.12. For all $n, l, l' \in \mathbb{Z}$, $l' \leqslant l$, we define $N(n, l') \in \mathbb{Z}$ by the condition*

$$I(n, -l) \subseteq I(N(n, l'), -l'),$$

and, for every $x \in \mathbb{R}$, we define $\lfloor x \rfloor \in \mathbb{Z}$ to be the biggest integer number smaller than or equal to x.

For all $m, n, l \in \mathbb{Z}$, we define the dyadic tree of bitiles $T_2(m, n, l)$ *in the upper half 3-space $\mathbb{R}^2 \times (0, \infty)$ by*

$$\begin{aligned} T_2(m, n, l) &= T_2(I(m, l), I(n, -l)) = T_2(H(m, n, l)) \\ &\coloneqq \bigcup_{l' \leqslant l} \Big(I(m, l) \times I\Big(\Big\lfloor \frac{N(n, l')}{2} \Big\rfloor, 1 - l'\Big) \times (0, 2^{l'}] \Big). \end{aligned}$$

Next, let

$$\begin{aligned} X &= \mathbb{R} \times \mathbb{R} \times (0,\infty), \\ \mathcal{T}_2 &= \Big\{ T_2(m,n,l) \colon m,n,l \in \mathbb{Z} \Big\}, \\ \tau_2(T_2(m,n,l)) &= 2^l, && \text{for all } m,n,l \in \mathbb{Z}, \\ \mathrm{d}\omega(y,\eta,t) &= \mathrm{d}y\,\mathrm{d}\eta\,\mathrm{d}t, && \text{for all } y,\eta \in \mathbb{R},\ t \in (0,\infty), \end{aligned}$$

and let ν_2 be the outer measure generated via minimal coverings as in (1.1.1) *by the pre-measure τ_2 on the collection $\mathcal{T}_2$ of dyadic trees of bitiles.*

The outer measures ν_2 and $\bar{\nu}$ are not equivalent. In fact, we can prove that, for every subset $A \subseteq X$, we have

$$\nu_2(A) \leqslant 6\bar{\nu}(A),$$

but, for every dyadic tree of bitiles $T_2 \in \mathcal{T}_2$, we have

$$\nu_2(T_2) < \bar{\nu}(T_2) = \infty.$$

Moreover, the single iterated outer L^p spaces associated with the outer measures ν_2 and $\bar{\nu}$ are different, and we can prove a result analogous to Theorem 4.4.2.

Theorem 4.4.15. *For all $p, r \in (0,\infty]$, $p > r$, we have*

$$\begin{aligned} L^r_{\bar{\nu}}(\ell^p_\omega) &\hookrightarrow L^r_{\nu_2}(\ell^p_\omega) && L^r_{\bar{\nu}}(\ell^p_\omega) \neq L^r_{\nu_2}(\ell^p_\omega), \\ L^p_{\nu_2}(\ell^r_\omega) &\hookrightarrow L^p_{\bar{\nu}}(\ell^r_\omega) && L^p_{\nu_2}(\ell^r_\omega) \neq L^p_{\bar{\nu}}(\ell^r_\omega). \end{aligned}$$

4.5 Outer L^p spaces with respect to a size with variable exponent on the upper half 3-space

In this section, we study the Banach space properties of the outer L^p spaces with respect to the size S on the upper half 3-space appearing in the article of Do and Thiele [DT15]. The size S is of the form of a sum of sizes ℓ^∞_ω and ℓ^2_ω restricted to certain subsets of each tree in the upper half 3-space. In Subsection 4.5.1, we recall the definition of S in details in the case of the settings on the upper half 3-space described in Subsections 1.2.12 – 1.2.13. In particular, we prove that the outer L^p quasi-norms with respect to the size S appearing in [DT15] do not satisfy a result analogous to Köthe duality for an appropriate dual size, see Lemma 4.5.2.

In Subsection 4.5.2, we start our analysis by the Banach space properties of the outer L^p spaces with respect to a single size ℓ^r_ω restricted to a certain subset of each tree in the upper half 3-space. We comment on why we cannot replicate the same argument we used in the cases studied in the previous section, where the size ℓ^r_ω is not restricted to certain subsets of each tree. After that, in Subsection 4.5.3, we pass to the case of the size with variable exponent appearing in [DT15].

Throughout this section, we focus on the case of the setting on the upper half 3-space described in Subsection 1.2.12, where the outer measure is generated via minimal coverings by the collection of dyadic trees. With the appropriate modifications in the spirit of those appearing in Subsection 4.4.3, we can adapt every argument we use to the case of the setting described in Subsection 1.2.13, where the outer measure is generated via minimal coverings by the collection of continuous trees.

In Subsection 4.5.4, we conclude the section by studying the case of the setting on the collection of Heisenberg upper half dyadic tiles described in Subsection 1.2.11. First, we define the size S_2 analogous to that appearing in [DT15], namely of the form of a sum of sizes ℓ^∞_ω and ℓ^2_ω restricted to certain subsets of each tree. Next, we prove that, contrary to the case of the settings on the upper half 3-space, the outer L^p spaces with respect to S_2 are equivalent to those with respect to just the ℓ^2_ω part of it.

4.5.1 Sizes on the inner and outer parts of trees, and sizes with variable exponent

Let (X, ν, ω) be the setting on the upper half 3-space described in Subsection 1.2.12. We recall that the outer measure ν is generated via minimal coverings by the pre-measure τ on the collection $\mathcal{T}$ of dyadic trees, and each dyadic tree is denoted by $T(m, n, l)$ for certain $m, n, l \in \mathbb{Z}$. We define the collection $\mathcal{T}_2$ of dyadic trees by

$$\mathcal{T}_2 := \Big\{T, \widetilde{T} \colon T, \widetilde{T} \in \mathcal{T}\Big\},$$

namely $\mathcal{T}_2$ contains two copies of every dyadic tree in $\mathcal{T}$.

For every Heisenberg upper half dyadic tile $H(m, n, l) \in \mathcal{H}$, we define its *lower and upper children* $H_{\text{low}}(m, n, l)$ and $H_{\text{upp}}(m, n, l)$ by

$$\begin{aligned} H_{\text{low}}(m,n,l) &= H_{\text{low}}(I(m,l), I(n,-l)) := I(m,l) \times I(2n,-l-1) \times (2^{l-1}, 2^l], \\ H_{\text{upp}}(m,n,l) &= H_{\text{upp}}(I(m,l), I(n,-l)) := H(m,n,l) \backslash H_{\text{low}}(m,n,l). \end{aligned}$$

For every dyadic tree $T(m, n, l) \in \mathcal{T}_2$, we define its *inner and outer parts* $T_{\text{inn}}(m, n, l)$ and $T_{\text{out}}(m, n, l)$ by

$$\begin{aligned} T_{\text{inn}}(m,n,l) &:= H_{\text{low}}(m,n,l) \cup \bigcup_{l' < l} \Big(\bigcup_{m'=2^{l-l'}m}^{2^{l-l'}(m+1)-1} H_*(m', N(n,l'), l') \Big), \\ T_{\text{out}}(m,n,l) &:= T(m,n,l) \backslash T_{\text{inn}}(m,n,l), \end{aligned}$$

and for every dyadic tree $\widetilde{T}(m,n,l) \in \mathcal{T}_2$, we define its *inner and outer parts* $\widetilde{T}(m,n,l)_{\text{inn}}$ and $\widetilde{T}(m,n,l)_{\text{out}}$ by

$$\begin{aligned}
\widetilde{T}_{\text{inn}}(m,n,l) &\coloneqq H_{\text{upp}}(m,n,l) \cup \bigcup_{l'<l} \Big(\bigcup_{m'=2^{l-l'}m}^{2^{l-l'}(m+1)-1} H_*(m', N(n,l'), l') \Big), \\
\widetilde{T}_{\text{out}}(m,n,l) &\coloneqq \widetilde{T}(m,n,l) \backslash \widetilde{T}_{\text{inn}}(m,n,l),
\end{aligned}$$

where, for every $l' \in \mathbb{Z}$, $l' \leqslant l$, we define $N(n,l') \in \mathbb{Z}$ by the condition

$$I(n,-l) \subseteq I(N(n,l'),-l'),$$

and

$$H_* = \begin{cases} H_{\text{low}}, & \text{if } 2N(n,l') = N(n,l'+1), \\ H_{\text{upp}}, & \text{if } 2N(n,l')+1 = N(n,l'+1). \end{cases}$$

For every $r \in (0,\infty]$, we define the sizes $(\ell^r_{\omega,\text{inn}}, \mathcal{T}_2)$ and $(\ell^r_{\omega,\text{out}}, \mathcal{T}_2)$. For every dyadic tree $T \in \mathcal{T}_2$, for every measurable function f on X, we define

$$\begin{aligned}
\ell^r_{\omega,\text{inn}}(f)(T) &\coloneqq \nu(T)^{-\frac{1}{r}} \|f 1_{T_{\text{inn}}}\|_{L^r(X,\omega)}, \\
\ell^r_{\omega,\text{out}}(f)(T) &\coloneqq \nu(T)^{-\frac{1}{r}} \|f 1_{T_{\text{out}}}\|_{L^r(X,\omega)},
\end{aligned} \tag{4.5.1}$$

where the exponent ∞^{-1} is understood to be 0. Moreover, we recall the definition of two variants $(S, \mathcal{T}_2), (\widetilde{S}, \mathcal{T}_2)$ of the size appearing in [DT15]. For every dyadic tree $T \in \mathcal{T}_2$, for every measurable function f on X, we define

$$\begin{aligned}
S(f)(T) &\coloneqq \ell^\infty_{\omega,\text{inn}}(f)(T) + \ell^2_{\omega,\text{out}}(f)(T), \\
\widetilde{S}(f)(T) &\coloneqq \ell^\infty_{\omega}(f)(T) + \ell^2_{\omega,\text{out}}(f)(T).
\end{aligned} \tag{4.5.2}$$

For every $p \in (0,\infty]$, we define the outer L^p quasi-norms and spaces with respect to each of the sizes appearing in the previous two displays as in Definition 1.1.4 and Definition 1.1.6.

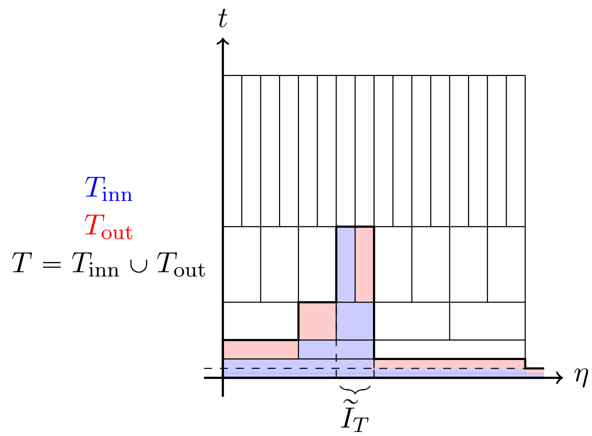

Figure 4.3: Inner and outer parts T_{inn}, T_{out} of a dyadic tree $T \in \mathcal{T}_2$ in the upper half 3-space with coordinates (y, η, t) projected onto the upper half plane with coordinates (η, t).

Next, let $(X, \bar{\nu}, \omega)$ be the setting on the upper half 3-space described in Subsection 1.2.13. We recall that the outer measure $\bar{\nu}$ is generated via minimal coverings by the pre-measure $\bar{\tau}$ on the collection $\bar{\mathcal{T}}$ of continuous trees, and each continuous tree is denoted by $\bar{T}(x, \xi, s)$ for certain $x, \xi \in \mathbb{R}$, $s \in (0, \infty)$.

For every continuous tree $\bar{T} = \bar{T}(x, \xi, s) \in \bar{\mathcal{T}}$, we define its *inner and outer parts* $\bar{T}_{\text{inn}}$ and $\bar{T}_{\text{out}}$ by

$$\bar{T}_{\text{inn}} := \bigcup_{s' \leqslant s} \Big((x, x+s] \times \Big(\xi - \frac{1}{2s'}, \xi + \frac{1}{2s'} \Big] \times (0, s'] \Big),$$
$$\bar{T}_{\text{out}} := \bar{T} \backslash \bar{T}_{\text{inn}}.$$

For every $r \in (0, \infty]$, we define the sizes $(\ell^r_{\omega,\text{inn}}, \bar{\mathcal{T}})$, $(\ell^r_{\omega,\text{out}}, \bar{\mathcal{T}})$, $(S, \bar{\mathcal{T}})$, and $(\widetilde{S}, \bar{\mathcal{T}})$ analogously to those in (4.5.1) and (4.5.2), replacing the dyadic trees in $\mathcal{T}_2$ with the continuous trees in $\bar{\mathcal{T}}$. For every $p \in (0, \infty]$, we define the outer L^p quasi-norms and spaces with respect to these sizes as in Definition 1.1.4 and Definition 1.1.6.

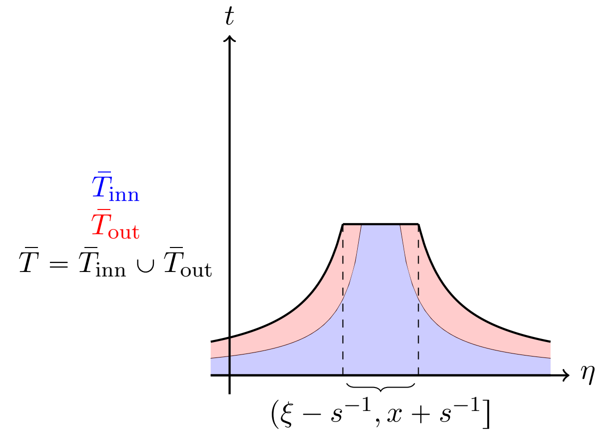

Figure 4.4: Inner and outer parts $\bar{T}_{\mathrm{inn}}$, $\bar{T}_{\mathrm{out}}$ of a continuous tree $\bar{T} \in \bar{\mathcal{T}}$ in the upper half 3-space with coordinates (y, η, t) projected onto the upper half plane with coordinates (η, t).

4.5.2 Outer L^p spaces with respect to the sizes in (4.5.1)

In this subsection, we consider the Banach space properties of the outer L^p spaces with respect to a size ℓ^r_ω restricted to a certain subset of each dyadic tree. We comment on why proving positive results about them is more complicated than in the cases studied in the previous section. In the proof of Theorem 4.4.1, we showed the quasi-triangle inequality for the single iterated outer L^p spaces on the upper half 3-space as a corollary of Köthe duality for them and the triangle inequality for the classical $L^1(X, \omega)$ space. Following the same argument, we would start by proving a version of Köthe duality for the outer $L^p_\nu(\ell^r_{\omega,\mathrm{inn}})$ spaces. In particular, we would show that there exists a constant $C = C(p, r)$ such that, for every measurable function $f \in L^p_\nu(\ell^r_{\omega,\mathrm{inn}})$ on X, we have

$$C^{-1} \|f\|_{L^p_\nu(\ell^r_{\omega,\mathrm{inn}})} \leqslant \sup \Big\{ \|fg\|_{L^1(\ell^1_{\omega,\mathrm{inn}})} \colon \|g\|_{L^{p'}_\nu(\ell^{r'}_{\omega,\mathrm{inn}})} = 1 \Big\} \leqslant C \|f\|_{L^p_\nu(\ell^r_{\omega,\mathrm{inn}})}.$$

We point out that we substituted the classical $L^1(X, \omega)$ norm with the outer $L^1_\nu(\ell^1_{\omega,\mathrm{inn}})$ quasi-norm to measure the product of the functions. This substitution is dictated by the outer Hölder inequality we would apply to obtain the second inequality in the previous display. However, the outer $L^1_\nu(\ell^1_{\omega,\mathrm{inn}})$ space does not satisfy the quasi-triangle inequality for countably many summands, as exhibited in the following result. In fact, we can comment in an analogous way also in the case of the outer $L^p_\nu(\ell^r_{\omega,\mathrm{out}})$ spaces.

Lemma 4.5.1. *For every $M > 0$, there exists a collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\nu(\ell^1_{\omega,\mathrm{inn}})$ of measurable functions on X such that*

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^1_\nu(\ell^1_{\omega,\mathrm{inn}})} \geqslant M \sum_{n \in \mathbb{N}} \|f_n\|_{L^1_\nu(\ell^1_{\omega,\mathrm{inn}})}.$$

Replacing $\ell^1_{\omega,\mathrm{inn}}$ *with* $\ell^1_{\omega,\mathrm{out}}$*, we have the same statement for the outer* $L^1_\nu(\ell^1_{\omega,\mathrm{out}})$ *spaces.*

Proof. Fix $j \in \mathbb{N}$. For all $n, l \in \mathbb{Z}$, $0 \leqslant l \leqslant j$, we define $\widetilde{N}(n,l) \in \mathbb{Z}$ by the condition

$$I(n,-j) \subseteq I(\widetilde{N}(n,l),-l).$$

Moreover, for all $n, l \in \mathbb{Z}$, $0 \leqslant l \leqslant j$, we refer to (4.4.15) for the definition of the measurable subset $\widetilde{H}(n,l)$. Next, for every $n' \in \mathbb{Z}$, $0 \leqslant n' < 2^j$, we define the measurable subset $E(n'), \widetilde{E}(n') \subseteq X$ by

$$E(n') = \bigcup_{l=0}^{j} \widetilde{H}(\widetilde{N}(n',l),l) \cap T_{\mathrm{out}}(0,n',j),$$
$$\widetilde{E}(n') = \bigcup_{l=0}^{j} \widetilde{H}(\widetilde{N}(n',l),l) \cap \widetilde{T}_{\mathrm{out}}(0,n',j).$$

Next, we define the measurable functions $f_{n'}$, $\widetilde{f}_{n'}$, and f on X by

$$f_{n'}(x,\xi,s) = 1_{E(n')}(x,\xi,s),$$
$$\widetilde{f}_{n'}(x,\xi,s) = 1_{\widetilde{E}(n')}(x,\xi,s),$$
$$f(x,\xi,s) = \sum_{n'=0}^{2^j-1} \Big(f_{n'}(x,\xi,s) + \widetilde{f}_{n'}(x,\xi,s) \Big) = \sum_{l=0}^{j} \sum_{n=0}^{2^l-1} 2^{j-l} 1_{\widetilde{H}(n,l)}(x,\xi,s).$$

There exist constants c and C such that

$$\|f\|_{L^1_\nu(\ell^1_{\omega,\mathrm{inn}})} \geqslant c 2^{2j}(j+1),$$
$$\sum_{n'=0}^{2^j-1} \|f_{n'}\|_{L^1_\nu(\ell^1_{\omega,\mathrm{inn}})} = \sum_{n'=0}^{2^j-1} \|\widetilde{f}_{n'}\|_{L^1_\nu(\ell^1_{\omega,\mathrm{inn}})} = 2^j 2^j.$$

Taking $j \in \mathbb{N}$ big enough, we obtain the desired inequality. □

4.5.3 Outer L^p spaces with respect to sizes in (4.5.2)

We exhibit a counterexample to Köthe duality for the outer $L^p_\nu(S)$ and $L^p_\nu(\widetilde{S})$ spaces with $p \in [1,\infty)$.

First, we define the auxiliary size $(S', \mathcal{T}_2)$ as follows. For every dyadic tree $T \in \mathcal{T}_2$, for every measurable function f on X,

$$S'(f)(T) \coloneqq \ell^1_{\omega,\mathrm{inn}}(f)(T) + \ell^2_{\omega,\mathrm{out}}(f)(T).$$

Therefore, for every dyadic tree $T \in \mathcal{T}_2$, we have

$$\ell^1_\omega(fg)(T) \sim \sup\Big\{S(f)(T)S'(g)(T) \colon S'(g)(T) = 1\Big\}.$$

By the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) for the setting (X, ν, ω) and outer Hölder's inequality (Theorem 1.1.7) for the sizes S and S', we have

$$\|fg\|_{L^1(X,\omega)} \leqslant C\|fg\|_{L^1_\nu(\ell^1_\omega)} \leqslant C\|f\|_{L^p_\nu(S)}\|g\|_{L^{p'}_\nu(S')}.$$

Next, we define the auxiliary size $(\widetilde{S}', \mathcal{T}_2)$ as follows. For every dyadic tree $T \in \mathcal{T}_2$, for every measurable function f on X,

$$\widetilde{S}'(f)(T) \coloneqq \ell^1_\omega(f)(T) + \ell^2_{\omega,\mathrm{out}}(f)(T).$$

Therefore, for every dyadic tree $T \in \mathcal{T}_2$, for every measurable function g on X, we have

$$\ell^1_\omega(fg)(T) \leqslant \widetilde{S}(f)(T)\widetilde{S}'(g)(T).$$

By the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) for the setting (X, ν, ω) and outer Hölder's inequality (Theorem 1.1.7) for the sizes $\widetilde{S}$ and $\widetilde{S}'$, we have

$$\|fg\|_{L^1(X,\omega)} \leqslant C\|fg\|_{L^1_\nu(\ell^1_\omega)} \leqslant C\|f\|_{L^p_\nu(\widetilde{S})}\|g\|_{L^{p'}_\nu(\widetilde{S}')}.$$

Lemma 4.5.2. *Let $p \in [1, \infty]$. For every $M > 0$, there exists a measurable function $f \in L^p_\nu(S)$ on X such that*

$$\|f\|_{L^p_\nu(S)} \geqslant M \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\nu(S')} = 1\Big\}.$$

Replacing S with $\widetilde{S}$, we have the same statement for the outer $L^p_\nu(\widetilde{S})$ spaces.

Proof. For every $\varepsilon \in (0, 1]$, we define the measurable subset $A_\varepsilon \subseteq X$ by

$$A_\varepsilon = \Big(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\Big] \times \Big(\frac{3-\varepsilon}{4}, \frac{3+\varepsilon}{4}\Big] \times \Big(\frac{3-\varepsilon}{4}, \frac{3+\varepsilon}{4}\Big].$$

In particular, we have

$$A_\varepsilon \subseteq H_{\mathrm{upp}}(0,0,0) \subseteq H(0,0,0) \in \mathcal{H}.$$

We have

$$\|1_{A_\varepsilon}\|_{L^p_\nu(S)} = \|1_{A_\varepsilon}\|_{L^p_\nu(\widetilde{S})} = \|1_{A_\varepsilon}\|_{L^p_\nu(\ell^\infty_{\omega,\mathrm{inn}})} = \nu(A_\varepsilon)^{\frac{1}{p}} = 1.$$

Moreover, for every measurable function g on X, we have

$$\|1_{A_\varepsilon} g\|_{L^1(X,\omega)} = \|1_{A_\varepsilon} g\|_{L^1(X,\omega)} \leqslant \omega(A_\varepsilon)^{\frac{1}{2}}\|1_{A_\varepsilon} g\|_{L^2(X,\omega)} = \frac{\varepsilon\sqrt{\varepsilon}}{2}\|1_{A_\varepsilon} g\|_{L^2(X,\omega)},$$
$$\|g\|_{L^{p'}_\nu(S')} = \|g\|_{L^{p'}_\nu(\widetilde{S}')} \geqslant \ell^2_{\omega,\mathrm{out}}(1_{A_\varepsilon} g)(T(0,0,0)) = \|1_{A_\varepsilon} g\|_{L^2(X,\omega)}.$$

Taking $\varepsilon \in (0, 1]$ small enough, we obtain the desired inequality. □

In fact, we have the following equivalence between outer L^p quasi-norms with respect to the sizes with variable exponent defined in (4.5.2).

Lemma 4.5.3. *For every $p \in (0, \infty]$, there exists a constant $C = C(p)$ such that, for every measurable function f on X, we have*

$$\|f\|_{L^p_\nu(S)} \leqslant \|f\|_{L^p_\nu(\widetilde{S})} \leqslant C\|f\|_{L^p_\nu(S)}.$$

Proof. For every tree $T \in \mathcal{T}_2$, for every function f on X, we have

$$S(f)(T) \leqslant \widetilde{S}(f)(T).$$

Moreover, for every function f on X, we have

$$\begin{aligned}\|f\|_{L^\infty_\nu(\widetilde{S})} &\leqslant \|f\|_{L^\infty_\nu(\ell^2_{\omega,\mathrm{out}})} + \|f\|_{L^\infty_\nu(\ell^\infty_\omega)} \\ &= \|f\|_{L^\infty_\nu(\ell^2_{\omega,\mathrm{out}})} + \|f\|_{L^\infty(X,\omega)} \\ &\leqslant 2\|f\|_{L^\infty_\nu(S)},\end{aligned}$$

where we used the fact that, for all $m, n, l \in \mathbb{Z}$, we have

$$\begin{aligned}\|f 1_{H(m,n,l)}\|_{L^\infty(X,\omega)} &\leqslant \|f 1_{H_{\mathrm{low}}(m,n,l)}\|_{L^\infty(X,\omega)} + \|f 1_{H_{\mathrm{upp}}(m,n,l)}\|_{L^\infty(X,\omega)} \\ &\leqslant \ell^\infty_{\omega,\mathrm{inn}}(f)(T(m,n,l)) + \ell^\infty_{\omega,\mathrm{inn}}(f)(\widetilde{T}(m,n,l)),\end{aligned}$$

where $T(m,n,l), \widetilde{T}(m,n,l) \in \mathcal{T}_2$. Together with the definition of the outer L^p quasi-norms in Definition 1.1.4 and Definition 1.1.6, the previous two displays yield the desired chain of inequalities. □

4.5.4 Sizes with variable exponent on the discrete model of the upper half 3-space

We start by defining a variant of the setting (X, ν, ω) on the collection of Heisenberg upper half dyadic tiles described in Subsection 1.2.11. For all $n, l, l' \in \mathbb{Z}$, $l' \leqslant l$, we define $N(n, l') \in \mathbb{Z}$ by the condition

$$I(n, -l) \subseteq I(N(n, l'), -l'),$$

and, for every $x \in \mathbb{R}$, we define $\lfloor x \rfloor \in \mathbb{Z}$ to be the biggest integer number smaller than or equal to x.

For all $m, n, l \in \mathbb{Z}$, we recall the definition of the subset $T(m,n,l) \subseteq \mathcal{H}$ by

$$\begin{aligned}T(m,n,l) &= T(I(m,l), I(n,-l)) = T(H(m,n,l)) \\ &\coloneqq \Big\{H \in \mathcal{H} \colon H \subseteq \bigcup_{l' \leqslant l} \Big(I(m,l) \times I(N(n,l'),-l') \times (0, 2^{l'}]\Big)\Big\},\end{aligned}$$

and we define the subset $T_2(m,n,l) \subseteq \mathcal{H}$ by

$$\begin{aligned} T_2(m,n,l) &= T_2(I(m,l), I(n,-l)) = T_2(H(m,n,l)) \\ &\coloneqq \Big\{ H \in \mathcal{H} \colon H \subseteq \bigcup_{l' \leqslant l} \Big(I(m,l) \times I\Big(\Big\lfloor \frac{N(n,l')}{2} \Big\rfloor, 1-l'\Big) \times (0, 2^{l'}] \Big) \Big\}. \end{aligned}$$

Next, let

$$\begin{aligned} X &= \mathcal{H}, \\ \mathcal{T}_2 &= \Big\{ T_2(m,n,l) \colon m,n,l \in \mathbb{Z} \Big\}, \\ \tau_2(T_2(m,n,l)) &= 2^l, && \text{for all } m,n,l \in \mathbb{Z}, \\ \omega(H(m,n,l)) &= 2^l, && \text{for all } m,n,l \in \mathbb{Z}, \end{aligned}$$

and let ν_2 be the outer measure generated via minimal coverings as in (1.1.1) by the pre-measure τ_2 on the collection $\mathcal{T}_2$ of subsets of X.

For every subset $T_2 = T_2(m,n,l) \in \mathcal{T}_2$, we define its *inner and outer parts* $T_{2,\mathrm{inn}}$ and $T_{2,\mathrm{out}}$ by

$$T_{2,\mathrm{inn}} \coloneqq T(m,n,l), \qquad T_{2,\mathrm{out}} \coloneqq T_2 \backslash T_{2,\mathrm{inn}}.$$

For every $r \in (0,\infty]$, we define the sizes $(\ell^r_{2,\omega,\mathrm{inn}}, \mathcal{T}_2)$ and $(\ell^r_{2,\omega,\mathrm{out}}, \mathcal{T}_2)$. For every subset $T_2 = T_2(m,n,l) \in \mathcal{T}_2$, for every function f on X, we define

$$\begin{aligned} \ell^r_{2,\omega,\mathrm{inn}}(f)(T_2) &\coloneqq \nu_2(T_2)^{-\frac{1}{r}} \| f 1_{T_{2,\mathrm{inn}}} \|_{L^r(X,\omega)}, \\ \ell^r_{2,\omega,\mathrm{out}}(f)(T_2) &\coloneqq \nu_2(T_2)^{-\frac{1}{r}} \| f 1_{T_{2,\mathrm{out}}} \|_{L^r(X,\omega)}, \end{aligned} \tag{4.5.3}$$

where the exponent ∞^{-1} is understood to be 0. Moreover, we define two additional sizes $(S_2, \mathcal{T}_2), (\widetilde{S}_2, \mathcal{T}_2)$ analogous to those appearing in [DT15]. For every subset $T_2 \in \mathcal{T}_2$, for every function f on X, we define

$$\begin{aligned} S_2(f)(T_2) &\coloneqq \ell^\infty_{2,\omega,\mathrm{inn}}(f)(T_2) + \ell^2_{2,\omega,\mathrm{out}}(f)(T_2), \\ \widetilde{S}_2(f)(T_2) &\coloneqq \ell^\infty_{2,\omega}(f)(T_2) + \ell^2_{2,\omega,\mathrm{out}}(f)(T_2). \end{aligned}$$

For every $p \in (0,\infty]$, we define the outer L^p quasi-norms and spaces with respect to the sizes appearing in the previous two displays as in Definition 1.1.4 and Definition 1.1.6.

On the setting (X, ν_2, ω), we have the following equivalence between outer L^p quasi-norms with respect to the different sizes $\ell^2_{2,\omega,\mathrm{out}}$, S_2, and $\widetilde{S}_2$.

Lemma 4.5.4. *For every $p \in (0,\infty]$, there exists a constant $C = C(p)$ such that, for every function f on X, we have*

$$\|f\|_{L^p_{\nu_2}(\ell^2_{2,\omega,\mathrm{out}})} \leqslant \|f\|_{L^p_{\nu_2}(S_2)} \leqslant \|f\|_{L^p_{\nu_2}(\widetilde{S}_2)} \leqslant C \|f\|_{L^p_{\nu_2}(\ell^2_{2,\omega,\mathrm{out}})}.$$

Proof. For every subset $T_2 \in \mathcal{T}_2$, for every function f on X, we have

$$\ell^2_{2,\omega,\mathrm{out}}(f)(T_2) \leqslant S_2(f)(T_2) \leqslant \widetilde{S}_2(f)(T_2).$$

Moreover, for every function f on X, we have

$$\begin{aligned}\|f\|_{L^\infty_{\nu_2}(\widetilde{S}_2)} &\leqslant \|f\|_{L^\infty_{\nu_2}(\ell^2_{2,\omega,\mathrm{out}})} + \|f\|_{L^\infty_{\nu_2}(\ell^\infty_{2,\omega})} \\ &= \|f\|_{L^\infty_{\nu_2}(\ell^2_{2,\omega,\mathrm{out}})} + \|f\|_{L^\infty(X,\omega)} \\ &\leqslant 2\|f\|_{L^\infty_{\nu_2}(\ell^2_{2,\omega,\mathrm{out}})},\end{aligned}$$

where we used the fact that, for every $H \in X$, we have

$$|f(H)| = \nu_2(T_2(H))^{-\frac{1}{2}}\|f 1_H\|_{L^2(X,\omega)} \leqslant \ell^2_{2,\omega,\mathrm{out}}(f)(T_2(H')).$$

In the previous display, for every $H = H(m,n,l) \in X$, we define $H' \in X$ by

$$H' = H(m, n_*, l),$$

and

$$n_* = \begin{cases} n+1, & \text{if } n \text{ is even}, \\ n-1, & \text{if } n \text{ is odd}. \end{cases}$$

Together with the definition of the outer L^p quasi-norms in Definition 1.1.4 and Definition 1.1.6, the previous two displays yield the desired chain of inequalities. □

Remark 4.5.5. *We point out that the outer L^p quasi-norms with respect to the sizes $\ell^2_{\omega,\mathrm{out}}$, S, and $\widetilde{S}$ in the case of the settings on the upper half 3-spaces described in Subsections 1.2.12 – 1.2.13 are not equivalent. In fact, for every measurable function f on X, we only have the inequalities*

$$\|f\|_{L^p_\nu(\ell^2_{\omega,\mathrm{out}})} \leqslant C\|f\|_{L^p_\nu(S)} \sim_p C\|f\|_{L^p_\nu(\widetilde{S})},$$

where we used the inequalities between the sizes. In general, the first inequality in the previous display is strict, for example for the measurable subset $A_\varepsilon \subseteq X$ defined in the proof of Lemma 4.5.2, we have

$$\|1_{A_\varepsilon}\|_{L^p_\nu(\ell^2_{\omega,\mathrm{out}})} = \frac{\varepsilon\sqrt{\varepsilon}}{2}, \qquad \|1_{A_\varepsilon}\|_{L^p_\nu(S)} = \|1_{A_\varepsilon}\|_{L^p_\nu(\widetilde{S})} = 1.$$

4.6 Double iterated outer $L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ spaces on finite settings

In this section, we study the uniformity in the finite setting (X,μ,ν,ω) of the constants appearing in the Banach space properties of the outer $L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ spaces. We prove a result about Köthe duality in a partial range of exponents $q, r \in (1,\infty)$, and a result about quasi-triangle inequality for countably many summands in the full range of exponents $q, r \in (1,\infty)$.

Theorem 4.6.1. *For all $q, r \in [1, \infty]$, there exists a constant $C = C(q, r)$ such that, for every finite setting (X, μ, ν, ω), the following properties hold true.*

(i) For all $q, r \in (1, \infty)$, $q < r$ or $q = r \in [1, \infty]$, for every function $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$\begin{aligned} C^{-1}\|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} &\leqslant \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^1_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1\Big\}, \\ \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^1_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1\Big\} &\leqslant C\|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))}. \end{aligned} \tag{4.6.1}$$

(ii) For all $q \in (1, \infty]$, $r \in [1, \infty)$ or $q = r \in \{1, \infty\}$, for every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ of functions on X, we have

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \sum_{n\in\mathbb{N}} \|f_n\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))}. \tag{4.6.2}$$

Proof. **Proof of property** (i)**.** If $q = r \in [1, \infty]$, by collapsing of exponents, property (i) in Theorem 4.2.1, for the setting (X, ν, ω), then we reduce to the case of the single iterated outer $L^\infty_\mu(\ell^r_\omega)$ quasi-norms on the setting (X, μ, ω).

Therefore, without loss of generality, we assume $q, r \in (1, \infty)$, $q < r$. Let $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$, $f \neq 0$. The second inequality in (4.6.1) follows from the Radon-Nikodym type result for the outer L^1 quasi-norms (Theorem 1.1.8) and outer Hölder's inequality (Theorem 1.1.7) for the finite setting (X, μ, ν, ω). To prove the first inequality in (4.6.1), let A be a subset of X such that

$$\ell^q_\nu(\ell^r_\omega)(f)(A) = \|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} > 0.$$

Let $\{A_k \colon k \in \mathbb{Z}\}$ be the partition of A produced by the decomposition of $f1_A$ with respect to the size ℓ^r_ω at levels $\{2^k \colon k \in \mathbb{Z}\}$ provided by Proposition 2.2.1 in Chapter 2. We define the function g on X by

$$g(x) = \sum_{k\in\mathbb{Z}} 2^{k(q-r)} 1_{A_k}(x) |f(x)|^{r-1}.$$

There exists a constant $c = c(q, r)$ such that

$$\|fg\|_{L^1(X,\omega)} = \sum_{k\in\mathbb{Z}} 2^{kq} \nu(A_k) \geqslant c\|f1_A\|^q_{L^q_\nu(\ell^r_\omega)} = c\mu(A)\|f\|^q_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

Next, by outer Hölder's inequality (Theorem 1.1.7) for the finite setting (X, μ, ν, ω), we have

$$\|g\|_{L^1_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} \leqslant \|1_A\|_{L^1_\mu(\ell^\infty_\nu(\ell^\infty_\omega))} \|g\|_{L^\infty_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = \mu(A)\|g\|_{L^\infty_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))}.$$

Moreover, there exists a constant $C = C(q, r)$ such that, for every subset $B \subseteq X$, we have

$$\|g1_B\|^{q'}_{L^{q'}_\nu(\ell^{r'}_\omega)} \leqslant C\|f1_B\|^q_{L^q_\nu(\ell^r_\omega)}.$$

The proof of the inequality in the previous display is the same of that of **Case II** in the proof of Lemma 3.3.9 in Chapter 3, and we refer to it for the details. Therefore, we have

$$\begin{aligned}
\|g\|_{L^\infty_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} &= \sup\Big\{\Big(\mu(B)^{-1}\|g 1_B\|^{q'}_{L^{q'}_\nu(\ell^{r'}_\omega)}\Big)^{\frac{1}{q'}} : B \subseteq A, B \neq \varnothing\Big\} \\
&\leqslant C\Big(\sup\Big\{\mu(B)^{-1}\|f 1_B\|^{q}_{L^{q}_\nu(\ell^{r}_\omega)} : B \subseteq A, B \neq \varnothing\Big\}\Big)^{\frac{1}{q'}} \\
&= C\mu(A)^{-\frac{1}{q'}}\|f 1_A\|^{q-1}_{L^{q}_\nu(\ell^{r}_\omega)} \\
&= C\|f\|^{q-1}_{L^\infty_\mu(\ell^{q}_\nu(\ell^{r}_\omega))},
\end{aligned}$$

yielding the inequality

$$\|g\|_{L^1_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} \leqslant C\mu(A)\|f\|^{q-1}_{L^\infty_\mu(\ell^{q}_\nu(\ell^{r}_\omega))}.$$

Proof of property (ii)**.** The inequality in (4.6.2) follows from the chain of inequalities

$$\begin{aligned}
\Big\|\sum_{n\in\mathbb{N}} f_n 1_A\Big\|_{L^\infty_\mu(\ell^{q}_\nu(\ell^{r}_\omega))} &= \sup\Big\{\mu(A)^{-\frac{1}{q}}\Big\|\sum_{n\in\mathbb{N}} f_n 1_A\Big\|_{L^{q}_\nu(\ell^{r}_\omega)} : A \subseteq X, A \neq \varnothing\Big\} \\
&\leqslant C\sum_{n\in\mathbb{N}}\sup\Big\{\mu(A)^{-\frac{1}{q}}\|f_n 1_A\|_{L^{q}_\nu(\ell^{r}_\omega)} : A \subseteq X, A \neq \varnothing\Big\} \\
&= C\sum_{n\in\mathbb{N}}\|f_n\|_{L^\infty_\mu(\ell^{q}_\nu(\ell^{r}_\omega))},
\end{aligned}$$

where we used the quasi-triangle inequality for the single iterated outer $L^q_\nu(\ell^r_\omega)$ quasi-norm, property (iii) in Theorem 2.1.1 in Chapter 2, for the finite setting (X, ν, ω) in the inequality.

□

4.7 Embedding maps in the upper half space via cancellative wavelets

In this section, we study embedding maps from the classical $L^p(\mathbb{R}^d, \mathrm{d}x)$ spaces to the outer $L^p_{\mu_d}(\ell^r_{\omega_d})$ spaces with $r \in (0, \infty]$ on the settings on the upper half space or its discrete model described in Subsections 1.2.8 – 1.2.10. In particular, the embedding maps are defined by convolving a function on $\mathbb{R}^d$ with dilated and translated copies of a wavelet with additional hypotheses of cancellation and Hölder continuity. We recall a classical result about the boundedness of the embedding maps for $r \in [2, \infty]$, and we exhibit counterexamples to it for $r \in (0, 2)$.

Fix a function $\phi \in L^1(\mathbb{R}^d, \mathrm{d}x)$. For every function $f \in L^\infty(\mathbb{R}^d, \mathrm{d}x)$, we define the embedded function $F_\phi(f)$ on the upper half space $\mathbb{R}^d \times (0, \infty)$ by

$$F_\phi(f)(y, t) = \int_{\mathbb{R}^d} f(x) t^{-d} \phi\Big(\frac{y - x}{t}\Big)\,\mathrm{d}x.$$

We have the following boundedness properties of the embedding maps.

Theorem 4.7.1. *For all $d \in \mathbb{N}$, $p \in [1,\infty]$, $r \in [2,\infty]$, $K \geqslant 0$, $\varepsilon > 0$, there exists a constant $C = C(d,p,r,K,\varepsilon)$ such that, for every function $\phi \in L^1(\mathbb{R}^d, \mathrm{d}x)$ satisfying the conditions*

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 0, \\ &|\phi(x) - \phi(y)| \leqslant K|x-y|^{\varepsilon}(1+|x|)^{-d-\varepsilon}, \qquad \text{for all } x,y \in \mathbb{R}^d,\ |x-y| \leqslant 1, \end{aligned} \tag{4.7.1}$$

the following properties hold true.

(i) For all $p \in (1,\infty]$, $r \in [2,\infty]$, for every measurable function $f \in L^p(\mathbb{R}^d, \mathrm{d}x)$ on $\mathbb{R}^d$, we have

$$\|F_\phi(f)\|_{L^p_{\mu_d}(\ell^r_{\omega_d})} \leqslant C\|f\|_{L^p(\mathbb{R}^d,\mathrm{d}x)}.$$

(ii) For every $r \in [2,\infty]$, for every measurable function $f \in L^1(\mathbb{R}^d, \mathrm{d}x)$ on $\mathbb{R}^d$, we have

$$\|F_\phi(f)\|_{L^{1,\infty}_{\mu_d}(\ell^r_{\omega_d})} \leqslant C\|f\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}.$$

(iii) For every $r \in [2,\infty]$, for every measurable function $f \in H^1(\mathbb{R}^d, \mathrm{d}x)$ on $\mathbb{R}^d$, we have

$$\|F_\phi(f)\|_{L^1_{\mu_d}(\ell^r_{\omega_d})} \leqslant C\|f\|_{H^1(\mathbb{R}^d,\mathrm{d}x)}.$$

(iv) For every $r \in [2,\infty]$, for every measurable function $f \in \mathrm{BMO}(\mathbb{R}^d, \mathrm{d}x)$ on $\mathbb{R}^d$, we have

$$\|F_\phi(f)\|_{L^\infty_{\mu_d}(\ell^{r'}_{\omega_d})} \leqslant C\|f\|_{\mathrm{BMO}(\mathbb{R}^d,\mathrm{d}x)}.$$

We briefly comment on the proof. First, we recall the equivalence between the outer $L^p_{\mu_d}(\ell^r_{\omega_d})$ spaces and tent T^p_r spaces stated in Theorem 2.1.3 in Chapter 2. Next, we recall the interpretation of the tent T^p_2 and T^p_∞ norms of the embedded function $F_\phi(f)$ as the classical L^p norms of the square and maximal functions associated with f respectively. Then, for $r \in \{2,\infty\}$, the statements in the previous theorem are classical results, see for example the book of Grafakos [Gra09] and Stein [Ste93]. By the logarithmic convexity of the sizes (Proposition 2.A.8 in Chapter 2), the extension to the case $r \in (2,\infty)$ follows.

We pass to the case $r \in (0,2)$. We exhibit counterexamples to the boundedness of the embedding maps via a bounded cancellative wavelet with compact support from $L^p(\mathbb{R}^d, \mathrm{d}x)$ to the outer $L^{p,\infty}_{\mu_d}(\ell^r_{\omega_d})$ spaces. We make two additional assumptions. First, we restrict to the case of dimension $d = 1$. Next, we restrict to the case of the setting (X,μ,ω) on the discrete model of the upper half plane. The counterexamples we exhibit provide a prototype to extend the result about the unboundedness of the embedding maps in the following directions. First, to the case of the settings on the upper half plane described in Subsections 1.2.9 – 1.2.10. Next, to the case of the settings on the upper half space of arbitrary dimension $d \in \mathbb{N}$. Finally, to the case of embedding maps via wavelets satisfying the conditions in (4.7.1).

Let $\mathcal{I}$ be the collection of all the dyadic intervals in $\mathbb{R}$. For every dyadic interval $I \in \mathcal{I}$, let $I_l, I_r \subseteq I$ be the left and right dyadic sibling of I respectively. We define the Haar function h_I associated with I and normalized in $L^\infty(\mathbb{R}, \mathrm{d}x)$ by

$$h_I = \chi_{I_l} - \chi_{I_r},$$

and the Haar function $\hbar_I$ associated with I and normalized in $L^1(\mathbb{R}, \mathrm{d}x)$ by

$$\hbar_I = |I|^{-1} h_I.$$

The function $\hbar = \hbar_{(0,1]}$ is the prototype of the cancellative wavelet.

Next, let (X, μ, ω) be the setting on the collection of upper half dyadic cubic boxes in the upper half plane $\mathbb{R} \times (0, \infty)$ described in Subsection 1.2.8. There exists a bijective correspondence between the elements of X and $\mathcal{I}$ defined as follows. For every $B \in X$, there exists a unique dyadic interval $I(B) \in \mathcal{I}$ such that

$$B = I(B) \times \Big(\frac{|I(B)|}{2}, |I(B)|\Big].$$

Then, for every measurable function $f \in L^\infty(\mathbb{R}, \mathrm{d}x)$, we define the function $F_\hbar(f)$ on X by

$$F_\hbar(f)(B) = \int_{\mathbb{R}} f(x) \hbar_{I(B)}(x) \,\mathrm{d}x.$$

Lemma 4.7.2. *Let $p \in [1, \infty]$, $r \in (0, 2)$. For every $M > 0$, there exists a measurable function $f \in L^p(\mathbb{R}, \mathrm{d}x)$ on $\mathbb{R}$ such that*

$$\|F_\hbar(f)\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \geqslant M \|f\|_{L^p(\mathbb{R}, \mathrm{d}x)}.$$

Proof. **Case I: $p \neq \infty$.** For every $L \in \mathbb{N}$, we define the collection $\mathcal{I}_L \subseteq \mathcal{I}$ of dyadic intervals by

$$\mathcal{I}_L = \Big\{ I(m, -l) \in \mathcal{I} \colon l \in \mathbb{Z}, 0 \leqslant l < L, m \in \mathbb{Z}, 0 \leqslant m < 2^l \Big\},$$

and the measurable function f_L on $\mathbb{R}$ by

$$f_L = \sum_{I \in \mathcal{I}_L} h_I.$$

Therefore, we have

$$F_\hbar(f_L)(B) = 1_{\mathcal{I}_L}(I(B)).$$

For every $p \in [1, \infty)$, by Khintchin's inequality, see for example the book of Grafakos [Gra08], there exists a constant $C = C(p)$ such that

$$\|f_L\|_{L^p(\mathbb{R}, \mathrm{d}x)} \leqslant C L^{\frac{1}{2}}. \tag{4.7.2}$$

Next, for every collection $\mathcal{J} \subseteq \mathcal{I}$ of dyadic intervals, we define the measurable subset $A = A(\mathcal{J}) \subseteq \mathbb{R}$ by

$$A = \bigcup_{J \in \mathcal{J}} J,$$

and the measurable subset $K = K(\mathcal{J}) \subseteq X$ by

$$K = \bigcup_{J \in \mathcal{J}} E(J).$$

Moreover, we define $\widetilde{\mathcal{J}} \subseteq \mathcal{I}$ to be the collection of maximal dyadic intervals contained in $(0,1]\backslash A$, maximal in terms of set inclusion. Furthermore, we define the measurable subsets $D = D(\mathcal{J}), \widetilde{D} = \widetilde{D}(\mathcal{J}) \subseteq (0,1]$ by

$$D = A \cap (0,1], \qquad \widetilde{D} = (0,1]\backslash A = \bigcup_{\widetilde{J} \in \widetilde{\mathcal{J}}} \widetilde{J}.$$

Finally, we define $\mathcal{L} \subseteq \mathcal{I}$ to be the collection of dyadic intervals defined by

$$\mathcal{L} = \Big\{ I \in \mathcal{I} \colon I \subseteq (0,1], I \nsubseteq D, I \nsubseteq \widetilde{D} \Big\}.$$

In particular, for every $\widetilde{J} \in \widetilde{\mathcal{J}}$, for every $I \in \mathcal{I}$, $I \subseteq (0,1]$ such that $\widetilde{J} \subseteq I$, $\widetilde{J} \neq I$, we have $I \in \mathcal{L}$. Hence, for every $l \in \mathbb{Z}$, $0 \leqslant l < L$, we have

$$\begin{aligned} \sum_{I \in \mathcal{I}_L, |I| = 2^{-l}, I \nsubseteq D} |I| &= \sum_{I \in \mathcal{I}_L, |I| = 2^{-l}, I \subseteq \widetilde{D}} |I| + \sum_{I \in \mathcal{I}_L, |I| = 2^{-l}, I \in \mathcal{L}} |I| \\ &\geqslant \sum_{I \in \mathcal{I}_L, |I| = 2^{-l}, I \subseteq \widetilde{D}} |I| + \sum_{I \in \mathcal{I}_L, |I| < 2^{-l}, I \in \widetilde{\mathcal{J}}} |I| \qquad (4.7.3) \\ &= |\widetilde{D}|. \end{aligned}$$

For every $\alpha \in [0,1]$, for every collection $\mathcal{J} \subseteq \mathcal{I}$ of pairwise disjoint dyadic intervals such that

$$|A| = \sum_{J \in \mathcal{J}} |J| = \alpha,$$

we have

$$\sum_{\widetilde{J} \in \widetilde{\mathcal{J}}} |\widetilde{J}| = |\widetilde{D}| = 1 - |D| \geqslant 1 - |A| = 1 - \alpha.$$

Therefore, by the inequality in (4.7.3), we have

$$\ell^r_\omega(F_\hbar(f_L) 1_{K^c})(E((0,1])) = \Big(\sum_{I \in \mathcal{I}_L, I \nsubseteq D} |I| \Big)^{\frac{1}{r}} \geqslant L^{\frac{1}{r}} (1-\alpha)^{\frac{1}{r}}.$$

Hence, for all $p \in [1,\infty)$, $r \in (0,2)$, there exists a constant $c = c(p,r)$ such that

$$\|F_\hbar(f_L)\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \geqslant \sup\left\{L^{\frac{1}{r}}(1-\alpha)^{\frac{1}{r}}\alpha^{\frac{1}{p}} : \alpha \in [0,1]\right\} \geqslant cL^{\frac{1}{r}}.$$

Taking $L \in \mathbb{N}$ big enough, the inequalities in (4.7.2) and (4.7) yield the desired inequality.

Case II: $p = \infty$**.** For every dyadic interval $I = I(m,l) \in \mathcal{I}$, we define its centre $c_I \in I$ by

$$c_I = m2^l + 2^{l-1}.$$

For every $L \in \mathbb{N}$, we define the collection $\mathcal{J}_L \subseteq \mathcal{I}$ of dyadic intervals by

$$\mathcal{J}_L = \left\{I(m,-l) \in \mathcal{I} \colon l \in \mathbb{Z}, 0 \leqslant l < 100L^2, m \in \mathbb{Z}, 0 \leqslant m < 2^l\right\},$$

and the measurable function f_L on $\mathbb{R}$ by

$$f_L = \sum_{I \in \mathcal{J}_L} a(I)h_I,$$

where, for every $I \in \mathcal{J}_L$, we define

$$a(I) = \begin{cases} 0, & \text{if } \Big|\displaystyle\sum_{J \in \mathcal{J}_L \setminus \{I\}, I \subseteq J} a(J)\operatorname{sgn}(h_J(c_I))\Big| = L, \\ 1, & \text{otherwise,} \end{cases}$$

and sgn is the signum function. Therefore, we have

$$F_\hbar(f_L)(B) = 1_{\mathcal{J}_L}(I(B))a(I(B)).$$

By definition, the function f_L is constant on the elements of $\mathcal{L}_L \subseteq \mathcal{I}$, the collection of dyadic intervals defined by

$$\mathcal{L}_L = \left\{I \in \mathcal{I} \colon I \subseteq (0,1], |I| = 2^{-100L^2}\right\}.$$

Moreover, there exists a bijective correspondence between the dyadic intervals in $\mathcal{L}_L$ and the sequences in the collection

$$\mathcal{A}_L = \left\{\vec{a} = (a_1,\dots,a_{100L^2}) \colon a_i \in \{-1,1\}\right\},$$

defined as follows. For every sequence $\vec{a} \in \mathcal{A}_L$, we define the dyadic interval $I(\vec{a}) \in \mathcal{L}_L$ by

$$\begin{aligned} I(\vec{a}) &= I\Big(\sum_{l=1}^{100L^2}(1-a_l)2^{100L^2-l-1}, -100L^2\Big) \\ &= \Big(\sum_{l=1}^{100L^2}(1-a_l)2^{-l-1}, 2^{-100L^2} + \sum_{l=1}^{100L^2}(1-a_l)2^{-l-1}\Big]. \end{aligned}$$

Next, for every sequence $\vec{a} \in \mathcal{A}_L$, we define a_+ to be the number of coordinates attaining the value 1, and a_- the number of coordinates attaining the value -1. In particular, for a fixed sequence $\vec{a} \in \mathcal{A}_L$, if $|a_+ - a_-| \geqslant L$, then the function $|f_L|$ attains the constant value L on $I(\vec{a})$. Therefore, we have

$$\Big\{x \in (0,1] \colon |f_L(x)| < L\Big\} \subseteq \widetilde{A}_L,$$

where the subset $\widetilde{A}_L \subseteq (0,1]$ is defined by

$$\widetilde{A}_L = \bigcup_{\vec{A} \in \widetilde{\mathcal{A}}_L} I(\vec{A}),$$

and the collection $\widetilde{\mathcal{A}}_L \subseteq \mathcal{A}_L$ is defined by

$$\widetilde{\mathcal{A}}_L = \Big\{\vec{a} \in \mathcal{A}_L \colon |a_+ - a_-| < L\Big\}.$$

By the upper and lower bounds for the factorials provided by Stirling's approximation, we have

$$\begin{aligned}
|\widetilde{\mathcal{A}}_L| &= \sum_{l=-\lfloor L/2 \rfloor}^{\lfloor L/2 \rfloor} \binom{100L^2}{50L^2 + l} \\
&\leqslant 2L \frac{100L^2!}{(50L^2!)^2} \\
&\leqslant 2L \frac{\sqrt{2\pi 100L^2}(100L^2/e)^{100L^2} e^{1/(1200L^2)}}{(\sqrt{2\pi 50L^2}(50L^2/e)^{50L^2} e^{1/(600L^2+1)})^2} \\
&\leqslant 2^{100L^2-1},
\end{aligned}$$

where, for every $x \in \mathbb{R}$, we define $\lfloor x \rfloor \in \mathbb{Z}$ to be the biggest integer number smaller than or equal to x. Therefore, we have

$$\begin{aligned}
\Big|\Big\{x \in (0,1] \colon |f_L(x)| = L\Big\}\Big| &\geqslant 1 - \Big|\Big\{x \in (0,1] \colon |f_L(x)| < L\Big\}\Big| \\
&\geqslant 1 - |\widetilde{A}_L| \\
&= 1 - 2^{-100L^2} |\widetilde{\mathcal{A}}_L| \\
&\geqslant 2^{-1},
\end{aligned}$$

hence, for every $p \in [1, \infty]$, we have

$$2^{-\frac{1}{p}} L \leqslant \|f_L\|_{L^p(\mathbb{R}, \mathrm{d}x)} \leqslant L, \tag{4.7.4}$$

where the exponent ∞^{-1} is understood to be 0.

Moreover, since the function a attains only values in $\{0,1\}$, we have

$$\begin{aligned}\|F_\hbar(f_L)\|_{L^\infty_\mu(\ell^r_\omega)} &\geqslant \ell^r_\omega(F_\hbar(f_L))(E(0,1]) \\ &= \Big(\sum_{I\in\mathcal{J}_L} a(I)^r|I|\Big)^{\frac{1}{r}} \\ &= \Big(\sum_{I\in\mathcal{J}_L} a(I)^2|I|\Big)^{\frac{1}{r}} \\ &= \|f_L\|^{\frac{2}{r}}_{L^2(\mathbb{R},\mathrm{d}x)} \\ &\geqslant 2^{-\frac{1}{r}}L^{\frac{2}{r}},\end{aligned} \tag{4.7.5}$$

where we used the $L^2(\mathbb{R},\mathrm{d}x)$ orthogonality between Haar functions associated with different dyadic intervals in the third equality, and the inequality in (4.7.4) in the second inequality.

Taking $L \in \mathbb{N}$ big enough, the inequalities in (4.7.4) for $p=\infty$ and (4.7.5) yield desired inequality. □

Analogous counterexamples in the case of the settings on the upper half space described in Subsections 1.2.9 – 1.2.10 are obtained by considering a Hölder continuous version of the Haar function $\hbar$, namely, for $d=1$, the function

$$\begin{aligned}\phi(x) = 8x1_{[0,\frac{1}{8})}(x) + 1_{[\frac{1}{8},\frac{3}{8})}(x) + (4-8x)1_{[\frac{3}{8},\frac{5}{8})}(x) - \\ - 1_{[\frac{5}{8},\frac{7}{8})}(x) + (-8+8x)1_{[\frac{7}{8},1)}(x),\end{aligned}$$

and its appropriate generalizations for arbitrary dimension $d \in \mathbb{N}$.

4.8 Conjectures

We split the collection of conjectures into two subsections, the first about the single iterated outer L^p quasi-norms and spaces, the second about the double iterated outer ones.

4.8.1 Conjectures for single iterated outer L^p spaces

First, we start with three conjectures about conditions on the setting (X,μ,ω) to recover the uniformity of the constant in Köthe duality for certain single iterated outer L^p spaces. As a corollary, we would recover the uniformity of the constant in the quasi-triangle inequality for countably many summands for the respective spaces. We consider the following cases.

- Sufficient conditions for the outer $L^p_\mu(\ell^\infty_\omega)$ and $L^1_\mu(\ell^r_\omega)$ spaces.
- Necessary and sufficient condition for the outer $L^{1,\infty}_\mu(\ell^\infty_\omega)$ spaces.

Conjecture 4.8.1. *For all $p \in (1,\infty]$, $K \geqslant 1$, there exists a constant $C = C(p,K)$ such that, for every σ-finite setting (X,μ,ω) described in Subsection 1.2.1 satisfying the condition associated with the inequality in* (4.2.10) *with constant K, for every measurable function $f \in L^p_\mu(\ell^\infty_\omega)$ on X, we have*

$$C^{-1}\|f\|_{L^p_\mu(\ell^\infty_\omega)} \leqslant \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^1_\omega)} = 1\Big\} \leqslant C\|f\|_{L^p_\mu(\ell^\infty_\omega)}.$$

Conjecture 4.8.2. *For all $r \in [1,\infty]$, $K \geqslant 1$, there exists a constant $C = C(r,K)$ such that, for every finite setting (X,μ,ω) described in Subsection 1.2.2 and satisfying the condition associated with the inequality in* (4.2.18) *with constant K, for every measurable function $f \in L^1_\mu(\ell^r_\omega)$ on X, we have*

$$C^{-1}\|f\|_{L^1_\mu(\ell^r_\omega)} \leqslant \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^\infty_\mu(\ell^{r'}_\omega)} = 1\Big\} \leqslant C\|f\|_{L^1_\mu(\ell^r_\omega)}.$$

Conjecture 4.8.3. *There exist two maps $c, \widetilde{c} \colon (0,\infty) \to (0,\infty)$ such that, for every σ-finite setting (X,μ,ω) described in Subsection 1.2.1, the following properties hold true.*

(i) If there exists a constant $\widetilde{C}$ such that, for every measurable function $f \in L^{1,\infty}_\mu(\ell^\infty_\omega)$ on X, we have

$$\widetilde{C}^{-1}\|f\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)} \leqslant \sup\Big\{\inf\Big\{\|f1_{E'}\|_{L^1(X,\omega)}\|1_{E'}\|^{-1}_{L^\infty_\mu(\ell^1_\omega)} \colon E' \in \Upsilon(\mu,E)\Big\} \colon E \in \Sigma\Big\},$$
$$\sup\Big\{\inf\Big\{\|f1_{E'}\|_{L^1(X,\omega)}\|1_{E'}\|^{-1}_{L^\infty_\mu(\ell^1_\omega)} \colon E' \in \Upsilon(\mu,E)\Big\} \colon E \in \Sigma\Big\} \leqslant \widetilde{C}\|f\|_{L^{1,\infty}_\mu(\ell^\infty_\omega)},$$

where, for every measurable subset $E \in \Sigma$, we define the collection $\Upsilon(\mu,E) \subseteq \Sigma$ of measurable subsets of E by

$$\Upsilon(\mu,E) = \Big\{E' \in \Sigma \colon E' \subseteq E, \mu(E') \geqslant \frac{\mu(E)}{2}\Big\},$$

then (X,μ,ω) satisfies the condition associated with the inequality in (4.2.10) *with constant $C = c(\widetilde{C})$.*

(ii) If (X,μ,ω) satisfies the condition associated with the inequality in (4.2.10) *with constant C, then there exists a constant $\widetilde{C} = \widetilde{c}(C)$ such that, for every measurable function $f \in L^{1,\infty}_\mu(\ell^\infty_\omega)$ on X, we have the inequalities in the previous display with constant $\widetilde{C}$.*

Next, we have a conjecture about the uniformity of the constant in a weak version of the quasi-triangle inequality for countably many summands for the outer $L^1_\mu(\ell^r_\omega)$ space.

Conjecture 4.8.4. *For every $r \in [1,\infty)$, there exists a constant $C = C(r)$ such that, for every σ-finite setting (X,μ,ω) described in Subsection 1.2.1, for every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^1_\mu(\ell^r_\omega)$ of measurable functions on X, we have*

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^{1,\infty}_\mu(\ell^r_\omega)} \leqslant C\sum_{n\in\mathbb{N}}\|f_n\|_{L^1_\mu(\ell^r_\omega)}.$$

When the uniformity of the constant in the quasi-triangle inequality for countably many summands fails, we have a conjecture about the dependence of the constant on the number of summands.

Conjecture 4.8.5. *For all $p \in [1, \infty)$, $r \in (1, \infty]$, there exist two constants $C_1 = C_1(p)$ and $C_2 = C_2(r)$ such that, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, for every $N \in \mathbb{N}$, for every collection $\{f_n : 1 \leqslant n \leqslant N\}$ of measurable functions on X, we have*

$$\Big\| \sum_{n=1}^{N} f_n \Big\|_{L^p_\mu(\ell^\infty_\omega)} \leqslant C_1 (1 + \ln(N))^{\frac{1}{p}} \sum_{n=1}^{N} \| f_n \|_{L^p_\mu(\ell^\infty_\omega)},$$
$$\Big\| \sum_{n=1}^{N} f_n \Big\|_{L^1_\mu(\ell^r_\omega)} \leqslant C_2 (1 + \ln(N))^{1-\frac{1}{r}} \sum_{n=1}^{N} \| f_n \|_{L^1_\mu(\ell^r_\omega)}.$$

Then, we have a conjecture about a condition equivalent to the uniformity of the constant in the quasi-triangle inequality for countably many summands for the outer $L^1_\mu(\ell^\infty)$ space.

Conjecture 4.8.6. *There exist three maps $\phi, k : (0, \infty) \to (1, \infty)$, and $c : (1, \infty)^2 \to (0, \infty)$ such that, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the following properties hold true.*

(i) If the outer $L^1_\mu(\ell^\infty_\omega)$ quasi-norm satisfies the quasi-triangle inequality for countably many summands with constant C, then the quadruple $(X, \mu, \mu, \mathrm{Id})$ satisfies the canopy and the crop conditions 1.3.4 – 1.3.5 with parameters $\Phi = \phi(C)$, $K = k(C)$.

(ii) If the quadruple $(X, \mu, \mu, \mathrm{Id})$ satisfies the canopy and the crop conditions 1.3.4 – 1.3.5 with parameters Φ, K, then the outer $L^1_\mu(\ell^\infty_\omega)$ quasi-norm satisfies the quasi-triangle inequality for countably many summands with constant $C = c(\Phi, K)$.

After that, we have a conjecture about the atomic decompositions of the outer $L^p_\mu(\ell^r_\omega)$ spaces. In the statement, ρ denotes the counting measure on $\mathbb{N}$. Moreover, for every measurable function f on X, we define $\mu(\mathrm{supp}(f)) \in [0, \infty]$ by

$$\mu(\mathrm{supp}(f)) = \inf \Big\{ \mu(A) : A \subseteq X, \| f 1_{A^c} \|_{L^\infty(X, \omega)} = 0 \Big\}.$$

Conjecture 4.8.7. *For all $p_0, p, r \in (0, \infty]$, there exists a constant $C = C(p_0, p, r)$ such that, for every σ-finite setting (X, μ, ω) described in Subsection 1.2.1, the following properties hold true.*

(i) If $p \geqslant r$, $p \geqslant p_0$, then, for every measurable function $f \in L^p_\mu(\ell^r_\omega)$ on X, we have

$$C^{-1} \| f \|_{L^p_\mu(\ell^r_\omega)} \leqslant \sup \Big\{ \| a_n \|_{L^p(\mathbb{N}, \rho)} : \exists \{ f_n : n \in \mathbb{N} \} \subseteq \mathcal{A}^{p,r}, |f| = \| a_n f_n \|_{L^{p_0}(\mathbb{N}, \rho)} \Big\},$$
$$\sup \Big\{ \| a_n \|_{L^p(\mathbb{N}, \rho)} : \exists \{ f_n : n \in \mathbb{N} \} \subseteq \mathcal{A}^{p,r}, |f| = \| a_n f_n \|_{L^{p_0}(\mathbb{N}, \rho)} \Big\} \leqslant C \| f \|_{L^p_\mu(\ell^r_\omega)},$$

where a is an outer $L^p_\mu(\ell^r_\omega)$ atom *in $\mathcal{A}^{p,r}$ with $p \geqslant r$ if*

$$\|f\|_{L^r(X,\omega)} \geqslant \mu(\mathrm{supp}(f))^{\frac{1}{r}-\frac{1}{p}}.$$

(ii) If $p \leqslant r$, $p \leqslant p_0$, then, for every measurable function $f \in L^p_\mu(\ell^r_\omega)$ on X, we have

$$C^{-1}\|f\|_{L^p_\mu(\ell^r_\omega)} \leqslant \inf\Big\{\|a_n\|_{L^p(\mathbb{N},\rho)} \colon \exists\{f_n \colon n \in \mathbb{N}\} \subseteq \mathcal{A}^{p,r}, |f| = \|a_n f_n\|_{L^{p_0}(\mathbb{N},\rho)}\Big\},$$

$$\inf\Big\{\|a_n\|_{L^p(\mathbb{N},\rho)} \colon \exists\{f_n \colon n \in \mathbb{N}\} \subseteq \mathcal{A}^{p,r}, |f| = \|a_n f_n\|_{L^{p_0}(\mathbb{N},\rho)}\Big\} \leqslant C\|f\|_{L^p_\mu(\ell^r_\omega)},$$

where a is an outer $L^p_\mu(\ell^r_\omega)$ atom *in $\mathcal{A}^{p,r}$ with $p \leqslant r$ if*

$$\|f\|_{L^r(X,\omega)} \leqslant \mu(\mathrm{supp}(f))^{\frac{1}{r}-\frac{1}{p}}.$$

Moreover, we have a conjecture about the quasi-triangle inequality for countably many summands for the outer L^p spaces with respect to the sizes $\ell^r_{\omega,\mathrm{inn}}$, $\ell^r_{\omega,\mathrm{out}}$, S, and $\widetilde{S}$ on the settings on the upper half 3-space defined in Subsection 4.5.1.

Conjecture 4.8.8. *For all $p \in (1,\infty]$, $r \in [1,\infty]$, there exists a constant $C = C(p,r)$ such that, for every setting (X,ν,ω) on the upper half 3-space described in Subsections 1.2.12 – 1.2.13, the following properties hold true.*

For every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\nu(\ell^r_{\omega,\mathrm{inn}})$ of measurable functions on X, we have

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{L^p_\nu(\ell^r_{\omega,\mathrm{inn}})} \leqslant C \sum_{n\in\mathbb{N}} \|f_n\|_{L^p_\nu(\ell^r_{\omega,\mathrm{inn}})}.$$

Replacing $\ell^r_{\omega,\mathrm{inn}}$ with $\ell^r_{\omega,\mathrm{out}}$, S, and $\widetilde{S}$, we have the same statements for the outer $L^p_\nu(\ell^r_{\omega,\mathrm{out}})$, $L^p_\nu(S)$, and $L^p_\nu(\widetilde{S})$ spaces respectively.

Finally, we have a conjecture about the equivalence of the outer L^p spaces with respect to different sizes for embedded function in the case of the setting (X,ν_2,ω) on the collection of Heisenberg upper half dyadic tiles described in Subsection 4.5.4.

Conjecture 4.8.9. *Let $M,N \in \mathbb{N}$. Let $\Phi = \Phi(M,N)$ be the collection of Schwartz functions ϕ on $\mathbb{R}$ such that*

$$1_{[-1,1]} \leqslant \widehat{\phi} \leqslant 1_{[-2,2]},$$

and, for every $k \in \mathbb{N}$, $k \leqslant N$, we have

$$|\partial^k \widehat{\phi}(\xi)| \leqslant M.$$

For every $p \in (0,\infty]$, there exists a constant $C = C(\Phi,p)$ such that, for every Schwartz function f on $\mathbb{R}$, we have

$$\|F_\Phi(f)\|_{L^p_{\nu_2}(\ell^\infty_{2,\omega})} \leqslant \|F_\Phi(f)\|_{L^p_{\nu_2}(\ell^2_{2,\omega,\mathrm{out}})} \leqslant C\|F_\Phi(f)\|_{L^p_{\nu_2}(\ell^\infty_{2,\omega})},$$

where $F_\Phi(f)$ is the function on X defined by

$$F_\Phi(f)(H) = \sup_{\phi\in\Phi} \sup_{(y,\eta,t)\in H} \Big| \int_{\mathbb{R}} f(z) \frac{1}{t} e^{2\pi i (y-z)\eta} \phi\Big(\frac{y-z}{t}\Big) \,\mathrm{d}z \Big|.$$

4.8.2 Conjectures for double iterated outer L^p spaces

First, we start with a conjecture about the exchange between the canopy and the crop conditions in the statement of Theorem 1.3.6.

Conjecture 4.8.10. *For all $p, q, r \in (0, \infty]$, $\Phi, K \geqslant 1$, there exist three constants $C_1 = C_1(q, r, \Phi, K)$, $C_2 = C_2(q, r, \Phi, K)$, and $C = C(p, q, r, \Phi, K)$ such that, for every finite setting (X, μ, ν, ω) described in Subsection 1.2.2, for every μ-covering function $\mathcal{C}$, the following property holds true.*

(i) If $(X, \mu, \nu, \mathcal{C})$ satisfies the crop condition 1.3.5, then for every function f on X, we have

$$C_1^{-1} \|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C_2 \|f\|_{L^q_\nu(\ell^r_\omega)}.$$

(ii) If $(X, \mu, \nu, \mathcal{C})$ satisfies the crop condition 1.3.5, then for all $p, q, r \in (1, \infty)$, $q \leqslant r$, for every function f on X, we have

$$C^{-1} \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \sup \left\{ \|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1 \right\} \leqslant C \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

If $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 1.3.4, then for all $p, q, r \in (1, \infty)$, $q \geqslant r$, the same inequalities hold true.

(iii) If $(X, \mu, \nu, \mathcal{C})$ satisfies the crop condition 1.3.5, then for all $p, q, r \in (1, \infty)$, $q \leqslant r$, for every collection $\{f_n \colon n \in \mathbb{N}\}$ of functions on X, we have

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \sum_{n \in \mathbb{N}} \|f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

If $(X, \mu, \nu, \mathcal{C})$ satisfies the canopy condition 1.3.4, then for all $p, q, r \in (1, \infty)$, $q \geqslant r$, the same inequality holds true.

Next, we have a conjecture about the Banach space properties of the double iterated outer L^p spaces with at least one exponent in $\{1, \infty\}$ in the case of the settings on the upper half 3-space or its discrete model described in Subsections 1.2.11 – 1.2.13. To make the statement of the conjecture cleaner, we include the whole range of exponents p, q, r. The actual conjectured results are those in the cases not already covered by Theorem 3.1.5 in Chapter 3.

Conjecture 4.8.11. *For all $p, q, r \in (0, \infty]$, there exists a constant $C = C(p, q, r)$ such that, for every setting (X, μ, ν, ω) on the upper half 3-space or its discrete model described in Subsections 1.2.11 – 1.2.13, the following property holds true.*

(i) For all $q, r \in (0, \infty]$, for every measurable function $f \in L^q_\mu(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$C^{-1} \|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \|f\|_{L^q_\nu(\ell^r_\omega)}.$$

(ii) *For all* $p, r \in [1, \infty]$, $q \in (1, \infty]$ *or* $p \in [1, \infty]$, $q = r = 1$, *for every measurable function* $f \in L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ *on* X, *we have*

$$C^{-1} \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant \sup \Big\{ \|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1 \Big\} \leqslant C \|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

(iii) *For all* $p, r \in [1, \infty]$, $q \in (1, \infty]$ *or* $p \in [1, \infty]$, $q = r = 1$, *for every collection* $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^q_\nu(\ell^r_\omega))$ *of measurable functions on* X, *we have*

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C \sum_{n \in \mathbb{N}} \|f_n\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))}.$$

(iv) *Let* $p \in [1, \infty]$, $q = 1$, $r \in (1, \infty]$. *For every* $M > 0$, *there exists a collection* $\{f_n \colon n \in \mathbb{N}\} \subseteq L^p_\mu(\ell^1_\nu(\ell^r_\omega))$ *of measurable functions on* X *such that*

$$\Big\| \sum_{n \in \mathbb{N}} f_n \Big\|_{L^p_\mu(\ell^1_\nu(\ell^r_\omega))} \geqslant M \sum_{n \in \mathbb{N}} \|f_n\|_{L^p_\mu(\ell^1_\nu(\ell^r_\omega))}.$$

Then, we have a conjecture about the failure of the uniformity of the constant in Köthe duality for outer $L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ spaces with $q > r$, addressing the remaining case in property (i) in Theorem 4.6.1.

Conjecture 4.8.12. *Let* $q, r \in [1, \infty]$, $q > r$. *For every* $M > 0$, *there exist a finite setting* (X, μ, ν, ω) *and a function* $f \in L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))$ *on* X *such that*

$$\|f\|_{L^\infty_\mu(\ell^q_\nu(\ell^r_\omega))} \geqslant M \sup \Big\{ \|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^1_\mu(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1 \Big\}.$$

After that, we have a conjecture about Minkowski's inequality for double iterated outer L^p spaces. Before stating the conjecture, we introduce some auxiliary definitions.

For every finite setting (X, μ, ν, ω), let $\upsilon = \upsilon(\mu, \nu)$ be the outer measure on X defined by

$$\upsilon \colon \mathcal{P}(X) \to (0, \infty), \qquad \upsilon(A) = \|1_A\|_{L^\infty_\mu(\ell^1_\nu(\ell^\infty_\omega))},$$

let $\rho = \rho(\mu, \omega)$ be the outer measure on X defined by

$$\rho \colon \mathcal{P}(X) \to (0, \infty), \qquad \rho(A) = \|1_A\|_{L^\infty_\mu(\ell^1_\omega)},$$

let $\zeta = \zeta(\mu, \nu, \omega)$ be the outer measure on X defined by

$$\zeta \colon \mathcal{P}(X) \to (0, \infty), \qquad \zeta(A) = \|1_A\|_{L^\infty_\upsilon(\ell^1_\omega)},$$

and let $\kappa = \kappa(\nu, \omega)$ be the outer measure on X defined by

$$\kappa \colon \mathcal{P}(X) \to (0, \infty), \qquad \kappa(A) = \|1_A\|_{L^\infty_\nu(\ell^1_\omega)}.$$

Conjecture 4.8.13. *For all $p, q, r \in (0, \infty]$, there exists a constant $C = C(p, q, r)$ such that, for every finite setting (X, μ, ν, ω), for every measurable function f on X, the following properties hold true.*

(i) If $p \geqslant q$, we have

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^q_\upsilon(\ell^p_\nu(\ell^r_\omega))}, \qquad \|f\|_{L^p_\upsilon(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^q_\mu(\ell^p_\nu(\ell^r_\omega))}.$$

(ii) If $p \geqslant q, r$, we have

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^q_\upsilon(\ell^r_\rho(\ell^p_\omega))}, \qquad \|f\|_{L^p_\upsilon(\ell^q_\rho(\ell^r_\omega))} \leqslant C\|f\|_{L^q_\mu(\ell^r_\nu(\ell^p_\omega))}.$$

(iii) If $q \geqslant r$, we have

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^p_\mu(\ell^r_\zeta(\ell^q_\omega))}, \qquad \|f\|_{L^p_\mu(\ell^q_\zeta(\ell^r_\omega))} \leqslant C\|f\|_{L^p_\mu(\ell^r_\nu(\ell^q_\omega))}.$$

(iv) If $p, q \geqslant r$, we have

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^r_\kappa(\ell^p_\zeta(\ell^q_\omega))}, \qquad \|f\|_{L^p_\kappa(\ell^q_\zeta(\ell^r_\omega))} \leqslant C\|f\|_{L^r_\mu(\ell^p_\nu(\ell^q_\omega))}.$$

(v) If $p \geqslant q \geqslant r$, we have

$$\|f\|_{L^p_\mu(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^r_\kappa(\ell^q_\rho(\ell^p_\omega))}, \qquad \|f\|_{L^p_\kappa(\ell^q_\rho(\ell^r_\omega))} \leqslant C\|f\|_{L^r_\mu(\ell^q_\nu(\ell^p_\omega))}.$$

Finally, we have a conjecture about the improved regularity of the outer measures associated with Minkowski's inequality in terms of the Banach space properties of the double iterated outer L^p spaces.

Conjecture 4.8.14. *For all $p, q, r \in (0, \infty]$, there exists a constant $C = C(p, q, r)$ such that, for every finite setting (X, μ, ν, ω) described in Subsection 1.2.2, the following properties hold true for the finite setting $(X, \upsilon, \nu, \omega)$ and for the outer measure υ defined as in Conjecture 4.8.13.*

(i) For all $q, r \in (0, \infty]$, for every function $f \in L^q_\upsilon(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$C^{-1}\|f\|_{L^q_\nu(\ell^r_\omega)} \leqslant \|f\|_{L^q_\upsilon(\ell^q_\nu(\ell^r_\omega))} \leqslant C\|f\|_{L^q_\nu(\ell^r_\omega)}.$$

(ii) For all $p, q, r \in (1, \infty)$, for every function $f \in L^p_\upsilon(\ell^q_\nu(\ell^r_\omega))$ on X, we have

$$C^{-1}\|f\|_{L^p_\upsilon(\ell^q_\nu(\ell^r_\omega))} \leqslant \sup\Big\{\|fg\|_{L^1(X,\omega)} \colon \|g\|_{L^{p'}_\upsilon(\ell^{q'}_\nu(\ell^{r'}_\omega))} = 1\Big\} \leqslant C\|f\|_{L^p_\upsilon(\ell^q_\nu(\ell^r_\omega))}.$$

(iii) For all $p, q, r \in (1, \infty)$, for every collection $\{f_n \colon n \in \mathbb{N}\} \subseteq\in L^p_\upsilon(\ell^q_\nu(\ell^r_\omega))$ of functions on X, we have

$$\Big\|\sum_{n\in\mathbb{N}} f_n\Big\|_{L^p_\upsilon(\ell^q_\nu(\ell^r_\omega))} \leqslant C\sum_{n\in\mathbb{N}}\|f_n\|_{L^p_\upsilon(\ell^q_\nu(\ell^r_\omega))}.$$

We have the same statements replacing the finite setting $(X, \upsilon, \nu, \omega)$ with the finite settings $(X, \upsilon, \rho, \omega)$, (X, μ, ζ, ω), $(X, \kappa, \zeta, \omega)$, and $(X, \kappa, \rho, \omega)$, for the outer measures υ, ρ, ζ, and κ defined as in Conjecture 4.8.13.

Index

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