Geometry and derived categories of holomorphic symplectic manifolds

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Summary

In this thesis we study various aspects of hyper-Kähler manifolds and abelian varieties such as their derived categories, sheaves, cycles, and topology. The thesis consists of six parts, which have appeared in [24–28, 30].

The first part is mostly a survey of [201]. We show that the LLV algebra is a derived invariant. Therefore, derived equivalences between hyper-Kähler manifolds yield Hodge isometries between their Verbitsky components, which come from isometries between their Mukai lattices. Moreover, derived equivalent hyper-Kähler manifolds have isomorphic $\mathbb{Q}$-Hodge structures.

The study of derived categories of hyper-Kähler manifolds is further refined in the next section. We introduce an extended Mukai vector with values in the Mukai lattice. This yields a structural result dividing derived equivalences into three cases with different geometric meaning. Moreover, for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds we define an integral lattice which is a derived invariant giving a higher-dimensional analogue to Mukai's results [155] for K3 surfaces. This has many consequences such as finiteness of $\mathrm{K3}^{[n]}$-type Fourier–Mukai partners of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds.

The subsequent section conceptualizes the extended Mukai vector by introducing the notion of atomic objects on hyper-Kähler manifolds. We relate the notion of atomicity to different obstruction maps. Stable atomic bundles are shown to be projectively hyperholomorphic, a class of bundles for which we prove formality of the dg algebra of derived endomorphisms. A thorough study of atomic Lagrangian submanifolds yields a structural result and expectations for the general behaviour of atomic objects. Our methods also yield that there do not exist spherical sheaves on any hyper-Kähler manifold of dimension at least four.

The question of topological properties of hyper-Kähler manifolds is discussed in the fourth part, which is a joint work with Jieao Song. The main result is a conditional bound on the second Betti number for hyper-Kähler manifolds X in terms of the two characteristic classes $\mathrm{c}_2(X)^2$ and $\mathrm{c}_4(X)$. In the known examples this bound is better than previous ones [84, 122] and remains true for orbifolds of dimension four. We further investigate (conjectural) properties that the generalized Fujiki constants and Riemann–Roch polynomials possess and discuss implications.

The fifth part studies group actions on hyper-Kähler manifolds and derived categories and is joint work with Georg Oberdieck. We show that fixed loci of actions by a finite group G on moduli spaces of stable objects on certain smooth projective varieties X which are induced by an action of G on the derived category $\mathrm{D^b}(X)$ are covered by moduli spaces of semistable objects of the equivariant category $\mathrm{D^b}(X)_G$. This yields a generalization of the derived McKay correspondence for symplectic surfaces and completely determines the fixed locus of a symplectic automorphism acting on a moduli space of stable objects on a K3 surface.

The final part of this thesis is a joint work with Olivier de Gaay Fortman and studies one-cycles on abelian varieties. We link the integral Hodge conjecture for one-cycles to lifts of the correspondence obtained from the Poincaré bundle. This implies the integral Hodge conjecture for one-cycles for products of Jacobians of curves. The arguments also work over other fields and yield the integral Tate conjecture for one-cycles for Jacobians of curves over the separable closure of a finitely generated field.

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1. Introduction

This thesis consists of six parts, which are based on [24–28, 30]. Each part constitutes a chapter of the appendix. In this chapter we give a global introduction to the topics of the six parts, thereby also linking the individual chapters. The next six chapters are summaries of the results of the respective parts. The eighth chapter serves as a conclusion of this thesis, relating it to current research and discussing possible next directions.

1.1. Holomorphic symplectic manifolds

1.1.1. Generalities

We will start by introducing the manifolds which will be primarily studied in this thesis. For the necessary backgrounds on complex geometry we refer to [96].

Let X be a compact Kähler manifold. For most of what follows the algebraically inclined reader can assume X to be a projective manifold. To X we can associate its *canonical bundle* ω_X, which is the determinant line bundle $\omega_X \coloneqq \det \Omega_X$ of the cotangent bundle $\Omega_X = \mathcal{T}_X^\vee$ of X. An important class of compact Kähler manifolds X are those for which the canonical bundle is trivial, i.e. ω_X is isomorphic to the trivial holomorphic line bundle $\mathcal{O}_X$. These are called *Calabi–Yau* manifolds.

The Beauville–Bogomolov decomposition theorem [21, Thm. 2] asserts that for a Calabi–Yau manifold there exists a finite étale cover $\pi\colon Z \to X$ such that Z decomposes

$$Z \cong T \times \prod_i Y_i \times \prod_j M_j. \tag{1.1.1}$$

Here, $T = \mathbb{C}^k/\Lambda$ is a complex torus for some lattice $\Lambda \subset \mathbb{C}^k$ of full rank and the Y_i are strict Calabi–Yau manifolds, i.e. each Y_i is a Calabi–Yau manifold such that $\Omega_{Y_i}^r$ admits holomorphic sections only for $r \in \{0, \dim Y_i\}$. The last factors M_j appearing in the decomposition are hyper-Kähler manifolds. A compact Kähler manifold X is called *hyper-Kähler* if it is simply connected and its space $\mathrm{H}^0(X, \Omega_X^2)$ of holomorphic two-forms is spanned by a nowhere-degenerate symplectic form.

We say that a manifold X is *holomorphic symplectic* if it admits a holomorphic two-form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ which defines a non-degenerate symplectic form at each point $x \in X$. In particular, for such X the symplectic form σ induces an isomorphism

$$\sigma\colon \mathcal{T}_X \cong \Omega_X.$$

Since strict Calabi–Yau manifolds of dimension at least three do not admit holomorphic two-forms, the factors in the decomposition (1.1.1) for holomorphic symplectic manifolds are either complex tori or hyper-Kähler manifolds. Most of the time, we will investigate holomorphic symplectic manifolds in this thesis.

1.1.2. Examples

The existence of a non-degenerate holomorphic symplectic form forces the dimension of a holomorphic symplectic manifold X to be even. Thus, the lowest complex dimension in which non-trivial examples of such manifolds can exist is two.

There are two types of surfaces which are holomorphic symplectic. The first one are abelian surfaces (or, more generally, two-dimensional complex tori) and the second one K3 surfaces. For an introduction to complex tori in general we recommend [31] and for the theory of K3 surfaces [100].

A classic example of a K3 surface is the Fermat quartic

$$Z(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3.$$

Any two K3 surfaces are deformation-equivalent to another showing that the topological invariants of K3 surfaces are always the same. The most prominent result for K3 surfaces is arguably the Global Torelli theorem stating that two K3 surfaces S and S' are isomorphic if and only if the second integral cohomology groups $\mathrm{H}^2(S, \mathbb{Z})$ and $\mathrm{H}^2(S', \mathbb{Z})$ are Hodge isometric. We will introduce the intersection form and Hodge structure in Section 1.2.3.

Let us now concentrate on the case of complex dimension greater than two. We again have even-dimensional complex tori, which are quite well understood and examples are easy to construct. For hyper-Kähler manifolds, only a few constructions are known, some of which we briefly explain.

Firstly, one can start with a K3 surface S and consider the Hilbert scheme $S^{[n]}$ of subschemes of length n. It is a crepant resolution of the symmetric product $S^{(n)} = S^n/\mathfrak{S}_n$ and was proven by Fujiki [74] for $n = 2$ and in full generality by Beauville [21] to be a hyper-Kähler manifold. This yields examples in all even complex dimensions starting with four. Other constructions such as the Fano variety of lines of a cubic fourfold turn out to be deformation-equivalent to the Hilbert scheme [23].

Similarly, Beauville constructed hyper-Kähler manifolds starting from an abelian surface (or complex two-dimensional torus) A. In this case the Hilbert scheme $A^{[n]}$ is not simply connected and admits an isotrivial fibration

$$\Sigma\colon A^{[n]} \to A$$

by summing over the support of the corresponding subscheme. The fibres of Σ are again hyper-Kähler manifolds called *generalized Kummer manifolds*. This yields again examples in all even dimensions greater than two.

There are two more examples in dimension six respectively ten. These were discovered by O'Grady [170, 171] as desingularizations of certain singular moduli spaces of stable sheaves on abelian respectively K3 surfaces. Up to deformations this list is for the time being exhaustive.

1.2. Hyper-Kähler manifolds

Throughout the thesis a particular emphasis is put on the study of hyper-Kähler manifolds. Therefore, we will discuss this class of manifolds in more detail in this chapter.

1.2.1. Second cohomology

Let X be a hyper-Kähler manifold of dimension $2n$. Its cohomology $\mathrm{H}^*(X, \mathbb{Q})$ has been investigated by many authors, for example [21, 33, 34, 75, 130, 207]. In this subsection we focus on the cohomological degree two part.

The second integral cohomology $\mathrm{H}^2(X, \mathbb{Z})$ of a hyper-Kähler manifold X is endowed with a quadratic form $\mathrm{q} = \mathrm{q}_X$ called the *Beauville–Bogomolov–Fujiki (BBF) form*. It is the unique primitive integral quadratic form which can be defined up to sign by satisfying

$$\int_X \lambda^{2n} = C_X \mathrm{q}(\lambda)^n$$

for all $\lambda \in \mathrm{H}^2(X, \mathbb{Z})$. Here, $C_X > 0$ is a positive rational constant only depending on the deformation class of X. It is called the *Fujiki constant* of X. The BBF form has signature $(3, b_2(X) - 3)$.

The space $\mathrm{H}^2(X, \mathbb{Z})$ together with the Beauville–Bogomolov–Fujiki form and its Hodge structure inherited from X being Kähler encodes a lot of information about the manifold. Here are some examples of this phenomenon.

(i) The very general hyper-Kähler manifold is not projective. However, those that are are dense in moduli. To determine whether or not a given hyper-Kähler manifold X is projective there is the following criterion due to Huybrechts [95, Thm. 3.11]: X is projective if and only if there exists a class

$$\lambda \in \mathrm{H}^{1,1}(X, \mathbb{Z}) := \mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X)$$

with $\mathrm{q}(\lambda) > 0$.

(ii) The Kähler cone of a very general hyper-Kähler manifold equals its positive cone $\mathcal{C}_X \subset \mathrm{H}^2(X, \mathbb{R})$, where $\mathcal{C}_X$ is the connected component of the open subset

$$\{\omega \mid \mathrm{q}(\omega) > 0\} \subset \mathrm{H}^2(X, \mathbb{R})$$

containing a Kähler class [95, Cor. 5.7].

(iii) Deformations of hyper-Kähler manifolds are governed by the period map taking values in the open subset

$$\{\sigma \in \mathbb{P}(\mathrm{H}^2(X, \mathbb{C})) \mid \mathrm{q}(\sigma) = 0, \mathrm{q}(\sigma, \bar{\sigma}) > 0\} \subset \mathbb{P}(\mathrm{H}^2(X, \mathbb{C}))$$

of the quadric defined by q. An extensive study of the period map has lead to the Global Torelli theorem for hyper-Kähler manifolds. For more details we refer to [99, 211].

Due to its importance, the second cohomology has been intensively studied. In all known examples the isomorphism type of $(\mathrm{H}^2(X, \mathbb{Z}), \mathrm{q})$ has been determined [21, 187]. For example in the case of K3 surfaces S we have

$$(\mathrm{H}^2(S, \mathbb{Z}), \mathrm{q}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

for U the hyperbolic plane and $E_8(-1)$ the unique even unimodular negative definite lattice of rank eight. In complex dimension four Guan [84] has determined the general bound $b_2(X) \leq 23$ for the second Betti number. In higher dimensions, a conjectural bound has been put forward [122]. In Chapter D we also obtain a bound on the second Betti number. To explain its ingredients, we need to recall more facts.

1.2.2. Verbitsky component and Fujiki classes

We now turn our attention to a subalgebra of the cohomology $\mathrm{H}^*(X, \mathbb{Q})$ of a $2n$-dimensional hyper-Kähler manifold X.

Starting from the second rational cohomology group $\mathrm{H}^2(X, \mathbb{Q})$ one can define the *Verbitsky component* $\mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{H}^*(X, \mathbb{Q})$ of X as the subalgebra generated by $\mathrm{H}^2(X, \mathbb{Q})$. Bogomolov [34] and Verbitsky [207] have determined the ring structure of $\mathrm{SH}(X, \mathbb{C}) = \mathrm{SH}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ and proven the ring isomorphism

$$\mathrm{SH}(X, \mathbb{C}) \cong \mathrm{Sym}^\bullet \mathrm{H}^2(X, \mathbb{C}) / \langle x^{n+1} \mid \mathrm{q}(x) = 0 \rangle.$$

In particular, the intersection product on the full cohomology $\mathrm{H}^*(X, \mathbb{C})$ restricts to a non-degenerate pairing on $\mathrm{SH}(X, \mathbb{C})$. This gives an orthogonal decomposition

$$\mathrm{H}^*(X, \mathbb{C}) \cong \mathrm{SH}(X, \mathbb{C}) \oplus \mathrm{SH}(X, \mathbb{C})^\perp$$

yielding an analogous decomposition

$$\mathrm{H}^*(X, \mathbb{Q}) \cong \mathrm{SH}(X, \mathbb{Q}) \oplus \mathrm{SH}(X, \mathbb{Q})^\perp \tag{1.2.1}$$

over the rational numbers.

The dual of the BBF form $\mathsf{q} \in \mathrm{H}^4(X, \mathbb{Q})$ lies in the Verbitsky component $\mathrm{SH}^4(X, \mathbb{Q})$. Its powers $\mathsf{q}^k \in \mathrm{SH}^{4k}(X, \mathbb{Q})$ span the subspace of elements of the Verbitsky component which stay of type $(2k, 2k)$ for all first-order deformations of X [130, Prop. 2.14]. Thus, for an arbitrary class $\gamma \in \mathrm{H}^{4k}(X, \mathbb{Q})$ which stays of type $(2k, 2k)$ on all first-order deformations of X the projection of γ via (1.2.1) to the Verbitsky component is a multiple of q^k. This implies that there exists a constant $C(\gamma) \in \mathbb{Q}$ such that for all $\omega \in \mathrm{H}^2(X, \mathbb{Q})$ we must have

$$\int_X \gamma \omega^{2n-2k} = C(\gamma) \mathrm{q}(\omega)^{n-k}. \tag{1.2.2}$$

The constant $C(\gamma)$ is called the *generalized Fujiki constant* of γ. The case $\gamma = 1 := [X]$ recovers the Fujiki constant $C(1) = C_X$. Fujiki [75] and Huybrechts [95] introduced these numbers.

The property (1.2.2) enables us to define the *Riemann–Roch polynomial* $\mathrm{RR}_X(q)$ of X. Namely, recall that the Hirzebruch–Riemann–Roch theorem asserts that for a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ we have

$$\chi(X, \mathcal{L}) = \sum_i (-1)^i \dim \mathrm{H}^i(X, \mathcal{L}) = \int_X \mathrm{ch}(L) \mathsf{td} \tag{1.2.3}$$

for $\mathsf{td} := \mathsf{td}_X$ the Todd class of X. The right hand side of (1.2.3) can be rewritten as

$$\int_X \mathrm{ch}(L)\mathsf{td} = \sum_{i=0}^n \int_X \frac{\mathrm{c}_1(\mathcal{L})^{2i}}{(2i)!} \mathsf{td}_{2n-2i} = \sum_{i=0}^n \frac{C(\mathsf{td}_{2n-2i})}{(2i)!} \mathrm{q}(\mathrm{c}_1(\mathcal{L}))^i$$

where $\mathsf{td}_{2n-2i} \in \mathrm{H}^{4n-4i}(X, \mathbb{Q})$ denotes the degree $4n - 4i$ component of td. The Riemann–Roch polynomial of X is now defined as

$$\mathrm{RR}_X(q) := \sum_{i=0}^n \frac{C(\mathsf{td}_{2n-2i})}{(2i)!} q^i.$$

The above discussion implies that the Riemann–Roch polynomial satisfies

$$\mathrm{RR}_X(\mathrm{q}(\mathrm{c}_1(\mathcal{L}))) = \chi(X, \mathcal{L})$$

for all line bundles $\mathcal{L} \in \mathrm{Pic}(X)$.

One of our main results in Chapter D is a bound on the second Betti number in all dimensions depending only on the two generalized Fujiki constants of $\mathrm{c}_2(X)^2$ and $\mathrm{c}_4(X)$. It also determines whether or not the second Chern class $\mathrm{c}_2(X)$ lies inside the Verbitsky component $\mathrm{SH}(X, \mathbb{Q})$. We relate these questions to the coefficients of the Riemann–Roch polynomial of X and discuss potential restrictions on its coefficients as well as on the generalized Fujiki constants of (products of) Chern class. For more information on the results of [30] we refer to Chapter 5.

1.2.3. LLV algebra and Mukai lattice

We now consider the full cohomology $\mathrm{H}^*(X, \mathbb{Q})$ of a hyper-Kähler manifold of dimension $2n$. This space is equipped with an action of a Lie algebra, which will also yield another perspective on the Verbitsky component. The results are mostly due to Looijenga–Lunts [130] and [207], see [81] and [199] for a modern account.

Let $\omega \in \mathrm{H}^2(X, \mathbb{R})$ be a Kähler class of a hyper-Kähler manifold of dimension $2n$. The Hard Lefschetz theorem asserts that the operator

$$e_\omega := \omega \cup \_ \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{R}))$$

given by cupping with ω induces isomorphisms

$$e_\omega^k \colon \mathrm{H}^{2n-k}(X, \mathbb{R}) \cong \mathrm{H}^{2n+k}(X, \mathbb{R}) \tag{1.2.4}$$

for all $k \geq 0$. Recall the dual Lefschetz operator $\Lambda_\omega \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{R}))$ and the cohomological grading operator $h \in \mathrm{End}(X, \mathbb{R})$ defined by acting on $\mathrm{H}^{2n+k}(X, \mathbb{R})$ via $k \cdot \mathrm{id}$. The Hard Lefschetz isomorphisms (1.2.4) are equivalent to saying that the three elements $(e_\omega, h, \Lambda_\omega)$ generate a Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{R}))$ isomorphic to the real Lie algebra $\mathfrak{sl}_2$.

What happens if we do not consider only one Kähler class at a time? This question has been investigated by Looijenga–Lunts [130] and Verbitsky [207]. It has been observed in [199] that their results descend to cohomology with rational coefficients, which is the setting we will now explain.

More generally, we can not only look at Kähler classes, but at any class $\omega \in \mathrm{H}^2(X, \mathbb{Q})$ satisfying (1.2.4) for all $k \geq 0$, that is

$$e_\omega^k \colon \mathrm{H}^{2n-k}(X, \mathbb{Q}) \cong \mathrm{H}^{2n+k}(X, \mathbb{Q}).$$

We say that such a class ω is a *Hard Lefschetz class* or satisfies the *Hard Lefschetz property.* For every such class ω we obtain a rational $\mathfrak{sl}_2$-triple $(e_\omega, h, \Lambda_\omega)$ generating a Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ isomorphic to the rational Lie algebra $\mathfrak{sl}_2$. We define the *Looijenga–Lunts–Verbitsky (LLV)* Lie algebra

$$\mathfrak{g}(X) \subset \mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$$

of X as the Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ generated by all such $\mathfrak{sl}_2$-triples $(e_\omega, h, \Lambda_\omega)$ for all Hard Lefschetz classes $\omega \in \mathrm{H}^2(X, \mathbb{Q})$.

The LLV algebra and its applications to the cohomology of hyper-Kähler manifolds will be fundamental in Chapters A, B and C. Let us, therefore, study this Lie algebra further.

For this we consider the vector space

$$\tilde{\mathrm{H}}(X,\mathbb{Q}) \coloneqq \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta$$

of dimension $b_2(X)+2$. We upgrade it to a quadratic space by equipping it with the quadratic form $\tilde{\mathrm{q}}$ which restricts to $\mathrm{H}^2(X,\mathbb{Q})$ as the BBF form q. Moreover, the classes α and β are orthogonal to $\mathrm{H}^2(X,\mathbb{Q})$ and satisfy

$$\tilde{\mathrm{q}}(\alpha) = \tilde{\mathrm{q}}(\beta) = 0, \quad \tilde{\mathrm{q}}(\alpha,\beta) = -1.$$

This vector space also has a grading by declaring α of degree zero, $\mathrm{H}^2(X,\mathbb{Q})$ of degree two, and β of degree four. Finally, $\tilde{\mathrm{H}}(X,\mathbb{Q})$ can also be endowed with two Hodge structures. One is pure of weight two and extends the known weight-two Hodge structure on $\mathrm{H}^2(X,\mathbb{Q})$. That is

$$\tilde{\mathrm{H}}^{2,0}(X) \coloneqq \mathrm{H}^{2,0}(X), \quad \tilde{\mathrm{H}}^{0,2}(X) \coloneqq \mathrm{H}^{0,2}(X), \quad \tilde{\mathrm{H}}^{1,1}(X) \coloneqq \mathrm{H}^{1,1}(X) \oplus \mathbb{C}\alpha \oplus \mathbb{C}\beta. \tag{1.2.5}$$

On the other hand, we can include the grading into the picture and consider $\tilde{\mathrm{H}}(X,\mathbb{Q})$ as the sum of three Hodge structures of weight zero, two, and four respectively. In this case $\tilde{\mathrm{H}}(X,\mathbb{Q})$ admits a Hodge diamond

$$\begin{array}{ccc} & \mathbb{C}\alpha & \\ \mathrm{H}^{2,0}(X) & \mathrm{H}^{1,1}(X) & \mathrm{H}^{0,2}(X) \\ & \mathbb{C}\beta & \end{array} \tag{1.2.6}$$

mimicking the one of a K3 surface.

In both cases the sub-Hodge structure $\mathrm{H}^2(X,\mathbb{Q})$ is equipped with its standard weight-two Hodge structure and α and β are both Hodge classes, i.e. they are both of type (p,p) independent of the Hodge structure of X. Each of the two perspectives will be used in this thesis and it will be clear from the context which one is meant. Most of the time we will consider the Hodge structure (1.2.5).

The quadratic space $\tilde{\mathrm{H}}(X,\mathbb{Q})$ appears with different names in the literature such as Mukai completion, rational Mukai lattice or extended Mukai lattice. We will call it the *Mukai lattice* of X throughout the next chapters.

Let us come back to the LLV algebra $\mathfrak{g}(X)$. We introduced the Mukai lattice at this point to explain the structure of $\mathfrak{g}(X)$. Namely, Looijenga–Lunts, and Verbitsky have shown that there is a Lie algebra isomorphism

$$\mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \tag{1.2.7}$$

(again, in [130, 207] this is only proven over the real numbers; the result holds true with rational coefficients as explained in [199]). Let us make the isomorphism (1.2.7) explicit by explaining how operators of $\mathfrak{g}(X)$ act on the Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Q})$, which is the standard representation of $\mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$.

For $\omega \in \mathrm{H}^2(X, \mathbb{Q})$ the induced operator e_ω satisfies

$$\alpha \mapsto \omega, \quad \lambda \mapsto \mathrm{q}(\lambda, \omega)\beta, \quad \beta \mapsto 0$$

for all $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$. The grading operator h acts trivially on $\mathrm{H}^2(X, \mathbb{Q})$. It multiplies α with -2 and β with 2.

Note that the adjoint action of h equips $\mathfrak{g}(X)$ with a grading compatible with the action on $\mathrm{H}^*(X, \mathbb{Q})$. That is, an element $f \in \mathfrak{g}(X)$ is of degree k if and only if it sends elements $x \in \mathrm{H}^j(X, \mathbb{Q})$ to elements contained inside $\mathrm{H}^{j+k}(X, \mathbb{Q})$. We will denote the operators of degree k by $\mathfrak{g}(X)_k$. One consequence of the isomorphism (1.2.7) is that $\mathfrak{g}(X)$ decomposes

$$\mathfrak{g}(X) = \mathfrak{g}(X)_2 \oplus \mathfrak{g}(X)_0 \oplus \mathfrak{g}(X)_{-2}.$$

In particular, for two Lefschetz classes $\omega, \omega' \in \mathrm{H}^2(X, \mathbb{Q})$ the corresponding dual Lefschetz operators $\Lambda_\omega, \Lambda_{\omega'}$ commute, that is

$$[\Lambda_\omega, \Lambda_{\omega'}] = 0.$$

Moreover, the degree zero part further decomposes

$$\mathfrak{g}(X)_0 = \bar{\mathfrak{g}}(X) \oplus \mathbb{Q}h$$

where $\bar{\mathfrak{g}}(X) = [\mathfrak{g}(X)_0, \mathfrak{g}(X)_0]$ is the semisimple part. The Lie subalgebra $\bar{\mathfrak{g}}(X)$ acts on $\mathrm{H}^*(X, \mathbb{Q})$ via derivations and we have

$$\bar{\mathfrak{g}}(X) \cong \mathfrak{so}(\mathrm{H}^2(X, \mathbb{Q})).$$

The existence of the action of the LLV algebra $\mathfrak{g}(X)$ on the cohomology $\mathrm{H}^*(X, \mathbb{Q})$ implies that the latter decomposes

$$\mathrm{H}^*(X, \mathbb{Q}) \cong \bigoplus_\lambda V_\lambda \tag{1.2.8}$$

into irreducible $\mathfrak{g}(X)$-representations V_λ. This has been recently investigated in [81]. The most prominent irreducible representation is the Verbitsky component. Let us discuss the subalgebra $\mathrm{SH}(X, \mathbb{Q})$ from the viewpoint of the LLV algebra.

The Verbitsky component comes equipped with a Hodge structure as well as a grading inherited from the inclusion $\mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{H}^*(X, \mathbb{Q})$. We will upgrade it to a quadratic space by introducing the *Mukai pairing* q_{SH} defined via

$$\mathrm{q}_{\mathrm{SH}}(\omega_1 \cdots \omega_m, \mu_1 \cdots \mu_{2n-m}) = (-1)^m \int_X \omega_1 \cdots \omega_m \mu_1 \cdots \mu_{2n-m}$$

for all $\omega_i, \mu_j \in \mathrm{H}^2(X, \mathbb{Q})$. Similarly, the space $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ is equipped with a bilinear form

$$\mathrm{q}_{[n]}(x_1 \cdots x_n, y_1 \cdots y_n) = (-1)^n c_X \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \tilde{\mathrm{q}}(x_i, y_{\sigma(i)}),$$

where

$$c_X \coloneqq C_X \frac{2^n n!}{(2n)!}$$

denotes the *small Fujiki constant.* The action of the LLV algebra $\mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ on $\tilde{\mathrm{H}}(X,\mathbb{Q})$ extends to one on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ by derivations. The important observation now is that there exists an injective morphism

$$\psi\colon \mathrm{SH}(X,\mathbb{Q}) \hookrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

of $\mathfrak{g}(X)$-modules sitting inside a short exact sequence

$$0 \to \mathrm{SH}(X,\mathbb{Q}) \xrightarrow{\psi} \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \xrightarrow{\Delta} \mathrm{Sym}^{n-2}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to 0 \tag{1.2.9}$$

of $\mathfrak{g}(X)$-modules. Here, Δ is the Laplace operator given by

$$v_1 \dots v_n \mapsto \sum_{i<j} \tilde{\mathrm{q}}(v_i, v_j) v_1 \dots \hat{v_i} \dots \hat{v_j} \dots v_n.$$

The morphism ψ respects the structures we have introduced. That is, ψ is an isometric inclusion of quadratic spaces, it is a morphism of $\mathfrak{g}(X)$-modules, and it is a graded morphism which respects the Hodge structures present on $\mathrm{SH}(X,\mathbb{Q})$ and $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$.

Although it first appeared more than two decades ago, the LLV algebra has experienced a re-emergence in recent years. Applications of $\mathfrak{g}(X)$ in the realm of hyper-Kähler geometry vary such as computations of Hodge numbers [81], topology of Lagrangian fibrations [198], or proving parts of Beauville's conjecture on the behaviour of the cycle class map [161, 166] to name a few.

To relate the LLV algebra to our results and bounds on invariants of hyper-Kähler manifolds in Chapter D, note that the existence of the decomposition (1.2.8) forces restrictions on the possible Hodge and Betti numbers. This has been studied and refined for example in [122].

Moreover, as mentioned before, the Lie algebra $\mathfrak{g}(X)$ also takes up a prominent role in Chapters A, B and C. For example, in Chapter C we introduce a new class of sheaves and objects on hyper-Kähler manifolds. The definition is given purely in terms of the LLV algebra and the Mukai lattice. In a nutshell, the Hodge diamond (1.2.6) of the Mukai lattice does not only mimic the shape of the Hodge diamond of a K3 surface, but also enables one to obtain similar statements valid on K3 surfaces also on higher-dimensional hyper-Kähler manifolds.

To name already one example at this point, the Mukai vector morphism on a K3 surface S induces a morphism

$$\mathrm{Coh}(S) \to \tilde{\mathrm{H}}(S,\mathbb{Q}), \quad \mathcal{E} \mapsto v(\mathcal{E}) \coloneqq \mathrm{ch}(\mathcal{E})\mathsf{td}^{1/2}.$$

This association is used frequently when investigating K3 surfaces, for example in the theory of moduli spaces of stable sheaves or in the study of derived categories. In Chapters B and C we explore the question of whether and when such a morphism with image in the Mukai lattice exists for hyper-Kähler manifolds of dimension greater than two. For the precise statements and implications we refer to the separate introductions in Chapters 3 and 4.

1.3. Derived categories

To relate and explain the contents of the other chapters we will introduce some background on derived categories of smooth projective manifolds over the complex numbers. For a thorough introduction see [97]. In this section all functors are implicitly derived.

1.3.1. Fourier–Mukai transforms and cohomology

Let X be a smooth projective variety. We associate to it its *derived category* $\mathrm{D^b}(X)$ which is the bounded derived category of coherent sheaves $\mathrm{D^b}(X) \coloneqq \mathrm{D^b}(\mathrm{Coh}(X))$ on X. This algebraic invariant attached to a variety X captures many (geometric) features of X as well as reveals certain new ones.

We now consider X and Y two smooth projective varieties together with an object $\mathcal{E} \in \mathrm{D^b}(X \times Y)$. We associate to $\mathcal{E}$ the *Fourier–Mukai transform* $\mathsf{FM}_{\mathcal{E}}$ with *Fourier–Mukai kernel* $\mathcal{E}$ which is the exact functor $\mathsf{FM}_{\mathcal{E}} \colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ defined via

$$\mathcal{F} \mapsto p_{Y*}\left(p_X^* \mathcal{F} \otimes \mathcal{E}\right)$$

with p_X and p_Y the respective projections. For example, taking $X = Y$ and $\mathcal{E} = \mathcal{O}_\Delta$ the structure sheaf of the diagonal yields the identity functor $\mathsf{FM}_{\mathcal{E}} \simeq \mathrm{id}_X$. Another example is the Fourier–Mukai kernel $\Delta_* \mathcal{L}$ for some line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ which induces the autoequivalence

$$\mathsf{FM}_{\Delta_* \mathcal{L}} \simeq \mathcal{L} \otimes \_$$

given by tensoring with $\mathcal{L}$.

The importance of Fourier–Mukai functors stems from a result due to Orlov [175] saying that any fully faithful exact functor

$$\Phi \colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$$

is isomorphic to a Fourier–Mukai functor $\Phi \simeq \mathsf{FM}_{\mathcal{E}}$ with Fourier–Mukai kernel

$$\mathcal{E} \in \mathrm{D^b}(X \times Y),$$

which is unique up to isomorphism (Orlov required the existence of right and left adjoint functors, but this assumption can be dropped using [36]). In particular, any derived equivalence is a Fourier–Mukai transform.

The formalism of Fourier–Mukai transforms $\mathsf{FM}_{\mathcal{E}} \colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ between varieties X and Y induce maps on several invariants associated to the varieties. Most important to us is the *cohomological Fourier–Mukai transform* $\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}}$. Namely, a class $e \in \mathrm{H}^*(X \times Y, \mathbb{Q})$ induces similarly to above a morphism

$$\varphi_e \colon \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(Y, \mathbb{Q}), \quad x \mapsto p_{Y*}(p_X^* x \cdot e).$$

We then define $\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}} \coloneqq \varphi_{v(\mathcal{E})}$. The reason for using the Mukai vector $v(\mathcal{E})$ instead of only the Chern character $\mathrm{ch}(\mathcal{E})$ is justified by the following diagram

$$\begin{tikzcd}
\mathrm{D^b}(X) \arrow[r, "\mathsf{FM}_{\mathcal{E}}"] \arrow[d, "v"] & \mathrm{D^b}(Y) \arrow[d, "v"] \\
\mathrm{H}^*(X, \mathbb{Q}) \arrow[r, "\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}}"] & \mathrm{H}^*(Y, \mathbb{Q})
\end{tikzcd}$$

which is commutative by the Grothendieck–Riemann–Roch theorem.

In particular, a derived equivalence

$$\mathsf{FM}_{\mathcal{E}} \colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$$

induces naturally an isomorphism

$$\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}}\colon \mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(Y,\mathbb{Q})$$

of vector spaces. Note that $\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}}$ does neither preserve the cup product nor the grading of the cohomology rings. On the positive side, since

$$v(\mathcal{E}) \in \bigoplus_p \mathrm{H}^{p,p}(X \times Y) \cap \mathrm{H}^*(X \times Y, \mathbb{Q}),$$

the cohomological Fourier–Mukai transform induces for all i an isomorphism

$$\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}}\colon \bigoplus_{q-p=i} \mathrm{H}^{p,q}(X) \cong \bigoplus_{q-p=i} \mathrm{H}^{p,q}(Y). \tag{1.3.1}$$

Furthermore, Căldăraru [59] extended Mukai's observation [155] that the cohomological Fourier–Mukai transform preserves a certain quadratic form on the cohomology. In general, for an element

$$v = \sum_j v_j \in \bigoplus_j \mathrm{H}^j(X,\mathbb{C})$$

one defines the *dual of* v as

$$v^\vee := \sum_j \sqrt{-1}^j v_j \in \mathrm{H}^*(X,\mathbb{C}).$$

The *Mukai pairing* on $\mathrm{H}^*(X,\mathbb{C})$ is then defined as the quadratic form

$$\langle v, v' \rangle := \int_X \exp(\mathrm{c}_1(X)/2) \cdot v^\vee \cdot v'.$$

Let us comment on the definition. Firstly, the dual of an element $v \in \mathrm{H}^{2*}(X,\mathbb{Q})$ in the even cohomology can be defined already over the rational numbers. Moreover, for two elements $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ their *Euler pairing* $\chi(\mathcal{E},\mathcal{F})$ is defined via

$$\chi(\mathcal{E},\mathcal{F}) := \sum_i (-1)^i \dim \mathrm{Ext}^i(\mathcal{E},\mathcal{F}).$$

The Hirzebruch–Riemann–Roch formula then yields the equality

$$\chi(\mathcal{E},\mathcal{F}) = \langle v(\mathcal{E}), v(\mathcal{F}) \rangle.$$

Let us consider one case of interest to us, namely Calabi–Yau manifolds X. Since $\mathrm{c}_1(X) = \mathrm{c}_1(\mathcal{T}_X) = 0$, the Mukai pairing becomes in this case a non-degenerate symmetric pairing on the rational cohomology groups $\mathrm{H}^*(X,\mathbb{Q})$.

1.3.2. Geometric considerations

The derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a smooth projective variety is, furthermore, equipped with a *Serre functor* S_X. By definition, a Serre functor is an exact equivalence $S_X\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X)$ such that for all objects $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ there exist bi-functorial isomorphisms

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{E},\mathcal{F}) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{F}, S_X(\mathcal{E}))^\vee.$$

Serre Duality implies that we have $S_X \simeq \_ \otimes \omega_X[\dim X]$. All derived equivalences commute with the Serre functor.

How much information about X does the derived category $\mathrm{D}^{\mathrm{b}}(X)$ contain? An answer to this question in the case that ω_X or its dual is ample was given by Bondal and Orlov [35]. More precisely, consider any X for which ω_X is (anti-)ample and a smooth projective variety Y which is derived equivalent to X, i.e. there exists an exact equivalence $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$. Then, Y must in fact already be isomorphic to X. This does not hold true for all smooth projective varieties and we will encounter interesting equivalences between non-isomorphic (even non-birational) varieties in subsequent sections.

There are also other geometric information which are preserved under derived equivalences for arbitrary smooth projective varieties. For example, if X and Y are smooth projective varieties, then $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ implies that $\dim X = \dim Y$. This follows for example easily from Orlov's representability result together with the explicit form of the Serre functors which commute with Φ.

Moreover, a similar argument again involving the explicit form of the Serre functors shows that a derived equivalence $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ induces an isomorphism of canonical rings

$$R(X) \cong R(Y),$$

where we recall

$$R(X) \coloneqq \bigoplus_{i \geq 0} \mathrm{H}^0(X, \omega_X^i)$$

is the graded ring associated to non-zero sections of powers of the canonical bundle. In particular, one deduces equality of Kodaira dimensions of X and Y. Another related fact is that in the case of Kodaira dimension zero the order of the canonical line bundle viewed as an element in the Picard group is also a derived invariant. Therefore, varieties Y which are derived equivalent to Calabi–Yau manifolds X must also be Calabi–Yau.

1.3.3. Auto-equivalences

To the derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a variety X we can associate its *group of auto-equivalences* $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ which is the group of isomorphism classes of exact equivalences

$$\Phi \simeq \mathsf{FM}_{\mathcal{E}}\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X).$$

This group will play an important role in Chapters C and E.

There is for any variety X an injective group homomorphism

$$\mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \hookrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)).$$

Therefore, the elements of this subgroup are frequently called *standard (auto-)equivalences*. Looking again at varieties X with (anti-)ample canonical bundle, Bondal and Orlov have shown that the group of standard equivalences for X agrees with $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ [35]. In this sense, their derived categories are well understood.

The situation can become more interesting, for example, in the case of Calabi–Yau manifolds. Let us introduce one example of a derived equivalence which will appear frequently in the subsequent chapters. We say that an object $\mathcal{E} \in \mathrm{D^b}(X)$ is *spherical* if its Ext algebra satisfies

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \cong \mathrm{H}^*(S^{\dim X}, \mathbb{C}),$$

i.e. the Ext algebra is as a ring isomorphic to the complex cohomology of a sphere of the same dimension (therefore the name). Any line bundle on a strict Calabi–Yau manifold is spherical. Seidel and Thomas [196] have constructed for each spherical object $\mathcal{E}$ an auto-equivalence $\mathsf{ST}_{\mathcal{E}}$ called the *spherical twist* associated to $\mathcal{E}$. The Fourier–Mukai kernel $\mathcal{F}$ of $\mathsf{ST}_{\mathcal{E}} \simeq \mathsf{FM}_{\mathcal{F}}$ can be defined by the distinguished triangle

$$\mathcal{E}^\vee \boxtimes \mathcal{E} \xrightarrow{\mathrm{ev}} \Delta_* \mathcal{O}_X \to \mathcal{F} \tag{1.3.2}$$

in $\mathrm{D^b}(X \times X)$. That is, $\mathcal{F}$ is isomorphic to the cone of the natural evaluation morphism. The defining distinguished triangle (1.3.2) lets one also easily calculate the corresponding action on cohomology

$$\mathsf{ST}^{\mathrm{H}}_{\mathcal{E}} \colon \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(X, \mathbb{Q}), \quad v \mapsto v - \langle v(\mathcal{E}), v \rangle \cdot v(\mathcal{E}).$$

In particular, since

$$\langle v(\mathcal{E}), v(\mathcal{E}) \rangle = 1 + (-1)^{\dim X},$$

for even-dimensional varieties the spherical twist acts on $\mathrm{H}^*(X, \mathbb{Q})$ as the reflection in the hyperplane orthogonal to $v(\mathcal{E})$. One can deduce from this that in the even-dimensional case

$$\mathsf{ST}^{\mathrm{H}}_{\mathcal{E}} \circ \mathsf{ST}^{\mathrm{H}}_{\mathcal{E}} = \mathrm{id} \in \mathrm{Aut}(\mathrm{H}^*(X, \mathbb{Q}))$$

and $\mathsf{ST}_{\mathcal{E}} \circ \mathsf{ST}_{\mathcal{E}}$ is not isomorphic to a shift.

1.3.4. Hochschild (co)homology and polyvector fields

There are other cohomology groups and vector spaces attached to (the derived category of) X, which will be introduced next. This will play an important role in Chapters A and C. For more information we refer to [58–60].

Let X again be a smooth projective variety. We can associate to it its *Hochschild cohomology* $\mathrm{HH}^*(X)$. The Hochschild cohomology groups are defined via

$$\mathrm{HH}^*(X) \coloneqq \mathrm{Ext}^*_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X),$$

where $\Delta \colon X \hookrightarrow X \times X$ is the diagonal embedding. Composition of morphisms turns the Hochschild cohomology of X into a graded ring. Next we introduce the *Hochschild homology* $\mathrm{HH}_*(X)$ of X as

$$\mathrm{HH}_*(X) \coloneqq \mathrm{Ext}^*_{X \times X}(\Delta_* \omega_X^{-1}[-\dim X], \Delta_* \mathcal{O}_X).$$

Composition of morphisms turns $\mathrm{HH}_*(X)$ into a graded module over $\mathrm{HH}^*(X)$. Note that $\Delta_* \mathcal{O}_X$ as well as $\Delta_* \omega_X^{-1}[-\dim X]$ are Fourier–Mukai kernels of auto-equivalences of X. This allows to interpret elements in the Hochschild (co)homology as natural transformations giving rise to a ring morphism

$$\chi_{\mathcal{E}} \colon \mathrm{HH}^*(X) \to \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{E}, \mathcal{E}[*])$$

called the *obstruction map* of $\mathcal{E}$. Moreover, associating to X its Hochschild (co)homology is via conjugation functorial for derived equivalences. That is, for a derived equivalence

$$\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$$

we obtain a ring isomorphism

$$\Phi^{\mathrm{HH}}\colon \mathrm{HH}^*(X) \cong \mathrm{HH}^*(Y), \quad f \mapsto \Phi \circ f \circ \Phi^{-1}$$

as well as an isomorphism

$$\Phi_{\mathrm{HH}}\colon \mathrm{HH}_*(X) \cong \mathrm{HH}_*(Y), \quad f \mapsto \Phi \circ f \circ \Phi^{-1}$$

compatible with the module structure.

How do these two invariants look like? This is answered by the *Hochschild–Konstant–Rosenberg (HKR)* isomorphisms. For this let us introduce the *ring of polyvector fields*

$$\mathrm{HT}^*(X) \coloneqq \bigoplus_{p+q=*} \mathrm{H}^q(X, \Lambda^p \mathcal{T}_X).$$

The HKR isomorphisms now identify the Hochschild cohomology of X with the ring of polyvector fields, i.e. there exists an isomorphism of vector spaces

$$I^{\mathrm{HKR}}\colon \mathrm{HH}^*(X) \cong \mathrm{HT}^*(X),$$

as well as the Hochschild homology groups of X with the de Rham cohomology groups

$$I_{\mathrm{HKR}}\colon \mathrm{HH}_*(X) \cong \mathrm{H}\Omega_*(X) \coloneqq \bigoplus_{q-p=*} \mathrm{H}^q(X, \Omega_X^p).$$

Note that $\mathrm{HT}^*(X)$ is also a ring and the de Rham cohomology is naturally a module over $\mathrm{HT}^*(X)$ via contraction of vector fields

$$\mathrm{HT}^*(X) \times \mathrm{H}\Omega_*(X) \to \mathrm{H}\Omega_*(X), \quad (v, x) \mapsto v \lrcorner x.$$

However, in general the HKR isomorphisms do not respect the ring and module structure present on both sides. It was conjectured by Căldăraru and proven in [47] that this holds true for the *modified HKR isomorphisms* I^{K} respectively I_{K}. These are obtained from the standard HKR isomorphisms by twisting with the square root of the Todd class $\mathsf{td}^{1/2}$, i.e.

$$I^{\mathrm{K}}\colon \mathrm{HH}^*(X) \xrightarrow{I^{\mathrm{HKR}}} \mathrm{HT}^*(X) \xrightarrow{\mathsf{td}^{-1/2} \lrcorner} \mathrm{HT}^*(X);$$

$$I_{\mathrm{K}}\colon \mathrm{HH}_*(X) \xrightarrow{I_{\mathrm{HKR}}} \mathrm{H}\Omega_*(X) \xrightarrow{\mathsf{td}^{1/2} \wedge} \mathrm{H}\Omega_*(X).$$

The compatibility of the ring and module structure means that for $x, y \in \mathrm{HH}^*(X)$ and $z \in \mathrm{HH}_*(X)$ we have

$$I^{\mathrm{K}}(x \circ y) = I^{\mathrm{K}}(x) \wedge I^{\mathrm{K}}(y), \quad I_{\mathrm{K}}(x \circ z) = I^{\mathrm{K}}(x) \lrcorner I_{\mathrm{K}}(z).$$

The situation becomes more accessible in the case that X is holomorphic symplectic of dimension $2n$. That is, a non-degenerate holomorphic two-form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ induces an isomorphism

$$\sigma\colon \mathcal{T}_X \cong \Omega_X.$$

This leads to the following chain of ring isomorphisms

$$\varphi\colon \mathrm{HT}^*(X) = \bigoplus_{p+q=*} \mathrm{H}^q(X, \Lambda^p \mathcal{T}_X) \cong \bigoplus_{p+q=*} \mathrm{H}^q(X, \Omega_X^p) \cong \mathrm{H}^*(X, \mathbb{C}). \tag{1.3.3}$$

Note that the composition φ is, moreover, graded. Furthermore, the top power $\sigma^n \in \mathrm{H}^0(X, \Omega_X^{2n})$ induces the graded isomorphism

$$\mathrm{HT}^*(X) \cong \mathrm{H}\Omega_*(X), \quad v \mapsto v \lrcorner \sigma^n$$

realizing the de Rham cohomology as a free module of rank one over $\mathrm{HT}^*(X)$ with generator σ^n. Thus, also $\mathrm{HH}_*(X)$ is in this case a free module of rank one over the Hochschild cohomology generated by any non-zero element in $\mathrm{HH}_{-2n}(X)$.

Another feature that this structure possesses is an analogue of the Chern character. More precisely, Căldăraru introduced for any object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ on an arbitrary smooth projective variety X its *Hochschild Chern character* $\mathrm{ch}^{\mathrm{HH}}(\mathcal{E}) \in \mathrm{HH}_0(X)$. It can be uniquely characterized by the property

$$\mathrm{Tr}_{X \times X}(\mu \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{Tr}_X(\mu_{\mathcal{E}})$$

for all elements $\mu \in \mathrm{HH}^*(X)$, where

$$\mu_{\mathcal{E}} \coloneqq \chi(\mathcal{E}) \in \mathrm{Ext}^*(\mathcal{E}, \mathcal{E}).$$

Here, $\mathrm{Tr}_{X \times X}$ and Tr_X are the trace morphisms on $X \times X$ respectively X obtained from the Serre Duality pairing. In addition, the HKR isomorphism identifies the Hochschild Chern character $\mathrm{ch}^{\mathrm{HH}}(\mathcal{E})$ with the usual Chern character $\mathrm{ch}(\mathcal{E})$, that is

$$I_{\mathrm{HKR}}(\mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{ch}(\mathcal{E}) \in \mathrm{H}^*(X, \mathbb{Q}).$$

This implies, in particular, the identity

$$I_{\mathrm{K}}(\mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = v(\mathcal{E}) \in \mathrm{H}^*(X, \mathbb{Q}).$$

Introducing Hochschild cohomology to the picture is the main point in Chapter A. We will consider following [201] a version of the LLV algebra for Hochschild cohomology for hyper-Kähler manifolds X. This will imply that the usual LLV algebra $\mathfrak{g}(X)$ is in a certain sense preserved by any auto-equivalence. Our results in Chapter B are building upon this observation. In addition, the Hochschild Chern character is frequently used in Chapter C. One result we obtain by applying the above is that there exist no spherical sheaves on any higher-dimensional hyper-Kähler manifold. Again, for more details we refer to the later chapters.

1.3.5. Abelian varieties and the Fourier transform

After having discussed the general structure and properties of derived categories we will now turn our attention to the derived categories of certain Calabi–Yau manifolds. In this section we will briefly explain a particular example of a derived equivalence first discovered by Mukai [154], which will be fundamental in Chapter F.

Let A be an abelian variety of dimension g, i.e. a projective complex torus. Attached to A is the *dual* abelian variety

$$\widehat{A} \coloneqq A^\vee \cong \mathrm{Pic}^0(A).$$

One can view $\widehat{A}$ as the moduli space of all line bundles of degree zero on A. Using this definition it can be shown that on the product

$$A \times \widehat{A}$$

there exists a (normalized) universal family $\mathcal{P}_A$ called the *Poincaré (line) bundle*. As such it satisfies the following universal property. Consider a variety Y and a line bundle

$$M \in \mathrm{Pic}(A \times Y).$$

Assume that M restricts to $\{0_A\} \times Y$ as the trivial line bundle and for all closed points $y \in Y$ the restricted line bundle $M_y \in \mathrm{Pic}(A)$ is of degree zero. Then, there exists a unique morphism

$$f_M \colon Y \to \widehat{A}$$

which satisfies

$$M \cong (\mathrm{id}_A \times f_M)^* \mathcal{P}_A.$$

The Fourier–Mukai transform $\mathcal{F}_A \coloneqq \mathsf{FM}_{\mathcal{P}_A}$ with Fourier–Mukai kernel the Poincaré bundle $\mathcal{P}_A$ is called the *Fourier transform*. Mukai first observed that the Fourier transform induces a derived equivalence

$$\mathcal{F}_A \colon \mathrm{D^b}(A) \cong \mathrm{D^b}(\widehat{A}).$$

Moreover, the composition

$$\mathrm{D^b}(A) \xrightarrow{\mathcal{F}_A} \mathrm{D^b}(\widehat{A}) \xrightarrow{\mathcal{F}_A} \mathrm{D^b}(A) \tag{1.3.4}$$

is isomorphic to $(-1)^*[-g]$, where -1 denotes the inversion coming from the group structure. For the second equivalence in (1.3.4) we interpreted $\mathcal{P}_A$ as a bundle on $\widehat{A} \times A$ via the natural isomorphism which interchanges the factors.

Thus, the Fourier transform yields many examples of derived equivalent varieties which may not even be birational. Furthermore, the induced cohomological Fourier transform

$$\mathsf{FM}^{\mathrm{H}}_{\mathcal{P}_A} \colon \mathrm{H}^*(A, \mathbb{Q}) \cong \mathrm{H}^*(\widehat{A}, \mathbb{Q})$$

restricts for all k to an isomorphism of integral cohomology groups

$$\mathsf{FM}^{\mathrm{H}}_{\mathcal{P}_A} \colon \mathrm{H}^k(A, \mathbb{Z}) \cong \mathrm{H}^{2g-k}(\widehat{A}, \mathbb{Z}),$$

which up to sign can be identified with the Poincaré duality isomorphism.

There are many other results on derived categories of abelian varieties and equivalences between them. We refer to [97] for an overview. We use the Poincaré bundle $\mathcal{P}_A$ and the Fourier transform in Chapter F. We briefly explain already here what we will study.

Recall the diagram (1.3.1) which shows the compatibility of a Fourier–Mukai transform

$$\mathsf{FM}_{\mathcal{E}} \colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$$

between derived categories and the corresponding cohomological Fourier–Mukai transform

$$\mathsf{FM}_{\mathcal{E}}^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(Y,\mathbb{Q})$$

on cohomology. One can consider as an intermediate step the morphism

$$\mathsf{FM}_{\mathcal{E}}^{\mathsf{A}}\colon \mathsf{A}(X) \cong \mathsf{A}(Y) \tag{1.3.5}$$

between the rational Chow groups of X and Y induced by the element

$$\mathrm{ch}(\mathcal{E}) \in \mathsf{A}(X \times Y).$$

This morphism is compatible as in (1.3.1) with $\mathsf{FM}_{\mathcal{E}}$ as well as $\mathsf{FM}_{\mathcal{E}}^{\mathrm{H}}$. Since the Chern character of $\mathcal{P}_A$ is an element in the integral cohomology $\mathrm{H}^*(A \times \widehat{A}, \mathbb{Z})$ and, in particular, induces an isomorphism between integral cohomology groups, a natural question is whether (1.3.5) in the case of the Poincaré bundle also already exists over the integers. We link this question in Chapter F with the integral Hodge conjecture for one-cycles. This implies, among other things, that the integral Hodge conjecture for one-cycles holds for (products of) Jacobians of smooth curves. Again, for a more thorough introduction to our results we refer to Chapter 7.

1.3.6. K3 surfaces

We now consider derived categories $\mathrm{D}^{\mathrm{b}}(S)$ of projective K3 surfaces S.

Recall that we have associated to S its Mukai lattice

$$\tilde{\mathrm{H}}(S,\mathbb{Q}),$$

which, in the surface case, is the usual cohomology $\mathrm{H}^*(S,\mathbb{Q})$ by identifying $\alpha = 1$ and $\beta = \mathsf{p}$. It is equipped with the usual grading and Hodge structure and we modified the cup product by a sign. In the following, we will diverge from the convention in Section 1.2.3 by a global sign and consider the bilinear form

$$\mathrm{q}((r\alpha, \lambda, s\beta), (r'\alpha, \lambda', s'\beta)) = \lambda\lambda' - rs' - sr'.$$

This is the standard notation for K3 surfaces, which is commonly used and stems from the fact that Mukai [155] introduced it in this form. We will also write v^2 for $\mathrm{q}(v,v)$.

Let us now consider two K3 surfaces S and S' together with a derived equivalence

$$\Phi\colon \mathrm{D}^{\mathrm{b}}(S) \cong \mathrm{D}^{\mathrm{b}}(S').$$

Mukai showed that the associated isomorphism

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(S,\mathbb{Q}) \cong \tilde{\mathrm{H}}(S',\mathbb{Q})$$

restricts to a Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(S,\mathbb{Z}) \cong \tilde{\mathrm{H}}(S',\mathbb{Z}) \tag{1.3.6}$$

between the integral Mukai lattices. Orlov in [175] proved the converse. That is, two K3 surfaces S and S' are derived equivalent if and only if there exists a Hodge isometry as in

(1.3.6). Thus, the Mukai lattice $\tilde{\mathrm{H}}(S,\mathbb{Z})$ determines the derived category completely similar to $\mathrm{H}^2(S,\mathbb{Z})$ determining the isomorphism type of S. This result is, therefore, sometimes called the Derived Global Torelli theorem.

One ingredient in Orlov's proof is the following construction of derived equivalences between K3 surfaces. If we start with S we can consider the moduli space $M_H^S(v)$ of H-stable sheaves $\mathcal{E}$ on S with Mukai vector $v(\mathcal{E}) = v$ for an ample line bundle H. For background on stability and moduli spaces in general we refer to [103] and for a survey in the case of K3 surfaces [100, Sec. 10]. We take a vector $v \in \tilde{\mathrm{H}}(S,\mathbb{Z})$ which is *primitive*, i.e. the quotient

$$\tilde{\mathrm{H}}(S,\mathbb{Z})/\mathbb{Z}v$$

is torsion-free, and assume the polarization H to be generic with respect to v. Under these circumstances the moduli space $M_H^S(v)$ is a smooth projective manifold of dimension $v^2 + 2$. In particular, in the case that v is isotropic meaning $v^2 = 0$, the space $M_H^S(v)$ is a surface, which turns out to be again a K3 surface. In some cases (depending on the Mukai vector v) there exists a universal family $\mathcal{E}$ on the product

$$M_H^S(v) \times S$$

and the moduli space is fine. When this happens the Fourier–Mukai transform $\mathsf{FM}_{\mathcal{E}}$ with Fourier–Mukai kernel the universal family $\mathcal{E}$ induces a derived equivalence

$$\mathsf{FM}_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(M_H^S(v)) \cong \mathrm{D}^{\mathrm{b}}(S).$$

This equivalence is in many cases non-trivial, i.e. $M_H^S(v)$ is frequently not isomorphic to S.

One consequence of the Derived Global Torelli theorem is that the derived category is governed by an even unimodular lattice together with a weight-two Hodge structure. This has been used by Bridgeland [43] to show that up to isomorphism there are only finitely many K3 surfaces S' being derived equivalent to a given K3 surface S. Such K3 surfaces S' are called *Fourier–Mukai partners* of S. Moreover, any Fourier–Mukai partner of S is isomorphic to a moduli space $M_H^S(v)$ of stable sheaves on S.

We now consider the group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S))$ of auto-equivalences of a K3 surface S. The above implies that it admits a representation

$$\rho\colon \ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S)) \to \tilde{\mathrm{H}}(S,\mathbb{Z}).$$

A similar strategy as in the proof of the Derived Global Torelli theorem can be used to obtain a lower bound for the image $\mathrm{Im}(\rho)$ of ρ. Namely, the group

$$\mathrm{Aut}^+(\tilde{\mathrm{H}}(S,\mathbb{Z})) \subset \mathrm{O}(\tilde{\mathrm{H}}(S,\mathbb{Z}))$$

of Hodge isometries of $\tilde{\mathrm{H}}(S,\mathbb{Z})$ with real spinor norm one is always contained in $\mathrm{Im}(\rho)$. It has been proven that, in fact, the image coincides with $\mathrm{Aut}^+(\tilde{\mathrm{H}}(S,\mathbb{Z}))$ [105]. Hence, the study of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S))$ is now reduced to understanding the kernel $\mathrm{Ker}(\rho)$.

A conjecture describing this kernel has been put forward by Bridgeland [41]. It involves the (connected component of the) space $\mathrm{Stab}^\dagger(S)$ of stability conditions on S. The concept of stability conditions on derived categories has been introduced by Bridgeland [40] as a generalization of the notion of slope-stability for coherent sheaves. The space $\mathrm{Stab}^\dagger(S)$ admits a covering map

$$\pi\colon \mathrm{Stab}^\dagger(S) \to \mathcal{P}_0^+(S),$$

where $\mathcal{P}_0^+(S) \subset \mathrm{NS}(S) \otimes \mathbb{C}$ is a certain open subset. Furthermore, the subgroup of $\mathrm{Ker}(\rho)$ preserving the connected component $\mathrm{Stab}^\dagger(S)$ is the group of deck transformations of π. The conjecture then says that $\mathrm{Aut}(\mathrm{D}^\mathrm{b}(S))$ preserves $\mathrm{Stab}^\dagger(S)$ and, moreover, that $\mathrm{Stab}^\dagger(S)$ is simply connected. This would yield an isomorphism

$$\pi_1(\mathcal{P}_0^+(S)) \cong \mathrm{Ker}(\rho)$$

and the short exact sequence

$$1 \to \pi_1(\mathcal{P}_0^+(S)) \to \mathrm{Aut}(\mathrm{D}^\mathrm{b}(S)) \to \mathrm{Aut}^+(\tilde{\mathrm{H}}(S,\mathbb{Z})) \to 1.$$

The conjecture has been verified for K3 surfaces with Picard rank one [15].

The theory of derived categories of K3 surfaces will be used as well as serves as a motivation for many of the subsequent chapters. To elaborate a bit on this, in Chapter B we establish analogues of the above results for higher-dimensional hyper-Kähler manifolds. For example, in the case of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds we show the existence of a lattice Λ of rank $b_2(S^{[n]}) + 2 = 25$ which satisfies the conclusion of Mukai's results in [155]. This yields for example finiteness of $\mathrm{K3}^{[n]}$-type Fourier–Mukai partners for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds. Moreover, Fourier–Mukai partners of moduli spaces of sheaves $M_H^S(v)$ on a K3 surface are again moduli spaces on the same K3 surface S. In addition, we obtain a lower bound for the image of the corresponding representation

$$\mathrm{Aut}(\mathrm{D}^\mathrm{b}(S^{[n]})) \to \mathrm{Aut}(\Lambda)$$

for Hilbert schemes $S^{[n]}$ of elliptic K3 surfaces.

Furthermore, in Chapter E we study derived equivalences of finite order of K3 and abelian surfaces which are *symplectic* meaning that they act trivially when restricted to the symplectic form. We relate the resulting equivariant derived categories to fixed loci on moduli spaces of stable objects using among other things the above results.

1.3.7. Hyper-Kähler manifolds

We also want to mention results about the derived category $\mathrm{D}^\mathrm{b}(X)$ of a hyper-Kähler manifold X obtained before the article [201].

Let, therefore, X be a projective hyper-Kähler manifold. By the general theory recalled above any smooth projective variety Y which is derived equivalent to X must again be Calabi–Yau. In [107] the authors refined this statement and showed that, in fact, Y must also be a hyper-Kähler manifold. The proof uses the decomposition theorem (1.1.1) and Hochschild cohomology to exclude the other factors.

There are also known results for certain special hyper-Kähler manifolds such as Hilbert schemes. Bridgeland–King–Reid [42] established the equivalence

$$\mathrm{D}^\mathrm{b}(S^{[n]}) \cong \mathrm{D}^\mathrm{b}(S^n)_{\mathfrak{S}_n}$$

between the derived category of the Hilbert scheme $S^{[n]}$ of a surface S and the $\mathfrak{S}_n$-equivariant derived category of S^n. Ploog [185] used this to lift derived equivalences between surfaces to their Hilbert schemes. In particular, one obtains a group homomorphism

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S)) \to \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S^{[n]})), \tag{1.3.7}$$

which, in the case that S is a K3 surface, provides many examples of interesting autoequivalences of higher-dimensional hyper-Kähler manifolds. Moreover, one deduces that two derived equivalent surfaces

$$\mathrm{D}^{\mathrm{b}}(S) \cong \mathrm{D}^{\mathrm{b}}(S')$$

have derived equivalent Hilbert schemes

$$\mathrm{D}^{\mathrm{b}}(S^{[n]}) \cong \mathrm{D}^{\mathrm{b}}(S'^{[n]}). \tag{1.3.8}$$

Not much more has been known previously about derived categories of hyper-Kähler manifolds X. Taelman in [201] recently showed that the LLV algebra $\mathfrak{g}(X)$ is a derived invariant. We explain his results and, for example, consequences for Hodge structures of derived equivalent hyper-Kähler manifolds in Chapter A.

Taelman's results will be heavily used in Chapter B. There, we also exploit Ploog's map (1.3.7) to obtain examples of derived equivalences on $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds. This is used to compute the derived monodromy group of X. Another result in Chapter B is a converse to (1.3.8) saying that two Hilbert schemes $S^{[n]}$ and $S'^{[n]}$ are derived equivalent only if the corresponding K3 surfaces S and S' are derived equivalent. For more details we refer to Chapter 2.

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2. Introduction to derived categories of hyper-Kähler manifolds via the LLV algebra

In this chapter we give an overview of Chapter A, which has appeared in [24].

In the first chapter of the appendix we further investigate the derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a hyper-Kähler manifold X and equivalences between them. It can mostly be seen as a review of the first half of the article [201].

2.1. Hochschild LLV algebra

Recall that given a derived equivalence

$$\Phi \colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$$

between hyper-Kähler manifolds we obtain an isomorphism

$$\Phi^{\mathrm{H}} \colon \mathrm{H}^*(X, \mathbb{Q}) \cong \mathrm{H}^*(Y, \mathbb{Q}) \tag{2.1.1}$$

between the cohomology groups. The main idea in this chapter is to introduce the LLV algebras $\mathfrak{g}(X)$ respectively $\mathfrak{g}(Y)$ into the picture. Recall that this Lie algebra is defined in terms of the cup product and the grading neither of which are preserved under derived equivalences.

To circumvent this problem, we focus our attention to the isomorphism

$$\Phi^{\mathrm{HT}} \colon \mathrm{HT}^*(X) \cong \mathrm{HT}^*(Y) \tag{2.1.2}$$

between the rings of polyvector fields of X respectively Y. This has the advantage of preserving the ring structure as well as the grading, but is only defined over the complex numbers. As a first step we introduce another complex Lie algebra

$$\mathfrak{g}'(X) \subset \mathrm{End}(\mathrm{HT}^*(X)).$$

Namely, as in Section 1.2.3 we say that $\mu \in \mathrm{HT}^2(X)$ is *Hard Lefschetz* if the operator

$$e_\mu \coloneqq \mu \wedge \_ \in \mathrm{End}(\mathrm{HT}^*(X))$$

induces for all $k \geq 0$ isomorphisms

$$e_\mu^k \colon \mathrm{HT}^{2n-k}(X) \cong \mathrm{HT}^{2n+k}(X).$$

This is as before equivalent to the existence of an action of the complex Lie algebra $\mathfrak{sl}_2$ on $\mathrm{HT}^*(X)$ such that the subspace of degree two operators is generated by e_μ. The Lie algebra

$\mathfrak{g}'(X)$ is then defined as the smallest Lie subalgebra of $\mathrm{End}(\mathrm{HT}^*(X))$ containing all such complex Lie algebras $\mathfrak{sl}_2$ for all Hard Lefschetz elements $\mu \in \mathrm{HT}^2(X)$.

The graded ring isomorphism $\mathrm{HT}^*(X) \cong \mathrm{H}^*(X, \mathbb{C})$ from (1.3.3) induced by choosing a symplectic form σ implies that there exists an abstract isomorphism

$$\mathfrak{g}'(X) \cong \mathfrak{g}(X)_{\mathbb{C}} \coloneqq \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

of complex Lie algebras.

2.2. Comparison and main result

The next important step is to relate the Lie algebra $\mathfrak{g}'(X)$ with the isomorphism (2.1.1). This is done by considering the free action of the ring of polyvector fields $\mathrm{HT}^*(X)$ on the de Rham cohomology $\mathrm{H}^*(X, \Omega_X^*)$. Since in the case of hyper-Kähler manifolds, $\mathrm{H}^*(X, \Omega_X^*) \cong \mathrm{H}^*(X, \mathbb{C})$ is a free module of rank one over $\mathrm{HT}^*(X)$, we can use the isomorphism

$$\mathrm{HT}^*(X) \cong \mathrm{H}^*(X, \Omega_X^*), \quad \mu \mapsto \mu \lrcorner \sigma^n$$

obtained from the symplectic form σ to view $\mathfrak{g}'(X)$ as a subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$. With these preparations the following is the key ingredient for the proof of the main result of Chapter A.

Theorem 2.2.1 (Theorem A.5.4)**.** *The Lie algebras $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ and $\mathfrak{g}'(X)$ are equal as Lie subalgebras of the Lie algebra* $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$.

This result will be proven in Section A.5.

Going back to our discussions about derived equivalences between hyper-Kähler manifolds we obtain as an immediate consequence the main result due to Taelman [201, Thm. A].

Theorem 2.2.2 (Theorem A.1.1)**.** *A derived equivalence $\Phi\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ between projective hyper-Kähler manifolds induces naturally a Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \cong \mathfrak{g}(Y).$$

The induced isomorphism of quadratic spaces

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X, \mathbb{Q}) \cong \mathrm{H}^*(Y, \mathbb{Q})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$.

Spelling this out this means that for $x \in \mathrm{H}^*(X, \mathbb{Q})$ and $f \in \mathfrak{g}(X)$ we have

$$\Phi^{\mathrm{H}}(f.x) = \Phi^{\mathfrak{g}}(f).\Phi^{\mathrm{H}}(x). \tag{2.2.1}$$

2.3. Induced action on the Mukai lattice

The rest of the article is devoted to applications of this result. To start, recall that the Verbitsky component $\mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{H}^*(X, \mathbb{Q})$ is the unique irreducible representation of the LLV algebra whose complexification contains $\sigma^n \in \mathrm{H}^0(X, \Omega_X^{2n})$. Using (1.3.1) we deduce the following consequence.

Corollary 2.3.1 (Corollary A.6.2). *For a derived equivalence* $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ *between hyper-Kähler manifolds the induced isomorphism* Φ^{H} *restricts to a Hodge isometry*

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X,\mathbb{Q}) \cong \mathrm{SH}(Y,\mathbb{Q}).$$

Thus, instead of the full cohomology $\mathrm{H}^*(X,\mathbb{Q})$ one can in the study of derived equivalences restrict to the Verbitsky component $\mathrm{SH}(X,\mathbb{Q})$ of X.

Next, the Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Q})$ and the LLV isomorphism

$$\mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

enter the discussion in Section A.7. The short exact sequence (1.2.9) implies that any isometry

$$g \in \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

induces an isometry g^n of $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ which restricts to one on the Verbitsky component. Moreover, this isometry normalizes by construction the action of the LLV algebra as in (2.2.1). Proposition A.7.2 establishes the converse as long as n is odd or the second Betti number $b_2(X)$ is odd, where we recall that X is of dimension $2n$.

In conclusion, the natural representation

$$\rho^{\mathrm{H}}\colon\ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\mathrm{H}^*(X,\mathbb{Q}))$$

factors by Corollary 2.3.1 over the representation

$$\rho^{\mathrm{SH}}\colon\ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\mathrm{SH}(X,\mathbb{Q})).$$

The above implies that, furthermore, the representation ρ^{SH} itself also factors over a representation

$$\rho^{\tilde{\mathrm{H}}}\colon\ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

when n or $b_2(X)$ is odd. We also discuss a version of this for equivalences $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ between different hyper-Kähler manifolds. As long as X and Y are deformation equivalent, an analogous statement can be deduced also in the relative setting implying the existence of a Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \cong \tilde{\mathrm{H}}(Y,\mathbb{Q}), \tag{2.3.1}$$

which is functorial for derived equivalences.

It is not known whether or not derived equivalent hyper-Kähler manifolds are deformation-equivalent. Nevertheless, we always have.

Theorem 2.3.2 (Theorem A.7.4). *Let* X *and* Y *be arbitrary hyper-Kähler manifolds and* $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ *be a derived equivalence. Then, there exists a Hodge similitude*

$$\phi\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \to \tilde{\mathrm{H}}(Y,\mathbb{Q})$$

and a scalar $\lambda \in \mathbb{Q}^*$ *such that*

$$\begin{tikzcd}
\mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"'] & \mathrm{SH}(Y,\mathbb{Q}) \arrow[d, "\psi"] \\
\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\lambda\cdot\mathrm{Sym}^n\phi"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(Y,\mathbb{Q}))
\end{tikzcd}$$

commutes.

2.4. Derived invariance of Hodge numbers

Soldatenkov [199] recently studied Hodge structures of hyper-Kähler manifolds. Motivated by the proof of [199, Thm. 3.6] we have included Theorem A.8.1 showing that whether or not an algebra automorphism

$$\phi\colon \mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(X,\mathbb{Q})$$

respects the Hodge structures can be determined on the restriction

$$\phi\colon \mathrm{H}^2(X,\mathbb{Q}) \cong \mathrm{H}^2(X,\mathbb{Q})$$

to the second cohomology groups.

In [176] Orlov conjectured that derived equivalent varieties have the same Hodge numbers. A similar strategy as in the proof of Theorem A.8.1 together with Theorem 2.3.2 is used to prove the final result in the chapter which establishes Orlov's conjecture for projective hyper-Kähler manifolds.

Theorem 2.4.1 (Theorem A.8.2)**.** *Let X and Y be derived equivalent hyper-Kähler manifolds. Then, for all $i \in \mathbb{Z}$ we have an isomorphism*

$$\mathrm{H}^i(X,\mathbb{Q}) \cong \mathrm{H}^i(Y,\mathbb{Q})$$

of $\mathbb{Q}$-Hodge structures.

Contribution by the author of the thesis

Chapter A is based on a talk, which the author delivered in the Bonn–Paris seminar in the summer term 2021 and has been written solely by the author. Its mathematical content is based on the work of Taelman [201]. We do not claim mathematical originality in this chapter.

Notation

Chapter A follows mostly the notation we have introduced and used so far. We briefly highlight the differences that occur.

The BBF form q on the second cohomology is denoted by b and the Mukai pairing q_{SH} on the Verbitsky component by b_{SH}. The Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Q})$ of a hyper-Kähler manifold is called extended Mukai lattice and it is equipped with the quadratic form $\tilde{b}$. The pairing $\mathrm{q}_{[n]}$ on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ is denoted by $b_{[n]}$.

3. Introduction to derived categories of hyper-Kähler manifolds: extended Mukai vector and integral structure

This chapter is an introduction to the results of Chapter B, which has appeared in [25]. It is a direct continuation of the preceding chapter.

The main focus of this chapter is again the derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a hyper-Kähler manifold and equivalences between them. The main ingredient in our study is the Mukai lattice $\tilde{\mathrm{H}}(X, \mathbb{Q})$, which by the results of the previous chapter governs in a certain sense the derived category.

3.1. Square root of Todd class

Our starting point is, however, a purely topological consideration. Namely, we consider the formal square root $\mathsf{td}^{1/2}$ of the Todd class td of X, which can also be thought of as the Mukai vector $v(\mathcal{O}_X)$ of the trivial line bundle $\mathcal{O}_X$. It is a linear combination

$$\mathsf{td}^{1/2} = 1 + \frac{1}{24}\mathrm{c}_2(X) + \frac{7}{5760}\mathrm{c}_2(X)^2 - \frac{1}{1440}\mathrm{c}_4(X) + \dots$$

of products of Chern classes. Let us introduce the number

$$r_X \coloneqq \frac{C(\mathrm{c}_2(X))2^n n!(2n-1)}{(2n)!24c_X} = \frac{C(\mathrm{c}_2(X))(2n-1)}{24C_X}.$$

Nieper-Wißkirchen [162, p. 738] has proven the identity

$$\int_X \mathsf{td}^{1/2}\exp(\omega) = \left(1 + \frac{\mathrm{q}(\omega)}{2r_X}\right)^n \int_X \mathsf{td}^{1/2} \tag{3.1.1}$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{Q})$. We will reprove this formula in Section D.5.3. Observe that (3.1.1) only depends on the projection $(\mathsf{td}^{1/2})_{\mathrm{SH}} \in \mathrm{SH}(X, \mathbb{Q})$ of $\mathsf{td}^{1/2}$ under the orthogonal decomposition

$$\mathrm{H}^*(X, \mathbb{Q}) \cong \mathrm{SH}(X, \mathbb{Q}) \oplus \mathrm{SH}(X, \mathbb{Q})^{\perp}$$

to the Verbitsky component.

Let us relate this picture with the isometric inclusion

$$\psi \colon \mathrm{SH}(X, \mathbb{Q}) \hookrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})).$$

We denote the corresponding orthogonal split

$$T \colon \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})) \twoheadrightarrow \mathrm{SH}(X, \mathbb{Q}).$$

The above discussion then yields the following.

Proposition 3.1.1 (Proposition B.3.4)**.** *Let X be a hyper-Kähler manifold of dimension $2n$. Then,*

$$(\mathsf{td}^{1/2})_{\mathrm{SH}} = T\left(\frac{(\alpha + r_X\beta)^n}{n!}\right) \in \mathrm{SH}(X, \mathbb{Q}).$$

Thus, the square root of the Todd class when projected to the Verbitsky component can be expressed as the n-th power of a linear polynomial.

3.2. Extended Mukai vector

If we use again the interpretation $\mathsf{td}^{1/2} = v(\mathcal{O}_X)$, Proposition 3.1.1 can be rephrased by the equality

$$v(\mathcal{O}_X)_{\mathrm{SH}} = T\left(\frac{(\alpha + r_X\beta)^n}{n!}\right) \in \mathrm{SH}(X, \mathbb{Q}). \tag{3.2.1}$$

In particular, the (projection of the) Mukai vector $v(\mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$ can be related with a vector

$$\tilde{v} \coloneqq \alpha + r_X\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

in the Mukai lattice. This prompts the question whether there are other sheaves or objects for which we can relate their classical Mukai vector with elements in the Mukai lattice. This leads to the definition of the *extended Mukai vector*. In the most general form this is given in Definition B.4.16. Note that this notion will be refined and further studied in the next chapter.

Let us, therefore, consider in this introduction the two cases of interest for our study of derived categories. The first one are objects $\mathcal{E} \in \mathrm{D^b}(X)$ such that there exists an equivalence $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))$ which satisfies

$$\Phi(\mathcal{O}_X) \cong \mathcal{E}.$$

Starting from (3.2.1) the functoriality of the assignment

$$\Phi \mapsto \Phi^{\tilde{\mathrm{H}}}$$

allows us to define in the case that n or $b_2(X)$ is odd an extended Mukai $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$, which up to sign equals $\Phi^{\tilde{\mathrm{H}}}(\tilde{v})$[1]. For the precise statements see Definitions B.4.3 and B.4.6. As an example, to a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ we associate the vector

$$\tilde{v}(\mathcal{L}) \coloneqq \alpha + \mathrm{c}_1(\mathcal{L}) + \left(\frac{\mathrm{q}(\mathrm{c}_1(\mathcal{L}))}{2} + r_X\right)\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q}).$$

We gain have the compatibility

$$v(\mathcal{L})_{\mathrm{SH}} = T\left(\frac{\tilde{v}(\mathcal{L})^n}{n!}\right) \in \mathrm{SH}(X, \mathbb{Q}).$$

Other examples of this kind we consider are Lagrangian projective spaces $\mathbb{P}^n \subset X$ inside hyper-Kähler manifolds, see Example B.4.17.

---

[1] The sign only occurs in the case n even and $b_2(X)$ odd and is due to a certain sign convention in [201]. For everything that follows the explicit choice of the sign can be ignored.

In the second case the starting point is the skyscraper sheaf $k(x)$ for a point $x \in X$ which satisfies

$$v(k(x)) = T\left(\frac{\beta^n}{c_X}\right) \in \mathrm{SH}(X, \mathbb{Q}).$$

As above, objects $\mathcal{E} \in \mathrm{D^b}(X)$ for which there exists an equivalence $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))$ with $\Phi(k(x)) \cong \mathcal{E}$ for some $x \in X$ admit an extended Mukai vector $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$. Other examples we consider which are of this type are line bundles of degree zero supported on Lagrangian tori. Objects in one of the two classes and their Mukai vector satisfy many intriguing properties, see for example Lemma B.4.13.

3.3. Structural result

In the previous chapter we have seen that a derived equivalence

$$\Phi\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$$

between deformation-equivalent hyper-Kähler manifolds induces a Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X, \mathbb{Q}) \cong \tilde{\mathrm{H}}(Y, \mathbb{Q})$$

between the corresponding Mukai lattices. Our interest in the extended Mukai vector stems from the fact that it allows us to easily deduce properties or often directly compute the induced Hodge isometry $\Phi^{\tilde{\mathrm{H}}}$. This principle is exploited to establish the following structural result for derived equivalences between hyper-Kähler manifolds.

Theorem 3.3.1 (Theorem B.4.15)**.** *Let X and Y be deformation-equivalent projective hyper-Kähler manifolds and $\Phi\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ an equivalence with Fourier–Mukai kernel $\mathcal{E}$. The rank r of $\mathcal{E}$ is of the form $\frac{a^n n!}{c_X}$ for $a \in \mathbb{Q}$. If $r = 0$, then $\mathcal{E}$ induces a covering of X and Y with Lagrangian cycles, or there exists a Hodge isometry $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{H}^2(Y, \mathbb{Z})$.*

Let us explain this result a bit further.

Theorem 3.3.1 divides derived equivalences $\Phi \simeq \mathsf{FM}_{\mathcal{E}}\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ into three cases. In the first case the rank r of the Fourier–Mukai kernel $\mathcal{E}$ is non-zero and the theorem asserts that the possible values for r are severely restricted. Note that in all known cases $c_X \in \mathbb{Z}$ which implies that in the expression

$$\frac{a^n n!}{c_X}$$

a must also already be an integer.

In the other cases, the rank of $\mathcal{E}$ is zero. If we assume, for example, that $\mathcal{E}$ is an X-flat sheaf, then the second case yields that the codimension n component of $\mathrm{supp}(\mathcal{E})$ is a flat family of Lagrangian subvarieties of Y parameterized by X which dominates Y.

In the final case, the derived equivalence Φ yields the existence of a Hodge isometry

$$\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{H}^2(Y, \mathbb{Z}). \tag{3.3.1}$$

If this isometry equals the action of a parallel transport operator, the Global Torelli theorem asserts that X and Y must be birational. Up to finite index, any Hodge isometry (3.3.1) is induced from a parallel transport operator [137, Lem. 6.23].

3.4. Derived monodromy group

The rest of the chapter focuses on hyper-Kähler manifolds X of $\mathrm{K3}^{[n]}$-type, i.e. manifolds which are deformation-equivalent to the Hilbert scheme $S^{[n]}$ of n points on a K3 surface S.

Our first goal is to describe how derived equivalences can act on the extended Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Q})$ of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds. The conceptual tool we use is the following.

Definition 3.4.1 (Definition B.6.1)**.** The *derived monodromy group* $\mathrm{DMon}(X)$ is the subgroup of $\mathrm{O}(\mathrm{SH}(X,\mathbb{Q}))$ generated by all isometries of the form

$$\mathrm{SH}(X,\mathbb{Q}) \xrightarrow{\gamma_1} \mathrm{SH}(X_1,\mathbb{Q}) \xrightarrow{F} \mathrm{SH}(X_2,\mathbb{Q}) \xrightarrow{\gamma_2^{-1}} \mathrm{SH}(X,\mathbb{Q}).$$

The isomorphisms γ_i are induced from parallel transport operators between X and X_i and $F = \Phi^{\mathrm{SH}}$ for a derived equivalence $\Phi\colon \mathrm{D^b}(X_1) \cong \mathrm{D^b}(X_2)$, see Section B.6 for further details. The results from the last chapter implies that we can (and will) consider the derived monodromy group $\mathrm{DMon}(X)$ as a subgroup of $\mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$.

When studying the derived monodromy group one looks at the induced isometries of derived equivalences of all deformations of X. Determining the group $\mathrm{DMon}(X)$, therefore, in particular yields an upper bound on the image of the representation

$$\rho^{\tilde{\mathrm{H}}}\colon \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})).$$

As summarized in Section 1.3.6 in the case of K3 surfaces S this image is understood and one obtains that the derived monodromy group

$$\mathrm{DMon}(S) = \mathrm{O}^+(\tilde{\mathrm{H}}(S,\mathbb{Z}))$$

is the group of isometries of the integral Mukai lattice with real spinor norm one.

To describe $\mathrm{DMon}(X)$ for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds, recall from (1.3.7) the group homomorphism

$$\mathrm{Aut}(\mathrm{D^b}(S)) \to \mathrm{Aut}(\mathrm{D^b}(S^{[n]}))$$

obtained using the Bridgeland–King–Reid equivalence. We observe in Proposition B.6.3 that this yields a group homomorphism

$$d_n\colon \mathrm{DMon}(S) \to \mathrm{DMon}(S^{[n]}).$$

Our first step towards understanding the derived category of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds is the computation of the morphism d_n.

Theorem 3.4.2 (Theorem B.7.4)**.** *The homomorphism* $d_n\colon \mathrm{DMon}(S) \to \mathrm{DMon}(S^{[n]})$ *is given by*

$$g \mapsto \det(g)^{n+1} B_{-\delta/2} \circ \iota(g) \circ B_{\delta/2}.$$

The group homomorphism

$$\iota\colon \mathrm{O}(\tilde{\mathrm{H}}(S,\mathbb{Q})) \to \mathrm{O}(\tilde{\mathrm{H}}(S^{[n]},\mathbb{Q}))$$

is obtained from the natural inclusion $\tilde{\mathrm{H}}(S,\mathbb{Q}) \subset \tilde{\mathrm{H}}(S^{[n]},\mathbb{Q})$. Furthermore, the class $\delta \in \mathrm{H}^2(S^{[n]},\mathbb{Z})$ is half the exceptional divisor of the Hilbert–Chow morphism $S^{[n]} \to S^{(n)}$ and for $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ the isometry $B_\lambda \in \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ is defined via

$$B_\lambda(r\alpha + \mu + s\beta) = r\alpha + \mu + r\lambda + \left(s + \mathrm{q}(\lambda,\mu) + r\frac{\mathrm{q}(\lambda,\lambda)}{2}\right)\beta.$$

3.5. Derived categories of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds

As in the case of K3 surfaces we would like to find an even lattice which governs the derived category of $\mathrm{K3}^{[n]}$-type hyper Kähler manifolds X of dimension $2n$ and equivalences between them. For this purpose we fix a class $\delta \in \mathrm{H}^2(X, \mathbb{Q})$ with square $2-2n$ and divisibility $2n-2$.

Definition 3.5.1 (Theorem B.5.2)**.** For $\delta \in \mathrm{H}^2(X, \mathbb{Z})$ as above we define the $\mathrm{K3}^{[n]}$ *lattice* as

$$\Lambda := B_{-\delta/2}(\tilde{\mathrm{H}}(X, \mathbb{Z})) \subset \tilde{\mathrm{H}}(X, \mathbb{Q}).$$

A detailed study of this lattice together with properties of the extended Mukai vectors yields the following bounds for the derived monodromy group.

Theorem 3.5.2 (Theorem B.8.1)**.** *Let X be a $\mathrm{K3}^{[n]}$-type hyper-Kähler manifold. There are inclusions*

$$\hat{\mathrm{O}}^+(\Lambda) \subset \mathrm{DMon}(X) \subset \mathrm{O}(\Lambda).$$

In particular, the $\mathrm{K3}^{[n]}$ lattice Λ is fixed by all derived equivalences.

The group $\hat{\mathrm{O}}^+(\Lambda)$ is the group of isometries of Λ with real spinor norm one which act via $\pm\mathrm{id}$ on the discriminant group. For details on these groups and lattice theory in general, see [100, Sec. 14] and [82].

As an immediate consequence we find that the representation

$$\rho^{\tilde{\mathrm{H}}} \colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$$

factors over a representation

$$\rho^{\tilde{\mathrm{H}}} \colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{Aut}(\Lambda_X), \tag{3.5.1}$$

where $\mathrm{Aut}(\Lambda_X)$ is the group of Hodge isometries of the lattice Λ_X equipped with the Hodge structure coming from the inclusion $\Lambda_X \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$. We also obtain a relative version of this statement, which yields a complete analogue for Mukai's result [155] for derived equivalences between K3 surfaces.

Theorem 3.5.3 (Theorem B.9.2)**.** *Let X and Y be projective $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds and $\Phi \colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ a derived equivalence. Then, $\Phi^{\tilde{\mathrm{H}}}$ restricts to a Hodge isometry*

$$\Phi^{\tilde{\mathrm{H}}} \colon \Lambda_X \cong \Lambda_Y.$$

Hence, the $\mathrm{K3}^{[n]}$ lattice governs in a similar sense to the surface case derived equivalences between $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds. This has strong implications. Here is one example.

Theorem 3.5.4 (Theorem B.9.4)**.** *For a fixed projective $\mathrm{K3}^{[n]}$-type hyper-Kähler manifold X the number of projective $\mathrm{K3}^{[n]}$-type manifolds Y up to isomorphism with $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ is finite.*

Another consequences regards smooth moduli spaces $M_\sigma^S(v)$ of σ-stable objects on the K3 surface S with Mukai vector v. In [17] it is shown that a hyper-Kähler manifold X which is birational to $M_\sigma^S(v)$ is isomorphic to a moduli space of stable objects on the same K3 surfaces. The $\mathrm{K3}^{[n]}$ lattice allows us to strengthen this result to consider derived equivalences.

Corollary 3.5.5 (Corollary B.9.6)**.** *Let* $M_\sigma^S(v)$ *be a smooth moduli space of stable objects on a projective K3 surface* S *and* X *a projective* $\mathrm{K3}^{[n]}$*-type hyper-Kähler manifold such that* $\mathrm{D^b}(X) \cong \mathrm{D^b}(M_\sigma^S(v))$*. Then,* X *is itself a moduli space of stable objects on* S*.*

This also immediately yields.

Corollary 3.5.6 (Corollary B.9.7)**.** *For two smooth moduli spaces* $M_\sigma^S(v)$ *and* $M_{\sigma'}^{S'}(v')$ *of stable objects on projective K3 surfaces* S *and* S' *with* $\mathrm{D^b}(M_\sigma^S(v)) \cong \mathrm{D^b}(M_{\sigma'}^{S'}(v'))$ *we have* $\mathrm{D^b}(S) \cong \mathrm{D^b}(S')$*. Furthermore,* S *and* S' *are derived equivalent if and only if their Hilbert schemes* $S^{[n]}$ *and* $S'^{[n]}$ *are derived equivalent.*

A final consequence of the above we would like to mention in the introduction is a bound on the image $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$ of the representation (3.5.1) for Hilbert schemes $S^{[n]}$ of elliptic K3 surfaces.

Theorem 3.5.7 (Theorem B.9.8)**.** *For the Hilbert scheme* $S^{[n]}$ *of a K3 surface with* $U \subset \mathrm{NS}(S)$ *the image* $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$ *of the representation* $\rho^{\tilde{\mathrm{H}}}$ *satisfies*

$$\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}}) \subset \mathrm{Im}(\rho^{\tilde{\mathrm{H}}}) \subset \mathrm{Aut}(\Lambda_{S^{[n]}}).$$

The group $\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}})$ is the group of all Hodge isometries of $\Lambda_{S^{[n]}}$ with real spinor norm one which act via ±id on the discriminant group. When $n-1$ is a prime power, the inclusion

$$\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}}) \subset \mathrm{Aut}^+(\Lambda_{S^{[n]}})$$

is an equality. In these cases Theorem 3.5.7 determines $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$ up to index two.

Notation

Chapter B uses the same notation as Chapter A. For convenience let us again mention the notational differences that occur.

The BBF form q on the second cohomology is denoted by b and the Mukai pairing $\mathrm{q_{SH}}$ on the Verbitsky component by b_{SH}. The orthogonal projection

$$\mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{SH}(X, \mathbb{Q})$$

to the Verbitsky component is denoted by $\overline{(\_)}$ in this chapter. The Mukai lattice $\tilde{\mathrm{H}}(X, \mathbb{Q})$ of a hyper-Kähler manifold is again called extended Mukai lattice and it is equipped with the quadratic form $\tilde{b}$. The pairing $\mathrm{q}_{[n]}$ on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ is denoted by $b_{[n]}$.

4. Introduction to atomic objects on hyper-Kähler manifolds

In this chapter we present an overview of the content of Chapter C, which has appeared in [26].

The impetus for this part was to better understand the extended Mukai vector of the preceding chapter with an aim to obtain an improved conceptual picture.

4.1. Extended Mukai vector revisited

We consider again a hyper-Kähler manifold X of dimension $2n$. Recall that for a sheaf or an object $\mathcal{E}$ we said that it admits an extended Mukai vector $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ if there exists $c \in \mathbb{Q}$ such that

$$v(\mathcal{E})_{\mathrm{SH}} = c \cdot T(\tilde{v}(\mathcal{E})^n) \in \mathrm{SH}(X, \mathbb{Q}).$$

Here, $(\_)_{\mathrm{SH}}$ denotes again the orthogonal projection to the Verbitsky component obtained from the decomposition

$$\mathrm{H}^*(X, \mathbb{Q}) = \mathrm{SH}(X, \mathbb{Q}) \oplus \mathrm{SH}(X, \mathbb{Q})^{\perp}$$

and, as before, T is the orthogonal split for the isometric inclusion

$$\psi \colon \mathrm{SH}(X, \mathbb{Q}) \hookrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})).$$

For our purposes in Chapter B this definition is sufficient. However, it relates only the Verbitsky component $\mathrm{SH}(X, \mathbb{Q})$ with the Mukai $v(\mathcal{E})$ and ignores all other irreducible representations V_λ of the LLV algebra from the decomposition

$$\mathrm{H}^*(X, \mathbb{Q}) \cong \bigoplus_{\lambda} V_\lambda. \tag{4.1.1}$$

4.2. Atomic objects

Instead of only looking at $\mathrm{SH}(X, \mathbb{Q})$ we can use (4.1.1) to decompose the Mukai vector $v(\mathcal{E})$ of a sheaf respective object $\mathcal{E}$ as

$$v(\mathcal{E}) = \sum_{\lambda} v(\mathcal{E})_\lambda \in \bigoplus_{\lambda} V_\lambda. \tag{4.2.1}$$

We want a notion which relates the Mukai vector $v(\mathcal{E})$ with the Mukai lattice $\tilde{\mathrm{H}}(X, \mathbb{Q})$ and takes into account all summands $v(\mathcal{E})_\lambda$. This leads to the central notion of this chapter.

Definition 4.2.1 (Definition C.1.1)**.** A sheaf $\mathcal{E} \in \mathrm{Coh}(X)$ or an object $\mathcal{E} \in \mathrm{D^b}(X)$ is called *atomic* if there exists a non-zero vector $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ such that the annihilator Lie subalgebra $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ of the representation of $\mathfrak{g}(X)$ on $\mathrm{H}^*(X, \mathbb{Q})$ equals the annihilator Lie subalgebra $\mathrm{Ann}(\tilde{v}) \subset \mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ of the representation of $\mathfrak{g}(X)$ on $\tilde{\mathrm{H}}(X, \mathbb{Q})$.

Let us explain this notion a bit more. A first observation is that any non-zero sheaf on a K3 surface is atomic. In this sense, atomic sheaves and objects can be considered as generalizations of sheaves on K3 surfaces. We will explore this philosophy in more depth in the next sections.

Next, recall that the Mukai vector $v(\mathcal{E})$ of any object is a Hodge class, i.e. contained in the subspace

$$\bigoplus_p \mathrm{H}^{p,p}(X, \mathbb{Q}) := \bigoplus_p \mathrm{H}^{p,p}(X) \cap \mathrm{H}^*(X, \mathbb{Q}).$$

For a general element $x \in \mathrm{H}^*(X, \mathbb{Q})$ which is Hodge one can consider two cases. The first one is when x is annihilated by the LLV algebra $\mathfrak{g}(X)$. This is equivalent to the inclusion of the annihilator Lie subalgebra $\mathrm{Ann}(x)$ of the LLV algebra to be an equality

$$\mathrm{Ann}(x) = \mathfrak{g}(X).$$

The second case is that the inclusion

$$\mathrm{Ann}(x) \subset \mathfrak{g}(X)$$

is strict and the annihilator is a proper subalgebra. With this in mind an equivalent definition for objects to be atomic is the following.

Proposition 4.2.2 (Proposition C.3.1)**.** *An object* $\mathcal{E} \in \mathrm{D^b}(X)$ *is atomic if and only if* $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ *is a Lie subalgebra of codimension* $b_2(X) + 1$*, which is the smallest positive codimension possible.*

Thus, atomic objects $\mathcal{E}$ are exactly those, for which their annihilator Lie subalgebra $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ is a proper subalgebra of the largest dimension possible. In particular, since by Lemma C.3.7 the Mukai vector of any non-zero sheaf projects non-trivially to the Verbitsky component, a sheaf $\mathcal{E}$ is atomic if and only if the Lie algebra $\mathrm{Ann}(v(\mathcal{E}))$ is as large as possible.

Furthermore, atomic objects admit an extended Mukai vector in the sense of Section 4.1. Therefore, atomic objects inherit all the properties we discussed in Section B.4. For example, if the rank $\mathrm{rk}(\mathcal{E})$ of $\mathcal{E}$ is non-zero, then the atomic object $\mathcal{E}$ admits the Mukai vector

$$\tilde{v}(\mathcal{E}) = \mathrm{rk}(\mathcal{E})\alpha + \mathrm{c}_1(\mathcal{E}) + s\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

in the Mukai lattice for some $s \in \mathbb{Q}$.

Moreover, since being atomic is a purely cohomological notion involving only the LLV algebra, the property of being atomic is stable under deformations and derived equivalences, see Proposition C.3.10. In addition, many summands in the decomposition (4.2.1) must vanish for atomic objects. That is, irreducible representations V_λ which admit elements satisfying Definition 4.2.1 are severely restricted. In particular, the Mukai vector of an atomic object can only lie in certain subspaces V_λ.

4.3. Obstruction maps

As has been observed above every non-zero sheaf on a K3 surface is atomic. In the remaining part we would like to convey our intuition that atomic objects on hyper-Kähler manifolds behave in many aspects like sheaves on K3 surfaces.

We start by studying aspects of deformation theory for atomic objects. More precisely, let us introduce the *cohomological obstruction map*

$$\mathrm{obs}_{\mathcal{E}} \colon \mathrm{HT}^2(X) \to \mathrm{H}^*(X, \mathbb{C}), \quad \mu \mapsto \mu \lrcorner v(\mathcal{E}).$$

Recall that

$$\mathrm{HT}^2(X) = \mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X) \oplus \mathrm{H}^1(X, \mathcal{T}_X) \oplus \mathrm{H}^2(X, \mathcal{O}_X).$$

The space $\mathrm{H}^1(X, \mathcal{T}_X)$ parameterizes first-order deformations of the manifold X. For an element $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ the cohomological obstruction map $\mathrm{obs}_{\mathcal{E}}$ measures whether the Mukai vector $v(\mathcal{E})$ stays a Hodge class along the given first-order deformation direction. More generally, the space $\mathrm{HT}^2(X)$ parameterizes by [204] first-order deformations of the category $\mathrm{D}^{\mathrm{b}}(X)$ and $\mathrm{obs}_{\mathcal{E}}$ again measures whether the element $v(\mathcal{E})$ deforms along.

Let us relate the cohomological obstruction map with the notion of atomicity.

Theorem 4.3.1 (Theorem C.1.2)**.** *Let X be a hyper-Kähler manifold and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$. Then, $\mathcal{E}$ is atomic if and only if the cohomological obstruction map $\mathrm{obs}_{\mathcal{E}}$ has a one-dimensional image.*

The theorem says that atomic objects $\mathcal{E}$ are precisely those objects for which the subspace of $\mathrm{HT}^2(X)$ of first-order deformations for which $v(\mathcal{E})$ stays a Hodge class has codimension one. This result relates the (symplectic) geometry of polyvector fields and deformation theory with the representation theory of the LLV algebra. This relation will be used throughout the rest and is often crucially used in the proof of later results.

Instead of looking only at cohomology, let us now consider the question whether or not the object $\mathcal{E}$ itself deforms along a given first-order deformation direction in $\mathrm{HH}^2(X) \cong \mathrm{HT}^2(X)$. As shown in [204, Prop. 6.1] this is governed by the obstruction map

$$\chi_{\mathcal{E}} \colon \mathrm{HH}^2(X) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}),$$

which we introduced in Section 1.3.4. The obstruction map $\chi_{\mathcal{E}}$ for the object $\mathcal{E}$ sits in the commutative diagram

$$\begin{tikzcd}
\mathrm{HH}^*(X) \arrow[r, "\chi_{\mathcal{E}}"] \arrow[d, "I^{\mathrm{HKR}}"'] & \mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \\
\mathrm{HT}^*(X), \arrow[ur, "\lrcorner \exp(\mathrm{At}_{\mathcal{E}})"'] &
\end{tikzcd}$$

where $\exp(\mathrm{At}_{\mathcal{E}})$ is the exponential of the *Atiyah class* $\mathrm{At}_{\mathcal{E}} \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$ of $\mathcal{E}$ [94]. We will call objects $\mathcal{E}$ for which the kernel $\mathrm{Ker}(\chi_{\mathcal{E}})$ has codimension one *1-obstructed.* These objects were recently investigated by Markman [139]. We want to relate them to the notion of atomicity.

Theorem 4.3.2 (Theorem C.1.3)**.** *If $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is a 1-obstructed object such that $v(\mathcal{E})$ is not annihilated by the LLV algebra $\mathfrak{g}(X)$, then $\mathcal{E}$ is atomic. In particular, 1-obstructed sheaves are atomic.*

Observe that the Mukai vector of atomic objects is not annihilated by the LLV algebra $\mathfrak{g}(X)$. In particular, for a 1-obstructed object $\mathcal{E}$ its Mukai vector $v(\mathcal{E})$ not being annihilated by the LLV algebra is necessary and sufficient for $\mathcal{E}$ to be atomic. In fact, we believe that this extra assumption is vacuous. That is, we conjecture that Mukai vectors of 1-obstructed objects are never annihilated by the LLV algebra.

What about the converse in Theorem 4.3.2? This already fails for vector bundles on K3 surfaces. More precisely, we observe in Example C.4.4 that for a non-trivial line bundle $\mathcal{L} \in \operatorname{Pic}(S)$ on a K3 surface the bundle

$$\mathcal{O}_S \oplus \mathcal{L}$$

is not 1-obstructed. However, if we restrict to simple atomic sheaves, we speculate the following.

Conjecture A (Conjecture E)**.** *Let X be a hyper-Kähler manifold and $\mathcal{E}$ a simple atomic object. For each $\gamma \in \mathrm{HH}^2(X)$ with $0 \neq \chi_{\mathcal{E}}(\gamma) = \gamma_{\mathcal{E}} \in \operatorname{Ext}^2(\mathcal{E},\mathcal{E})$ there exists $\mu \in \mathrm{HH}^{2n-2}(X)$ such that the composition $\mu_{\mathcal{E}} \circ \gamma_{\mathcal{E}} \in \operatorname{Ext}^{2n}(\mathcal{E},\mathcal{E})$ is non-zero.*

Note that for a simple object $\mathcal{E} \in \mathrm{D^b}(X)$ Serre Duality equips $\operatorname{Ext}^*(\mathcal{E},\mathcal{E})$ with a perfect pairing by identifying

$$\operatorname{Ext}^{2n}(\mathcal{E},\mathcal{E}) \cong \mathbb{C}.$$

From this viewpoint one may ask whether this pairing restricts to a non-degenerate pairing on the subalgebra given by the image

$$\operatorname{Im}(\chi_{\mathcal{E}}) \subset \operatorname{Ext}^*(\mathcal{E},\mathcal{E})$$

of the obstruction map

$$\chi_{\mathcal{E}} \colon \mathrm{HH}^*(X) \to \operatorname{Ext}^*(\mathcal{E},\mathcal{E}).$$

The above conjecture is a special case of this question.

As evidence, note that Conjecture A holds for K3 surfaces. Moreover, Proposition C.4.6 shows that simple 1-obstructed sheaves satisfy the conclusion of the conjecture. Proving it would establish the following connection between 1-obstructed and atomic objects.

Theorem 4.3.3 (Theorem C.1.4)**.** *If $\mathcal{E} \in \mathrm{D^b}(X)$ is a simple object satisfying the conclusion of Conjecture E, then $\mathcal{E}$ is 1-obstructed if and only if $\mathcal{E}$ is atomic.*

In particular, for a simple object $\mathcal{E}$ consider the three properties: $\mathcal{E}$ is atomic, $\mathcal{E}$ is 1-obstructed, $\mathcal{E}$ satisfies the conclusion of Conjecture A. Then, any two of these properties imply the remaining one.

4.4. Comparison to other notions

Prior to our study of atomic sheaves there were two prominent notions of bundles on higher-dimensional hyper-Kähler manifolds. The first one is that of (projectively) hyperholomorphic bundles due to Verbitsky [208]. We recall their definitions and properties in Section C.5. O'Grady proposed in [172] the notion of modular sheaves and bundles. This is also a completely topological notion asking that the discriminant

$$\Delta(\mathcal{E}) := -2r\mathrm{ch}_2(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E})^2 \in \mathrm{H}^4(X,\mathbb{Q})$$

of the sheaf $\mathcal{E}$ when projected to the Verbitsky component is a multiple of the dual $\mathfrak{q}$ of the BBF form.

Let us relate these with the notion of atomic sheaves and objects.

Proposition 4.4.1 (Proposition C.1.5)**.** *Let $\mathcal{E}$ be a torsion-free atomic sheaf. Then, $\mathcal{E}$ is modular.*

The converse does not hold. An easy counterexample is given by the ideal sheaf $\mathcal{I}$ of any point $x \in X$ on a hyper-Kähler manifold of dimension at least four.

Proposition 4.4.2 (Proposition C.1.6)**.** *Let $\mathcal{E}$ be a slope polystable atomic vector bundle. Then, $\mathcal{E}$ is projectively hyperholomorphic.*

We note that since projectively hyperholomorphic bundles are polystable [208, Thm. 2.3], atomic bundles are projectively hyperholomorphic if and only if they are polystable.

Again, the converse does not hold. A counterexample is provided by the tangent bundle $\mathcal{T}_X$ on higher dimensional hyper-Kähler manifolds, see Proposition C.8.3.

4.5. Deformations of stable atomic bundles

The deformation behaviour of slope stable bundles on K3 surfaces is particularly well-behaved. We show that this conclusion remains true for stable atomic bundles on hyper-Kähler manifolds.

Proposition 4.5.1 (Proposition C.5.5)**.** *Let $\mathcal{E}$ be a slope stable atomic bundle. Then, $\mathbb{P}(\mathcal{E})$ deforms over the whole moduli space of Kähler deformations of X.*

This is a consequence of Proposition 4.4.2. Another way of phrasing the above is that the endomorphism bundle $\mathcal{E}nd(\mathcal{E}, \mathcal{E})$ is hyperholomorphic for any hyper-Kähler metric. In particular, also $\mathcal{E}nd(\mathcal{E}, \mathcal{E})$ deforms along with any Kähler deformation of X.

Fixing the manifold X let us consider the deformation behaviour of a stable atomic bundle $\mathcal{E}$ on X. This question is governed by the dg Lie algebra

$$\mathrm{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}),$$

see [134, 197] for the necessary background. We recall that a dg (Lie) algebra L is *formal* if it is isomorphic

$$L \cong \mathrm{H}^*(L)$$

as a dg (Lie) algebra to its cohomology algebra.

Theorem 4.5.2 (Theorem C.1.7)**.** *Let $\mathcal{E}$ be an atomic slope stable vector bundle. Then, the dg algebra $\mathrm{R}\mathcal{H}om(\mathcal{E}^{\oplus k}, \mathcal{E}^{\oplus k})$ is formal for any $k > 0$.*

More generally, we establish the above result under the weaker assumption that $\mathcal{E}$ is a stable projectively hyperholomorphic bundle. Thus, also for these bundles the associated dg algebra is formal.

To relate this to the deformation behaviour of atomic bundles $\mathcal{E}$, formality for the associated dg Lie algebra implies that the local versal deformation space associated to the deformations of $\mathcal{E}$ on X is isomorphic to the preimage of 0 under

$$\kappa_2 \colon \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}), \quad f \mapsto f \circ f.$$

In particular, this space has at most quadratic singularities, see Corollary C.6.3. For more details and a thorough discussion we refer to Section C.6.

4.6. Atomic Lagrangians

An import class of submanifolds of hyper-Kähler manifolds are Lagrangian submanifolds. We take a closer look at these manifolds using the notion of atomic sheaves.

Definition 4.6.1 (Definition C.7.1)**.** We call a connected Lagrangian submanifold $\iota\colon L \subset X$ *atomic* if $\iota_* \mathcal{O}_L$ is an atomic sheaf.

The main result about atomic Lagrangians is the following.

Theorem 4.6.2 (Theorem C.1.8)**.** *Let $\iota\colon L \subset X$ be a connected Lagrangian submanifold. Then $\iota_* \mathcal{O}_L$ is atomic if and only if the restriction map $\iota^*\colon \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(L, \mathbb{Q})$ has a one-dimensional image and $\mathrm{c}_1(L) = \mathrm{c}_1(\mathcal{T}_L) \in \mathrm{Im}(\iota^*) \subset \mathrm{H}^2(L, \mathbb{Q})$.*

Let us consider this result from the viewpoint of obstructions to deformations via Theorem 4.3.1. Namely, since the sheaf $\iota_* \mathcal{O}_L$ is supported on a Lagrangian submanifold, the cohomological obstruction map $\mathrm{obs}_{\iota_* \mathcal{O}_L}$ vanishes when restricted to $\mathrm{H}^2(X, \mathcal{O}_X)$. It, therefore, remains to control first-order deformations parameterized by $\mathrm{H}^1(X, \mathcal{T}_X)$ and $\mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X)$. The fact that the restriction map

$$\iota^*\colon \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(L, \mathbb{Q})$$

of the pullback homomorphism has a one-dimensional image translates into the the kernel of $\mathrm{obs}_{\iota_* \mathcal{O}_L}$ having a one-dimensional image when restricted to $\mathrm{H}^1(X, \mathcal{T}_X)$. The condition that $\mathrm{c}_1(L)$ is supposed to be contained in the image $\mathrm{Im}(\iota^*)$ means that the obstruction coming from the Poisson deformation direction agrees with the obstructions coming from $\mathrm{H}^1(X, \mathcal{T}_X)$.

This again establishes a nice analogy to the surface case. Namely, Lagrangian submanifolds C of a K3 surface are smooth curves of any genera. In particular, they are either Fano manifolds, are K-trivial, or their canonical bundle ω_C is ample. This conclusion remains true for atomic Lagrangians inside hyper-Kähler manifolds.

Atomic Lagrangians provide a convenient class of examples, which we further study in Section C.7. Among other things we study the question of formality for the associated derived endomorphisms. Moreover, the ring isomorphism

$$\mathrm{Ext}^*(\iota_* \mathcal{O}_L, \iota_* \mathcal{O}_L) \cong \mathrm{H}^*(L, \mathbb{C})$$

shows that their Ext algebra are of topological nature and, in particular, graded-commutative. As demonstrated in Proposition C.7.7, one may compare this with the case of simple objects $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(S)$ on K3 surfaces S, where we always have

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \cong \mathrm{H}^*(C, \mathbb{C})$$

for some Riemann surface C. We speculate that this topological nature of the Ext algebra $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$ for simple atomic objects $\mathcal{E}$ remains true. This would, in particular, mean that $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$ is graded-commutative. We refer to Conjecture F and the surrounding discussion for a partial form of this statement.

4.7. Spherical sheaves and objects

In the appendix we study spherical sheaves and objects on hyper-Kähler manifolds. This part is independent of our discussion of atomic objects.

For many of our results in this chapter, such as Theorem 4.3.3, we related the Mukai vector $v(\mathcal{E})$ of an object $\mathcal{E}$ to properties of the Ext algebra $\mathrm{Ext}^*(\mathcal{E},\mathcal{E})$. A refined study of this relationship yields the following result, which seems to have been expected, but no proof existed so far in the literature.

Theorem 4.7.1 (Theorem C.1.9)**.** *There exist no spherical sheaves on a hyper-Kähler manifold X of dimension greater than two. Moreover, if X is of* $\mathrm{K3}^{[n]}$ *with* $n > 1$ *or* OG10*-type, then* $\mathrm{D^b}(X)$ *contains no spherical objects.*

This result is deduced from Theorem C.A.2 which relates non-vanishing results for projections of the Mukai vector $v(\mathcal{E})$ of an object $\mathcal{E}$ to non-vanishing of certain Ext groups $\mathrm{Ext}^j(\mathcal{E},\mathcal{E})$. For example, any non-zero sheaf $\mathcal{E}$ on a hyper-Kähler manifold satisfies

$$\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \neq 0,$$

see Corollary C.A.3.

In general, we show that the Mukai vector of a spherical object must be contained in the subspace U of $\mathrm{H}^{2n}(X,\mathbb{Q})$ which is the orthogonal complement of the subalgebra generated by all cohomology classes of degree at most $2n-1$. In particular, U is annihilated by the LLV algebra $\mathfrak{g}(X)$. We expect that spherical objects on hyper-Kähler manifolds can only exist for K3 surfaces.

5. Introduction to second Chern class and Fujiki constants of hyper-Kähler manifolds

We will give an introduction to the results of Chapter D. It has appeared as [30] and is a joint work with Jieao Song.

In this part we study the (conjectural) behaviour of the generalized Fujiki constants and the Riemann–Roch polynomial of hyper-Kähler manifolds X. At first sight quite surprisingly, this leads to a conditional bound on the second Betti number $b_2(X)$.

5.1. Topological restrictions

In Section 1.1.2 we have listed up to deformations all currently known examples of hyper-Kähler manifolds. The scarcity of examples and the difficulty to construct hyper-Kähler manifolds naturally prompts the question what restrictions this class of manifolds must obey.

There are several results in this direction of which we want to name a few. The Verbitsky component $\mathrm{SH}(X,\mathbb{Q}) \subset \mathrm{H}^*(X,\mathbb{Q})$ is always a subalgebra of the cohomology. The isomorphism

$$\mathrm{SH}^{2k}(X,\mathbb{C}) \cong \mathrm{Sym}^k(\mathrm{H}^2(X,\mathbb{C}))$$

for $k \leq n$ due to Verbitsky [207] gives a lower bound on the even Betti numbers. Furthermore, Fujiki [75] proved that the odd Betti numbers $b_{2k+1}(X)$ are always divisible by four.

In dimension four Guan [84] obtained the general bound

$$b_2(X) \leq 23$$

and, moreover, if $b_2(X) < 23$, then the second Betti number is at most eight. The main ingredient for these result is the following relation

$$\sum_{j=0}^{4n}(-1)^j(3j^2 - n(12n+1))b_j(X) = 0$$

for hyper-Kähler manifolds X of dimension $2n$ due to Salamon [191].

There are also other constraints apart from the Betti numbers. For example, the Bogomolov inequality implies that the generalized Fujiki constant of the second Chern class $c_2 := c_2(X)$ of X satisfies

$$C(c_2) > 0.$$

The result of Nieper-Wißkirchen (3.1.1) shows that the generalized Fujiki constants $C(\mathsf{td}_{2k}^{1/2})$ of the square root of the Todd class in each degree $4k$ are also always positive. Jiang [113] recently showed the positivity

$$C(\mathsf{td}_{2k}) > 0$$

for the Todd class itself. In particular, the coefficients of the Riemann–Roch polynomial $\mathrm{RR}_X(q)$ of X must be positive.

5.2. Characteristic classes and Verbitsky component

The point of departure for this part is the following. Can we determine when certain characteristic classes lie in the Verbitsky component? A first candidate to investigate is the second Chern class c_2 of X. It was known before [138, Lem. 1.5] that it is not always contained in $\mathrm{SH}(X, \mathbb{Q})$.

In general, we decompose

$$\mathrm{H}^4(X, \mathbb{Q}) \cong \mathrm{SH}^4(X, \mathbb{Q}) \oplus \mathrm{SH}^4(X, \mathbb{Q})^\perp$$

which allows us to write

$$c_2 = a\mathfrak{q} + z \tag{5.2.1}$$

with $z \in \mathrm{SH}^4(X, \mathbb{Q})^\perp$ and $a \neq 0$. The first result towards an answer to the above question is the following which is obtained from squaring both sides of (5.2.1) and investigating the factors.

Proposition 5.2.1 (Proposition D.2.3)**.** *We have the following inequality*

$$C(c_2^2) \geq \frac{C(c_2)^2}{C(\mathfrak{q})^2} C(\mathfrak{q}^2), \tag{5.2.2}$$

where equality holds if and only if $c_2 \in \mathrm{Sym}^2(\mathrm{H}^2(X, \mathbb{Q}))$.

Since $\mathfrak{q}^k \in \mathrm{SH}(X, \mathbb{Q})$, the generalized Fujiki constants $C(\mathfrak{q}^k)$ can be expressed in terms of the second Betti number $b_2(X)$ and the Fujiki constant $C_X = C(1)$, see Proposition D.2.4. This allows to rewrite (5.2.2) as

$$C(c_2^2) \geq \frac{(2n-1)(b_2(X)+2n-4)C(c_2)^2}{(2n-3)(b_2(X)+2n-2)C(1)}.$$

5.3. Bounding the second Betti number

We want to relate the preceding section with our discussion about constraints for hyper-Kähler manifolds. Nieper-Wißkirchen's result (3.1.1) about the factorization of $\mathsf{td}^{1/2}$ when restricted to the Verbitsky component yields the relation

$$7C(c_2^2) - 4C(c_4) = \frac{5(2n-1)C(c_2)^2}{(2n-3)C(1)}$$

between generalized Fujiki constants, see Corollary D.2.6. Combining everything we said so far leads to the following reformulation of the main result of the present part, which also determines precisely whether or not the second Chern class lies in the Verbitsky component.

Theorem 5.3.1 (Remark D.2.8)**.** *Let X be a hyper-Kähler manifold of dimension $2n$ with second Betti number $b_2(X)$ and let us write $C(c_2^2) = \mu C(c_4)$. If*

$$\mu > 2, \tag{5.3.1}$$

then we have the inequality

$$b_2(X) \leq 9 - 2n + \frac{10}{\mu - 2} \tag{5.3.2}$$

and equality holds if and only if $c_2 \in \mathrm{Sym}^2 H^2(X, \mathbb{Q})$. *If condition* (5.3.1) *does not hold, then* c_2 *is not contained in the Verbitsky component.*

Let us discuss this statement further.

Firstly, the condition (5.3.1) is equivalent to

$$C(\mathrm{ch}_4) > 0.$$

This holds true in dimension four [169, Lem. 4.6]. We will comment more on the expected positivity behaviour of generalized Fujiki constants in the next section.

Moreover, in all known examples we have $C(\mathrm{ch}_4) > 0$. That is, Theorem 5.3.1 applies in theses cases and yields a bound on the second Betti number. In the case that X is of $\mathrm{K3}^{[n]}$ or OG_{10}-type we have

$$b_2(X) \leq n + 17 + \frac{12}{n+1}$$

and for manifolds of Kum_n or OG_6-type the bound is

$$b_2(X) \leq n + 5,$$

see Examples D.2.12 and D.2.13. In particular, for Kum_2-type hyper-Kähler manifolds this bound is stronger than the one by Guan. Furthermore, note that the bound is attained for $\mathrm{K3}^{[2]}$, $\mathrm{K3}^{[3]}$, OG_{10}, Kum_2 as well as OG_6-type hyper-Kähler manifolds and is, therefore, sharp. In addition, for these manifolds the theorem implies that the second Chern class lies in the Verbitsky component.

Theorem 5.3.1 can equivalently be formulated in terms of the coefficients of the Riemann–Roch polynomial

$$\mathrm{RR}_X(q) = A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \dots,$$

which is Theorem D.1.1. The condition (5.3.1) becomes

$$2nA_0A_2 < (n-1)A_1^2 \tag{5.3.3}$$

and the bound (5.3.2) translates into the inequality

$$b_2(X) \leq \frac{1}{1 - \frac{2nA_0A_2}{(n-1)A_1^2}} - (2n-2).$$

If the Riemann–Roch polynomial factorizes

$$\mathrm{RR}_X(q) = A_0 \prod_{i=1}^{n} (q + \lambda_i)$$

with n distinct roots λ_i, then (5.3.3) is satisfied and the bound on the second Betti number reads

$$b_2(X) \leq \frac{n-1}{\frac{n \sum \lambda_i^2}{(\sum \lambda_i)^2} - 1} - (2n-2),$$

see Remark D.2.9.

5.4. Generalized Fujiki constants

Theorem 5.3.1 remains true for primitive symplectic orbifolds as in [73, Def. 3.1]. Generalized Fujiki constants still exist for this class of varieties [146, Lem. 4.6]. Moreover, an orbifold version of the Hirzebruch–Riemann–Roch theorem has been established by Blache [32].

We explore Theorem 5.3.1 for many examples of orbifolds in Section D.3. In all cases we check, condition (5.3.1) is satisfied and Theorem 5.3.1 yields bounds on the second Betti number which are frequently attained. The reason why (5.3.1) remains true in the singular case is that one expects the inequality

$$C(\mathrm{ch}_4) > 0$$

to hold true pointwise on the level of forms for the right representative of ch_4.

More generally, we expect the following positivity behaviour for generalized Fujiki constants of products of Chern classes and characters.

Conjecture B (Conjecture K)**.** *For $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$ with $k \coloneqq \sum_i k_i \leq n$ we have*

$$(-1)^k C(\mathrm{ch}_{2k_1} \cdots \mathrm{ch}_{2k_r}) > 0 \quad \textit{as well as} \quad C(c_{2k_1} \cdots c_{2k_r}) > 0.$$

This question was independently asked in [50] and generalizes [169, Qu. 4.7, 4.8] to products for which $k < n$. The combination of the consistent positivity for products of Chern classes together with the alternating behaviour in the case of products of Chern characters would yield many constraints and relations between these numbers. As above, one expects the positivity conditions in Conjecture B to hold true pointwise.

In Section D.4 we discuss generalized Fujiki constants for the known smooth hyper-Kähler manifolds. The case of $\mathrm{K3}^{[n]}$ and Kum_n-type are well-known [68, 163]. For X one of the two sporadic examples, we show that all Chern classes satisfy

$$c_{2i} \in \mathrm{SH}(X, \mathbb{Q}),$$

i.e. they are all contained inside the Verbitsky component. Together with the knowledge of the Riemann–Roch polynomial polynomial of X, this easily allows to compute all generalized Fujiki constants.

5.5. Conjectural form of the Riemann–Roch polynomial

In the final section we further explore the possible shape of the Riemann–Roch polynomial $\mathrm{RR}_X(q)$ of X and the resulting constraints for other invariants.

The main expectation for the shape of $\mathrm{RR}_X(q)$ is the following.

Conjecture C (Conjecture I)**.** *Let X be a primitively symplectic orbifold of dimension $2n$.*

(i) *The Riemann–Roch polynomial $\mathrm{RR}_X(q)$ has n distinct negative real roots forming an arithmetic sequence.*

(ii) *If X is smooth, then its Riemann–Roch polynomial $\mathrm{RR}_X(q)$ has even negative integer roots $\lambda_1, \ldots, \lambda_n$ satisfying $\lambda_i - \lambda_{i-1} = 2$.*

The second point necessarily involves the smoothness assumption and fails already for many examples we discuss in Section D.3. It is a slight strengthening of [113, Conj. 1.3].

We study these questions using Rozansky–Witten theory. As a warm up, we give a shorter proof of Nieper-Wißkirchen's result (3.1.1), see Section D.5.3. The main input is the Wheeling theorem, which we recall in Theorem D.5.1. We explore a similar strategy for the Riemann–Roch polynomial in Section D.5.4. It leads to the following expectation.

Conjecture D (Conjecture H)**.** *Let X be a hyper-Kähler manifold of dimension $2n > 2$. We have*

$$\frac{C(\mathrm{ch}_4)}{C(1)} = \frac{5(n+1)}{(2n-1)(2n-3)}. \tag{5.5.1}$$

In particular, this would yield the expected positivity $C(\mathrm{ch}_4) > 0$. We prove in Proposition D.5.3 that Conjecture C (ii) implies (5.5.1). See Conjecture J for another expectation obtained from the above consideration involving $C(\mathrm{ch}_8)$.

To make the above conjectures more explicit, let us consider their consequences for four-dimensional hyper-Kähler manifolds. One implication is that they limit the possible values of generalized Fujiki constants to two.

Proposition 5.5.1 (Proposition D.5.4)**.** *Assuming Conjecture D, for $n = 2$ the following are the only possibilities for the generalized Fujiki constants of a hyper-Kähler fourfold.*

$C(1)$	$C(c_2)$	$C(c_2^2)$	$C(c_4)$
3	30	828	324
9	54	756	108

Moreover, there are, therefore, only two possible Riemann–Roch polynomials $\mathrm{RR}_X(q)$ for hyper-Kähler fourfolds. This also drastically lowers the possible Betti and Hodge numbers in dimension four.

Corollary 5.5.2 (Corollary D.1.2)**.** *Assuming Conjecture D in dimension 4, the Betti numbers of a hyper-Kähler fourfold X are one of the following:*

- $b_2(X) = 5, b_3(X) = 0, b_4(X) = 96$;
- $b_2(X) = 6, b_3(X) = 4, b_4(X) = 102$;
- $b_2(X) = 7, b_3(X) = 8, b_4(X) = 108$;
- $b_2(X) = 23, b_3(X) = 0, b_4(X) = 276$.

Contribution by the author of the thesis

This chapter and all the results were obtained in collaboration with Jieao Song. We started a mail conversation about the question whether and when the second Chern class lies inside the Verbitsky component. We realized together that this question has connections to bounding the second Betti numbers. From then on the collaboration was intensified and the author of this thesis also visited Song in Paris for a week. All the results and conjectures were obtained together and are shared equally between both authors.

Notation

We want to again mention the few differences in the notation between the previous chapters and Chapter D.

Hyper-Kähler manifolds are called hyperkähler manifolds in this chapter. The BBF form q is denoted by q_X and the fundamental class $[X] \in \mathrm{H}^0(X, \mathbb{Z})$ of a manifold X by 1.

6. Introduction to equivariant categories of symplectic surfaces and fixed loci of Bridgeland moduli spaces

In this chapter we discuss Chapter E and the results therein. The research was done jointly with Georg Oberdieck and has appeared in [28].

In this chapter, we consider symplectic group actions on hyper-Kähler manifolds and derived categories. The fixed locus of the geometric action is shown to be governed by the equivariant category of the categorical action.

6.1. Symplectic actions on hyper-Kähler manifolds

Let us start with a K3 surface S and an automorphism $f \in \mathrm{Aut}(S)$ in the group of automorphisms of S. We assume that f act *symplectically*, i.e. that the induced action on the symplectic form is trivial. This, in particular, implies that the fixed locus

$$S^f \subset S$$

is a symplectic submanifold and is, therefore, discrete. Mukai [156] studied (groups of) symplectic automorphisms of finite order and showed that one has

$$|S^f| \in \{2, 3, 4, 6, 8\}.$$

If we consider the action of a symplectic automorphism f on a higher-dimensional hyper-Kähler manifold X, then the fixed locus X^f is again a union of symplectic submanifolds. For example, if we consider a symplectic involution on a hyper-Kähler manifold of $\mathrm{K3}^{[2]}$-type, then the fixed locus consists always of 28 isolated fixed points and one K3 surface [149]. Fixed loci for involutions on higher-dimensional known hyper-Kähler manifolds were investigated in [118]. The idea in both articles is to deform a given pair (X, f) consisting of a hyper-Kähler manifold X together with an involution f to another pair (X', f') on which one can compute the fixed locus explicitly, see [150, Sec. 5] for an introduction to deformation of such pairs. In the above cases X' is either $S^{[n]}$ for S a K3 surface or $\mathrm{Kum}_n(A)$ for an abelian surface A. The involution f' is induced from a symplectic involution on the surface itself.

6.2. Fixed loci for moduli spaces

Thus, in order to study the fixed locus of a group

$$G \subset \mathrm{Aut}(X)$$

of symplectic automorphisms of a hyper-Kähler manifold one is lead to study this question for a specific representative of the deformation class of the pair (X, G). Natural candidates for the known examples are smooth moduli spaces $M^S_\sigma(v)$ of σ-stable objects on holomorphic symplectic surfaces S. Using also the derived category $\mathrm{D}^\mathrm{b}(S)$ one can consider not only automorphisms, but any finite subgroup

$$G \subset \mathrm{Aut}(\mathrm{D}^\mathrm{b}(S))$$

of symplectic auto-equivalences such that the induced action of G leaves $\sigma \in \mathrm{Stab}^\dagger(S)$ and $v \in \tilde{\mathrm{H}}^{1,1}(S, \mathbb{Z})$ invariant. This, in particular, then implies that G acts via automorphisms on the moduli space $M^S_\sigma(v)$.

More generally, we study the fixed locus of induced automorphisms on Bridgeland moduli spaces for a larger class of smooth projective varieties X and relate the fixed locus with moduli spaces on the equivariant category. We now introduce the framework to state our results.

Let us consider X for which there exists a connected component

$$\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$$

of the space of stability conditions which satisfies the technical condition (†) of Section E.3.6. For example, X could be any smooth projective curve or surface. It is known that for any stability condition $\sigma \in \mathrm{Stab}^*(X)$ and element $v \in K(\mathrm{D}^\mathrm{b}(X))$ there exists good moduli spaces $M^X_\sigma(v)$ of σ-semistable objects with class v [6].

Moreover, let us assume that there exists a finite group G acting on $\mathrm{D}^\mathrm{b}(X)$. We refer to the accompanying paper [29] for an introduction to categorical actions and equivariant categories. For example, G could be a finite group of automorphisms of X. We consider the equivariant category

$$\mathrm{D}^\mathrm{b}(X)_G,$$

which can be interpreted as a noncommutative way of taking a quotient by a group action. The G-action on $\mathrm{D}^\mathrm{b}(X)$ also yields an action on $\mathrm{Stab}(X)$ as well as $K(\mathrm{D}^\mathrm{b}(X))$. Any G-invariant stability condition $\sigma \in \mathrm{Stab}(X)$ induces by [133] a stability condition $\sigma_G \in \mathrm{Stab}(\mathrm{D}^\mathrm{b}(X)_G)$ for the equivariant category. One of our main results is the existence of proper good moduli spaces $M_{\sigma_G}(v')$ of σ_G-semistable objects in $\mathrm{D}^\mathrm{b}(X)_G$ for some class $v' \in K(\mathrm{D}^\mathrm{b}(X)_G)$, see Theorem E.3.22. This is used to describe the fixed locus in the following way.

Theorem 6.2.1 (Theorem E.1.2). *Let $\sigma \in \mathrm{Stab}^*(X)$ be G-invariant and let $M = M^X_\sigma(v)$ be a smooth good moduli space of σ-stable objects in $D^b(X)$ of class $v \in K(D^b(X))^G$. Then, the natural morphism*

$$\bigsqcup_{v' \mapsto v} M_{\sigma_G}(v') \to M^G \tag{6.2.1}$$

is a $G^\vee$-torsor over the union of all G-linearizable connected components of M^G. Here, v' runs over all classes in $K(D^b(X)_G)$ mapping to v under the forgetful functor.

Furthermore, (6.2.1) *is surjective if $H^2(G, \mathbb{C}^*) = 0$ or, more generally, if the G-action on $D^b(X)$ factors through the action of a quotient $G \twoheadrightarrow Q$ such that G is a Schur covering group of Q.*

Schur covering groups are discussed in Section E.2.1. The *dual group* $G^\vee$ is defined as

$$G^\vee \coloneqq \mathrm{Hom}(G, \mathbb{C}^*).$$

We say that a connected component $N \subset M^X_\sigma(v)^G$ is *G-linearizable* if there exists a point $[E] \in N$ such that E admits a G-linearization. The obstruction for the existence of a linearization is an element in the group cohomology group $\mathrm{H}^2(G, \mathbb{C}^*)$ [185]. Thus, if G is, for example, cyclic, the above result determines the entire fixed locus up to possibly an étale cover.

To establish the result, we investigate group actions on (derived) categories in Section E.2. Orlov's representability result [175] allows to lift the action on $\mathrm{D^b}(X)$ to one on the (moduli) stacks. On the level of (moduli) stacks the close relation between fixed stack and equivariant category follows readily from the definitions, see Proposition E.3.9 and [189]. One then uses the moduli theory for objects in $\mathrm{D^b}(X)$ to deduce Theorem E.3.22 saying that also the moduli functor for objects in the equivariant category is well-behaved. The map (6.2.1) is obtained from passing to good moduli spaces.

6.3. Equivariant categories of symplectic surfaces

We now return to our setting $X = S$ a symplectic surface. Theorem 6.2.1 shows that (the G-linearizable components of the) fixed locus of Bridgeland moduli spaces for symplectic group actions induced from $\mathrm{D^b}(S)$ are covered by Bridgeland moduli spaces on the equivariant category. This can be used in two different ways.

Firstly, it helps to understand equivariant categories. That is, let us again consider a group

$$G \subset \mathrm{Aut}(\mathrm{D^b}(S))$$

of symplectic equivalences which we assume acts on $\mathrm{D^b}(S)$. Take a G-invariant stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ and an element

$$v \in \tilde{\mathrm{H}}^{1,1}(S, \mathbb{Z})$$

fixed by the action of G on cohomology. This again leads to an induced action by automorphisms of G on $M^S_\sigma(v)$.

Theorem 6.3.1 (Theorem E.1.1)**.** *Assume that $M^S_\sigma(v)$ is a fine moduli space and that the fixed locus $M_\sigma(v)^G$ has a 2-dimensional G-linearizable connected component F. Then, there exists a subgroup $H \subset G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$, a connected H-torsor $S' \to F$, and an equivalence*

$$D^b(S') \xrightarrow{\cong} D^b(S)_G.$$

Recall that the derived McKay correspondences yields for a finite group $G \subset \mathrm{Aut}(S)$ acting symplectically on S a derived equivalence

$$\mathrm{D^b}(S)_G \cong \mathrm{D^b}(S'), \tag{6.3.1}$$

where S' is a minimal resolution of the quotient S/G. The above result recovers (6.3.1) by considering

$$M^S_\sigma(v) = S^{[|G|]}$$

with the natural induced action of G on $S^{[|G|]}$. Thus, Theorem 6.3.1 can be regarded as a generalization of the derived McKay correspondence for symplectic surfaces. For a more general version allowing coarse moduli spaces with strictly semistable objects, see Theorem E.5.4.

6.4. Fixed loci of Bridgeland moduli spaces for K3 surfaces

In the last section we used Theorem 6.2.1 to describe the equivariant category $\mathrm{D}^{\mathrm{b}}(S)_G$. On the other hand, once we understand $\mathrm{D}^{\mathrm{b}}(S)_G$ we can use Theorem 6.2.1 to describe fixed loci of Bridgeland moduli spaces.

We consider a symplectic G-action on $\mathrm{D}^{\mathrm{b}}(S)$ with S a symplectic surface. Let us look at the moduli space $M_\sigma^S(v)$ for G-invariant $\sigma \in \mathrm{Stab}^\dagger(S)$ and $v \in \tilde{\mathrm{H}}^{1,1}(X, \mathbb{Z})$. We assume that there exists an equivalence

$$\Phi\colon \mathrm{D}^{\mathrm{b}}(S') \cong \mathrm{D}^{\mathrm{b}}(S)_G,$$

where S' is a symplectic surface (one can also, more generally, allow twisted symplectic surfaces using Brauer classes). Hence, Theorem E.1.2 shows that

$$\bigsqcup_{v' \in R_v} M_{\sigma_G}^{S'}(v') \to M_\sigma^S(v)^G$$

is a $G^\vee$-torsor over the union of all G-linearizable components. Here,

$$R_v \subset \tilde{\mathrm{H}}^{1,1}(S', \mathbb{Z})$$

is the set of all elements which get mapped to v under $(p \circ \Phi)^{\tilde{\mathrm{H}}}$ and

$$p\colon \mathrm{D}^{\mathrm{b}}(S)_G \to \mathrm{D}^{\mathrm{b}}(S)$$

is the forgetful functor. In the following special case we can determine the fixed locus completely.

Theorem 6.4.1 (Theorem E.1.3)**.** *Suppose that G is cyclic and that S' is a K3 surface. If $M_\sigma^S(v)$ is a moduli space of stable objects, then we have an isomorphism*

$$M_\sigma^S(v)^G \cong \bigsqcup_{v' \in R_v/G^\vee} M_{\sigma_G}(v').$$

Phrased differently, in the above situation the torsor from Theorem E.1.2 is shown to be trivial leading, therefore, to an explicit description of the fixed locus.

6.5. Symplectic automorphisms of Bridgeland moduli spaces

The previous section yields a general procedure how to determine fixed loci of moduli spaces of stable objects $M_\sigma^S(v)$ for symplectic G-actions which are induced by an action of the group G on $\mathrm{D}^{\mathrm{b}}(S)$. For K3 surfaces, the condition that the action is induced from an action on $\mathrm{D}^{\mathrm{b}}(S)$ is, in fact, no restriction.

Proposition 6.5.1 (Proposition E.1.4)**.** *Let S be a K3 surface and let $\sigma' \in \mathrm{Stab}^\dagger(S)$ be a stability condition. Let G be a finite group which acts faithfully and symplectically on a moduli space $M = M_{\sigma'}^S(v)$ of σ'-stable objects. Then the following holds:*

(a) There exists a surjection $G' \to G$ from a finite group G' and a symplectic action of G' on $D^b(S)$ fixing some stability condition inside $\mathrm{Stab}^\dagger(S)$ *which induces the given G-action on M.*

(b) If G is cyclic, then we can take $G' = G$ in part (a).

We illustrate and apply the obtained results in Section E.7 and explicitly study fixed loci and derived categories for low-dimensional examples. The chapter ends with two appendices. In the first one we discuss properties of hearts of bounded t-structures that are part of a stability condition inside $\mathrm{Stab}^\dagger(S)$ for a symplectic surface S. The second section of the appendix establishes a formula for the Euler characteristic of the fixed locus of a moduli space of stable objects on a K3 surface S for a symplectic automorphism of finite order.

Contribution by the author of the thesis

The results of this part are obtained jointly with Georg Oberdieck. We discussed fixed loci of symplectic automorphisms and wanted to understand the symplectic manifolds that can appear as such. We realized that fixed loci can be related to moduli spaces in the equivariant category which lead to the here presented collaboration. All the results are shared equally between the two authors.

Notation

The notation in Chapter E differs in three instances from the one used so far.

Firstly, a Fourier–Mukai functor with kernel $\mathcal{E}$ is denoted by $\mathsf{F}_{\mathcal{E}}$ instead of $\mathsf{FM}_{\mathcal{E}}$. Furthermore, the square root of the Todd class of a K3 surface S is called $\sqrt{\mathsf{td}(S)}$. For moduli spaces $M_\sigma^X(v)$ of σ-stable objects with class v on the smooth projective variety X the underlying manifold is dropped from the notation and the moduli space is denoted by $M_\sigma(v)$.

7. Introduction to integral Fourier transforms and the integral Hodge conjecture for one-cycles on abelian varieties

This chapter gives an overview over the results obtained in Chapter F. It has appeared as [27] and is a joint work with Olivier de Gaay Fortman.

In this part we study lifts of the Fourier transform on rational Chow groups to integral Chow groups. It turns out that such lifts are linked with the integral Hodge conjecture for one-cycles, which we prove for products of Jacobians of smooth projective curves.

7.1. Cycles and Hodge conjecture

We will leave the realm of hyper-Kähler manifolds and will consider the non-simply connected factors that can appear in the Beauville–Bogomolov decomposition (1.1.1). That is, this part concerns abelian varieties A and cycles on them.

Recall that a subvariety $Z \subset A$ determines a class

$$[Z] \in \mathrm{H}^{2k}(A, \mathbb{Z}),$$

which is known to be contained in the subspace

$$[Z] \in \mathrm{H}^{k,k}(A, \mathbb{Z}) := \mathrm{H}^{2k}(A, \mathbb{Z}) \cap \mathrm{H}^{k,k}(A).$$

This association can be enlarged to the natural graded ring homomorphism

$$cl\colon \mathsf{A}(A)_{\mathbb{Z}} \to \mathrm{H}^*(A, \mathbb{Z}),$$

called the *cycle class map*, from the integral Chow groups $\mathsf{A}(A)_{\mathbb{Z}}$ of A to the integral cohomology (in this chapter, to avoid confusions, we will always indicate whether we work with integral or rational Chow groups). The famous Hodge conjecture for k-cycles states that the induced morphism

$$cl\colon \mathsf{A}^k(A)_{\mathbb{Q}} \to \mathrm{H}^{2k}(A, \mathbb{Q})$$

with rational coefficients is supposed to be surjective. The integral Hodge conjecture for k-cycles is asking for the corresponding morphism

$$cl\colon \mathsf{A}^k(A)_{\mathbb{Z}} \to \mathrm{H}^{2k}(A, \mathbb{Z}) \tag{7.1.1}$$

with integral coefficients to be surjective. A cohomology class is called *algebraic*, if it lies in the image of (7.1.1). It is known that the integral Hodge conjecture fails in general [12, 14, 205].

7.2. Abelian varieties and Fourier transform

For complex abelian varieties A, however, no counterexample to the integral Hodge conjecture is known. The main motivation was to study its validity for this class of manifolds. In particular, we discuss this question for one-cycles. Note that the (rational) Hodge conjecture for one-cycles holds for any smooth projective variety X by the Lefschetz theorem on $(1,1)$-classes, saying that the map

$$cl\colon \mathsf{A}^1(X)_{\mathbb{Z}} \to \mathrm{H}^2(X,\mathbb{Z})$$

is surjective, in combination with the Hard Lefschetz theorem.

We now focus on the case of an abelian variety A of dimension g. Recall from Section 1.3.5 that the Poincare bundle $\mathcal{P}_A$ on $A \times \widehat{A}$ induces an isomorphism

$$\mathcal{F}_A \coloneqq \mathsf{FM}^{\mathrm{H}}_{\mathcal{P}_A}\colon \mathrm{H}^2(A,\mathbb{Z}) \cong \mathrm{H}^{2g-2}(A,\mathbb{Z}), \tag{7.2.1}$$

where we remark that $v(\mathcal{P}_A) = \mathrm{ch}(\mathcal{P}_A)$ since the tangent bundle of any abelian variety is trivial. Moreover, since

$$\mathrm{ch}(\mathcal{P}_A) \in \bigoplus_p \mathrm{H}^{p,p}(A \times \widehat{A}, \mathbb{Z}),$$

the isomorphism (7.2.1) restricts to an isomorphism between Hodge classes. However, one cannot deduce from this the integral Hodge conjecture for one-cycles on $\widehat{A}$, as the class $\mathrm{ch}(\mathcal{P}_A)$ is, a priori, not algebraic[1].

Rephrasing the above, if there exists a group homomorphism

$$\mathcal{F}\colon \mathsf{A}(A)_{\mathbb{Z}} \to \mathsf{A}(\widehat{A})_{\mathbb{Z}} \tag{7.2.2}$$

inducing a commutative diagram

$$\begin{tikzcd}
\mathsf{A}(A)_{\mathbb{Z}} \arrow[r, "\mathcal{F}"] \arrow[d, "cl"] & \mathsf{A}(\widehat{A})_{\mathbb{Z}} \arrow[d, "cl"] \\
\mathrm{H}^*(A,\mathbb{Z}) \arrow[r, "\mathcal{F}_A"] & \mathrm{H}^*(\widehat{A},\mathbb{Z}),
\end{tikzcd} \tag{7.2.3}$$

then the (known) integral Hodge conjecture for codimension one cycles on A implies the integral Hodge conjecture for one-cycles on $\widehat{A}$. More generally, one could ask for a morphism $\mathcal{F}$ as in (7.2.2) which does not only lift the morphism $\mathcal{F}_A$ on cohomology, but also yields a commutative diagram

$$\begin{tikzcd}
\mathsf{A}(A)_{\mathbb{Z}} \arrow[r, "\mathcal{F}"] \arrow[d, "cl"] & \mathsf{A}(\widehat{A})_{\mathbb{Z}} \arrow[d, "cl"] \\
\mathsf{A}(A)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}(\mathcal{P}_A)"] & \mathsf{A}(\widehat{A})_{\mathbb{Q}}
\end{tikzcd} \tag{7.2.4}$$

lifting the homomorphism obtained from the (ungraded) correspondence $\mathrm{ch}(\mathcal{P}_A) \in \mathsf{A}(A \times \widehat{A})$ to integral Chow groups.

The remarkable fact is, that not only does a lift $\mathcal{F}$ as in (7.2.3) yield consequences for the integral Hodge conjecture, but also, conversely, the integral Hodge conjecture for one-cycles

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[1] Recall that this asks for a class to be an integral linear combination of cycles.

implies the existence of a lift $\mathcal{F}$ satisfying the stronger compatibility (7.2.4). Moreover, such an $\mathcal{F}$ is then again induced by an (ungraded) correspondence. This is the content of the following theorem.

Theorem 7.2.1 (Theorem F.1.1)**.** *Let A be a complex abelian variety of dimension g with Poincaré bundle $\mathcal{P}_A$. The following three statements are equivalent:*

(i) *The cohomology class* $\mathrm{c}_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ *is algebraic.*

(ii) *The Chern character* $\mathrm{ch}(\mathcal{P}_A) = \exp(\mathrm{c}_1(\mathcal{P}_A)) \in \mathrm{H}^*(A \times \widehat{A}, \mathbb{Z})$ *is algebraic.*

(iii) *The integral Hodge conjecture for one-cycles holds for $A \times \widehat{A}$.*

Any of these statements implies that

(iv) *The integral Hodge conjecture for one-cycles holds for A and $\widehat{A}$.*

Suppose that A is principally polarized by $\theta \in \mathrm{H}^{1,1}(A, \mathbb{Z})$ and consider the following statements:

(v) *The minimal cohomology class* $\gamma_\theta := \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ *is algebraic.*

(vi) *The cohomology class* $\mathrm{c}_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}(A \times \widehat{A}, \mathbb{Z})$ *is algebraic.*

Then, statements (i) – (vi) *are equivalent. If they hold, the class* $\theta^i/i! \in \mathrm{H}^{2i}(A, \mathbb{Z})$ *is algebraic for* $i \geq 1$.

The minimal cohomology class γ_θ attached to a principally polarized abelian variety (A, θ) has been frequently investigated, see for example [20] where some properties of this class are being proven. Theorem 7.2.1 makes it worthwhile to study the class

$$\mathcal{R}_A := \mathrm{c}_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z}).$$

This class has the advantage that it can be defined for arbitrary (not necessarily principally polarized) abelian varieties. We show in Section F.3.2 that the cycle $\mathcal{R}_A$ satisfies similar properties mimicking the ones shown for γ_θ by Beauville.

The crucial ingredient needed to prove the above result is to relate the algebraicity of the class $\mathcal{R}_A$ with the algebraicity of the whole Chern character $\mathrm{ch}(\mathcal{P}_A)$. This uses a result due to Moonen–Polishchuk [153] showing that a certain ideal in the Chow ring (equipped with the Pontryagin product) admits a divided power structure.

7.3. Jacobians, density, and torsion bounds

An immediate application of Theorem 7.2.1 yields the following.

Theorem 7.3.1 (Theorem F.1.2)**.** *Let $C_1, \ldots, C_n$ be smooth projective curves over $\mathbb{C}$. Then, the integral Hodge conjecture for one-cycles holds for the product of Jacobians $J(C_1) \times \cdots \times J(C_n)$.*

Note that passing from studying the integral Hodge conjecture for one-cycles for one Jacobian to products of Jacobians is, a priori, not clear and crucially uses Theorem 7.2.1.

This yields a large set of principally polarized abelian varieties satisfying the integral Hodge conjecture for one-cycles. We use it together with Theorem 7.2.1 to obtain density of polarized abelian varieties in their moduli space for arbitrary polarization type.

Theorem 7.3.2 (Theorem F.1.3)**.** *Let $\delta = (\delta_1, \ldots, \delta_g)$ be positive integers such that $\delta_i | \delta_{i+1}$ and let $\mathsf{A}_{g,\delta}(\mathbb{C})$ be the coarse moduli space of polarized abelian varieties over $\mathbb{C}$ with polarization type δ. There is a countable union $X \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ of closed algebraic subvarieties of dimension at least g, that satisfies the following property: X is dense in the analytic topology and the integral Hodge conjecture for one-cycles holds for those polarized abelian varieties whose isomorphism class lies in X.*

The lower bound on the dimension of the components of X depends on the polarization type δ and is often greater than g. See Remark F.4.7 for a more precise discussion on this.

The integral Hodge conjecture for one-cycles is equivalent to asking that the abelian group

$$Z^{2g-2}(A) := \mathrm{H}^{g-1,g-1}(A, \mathbb{Z})/\mathrm{Im}(cl),$$

called the degree $2g-2$ *Voisin group*, is trivial. Note that $Z^{2g-2}(A)$ is always torsion (for any smooth projective variety). In Section F.5 we discuss how one can use Theorem 7.2.1 to bound the torsion of the degree $2g-2$ Voisin group. The most general result in this direction is Theorem F.5.3. There, we consider the smallest integer n such that the cycle $n \cdot \mathcal{R}_A$ (respectively $n \cdot \theta_A$ for principally polarized A) is algebraic and show that

$$\gcd(n^2, (2g-2)!) \cdot Z^{2g-2}(A) = (0).$$

For Prym varieties this gives the following.

Theorem 7.3.3 (Theorem F.1.5)**.** *Let A be a g-dimensional Prym variety over $\mathbb{C}$. Then, $4 \cdot \mathrm{Z}^{2g-2}(A) = (0)$.*

7.4. Finitely generated fields and Tate conjecture

Since the Poincaré line bundle $\mathcal{P}_A$ attached to an abelian variety A is universal, it is natural to wonder whether the above results can be generalized to abelian varieties over other fields than the complex numbers.

This is, indeed, the case and is studied in Section F.6. There, we consider fields k which are the separable closure of a finitely generated field. The idea is to replace Betti cohomology with étale cohomology and study, instead of the (integral) Hodge conjecture, the (integral) Tate conjecture for one-cycles.

Let X be a smooth projective variety of dimension d over the separable closure k of a finitely generated field. Recall that X satisfies the integral Tate conjecture for one-cycles if for every prime number ℓ different from $\mathrm{char}(k)$ and for some finitely generated field of definition $k_0 \subset k$ of X the (ℓ-adic) cycle class map

$$cl\colon \mathsf{A}_1(X)_{\mathbb{Z}_\ell} := \mathsf{A}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \bigcup_U \mathrm{H}^{2d-2}_{\text{ét}}(X, \mathbb{Z}_\ell(d-1))^U \tag{7.4.1}$$

is surjective, where U ranges over the open subgroups of $\mathrm{Gal}(k/k_0)$. Elements in the target of (7.4.1) are called *Tate classes.*

If we now again consider $X = A$ an abelian variety, then the Poincaré bundle $\mathcal{P}_A$ is always defined over any field of definition k_0. In particular, its Chern classes are Galois-invariant and, therefore, integral Tate classes. Thus, the induced correspondence

$$\mathcal{F}_A\colon \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^{2g-2}_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell(g-1)) \tag{7.4.2}$$

again sends integral Tate classes to integral Tate classes.

Tate [202], Faltings [70,71], and Zarhin [218,219] have shown the (usual) Tate conjecture for codimension one cycles holds true for abelian varieties. Totaro [206, Lemma 6.2] established the integral version. Hence, to obtain the integral Tate conjecture for one-cycles on an abelian variety A one can proceed similarly as over the complex numbers inspecting this time (7.4.2).

Theorem 7.4.1 (Theorem F.1.6). *Let A be an abelian variety of dimension g over the separable closure k of a finitely generated field.*

- *The abelian variety A satisfies the integral Tate conjecture for one-cycles over k if the cohomology class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}_{\text{ét}}(A \times \widehat{A}, \mathbb{Z}_\ell(2g-1))$ is the class of a one-cycle with $\mathbb{Z}_\ell$-coefficients for every prime number $\ell < (2g-1)!$ unequal to* $\mathrm{char}(k)$.

- *Suppose that A is principally polarized and let $\theta_\ell \in \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1))$ be the class of the polarization. The map (7.4.1) is surjective if $\gamma_{\theta_\ell} := \theta_\ell^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A, \mathbb{Z}_\ell(g-1))$ is in its image. In particular, if $\ell > (g-1)!$, then this always holds. Thus, A satisfies the integral Tate conjecture for one-cycles if γ_{θ_ℓ} is in the image of (7.4.1) for every prime number $\ell < (g-1)!$ unequal to* $\mathrm{char}(k)$.

We again obtain the immediate corollary that products of Jacobians of smooth projective curves over k satisfy the integral Tate conjecture for one-cycles. Moreover, the density result established in Theorem 7.3.2 also has an analogue in positive characteristic.

Theorem 7.4.2 (Theorem F.1.7). *Let k be the algebraic closure of a finitely generated field of characteristic $p > 0$. Let A_g be the coarse moduli space over k of principally polarized abelian varieties of dimension g over k. The subset of $\mathsf{A}_g(k)$ of isomorphism classes of principally polarized abelian varieties over k that satisfy the integral Tate conjecture for one-cycles over k is Zariski dense in A_g.*

Contribution by the author of the thesis

The results in this part are obtained in collaboration with Olivier de Gaay Fortman. The project was initiated through a research stay of de Gaay Fortman in Bonn. We discussed open questions in the study of cycles on abelian varieties and observed that the Fourier transform on cohomology relates different degree cycles. We discovered the paper [153] and realized that it could be applied to these questions. From there on, we elaborated jointly on lifts of Fourier transforms and their consequences. All the results are shared equally between the two authors.

Notation

We want to again mention the notational change which occurs in Chapter F in comparison to the previous parts.

Integral Chow groups for a variety X are denoted by $\mathrm{CH}(X)$ instead of $\mathsf{A}(X)_{\mathbb{Z}}$ and rational Chow groups by $\mathrm{CH}(X)_{\mathbb{Q}}$. The symbol $\mathsf{A}_{g,\delta}$ is reserved to denote exclusively moduli spaces for δ-polarized abelian varieties of dimension g.

8. Summary and outlook

The final chapter of this thesis serves as a conclusion as well as an outlook. We will relate some of our results to current research. Moreover, we discuss possible further questions and research directions, which evolve from and complement our results presented in this thesis[1].

8.1. Derived categories of hyper-Kähler manifolds

In this thesis we established many new results about the derived category $\mathrm{D}^{\mathrm{b}}(X)$ of hyper-Kähler manifolds X. Here, we would like to discuss questions and problems which would enhance the understanding of these categories and equivalences between them.

8.1.1. Relation with other notions

Let us consider two projective hyper-Kähler manifolds X and Y. We want to understand the relations and implications between isomorphisms of different invariants attached to X and Y.

For example, if X and Y are birational, then we know that there exists a Hodge isometry

$$\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(Y,\mathbb{Z}). \tag{8.1.1}$$

Moreover, by [83, Prop. 4.6] any two birational hyper-Kähler manifolds are deformation-equivalent. Furthermore, for two hyper-Kähler manifolds X and Y which are birational there exists a graded ring isomorphism

$$\mathsf{A}^*(X) \cong \mathsf{A}^*(Y)$$

of the Chow rings [188].

How do derived categories fit into this picture? In dimension two, that is K3 surfaces, the notion of birational and isomorphic surfaces agree. The Global Torelli theorem shows that two K3 surfaces S and S' are isomorphic if and only if there exists a Hodge isometry

$$\mathrm{H}^2(S,\mathbb{Z}) \cong \mathrm{H}^2(S',\mathbb{Z}).$$

Of course, this implies that their derived categories $\mathrm{D}^{\mathrm{b}}(S) \cong \mathrm{D}^{\mathrm{b}}(S')$ are equivalent. In [102] it is shown that derived equivalent K3 surfaces have isomorphic Chow motives

$$\mathfrak{h}(S) \cong \mathfrak{h}(S').$$

All K3 surfaces are deformation-equivalent.

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[1] I wish to thank Daniel Huybrechts for stimulating conversations on many of the problems discussed in this chapter.

In higher dimensions, this picture is not yet completely understood, but there exist some conjectures linking these notions. If we start with two projective hyper-Kähler manifolds X and Y which are birational, then it is conjectured [120, Conj. 1.2] that there exists a derived equivalence

$$\mathrm{D^b}(X) \cong \mathrm{D^b}(Y).$$

In general, this conjecture is open. It has been proven for the Mukai flop, a special construction of a birational correspondence involving a Lagrangian $\mathbb{P}^n \subset X$, in [160]. Recently, Halpern-Leistner [86], building upon [17], showed that the conjecture holds if X is isomorphic to a smooth moduli space $M^S_\sigma(v)$ of stable objects on a K3 surface S. Recall that we prove in Proposition B.9.9 a converse of this statement for Hilbert schemes of elliptic K3 surfaces.

Next, let us assume we have two hyper-Kähler manifolds X and Y together with a Hodge isometry as in (8.1.1). We know that this does not necessarily imply that X and Y are birational [159]. However, it is not known what the existence of a Hodge isometry between the second cohomology groups of X and Y implies for their derived categories. Note that if the hyper-Kähler manifolds are of $\mathrm{K3}^{[n]}$-type, a Hodge isometry (8.1.1) induces a Hodge isometry

$$\Lambda_X \cong \Lambda_Y$$

between the associated $\mathrm{K3}^{[n]}$ lattices. It is tempting to speculate that in this case the varieties are derived equivalent, see also [137, Qu. 10.8]. We have already observed in Section B.10.2 that the converse does not hold, see also [4, 145].

Orlov [176] conjectured that derived equivalent hyper-Kähler manifolds have isomorphic Chow motives. This seems to be largely open. Another question is the relation between the derived categories of hyper-Kähler manifolds X and Y and their deformation types. Does a derived equivalence

$$\Phi\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$$

imply that X and Y are deformation-equivalent? The Hodge similitude

$$\phi\colon \tilde{\mathrm{H}}(X, \mathbb{Q}) \to \tilde{\mathrm{H}}(Y, \mathbb{Q})$$

from Theorem 2.3.2 implies, by Witt cancellation, that there exists a Hodge similitude

$$\phi'\colon \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(Y, \mathbb{Q}).$$

However, in general it is not even known whether hyper-Kähler manifolds X and Y for which there exists a Hodge isometry

$$\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{H}^2(Y, \mathbb{Z})$$

are deformation-equivalent (even if we assume, for example, $c_X = c_Y$).

Another idea to link derived categories and deformation types would be to use that the derived equivalence Φ is equivalent to a Fourier–Mukai transform $\Phi \simeq \mathsf{FM}_{\mathcal{E}}$ with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D^b}(X \times Y)$. Starting from this one may try to deform X and Y together with the kernel $\mathcal{E}$ to end up with an element

$$\mathcal{E}' \in \mathrm{D^b}(X' \times Y').$$

Can one find such a deformation such that the induced equivalence $\mathsf{FM}_{\mathcal{E}'}$ is as in the third case of Theorem 3.3.1? One could hope to then modify $\mathcal{E}'$ by pre and postcomposing with auto-equivalences on X' and Y' to obtain an irreducible component $Z \subset \mathrm{supp}(\mathcal{E}')$ mapping birationally onto X' respectively Y'.

8.1.2. Derived Torelli for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds

Using our results from Chapter B one can ask more refined questions for the derived category of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds.

A natural problem prompted by Theorem 3.5.3 is whether, as in the case of K3 surfaces, the converse holds. That is, are $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds X and Y which admit a Hodge isometry

$$\Lambda_X \cong \Lambda_Y$$

derived equivalent? One may look at the proof in the two-dimensional case and see what is missing in higher dimensions.

Starting with a Hodge isometry

$$\varphi\colon \Lambda_X \cong \Lambda_Y$$

one may first assume that $\varphi(\beta) = \pm\beta$. As in Proposition B.9.9 after applying the Hodge isometry B_λ for some $\lambda \in \mathrm{H}^{1,1}(Y,\mathbb{Z})$ one ends up with a Hodge isometry

$$\varphi'\colon \mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(Y,\mathbb{Z}).$$

This leads again to the question whether this implies that X and Y are derived equivalent. If φ' agrees with the action of a parallel transport operator, e.g. this is up to sign true if $n-1$ is a prime power [137, Lem. 9.2], then X and Y are birational and a positive answer to Kawamata's conjecture [120, Conj. 1.2] would yield the desired result.

For general φ the image of β is some arbitrary class

$$\tilde{v} := r\alpha + \lambda + s\beta \in \Lambda_Y.$$

For $r \neq 0$ one uses in the case of K3 surfaces S that moduli spaces $M_H^S(v)$ of stable sheaves with Mukai vector $\tilde{v}$ is a smooth projective K3 surface and its universal family yields a derived equivalence reducing this case to the previous one. For $r = 0$ one uses the auto-equivalences given by tensoring with high powers of an ample line bundle together with the spherical twist $\mathsf{ST}_{\mathcal{O}_S}$ to reduce to the case $r \neq 0$.

8.1.3. Equivalences of fine moduli spaces of stable sheaves

As we have seen, there are a lot of ingredients missing even for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds. If we assume that $X = S^{[n]}$ is the Hilbert scheme of n points, the situation is slightly better. Firstly, as demonstrated in Lemma B.9.10 a Hodge isometry

$$\mathrm{H}^2(S^{[n]},\mathbb{Z}) \cong \mathrm{H}^2(Y,\mathbb{Z}) \tag{8.1.2}$$

implies that Y is as well a fine moduli space of stable sheaves on S. If $n-1$ is a prime power [137, Lem. 9.2] or, by Proposition B.9.9, S is an elliptic K3 surface with a section, then (8.1.2) yields that Y must be birational to $S^{[n]}$. Halpern-Leistner's result [86] yields the desired derived equivalence.

In the general case, i.e. $n-1$ is not a prime power and S is not elliptic, one would need to answer the question whether a fine moduli space $M_H^S(v)$ of stable sheaves on a K3 surface S of dimension $2n$ is derived equivalent to $S^{[n]}$. It is tempting to use the universal family $\mathcal{E}$ on the product

$$M_H^S(v) \times S$$

together with the Bridgeland–King–Reid equivalence

$$\mathrm{D^b}(S^{[n]}) \cong \mathrm{D^b}(S^n)_{\mathfrak{S}_n} \tag{8.1.3}$$

and consider the object

$$p_1^*\mathcal{E} \otimes p_2^*\mathcal{E} \otimes \cdots \otimes p_n^*\mathcal{E} \in \mathrm{D^b}(M_H^s(v) \times S^n)$$

equipped with the natural $1 \times \mathfrak{S}_n$-linearization, where

$$p_i \colon M_H^S(v) \times S^n \to S$$

denotes the projection to the i-th K3 surface. However, the induced Fourier–Mukai transform is not an equivalence.

Similarly, one may try to use the universal ideal sheaf $\mathcal{I}$ on $S \times S^{[n]}$ for a K3 surface S. The object

$$\mathcal{T} \coloneqq \pi_{12}^*\mathcal{E} \otimes \pi_{23}^*\mathcal{I} \in \mathrm{D^b}(M_H^S(v) \times S \times S^{[n]})$$

pushed forward to $M_H^S(v) \times S^{[n]}$ is the composition of the Fourier–Mukai transforms with kernels $\mathcal{E}$ respectively $\mathcal{I}$. As such, it cannot be an equivalence. Can one modify $\mathcal{T}$ in a certain way to make it an equivalence? For example, for $M_H^S(v) = S^{[n]}$, the relative Ext^1 sheaf discussed in Section B.10.1 yields a derived equivalence. Can viewing $\mathcal{I}$ as a $\mathbb{P}^{n-1}$ functor

$$\mathsf{FM}_{\mathcal{I}} \colon \mathrm{D^b}(S) \to \mathrm{D^b}(S^{[n]})$$

as in [2] help for such a construction?

8.1.4. Deformation and derived equivalence for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds

In another direction, Theorem B.9.4 raises the question whether one can still obtain finiteness of Fourier–Mukai partners without restricting to $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds. Let us consider the case $n = 2$. If Y is a hyper-Kähler manifold derived equivalent to a $\mathrm{K3}^{[2]}$-type hyper-Kähler manifold X, then, by Theorem A.8.2, Y has the same Hodge numbers as X. As in the proof of Proposition D.5.4, the Hodge numbers of a hyper-Kähler fourfold determine $\mathrm{c}_4(Y)$ as well as $\mathrm{c}_2(Y)^2$. Since $\mathrm{H}^*(Y, \mathbb{Q}) = \mathrm{SH}(Y, \mathbb{Q})$ for dimension reasons, we have $\mathrm{c}_2(Y) \in \mathrm{SH}(Y, \mathbb{Q})$. Therefore, $\mathrm{c}_2(Y)^2$ determines $C(\mathrm{c}_2(Y))$. The relation from Corollary D.2.6 shows that the Riemann–Roch polynomial $\mathrm{RR}_Y(q)$ is the same as the one for X and, therefore, $c_Y = 1$.

We now specialize to $X = S^{[2]}$ with $U \subset \mathrm{NS}(S)$ and assume that the induced Hodge similitude

$$\varphi \colon \tilde{\mathrm{H}}(S^{[2]}, \mathbb{Q}) \to \tilde{\mathrm{H}}(Y, \mathbb{Q}) \tag{8.1.4}$$

from Theorem A.7.4 is a Hodge isometry. Theorem B.9.8 shows that we can precompose φ by an equivalence

$$\Phi \in \mathrm{Aut}(\mathrm{D^b}(S^{[2]}))$$

such that the composite Hodge isometry

$$\eta \coloneqq \varphi \circ \Phi^{\tilde{\mathrm{H}}} \colon \tilde{\mathrm{H}}(S^{[2]}, \mathbb{Q}) \cong \tilde{\mathrm{H}}(Y, \mathbb{Q})$$

satisfies $\eta(\beta) = k\beta$ for some $k \in \mathbb{Q}$. As in the proof of Theorem B.4.15, we can use topological K-theory and the equality $c_{S^{[2]}} = c_Y$ to deduce $k \in \{\pm 1\}$. Then, following the proof of the third case of Theorem B.4.15, we find that there exists a Hodge isometry

$$\mathrm{H}^2(S^{[2]}, \mathbb{Z}) \cong \mathrm{H}^2(Y, \mathbb{Z})$$

of the integral second cohomology groups. Employing [64, Cor. 1.6], we see that in fact Y is of K3$^{[2]}$-type.

There are two obstacles to make this rigorous. The first one is that we do not know whether (8.1.4) is always a Hodge isometry. For two hyper-Kähler manifolds X and Y the existence of a Hodge isometry

$$\mathrm{SH}(X, \mathbb{Q}) \cong \mathrm{SH}(Y, \mathbb{Q})$$

in general only implies the existence of an isometry

$$(\tilde{\mathrm{H}}(X, \mathbb{Q}), \mathrm{q}) \cong (\tilde{\mathrm{H}}(Y, \mathbb{Q}), \mu\mathrm{q}) \tag{8.1.5}$$

for some $\mu \in \mathbb{Q}^*$, see [201, Sec. 4]. For even n, there exist examples with $\mu \in \mathbb{Q}^* \setminus \mathbb{Q}^2$ not a square such that the associated kernels of the Laplacians as in (1.2.9) are isometric. To exclude such cases, one could try to use some integral structure, such as topological K-theory, which is preserved under derived equivalences. Can this be combined with the indivisibility of the BBF form to exclude (8.1.5) with $\mu \neq 1$?

The other obstacle is that we needed $X = S^{[2]}$ with $U \subset \mathrm{NS}(S)$ to modify the Hodge isometry (8.1.4). A naive hope would be to use deformation theory to reduce to this case. Namely, assume we start with a general X of K3$^{[2]}$-type and an equivalence

$$\mathsf{FM}_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y).$$

This induces isomorphisms

$$\mathsf{FM}_{\mathcal{E}}^{\mathrm{HT}} \colon \mathrm{HT}^2(X) \cong \mathrm{HT}^2(Y)$$

and we can consider first-order deformation directions inside $\mathrm{H}^1(X, \mathcal{T}_X)$ which are mapped by $\mathsf{FM}_{\mathcal{E}}^{\mathrm{HT}}$ to elements inside $\mathrm{H}^1(Y, \mathcal{T}_Y)$. In [204, Thm. 1.1] it is shown that the equivalence $\mathsf{FM}_{\mathcal{E}}$ lifts to first-order. It would be highly desirable to be able to upgrade this to algebraic families. Roughly, for two families

$$\pi_X \colon \mathcal{X} \to B, \quad \pi_Y \colon \mathcal{Y} \to B$$

over a one dimensional base B representing the above first-order deformation directions one could hope to find an element $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathcal{X} \times_B \mathcal{Y})$ lifting $\mathcal{E}$.

8.1.5. Study of the representation $\rho^{\tilde{\mathrm{H}}}$

A way to circumvent the deformation theory of Fourier–Mukai transforms would be to study the representation

$$\rho^{\tilde{\mathrm{H}}} \colon \ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{Aut}(\Lambda_X)$$

for arbitrary K3$^{[n]}$-type hyper-Kähler manifolds X. What can we say about the image $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$? This has two parts.

The first is to construct auto-equivalences of K3$^{[n]}$-type hyper-Kähler manifolds, especially when X is not isomorphic to a Hilbert scheme $S^{[n]}$. For example, let us consider Hodge isometries of $\mathrm{H}^2(X,\mathbb{Z})$ which are parallel transport operators. By [137, Thm. 1.6] these are composed of two classes. One is given by the induced action of birational self-maps of X. Can one modify the closure of the graph of such a map to obtain an auto-equivalence?

The other case is the subgroup generated by reflections along *prime exceptional divisors*, which are integral effective divisors with negative BBF-square. In the case of K3 surfaces S, these correspond to smooth rational curves $C \subset S$. The reflection $s_{[C]}$ along the -2-class $[C] \in \mathrm{H}^2(S,\mathbb{Z})$ can be obtained as the induced action of an auto-equivalence. Indeed, the spherical twist

$$\mathsf{ST}_{\mathcal{O}_C(-C)} \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S))$$

associated to the spherical sheaf $\mathcal{O}_C(-C)$ satisfies

$$\mathsf{ST}^{\tilde{\mathrm{H}}}_{\mathcal{O}_C(-C)} = s_{[C]}.$$

Consider the diagram

$$\begin{tikzcd} C \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & S \\ \mathsf{p}. & \end{tikzcd}$$

One can interpret the auto-equivalence $\mathsf{ST}_{\mathcal{O}_C(-C)}$ (up to line bundle twists) as the induced equivalence associated to the spherical functor

$$\iota_* \circ \pi^* \colon \mathrm{D}^{\mathrm{b}}(\mathsf{p}) \to \mathrm{D}^{\mathrm{b}}(S).$$

Can this approach be generalized to prime exceptional divisors E to obtain that the induced reflection $s_{[E]}$ lies in the image of $\rho^{\tilde{\mathrm{H}}}$? For example, in dimension four the exceptional divisor $E \subset S^{[2]}$ of the Hilbert–Chow morphism is prime exceptional and sits in a diagram

$$\begin{tikzcd} E \cong \mathbb{P}(\Omega_S) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & S^{[2]} \\ S. & \end{tikzcd}$$

We have already encountered in Section B.10.1 the corresponding spherical twist associated to the spherical functor

$$\iota_* \circ \pi^* \colon \mathrm{D}^{\mathrm{b}}(S) \to \mathrm{D}^{\mathrm{b}}(S^{[2]})$$

and seen that, indeed, it acts on the Mukai lattice $\tilde{\mathrm{H}}(S^{[2]},\mathbb{Q})$, up to conjugation by twisting with the line bundle $\mathcal{O}_{S^{[2]}}(\delta)$, as the reflection $s_{[E]}$.

This example can easily be generalized to smooth prime exceptional divisors $\iota \colon E \hookrightarrow X$ which admit a fibration

$$\pi \colon E \to B$$

onto a smooth $2n-2$-dimensional base such that all fibres of π are projective lines. That is, in this case we again obtain a spherical functor

$$\iota_* \circ \pi^* \colon \mathrm{D}^{\mathrm{b}}(B) \to \mathrm{D}^{\mathrm{b}}(X)$$

and the induced spherical twist acts on the extended Mukai lattice up to conjugation with a line bundle twist as the reflection $s_{[E]}$ along E. In general, if one wants to get rid of the smoothness assumption on X and B, one could try to work with the category of perfect complexes and categorical resolutions.

Another question related to the image of the representation $\rho^{\tilde{\mathrm{H}}}$ is to find a sharp upper bound. For example, in Theorem B.9.8 we have obtained the lower bound

$$\hat{\mathrm{Aut}}^{+}(\Lambda_{S^{[n]}}) \subset \mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$$

for Hilbert schemes $S^{[n]}$ of elliptic K3 surfaces with a section. Is this inclusion an equality? For example, in the case $n = 2$, the group $\hat{\mathrm{Aut}}^{+}(\Lambda_{S^{[2]}})$ equals $\mathrm{Aut}^{+}(\Lambda_{S^{[2]}})$ and the question becomes whether all equivalences respect the natural orientation of the four positive directions. It is not clear if the approach in [105], where this question was answered for K3 surfaces, can be generalized to higher dimensions.

8.1.6. Generalized Kummer manifolds

What we have not discussed so far are derived categories of Kum_n-type hyper-Kähler manifolds and equivalences between them. We briefly mention some difficulties that occur in this case as well as a specific open problem.

A major tool used in Chapter B was the derived monodromy group $\mathrm{DMon}(S^{[n]})$, which we studied using Ploog's map

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S)) \to \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S^{[n]})).$$

Underlying it is the Bridgeland–King–Reid isomorphism (8.1.3). For the generalized Kummer manifold $\mathrm{Kum}_n(A)$ of dimension $2n$ associated to an abelian surface A there exists an analogue of the above. Namely, let us consider the group $\mathfrak{S}_{n+1}$ acting on A^n. The inclusion

$$\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$$

of the first n-factors acts via permutation of factors on A^n. The transposition $(1\ \ n+1)$ gives the automorphism

$$(1\ \ n+1)\colon A^n \cong A^n, \quad (x_1, x_2, x_3, \dots x_n) \mapsto (-\sum_{i=1}^{n} x_i, x_2, x_3, \dots, x_n).$$

Then, by [144, Thm. 6.2] there exists a derived equivalence

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Kum}_n(A)) \cong \mathrm{D}^{\mathrm{b}}(A^n)_{\mathfrak{S}_{n+1}}. \tag{8.1.6}$$

One sees already from the definition of the action of $\mathfrak{S}_{n+1}$ that this case is more involved. In particular, a Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(A \times A)$ giving a derived equivalence

$$\mathsf{FM}_{\mathcal{E}}\colon \mathrm{D}^{\mathrm{b}}(A) \cong \mathrm{D}^{\mathrm{b}}(A)$$

lifts to the object

$$\mathcal{E}^{\boxtimes n} \in \mathrm{D}^{\mathrm{b}}(A^n \times A^n)$$

which, in general, does not admit a $\mathfrak{S}_{n+1} \times \mathfrak{S}_{n+1}$-linearization, see [184]. As we saw in Chapter B, one does not need the knowledge of the full group of auto-equivalences

$\mathrm{Aut}(\mathrm{D^b}(\mathrm{Kum}_n(A)))$, but rather one equivalence whose induced action on $\tilde{\mathrm{H}}(\mathrm{Kum}_n(A), \mathbb{Z})$ sends β to a vector with non-zero minimal rank.

A particular example of the above failure of lifting derived equivalences between abelian surfaces to the associated generalized Kummer manifolds is given by the Poincaré bundle $\mathcal{P}_A$ [184, Sec. 4.4]. It is not known whether, in general, the two derived categories $\mathrm{D^b}(\mathrm{Kum}_n(A))$ and $\mathrm{D^b}(\mathrm{Kum}_n(\widehat{A}))$ are equivalent. Note that we always have a Hodge isometry

$$\mathrm{H}^2(\mathrm{Kum}_n(A), \mathbb{Z}) \cong \mathrm{H}^2(\mathrm{Kum}_n(\widehat{A}), \mathbb{Z}).$$

The group $G = A[n+1]$ of $(n+1)$-torsion points of an abelian surface A acts via automorphisms on the manifold $\mathrm{Kum}_n(A)$. The quotient

$$\mathrm{Kum}_n(A)/G$$

does not admit a symplectic resolution for $n > 1$, since the fixed locus has components of codimension greater than two. Thus, one is instead lead to consider the equivariant category $\mathrm{D^b}(\mathrm{Kum}_n(A))_G$. We can show that there exists a derived equivalence

$$\mathrm{D^b}(\mathrm{Kum}_n(A))_G \cong \mathrm{D^b}(\mathrm{Kum}_n(\widehat{A}))$$

relating the derived category of the dual abelian surface with the equivariant category. Hence, one can recover $\mathrm{D^b}(\mathrm{Kum}_n(\widehat{A}))$ from $\mathrm{D^b}(\mathrm{Kum}_n(A))$ using the G-action.

8.2. Atomic objects

In Chapter C we have defined the notion of atomic sheaves and objects and initiated the study of their properties. It would be beneficial if these sheaves and objects would be investigated in more depth.

Two open problems in this realm we have already mentioned are Conjectures E and F. Let us elaborate a bit on possible attempts to prove the latter.

8.2.1. Actions on Ext algebra

For a slope stable atomic bundle $\mathcal{E}$ we would like to investigate the pairing

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}). \tag{8.2.1}$$

One approach could be to construct an action of a group G (or Lie algebra) on the Ext algebra $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$ which is equivariant for the algebra structure. The tensor product of the G-representation $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ would then decompose

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \otimes \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \cong \mathrm{Sym}^2(\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})) \oplus \Lambda^2(\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}))$$

respecting the G-action. Can G and the action of G on $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ and $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ be chosen in a way that the only possible G-equivariant map

$$\mathrm{Sym}^2(\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \tag{8.2.2}$$

is the zero morphism? This would yield the desired skew-symmetry. In [208, Sec. 3] Verbitsky constructed an action of the group SU(2) on

$$\mathrm{H}^*(X, \mathcal{E}nd(\mathcal{E}))$$

using real structures.

One may try to construct the action not on the Ext algebra itself, but on the larger algebra

$$\mathrm{H}^{*,*}(X, \mathcal{E}nd(\mathcal{E})),$$

i.e. the de Rham cohomology with values in the endomorphism bundle $\mathcal{E}nd(\mathcal{E})$. We have the natural identification

$$\mathrm{H}^{0,*}(X, \mathcal{E}nd(\mathcal{E})) \cong \mathrm{Ext}^*(\mathcal{E}, \mathcal{E}).$$

Thus, the same approach may be conducted for the larger algebra $\mathrm{H}^{*,*}(X, \mathcal{E}nd(\mathcal{E}))$. Note that for a hyper-Kähler metric the induced SU(2)-action (equivalently $\mathfrak{su}(2)$-action) on the level of forms with values in $\mathcal{E}nd(\mathcal{E})$ yields an action of SU(2) on $\mathrm{H}^{*,*}(X, \mathcal{E}nd(\mathcal{E}))$. However, inspecting the decomposition of the degree one and two components of this algebra into irreducible SU(2)-representations, the analogous map to (8.2.2) does not vanish. In the case of the atomic bundle being a line bundle $\mathcal{E} = \mathcal{L} \in \mathrm{Pic}(X)$ one can combine all possible $\mathfrak{su}(2)$-actions on

$$\mathrm{H}^{*,*}(X, \mathcal{E}nd(\mathcal{E})) \cong \mathrm{H}^{*,*}(X, \mathcal{O}_X) \cong \mathrm{H}^*(X, \Omega_X^*)$$

to obtain the action of the reduced LLV algebra $\bar{\mathfrak{g}}(X)$ acting on the de Rham cohomology. Recall that this acts by derivations and, therefore, the integrated action is multiplicative. Of course, in this case we know already that the multiplication on the usual cohomology is graded-commutative. Can one, in general, combine the $\mathfrak{su}(2)$-actions for a stable atomic bundle $\mathcal{E}$ to obtain a larger Lie algebra acting on $\mathrm{H}^{*,*}(X, \mathcal{E}nd(\mathcal{E}))$?

8.2.2. Relation to other deformation problems

Another approach to Conjecture F may be to relate the deformation problem for the stable bundle $\mathcal{E}$ on X to another deformation problem, which then might be easier to solve.

To have an example in mind, let us consider the sheaf of first-order differential operators $D(X, \mathcal{E})$ with scalar symbol associated to $\mathcal{E}$, see [112, Sec. 2] for an introduction to this notion and its properties. It sits inside the short exact sequence

$$0 \to \mathcal{E}nd(\mathcal{E}, \mathcal{E}) \to D(X, \mathcal{E}) \to \mathcal{T}_X \to 0 \tag{8.2.3}$$

and the associated extension class inside

$$\mathrm{Ext}^1(\mathcal{T}_X, \mathcal{E}nd(\mathcal{E})) \cong \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_X)$$

is exactly the Atiyah class. To the vector bundle $D(X, \mathcal{E})$ we can associate a dg Lie algebra L whose associated deformation functor is isomorphic to the deformation functor of the pair $(X, \mathcal{E})$, see [111]. That is, it controls deformations of the manifold X together with the vector bundle $\mathcal{E}$. In [112] this deformation problem was studied for Calabi–Yau varieties together with a line bundle.

Let us discuss a possible analogue for atomic bundles $\mathcal{E}$ on hyper-Kähler manifolds X. One can complete (8.2.3) to the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & \mathcal{E}nd(\mathcal{E}) \arrow[r] \arrow[d] & D(X,\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{E}nd(\mathcal{E})_0 \arrow[r] & D(X,\mathcal{E})_0 \arrow[r] & \mathcal{T}_X \arrow[r] & 0,
\end{tikzcd} \tag{8.2.4}$$

where again $\mathcal{E}nd(\mathcal{E})_0$ denotes the bundle of traceless endomorphisms and $D(X,\mathcal{E})_0$ the corresponding cokernel. The lower short exact sequence equals the derived pushforward of the short exact sequence

$$0 \to \mathcal{T}_{\mathbb{P}(\mathcal{E})|X} \to \mathcal{T}_{\mathbb{P}(\mathcal{E})} \to \pi^*\mathcal{T}_X \to 0$$

under the projective bundle morphism $\pi\colon \mathbb{P}(\mathcal{E}) \to X$. In particular, the dg Lie algebra associated to $\mathcal{T}_{\mathbb{P}(\mathcal{E})}$ is quasi-isomorphic to the one associated to $D(X,\mathcal{E})_0$. This implies that the deformation functor associated to the latter governs the deformation problem of deforming $\mathbb{P}(\mathcal{E})$. The long exact sequence in cohomology of the lower short exact sequence in (8.2.4) reads

$$\mathrm{H}^1(X,\mathcal{E}nd(\mathcal{E})) \hookrightarrow \mathrm{H}^1(X,D(X,\mathcal{E})_0) \to \mathrm{H}^1(X,\mathcal{T}_X) \xrightarrow{\tau} \mathrm{H}^2(X,\mathcal{E}nd(\mathcal{E})_0) \to \mathrm{H}^2(X,D(X,\mathcal{E})_0).$$

Note that if $\mathcal{E}$ is 1-obstructed, then the morphism τ is trivial and one can deduce that, in order to study the pairing (8.2.1), it suffices to study the deformation problem associated to $\mathbb{P}(\mathcal{E})$.

As a closing note to this subsection we want to mention that in [139, Sec. 13] Markman studies certain 1-obstructed sheaves and bundles $\mathcal{E}$ on the Fano variety of lines $F(X)$ of a cubic fourfold $X \subset \mathbb{P}^5$. The specific choice is motivated from [128], where, starting from the Fano variety of lines $F(X)$, a hyper-Kähler manifold M_X of OG_{10}-type is constructed via a fibration into intermediate Jacobians. If we denote by $\tilde{v} \in \Lambda_{F(X)}$ (a representative of) the Mukai vector of the sheaf $\mathcal{E}$ in the Mukai lattice, one obtains a rank 24 integral lattice

$$\tilde{v}^{\perp} \subset \Lambda_{F(X)}.$$

One can check that, in fact, there exists a Hodge isometry

$$\tilde{v}^{\perp} \cong \mathrm{H}^2(M_X,\mathbb{Z})$$

mimicking the Hodge isometry

$$\mathrm{H}^2(M_H^S(v)) \cong v^{\perp} \subset \mathrm{H}^*(S,\mathbb{Z})$$

for smooth moduli spaces $M_H^S(v)$ of stable sheaves on a K3 surface S.

8.2.3. Mukai vector

The definition of atomic objects $\mathcal{E}$ yields an element $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ relating the two annihilator Lie subalgebras $\mathrm{Ann}(v(\mathcal{E}))$ and $\mathrm{Ann}(\tilde{v})$ of the LLV algebra $\mathfrak{g}(X)$. The element $\tilde{v}$ is only unique up to a scalar. For an arbitrary atomic object $\mathcal{E}$ it is not clear what the precise multiple of $\tilde{v}$ should be that one could call the extended Mukai vector $\tilde{v}(\mathcal{E})$ of $\mathcal{E}$. In Section B.4 we studied the two cases when $\mathcal{E}$ is in the orbit of $\mathcal{O}_X$ respectively $k(x)$ under the group $\mathrm{Aut}(\mathrm{D^b}(X))$.

As done in Section C.3 it is tempting to declare

$$\tilde{v}(\mathcal{E}) = \mathrm{rk}(\mathcal{E})\alpha + \mathrm{c}_1(\mathcal{E}) + s\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

for some $s \in \mathbb{Q}$ if $\mathrm{rk}(\mathcal{E}) \neq 0$. However, this does not always seem to be the best choice. For example, consider the $\mathbb{P}^2$-vector bundles $\mathcal{E}$ on $\mathrm{K3}^{[2]}$-type hyper-Kähler manifolds studied in [172, Thm. 1.4]. In accordance with objects in the $\mathcal{O}_X$-orbit, one may demand the equality

$$v(\mathcal{E}) = \frac{T(\tilde{v}(\mathcal{E})^2)}{2}.$$

However, this would mean that one would normalize the vector in the Mukai lattice to be

$$\tilde{v}(\mathcal{E}) = \sqrt{\mathrm{rk}(\mathcal{E})}\alpha + \frac{\mathrm{c}_1(\mathcal{E})}{\sqrt{\mathrm{rk}(\mathcal{E})}} + \frac{s}{\sqrt{\mathrm{rk}(\mathcal{E})}}\beta. \tag{8.2.5}$$

Note that the examples in [172, Thm. 1.4] all satisfy $\mathrm{rk}(\mathcal{E}) = r_0^2$ for some positive integer $r_0 \in \mathbb{Z}$. Moreover, the multiple $\tilde{v}(\mathcal{E})$ in (8.2.5) satisfies

$$\tilde{v}(\mathcal{E})^2 = -\frac{5}{2}$$

just like the $\mathbb{P}^2$-objects in the orbit of $\mathcal{O}_X$.

8.2.4. Examples

Of course, more examples or existence results for atomic sheaves on higher-dimensional hyper-Kähler manifolds would be highly desirable. For example, as in [172], can one use Lagrangian fibrations and the knowledge of semi-homogeneous bundles on abelian varieties to construct new examples of stable atomic bundles? These would then deform to nearby complex structures not admitting Lagrangian fibrations using the results of Chapter C.

8.3. Restrictions on generalized Fujiki constants and Riemann–Roch polynomials

In this section we discuss open questions concerning properties of generalized Fujiki constants of hyper-Kähler manifolds[2].

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[2] I thank Gebhard Martin, Mirko Mauri, and Jieao Song for interesting discussions about the topics of this section.

8.3.1. Positivity

It would be worthwhile to establish Conjecture D or one of its higher degree analogues. Already the positivity in Conjecture B would yield many restrictions on generalized Fujiki constants.

In principle, the latter conjecture could be proven purely via linear algebra. Namely, one could use that the Chern character $\mathrm{ch}(\mathcal{T}_X)$ of the tangent bundle of a hyper-Kähler manifold X can be represented as the trace of the exponential of the Atiyah class

$$\mathrm{At}_{\mathcal{T}_X} \in \mathrm{Ext}^1(\mathcal{T}_X, \mathcal{T}_X \otimes \Omega_X^1).$$

The symplectic structure yields additional symmetries which implies that this class is, in fact, contained in $\mathrm{H}^1(X, \mathrm{Sym}^3 \mathcal{T}_X)$ [163, Cor. 1.3].

The positivity $C(-\mathrm{ch}_2(X)) > 0$ can be deduced directly on the level of forms. In [193, Sec. 4] this strategy is being tested for higher degree Chern characters. An idea would be to group the terms that appear in an expression of the exponential of the Atiyah class into terms which, when compared correctly with each other, yield the desired positivity pointwise.

8.3.2. Small Fujiki constant

One can also consider the small Fujiki constant c_X. In general, it is only known that this is a positive rational number. Using the Riemann–Roch polynomial, this can be refined. Namely, the leading coefficient of $\mathrm{RR}_X(q)$ is equal to

$$\frac{c_X}{2^n n!}.$$

We now employ the fact that $\mathrm{RR}_X(q)$ sends all integers which are attained by the BBF form q again to integers. Thus, if $\mathrm{H}^2(X, \mathbb{Z})$ is known to be an even lattice and

$$\mathrm{Im}(\mathrm{q}) = 2\mathbb{Z},$$

then c_X must already be an integer. Indeed, this would follow under these assumptions from the property of integer-valued polynomials that the binomial coefficients

$$q(q-1)\dots(q-k+1)/k!$$

yield a basis of this class of polynomials.

8.3.3. Riemann–Roch polynomial via Lagrangian fibrations

Another approach to obtain more information on generalized Fujiki constants is to try to exploit the relationship of these numbers with the geometry of the hyper-Kähler manifold X.

One example of this kind was recently explored in [64], in particular Theorem 3.1 in *loc. cit.* Namely, the existence of a Lagrangian fibration

$$\pi \colon X \to \mathbb{P}^n$$

with certain numerical properties is used to compute the Riemann–Roch polynomial $\mathrm{RR}_X(q)$ of X. This idea goes back to [190], where this strategy was used to determine the Riemann–Roch polynomial of OG_{10}-type hyper-Kähler manifolds. The main point is to consider the line bundle $\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ and another line bundle $\mathcal{M}$ which restricts to smooth fibres to a principal polarization. One then shows that

$$\mathrm{R}\pi_*\mathcal{M} \cong \pi_*\mathcal{M}$$

is, in fact, a line bundle. The projection formula for the line bundles

$$\mathcal{M} \otimes \mathcal{L}^{\otimes i}$$

together with the Hirzebruch–Riemann–Roch theorem implies that the Riemann–Roch polynomial of X equals that of $S^{[n]}$ with S a K3 surface, where $\dim X = 2n$.

One could consider the above strategy for Lagrangian fibrations whose fibres do not carry a principal polarization. We sketch here a possible example.

Consider X a hyper-Kähler manifold with two different Lagrangian fibrations

$$\pi_1 \colon X \to \mathbb{P}^n, \quad \pi_2 \colon X \to \mathbb{P}^n$$

and let us denote the corresponding non-isomorphic nef line bundles $\mathcal{L}_i = \pi_i^*\mathcal{O}_{\mathbb{P}^n}(1)$. The product

$$\mathcal{M} \coloneqq \mathcal{L}_1 \otimes \mathcal{L}_2$$

must then be an ample line bundle. In particular, it restricts to a polarization on smooth fibres for both Lagrangian fibrations π_i. Hence, $\mathcal{L}_2$ is π_1-relatively ample and vice versa. As in the proof of [64, Thm. 3.1] we get a vector bundle

$$\mathcal{E} \coloneqq \mathrm{R}\pi_{1*}\mathcal{L}_2 \cong \pi_{1*}\mathcal{L}_2$$

of rank

$$d \coloneqq \prod_i d_i,$$

where $\mathcal{L}_2$ restricts to the general fibre of π_1 as a polarization of type $(d_1, \ldots, d_n)$. Moreover, we can compute for a smooth fibre $A \subset X$ of π_1

$$d \cdot n! = \int_A \mathrm{c}_1(\mathcal{L}_2)^n = \int_X \mathrm{c}_1(\mathcal{L}_1)^n \mathrm{c}_1(\mathcal{L}_2)^n = n! \cdot c_X \cdot \mathrm{q}(\mathrm{c}_1(\mathcal{L}_1), \mathrm{c}_1(\mathcal{L}_2))^n.$$

It seems reasonable to expect

$$\mathrm{q}(\mathrm{c}_1(\mathcal{L}_1), \mathrm{c}_1(\mathcal{L}_2)) = d_1$$

so that there exists a polarization $\mathcal{L}'$ on a generic fibre A whose d_1-th power is isomorphic to $\mathcal{L}_2|_A$. This would give

$$c_X = \prod_i \frac{d_i}{d_1}.$$

One can also use the two Lagrangian fibrations to show that

$$\mathrm{H}^i(X, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(k))$$

vanishes for $0 < i < n$ and all $k \in \mathbb{Z}$. In particular, by [173, Thm. 2.3.1], the bundle $\mathcal{E}$ decomposes into line bundles

$$\mathcal{E} \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(a_i).$$

Using, for example, $\mathrm{RR}_X(0) = n + 1$ and the positivity of the coefficients of the Riemann–Roch polynomial, one can bound the integers a_i.

Hence, also in the above case one can relate the Riemann–Roch polynomial of X with (sums of) Euler characteristics of line bundles on the projective space. It is, however, not clear whether one can deform an arbitrary (unknown) hyper-Kähler manifold X to one with two Lagrangian fibrations as discussed above. The result [119, Thm. 3.3] shows that this is predicted by the SYZ conjecture as soon as $b_2(X) \geq 5$.

8.3.4. Hyper-Kähler fourfolds with largest second Betti number

We want to close this section with the following question. Is any hyper-Kähler fourfold X with $b_2(X) = 23$ of $\mathrm{K3}^{[2]}$-type?

As already discussed, any such manifold X shares the Hodge numbers and the Riemann–Roch polynomial with that of $S^{[2]}$ for S a K3 surface. Moreover, [64, Lem. 3.2] shows that the BBF form q of X must be even. It would be enough to establish the inclusion

$$U \subset \mathrm{H}^2(X, \mathbb{Z})$$

to apply [64, Thm. 1.5].

Expressed differently, for an isotropic class $e \in \mathrm{H}^2(X, \mathbb{Z})$ (which exists since $b_2(X) = 23$) we would need to find another class $f \in \mathrm{H}^2(X, \mathbb{Z})$ such that

$$\mathrm{q}(e, f) = 1,$$

which, using $c_X = 1$, is equivalent to

$$\int_X e^2 f^2 = 2.$$

For K3 surfaces S Poincaré Duality implies that the quadratic form on $\mathrm{H}^2(S, \mathbb{Z})$ is unimodular. This allows one to obtain the inclusion $U \subset \mathrm{H}^2(S, \mathbb{Z})$.

For the fourfold X under consideration, we still have that the pairing

$$\mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^6(X, \mathbb{Z}) \to \mathrm{H}^8(X, \mathbb{Z}) \cong \mathbb{Z} \tag{8.3.1}$$

is perfect. Note that the polarized Fujiki relations

$$\int_X \lambda_1\lambda_2\lambda_3\lambda_4 = \mathrm{q}(\lambda_1, \lambda_2)\mathrm{q}(\lambda_3, \lambda_4) + \mathrm{q}(\lambda_1, \lambda_3)\mathrm{q}(\lambda_2, \lambda_4) + \mathrm{q}(\lambda_1, \lambda_4)\mathrm{q}(\lambda_2, \lambda_3)$$

together with the fact that $\mathrm{H}^2(X, \mathbb{Z})$ is an even lattice show that for $\omega \in \mathrm{H}^2(X, \mathbb{Z})$ the element

$$\omega^3 \in \mathrm{H}^6(X, \mathbb{Z})$$

is divisible as an element in $\mathrm{H}^6(X, \mathbb{Z})$ by at least six. Let us assume that $\omega^3/6$ is primitive inside $\mathrm{H}^6(X, \mathbb{Z})$ and that this element is dual to our isotropic class e under (8.3.1). This would yield

$$1 = \frac{1}{6} \int_X e\omega^3 = \mathrm{q}(e, \omega) \frac{\mathrm{q}(\omega, \omega)}{2}.$$

Since the right hand side is a product of integers, we would have obtained the desired element which pairs to one with e. Any element in $\mathrm{H}^6(X, \mathbb{Z})$ can up to scaling with a rational number be written as the product of three elements of $\mathrm{H}^2(X, \mathbb{Z})$.

A. Derived categories of hyper-Kähler manifolds via the LLV algebra

Abstract. We mostly review work of Taelman [201] on derived categories of hyper-Kähler manifolds. We study the LLV algebra using polyvector fields to prove that it is a derived invariant. Applications to the action of derived equivalences on cohomology and to the study of their Hodge structures are given.

A.1. Introduction

In this note we discuss the *(bounded) derived category* $\mathrm{D}^b(X) \coloneqq \mathrm{D}^b(\mathrm{Coh}(X))$ and its group of auto-equivalences $\mathrm{Aut}(\mathrm{D}^b(X))$ for projective hyper-Kähler manifolds X. The situation in dimension two, that is for K3 surfaces, is fairly well understood and we refer to [97, Sec. 10] for an overview. Therefore, we will only concentrate on the higher-dimensional case. More precisely, we mainly present the first part of Taelman's paper [201].

These notes are, for the most part, light on derived categories and focus more on a different perspective of the Looijenga–Lunts–Verbitsky (LLV) Lie algebra $\mathfrak{g}(X)$ [130, 207] which will allow us to show the following.

Theorem A.1.1 (Taelman)**.** *A derived equivalence* $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ *between projective hyper-Kähler manifolds induces naturally a Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(Y).$$

The induced isomorphism of quadratic spaces

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$*.*

The theorem will be proven in Section A.5.

We start these notes by introducing the main objects of study and a collection of known results prior to [201]. Afterwards, we introduce a new Lie subalgebra of the (ungraded) endomorphism algebra $\mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$ which is better suited for the study of derived categories. In the subsequent section we establish Theorem A.1.1 via proving that the newly defined Lie subalgebra coincides with the well-known LLV Lie algebra $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ with scalars extended to the complex numbers. The next three sections will draw consequences from this result for the action of derived equivalences on cohomology and for Hodge structures of derived equivalent hyper-Kähler manifolds.

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Notation

We work over the complex numbers. Throughout these notes X and Y will be projective hyper-Kähler manifolds of dimension $2n$. All functors will be implicitely derived.

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A.2. Derived categories

A.2.1. General theory

For a thorough introduction to derived categories we recommend [97]. Let us recall one of the most important results in the study of derived equivalences proved by Orlov [175].

Theorem A.2.1. *Let Z and T be smooth projective varieties and $\Phi\colon \mathrm{D}^b(Z) \xrightarrow{\sim} \mathrm{D}^b(T)$ be an exact derived equivalence. Then Φ is isomorphic to a Fourier–Mukai functor, i.e. there exists $\mathcal{E} \in \mathrm{D}^b(Z \times T)$ such that*

$$\Phi \cong \mathrm{FM}_{\mathcal{E}} \coloneqq p_{T*} \circ (\mathcal{E} \otimes \_) \circ p_Z^*.$$

Orlov's result is in fact stronger in that it applies also to fully faithful exact functors between the derived categories of smooth projective varieties. The resulting isomorphism is an isomorphism of exact functors.

Moreover, a derived equivalence as in the theorem naturally induces isomorphisms of several invariants associated with the varieties such as (topological) K-theory [97, Sec. 5.2]. For us the most important invariant will be singular cohomology. Namely, every derived equivalence $\mathrm{FM}_{\mathcal{E}}$ induces a *cohomological Fourier–Mukai transform* $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ given by the correspondence $v(\mathcal{E}) \in \mathrm{H}^*(Z \times T)$ where $v = \mathrm{ch}(\_)\sqrt{\mathrm{td}}$ is the Mukai vector. These are compatible via the Mukai vector, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathrm{D}^b(Z) \arrow[r, "\mathrm{FM}_{\mathcal{E}}"] \arrow[d, "v"] & \mathrm{D}^b(T) \arrow[d, "v"] \\
\mathrm{H}^*(Z, \mathbb{Q}) \arrow[r, "\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}"] & \mathrm{H}^*(T, \mathbb{Q}).
\end{tikzcd} \tag{A.2.1}$$

Hence, the study of derived categories leads naturally to cycles on hyper-Kähler manifolds.

Remark A.2.2. Let us mention properties of the cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$.

- Since $v(\mathcal{E}) \in \oplus_p \mathrm{H}^{p,p}(Z \times T)$ is algebraic, the isomorphism $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects the weight-zero Hodge structure on $\mathrm{H}^*(Z)$ (respectively $\mathrm{H}^*(T)$) given by

  $$\mathrm{H}^{-i,i}(Z) = \bigoplus_{q-p=i} \mathrm{H}^{p,q}(Z)$$

  for $i \in \mathbb{Z}$ where the Hodge structure on the right-hand side is the usual one [97, Prop. 5.39].

- The isomorphism $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects the generalized Mukai pairing, see [59].
- The cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects neither the cup product structure on cohomology nor the cohomological grading as can be seen by considering the equivalence given by tensoring with a non-trivial line bundle.

A.2.2. Case of hyper-Kähler manifolds

We know that if a smooth projective variety Z is derived equivalent to a hyper-Kähler manifold X, then the dimensions of X and Z coincide and the canonical bundle ω_Z is trivial [97, Sec. 4]. Huybrechts and Nieper-Wißkirchen [107] have proven that Z must in fact also be an irreducible hyper-Kähler manifold.

A.3. Recollection of the LLV Lie algebra

We quickly recall the definition of the LLV Lie algebra introduced independently by Looijenga–Lunts [130] and Verbitsky [207]. For a more thorough discussion we refer to [39].

Let X be a hyper-Kähler manifold and $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ be a cohomology class. We attach to it the operator

$$e_\lambda := \lambda \cup \_ \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$$

given by cup product with the class λ. We say that λ has the *Hard Lefschetz property*, if for all i the maps

$$e_\lambda^i \colon \mathrm{H}^{2n-i}(X, \mathbb{Q}) \to \mathrm{H}^{2n+i}(X, \mathbb{Q})$$

are isomorphisms. The class λ is often called a *Hard Lefschetz class.* We denote by $h \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ the grading operator acting on $\mathrm{H}^i(X, \mathbb{Q})$ via $(i - 2n)\mathrm{id}$. For a Hard Lefschetz class $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$, the triple

$$(e_\lambda, h, f_\lambda),$$

where f_λ is the dual Lefschetz operator, spans a Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ isomorphic to the Lie algebra $\mathfrak{sl}_2$.

Definition A.3.1. The *LLV Lie algebra* $\mathfrak{g}(X)$ is the Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ generated by all $\mathfrak{sl}_2$-triples $(e_\lambda, h, f_\lambda)$ for $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ Hard Lefschetz.

As said in the beginning, we refer to [39] or [130, 207] for more details and properties of $\mathfrak{g}(X)$. Our main goal is to relate the Lie algebra $\mathfrak{g}(X)$ to $\mathrm{D}^b(X)$. Note that since a cohomological Fourier–Mukai functor does not respect cup product nor grading, which are the defining properties of the LLV algebra, it is a priori not clear how this can be done. The main ingredient for it is the ring of polyvector fields, to be introduced now.

A.4. Polyvector fields

Definition A.4.1. The *ring of polyvector fields* $\mathrm{HT}^*(X)$ is the graded $\mathbb{C}$-algebra whose degree k part is

$$\mathrm{HT}^k(X) := \oplus_{p+q=k} \mathrm{H}^q(X, \Lambda^p \mathcal{T}_X).$$

The ring structure is induced from the exterior algebra.

For X a hyper-Kähler manifold we can choose a symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ which induces isomorphisms

$$\Lambda^p \mathcal{T}_X \cong \Omega_X^p$$

which, in turn, induce a graded $\mathbb{C}$-algebra isomorphism

$$\mathrm{HT}^*(X) = \mathrm{H}^*(X, \Lambda^* \mathcal{T}_X) \cong \mathrm{H}^*(X, \Omega_X^*) \cong \mathrm{H}^*(X, \mathbb{C}). \tag{A.4.1}$$

Thus, as a graded $\mathbb{C}$-algebra, the ring of polyvectors is isomorphic to the de Rham cohomology.

In this note, we are mostly interested in another viewpoint of the polyvector fields. Namely, the ring of polyvectors acts on the de Rham cohomology by contraction. That is, given $v \in \mathrm{H}^q(X, \Lambda^p \mathcal{T}_X)$ and $x \in \mathrm{H}^{q'}(X, \Omega_X^{p'})$ the action is defined as

$$v \lrcorner x \in \mathrm{H}^{q+q'}(X, \Omega_X^{p'-p}).$$

The following is immediate, see also [201, Lem. 2.4].

Lemma A.4.2. *For X a hyper-Kähler manifold the de Rham cohomology is a free module of rank one over the polyvector fields generated by a Calabi–Yau form $\sigma^n \in \mathrm{H}^0(X, \Omega_X^{2n})$.*

The reason why the ring of polyvectors is of interest to us is the following crucial result. It relies on the modified Hochschild–Konstant–Rosenberg isomorphism identifying Hochschild (co)homology with polyvectors and the de Rham cohomology [47].

Theorem A.4.3. *A derived equivalence $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ induces naturally a $\mathbb{C}$-algebra isomorphism $\Phi^{\mathrm{HT}}\colon \mathrm{HT}^*(X) \xrightarrow{\sim} \mathrm{HT}^*(Y)$ such that the action of the polyvector fields is equivariant for the induced isomorphism $\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X, \mathbb{C}) \xrightarrow{\sim} \mathrm{H}^*(Y, \mathbb{C})$.*

Spelling this out, for $v \in \mathrm{HT}^*(X)$ and $x \in \mathrm{H}^*(X, \mathbb{C})$ we have

$$\Phi^{\mathrm{H}}(v \lrcorner x) = \Phi^{\mathrm{HT}}(v) \lrcorner \Phi^{\mathrm{H}}(x) \in \mathrm{H}^*(Y, \mathbb{C}).$$

A.5. Reinventing the LLV Lie algebra

We will define a new Lie algebra, which will turn out to be isomorphic to $\mathfrak{g}(X)$ with scalars extended to $\mathbb{C}$. This will prove Theorem A.1.1 from the introduction.

Recall that X is a hyper-Kähler manifold of dimension $2n$. We consider the holomorphic grading operator h_p and the antihomolorphic grading operator h_q defined by acting on $\mathrm{H}^{k,l}(X)$ via

$$h_p = (k - n)\mathrm{id}, \quad h_q = (l - n)\mathrm{id}.$$

To avoid confusions, the indices p and q do not relate to k or l in any way, but just refer to the standard convention that the holomorphic degree of a smooth form is usually denoted by p and the antiholomorphic degree of a form by q.

With these definitions the usual grading operator h for the cohomological grading is just $h = h_p + h_q$. We define the Hodge grading operator $h' := h_q - h_p$.

$$\begin{array}{cccccccc}
 & & & \overset{h'\text{-grading}}{\longleftrightarrow} & & & & \\
 & & & \mathrm{H}^{0,0} & & & & \\
 & & \mathrm{H}^{1,0} & & \mathrm{H}^{0,1} & & & \\
 & \mathrm{H}^{2,0} & & \mathrm{H}^{1,1} & & \mathrm{H}^{0,2} & & \\
 & & & \vdots & & & & \\
\mathrm{H}^{2n,0} & \mathrm{H}^{2n-1,1} & \dots & \mathrm{H}^{n,n} & \dots & \mathrm{H}^{1,2n-1} & \mathrm{H}^{0,2n} & \updownarrow h\text{-grading} \\
 & & & \vdots & & & & \\
 & \mathrm{H}^{2n,2n-2} & & \mathrm{H}^{2n-1,2n-1} & & \mathrm{H}^{2n,2n-2} & & \\
 & & \mathrm{H}^{2n,2n-1} & & \mathrm{H}^{2n-1,2n} & & & \\
 & & & \mathrm{H}^{2n,2n} & & & &
\end{array}$$

With this definition the action of the polyvector fields $\mathrm{HT}^*(X)$ on the de Rham cohomology $\mathrm{H}^*(X, \mathbb{C})$ alluded to in Lemma A.4.2 has degree two with respect to the grading h'.

For $\mu \in \mathrm{HT}^2(X)$ we define the operator

$$e_\mu := \mu \lrcorner \_ \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{C})).$$

We say that μ is Hard Lefschetz if the operator e_μ satisfies the Hard Lefschetz isomorphisms with respect to the grading operator h'. The Jacobson–Morozov theorem asserts that this is equivalent to the existence of an operator $f_\mu \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ such that

$$(e_\mu, h', f_\mu)$$

generates a Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ isomorphic to $\mathfrak{sl}_2$.

Definition A.5.1. The complex Lie algebra $\mathfrak{g}'(X)$ is defined to be the smallest Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ containing all $\mathfrak{sl}_2$-triples (e_μ, h', f_μ) for all Hard Lefschetz $\mu \in \mathrm{HT}^2(X)$.

Equivalently, one could have defined the Lie algebra $\mathfrak{g}'(X)$ as the Lie subalgebra of the endomorphism algebra $\mathrm{End}(\mathrm{HT}^*(X))$ containing all $\mathfrak{sl}_2$-triples with μ Hard Lefschetz. Through the isomorphism

$$\mathrm{HT}^*(X) \lrcorner \sigma^n \cong \mathrm{H}^*(X, \mathbb{C})$$

these two definitions are identified.

Recall from (A.4.1) that the choice of a symplectic form produces an abstract graded $\mathbb{C}$-algebra isomorphism

$$\mathrm{HT}^*(X) \cong \mathrm{H}^*(X, \Omega_X^*) \cong \mathrm{H}^*(X, \mathbb{C}).$$

Thus, the choice of a symplectic form leads to the following result.

Lemma A.5.2. *There is an isomorphism of complex Lie algebras*

$$\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathfrak{g}'(X).$$

We also deduce the following consequence from Theorem A.4.3.

Proposition A.5.3. *For a derived equivalence between hyper-Kähler manifolds* $\Phi\colon \mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ *the isomorphism*

$$\Phi^{\mathrm{HT}}\colon \mathrm{HT}^2(X) \xrightarrow{\sim} \mathrm{HT}^2(Y)$$

induces naturally a Lie algebra isomorphism

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}'(X) \xrightarrow{\sim} \mathfrak{g}'(Y)$$

such that the induced isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{C}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{C})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$.

Spelling this again out means that for $j \in \mathfrak{g}'(X)$ and $x \in \mathrm{H}^*(X,\mathbb{C})$ we have

$$\Phi^{\mathrm{H}}(j.x) = \Phi^{\mathfrak{g}}(j).\Phi^{\mathrm{H}}(x) \in \mathrm{H}^*(Y,\mathbb{C}).$$

The connection between all that has been said so far and the main tool for all the applications we will present is the following main theorem of [201] which was also implicitly proven (but not stated in the form below) by Verbitsky [209].

Theorem A.5.4. *The Lie algebras* $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ *and* $\mathfrak{g}'(X)$ *are equal as Lie subalgebras of the Lie algebra* $\mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$.

Proof. Verbitsky showed that there is an isomorphism of ungraded vector spaces

$$\eta\colon \mathrm{H}^*(X,\mathbb{C}) \xrightarrow{\sim} \mathrm{H}^*(X,\mathbb{C}).$$

The explicit description of η is not import, we only need the following two properties shown by Verbitsky. Firstly, η conjugates the two Lie algebras, i.e.

$$\eta\,(\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C})\,\eta^{-1} = \mathfrak{g}'(X).$$

Secondly, the isomorphism η is obtained by integrating the action of the Lie algebra $\mathfrak{g}(X)$, that is it lies in the subgroup of automorphism $\mathrm{Aut}(\mathrm{H}^*(X,\mathbb{C}))$ generated by integrated operators of $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{C}$. Since all such operators μ contained in the above subgroup satisfy

$$\mu\,(\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C})\,\mu^{-1} = \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

one can conclude the proof.

We will, however, follow Taelman's proof. From Lemma A.5.2 we infer that it is enough to show only the inclusion

$$\mathfrak{g}'(X) \subset \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

A straightforward calculation shows that

$$(e_\sigma, h_p, e_{\check{\sigma}})$$

is an $\mathfrak{sl}_2$-triple, where $\check{\sigma} \in \mathrm{H}^0(\Lambda^2\mathcal{T}_X)$ is the dual symplectic form (note that the Lefschetz operator e_σ acts via cup product, whereas $e_{\check{\sigma}}$ acts by contraction of polyvector fields).

Analogously or using Hodge symmetry, for the complex conjugate form $\bar{\sigma} \in \mathrm{H}^2(X, \mathcal{O}_X)$ the operator $e_{\bar{\sigma}}$ has the Hard Lefschetz property for the grading operator h_q. The Jacobson–Morozov Theorem grants the existence of an operator $g \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ such that

$$(e_{\bar{\sigma}}, h_q, g)$$

forms an $\mathfrak{sl}_2$-triple. An easy check shows that all elements from the $\mathfrak{sl}_2$-triple $(e_\sigma, h_p, e_{\check{\sigma}})$ commute with all elements from the $\mathfrak{sl}_2$-triple $(e_{\bar{\sigma}}, h_q, g)$. For example, e_σ and $e_{\bar{\sigma}}$ commute as the de Rham cohomology is graded-commutative and the operators e_σ and $e_{\check{\sigma}}$ commute with h_q, because they do not change the antiholomorphic degree of a form. Similar arguments apply to the other operators. Thus we obtain two new $\mathfrak{sl}_2$-triples

$$(e_\sigma + e_{\bar{\sigma}}, h, e_{\check{\sigma}} + g), \quad (e_\sigma - e_{\bar{\sigma}}, h, e_{\check{\sigma}} - g).$$

This gives that $e_{\check{\sigma}} \in \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. Since $[e_\sigma, e_{\check{\sigma}}] = h_p$ and $h_p + h_q = h$, we deduce furthermore that h_p, h_q and therefore $h' = h_q - h_p$ are all contained inside $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.

Since evidently $e_{\bar{\sigma}}$ is also contained in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ (the action via contraction of polyvector fields agrees with the cup product), it is left to show that for almost all $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ the operator e_μ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. This follows from the identity

$$[e_{\check{\sigma}}, e_\eta] = e_\mu$$

for $\eta \in \mathrm{H}^1(X, \Omega_X)$ satisfying

$$\mu = \check{\sigma} \lrcorner \eta \in \mathrm{H}^1(X, \mathcal{T}_X)$$

which follows from a straightforward calculation, see [201, Lem. 2.13]. ☐

The theorem implies that the isomorphism $\Phi^{\mathfrak{g}}$ from Proposition A.5.3 is already defined over $\mathbb{Q}$, since the same holds for the induced isomorphism on singular cohomology. We thus have proved Theorem A.1.1 which we state her again for the reader's convenience.

Corollary A.5.5. *A derived equivalence* $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ *between hyper-Kähler manifolds induces naturally a Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(Y)$$

such that the induced isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y, \mathbb{Q})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$.

A.6. Verbitsky component and extended Mukai lattice

We want to draw consequences of Theorem A.5.4 for the study of derived equivalences of hyper-Kähler manifolds and their induced actions on cohomology.

Definition A.6.1. The Verbitsky component $\mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{H}^*(X, \mathbb{Q})$ is the subalgebra generated by $\mathrm{H}^2(X, \mathbb{Q})$.

It is easy to see that the Verbitsky component is an irreducible representation of the LLV Lie algebra $\mathfrak{g}(X)$ and it is characterized as such as the irreducible representation whose Hodge structure attains the maximal possible width. It is equipped with the Mukai pairing b_{SH} defined via

$$b_{\mathrm{SH}}(\lambda_1 \dots \lambda_m, \mu_1 \dots \mu_{2n-m}) \coloneqq (-1)^m \int_X \lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_{2n-m}$$

for classes $\lambda_i, \mu_j \in \mathrm{H}^2(X, \mathbb{Q})$ which agrees with the generalized Mukai pairing alluded to in Remark A.2.2.

Corollary A.6.2. *For a derived equivalence $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ between hyper-Kähler manifolds the induced isomorphism Φ^{H} restricts to a Hodge isometry*

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{SH}(Y, \mathbb{Q}).$$

Proof. Since the Verbitsky component is the unique irreducible representation whose Hodge structure attains the maximal possible width and by Theorem A.1.1 the isomorphism Φ^{H} respects the LLV algebra, we conclude that Φ^{H} must restrict to an isomorphism of the Verbitsky component. The Mukai pairing on the Verbitsky component agrees with the generalized Mukai pairing, which is a derived invariant. ☐

We want to study the Verbitsky component and the LLV Lie algebra more closely to further refine the study of $\mathrm{Aut}(\mathrm{D}^b(X))$.

Definition A.6.3. The rational quadratic vector space defined by

$$\tilde{\mathrm{H}}(X, \mathbb{Q}) \coloneqq \mathbb{Q}\alpha \oplus \mathrm{H}^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta.$$

is called the extended Mukai lattice. Its quadratic form $\tilde{b}$ restricts to the Beauville–Bogomolov–Fujiki form b on $\mathrm{H}^2(X, \mathbb{Q})$ [83, Sec. 23] and the two classes α and β are orthogonal to $\mathrm{H}^2(X, \mathbb{Q})$ and satisfy $\tilde{b}(\alpha, \beta) = -1$ as well as $\tilde{b}(\alpha, \alpha) = \tilde{b}(\beta, \beta) = 0$.

Furthermore, we define on $\tilde{\mathrm{H}}(X, \mathbb{Q})$ a grading by declaring α to be of degree -2, $\mathrm{H}^2(X, \mathbb{Q})$ sits in degree zero and β is of degree two. Finally, the extended Mukai lattice is equipped with a weight-two Hodge structure

$$\begin{aligned}
(\tilde{\mathrm{H}}(X, \mathbb{Q}) \otimes \mathbb{C})^{2,0} &\coloneqq \mathrm{H}^{2,0}(X) \\
(\tilde{\mathrm{H}}(X, \mathbb{Q}) \otimes \mathbb{C})^{0,2} &\coloneqq \mathrm{H}^{0,2}(X) \\
(\tilde{\mathrm{H}}(X, \mathbb{Q}) \otimes \mathbb{C})^{1,1} &\coloneqq \mathrm{H}^{1,1}(X) \oplus \mathbb{C}\alpha \oplus \mathbb{C}\beta.
\end{aligned}$$

There exists a graded morphism $\psi\colon \mathrm{SH}(X, \mathbb{Q})[-2n] \to \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ sitting in the following short exact sequence

$$0 \to \mathrm{SH}(X, \mathbb{Q})[-2n] \xrightarrow{\psi} \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})) \xrightarrow{\Delta_n} \mathrm{Sym}^{n-2}(\tilde{\mathrm{H}}(X, \mathbb{Q})) \to 0.$$

Here, the map Δ_n is the Laplacian operator defined on pure tensors via

$$v_1 \cdots v_n \mapsto \sum_{i<j} \tilde{b}(v_i, v_j) v_1 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n.$$

Surjectivity follows easily from the fact that the symmetric power $\mathrm{Sym}^k V$ of a vector space V is generated by $v \cdots v$ for all $v \in V$. The map ψ is uniquely determined (up to scaling) by the condition that it is a morphism of $\mathfrak{g}(X)$-modules. The $\mathfrak{g}(X)$-structure of $\tilde{\mathrm{H}}(X, \mathbb{Q})$ is defined by $e_\omega(\alpha) = \omega$, $e_\omega(\mu) = b(\omega, \mu)\beta$ and $e_\omega(\beta) = 0$ for all classes $\omega, \mu \in \mathrm{H}^2(X, \mathbb{Q})$. The n-th symmetric power $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ then inherits the structure of a $\mathfrak{g}(X)$-module by letting $\mathfrak{g}(X)$ act by derivations. We fix once and for all a choice of ψ by setting $\psi(1) = \alpha^n/n!$. By Schur's lemma, ψ is injective.

Taelman [201, Sec. 3] showed that the map ψ is an isometry with respect to the Mukai pairing on $\mathrm{SH}(X, \mathbb{Q})$ and the pairing

$$b_{[n]}(x_1 \cdots x_n, y_1 \cdots y_n) = (-1)^n c_X \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \tilde{b}(x_i, y_{\sigma(i)})$$

on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$, where c_X is the Fujiki constant characterized by the property

$$\int_X \omega^{2n} = c_X \frac{(2n)!}{2^n n!} b(\omega, \omega)^n$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{Q})$. Note that our definition of $b_{[n]}$ differs from Taelman's definition by the Fujiki constant. Ours has the advantage that ψ is always an isometry.

Summing up, the inclusion ψ respects the

- $\mathfrak{g}(X)$-module structure,
- quadratic forms,
- Hodge structures,
- gradings.

A.7. Action of derived equivalences on the extended Mukai lattice

Recall that we have deduced the existence of a representation

$$\rho^{\mathrm{SH}} \colon \mathrm{Aut}(\mathrm{D}^b(X)) \to \mathrm{O}(\mathrm{SH}(X, \mathbb{Q})) \tag{A.7.1}$$

and the isometries in the image of this representation normalize the action of the LLV algebra $\mathfrak{g}(X)$, i.e. for these $g \in \mathrm{O}(\mathrm{SH}(X, \mathbb{Q}))$ we have

$$g\mathfrak{g}(X)g^{-1} = \mathfrak{g}(X) \subset \mathrm{End}(\mathrm{SH}(X, \mathbb{Q})).$$

Let us study these automorphisms a bit further.

Definition A.7.1. The group $\mathrm{Aut}(\mathrm{SH}(X, \mathbb{Q}), b_{\mathrm{SH}}, \mathfrak{g}(X))$ is the group of all isometries of the Verbitsky component that normalize the action of the LLV algebra.

The main representation-theoretic input for our discussion is the following result [201, Sec. 4].

Proposition A.7.2. *If n is odd or the second Betti number is odd, then*

$$\mathrm{Aut}(\mathrm{SH}(X,\mathbb{Q}), b_{\mathrm{SH}}, \mathfrak{g}(X)) \cong \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})).$$

We make this isomorphism more explicit. Let X and Y be deformation-equivalent hyper-Kähler manifolds together with a derived equivalence $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$. Then there exists a unique Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \xrightarrow{\sim} \tilde{\mathrm{H}}(Y,\mathbb{Q})$$

inducing the following commutative diagram

$$\begin{tikzcd}
\mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\epsilon(\Phi^{\tilde{\mathrm{H}}})\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"'] & \mathrm{SH}(Y,\mathbb{Q}) \arrow[d, "\psi"] \\
\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(Y,\mathbb{Q})).
\end{tikzcd}$$

The scalar $\epsilon(\Phi^{\tilde{\mathrm{H}}}) \in \{\pm 1\}$ depends on defining orientations on the vector spaces $\tilde{\mathrm{H}}(X,\mathbb{Q})$ respectively $\tilde{\mathrm{H}}(Y,\mathbb{Q})$ and for $X = Y$ we simply have $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = \det(\Phi^{\tilde{\mathrm{H}}})^{n+1}$. In particular, in the case $X = Y$, the representation (A.7.1) factors via the commutative diagram

$$\begin{tikzcd}
 & \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[dd] \\
\mathrm{Aut}(\mathrm{D}^b(X)) \arrow[ur, "\rho^{\tilde{\mathrm{H}}}"] \arrow[dr, "\rho^{\mathrm{SH}}"'] & \\
 & \mathrm{O}(\mathrm{SH}(X,\mathbb{Q})).
\end{tikzcd} \tag{A.7.2}$$

Remark A.7.3. In all known examples, derived equivalent hyper-Kähler manifolds are deformation-equivalent, but this is not known in general. Without this assumption, the above proposition has to be weakened as we shall demonstrate.

One can, using similitudes, still formulate a version of Proposition A.7.2 in the general case. This will be needed in the next section for the application to Hodge structures.

Theorem A.7.4. *Let X and Y be arbitrary hyper-Kähler manifolds and $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ be a derived equivalence. Then there exists a Hodge similitude $\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \to \tilde{\mathrm{H}}(Y,\mathbb{Q})$ and a scalar $\lambda \in \mathbb{Q}^*$ such that*

$$\begin{tikzcd}
\mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"'] & \mathrm{SH}(Y,\mathbb{Q}) \arrow[d, "\psi"] \\
\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\lambda\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(Y,\mathbb{Q}))
\end{tikzcd}$$

commutes.

A.8. Hodge structures

In this section we want to give one application of the results presented so far regarding derived equivalent hyper-Kähler manifolds and their Hodge structures. We first want to recall a recent result of Soldatenkov [199][1], whose statement and proof are similar in flavour to what we will discuss afterwards for derived equivalences.

Theorem A.8.1. *Let X and Y be arbitrary hyper-Kähler manifolds and $\varphi\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})$ be an isomorphism of $\mathbb{Q}$-Hodge structures, which is the restriction of a global algebra automorphism $\phi\colon \mathrm{H}^*(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q})$. Then for all $i \in \mathbb{Z}$ the restrictions*

$$\phi\colon \mathrm{H}^i(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^i(Y,\mathbb{Q})$$

are isomorphisms of $\mathbb{Q}$-Hodge structures.

Proof. We briefly sketch the argument. Since ϕ is a graded algebra automorphism, the adjoint action produces an isomorphism

$$\mathrm{ad}(\phi)\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(Y).$$

The fact that ϕ is graded implies that $\mathrm{ad}(\phi)(h) = h$. Moreover, the restriction of ϕ to $\mathrm{H}^2(X,\mathbb{Q})$ respects the Hodge structures. This implies that $\mathrm{ad}(\phi)(h') = h'$, where again $h' = h_q - h_p$. Indeed, the adjoint action of ϕ is determined by its restriction to the degree two component [199, Prop. 2.11]. As the morphism ϕ respects the Hodge structure on the second cohomology, the claim follows.

Since $h + h' = 2h_q$ and $h - h' = 2h_p$ we deduce $\mathrm{ad}(\phi)(h_p) = h_p$ and $\mathrm{ad}(\phi)(h_q) = h_q$. This is equivalent to ϕ being a morphism of $\mathbb{Q}$-Hodge structures. ☐

The assertion that the isomorphism of Hodge structures is the restriction of a global algebra automorphism is frequently met. For example, Hodge isometries with positive determinant can be extended to algebra automorphisms of the even cohomology by integrating the LLV action. For more details and examples we refer to [199].

With this in mind, we can now prove the following result of Taelman [201, Sec. 5]. It also establishes a conjecture of Orlov in the case of hyper-Kähler manifolds [176] stating that derived equivalent varieties have the same Hodge numbers.

Theorem A.8.2. *Let X and Y be derived equivalent hyper-Kähler manifolds. Then for all $i \in \mathbb{Z}$ we have an isomorphism*

$$\mathrm{H}^i(X,\mathbb{Q}) \cong \mathrm{H}^i(Y,\mathbb{Q})$$

of $\mathbb{Q}$-Hodge structures.

Proof. Let us denote by Φ a derived equivalence between X and Y. Recall from [130, 207] the Lie algebra isomorphism $\mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ (in loc. cit. the isomorphism is only stated over $\mathbb{R}$. For the statement with rational coefficients, see [199, Prop. 2.9].). Composing this isomorphism with $\Phi^{\mathfrak{g}}$ we obtain a Lie algebra isomorphism

$$\mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \cong \mathfrak{so}(\tilde{\mathrm{H}}(Y,\mathbb{Q})).$$

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[1] We thank Andrey Soldatenkov for a stimulating conversation about his results.

Every such Lie algebra isomorphism is equal to $\mathrm{ad}(\phi)$ for some $\phi\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \to \tilde{\mathrm{H}}(Y,\mathbb{Q})$, see [201, Prop. 4.1] which is the analogue of Proposition A.7.2 in this case. Theorem A.7.4 now implies that ϕ must be a Hodge similitude. More precisely, it differs from $\Phi^{\tilde{\mathrm{H}}}$ only by a scalar.

Using

$$\tilde{\mathrm{H}}(X,\mathbb{Q}) \cong \mathbb{Q}\alpha \oplus \mathbb{Q}\beta \oplus \mathrm{NS}(X)_{\mathbb{Q}} \oplus \mathrm{T}(X)_{\mathbb{Q}}$$

and Witt cancellation for quadratic spaces, one easily shows that there exists a Hodge isometry $\gamma \in \mathrm{SO}(\tilde{\mathrm{H}}(Y,\mathbb{Q}))$ such that the composition $\gamma \circ \phi$ is now a graded Hodge similitude, i.e. α and β are mapped to multiples of themselves. By definition, this implies that the adjoint morphism of $\gamma \circ \phi$ satisfies

$$\mathrm{ad}(\gamma \circ \phi)(h) = h, \quad \mathrm{ad}(\gamma \circ \phi)(h') = h'. \tag{A.8.1}$$

Let us for the moment assume that we can find a global algebra isomorphism

$$\eta\colon \mathrm{H}^*(Y,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q})$$

whose adjoint action equals γ as isomorphisms of the LLV Lie algebra $\mathfrak{g}(Y)$. Then we can consider the composition

$$\eta \circ \Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q}).$$

From (A.8.1) we infer again that $\mathrm{ad}(\eta \circ \Phi^{\mathrm{H}})(h) = h$ and $\mathrm{ad}(\eta \circ \Phi^{\mathrm{H}})(h') = h'$. As in the proof of Theorem A.8.1 this implies that $\eta \circ \Phi^{\mathrm{H}}$ induces in each degree the desired isomorphism of Hodge structures.

It is left to prove the existence of the global algebra isomorphism η. In general, integrating the action of the LLV algebra $\mathfrak{g}(X)$ produces an action of $\mathrm{SO}(\tilde{\mathrm{H}}(Y,\mathbb{Q}))$ on the even cohomology $\mathrm{H}^{2*}(Y,\mathbb{Q})$ [199, Prop. 2.10]. To construct an algebra automorphism of the full cohomology $\mathrm{H}^*(Y,\mathbb{Q})$ one uses the $\mathbb{Q}$-algebraic group GSpin. More precisely, one uses the natural surjection

$$\mathrm{GSpin}(\tilde{\mathrm{H}}(Y,\mathbb{Q})) \twoheadrightarrow \mathrm{SO}(\tilde{\mathrm{H}}(Y,\mathbb{Q}))$$

to lift γ and constructs an action of $\mathrm{GSpin}(\tilde{\mathrm{H}}(Y,\mathbb{Q}))$ on the full cohomology such that the induced action of $\mathrm{Spin}(\tilde{\mathrm{H}}(Y,\mathbb{Q})) \subset \mathrm{GSpin}(\tilde{\mathrm{H}}(Y,\mathbb{Q}))$ is the integrated action of the LLV algebra. For details we refer to [201, Sec. 5]. ☐

A.9. Further results

We have presented the first six sections of [201]. In the remaining part of loc. cit. the representation $\rho^{\tilde{\mathrm{H}}}$ from (A.7.2) is further studied. The main result is a bound on the image of $\rho^{\tilde{\mathrm{H}}}$ in terms of (subgroups) of the orthogonal group $\mathrm{O}(\Lambda)$ some lattice

$$\Lambda \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$$

for X (a deformation of) the Hilbert scheme of two points on a K3 surface.

In [25], building upon the results presented so far, the study of derived categories of projective hyper-Kähler manifolds is further refined. The main technical tool is a Mukai vector taking values in the extended Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Q})$. This yields structural results for derived categories and derived equivalences for general hyper-Kähler varieties as well as many generalisations of results known for derived categories of K3 surfaces to the case of higher-dimensional deformations of Hilbert schemes.

B. Derived categories of hyper-Kähler manifolds: extended Mukai vector and integral structure

ABSTRACT. We introduce a linearised form of the square root of the Todd class inside the Verbitsky component of a hyper-Kähler manifold using the extended Mukai lattice. This enables us to define a Mukai vector for certain objects in the derived category taking values inside the extended Mukai lattice which is functorial for derived equivalences. As applications, we obtain a structure theorem for derived equivalences between hyper-Kähler manifolds as well as an integral lattice associated to the derived category of hyper-Kähler manifolds deformation equivalent to the Hilbert scheme of a K3 surface mimicking the surface case.

B.1. Introduction

B.1.1. Background: Derived categories of K3 surfaces

The study of derived categories of smooth projective varieties goes back to the works of Mukai [154, 155]. Over the years derived categories and equivalences between them have attracted great attention and culminated in many results, see for example [35, 40–42, 86, 175].

Let X and Y be smooth projective varieties. We denote by $\mathrm{D}^{\mathrm{b}}(X) \coloneqq \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ the bounded derived category of coherent sheaves on X. Orlov [175] showed that any derived equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ is isomorphic to a Fourier–Mukai functor $\mathsf{FM}_{\mathcal{E}}$ with Fourier–Mukai kernel $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$. In particular, using the Mukai vector

$$v = \mathrm{ch}(\_)\mathsf{td}^{1/2} \colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{H}^*(X, \mathbb{Q})$$

with $\mathsf{td}^{1/2}$ the formal square root of the Todd class $\mathsf{td} \in \mathrm{H}^*(X, \mathbb{Q})$ the Fourier–Mukai functor $\mathsf{FM}_{\mathcal{E}}$ induces an isomorphism

$$\Phi^{\mathrm{H}} = \mathsf{FM}_{\mathcal{E}}^{\mathrm{H}} \colon \mathrm{H}^*(X, \mathbb{Q}) \cong \mathrm{H}^*(Y, \mathbb{Q}).$$

Let us specialize the above to the case of K3 surfaces S. Any smooth variety Y which is derived equivalent to S is again a K3 surface [43]. The integral cohomology groups $\mathrm{H}^*(S, \mathbb{Z})$ are equipped with the *Mukai pairing* $\tilde{b}$ which is equal to the intersection pairing up to a sign $\tilde{b}(\mathsf{1}, \mathsf{p}) = -1$ for $\mathsf{1} \in \mathrm{H}^0(S, \mathbb{Z})$ the fundamental class and $\mathsf{p} \in \mathrm{H}^4(S, \mathbb{Z})$ the point class.

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Moreover, the lattice $\mathrm{H}^*(S,\mathbb{Z})$ carries a weight-two Hodge structure inherited from $\mathrm{H}^*(S,\mathbb{Z})$. As alluded to above for a derived equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(S) \cong \mathrm{D}^{\mathrm{b}}(S')$ between two K3 surfaces the following diagram commutes

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(S) \arrow[r, "\Phi"] \arrow[d, "v"] & \mathrm{D}^{\mathrm{b}}(S') \arrow[d, "v"] \\ \mathrm{H}^*(S,\mathbb{Z}) \arrow[r, "\Phi^{\mathrm{H}}"] & \mathrm{H}^*(S',\mathbb{Z}). \end{tikzcd} \tag{B.1.1}$$

Mukai has shown that the morphism Φ^{H} associated to Φ is a Hodge isometry [155]. Furthermore, the lattice $\mathrm{H}^*(S,\mathbb{Z})$ together with its Hodge structure determines the derived category completely. That is, two K3 surfaces S and S' are derived equivalent if and only if $\mathrm{H}^*(S,\mathbb{Z})$ and $\mathrm{H}^*(S',\mathbb{Z})$ are Hodge isometric [175].

In particular, many properties of the derived category of a K3 surface S are encoded by the lattice $\mathrm{H}^*(S,\mathbb{Z})$ of rank $b_2(S)+2$ together with its Hodge structure. For example, this can be used to show that the number of Fourier–Mukai partners of S, that is the number of non-isomorphic K3 surfaces S' which are derived equivalent to S, is finite.

In addition, the group of auto-equivalences $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S))$ of K3 surfaces admits a representation

$$\rho^{\mathrm{H}}\colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S)) \to \mathrm{H}^*(S,\mathbb{Z}).$$

The group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S))$ contains elements such as spherical twists $\mathsf{ST}_{\mathcal{E}}$ along spherical objects $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(S)$. These are symmetries which become only visible in the derived category. The image of ρ^{H} has been computed to be $\mathrm{Aut}^+(\mathrm{H}^*(X,\mathbb{Z}))$, the group of Hodge isometries with real spinor norm one [93, 105, 155, 175]. A conjecture describing the kernel of ρ^{H} has been put forward by Bridgeland [41] and has been proven for K3 surfaces with Picard rank one [15, Thm. 1.3].

The main goal of this paper is to find suitable analogues for the above results for the higher-dimensional analogues of K3 surfaces, that is hyper-Kähler manifolds X. For example, we want to study their derived categories and equivalences between them by means of an integral lattice of rank $b_2(X)+2$.

B.1.2. Hyper-Kähler manifolds

Let X be a compact irreducible hyper-Kähler manifold of dimension $2n$, that is a compact simply connected Kähler manifold whose space of holomorphic two-forms is spanned by a non-degenerate symplectic form. We briefly recall properties of X needed to state our results, see Section B.2 for a more thorough recollection.

The second cohomology $\mathrm{H}^2(X,\mathbb{Q})$ of X is endowed with a quadratic form called the Beauville–Bogomolov–Fujiki (BBF) form b. Its Verbitsky component $\mathrm{SH}(X,\mathbb{Q}) \subset \mathrm{H}^*(X,\mathbb{Q})$ of X is the subalgebra generated by all cohomology classes of degree two. It inherits a bilinear form

$$b_{\mathrm{SH}}(\omega_1\cdots\omega_m, \mu_1\cdots\mu_{2n-m}) = (-1)^m \int_X \omega_1\cdots\omega_m\mu_1\cdots\mu_{2n-m}$$

called *Mukai pairing* from the intersection pairing, where $\omega_i, \mu_j \in \mathrm{H}^2(X,\mathbb{Q})$. The extended rational Mukai lattice of X is the graded vector space

$$\tilde{\mathrm{H}}(X,\mathbb{Q}) := \mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta.$$

This space is endowed with a bilinear form $\tilde{b}$ as well as a Hodge structure which both restrict to the BBF form and the Hodge structure on $\mathrm{H}^2(X,\mathbb{Q})$, respectively. The classes α and β are of Hodge type, orthogonal to $\mathrm{H}^2(X,\mathbb{Q})$ and satisfy $\tilde{b}(\alpha,\alpha) = \tilde{b}(\beta,\beta) = 0$ as well as $\tilde{b}(\alpha,\beta) = -1$ such that $\tilde{\mathrm{H}}(X,\mathbb{Q})$ resembles $\mathrm{H}(S,\mathbb{Q})$ for S a K3 surface, see Section B.2.2 for more details. There exists an embedding

$$\begin{tikzcd}\psi\colon \mathrm{SH}(X,\mathbb{Q}) \arrow[r, hook] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[l, bend right, "T"']\end{tikzcd}$$

of quadratic spaces and T denotes the the orthogonal projection onto the subspace $\mathrm{SH}(X,\mathbb{Q})$. The morphism ψ realizes the Verbitsky component as an irreducible representation of the Looijenga–Lunts–Verbitsky (LLV) algebra $\mathfrak{g}(X)$, see [130, 201, 207] and Section B.2.2.

Let us turn now to derived categories of hyper-Kähler manifolds X. Until recently not much has been known about $\mathrm{D^b}(X)$. Huybrechts–Nieper-Wißkirchen have shown that any Fourier–Mukai partner of X is again a hyper-Kähler manifold [107, Thm. 0.4]. Taelman in [201] has refined the study of the derived category of X. He showed that a derived equivalence $\Phi\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ between hyper-Kähler manifolds restricts to a Hodge isometry

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X,\mathbb{Q}) \cong \mathrm{SH}(Y,\mathbb{Q})$$

which is functorially induced by a Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \cong \tilde{\mathrm{H}}(Y,\mathbb{Q}), \tag{B.1.2}$$

see [201, Sec. 4] or Section B.2.3. This can be used to show that for derived equivalent hyper-Kähler manifolds X and Y there is an isomorphism

$$\mathrm{H}^i(X,\mathbb{Q}) \cong \mathrm{H}^i(Y,\mathbb{Q})$$

of $\mathbb{Q}$-Hodge structures for all i [201, Thm. D].

B.1.3. Extended Mukai vector

The starting point of this paper is the following observation.

Proposition B.3.4. *Let X be a hyper-Kähler manifold of dimension $2n$. Then*

$$\overline{\mathsf{td}^{1/2}} = T\left(\frac{(\alpha + r_X\beta)^n}{n!}\right) \in \mathrm{SH}(X,\mathbb{Q}).$$

Here, we decompose

$$\mathrm{H}^*(X,\mathbb{Q}) = \mathrm{SH}(X,\mathbb{Q}) \oplus \mathrm{SH}(X,\mathbb{Q})^{\perp}$$

orthogonally with respect to the intersection product on cohomology and $\overline{(\_)}$ denotes the projection onto the subspace $\mathrm{SH}(X,\mathbb{Q})$. The number $r_X \in \mathbb{Q}$ is an explicit constant depending on n, the second Chern class $\mathrm{c}_2(X)$, and the Fujiki constant c_X of X, see (B.3.1). In particular, it only depends on the deformation type of X. For K3 surfaces the proposition reads

$$\mathsf{td}^{1/2} = T(\alpha + \beta) = 1 + \mathsf{p} \in \mathrm{H}^*(X,\mathbb{Z})$$

and for X of $\mathrm{K3}^{[n]}$-type we have

$$\overline{\mathsf{td}^{1/2}} = T\left(\frac{(\alpha + \frac{n+3}{4}\beta)^n}{n!}\right) \in \mathrm{SH}(X, \mathbb{Q}).$$

This enables us to define in Section B.4 an *extended Mukai vector*

$$\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

for certain objects $\mathcal{E} \in \mathrm{D^b}(X)$. It relates to the classical Mukai vector $v(\mathcal{E}) \in \mathrm{H}^*(X, \mathbb{Q})$ via

$$T(\tilde{v}(\mathcal{E})^n) = c\overline{v(\mathcal{E})} \in \mathrm{SH}(X, \mathbb{Q})$$

for $c \in \mathbb{Q}$[1] and this property characterizes the line spanned by $\tilde{v}(\mathcal{E})$ in $\tilde{\mathrm{H}}(X, \mathbb{Q})$. For example, to a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ with first Chern class $\mathrm{c}_1(\mathcal{L}) = \lambda$ we assign

$$\tilde{v}(\mathcal{L}) = \alpha + \lambda + \left(r_X + \frac{b(\lambda, \lambda)}{2}\right)\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q}).$$

Moreover, the formation of the extended Mukai vector is functorial[2] for derived equivalences, i.e. for a derived equivalence Φ the extended Mukai vector of $\Phi(\mathcal{E})$ equals $\pm\Phi^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{E}))$. For the details and precise definitions we refer to Section B.4.

B.1.4. Derived equivalences of hyper-Kähler manifolds

Consider a derived equivalence $\Phi\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ between projective hyper-Kähler manifolds X and Y. Associated to it we have the induced Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X, \mathbb{Q}) \cong \tilde{\mathrm{H}}(Y, \mathbb{Q}).$$

It is a priori very hard to calculate this isometry for a given derived equivalence. However, the above defined extended Mukai vector allows us now to easily compute $\Phi^{\tilde{\mathrm{H}}}$ for most known examples of derived equivalences between hyper-Kähler manifolds. We demonstrate this in Sections B.7 and B.10.

Moreover, its properties lead to the following structural result for derived equivalences between hyper-Kähler manifolds.

Theorem B.4.15. *Let X and Y be deformation-equivalent projective hyper-Kähler manifolds and $\Phi\colon \mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ an equivalence with Fourier–Mukai kernel $\mathcal{E}$. The rank r of $\mathcal{E}$ is of the form $\frac{a^n n!}{c_X}$ for $a \in \mathbb{Q}$. If $r = 0$, then $\mathcal{E}$ induces coverings of X and Y with Lagrangian cycles or there exists a Hodge isometry $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{H}^2(Y, \mathbb{Z})$.*

If, for example, $\mathcal{E}$ is an X-flat sheaf on $X \times Y$, then the second statement of the theorem means that the codimension n component of $\mathrm{supp}(\mathcal{E})$ is a flat family of Lagrangian subvarieties of Y which dominates Y.

---

[1] For all known deformation types of hyper-Kähler manifolds we actually have $c \in \mathbb{Z}$.

[2] The (at first sight surprising) possible extra sign comes from a sign convention in [201, Thm. 4.9] for certain hyper-Kähler manifolds. For all applications this issue can be ignored.

Note that the number in the above theorem

$$\frac{a^n n!}{c_X}$$

must in particular be an integer. For all known examples of hyper-Kähler manifolds $c_X \in \mathbb{Z}$ and therefore we must already have $a \in \mathbb{Z}$ using Legendre's or de Polignac's formula.

The theorem splits derived equivalences $\Phi = \mathsf{FM}_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ between hyper-Kähler manifolds into three cases. If the rank of the Fourier–Mukai kernel $\mathcal{E}$ is non-zero, we are in the first case and the theorem asserts that the possible ranks of $\mathcal{E}$ are severly restricted. In the second case, the geometries of X and Y are related by a correspondence in the n-th Chow group $\mathsf{A}^n(X \times Y)$ which induces coverings of both manifolds by Lagrangian cycles. In the last case the derived equivalence implies the existence of a Hodge isometry $\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(Y,\mathbb{Z})$. To obtain a geometric interpretation of this conclusion, recall that up to finite index any Hodge isometry is induced from a parallel transport operator [137, Lem. 6.23]. The Global Torelli Theorem [211] states that the existence of a Hodge isometry $\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(Y,\mathbb{Z})$ induced from a parallel transport operator is equivalent to X and Y being birational.

B.1.5. Integral structure

We now specialize for the rest of the introduction to the case of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds X, that is hyper-Kähler manifolds which are deformation-equivalent to Hilbert scheme of length n subschemes on a K3 surface. In this case we are able to obtain an integral lattice of rank $b_2(X) + 2$ invariant under derived equivalences mimicking the situation for K3 surfaces.

More explicitly, we define in Section B.5 an integral lattice

$$\Lambda \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$$

called $\mathrm{K3}^{[n]}$ *lattice* which inherits a Hodge structure Λ_X from X through the embedding. As an abstract lattice it is isometric to $\mathrm{H}^2(X,\mathbb{Z}) \oplus U$ with U the hyperbolic plane, but its weight-two Hodge structure differs from the one induced from $\mathrm{H}^2(X,\mathbb{Z})$ by a B-field twist, see Remark B.5.8. The main difference in the higher-dimensional situation compared to the case of K3 surfaces is that $\mathrm{H}^2(X,\mathbb{Z})$ is not unimodular and the B-field twist compensates for the non-trivial discriminant.

The $\mathrm{K3}^{[n]}$ lattice is a sublattice $\Lambda \subset \Lambda_g$ of index two of the lattice Λ_g generated by all extended Mukai vectors of objects in $\mathrm{D}^{\mathrm{b}}(X)$. We refer to Section B.5 for a discussion of all the lattices that appear and their relations.

Our main result now is the following yielding a complete analogue of Mukai's results [155] for derived equivalences of K3 surfaces.

Theorem B.9.2. *Let X and Y be projective $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds and $\Phi \colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ a derived equivalence. Then $\Phi^{\tilde{\mathrm{H}}}$ restricts to a Hodge isometry*

$$\Phi^{\tilde{\mathrm{H}}} \colon \Lambda_X \cong \Lambda_Y.$$

Even stronger, the $\mathrm{K3}^{[n]}$ lattice is invariant by the action of all (compositions of) parallel transport operators and derived equivalences acting on the extended Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Q})$. The precise statement is Theorem B.8.1.

As in the surface case, the existence of a lattice together with a Hodge structure governing properties of the derived category has strong implications. Here is one example.

Theorem B.9.4. *For a fixed projective* $\mathrm{K3}^{[n]}$*-type hyper-Kähler manifold* X *the number of projective* $\mathrm{K3}^{[n]}$*-type manifolds* Y *up to isomorphism with* $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ *is finite.*

For all currently known deformation types of hyper-Kähler manifolds a derived equivalence $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$ implies that X and Y must, in fact, be deformation-equivalent. In general, it is not known whether this conclusion remains true for arbitrary hyper-Kähler manifolds.

In [17, Thm. 1.2] it is shown that any hyper-Kähler manifold which is birational to a moduli space of stable objects on a K3 surface S is itself a moduli space of stable objects on S. Using the $\mathrm{K3}^{[n]}$ lattice we are able to upgrade this result to derived categories.

Corollary B.9.6. *Let* $M_\sigma^S(v)$ *be a smooth moduli space of stable objects on a projective K3 surface* S *and* X *a projective* $\mathrm{K3}^{[n]}$*-type hyper-Kähler manifold such that* $\mathrm{D^b}(X) \cong \mathrm{D^b}(M_\sigma^S(v))$*. Then* X *is itself a moduli space of stable objects on* S*.*

The corollary also yields the following.

Corollary B.9.7. *For two smooth moduli spaces* $M_\sigma^S(v)$ *and* $M_{\sigma'}^{S'}(v')$ *of stable objects on projective K3 surfaces* S *and* S' *with* $\mathrm{D^b}(M_\sigma^S(v)) \cong \mathrm{D^b}(M_{\sigma'}^{S'}(v'))$ *we have* $\mathrm{D^b}(S) \cong \mathrm{D^b}(S')$*. Furthermore,* S *and* S' *are derived equivalent if and only if their Hilbert schemes* $S^{[n]}$ *and* $S'^{[n]}$ *are derived equivalent.*

Finally, considering a single hyper-Kähler manifold X of $\mathrm{K3}^{[n]}$-type Theorem B.9.2 implies that the representation

$$\rho^{\tilde{\mathrm{H}}} \colon \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{O}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$$

induced from (B.1.2) factors via a representation

$$\rho^{\tilde{\mathrm{H}}} \colon \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}(\Lambda_X).$$

Here, $\mathrm{Aut}(\Lambda_X)$ denotes the group of all Hodge isometries of the $\mathrm{K3}^{[n]}$ lattice Λ_X. Specializing to Hilbert schemes $X = S^{[n]}$ of elliptic K3 surfaces S with a section, we are able to give a lower bound on the image of $\rho^{\tilde{\mathrm{H}}}$.

Theorem B.9.8. *For the Hilbert scheme* $S^{[n]}$ *of a K3 surface with* $U \subset \mathrm{NS}(S)$ *the image* $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$ *of the representation* $\rho^{\tilde{\mathrm{H}}}$ *satisfies*

$$\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}}) \subset \mathrm{Im}(\rho^{\tilde{\mathrm{H}}}) \subset \mathrm{Aut}(\Lambda_{S^{[n]}}).$$

The group $\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}})$ is the group of all Hodge isometries with real spinor norm one which act via ±id on the discriminant group.

Theorem B.9.2 as well as the existence of the extended Mukai vectors yield several further strong consequences for the derived category and derived equivalences of hyper-Kähler manifolds. Instead of reciting all of them here, we invite the reader to directly go to Sections B.4 and B.9.

B.1.6. Related work

While finishing writing this paper, Eyal Markman informed us that he has also constructed in [139] a Mukai vector with image in the extended Mukai lattice for certain objects in the derived category. His approach uses Hochschild (co)homology and obstruction maps and is independent and different from ours.

While working on this paper we also realised that a broader definition of the extended Mukai vector is possible. This has lead to the definition of atomic objects on hyper-Kähler manifolds studied in [26].

B.1.7. Structure of the paper

In Section B.2 we recall results for hyper-Kähler manifolds and their derived categories.

In the first part of this paper we study arbitrary hyper-Kähler manifolds. In Section B.3 we prove Proposition B.3.4 using results from Rozansky–Witten theory. In Section B.4 we define the extended Mukai vector. There are two different classes of objects for which this can be done and we discuss their properties and give examples.

In the second part we specialize to hyper-Kähler manifolds deformation-equivalent to the Hilbert scheme of a K3 surface. In Sections B.5 and B.6 we introduce the lattices that will play a role as well as the derived monodromy group. In Section B.7 we study derived equivalences of the Hilbert scheme on the extended Mukai lattice using the extended Mukai vector. With these preparations we prove in the subsequent section the invariance under derived equivalences of the lattice Λ. Consequences of the previous results will be drawn in Section B.9. We conclude by demonstrating how known derived equivalences of hyper-Kähler manifolds fit into the new set-up.

B.1.8. Acknowledgements

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B.1.9. Notation

We will always work over the complex numbers. The derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a smooth projective variety X is the bounded derived category of coherent sheaves on X. All functors will be implicitly derived.

A lattice is a free $\mathbb{Z}$-module of finite rank with an integral (mostly even) quadratic form. We use the notations from [100, Sec. 14] and [82]. We remark that in Section B.8 we use the word lattice as well to denote a full rank discrete subset W inside a finite dimensional rational vector space V with a specified embedding $W \hookrightarrow V$. It will be clear from the context what is meant.

B.2. Recollections

We recollect facts and results and introduce the notation we will employ throughout the paper.

B.2.1. Hyper-Kähler manifolds and their cohomology

Let X be a hyper-Kähler manifold of complex dimension $2n$, i.e. a simply connected compact Kähler manifold such that $\mathrm{H}^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form. The second cohomology $\mathrm{H}^2(X, \mathbb{Z})$ possesses an integral primitive quadratic form b called the *Beauville–Bogomolov–Fujiki (BBF)* form. It is characterized up to sign by the property that there exists a constant c_X, called the *Fujiki constant*, that only depends on the deformation type of X such that

$$\int_X \omega^{2n} = c_X \frac{(2n)!}{2^n n!} b(\omega, \omega)^n$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{Z})$. For the known examples of hyper-Kähler manifolds we have

$$c_X = \begin{cases} 1 & \mathrm{K3}^{[n]} \text{ or } \mathrm{OG}^{10}\text{-type}, \\ n+1 & \mathrm{Kum}^n \text{ or } \mathrm{OG}^6\text{-type}. \end{cases}$$

For the following, see [83, Cor. 23.17].

Proposition B.2.1. *Let X be a hyper-Kähler manifold of dimension $2n$ and consider a class $\mu \in \mathrm{H}^{4p}(X, \mathbb{R})$ which is of type $(2p, 2p)$ on all small deformations of X. Then there exists a constant $C(\mu) \in \mathbb{R}$ such that*

$$\int_X \mu \omega^{2n-2p} = C(\mu) b(\omega, \omega)^{n-p}$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{R})$.

Using Rozansky–Witten theory Nieper-Wißkirchen [162] established results on characteristic classes and Riemann–Roch formulae for hyper-Kähler manifolds.

Definition B.2.2. For $\omega \in \mathrm{H}^2(X, \mathbb{R})$ define its *characteristic value* as

$$\lambda(\omega) := \frac{(2n)! 12 c_X}{2^n n! (2n-1) C(\mathrm{c}_2(X))} b(\omega, \omega).$$

Using the Fujiki relations, one can check that the above definition agrees with [162, Def. 17]. For the formula of the square root of the Todd class we will need the following result, cf. [162, p. 738].

Proposition B.2.3. *For X a hyper-Kähler manifold of dimension $2n$ and arbitrary $\omega \in \mathrm{H}^2(X, \mathbb{R})$ it holds*

$$\int_X \mathsf{td}^{1/2} \exp(\omega) = (1 + \lambda(\omega))^n \int_X \mathsf{td}^{1/2}.$$

B.2.2. Verbitsky component

Denote by $(\tilde{\mathrm{H}}(X,\mathbb{Q}),\tilde{b})$ the rational quadratic vector space defined by

$$\mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta.$$

The quadratic form $\tilde{b}$ on $\tilde{\mathrm{H}}(X,\mathbb{Q})$ restricts to the BBF form b on $\mathrm{H}^2(X,\mathbb{Q})$ and the two classes α and β are orthogonal to $\mathrm{H}^2(X,\mathbb{Q})$ and satisfy $\tilde{b}(\alpha,\beta)=-1$ as well as $\tilde{b}(\alpha,\alpha)=\tilde{b}(\beta,\beta)=0$. Although it is not integral, we call $\tilde{\mathrm{H}}(X,\mathbb{Q})$ the *extended Mukai lattice* of X.

Furthermore, we define on $\tilde{\mathrm{H}}(X,\mathbb{Q})$ a grading by declaring α to be of degree zero, $\mathrm{H}^2(X,\mathbb{Q})$ remains in degree two and β is of degree four. Finally, the extended Mukai lattice is equipped with a weight-two Hodge structure

$$\begin{aligned}
(\tilde{\mathrm{H}}(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C})^{2,0} &:= \mathrm{H}^{2,0}(X)\\
(\tilde{\mathrm{H}}(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C})^{0,2} &:= \mathrm{H}^{0,2}(X)\\
(\tilde{\mathrm{H}}(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C})^{1,1} &:= \mathrm{H}^{1,1}(X)\oplus\mathbb{C}\alpha\oplus\mathbb{C}\beta.
\end{aligned}$$

Let $\mathrm{SH}(X,\mathbb{Q})$ be the *Verbitsky component*, i.e. the graded subalgebra of $\mathrm{H}^*(X,\mathbb{Q})$ generated by $\mathrm{H}^2(X,\mathbb{Q})$. Verbitsky [34, 207] proved the existence of a graded morphism $\psi\colon \mathrm{SH}(X,\mathbb{Q}) \to \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ sitting in a short exact sequence

$$0 \to \mathrm{SH}(X,\mathbb{Q}) \xrightarrow{\psi} \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \xrightarrow{\Delta} \mathrm{Sym}^{n-2}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to 0.$$

Here, the map Δ is the Laplacian operator defined on pure tensors via

$$v_1\cdots v_n \mapsto \sum_{i<j}\tilde{b}(v_i,v_j)v_1\cdots\hat{v_i}\cdots\hat{v_j}\cdots v_n.$$

The map ψ is uniquely determined (up to scaling) by the condition that it is a morphism of $\mathfrak{g}(X)$-modules, where $\mathfrak{g}(X)$ denotes the *Looijenga–Lunts–Verbitsky (LLV)* algebra, see [130] or [81]. Recall that $\mathfrak{g}(X)$ is the Lie algebra generated by all $\mathfrak{sl}_2$-triples (e_ω,h,f_ω) with e_ω the Lefschetz operator for $\omega\in\mathrm{H}^2(X,\mathbb{Q})$ satisfying the Hard Lefschetz property, h the grading operator and f_ω the dual Lefschetz operator.

The $\mathfrak{g}(X)$-structure of $\tilde{\mathrm{H}}(X,\mathbb{Q})$ is defined by the conditions $e_\omega(\alpha)=\omega$, $e_\omega(\mu)=b(\omega,\mu)\beta$ and $e_\omega(\beta)=0$ for all classes $\omega,\mu\in\mathrm{H}^2(X,\mathbb{Q})$. The n-th symmetric power $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ then inherits the structure of a $\mathfrak{g}(X)$-module by letting $\mathfrak{g}(X)$ act by derivations. The inclusion realizes $\mathrm{SH}(X,\mathbb{Q})$ as an irreducible Lefschetz module [207]. We fix once and for all a choice of ψ by setting $\psi(1)=\alpha^n/n!$.

Taelman [201, Sec. 3] showed that the map ψ is an isometry with respect to the *Mukai pairing*

$$b_{\mathrm{SH}}(\omega_1\cdots\omega_m,\mu_1\cdots\mu_{2n-m}) = (-1)^m\int_X \omega_1\cdots\omega_m\mu_1\cdots\mu_{2n-m}$$

on $\mathrm{SH}(X,\mathbb{Q})$ and the pairing

$$b_{[n]}(x_1\cdots x_n,y_1\cdots y_n) = (-1)^n c_X \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}\tilde{b}(x_i,y_{\sigma(i)})$$

on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$. Note that our definition of $b_{[n]}$ differs from Taelman's definition by the Fujiki constant. Ours has the advantage that ψ is always an isometry. The orthogonal projection onto the subspace $\mathrm{SH}(X,\mathbb{Q})$ will be denoted by

$$T\colon \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to \mathrm{SH}(X,\mathbb{Q}).$$

Remark B.2.4. Observe that ψ is surjective in cohomological degrees $0, 2, 4n-2$ and $4n$. Equivalently, the projection T is injective restricted to these degrees.

Bogomolov and Verbitsky [34, 207] showed that the Verbitsky component can also be described via

$$\mathrm{SH}(X,\mathbb{C}) \cong \mathrm{Sym}^{\bullet}\mathrm{H}^2(X,\mathbb{C})/\langle \mu^{n+1} \mid b(\mu,\mu)=0\rangle. \tag{B.2.1}$$

B.2.3. Derived equivalences

The following is [201, Thm. A].

Theorem B.2.5 (Taelman)**.** *Let X and Y be projective hyper-Kähler manifolds together with an equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$. Then Φ induces a canonical Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \cong \mathfrak{g}(Y)$$

which is equivariant for the induced isometry $\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(Y,\mathbb{Q})$.

We reproduce some consequences of this result from [201, Sec. 4] needed below. The above theorem implies that given an auto-equivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$, the induced action on cohomology

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(X,\mathbb{Q})$$

restricts to a Hodge isometry

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X,\mathbb{Q}) \cong \mathrm{SH}(X,\mathbb{Q})$$

which is equivariant with respect to $\Phi^{\mathfrak{g}}$. This yields a representation

$$\rho^{\mathrm{SH}}\colon\ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\mathrm{SH}(X,\mathbb{Q})).$$

Moreover, the above represention ρ^{SH} factors over a representation

$$\rho^{\tilde{\mathrm{H}}}\colon\ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

under the assumption that n is odd, or having n even and $b_2(X)$ odd. Note that all known examples of hyper-Kähler manifolds satisfy one of the two conditions.

More precisely, for odd n every Hodge isometry Φ^{SH} of $\mathrm{SH}(X,\mathbb{Q})$ is induced by an isometry of $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ which comes from a unique Hodge isometry $\Phi^{\tilde{\mathrm{H}}}$ of $\tilde{\mathrm{H}}(X,\mathbb{Q})$ [201, Prop. 4.1], i.e. the following diagram

$$\begin{tikzcd}
\mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"] & \mathrm{SH}(X,\mathbb{Q}) \arrow[d, "\psi"] \\
\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))
\end{tikzcd}$$

commutes. For even n and odd $b_2(X)$, there is an extra sign $\det(\Phi^{\tilde{\mathrm{H}}}) = \epsilon(\Phi^{\tilde{\mathrm{H}}}) \in \{\pm 1\}$ such that

$$\begin{tikzcd}
\mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\epsilon(\Phi^{\tilde{\mathrm{H}}})\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"] & \mathrm{SH}(X,\mathbb{Q}) \arrow[d, "\psi"] \\
\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))
\end{tikzcd} \tag{B.2.2}$$

commutes. We refer to [201, Sec. 4] for more details and proofs.

The process of associating to Φ^{SH} the isometry $\Phi^{\tilde{\mathrm{H}}}$ is non-trivial. Given an equivalence we cannot say directly how it will act on $\tilde{\mathrm{H}}(X,\mathbb{Q})$, e.g. there is no obvious cycle associated to the kernel of the equivalence. We circumvent this obstacle by using the extended Mukai vector.

B.3. Square root of the Todd class

Denote by $\overline{\mathsf{td}^{1/2}}$ the projection of the square root of the Todd class to the Verbitsky component $\mathrm{SH}(X,\mathbb{Q})$. The main goal of this section is to express this class in terms of the extended Mukai lattice. Throughout this section X will be a fixed hyper-Kähler manifold of dimension $2n$ of arbitrary deformation type.

Let us define a number

$$r_X := \frac{C(\mathrm{c}_2(X))2^n n!(2n-1)}{(2n)!24c_X} \tag{B.3.1}$$

where $C(\mathrm{c}_2(X))$ is the constant from Proposition B.2.1 associated to the second Chern class $\mathrm{c}_2(X)$. The number relates the BBF form and the characteristic value via

$$b(\omega,\omega) = 2r_X\lambda(\omega) \tag{B.3.2}$$

for all $\omega \in \mathrm{H}^2(X,\mathbb{Q})$.

Lemma B.3.1. *The following equality holds*

$$\int_X \mathsf{td}^{1/2} = c_X\frac{r_X^n}{n!}.$$

Proof. Let $\omega \in \mathrm{H}^2(X,\mathbb{R})$ be a Kähler class and t a formal variable. Proposition B.2.3 gives

$$\int_X \mathsf{td}^{1/2}\exp(t\omega) = (1+\lambda(t\omega))^n\int_X \mathsf{td}^{1/2}. \tag{B.3.3}$$

Both sides are even polynomials in t of degree $2n$. Comparing the coefficient in front of t^{2n} in (B.3.3) and using

$$\mathsf{td}^{1/2} = 1 + \text{terms of higher degree}$$

we obtain

$$c_X\frac{(2n)!}{2^n n!}\frac{b(\omega,\omega)^n}{(2n)!} = \lambda(\omega)^n\int_X \mathsf{td}^{1/2}.$$

Solving for $\int_X \mathsf{td}^{1/2}$ and employing (B.3.2) yields the assertion. □

For the known examples of hyper-Kähler manifolds of dimension $2n$ we have

$$r_X = \begin{cases} \frac{n+3}{4} & \mathrm{K3}^{[n]} \text{ or } \mathrm{OG}^{10}\text{-type}, \\ \frac{n+1}{4} & \mathrm{Kum}^n \text{ or } \mathrm{OG}^6\text{-type}. \end{cases}$$

Lemma B.3.1 for $\mathrm{K3}^{[n]}$- and Kum^n-type hyper-Kähler manifolds was also obtained by Sawon [194].

Denote for $0 \le i \le n$ by $\mathsf{q}_{2i} \in \mathrm{SH}^{4i}(X, \mathbb{Q})$ the class defined by the property

$$\int_X \mathsf{q}_{2i}\omega^{2n-2i} = c_X \frac{(2n-2i)!}{2^{n-i}(n-i)!} b(\omega,\omega)^{n-i}$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{Q})$.

Lemma B.3.2. *Let X be a hyper-Kähler manifold. Then the subspace of $\mathrm{SH}^{2i}(X,\mathbb{Q})$ being of type (i,i) on all small deformations is one-dimensional if i is even and zero otherwise. These subspaces are generated by q_{2i}.*

Proof. This follows from Proposition B.2.1. ☐

Using Lemma B.3.2 let us write

$$\overline{\mathsf{td}^{1/2}} = \mathsf{q}_0 + a_2\mathsf{q}_2 + \cdots + a_{2n-2}\mathsf{q}_{2n-2} + \frac{r_X^n}{n!}\mathsf{q}_{2n} \in \mathrm{SH}(X, \mathbb{Q})$$

for $a_{2i} \in \mathbb{Q}$. We will now determine the remaining coefficients.

Lemma B.3.3. *For $1 \le i \le n$ we have*

$$a_{2i} = \frac{r_X^i}{i!}.$$

Proof. We use again (B.3.3) and this time compare the coefficients in front of t^{2n-2i}. This reads

$$a_{2i}c_X \frac{b(\omega,\omega)^{n-i}}{2^{n-i}(n-i)!} = \binom{n}{n-i} \frac{r_X^i c_X}{2^{n-i} n!} b(\omega,\omega)^{n-i}.$$ ☐

Recall the isometric embedding $\psi\colon \mathrm{SH}(X,\mathbb{Q}) \hookrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ and the orthogonal projection $T\colon \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to \mathrm{SH}(X,\mathbb{Q})$. The class $\overline{\mathsf{td}^{1/2}}$ has the following expression.

Proposition B.3.4. *Let X be a hyper-Kähler manifold of dimension $2n$. Then*

$$\overline{\mathsf{td}^{1/2}} = T\left(\frac{(\alpha + r_X\beta)^n}{n!}\right) \in \mathrm{SH}(X, \mathbb{Q}).$$

If one ignores the orthogonal projection T for the moment, then the proposition says that (the projection of) the square root of the Todd class can be expressed as the n-th power of a linear polynomial. Note that the orthogonal projection T really is necessary since $\alpha^{n-i}\beta^i$ is not in the kernel of the Laplacian operator Δ for $1 \le i \le n-1$.

The key step to prove Proposition B.3.4 is to relate $\alpha^{n-i}\beta^i$ with the classes q_{2i}. By definition we have

$$T(\alpha^n) = n!1.$$

In general the connection is given by the following.

Lemma B.3.5. *For* $1 \leq i \leq n$ *we have*

$$T(\alpha^{n-i}\beta^i) = (n-i)!\mathsf{q}_{2i}.$$

Proof. By definition of the Mukai pairing on $\mathrm{SH}(X, \mathbb{Q})$ and the defining property of q_{2i} the assertion of the lemma is equivalent to

$$b_{\mathrm{SH}}(\omega^{2n-2i}, T(\alpha^{n-i}\beta^i)) = \int_X \omega^{2n-2i} T(\alpha^{n-i}\beta^i) = c_X \frac{(2n-2i)!}{2^{n-i}} b(\omega, \omega)^{n-i}$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{Q})$. The embedding $\psi \colon \mathrm{SH}(X, \mathbb{Q}) \hookrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ is an isometry and T is its orthogonal split. The above is therefore equivalent to

$$b_{[n]}(\psi(\omega^{2n-2i}), \alpha^{n-i}\beta^i) = c_X \frac{(2n-2i)!}{2^{n-i}} b(\omega, \omega)^{n-i}$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{Q})$. Since ψ is a morphism of $\mathfrak{g}(X)$-modules, we have

$$\psi(\omega^{2n-2i}) = \psi(e_\omega^{2n-2i}(1)) = e_\omega^{2n-2i}(\psi(1)) \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$$

which is a $\mathbb{Q}$-linear combination of tensors of the form $\alpha^r \omega^s \beta^t$. Only the tensor $\alpha^i \beta^{n-i}$ pairs non-trivially with $\alpha^{n-i}\beta^i$. We claim that

$$e_\omega^{2n-2i}(\psi(1)) = \frac{(2n-2i)!}{2^{n-i}(n-i)!i!} b(\omega, \omega)^{n-i} \alpha^i \beta^{n-i} + \dots .$$

Indeed, let us choose $N \gg n$ and consider the terms $\alpha^r \omega^s \beta^t$ for $r + s + t = N$. We define the order of such a term as r, i.e. the exponent of α. For general $j \geq 0$ the element $e_\omega^j(\psi(1))$ decomposes

$$e_\omega^j(\psi(1)) = a_1^j \left(\frac{\alpha^{N-j}\omega^j}{(n-j)!} \right) + a_2^j \left(\frac{\alpha^{N-j+1}\omega^{j-2}\beta}{(n-j+21)!} \right) + \dots + a_{\lfloor \frac{j}{2} \rfloor}^j \left(\frac{\alpha^{N-\lfloor \frac{j}{2} \rfloor}\omega^{j-2\lfloor \frac{j}{2} \rfloor}\beta^{\lfloor \frac{j}{2} \rfloor}}{(N - \lfloor \frac{j}{2} \rfloor)!} \right)$$

according to the order of the terms that appear. A straight forward proof by induction shows that

$$a_{k+1}^j = \frac{(\lfloor \frac{j}{2} \rfloor + k)!}{(\lfloor \frac{j}{2} \rfloor - k)!k!2^k}$$

which are the coefficients of the Besse polynomials. This then yields the claim.

Observing that

$$b_{[n]}(\alpha^i \beta^{n-i}, \alpha^{n-i}\beta^i) = c_X (n-i)!i!$$

finishes the proof. □

The above lemma helps to understand the two extra classes α and β in the extended Mukai lattice $\tilde{\mathrm{H}}(X, \mathbb{Q})$. Rather than trying to identify the classes α and β in $\tilde{\mathrm{H}}(X, \mathbb{Q})$ with a single element in $\mathrm{SH}(X, \mathbb{Q})$ as in the case of K3 surfaces one should think simultaneously of all the powers $\alpha^{n-i}\beta^i$ as (multiples of) the powers of the BBF-form $\mathsf{q}_2^i \in \mathrm{SH}^{4i}(X, \mathbb{Q})$.

Proof of Proposition B.3.4. This follows immediately from Lemma B.3.1, Lemma B.3.3 and Lemma B.3.5. □

Remark B.3.6. Proposition B.3.4 raises the analogous question for $\overline{\mathsf{td}}$. The formulas in the known examples are as follows. If X is of $\mathrm{K3}^{[n]}$ or OG^{10}-type of dimension $2n$, then

$$\overline{\mathsf{td}} = T\left(\frac{(\alpha+2\beta)\cdots(\alpha+(n+1)\beta)}{n!}\right) \in \mathrm{SH}(X,\mathbb{Q})$$

and if X is of Kum^n or OG^6-type of dimension $2n$, then

$$\overline{\mathsf{td}} = T\left(\frac{(\alpha+\beta)\cdots(\alpha+n\beta)}{n!}\right) \in \mathrm{SH}(X,\mathbb{Q}).$$

These expressions are obtained by an analogous approach as above using this time the known forms of the Riemann–Roch polynomials [68, 162, 190]. There is a formula for td similar in fashion as the one given in Proposition B.2.3 using Chebyshev polynomials, see [162, Thm. 5.2].

B.4. Extended Mukai vector

The previous section enables us to define a Mukai vector for interesting objects with image in the extended Mukai lattice. We will distinguish two cases using the self-intersection of the vector under consideration.

B.4.1. Square $-2r_X$

Let X be a hyper-Kähler manifold of dimension $2n$. A line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ naturally induces an auto-equivalence

$$\mathsf{M}_{\mathcal{L}} := \_ \otimes \mathcal{L} \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)).$$

Its action $\mathsf{M}_{\mathcal{L}}^{\mathrm{H}}$ on singular cohomology is given by multiplication with the Chern character of $\mathcal{L}$, i.e.

$$\mathsf{M}_{\mathcal{L}}^{\mathrm{H}} = \_ \cdot \exp(\lambda) \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{Q}))$$

where $\lambda = \mathrm{c}_1(\mathcal{L}) \in \mathrm{H}^2(X,\mathbb{Z})$. Furthermore, denote by $B_\lambda \in \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ the isometry defined by

$$B_\lambda(r\alpha+\mu+s\beta) = r\alpha+\mu+r\lambda+\left(s+b(\lambda,\mu)+r\frac{b(\lambda,\lambda)}{2}\right)\beta.$$

As checked in [201, Prop. 3.2] we have $\mathsf{M}_{\mathcal{L}}^{\tilde{\mathrm{H}}} = B_\lambda$, i.e. if we restrict $\mathsf{M}_{\mathcal{L}}^{\mathrm{H}}$ to $\mathrm{SH}(X,\mathbb{Q}) \subset \mathrm{H}^*(X,\mathbb{Q})$, then it is given by the natural action of B_λ on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ via the diagram

$$\begin{tikzcd}
\mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\epsilon(\mathsf{M}_{\mathcal{L}}^{\tilde{\mathrm{H}}})\mathsf{M}_{\mathcal{L}}^{\mathrm{SH}}"] \arrow[d, "\psi"] & \mathrm{SH}(X,\mathbb{Q}) \arrow[d, "\psi"] \\
\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\mathrm{Sym}^n B_\lambda"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))
\end{tikzcd} \tag{B.4.1}$$

(where we set $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = 1$ for all equivalences if n is odd). By definition

$$\alpha^n = \psi(n!1) = \psi(n!\mathrm{ch}(\mathcal{O}_X)) \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

which yields

$$\psi(\mathrm{ch}(\mathcal{L})) = \frac{B_\lambda(\alpha)^n}{n!} = \frac{\left(\alpha + \lambda + \frac{b(\lambda,\lambda)}{2}\beta\right)^n}{n!} \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})).$$

We will upgrade this to the Mukai vector. An immediate consequence from Proposition B.3.4 is that for the trivial line bundle $\mathcal{O}_X \in \mathrm{Pic}(X)$ and its Mukai vector $v(\mathcal{O}_X)$ one has

$$\overline{v(\mathcal{O}_X)} = T\left(\frac{(\alpha + r_X\beta)^n}{n!}\right) \in \mathrm{SH}(X,\mathbb{Q})$$

where $\overline{(\_)}$ denotes again the projection from the cohomology $\mathrm{H}^*(X,\mathbb{Q})$ to the Verbitsky component $\mathrm{SH}(X,\mathbb{Q})$. Note that (B.2.2) also induces a commutative diagram

$$\begin{tikzcd}
\mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\epsilon(\Phi^{\tilde{\mathrm{H}}})\Phi^{\mathrm{SH}}"] & \mathrm{SH}(X,\mathbb{Q}) \\
\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[u, "T"] \arrow[r, "\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[u, "T"']
\end{tikzcd} \tag{B.4.2}$$

for all equivalences $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$. Since $\mathsf{M}_{\mathcal{L}}(\mathcal{O}_X) = \mathcal{L}$ and by the compatibility of the cohomological Fourier–Mukai transform $\mathsf{M}^{\mathrm{H}}_{\mathcal{L}}(v(\mathcal{O}_X)) = v(\mathcal{L})$ we infer that

$$\overline{v(\mathcal{L})} = T\left(\frac{(\alpha + \lambda + (r_X + \frac{b(\lambda,\lambda)}{2})\beta)^n}{n!}\right) \in \mathrm{SH}(X,\mathbb{Q}) \tag{B.4.3}$$

for all line bundles $\mathcal{L} \in \mathrm{Pic}(X)$.

Definition B.4.1. For $\mathcal{L} \in \mathrm{Pic}(X)$ we define the *extended Mukai vector* of $\mathcal{L}$ with $\mathrm{c}_1(\mathcal{L}) = \lambda$ as

$$\tilde{v}(\mathcal{L}) = \alpha + \lambda + \left(r_X + \frac{b(\lambda,\lambda)}{2}\right)\beta \in \tilde{\mathrm{H}}(X,\mathbb{Q}).$$

With this definition, we have

$$\overline{v(\mathcal{L})} = T\left(\frac{\tilde{v}(\mathcal{L})^n}{n!}\right) \in \mathrm{SH}(X,\mathbb{Q}). \tag{B.4.4}$$

Formula (B.4.4) is a helpful tool to deduce properties of $\Phi^{\tilde{\mathrm{H}}}$ and compute its action on $\tilde{\mathrm{H}}(X,\mathbb{Q})$ for an auto-equivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$. Here is one example.

If n is even, then the functoriality of the representation $\rho^{\tilde{\mathrm{H}}}\colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ depends on the determinant of the isometry of $\tilde{\mathrm{H}}(X,\mathbb{Q})$. A useful criterion to calculate $\epsilon(\Phi^{\tilde{\mathrm{H}}})$ for an auto-equivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ is the following.

Lemma B.4.2. *Let X be a projective hyper-Kähler manifold with n even and $b_2(X)$ odd. Assume that a line bundle $\mathcal{L}$ is sent under an auto-equivalence Φ to an object $\mathcal{F}$ with positive rank. Then $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = \det(\Phi^{\tilde{\mathrm{H}}}) = 1$.*

Proof. The class $\tilde{v}(\mathcal{L})^n/n! \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ maps under T to $\overline{v(\mathcal{L})}$. We know that Φ^{H} sends $v(\mathcal{L})$ to $v(\mathcal{F})$. Therefore $\overline{v(\mathcal{L})}$ is sent to $\overline{v(\mathcal{F})}$ under Φ^{SH}. Since n is even, the coefficient in front of α^n in the expression $(\Phi^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{L})))^n$ must be positive. The commutativity of (B.4.2) forces $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = 1$. ☐

Instead of twists with line bundles, we can also use other auto-equivalences to define an extended Mukai vector for a larger set of objects.

Definition B.4.3. Let X be a projective hyper-Kähler manifold of dimension $2n$ with n odd and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ an object such that there exists an auto-equivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ with $\Phi(\mathcal{O}_X) \cong \mathcal{E}$. We define the *extended Mukai vector* of $\mathcal{E}$ as

$$\tilde{v}(\mathcal{E}) := \Phi^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{O}_X)) \in \tilde{\mathrm{H}}(X, \mathbb{Q}).$$

This definition does not depend on the chosen equivalence. For such an object $\mathcal{E}$ we have the equality

$$\overline{v(\mathcal{E})} = T\left(\frac{\tilde{v}(\mathcal{E})^n}{n!}\right). \tag{B.4.5}$$

We will say that such objects are in the *$\mathcal{O}_X$-orbit.* With this terminology, objects in the $\mathcal{O}_X$-orbit are *cohomologically linearisable* as in (B.4.5), which means that they admit an extended Mukai vector in the extended Mukai lattice, see also Definition B.4.16.

Remark B.4.4. Ideally one would like to give an analogous definition in the case that n is even and $b_2(X)$ is odd. However, one must be cautious since Definition B.4.3 is not well-defined in this case and (B.4.5) may not serve as a defining property ($v^n = (-v)^n$ for all elements $v \in \tilde{\mathrm{H}}(X, \mathbb{Q})$). The problem is the extra sign discussed in Section B.2.3. In other words, associating to the natural isometry Φ^{SH} an isometry $\Phi^{\tilde{\mathrm{H}}}$ inducing Φ^{SH} as done in Section B.2.3 is not natural and leads to considering sign conventions when defining an extended Mukai vector. We will give an adhoc definition.

Let X be a projective hyper-Kähler manifold of dimension $2n$ with n even and $b_2(X)$ odd and choose once and for all a very general Kähler class $\omega \in \mathrm{H}^2(X, \mathbb{R})$. Consider an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ such that there is an equivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ satisfying

$$\Phi(\mathcal{O}_X) \cong \mathcal{E}.$$

If the rank of $\mathcal{E}$ is strictly positive, then Lemma B.4.2 forces $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = 1$ and for negative rank we obtain $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = -1$. This motivates the following.

Definition B.4.5. We say that $\mathcal{E}$ is *positive* if $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = 1$ and *negative* if $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = -1$.

This definition is well-defined, i.e. it is independent of the chosen equivalence Φ. Keeping the above notation, let us denote

$$v = \Phi^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{O}_X)) = r\alpha + \lambda + s\beta.$$

We define the *signum* $\mathrm{sgn}(v) \in \{\pm 1\}$ of the vector v. For $r \neq 0$ we set $\mathrm{sgn}(v) := \mathrm{sgn}(r)$ as the signum of the number r. If $r = 0$, then the self-pairing of $\tilde{v}(\mathcal{O}_X)$ forces $\lambda \neq 0$ and $c = b(\omega, \lambda) \neq 0$ as the Kähler class ω was assumed to be very general. We define in this case $\mathrm{sgn}(v) := \mathrm{sgn}(c)$.

Definition B.4.6. The *extended Mukai vector* of $\mathcal{E}$ is

$$\tilde{v}(\mathcal{E}) := \epsilon(\Phi^{\tilde{\mathrm{H}}})\mathrm{sgn}(v)v.$$

We also say that such objects are in the $\mathcal{O}_X$*-orbit.* The extended Mukai vector $\tilde{v}(\mathcal{E})$ satisfies a version of (B.4.5) namely

$$\overline{v(\mathcal{E})} = \epsilon(\Phi^{\tilde{\mathrm{H}}})T\left(\frac{\tilde{v}(\mathcal{E})^n}{n!}\right). \tag{B.4.6}$$

That is, objects in the $\mathcal{O}_X$-orbit are cohomologically linearisable, but for negative objects we have to add an extra sign.

Remark B.4.7. The definition agrees with Definition B.4.1 for line bundles. The motivation for this definition comes from the notion of a positive vector in the theory of moduli spaces of stable sheaves for K3 surfaces as in [217, Def. 0.1]. Moreover, we expect that the choice of a Kähler class is not important, i.e. the sign of $b(\omega, \lambda)$ is independent of the chosen Kähler class. In all examples that we have calculated for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds X the class λ is always a multiple of the class Poincaré dual to a line in a projective space $\mathbb{P}^n \subset X$, therefore bounding the Kähler cone. For all our applications in subsequent chapters the sign choices will not matter.

Definition B.4.6 for the case of even n is up to sign compatible with derived equivalences, i.e. for $\mathcal{E}$ and $\mathcal{F}$ two objects in the $\mathcal{O}_X$-orbit and a derived equivalence $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))$ with $\Phi(\mathcal{E}) \cong \mathcal{F}$ we have

$$\Phi^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{E})) = \pm\tilde{v}(\mathcal{F}) \in \tilde{\mathrm{H}}(X, \mathbb{Q}) \tag{B.4.7}$$

respectively

$$(\Phi^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{E})))^n = (\tilde{v}(\mathcal{F}))^n \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})). \tag{B.4.8}$$

We list some easy properties.

Lemma B.4.8. *Let* $\mathcal{E}$ *be an object in the* $\mathcal{O}_X$*-orbit.*

(i) *The object* $\mathcal{E}$ *is a* $\mathbb{P}^n$*-object.*

(ii) *Its Mukai vector satisfies* $\langle v(\mathcal{E}), v(\mathcal{E})\rangle = n + 1$.

(iii) *Its extended Mukai vector satisfies* $\tilde{b}(\tilde{v}(\mathcal{E}), \tilde{v}(\mathcal{E})) = -2r_X$.

(iv) *The rank of* $\mathcal{E}$ *is of the form* $\pm a^n$ *for* $a \in \mathbb{Z}$.

(v) *The rank and determinant of* $\mathcal{E}$ *determine* $\overline{v(\mathcal{E})}$ *completely.*

Proof. For the notion of $\mathbb{P}^n$-object and their properties see [110]. The pairing in (ii) is the generalized Mukai product introduced in [58].

The first four points follow easily from the definitions and the fact that for an equivalence $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))$ the induced isomorphism $\Phi^{\tilde{\mathrm{H}}}$ of $\tilde{\mathrm{H}}(X, \mathbb{Q})$ is an isometry. Let $r = a^n$ be the rank of $\mathcal{E}$ and $\lambda = \mathrm{c}_1(\mathcal{E})$ be its determinant. Equation (B.4.5) implies that we only have to determine the extended Mukai vector of $\mathcal{E}$. Using that the orthogonal projection T is injective in degrees 0 and 2 we deduce that up to sign

$$\tilde{v}(\mathcal{E}) = a\alpha + \frac{\lambda}{a^{n-1}} + c\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

for $c \in \mathbb{Q}$ uniquely determined by the property $\tilde{b}(\tilde{v}(\mathcal{E}), \tilde{v}(\mathcal{E})) = -2r_X$. □

The lemma implies that the Chern classes of such objects are severely restricted.

Remark B.4.9. For K3 surfaces S the definition of the extended Mukai vector agrees with the usual Mukai vector for line bundles if we identify $\tilde{\mathrm{H}}(S,\mathbb{Z}) := \mathrm{H}^2(S,\mathbb{Z}) \oplus U$ with $\mathrm{H}^*(S,\mathbb{Z})$ via $\alpha \mapsto 1$ and $\beta \mapsto \mathsf{p}$. Note that the Mukai vectors of topological line bundles generate $\mathrm{H}^*(S,\mathbb{Z})$. For certain K3 surfaces (e.g. very general projective K3 surfaces of low degree [15, Rem. 6.10]) we know that all spherical objects are in the orbit of the structure sheaf $\mathcal{O}_S$ under the action of the group of auto-equivalences.

B.4.2. Square 0

There is another class of objects for which one can naturally define an extended Mukai vector. This does not involve Proposition B.3.4.

Lemma B.3.5 yields that the element $\beta \in \tilde{\mathrm{H}}(X,\mathbb{Q})$ has the property

$$\beta^n = \psi(c_X \mathsf{p}) \in \mathrm{SH}(X,\mathbb{Q}).$$

Since we have for a point $x \in X$

$$v(k(x)) = \mathrm{ch}(k(x)) = \mathsf{p} \in \mathrm{H}^{4n}(X,\mathbb{Q})$$

we obtain the relation

$$\psi(v(k(x))) = \frac{\beta^n}{c_X} \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

respectively

$$v(k(x)) = T\left(\frac{\beta^n}{c_X}\right) \in \mathrm{SH}(X,\mathbb{Q}). \tag{B.4.9}$$

Definition B.4.10. For a point $x \in X$ and the associated skyscraper sheaf $k(x)$ we define its *extended Mukai vector* as

$$\tilde{v}(k(x)) := \beta.$$

As in the case of objects in the $\mathcal{O}_X$-orbit this definition can be extended using derived equivalences.

Definition B.4.11. Let X be a projective hyper-Kähler manifold of dimension $2n$ with n odd, $\mathcal{E} \in \mathrm{D^b}(X)$ an object and $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))$ such that $\Phi(k(x)) \cong \mathcal{E}$ for some $x \in X$. We define the *extended Mukai vector* of $\mathcal{E}$ as

$$\tilde{v}(\mathcal{E}) := \Phi^{\tilde{\mathrm{H}}}(\beta) \in \tilde{\mathrm{H}}(X,\mathbb{Q}).$$

We will say that such objects are in the $k(x)$-*orbit.* The analogous relation to (B.4.9) for objects in the $k(x)$-orbit reads

$$v(\mathcal{E}) = T\left(\frac{\tilde{v}(\mathcal{E})^n}{c_X}\right) \in \mathrm{SH}(X,\mathbb{Q}). \tag{B.4.10}$$

Again in the case n even and $b_2(X)$ odd one has to be more careful. Let $\mathcal{E} \in \mathrm{D^b}(X)$ be such that there exists $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))$ with $\Phi(k(x)) \cong \mathcal{E}$ for some $x \in X$. Let us again write

$$v = \Phi^{\tilde{\mathrm{H}}}(\beta) = r\alpha + \lambda + s\beta.$$

We define again the *signum* $\mathrm{sgn}(v)$ of the vector v. As before for $r \neq 0$ we set $\mathrm{sgn}(v) \coloneqq \mathrm{sgn}(r)$. In the case $r = 0$ and $\lambda \neq 0$ the Hodge Index Theorem asserts that $c = b(\lambda, \omega) \neq 0$ for all Kähler classes ω. We assign $\mathrm{sgn}(v) \coloneqq \mathrm{sgn}(c)$. Finally for $r = \lambda = 0$ we define $\mathrm{sgn}(v) \coloneqq \mathrm{sgn}(s)$.

Definition B.4.12. The *extended Mukai vector* of $\mathcal{E}$ is defined as

$$\tilde{v}(\mathcal{E}) \coloneqq \epsilon(\Phi^{\tilde{\mathrm{H}}})\mathrm{sgn}(v)v.$$

We also say that such objects are in the *$k(x)$-orbit.* As above we have

$$v(\mathcal{E}) = \epsilon(\Phi^{\tilde{\mathrm{H}}})T\left(\frac{\tilde{v}(\mathcal{E})^n}{c_X}\right) \in \mathrm{SH}(X, \mathbb{Q}) \tag{B.4.11}$$

and the formation of the extended Mukai vector is as in (B.4.7) and (B.4.8) functorial for derived equivalences.

Lemma B.4.13. *Let $\mathcal{E}$ be an object in the $k(x)$-orbit.*

(i) *Its Mukai vector $v(\mathcal{E})$ lies in $\mathrm{SH}(X, \mathbb{Q})$ and satisfies $b_{\mathrm{SH}}(v(\mathcal{E}), v(\mathcal{E})) = 0$.*

(ii) *Its extended Mukai vector satisfies $\tilde{b}(\tilde{v}(\mathcal{E}), \tilde{v}(\mathcal{E})) = 0$.*

(iii) *The rank of $\mathcal{E}$ is of the form $\pm\frac{a^n n!}{c_X}$ for $a \in \mathbb{Q}$.*

(iv) *The rank and determinant of $\mathcal{E}$ determine $v(\mathcal{E})$ completely.*

(v) *If the rank of $\mathcal{E}$ is zero, then all Chern classes $\mathrm{c}_i(\mathcal{E})$ are isotropic, that is $\sigma|_{\mathrm{c}_i(\mathcal{E})} = \sigma\mathrm{c}_i(\mathcal{E}) = 0 \in \mathrm{H}^{2i+2}(X, \mathbb{C})$ with $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ a symplectic form.*

Proof. The first three points follow easily from the definition and the fourth point is analogous to Lemma B.4.8.

Suppose that the rank of $\mathcal{E}$ is zero and write

$$\tilde{v}(\mathcal{E}) = \lambda + s\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

with $\lambda \in \mathrm{H}^{1,1}(X, \mathbb{Q})$ and $s \in \mathbb{Q}$. We will assume that $\lambda \neq 0$, the other case being trivial. Since

$$\tilde{b}(\tilde{v}(\mathcal{E}), \tilde{v}(\mathcal{E})) = 0$$

we infer that $b(\lambda, \lambda) = 0$. Equation (B.4.10) gives

$$v(\mathcal{E}) = T\left(\frac{(\lambda + s\beta)^n}{c_X}\right) \in \mathrm{SH}(X, \mathbb{Q}) \tag{B.4.12}$$

which is supported in cohomological degrees ranging from $2n$ to $4n$. Up to a constant, the degree $2n$ component of $v(\mathcal{E})$ equals the n-th Chern character $\mathrm{ch}_n(\mathcal{E})$ which again up to a constant equals the Chern class $\mathrm{c}_n(\mathcal{E})$. From (B.4.12) we obtain

$$\mathrm{c}_n(\mathcal{E}) = dT(\lambda^n) \in \mathrm{SH}^{2n}(X, \mathbb{Q})$$

for some $d \in \mathbb{Q}$. Since ψ is a morphism of $\mathfrak{g}(X)$-modules we get

$$\psi(\lambda^n) = \psi(e_\lambda^n(1)) = e_\lambda^n\left(\frac{\alpha^n}{n!}\right) = \lambda^n \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

where the last equality used that $b(\lambda,\lambda) = 0$. Therefore $T(\lambda^n) = \lambda^n \in \mathrm{SH}(X,\mathbb{Q})$ and so in particular

$$\mathrm{c}_n(\mathcal{E}) = d\lambda^n.$$

The assertion of the lemma is therefore that $\sigma\mathrm{c}_n(\mathcal{E}) = d\sigma\lambda^n = 0$. We have

$$b(\lambda,\lambda) = b(\sigma,\sigma) = b(\lambda,\sigma) = 0 \tag{B.4.13}$$

which shows that

$$(\lambda+\sigma)^{n+1} \in \mathrm{SH}^{2n+2}(X,\mathbb{C}) \tag{B.4.14}$$

must vanish by using (B.2.1). As each summand in (B.4.14) lies in a different piece of the Hodge decomposition, we deduce that $\lambda^n\sigma = 0$.

For $k > n$, induction on k shows that $\sigma\mathrm{c}_k(\mathcal{E}) = 0$ if and only if $\sigma v(\mathcal{E})_{2k} = 0$ where $v(\mathcal{E})_{2k}$ denotes the cohomological degree $2k$ part of the Mukai vector. Equation (B.4.12) gives that the degree $2k$ part of $v(\mathcal{E})$ equals

$$v(\mathcal{E})_{2k} = \frac{s^{k-n}}{c_X}\binom{n}{k-n}T(\lambda^{2n-k}\beta^{k-n}) \in \mathrm{SH}^{2k}(X,\mathbb{Q}).$$

To determine the image of $\lambda^{2n-k}\beta^{k-n}$ under the orthogonal projection note that T is as well a morphism of $\mathfrak{g}(X)$-modules. Similarly to above, one shows

$$\lambda^{2n-k}\beta^{k-n} = e_\lambda^{2n-k}\left(\frac{\alpha^{2n-k}\beta^{k-n}}{(2n-k)!}\right) \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$$

using again that $b(\lambda,\lambda) = 0$. Lemma B.3.5 implies that we have

$$T(\alpha^{2n-k}\beta^{k-n}) = (2n-k)!\mathsf{q}_{2k-2n}$$

which yields

$$v(\mathcal{E})_{2k} = \frac{s^{k-n}}{c_X}\binom{n}{k-n}\lambda^{2n-k}\mathsf{q}_{2k-2n} \in \mathrm{SH}(X,\mathbb{Q}).$$

Ignoring constants we have to show that $\lambda^{2n-k}\mathsf{q}_{2k-2n}\sigma \in \mathrm{SH}(X,\mathbb{C})$ vanishes which is equivalent to

$$\lambda^{2n-k}\mathsf{q}_{2k-2n}\sigma\mu^{2n-k-1} = 0 \in \mathrm{SH}^{4n}(X,\mathbb{C})$$

for all $\mu \in \mathrm{H}^2(X,\mathbb{Q})$. This follows similar to above using again (B.4.13) and the polarized version of the Fujiki relations, see Proposition B.2.1. □

Note that the number from Lemma B.4.13 (iii)

$$\frac{a^n n!}{c_X}$$

must in particular be an integer. For all known examples of hyper-Kähler manifolds $c_X \in \mathbb{Z}$ and therefore we must already have $a \in \mathbb{Z}$ using Legendre's or de Polignac's formula.

Remark B.4.14. We want to comment on the denominators appearing in (B.4.5) and (B.4.10) and their relation.

The Chern character in degree $4n$ is of the form

$$\mathrm{ch}_{2n}(\mathcal{E}) = \frac{1}{(2n)!}(\mathrm{c}_1(\mathcal{E})^{2n} + \dots).$$

For a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ we have by using the Fujiki relation

$$\frac{\mathrm{c}_1(\mathcal{L})^{2n}}{(2n)!} = c_X \frac{(2n)! b(\lambda, \lambda)^n}{(2n)! n! 2^n} = \frac{c_X}{n!} \frac{b(\lambda, \lambda)^n}{2^n} \in \mathrm{SH}^{4n}(X, \mathbb{Q}).$$

For the known examples of hyper-Kähler manifolds the BBF form is even and the Fujiki constant is integral. In these cases we have

$$\frac{\mathrm{c}_1(\mathcal{L})^{2n}}{(2n)!} \in \frac{c_X}{n!}\mathbb{Z} \tag{B.4.15}$$

by identifying the class p with $1 \in \mathbb{Z}$. The factor $\frac{1}{c_X}$ for objects in the $k(x)$-orbit paired with $\frac{c_X}{n!}$ from (B.4.15) gives the factor $\frac{1}{n!}$ for objects in the $\mathcal{O}_X$-orbit. Note that all our computations take place in the Verbitsky component $\mathrm{SH}(X, \mathbb{Q})$ which is over $\mathbb{Q}$ generated by Chern characters of line bundles.

B.4.3. Structural result

The discussion of the two previous subsections enables us to prove the following general structural result for derived equivalences between hyper-Kähler manifolds.

Theorem B.4.15. *Let X and Y be deformation-equivalent projective hyper-Kähler manifolds and $\Phi \colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ an equivalence with Fourier–Mukai kernel $\mathcal{E}$. The rank r of $\mathcal{E}$ is of the form $\frac{a^n n!}{c_X}$ for $a \in \mathbb{Q}$. If $r = 0$, then $\mathcal{E}$ induces a covering of X and Y with Lagrangian cycles, or there exists a Hodge isometry $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{H}^2(Y, \mathbb{Z})$.*

Proof. Let us first assume that either n is odd or that n is even and $b_2(X)$ odd.

We distinguish three cases depending on the image vector

$$v = a\alpha + \lambda + s\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

of β under $\Phi^{\tilde{\mathrm{H}}}$. For $a \neq 0$ the assertion on the rank follows from Lemma B.4.13.

In the case that $a = 0$, but $\lambda \neq 0$, we consider the the n-th Chern class $\mathrm{c}_n(\mathcal{E}) \in \mathsf{A}^n(X \times Y)$ in the Chow ring with rational coefficients. The compatibility of derived equivalences with the induced maps between Chow and cohomology groups shows that for all $x \in X$ and all $y \in Y$ the cycles $\mathrm{c}_n(\mathcal{E})|_{x \times Y} \in \mathsf{A}^n(Y)$ respectively $\mathrm{c}_n(\mathcal{E})|_{X \times y} \in \mathsf{A}^n(X)$ are non-zero. Indeed the cohomological degree $2n$ component of $v(\mathcal{E}_x)$ is equal to $\mathrm{c}_n(\mathcal{E}_x)$ considered in cohomology which by assumption equals $\lambda^n \in \mathrm{SH}(Y, \mathbb{Q})$ and similarly for X. This shows that viewing the cycle $\mathrm{c}_n(\mathcal{E})$ as a family of cycles on Y parametrized by points $x \in X$ these cycles cover Y in the sense that for each $y \in Y$ the family of cycles $\mathrm{c}_n(\mathcal{E})$ restricts non-trivially to the subvariety $X \times y$. Since being isotropic is a cohomological property, the assertion follows now from Lemma B.4.13 (v).

Lastly, we assume that $v = s\beta$ for $s \in \mathbb{Q}$. This assumption implies that the element $\mathsf{p} \in \mathrm{H}^{4n}(X,\mathbb{Q})$ gets sent to $\pm s^n\mathsf{p} \in \mathrm{H}^{4n}(Y,\mathbb{Q})$ under the induced cohomological Fourier–Mukai transform Φ^{H}. We can view the image of topological K-theory under the Mukai vector map $v(K_{\mathrm{top}}(X))$ as a lattice inside the full cohomology $\mathrm{H}^*(X,\mathbb{Q})$ equipped with the generalized Mukai pairing. We refer to [5] for a recollection of topological K-theory and its relationship to derived categories and Fourier–Mukai transforms. The isomorphism Φ^{H} then induces an isometry between the lattices $v(K_{\mathrm{top}}(X))$ and $v(K_{\mathrm{top}}(Y))$. Since $\mathsf{p} \in v(K_{\mathrm{top}}(X))$ is a primitive element, the same must be true for $\Phi^{\mathrm{H}}(\mathsf{p}) = \pm s^n\mathsf{p}$. Therefore $s \in \{\pm 1\}$.

We may assume without loss of generality that $s = 1$. Since $\Phi^{\tilde{\mathrm{H}}}$ is an isometry, we infer that

$$\Phi^{\tilde{\mathrm{H}}}(\alpha) = \alpha + \lambda + t\beta \in \tilde{\mathrm{H}}(Y,\mathbb{Q}).$$

We claim that already $\lambda \in \mathrm{H}^2(Y,\mathbb{Z})$. To see this, consider $\tilde{v}(\mathcal{O}_X) = \alpha + r_X\beta$ and its image

$$\Phi^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{O}_X)) = \alpha + \lambda + (t + r_X)\beta \in \tilde{\mathrm{H}}(Y,\mathbb{Q}).$$

As above the element $v(\mathcal{O}_X)$ belongs to $v(K_{\mathrm{top}}(X))$ and therefore $\Phi^{\mathrm{H}}(v(\mathcal{O}_X))$ must be contained in $v(K_{\mathrm{top}}(Y))$. Using that the projection of $v(K_{\mathrm{top}}(Y))$ to its degree two component lands inside $\mathrm{H}^2(Y,\mathbb{Z})$ and the compatibility (B.4.2) we infer that also λ belongs to $\mathrm{H}^2(Y,\mathbb{Z})$.

Employing that $\Phi^{\tilde{\mathrm{H}}}$ is a Hodge isometry we furthermore conclude that $\lambda \in \mathrm{H}^{1,1}(Y,\mathbb{Z})$. Hence, there exists a line bundle $\mathcal{L} \in \mathrm{Pic}(Y)$ with first Chern class $-\lambda$. Changing Φ by postcomposing it with $\mathsf{M}_{\mathcal{L}}$ we obtain a derived equivalence $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ still denoted by Φ which satisfies $\Phi^{\tilde{\mathrm{H}}}(\alpha) = \alpha$ and $\Phi^{\tilde{\mathrm{H}}}(\beta) = \beta$. Thus, $\Phi^{\tilde{\mathrm{H}}}$ restricts to a Hodge isometry

$$\mathrm{H}^2(X,\mathbb{Q}) \cong \mathrm{H}^2(Y,\mathbb{Q}).$$

It remains to show that this isometry sends $\mathrm{H}^2(X,\mathbb{Z})$ to $\mathrm{H}^2(Y,\mathbb{Z})$. We employ the same strategy as above. For $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$ the vector

$$\tilde{v}(\mathcal{L}) = \alpha + \lambda + \left(r_X + \frac{b(\lambda,\lambda)}{2}\right)\beta$$

can be viewed as the extended Mukai vector of the (topological) line bundle $\mathcal{L}$ with first Chern class λ. The Mukai vector $v(\mathcal{L})$ lies inside $v(K_{\mathrm{top}}(X))$ and $\tilde{v}(\mathcal{L})$ is mapped under $\Phi^{\tilde{\mathrm{H}}}$ to

$$\alpha + \Phi^{\tilde{\mathrm{H}}}(\lambda) + \left(r_X + \frac{b(\lambda,\lambda)}{2}\right)\beta.$$

As before, the compatibility with topological K-theory forces $\Phi^{\tilde{\mathrm{H}}}(\lambda)$ to lie inside $\mathrm{H}^2(Y,\mathbb{Z})$. This finishes the proof in the case n odd or n even and $b_2(X)$ odd.

If we now assume that n as well as $b_2(X)$ are odd, then we cannot apply the results from [201] as explained in Section B.2.3 directly. That is, given a derived equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ the induced cohomological Fourier–Mukai transform $\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \cong \mathrm{H}^*(Y,\mathbb{Q})$ still restricts to a Hodge isometry

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X,\mathbb{Q}) \cong \mathrm{SH}(Y,\mathbb{Q}),$$

but there may not exist a Hodge isometry $\varphi \in \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ such that (B.2.2) commutes. However, inspecting [201, Prop. 4.1] and its proof we see that there exists a Hodge isometry $\varphi \in \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ unique up to sign such that via (B.2.2) either Φ^{SH} or $-\Phi^{\mathrm{SH}}$ agrees with φ^n. Reinspecting the above proof we see that this sign discrepancy does not affect the arguments and the proof remains valid also in the case n even and $b_2(X)$ even. □

B.4.4. Concluding remarks and further examples

For general objects $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ we make the following definition.

Definition B.4.16. An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ admits an *extended Mukai vector* $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ if there exists $c \in \mathbb{Q}$ such that

$$\overline{v(\mathcal{E})} = cT(\tilde{v}(\mathcal{E})^n) \in \mathrm{SH}(X, \mathbb{Q}).$$

With this definition, the vector $\tilde{v}(\mathcal{E})$ is not uniquely defined. One rather considers a one-dimensional subspace $V \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$ and demands that $\overline{v(\mathcal{E})}$ lies in the one-dimensional subspace $T(V^n)$. In the above two series of examples we considered a certain natural choice of $c \in \mathbb{Q}$ which then enabled us to define the extended Mukai vector as a uniquely determined vector in $\tilde{\mathrm{H}}(X, \mathbb{Q})$.

Here are two observations how one can generate new examples of objects admitting an extended Mukai vector from known ones:

- If $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ is an equivalence, then $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ admits an extended Mukai vector if and only if $\Phi(\mathcal{E}) \in \mathrm{D}^{\mathrm{b}}(Y)$ admits an extended Mukai vector.
- Let $\pi\colon \mathcal{X} \to B$ be a smooth and projective morphism with hyper-Kähler manifolds as fibres and $\mathcal{E}$ on $\mathcal{X}$ a B-flat sheaf or a B-perfect complex. For two points $b, b' \in B$ we have that $\mathcal{E}|_{\mathcal{X}_b}$ admits an extended Mukai vector if and only if $\mathcal{E}|_{\mathcal{X}_{b'}}$ admits an extended Mukai vector.

One can prove similar results as in Lemmas B.4.8 and B.4.13 for objects $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ satisfying Definition B.4.16 for a fixed $c \in \mathbb{Q}$. We just mention that if $\mathcal{E}$ has zero rank, then the projections of all Chern classes of $\mathcal{E}$ to $\mathrm{SH}(X, \mathbb{Q})$ are isotropic. To see this one writes $\tilde{v}(\mathcal{E}) = \lambda + s\beta$ for $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ and $s \in \mathbb{Q}$ and uses that $e_\sigma(\tilde{v}(\mathcal{E})) = e_{\bar{\sigma}}(\tilde{v}(\mathcal{E})) = 0$ for σ and $\bar{\sigma}$ the (anti-)holomorphic two-form.

An important class of cohomologically linearisable objects are line bundles and skyscraper sheaves. We want to give further examples.

Example B.4.17. Let S be a projective K3 surface and $\mathbb{P}^1 \cong C \subset S$ a smooth rational curve with class $l = [C] \in \mathrm{H}^2(S, \mathbb{Z})$. This yields a Lagrangian projective space $\mathbb{P}^n \cong C^{[n]} \subset S^{[n]}$ inside the Hilbert scheme of n points. The proof of Proposition B.7.2 will imply that the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$ is in the $\mathcal{O}_{S^{[n]}}$-orbit. If we write $\mathrm{H}^2(S^{[n]}, \mathbb{Z}) = \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$ where 2δ is the class of the exceptional divisor, one has

$$\tilde{v}(\mathcal{O}_{\mathbb{P}^n}) = l + \frac{\delta}{2} + \frac{n+1}{2}\beta \in \tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q}).$$

In particular, the projection $\overline{[\mathbb{P}^n]}$ of the class $[\mathbb{P}^n]$ to $\mathrm{SH}^{2n}(S^{[n]}, \mathbb{Q})$ equals

$$\overline{[\mathbb{P}^n]} = T\left(\frac{(l + \frac{\delta}{2})^n}{n!}\right) \in \mathrm{SH}^{2n}(S^{[n]}, \mathbb{Q}).$$

This yields a partial answer to a question posed by Bakker [13, Q. 29]. For more on this example, we refer to Proposition B.7.2 and Remark B.7.3.

Example B.4.18. For a very general projective K3 surface S of degree $2g-2$ we will study in Section B.10.2 the case of the moduli space of stable sheaves $M = M_H^S(0,1,d+1-g)$ which admits naturally a Lagrangian fibration $\pi\colon M \to \mathbb{P}^g = |H|$. The general fibre A is a smooth abelian variety and a degree zero line bundle $\mathcal{L}$ supported on A is an example of an object in the $k(x)$-orbit with

$$\tilde{v}(\mathcal{L}) = f \in \tilde{\mathrm{H}}(M,\mathbb{Q})$$

where $f \in \mathrm{H}^2(M,\mathbb{Z})$ is the image of the ample generator of $\mathrm{Pic}(\mathbb{P}^g)$ under pullback via π. For $d=0$ the section $\mathbb{P}^g \subset M$ again yields an object $\mathcal{O}_{\mathbb{P}^g} \in \mathrm{D}^{\mathrm{b}}(M)$ in the $\mathcal{O}_M$-orbit.

Example B.4.19. To the universal ideal sheaf $\mathcal{I}$ on $S \times S^{[2]}$ one associates the Fourier–Mukai kernel [2, 140]

$$\mathcal{E}^1 := \mathcal{E}xt^1_{\pi_{13}}(\pi_{12}^*(\mathcal{I}), \pi_{23}^*(\mathcal{I})) \in \mathrm{Coh}(S^{[2]} \times S^{[2]})$$

where π_{ij} denote the projections from $S^{[2]} \times S \times S^{[2]}$. Consider a point $p \in S^{[2]}$ parametrizing two distinct points $x, y \in S$ and denote by Z_x respectively Z_y the subvarieties of $S^{[2]}$ parametrizing subschemes whose support contains x respectively y. The derived equivalence $\mathsf{FM}_{\mathcal{E}^1}$ sends $k(p)$ to the sheaf $\mathcal{E}^1_{p\times S^{[2]}}$ which sits in a short exact sequence

$$0 \to \mathcal{O}(-\delta) \to \mathcal{E}^1_{p\times S^{[2]}} \to I_{Z_x \cup Z_y} \to 0$$

and is an example of an object in the $k(x)$-orbit with extended Mukai vector

$$\tilde{v}(\mathcal{E}^1_{p\times S^{[2]}}) = \alpha - \frac{\delta}{2} - \frac{1}{4}\beta.$$

For more on this example, see Section B.10.1.

Remark B.4.20. Ideally, one would like to define a vector $\tilde{w}$ for all elements $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ in a coherent way. By this we mean for example that its formation should factor through the K-group $K(X)$ and is compatible with derived equivalences, i.e. the following diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(X) \arrow[r, "\Phi"] \arrow[d, "\tilde{w}"] & \mathrm{D}^{\mathrm{b}}(Y) \arrow[d, "\tilde{w}"] \\
\tilde{\mathrm{H}}(X,\mathbb{Q}) \arrow[r, "\Phi^{\tilde{\mathrm{H}}}"] & \tilde{\mathrm{H}}(Y,\mathbb{Q})
\end{tikzcd}$$

should commute. However, this is too much to ask for.

Indeed, consider for example the case $X = S^{[2]}$ of the Hilbert scheme of two points on a projective K3 surface S. Using the Koszul resolution, one can check that the structure sheaf of a complete intersection of divisors of codimension larger than two must have trivial image under $\tilde{w}$. In particular, all sheaves supported on a zero-dimensional subscheme must have trivial image under $\tilde{w}$. Therefore all previously defined objects in the $k(x)$-orbit must map to zero under $\tilde{w}$. Hence, the vector $\tilde{w}$ vanishes for $\mathcal{E}^1_{p\times S^{[2]}} \otimes \mathcal{L}$ for all line bundles $\mathcal{L} \in \mathrm{Pic}(X)$. It therefore also has to vanish on all divisors.

The same argument as above also shows that any vector

$$K^0_{\mathrm{top}}(S^{[2]}) \xrightarrow{\tilde{w}} \tilde{\mathrm{H}}(S^{[2]},\mathbb{Q})$$

compatible with derived equivalences as above must vanish on all topological line bundles.

B.5. Integral lattices for $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds

We want to apply the results and definitions from the previous sections. From now on, X will denote a hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-type with $n > 1$.

B.5.1. Lattices

In this section, we want to discuss the (potential) integral lattices inside the extended Mukai lattice that appear and set up notation which will be used throughout the rest of the paper. We will fix for X once and for all an isometry

$$\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(S^{[n]},\mathbb{Z}) \cong \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta, \tag{B.5.1}$$

where S is a K3 surface, $S^{[n]}$ is the n-th Hilbert scheme with 2δ the class of the exceptional divisor of the Hilbert–Chow morphism and the second isometry is given by (B.6.1) (for $X = S^{[n]}$ we choose the first isometry to be the identity).

Let us first quickly review the case of a K3 surface S. There, the full integral cohomology $\mathrm{H}^*(S,\mathbb{Z}) = \tilde{\mathrm{H}}(S,\mathbb{Z})$ with the Mukai pairing is governing the derived category [155,175]. That is, an equivalence of K3 surfaces yields a Hodge isometry between the full integral cohomologies and two K3 surfaces are derived equivalent if and only if their integral cohomologies are Hodge isometric. Moreover, the lattice spanned by the Mukai vectors of topological line bundles equals the full integral cohomology.

In higher dimensions, the situation changes. There are several relevant lattices.

Definition B.5.1. We define the *integral extended Mukai lattice* as the lattice

$$\tilde{\mathrm{H}}(X,\mathbb{Z}) := \mathbb{Z}\alpha \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathbb{Z}\beta \subset \tilde{\mathrm{H}}(X,\mathbb{Q}).$$

Several facts suggest that in higher dimensions this is the wrong lattice to look at. Firstly, Definition B.4.1 and Examples B.4.17 and B.4.19 suggest that one should allow certain denominators. Secondly, we will see in Proposition B.7.2 that derived equivalences do not send integral elements to integral elements.

Definition B.5.2. For $\delta \in \mathrm{H}^2(X,\mathbb{Z})$ as above we define the $\mathrm{K3}^{[n]}$ *lattice* as

$$\Lambda := B_{-\delta/2}(\tilde{\mathrm{H}}(X,\mathbb{Z})) \subset \tilde{\mathrm{H}}(X,\mathbb{Q}).$$

This is independent of our fixed choice of δ, i.e. for any class $\gamma \in \mathrm{H}^2(X,\mathbb{Z})$ of square $2-2n$ and divisibility $2n-2$ one has

$$\Lambda = B_{-\gamma/2}(\tilde{\mathrm{H}}(X,\mathbb{Z})) \subset \tilde{\mathrm{H}}(X,\mathbb{Q}).$$

In dimension 4 this lattice was also considered by Taelman [201, Thm. E].

We introduce the notation

$$\tilde{\alpha} := B_{-\delta/2}(\alpha) = \alpha - \frac{\delta}{2} + \frac{1-n}{4}\beta, \quad \tilde{\delta} := B_{-\delta/2}(\delta) = \delta + (n-1)\beta.$$

With it one can define equivalently the $\mathrm{K3}^{[n]}$ lattice as the lattice

$$\Lambda = \mathbb{Z}\tilde{\alpha} \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\tilde{\delta} \oplus \mathbb{Z}\beta = \Lambda_S \oplus \mathbb{Z}\tilde{\delta} \tag{B.5.2}$$

where

$$\Lambda_S = \mathbb{Z}\tilde{\alpha} \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\beta.$$

Note that $\tilde{\alpha}$ and β still generate an integral hyperbolic plane and that the decomposition $\Lambda_S \oplus \mathbb{Z}\tilde{\delta}$ is orthogonal. The integral extended Mukai lattice and the $\mathrm{K3}^{[n]}$ lattice are isometric as abstract lattices and neither is included in the other when seen inside $\tilde{\mathrm{H}}(X,\mathbb{Q})$.

Definition B.5.3. The *geometric lattice* Λ_g is defined as

$$\Lambda_g \coloneqq \Lambda_S \oplus \mathbb{Z}\frac{\tilde{\delta}}{2} \subset \tilde{\mathrm{H}}(X,\mathbb{Q}).$$

Be aware that the quadratic form of Λ_g inherited from $\tilde{\mathrm{H}}(X,\mathbb{Q})$ may not be integral.

We want to motivate this definition. Recall that $r_X = \frac{n+3}{4}$ and let us look at the lattice generated by all extended Mukai vectors of topological line bundles, i.e.

$$\Lambda_{LB} \coloneqq \left\langle \left\{ \tilde{v}(\lambda) \coloneqq \alpha + \lambda + \left(\frac{n+3}{4} + \frac{b(\lambda,\lambda)}{2}\right)\beta \;\middle|\; \lambda \in \mathrm{H}^2(X,\mathbb{Z}) \right\} \right\rangle.$$

Note that one can write the generators equivalently as

$$\tilde{v}(\lambda) = \tilde{\alpha} + \frac{\tilde{\delta}}{2} + \lambda + \left(1 + \frac{b(\lambda,\lambda)}{2}\right)\beta.$$

If one ignores the term $\frac{\tilde{\delta}}{2}$ for a moment, then the expression resembles the Mukai vector on a K3 surface (where $\mathsf{td}^{1/2} = 1 + \mathsf{p}$). One can check that as an abstract lattice Λ_{LB} is isometric to $\tilde{\mathrm{H}}(X,\mathbb{Z})$.

In Section B.4.4 we saw that there are more objects than line bundles and skyscraper sheaves of points for which we can define an extended Mukai vector. We have

$$\tilde{v}(\mathcal{O}_{\mathbb{P}^n}) = l + \frac{\tilde{\delta}}{2} + \beta, \quad \tilde{v}(\mathcal{E}^1_{p\times S^{[n]}}) = \tilde{\alpha}.$$

Lemma B.5.4. *The geometric lattice Λ_g equals the lattice spanned by Λ_{LB} as well as all extended Mukai vectors from Section B.4.4.*

Proof. This follows from a straightforward calculation. ☐

Remark B.5.5. We expect that for all elements $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ for which a (meaningful) extended Mukai vector $\tilde{v}(\mathcal{E})$ can be defined, one has $\tilde{v}(\mathcal{E}) \in \Lambda_g$. We will prove in Corollary B.8.7 that Λ_g is invariant under all parallel transport isometries as well as derived equivalences.

B.5.2. Hodge structures

All the above defined lattices carry a weight-two Hodge structure from their inclusion into $\tilde{\mathrm{H}}(X,\mathbb{Q})$.

Definition B.5.6. For a lattice $\Gamma \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$ we define its *algebraic part* as

$$\Gamma_{\mathrm{alg}} \coloneqq \Gamma \cap \tilde{\mathrm{H}}^{1,1}(X,\mathbb{C}) \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$$

and its *transcendental part* as

$$\Gamma_{\mathrm{tr}} \coloneqq \Gamma_{\mathrm{alg}}^{\perp} \cap \Gamma \subset \tilde{\mathrm{H}}(X,\mathbb{Q}).$$

With this definition the transcendental part of the integral extended Mukai lattice equals the *transcendental lattice* of the hyper-Kähler manifold X, i.e.

$$\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tr}} \coloneqq \mathrm{H}^2(X,\mathbb{Z}) \cap \mathrm{NS(X)}^{\perp} = \tilde{\mathrm{H}}(X,\mathbb{Z})_{\mathrm{tr}} \subset \mathrm{H}^2(X,\mathbb{Z}).$$

Lemma B.5.7. *The transcendental part* Λ_{tr} *of the* $\mathrm{K3}^{[n]}$ *lattice* Λ *equals the transcendental lattice of* X*.*

Proof. Both inclusions follow from (B.5.2). □

Remark B.5.8. The isometry $B_{-\delta/2}$ yields an isometry between the integral extended Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Z})$ and the $\mathrm{K3}^{[n]}$ lattice Λ, which in general does not respect the Hodge structures. However, if we endow $\tilde{\mathrm{H}}(X,\mathbb{Z})$ with the twisted Hodge structure associated to the B-field $\delta/2 \in \mathrm{H}^2(X,\mathbb{Q})$ as defined in [109, Def. 2.3], then $B_{-\delta/2}$ induces a Hodge isometry between $\tilde{\mathrm{H}}(X,\mathbb{Z})$ endowed with the twisted Hodge structure and Λ equipped with the Hodge structure coming from the embedding $\Lambda \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$.

To see this consider a symplectic form $\sigma \in \mathrm{H}^2(X,\mathbb{C})$. The twisted Hodge structure is determined by the element $\sigma + \frac{1}{2}b(\sigma,\delta)\beta$ and this is sent under $B_{-\delta/2}$ to the symplectic form σ. The untwisted and the twisted Hodge structure on $\tilde{\mathrm{H}}(X,\mathbb{Z})$ have the same transcendental lattice, whereas in the case of K3 surfaces the transcendental lattice of a twisted Hodge structure associated to a non-trivial Brauer class is always a proper sublattice of the transcendental lattice of the untwisted Hodge structure [109, Sec. 2].

B.6. Derived Monodromy group

Let X be a hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-type and let X_1 and X_2 be deformations of X. By this we mean smooth and proper morphisms $\pi_i\colon \mathcal{X}_i \to B_i$ for $i \in \{1,2\}$ with B_i connected such that there is one point $0_i \in B_i$ with $\pi_i^{-1}(0_i) \cong X$ and another point $b_i \in B_i$ such that $\pi_i^{-1}(b_i) \cong X_i$. Let $\gamma_i\colon \mathrm{SH}(X,\mathbb{Q}) \cong \mathrm{SH}(X_i,\mathbb{Q})$ be parallel transport isometries obtained from choosing a path between 0 and b_i in B_i. Moreover, consider a Fourier–Mukai equivalence $f\colon \mathrm{D^b}(X_1) \cong \mathrm{D^b}(X_2)$ and denote by $F = f^{\mathrm{SH}}$ the induced isometry.

Definition B.6.1 (Taelman)**.** The *derived monodromy group* $\mathrm{DMon}(X)$ is the subgroup of $\mathrm{O}(\mathrm{SH}(X,\mathbb{Q}))$ generated by all isometries of the form

$$\mathrm{SH}(X,\mathbb{Q}) \xrightarrow{\gamma_1} \mathrm{SH}(X_1,\mathbb{Q}) \xrightarrow{F} \mathrm{SH}(X_2,\mathbb{Q}) \xrightarrow{\gamma_2^{-1}} \mathrm{SH}(X,\mathbb{Q}).$$

We know from Section B.2.3 that the isometry F is induced from an isometry $f^{\tilde{\mathrm{H}}}$. Similarly, the isometries γ_i are by [201, Prop. 4.1] induced by unique Hodge isometries $\gamma_i^{\tilde{\mathrm{H}}} \in \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$. This implies that the derived monodromy group has an inclusion $\mathrm{DMon}(X) \subset \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$. Throughout this paper we will consider the elements of $\mathrm{DMon}(X)$ always as isometries of the extended Mukai lattice.

Let now $S^{[n]}$ be the Hilbert scheme of n points on a projective K3 surface S. Bridgeland, King, and Reid [42] proved the existence of a derived equivalence

$$\mathrm{D^b}(S^{[n]}) \cong \mathrm{D}^{\mathrm{b}}_{\mathfrak{S}_n}(S^n)$$

where the latter is the $\mathfrak{S}_n$-equivariant derived category of the product variety S^n. For an introduction and notation regarding equivariant categories we refer to [29]. For our purposes, we will not take the equivalence from [42], but the one considered by Krug in [124]

$$\Psi\colon \mathrm{D}^{\mathrm{b}}_{\mathfrak{S}_n}(S^n) \cong \mathrm{D}^{\mathrm{b}}(S^{[n]}),$$

since it has nice properties for the computations we want to perform. Consider a line bundle $\mathcal{L}$ on S. There is a natural line bundle $\mathcal{L}_n$ on $S^{[n]}$ associated to $\mathcal{L}$ which satisfies $\Psi((\mathcal{L}^{\boxtimes n}, 1)) = \mathcal{L}_n$ [124, Thm. 1.1]. This yields the well-known isomorphisms

$$\operatorname{Pic}(S^{[n]}) \cong \operatorname{Pic}(S) \oplus \mathbb{Z}\delta, \quad \mathrm{H}^2(S^{[n]}, \mathbb{Z}) \cong \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta \tag{B.6.1}$$

where $2\delta = [E]$ is the class of the exceptional divisor of the Hilbert–Chow morphism. Since $\mathrm{H}^1(\mathfrak{S}_n, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$, the simple object $\mathcal{L}^{\boxtimes n} \in \mathrm{D}^{\mathrm{b}}(S^n)$ possesses another linearisation given by tensoring with the sign-representation. It holds

$$\Psi(\mathcal{L}^{\boxtimes n}, -1) = \mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta), \tag{B.6.2}$$

where $\mathcal{O}_{S^{[n]}}(-\delta) \in \operatorname{Pic}(S^{[n]})$ is the line bundle with first Chern class $-\delta$ [124, Rem. 3.10].

Ploog [185], later generalized by Ploog–Sosna [186], observed that there is an injective group homomorphism

$$\phi_{[n]}\colon \operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(S)) \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow \operatorname{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathfrak{S}_n}(S^n)).$$

More precisely, Orlov's Theorem [175] asserts that every auto-equivalence of $\mathrm{D}^{\mathrm{b}}(S)$ is given by a Fourier–Mukai functor with kernel $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(S \times S)$. The kernel $\mathcal{E}^{\boxtimes n}$ can be canonically equipped with a $\mathfrak{S}_n$-linearisation, where $\mathfrak{S}_n$ acts diagonally on $S^n \times S^n$. The factor $\mathrm{H}^1(\mathfrak{S}_n, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$ corresponds to the two possible linearisations of the kernel. We will often write $\phi_{[n]}(\Phi)$ instead of $\phi_{[n]}((\Phi, 1))$. Using the equivalence Ψ we also denote the resulting homomorphism

$$\operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(S)) \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow \operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(S^{[n]}))$$

obtained via conjugation by $\phi_{[n]}$.

Lemma B.6.2. *Let $\Phi \in \operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(S))$ such that $\Phi^{\mathrm{H}} \in \mathrm{O}(\tilde{\mathrm{H}}(S, \mathbb{Z}))$ is the identity. Then $\phi_{[n]}(\Phi)$ acts trivially on the extended Mukai lattice $\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$.*

Proof. Let $\Phi = \mathsf{FM}_{\mathcal{E}}$ and let us consider $\mathsf{FM}_{\mathcal{E}^{\boxtimes n}} \in \operatorname{Aut}(\mathrm{D}^{\mathrm{b}}(S^n))$. Using [97, Exc. 5.13] and the Künneth formula, one sees that $\mathsf{FM}_{\mathcal{E}^{\boxtimes n}}$ acts trivial on singular cohomology $\mathrm{H}^*(S^n, \mathbb{Q})$.

The line bundle $\mathcal{L}_n \in \operatorname{Pic}(S^{[n]})$ corresponds to the equivariant object $(\mathcal{L}^{\boxtimes n}, 1)$ in $\mathrm{D}^{\mathrm{b}}_{\mathfrak{S}_n}(S^n)$. By [186, Prop. 2.3] the equivalence $\phi_{[n]}(\Phi)$ sends $(\mathcal{L}^{\boxtimes n}, \pm 1)$ to the objects $(\Phi(\mathcal{L})^{\boxtimes n}, \pm 1)$. Using the compatibility of Fourier–Mukai transforms with (equivariant) topological K-theory [201, Sec. 6], one sees that $\phi_{[n]}(\Phi)$ induces an isomorphism of equivariant topological K-theory $\mathrm{K}^0_{\mathrm{top},\mathfrak{S}_n}(S^n)$ which fixes the classes $[(\mathcal{L}^{\boxtimes n}, \pm 1)]$.

Moreover, the equivalence Ψ induces an isomorphism $\mathrm{K}^0_{\mathrm{top},\mathfrak{S}_n}(S^n) \cong \mathrm{K}^0_{\mathrm{top}}(S^{[n]})$, see [42, Ch. 10] or [201, Thm. 8.2]. This implies that $\phi_{[n]}(\Phi)$ leaves the classes $v(\mathcal{L}_n)$ and $v(\mathcal{L}_n \otimes \mathcal{O}_X(-\delta))$ in $\mathrm{H}^*(X, \mathbb{Q})$ invariant. Using the compatibility (B.4.2) we see that the classes $\tilde{v}(\mathcal{L}_n)$ and $\tilde{v}(\mathcal{L}_n \otimes \mathcal{O}(-\delta))$ are fixed by the action of $\phi_{[n]}(\Phi)$ on the extended Mukai lattice. To conclude the proof, simply observe that these classes generate $\tilde{\mathrm{H}}(X, \mathbb{Q})$ as a $\mathbb{Q}$-vector space, since Λ_{LB} from Section B.5.1 is a full rank lattice. □

Let $\pi\colon \mathcal{S} \to B$ be a smooth and proper family of K3 surfaces and consider a path $\gamma\colon [0,1] \to B$. This yields a parallel transport isometry $\mathrm{H}^*(\mathcal{S}_{\gamma^{-1}(0)}, \mathbb{Z}) \cong \mathrm{H}^*(\mathcal{S}_{\gamma^{-1}(1)}, \mathbb{Z})$ of the fibres which we will also denote by γ. The family π induces naturally a corresponding family $\pi^{[n]}\colon \mathcal{S}^{[n]} \to B$ of relative Hilbert schemes over B. The path γ in B then gives for this deformation a corresponding parallel transport isometry $\gamma^{[n]}\colon \mathrm{H}^*(\mathcal{S}^{[n]}_{\gamma^{-1}(0)}, \mathbb{Q}) \cong \mathrm{H}^*(\mathcal{S}^{[n]}_{\gamma^{-1}(1)}, \mathbb{Q})$.

Consider an element $g \in \mathrm{DMon}(S)$ of the form $g = \gamma' \circ F \circ \gamma$. Here γ respectively γ' are as above parallel transport isometries obtained from deforming S to S' respectively S'' to S and $F = f^{\mathrm{H}}$ for a Fourier–Mukai equivalence $f\colon \mathrm{D}^{\mathrm{b}}(S') \cong \mathrm{D}^{\mathrm{b}}(S'')$. We associate to g the element $g^{[n]} := \gamma'^{[n]} \circ \phi_{[n]}(f)^{\tilde{\mathrm{H}}} \circ \gamma^{[n]}$.

Proposition B.6.3. *The association $g \mapsto g^{[n]}$ yields a well-defined group homomorphism*

$$d_n\colon \mathrm{DMon}(S) \to \mathrm{DMon}(S^{[n]}).$$

Proof. This follows as in the proof of Lemma B.6.2 together with the assertion of Lemma B.6.2. $\square$

B.7. Auto-equivalences of Hilbert schemes

Let S be a projective K3 surface and $S^{[n]}$ be the n-th punctual Hilbert scheme. In this section we will calculate the action of certain auto-equivalences on $\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$.

B.7.1. Sign equivalence

Denote by $F \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}_{\mathfrak{S}_n}(S^n))$ the auto-equivalence given by tensoring with the sign-representation. It is the image of the generator of $\mathbb{Z}/2\mathbb{Z}$ under $\phi_{[n]}$. We will also denote by F the auto-equivalence of $\mathrm{D}^{\mathrm{b}}(S^{[n]})$ induced via the equivalence Ψ. For a vector $v \in \tilde{\mathrm{H}}^{1,1}(S^{[n]}, \mathbb{Q})$ we denote by

$$s_v \in \mathrm{O}(\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})), \quad x \mapsto x - 2\frac{\tilde{b}(x,v)}{\tilde{b}(v,v)} v$$

the Hodge isometry given by reflection along v.

Proposition B.7.1. *The action of F on $\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$ is given by $(-1)^{n+1} s_{\tilde{\delta}}$.*

Proof. For all topological line bundles $\mathcal{L} \in K^0_{\mathrm{top}}(S)$ the involution F exchanges the equivariant objects $(\mathcal{L}^{\boxtimes n}, 1)$ and $(\mathcal{L}^{\boxtimes n}, -1)$ viewed as elements in equivariant topological K-theory $K^0_{\mathrm{top},\mathfrak{S}_n}(S^n)$. Thus, by (B.6.2) the induced isometry $F^{\tilde{\mathrm{H}}}$ on the extended Mukai lattice exchanges $\tilde{v}(\mathcal{L}_n)$ and $\tilde{v}(\mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta))$.

If n is odd, then we conclude from the above that for all $\lambda \in \mathrm{H}^2(S, \mathbb{Z}) \subset \mathrm{H}^2(S^{[n]}, \mathbb{Z})$ the action on the extended Mukai lattice $F^{\tilde{\mathrm{H}}}$ satisfies

$$\tilde{\alpha} + \frac{\tilde{\delta}}{2} + \lambda + \left(1 + \frac{\tilde{b}(\lambda,\lambda)}{2}\right)\beta \mapsto \tilde{\alpha} - \frac{\tilde{\delta}}{2} + \lambda + \left(1 + \frac{\tilde{b}(\lambda,\lambda)}{2}\right)\beta.$$

This property completely characterizes $F^{\tilde{\mathrm{H}}}$.

If n is even, Lemma B.4.2 implies that the determinant of $F^{\tilde{\mathrm{H}}}$ must be one, because F preserves the rank of objects. The result then follows as for n odd. $\square$

B.7.2. Spherical twist

An object $\mathcal{E} \in \mathrm{D^b}(S)$ is called spherical if its Ext-algebra satisfies $\mathrm{Ext}^*(\mathcal{E},\mathcal{E}) \cong \mathrm{H}^*(S^2,\mathbb{C})$. The auto-equivalence $\mathsf{ST}_{\mathcal{E}}$ given by the Fourier–Mukai functor $\mathsf{FM}_{\mathcal{G}}$ with Fourier–Mukai kernel defined via the distinguished triangle

$$\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta \to \mathcal{G}$$

in $\mathrm{D^b}(S \times S)$ is called the spherical twist [196]. Its action on the Mukai lattice $\tilde{\mathrm{H}}(S,\mathbb{Z})$ is given by the reflection $s_{v(\mathcal{E})}$.

An important example is the spherical twist $\mathsf{ST}_{\mathcal{O}_S}$ along the structure sheaf $\mathcal{O}_S$. It induces on cohomology the reflection along the vector $\mathsf{1}+\mathsf{p}$. The morphism $\phi_{[n]}$ yields an equivalence $P \in \mathrm{Aut}(\mathrm{D^b}(S^{[n]}))$.

Proposition B.7.2. *The equivalence P acts on $\tilde{\mathrm{H}}(S^{[n]},\mathbb{Q})$ via the isometry $(-1)^{n+1}s_v$, where $v = \tilde{\alpha}+\beta$.*

Proof. We want to understand the images of line bundles under P. The spherical twist $\mathsf{ST}_{\mathcal{O}_S}$ sends the structure sheaf $\mathcal{O}_S$ to $\mathcal{O}_S[-1]$. Applying [186, Prop. 2.3] we see that $P(\mathcal{O}_{S^n},1) = (\mathcal{O}_{S^n},-1)[-n]$ and $P(\mathcal{O}_{S^n},-1) = (\mathcal{O}_{S^n},1)[-n]$[3]. Lemma B.4.2 shows that $\epsilon(P^{\tilde{\mathrm{H}}}) = 1$ if n is even.

We first consider the case when n is odd. Assume there exists a smooth rational curve $C \subset S$ and let $\mathcal{L} = \mathcal{O}_S(C)$ be the corresponding line bundle with first Chern class $l \coloneqq \mathrm{c}_1(\mathcal{L})$. Then by Riemann–Roch $\mathcal{L}$ has a unique section up to scaling and the higher cohomologies of $\mathcal{L}$ vanish. The auto-equivalence $\mathsf{ST}_{\mathcal{O}_S}$ sends the line bundle $\mathcal{L}$ to $\mathcal{L}|_C$. We infer that the equivariant object $(\mathcal{L}^{\boxtimes n},-1)$ is being sent to $((\mathcal{L}|_C)^{\boxtimes n},-1)$ under the auto-equivalence P. We want to transfer this identity to the Hilbert scheme via Ψ. From (B.6.2) we know that $\Psi((\mathcal{L}^{\boxtimes n}),-1) \cong \mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta)$.

It is left to compute $\Psi((\mathcal{L}|_C)^{\boxtimes n},-1)$[4]. We claim $\Psi((\mathcal{L}|_C)^{\boxtimes n},-1) \cong \iota_*\omega_Z$ for $\iota\colon Z = C^{[n]} \cong \mathbb{P}^n \subset S^{[n]}$. We will sketch the arguments, see also [168, Sec. 3.2] for a thorough computation of this identity.

The sheaf $\mathcal{O}_C$ admits the resolution $\mathcal{L}^\vee \to \mathcal{O}_X$. Taking the n-th box product we obtain

$$\Big[W^n(\mathcal{L}^\vee) \to W^{n-1}(\mathcal{L}^\vee) \to \cdots \to W^1(\mathcal{L}^\vee) \to W^0(\mathcal{L}^\vee)\Big] \cong (\mathcal{O}_C^{\boxtimes n},1)$$

where we used the notation as in [124, Def. 3.4]. Tensoring with the sign-representation and envoking [168, Lem. 3.3]

$$\Big[W^0(\mathcal{L}) \to W^1(\mathcal{L}) \to \cdots \to W^{n-1}(\mathcal{L}) \to W^n(\mathcal{L})\Big] \cong ((\mathcal{L}|_C)^{\boxtimes n},-1). \tag{B.7.1}$$

Applying Ψ to (B.7.1) and using [124, Thm. 1.1] we find

$$\left[\mathcal{O}_{S^{[n]}} \to \mathcal{L}^{[n]} \to \cdots \to \bigwedge^{n-1} \mathcal{L}^{[n]} \to \det(\mathcal{L}^{[n]})\right] \cong \Psi((\mathcal{L}|_C)^{\boxtimes n},-1). \tag{B.7.2}$$

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[3] The fact that the linearisations get exchanged follows from the Koszul sign convention for graded tensor products.

[4] Thanks to the anonymous referee and Georg Oberdieck for spotting a mistake in an earlier version and Georg Oberdieck for discussions on computing images under Ψ.

In particular, the derived dual of $\Psi(\mathcal{O}_X^{\boxtimes n}, -1)$ is via (B.7.2) identified with the Koszul resolution of a regular section of the bundle $\mathcal{L}^{[n]}$ shifted by $[n]$. As the zero locus of this section is exactly $Z = C^{[n]}$ the claim follows from Grothendieck–Verdier duality.

Taking extended Mukai vectors and using Lemma B.4.8 we see that $P^{\tilde{\mathrm{H}}}$ sends the vector $\tilde{v}(\mathcal{L}_n \otimes \mathcal{O}_{S^n}(-\delta)) = \tilde{\alpha} - \frac{\tilde{\delta}}{2} + l$ to $w = \lambda + c\beta$ with $\lambda \in \mathrm{H}^2(S^{[n]}, \mathbb{Q})$, because $\iota_*\omega_Z$ has rank 0. We already know $P^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{O}_{S^{[n]}}(-\delta))) = -\tilde{v}(\mathcal{O}_{S^{[n]}})$ (we assume n odd) and since $P^{\tilde{\mathrm{H}}}$ is an isometry we conclude that

$$c = \tilde{b}(w, -\tilde{v}(\mathcal{O}_{S^{[n]}})) = \tilde{b}(\tilde{v}(\mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta)), \tilde{v}(\mathcal{O}_{S^{[n]}}(-\delta))) = \frac{-1-n}{2}.$$

Similarly we can use that $P^{\tilde{\mathrm{H}}}(\tilde{v}(\mathcal{O}_{S^{[n]}})) = -\tilde{v}(\mathcal{O}_{S^{[n]}}(-\delta))$ to infer

$$w = \lambda' - \frac{\delta}{2} + \frac{-1-n}{2}\beta = \lambda' - \frac{\tilde{\delta}}{2} - \beta$$

with $\lambda' \in \mathrm{H}^2(S, \mathbb{Q}) \subset \mathrm{H}^2(S^{[n]}, \mathbb{Q})$.

Since $Z \cong \mathbb{P}^n$, all curve classes on Z are multiples of each other. A line in Z is known to have homology class $l + (n-1)\delta^\vee \in \mathrm{H}_2(X, \mathbb{Z})$ [88, Ex. 4.11], where $\delta^\vee$ is the dual class to δ satisfying $\int_{S^{[n]}} \delta\delta^\vee = 1$. Denoting $s = l - \frac{\delta}{2}$ the cohomology class $s^{2n-1} \in \mathrm{SH}^{4n-2}(S^{[n]}, \mathbb{Q})$ is Poincaré dual to a multiple of the homology class $l + (n-1)\delta^\vee$. Therefore, the degree $4n-2$ part in $\mathrm{SH}(S^{[n]}, \mathbb{Q})$ of $\overline{v(\iota_*\omega_Z)}$ must be a multiple of s^{2n-1}. Since for $\mu \in \mathrm{H}^2(S^{[n]}, \mathbb{Q})$ we have

$$\psi(\mu^{2n-1}) = \psi(e_\mu^{2n-1}(1)) = e_\mu^{2n-1}(\psi(1)) = b(\mu, \mu)^{n-1}\frac{(2n)!}{2^n n!}\mu\beta^{n-1} \in \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$$

we conclude that $\lambda' = l$ and

$$\tilde{\alpha} + \frac{\tilde{\delta}}{2} + l \mapsto \frac{\tilde{\delta}}{2} + l - \beta.$$

In general, for a class $l \in \mathrm{H}^2(S, \mathbb{Z})$ of square -2 there exists a deformation S' of S such that either l or $-l$ is the class of a smooth rational curve $C' \subset S'$. Using Proposition B.6.3 we can assume that the topological line bundle $\mathcal{L}$ on S with first Chern class l is algebraic and that $\mathcal{L} \cong \mathcal{O}_S(C)$, where $C \subset S$ is a smooth rational curve. By the above, we therefore know the image of $\tilde{v}(\mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta))$ under $P^{\tilde{\mathrm{H}}}$. Since the vectors $\tilde{v}(\mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta))$ for $\mathcal{L}$ a topological line bundle on S whose first Chern class has self-intersection -2 together with $\tilde{v}(\mathcal{O}_{S^{[n]}})$ and $\tilde{v}(\mathcal{O}_{S^{[n]}}(-\delta))$ generate the vector space $\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$, we have proven the assertion in the case that n is odd.

If n is even, then the above shows that $P^{\tilde{\mathrm{H}}}$ must be either s_v or $-s_v$. Using that $\epsilon(P^{\tilde{\mathrm{H}}}) = \det(P^{\tilde{\mathrm{H}}}) = 1$ yields the assertion. ☐

Remark B.7.3. Here is one observation from the proof which might help to understand the extended Mukai lattice $\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$.

Given a smooth rational curve $C \subset S$ inside a K3 surface and the corresponding line bundle $\mathcal{L} = \mathcal{O}_S(C) \in \mathrm{Pic}(S)$ we have associated to it a line bundle $\mathcal{L}_n \in \mathrm{Pic}(S^{[n]})$. Its Mukai vector $v(\mathcal{L}_n)$ has self-pairing $n+1$ under the generalized Mukai pairing. We also associate to $\mathcal{L}_n$ the class $\tilde{v}(\mathcal{L}_n) \in \tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$. This class has self-pairing $-(n+3)/2$.

The auto-equivalence P induced from the spherical twist $\mathsf{ST}_{\mathcal{O}_S}$ via Ploog's map $\phi_{[n]}$ sends the line bundle $\mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta)$ to $\iota_*\omega_Z$. This is compatible with the pairings since the self-intersection of the projective space $\mathbb{P}^n \cong C^{[n]} \subset S^{[n]}$ is $(-1)^n(n+1)$. The image of $\tilde{v}(\mathcal{L}_n \otimes \mathcal{O}_{S^{[n]}}(-\delta))$ under $P^{\tilde{\mathrm{H}}}$ is

$$l - \frac{\delta}{2} + \frac{-1-n}{2}\beta = l - \frac{\tilde{\delta}}{2} - \beta.$$

Its self-intersection is equal to $-(n+3)/2$ which is exactly the value of $b(\ell, \ell)$, where ℓ is the class of a line in the projective space $C^{[n]}$ and we view a curve class as an element in $\mathrm{H}^2(S^{[n]}, \mathbb{Q})$ via Poincaré duality.

In the above prove we have calculated $\Psi(\mathcal{O}_C^{\boxtimes n}, -1)$. One can also consider the image of $(\mathcal{O}_C^{\boxtimes n}, 1)$ under Ψ, i.e. with the canonical linearization. One can show that this is $\mathcal{O}_{Y_C}$, where $Y_C \subset S^{[n]}$ is the reducible subscheme which is the preimage of $C^{(n)} \subset S^{(n)}$ under the Hilbert–Chow morphism.

B.7.3. From K3 surfaces to Hilbert schemes

We can now describe the homomorphism d_n from Proposition B.6.3. Consider the natural inclusion $\tilde{\mathrm{H}}(S, \mathbb{Q}) \hookrightarrow \tilde{\mathrm{H}}(S, \mathbb{Q}) \oplus \mathbb{Q}\delta = \tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$ of quadratic spaces. For $g \in \mathrm{O}(\tilde{\mathrm{H}}(S, \mathbb{Q}))$ we define $\iota(g) \in \mathrm{O}(\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q}))$ via $\iota(g)(\lambda) = g(\lambda)$ for $\lambda \in \tilde{\mathrm{H}}(S, \mathbb{Q}) \subset \tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})$ and $\iota(g)(\delta) = \delta$. This yields a group homomorphism

$$\iota \colon \mathrm{O}(\tilde{\mathrm{H}}(S, \mathbb{Q})) \to \mathrm{O}(\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q})).$$

Theorem B.7.4. *The homomorphism* $d_n \colon \mathrm{DMon}(S) \to \mathrm{DMon}(S^{[n]})$ *is given by*

$$g \mapsto \det(g)^{n+1} B_{-\delta/2} \circ \iota(g) \circ B_{\delta/2}.$$

Proof. The group $\mathrm{DMon}(S)$ is equal to the group of orientation-preserving isometries $\mathrm{O}^+(\tilde{\mathrm{H}}(S, \mathbb{Z}))$ of the full integral cohomology [93, 105]. This group is generated by the reflection along the -2-vector $1 + \mathsf{p}$ and the isometries B_λ for $\lambda \in \mathrm{H}^2(S, \mathbb{Z})$ [82, Prop. 3.4]. Note that the assignment of the statement of the theorem does define a group homomorphism $\mathrm{O}(\tilde{\mathrm{H}}(S, \mathbb{Q})) \to \mathrm{O}(\tilde{\mathrm{H}}(S^{[n]}, \mathbb{Q}))$. Hence, one can check on generators of $\mathrm{DMon}(S)$ that this morphism agrees with d_n. This is a straightforward calculation. □

B.8. Invariant Lattice

Let X be again a $\mathrm{K3}^{[n]}$-type hyper-Kähler manifold with $n > 1$. Any $\Gamma \cong \mathbb{Z}^{25}$ with an inclusion $\Gamma \hookrightarrow \tilde{\mathrm{H}}(X, \mathbb{Q})$ inherits a quadratic form which takes values in the rational numbers. We will denote by $\mathrm{O}(\Gamma) \subset \mathrm{O}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ the group of all isometries γ satisfying $\gamma(\Gamma) = \Gamma$.

The main goal of this section is to prove the following result.

Theorem B.8.1. *Let X be a* $\mathrm{K3}^{[n]}$*-type hyper-Kähler manifold. There are inclusions*

$$\hat{\mathrm{O}}^+(\Lambda) \subset \mathrm{DMon}(X) \subset \mathrm{O}(\Lambda).$$

In particular, the $\mathrm{K3}^{[n]}$ *lattice Λ is fixed by all derived equivalences.*

The group $\hat{\mathrm{O}}^+(\Lambda)$ is the group of all isometries with spinor norm 1 and which act via ±id on the discriminant group. For $n = 2$ this result was also obtained by Taelman [201, Thm. 9.8].

B.8.1. Realizing orthogonal transformations as derived equivalences

The first inclusion follows easily from the results of the last sections.

Proposition B.8.2. *There is an inclusion*

$$\hat{\mathrm{O}}^+(\Lambda) \subset \mathrm{DMon}(X).$$

Proof. The shift [1] acts on the extended Mukai lattice by $-\mathrm{id}$ and therefore acts non-trivially on the discriminant lattice and has determinant -1. Proposition B.7.2 endows us with an isometry whose action on the discriminant lattice is trivial if and only if its determinant is non-trivial and vice versa. Hence, it suffices to show that $\widetilde{\mathrm{SO}}^+(\Lambda)$, i.e. the group of all isometries with spinor norm and determinant 1 acting trivially on the discriminant, is contained in $\mathrm{DMon}(X)$. For this we will use the notion of Eichler transvections, for details and notations see [82, Sec. 3].

Let us orthogonally decompose

$$\Lambda = U \oplus \Lambda'$$

where the hyperbolic plane U is spanned by $\tilde{\alpha}$ and $-\beta$. The group $\widetilde{\mathrm{SO}}^+(\Lambda)$ equals the group $E_U(\Lambda')$ of unimodular transvections [82, Prop. 3.4]. For $\lambda \in \Lambda'$ the Eichler transvection $t(-\beta, \lambda)$ equals B_λ (note that for $\tilde{\delta} \in \Lambda'$ the transvection $t(-\beta, \tilde{\delta})$ also equals B_δ). Using tensoring with line bundles we see that all these isometries are contained in $\mathrm{DMon}(X)$. Furthermore, we infer from Proposition B.7.2 that the reflection s_v along the vector $v = \tilde{\alpha} + \beta$ lies in $\mathrm{DMon}(X)$. This involution exchanges $\tilde{\alpha}$ and $-\beta$ and acts trivially on Λ'. Using [82, Eq. (6)] we deduce that the transvections $t(\tilde{\alpha}, \lambda)$ for $\lambda \in \Lambda'$ are contained in $\mathrm{DMon}(X)$. By [82, Prop. 3.4] these isometries generate $\widetilde{\mathrm{SO}}^+(\Lambda)$ yielding the assertion. □

B.8.2. Finding derived invariant lattices

The proof of the other inclusion in Theorem B.8.1 will occupy the remainder of this section.

Lemma B.8.3. *There exists a lattice* $\Gamma \hookrightarrow \tilde{\mathrm{H}}(X, \mathbb{Q})$ *of rank* 25 *such that* $\mathrm{DMon}(X) \subset \mathrm{O}(\Gamma)$.

Proof. The group $\mathrm{DMon}(X)$ has a natural and faithful action on $\mathrm{SH}(X, \mathbb{Q})$ via the embedding $\mathrm{DMon}(X) \subset \mathrm{O}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$. Moreover, $\mathrm{DMon}(X)$ preserves the integral lattice $v(K_{\mathrm{top}}(X))) \cap \mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{SH}(X, \mathbb{Q})$ in this representation. Therefore it is contained in an arithmetic subgroup of $\mathrm{O}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$. □

We want to classify lattices Γ with the property $\mathrm{DMon}(X) \subset \mathrm{O}(\Gamma)$. We know by Proposition B.8.2 that for any such lattice Γ there is an inclusion $\hat{\mathrm{O}}^+(\Lambda) \subset \mathrm{O}(\Gamma)$. This yields strong restrictions.

Lemma B.8.4. *Let* $\tilde{\Gamma}$ *be a lattice preserved by* $\mathrm{DMon}(X)$ *as in Lemma B.8.3. Up to replacing* $\tilde{\Gamma}$ *by* $a\tilde{\Gamma} \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$ *for* $a \in \mathbb{Q}$ *the lattice* $\tilde{\Gamma}$ *is equal (as subsets) to* $k\Lambda_S \oplus \mathbb{Z}\tilde{\delta} \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$ *for some* $k \in \mathbb{Z}$ *satisfying* $k|(2n-2)$.

Proof. Let us replace $\tilde{\Gamma}$ with $a\tilde{\Gamma}$ for $a \in \mathbb{Q}_{>0}$ such that $\tilde{\Gamma} \subset \Lambda$ and a is the smallest positive rational number with that property.

Consider $v \in \tilde{\Gamma}$ and write $v = x + b\tilde{\delta}$ with $x \in \Lambda_S$ and $b \in \mathbb{Z}$. If $x \neq 0$, then its divisibility agrees with the largest integer $t \in \mathbb{Z}_{>0}$ such that $x \in t\Lambda_S$, since Λ_S is unimodular. Consider now all $v \in \tilde{\Gamma}$ such that in the above decomposition $x \neq 0$ and let k be the minimum of all integers t as above. Then $k\Lambda_S \subset \tilde{\Gamma}$.

Indeed, take an element $v \in \tilde{\Gamma}$ such that $v = kx + c\tilde{\delta}$ for some $c \in \mathbb{Z}$ and $x \in \Lambda_S$ is primitive. One immediately sees that $\mathrm{O}^+(\Lambda_S)$ can be embedded into $\hat{\mathrm{O}}^+(\Lambda)$ as the group of all isometries fixing $\tilde{\delta}$. Using [82, Prop. 3.3] we see that for every primitive $y \in \Lambda_S$ with $\tilde{b}(y,y) = \tilde{b}(x,x)$ the element $ky + c\tilde{\delta}$ is contained in $\tilde{\Gamma}$. This yields $k\Lambda_S \subset \tilde{\Gamma}$.

Consider $(k\Lambda_S)^\perp \subset \tilde{\Gamma}$ and take the positive integer $s \in \mathbb{Z}$ such that $(k\Lambda_S)^\perp = s\mathbb{Z}\tilde{\delta} \subset \tilde{\Gamma}$. We claim $\tilde{\Gamma} = k\Lambda_S \oplus s\mathbb{Z}\tilde{\delta}$. For this take an arbitrary $v \in \tilde{\Gamma}$ and write $v = dx + e\tilde{\delta}$ for $d, e \in \mathbb{Z}$. The definition of the integer k implies that k divides d and by the above we therefore have that $dx \in k\Lambda_S \subset \tilde{\Gamma}$. Hence $v - dx = e\tilde{\delta}$ is an element of $\tilde{\Gamma}$ orthogonal to $k\Lambda_S$. By definition of the integer s we have that s divides e and so $v \in k\Lambda_S \oplus s\mathbb{Z}\tilde{\delta}$.

The minimality assumption of a yields that the integers k and s do not have a common divisor. On the other hand, we know that $k\tilde{\alpha} \in \tilde{\Gamma}$ and $B_\delta(k\tilde{\alpha}) = k\tilde{\alpha} + k\tilde{\delta} + k(1-n)\beta$. This implies that $k\tilde{\delta} \in \tilde{\Gamma}$ and therefore $s = 1$. Finally, one sees that $B_\delta(\tilde{\delta}) = \tilde{\delta} + (2-2n)\beta$ which finishes the proof. ☐

Remark B.8.5. Ideally, one would like to conclude in the above situation directly that $k = 1$ and therefore (up to scaling) $\tilde{\Gamma}$ must equal Λ. However, this is in general not true.

For example, let us consider the case of $\mathrm{K3}^{[10]}$-type hyper-Kähler manifolds. The lemma below implies that for the lattice $\tilde{\Gamma} = 3\Lambda_S \oplus \mathbb{Z}\tilde{\delta} \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$ there is an inclusion $\mathrm{O}(\Lambda) \subset \mathrm{O}(\tilde{\Gamma})$. Moreover, the isometry $B_{\delta/3}$ lies in $\mathrm{O}(\tilde{\Gamma})$ but not in $\mathrm{O}(\Lambda)$. Therefore additional (geometric) input is necessary for the proof of Theorem B.8.1.

We make some further reductions.

Lemma B.8.6. *Let $l \in \mathbb{Z}_{>0}$ be the largest integer such that $l^2 | (n-1)$. For every lattice $\tilde{\Gamma}$ as in Lemma B.8.4 there is an inclusion*

$$\mathrm{O}(\tilde{\Gamma}) \subset \mathrm{O}(\Gamma)$$

with $\Gamma \coloneqq l\Lambda_S \oplus \mathbb{Z}\tilde{\delta} \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$.

Proof. Write $\tilde{\Gamma} = k\Lambda_S \oplus \mathbb{Z}\tilde{\delta}$ with $k | (2n-2)$. Let t be the greatest common divisor of l and k and denote by Γ_t the lattice $t\Lambda_S \oplus \mathbb{Z}\tilde{\delta} \subset \tilde{\mathrm{H}}(X,\mathbb{Q})$. The proof consists of showing the following two inclusions

$$\mathrm{O}(\tilde{\Gamma}) \subset \mathrm{O}(\Gamma_t) \subset \mathrm{O}(\Gamma).$$

Let us prove the first inclusion. Take an isometry $\gamma \in \mathrm{O}(\tilde{\Gamma})$ and write $k = k't$. Since $\gamma(\tilde{\delta}) \in \tilde{\Gamma} \subset \Gamma_t$ as subsets of the extended Mukai lattice it suffices to show that for every $\lambda \in \Lambda_S$ we have $\gamma(t\lambda) \in \Gamma_t$. By definition we have $\gamma(k\lambda) \in \tilde{\Gamma}$. Therefore we can write $\gamma(k\lambda) = ak\mu + b\tilde{\delta}$ for some integers a and b and $\mu \in \Lambda_S$. Dividing this equation by k' we obtain

$$\gamma(t\lambda) = at\mu + \frac{b}{k'}\tilde{\delta}.$$

As the self-pairing of $t\lambda$ is an even integer, the same must hold true for $\gamma(t\lambda)$. In particular, we find that

$$2b^2\frac{1-n}{k'^2} \in 2\mathbb{Z}.$$

The defining property of l together with the fact that l and k' are coprime implies that k' must divide b. This gives the first inclusion.

For the second inclusion consider an isometry $\gamma \in \mathrm{O}(\Gamma_t)$ and observe that for every $\lambda \in \Lambda_S$ we have $t\gamma(\lambda) \in \Gamma_t = t\Lambda_S \oplus \mathbb{Z}\tilde{\delta}$. This yields

$$l\gamma(\lambda) = (l/t)t\gamma(\lambda) \in l\Lambda_S \oplus \mathbb{Z}\tilde{\delta} = \Gamma.$$

It is left to show that $\gamma(\tilde{\delta}) \in \Gamma$. This follows immediately from the fact that $\tilde{\delta}$ as an element in the lattice Γ_t has divisibility $2n-2$. ☐

We therefore have an upper bound for the lattice from Lemma B.8.3, i.e. for $\Gamma = l\Lambda_S \oplus \mathbb{Z}\tilde{\delta}$ as above we have $\mathrm{DMon}(X) \subset \mathrm{O}(\Gamma)$. In particular, if $n-1$ is square-free, then we have already obtained $\mathrm{DMon}(X) \subset \mathrm{O}(\Lambda)$.

B.8.3. Conclusion of proof

Proof of Theorem B.8.1. From Lemma B.8.6 we know that for the lattice $\Gamma = l\Lambda_S \oplus \mathbb{Z}\tilde{\delta}$ with l maximal such that $l^2|(n-1)$ there is an inclusion $\mathrm{DMon}(X) \subset \mathrm{O}(\Gamma)$.

Suppose there exists an isometry $\gamma \in \mathrm{DMon}(X)$ which does not lie in $\mathrm{O}(\Lambda)$. Consider the composition

$$\varphi \colon \Lambda_S \xrightarrow{\gamma} \tilde{\mathrm{H}}(X, \mathbb{Q}) \xrightarrow{p} \mathbb{Q}\tilde{\delta}$$

where p is the orthogonal projection and denote $K = \mathrm{Ker}(\varphi)$. Let v be a generator of $K^{\perp} \subset \Lambda_S$ and let us write $\frac{k}{l}\tilde{\delta}$ for its image under γ. By assumption $\frac{k}{l}$ is not an integer. Note that there are two hyperbolic planes $U_1 \oplus U_2$ contained in K.

Indeed, since $\mathrm{O}^+(\Lambda_S)$ acts transitively on primitive elements with the same square [82, Prop. 3.3], one can send v into a hyperbolic plane $U \subset U^4 \oplus E_8(-1)^2 \cong \Lambda_S$. Since $K = v^{\perp}$ we know there are two (in fact at least three) hyperbolic planes contained in K.

Changing v to w by adding an element of U_1 we can assume that $\tilde{b}(w, w) = -2$ and the image of w generates the image of φ. Moreover, we know that there is a primitive isotropic element $z \in U_2$ which is orthogonal to w and which under γ is mapped to a primitive element $u \in \Lambda_S$.

Let us write

$$\gamma(w) = x + \frac{k}{l}\tilde{\delta}$$

with $x \in \Lambda_S$ and

$$\gamma(\tilde{\delta}) = \frac{2n-2}{l}y + s\tilde{\delta} \tag{B.8.1}$$

for $y \in \Lambda_S$ and some $s \in \mathbb{Z}$, because $\tilde{\delta} \in \Gamma$ has divisibility $2n-2$. Recall $u = \gamma(z)$ and note that there exists $a \in \mathbb{Z}$ such that

$$x' = x + \frac{n-1}{l}y + au \in \Lambda_S$$

is primitive since $u \in \Lambda_S$ is itself primitive. We define the element $w' = w + az \in \Lambda_S$ which is still primitive, has self-pairing -2 and its image generates the image of φ.

The group $\mathrm{O}^+(\Lambda_S)$ can be included into $\hat{\mathrm{O}}^+(\Lambda) \subset \mathrm{DMon}(X)$ by letting isometries act trivially on $\tilde{\delta}$. Hence, there exists an element $\gamma' \in \hat{\mathrm{O}}^+(\Lambda)$ which maps $\tilde{\alpha} + \beta$ to $w' \in \Lambda_S$ and fixes $\tilde{\delta}$, since $\mathrm{O}^+(\Lambda_S)$ acts transitively on the set of primitive vectors with prescribed self-pairing. Furthermore, there exists an isometry $\gamma'' \in \hat{\mathrm{O}}^+(\Lambda)$ such that the primitive element x' is mapped to $\tilde{\alpha} + b\beta$ for some $b \in \mathbb{Z}$ and $\tilde{\delta}$ to itself. Precomposing γ with γ' and postcomposing with γ'' we therefore obtain an isometry in $\mathrm{DMon}(X)$ which satisfies

$$\tilde{v}(\mathcal{O}_X) = \tilde{\alpha} + \frac{\tilde{\delta}}{2} + \beta \mapsto h := \tilde{\alpha} + \left(\frac{s}{2} + \frac{k}{l}\right)\tilde{\delta} + b\beta = \alpha + \left(\frac{s-1}{2} + \frac{k}{l}\right)\delta + c\beta$$

for some $c \in \mathbb{Q}$.

The extended Mukai vector of $\mathcal{O}_X$ satisfies

$$T\left(\frac{\tilde{v}(\mathcal{O}_X)^n}{n!}\right) = \overline{v(\mathcal{O}_X)} \in \mathrm{SH}(X,\mathbb{Q}) \subset \mathrm{H}^*(X,\mathbb{Q})$$

which is in particular an element in the image of the Mukai vector morphism

$$\overline{v} = \overline{\mathrm{ch}(\_)\mathsf{td}^{1/2}}\colon K^0_{\mathrm{top}}(X) \to \mathrm{SH}(X,\mathbb{Q}) \subset \mathrm{H}^*(X,\mathbb{Q})$$

projected to $\mathrm{SH}(X,\mathbb{Q})$. Since parallel transport operators as well as derived equivalences preserve the image of topological K-theory under the Mukai vector morphism in cohomology, the same must hold true for $\gamma'' \circ \gamma \circ \gamma'$, so in particular

$$T\left(\frac{h^n}{n!}\right) \in \overline{v(K^0_{\mathrm{top}}(X))}.$$

Let us write

$$T\left(\frac{h^n}{n!}\right) = 1 + \left(\frac{s-1}{2} + \frac{k}{l}\right)\delta + \mu$$

with $\mu \in \mathrm{SH}^{>2}(X,\mathbb{Q})$. Applying the quadratic form to the equality (B.8.1) we see that s must be odd and therefore $(s-1)/2$ is an integer. Note that the degree 2 component of the image of elements of topological K-theory under the Mukai vector morphism always lies inside $\mathrm{H}^2(X,\mathbb{Z})$. This yields a contradiction and finishes the proof. □

Corollary B.8.7. *There is an inclusion*

$$\mathrm{DMon}(X) \subset \mathrm{O}(\Lambda_g).$$

Proof. Take $\gamma \in \mathrm{DMon}(X)$ and recall that

$$\Lambda \subset \Lambda_g = \Lambda_S \oplus \mathbb{Z}\frac{\tilde{\delta}}{2}.$$

The inclusion $\mathrm{DMon}(X) \subset \mathrm{O}(\Lambda)$ yields that every element of Λ_S is mapped under γ again to $\Lambda \subset \Lambda_g$. Moreover,

$$\gamma(\tilde{\delta}) = (2n-2)x + s\tilde{\delta}$$

for some $s \in \mathbb{Z}$ and $x \in \Lambda_S$, because $\tilde{\delta} \in \Lambda$ has divisibility $2n-2$. This implies $\gamma(\tilde{\delta}/2) \in \Lambda_g$. □

B.9. Derived equivalences of $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds

We will draw some consequences from the results of the previous sections.

B.9.1. General results

Let X be a projective hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-type. We denote by Λ_X the Hodge structure obtained from the inclusion $\Lambda \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$ and by $\mathrm{Aut}(\Lambda_X)$ the group of all Hodge isometries of Λ_X. Recall the representation

$$\rho^{\tilde{\mathrm{H}}} \colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$$

from Section B.2.3.

Corollary B.9.1. *The representation* $\rho^{\tilde{\mathrm{H}}}$ *of the group of auto-equivalences* $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ *factors via a representation*

$$\rho^{\tilde{\mathrm{H}}} \colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{Aut}(\Lambda_X) \subset \mathrm{O}(\tilde{\mathrm{H}}(X, \mathbb{Q})).$$

One can also formulate the following more general version of the above statement.

Theorem B.9.2. *Let* X *and* Y *be projective* $\mathrm{K3}^{[n]}$*-type hyper-Kähler manifolds and* $\Phi \colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ *a derived equivalence. Then* $\Phi^{\tilde{\mathrm{H}}}$ *restricts to a Hodge isometry*

$$\Phi^{\tilde{\mathrm{H}}} \colon \Lambda_X \cong \Lambda_Y.$$

Proof. Since X and Y are deformation-equivalent, there exists a parallel transport isometry $\gamma \colon \tilde{\mathrm{H}}(Y, \mathbb{Q}) \cong \tilde{\mathrm{H}}(X, \mathbb{Q})$. The composition $\Phi^{\tilde{\mathrm{H}}} \circ \gamma$ lies in $\mathrm{DMon}(Y)$ and, therefore, satisfies $\Phi^{\tilde{\mathrm{H}}} \circ \gamma(\Lambda) = \Lambda$ by Theorem B.8.1. Now $\gamma(\Lambda) \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$ is a lattice invariant under $\mathrm{DMon}(X)$.

Indeed, let $\gamma_2 \circ F \circ \gamma_1$ be one of the generators of $\mathrm{DMon}(X)$ as in Definition B.6.1. Then $\gamma^{-1} \circ \gamma_2 \circ F \circ \gamma_1 \circ \gamma \in \mathrm{DMon}(Y)$, so Theorem B.8.1 gives

$$\gamma^{-1} \circ \gamma_2 \circ F \circ \gamma_1 \circ \gamma(\Lambda) = \Lambda$$

which yields

$$\gamma_2 \circ F \circ \gamma_1(\gamma(\Lambda)) = \gamma(\Lambda).$$

Using Lemma B.8.4 and that γ is an isometry we see that the subset $\gamma(\Lambda) \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$ is equal to $\Lambda \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$. Combining everything yields the assertion. ☐

We can use Lemma B.5.7 to obtain the following form of Theorem B.9.2.

Corollary B.9.3. *Let* X, Y *and* Φ *be as above. Then* $\Phi^{\tilde{\mathrm{H}}}$ *restricts to a Hodge isometry*

$$\Phi^{\tilde{\mathrm{H}}} \colon \mathrm{H}^2(X, \mathbb{Z})_{\mathrm{tr}} \cong \mathrm{H}^2(Y, \mathbb{Z})_{\mathrm{tr}}$$

between the transcendental lattices of X *and* Y*.* ☐

An immediate consequence is the following.

Theorem B.9.4. *For a fixed projective* $\mathrm{K3}^{[n]}$*-type hyper-Kähler manifold* X *the number of projective* $\mathrm{K3}^{[n]}$*-type manifolds* Y *up to isomorphism with* $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ *is finite.*

Proof. The proof is similar in flavour to [43, Prop. 5.3].

Corollary B.9.3 implies that for any Y as in the assertion its transcendental lattice $\mathrm{H}^2(Y,\mathbb{Z})_{\mathrm{tr}}$ is Hodge isometric to $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tr}}$. As abstract lattices the number of embeddings

$$\mathrm{H}^2(Y,\mathbb{Z})_{\mathrm{tr}} \hookrightarrow \mathrm{H}^2(Y,\mathbb{Z})$$

is finite up to isometries of $\mathrm{H}^2(Y,\mathbb{Z})$, see [123, Satz 30.2]. Therefore, the set of lattices appearing as $\mathrm{NS}(Y)$ for any such Y is as well finite.

As in [43, Prop. 5.3] we conclude that there are only finitely many Hodge structures on the lattice $\mathrm{H}^2(Y,\mathbb{Z})$ being realized by $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds Y derived equivalent to our fixed X. Since the monodromy group $\mathrm{Mon}^2(Y)$ is a finite index subgroup of $\mathrm{O}(\mathrm{H}^2(Y,\mathbb{Z}))$ [136, Cor. 1.8] the Global Torelli Theorem [211] shows that up to birational equivalence there are only finitely many hyper-Kähler manifolds realizing a given Hodge structure on $\mathrm{H}^2(Y,\mathbb{Z})$. The assertion now follows from [141, Cor. 1.5].

□

We also have the following structural result.

Corollary B.9.5. *Let X, Y and Φ be as in Theorem B.9.2 and let $\mathcal{E}$ be the Fourier–Mukai kernel of Φ. Then the rank of $\mathcal{E}$ is of the form $n!a^n$ for $a \in \mathbb{Z}$ and the smallest non-zero cohomological degree of the Mukai vector of the image of $k(x)$ under Φ for all $x \in X$ is $0, 2n$ or $4n$. In the second case Y admits a rational Lagrangian fibration.*

Proof. The first statements follow from Lemma B.4.13 and Theorem B.9.2.

For the last assertion, we know that $\beta \in \Lambda_X$ is mapped to $\lambda + c\beta \in \Lambda_Y$ for $c \in \mathbb{Z}$ and $\lambda \in \mathrm{H}^{1,1}(Y,\mathbb{Z})$ satisfying $b(\lambda,\lambda) = 0$. Let $\mathcal{C}_Y \subset \mathrm{H}^2(Y,\mathbb{R})$ the the positive cone of Y. Then $\lambda^\perp \cap \mathcal{C}_Y \neq 0$. By [137] there exist an isometry mapping λ into the closure of the birational Kähler cone. The result now follows from [143, Cor. 1.1]. □

This yields strong restrictions on Fourier–Mukai kernels of derived equivalences between hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$-type. Note that all three cases $0, 2n$ and $4n$ occur, see Proposition B.7.2 and Section B.10.2. Furthermore, Lemma B.4.13 implies that if $\mathcal{E}$ if of rank zero, then for all $x \in X$ all Chern classes of $\mathcal{E}_x$ are isotropic as in Lemma B.4.13.

B.9.2. Moduli spaces

We demonstrate consequences for smooth moduli spaces of stable objects, see [100, Ch. 10] and [17] for the necessary background and notation.

Corollary B.9.6. *Let $M^S_\sigma(v)$ be a smooth moduli space of stable objects on a projective K3 surface S and X a projective $\mathrm{K3}^{[n]}$-type hyper-Kähler manifold such that $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(M^S_\sigma(v))$. Then X is itself a moduli space of stable objects on S.*

Proof. For a K3 surface S and a primitive Mukai vector v with generic stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ one has a Hodge isometry

$$\mathrm{H}^2(M^S_\sigma(v),\mathbb{Z}) \cong v^\perp \subset \tilde{\mathrm{H}}(S,\mathbb{Z}),$$

see [217], [18, Thm. 6.10] and [38, Thm. 1.1 (2)]. Since v is in the algebraic part of the Mukai lattice of the K3 surface the restriction of the above Hodge isometry yields

$$\mathrm{H}^2(M^S_\sigma(v),\mathbb{Z})_{\mathrm{tr}} \cong \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tr}}.$$

Corollary B.9.3 together with [3, Prop. 4] and [17, Thm. 1.2(c)] imply that X is a moduli space $M^{S'}_{\sigma'}(v')$ of stable objects on a K3 surface S' such that

$$\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tr}} \cong \mathrm{H}^2(S',\mathbb{Z})_{\mathrm{tr}}.$$

From [175, Thm. 3.3] we infer that S and S' are derived equivalent. Choosing one such equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(S') \cong \mathrm{D}^{\mathrm{b}}(S)$ yields an isomorphism

$$\Phi\colon M^{S'}_{\sigma'}(v') \cong M^S_{\Phi.\sigma'}(\Phi^{\mathrm{H}}(v'))$$

where the latter variety is a moduli space of stable objects on S. □

The proof of the corollary also shows the following.

Corollary B.9.7. *For two smooth moduli spaces $M^S_\sigma(v)$ and $M^{S'}_{\sigma'}(v')$ of stable objects on projective K3 surfaces S and S' with $\mathrm{D}^{\mathrm{b}}(M^S_\sigma(v)) \cong \mathrm{D}^{\mathrm{b}}(M^{S'}_{\sigma'}(v'))$ we have $\mathrm{D}^{\mathrm{b}}(S) \cong \mathrm{D}^{\mathrm{b}}(S')$. Furthermore, S and S' are derived equivalent if and only if their Hilbert schemes $S^{[n]}$ and $S'^{[n]}$ are derived equivalent.*

Proof. The first part follows from the above and that derived equivalent K3 surfaces have derived equivalent Hilbert schemes was proven in [185, Prop. 8]. □

B.9.3. Hilbert schemes

We specialize to elliptic K3 surfaces S with a section and their Hilbert schemes. Recall that an elliptic K3 surface S has a section if and only if $U \subset \mathrm{NS}(S)$ [100, Rem. 11.1.4]. Theorem B.7.4 allows us to determine in this situation the image of the representation $\rho^{\tilde{\mathrm{H}}}$ up to finite index.

Theorem B.9.8. *For the Hilbert scheme $S^{[n]}$ of a K3 surface with $U \subset \mathrm{NS}(S)$ the image $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$ of the representation $\rho^{\tilde{\mathrm{H}}}$ satisfies*

$$\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}}) \subset \mathrm{Im}(\rho^{\tilde{\mathrm{H}}}) \subset \mathrm{Aut}(\Lambda_{S^{[n]}}).$$

The group $\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}})$ is the group of all Hodge isometries with real spinor norm one which act via ±id on the discriminant group.

Proof. Let $\gamma \in \mathrm{Aut}^+(\Lambda_{S^{[n]}})$ be a Hodge isometry with real spinor norm one which acts trivially on the discriminant group. We want to show $\gamma \in (Im)(\rho^{\tilde{\mathrm{H}}})$.

For line bundles $\mathcal{L} \in \mathrm{Pic}(S^{[n]})$ the auto-equivalence $\mathsf{M}_{\mathcal{L}}$ given by tensoring with $\mathcal{L}$ as well as the equivalence $\phi_{[n]}(\mathsf{ST}_{\mathcal{O}_S})$ from Proposition B.7.2 are contained in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S^{[n]}))$. The assumption $U \subset \mathrm{NS}(S)$ implies that $\Lambda_{S^{[n]},\mathrm{alg}}$ contains two copies of the hyperbolic plane U. The elements $\tilde{\delta}$ and $\gamma(\tilde{\delta})$ are both contained in $\Lambda_{S^{[n]},\mathrm{alg}}$ and have the same self-pairing as well as divisibility. As explained in the proof of Proposition B.8.2 using [82, Sec. 3] we conclude that there exists a derived equivalence $\Phi \in \mathrm{Aut}(\mathrm{D^b}(S^{[n]}))$ whose induced action $\Phi^{\tilde{\mathrm{H}}}$ is trivial on the discriminant group, has real spinor norm one and sends $\gamma(\tilde{\delta})$ to $\tilde{\delta}$, i.e.

$$\Phi^{\tilde{\mathrm{H}}} \circ \gamma(\tilde{\delta}) = \tilde{\delta}.$$

In particular, the isometry $\Phi^{\tilde{\mathrm{H}}} \circ \gamma$ restricts to a Hodge isometry of

$$\tilde{\delta}^{\perp} = B_{-\delta/2}(\mathrm{H}^2(S,\mathbb{Z})) \subset \Lambda_{S^{[n]}}$$

with real spinor norm one. Using [105, Cor. 3] there is an auto-equivalence $\eta \in \mathrm{Aut}(\mathrm{D^b}(S))$ such that

$$B_{-\delta/2} \circ \Phi^{\tilde{\mathrm{H}}} \circ \gamma \circ B_{\delta/2}$$

restricted to a Hodge isometry of $\tilde{\mathrm{H}}(S,\mathbb{Z})$ agrees with $\eta^{\tilde{\mathrm{H}}}$. Theorem B.7.4 implies that $\Phi^{\tilde{\mathrm{H}}}\circ\gamma$ or $-(\Phi^{\tilde{\mathrm{H}}}\circ\gamma)$ lies in $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$. As the shift functor [1] acts as $-\mathrm{id}$, we conclude that $\Phi^{\tilde{\mathrm{H}}}\circ\gamma \in \mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$ and, therefore, $\gamma \in \mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$.

Hence, we have proven that all Hodge isometries with real spinor norm one which act trivially on the discriminant lattice are contained in $\mathrm{Im}(\rho^{\tilde{\mathrm{H}}})$. The assertion now follows from Proposition B.7.1 which yields an isometry acting as $-\mathrm{id}$ on the discriminant group. □

Proposition B.9.9. *Let X be a projective $\mathrm{K3}^{[n]}$-type hyper-Kähler manifold such that $\mathrm{D^b}(X) \cong \mathrm{D^b}(S^{[n]})$ for a K3 surface S with $U \subset \mathrm{NS}(S)$. Then X and $S^{[n]}$ are birational.*

Proof. The derived equivalence yields a Hodge isometry

$$\varphi \colon \Lambda_X \cong \Lambda_{S^{[n]}}$$

which by [82, Prop. 3.3] and $U \subset \mathrm{NS}(S)$ we can postcompose by a Hodge isometry to assume $\varphi(\beta) = \beta$. Therefore the preimage of $\tilde{\delta} \in \Lambda_{S^{[n]},\mathrm{alg}}$ under the isometry φ must be of the form

$$\gamma + c(2n-2)\beta \in \Lambda_{X,\mathrm{alg}}$$

for some $c \in \mathbb{Z}$ and $\gamma \in \mathrm{H}^{1,1}(X,\mathbb{Z})$ of divisibility $2n-2$ with $b(\gamma,\gamma) = 2-2n$. We can choose the isometry (B.5.1) for X in such a way that γ maps to δ. With this choice the image of $\tilde{\alpha}$ under φ is of the form $B_{\mu}(\tilde{\alpha})$ for $\mu \in \mathrm{H}^2(S^{[n]},\mathbb{Z})_{\mathrm{alg}}$.

Indeed, since $\tilde{b}(\tilde{\alpha},\beta) = -1$ we must have

$$\tilde{b}(\varphi(\tilde{\alpha}),\varphi(\beta)) = \tilde{b}(\varphi(\tilde{\alpha}),\beta) = -1$$

and similarly $\tilde{b}(\varphi(\tilde{\alpha}),\tilde{\delta}) = 0$. Using the orthogonal decomposition

$$\Lambda_{S^{[n]}} \cong (\mathbb{Z}\tilde{\alpha} \oplus \mathbb{Z}\beta) \oplus^{\perp} \mathbb{Z}\tilde{\delta} \oplus^{\perp} \mathrm{H}^2(S,\mathbb{Z})$$

we see that $\varphi(\tilde{\alpha})$ is of the form

$$\varphi(\tilde{\alpha}) = \tilde{\alpha} + \mu + d\beta$$

for some $\mu \in \mathrm{H}^2(S,\mathbb{Z})$ and $d \in \mathbb{Z}$. As φ is a Hodge isometry, we furthermore have $\mu \in \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{alg}}$ and $b(\mu,\mu) = 2d$ which implies $\varphi(\tilde{\alpha}) = B_{\mu}(\tilde{\alpha})$. Postcomposing φ with $B_{-\mu}$ and using (B.5.2) we obtain a Hodge isometry

$$\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(S^{[n]},\mathbb{Z}). \tag{B.9.1}$$

Corollary B.9.6 implies that X is a moduli space $M_\sigma^S(v)$ of stable objects on S. Moreover, from (B.9.1) and the lemma further below we infer that $M_\sigma^S(v)$ is a fine moduli space, i.e. there exists $w \in \tilde{\mathrm{H}}(S,\mathbb{Z})_{\mathrm{alg}}$ such that $b(v,w) = 1$. Invoking again [82, Prop. 3.3] we see that there exists a Hodge isometry $\gamma \in \mathrm{O}^+(\tilde{\mathrm{H}}(S,\mathbb{Z}))$ such that $\gamma(v) = (1,0,1-n)$. The assertion follows now from [137, Cor. 9.9]. ☐

In [4, Thm. B] the authors found an example of derived equivalent hyper-Kähler manifolds such that their second integral cohomology is not Hodge isometric. In particular, the above proposition does not always hold. As granted by Theorem B.9.2 their $\mathrm{K3}^{[n]}$ lattices are Hodge isometric, see also Remark B.10.5.

Summarising and using [86] we have for a projective $\mathrm{K3}^{[n]}$-type hyper-Kähler manifold X and an elliptic K3 surface S with section: X and $S^{[n]}$ are birational if and only if $\mathrm{H}^2(X,\mathbb{Z})$ is Hodge isometric to $\mathrm{H}^2(S^{[n]},\mathbb{Z})$ if and only if $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(S^{[n]})$.

We finish the section with the following result used in the above proof. We will need some lattice theory and refer once more to [100, Sec. 14] for notations and results. Recall that a moduli space $M_\sigma^S(v)$ of stable sheaves or objects on a K3 surface S is *fine* if there exists a universal family $\mathcal{E}$ on $M_\sigma^S(v) \times S$. This is equivalent to the existence of some $w \in \tilde{\mathrm{H}}(S,\mathbb{Z})_{\mathrm{alg}}$ such that $\tilde{b}(v,w) = 1$, see [100, Sec. 10.2.2].

Lemma B.9.10. *Let M and M' be smooth moduli spaces of stable objects on a projective K3 surface S such that* $\mathrm{NS}(M)$ *and* $\mathrm{NS}(M')$ *are isometric with respect to the BBF pairing. Then M is a fine moduli space if and only if M' is.*

Proof. Let $N \subset L$ be a saturated sublattice of an even lattice $(L, (\_,\_))$, i.e. L/N is torsion-free. Consider the diagram

$$\begin{tikzcd}
& 0 \arrow[r] & N^\perp \arrow[r] \arrow[d] & N^\perp \arrow[r] \arrow[d] & 0 \arrow[d] & \\
& 0 \arrow[r] & N \oplus N^\perp \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & N \arrow[r] \arrow[d] & N^\vee \arrow[r] \arrow[d] & A(N) \arrow[r] \arrow[d] & 0 \\
& & 0 \arrow[r] & P \arrow[r] & P \arrow[r] & 0.
\end{tikzcd} \tag{B.9.2}$$

Here, $N^\vee := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ is the dual lattice, $N \to N^\vee$ and $L \to N^\vee$ denote the natural maps $v \mapsto (x \mapsto (x,v))$, $A(N)$ is the discriminant group of N, and K and P denote the cokernel of the corresponding morphisms.

As recalled above, for a moduli space $M = M_\sigma^S(v)$ we have

$$\mathrm{H}^2(M,\mathbb{Z}) \cong v^\perp \subset \tilde{\mathrm{H}}(S,\mathbb{Z}).$$

In particular,

$$\mathrm{NS}(M) \oplus \mathbb{Z}v \subset \tilde{\mathrm{H}}(S,\mathbb{Z})_{\mathrm{alg}}$$

is an orthogonal decomposition of a finite index sublattice of $\tilde{\mathrm{H}}(S,\mathbb{Z})_{\mathrm{alg}}$.

Let us assume that $M = M_\sigma^S(v)$ is fine and apply diagram (B.9.2) for $L = \tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}}$ and $N = \mathbb{Z}v$. The moduli space M being fine is equivalent to surjectivity of the map

$$\tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}} \to (\mathbb{Z}v)^\vee, \quad x \mapsto (kv \mapsto \tilde{b}(x, kv)).$$

Hence, in our situation we have $P \cong 0$ and, therefore, $K \cong A(N) \cong \mathbb{Z}/(2n-2)\mathbb{Z}$ for $2n$ the dimension of M. Using [100, Eq. (0.2)] we find

$$\mathrm{disc}(\mathrm{NS}(M)) = |K|^2 \cdot \mathrm{disc}(\tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}})/\mathrm{disc}(\mathbb{Z}v) = (2n-2) \cdot \mathrm{disc}(\tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}}) \qquad \text{(B.9.3)}$$

Let us now consider $M' = M_{\sigma'}^S(v')$ and inspect diagram (B.9.2) for $L = \tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}}$ and $N = \mathbb{Z}v'$ such that $K = L/(N \oplus N^\perp)$. We employ again [100, Eq. (0.2) in Ch. 14] and find

$$\mathrm{disc}(\mathrm{NS}(M')) = |K|^2 \cdot \mathrm{disc}(\tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}})/(2n-2). \qquad \text{(B.9.4)}$$

By assumption, $\mathrm{NS}(M)$ and $\mathrm{NS}(M')$ are isometric, thus $\mathrm{disc}(\mathrm{NS}(M)) = \mathrm{disc}(\mathrm{NS}(M'))$. Combining (B.9.3) and (B.9.4) we find $|K| = 2n-2$. In particular, in the situation $L = \tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}}$ and $N = \mathbb{Z}v'$ we find that $K \cong A(N)$ and, therefore, $P \cong 0$ in (B.9.2). This implies that

$$\tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}} \to (\mathbb{Z}v')^\vee, x \mapsto (kv' \mapsto \tilde{b}(x, kv'))$$

is surjective which shows that M' is a fine moduli space as well. □

We remark that the proof also applies for non-fine moduli spaces M and M'. That is, in general, there always exists Brauer classes $\alpha, \alpha' \in \mathrm{Br}(S)$ such that α respectively α' twisted universal families exist over $M \times S$ respectively $M' \times S$. The proof then shows that $\mathrm{ord}(\alpha) = \mathrm{ord}(\alpha')$ if $\mathrm{NS}(M) \cong \mathrm{NS}(M')$.

B.10. Further examples of derived equivalences

We complement the previous sections by integrating some derived equivalences of hyper-Kähler manifolds into the framework of the extended Mukai lattice.

B.10.1. Dimension four

We come back to Example B.4.19. Addington [2] as well as Markman–Mehrotra [140] considered the sheaf

$$\mathcal{E}^1 := \mathcal{E}xt^1_{\pi_{13}}(\pi_{12}^*(\mathcal{I}), \pi_{23}^*(\mathcal{I})) \in \mathrm{Coh}(S^{[2]} \times S^{[2]})$$

which is reflexive, of rank 2 and locally free away from the diagonal. Here, $\mathcal{I}$ is the universal ideal sheaf on $S \times S^{[2]}$ and π_{ij} are the projections from $S^{[2]} \times S \times S^{[2]}$. The Fourier–Mukai transform $\mathsf{FM}_{\mathcal{E}^1}$ with kernel $\mathcal{E}^1$ was shown to yield an auto-equivalence $\mathsf{FM}_{\mathcal{E}^1} \in \mathrm{Aut}(\mathrm{D^b}(S^{[2]}))$. More conceptually, the functor $\mathsf{FM}_{\mathcal{I}}$ is shown to be a spherical functor with $\mathcal{E}^1[1]$ the corresponding twist auto-equivalence.

Proposition B.10.1. *The equivalence* $\mathsf{FM}_{\mathcal{E}^1}$ *acts on the extended Mukai lattice via* $-s_v$*, where* v *is the vector* $\tilde{\alpha} + \beta$*.*

Proof. One way to prove the assertion is to use general results on the action of the twist equivalence associated to a spherical functor [2, Sec. 1.4]. Instead, we will calculate directly the images of line bundles using the definition of the twist auto-equivalence associated to a spherical functor.

The relative Ext complex

$$\mathcal{E} := \mathcal{E}\mathit{xt}^{\bullet}_{\pi_{13}}(\pi_{12}^*(\mathcal{I}), \pi_{23}^*(\mathcal{I})) = \pi_{13*}(\pi_{12}^*(\mathcal{I}^\vee) \otimes \pi_{23}^*(\mathcal{I})) \in \mathrm{D}^{\mathrm{b}}(S^{[2]} \times S^{[2]})$$

describes (up to the shift [2]) the composition of the right adjoint of

$$\mathsf{FM}_{\mathcal{I}} \colon \mathrm{D}^{\mathrm{b}}(S^{[2]}) \to \mathrm{D}^{\mathrm{b}}(S)$$

with $\mathsf{FM}_{\mathcal{I}}$ and sits in a distinguished triangle

$$\mathcal{E}^1[-1] \to \mathcal{E} \to \mathcal{O}_\Delta[-2]$$

in $\mathrm{D}^{\mathrm{b}}(S^{[2]} \times S^{[2]})$. This yields the identity

$$[\mathsf{FM}_{\mathcal{E}^1}(\mathcal{L}_2)] = [\mathcal{L}_2] - [\mathsf{FM}_{\mathcal{E}}(\mathcal{L}_2)] = [\mathcal{L}_2] - [\mathsf{FM}_{\mathcal{I}}(\mathsf{FM}_{\mathcal{I}^\vee}(\mathcal{L}_2))]$$

in topological K-theory for all topological line bundles $\mathcal{L}$ on S.

There is a natural short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{S \times S^{[2]}} \to \mathcal{O}_{\mathcal{Z}} \to 0$$

on $S \times S^{[2]}$, where $\mathcal{Z} \subset S \times S^{[2]}$ is the universal subscheme, which we can dualize to obtain the distinguished triangle

$$\omega_{\mathcal{Z}}[-2] \to \mathcal{O}_{S \times S^{[2]}} \to \mathcal{I}^\vee$$

in $\mathrm{D}^{\mathrm{b}}(S \times S^{[2]})$. From these sequences we obtain the identities

$$\begin{aligned} [\mathsf{FM}_{\mathcal{I}^\vee}(\mathcal{L}_2)] &= \chi(\mathcal{L}_2)[\mathcal{O}_S] - \chi(\mathcal{L})[\mathcal{L}], \\ [\mathsf{FM}_{\mathcal{I}}(\mathcal{O}_S)] &= 2[\mathcal{O}_{S^{[2]}}] - [\mathcal{O}_{S^{[2]}}] - [\mathcal{O}_{S^{[2]}}(-\delta)], \\ [\mathsf{FM}_{\mathcal{I}}(\mathcal{L})] &= \chi(\mathcal{L})[\mathcal{O}_{S^{[2]}}] - [\mathcal{L}^{[2]}] \end{aligned}$$

in topological K-theory, where $\mathcal{L}^{[2]} = \mathsf{FM}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{L})$ is the tautological rank 2 bundle associated to $\mathcal{L}$ and the second identity is a special case of the third one using $\mathcal{O}_S^{[2]} \cong \mathcal{O}_{S^{[2]}} \oplus \mathcal{O}_{S^{[2]}}(-\delta)$. The class of a point p is sent to a sheaf of rank 2. By an analogous argument to Lemma B.4.2 using the object $k(x)$ one concludes $\epsilon(\mathsf{FM}^{\tilde{\mathrm{H}}}_{\mathcal{E}^1}) = 1$.

To finish the proof we need to use the above to calculate how $\mathsf{FM}_{\mathcal{E}^1}$ acts on the extended Mukai lattice $\tilde{\mathrm{H}}(S^{[2]}, \mathbb{Q})$. We have $[\mathsf{FM}_{\mathcal{E}^1}(\mathcal{O}_{S^{[2]}})] = [\mathcal{O}_{S^{[2]}}(-\delta)]$ as well as $[\mathsf{FM}_{\mathcal{E}^1}(\mathcal{O}_{S^{[2]}}(-\delta))] = [\mathcal{O}_{S^{[2]}}]$, since spherical functors always induce involutions on cohomology. Applying extended Mukai vectors to this equality we find $\mathsf{FM}^{\tilde{\mathrm{H}}}_{\mathcal{E}^1}(\tilde{v}(\mathcal{O}_{S^{[2]}})) = \tilde{v}(\mathcal{O}_{S^{[2]}}(-\delta))$ and vice versa.

For a general topological line bundle $\mathcal{L}$ we find

$$[\mathsf{FM}_{\mathcal{E}^1}(\mathcal{L}_2)] = [\mathcal{L}_2] - \chi(\mathcal{L}_2)([\mathcal{O}_{S^{[2]}}] - [\mathcal{O}_{S^{[2]}}(-\delta)]) - \chi(\mathcal{L})^2[\mathcal{O}_{S^{[2]}}] + \chi(\mathcal{L})[\mathcal{L}^{[2]}].$$

If $b(\mathrm{c}_1(\mathcal{L}), \mathrm{c}_1(\mathcal{L})) = \ell$, then $\chi(\mathcal{L}) = \ell/2 + 2$ and $\chi(\mathcal{L}_2) = \ell^2/8 + 5\ell/2 + 3$, see [68, Lem. 5.1]. Moreover, $\mathcal{L}^{[2]}$ is a bundle of rank two and $\mathrm{c}_1(\mathcal{L}^{[2]}) = \mathrm{c}_1(\mathcal{L}) - \delta$. Taking extended Mukai vectors an explicit calculation shows that $\mathsf{FM}^{\tilde{\mathrm{H}}}_{\mathcal{E}^1}$ agrees with $-s_v$ for $v = \tilde{\alpha} + \beta$. ☐

Thus, the functors $\mathsf{FM}_{\mathcal{E}^1}$ and $\phi_{[2]}(\mathsf{ST}_{\mathcal{O}_S})$ induce the same isometry on the extended Mukai lattice and therefore also on the whole cohomology.

For $S^{[2]}$ there are other auto-equivalences given as the twist of a spherical functor. One example is Horja's EZ-spherical twist [92]. The exceptional divisor $i\colon \mathbb{P}(\Omega^1_S) \cong E \hookrightarrow S^{[2]}$ fibres over the K3 surface $\pi\colon E \to S$. One obtains the spherical functor $i_*(\pi^*(\_))\colon \mathrm{D^b}(S) \to \mathrm{D^b}(S^{[2]})$ and an auto-equivalence $T_{i_*\pi^*} \in \mathrm{Aut}(\mathrm{D^b}(S^{[2]}))$ characterized for $\mathcal{F} \in \mathrm{D^b}(S^{[2]})$ by the distinguished triangle

$$i_*\pi^*\pi_*i^!(\mathcal{F}) \to \mathcal{F} \to T_{i_*\pi^*}(\mathcal{F}).$$

Proposition B.10.2. *The auto-equivalence $T_{i_*\pi^*}$ acts on the extended Mukai lattice via the isometry $-s_v$ for the vector $v = \tilde{\delta} + \beta$.*

Proof. We employ [2, Sec. 2.4]. Lemma B.4.2 gives once more $\epsilon(T^{\tilde{\mathrm{H}}}_{i_*\pi^*}) = 1$. For $\mathcal{L}$ a line bundle on S one easily obtains

$$i_*\pi^*(\mathcal{L}^{\otimes 2}) \cong \mathcal{L}_2|_E.$$

This means that the classes $[\mathcal{L}_2] - [\mathcal{L}_2 \otimes \mathcal{O}_{S^{[2]}}(-E)]$ in $K^0_{\mathrm{top}}(S^{[2]})$ are being multiplied by -1 under the action of $T_{i_*\pi^*}$ and their orthogonal complement is left invariant. We have an equality

$$2(v(\mathcal{L}_2) - v(\mathcal{L}_2 \otimes \mathcal{O}_{S^{[2]}}(-E)) = T(\tilde{v}(\mathcal{L}_2)^2 - \tilde{v}(\mathcal{L}_2 \otimes \mathcal{O}_{S^{[2]}}(-E))^2)$$

in $\mathrm{SH}(S^{[2]}, \mathbb{Q})$. A computation finishes the proof. ☐

Alternatively, one could have proven the proposition using [126, Thm. 4.26] and Proposition B.7.1.

Remark B.10.3. All cohomological involutions we encountered (Proposition B.7.1, Proposition B.7.2, Proposition B.10.1 and Proposition B.10.2) had an extra $-\mathrm{id}$ in the case n even which is due to the sign convention from Section B.2.3.

B.10.2. Relative Poincaré

In [4] the authors study derived equivalences between certain moduli spaces of stable sheaves on K3 surfaces. Let S be a very general projective K3 surface with polarization H of degree $2g-2$ and consider the moduli spaces of stable sheaves

$$M^S_H(0, 1, d+1-g).$$

One can equivalently consider these varieties as the relative compactified Jacobians $\overline{\mathrm{Pic}}^d := \overline{\mathrm{Pic}}^d(\mathcal{C}/\mathbb{P}^g)$ of degree d of the universal curve

$$\mathcal{C} \to \mathbb{P}^g = |H|.$$

In [4, Prop. 3.1] following Arinkin [9] the authors construct a relative (twisted) Poincaré sheaf $\mathcal{P}_{dd'}$ on

$$\overline{\mathrm{Pic}}^d \times_{\mathbb{P}^g} \overline{\mathrm{Pic}}^{d'}$$

inducing a (twisted) derived equivalence. For simplicity we will consider the untwisted case $d = d' = 0$ and denote $\mathcal{P} := \mathcal{P}_{00}$, $M := M^S_H(0, 1, 1-g)$ together with the Lagrangian fibration

$$\pi\colon M \to \mathbb{P}^g.$$

We have the well-known Hodge isometry

$$\mathrm{H}^2(M,\mathbb{Z}) \cong (0,1,1-g)^\perp \subset \tilde{\mathrm{H}}(S,\mathbb{Z})$$

and the algebraic part of $\mathrm{H}^2(M,\mathbb{Z})$ has a basis λ, f with intersection form

$$\begin{pmatrix} 2g-2 & 2 \\ 2 & 0 \end{pmatrix}$$

where f is the first Chern class of $\mathcal{O}_{\mathbb{P}^g}(1)$ pulled back to M.

Let us determine the action of $\mathcal{P}$ on the extended Mukai lattice. We will consider the case g even, the case g even is similar. The skyscraper sheaf $k(x)$ of a point $x \in A \subset M$ contained in a smooth fibre A of the Lagrangian fibration is by definition sent under $\mathsf{FM}_{\mathcal{P}}$ to a degree 0 line bundle $\mathcal{L}$ on the abelian variety A whose Mukai vector is of the form $v(\mathcal{L}) = f^g \in \mathrm{SH}(X,\mathbb{Q})$. The duality property of the Poincaré sheaf [9, Sec. 6.2] implies that $\mathcal{L}$ is sent under $\mathsf{FM}_{\mathcal{P}}$ to the object $k(x^\vee)[-g]$, where $x^\vee \in A$ parametrizes $\mathcal{L}^\vee$. This gives

$$\beta \mapsto f, \quad f \mapsto \beta.$$

Moreover, the Lagrangian fibration $M \to \mathbb{P}^g$ admits a section $\mathbb{P}^g \hookrightarrow M$ given by the trivial line bundle on each fibre. Using Remark B.7.3 and $\int_X [A][\mathbb{P}^g] = 1$ we see that the Mukai vector of $\mathcal{O}_{\mathbb{P}^g} \in \mathrm{D}^{\mathrm{b}}(M)$ satisfies

$$\overline{v(\mathcal{O}_{\mathbb{P}^g})} = T\left(\frac{(\frac{1}{2}\lambda - \frac{g+1}{2}f + \frac{g+1}{2}\beta)^g}{g!}\right) \in \mathrm{SH}(M,\mathbb{Q}).$$

The definition of $\mathcal{P}$ [4, Eq. (3.1)] yields that $\mathsf{FM}_{\mathcal{P}}$ sends $\mathcal{O}_{\mathbb{P}^g}$ to a line bundle $\mathcal{M} \in \mathrm{Pic}(M)$. The duality property of $\mathcal{P}$ for families of curves [9, Eq. (7.8)] implies that $\mathcal{M}$ is mapped under $\mathsf{FM}_{\mathcal{P}}$ to $\mathcal{O}_{\mathbb{P}^g}[-g] \otimes \mathcal{K}$. Here, $\mathcal{K}$ is the line bundle $\pi^* \det(\mathrm{R}^1\pi_*\mathcal{O}_M)$ which, using [142, Thm. 1.3], has first Chern class $-(g+1)f$. Let us denote

$$h = -\frac{\lambda}{2} + \frac{g-1}{4}f + \frac{g+1}{2}\beta \in \tilde{\mathrm{H}}(M,\mathbb{Q})$$

and note that f and $-h$ span a rational hyperbolic plane. Summarizing the above discussion and using the extended Mukai vector we have the following.

Proposition B.10.4. *The equivalence* $\mathsf{FM}_{\mathcal{P}}$ *acts on the extended Mukai lattice via*

$$\alpha \mapsto h, \quad h \mapsto \alpha, \quad \beta \mapsto f, \quad f \mapsto \beta.$$

Expressed differently, the derived equivalence $\mathsf{FM}_{\mathcal{P}}$ exchanges the two rational hyperbolic planes given by α, β and f, h.

Remark B.10.5. In [4] it was observed that the case $d = 0$ and $d' = g-1$ yields an example of derived equivalent hyper-Kähler manifolds M and M' such that their second integral cohomology groups are not isometric [4, Thm. B]. The intersection form on $\mathrm{NS}(M)$ has discriminant -4 whereas the lattice $\mathrm{NS}(M')$ is isometric to the hyperbolic plane. Let us denote the generators of $\mathrm{NS}(M')$ inside $\mathrm{H}^2(M',\mathbb{Z})$ by e', f' such that $b(e',f') = 1$ and f'

denotes again the fibre class. As above one can show that the derived equivalence induces an isometry $\tilde{\mathrm{H}}(M, \mathbb{Q}) \cong \tilde{\mathrm{H}}(M', \mathbb{Q})$ given by

$$\beta \mapsto f', \quad f \mapsto \beta, \quad \alpha \mapsto -e' + \frac{g+1}{2}\beta, \quad h \mapsto \alpha.$$

This is compatible with the $\mathrm{K3}^{[n]}$ lattices, i.e. the above induces a Hodge isometry

$$\Lambda_M \cong \Lambda_{M'}$$

in accordance with Theorem B.9.2. Geometrically the variety M admits a section whereas the variety M' admits a line bundle with first Chern class e' which restricts to a principal polarization on each fibre. The derived equivalence $\mathcal{P}_{0_{g-1}}$ relates these different geometric properties.

C. Atomic objects on hyper-Kähler manifolds

Abstract. We introduce and study the notion of atomic sheaves and complexes on higher-dimensional hyper-Kähler manifolds and show that they share many of the intriguing properties of simple sheaves on K3 surfaces. For example, we prove formality of the dg algebra of derived endomorphisms for stable atomic bundles. We further demonstrate the characteristics of atomic objects by studying atomic Lagrangian submanifolds. In the appendix, we prove non-existence results for spherical objects on hyper-Kähler manifolds.

C.1. Introduction

C.1.1. K3 surfaces and Mukai vectors

Since the seminal work of Mukai [155], simple bundles on a K3 surface X and, more generally, simple complexes in its bounded derived category $\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ have been studied intensively. One is therefore led to look for an analogue of these objects on higher-dimensional compact hyper-Kähler manifolds.

Again motivated by the case of K3 surfaces, we introduced in [25] the notion of an (extended) Mukai vector taking values in the (extended) Mukai lattice

$$\tilde{\mathrm{H}}(X,\mathbb{Q}) := \mathrm{H}^2(X,\mathbb{Q}) \oplus \mathbb{Q}^{\oplus 2}$$

for certain objects $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ on hyper-Kähler manifolds X. In this paper, we consider a natural refinement of this construction which leads to the notion of atomic sheaves and complexes. It turns out that these objects possess many of the properties of simple sheaves and complexes on K3 surfaces.

C.1.2. Cohomology and LLV algebra

From now on, X will denote a compact irreducible hyper-Kähler manifold of dimension $2n$. The second cohomology $\mathrm{H}^2(X,\mathbb{Q})$ of a hyper-Kähler manifold is endowed with the Beauville–Bogomolov–Fujiki (BBF) form $q = q_X$ making it into a quadratic space. Moreover, the full cohomology $\mathrm{H}^*(X,\mathbb{Q})$ is naturally a module for the Looijenga–Lunts–Verbitsky (LLV) Lie algebra $\mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ generated by all $\mathfrak{sl}_2$-triples for all elements in $\mathrm{H}^2(X,\mathbb{Q})$ having the Hard Lefschetz property, see [81, 130, 207] for more details. This leads naturally to a decomposition

$$\mathrm{H}^*(X,\mathbb{Q}) \cong \bigoplus_{\lambda} V_{\lambda} \tag{C.1.1}$$

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of the cohomology into irreducible $\mathfrak{g}(X)$-representations. The most prominent irreducible representation is the Verbitsky component $\mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{H}^*(X, \mathbb{Q})$ which is the subalgebra generated by $\mathrm{H}^2(X, \mathbb{Q})$.

C.1.3. Atomic objects

Recall the definition of the Mukai vector

$$v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\mathsf{td}^{1/2} \in \mathrm{H}^*(X, \mathbb{Q})$$

for a sheaf $\mathcal{E} \in \mathrm{Coh}(X)$ or an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$, where $\mathsf{td}^{1/2} = \sqrt{\mathsf{td}}$ is the formal root of the Todd class $\mathsf{td} \coloneqq \mathsf{td}_X$ of X. The idea in [25, Sec. 4] was to compare the projection $v(\mathcal{E})_{\mathrm{SH}}$ of the Mukai vector $v(\mathcal{E})$ of an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ to the Verbitsky component

$$(\_)_{\mathrm{SH}} \colon \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{SH}(X, \mathbb{Q})$$

with some vector $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ by means of the short exact sequence

$$0 \to \mathrm{SH}(X, \mathbb{Q}) \to \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})) \to \mathrm{Sym}^{n-2}(\tilde{\mathrm{H}}(X, \mathbb{Q})) \to 0.$$

This definition has the disadvantage that it only concerns the Verbitsky component and ignores all other irreducible representations of the LLV algebra $\mathfrak{g}(X)$, but for many applications, such as in [25], this is sufficient.

Instead of only focusing on the projection to the Verbitsky component, one can consider more generally the decomposition

$$v(\mathcal{E}) = \sum_{\lambda} v(\mathcal{E})_{\lambda} \tag{C.1.2}$$

obtained from the decomposition (C.1.1). In particular, one may demand a compatibility of the Mukai vector $v(\mathcal{E})$ of $\mathcal{E}$ not only with its projection to the Verbitsky component, but with respect to the entire decomposition (C.1.2). This leads naturally to the central notion of this paper.

Definition C.1.1. A sheaf $\mathcal{E} \in \mathrm{Coh}(X)$ or an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is called *atomic* if there exists a non-zero vector $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ such that the annihilator Lie subalgebra $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ of the representation of $\mathfrak{g}(X)$ on $\mathrm{H}^*(X, \mathbb{Q})$ equals the annihilator Lie subalgebra $\mathrm{Ann}(\tilde{v}) \subset \mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ of the representation of $\mathfrak{g}(X)$ on $\tilde{\mathrm{H}}(X, \mathbb{Q})$.

Let us comment on the definition. First, every non-zero sheaf on a K3 surface is atomic. Moreover, a sheaf $\mathcal{E}$ being atomic is equivalent to $\mathrm{Ann}(\mathcal{E})$ having the largest possible dimension, see Proposition C.3.1 and Lemma C.3.7. This should be interpreted as its Mukai vector behaving just as in the case of K3 surfaces. As demonstrated in Proposition C.3.10 the property of being atomic is invariant under derived equivalences as well as deformations.

Furthermore, Definition C.1.1 recovers [25, Def. 4.16] when restricted to the Verbitsky component. That is, denoting by T the orthogonal projection to the isometric embedding $\mathrm{SH}(X, \mathbb{Q}) \hookrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$, the condition

$$v(\mathcal{E})_{\mathrm{SH}} \in \mathbb{Q}\langle T(\tilde{v}^n)\rangle$$

is by Proposition C.3.3 equivalent to the equality

$$\operatorname{Ann}(v(\mathcal{E})_{\mathrm{SH}}) = \operatorname{Ann}(\tilde{v}) \subset \mathfrak{g}(X).$$

In particular, as discussed in Section C.3.2, these objects possess a Mukai vector in $\widetilde{\mathrm{H}}(X,\mathbb{Q})$ which for a torsion-free atomic sheaf $\mathcal{E}$ is of the form $\mathrm{rk}(\mathcal{E})\alpha + \mathrm{c}_1(\mathcal{E}) + s\beta$ for some $s \in \mathbb{Q}$. Let us also remark that we show in Section C.3.1 that many summands in (C.1.2) must vanish for atomic objects. See Section C.3 for a thorough discussion of the definition.

C.1.4. Obstruction maps

One of the key results exploited throughout the whole paper is the relation and interplay for a sheaf or an object $\mathcal{E}$ between the (a priori topological) property of being atomic, (noncommutative) deformations parametrized by Hochschild cohomology $\mathrm{HH}^*(X)$ respectively polyvector fields $\mathrm{HT}^*(X)$, and its extension groups $\mathrm{Ext}^*(\mathcal{E},\mathcal{E})$. This relationship is established through the use of two so called obstruction maps, which we now elaborate on. The name obstruction maps refers to their appearance and application in deformation theory, see also Remark C.4.5.

We recall here

$$\mathrm{HT}^2(X) \coloneqq \mathrm{H}^2(X,\mathcal{O}_X) \oplus \mathrm{H}^1(X,\mathcal{T}_X) \oplus \mathrm{H}^0(X,\Lambda^2\mathcal{T}_X)$$

and refer to Section C.2.1 for a thorough definition of the ring of polyvector fields. To every object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ we associate a natural morphism

$$\mathrm{obs}_{\mathcal{E}} \colon \mathrm{HT}^2(X) \to \mathrm{H}^*(X,\Omega_X^*), \quad \mu \mapsto \mu \lrcorner v(\mathcal{E})$$

defined by contraction of vector fields. We call it the *cohomological obstruction map* for $\mathcal{E}$.

We have the first result.

Theorem C.1.2. *Let X be a hyper-Kähler manifold and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$. Then $\mathcal{E}$ is atomic if and only if the cohomological obstruction map* $\mathrm{obs}_{\mathcal{E}}$ *has a one-dimensional image.*

This result enables us to freely intertwine the representation theory of the LLV algebra with the (symplectic) geometry of vector fields on hyper-Kähler manifolds. We remark that Markman has obtained the if direction in the above theorem in [139, Thm. 6.13] under the extra assumption that $v(\mathcal{E})_{\mathrm{SH}} \neq 0$.

Next, to any $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ we can associate the natural homomorphism

$$\chi_{\mathcal{E}} \colon \mathrm{HH}^2(X) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$$

via evaluation at the natural transformation called the *obstruction map.* See Section C.2.1 for a brief recollection on the notions of Hochschild (co)homology. The map $\chi_{\mathcal{E}}$ parametrizes the obstruction to lifting the complex $\mathcal{E}$ to first order along the (noncommutative) first-order deformations given by $\mathrm{HH}^2(X)$ [204, Prop. 6.1]. For an element $\gamma \in \mathrm{HH}^2(X)$ we will often denote its image $\chi_{\mathcal{E}}(\gamma)$ as $\gamma_{\mathcal{E}}$. By [94] the following diagram

$$\begin{tikzcd} \mathrm{HH}^*(X) \arrow[r, "\chi_{\mathcal{E}}"] \arrow[d, "I^{\mathrm{HKR}}"'] & \mathrm{Ext}^*(\mathcal{E},\mathcal{E}) \\ \mathrm{HT}^*(X) \arrow[ur, "\lrcorner\exp(\mathrm{At}_{\mathcal{E}})"'] & \end{tikzcd} \tag{C.1.3}$$

commutes, where $\exp(\mathrm{At}_{\mathcal{E}})$ is the exponential of the *Atiyah class* $\mathrm{At}_{\mathcal{E}} \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_X^1)$ of $\mathcal{E}$ and $I^{\mathrm{HKR}} \colon \mathrm{HH}^*(X) \cong \mathrm{HT}^*(X)$ is the Hochschild–Konstant–Rosenberg (HKR) isomorphism. Markman [139] recently studied objects for which the obstruction map has a one-dimensional image. We will call such objects *1-obstructed.* The following result is a strengthening of [139, Thm. 6.13 (1)].

Theorem C.1.3. *If $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is a 1-obstructed object such that $v(\mathcal{E})$ is not annihilated by the LLV algebra $\mathfrak{g}(X)$, then $\mathcal{E}$ is atomic. In particular, 1-obstructed sheaves are atomic.*

We note that if $\mathcal{E}$ satisfies the conclusion of the theorem, i.e. if $\mathcal{E}$ is atomic, then its Mukai vector $v(\mathcal{E})$ satisfies the assumption in the theorem of not being annihilated by the LLV algebra, see Section C.4.2. Under a certain non-degeneracy condition for the Serre duality trace map, the implication that 1-obstructed objects are atomic holds unconditionally, see Conjecture E.

It is, however, not true that the converse implication always holds. As shown by Example C.4.4, there are vector bundles on K3 surfaces which are not 1-obstructed. However, for K3 surfaces, 1-obstructedness and atomicity are equivalent for simple sheaves and complexes. We show that under the above alluded to non-degeneracy condition of the Serre duality trace morphism restricted to the image of the obstruction map, this statement remains valid for simple atomic objects on higher-dimensional hyper-Kähler manifolds.

Theorem C.1.4. *If $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is a simple object satisfying Conjecture E, then $\mathcal{E}$ is 1-obstructed if and only if $\mathcal{E}$ is atomic.*

We want to emphasize that we view the property of being 1-obstructed as a (conjectural) feature of simple atomic objects and not vice versa.

C.1.5. Modular & projectively hyperholomorphic bundles and deformations

Stable vector bundles are the easiest examples of simple objects on K3 surfaces. On higher-dimensional hyper-Kähler manifolds, there exists the notion of (projectively) hyperholomorphic bundles due to Verbitsky [208]. Recently, O'Grady proposed the notion of modular sheaves and bundles in [172].

We discuss their relation and, in particular, how atomic sheaves and bundles fit into the picture. The discussion can be summarized by the following two results.

Proposition C.1.5. *Let $\mathcal{E}$ be a torsion-free atomic sheaf. Then $\mathcal{E}$ is modular.*

In particular, for torsion-free atomic sheaves the ample cone admits a wall and chamber decomposition similar to the case of K3 surfaces as proven in [172, Prop. 3.4].

In [139, Thm. 1.2], the author obtained a weaker form of the above result, where it is also assumed that the sheaf is reflexive as well as slope stable for ample classes in an open subcone of the ample cone. Our result does not require these assumptions and our proof is independent and shorter.

Proposition C.1.6. *Let $\mathcal{E}$ be a slope polystable atomic vector bundle. Then $\mathcal{E}$ is projectively hyperholomorphic.*

We will recall the relevant details on (projectively) hyperholomorphic bundles in Section C.5. However, quite intriguingly, the tangent bundle $\mathcal{T}_X$ on higher-dimensional hyper-Kähler manifolds, which is hyperholomorphic as well as modular, fails to be atomic, see Proposition C.8.3.

One remarkable property of stable bundles on K3 surfaces is their deformation behavior. We investigate the deformation theory of (poly)stable atomic bundles.

We obtain two results. From Theorem C.1.6 one can deduce that for stable atomic bundle $\mathcal{E}$ the associated projective bundle $\mathbb{P}(\mathcal{E})$ deforms over the whole moduli space which is the content of Proposition C.5.5. The other result is the following.

Theorem C.1.7. *Let $\mathcal{E}$ be an atomic slope stable vector bundle. Then the dg algebra $\mathrm{R}\mathcal{H}om(\mathcal{E}^{\oplus k}, \mathcal{E}^{\oplus k})$ is formal for any $k > 0$.*

More precisely, in Theorem C.6.1 we prove formality of the dg algebra of derived endomorphisms for the bigger class of projectively hyperholomorphic bundles. The above result then follows immediately from Proposition C.1.6. One consequence of this is that the local Kuranishi space of infinitesimal deformations is cut out by quadrics. For the details and further consequences for moduli spaces of stable sheaves we refer to Section C.6.

C.1.6. Lagrangians

It follows easily from the definitions that atomic sheaves $\mathcal{E}$ which are torsion must be skyscraper sheaves or supported on Lagrangian subvarieties. This raises the question which Lagrangian submanifolds $\iota\colon L \subset X$ can support atomic sheaves.

Theorem C.1.8. *Let $\iota\colon L \subset X$ be a connected Lagrangian submanifold. Then $\iota_*\mathcal{O}_L$ is atomic if and only if the restriction map $\iota^*\colon \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(L, \mathbb{Q})$ has a one-dimensional image and $\mathrm{c}_1(L) = \mathrm{c}_1(\mathcal{T}_L) \in \mathrm{Im}(\iota^*) \subset \mathrm{H}^2(L, \mathbb{Q})$.*

If one uses the interplay of (obstructions to) deformations and atomicity derived from Theorem C.1.2, the first condition in the above theorem controls the behaviour with respect to geometric deformations parametrized by $\mathrm{H}^1(X, \mathcal{T}_X)$ and the second condition controls Poisson deformations parametrized by $\mathrm{H}^0(X, \Lambda^2\mathcal{T}_X)$. For the special case of $\mathrm{K3}^{[2]}$-type hyper-Kähler manifolds, where only the Verbitsky component is present, this result was obtained in [139, Lem. 7.3].

We call submanifolds which satisfy one of the equivalent conditions from Theorem C.1.8 *atomic Lagrangians.* Since being atomic is stable under derived equivalences, we get many examples of atomic sheaves supported on atomic Lagrangians.

Theorem C.1.8 displays once more that atomic objects behave similarly to those on K3 surface. Namely, smooth Lagrangian submanifolds of K3 surfaces correspond to Riemann surfaces and are therefore either Fano, of Kodaira dimension zero, or have ample canonical bundle. This conclusion remains true for atomic Lagrangians, that is the canonical bundle ω_L of an atomic Lagrangian $L \subset X$ is also (anti-)ample or numerically trivial.

We also discuss the question of formality of the derived endomorphisms for the sheaf $\iota_*\mathcal{O}_L$ in Section C.7.4. Moreover, it follows from recent results of Mladenov [147] that for many simple sheaves on atomic Lagrangians the Ext algebra is of topological nature, that is, there is a ring isomorphism

$$\mathrm{Ext}^*(\iota_*\mathcal{O}_L, \iota_*\mathcal{O}_L) \cong \mathrm{H}^*(L, \mathbb{C}).$$

This implies, in particular, that the Ext algebra is graded-commutative. As is shown in Proposition C.7.7, this compares nicely with the case of simple objects $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(S)$ on K3 surfaces S, where we always have

$$\mathrm{Ext}^*(\mathcal{E},\mathcal{E}) \cong \mathrm{H}^*(C,\mathbb{C})$$

for some Riemann surface C. We expect this topological nature to remain true for simple atomic objects on higher dimensional hyper-Kähler manifolds, see also Conjecture F for a weaker version of this statement.

C.1.7. Spherical sheaves and objects

To study the interplay between the different obstruction maps alluded to in Section C.1.4, we study how the Mukai vector $v(\mathcal{E})$ of an object $\mathcal{E}$ forces restrictions on the Ext algebra $\mathrm{Ext}^*(\mathcal{E},\mathcal{E})$. We refine this study in the appendix, which is logically independent from the rest of the paper. The general structural result is Theorem C.A.2.

Recall that a sheaf or an object $\mathcal{E}$ is called spherical, if there is a ring isomorphism

$$\mathrm{Ext}^*(\mathcal{E},\mathcal{E}) \cong \mathrm{H}^*(S^{\dim X},\mathbb{C}).$$

One of the consequences of the above result is the following, which has been expected, but a proof has been missing in the literature.

Theorem C.1.9. *There exist no spherical sheaves on a hyper-Kähler manifold X of dimension greater than two. Moreover, if X is of* $\mathrm{K3}^{[n]}$ *with $n > 1$ or* OG10*-type, then* $\mathrm{D}^{\mathrm{b}}(X)$ *contains no spherical objects.*

In general, we show that spherical objects on hyper-Kähler manifolds, if existent, are severely restricted. For example, their Mukai vectors must be contained in a subspace of the subspace annihilated by the LLV algebra, see Remark C.A.6.

C.1.8. Organization of results

We provide in the next section results about Hochschild (co)homology, polyvector fields and the LLV algebra that we will employ throughout the paper.

In Section C.3 we deduce consequences and properties from Definition C.1.1 for atomic objects. The relation between atomic objects and the different obstruction maps is discussed in Section C.4.

The next two sections are devoted to the study of vector bundles on hyper-Kähler manifolds and their deformation theory. Section C.7 discusses the structure of atomic Lagrangians such as formality aspects, obstruction maps and Yoneda multiplication.

The last section discusses examples of atomic sheaves and complexes. We also discuss further properties of atomic objects such as an $\mathfrak{sl}_2$-action on its extension groups. In the appendix, we establish the above mentioned restriction results for spherical objects on higher-dimensional hyper-Kähler manifolds.

C.1.9. Relation to other work

We independently obtained the notion of atomic sheaves and complexes naturally from a thorough inspection of our work [25, Sec. 4].

In [139], Markman studies sheaves and complexes on hyper-Kähler manifolds whose obstruction map or cohomological obstruction map has a one-dimensional image. The notion of atomicity appears implicitely in [139, Thm. 6.13] and is related to the obstruction maps under the extra assumption $v(\mathcal{E})_{\mathrm{SH}} \neq 0$.

However, in [139] being atomic is seen as a consequence of (cohomologically) 1-obstructed objects. On the other hand, we see atomicity as the central notion. We show in Theorem C.1.2 that being atomic and having a one-dimensional cohomological obstruction map is equivalent, which, a posteriori, also strengthens some results of [139]. Nevertheless, we remark that [139] helped us in shaping our exposition and directing our attention.

As has been mentioned at a few places in the introduction, a few of our results have appeared in weaker forms in [139] for (cohomologically) 1-obstructed objects. It is the notion of atomicity and making use of the full force of the LLV algebra in combination with Theorem C.1.2 which allows us to give independent proofs of our stronger results which are more general and need less assumptions.

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Conventions

We will work throughout over the complex numbers.

C.2. Recollections

C.2.1. Hochschild (co)homology

We briefly recall the notions of Hochschild homology and cohomology and related results relevant for our purposes. For more details we refer to [58–60].

Let X be a smooth projective variety of dimension n. The Hochschild cohomology $\mathrm{HH}^*(X)$ and Hochschild homology $\mathrm{HH}_*(X)$ of X are defined as

$$\mathrm{HH}^*(X) \coloneqq \mathrm{Ext}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X), \quad \mathrm{HH}_*(X) \coloneqq \mathrm{Ext}^*_{X\times X}(\Delta_*\omega_X^{-1}[-n], \Delta_*\mathcal{O}_X)$$

with $\Delta\colon X \hookrightarrow X \times X$ the diagonal embedding. Composition of morphisms turns $\mathrm{HH}^*(X)$ into a graded ring and $\mathrm{HH}_*(X)$ into a module over $\mathrm{HH}^*(X)$. Elements in the Hochschild (co)homology can be interpreted as natural transformations and, therefore, be evaluated at elements $\mathcal{E} \in \mathrm{D^b}(X)$. The Hochschild–Konstant–Rosenberg (HKR) isomorphisms identify the Hochschild cohomology of X with the ring of polyvector fields

$$I^{\mathrm{HKR}} \colon \mathrm{HH}^*(X) \cong \mathrm{HT}^*(X) \coloneqq \bigoplus_{p+q=*} \mathrm{H}^q(X, \Lambda^p \mathcal{T}_X)$$

as well as the Hochschild homology of X with the de Rham cohomology

$$I_{\mathrm{HKR}} \colon \mathrm{HH}_*(X) \cong \mathrm{H}\Omega_*(X) \coloneqq \bigoplus_{q-p=*} \mathrm{H}^q(X, \Omega_X^p),$$

see [58, Cor. 4.2]. If these are twisted by the square root of the Todd class $\mathsf{td}^{1/2}$, the graded isomorphisms

$$I^{\mathrm{K}} \colon \mathrm{HH}^*(X) \xrightarrow{I^{\mathrm{HKR}}} \mathrm{HT}^*(X) \xrightarrow{\mathsf{td}^{-1/2} \lrcorner} \mathrm{HT}^*(X)$$

$$I_{\mathrm{K}} \colon \mathrm{HH}_*(X) \xrightarrow{I_{\mathrm{HKR}}} \mathrm{H}\Omega_*(X) \xrightarrow{\mathsf{td}^{1/2} \wedge} \mathrm{H}\Omega_*(X)$$

respect the ring and module structure [47]. We will often use implicitly the degeneration of the Hodge–de Rham spectral sequence to identify non gradedly $\mathrm{H}\Omega_*(X) \cong \mathrm{H}^*(X, \Omega_X^*) \cong \mathrm{H}^*(X, \mathbb{C})$.

Let now X be a hyper-Kähler manifold of dimension $2n$. The choice of a non-degenerate symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ yields a generator $\sigma^n \in \mathrm{H}\Omega_{-2n}(X)$ realizing $\mathrm{HH}_*(X)$ as a free $\mathrm{HH}^*(X)$-module of rank one [201, Lem. 2.5]. Moreover, the symplectic form induces an isomorphism $\sigma \colon \Omega_X^1 \cong \mathcal{T}_X$ such that the composite isomorphism

$$\mathrm{HH}^*(X) \xrightarrow{I^{\mathrm{K}}} \mathrm{HT}^*(X) \xrightarrow{\sigma} \mathrm{H}\Omega_*(X) \xrightarrow{\cong} \mathrm{H}^*(X, \mathbb{C}) \tag{C.2.1}$$

is a graded ring isomorphism, where the last isomorphism comes from the degeneration of the Hodge–de Rham spectral sequence.

For an object $\mathcal{E} \in \mathrm{D^b}(X)$ Căldăraru [59] introduced the Hochschild Chern character $\mathrm{ch}^{\mathrm{HH}}(\mathcal{E}) \in \mathrm{HH}_0(X)$. It is uniquely defined by satisfying the equality

$$\mathrm{Tr}_{X \times X}(\mu \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{Tr}_X(\mu_{\mathcal{E}}) \tag{C.2.2}$$

for all $\mu \in \mathrm{HH}^*(X)$, where $\mathrm{Tr}_{X \times X}$ and Tr_X are the trace morphisms on $X \times X$ and X obtained from the Serre duality pairing. It is shown in [58, Thm. 4.5] that the HKR isomorphism identifies the Hochschild Chern character with the classical Chern character, i.e. $I_{\mathrm{HKR}}(\mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{ch}(\mathcal{E}) \in \mathrm{H}^*(X, \mathbb{C})$. Therefore, we also have $I_{\mathrm{K}}(\mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = v(\mathcal{E}) \in \mathrm{H}^*(X, \mathbb{C})$.

C.2.2. Hyper-Kähler cohomology and LLV algebra

Let X be a hyper-Kähler manifold of complex dimension $2n$, i.e. a simply connected compact Kähler manifold such that $\mathrm{H}^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form. The second cohomology $\mathrm{H}^2(X, \mathbb{Z})$ possesses an integral primitive quadratic form $\mathrm{q} = \mathrm{q}_X$ called the *Beauville–Bogomolov–Fujiki (BBF)* form and has rank $b_2(X)$. We associate to X its *Mukai lattice*

$$(\tilde{\mathrm{H}}(X, \mathbb{Q}) \coloneqq \mathbb{Q}\alpha \oplus \mathrm{H}^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta, \tilde{\mathrm{q}})$$

which is a quadratic space with a grading and Hodge structure. More precisely, the quadratic form $\tilde{\mathrm{q}}$ restricts on $\mathrm{H}^2(X,\mathbb{Q})$ to the BBF form q and α and β are isotropic elements orthogonal to $\mathrm{H}^2(X,\mathbb{Q})$ and satisfy $\tilde{\mathrm{q}}(\alpha,\beta) = -1$. The elements α and β are of degree -2 and 2 respectively and carry the trivial rational Hodge structure. The space $\mathrm{H}^2(X,\mathbb{Q})$ has degree zero and carries the corresponding Tate twist of its usual Hodge structure. See [25, Sec. 2.2] for more details.

Looijenga–Lunts [130] and Verbitsky [207] introduced the *Looijenga–Lunts–Verbitsky (LLV)* algebra $\mathfrak{g}(X)$ naturally associated to the cohomology $\mathrm{H}^*(X,\mathbb{Q})$ of a hyper-Kähler manifold. For another account, see [81].

We denote by $h \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{Q}))$ the *cohomological grading operator* acting on $\mathrm{H}^k(X,\mathbb{Q})$ via $(k-2n)\mathrm{id}$. To an element $\omega \in \mathrm{H}^2(X,\mathbb{Q})$ we associate the operator $e_\omega = \omega \cup \_ \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{Q}))$ of cupping with ω. We say that ω has the *Hard Lefschetz property* if there exists an operator $\Lambda_\omega \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{Q}))$ such that $(e_\omega, h, \Lambda_\omega)$ forms an $\mathfrak{sl}_2$-triple.

The LLV algebra $\mathfrak{g}(X) \subset \mathrm{End}(\mathrm{H}^*(X,\mathbb{Q}))$ is the Lie subalgebra generated by all such $\mathfrak{sl}_2$-triples for all ω having the Hard Lefschetz property. The main result of Looijenga–Lunts and Verbitsky is then the Lie algebra isomorphism

$$\mathfrak{g}(X) \cong \mathfrak{so}(\tilde{\mathrm{H}}(X,\mathbb{Q})).$$

The $\mathfrak{g}(X)$-structure of $\tilde{\mathrm{H}}(X,\mathbb{Q})$ is defined by the conditions $e_\omega(\alpha) = \omega$, $e_\omega(\mu) = q(\omega,\mu)\beta$ and $e_\omega(\beta) = 0$ for all classes $\omega, \mu \in \mathrm{H}^2(X,\mathbb{Q})$.

Let $\mathrm{SH}(X,\mathbb{Q})$ be the *Verbitsky component*, i.e. the graded subalgebra of $\mathrm{H}^*(X,\mathbb{Q})$ generated by $\mathrm{H}^2(X,\mathbb{Q})$. Verbitsky [34, 207] proved the existence of a graded morphism $\psi\colon \mathrm{SH}(X,\mathbb{Q}) \to \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ sitting in a short exact sequence

$$0 \to \mathrm{SH}(X,\mathbb{Q}) \xrightarrow{\psi} \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \xrightarrow{\Delta} \mathrm{Sym}^{n-2}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to 0. \tag{C.2.3}$$

Here, the map Δ is the Laplacian operator defined on pure tensors via

$$v_1 \cdots v_n \mapsto \sum_{i<j} \tilde{\mathrm{q}}(v_i, v_j) v_1 \cdots \hat{v_i} \cdots \hat{v_j} \cdots v_n.$$

The map ψ is uniquely determined (up to scaling) by the condition that it is a morphism of $\mathfrak{g}(X)$-modules.

The n-th symmetric power $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ inherits the structure of a $\mathfrak{g}(X)$-module by letting $\mathfrak{g}(X)$ act by derivations. The inclusion realizes $\mathrm{SH}(X,\mathbb{Q})$ as an irreducible Lefschetz module [207]. We fix once and for all a choice of ψ by setting $\psi(1) = \alpha^n/n!$. The orthogonal projection onto the subspace $\mathrm{SH}(X,\mathbb{Q})$ will be denoted by

$$T\colon \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \to \mathrm{SH}(X,\mathbb{Q}).$$

C.2.3. Hochschild LLV algebra

The two previous subsections have a common ground which will be frequently used.

Let us consider the *Hodge grading operator* $h' \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$ defined via

$$h'|_{\mathrm{H}^{p,q}(X)} = (q-p)\mathrm{id},$$

i.e. the graded pieces of $\mathrm{H}^*(X,\mathbb{C})$ induced from the grading given by h' agree with the columns of the Hodge diamond. We will say that an element x is of *Hodge type* if $h'(x)=0$, i.e. if

$$x \in \bigoplus_p \mathrm{H}^{p,p}(X).$$

An element $\mu \in \mathrm{HT}^2(X)$ induces an operator $e_\mu \coloneqq \mu \lrcorner \_ \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$ by contraction. As before, we say that μ has the *Hard Lefschetz property*, if there exists an operator Λ_μ such that (e_μ, h', Λ_μ) forms a complex $\mathfrak{sl}_2$-triple.

Analogously to the previous case, we can consider the complex Lie subalgebra $\mathfrak{g}'(X) \subset \mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$ generated by all $\mathfrak{sl}_2$-triples for all μ having the Hard Lefschetz property. The following is [201, Prop. 2.8], see also [209, Sec. 9] for an earlier account, where the result is essentially already proved.

Theorem C.2.1 (Taelman, Verbitsky)**.** *There is an equality*

$$\mathfrak{g}(X)_{\mathbb{C}} = \mathfrak{g}'(X) \subset \mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$$

of complex Lie subalgebras.

This result sheds new light on the LLV algebra. For example, the operators in $\mathfrak{g}(X)_{\mathbb{C}}$ having degree two for the grading given by h' are exactly given by contraction with elements in $\mathrm{HT}^2(X)$. Throughout the paper, we will frequently use the above identification and switch between the gradings h and h'.

C.3. Atomic objects

We discuss Definition C.1.1 and general results about atomic objects. We fix a hyper-Kähler manifold X of dimension $2n>2$.

C.3.1. Lie theoretic properties

Let $\mathcal{E}$ be a sheaf on X or an object in $\mathrm{D}^{\mathrm{b}}(X)$. Recall that the property of $\mathcal{E}$ being atomic is a condition on the Lie subalgebra $\mathrm{Ann}(v(\mathcal{E}))$.

Proposition C.3.1. *An object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is atomic if and only if* $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ *is a Lie subalgebra of codimension $b_2(X)+1$ which is the smallest positive codimension possible.*

Proof. If $\mathcal{E}$ is atomic, then $\mathrm{Ann}(v(\mathcal{E})) = \mathrm{Ann}(\tilde{v})$ for some non-zero $\tilde{v} \in \tilde{\mathrm{H}}(X,\mathbb{Q})$. Recall that $\mathfrak{g}(X)_{\mathbb{C}} \cong \mathfrak{so}(b_2(X)+2)$. If $\tilde{q}(\tilde{v}) \neq 0$, we immediately get that $\mathrm{Ann}(\tilde{v}) \cong \mathfrak{so}(b_2(X)+1)$. It follows from a straightforward calculation that the condition on the codimension remains valid also in the case $\tilde{q}(\tilde{v})=0$, see also the proof of the lemma below.

Let us now assume that $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ has codimension $b_2(X)+1$. We will study the cohomological obstruction map

$$\mathrm{obs}_{\mathcal{E}} \colon \mathrm{HT}^2(X) \to \mathrm{H}^*(X,\Omega_X^*), \quad \mu \mapsto \mu \lrcorner v(\mathcal{E}).$$

Since $v(\mathcal{E})$ is of Hodge type, we have $h' \in \mathrm{Ann}(v(\mathcal{E}))$. If $\mathrm{obs}_{\mathcal{E}}$ would vanish identically, i.e. $\mathrm{Ker}(\mathrm{obs}_{\mathcal{E}}) = \mathrm{HT}^2(X)$, we would know from Theorem C.2.1 that for all $\mu \in \mathrm{HT}^2(X)$ we have $e_\mu \in \mathrm{Ann}(v(\mathcal{E}))_{\mathbb{C}}$.

In particular, for any such μ having the Hard Lefschetz property with respect to h', we would have

$$0 = h'(v(\mathcal{E})) = [e_\mu, \Lambda_\mu](v(\mathcal{E})) = e_\mu(\Lambda_\mu(v(\mathcal{E}))) - \Lambda_\mu(e_\mu(v(\mathcal{E}))) = e_\mu(\Lambda_\mu(v(\mathcal{E}))). \qquad \text{(C.3.1)}$$

Since e_μ is injective when restricted to $\mathrm{H}\Omega_{-2}(X)$, we deduce that $\Lambda_\mu(v(\mathcal{E})) = 0$ for all such $\mu \in \mathrm{HT}^2(X)$. However, as by Theorem C.2.1 $\mathfrak{g}(X)_\mathbb{C}$ is generated by all $\mathfrak{sl}_2$-triples associated to all $\mu \in \mathrm{HT}^2(X)$ having the Hard Lefschetz property, we would deduce that $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C} = \mathfrak{g}(X)_\mathbb{C}$ which contradicts our assumption.

Hence, the cohomological obstruction map $\mathrm{obs}_\mathcal{E}$ does not vanish identically. If $W = \mathrm{Ker}(\mathrm{obs}_\mathcal{E}) \subset \mathrm{HT}^2(X)$ has codimension one, then the arguments above imply that for all Hard Lefschetz elements $\mu \in W$ we have that $e_\mu, \Lambda_\mu \in \mathrm{Ann}(v(\mathcal{E}))_\mathbb{C}$. The Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}(X)_\mathbb{C}$ generated by h' and all e_μ, Λ_μ for all $\mu \in W$ having the Hard Lefschetz property has dimension $(b_2(X)^2 + b_2(X))/2$ as follows from [81, Thm. 2.7]. Moreover, from (C.3.1) we infer the inclusion $\mathfrak{h} \subset \mathrm{Ann}(v(\mathcal{E}))_\mathbb{C}$ of Lie algebras. The assumption on the codimension of $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C} \subset \mathfrak{g}(X)_\mathbb{C}$ yields that the inclusion $\mathfrak{h} \subset \mathrm{Ann}(v(\mathcal{E}))_\mathbb{C}$ must already be an equality.

Furthermore, let us consider the pairing

$$\mathrm{HT}^2(X) \times \left(\mathbb{C}\alpha \oplus \tilde{\mathrm{H}}^{1,1}(X,\mathbb{C}) \oplus \mathbb{C}\beta\right) \to \mathbb{C}\bar{\sigma}, \quad (\mu, x) \mapsto \mu \lrcorner x$$

obtained from considering $\tilde{\mathrm{H}}(X,\mathbb{C})$ as a $\mathfrak{g}(X)_\mathbb{C}$-module. Since this pairing is non-degenerate, see for example [139, Lem. 6.3], we obtain that there is an element $\tilde{v} \in \mathbb{C}\alpha \oplus \tilde{\mathrm{H}}^{1,1}(X,\mathbb{C}) \oplus \mathbb{C}\beta$ unique up to scaling with the property that it pairs trivially with the subspace W. Since $h'(\tilde{v}) = 0$, the above discussion shows $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C} = \mathfrak{h} \subset \mathrm{Ann}(\tilde{v})$. We claim that the inclusion is an equality.

Indeed, we know by assumption that there exists an element $\mu \in \mathrm{HT}^2(X)$ having the Hard Lefschetz property such that e_μ is not contained in $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C}$. Moreover, the dual operator Λ_μ to e_μ satisfying $[e_\mu, \Lambda_\mu] = h'$ is by (C.3.1) as well not contained in $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C}$. Furthermore, the $b_2(X) - 1$-dimensional subspace of operators generated as a vector space by $[e_\tau, \Lambda_\mu]$ for all $\tau \in W$ intersects the subspace $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C} \subset \mathfrak{g}(X)_\mathbb{C}$ trivially. This implies that the inclusion

$$\mathrm{Ann}(\tilde{v}) \subset \mathfrak{g}(X)_\mathbb{C}$$

has codimension at least $b_2(X) + 1$, which is exactly the codimension of the inclusion $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C} \subset \mathfrak{g}(X)$. This yields the assertion.

From Lemma C.3.2 we can now deduce that $\tilde{v}$ is already defined over $\mathbb{Q}$ and $\mathcal{E}$ is, therefore, atomic.

The case of $\mathrm{Ker}(\mathrm{obs}_\mathcal{E}) \subset \mathrm{HT}^2(X)$ having higher codimension can be excluded using the same line of arguments. We leave the details to the reader. ☐

Lemma C.3.2. *If* $\mathfrak{h} \subset \mathfrak{g}(X)$ *is a Lie subalgebra and* $\tilde{v} \in \tilde{\mathrm{H}}(X,\mathbb{C})$ *is such that* $\mathfrak{h}_\mathbb{C} = \mathrm{Ann}(\tilde{v}) \subset \mathfrak{g}(X)_\mathbb{C}$*, then* $\tilde{v} \in \tilde{\mathrm{H}}(X,\mathbb{Q})$.

Proof. We extend the beautiful argument from the proof of [139, Lem. 6.9].

Consider the natural map

$$\varphi \colon \mathbb{P}(\tilde{\mathrm{H}}(X,\mathbb{C})) \to \mathrm{Gr}\left(\binom{b_2(X)+1}{2}, \mathfrak{g}(X)_\mathbb{C}\right), \quad \ell \mapsto \mathrm{Ann}(\ell) \subset \mathfrak{g}(X)_\mathbb{C}.$$

This morphism is well-defined, i.e. for each $0 \neq \ell \in \tilde{\mathrm{H}}(X, \mathbb{C})$ the Lie subalgebra $\mathrm{Ann}(\ell) \subset \mathfrak{g}(X)_{\mathbb{C}}$ has codimension $b_2(X)+1$. Indeed, if $\tilde{\mathrm{q}}(\ell) \neq 0$, then we have the natural isomorphism

$$\mathrm{Ann}(\ell) \cong \mathfrak{so}(\ell^{\perp}) \cong \mathfrak{so}(b_2(X)+1).$$

In the case $\tilde{\mathrm{q}}(\ell) = 0$, the natural map of Lie groups

$$\mathrm{Fix}(\ell) \twoheadrightarrow \mathrm{SO}(\ell^{\perp}/\langle \ell \rangle) \cong \mathrm{SO}(b_2(X))$$

reveals that the Lie subgroup $\mathrm{Fix}(\ell) \subset \mathrm{SO}(b_2(X)+2)$ splits as a semidirect product. A straightforward calculation shows that the other factor consists of unipotent matrices acting trivially on $\ell^{\perp}/\langle \ell \rangle$ and ℓ and is of dimension $b_2(X)$.

Since φ is injective as well as defined over $\mathbb{Q}$, we obtain the assertion. ☐

As shown in the proof of Proposition C.3.1, if $\mathcal{E}$ is atomic, then its annihilator $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ is the largest non-trivial proper Lie subalgebra of the LLV algebra of the form $\mathrm{Ann}(v)$ for an element $v \in \mathrm{H}^*(X, \mathbb{Q})$ with $h'(v) = 0$.

The annihilator $\mathrm{Ann}(v(\mathcal{E}))$ measures, in some sense, the complexity of the Mukai vector $v(\mathcal{E})$. For example, if $\mathcal{E}$ is atomic, to its Mukai vector one can associate a vector $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ inside the much smaller vector space $\tilde{\mathrm{H}}(X, \mathbb{Q})$ still encoding most information about the vector. In that sense, the annihilator $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ having low codimension corresponds to the Mukai vector of $\mathcal{E}$ having low complexity.

However, it is not in general true that one can recover (the $\mathbb{Q}$-line spanned by) $v(\mathcal{E})$ from the knowledge of $\mathrm{Ann}(v(\mathcal{E}))$ even if $\mathcal{E}$ is atomic. The naive idea would be to consider $\mathrm{H}^*(X, \mathbb{Q})$ as a representation of $\mathrm{Ann}(v(\mathcal{E}))$ and study its trivial representations. However, viewing $\mathrm{H}^*(X, \mathbb{Q})$ as a module over the larger Lie algebra $\mathfrak{g}(X)$, there can already be (many) trivial representations.

On the positive side, the Mukai vector of an atomic object is still severely restricted, as we will demonstrate now. As alluded to in the introduction, if we restrict for $\mathcal{E}$ atomic the action of $\mathrm{Ann}(v(\mathcal{E}))$ to the Verbitsky component, there exists a unique one-dimensional trivial representation.

Proposition C.3.3. *Let $\mathcal{E}$ be an atomic object and $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ an element such that* $\mathrm{Ann}(v(\mathcal{E})) = \mathrm{Ann}(\tilde{v})$. *Consider the Verbitsky component* $\mathrm{SH}(X, \mathbb{Q})$ *as an* $\mathrm{Ann}(v(\mathcal{E}))$*-module. This representation has a unique trivial subrepresentation, which is spanned by $T(\tilde{v}^n) \in$* $\mathrm{SH}(X, \mathbb{Q})$ *and $v(\mathcal{E})_{\mathrm{SH}} \in \mathbb{Q}\langle T(\tilde{v}^n) \rangle$.*

Proof. It is easy to see that $0 \neq T(\tilde{v}^n) \in \mathrm{SH}(X, \mathbb{Q})$ is annihilated by $\mathrm{Ann}(\tilde{v}) = \mathrm{Ann}(v(\mathcal{E}))$. Moreover, the first part of the assertion then also gives $v(\mathcal{E})_{\mathrm{SH}} \in \mathbb{Q}\langle T(\tilde{v}^n) \rangle$, because $v(\mathcal{E})$ is annihilated by $\mathrm{Ann}(v(\mathcal{E}))$.

Hence, let us prove that there is a unique trivial subrepresentation. This statement is independent of the complex structure for which $v(\mathcal{E})$ remains algebraic. Furthermore, it is invariant under an integrated automorphism of $\mathfrak{g}(X)$ acting on $\mathrm{SH}(X, \mathbb{Q})$ and respecting the Hodge structure. We can therefore assume that $\tilde{v}$ in Definition C.1.1 is of the form

$$\tilde{v} = \alpha + k\beta$$

for $k \in \mathbb{Q}$.

Let $x \in \mathrm{SH}(X, \mathbb{Q})$ be an element being annihilated by $\mathrm{Ann}(v(\mathcal{E}))$. Since $h' \in \mathrm{Ann}(\tilde{v}) = \mathrm{Ann}(v(\mathcal{E}))$, we know that $h'(x) = 0$. Moreover, for any element $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ we have

$$\mu \lrcorner \tilde{v} = 0$$

by bidegree reasons and, therefore, applying Theorem C.2.1 we have $\mu \lrcorner x = 0$. In particular, the element x is of Hodge type for all possible complex structures of X. By [130, Prop. 2.14], the subalgebra of elements satisfying these properties is generated by powers ${\mathsf{q}_2}^i$ of the dual of the BBF form $\mathsf{q}_2 \in \mathrm{SH}^4(X, \mathbb{Q})$.

It remains to determine the coefficients in front of each ${\mathsf{q}_2}^i$. For $\omega \in \mathrm{H}^2(X, \mathbb{Q})$ having the Hard Lefschetz property for the grading operator h we have

$$\Lambda_\omega(\beta) = \frac{2}{q(\omega)}\omega \in \tilde{\mathrm{H}}(X, \mathbb{Q}).$$

This implies that $2ke_\omega - q(\omega)\Lambda_\omega \in \mathrm{Ann}(\tilde{v}) = \mathrm{Ann}(v(\mathcal{E}))$. Moreover, using that $\mathsf{td}^{1/2}$ projects non-trivially to the Verbitsky component and [113, Cor. 3.20] we deduce

$$0 \neq \Lambda_\omega {\mathsf{q}_2}^{i+1} \in \mathbb{Q}\langle {\mathsf{q}_2}^i \wedge \omega \rangle$$

which immediately yields that up to scaling $x = T(\tilde{v}^n)$. □

Remark C.3.4. In [25, Sec. 4] we assigned to certain coherent sheaves $\mathcal{E}$ or, more generally, certain objects $\mathcal{E} \in \mathrm{D^b}(X)$ a so-called extended Mukai vector $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$. More precisely, we asked for the existence of a non-zero rational number a such that

$$v(\mathcal{E})_{\mathrm{SH}} = aT(\tilde{v}(\mathcal{E})^n) \in \mathrm{SH}(X, \mathbb{Q}). \tag{C.3.2}$$

The proposition shows that atomic objects fulfill this definition.

The proof and, therefore, conclusion of the proposition remains true for all irreducible representations $V_\lambda \subset \mathrm{H}^*(X, \mathbb{Q})$ of the LLV algebra of the form $V_\lambda = V_{(k)} = V_{k\epsilon_1}$ where we use the notation of [81, App. A].

We note that the branching rules discussed in [81, App. B.2] immediately yield the same result for atomic objects $\mathcal{E} \in \mathrm{D^b}(X)$ such that the associated elemet $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ satisfies $\tilde{\mathrm{q}}(\tilde{v}) \neq 0$. The branching rules also imply the following.

Proposition C.3.5. *Let $\mathcal{E}$ be an atomic object with $\tilde{\mathrm{q}}(\tilde{v}) \neq 0$. Then $v(\mathcal{E})$ projects trivially to all irreducible representations which are not of the form $V_{(k)}$ with $k \in \mathbb{Z}_{\geq 0}$.*

We expect the conclusion of the proposition to remain true for all atomic complexes.

The last two propositions imply that for an atomic object with $\tilde{\mathrm{q}}(\tilde{v}) \neq 0$ the number of trivial $\mathrm{Ann}(v(\mathcal{E}))$ representations of $\mathrm{H}^*(X, \mathbb{Q})$ is the number of irreducible $\mathfrak{g}(X)$-representations of the form $V_{(k)}$ for $k \in \mathbb{Z}_{\geq 0}$. This shows that the Mukai vector $v(\mathcal{E})$ of an atomic object is severly restricted.

Remark C.3.6. The definition of the extended Mukai vector in [25] was inspired by the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{D^b}(S) \arrow[r, "\Phi"] \arrow[d, "v"] & \mathrm{D^b}(S') \arrow[d, "v"] \\
\mathrm{H}^*(S, \mathbb{Z}) \arrow[r, "\Phi^{\mathrm{H}}"] & \mathrm{H}^*(S', \mathbb{Z})
\end{tikzcd} \tag{C.3.3}$$

for derived equivalences between K3 surfaces. That is, we wanted to study complexes for which this diagram had a higher-dimensional counterpart. For this, restricting to the Verbitsky component was sufficient.

Inspecting the decomposition (C.1.2) leads naturally to Definition C.1.1, i.e. of atomic sheaves and complexes. While studying atomic complexes and their properties we came to the conclusion that these are complexes on higher dimensional hyper-Kähler manifolds which behave much like stable respectively simple sheaves on K3 surfaces. In what follows, we want to convey the reader this intuition.

C.3.2. Mukai vector and general properties of atomic objects

In this subsection we discuss general properties of atomic objects that follow easily from [25].

Lemma C.3.7. *Let* $\mathcal{E}$ *be a non-zero sheaf. Then* $0 \neq v(\mathcal{E})_{\mathrm{SH}} \in \mathrm{SH}(X, \mathbb{Q})$.

Proof. The Verbitsky component exhausts the subspaces of degree $0, 2, 4n-2$ and $4n$ of the cohomology $\mathrm{H}^*(X, \mathbb{Q})$. Therefore, if the Mukai vector does not project trivially to these subspaces, the assertion is proven.

In general, let us consider the decomposition of the support

$$\mathrm{supp}(\mathcal{E}) = \bigcup_i Z_i$$

of the sheaf $\mathcal{E}$ into irreducible components. Let j be an index such that V_j has maximal dimension k in the above decomposition. For a Kähler class $\omega \in \mathrm{H}^{1,1}(X)$ we have

$$\int_X [Z_i]\omega^{2n-k} \geq 0, \quad \int_X [Z_j]\omega^{2n-k} > 0.$$

In particular, $0 \neq v(\mathcal{E})\omega^{2n-k} \in \mathrm{H}^{4n}(X, \mathbb{R})$ which proves the assertion. □

We believe that all simple atomic objects $\mathcal{E} \in \mathrm{D^b}(X)$ satisfy $v(\mathcal{E})_{\mathrm{SH}} \neq 0$.

Proposition C.3.8. *Let* $\mathcal{E}$ *be an atomic object such that* $\mathrm{rk}(\mathcal{E}) \neq 0$ *or* $\mathrm{c}_1(\mathcal{E}) \neq 0$. *Then there exists* $s \in \mathbb{Q}$ *such that* $\tilde{v}$ *from Definition C.1.1 can be assumed to be*

$$\tilde{v} = \mathrm{rk}(\mathcal{E})\alpha + \mathrm{c}_1(\mathcal{E}) + s\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q}).$$

Proof. The assumptions imply that in particular $v(\mathcal{E})_{\mathrm{SH}} \neq 0$. This is then the same computation as in the proof of [25, Lem. 4.8(v)]. □

Hence, there is a particular element in the line spanned by $\tilde{v}$ which gives the following.

Definition C.3.9. Let $\mathcal{E} \in \mathrm{D^b}(X)$ be an atomic object such that $\mathrm{rk}(\mathcal{E}) \neq 0$. Then its *Mukai vector* $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ is defined as

$$\tilde{v}(\mathcal{E}) = \mathrm{rk}(\mathcal{E})\alpha + \mathrm{c}_1(\mathcal{E}) + s\beta \in \tilde{\mathrm{H}}(X, \mathbb{Q})$$

for the unique $s \in \mathbb{Q}$ such that $\mathrm{Ann}(v(\mathcal{E})) = \mathrm{Ann}(\tilde{v}(\mathcal{E})) \subset \mathfrak{g}(X)$.

If $\mathcal{E}$ is an atomic sheaf, we know by Lemma C.3.7 that $v(\mathcal{E})_{\mathrm{SH}} \neq 0$. From Proposition C.3.11 below, we know that if $\mathrm{rk}(\mathcal{E}) = 0$, then the support of $\mathcal{E}$ is a union of Lagrangian subvarieties or points. In the former case, taking $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ associated to $\mathcal{E}$ from Definition C.1.1, its projection $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ to the component in $\mathrm{H}^2(X, \mathbb{Q}) \subset \tilde{\mathrm{H}}(X, \mathbb{Q})$ is non-zero. Normalize λ in such a way that $q(\lambda, \omega) > 0$ for a Kähler class ω and such that $\lambda \in \mathrm{H}^2(X, \mathbb{Z})^\vee \subset \mathrm{H}^2(X, \mathbb{Q})$ is a primitive element in the dual lattice of $\mathrm{H}^2(X, \mathbb{Z})$. We define the corresponding multiple of $\tilde{v}$ to be the Mukai vector $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ of $\mathcal{E}$.

We note that in the rest of the text, the precise multiple of $\tilde{v}$ in Definition C.1.1 will not play a role. See [25, Sec. 4] for another discussion of the question which element of the line $\mathbb{Q}\langle\tilde{v}\rangle$ is a candidate for the Mukai vector $\tilde{v}(\mathcal{E})$ of an atomic sheaf or complex $\mathcal{E}$ when its rank and determinant are zero.

Proposition C.3.10. *Let $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ be a derived equivalence between projective hyper-Kähler manifolds and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$. Then $\mathcal{E}$ is atomic if and only if $\Phi(\mathcal{E})$ is. Similarly, for $\mathcal{X} \to B$ a family of hyper-Kähler and $\mathcal{E}$ a B-perfect complex on $\mathcal{X}$ we have for two points $b, b' \in B$ that $\mathcal{E}_b$ is atomic if and only if $\mathcal{E}_{b'}$ is.*

Proof. This is immediate from the definitions. □

To finish this section let us mention one more property of atomic sheaves and complexes similar to [25, Lem. 4.13(v)].

Proposition C.3.11. *Let $\mathcal{E}$ be an atomic object with $v(\mathcal{E})_{\mathrm{SH}} \neq 0$, e.g. $\mathcal{E}$ is a sheaf, such that $\mathrm{rk}(\mathcal{E}) = 0$ or $\mathrm{c}_1(\mathcal{E}) = 0$. Then all Chern classes of $\mathcal{E}$ are isotropic, that is $\mathrm{c}_i(\mathcal{E})\sigma = 0$ for all i and σ a symplectic form.*

Proof. This follows already from the definition of atomicity, see also [25, Sec. 4.4]. The vector $\tilde{v}$ as in Definition C.1.1 projects by assumption trivially onto the subspace spanned by $\alpha \in \tilde{\mathrm{H}}(X, \mathbb{Q})$. But for all such elements we have $e_\sigma(\tilde{v}) = 0$. This means that $e_\sigma \in \mathrm{Ann}(v(\mathcal{E}))$ from which the assertion immediately follows. □

We recall here that for $\mathcal{E}$ as in the proposition $\mathrm{ch}_0(\mathcal{E}) = 0$ or $\mathrm{ch}_1(\mathcal{E}) = 0$ already implies that $\mathrm{ch}_i(\mathcal{E}) = 0$ for $i < n$, see [25, Lem. 4.8(v)]. If, moreover, $\mathrm{ch}_n(\mathcal{E}) = 0$, then we have that $\mathrm{ch}_i(\mathcal{E}) = 0$ for $i < 2n$.

C.4. Obstruction Maps

In this section we will discuss the implications between the various obstruction maps from the introduction and atomicity. In particular, we will prove Theorem C.1.2 and Theorem C.1.3.

C.4.1. Cohomological Obstruction map and Atomicity

We show here that being atomic is equivalent to having a cohomological obstruction map with kernel of codimension one.

Proof of Theorem C.1.2. Let us assume first that $\mathcal{E}$ is atomic. We know that

$$\mathrm{Ann}(v(\mathcal{E})) = \mathrm{Ann}(\tilde{v}) \subset \mathfrak{g}(X)$$

for some $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$. Since $v(\mathcal{E})$ is algebraic and, therefore, $h'(v(\mathcal{E})) = 0$ we conclude $h' \in \mathrm{Ann}(\tilde{v})$. Thus, we find that $h'(\tilde{v}) = 0$ which implies $\tilde{v} \in \tilde{\mathrm{H}}^{1,1}(X, \mathbb{Q})$.

An element $\mu \in \mathrm{HT}^2(X)$ induces the operator $e_\mu \in \mathfrak{g}(X)_\mathbb{C}$ which has degree two for the grading operator h'. Moreover, we have the perfect pairing

$$\mathrm{HT}^2(X) \times \left(\mathbb{C}\alpha \oplus \mathrm{H}^{1,1}(X, \mathbb{C}) \oplus \mathbb{C}\beta\right) \to \mathrm{H}^{0,2}(X), \quad (\mu, x) \mapsto e_\mu(x) = \mu \lrcorner x$$

obtained from viewing $\tilde{\mathrm{H}}(X, \mathbb{C})$ as a $\mathfrak{g}(X)_\mathbb{C}$-module. In particular, restricting the perfect pairing to $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{C})$ we see that under the embedding

$$\mathrm{HT}^2(X) \hookrightarrow \mathfrak{g}(X)_\mathbb{C}, \quad \mu \mapsto e_\mu$$

the intersection $\mathrm{Ann}(\tilde{v})_\mathbb{C} \cap \mathrm{HT}^2(X) \subset \mathfrak{g}(X)_\mathbb{C}$ is $b_2(X) - 1$-dimensional. Since $\mathrm{Ann}(v(\mathcal{E}))_\mathbb{C} \cap \mathrm{HT}^2(X)$ equals the kernel $\mathrm{Ker}(\mathrm{obs}_\mathcal{E})$ of the cohomological obstruction map, the equality $\mathrm{Ann}(\tilde{v}) = \mathrm{Ann}(v(\mathcal{E}))$ shows that $\mathrm{obs}_\mathcal{E}$ has a one-dimensional image.

For the converse implication let us reinspect the proof of Proposition C.3.1. There, we studied the codimension of $\mathrm{Ann}(v(\mathcal{E})) \subset \mathfrak{g}(X)$ in terms of the kernel of the cohomological obstruction map. In particular, in the case of interest of us, that is, the kernel having codimension one, we already deduced that $\mathcal{E}$ must be atomic, which finishes the proof. ☐

Remark C.4.1. The statement and the proof of the above theorem are purely cohomological. That is, we actually proved the following for an element $x \in \mathrm{H}^*(X, \mathbb{Q})$ of Hodge type, i.e. $h'(x) = 0$:

The annihilator Lie subalgebra $\mathrm{Ann}(x) \subset \mathfrak{g}(X)$ is equal to $\mathrm{Ann}(\tilde{v}) \subset \mathfrak{g}(X)$ for a non-zero element $\tilde{v} \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ if and only if the morphism

$$\mathrm{HT}^2(X) \to \mathrm{H}^*(X, \mathbb{C}), \quad \mu \mapsto \mu \lrcorner x$$

has a one-dimensional image.

In [107, Prop. 2.6] the authors have shown that for $\mu \in \mathrm{H}^1(X, \mathcal{T}_X) \oplus \mathrm{H}^2(X, \mathcal{O}_X)$ the vanishing

$$\mu \lrcorner v(\mathcal{E}) = 0$$

is equivalent to the vanishing

$$\mu \lrcorner \mathrm{ch}(\mathcal{E}) = 0.$$

However, this does not remain true for the total space $\mathrm{HT}^2(X)$, i.e. the cohomological obstruction map having a one-dimensional image is not equivalent to the map

$$\mathrm{HT}^2(X) \to \mathrm{H}\Omega_2(X), \quad \mu \mapsto \mu \lrcorner \mathrm{ch}(\mathcal{E})$$

having a one-dimensional image. An example for this phenomenon is any complex $\mathcal{E} \in \mathrm{D}^\mathrm{b}(S^{[2]})$ in the derived category of the second Hilbert scheme $S^{[2]}$ for S a K3 surface such that $\mathrm{ch}(\mathcal{E}) \in \mathbb{Q}\langle v(\mathcal{O}_{S^{[2]}})\rangle$.

C.4.2. Obstruction Map and Atomicity

Let us recall the observation [94, Lem. 3.2] which relates the obstruction and the cohomological obstruction map for $\mathcal{E}$.

Lemma C.4.2. *Let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ be an object and $\gamma \in \mathrm{HH}^2(X)$. Then $0 = \chi_{\mathcal{E}}(\gamma) = \gamma_{\mathcal{E}} \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ implies $0 = \gamma \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E}) \in \mathrm{HH}_2(X)$. In particular,*

$$I^{\mathrm{K}}(\mathrm{Ker}(\chi_{\mathcal{E}})) \subset \mathrm{Ker}(\mathrm{obs}_{\mathcal{E}}).$$

The proof is an application of the defining property of the Hochschild Chern character and the non-degeneracy of the Serre duality trace. We can use this and the relation between the cohomological obstruction map and atomicity to give a proof of Theorem C.1.3.

Recall that Theorem C.1.3 asserts a relationship between obstructions to first-order (noncommutative) deformations of $\mathcal{E}$ and atomicity of $\mathcal{E}$ when the object $\mathcal{E}$ is 1-obstructed. Employing Theorem C.1.2 this is equivalent to establishing a relationship between obstructions to first-order (noncommutative) deformations of $\mathcal{E}$ and obstructions to the Mukay vector $v(\mathcal{E})$ of $\mathcal{E}$ staying of Hodge type.

Proof of Theorem C.1.3. As recalled above we need to relate (the dimensions of the vector spaces) $\mathrm{Ker}(\chi_{\mathcal{E}})$ and $\mathrm{Ker}(\mathrm{obs}_{\mathcal{E}})$ for $\mathcal{E}$ 1-obstructed. This is done using Theorem C.1.2 and Lemma C.4.2.

More precisely, Lemma C.4.2 gives

$$I^{\mathrm{K}}(\mathrm{Ker}(\chi_{\mathcal{E}})) \subset \mathrm{Ker}(\mathrm{obs}_{\mathcal{E}})$$

which implies that the cohomological obstruction map $\mathrm{obs}_{\mathcal{E}}$ must have one or zero-dimensional image. If it is one-dimensional, Theorem C.1.2 gives that $\mathcal{E}$ is atomic.

To conclude, it is left to show that the image of $\mathrm{obs}_{\mathcal{E}}$ is not zero-dimensional. This follows from the lemma below. ☐

Lemma C.4.3. *The radical $W \subset \mathrm{H}\Omega_0(X)$ of the pairing*

$$\mathrm{HT}^2(X) \times \mathrm{H}\Omega_0(X) \to \mathrm{H}\Omega_2(X)$$

corresponds under the isomorphism $\mathrm{H}^*(X, \Omega_X^*) \cong \mathrm{H}^*(X, \mathbb{C})$ to the subspace spanned by trivial representations of the LLV algebra.

Proof. Since by Theorem C.2.1 the operator e_μ for $\mu \in \mathrm{HT}^2(X)$ is contained in $\mathfrak{g}(X)_{\mathbb{C}}$ it is immediate that elements in the subspace spanned by trivial representations lie in W.

For the converse inclusion, note that $\mathrm{H}\Omega_0(X)$ is by definition the subspace of elements x satisfying $h'(x) = 0$. If x is contained in the radical W, we infer from (C.3.1) that for all elements $\mu \in \mathrm{HT}^2(X)$ having the Hard Lefschetz property the operators Λ_μ also satisfy $\Lambda_\mu(x) = 0$. As the set of all these operators generate $\mathfrak{g}(X)_{\mathbb{C}}$, we conclude that x is annihilated by the LLV algebra. ☐

We now discuss the converse implication of whether atomic sheaves and complexes are 1-obstructed. The following shows that it does not always hold.

Example C.4.4. Consider a K3 surface X and a non-trivial line bundle $\mathcal{L} \in \operatorname{Pic}(X)$. The bundle $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{L}$ is atomic, but not 1-obstructed.

Indeed, any non-zero sheaf on a K3 surface is atomic. The Atiyah class $\mathrm{At}_{\mathcal{E}}$ decomposes

$$\mathrm{At}_{\mathcal{E}} = \mathrm{At}_{\mathcal{O}_X} + \mathrm{At}_{\mathcal{L}} \in \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X \otimes \Omega^1_X) \oplus \operatorname{Ext}^1(\mathcal{L}, \mathcal{L} \otimes \Omega^1_X) \subset \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$$

which can be simplified using $\mathrm{At}_{\mathcal{O}_X} = 0$. Since $\mathcal{L}$ is non-trivial, there exists $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ such that $\mu \lrcorner \mathrm{c}_1(\mathcal{L}) \neq 0$. In particular, the element

$$x := \mu \lrcorner \mathrm{At}_{\mathcal{E}} \in \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$$

projects non-trivially to the subspace $\operatorname{Ext}^2(\mathcal{L}, \mathcal{L}) \subset \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$, but trivially to the subspace $\operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X) \subset \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$. Moreover, any non-trivial $\mu' \in \mathrm{H}^2(X, \mathcal{O}_X)$ induces a non-trivial element

$$y := \mu' \lrcorner \mathrm{At}^0_{\mathcal{E}} \in \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$$

which projects non-trivially to $\operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X) \subset \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$ (more precisely, after identifying $\operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X) \cong \mathrm{H}^2(X, \mathcal{O}_X)$ we have that the projection of y equals $2\mu'$). This shows that x and y must be linearly independent.

Note, however, that every simple sheaf or complex on a K3 surface with non-zero Mukai vector is 1-obstructed. A natural question therefore is whether this also holds true in higher dimensions.

We state here the following.

Conjecture E. *Let X be a hyper-Kähler manifold and $\mathcal{E}$ a simple atomic object. For each $\gamma \in \mathrm{HH}^2(X)$ with $0 \neq \chi_{\mathcal{E}}(\gamma) = \gamma_{\mathcal{E}} \in \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$ there exists $\mu \in \mathrm{HH}^{2n-2}(X)$ such that the composition $0 \neq \mu_{\mathcal{E}} \circ \gamma_{\mathcal{E}} \in \operatorname{Ext}^{2n}(\mathcal{E}, \mathcal{E})$.*

Since $\mathcal{E}$ is assumed to be simple, this is equivalent to asking $\operatorname{Tr}_X(\mu_{\mathcal{E}} \circ \gamma_{\mathcal{E}}) \neq 0$. One could formulate an even stronger conjecture by asking that for each $\gamma \in \mathrm{HH}^k(X)$ with $0 \neq \chi_{\mathcal{E}}(\gamma) = \gamma_{\mathcal{E}} \in \operatorname{Ext}^k(\mathcal{E}, \mathcal{E})$ there exists $\mu \in \mathrm{HH}^{2n-k}(X)$ such that $\operatorname{Tr}_X(\mu_{\mathcal{E}} \circ \gamma_{\mathcal{E}}) \neq 0$. Using that X is Calabi–Yau and, therefore, $\operatorname{Ext}^*(\mathcal{E}, \mathcal{E})$ is via Serre duality equipped with a non-degenerate pairing, this could be rephrased by saying that the this non-degenerate pairing on $\operatorname{Ext}^*(\mathcal{E}, \mathcal{E})$ restricts to a non-degenerate pairing on the image subalgebra $\operatorname{Im}(\chi_{\mathcal{E}}) \subset \operatorname{Ext}^*(\mathcal{E}, \mathcal{E})$.

The following concerns the reverse implication in Theorem C.1.3 assuming Conjecture E and establishes a complete relationship between the notion of 1-obstructedness and atomicity.

Proof of Theorem C.1.4. Recall the defining property of the Hochschild Chern character $\mathrm{ch}^{\mathrm{HH}}(\mathcal{E}) \in \mathrm{HH}_0(X)$

$$\operatorname{Tr}_{X \times X}(\delta \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \operatorname{Tr}_X(\delta_{\mathcal{E}})$$

for all $\delta \in \mathrm{HH}^*(X)$. For $\mu \in \mathrm{HT}^2(X)$ we have that

$$\mu \lrcorner v(\mathcal{E}) = 0$$

is equivalent to

$$(I^{\mathrm{K}})^{-1}(\mu) \circ (I_{\mathrm{K}})^{-1}(v(\mathcal{E})) = (I^{\mathrm{K}})^{-1}(\mu) \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E}) = 0.$$

If we denote $\gamma \coloneqq (I^{\mathrm{K}})^{-1}(\mu) \in \mathrm{HH}^2(X)$, then the above vanishing implies for arbitrary $\gamma' \in \mathrm{HH}^{2n-2}(X)$

$$0 = \mathrm{Tr}_{X\times X}(\gamma' \circ \gamma \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{Tr}_X((\gamma' \circ \gamma)_{\mathcal{E}}) = \mathrm{Tr}_X(\gamma'_{\mathcal{E}} \circ \gamma_{\mathcal{E}}).$$

Conjecture E now gives that we can deduce from this the vanishing $\gamma_{\mathcal{E}} = 0$. This gives

$$\mathrm{Ker}(\mathrm{obs}_{\mathcal{E}}) \subset I^{\mathrm{K}}(\mathrm{Ker}(\chi_{\mathcal{E}})).$$

Combined with Lemma C.4.2 we therefore obtain the equality

$$I^{\mathrm{K}}(\mathrm{Ker}(\chi_{\mathcal{E}})) = \mathrm{Ker}(\mathrm{obs}_{\mathcal{E}}) \tag{C.4.1}$$

which, together with Theorem C.1.2 yields the assertion. □

Note that the above also strengthens Theorem C.1.3. Namely, assuming that an object $\mathcal{E}$ satisfies Conjecture E, one concludes that $\mathcal{E}$ is atomic without the condition on its Mukai vector not lying in the subspace generated by trivial representations of the LLV algebra. That is, Conjecture E implies that Mukai vectors of 1-obstructed objects cannot cannot be annihilated by the LLV algebra as the equality (C.4.1) forces a non-trivial radical.

Remark C.4.5. The obstruction map

$$\chi_{\mathcal{E}} \colon \mathrm{HH}^2(X) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$$

measures the obstruction to deform $\mathcal{E}$ to first order along the first order deformation corresponding to the element in $\mathrm{HH}^2(X)$.

On the other hand, the cohomological obstruction map

$$\mathrm{obs}_{\mathcal{E}} \colon \mathrm{HT}^2(X) \to \mathrm{H}\Omega_2(X)$$

concerns only the Mukai vector of the corresponding object and measures whether the Mukai vector stays of Hodge-type along the given first order deformation.

From this viewpoint, Theorem C.1.4 says that under a certain condition, if the Mukai vector stays algebraic along a given first order deformation direction, then the object can be lifted to this first order deformation.

The following is evidence supporting Conjecture E.

Proposition C.4.6. *Let $\mathcal{E} \in \mathrm{D^b}(X)$ be a simple 1-obstructed object such that its Mukai vector is not annihilated by the LLV algebra, e.g. $\mathcal{E}$ is a sheaf. Then $\mathcal{E}$ satisfies the conclusion of Conjecture E.*

Proof. Since $\mathcal{E}$ is 1-obstructed we only need to show Conjecture E for one non-zero representative of the image of $\chi_{\mathcal{E}}$ in $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$. This means we need to find one element in the image of

$$\chi_{\mathcal{E}} \colon \mathrm{HH}^{2n-2}(X) \to \mathrm{Ext}^{2n-2}(\mathcal{E}, \mathcal{E})$$

which pairs non trivially with the one-dimensional subspace of $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ given by the image of $\chi_{\mathcal{E}}$.

By assumption, $v(\mathcal{E})$ is not annihilated by the LLV algebra $\mathfrak{g}(X)$. As demonstrated in the proof of Proposition C.3.1 this means that there exists $\mu \in \mathrm{HT}^2(X)$ such that $e_\mu(v(\mathcal{E})) = \mu \lrcorner v(\mathcal{E}) \neq 0$. Using (C.2.1), [201, Lem. 2.5] and the fact that the intersection pairing on $\mathrm{H}^*(X, \mathbb{C})$ is non-degenerate, we see that there exists $\gamma \in \mathrm{HT}^{2n-2}(X)$ such that $0 \neq \gamma \lrcorner e_\mu(v(\mathcal{E})) = (\gamma \wedge \mu) \lrcorner v(\mathcal{E}) \in \mathrm{H}^{2n}(X, \mathcal{O}_X)$.

Defining $\tau = (I^{\mathrm{K}})^{-1}(\gamma \wedge \mu)$ and employing the defining property of the Hochschild Chern character we obtain

$$0 \neq \mathrm{Tr}_{X \times X}(\tau \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{Tr}_X(\tau_{\mathcal{E}}) = \mathrm{Tr}_X((I^{\mathrm{K}})^{-1}(\gamma)_{\mathcal{E}} \circ (I^{\mathrm{K}})^{-1}(\mu)_{\mathcal{E}}).$$

This proves the proposition. □

Thus, 1-obstructed sheaves satisfy Conjecture E by Lemma C.3.7. Moreover, if the 1-obstructed object $\mathcal{E}$ satisfies the conclusion of Conjecture E, then by Theorem C.1.4 its Mukai vector $v(\mathcal{E})$ does not lie inside the subspace of trivial representations of the LLV algebra.

We get the following consequence.

Corollary C.4.7. *Let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ be a simple atomic object. Then $\mathcal{E}$ is 1-obstructed if and only if it satisfies the conclusion of Conjecture E.*

In particular, for a simple object $\mathcal{E}$ consider the three properties: $\mathcal{E}$ is atomic, $\mathcal{E}$ is 1-obstructed, $\mathcal{E}$ satisfies Conjecture E. Then any two of these properties imply the remaining one.

C.5. Vector bundles and torsion-free sheaves

We will recall the notion and relevant results of Verbitsky concerning (projectively) hyperholomorphic bundles. This will be applied in the next section to study the deformation theory of slope (poly)stable bundles. We will compare this notion as well as the notion of a modular sheaf of O'Grady with being atomic.

C.5.1. Hyperholomorphicity

Let $\mathcal{E}$ be a vector bundle on a hyper-Kähler manifold X. For every Kähler class ω in the Kähler cone $\mathcal{K}_X$ there exists by Yau's solution to Calabi's conjecture [83, Thm. 23.5] a unique hyper-Kähler metric g on the underlying real manifold such that $\omega = [\omega_I]$, where $\omega_I = g(I(\_), \_)$. We denote the complex structures corresponding to the hyper-Kähler metric g by I, J, K. We denote the resulting twistor deformation by $\pi \colon \mathcal{X} \to \mathbb{P}^1_\omega$.

Definition C.5.1. A Hermitian connection ∇ on $\mathcal{E}$ is called (ω)-*hyperholomorphic*, if ∇ is integrable with respect to each complex structure induced by the hyper-Kähler metric g.

The three complex structures I, J, K induce naturally an $\mathrm{SU}(2)$-action on the cohomology $\mathrm{H}^*(X, \mathbb{C})$. Note that the associated Lie algebra $\mathfrak{su}(2)$ is contained in the LLV algebra $\mathfrak{g}(X)_\mathbb{C}$ and its action has degree zero with respect to the grading given by h. A cohomology class $x \in \mathrm{H}^*(X, \mathbb{C})$ is $\mathrm{SU}(2)$-invariant if and only if it is of type (p, p) for all Hodge structures induced by all complex structures obtained from the hyper-Kähler metric g (for more see [208, Sec. 1]). Here are several results related to hyperholomorphic bundles which we will need later on:

- Every ω-hyperholomorphic bundle $\mathcal{E}$ is ω-slope polystable[1] [208, Thm. 2.3]. For the induced curvature Θ we have $\Lambda_\omega(\Theta) = 0$.
- A Hermitian connection ∇ on a holomorphic bundle $\mathcal{E}$ is ω-hyperholomorphic if and only if its curvature Θ is SU(2)-invariant. Furthermore, a polystable bundle $\mathcal{E}$ is hyperholomorphic if and only if $\mathrm{c}_1(\mathcal{E})$ and $\mathrm{c}_2(\mathcal{E})$ are SU(2)-invariant [210, Thm. 3.9].
- The pullback of a hyperholomorphic bundle $\mathcal{E}$ to the associated twistor line admits a holomorphic structure over the twistor space $\pi\colon \mathcal{X} \to \mathbb{P}^1_\omega$ [117, Lem. 1.1]. A bundle $\mathcal{E}$ is hyperholomorphic if and only if there exists a holomorphic bundle $\mathcal{F}$ on the twistor space $\mathcal{X}$ such that the restriction to X of $\mathcal{F}$ is $\mathcal{E}$, see [108, Def. 2.2] and the paragraph afterwards.

Definition C.5.2. A bundle $\mathcal{E}$ is called (ω-)*projectively hyperholomorphic*, if the traceless curvature Θ_{tl} is SU(2)-invariant for the induced hyper-Kähler structure.

Equivalently, $\mathcal{E}$ is projectively hyperholomorphic if and only if $\mathcal{E}nd(\mathcal{E})$ is hyperholomorphic [208, Prop. 11.1].

C.5.2. Comparison of notions for bundles on hyper-Kähler manifolds

We recall here the element

$$\kappa(\mathcal{E}) \coloneqq \mathrm{ch}(\mathcal{E}) \exp\left(-\frac{\mathrm{c}_1(\mathcal{E})}{r}\right) \in \mathrm{H}^*(X, \mathbb{Q})$$

for a torsion-free sheaf $\mathcal{E}$ of rank $\mathrm{rk}(\mathcal{E}) = r$ and its discriminant

$$\Delta(\mathcal{E}) \coloneqq -2r\mathrm{ch}_2(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E})^2.$$

In [172], O'Grady proposed a notion of modular sheaves.

Definition C.5.3. A torsion-free sheaf $\mathcal{E}$ is *modular* if the projection of $\Delta(\mathcal{E})$ to the Verbitsky component is a multiple of the dual of the BBF form $\mathsf{q}_2 \in \mathrm{SH}^4(X, \mathbb{Q})$.

Let us compare the notions of atomicity, (projective) hyperholomorphicity and modularity for a bundle $\mathcal{E}$.

Lemma C.5.4. *Let $\mathcal{E}$ be a torsion-free atomic sheaf. Then $\kappa(\mathcal{E})$ and $\Delta(\mathcal{E})$ remain of Hodge type for all Kähler deformations of X. If $\mathcal{E}$ is a vector bundle, the same is true for* $\mathrm{ch}(\mathcal{E} \otimes \mathcal{E}^\vee)$.

Proof. The sheaf $\mathcal{E}$ is atomic and by Proposition C.3.8 there exists $\tilde{v}(\mathcal{E}) \in \tilde{\mathrm{H}}(X, \mathbb{Q})$ such that

$$\mathrm{Ann}(v(\mathcal{E})) = \mathrm{Ann}(\tilde{v}(\mathcal{E})) \subset \mathfrak{g}(X).$$

Note that $\kappa(\mathcal{E})$ is of type Hodge type if and only if the class

$$\tilde{\kappa}(\mathcal{E}) \coloneqq \mathrm{ch}(\mathcal{E})\mathsf{td}^{1/2} \exp\left(-\frac{\mathrm{c}_1(\mathcal{E})}{r}\right) = v(\mathcal{E}) \exp\left(-\frac{\mathrm{c}_1(\mathcal{E})}{r}\right) \in \mathrm{H}^*(X, \mathbb{Q})$$

---

[1] Sum of slope stable bundles with the same slope.

is of Hodge type. The isometry given by multiplication with $\exp(-\mathrm{c}_1(\mathcal{E})/r)$ is the integrated action of the operator $e_{-\mathrm{c}_1(\mathcal{E})/r}$ given by cup product with the class $-\mathrm{c}_1(\mathcal{E})/r \in \mathrm{H}^{1,1}(X,\mathbb{Q})$. We therefore obtain the equality

$$\mathrm{Ann}(\tilde{\kappa}(\mathcal{E})) = \mathrm{Ann}\left(v(\mathcal{E})\exp\left(-\frac{\mathrm{c}_1(\mathcal{E})}{r}\right)\right) = \mathrm{Ann}\left(\tilde{v}(\mathcal{E})\exp\left(-\frac{\mathrm{c}_1(\mathcal{E})}{r}\right)\right).$$

From Proposition C.3.8 we infer

$$\tilde{v} := \tilde{v}(\mathcal{E})\exp\left(-\frac{\mathrm{c}_1(\mathcal{E})}{r}\right) = r\alpha + t\beta \in \tilde{\mathrm{H}}(X,\mathbb{Q})$$

for some $t \in \mathbb{Q}$. In particular, for every possible complex structure I and associated Weil operator W_I we have $W_I \in \mathrm{Ann}(\tilde{v}) = \mathrm{Ann}(\tilde{\kappa}(\mathcal{E}))$. This proves that $\kappa(\mathcal{E})$ remains of Hodge type. The assertion for $\Delta(\mathcal{E})$ follows from the identity

$$-2r\kappa(\mathcal{E})_4 = \Delta(\mathcal{E}),$$

where $\kappa(\mathcal{E})_4 \in \mathrm{H}^4(X,\mathbb{Q})$ is the degree four component of $\kappa(\mathcal{E})$.

If $\mathcal{E}$ is a vector bundle, we use

$$\mathrm{ch}(\mathcal{E}\otimes\mathcal{E}^\vee) = \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{E}^\vee) = \left(\mathrm{ch}(\mathcal{E})\exp\left(-\frac{\mathrm{c}_1(\mathcal{E})}{r}\right)\right)\left(\mathrm{ch}(\mathcal{E}^\vee)\exp\left(\frac{\mathrm{c}_1(\mathcal{E})}{r}\right)\right).$$

By what we have already proven, the right hand side is the product of two classes which are of Hodge type for all Kähler deformations. This finishes the proof. ☐

For an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ which is atomic the proof also shows that the class $\mathrm{ch}(\mathcal{E} \otimes^{\mathrm{L}} \mathrm{R}\mathcal{H}om(\mathcal{E},\mathcal{O}_X))$ stays algebraic for all possible complex structures.

The lemma immediately implies Proposition C.1.5 which is a strengthening of [139, Thm. 3.4]. We can also now proof the relationship with projectively hyperholomorphic bundles alluded to in the introduction.

Proof of Proposition C.1.6. Since $\mathcal{E}$ is ω-polystable so is the bundle $\mathcal{E}nd(\mathcal{E})$, i.e. $\mathcal{E}nd(\mathcal{E})$ decomposes into the direct sum of indecomposable ω-slope stable bundles of the same slope. Now $\mathcal{E}$ is ω-projectively hyperholomorphic if and only if $\mathcal{E}nd(\mathcal{E})$ is ω-hyperholomorphic [208, Prop. 11.1]. By [208, Thm. 2.5] we know that $\mathcal{E}nd(\mathcal{E})$ is hyperholomorphic if and only if $\mathrm{c}_1(\mathcal{E}nd(\mathcal{E}))$ and $\mathrm{c}_2(\mathcal{E}nd(\mathcal{E}))$ remain of Hodge type (p,p) for all complex structures induced by the twistor space associated to ω. This follows from Lemma C.5.4. ☐

The converse in the above statements does not hold. A counterexample is given by the tangent bundle $\mathcal{T}_X$ on higher-dimensional hyper-Kähler manifolds X, see Proposition C.8.3.

We obtain also the following which is similar to [139, Thm. 3.4] where the statement is also essentially proved under stronger assumptions.

Proposition C.5.5. *Let $\mathcal{E}$ be a slope stable atomic bundle. Then $\mathbb{P}(\mathcal{E})$ deforms over the whole moduli space of Kähler deformations of X.*

Proof. From what has just been proven we know that $\mathcal{E}$ is also modular as well as projectively hyperholomorphic. By [172, Sec. 3] we know that there is an open subcone of the ample cone for which $\mathcal{E}$ remains slope stable and projectively hyperholomorphic. Moreover, from Lemma C.5.4 we know that the traceless curvature Θ_{tl} is of type $(2,2)$ for all possible complex structures. The result follows now from [108, Prop. 2.3] and the fact that each two points in the moduli space are connected by twistor lines, see [207, Thm. 3.2]. □

We note that in the proof of Proposition C.1.6 we did not use the condition of $\mathcal{E}$ being atomic explicitly, but only the consequence that all Chern classes (we only needed c_2) of $\mathcal{E}nd(\mathcal{E})$ stay of Hodge type. This leads to the following.

Proposition C.5.6. *A modular vector bundle $\mathcal{E}$ is ω-projectively hyperholomorphic if and only if $\mathcal{E}$ is ω-slope polystable and the projection of $\mathrm{c}_2(\mathcal{E})$ to the complement $\mathrm{SH}(X,\mathbb{Q})^{\perp}$ of the Verbitsky component stays of type $(2,2)$ for all induced complex structures of the hyper-Kähler structure.*

For example if $\mathcal{E}$ is a ω-slope polystable modular vector bundle such that $\mathrm{c}_2(\mathcal{E}) \in \mathrm{SH}(X,\mathbb{Q})$, then $\mathcal{E}$ is ω-projectively hyperholomorphic.

C.6. Deformation theory of stable atomic vector bundles

Let X be a hyper-Kähler manifold of dimension $2n$. Throughout this section we fix an H-projectively hyperholomorphic vector bundle $\mathcal{E}$ on X which is slope stable for some ample line bundle H. In particular, $\mathcal{E}$ is simple, i.e. $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}\mathrm{id}$. In this section we want to study the deformation theory of the bundle $\mathcal{E}$ on X.

C.6.1. Deformation theory

We introduce the functor and notions we want to study. For more details we refer to [197].

The deformation functor we consider is the covariant functor

$$\mathrm{Def}_{\mathcal{E}} \colon \mathrm{Art}/\mathbb{C} \to \mathrm{Sets}$$

from Artinian local $\mathbb{C}$-algebras with residue field $\mathbb{C}$ to sets which assigns to $A \in \mathrm{Art}/\mathbb{C}$ the isomorphism classes of pairs $(\mathcal{F}, t)$, where $\mathcal{F}$ is a coherent sheaf on $X \times \mathrm{Spec}(A)$ flat over $\mathrm{Spec}(A)$ and t is an isomorphism between the restriction of $\mathcal{F}$ to $X \times \mathrm{Spec}(\mathbb{C})$ and $\mathcal{E}$. The deformation functor $\mathrm{Def}_{\mathcal{E}}$ has a tangent-obstruction theory given by $T^1 = \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ and $T^2 = \mathrm{Ext}^2(\mathcal{E},\mathcal{E})_0$, where $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})_0$ denotes the kernel of the natural trace morphism

$$\mathrm{Tr} \colon \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \to \mathrm{H}^2(X,\mathcal{O}_X).$$

One can define a formal map

$$\kappa = \kappa_2 + \kappa_3 + \ldots \colon \widehat{\mathrm{Ext}^1(\mathcal{E},\mathcal{E})} \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})_0,$$

called *Kuranishi map*, whose scheme-theoretic fibre $\kappa^{-1}(0)$ is the base space of the formal semiuniversal deformation of $\mathcal{E}$. The quadratic part κ_2 is the usual Yoneda pairing.

C.6.2. Formality

The main result of this section is the following, which will also imply Theorem C.6.1 from the introduction.

Theorem C.6.1. *Let $\mathcal{E}$ be an H-projectively hyperholomorphic vector bundle on a hyper-Kähler manifold which is H-slope stable. Then the dg algebra* $\mathrm{R}\mathcal{H}om(\mathcal{E}^{\oplus k}, \mathcal{E}^{\oplus k})$ *is formal for all* $k > 0$.

Recall that a dg algebra is formal if it is quasi-isomorphic to its cohomology algebra.

For K3 surfaces, the study of formality of the endomorphism algebra goes back to work of Kaledin–Lehn [115] and Kaledin–Lehn–Sorger [116]. They proved the result for direct sums of ideal sheaves of zero-dimensional subvarieties. Zhang [220] and later Budur–Zhang [46] extended it to all slope polystable sheaves on K3 surfaces. The main ingredient in all of the proofs is the following result of Kaledin [114, Thm. 4.2].

Theorem C.6.2. *Let $\mathcal{A}^\bullet$ be a dg algebra of quasi-coherent and flat sheaves on an integral scheme X and denote by $\mathcal{B}^\bullet$ its cohomology algebra. Assume that the sheaves $\mathcal{B}^\bullet$ are coherent and flat on X and that for all $i, l \in \mathbb{Z}$ the degree l component $\mathcal{HH}^i_l(\mathcal{B}^\bullet)$ of the i-th Hochschild cohomology sheaf $\mathcal{HH}^i(\mathcal{B}^\bullet)$ is also coherent and flat.*

(i) *For X affine, formality of $\mathcal{A}^\bullet_x$ over a generic point $x \in X$ implies formality for all points $x \in X$.*

(ii) *If $\mathcal{HH}^2_l(\mathcal{B}^\bullet)$ has no global sections for all $l \leq -1$, then the dg algebra $\mathcal{A}^\bullet_x$ is formal for all $x \in X$.*

We will also apply this statement to prove the main result. Our proof follows roughly the arguments of [115, Prop. 3.1] and [220, Thm. 1.3] with the necessary modifications.

Proof of Theorem C.6.1. We consider the induced hyper-Kähler metric on X and the induced twistor line $\pi \colon \mathcal{X} \to \mathbb{P}^1$. We can lift the bundle $\mathcal{E}nd(\mathcal{E}^{\oplus k}, \mathcal{E}^{\oplus k})$ to a holomorphic bundle $\mathcal{F}$ on $\mathcal{X}$ [117, Thm. 5.12]. Consider the sheaf of dg algebras

$$\mathrm{R}\mathcal{H}om_{\mathcal{X}/\mathbb{P}^1}(\mathcal{O}_{\mathcal{X}}, \mathcal{F}) = \mathrm{R}\pi_*\mathrm{R}\mathcal{H}om(\mathcal{O}_{\mathcal{X}}, \mathcal{F})$$

on $\mathbb{P}^1$ and the sheaf of algebras $\mathcal{B}^\bullet = \mathcal{E}xt^\bullet_{\mathcal{X}/\mathbb{P}^1}(\mathcal{O}_{\mathcal{X}}, \mathcal{F})$ associated to the dg algebra by taking cohomology.

Verbitsky [210, Prop. 6.3] proved that

$$\mathrm{R}^i\pi_*(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^1}(i) \otimes_{\mathbb{C}} \mathrm{H}^i(X, \mathcal{F}) \tag{C.6.1}$$

for all $i \in \mathbb{Z}$. Since $\mathcal{B}^i = \mathrm{R}^i\pi_*(\mathcal{F})$, we conclude that the sheaves of algebras $\mathcal{B}^\bullet$ are coherent and flat. Moreover, (C.6.1) shows that $\mathcal{B}^\bullet$ is locally constant as a sheaf of algebras. This implies that its Hochschild cohomology sheaves $\mathcal{HH}^\bullet(\mathcal{B}^\bullet)$ are locally trivial and we can apply Theorem C.6.2. The proof proceeds now as the proof of [115, Prop. 3.1]. ☐

Using Proposition C.1.6 we see that Theorem C.6.1 also proves Theorem C.1.7 from the introduction.

C.6.3. Moduli spaces

For a slope stable projectively hyperholomorphic vector bundle $\mathcal{E}$ Verbitsky showed that $\mathcal{E}$ satisfies the quadraticity property [208, Thm. 6.2, 11.2]. That is, the scheme-theoretic fibre $\kappa^{-1}(0)$ of the Kuranishi map is isomorphic to the fibre κ_2^{-1} of its quadratic part.

Note that formality for the dg algebra $\mathrm{R}\mathcal{H}om(\mathcal{E},\mathcal{E})$ implies formality of the dg Lie algebra associated to $\mathrm{R}\mathcal{H}om(\mathcal{E},\mathcal{E})$. If a dg Lie Algebra has trivial differential $d = 0$, then it is well-known that the equations defining the versal deformation space are quadratic [79]. In particular, if $\mathrm{R}\mathcal{H}om(\mathcal{E},\mathcal{E})$ is formal, then its versal deformation space is cut out by quadrics. Hence, we recover the above result of Verbitsky.

Corollary C.6.3. *Let $\mathcal{E}$ be a slope stable projectively hyperholomorphic vector bundle. Then its associated versal deformation space $\kappa^{-1}(0)$ is isomorphic to $\kappa_2^{-1}(0)$ and has at most quadratic singularities.*

Thus, to study (locally) the moduli space of slope stable atomic vector bundles $\mathcal{E}$ one is lead to the study of the pairing

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$$

whose induced quadratic map $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ yields κ_2. We state here the following.

Conjecture F. *Let $\mathcal{E}$ be a slope stable atomic vector bundle. Then the pairing*

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$$

is skew-symmetric.

Conjecture F implies that the moduli space of slope stable torsion-free sheaves with Mukai vector $v = v(\mathcal{E})$ is smooth at the point $[E] \in M(v)$ corresponding to the stable atomic bundle $\mathcal{E}$.

We could prove formality using the concept of (projective) hyperholomorphicity. Considering Conjecture E we see that the bundle $\mathcal{E}$ in Conjecture F is speculated to be 1-obstructed. We believe that this property could enable one to prove smoothness at the point $[\mathcal{E}]$ of the moduli space corresponding to the stable atomic bundle.

We note here the following.

Corollary C.6.4. *Let X be a hyper-Kähler manifold and $\mathcal{E}$ a projectively hyperholomorphic bundle such that $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \cong \mathbb{C}$. Then $\mathcal{E}$ satisfies Conjecture F.*

Proof. By assumption the trace morphism

$$\mathrm{Tr}_{\mathcal{E}} \colon \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \to \mathrm{H}^2(X,\mathcal{O}_X)$$

is an isomorphism in this case and the composition

$$\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^j(\mathcal{E},\mathcal{E}) \xrightarrow{\circ} \mathrm{Ext}^{i+j}(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{Tr}_{\mathcal{E}}} \mathrm{H}^{i+j}(X,\mathcal{O}_X)$$

is well-known to be graded-commutative. □

For more evidence for Conjecture F see Proposition C.7.6.

C.7. Atomic Lagrangian

Lagrangian submanifolds inside hyper-Kähler manifolds are an active part of current research. We recommend [106] for an account of some of the known results and questions. We want to discuss in this section Lagrangian submanifolds with a view towards atomicity.

C.7.1. Definition and structural result

We make the following definition.

Definition C.7.1. We call a connected Lagrangian submanifold $\iota\colon L \subset X$ *atomic* if $\iota_*\mathcal{O}_L$ is an atomic sheaf.

The main goal of this section is to prove Theorem C.1.8 from the introduction which completely determines when a Lagrangian submanifold is atomic.

In what follows, we will frequently implicitly use a result due to Voisin [212, Lem. 1.5]. It says that the kernel $\mathrm{Ker}(\iota^*) \subset \mathrm{H}^2(X,\mathbb{Q})$ of the pullback morphism

$$\iota^*\colon \mathrm{H}^2(X,\mathbb{Q}) \to \mathrm{H}^2(L,\mathbb{Q})$$

is equal to the kernel of the composition

$$\iota_*[L] \wedge \_\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\iota^*} \mathrm{H}^2(L,\mathbb{Q}) \xrightarrow{\iota_*} \mathrm{H}^{2n+2}(X,\mathbb{Q})$$

given by cupping with the fundamental class $\iota_*[L] \in \mathrm{H}^{2n}(X,\mathbb{Q})$ for a Lagrangian submanifold $L \subset X$.

Proposition C.7.2. *Let $\iota\colon L \subset X$ be a connected Lagrangian submanifold and denote by $W \subset \mathrm{HT}^2(X)$ the kernel of the contraction morphism*

$$\mathrm{HT}^2(X) \to \mathrm{H}^*(X,\mathbb{C}), \quad \mu \mapsto \mu \lrcorner \iota_*[L]$$

acting on the fundamental class $\iota_*[L] \in \mathrm{H}^{2n}(X,\mathbb{Q})$. Then, there is an isomorphism

$$W \cong \mathrm{Ker}(\iota^*)$$

of vector spaces with the kernel $\mathrm{Ker}(\iota^*) \subset \mathrm{H}^2(X,\mathbb{C})$ *of the pullback morphism*

$$\iota^*\colon \mathrm{H}^2(X,\mathbb{C}) \to \mathrm{H}^2(L,\mathbb{C}).$$

Proof. First, observe that the subspace $\mathrm{H}^2(X,\mathcal{O}_X)$ is naturally contained in $\mathrm{HT}^2(X)$ as well as $\mathrm{H}^2(X,\mathbb{C})$ and the action given by contraction agrees with the cup product. Since L is Lagrangian, we therefore have

$$W \supset \mathrm{H}^2(X,\mathcal{O}_X) \subset \mathrm{Ker}(\iota^*).$$

Moreover, for a symplectic form $\sigma \in \mathrm{H}^0(X,\Omega_X^2)$ there is an $\mathfrak{sl}_2$-triple

$$(e_\sigma, h_\sigma, \Lambda_\sigma) \subset \mathfrak{g}(X)_\mathbb{C},$$

where $e_\sigma = \sigma \wedge \_$ is the operator given by cupping with σ and $h_{\sigma|_{\mathrm{H}^{p,q}}} = (p-n)\mathrm{id}$, see [75] and [201, Sec. 2]. The action of $\mathrm{H}^0(X, \wedge^2\mathcal{T}_X)$ on $\mathrm{H}^*(X,\mathbb{C})$ via contraction agrees with the action of Λ_σ up to a constant.

Indeed, both operators are contained in the LLV algebra $\mathfrak{g}(X)_\mathbb{C}$ and the subspace of the LLV algebra consisting of operators sending $\mathrm{H}^{p,q}(X)$ to $\mathrm{H}^{p-2,q}(X)$ is one-dimensional. That is, up to scaling, there exists a unique operator having degree -2 for the grading given by h and degree 2 for the grading given by h'.

Since $L \subset X$ is Lagrangian we have $e_\sigma(\iota_*[L]) = 0$. This yields

$$0 = h_\sigma(\iota_*[L]) = [e_\sigma, \Lambda_\sigma](\iota_*[L]) = e_\sigma(\Lambda_\sigma(\iota_*[L])) - \Lambda_\sigma(e_\sigma(\iota_*[L])) = e_\sigma(\Lambda_\sigma(\iota_*[L])). \quad \text{(C.7.1)}$$

As e_σ has the Hard Lefschetz property for the grading given by h_σ, we conclude that $e_\sigma(\Lambda_\sigma(\iota_*[L])) = 0$ is equivalent to $\Lambda_\sigma(\iota_*[L]) = 0$.

It remains to identify $\mathrm{H}^1(X,\Omega^1_X) \cap \mathrm{Ker}(\iota^*)$ and $\mathrm{H}^1(X,\mathcal{T}_X) \cap W$. The image of the contraction map

$$\mathrm{H}^1(X,\mathcal{T}_X) \to \mathrm{H}^{2n}(X,\mathbb{C}), \quad \mu \mapsto \mu \lrcorner \iota_*[L]$$

is contained in $\mathrm{H}^{n-1,n+1}(X)$. As recalled above, the operator e_σ is injective when restricted to the subspace $\mathrm{H}^{n-1,n+1}(X)$. Hence, the subspace $\mathrm{H}^1(X,\mathcal{T}_X) \cap W$ is equal to the kernel of the morphism

$$\mathrm{H}^1(X,\mathcal{T}_X) \to \mathrm{H}^{n+1,n+1}(X), \quad \mu \mapsto e_\sigma(e_\mu(\iota_*[L])) \quad \text{(C.7.2)}$$

where as before $e_\mu \in \mathfrak{g}(X)_\mathbb{C}$ denotes the operator given by contraction with μ. Since L is Lagrangian we have that

$$[e_\sigma, e_\mu](\iota_*[L]) = e_\sigma(e_\mu(\iota_*[L]))$$

which means that the kernel of (C.7.2) is equal to the kernel of the morphism

$$\mathrm{H}^1(X,\mathcal{T}_X) \to \mathrm{H}^{n+1,n+1}(X), \quad \mu \mapsto [e_\sigma, e_\mu](\iota_*[L]). \quad \text{(C.7.3)}$$

Lemma C.7.3 below shows that the symplectic form σ induces the isomorphism

$$[e_\sigma, \_] \colon \mathrm{H}^1(X,\mathcal{T}_X) \cong \mathrm{H}^1(X,\Omega^1_X), \quad \mu \mapsto -\mu \lrcorner \sigma, \quad \text{(C.7.4)}$$

where we identified the spaces $\mathrm{H}^1(X,\mathcal{T}_X)$ and $\mathrm{H}^1(X,\Omega^1_X)$ with the operators they induce inside $\mathfrak{g}(X)_\mathbb{C}$.

In particular, this implies that the kernel of (C.7.3), which is equal to $W \cap \mathrm{H}^1(X,\mathcal{T}_X)$, is via (C.7.4) identified with the kernel of

$$\mathrm{H}^1(X,\Omega^1_X) \to \mathrm{H}^{n+1,n+1}(X), \quad \omega \mapsto \omega \wedge \iota_*[L].$$

Recalling the result due to Voisin alluded to above finishes the proof. ☐

Lemma C.7.3. *Consider a symplectic form* $\sigma \in \mathrm{H}^0(X,\Omega^2_X)$ *and let us identify the subspaces* $\mathrm{H}^1(X,\mathcal{T}_X)$ *and* $\mathrm{H}^1(X,\Omega^1_X)$ *with the subspaces*

$$\mathrm{H}^1(X,\mathcal{T}_X) \hookrightarrow \mathfrak{g}(X)_\mathbb{C}, \quad \mu \mapsto e_\mu \quad \textit{and} \quad \mathrm{H}^1(X,\Omega^1_X) \hookrightarrow \mathfrak{g}(X)_\mathbb{C}, \quad \omega \mapsto e_\omega$$

via the corresponding operators they induce. Then, the morphism

$$[e_\sigma, \_] \colon \mathfrak{g}(X)_\mathbb{C} \to \mathfrak{g}(X)_\mathbb{C}, \quad f \mapsto [e_\sigma, f]$$

induces the isomorphism

$$\mathrm{H}^1(X,\mathcal{T}_X) \cong \mathrm{H}^1(X,\Omega^1_X), \quad \mu \mapsto -\mu \lrcorner \sigma.$$

Proof. Note first that the morphism is well-defined, as the operator $[e_\sigma, e_\mu]$ has degree 2 for the grading given by h and degree 0 for the grading given by h' and is, therefore, contained in $\mathrm{H}^1(X, \Omega_X^1) \subset \mathfrak{g}(X)_{\mathbb{C}}$. Moreover, this subspace acts faithfully on the fundamental class $1 \in \mathrm{H}^0(X, \mathbb{C})$. Thus, we can compute

$$[e_\sigma, e_\mu](1) = e_\sigma(e_\mu(1)) - e_\mu(e_\sigma(1)) = -\mu \lrcorner \sigma \in \mathrm{H}^1(X, \Omega_X^1)$$

which yields the assertion. □

With these preparations we are now ready to give the promised proof of the main result of this section.

Proof of Theorem C.1.8. Step 1. Let us first show that the conditions in the theorem are sufficient for a connected Lagrangian submanifold to be atomic.

By Proposition C.3.1 the sheaf $\iota_*\mathcal{O}_L$ is atomic if and only if $\mathrm{Ann}(v(\iota_*\mathcal{O}_L))$ has the right dimension. An element $\omega \in \mathrm{H}^{1,1}(X, \mathbb{Q})$ yields an operator $e_\omega \in \mathfrak{g}(X)$ which can be integrated to the isomorphism $\exp(\omega)$. Moreover, the Lie subalgebras $\mathrm{Ann}(v(\iota_*\mathcal{O}_L))$ and $\mathrm{Ann}(v(\iota_*\mathcal{O}_L)\exp(\omega))$ are adjoint to each other and have, therefore, the same dimension.

By assumption, there exists $\omega \in \mathrm{H}^{1,1}(X, \mathbb{Q})$ with the property that $\iota^*(\omega) = -\mathrm{c}_1(L)/2$. From Lemma C.7.4 below we infer

$$v(\iota_*\mathcal{O}_L)\exp(\omega) = \iota_*[L].$$

Using Theorem C.1.2 and Remark C.4.1 the above discussion shows that $\mathcal{E}$ is atomic if and only if the map

$$\mathrm{HT}^2(X) \to \mathrm{H}\Omega_2(X), \quad \mu \mapsto \mu \lrcorner \iota_*[L]$$

has a one-dimensional image. This follows by assumption employing Proposition C.7.2.

Step 2. Conversely, let us assume that $\iota_*\mathcal{O}_L$ is atomic. The degree $2n$ component of $v(\iota_*\mathcal{O}_L)$ is equal to $\iota_*[L]$. Therefore, as $\iota_*\mathcal{O}_L$ is atomic, the $b_2(X) - 1$-dimensional kernel of the cohomological obstruction map $\mathrm{obs}_{\iota_*\mathcal{O}_L}$ is contained in the kernel of

$$\varphi\colon \mathrm{HT}^2(X) \to \mathrm{H}\Omega_2(X), \quad \mu \mapsto \mu \lrcorner \iota_*[L].$$

Note that the kernel $\mathrm{Ker}(\varphi) \subset \mathrm{HT}^2(X)$ of φ has codimension at least one, because

$$\varphi|_{\mathrm{H}^1(X,\mathcal{T}_X)}\colon \mathrm{H}^1(X, \mathcal{T}_X) \to \mathrm{H}^{2n}(X, \mathbb{C}), \quad \mu \mapsto \mu \lrcorner \iota_*[L]$$

is non-trivial by Lemma C.7.5 below. Using Proposition C.7.2 we see that the image $\mathrm{Im}(\iota^*)$ of the pullback morphism is one-dimensional.

Step 3. It remains to show that $\mathrm{c}_1(L) \in \mathrm{H}^2(L, \mathbb{Q})$ is contained in the image of ι^*. This uses a variant of the proof of [198, Prop. B.2]. We first consider the case that $\iota_*\mathrm{c}_1(L) = 0$, which is a guideline for the general case.

Since L is Lagrangian, the operator e_σ acts trivially on $v(\iota_*\mathcal{O}_L)$. Using that $\iota_*\mathcal{O}_L$ is atomic, we know from Theorem C.1.2 that there exists $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ such that $\Lambda_\sigma - e_\mu \in \mathrm{Ker}(\mathrm{obs}_{\iota_*\mathcal{O}_L}) \subset \mathrm{HT}^2(X)$, where we used again that for a symplectic form σ the action of the operator Λ_σ agrees up to a constant with the action of $\mathrm{H}^0(X, \Lambda^2\mathcal{T}_X)$. By Lemma C.7.4 this yields

$$\Lambda_\sigma(\iota_*\mathrm{c}_1(L)^2/8) = e_\mu(\iota_*\mathrm{c}_1(L)/2.) = \mu \lrcorner \iota_*\mathrm{c}_1(L)/2 \in \mathrm{H}^{n,n+2}(X).$$

Since Λ_σ is injective when restricted to $\mathrm{H}^{n+2,n+2}(X)$ it immediately follows that also $\iota_*\mathrm{c}_1(L)^2$ vanishes, because we assumed $\iota_*\mathrm{c}_1(L) = 0$.

Consider now a Kähler class $\omega \in \mathrm{H}^{1,1}(X)$ which restricts to a Kähler class on L. The projection formula yields

$$\iota_*(\mathrm{c}_1(L) \cdot \iota^*\omega^{n-1}) = \iota_*\mathrm{c}_1(L) \cdot \omega^{n-1} = 0$$

which, as ι_* is injective restricted to $\mathrm{H}^{2n}(L, \mathbb{C})$, implies that $\mathrm{c}_1(L)$ is $\iota^*\omega$-primitive. Applying once more the projection formula

$$\iota_*(\mathrm{c}_1(L)^2 \cdot \iota^*\omega^{n-2}) = \iota_*\mathrm{c}_1(L)^2 \cdot \omega^{n-2} = 0$$

together with the injectivity of ι_* on top degree and the Hodge–Riemann bilinear relations yields that $\mathrm{c}_1(L) = 0 \in \mathrm{H}^2(X, \mathbb{Q})$.

Step 4. Let us now consider the case $\iota_*\mathrm{c}_1(L) \neq 0$. The degree $2n+2$-component of the Mukai vector $v(\iota_*\mathcal{O}_L)$ of $\iota_*\mathcal{O}_L$ is by Lemma C.7.4 equal to $\iota_*\mathrm{c}_1(L)/2$. Since $\iota_*\mathcal{O}_L$ is atomic, by Theorem C.1.2 to a given symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ there exists as above $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ such that $e_\mu - \Lambda_\sigma \in \mathrm{Ker}(\mathrm{obs}_{\iota_*\mathcal{O}_L}) \subset \mathrm{HT}^2(X)$. This implies

$$e_\mu(\iota_*[L]) = \Lambda_\sigma(\iota_*\mathrm{c}_1(L)/2) \neq 0.$$

Applying e_σ to this equality and noting once more that this operator has trivial kernel restricted to $\mathrm{H}^{n-1,n+1}(X)$ we obtain the equality

$$e_\sigma(e_\mu(\iota_*[L])) = e_\sigma(\Lambda_\sigma(\iota_*\mathrm{c}_1(L)/2)).$$

Since L is Lagrangian, we know $e_\sigma(\iota_*\mathrm{c}_1(L)/2) = e_\sigma(\iota_*[L]) = 0$. The above equality can, therefore, be written as

$$[e_\sigma, e_\mu](\iota_*[L]) = [e_\sigma, \Lambda_\sigma](\iota_*\mathrm{c}_1(L)/2) = h_\sigma(\iota_*\mathrm{c}_1(L)/2) = \iota_*\mathrm{c}_1(L)/2.$$

Lemma C.7.3 shows that $[e_\sigma, e_\mu]$ is equal to e_ω for some $\omega \in \mathrm{H}^1(X, \Omega_X^1)$.

Step 5. We claim that we can assume that $\pm\omega$ is a Kähler class.

Indeed, we have already proven that the image of the restriction morphism

$$\iota^* \colon \mathrm{H}^1(X, \Omega_X^1) \to \mathrm{H}^1(L, \Omega_L^1)$$

is one-dimensional. Hence, there exists a Kähler class $\tilde{\omega} \in \mathrm{H}^1(X, \Omega_X^1)$ whose image $\iota^*\tilde{\omega}$ is a Kähler class and generates $\mathrm{Im}(\iota^*)$. Thus, there exists $k \in \mathbb{C}$ such that $\iota^*\omega = k\iota^*\tilde{\omega}$ for ω from above. Moreover, Lemma C.7.3 shows that there exists $\tilde{\mu} \in \mathrm{H}^1(X, \mathcal{T}_X)$ such that

$$-\tilde{\mu} \lrcorner \sigma = -e_{\tilde{\mu}}(\sigma) = k\tilde{\omega}.$$

In particular, using once more Lemma C.7.3 we obtain

$$[e_\sigma, e_\mu](\iota_*[L]) = e_\omega(\iota_*[L]) = \omega \wedge \iota_*[L] = k\tilde{\omega} \wedge \iota_*[L] = [e_\sigma, e_{\tilde{\mu}}](\iota_*[L]).$$

This shows that the element $\mu - \tilde{\mu} \in \mathrm{H}^1(X, \mathcal{T}_X)$ is contained in the kernel of $\mathrm{obs}_{\iota_*\mathcal{O}_L}$ and all the above arguments remain valid replacing μ with $\tilde{\mu}$.

Step 6. Summing up the above discussion, we obtain the equality

$$e_\omega(\iota_*[L]) = \iota_*[L] \wedge \omega = \iota_*([L] \wedge \iota^*\omega) = \iota_*\mathrm{c}_1(L)/2 \tag{C.7.5}$$

for $\omega = -\mu \lrcorner \sigma \in \mathrm{H}^1(X, \Omega^1_X)$ a (possibly negative) multiple of a Kähler class and $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ such that $\Lambda_\sigma - e_\mu \in \mathrm{Ker}(\mathrm{obs}_{\iota_*\mathcal{O}_L}) \subset \mathrm{HT}^2(X)$.

Repeating this argument with the same ω and μ we get again by Lemma C.7.4

$$\Lambda_\sigma(\iota_*\mathrm{c}_1(L)^2/8) = e_\mu(\iota_*\mathrm{c}_1(L)/2).$$

As before, applying e_σ we deduce

$$\iota_*\mathrm{c}_1(L)^2/4 = e_\omega(\iota_*\mathrm{c}_1(L)/2). \tag{C.7.6}$$

One now concludes the proof as in the case $\iota_*\mathrm{c}_1(L) = 0$. We sketch the argument. First, $\mathrm{c}_1(L)/2 - \iota^*\omega$ is $\iota^*\omega$-primitive using (C.7.5). Moreover

$$(\mathrm{c}_1(L)/2 - \iota^*\omega)^2 \iota^*\omega^{n-2} = (\mathrm{c}_1(L)^2/4 - \iota^*\omega \wedge \mathrm{c}_1(L) + \iota^*\omega^2)\iota^*\omega^{n-2}$$

vanishes by employing (C.7.6). Invoking the Hodge–Riemann bilinear relations yields $\mathrm{c}_1(L)/2 = \iota^*\omega$. This finishes the proof. □

It remains to prove the two lemmata used in the above proof.

Lemma C.7.4. *Let X be a smooth symplectic projective manifold and $\iota\colon L \subset X$ a smooth Lagrangian submanifold. Then $v(\iota_*\mathcal{O}_L) = \iota_*\exp(\mathrm{c}_1(L)/2)$.*

Proof. It is well-known that the normal bundle sequence

$$0 \to \mathcal{T}_L \to \mathcal{T}_X|_L \to \mathcal{N}_{L|X} \to 0$$

combined with the isomorphism $\sigma\colon \mathcal{T}_X \cong \Omega_X$, the short exact sequence

$$0 \to \mathcal{N}^\vee_{L|X} \to \Omega_X|_L \to \Omega_L \to 0$$

and the fact that L is Lagrangian yield $\mathcal{N}_{L|X} \cong \Omega_L$.

Using the Grothendieck–Riemann–Roch theorem we get

$$\mathrm{ch}(\iota_*(\mathcal{O}_L))\mathsf{td}(X) = \iota_*(\mathrm{ch}(\mathcal{O}_L)\mathsf{td}(L)) = \iota_*\mathsf{td}(L).$$

Multiplying the above equation by $\mathsf{td}(X)^{-1/2}$ we obtain

$$v(\iota_*\mathcal{O}_L) = \iota_*(\mathsf{td}(L) \cdot \iota^*\mathsf{td}(X)^{-1/2}).$$

The previous paragraph yields

$$\iota^*\mathsf{td}(X) = \mathsf{td}(\mathcal{T}_X|_L) = \mathsf{td}(L) \cdot \mathsf{td}(\Omega_L).$$

From this we obtain

$$v(\iota_*\mathcal{O}_L) = \iota_*(\mathsf{td}(L) \cdot \mathsf{td}(L)^{-1/2} \cdot \mathsf{td}(\Omega_L)^{-1/2}) = \iota_*(\mathsf{td}(L)^{1/2} \cdot \mathsf{td}(\Omega_L)^{-1/2}). \tag{C.7.7}$$

Recall that given the formal Chern roots e_i of a bundle $\mathcal{E}$ its Todd class is the product

$$\mathsf{td}(\mathcal{E}) = \prod_i Q(e_i)$$

where

$$Q(x) = \frac{x}{1 - e^{-x}}.$$

The assertion is now a consequence from the identity

$$\frac{x}{1-e^{-x}} \cdot \left(\frac{-x}{1-e^{x}}\right)^{-1} = \frac{x}{1-e^{-x}} \cdot \frac{e^x - 1}{x} = \frac{e^x-1}{1-e^{-x}} = e^x$$

applied to (C.7.7). □

Lemma C.7.5. *Let X be a hyper-Kähler manifold and $\iota\colon L \subset X$ a Lagrangian subvariety. Then the morphism*

$$\mathrm{H}^1(X, \mathcal{T}_X) \to \mathrm{H}^*(X, \mathbb{C}), \quad \mu \mapsto \mu \lrcorner \iota_*[L]$$

is non-trivial.

Proof. The assertion can be deduced from results of Voisin [212, Sec. 1]. We want to give another proof using the LLV algebra.

By assumption, as $\iota\colon L \subset X$ is Lagrangian, we know that

$$\sigma \wedge \iota_*[L] = 0 = \bar{\sigma} \wedge \iota_*[L] \in \mathrm{H}^*(X, \mathbb{C})$$

for $\sigma, \bar{\sigma}$ the (anti-)holomorphic two-form. Using again (C.7.1) we see that $\Lambda_\sigma(\iota_*[L]) = 0$. Hence, assuming

$$\mathrm{H}^1(X, \mathcal{T}_X) \to \mathrm{H}^*(X, \mathbb{C}), \quad \mu \mapsto \mu \lrcorner \iota_*[L]$$

to be trivial implies that

$$\mathrm{HT}^2(X) \to \mathrm{H}^*(X, \mathbb{C}), \quad \mu \mapsto \mu \lrcorner \iota_*[L]$$

is also trivial. As demonstrated in the proof of Proposition C.3.1 this would imply that $\iota_*[L]$ is annihilated by the LLV algebra. We obtain a contradiction, as there exists a Kähler class $\omega \in \mathrm{H}^2(X, \mathbb{C})$ which restricts non-trivially to L and, therefore, $e_\omega(\iota_*[L]) \neq 0$. □

The statement of the lemma can be interpreted by saying that no Lagrangian subvariety can be deformed (cohomologically) along with to all Kähler deformations of X.

C.7.2. 1-Obstructedness

Atomic Lagrangians $\iota\colon L \subset X$ and the sheaves $\iota_*\mathcal{L}$ for $\mathcal{L} \in \mathrm{Pic}^0(L)$ are a good testing ground for Conjecture E. By Corollary C.4.7 it is equivalent to study whether these sheaves are 1-obstructed. In this section, we discuss the obstruction map for atomic Lagrangians. See also [139, Sec. 3.1] for a related discussion.

Recall that by adjunction the group $\mathrm{Ext}^2(\iota_*\mathcal{O}_L, \iota_*\mathcal{O}_L)$ decomposes into

$$\mathrm{Ext}^2(\iota_*\mathcal{O}_L, \iota_*\mathcal{O}_L) \cong \mathrm{H}^2(L, \mathcal{O}_L) \oplus \mathrm{H}^1(L, \Omega^1_L) \oplus \mathrm{H}^0(L, \Omega^2_L).$$

Similarly, the degree two polyvector fields $\mathrm{HT}^2(X)$ decompose by definition as

$$\mathrm{HT}^2(X) = \mathrm{H}^2(X, \mathcal{O}_X) \oplus \mathrm{H}^1(X, \mathcal{T}_X) \oplus \mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X).$$

Using these decompositions we want to refine the study of the obstruction map

$$\_\lrcorner \left(\mathrm{At}^0_{\iota_*\mathcal{O}_L} + \mathrm{At}_{\iota_*\mathcal{O}_L} + \mathrm{At}^2_{\iota_*\mathcal{O}_L}/2\right) : \mathrm{HT}^2(X) \to \mathrm{Ext}^2(\iota_*\mathcal{O}_L, \iota_*\mathcal{O}_L).$$

The fact that L is Lagrangian implies immediately that $\mathrm{At}^0_{\iota_*\mathcal{O}_L} \lrcorner \bar{\sigma}$ vanishes for $\bar{\sigma} \in \mathrm{H}^2(X, \mathcal{O}_X)$. The induced map

$$\mathrm{H}^1(X, \mathcal{T}_X) \to \mathrm{H}^1(L, \Omega^1_L)$$

is induced by the morphism $\mathcal{T}_X \to \mathcal{N}_{L|X}$ together with the isomorphism $\mathcal{N}_{L|X} \cong \Omega^1_L$. Under the isomorphism $\Omega^1_X \cong \mathcal{T}_X$ the composition

$$\mathrm{H}^1(X, \Omega^1_X) \to \mathrm{H}^1(L, \Omega^1_L)$$

agrees (up to a constant) with the pullback map on cohomology.

The most difficult piece is to study the induced map

$$\psi \colon \mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X) \to \mathrm{H}^2(L, \mathcal{O}_L) \oplus \mathrm{H}^1(L, \Omega^1_L) \oplus \mathrm{H}^0(L, \Omega^2_L).$$

The morphism $\mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X) \to \mathrm{H}^0(L, \Omega^2_L)$ is again zero due to L being Lagrangian. However, the map ψ is not equal to the projection to this component.

Indeed, Lemma C.7.4 and Theorem C.1.8 show that as soon as $\mathrm{c}_1(\omega_L) \in \mathrm{H}^2(X, \mathbb{Q})$ is non-trivial, then the degree $4n$ component of $v(\iota_*\mathcal{O}_L)$ is non-trivial. In particular, the operator Λ_σ, whose action agrees with $\mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X)$ up to multiples, acts non-trivially on $v(\iota_*\mathcal{O}_L)$. Lemma C.4.2 then shows that ψ must also be non-zero.

From the proof of Theorem C.1.8 we deduce that the image of the morphism ψ projected onto the component $\mathrm{H}^1(L, \Omega^1_L)$ should be a multiple of $\mathrm{c}_1(L)$. This then would prove that the atomic sheaf $\iota_*\mathcal{O}_L$ is indeed 1-obstructed and, by Corollary C.4.7, would satisfy Conjecture E.

Note that in [139, Rem. 3.10] it is speculated that the map ψ is the zero morphism for the atomic sheaf $\iota_*\omega_L^{1/2}$. From Lemma C.7.4 we conclude that the Mukai vector of $\iota_*\omega_L^{1/2}$ is just $\iota_*[L] \in \mathrm{H}^{2n}(X, \mathbb{Q})$. In particular, the cohomological obstruction map $\mathrm{obs}_{\iota_*\omega_L^{1/2}}$ vanishes when restricted to $\mathrm{H}^0(X, \Lambda^2 \mathcal{T}_X)$. This shows that ψ is zero if and only if $\iota_*\omega_L^{1/2}$ satisfies Conjecture E. This seems to be suggested from [61] as discussed in [139, Rem. 3.10].

C.7.3. Graded Commutativity

The results from [147] imply that for an atomic Lagrangian $\iota \colon L \subset X$ we have a graded multiplicative isomorphism

$$\mathrm{Ext}^*(\iota_*\mathcal{O}_L, \iota_*\mathcal{O}_L) \cong \mathrm{H}^*(L, \mathbb{C}).$$

In particular, for all line bundles $\mathcal{L} \in \mathrm{Pic}(X)$ the above isomorphism remains valid for the atomic sheaf $\iota_*\iota^*\mathcal{L}$. This leads to the following immediate consequence.

Proposition C.7.6. *Let $\iota \colon L \subset X$ be an atomic Lagrangian and $\mathcal{L} \in \mathrm{Pic}(X)$. The algebra structure of $\mathrm{Ext}^*(\iota_*\iota^*\mathcal{L}, \iota_*\iota^*\mathcal{L})$ is graded-commutative. If X is of dimension at most four, then for all $\mathcal{M} \in \mathrm{Pic}(L)$ the algebra $\mathrm{Ext}^*(\iota_*\mathcal{M}, \iota_*\mathcal{M})$ is graded-commutative.*

Proof. The first part follows from the above discussion. For the second part we employ [147, Thm. 0.1.1] and the vanishing of $\mathrm{H}^3(L, \mathcal{O}_L)$ which implies that in the situation of *loc. cit.*

$$d_2^{1,1} \colon \mathrm{H}^1(L, \Omega_L^1) \to \mathrm{H}^3(L, \mathcal{O}_L)$$

is the zero map. □

We have stated Conjecture F only for vector bundles. The proposition shows that (a stronger form of) its conclusion holds true for line bundles supported on atomic Lagrangians.

Moreover, we see the above as evidence for Conjecture F. Let us elaborate how one might be able to prove the conjecture employing the above in the case of K3 surfaces.

Proposition C.7.7. *Let S be a K3 surface with a hyperbolic plane $U \subset \mathrm{Pic}(S)$ and $[\mathcal{E}] \in M_H(v)$ a generic point of a smooth moduli space corresponding to an H-slope stable bundle. Then there exists a smooth curve $C \subset S$, a line bundle $\mathcal{L} \in \mathrm{Pic}(C)$ and a derived equivalence $\Phi \in \mathrm{Aut}(\mathrm{D^b}(S))$ such that $\Phi(\mathcal{E}) \cong \iota_* \mathcal{L}$.*

Proof. The assumption on the Picard group of S implies that there exists an isometry of $\tilde{\mathrm{H}}(S, \mathbb{Z})$ with real spinor norm one sending $v = v(\mathcal{E})$ to the class $(0, [C], 0)$ for $C \subset S$ a smooth connected curve.

Indeed, we can write

$$\tilde{\mathrm{H}}(S, \mathbb{Z})_{\mathrm{alg}} = U \oplus U \oplus L_0$$

where the first hyperbolic plane is spanned by $\alpha = \mathsf{1}$ and $\beta = \mathsf{p}$. Using [82, Prop. 3.3] we can modify the part of v which lies in the first two hyperbolic planes as desired to have no contribution from the classes α and β.

From [109] we know that there exists an auto-equivalence $\Phi \in \mathrm{Aut}(\mathrm{D^b}(S))$ such that the induced action on cohomology agrees with the above isometry. This yields the isomorphism

$$\Phi \colon M_H(v) \cong M_\sigma(0, [C], 0)$$

for some stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$.

We consider now two cases. If $v^2 = -2$, where we use the usual convention on K3 surfaces that we multiply the generalized Mukai pairing with -1, then $M_H(v) = [\mathcal{E}]$ for the spherical bundle $\mathcal{E}$. We apply [17, Prop. 6.8] as explained in [15, Rem. 6.10] to obtain a derived equivalence Ψ acting trivially on cohomology and sending σ into the Gieseker chamber. The composition therefore satisfies

$$\Psi \circ \Phi(\mathcal{E}) \cong \mathcal{O}_C(-1)$$

for the smooth rational curve C.

If $v^2 \geq 0$ we can employ [17, Thm. 1.1] to find an equivalence Ψ sending σ into the Gieseker chamber such that the composition $\Psi \circ \Phi$ induces a birational map between $M_H(v)$ and $M_H(0, [C], 0)$. In particular, for a generic stable bundle $[\mathcal{E}] \in M_H(v)$ the composition $\Psi \circ \Phi$ sends $[\mathcal{E}]$ to a generic stable sheaf in $M_H(0, [C], 0)$, which is a line bundle supported on a curve with class $[C]$. □

The algebra structure of the Yoneda Ext algebra is invariant under derived equivalences. Using Proposition C.7.6 we get the multiplicative isomorphism

$$\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \cong \mathrm{Ext}^*(\iota_* \mathcal{L}, \iota_* \mathcal{L}) \cong \mathrm{H}^*(C, \mathbb{C}).$$

This gives another argument for the (well-known) fact that $\mathrm{Ext}^*(\mathcal{E},\mathcal{E})$ is graded-commutative. In particular, this reproves Conjecture F for the bundle $\mathcal{E}$. Note that if we start with a stable bundle $\mathcal{E}$ on an arbitrary projective K3 surface, we can always deform the surface together with $\mathcal{E}$ via twistor lines such that a hyperbolic plane is contained in its Picard group.

We expect that a similar approach could be pursued for higher-dimensional hyper-Kähler manifolds. A promising candidate would be the case of the Hilbert scheme of n points $S^{[n]}$ of a K3 surface using the results of [25].

Here is how this could be pursued. Using twistor lines and [210, Prop. 6.3] one can deform a stable atomic bundle $\mathcal{E}$ on $S^{[n]}$ to a bundle $\mathcal{E}'$ on $S'^{[n]}$ such that $U \subset \mathrm{Pic}(S')$ without modifying the Ext algebra structure. Employing [25, Prop. 9.8] we find a derived equivalence Φ mapping the Mukai vector $\tilde{v}(\mathcal{E}') = \mathrm{rk}(\mathcal{E}') + \mathrm{c}_1(\mathcal{E}') + s\beta$ of $\mathcal{E}'$ in the Mukai lattice $\tilde{\mathrm{H}}(X,\mathbb{Q})$ to one of the form $0\alpha + \lambda + k\beta$ for $\lambda \in \mathrm{H}^2(X,\mathbb{Q})$ the dual of a smooth curve $C \subset S'^{[n]}$ and some $k \in \mathbb{Q}$. However, the image $\Phi(\mathcal{E}')$ might be a priori an arbitrary complex. In the case of K3 surfaces, a solid knowledge of the stability manifold was employed to conclude. In higher-dimensions, a further study of the equivalences involved to construct Φ via [25, Prop. 9.8] could potentially shed more light on the situation.

C.7.4. Formality

We want to finish this section by discussing formality for atomic Lagrangians.

Employing [148, Thm. 0.1.2] and [46, Prop. 1.4] we get the following result.

Proposition C.7.8. *Let $\iota\colon L \subset X$ be an atomic Lagrangian and $\mathcal{L} \in \mathrm{Pic}(X)$. Assume that ω_L admits a square root. Then $\mathrm{R}\mathcal{H}om(\iota_*(\omega_L^{1/2} \otimes \iota^*\mathcal{L}), \iota_*(\omega_L^{1/2} \otimes \iota^*\mathcal{L}))$ is formal.*

Note that for a Lagrangian projective space $\mathbb{P}^n \subset X$ we know that by [91, Thm. A] $\mathrm{R}\mathcal{H}om(\iota_*\mathcal{L}, \iota_*\mathcal{L})$ is formal for all line bundles $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^n)$. See Section C.8 for further cases of line bundles on atomic Lagrangian whose associated derived endomorphism dg algebra is formal.

C.8. Examples and further properties

In this section, we discuss some example and further properties that are shared by atomic sheaves and complexes.

C.8.1. Examples of atomic objects

We will study some examples of atomic objects together with their properties. Recall that by Proposition C.3.10 being atomic is stable under derived equivalences as well as deformations. Therefore, every example produces via these two operations many more examples.

C.8.1.1. $\mathbb{P}^n$-objects

For the definition and properties of $\mathbb{P}^n$-objects, see [110].

From Theorem C.1.2 and Theorem C.1.3 we deduce.

Proposition C.8.1. *If $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is a $\mathbb{P}^n$-object, then $\mathcal{E}$ is atomic except if $v(\mathcal{E})$ is annihilated by the LLV algebra.*

Again, if Conjecture E holds, the above implication that $\mathbb{P}^n$-objects $\mathcal{E}$ are atomic holds unconditionally and their Mukai vectors $v(\mathcal{E})$ cannot be annihilated by $\mathfrak{g}(X)$.

Moreover, $\mathbb{P}^n$-objects $\mathcal{E}$ are simple by definition and the associated derived endomorphism dg algebra $\mathrm{R}\mathcal{H}om(\mathcal{E},\mathcal{E})$ is formal as shown in [91, Thm. A]. Moreover, they give further evidence for Conjecture E.

Corollary C.8.2. *Let $\mathcal{E}$ be an atomic $\mathbb{P}^n$-object. Then $\mathcal{E}$ is 1-obstructed and satisfies the conclusion of Conjecture E.*

Proof. As $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \cong \mathbb{C}$, the kernel of the obstruction map $\mathrm{Ker}(\chi_{\mathcal{E}})$ has at least dimension $b_2(X)-1$. Lemma C.4.2 shows that this kernel is contained under the modified HKR isomorphism in the kernel $\mathrm{Ker}(\mathrm{obs}_{\mathcal{E}})$ of the cohomological obstruction map. By Theorem C.1.2, this space is $b_2(X) - 1$-dimensional, which implies that $\mathcal{E}$ is 1-obstructed. The second assertion now follows from Corollary C.4.7. ☐

In particular, given an H-slope stable torsion free atomic sheaf $\mathcal{E}$ which is also a $\mathbb{P}^n$-object the connected component of the moduli space $M_H(v(\mathcal{E}))$ containing $[\mathcal{E}]$ is a smooth point. In [172], it is shown that in some examples such moduli spaces are connected.

Examples of atomic $\mathbb{P}^n$-objects are line bundles and the sheaves $\iota_*\mathcal{O}_{\mathbb{P}^n}(k)$ for $\iota\colon \mathbb{P}^n \subset X$. See also [172, Thm. 1.4] for many slope stable vector bundles on $\mathrm{K3}^{[2]}$-type hyper-Kähler manifolds which are $\mathbb{P}^n$-objects.

C.8.1.2. $k(x)$-orbit

Skyscraper sheaves of points $k(x)$ for $x \in X$ are also examples of atomic sheaves. They have the property

$$\mathrm{Ext}^*(k(x),k(x)) \cong \bigwedge^* \mathrm{Ext}^1(k(x),k(x))$$

and, therefore, the Yoneda multiplication is again graded-commutative.

Another example of this kind are Lagrangian tori in hyper-Kähler manifolds. Assume we are given a Lagrangian fibration $\pi\colon X \to \mathbb{P}^n$. A numerically trivial line bundle $\mathcal{L}$ on a generic fibre $\iota\colon A = \pi^{-1}(\mathsf{p}) \subset X$ induces the atomic sheaf $\iota_*\mathcal{L} \in \mathrm{D^b}(X)$. In [4] an example of a derived equivalence is being discussed, which extends the fibrewise Poincaré Fourier–Mukai transform. As explained in [25, Sec. 10.2] the generic skyscraper sheaf $k(x)$ for $x \in X$ is being mapped to $\iota_*\mathcal{L}$. In particular, in this situation the results of [147,148] as discussed in Section C.7 extend to all numerically trivial line bundles $\mathcal{L}$ on generic fibres $A \subset X$. That is, in these cases the local-to-global Ext spectral sequence degenerates multiplicatively and the associated derived endomorphism dg algebra is formal. Therefore, the irreducible component of the moduli space M of slope stable sheaves containing $\iota_*\mathcal{L}$ is in these cases generically smooth and an open subset of M possesses a non-degenerate symplectic form.

For examples of sheaves with positive rank being derived equivalent to skyscraper sheaves see [25, Prop. 10.1] or [139, Thm. 1.6].

C.8.1.3. Fano variety of lines on cubics

The Fano variety of lines $F(Y)$ of a smooth cubic fourfold $Y \subset \mathbb{P}^5$ admits for every smooth hyperplane section $Y \cap H$ a Lagrangian surface $\iota\colon F(Y\cap H) \subset F(Y)$. Powers $\mathcal{L}^i \in \mathrm{Pic}(F(Y\cap H))$ of the Plücker polarization yield atomic sheaves $\iota_*\mathcal{L}^i \in \mathrm{D^b}(F(Y))$.

Indeed, the cohomology $\mathrm{H}^*(F(Y), \mathbb{Q})$ agrees with the Verbitsky component in this case and applying Remark C.3.4 and the Grothendieck–Riemann–Roch Theorem, the claim follows from a straightforward Chern character computation. See also [139, Sec. 13] for images of these atomic sheaves under derived equivalences for special cubic fourfolds. Note that in this case we again have an isomorphism

$$\mathrm{Ext}^*(\iota_*\mathcal{L}^i, \iota_*\mathcal{L}^i) \cong \bigwedge^{*} \mathrm{Ext}^1(\iota_*\mathcal{L}^i, \iota_*\mathcal{L}^i) \cong \mathrm{H}^*(F(Y \cap H), \mathbb{C}).$$

C.8.1.4. Lagrangian plane in double EPW sextics

In the case of K3 surfaces, the structure of the Ext algebra of simple atomic objects only depends on one numerical value, namely the self-intersection of the Mukai vector or, equivalently, the dimension of the first extension group. The examples of atomic objects discussed above could convey the impression that Ext algebras of atomic objects on higher-dimensional hyper-Kähler manifolds may be as well easy to understand. We therefore want to give one more example where the Ext groups have interesting dimensions.

Let X be a double EPW sextic, see [72] for an overview of these varieties. The natural antisymplectic involution has a connected Lagrangian surface $\iota\colon Z \subset X$ as fixed locus, which is of general type [72, Cor. 2.9]. The relevant Hodge numbers are

$$h^{1,0} = 0, \quad h^{2,0} = 45, \quad h^{1,1} = 100,$$

see [72, Sec. 3.3]. In the proof of [72, Prop. 4.22] the following equalities

$$\iota_*[Z] = 5h^2 - \frac{\mathrm{c}_2(X)}{3}, \quad \mathrm{c}_3(\iota_*\omega_Z) = 9h \cdot \iota_*[Z], \quad \mathrm{c}_4(\iota_*\omega_Z) = \iota_*[Z]^2 - 63h^2 \cdot \iota_*[Z]$$

in $\mathrm{H}^*(X, \mathbb{Q})$ are obtained, where h is the canonical polarization on X obtained from the description as a double cover. Using $\mathrm{c}_1(Z) = -3\iota^*h \in \mathrm{H}^2(Z, \mathbb{Q})$, it is straightforward to verify that the cohomological obstruction map has one-dimensional image using Remark C.3.4.

In particular, we have that $\iota\colon Z \subset X$ is an atomic Lagrangian and $\iota_*\mathcal{O}_Z$ is an atomic sheaf. Via adjunction, we therefore have

$$\mathrm{Ext}^0(\iota_*\mathcal{O}_Z, \iota_*\mathcal{O}_Z) \cong \mathbb{C}, \quad \mathrm{Ext}^1(\iota_*\mathcal{O}_Z, \iota_*\mathcal{O}_Z) = 0, \quad \mathrm{Ext}^2(\iota_*\mathcal{O}_Z, \iota_*\mathcal{O}_Z) \cong \mathbb{C}^{190}.$$

From [72, Sec. 3.3] we know that $\mathrm{c}_1(Z) = -3\iota^*h + \tau \in \mathrm{H}^2(Z, \mathbb{Z})$ for a two-torsion class τ. Especially, in this example we have that $\mathrm{c}_1(Z)$ is not contained in the image of the restriction map

$$\iota^*\colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(Z, \mathbb{Z})$$

with integer coefficients, whereas this holds true with rational coefficients by Theorem C.1.8.

C.8.2. Tangent bundle

The following is the most prominent example of a bundle which is modular, slope stable and hyperholomorphic, but not atomic as soon as the dimension of the manifold is greater than two.

Proposition C.8.3. *Let $\mathcal{T}_X$ be the tangent bundle of a hyper-Kähler manifold X of dimension $2n > 2$ which is of* $\mathrm{K3}^{[n]}, \mathrm{Kum}_n, \mathrm{OG6}$ *or* $\mathrm{OG10}$*-type, or an arbitrary hyper-Kähler manifold of dimension four. Then $\mathcal{T}_X$ is not atomic.*

Proof. Let us assume that $\mathcal{T}_X$ is atomic. The projection $v(\mathcal{T}_X)_{\mathrm{SH}} \in \mathrm{SH}(X, \mathbb{Q})$ is non-zero and using Remark C.3.4 we must have

$$\begin{aligned} v(\mathcal{T}_X)_{\mathrm{SH}} &= \left(2n + \frac{2n-24}{24}\mathrm{c}_2(X) + \frac{120+7n}{2880}\mathrm{c}_2(X)^2 - \frac{120+n}{720}\mathrm{c}_4(X) + \dots\right)_{\mathrm{SH}} \\ &= \frac{2n}{n!} T\left(\alpha + k\beta\right)^n \end{aligned} \tag{C.8.1}$$

for some $k \in \mathbb{Q}$. From [25, Prop. 3.4] we know that there exists $r_X \in \mathbb{Q}$ such that

$$v(\mathcal{O}_X)_{\mathrm{SH}} = \frac{1}{n!} T(\alpha + r_X\beta)^n. \tag{C.8.2}$$

From equations (C.8.1) and (C.8.2) we infer that

$$k = \frac{2n-24}{2n} r_X \tag{C.8.3}$$

by comparing coefficients in degree four.

If now $n = 2$, we compare the coefficients in front of $T(\beta^2)$ in (C.8.1) and (C.8.2) to obtain the following equality in degree eight

$$\begin{aligned} 100\mathsf{td}_4^{1/2} &= \left(\frac{35}{288}\mathrm{c}_2(X)^2 - \frac{5}{72}\mathrm{c}_4(X)\right) \\ &= \left(\frac{67}{1440}\mathrm{c}_2(X)^2 - \frac{61}{360}\mathrm{c}_4(X)\right) = v(\mathcal{T}_X)_4 \in \mathrm{H}^8(X, \mathbb{Q}). \end{aligned}$$

Together with the relation $\int_X \mathsf{td} = 3$ involving $\mathrm{c}_2(X)^2$ and $\mathrm{c}_4(X)$ we obtain the unique solution

$$\int_X \mathrm{c}_2(X)^2 = 576, \quad \int_X \mathrm{c}_4(X) = -432$$

which violates the known bounds of Guan [84].

In the known examples, we proceed analogously making use of the fact that we know the generalized Fujiki constants $C(\mathrm{c}_2(X)^2)$ and $C(\mathrm{c}_4(X))$ through knowing the Riemann–Roch polynomial [30, Cor. 2.7]. Recall that the knowledge of the generalized Fujiki constant $C(\gamma)$ of a class $\gamma \in \mathrm{H}^{4s}(X, \mathbb{Q})$ is precisely knowing the projection $\gamma_{\mathrm{SH}} \in \mathrm{SH}^{4s}(X, \mathbb{Q})$ for a class γ which stays of type $(2s, 2s)$ on all deformations.

From (C.8.2) we infer

$$C(\mathsf{td}_4^{1/2})\mathsf{q}_4 = \frac{1}{n!}\binom{n}{2} r_x^2 T(\alpha^{n-2}\beta^2),$$

where $\mathsf{q}_4 \in \mathrm{SH}^8(X, \mathbb{Q})$ is defined by the property

$$\int_X \lambda^{2n-4}\mathsf{q}_4 = \mathrm{q}(\lambda)^{n-2}$$

for all $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$. Analogously to the four-dimensional case, using (C.8.1) and (C.8.3) we get

$$C(v(\mathcal{T}_X)_4)\mathfrak{q}_4 = \frac{2n}{n!}\binom{n}{2}\left(\frac{2n-24}{2n}\right)^2 r_X^2 T(\alpha^{n-2}\beta^2).$$

Combining these two equations, we obtain an equation involving $C(\mathrm{c}_2(X)^2)$ and $C(\mathrm{c}_4(X))$ which is violated in all the known examples, see [30, Sec. 4]. ☐

Remark C.8.4. In particular, in all of the above cases the tangent bundle is not 1-obstructed. We know that the tangent bundle does deform along to all geometric deformations coming from $\mathrm{H}^1(X, \mathcal{T}_X)$. Together with Lemma C.4.2 we infer that the two noncommutative first order deformation directions, namely the gerby and the Poisson deformations, yield different obstructions in $\mathrm{Ext}^2(\mathcal{T}_X, \mathcal{T}_X)$.

C.8.3. Hard Lefschetz

We discuss here a possible $\mathfrak{sl}_2$-structure on the Ext algebra $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$ for simple atomic sheaves and complexes.

Recall the following result due to Verbitsky [208, Thm. 4.2A].

Theorem C.8.5. *Let $\mathcal{E}$ be a slope stable (projectively) hyperholomorphic bundle. The image of $\bar{\sigma} \in \mathrm{H}^2(X, \mathcal{O}_X)$ under the obstruction map yields an element $f \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ which has the Hard Lefschetz property for the algebra* $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$.

The Hard Lefschetz property means that

$$f^i \circ \_ \colon \mathrm{Ext}^{n-i}(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^{n+i}(\mathcal{E}, \mathcal{E})$$

is an isomorphism for all $i > 0$. Note that $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \cong \mathrm{H}^*(\mathcal{E}nd(\mathcal{E}, \mathcal{E}))$ and

$$\mathcal{E}nd(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_X \oplus \mathcal{E}nd(\mathcal{E}, \mathcal{E})_0$$

via the trace morphism, where $\mathcal{E}nd(\mathcal{E}, \mathcal{E})_0$ is the bundle of traceless endomorphisms. The image of the subalgebra generated by the Hard Lefschetz element f corresponds under this isomorphism to $\mathrm{H}^*(\mathcal{O}_X)$.

Using Proposition C.1.6 we obtain.

Corollary C.8.6. *For a slope stable atomic bundle $\mathcal{E}$ there exists an element $f \in \mathrm{Im}(\chi_{\mathcal{E}})$ of degree two which has the Hard Lefschetz property.*

Assuming Conjecture E we have that the image of the obstruction map in degree two is spanned by a Hard Lefschetz element.

Similarly, for atomic Lagrangians $\iota\colon L \subset X$ we can consider the multiplicative isomorphism

$$\mathrm{Ext}^*(\iota_*\mathcal{O}_L, \iota_*\mathcal{O}_L) \cong \mathrm{H}^*(L, \mathbb{C}) \tag{C.8.4}$$

alluded to in Section C.7.3. By Theorem C.1.8 and the discussion in Section C.7.2, there exists an element $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ whose image under the obstruction map $\chi_{\iota_*\mathcal{O}_L}$ followed by the isomorphism (C.8.4) and projected to $\mathrm{H}^1(L, \Omega_L^1)$ yields an ample class. From this we deduce.

Proposition C.8.7. *For an atomic Lagrangian* $\iota\colon L \subset X$ *the image of* $\mathrm{H}^1(X, \mathcal{T}_X)$ *under the obstruction map is spanned by an element* $f \in \mathrm{Ext}^2(\iota_*\mathcal{O}_L, \iota_*\mathcal{O}_L)$ *having the Hard Lefschetz property.*

Again one can use auto-equivalences to obtain the same conclusion for a wider range of atomic objects.

Let $\mathcal{E}$ be a simple atomic object. The Hard Lefschetz property for an element $\chi_{\mathcal{E}}(\mu) = \mu_{\mathcal{E}} = f \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ in the image of $\chi_{\mathcal{E}}$ in degree two in particular implies that $0 \neq \mu_{\mathcal{E}}^n = f^n \in \mathrm{Ext}^{2n}(\mathcal{E}, \mathcal{E})$. Using once more the defining property of the Hochschild Chern character we get

$$\mathrm{Tr}_{X\times X}(\mu^n \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{Tr}_X(\mu_{\mathcal{E}}^n) \neq 0.$$

Thus, there must exist an element $\gamma \in \mathrm{HT}^2(X)$ such that $\gamma^n \lrcorner v(\mathcal{E}) \neq 0$. This implies that the projection $v(\mathcal{E})_{\mathrm{SH}}$ of $v(\mathcal{E})$ to the Verbitsky component $\mathrm{SH}(X, \mathbb{Q})$ is non-zero, as the Verbitsky component is the irreducible representation exhausting $\mathrm{H}^{0,2n}(X)$ which contains $\gamma^n \lrcorner v(\mathcal{E})$. In all examples of simple atomic objects $\mathcal{E}$ we are aware of, the condition $v(\mathcal{E})_{\mathrm{SH}} \neq 0$ is satisfied. For example, if $\mathcal{E}$ is a sheaf or derived equivalent to an object with non-zero rank, we know this holds true by Lemma C.3.7.

Assuming Conjecture E, we expect that the generator of the image of $\chi_{\mathcal{E}}$ in degree two for a simple atomic object always has the Hard Lefschetz property when $v(\mathcal{E})$ projects non-trivially to the Verbitsky component.

C.A. Spherical objects on hyper-Kähler manifolds

In Section C.4 we studied the interplay of the obstruction map and the cohomological obstruction map. In the appendix, we want to further use the relationship between topological properties of the Mukai vector $v(\mathcal{E})$ of an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ and its extension groups $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$. Throughout this section X is a fixed projective hyper-Kähler manifold of dimension $2n$.

Let us define the subalgebras

$$R_i \subset \mathrm{HH}^*(X)$$

generated by all elements of degree at most i for $2 \leq i \leq 2n$. Since the modified HKR isomorphism is graded as well as multiplicative there are analogous subalgebras

$$W_i := I^{\mathrm{K}}(R_i) \subset \mathrm{HT}^*(X).$$

Recall that $\mathrm{H}\Omega_*(X)$ is a free $\mathrm{HT}^*(X)$-module of rank one with generator σ^n leading to the isomorphism

$$\varphi\colon \mathrm{HT}^*(X) \cong \mathrm{H}\Omega_*(X), \quad \mu \mapsto \mu \lrcorner \sigma^n.$$

We denote $U_i := \varphi(W_i)$. One can check that this equals the subalgebra of the de Rham algebra $\mathrm{H}^*(X, \Omega_X^*)$ generated by elements of degree at most i. To illustrate the above, for $i = 2$ we have

$$\varphi(W_2) = U_2 = \mathrm{SH}(X, \mathbb{C}) \subset \mathrm{H}^*(X, \Omega_X^*) \cong \mathrm{H}^*(X, \mathbb{C}).$$

Similar comparisons can be made for larger i.

Proposition C.A.1. *Let* $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ *be an object and* $\mu \in R_i$ *such that* $\mu \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E}) \neq 0$. *Then there exists* $2 \leq j \leq i$ *such that* $0 \neq \mathrm{Ext}^j(\mathcal{E}, \mathcal{E})$.

Proof. The defining property of the Hochschild Chern character together with the non-degeneracy of the Serre duality trace shows that there exists $\gamma \in \mathrm{HH}^*(X)$ such that

$$0 \neq \mathrm{Tr}_{X\times X}(\gamma \circ \mu \circ \mathrm{ch}^{\mathrm{HH}}(\mathcal{E})) = \mathrm{Tr}_X(\gamma_{\mathcal{E}} \circ \mu_{\mathcal{E}}).$$

In particular, $0 \neq \mu_{\mathcal{E}} \in \mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$.

Since $\mu \in R_i$, we can write

$$\mu = \sum_k \gamma_k^1 \circ \cdots \circ \gamma_k^r$$

and each γ_k^l is contained in $\mathrm{HH}^s(X)$ for $2 \leq s \leq i$. Now, $\mu_{\mathcal{E}} \neq 0$ implies that there must exist k such that

$$0 \neq (\gamma_k^1)_{\mathcal{E}} \circ \cdots \circ (\gamma_k^r)_{\mathcal{E}} \in \mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$$

which implies that $0 \neq (\gamma_k^l)_{\mathcal{E}} \in \mathrm{Ext}^s(\mathcal{E}, \mathcal{E})$. □

We note that $\mathrm{HT}^*(X)$ is equipped with a non-degenerate pairing $\langle \_, \_ \rangle$ given by

$$\langle v, w \rangle \coloneqq \mathrm{pr}_{\mathrm{HT}^{4n}(X)}(v \wedge w) \in \mathrm{HT}^{4n}(X) \cong \mathbb{C},$$

i.e. one takes the normal product of two elements and projects it to the top degree component $\mathrm{HT}^{4n}(X)$. Note that under the multiplicative isomorphism

$$\mathrm{HT}^*(X) \cong \mathrm{H}^*(X, \mathbb{C})$$

from (C.2.1) induced by the isomorphism $\mathcal{T}_X \cong \Omega_X^1$ coming from a symplectic form (which is different than the isomorphism φ), the non-degenerate pairing $\langle \_, \_ \rangle$ corresponds to

$$(v, w) \mapsto \int_X vw$$

up to scaling. From Hard Lefschetz and the Hodge–Riemann bilinear relations we deduce that for each i we have a orthogonal decomposition

$$W_i \oplus W_i^{\perp} = \mathrm{HT}^*(X) \tag{C.A.1}$$

with respect to $\langle \_, \_ \rangle$ and, therefore, similarly

$$U_i \oplus U_i^{\perp} = \mathrm{H}\Omega_*(X), \tag{C.A.2}$$

where we define $U_i^{\perp} \coloneqq \varphi(W_i^{\perp})$.

Theorem C.A.2. *Let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ be an object such that $v(\mathcal{E})$ projects non-trivially to U_i. Then there exists $2 \leq j \leq i$ such that $\mathrm{Ext}^j(\mathcal{E}, \mathcal{E}) \neq 0$.*

Proof. Since the pairing $\langle \_, \_ \rangle$ is non-degenerate when restricted to W_i there exists by assumption an element $\mu \in W_i$ such that $\mu \lrcorner v(\mathcal{E}) \neq 0$. Using the modified HKR isomorphism we know there exists $\gamma = (I^{\mathrm{K}})^{-1}(\mu) \in R_i$ such that

$$\gamma \circ \mathrm{ch}(\mathcal{E}) \neq 0.$$

Proposition C.A.1 yields now the assertion. □

This result is already sufficient to prove one part of Theorem C.1.9.

Corollary C.A.3. *Let X be a hyper-Kähler manifold of dimension greater than two and $\mathcal{E}$ a non-zero sheaf. Then* $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \neq 0$ *and, in particular, $\mathcal{E}$ is not spherical.*

Proof. We know from Lemma C.3.7 that $v(\mathcal{E})_{\mathrm{SH}}$ is non-zero. Theorem C.A.5 then implies that $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \neq 0$. □

Remark C.A.4. We want to remark that there do exist non-zero objects in the bounded derived category of a hyper-Kähler manifold satisfying $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for all $0 < i < 2n$. For example, on a four-dimensional hyper-Kähler manifold X the object $\mathcal{E}$ defined as the cone of the natural morphism

$$\mathcal{O}_X \to \mathcal{O}_X[2]$$

satisfies $\mathrm{ch}(\mathcal{E}) = 0$ and $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $0 < i < 4$. In this example $\mathcal{E}$ is also simple, but not spherical, since $\mathrm{Ext}^{-1}(\mathcal{E},\mathcal{E}) \neq 0$.

An important class of auto-equivalences of a K3 surface S is given by spherical twists $\mathsf{ST}_{\mathcal{E}}$ along spherical objects $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(S)$. Recall that an object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(Y)$ is spherical, if its Ext algebra $\mathrm{Ext}^*(\mathcal{F},\mathcal{F})$ is isomorphic to the complex cohomology $\mathrm{H}^*(S^{\dim Y},\mathbb{C})$ of a sphere of dimension $\dim(Y)$.

It is notoriously hard to construct examples of interesting derived equivalences of higher-dimensional hyper-Kähler manifolds, see [2] for an account of some of the known constructions. The following is a partial explanation for this difficulty.

Theorem C.A.5. *Let X be a projective hyper-Kähler manifold of dimension $2n$ such that its cohomology is generated by elements of degree less than $2n-1$. Then $\mathrm{D}^{\mathrm{b}}(X)$ contains no spherical objects.*

Proof. If $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ is a spherical object, then $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $0 < i < 2n$. Theorem C.A.2 implies therefore that $v(\mathcal{E})$ must project trivially to U_{2n-1}.

Our assumptions imply that we have $U_{2n-1} = \mathrm{H}^*(X,\Omega_X^*)$ and therefore $v(\mathcal{E}) = 0$. This contradicts the equality

$$\langle v(\mathcal{E}), v(\mathcal{E})\rangle = \chi(\mathcal{E},\mathcal{E}) = \sum_i (-1)^i \mathrm{ext}^i(\mathcal{E},\mathcal{E}) = 2,$$

where $\langle \_, \_ \rangle$ denotes the generalized Mukai pairing on $\mathrm{H}^*(X,\Omega_X^*)$, see [60]. □

Proof of Theorem C.1.9. The first part is proven in Corollary C.A.3. The second part of the assertion is implied by Theorem C.A.5 and the fact that for these manifolds the cohomology is generated by classes of degree less than $2n$, see [136, Lem. 3.16] for the case of $\mathrm{K3}^{[n]}$-type and [81, Thm. 1.2] for the case of OG10-type hyper-Kähler manifolds. □

Remark C.A.6. (i) The proof of Theorem C.A.5 does not exclude the existence of spherical objects on hyper-Kähler manifolds in total generality. Still, the proof shows that for a potential spherical object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X)$ one has that its Mukai vector $v(\mathcal{E})$ must be contained in the subspace $U_{2n-1}^{\perp} \subset \mathrm{H}^*(X,\mathbb{Q})$, i.e. the orthogonal complement of the subalgebra generated by elements of degree $2n-1$. In particular, this subspace is a subspace of $\mathrm{H}^{n,n}(X)$. Moreover, the LLV algebra $\mathfrak{g}(X)$ acts trivially on the subspace $U_{2n-1}^{\perp}$. Thus, the induced derived equivalence of a potential spherical object would act trivially on all non-trivial representations of the LLV algebra such as the Verbitsky component.

(ii) Note that one can prove that if $\mathcal{E}$ is a spherical object, then its Mukai vector $v(\mathcal{E})$ must be contained in $\mathrm{SH}(X,\mathbb{Q})^{\perp} \subset \mathrm{H}^*(X,\mathbb{Q})$ without using Hochschild (co)homology. Indeed, the induced action of $\mathsf{ST}_{\mathcal{E}}$ on $\mathrm{SH}(X,\mathbb{Q})$ would be the reflection along the vector $v(\mathcal{E})_{\mathrm{SH}} \in \mathrm{SH}(X,\mathbb{Q})$. However, there is no isometry in $\mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ inducing the reflection along a one-dimensional subspace via [25, Eq. (2.2)].

Motivated by the above, we finish with the following.

Conjecture G. *Let X be a projective hyper-Kähler manifold of dimension greater than two. Then $\mathrm{D}^{\mathrm{b}}(X)$ contains no spherical objects.*

D. Second Chern class and Fujiki constants of hyperkähler manifolds

Abstract. We study characteristic classes on hyperkähler manifolds with a view towards the Verbitsky component. The case of the second Chern class leads to a conditional upper bound on the second Betti number in terms of the Riemann–Roch polynomial, which is also valid for singular examples. We discuss the general structure of characteristic classes and the Riemann–Roch polynomial on hyperkähler manifolds using among other things Rozansky–Witten theory.

D.1. Introduction

In the study of smooth projective varieties with trivial canonical bundle, irreducible compact hyperkähler manifolds take up a prominent place, partly due to the scarcity of examples. It is therefore natural to study a priori topological restrictions that such varieties must obey. There are several results in this direction, for example [84, 113, 122, 191, 193, 194].

Given an irreducible hyperkähler manifold X of dimension $2n$, its cohomology $H^*(X,\mathbb{R})$ is equipped with the Beauville–Bogomolov–Fujiki form q_X. Moreover, $H^*(X,\mathbb{R})$ is naturally a module under the Looijenga–Lunts–Verbitsky (LLV) Lie algebra $\mathfrak{g}(X)_{\mathbb{R}}$ [81, 130, 207]. This leads to a decomposition of $H^*(X,\mathbb{R})$ into irreducible representations. Arguably, the most important one is the Verbitsky component $\mathrm{SH}(X,\mathbb{R}) \subset H^*(X,\mathbb{R})$, which is the subalgebra generated by $H^2(X,\mathbb{R})$.

A natural question that arises is how much information this subalgebra encodes on the full cohomology. For example, one could ask which Chern classes of sheaves and, in particular, characteristic classes are contained inside the Verbitsky component.

One case we consider here is that of the second Chern class $c_2 \coloneqq c_2(X) \in H^4(X,\mathbb{R})$. Maybe a priori counter-intuitively, it is not always contained in the Verbitsky component, see for example [138, Lem. 1.5] for the case of the Hilbert scheme of n points on a K3 surface with $n > 3$. Note that c_2 lies in the Verbitsky component if and only if it is a multiple of the class $\mathfrak{q} \in H^4(X,\mathbb{Q})$, the dual of the Beauville–Bogomolov–Fujiki form.

We answer completely the question when c_2 lies inside the Verbitsky component using the Riemann–Roch polynomial of X. Recall that for a class $\alpha \in H^{4k}(X,\mathbb{R})$ which remains of type $(2k,2k)$ on all small deformations of X, there exists a number $C(\alpha)$, called the *generalized Fujiki constant* of α, such that

$$\forall \beta \in H^2(X,\mathbb{R}) \quad C(\alpha)\cdot q_X(\beta)^{n-k} = \int_X \alpha \cdot \beta^{2n-2k}. \tag{D.1.1}$$

Let td be the Todd class of X and let td_{2k} be its degree $2k$ part. The Riemann–Roch polynomial of X is defined as

$$\begin{aligned}\mathrm{RR}_X(q) := \sum_{i=0}^{n} \frac{C(\mathsf{td}_{2n-2i})}{(2i)!} q^i = \frac{C(1)}{(2n)!} q^n + \frac{C(\mathsf{td}_2)}{(2n-2)!} q^{n-1} + \cdots + \frac{C(\mathsf{td}_{2n})}{1} \\ =: A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots + A_n.\end{aligned}$$

The Hirzebruch–Riemann–Roch theorem, whence the name, together with the property of the generalized Fujiki constants assert that this polynomial satisfies

$$\mathrm{RR}_X(q_X(c_1(L))) = \chi(X, L)$$

for all line bundles $L \in \mathrm{Pic}(X)$. In particular, we have $A_n = n + 1$.

The following is the main result which, additionally, yields an upper bound on the second Betti number $b_2(X)$ under some conditions.

Theorem D.1.1. *Let X be a hyperkähler manifold of dimension $2n$ with second Betti number $b_2(X)$ and consider its Riemann–Roch polynomial*

$$\mathrm{RR}_X(q) = A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots .$$

If the first three coefficients satisfy the condition

$$2nA_0A_2 < (n-1)A_1^2, \tag{D.1.2}$$

then we have the inequality

$$b_2(X) \leq \frac{1}{1 - \dfrac{2nA_0A_2}{(n-1)A_1^2}} - (2n-2), \tag{D.1.3}$$

and equality holds if and only if $c_2 \in \mathrm{Sym}^2 H^2(X, \mathbb{R})$. If the condition (D.1.2) *does not hold, then c_2 is not contained in the Verbitsky component.*

The theorem can also be phrased using generalized Fujiki constants of (products of) Chern classes. Namely, inequality (D.1.2) is equivalent to the condition that the generalized Fujiki constant $C(\mathrm{ch}_4)$ is positive or, expressed differently, that

$$C(c_2^2) > 2C(c_4). \tag{D.1.4}$$

This is satisfied if the polynomial $\mathrm{RR}_X(q)$ has n distinct real roots, see Remark D.2.9. In the case $n = 2$ we always have $C(\mathrm{ch}_4) > 0$, see [169, Lem. 4.6]. Writing $C(c_2^2) = \mu C(c_4)$ condition (D.1.2) is equivalent to $\mu > 2$ and the bound (D.1.3) becomes

$$b_2(X) \leq 9 - 2n + \frac{10}{\mu - 2}. \tag{D.1.5}$$

We show in Corollary D.2.11 that the above conditions are also necessary and sufficient for $\mathsf{td}^{1/2}_{2n-2} \in H^{4n-4}(X, \mathbb{R})$, i.e. the degree $2n-2$ component of the square root of the Todd class, to be contained in the Verbitsky component.

Among known smooth hyperkähler manifolds, there are only two types of Riemann–Roch polynomials: the $\mathrm{K3}^{[n]}$-type and the Kum_n-type (OG_6 and OG_{10} fall into these two types, see [190]). On the other hand, Theorem D.1.1 can be generalized to singular symplectic varieties of dimension 4 and this gives rise to many more examples. We check that the inequality (D.1.4) is satisfied for all known smooth examples, as well as for many singular examples, in Sections D.2 and D.3 respectively.

In Section D.4, we give an account of all generalized Fujiki constants for the known examples of smooth hyperkähler manifolds. In particular, we prove that when X is of OG_6 or OG_{10}-deformation type, all Chern classes c_{2i} satisfy

$$c_{2i} \in \mathrm{SH}(X, \mathbb{R})$$

and, thus, all characteristic classes of X lie in the Verbitsky component. This easily leads to the determination of the generalized Fujiki constants for all characteristic classes on these manifolds.

In the final section, we further discuss generalized Fujiki constants and Riemann–Roch polynomials using Rozansky–Witten theory. We present a conceptual proof for the fact that the polynomial

$$\mathrm{RR}_{X,1/2}(q) := \sum_{i=0}^{n} \frac{C(\mathsf{td}^{1/2}_{2n-2i})}{(2i)!} q^i$$

factorizes as an n-th power using the Wheeling Theorem and discuss how this method could be used in general to analyze the Riemann–Roch polynomial. This leads to conjectural relations between the generalized Fujiki constants. We mention here the degree four case which yields a precise value of $C(\mathrm{ch}_4)$. For another instance of these conjectural relations, see Conjecture J.

Conjecture H. *Let X be a hyperkähler manifold of dimension $2n > 2$. We have*

$$\frac{C(\mathrm{ch}_4)}{C(1)} = \frac{5(n+1)}{(2n-1)(2n-3)}.$$

Note that, in particular, Conjecture H would imply (D.1.4). We prove in Proposition D.5.3 that the conjecture holds true if the Riemann–Roch polynomial satisfies certain expectations on its shape such as [113, Conj. 1.3 (3)] or Conjecture I. We present a possible strategy towards proving these conjectures.

We want to remark that we expect the inequality (D.1.4) to hold true pointwise on the level of forms for the right representative of ch_4 and therefore be of local nature. In contrast, Conjecture H is of global nature. The distinction between these two expectations will occur frequently in the paper.

If proven true, Conjecture H would imply that for hyperkähler fourfolds there are exactly two possible sets of values that the generalized Fujiki constants can take, see Proposition D.5.4. As a consequence, we obtain the following.

Corollary D.1.2. *Assuming Conjecture H in dimension 4, the Betti numbers of a hyperkähler fourfold X are one of the following:*

- $b_2(X) = 5, b_3(X) = 0, b_4(X) = 96$;

- $b_2(X) = 6, b_3(X) = 4, b_4(X) = 102$;
- $b_2(X) = 7, b_3(X) = 8, b_4(X) = 108$;
- $b_2(X) = 23, b_3(X) = 0, b_4(X) = 276$.

Hence, Conjecture H would reduce the number of possible Hodge diamonds and LLV decompositions of hyperkähler fourfolds to four. The two known cases are the ones where c_2 lies in the Verbitsky component. In the case $b_2(X) = 7$, there are 80 trivial representations of the LLV algebra in $H^{2,2}$, whereas there are 81 trivial representations when the second Betti number is smaller than seven.

In the recent work [64, Thm. 9.3], the authors obtained a similar result under a different assumption. We remark that the condition in our Conjecture H is stronger but makes no explicit assumption on the lattice $H^2(X, \mathbb{Z})$. It focuses only on numerical properties of the Riemann–Roch polynomial.

Relation to other work

While working on further results related to the topic of the paper we learned about the recent preprint of Justin Sawon [193] who independently obtained the same bound on the second Betti number as in Theorem D.1.1. The pointwise conjectural relations in Section D.5 have a similar flavor as the ones in [193, Sec. 2].

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D.2. The inequality

We prove Theorem D.1.1 in this section. Let X be a hyperkähler manifold of complex dimension $2n$ with $n \geq 2$. We first recall the following result by Fujiki [75] and Huybrechts [95].

Theorem D.2.1 (Fujiki, Huybrechts). *Let $\alpha \in H^{4k}(X, \mathbb{R})$ be a class that remains of type $(2k, 2k)$ on all small deformations of X (for example, all characteristic classes satisfy this condition). Then there exists a constant $C(\alpha) \in \mathbb{R}$, called the* generalized Fujiki constant of *α, such that*

$$\forall \beta \in H^2(X, \mathbb{R}) \quad C(\alpha) \cdot q_X(\beta)^{n-k} = \int_X \alpha \cdot \beta^{2n-2k}.$$

Remark D.2.2. The term *Fujiki constant* is reserved for the value $C(1) = C(1_X)$. There is also the notion of *small Fujiki constant* c_X: it differs from $C(1)$ by a constant multiple

$$C(1) = \frac{(2n)!}{2^n n!} c_X = (2n-1)!! \cdot c_X.$$

For example, it is known that $c_{\mathrm{K3}^{[n]}} = 1$ and $c_{\mathrm{Kum}_n} = n + 1$.

Denote by $\mathfrak{q} \in \mathrm{Sym}^2 H^2(X,\mathbb{R})$ the dual of the Beauville–Bogomolov–Fujiki form, and by $\mathrm{SH}(X,\mathbb{R}) \subset H^\bullet(X,\mathbb{R})$ the Verbitsky component, which is the subalgebra generated by $H^2(X,\mathbb{R})$. The key step to Theorem D.1.1 is the following result.

Proposition D.2.3. *We have the following inequality*

$$C(c_2^2) \geq \frac{C(c_2)^2}{C(\mathfrak{q})^2} C(\mathfrak{q}^2), \tag{D.2.1}$$

where equality holds if and only if $c_2 \in \mathrm{Sym}^2 H^2(X,\mathbb{R})$.

Proof. We write

$$c_2 = a\mathfrak{q} + z \quad \text{where } a \in \mathbb{R}, z \in \mathrm{SH}(X,\mathbb{R})^\perp.$$

In other words, we project c_2 orthogonally to the Verbitsky component and let $a\mathfrak{q}$ be its image. Then we have

$$C(c_2) = C(a\mathfrak{q}), \quad \text{so } a = \frac{C(c_2)}{C(\mathfrak{q})}.$$

Now we consider the square $c_2^2 = a^2\mathfrak{q}^2 + 2a\mathfrak{q}z + z^2 \in H^8(X,\mathbb{R})$. Since the class z is in $\mathrm{SH}(X,\mathbb{R})^\perp$, it is orthogonal to the image of $\mathrm{Sym}^{2n-2}H^2(X,\mathbb{R})$, so the class $\mathfrak{q}z$ is orthogonal to the image of $\mathrm{Sym}^{2n-4}H^2(X,\mathbb{R})$ and also lies in $\mathrm{SH}(X,\mathbb{R})^\perp$.

On the other hand, for any Kähler class $\omega \in H^2(X,\mathbb{R})$, since z lies $\mathrm{SH}(X,\mathbb{R})^\perp$, the class $z \cdot \omega^{2n-3} \in H^{4n-2}(X,\mathbb{R})$ is orthogonal to the entire $H^2(X,\mathbb{R})$ hence must vanish. So the class z is primitive of type $(2,2)$ with respect to all Kähler classes on X. By the Hodge–Riemann bilinear relations, for a Kähler class $\omega \in H^2(X,\mathbb{R})$ we have

$$\int_X z^2 \cdot \omega^{2n-4} \geq 0, \quad \text{hence } C(z^2) \geq 0,$$

where equality holds if and only if $z = 0$, i.e. $c_2 \in \mathrm{Sym}^2 H^2(X,\mathbb{R})$. In other words, the projection of z^2 to the Verbitsky component is non-trivial, unless z is itself trivial. Therefore we obtain the desired inequality

$$C(c_2^2) = a^2 C(\mathfrak{q}^2) + C(z^2) \geq a^2 C(\mathfrak{q}^2) = \frac{C(c_2)^2}{C(\mathfrak{q})^2} C(\mathfrak{q}^2),$$

where equality holds if and only if $c_2 \in \mathrm{Sym}^2 H^2(X,\mathbb{R})$. □

We now study the values of the various generalized Fujiki constants that appear in (D.2.1).

Proposition D.2.4. *Let X be a hyperkähler manifold of dimension $2n$ with second Betti number $b := b_2(X)$. For any $\alpha \in H^{4k}(X,\mathbb{R})$ that is of type $(2k,2k)$ on all small deformations of X, we have*

$$C(\mathfrak{q} \cdot \alpha) = \frac{b + 2n - 2k - 2}{2n - 2k - 1} C(\alpha).$$

In particular, we get

$$C(\mathfrak{q}^k) = \frac{b + 2n - 2k}{1 + 2n - 2k} C(\mathfrak{q}^{k-1}) = \prod_{i=1}^{k} \frac{b + 2n - 2i}{1 + 2n - 2i} \cdot C(1).$$

Proof. Take a basis $(e_1, \dots, e_b)$ of $H^2(X, \mathbb{R})$ such that

$$\mathfrak{q} = e_1^2 + e_2^2 + e_3^2 - e_4^2 - \cdots - e_b^2.$$

Writing $s_i \coloneqq q_X(e_i) \in \{\pm 1\}$, we have

$$\begin{aligned} C(\mathfrak{q} \cdot \alpha) = \int_X \mathfrak{q} \cdot \alpha \cdot e_1^{2n-2k-2} &= \int_X \alpha \cdot (e_1^{2n-2k} + e_1^{2n-2k-2} e_2^2 + \cdots - e_1^{2n-2k-2} e_b^2) \\ &= C(\alpha) + \sum_{i>1} s_i \int_X \alpha \cdot e_1^{2n-2k-2} e_i^2. \end{aligned}$$

For each term $e_1^{2n-2k-2} e_i^2$, consider the function

$$t \mapsto \int_X \alpha \cdot (e_1 + t e_i)^{2n-2k} = C(\alpha) \cdot (1 + t^2 s_i)^{n-k},$$

which is a polynomial in t. Comparing the coefficients of t^2, we get

$$\binom{2n-2k}{2} \int_X \alpha \cdot e_1^{2n-2k-2} e_i^2 = C(\alpha) \cdot (n-k) s_i.$$

So we have

$$C(\mathfrak{q} \cdot \alpha) = C(\alpha) + \sum_{i>1} s_i \frac{C(\alpha) s_i}{2n-2k-1} = C(\alpha) + (b-1) \frac{C(\alpha)}{2n-2k-1} = \frac{b+2n-2k-2}{2n-2k-1} C(\alpha),$$

where we used the fact that $s_i^2 = 1$. $\square$

We use the above description to replace $C(\mathfrak{q})$ and $C(\mathfrak{q}^2)$ in (D.2.1) and get

$$C(c_2^2) \geq \frac{(2n-1)(b_2(X) + 2n - 4) C(c_2)^2}{(2n-3)(b_2(X) + 2n - 2) C(1)}. \tag{D.2.2}$$

On the other hand, we have the following result by Nieper-Wißkirchen [162], which generalizes the work of Hitchin–Sawon [90]. In particular, it produces linear relations among certain generalized Fujiki constants. We will present a proof of the theorem in Section D.5.3.

Theorem D.2.5. *Let X be a hyperkähler manifold of dimension $2n$. Consider the following polynomial*

$$\begin{aligned} \mathrm{RR}_{X,1/2}(q) &\coloneqq \sum_{i=0}^{n} \frac{C(\mathsf{td}_{2n-2i}^{1/2})}{(2i)!} q^i \\ &= \frac{C(1)}{(2n)!} q^n + \frac{C(\frac{1}{24} c_2)}{(2n-2)!} q^{n-1} + \frac{C(\frac{7}{5760} c_2^2 - \frac{1}{1440} c_4)}{(2n-4)!} q^{n-2} + \cdots + \frac{C(\mathsf{td}_{2n}^{1/2})}{1}. \end{aligned}$$

There exists a constant r_X such that this polynomial factorizes as

$$\mathrm{RR}_{X,1/2}(q) = C(\mathsf{td}_{2n}^{1/2}) \left(1 + \frac{1}{2 r_X} q\right)^n.$$

In [25, Sec. 3], by comparing the first two coefficients, it is shown that

$$r_X = \frac{(2n-1)C(c_2)}{24C(1)} = \frac{(2n-1)2^n n! C(c_2)}{24(2n)! c_X},$$

and

$$C(\mathsf{td}_{2n}^{1/2}) = \frac{C(1)(2r_X)^n}{(2n)!} = c_X \frac{r_X^n}{n!},$$

where c_X is the small Fujiki constant. By comparing the third coefficients, we get the following relation.

Corollary D.2.6. *Let X be a hyperkähler manifold of dimension $2n > 2$. Then*

$$7C(c_2^2) - 4C(c_4) = \frac{5(2n-1)C(c_2)^2}{(2n-3)C(1)}.$$

Combining Corollary D.2.6 and (D.2.2) we obtain

$$C(c_2^2) \geq \frac{b_2(X) + 2n - 4}{5(b_2(X) + 2n - 2)} (7C(c_2^2) - 4C(c_4)) \tag{D.2.3}$$

which is equivalent to

$$(C(c_2^2) - 2C(c_4))(b_2(X) + 2n - 9) \leq 10C(c_4). \tag{D.2.4}$$

The last missing ingredient for proving Theorem D.1.1 is the connection of the above with the Riemann–Roch polynomial.

Corollary D.2.7. *All generalized Fujiki constants for characteristic classes of degree ≤ 4 are determined by the Riemann–Roch polynomial, or more precisely, by its first three coefficients*

$$\begin{aligned} \mathrm{RR}_X(q) = \sum_{i=0}^{n} \frac{C(\mathsf{td}_{2n-2i})}{(2i)!} q^i = \frac{C(1)}{(2n)!} q^n + \frac{C(\frac{1}{12}c_2)}{(2n-2)!} q^{n-1} + \frac{C(\frac{1}{240}c_2^2 - \frac{1}{720}c_4)}{(2n-4)!} q^{n-2} + \cdots \\ = A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots \end{aligned}$$

Proof. Clearly $C(1)$ and $C(c_2)$ appear as coefficients of the Riemann–Roch polynomial so we have

$$C(1) = (2n)! A_0, \quad C(c_2) = 12(2n-2)! A_1.$$

For $C(c_2^2)$ and $C(c_4)$, we already have one linear relation

$$7C(c_2^2) - 4C(c_4) = \frac{5(2n-1)C(c_2)^2}{(2n-3)C(1)} = 720(2n-4)! \frac{(n-1)A_1^2}{nA_0}.$$

The third coefficient gives another one

$$3C(c_2^2) - C(c_4) = 720(2n-4)! A_2,$$

which allows us to uniquely determine their values

$$C(c_2^2) = 144(2n-4)! \left(4A_2 - \frac{(n-1)A_1^2}{nA_0}\right),$$

$$C(c_4) = 144(2n-4)! \left(7A_2 - \frac{3(n-1)A_1^2}{nA_0}\right).$$

Hence we get all four generalized Fujiki constants of degree ≤ 4. □

Proof of Theorem D.1.1. We replace all generalized Fujiki constants in (D.2.3) by the coefficients of the Riemann–Roch polynomial. After some simplifications we get

$$\begin{aligned} 4A_2 &\geq \frac{(n-1)A_1^2}{nA_0}\left(1+\frac{b+2n-4}{b+2n-2}\right) \\ &= \frac{(n-1)A_1^2}{nA_0}\left(2-\frac{2}{b+2n-2}\right), \end{aligned} \tag{D.2.5}$$

or equivalently,

$$\frac{1}{b_2(X)+2n-2} \geq 1-\frac{2nA_0A_2}{(n-1)A_1^2}.$$

This yields the desired inequality provided

$$1-\frac{2nA_0A_2}{(n-1)A_1^2} > 0$$

which is exactly condition (D.1.2).

From Proposition D.2.3 we know that $c_2 \in \mathrm{SH}^4(X,\mathbb{R})$ if and only if the inequality (D.1.3) is in fact an equality. We claim that in this case (D.1.2) must be satisfied. Indeed, if we assume

$$1-\frac{2nA_0A_2}{(n-1)A_1^2} \leq 0$$

we obtain

$$\frac{1}{b_2(X)+2n-2} \leq 0$$

which is absurd. □

Remark D.2.8. As mentioned in the introduction we can use (D.2.4) to state Theorem D.1.1 in terms of the generalized Fujiki constants $C(c_2^2)$ and $C(c_4)$. Condition (D.1.2) then becomes

$$C(c_2^2) > 2C(c_4)$$

and writing $C(c_2^2) = \mu C(c_4)$ for some $\mu > 2$ the bound (D.1.3) becomes

$$b_2(X) \leq 9-2n+\frac{10}{\mu-2}.$$

So we still get a bound on $b_2(X)$ without knowing the values for $C(1)$ and $C(c_2)$.

Remark D.2.9. Suppose that the Riemann–Roch polynomial factorizes as a product of linear factors

$$\mathrm{RR}_X(q) = A_0\prod_i(q+\lambda_i).$$

It was shown in [113] that all the coefficients of $\mathrm{RR}_X(q)$ are positive. Hence the λ_i must all be positive. If, moreover, we assume that the λ_i are not all equal, then condition (D.1.2) is satisfied by Cauchy–Schwarz, and the inequality (D.1.3) can be written as

$$b_2(X) \leq \frac{n-1}{\frac{n\sum\lambda_i^2}{(\sum\lambda_i)^2}-1}-(2n-2).$$

This is homogeneous with respect to the λ_i and measures in a certain sense the dispersion of the roots.

The condition for c_2 to be contained inside the Verbitsky component actually also gives an equivalent condition for $\mathsf{td}^{1/2}_{2n-2}$ to lie inside $\mathrm{SH}(X,\mathbb{R})$, by the following result.

Proposition D.2.10. *For a hyperkähler manifold X of dimension $2n$, we have $\mathsf{td}^{1/2}_{2k} \in \mathrm{SH}(X,\mathbb{R})$ if and only if $\mathsf{td}^{1/2}_{2n-2k} \in \mathrm{SH}(X,\mathbb{R})$. Moreover, $\mathsf{td}^{1/2}_{2k} \in \mathrm{SH}(X,\mathbb{R})$ implies $\mathsf{td}^{1/2}_{2k'} \in \mathrm{SH}(X,\mathbb{R})$ for $k' < k < n$.*

Proof. For a class $\alpha \in H^2(X,\mathbb{C})$, denote by $e_\alpha \in \mathfrak{g}(X)_\mathbb{C}$ the operator $x \mapsto x \cdot \alpha$. Define h_p to be the holomorphic grading operator that acts on $H^{p,q}(X)$ as $(n-p)\,\mathrm{Id}$ (which is denoted by Π in [113]), and similarly the antiholomorphic grading operator h_q which acts on $H^{p,q}(X)$ as $(n-q)\,\mathrm{Id}$. Recall that for the class σ of a symplectic form, the operator e_σ has the Lefschetz property with respect to the grading given by h_p: there exists a dual Lefschetz operator $\Lambda_\sigma \in \mathfrak{g}(X)_\mathbb{C}$, such that together with the operator h_p, we get an $\mathfrak{sl}_2$-triple $(e_\sigma, h_p, \Lambda_\sigma)$ in the LLV algebra. The same result holds if we consider $e_{\overline{\sigma}}$ and h_q.

Jiang [113, Cor. 3.19] showed that there exists a constant $r_\sigma \in \mathbb{R}_{>0}$ such that

$$\Lambda_\sigma(\mathsf{td}^{1/2}_{2k}) = r_\sigma \mathsf{td}^{1/2}_{2k-2} \wedge \overline{\sigma}. \tag{D.2.6}$$

Furthermore, the operators $e_{\overline{\sigma}}$ and Λ_σ commute for degree reasons. Applying (D.2.6) repeatedly, we see that the following holds for all $k < n/2$

$$\Lambda_\sigma^{n-2k}(\mathsf{td}^{1/2}_{2n-2k}) = r_\sigma^{n-2k} \mathsf{td}^{1/2}_{2k} \wedge \overline{\sigma}^{n-2k}. \tag{D.2.7}$$

On the other hand, Fujiki [75] showed that the operators $e_{\overline{\sigma}}$ and Λ_σ yield isomorphisms

$$e_{\overline{\sigma}}^s \colon H^{l,n-s}(X) \cong H^{l,n+s}(X), \quad \Lambda_\sigma^s \colon H^{n+s,l}(X) \cong H^{n-s,l}(X).$$

Moreover, these isomorphisms are compatible with the decomposition of $H^*(X,\mathbb{C})$ into irreducible $\mathfrak{g}(X)_\mathbb{C}$-representations, i.e. for each irreducible representation $V \subset H^*(X,\mathbb{C})$, the isomorphism $e_{\overline{\sigma}}^s$ restricts to an isomorphism

$$e_{\overline{\sigma}}^s \colon H^{l,n-s}(X) \cap V \cong H^{l,n+s}(X) \cap V,$$

and similar for Λ_σ^s. Combining this with (D.2.7) yields the first assertion. The second statement also follows from (D.2.7) using the same line of arguments. ☐

Corollary D.2.11. *For a hyperkähler manifold X of dimension $2n$, the class $\mathsf{td}^{1/2}_{2n-2}$ lies in the Verbitsky component if and only if the condition* (D.1.2) *is satisfied and the equality in* (D.1.3) *holds.*

We now examine the bound (D.1.3) for the known deformation types of smooth hyperkähler manifolds. There are only two types of Riemann–Roch polynomials

$$\mathrm{RR}_{\mathrm{K3}^{[n]}}(q) = \binom{q/2+n+1}{n}, \quad \mathrm{RR}_{\mathrm{Kum}_n}(q) = (n+1)\binom{q/2+n}{n},$$

see [68, Lem. 5.1] and [162, Lem. 5.2]. Ríos Ortiz showed that O'Grady's sporadic examples satisfy $\mathrm{RR}_{\mathrm{OG}_{10}}(q) = \mathrm{RR}_{\mathrm{K3}^{[5]}}(q)$ and $\mathrm{RR}_{\mathrm{OG}_6}(q) = \mathrm{RR}_{\mathrm{Kum}_3}(q)$ in [190].

Example D.2.12 (K3$^{[n]}$-type)**.** We compute the first three coefficients

$$\begin{aligned}\mathrm{RR}_{\mathrm{K3}^{[n]}}(q) &= \binom{q/2+n+1}{n} \\ &= \frac{1}{2^n n!}q^n + \frac{n+3}{2^n(n-1)!}q^{n-1} + \frac{3n^2+17n+26}{3\cdot 2^{n+1}(n-2)!}q^{n-2} + \cdots\end{aligned}$$

Then by inserting the values A_0, A_1, A_2 into (D.1.3), we get the following upper bound

$$b_2(X) \le n + 17 + \frac{12}{n+1}.$$

Alternatively, we could also have used Remark (D.2.9) to obtain the expression. When $n = 2$ or $n = 3$, it evaluates to 23 and is attained by K3$^{[n]}$; when $n = 5$, it evaluates to 24 and is attained by OG_{10}. In particular, these are exactly the three known deformation types with this Riemann–Roch polynomial for which we have $c_2 \in \mathrm{Sym}^2 H^2(X, \mathbb{R})$.

Example D.2.13 (Kum$_n$-type)**.** We compute similarly the first three coefficients

$$\begin{aligned}\mathrm{RR}_{\mathrm{Kum}_n}(q) &= (n+1)\binom{q/2+n}{n} \\ &= \frac{n+1}{2^n n!}q^n + \frac{(n+1)^2}{2^n(n-1)!}q^{n-1} + \frac{(n+1)^2(3n+2)}{3\cdot 2^{n+1}(n-2)!}q^{n-2} + \cdots\end{aligned}$$

and insert these three coefficients into (D.1.3). In this case, the upper bound we get is

$$b_2(X) \le n + 5.$$

When $n = 2$, it is attained by Kum_2; when $n = 3$ it is attained by OG_6. Again, for these two types, we have $c_2 \in \mathrm{Sym}^2 H^2(X, \mathbb{R})$.

Note also that for $n = 2$, the bound $b_2(X) \le 7$ is much stronger than the general bound $b_2(X) \le 23$ by Guan.

Another consequence of the inequality is the positivity of the generalized Fujiki constants $C(c_2^2)$ and $C(c_4)$.

Proposition D.2.14. *Let X be a hyperkähler manifold of dimension $2n$. The generalized Fujiki constant $C(c_2^2)$ is always positive, and $C(c_4)$ is positive except possibly when $n = 2$ and $b_2(X) = 3, 4, 5$ or when $n = 3$ and $b_2(X) = 3$.*

Proof. From the inequality (D.2.2), it is clear that $C(c_2^2)$ is positive. For $C(c_4)$ to be positive, it is equivalent to have

$$\frac{nA_0A_2}{(n-1)A_1^2} > \frac{3}{7}.$$

By (D.2.5), we have

$$\frac{nA_0A_2}{(n-1)A_1^2} \ge \frac{1}{4}\left(2 - \frac{2}{b_2(X)+2n-2}\right).$$

So we want the inequality

$$\frac{1}{4}\left(2 - \frac{2}{b_2(X)+2n-2}\right) > \frac{3}{7}$$

which is equivalent to $b_2(X) + 2n > 9$, and is satisfied except when $n = 2$ and $b_2(X) \le 5$ or $n = 3$ and $b_2(X) = 3$. □

Remark D.2.15. When $n = 2$, these two generalized Fujiki constants are just Chern numbers, and this is already known by the results of Guan [84].

D.3. Orbifold examples

Theorem D.1.1 can also be generalized to the singular case, at least when $n = 2$. The proof is exactly the same as in Section D.2, so we only indicate the key ingredients. We follow the paper by Fu–Menet [73] and the notation therein.

- We consider *primitively symplectic orbifolds* [73, Def. 3.1]. In dimension 4, such orbifolds only contain isolated singular points.
- Generalized Fujiki constants still exist, as proved by Menet in [146, Lem. 4.6]. Hence we may still define the Riemman–Roch polynomial using the generalized Fujiki constants of the Todd class

$$\mathrm{RR}_X(q) \coloneqq \sum_{i=0}^{n} \frac{C(\mathsf{td}_{2n-2i})}{(2i)!} q^i = A_0 q^n + \cdots + A_n.$$

- Orbifold versions of the Gauss–Bonnet theorem and the Hirzebruch–Riemann–Roch theorem exist in dimension 4 (or more generally, for orbifolds with only isolated singularities), as proved by Blache in [32] (see [73, Thm. 2.12 and Thm. 2.13]): we have

$$\chi_{\mathrm{top}}(X) = \int_X c_4 + \sum_{x \in \mathrm{Sing}(X)} \left(1 - \frac{1}{|G_x|}\right),$$

  and for all $L \in \mathrm{Pic}(X)$,

$$\chi(X, L) = \int_X \mathrm{ch}(L) \cdot \mathsf{td}(X) + \sum_{x \in \mathrm{Sing}(X)} \frac{1}{|G_x|} \sum_{g \in G_x \setminus \{e\}} \frac{1}{\det(\mathrm{Id} - \rho_{x,T_X}(g))}.$$

  Beware that the Riemann–Roch polynomial as defined above no longer gives the correct Euler characteristic, due to the contribution from singular points: instead we have

$$\forall L \in \mathrm{Pic}(X) \quad \chi(X, L) = \mathrm{RR}_X(q_X(L)) + (3 - C(\mathsf{td}_4)).$$

- An orbifold version of the Hitchin–Sawon formula exists: this is [73, Prop. 4.2]. In particular, when $n = 2$, this gives the orbifold version of Corollary D.2.6. One would expect that the more general result of Nieper-Wißkirchen should also hold for the singular case.

Using these ingredients and repeating the proof in Section D.2, we obtain Theorem D.1.1 for primitively symplectic orbifolds in dimension 4. We apply it to examine the examples listed in [73, Sec. 5]. We will use a_m to denote the number of isolated cyclic quotient singularities of order m.

Remark D.3.1. The conceptual reason why Theorem D.1.1 remains valid also in the singular case is that this type of result holds pointwise and, therefore, generalizes to orbifolds.

Example D.3.2. Let M' be the irreducible symplectic orbifold of dimension 4 with second Betti number $b_2(M') = 16$, also known as a *Nikulin orbifold* (see [73, Sec. 5.11] and [48]). It has 28 isolated quotient singularities of order 2, i.e., $a_2 = 28$. The orbifold M' has topological Euler characteristic $\chi_{\mathrm{top}}(M') = 212$ and Fujiki constant $C(1) = 6$.

Using the orbifold Riemann–Roch and Gauss–Bonnet theorems, we get

$$\begin{aligned}\int_{M'} \mathsf{td}_4 = \int_{M'} \frac{3c_2^2 - c_4}{720} &= \chi(M', \mathcal{O}_{M'}) - \sum_{x \in \mathrm{Sing}(M')} \frac{1}{|G_x|} \sum_{g \in G_x \setminus \{e\}} \frac{1}{\det(\mathrm{Id} - \rho_{x,T_{M'}}(g))} \\ &= 3 - 28 \cdot \frac{1}{2} \cdot \frac{1}{16} \\ &= \frac{17}{8},\end{aligned}$$

and

$$\int_{M'} c_4 = \chi_{\mathrm{top}}(M') - \sum_{x \in \mathrm{Sing}(M')} \left(1 - \frac{1}{|G_x|}\right) = 198.$$

Therefore we may compute

$$C(c_2^2) = 576, \quad C(c_4) = 198.$$

The orbifold Hitchin–Sawon formula gives the relation in Corollary D.2.6, from which we deduce that $C(c_2) = 36$. Hence we have obtained the Riemann–Roch polynomial of M':

$$\mathrm{RR}_{M'}(q) = \frac{1}{4}q^2 + \frac{3}{2}q + \frac{17}{8}.$$

Note that this polynomial was also computed directly from the geometry of M' by Camere–Garbagnati–Kaputska–Kaputska [48, Thm. 1.3].

Now if we insert the values into (D.1.3), we get

$$b_2(X) \leq 16,$$

for any primitively symplectic orbifold X with the same Riemann–Roch polynomial as M'. The Nikulin orbifold M' attains the upper bound, and we have $c_2(M') \in \mathrm{Sym}^2 H^2(M', \mathbb{R})$. Note that the two roots of $\mathrm{RR}_{M'}(q)$ are $-3 \pm \frac{\sqrt{2}}{2}$, so they are not integers.

Example D.3.3. Let K' be the orbifold example in [73, Sec. 5.6] with second Betti number $b_2(K') = 8$ and $a_2 = 36$: we have $\chi_{\mathrm{top}}(K') = 108$ and $C(1) = 8$. Similarly, we compute

$$C(c_2) = 40, \quad C(c_2^2) = 480, \quad C(c_4) = 90,$$

and

$$\mathrm{RR}_{K'}(q) = \frac{1}{3}q^2 + \frac{5}{3}q + \frac{15}{8}.$$

Using (D.1.3), we get the bound

$$b_2(X) \leq 8,$$

which again holds for any primitively symplectic orbifold with the same Riemann–Roch polynomial. So the example K' also attains the upper bound. The two roots are $\frac{-10 \pm \sqrt{10}}{4}$.

Note that surprisingly, the Beauville–Bogomolov-Fujiki form of K' is odd and represents the value 1. If we take a line bundle H with $q(c_1(H)) = 1$, after adding the correction term, the Riemann–Roch formula tells us that $\chi(K', H) = 5$, so one could expect that the linear system $|H|$ gives a (rational) finite cover of $\mathbb{P}^4$.

Example D.3.4. The following examples are obtained as cyclic quotients of smooth hyperkähler manifolds of $\mathrm{K3}^{[2]}$-type [73, Sec. 5.2, 5.3, and 5.9], so they are primitively symplectic but not irreducible.

- Case $b_2(M_{11}^i) = 3$ for $i = 1, 2$ with $a_{11} = 5$: we have $\chi_{\mathrm{top}}(M_{11}^i) = 34$ and $C(1) = 33$ for both $i = 1, 2$, so
$$C(c_2) = 30, \quad C(c_2^2) = \frac{828}{11}, \quad C(c_4) = \frac{324}{11},$$
and
$$\mathrm{RR}_{M_{11}^i}(q) = \frac{11}{8}q^2 + \frac{5}{4}q + \frac{3}{11} = \frac{1}{11}\mathrm{RR}_{\mathrm{K3}^{[2]}}(11q).$$
- Case $b_2(M_7) = 5$ with $a_7 = 9$: we have $\chi_{\mathrm{top}}(M_7) = 54$ and $C(1) = 21$, so
$$C(c_2) = 30, \quad C(c_2^2) = \frac{828}{7}, \quad C(c_4) = \frac{324}{7},$$
and
$$\mathrm{RR}_{M_7}(q) = \frac{7}{8}q^2 + \frac{5}{4}q + \frac{3}{7} = \frac{1}{7}\mathrm{RR}_{\mathrm{K3}^{[2]}}(7q).$$
- Case $b_2(M_3) = 11$ with $a_3 = 27$: we have $\chi_{\mathrm{top}}(M_3) = 126$ and $C(1) = 9$, so
$$C(c_2) = 30, \quad C(c_2^2) = 276, \quad C(c_4) = 108,$$
and
$$\mathrm{RR}_{M_3}(q) = \frac{3}{8}q^2 + \frac{5}{4}q + 1 = \frac{1}{3}\mathrm{RR}_{\mathrm{K3}^{[2]}}(3q).$$

In all these cases, the bound we get is $b_2(X) \leq 23$, which is not attained. These are all equal to the bound for $\mathrm{K3}^{[2]}$, due to the fact that the expression in (D.1.3) is homogeneous in terms of the roots of $\mathrm{RR}_X(q)$, hence will remain invariant after a change of variables.

In some sense, taking cyclic quotient does not produce genuinely "new" examples or Riemann–Roch polynomials.

Example D.3.5. For the following examples, we could not find the values of the Fujiki constant $C(1)$ in the literature. But a bound on b_2 can still be given, due to the observation in Remark D.2.8. We will simply write the upper bound obtained as $b_2(X) \leq B$, where X is understood as a primitively symplectic orbifold with the same Riemann–Roch polynomial.

- Case $b_2(K_4') = 6$ with $a_2 = 45, a_4 = 2$ and $\chi_{\mathrm{top}}(K_4') = 69$ [73, Sec. 5.4]: we have
$$C(c_2) = \sqrt{142C(1)}, \quad C(c_2^2) = 330, \quad C(c_4) = 45,$$
and
$$b_2(X) \leq \frac{55}{8} = 6.875.$$
So $b_2(K_4') = 6$ is the maximal possible but does not attain the bound.

- Case $b_2(K_3') = 7$ with $a_3 = 12$ and $\chi_{\mathrm{top}}(K_3') = 108$ [73, Sec. 5.5]: we have

  $$C(c_2) = 26\sqrt{C(1)/3}, \quad C(c_2^2) = 540, \quad C(c_4) = 100,$$

  and

  $$b_2(X) \leq \frac{135}{17} \approx 7.94$$

  So $b_2(K_3') = 7$ is the maximal possible but does not attain the bound.

- Case $b_2(Y_{\mathrm{K3}}(D_3)) = 9$: the description of this example in [73] appears to be incorrect.[1]

- Case $b_2(Y_{\mathrm{K3}}(\mathbb{Z}/4\mathbb{Z})) = 10$ with $a_2 = 10, a_4 = 6$ and $\chi_{\mathrm{top}} = 140$ [74, Table 1]: we have

  $$C(c_2) = 8\sqrt{3C(1)}, \quad C(c_2^2) = 486, \quad C(c_4) = \frac{261}{2},$$

  and

  $$b_2(X) \leq \frac{54}{5} = 10.8.$$

  So $b_2(Y_{\mathrm{K3}}(\mathbb{Z}/4\mathbb{Z})) = 10$ is the maximal possible but does not attain the bound.

- Case $b_2\big(Y_{\mathrm{K3}}\big((\mathbb{Z}/2\mathbb{Z})^2\big)\big) = 14$ with $a_2 = 36$ and $\chi_{\mathrm{top}} = 180$ [74, Table 1]: we have

  $$C(c_2) = 8\sqrt{3C(1)}, \quad C(c_2^2) = 504, \quad C(c_4) = 162,$$

  and

  $$b_2(X) \leq 14.$$

  So the bound is attained in this example.

Example D.3.6 (Kim)**.** This example was studied by Kim in [121, Sec. 7]: let X be a hyperkähler fourfold of Kum_2-type admitting a Lagrangian fibration. We consider its dual Lagrangian fibration $\check{X}$. It is a singular hyperkähler orbifold with only isolated quotient singularities.

However, the analysis in *loc. cit.* of the singularities of $\check{X}$ contains an error: the group action admits 108 fixed points on X, and every other 3 of them are identified after the quotient. So one should have $a_3 = 36$, that is, $\check{X}$ admits 36 isolated cyclic quotient singularities of order 3, instead of just 18 of them as claimed in *loc. cit.* Since $\chi_{\mathrm{top}}(X) = 108$, we may conclude that $\chi(\check{X}) = 108/3 = 36$, which is consistent with the description of the cohomology.

We compute the numerical invariants. By the orbifold Gauss–Bonnet theorem, we have $C(c_4) = \chi_{\mathrm{top}} - a_3 \cdot \frac{2}{3} = 12$. Then by the orbifold Riemann–Roch theorem, we have $\frac{1}{720}(3C(c_2^2) - C(c_4)) = 3 - a_3 \cdot \frac{1}{3} \cdot \frac{2}{9} = \frac{1}{3}$, hence $C(c_2^2) = 84$. This already gives us the bound on the second Betti number

$$b_2(\check{X}) \leq \frac{10}{\frac{84}{12} - 2} - 2 \cdot 2 + 9 = 7,$$

which is attained by the dual Lagrangian fibration $\check{X}$.

---

[1] Namely, the orbifold is described as the quotient of an $S^{[2]}$ by some symplectic automorphisms forming the dihedral group D_3. But such a quotient would necessarily contain singularities in codimension 2.

Kim showed that the *small* Fujiki constant $c_{\check{X}}$ of the dual Lagrangian fibration $\check{X}$ is $1/c_X$, so $C(1_{\check{X}}) = \frac{1}{3}\cdot 3 = 1$ in the dual Kum$_2$ case. Then by the orbifold Hitchin–Sawon formula, we may compute that $C(c_2) = 6$. Hence the Riemann–Roch polynomial is given by

$$\mathrm{RR}_{\check{\mathrm{Kum}}_2}(q) = \frac{1}{24}q^2 + \frac{1}{4}q + \frac{1}{3} = \frac{1}{9}\mathrm{RR}_{\mathrm{Kum}_2}(q).$$

In particular, for a line bundle H with square $q(c_1(H)) = 6$, we can use the Riemann–Roch formula with the correction term to compute $\chi(\check{X}, H) = 6$. So one could expect that the linear system $|H|$ gives a hypersurface (or a cover thereof) in $\mathbb{P}^5$.

D.4. Generalized Fujiki constants for known smooth examples

In this section, we give an account for the generalized Fujiki constants $C(c_\lambda)$ of characteristic classes $c_\lambda := c_2^{\lambda_2}c_4^{\lambda_4}\cdots c_{2n}^{\lambda_{2n}}$ for all known deformation types of hyperkähler manifolds.

D.4.1. $\mathrm{K3}^{[\mathrm{n}]}$ and Kum_n

The results are classical for the two infinite families. In the $\mathrm{K3}^{[n]}$-case, the method in Ellingsrud–Göttsche–Lehn [68] can be used to compute all the generalized Fujiki constants using a computer for small n. A similar algorithmic method can be used to treat the Kum_n-case, with some slight modifications based on the work of Nieper-Wißkirchen [163, Sec. 4.2.3]. An implementation for these algorithms in *Sage* can be found on the second-named author's webpage. Closed formulae for the values $C(c_{2k})$ for both families were recently established in [50, Thm. 4.2].

D.4.2. OG_6

By Corollary D.2.7, the generalized Fujiki constants for characteristic classes of degree ≤ 4 for OG$_6$ are the same as those for Kum$_3$, since they share the same Riemann–Roch polynomial. Since the Chern numbers of OG$_6$ are also known [152, Prop. 6.8], we can obtain all of them:

α	1	c_2	c_4	c_2^2	c_6	c_4c_2	c_2^3
$C(\alpha)$	60	288	480	1920	1920	7680	30720

Alternatively, since for OG$_6$-type the second Chern class c_2 lies in the Verbitsky component (namely, $c_2(\mathrm{OG}_6) = 2\mathfrak{q}$), Corollary D.2.11 shows that the class $\mathsf{td}_4^{1/2}$ also lies in $\mathrm{SH}(X,\mathbb{R})$. Now $\mathsf{td}_4^{1/2}$ is a linear combination of c_2^2 and c_4, so the same may be said for the class c_4. Then we can use Proposition D.2.4 to determine that $c_4(\mathrm{OG}_6) = \mathfrak{q}^2$, which then allows us to also compute $C(c_4c_2)$ and $C(c_2^3)$. Finally we can use $C(\mathsf{td}_6) = 4$ to solve the Euler characteristic $C(c_6)$.

Proposition D.4.1. *For hyperkähler manifolds of* OG$_6$*-type, all Chern classes* c_2, c_4, c_6 *lie in the Verbitsky component. We have*

$$c_2(\mathrm{OG}_6) = 2\mathfrak{q}, \quad c_4(\mathrm{OG}_6) = \mathfrak{q}^2, \quad c_6(\mathrm{OG}_6) = \tfrac{1}{2}\mathfrak{q}^3.$$

D.4.3. OG_{10}

The question for OG_{10} might seem difficult at first, as there are many more unknown Fujiki constants to determine. It turns out to be quite easy, due to the following observation.

Proposition D.4.2. *For hyperkähler manifolds of* OG_{10}*-type, all Chern classes* $c_2, \dots, c_{10}$ *lie in the Verbitsky component. We have*

$$c_2(\mathrm{OG}_{10}) = \tfrac{3}{2}\mathfrak{q}, \quad c_4(\mathrm{OG}_{10}) = \tfrac{15}{16}\mathfrak{q}^2, \quad c_6(\mathrm{OG}_{10}) = \tfrac{21}{64}\mathfrak{q}^3,$$
$$c_8(\mathrm{OG}_{10}) = \tfrac{237}{3328}\mathfrak{q}^4, \quad c_{10}(\mathrm{OG}_{10}) = \tfrac{27}{2560}\mathfrak{q}^5.$$

Proof. We use the LLV decomposition of the cohomology obtained in [81, Thm 3.26]

$$H^*(\mathrm{OG}_{10}, \mathbb{Q}) = V_{(5)} \oplus V_{(2,2)} \quad \text{as } \mathfrak{so}(4,22)\text{-modules.}$$

We are interested in the second component, which only contributes to cohomological degree k for $k \in \{6, 8, 10, 12, 14\}$.

For a generic X in the moduli space, the (special) Mumford–Tate algebra is the maximal possible and is isomorphic to $\mathfrak{so}(3,21)$. Using the branching rules, we get the following decompositions of $\mathfrak{so}(3,21)$-modules/Hodge structures (H^{12} and H^{14} are omitted by symmetry)

$$\begin{aligned} H^6(X,\mathbb{Q}) &= \mathrm{SH}^6(X,\mathbb{Q}) \oplus V_{(2)}, \\ H^8(X,\mathbb{Q}) &= \mathrm{SH}^8(X,\mathbb{Q}) \oplus V_{(2,1)} \oplus V_{(1)}, \\ H^{10}(X,\mathbb{Q}) &= \mathrm{SH}^{10}(X,\mathbb{Q}) \oplus V_{(2,2)} \oplus V_{(2)} \oplus V_{(1,1)} \oplus \mathbb{Q}. \end{aligned}$$

In other words, up to multiplying by a non-zero scalar, there is only one Hodge class $\eta \in H^{10}(X,\mathbb{Q})$ that lies in $\mathrm{SH}(X,\mathbb{Q})^{\perp}$ for a generic X. In particular, this means that all the Chern classes $c_2, \dots, c_{10}$ lie in the Verbitsky component.

For a generic X, the only Hodge classes in the Verbitsky components are multiples of powers of $\mathfrak{q}$, so each Chern class c_{2k} is a multiple of $\mathfrak{q}^k$. We explain how to determine the scalars, starting from smaller k: we use Corollary D.2.7 to determine $C(c_2)$ and $C(c_4)$. Since the values of $C(\mathfrak{q}^k)$ are known by Proposition D.2.4, we have determined c_2 and c_4. Once all c_{2i} for $i < k$ are known, we study the class $\mathsf{td}^{1/2}_{2k}$, whose generalized Fujiki constant $C(\mathsf{td}^{1/2}_{2k})$ is known by Theorem D.2.5 and whose only unknown term is a given multiple of c_{2k}. Therefore we will be able to uniquely determine $C(c_{2k})$ and thus c_{2k} itself. □

It is then straightforward to compute the generalized Fujiki constants, which we include for the reader's convenience.

α	1	c_2	c_4	c_2^2	c_6	c_4c_2	c_2^3	c_8	c_6c_2	c_4^2	$c_4c_2^2$	c_2^4
$C(\alpha)$	945	5040	13500	32400	26460	113400	272160	49770	343980	614250	1474200	3538080

c_{10}	c_8c_2	c_6c_4	$c_6c_2^2$	$c_4^2c_2$	$c_4c_2^3$	c_2^5
176904	1791720	5159700	12383280	22113000	53071200	127370880

Note that the Chern numbers for OG_{10} have already been computed by Cao–Jiang in the appendix of [190].

It is remarkable that the knowledge of the Riemann–Roch polynomial together with the assumption that all Chern classes lie in the Verbitsky component allow us to completely determine the second Betti number as well as all the generalized Fujiki constants, in particular all the Chern numbers including the Euler characteristic $C(c_{2n}) = \int_X c_{2n}$.

D.5. Further discussions

We see that the Riemann–Roch polynomial $\mathrm{RR}_X(q)$ of a hyperkähler manifold X is a very important notion: it puts strong topological restriction on X, namely an upper bound for the second Betti number. We now formulate some conjectures on the shape of such polynomials and discuss some possible ways of studying them.

Recall from Theorem D.2.5 that the polynomial $\mathrm{RR}_{X,1/2}(q)$ factors as a n-th power. The proof by Nieper-Wißkirchen [162] uses the machinery of Rozansky–Witten invariants. We will briefly explain the proof, and discuss the possibility of using this method to study the Riemann–Roch polynomial $\mathrm{RR}_X(q)$.

D.5.1. Conjectural form of the Riemann–Roch polynomial

Motivated by the above discussions, we speculate about the general shape of the Riemann–Roch polynomial of certain symplectic varieties.

We make the following conjecture. Similar conjectures have already been formulated by Ríos Ortiz and Jiang in [113, Conj. 1.3].

Conjecture I. *Let X be a primitively symplectic orbifold of dimension $2n$.*

(i) *The Riemann–Roch polynomial $\mathrm{RR}_X(q)$ has n distinct negative real roots forming an arithmetic sequence.*

(ii) *If X is smooth, then its Riemann–Roch polynomial $\mathrm{RR}_X(q)$ has even negative integer roots $\lambda_1, \ldots, \lambda_n$ satisfying $\lambda_i - \lambda_{i-1} = 2$.*

The second point is a slight strengthening of [113, Conj. 1.3(3)]. Note that it fails already in the case of four-dimensional orbifolds as demonstrated in Section D.3 and should necessarily involve the smoothness assumption.

By Remark D.2.9, Conjecture I (i) would imply the inequality (D.1.2) and therefore yield the bound on the second Betti number.

D.5.2. Rozansky–Witten invariants

We give a very rough overview of parts of Rozansky–Witten theory that we want to employ. For proofs, details and a general overview we refer mainly to the book [163]. See also [90, 113, 194].

After choosing a symplectic form $\sigma \in H^0(X, \Omega_X^2)$, the *Rozansky–Witten weight system* RW_σ is a ring homomorphism

$$\mathrm{RW}_\sigma \colon B \to H^*(X, \mathbb{C}), \tag{D.5.1}$$

where B denotes the graph homology space, i.e., the $\mathbb{C}$-algebra spanned by all unitrivalent graphs modulo the antisymmetry and IHX relation. Important graphs are ℓ, the unique univalent graph with two vertices, Θ, the trivalent graph $\ominus$ with two vertices, and the $2k$-wheels w_{2k} which, for example, looks like

for $k = 4$.

Using the $2k$-wheels, we can define the wheeling element

$$\Omega := \exp\left(\sum_{k=1}^{\infty} b_{2k} w_{2k}\right)$$

contained in the completion $\hat{B}$ of B with b_{2k} the modified Bernoulli numbers. We have

- $\mathrm{RW}_\sigma(\ell) = 2\sigma$,
- $\mathrm{RW}_\sigma(\Theta) = b_\Theta \left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]$, where $b_\Theta = 48\, r_X = \frac{2(2n-1)C(c_2)}{C(1)}$ [162, Prop. 7],
- $\mathrm{RW}_\sigma(w_{2k}) = -(2k)!\,\mathrm{ch}_{2k}$,
- $\mathrm{RW}_\sigma(\Omega) = \mathsf{td}^{1/2}$.

There is a bilinear product $\langle -, - \rangle$ on the graph homology space defined by summing over all possible ways of gluing all univalent vertices of the graphs under consideration, see [163, Def. 2.39] for a precise account. One form of the Wheeling Theorem is the following [163, Cor. 2.3].

Theorem D.5.1. *The map*

$$\langle \Omega, - \rangle \colon x \mapsto \langle \Omega, x \rangle$$

respects the ring structure on B given by disjoint union.

There is also a bilinear product $\langle -, - \rangle_\sigma$ defined on the cohomology [163, Def. 3.9], which depends on the symplectic form σ chosen. We use the subscript σ to emphasize this dependence. The map RW_σ respects the two bilinear products [163, Prop. 3.4]

$$\mathrm{RW}_\sigma(\langle x, y \rangle) = \langle \mathrm{RW}_\sigma(x), \mathrm{RW}_\sigma(y) \rangle_\sigma.$$

This is the crucial result which allows us to transport relations present inside the graph homology space to the cohomology of X.

Generalized Fujiki constants naturally appear in the study of Rozansky–Witten invariants, which can already be seen in the above formula for $\mathrm{RW}_\sigma(\Theta)$. The key idea for the formula is that $\mathrm{RW}_\sigma(\Theta)$ is a class in $H^{0,2}(X)$, which is generated by $[\overline{\sigma}]$. So we can uniquely determine the class just by a scalar. To determine this number, one could cup the two classes with $\exp(\sigma + \overline{\sigma})$ and compare the integral.

To illustrate this method, we determine the value of $\mathrm{RW}_\sigma(\Theta_2)$, where Θ_2 is the necklace graph with two beads.

Proposition D.5.2. *We have*

$$\begin{aligned}\mathrm{RW}_\sigma(\Theta_2) &= -\frac{4\int (c_2^2 - 2c_4)\exp(\sigma+\overline{\sigma})}{5n(n-1)\int \exp(\sigma+\overline{\sigma})}[\overline{\sigma}]^2 \\ &= -\frac{4(2n-1)(2n-3)C(c_2^2 - 2c_4)}{5C(1)}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^2.\end{aligned}$$

Proof. Using the definition of the pairing $\langle -, - \rangle$ on the graph homology space, one can verify that

$$\left\langle w_4, \ell^2 \right\rangle = 20\Theta_2.$$

Hence

$$\begin{aligned}
\mathrm{RW}_\sigma(\Theta_2) &= \mathrm{RW}_\sigma\left(\frac{1}{20}\left\langle w_4, \ell^2\right\rangle\right) = \frac{1}{20}\left\langle \mathrm{RW}_\sigma(w_4), \mathrm{RW}_\sigma(\ell^2)\right\rangle_\sigma \\
&= \frac{1}{20}\left\langle -24\left(\tfrac{1}{12}c_2^2 - \tfrac{1}{6}c_4\right), 4\sigma^2\right\rangle_\sigma = -\frac{2}{5}\left\langle c_2^2 - 2c_4, \sigma^2\right\rangle_\sigma = -\frac{4}{5}\left\langle c_2^2 - 2c_4, \exp\sigma\right\rangle_\sigma.
\end{aligned}$$

Cupping it with $\exp(\sigma + \overline{\sigma})$ and comparing the integral, we get

$$\mathrm{RW}_\sigma(\Theta_2) = -\frac{4\int \left\langle c_2^2 - 2c_4, \exp\sigma\right\rangle_\sigma \exp(\sigma + \overline{\sigma})}{5\int \overline{\sigma}^2 \exp(\sigma + \overline{\sigma})}[\overline{\sigma}]^2.$$

For the denominator, we can simplify it as

$$\begin{aligned}
\int_X \overline{\sigma}^2 \exp(\sigma + \overline{\sigma}) &= \int_X \overline{\sigma}^2 \frac{1}{(2n-2)!}(\sigma + \overline{\sigma})^{2n-2} \\
&= \int_X \frac{1}{n!(n-2)!}(\sigma\overline{\sigma})^n \\
&= n(n-1)\int_X \exp(\sigma + \overline{\sigma}).
\end{aligned}$$

For the numerator, we use the following equality [163, Lem. 3.4]

$$\int_X \langle \alpha, \exp\sigma\rangle_\sigma \exp(\sigma + \overline{\sigma}) = \int_X \alpha \exp(\sigma + \overline{\sigma}).$$

This shows the first equality that we want to prove.

For the second equality, we note that for a class of type $(2j, 2j)$, the Fujiki relations give

$$\int_X \alpha \exp(\sigma + \overline{\sigma}) = \int_X \alpha \cdot \frac{1}{(2n-2j)!}(\sigma + \overline{\sigma})^{2n-2j} = \frac{C(\alpha)}{(2n-2j)!}q(\sigma + \overline{\sigma})^{n-j}.$$

Taking α to be 1_X and $c_2^2 - 2c_4$ respectively, we get the desired equality. $\square$

In general, for a trivalent graph Γ with $2k$ vertices, there is a number b_Γ independent of the symplectic form σ chosen, such that we have

$$\mathrm{RW}_\sigma(\Gamma) = b_\Gamma \left[\frac{2\overline{\sigma}}{q(\sigma + \overline{\sigma})}\right]^k \in H^{0,2k}(X).$$

For example, we have obtained that

$$b_\Theta = \frac{2(2n-1)C(c_2)}{C(1)}, \quad b_{\Theta_2} = -\frac{4(2n-1)(2n-3)C(c_2^2 - 2c_4)}{5C(1)}.$$

This is the same notation used by Sawon in [193, 194], although he only used the letter b_Γ for graphs with exactly $2n$ vertices and referred to those as the *Rozansky–Witten invariants* of X. By the properties of the map RW_σ, the values b_Γ are multiplicative with respect to disjoint union.

There is another way to obtain the value of $\mathrm{RW}_\sigma(\Theta_2)$. Namely

$$\mathrm{RW}_\sigma(2\Theta_2) = \mathrm{RW}_\sigma(\langle w_2, w_2\rangle) = 4\langle c_2, c_2\rangle_\sigma$$

where we used the relation $\mathrm{RW}_\sigma(w_2) = 2c_2$. We therefore obtain from Proposition D.5.2 the equality

$$\langle c_2, c_2\rangle_\sigma = \frac{b_{\Theta_2}}{2}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^2 \in H^4(X, \mathcal{O}_X).$$

We expect that this equality is equivalent to the equality obtained in Corollary D.2.6, but have not pursued this further.

D.5.3. Proof of Theorem D.2.5

Using the map RW_σ and the Wheeling Theorem, we can obtain a very conceptual proof and see why the polynomial $\mathrm{RR}_{X,1/2}(q)$ factorizes as an n-th power.

Proof of Theorem D.2.5. For a class α of degree $(2k, 2k)$ admitting a generalized Fujiki constant, we follow the same method as in the proof of Proposition D.5.2 to compute

$$\begin{aligned}
\left\langle \alpha, (2\sigma)^k \right\rangle_\sigma &= 2^k k! \langle \alpha, \exp\sigma\rangle_\sigma \\
&= 2^k k! \frac{\int \langle \alpha, \exp\sigma\rangle_\sigma \exp(\sigma+\overline{\sigma})}{n(n-1)\cdots(n-(k-1))\int \exp(\sigma+\overline{\sigma})}[\overline{\sigma}]^k \\
&= \frac{2^k}{\binom{n}{k}} \frac{\int \alpha \exp(\sigma+\overline{\sigma})}{\int \exp(\sigma+\overline{\sigma})}[\overline{\sigma}]^k \\
&= \frac{2^k}{\binom{n}{k}} \frac{\frac{C(\alpha)}{(2n-2k)!}q(\sigma+\overline{\sigma})^{n-k}}{\frac{C(1)}{(2n)!}q(\sigma+\overline{\sigma})^{n}}[\overline{\sigma}]^k \\
&= \frac{1}{\binom{n}{k}\frac{C(1)}{(2n)!}} \frac{C(\alpha)}{(2n-2k)!}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^k .
\end{aligned} \tag{D.5.2}$$

We can take α to be $\mathsf{td}^{1/2}_{2k}$, which gives us

$$\begin{aligned}
\frac{C(1)}{(2n)!}\left\langle \mathsf{td}^{1/2}, (1+2\sigma)^n \right\rangle_\sigma &= \frac{C(1)}{(2n)!}\left\langle \mathsf{td}^{1/2}, \sum_{k=0}^{n}\binom{n}{k}(2\sigma)^k \right\rangle_\sigma \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{C(1)}{(2n)!}\left\langle \mathsf{td}^{1/2}_{2k}, (2\sigma)^k \right\rangle_\sigma \\
&= \sum_{k=0}^{n}\frac{C(\mathsf{td}^{1/2}_{2k})}{(2n-2k)!}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^k \\
&= \mathrm{RR}'_{X,1/2}\left(\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]\right).
\end{aligned}$$

Here $\mathrm{RR}'_{X,1/2}(q) \coloneqq q^n \mathrm{RR}_{X,1/2}(1/q)$ is the polynomial obtained by reversing the coefficients. The polynomial evaluated at the class $\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]$ is an element in the cohomology ring, with terms in various degrees.

On the graph homology side, the Wheeling Theorem provides the relation

$$\langle \Omega, (1+\ell)^n \rangle = \langle \Omega, 1+\ell \rangle^n.$$

Since the Rozansky–Witten invariant RW_σ is a ring homomorphism respecting the bilinear form $\langle -, - \rangle$, we get

$$\left\langle \mathsf{td}^{1/2}, (1+2\sigma)^n \right\rangle_\sigma = \left\langle \mathsf{td}^{1/2}, 1+2\sigma \right\rangle_\sigma^n.$$

Hence the polynomial $\mathrm{RR}_{X,1/2}(q)$ must indeed factorize as an n-th power. □

D.5.4. Riemann–Roch polynomial via RW invariants

Following the idea of the proof of Theorem D.2.5, if we want to study the Riemann–Roch polynomial $\mathrm{RR}_X(q)$, we should replace α with td_{2k} in (D.5.2): summing over all k we get similarly

$$\frac{C(1)}{(2n)!} \langle \mathsf{td}, (1+2\sigma)^n \rangle_\sigma = \mathrm{RR}'_X \left(\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})} \right] \right).$$

So for the same strategy to work, we need to study how the graph homology element

$$\left\langle \Omega^2, (1+\ell)^n \right\rangle$$

might potentially factorize into linear terms. Since the multiplication for the graph homology classes is the disjoint union, this would unfortunately not be possible in general. Below we compute its value for $n \leq 4$:

$$\begin{aligned}
\left\langle \Omega^2, 1+\ell \right\rangle &= 1 + \frac{1}{12}\Theta, \\
\left\langle \Omega^2, (1+\ell)^2 \right\rangle &= 1 + \frac{1}{12}2\Theta + \frac{1}{12^2}(\Theta^2 + \Theta_2), \\
\left\langle \Omega^2, (1+\ell)^3 \right\rangle &= 1 + \frac{1}{12}3\Theta + \frac{1}{12^2}3(\Theta^2 + \Theta_2) + \frac{1}{12^3}(\Theta^3 + 3\Theta\Theta_2), \\
\left\langle \Omega^2, (1+\ell)^4 \right\rangle &= 1 + \frac{1}{12}4\Theta + \frac{1}{12^2}6(\Theta^2 + \Theta_2) + \frac{1}{12^3}4(\Theta^3 + 3\Theta\Theta_2) \\
&\quad + \frac{1}{12^4}(\Theta^4 + 6\Theta^2\Theta_2 + 3\Theta_2^2 + \tfrac{144}{25}\Xi - \tfrac{162}{25}\Theta_4),
\end{aligned}$$

where Ξ is the extra graph for $n = 4$. We study the implications on the Riemann–Roch polynomial.

- When $n = 2$, we get

  $$\mathrm{RR}_X(q) = \frac{C(1)}{(2 \cdot 2)!} \left(q^2 + \frac{1}{12} 2 b_\Theta q + \frac{1}{12^2}(b_\Theta^2 + b_{\Theta_2}) \right).$$

  For the polynomial to admit two real roots, the value b_{Θ_2} needs to be negative, or equivalently, the integral $C(\mathrm{ch}_4) = \int_X \mathrm{ch}_4$ needs to be positive. For smooth hyperkähler fourfolds, this indeed holds by the bound of Guan (see [169, Lem. 4.6] or [193, Thm. 7]).

- When $n = 3$, the graph homology class admits a factor $1 + \frac{1}{12}\Theta$, so we also get a factorization for the Riemann–Roch polynomial
$$\mathrm{RR}_X(q) = \frac{C(1)}{(2\cdot 3)!}\left(q + \frac{1}{12}b_\Theta\right)\left(q^2 + \frac{1}{12}2b_\Theta q + \frac{1}{12^2}(b_\Theta^2 + 3b_{\Theta_2})\right).$$
So if b_{Θ_2} is negative, the polynomial will indeed admit three real roots forming an arithmetic sequence, with difference $\frac{1}{12}\sqrt{-3b_{\Theta_2}}$.

- When $n = 4$, the graph homology class becomes more complicated due to the extra graph Ξ. If we expect the Riemann–Roch polynomial to admit four real roots forming an arithmetic sequence, this would lead to the following conjectural relations among certain generalized Fujiki constants.

  Conjecture J. *If X is of dimension $2n \geq 8$, then*
$$\frac{C(\mathrm{ch}_4^2 + 120\mathrm{ch}_8)\cdot C(1)}{C(\mathrm{ch}_4)^2} = \frac{(5n+7)(2n-1)(2n-3)}{5(n+1)(2n-5)(2n-7)}.$$

  Admitting this relation, we would then get
$$\left\langle \mathrm{ch}_4^2 + 120\mathrm{ch}_8, (2\sigma)^4\right\rangle_\sigma = \left(\tfrac{5}{3(n+1)} + \tfrac{25}{6}\right)\mathrm{RW}_\sigma(\Theta_2^2).$$

  On the other hand, based on the computation of Sawon [194], we have
$$\tfrac{1}{384}\left\langle w_4^2, \ell^4\right\rangle = 24\Xi + 48\Theta_4 + \tfrac{25}{4}\Theta_2^2,$$
$$\tfrac{1}{384}\left\langle w_8, \ell^4\right\rangle = 7\Xi + \tfrac{287}{8}\Theta_4.$$

  Taking a suitable linear combination and applying RW_σ, we get
$$\left\langle \mathrm{ch}_4^2 + 120\mathrm{ch}_8, (2\sigma)^4\right\rangle_\sigma = \mathrm{RW}_\sigma(8\Xi - 9\Theta_4 + \tfrac{25}{6}\Theta_2^2),$$
  so
$$\mathrm{RW}_\sigma(8\Xi - 9\Theta_4) = \tfrac{5}{3(n+1)}\mathrm{RW}_\sigma(\Theta_2^2).$$

  Hence we can express the Rozansky–Witten invariant of $\frac{144}{25}\Xi - \frac{162}{25}\Theta_4 = \frac{18}{25}(8\Xi - 9\Theta_4)$ in terms of b_{Θ_2}, so the Riemann–Roch polynomial has the following form
$$\mathrm{RR}_X(q) = \frac{C(1)}{(2\cdot 4)!}\left(q^2 + \tfrac{1}{12}2b_\Theta q + \tfrac{1}{12^2}(b_\Theta^2 + \tfrac{3}{5}b_{\Theta_2})\right)\left(q^2 + \tfrac{1}{12}2b_\Theta q + \tfrac{1}{12^2}(b_\Theta^2 + \tfrac{27}{5}b_{\Theta_2})\right).$$

  If b_{Θ_2} is negative, then it indeed admits four roots forming an arithmetic progression with difference $\frac{1}{6}\sqrt{-\frac{3}{5}b_{\Theta_2}}$.

D.5.5. Conjectural value for generalized Fujiki constants

In the above examples we see that the value b_{Θ_2} or equivalently $C(\mathrm{ch}_4)$ governs the differences between the roots of the Riemann–Roch polynomial. We speculate that the roots always form an arithmetic progression with difference 2. This is our main motivation for Conjecture H. Note that the conjectural value for $C(\mathrm{ch}_4)$ also predicts that one should always have $b_{\Theta_2} = -48(n+1)$ by Proposition D.5.2. It can also be seen as a weaker version of Conjecture I (ii), for purely algebraic reasons.

Proposition D.5.3. *Conjecture I (ii) implies Conjecture H.*

Proof. By assumption, the roots of $\mathrm{RR}_X(q)$ form an arithmetic progression with difference 2, so we have

$$\begin{aligned}\mathrm{RR}_X(q) &= \frac{C(1)}{(2n)!}(q+a)(q+a+2)\cdots(q+a+2n-2)\\ &= \frac{C(1)}{(2n)!}\Big(q^n + (na+n(n-1))q^{n-1}\\ &\quad + \Big(\tfrac{n(n-1)}{2}a^2 + (n-1)^2na + \tfrac{(3n-1)n(n-1)(n-2)}{6}\Big)q^{n-2} + \dots\Big)\end{aligned}$$

Then by the result of Corollary D.2.7, we may deduce the values for $C(c_2^2)$ and $C(c_4)$, and consequently $C(\mathrm{ch}_4)$, which turns out to depend only on $C(1)$ and n, and not on a. ☐

We also explore some consequences of Conjecture H.

Proposition D.5.4. *Assuming Conjecture H, for $n = 2$ the following are the only possibilities for the generalized Fujiki constants of a hyperkähler fourfold.*

$C(1)$	$C(c_2)$	$C(c_2^2)$	$C(c_4)$
3	30	828	324
9	54	756	108

Proof. We have the following three relations

$$\begin{aligned}7C(c_2^2) - 4C(c_4) &= 15\tfrac{C(c_2)^2}{C(1)},\\ C(c_2^2) - 2C(c_4) &= 60C(1),\\ 3C(c_2^2) - C(c_4) &= 2160,\end{aligned}$$

from which we may deduce that

$$\begin{aligned}C(c_2) &= 2\sqrt{C(1)^2 + 72C(1)},\\ C(c_2^2) &= -12C(1) + 864,\\ C(c_4) &= -36C(1) + 432.\end{aligned}$$

The top-degree ones are just Chern numbers, and using the relations on Betti numbers by Salamon, we have

$$c_2^2 = 736 + 4b_2(X) - b_3(X), \quad c_4 = 48 + 12b_2(X) - 3b_3(X).$$

Since $b_3(X)$ is a multiple of 4, the Chern number c_2^2 must also be a multiple of 4, so we have $C(1) \in \frac{1}{3}\mathbb{Z}$. By the bounds of Guan, we have $-120 \leq c_4 \leq 324$, hence $\frac{46}{3} \geq C(1) \geq 3$, so we only have a finite number of possibilities left.

By definition, the generalized Fujiki constant $C(c_2)$ should be rational. Using this property we may verify that only the listed two cases are possible, which are realized by $\mathrm{K3}^{[2]}$ and Kum_2 respectively. ☐

This further reduces the number of possibilities for Betti numbers to 4, as stated in the introduction.

Proof of Corollary D.1.2. From Salamon's relations [191] one obtains the formula

$$c_4 = 48 + 12b_2(X) - 3b_3(X).$$

By Proposition D.5.4 there are only two possible values for c_4 which together with previously obtained bounds from Guan yield the assertion. ☐

Finally, motivated by the degree 4 case, we conjecture the following behavior to be true for arbitrary dimensions. The question was also asked independently in [50].

Conjecture K. *For* $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$ *with* $k := \sum_i k_i \leq n$ *we have*

$$(-1)^k C(\mathrm{ch}_{2k_1} \cdots \mathrm{ch}_{2k_r}) > 0 \quad \textit{as well as} \quad C(c_{2k_1} \cdots c_{2k_r}) > 0.$$

This in particular generalizes the conjectures in [169, Questions 4.7 and 4.8] to products which do not necessarily live in top degree.

The conjectured alternating behaviour of products of Chern characters together with the positivity of products of Chern classes would yield in combination many restrictions and inequalities between these characteristic values. We expect the above positivity to hold pointwise and to be of local nature.

E. Equivariant categories of symplectic surfaces and fixed loci of Bridgeland moduli spaces

ABSTRACT. Given an action of a finite group G on the derived category of a smooth projective variety X we relate the fixed loci of the induced G-action on moduli spaces of stable objects in $D^b(\mathrm{Coh}(X))$ with moduli spaces of stable objects in the equivariant category $D^b(\mathrm{Coh}(X))_G$. As an application we obtain a criterion for the equivariant category of a symplectic action on the derived category of a symplectic surface to be equivalent to the derived category of a surface. This generalizes the derived McKay correspondence, and yields a general framework for describing fixed loci of symplectic group actions on moduli spaces of stable objects on symplectic surfaces.

E.1. Introduction

E.1.1. Equivariant categories

Let S be a smooth complex projective surface which is symplectic, hence either a K3 or abelian surface. Whenever a finite group G acts symplectically on S, the derived McKay correspondence provides an equivalence between the category $D^b(S)_G$ of G-equivariant objects in the derived category $D^b(S)$, and the derived category of the minimal resolution of the quotient S/G. The equivariant category $D^b(S)_G$ depends only on the action of G on the derived category and not on the underlying surface. Hence we may ask whether a similar correspondence can be formulated for group actions on the derived category which do not come from an action on the surface. Our first result considers this question under the following assumptions:

Let ρ be the action of a finite group G on $D^b(S)$ satisfying the following conditions:

(i) For every $g \in G$ the equivalence $\rho_g \colon D^b(S) \to D^b(S)$ is symplectic.

(ii) There exists a stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ which is fixed by every ρ_g.

(iii) The group G acts faithfully, *i.e.* the equivariant category is indecomposable.

Here an equivalence is *symplectic* if the induced action on singular cohomology $H^*(S, \mathbb{Z})$ preserves the class of the symplectic form. We let $\mathrm{Stab}^\dagger(S)$ be the distinguished connected component of the space of Bridgeland stability conditions of $D^b(S)$ introduced in [41]. The action ρ is faithful, if $\rho_g \not\cong \mathrm{id}$ for all $g \neq 1$. Also no generality is lost by assuming (iii) since for non-faithful actions the equivariant category decomposes as an orthogonal sum where each summand is determined by a faithful action on $D^b(S)$, see [29]. By the derived Torelli theorem for symplectic surfaces [101, Thm. 0.1], group actions satisfying these conditions can be constructed using lattice methods. In particular, there are many such group actions which do not arise from automorphisms of the surface even after deformation.

Write $\Lambda = H^{2*}(S,\mathbb{Z})$ for the even cohomology lattice and let Λ^G_{alg} be the invariant sublattice of the induced G-action on its algebraic part

$$\Lambda_{\mathrm{alg}} = \Lambda \cap (H^0(S,\mathbb{C}) \oplus H^{1,1}(S,\mathbb{C}) \oplus H^4(S,\mathbb{C})).$$

Let $M_\sigma(v)$ be a moduli space of σ-semistable objects of Mukai vector $v \in \Lambda^G_{\mathrm{alg}}$. For the induced G-action on $M_\sigma(v)$ we prove the following:

Theorem E.1.1. *Assume that $M_\sigma(v)$ is a fine moduli space and that the fixed locus $M_\sigma(v)^G$ has a 2-dimensional G-linearizable connected component F. Then there exists a subgroup $H \subset G^\vee = \mathrm{Hom}(G,\mathbb{C}^*)$, a connected H-torsor $S' \to F$ and an equivalence*

$$D^b(S') \xrightarrow{\cong} D^b(S)_G.$$

We say here that a connected component of $M_\sigma(v)^G$ is *G-linearizable* if for some (or equivalently any) point on it the corresponding G-invariant object in $D^b(S)$ admits a G-linearization. By a result of Ploog [185] the obstruction to such a linearization is an element in the second group cohomology $H^2(G,\mathbb{C}^*)$. Hence for groups where this cohomology vanishes, such as cyclic groups, the condition on F to be G-linearizable is automatically satisfied.

Recall from [18, 103] that every fine moduli space $M_\sigma(v)$ is smooth and inherits a symplectic form from the surface S. By assumption (i) the G-action preserves this symplectic form. Hence, its fixed locus is smooth and symplectic, so S' is a symplectic surface. If the action of G is induced by an action on the surface S, then Theorem E.1.1 recovers the usual derived McKay correspondence by taking the moduli space to be the Hilbert scheme of points $\mathsf{Hilb}^{|G|}(S)$ (the component F is the closure of the locus of free orbits).

Theorem E.1.1 applies also to coarse moduli spaces $M_\sigma(v)$ of stable objects with the only difference that $D^b(S')$ has to be replaced by the derived category of α-twisted coherent sheaves $D^b(S',\alpha)$, where $\alpha \in \mathrm{Br}(S')$ is the Brauer class obtain from the universal family of $M_\sigma(v)$ by restriction. For a more general version of the theorem which applies also to moduli spaces containing strictly semistable points, see Section E.5.4.

E.1.2. Fixed loci

The result above relies on a general relationship between fixed loci of moduli spaces of (semi)stable objects and the equivariant category.

Let X be a smooth projective variety and let

$$\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$$

be a connected component of the space of stability conditions satisfying the technical condition (†) of Section E.3.6. The existence of components $\mathrm{Stab}^*(X)$ satisfying (†) is known for arbitrary curves and surfaces, as well as for certain threefolds, see [16, Rem. 26.4] and references therein. Moreover, as shown in [6] there exists good moduli spaces of semistable objects with respect to any stability condition in $\mathrm{Stab}^*(X)$.

Consider an action on $D^b(X)$ by a finite group G. Any G-invariant stability condition $\sigma \in \operatorname{Stab}(X)$ yields an induced stability condition σ_G on the equivariant category [133]. If moreover $\sigma \in \operatorname{Stab}^*(X)$, then we will prove that there exists proper good moduli spaces $M_{\sigma_G}(v')$ of σ_G-semistable objects in $D^b(X)_G$, see Theorem E.3.22.

Theorem E.1.2. *Let $\sigma \in \operatorname{Stab}^*(X)$ be G-invariant and let M be a smooth good moduli space of σ-stable objects in $D^b(X)$ of class $v \in K(D^b(X))^G$. Then the natural morphism*

$$\bigsqcup_{v' \mapsto v} M_{\sigma_G}(v') \to M^G \tag{E.1.1}$$

is a $G^\vee$-torsor over the union of all G-linearizable connected components of M^G. Here v' runs over all classes in $K(D^b(X)_G)$ mapping to v under the forgetful functor.

Furthermore, (E.1.1) *is surjective if $H^2(G, \mathbb{C}^*) = 0$ or, more generally, if the G-action on $D^b(X)$ factors through the action of a quotient $G \twoheadrightarrow Q$, such that G is a Schur covering group of Q.*

The notion of a Schur covering group will be reviewed in Section E.2.1.

Theorem E.1.2 serves as a bridge between the geometry of the fixed locus M^G and the formal properties of the equivariant category. Information can flow in both ways: It can be used to describe moduli spaces of stable objects in the equivariant category in terms of the fixed loci, for example showing projectivity. This generalizes an approach of Nuer towards the moduli space of stable objects on an Enrique surface [165]. In the case of Theorem E.1.1 it is used to determine the equivariant category. In the opposite direction, if one knows that the equivariant category is equivalent to the derived category of a variety whose moduli spaces are well-understood (e.g. a curve, $\mathbb{P}^2$ or a symplectic surface[1]), then the left hand side of (E.1.1) determines the G-linearizable part of the fixed locus up to an étale cover.

E.1.3. Back to symplectic surfaces

Consider again a G-action on the derived category of a symplectic surface S satisfying (i)-(iii). Assume that we have an equivalence

$$D^b(S', \alpha) \xrightarrow{\cong} D^b(S)_G$$

for a symplectic surface S' with Brauer class $\alpha \in \operatorname{Br}(S')$. Let $v \in \Lambda^G_{\mathrm{alg}}$ and define

$$R_v = \{v' \in \Lambda_{(S',\alpha),\mathrm{alg}} \mid v' \mapsto v\},$$

where the algebraic part $\Lambda_{(S',\alpha),\mathrm{alg}}$ of the lattice $H^{2*}(S', \mathbb{Z})$ is taken with respect to α [109]. If $M_\sigma(v)$ is a moduli space of stable objects, then Theorem E.1.2 shows that

$$\bigsqcup_{v' \in R_v} M_{\sigma_G}(v') \to M_\sigma(v)^G$$

is a $G^\vee$-torsor over the union of all G-linearizable components.

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[1] Strictly speaking, for symplectic surfaces one also needs to know that the induced stability condition σ_G lies in the distinguished component. This is proven in Section E.6.3 if the equivalence is induced by a Fourier–Mukai kernel as in Theorem E.1.1.

In a special case we can be more precise. Consider a set of representatives

$$\overline{R}_v \subset \Lambda_{(S',\alpha),\mathrm{alg}}$$

for the coset $R_v/G^\vee$ where the $G^\vee$-action is induced by the action on the equivariant category by twisting the linearization, see Section E.2.1.

Theorem E.1.3. *Suppose that G is cyclic and that S' is a K3 surface. If $M_\sigma(v)$ is a moduli space of stable objects, then we have an isomorphism*

$$M_\sigma(v)^G \cong \bigsqcup_{v' \in \overline{R}_v} M_{\sigma_G}(v'). \tag{E.1.2}$$

Our description of fixed loci can be applied whenever a group action on a moduli space of stable objects is induced by a group action on the derived category. Fortunately, it is an immediate consequence of work of Mongardi [151], Huybrechts [101], and Bayer–Macrí [18] that for K3 surfaces every symplectic group action is of this type. One has the following:

Proposition E.1.4. *Let S be a K3 surface and let $\sigma' \in \mathrm{Stab}^\dagger(S)$ be a stability condition. Let G be a finite group which acts faithfully and symplectically on a moduli space $M = M_{\sigma'}(v)$ of σ'-stable objects. Then the following holds:*

(a) There exists a surjection $G' \to G$ from a finite group G' and an action of G' on $D^b(S)$ satisfying (i), (ii) of Section E.1.1 which induces the given G-action on M.

(b) If G is cyclic, then we can take $G' = G$ in part (a).

The results presented above yield a general framework to determine the fixed loci of any symplectic group action on a moduli space M of stable objects on a symplectic surface S. There are three steps that have to be taken:

Step 1. Find the group action on the derived category which induces the action on M (Proposition E.1.4).

Step 2. Determine the equivariant category[2], i.e. express it in terms of derived categories of symplectic surfaces (Theorem E.1.1).

Step 3. Apply Theorem E.1.2.

In other words, we have reduced the problem of describing fixed loci of such symplectic actions to determining the equivariant category. An example where the above process is applied in a non-trivial case can be found in Section E.7.4 below.

E.1.4. Related work

Kamenova, Mongardi, and Oblomkov determined in [118] the fixed loci of symplectic involutions of holomorphic symplectic varieties of $\mathrm{K3}^{[n]}$-type. Their argument proceeds by deforming to an involution of the Hilbert scheme of points of a K3 surface which is induced by an involution on the surface. For these actions a description of the fixed locus can be

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[2] In the non-cyclic case with respect to a Schur cover of the group

obtained by a local analysis near the fixed points. Our work here grew out of the desire to also describe fixed loci of more general (e.g. non-natural) automorphisms.

By work of Huybrechts [101] and Gaberdiel, Hohenegger, and Volpato [77] there is a bijection between finite groups of symplectic auto-equivalences of a K3 surface fixing a stability condition and subgroups of the Conway group with invariant lattice of rank at least four. The bijection generalizes classical work of Mukai [156] relating symplectic automorphism groups of a K3 surface with subgroups of the Mathieu group. Similar results for abelian surfaces have been obtained by Volpato [216]. In particular, the derived Torelli theorem in [101, Prop. 1.4] provides a large reservoir of symplectic group actions on the derived category, and thus a good testing ground for our ideas. We refer to Section E.7 for a series of examples. The auto-equivalences obtained in this way are described lattice-theoretically, but a concrete geometric description is often missing. By a criterion of Huybrechts [101] and Mongardi [151] some of these auto-equivalences induce an action on a moduli space of stable objects, but not all of them do (it is still an open question whether that criterion is sharp).

Group actions on the derived category also play an important role in the string theory of K3 surfaces. In physics the pair (S, σ) of a symplectic surface and a distinguished stability condition corresponds to a *non-singular sigma model* on S. Symplectic σ-preserving actions on the derived category correspond to supersymmetry-preserving discrete symmetries. The equivariant categories are the orbifold sigma models. Based partially on counting BPS states/dyons, string theory predicts that the orbifold models should be again either K3 or torus (i.e. abelian surface) models [181]. The relationship between auto-equivalences and the Conway group cited above provides the key link between BPS counting in equivariant sigma models and moonshine phenomena for the Conway group, see [178] and [77] for an introduction on the physical and mathematical side respectively.

E.1.5. Open questions

The equivariant categories $D^b(S)_G$ we have considered above are 2-Calabi–Yau categories. Moduli spaces of stable objects in them are holomorphic-symplectic varieties of yet unknown type, and hence provide potentially new examples of (irreducible) holomorphic symplectic varieties. The most pressing question is therefore the following:

Question E.1.5. Is the set of derived categories of (twisted) coherent sheaves on K3 and abelian surfaces closed under the operation of taking equivariant categories with respect to finite group actions satisfying (i)-(iii)?

In this set we should also include deformations of these categories in the sense of [16] such as the Kuznetsov category of a cubic fourfold. All evidence so far (as well as the expectation of physics) points to a positive answer. The parallel question in dimension 1 has an affirmative answer, see [29, Sec. 7].

E.1.6. Plan of the paper

The paper consists of two parts. The first part can be read independently and deals with the construction of moduli spaces of objects in the equivariant category. Section E.2 recalls basic properties of equivariant categories. In Section E.3 we consider the relation between fixed stacks and the equivariant category and prove Theorem E.1.2.

For the proof we first use Orlov's result on Fourier–Mukai functors [175] to construct a G-action on Lieblich's stack $\mathfrak{M}$ of universally gluable objects in $D^b(X)$ (Section E.3.3). The associated fixed stack $\mathfrak{M}^G$ defined in the categorical sense of Romagny is precisely the stack of objects in the equivariant category $D^b(X)_G$ (Proposition E.3.8). By transferring geometric properties from $\mathfrak{M}$ to its fixed stack this yields a well-behaved moduli theory for objects in the equivariant category (Section E.3.5). Theorem E.1.2 follows then simply by comparing the fixed stack of a $\mathbb{G}_m$-gerbe with the fixed locus of the underlying coarse moduli space.

The second part concerns equivariant categories of symplectic surfaces. In Section E.4 we first discuss Serre functors of equivariant categories and define equivariant Fourier–Mukai transforms. In Section E.5 we prove Theorem E.1.1 (including its more general form) and Theorem E.1.3. In Section E.6 we show that in good cases the induced stability condition lies again in the distinguished component and prove Proposition E.1.4. In Section E.7 we discuss a series of examples illustrating the general theory.

In Appendix E.A we prove that for every distinguished stability condition on a K3 surface after a shift the heart generates the derived category. In Appendix E.B we prove a formula for the topological Euler characteristic of the fixed locus of moduli spaces of stable objects on K3 surfaces under cyclic groups actions.

E.1.7. Conventions

We always work over $\mathbb{C}$. A variety is connected unless specified otherwise. All functors are derived unless mentioned otherwise. The K-group $K(\mathcal{D})$ of a triangulated category $\mathcal{D}$ with finite-dimensional Hom-spaces is always taken numerically, i.e. modulo the ideal generated by the kernel of the Euler pairing. Given a smooth projective variety X we let $D^b(X) = D^b(\mathrm{Coh}(X))$ denote the bounded derived category of coherent sheaves on X. If $\pi\colon X \to T$ is a smooth projective morphism with geometrically connected fibers to a $\mathbb{C}$-scheme T, then $D(X)$ or $D(X/T)$ will stand for the full triangulated subcategory of T-perfect complexes of the unbounded derived category of $\mathcal{O}_X$-modules. We refer to Sections 2 and 8.1 of [16] for definitions and further references. If $T = \mathrm{Spec}(\mathbb{C})$, then $D(X)$ is the bounded derived category of coherent sheaves as before.

E.1.8. Acknowledgements

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Part 1. Moduli spaces for the equivariant category

E.2. Equivariant categories

E.2.1. Categorical actions

An action (ρ, θ) of a finite group G on an additive $\mathbb{C}$-linear category $\mathcal{D}$ consists of

- for every $g \in G$ an auto-equivalence $\rho_g \colon \mathcal{D} \to \mathcal{D}$,
- for every pair $g, h \in G$ an isomorphism of functors $\theta_{g,h} \colon \rho_g \circ \rho_h \to \rho_{gh}$

such that for all $g, h, k \in G$ the following diagram commutes

$$\begin{tikzcd}
\rho_g\rho_h\rho_k \arrow[r, "\rho_g\theta_{h,k}"] \arrow[d, "\theta_{g,h}\rho_k"] & \rho_g\rho_{hk} \arrow[d, "\theta_{g,hk}"] \\
\rho_{gh}\rho_k \arrow[r, "\theta_{gh,k}"] & \rho_{ghk}.
\end{tikzcd} \tag{E.2.1}$$

A G-functor $(f, \sigma) \colon (\mathcal{D}, \rho, \theta) \to (\mathcal{D}', \rho', \theta')$ between categories with G-actions is a pair of a functor $f \colon \mathcal{D} \to \mathcal{D}'$ together with 2-isomorphisms $\sigma_g \colon f \circ \rho_g \to \rho'_g \circ f$ such that (f, σ) intertwines the associativity relations on both sides, i.e. such that the following diagram commutes:

$$\begin{tikzcd}
f\rho_g\rho_h \arrow[r, "\sigma_g\rho_h"] \arrow[d, "f\theta_{g,h}"] & \rho'_g f\rho_h \arrow[r, "\rho'_g\sigma_h"] & \rho'_g\rho'_h f \arrow[d, "\theta'_{g,h}f"] \\
f\rho_{gh} \arrow[rr, "\sigma_{gh}"] & & \rho'_{gh}f.
\end{tikzcd}$$

A 2-morphism of G-functors $(f, \sigma) \to (\tilde{f}, \tilde{\sigma})$ is a 2-morphism $t \colon f \to f'$ that intertwines the σ_g, i.e. $\tilde{\sigma}_g \circ t\rho_g = \rho'_g t \circ \sigma_g$.

Definition E.2.1. Given a G-action (ρ, θ) on the category $\mathcal{D}$ the equivariant category $\mathcal{D}_G$ is defined as follows:

- Objects of $\mathcal{D}_G$ are pairs (E, ϕ) where E is an object in $\mathcal{D}$ and $\phi = (\phi_g \colon E \to \rho_g E)_{g \in G}$ is a family of isomorphisms such that

$$\begin{tikzcd}
E \arrow[r, "\phi_g"] \arrow[rrr, bend right, "\phi_{gh}"'] & \rho_g E \arrow[r, "\rho_g\phi_h"] & \rho_g\rho_h E \arrow[r, "\theta^E_{g,h}"] & \rho_{gh}E
\end{tikzcd} \tag{E.2.2}$$

  commutes for all $g, h \in G$.

- A morphism from (E, ϕ) to (E', ϕ') is a morphism $f \colon E \to E'$ in $\mathcal{D}$ which commutes with linearizations, i.e. such that

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "\phi_g"] & E' \arrow[d, "\phi'_g"] \\
\rho_g E \arrow[r, "\rho_g f"] & \rho_g E'
\end{tikzcd}$$

commutes for every $g \in G$.

For all objects (E, ϕ) and (E', ϕ') in $\mathcal{D}_G$ the group G acts on $\operatorname{Hom}_{\mathcal{D}}(E, E')$ via $f \mapsto (\phi'_g)^{-1} \circ \rho_g(f) \circ \phi_g$. By definition,

$$\operatorname{Hom}_{\mathcal{D}_G}((E, \phi), (E, \phi')) = \operatorname{Hom}_{\mathcal{D}}(E, E')^G.$$

The equivariant category comes equipped with a forgetful functor

$$p \colon \mathcal{D}_G \to \mathcal{D}, \quad (E, \psi) \mapsto E$$

and a linearization functor

$$q \colon \mathcal{D} \to \mathcal{D}_G, \quad E \mapsto (\oplus_{g \in G} \rho_g E, \phi) \tag{E.2.3}$$

where the linearization ϕ is given by considering $\theta^{-1}_{h,h^{-1}g} \colon \rho_g E \to \rho_h \rho_{h^{-1}g} E$ and then taking the direct sum over all g,

$$\phi_h = \oplus_g \theta^{-1}_{h,h^{-1}g} \colon \oplus_g \rho_g E \to \rho_h \left(\oplus_g \rho_{h^{-1}g} E \right) = \rho_h \left(\oplus_g \rho_g E \right). \tag{E.2.4}$$

By [67, Lem. 3.8], p is both left and right adjoint to q.

We discuss several properties of equivariant categories. We will often write g for ρ_g.

Example E.2.2. The *trivial* G-action on $\mathcal{D}$ is defined by $\rho_g = \mathrm{id}$ and $\theta_{g,h} = \mathrm{id}$ for all $g, h \in G$. In this case the objects of $\mathcal{D}_G$ are pairs of an object $x \in \mathcal{D}$ and a homomorphism $\phi \colon G \to \operatorname{Aut}(x)$.

Remark E.2.3. Consider the 2-category G-$\mathfrak{Cats}$ whose objects are categories with a G-action and whose morphisms are G-functors. The equivariant category $\mathcal{D}_G$ satisfies the universal property that for all categories $\mathcal{A}$ we have the equivalence

$$\operatorname{Hom}_{\mathfrak{Cats}}(\mathcal{A}, \mathcal{D}_G) \cong \operatorname{Hom}_{G\text{-}\mathfrak{Cats}}(\iota(\mathcal{A}), \mathcal{D})$$

where $\iota(\mathcal{A})$ is the category $\mathcal{A}$ endowed with the trivial G-action. Hence, any G-functor from $\iota(\mathcal{A})$ to $\mathcal{D}$ factors over the forgetful functor p, see [78, Prop. 4.4] for more details.

If a triangulated category has a dg-enhancement, then the equivariant category is again triangulated [67, Cor. 6.10]. This is implied also more directly as follows.

Proposition E.2.4. *Let $\mathcal{D}$ be a triangulated category with an action of a group G. Suppose there is a full abelian subcategory $\mathcal{A} \subset \mathcal{D}$ such that $D^b(\mathcal{A}) = \mathcal{D}$ and G preserves $\mathcal{A}$, i.e. $\rho_g E \in \mathcal{A}$ for all $E \in \mathcal{A}$. Then the following holds.*

(i) There exist a dg-enhancement $\mathcal{D}_{dg}$ of $\mathcal{D}$ together with an action of G on $\mathcal{D}_{dg}$ which lifts the action of G on $\mathcal{D}$.

(ii) The equivariant category $\mathcal{D}_G$ is triangulated.

Proof. By [49, Sec. 1.2] the dg-quotient category

$$D_{dg}(\mathcal{A}) = C_{dg}(\mathcal{A})/\mathrm{Acyclic}_{dg}(\mathcal{A})$$

of the dg-category of bounded complexes in $\mathcal{A}$ by the dg-category of acyclic bounded complexes in $\mathcal{A}$ defines a dg-enhancement of $D^b(\mathcal{A})$. By hypothesis $D^b(\mathcal{A}) \cong \mathcal{D}$ hence $D_{dg}(\mathcal{A})$ is a dg-enhancement. Moreover, the G-action on $\mathcal{D}$ induces a G-action on $\mathcal{A}$. Since G preserves acyclic complexes we obtain a G-action on $D_{dg}(\mathcal{A})$ with the desired properties. This proves the first part. For the second part we apply [54], see also [67, Thm. 7.1], to get

$$\mathcal{D}_G = D^b(\mathcal{A})_G \cong D^b(\mathcal{A}_G)$$

and as a derived category the latter is naturally triangulated. □

Remark E.2.5. If X is a smooth projective variety, then $D^b(X)$ has (up to equivalence) a unique dg-enhancement [132].

The group of characters $G^\vee = \{\chi\colon G \to \mathbb{C}^* \mid \chi \text{ homomorphism}\}$ acts on the equivariant category $\mathcal{D}_G$ by the identity on morphisms and by

$$\chi \cdot (E, \phi) = (E, \chi\phi)$$

on objects, where we let $\chi\phi$ denote the linearization $\chi(g)\phi_g\colon E \to \rho_g E$.

An object $E \in \mathcal{D}$ is called *G-invariant* if for all $g \in G$ there exists an isomorphism $\rho_g E \cong E$. A G-linearization of E is an element $\tilde{E} \in \mathcal{D}_G$ such that $p\tilde{E} \cong E$. There is the following obstruction for a G-invariant simple object to be G-linearizable (which, since $H^2(\mathbb{Z}_n, \mathbb{C}^*) = 0$ for all n, is trivial for cyclic groups).

Lemma E.2.6 ([185, Lem. 1])**.** *Given a G-invariant simple object $E \in \mathcal{D}$, there exists a class in $H^2(G, \mathbb{C}^*)$ which vanishes if and only if there exists a G-linearization of E. The set of (isomorphism classes) of G-linearizations of E is a torsor under $G^\vee$.*

Example E.3.15 below shows that this obstruction is effective.

Recall that an extension of groups $1 \to K \to E \to G \to 1$ is *stem* if K is contained both in the commutator subgroup and the center of E. Any maximal stem extension $\tilde{G} \twoheadrightarrow G$ is called a Schur covering group of G. It has the property that the restriction morphism

$$H^2(G, \mathbb{C}^*) \to H^2(\tilde{G}, \mathbb{C}^*)$$

vanishes. Hence, by Lemma E.2.6 if we let $\tilde{G}$ act on $\mathcal{D}$ via the quotient map to G, then every invariant simple object admits a $\tilde{G}$-linearization.

Let $\operatorname{Aut}\mathcal{D}$ be the group of equivalences of $\mathcal{D}$. Every group action on $\mathcal{D}$ yields a subgroup of $\operatorname{Aut}\mathcal{D}$. For the converse one has the following obstruction (which because of $H^3(\mathbb{Z}_n, \mathbb{C}^*) = \mathbb{Z}_n$ is non-trivial even for cyclic groups).

Lemma E.2.7. ([29, Sec. 2.2]) *Assume that* $\mathrm{Hom}(\mathrm{id}_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}) = \mathbb{C}\mathrm{id}$ *and let* $G \subset \operatorname{Aut}\mathcal{D}$ *be a finite subgroup.*

(a) There exists a class in $H^3(G, \mathbb{C}^*)$ which vanishes if and only if there exists an action of G on $\mathcal{D}$ whose image in $\operatorname{Aut}\mathcal{D}$ *is G. Moreover, the set of isomorphism classes of such actions is a torsor under $H^2(G, \mathbb{C}^*)$.*

(b) *There exits a finite group G' and a surjection $G' \to G$ such that G' acts on $\mathcal{D}$ and the induced map $G' \to \operatorname{Aut}\mathcal{D}$ is the given quotient map to G.*

(c) *If $G = \mathbb{Z}_n$, then we can take $\mathbb{Z}_{n^2} \to \mathbb{Z}_n$ in (b).*

E.2.2. Stability conditions

A (Bridgeland) stability condition on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{A}, Z)$ consisting of

- the heart $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure on $\mathcal{D}$ and
- a stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$

satisfying several conditions, see [40]. Given an equivalence $\Phi\colon \mathcal{D} \to \mathcal{D}'$ of triangulated categories the image of σ under Φ is defined by

$$\Phi\sigma = (\Phi\mathcal{A}, Z \circ \Phi_*^{-1})$$

where $\Phi_*\colon K(\mathcal{D}) \to K(\mathcal{D}')$ is the induced map on K-groups. If $\Phi\colon \mathcal{D} \to \mathcal{D}$ is an autoequivalence, we say that Φ preserves (or fixes) σ if $\Phi\sigma = \sigma$.

Let X be a smooth projective variety together with an action of a finite group G on $D^b(X)$ which fixes a stability condition $\sigma = (\mathcal{A}, Z)$. By [133, Lem. 2.16] σ induces a stability condition on $D^b(X)_G$ defined by

$$\sigma_G = (\mathcal{A}_G, Z_G), \quad Z_G \coloneqq Z \circ p_*\colon K(\mathcal{A}_G) \to \mathbb{C}.$$

Lemma E.2.8. *Let $(E,\phi) \in \mathcal{A}_G$. Then (E,ϕ) is σ_G-semistable if and only if E is σ-semistable. If E is σ-stable, then (E,ϕ) is σ_G-stable.*

Proof. If an element $(E,\phi) \in \mathcal{A}_G$ is destabilized by (F,ψ), then $p(E,\phi)$ is destabilized by $p(F,\psi)$. Conversely, if $p(E,\phi)$ is destabilized by $F' \in \mathcal{A}$, then the image of the adjoint morphism $qF' \to (E,\phi)$ destabilizes (E,ϕ). This shows the first claim. A subobject of (E,ϕ) is given by a subobject $F \subset E$ such that ϕ restricts to a linearization of F. Hence any destabilizing subobject of (E,ϕ) yields a destabilizing subobject of E. This shows the second claim. ☐

Definition E.2.9. A class $v \in K(\mathcal{A})^G$ is *(G,σ)-generic* if it is primitive and for every splitting $v = v_0 + v_1$ with $v_i \in K(\mathcal{A})^G \setminus \mathbb{Z}v$ the summands have different slopes.

Lemma E.2.10. *Let $(E,\phi) \in \mathcal{A}_G$ such that E is σ-semistable and its class $[E] \in K(\mathcal{A})^G$ is (G,σ)-generic. Then (E,ϕ) is σ_G-stable. In particular,*

$$\operatorname{Hom}_{\mathcal{A}_G}((E,\phi),(E,\phi)) = \mathbb{C}\mathrm{id}.$$

Proof. As explained above the object (E,ϕ) is σ_G-semistable. If it is not stable, then there exists a short exact sequence in $\mathcal{A}_G$

$$0 \to (F_1,\phi) \to (E,\phi) \to (F_2,\phi) \to 0$$

with F_1, F_2 of the same phase as E. Applying the forgetful functor we obtain

$$0 \to F_1 \to E \to F_2 \to 0$$

in $\mathcal{A}$ with F_i semistable of the same phase as E. However, the classes $[F_i]$ are G-invariant which shows that $[E] = [F_1] + [F_2]$ is not (G,σ)-generic. ☐

E.2.3. Fourier–Mukai actions

Let $\pi\colon X \to T$ be a smooth projective morphism to a $\mathbb{C}$-scheme T with geometrically connected fibers. Let

$$p, q\colon X \times_T X \to X$$

be the projections to the factors. The Fourier–Mukai transform $\mathsf{FM}_{\mathcal{E}}\colon D(X) \to D(X)$ with kernel $\mathcal{E} \in D(X \times_T X)$ is defined by

$$\mathsf{FM}_{\mathcal{E}}(A) = q_*(p^*(A) \otimes \mathcal{E}).$$

Using a push-pull argument we have isomorphisms

$$\mathsf{FM}_{\mathcal{E}}(A \otimes \pi^* B) \cong \mathsf{FM}_{\mathcal{E}}(A) \otimes \pi^* B \tag{E.2.5}$$

for all $A \in D(X)$ and $B \in D(T)$, functorial in both A and B.

Definition E.2.11. A *Fourier–Mukai action* of G on $D(X)$ consists of[3]

- for every $g \in G$ a Fourier–Mukai kernel $\mathcal{E}_g \in D(X \times_T X)$,
- for every pair $g, h \in G$ an isomorphism $\theta_{g,h}\colon \mathcal{E}_g \circ \mathcal{E}_h \to \mathcal{E}_{gh}$

such that for all g, h, k the diagram (E.2.1) commutes with ρ_g replaced by $\mathcal{E}_g$.

For smooth projective varieties we have not defined anything new:

Lemma E.2.12. ([29, Sec. 2.3]) *Let X be smooth projective variety and let G be a finite group. Then any G-action on $D^b(X)$ is induced by a unique Fourier–Mukai action.*

Given a Fourier–Mukai action on the derived category of X/T our next goal is to define natural operations on the equivariant category. If G is induced by an action on X, this is discussed in [42, Sec. 4]. Since our G-action does not have to preserve the tensor product or the structure sheaf, some care is needed in the general case.

E.2.3.1. Pushforward and pullback

Consider a fiber product diagram

$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ T' \arrow[r, "\beta"] & T. \end{tikzcd} \tag{E.2.6}$$

The pullback of the kernels of the G-action on X,

$$(\alpha \times \alpha)^* \mathcal{E}_g \in D(X' \times_{T'} X'),$$

together with the pullback of the $\theta_{g,h}$ define a Fourier–Mukai G-action on $D(X')$. We say that the morphism α is *G-equivariant*.

---

[3] We write $\mathcal{E} \circ \mathcal{F}$ to indicate the composition of correspondences $\mathcal{E}, \mathcal{F}$.

Given an equivariant object (F, ϕ) in $D(X)_G$ we define its pullback by

$$\alpha^*(F, \phi) = (\alpha^* F, \phi') \in D(X')_G$$

where the G-linearization ϕ'_g is the composition

$$\alpha^* F \xrightarrow{\alpha^*\phi_g} \alpha^*(gF) = \alpha^* q_*(p^*(F) \otimes \mathcal{E}_g) \cong q'_*(\alpha \times \alpha)^*(p^*(F) \otimes \mathcal{E}_g)$$
$$\cong q'_*(p'^*(\alpha^* F) \otimes (\alpha \times \alpha)^* \mathcal{E}_g) = g\alpha^*(F)$$

with $p', q' \colon X' \times_{T'} X' \to X'$ the projections. The pullback α^* of an equivariant morphism is the pullback of the morphism in $D(X)$ (one checks that the pullback morphism is G-invariant). Taken together this yields a functor

$$\alpha^* \colon D(X)_G \to D(X')_G.$$

Similarly if β is proper and flat and $(E, \phi) \in D(X')_G$, we define the pushforward functor by

$$\alpha_*(E, \phi) \coloneqq (\alpha_* E, \phi')$$

where the G-linearization ϕ' is obtained as the composition

$$\alpha_* E \xrightarrow{\alpha_*\phi_g} \alpha_* gE = \alpha_* q'_*(p'^*(E) \otimes (\alpha \times \alpha)^*(\mathcal{E}_g))$$
$$\cong q_*(\alpha \times \alpha)_*(p'^*(E) \otimes (\alpha \times \alpha)^*(\mathcal{E}_g)) \cong q_*(p^*(\alpha_* E) \otimes \mathcal{E}_g) = g\alpha_*(E).$$

The pushforward of an equivariant morphism is the pushforward of the underlying morphism. The pullback functor α^* is left adjoint to α_*.

E.2.3.2. Hom and tensor product

Given a T-perfect object $B \in D(T)$ and an equivariant object $(E, \phi) \in D(X)_G$ we define the tensor product by

$$(E, \phi) \otimes \pi^* B \coloneqq (\pi^* B \otimes E, \phi')$$

where the linearization ϕ' is the composition

$$E \otimes \pi^*(B) \xrightarrow{\phi_g \otimes \mathrm{id}} \mathsf{FM}_{\mathcal{E}_g}(E) \otimes \pi^*(B) \overset{\text{(E.2.5)}}{\cong} \mathsf{FM}_{\mathcal{E}_g}(E \otimes \pi^*(B)) = g(E \otimes \pi^*(B)).$$

More generally, if $D(T)$ is equipped with the trivial G-action and $(B, \chi) \in D(T)_G$, we let

$$(B, \chi) \otimes (E, \phi) \coloneqq (\pi^* B \otimes E, \chi\phi')$$

Similarly, given two equivariant objects (E, ϕ) and (F, ψ) in $D(X)_G$ and an open subset $U \subset T$ the group G acts on $\mathrm{Hom}_{D(X_U)}(E|_U, F|_U)$ by $f \mapsto \phi_g|_U \circ \mathsf{FM}_{\mathcal{E}_g|_U}(f) \circ \psi_g^{-1}|_U$ where we use again that Fourier–Mukai actions induce actions after base change. Since this action is compatible with restrictions to smaller open subsets we obtain a G-action on $\mathscr{H}om_\pi(E, F) \coloneqq \pi_* \mathscr{H}om(E, F)$ and thus a bifunctor

$$\mathscr{H}om_\pi \colon D(X)_G \times D(X)_G \to D(T)_G.$$

It satisfies the usual adjunctions with respect to the tensor product.

For any (closed or non-closed) point $t \in T$ let $\iota_t \colon X_t \to X$ be the inclusion of the fiber of X over t. Given $(E, \phi) \in D(X)_G$ we write $(E, \phi)_t$ for the equivariant pullback $\iota_t^*(E, \phi)$.

Lemma E.2.13. *Let* $(E,\phi),(F,\psi)$ *be objects in* $D(X)_G$*. Then*

$$t \mapsto \chi((E,\phi)_t,(F,\psi)_t) := \sum_i \dim \operatorname{Ext}^i_{D(X_t)_G}((E,\phi)_t,(F,\psi)_t)$$

is locally constant in t*.*

Proof. By a push-pull argument we have that

$$\chi((E,\phi)_t,(F,\psi)_t) = \chi(k(t), \mathscr{H}om_\pi(E,F)^G \otimes k(t)).$$

Since $\mathscr{H}om_\pi(E,F)$ is perfect, the same holds for its invariant part which implies the claim. □

E.3. Moduli spaces

E.3.1. Group actions on stacks

Following [189] an action of a finite group G on a stack $\mathcal{M}$ over $\mathbb{C}$ consists of

- for every $g \in G$ an automorphism of stacks $\rho_g\colon \mathcal{M} \to \mathcal{M}$
- for every pair $g,h \in G$ an isomorphism of functors $\theta_{g,h}\colon \rho_g\rho_h \to \rho_{gh}$

such that for all $g,h,k \in G$ the diagram (E.2.1) commutes. In other words, if we view $\mathcal{M}$ as a category fibered in groupoids, then a G-action on $\mathcal{M}$ is precisely a G-action on the category $\mathcal{M}$ in the sense of Section E.2.1 with the additional assumption that every ρ_g is a morphism of stacks. A morphism of stacks with G-actions (also called a G-equivariant morphism) is a G-functor (f,σ) such that f is a morphism of stacks. A 2-morphism of such morphisms is a 2-morphism of G-functors.

Let $\mathfrak{St}$ and G-$\mathfrak{St}$ denote the 2-categories of stacks and stacks with a G-action respectively. There is a functor $\iota\colon \mathfrak{St} \to G\text{-}\mathfrak{St}$ which equips a stack with the trivial G-action. Let $\mathfrak{Grpds}$ be the category of groupoids.

Definition E.3.1 ([189, Def. 2.3])**.** Let G be a finite group acting on a stack $\mathcal{M}$. The fixed stack is the functor $\mathcal{M}^G\colon \mathfrak{St} \to \mathfrak{Grpds}$ defined by the condition that for all stacks T we have the equivalence

$$\operatorname{Hom}_{\mathfrak{St}}(T,\mathcal{M}^G) \cong \operatorname{Hom}_{G\text{-}\mathfrak{St}}(\iota(T),\mathcal{M}).$$

Hence there is a G-equivariant morphism $\epsilon\colon \iota(\mathcal{M}^G) \to \mathcal{M}$ satisfying the following universal property: For any stack T and for any G-equivariant morphism $f\colon \iota(T) \to \mathcal{M}$ there exists a unique morphism $\tilde{f}\colon T \to \mathcal{M}^G$ such that $\epsilon \circ \tilde{f} = f$.

Remark E.3.2. As explained in [189, Proof of Prop. 2.5] the objects of $\mathcal{M}^G$ are pairs $(x,\{\alpha_g\}_{g\in G})$ of an element $x \in \mathcal{M}$ and maps $\alpha_g\colon x \to g.x$ such that $\theta^x_{g,h} \circ g\alpha_h \circ \alpha_g = \alpha_{gh}$ for all $g,h \in G$. Morphisms are the morphisms in $\mathcal{M}$ which respect the linearizations. Hence, viewed as a category, the fixed stack $\mathcal{M}^G$ *is* the equivariant category $\mathcal{M}_G$ of the action (ρ,θ) in the sense of Definition E.2.1. This can be seen also more conceptually: By the universal property of the equivariant category (Remark E.2.3) we have a functor $\mathcal{M}^G \to \mathcal{M}_G$, but by the universal property of the fixed stack we also have an inverse.

Remark E.3.3. By the universal property, if $(f, \sigma)\colon \mathcal{N} \to \mathcal{M}$ is a G-equivariant morphism which is a monomorphism (e.g. an open or closed immersion), then we have a fiber diagram

$$\begin{tikzcd}
\mathcal{N}^G \arrow[r] \arrow[d, "\epsilon"] & \mathcal{M}^G \arrow[d, "\epsilon"] \\
\mathcal{N} \arrow[r, "f"] & \mathcal{M}.
\end{tikzcd}$$

Proposition E.3.4. [189, Thm. 3.3, Prop. 3.7] *Let G be a finite group acting on an Artin stack $\mathcal{M}$ (locally) of finite type over $\mathbb{C}$. Then $\mathcal{M}^G$ is an Artin stack (locally) of finite type over $\mathbb{C}$ and the classifying morphism $\epsilon\colon \mathcal{M}^G \to \mathcal{M}$ is representable, separated and quasi-compact. If $\mathcal{M}$ has affine diagonal, then so does $\mathcal{M}^G$.*

Furthermore, consider any property of morphisms of schemes that is satisfied by closed immersions and is stable under composition. Then, if the diagonal of $\mathcal{M}$ has this property, then ϵ has this property.

Proof. We prove that $\mathcal{M}^G$ has affine diagonal if $\mathcal{M}$ has. Everything else can be found in [189]. Assume that $\mathcal{M}$ has affine diagonal and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{M}^G \arrow[r, "\Delta_{\mathcal{M}^G}"] \arrow[dr, "\Delta_{\mathcal{M}} \circ \epsilon"'] & \mathcal{M}^G \times \mathcal{M}^G \arrow[d, "\epsilon \times \epsilon"] \\
& \mathcal{M} \times \mathcal{M}.
\end{tikzcd}$$

Since $\Delta_{\mathcal{M}}$ is affine, ϵ is affine by the second part, hence so is the composition $\Delta_{\mathcal{M}} \circ \epsilon$. Since $\epsilon \times \epsilon$ is separated, its diagonal is a closed immersion and hence affine. By the cancellation lemma it follows that $\Delta_{\mathcal{M}^G}$ is affine. ☐

If G acts on a separated scheme, then the fixed stack is a closed subscheme and equal to the fixed locus defined in the usual way. However, in general the map $\epsilon\colon \mathcal{M}^G \to \mathcal{M}$ may behave quite subtle. For example, taking fixed stacks usually does not commute with passing to the good or coarse moduli space (if it exists).

E.3.2. The fixed stack of a $\mathbb{G}_m$-gerbe

Consider a G-action (ρ, θ) on the stack $B\mathbb{G}_m$ such that $\rho_g = \mathrm{id}$ for all $g \in G$ but we allow the 2-isomorphisms θ to be arbitrary. According to Lemma E.2.7 there is an associated class

$$\alpha(\theta) \in H^2(G, \mathbb{C}^*)$$

where we let the trivial action correspond to the trivial class.[4] By a direct check using the universal property and Lemma E.2.6 one has that

$$(B\mathbb{G}_m)^G = \begin{cases} \bigsqcup_{\chi \in G^\vee} B\mathbb{G}_m & \text{if } \alpha(\theta) = 0, \\ \varnothing & \text{if } \alpha(\theta) \neq 0. \end{cases}$$

---

[4] We have stated Lemma E.2.7 only for additive $\mathbb{C}$-linear category, but since $\mathrm{Aut}(\mathrm{id}_{B\mathbb{G}_m}) = \mathbb{C}^*\mathrm{id}$ on which G acts trivially by conjugation, the result applies verbatim also in this case.

In this section we consider the following generalization: Let M be a complete variety, and consider the trivial $\mathbb{G}_m$-gerbe

$$\mathcal{M} = M \times B\mathbb{G}_m.$$

The projections to the factors and the section of the gerbe are denoted by

$$p_1\colon \mathcal{M} \to M, \quad p_2\colon \mathcal{M} \to B\mathbb{G}_m, \quad s = (\mathrm{id}_M, t)\colon M \to \mathcal{M}$$

where $t\colon M \to B\mathbb{G}_m$ corresponds to the trivial line bundle. We refer to [174, Def. 12.2.2] for a definition of gerbes and morphisms of gerbes.

Lemma E.3.5. *There is a natural bijection between the set of morphisms of $\mathbb{G}_m$-gerbes $f\colon \mathcal{M} \to \mathcal{M}$ and the set of pairs $(F, \mathcal{L})$ where $F\colon M \to M$ is an automorphism and $\mathcal{L} \in$* Pic(M)*. If the morphism f corresponds to $(F, \mathcal{L})$ and g corresponds to $(G, \mathcal{L}')$, then $f \circ g$ corresponds to $(F \circ G, \mathcal{L} \otimes F^*(\mathcal{L}'))$.*

Proof. For a more general statement of the lemma as an equivalence of categories see [89].

Let $f\colon \mathcal{M} \to \mathcal{M}$ be a morphism of gerbes. Define $F = p_1 \circ f \circ s$ and let $\mathcal{L} \in \mathrm{Pic}(M)$ be the line bundle corresponding to $p_2 \circ f \circ s\colon M \to B\mathbb{G}_m$. By [174, Lem. 12.2.4] F is an automorphism.

Conversely, let L_{univ} be the universal line bundle on $B\mathbb{G}_m$, and let $\widetilde{L}_{\mathrm{univ}} = p_2^*(L_{\mathrm{univ}})$. Since f is a morphism of gerbes, the restriction of $f^*\widetilde{L}_{\mathrm{univ}}$ to $x \times B\mathbb{G}_m$ for every $\mathbb{C}$-valued point $x \in M$ is isomorphic to L_{univ}. Hence we have $f^*\widetilde{L}_{\mathrm{univ}} = p_1^*(\mathcal{L}') \otimes \widetilde{L}_{\mathrm{univ}}$ for some $\mathcal{L}' \in \mathrm{Pic}(M)$. Restricting this equality to M shows $\mathcal{L}' = \mathcal{L}$ and hence

$$f^* p_2^*(L_{\mathrm{univ}}) = p_1^*(\mathcal{L}) \otimes p_2^*(L_{\mathrm{univ}}).$$

Hence given $(F, \mathcal{L})$ we can recover f as the product of $F \circ p_1$ and the morphism associated to $p_1^*(\mathcal{L}) \otimes p_2^*(L_{\mathrm{univ}})$. This yields the 1-to-1 correspondence.

For the last claim, we have that

$$g^*\widetilde{L}_{\mathrm{univ}} = p_1^*(\mathcal{L}') \otimes \widetilde{L}_{\mathrm{univ}}$$

hence

$$f^*g^*\widetilde{L}_{\mathrm{univ}} = p_1^*F^*(\mathcal{L}') \otimes f^*\widetilde{L}_{\mathrm{univ}} = p_1^*F^*(\mathcal{L}') \otimes p_1^*(\mathcal{L}) \otimes \widetilde{L}_{\mathrm{univ}}$$

which gives the claim by restriction to M. □

Let (ρ, θ) be a G-action on $\mathcal{M}$ such that for all $g \in G$:

- the morphism ρ_g is a morphism of $\mathbb{G}_m$-gerbes, and
- if $(F_g, \mathcal{L}_g)$ is the pair associated to ρ_g, then $F_g = \mathrm{id}$.[5]

For a $\mathbb{C}$-point $p \in M$ the G-action (ρ, θ) induces an action (ρ^p, θ^p) on $p \times B\mathbb{G}_m$ such that for all $g \in G$ we have $\rho_g^p \cong \mathrm{id}_{B\mathbb{G}_m}$ (since ρ_g acts by gerbe morphisms). Hence as before we have an associated class

$$\alpha(\theta^p) \in H^2(G, \mathbb{C}^*).$$

---

[5] One can always reduce to this case by replacing $\mathcal{M}$ with $\mathcal{M} \times_M F$ for an irreducible component F of M^G.

The class $\alpha(\theta^p)$ vanishes if and only if $(p \times B\mathbb{G}_m)^G$ is non-empty. In this case we say that $p \in M$ is *G-linearizable.*

By Remark E.3.3 the fixed stack $\mathcal{M}^G$ is non-empty if and only if M contains a G-linearizable point. Hence let $p \in M$ be G-linearizable. The 2-isomorphisms $\theta_{g,h} \colon \rho_g\rho_h \to \rho_{gh}$ induce isomorphisms

$$\theta_{g,h} \colon \mathcal{L}_g \otimes \mathcal{L}_h \xrightarrow{\cong} \mathcal{L}_{gh} \tag{E.3.1}$$

which satisfy the associativity relations (E.2.1). In particular, up to isomorphism the line bundles $\mathcal{L}_g$ only depend on the conjugacy class $\bar{g}$ of g and we obtain a group homomorphism

$$G_{\mathrm{ab}} \to \mathrm{Pic}(M),\ \bar{g} \mapsto [\mathcal{L}_g]$$

where G_{ab} is the abelianization of G, and $[\mathcal{L}]$ stands for the isomorphism class of a line bundle $\mathcal{L}$.

Claim. The G-action on $\mathcal{M}$ is isomorphic to an action which factors through G_{ab} and such that the isomorphisms (E.3.1) are commutative, i.e. $\theta_{g,h} = \theta_{h,g}$ where we identify $\mathcal{L}_g \otimes \mathcal{L}_h$ with $\mathcal{L}_h \otimes \mathcal{L}_g$ by swapping the factors.

Proof of Claim. Let $H = [G, G]$ and choose representatives $\{g_1, \dots, g_r\}$ for the cosets G/H where we take the identity element for the unit coset. Given any element $g \in g_iH$ we set $\rho'_g = \rho_{g_i}$. The isomorphisms $\mathcal{L}_g \cong \mathcal{L}_{g_i}$ induced by (E.3.1) yield isomorphisms $t_g \colon \rho_g \cong \rho_{g_i} = \rho'_g$. Consider the action (ρ'_g, θ') on $\mathcal{M}$ where θ' is determined by the commutative diagram

$$\begin{tikzcd}
\rho_g\rho_h \arrow[r, "\theta_{g,h}"] \arrow[d, "t_g t_h"] & \rho_{gh} \arrow[d, "t_{gh}"] \\
\rho'_g\rho'_h \arrow[r, "\theta'_{g,h}"] & \rho'_{gh}.
\end{tikzcd}$$

By construction, ρ'_g only depends on the image of g in G/H. We need to show that we can further modify θ' such that it also only depends on the image in G/H, and is commutative. The key idea is that since M is a complete variety, $\mathrm{Hom}(\mathcal{L}_g, \mathcal{L}_g) = \mathbb{C}$, and hence we may find and check all the required relations by restricting to the point $p \in M$ where the action is trivial. Concretely, we may first choose an identification $\mathcal{L}_g|_p \cong \mathbb{C}_p$ for every g. Since $\alpha(\theta^p) = 0$ we may then modify θ' (i.e. replace $\theta'_{g,h}$ by $\lambda_{g,h}\theta'_{g,h}$ for some $\lambda_{g,h} \in \mathbb{C}^*$ which is the derivative of a 1-cycle) such that the restrictions

$$\theta'_{g,h}|_p \colon \mathcal{L}_g|_p \otimes \mathcal{L}_h|_p \to \mathcal{L}_{gh}|_p$$

are the identity maps under the given identification. Since $\mathcal{L}_g$ only depends on G/H it follows that $\theta_{g,g'}$ only depends on the image of g and g' in G/H. (To spell this out: for any $g \in g_iH$, $g' \in g_jH$ and $h, h' \in H$ we have that $\theta_{g,g'}$ and $\theta_{gh,g'h'}$ are both morphisms $\mathcal{L}_{g_i} \otimes \mathcal{L}_{g_j} \to \mathcal{L}_{g_k}$ where $g_ig_j \in g_kH$; they agree after restriction to p hence they must agree.) Similarly, the commutativity $\theta'_{g,g'} = \theta'_{g',g}$ follows by restriction. □

After replacing (ρ, θ) with an isomorphic action as in the Claim, we obtain a commutative $\mathcal{O}_M$-algebra

$$\mathcal{A} = \bigoplus_{g \in G_{\mathrm{ab}}} \mathcal{L}_g,$$

where the multiplication is induced by θ. Consider the étale cover

$$\pi\colon Y \to M, \quad Y = \operatorname{Spec}(\mathcal{A}).$$

For every $g \in G$ the natural inclusion $\mathcal{L}_g \to \mathcal{A}$ yields a natural isomorphism

$$\phi_g\colon \pi^*(\mathcal{L}_g) \xrightarrow{\cong} \mathcal{O}_Y. \tag{E.3.2}$$

The composition

$$\pi^*(\mathcal{L}_g \otimes \mathcal{L}_h) \xrightarrow{\phi_g \otimes \mathrm{id}_{\mathcal{L}_h}} \pi^*(\mathcal{L}_h) \xrightarrow{\phi_h} \mathcal{O}_Y$$

is induced by $\mathcal{L}_g \otimes \mathcal{L}_h \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and hence isomorphic to

$$\pi^*(\mathcal{L}_g \otimes \mathcal{L}_h) \xrightarrow{\pi^*\theta_{g,h}} \pi^*\mathcal{L}_{gh} \xrightarrow{\phi_{gh}} \mathcal{O}_Y.$$

We see that ϕ_g gives $s \circ \pi\colon Y \to \mathcal{M}$ the structure of a G-equivariant morphism with respect to the trivial action on Y. This yields a morphism $Y \to \mathcal{M}^G$.

Define the product

$$\mathcal{Y} = Y \times B\mathbb{G}_m$$

and consider the morphism

$$f = \pi \times \mathrm{id}_{B\mathbb{G}_m}\colon \mathcal{Y} \to \mathcal{M}.$$

As before, the tensor product of ϕ_g with the identity on the universal bundle makes f equivariant with respect to the trivial action on $\mathcal{Y}$. We obtain a morphism $\mathcal{Y} \to \mathcal{M}^G$. This yields the following description of the fixed stack.

Proposition E.3.6. *In the setting above, if M contains a G-linearizable point, then $f\colon \mathcal{Y} \to \mathcal{M}$ is the fixed stack of the G-action on $\mathcal{M}$.*

Proof. We have seen above that there is a natural morphism $\mathcal{Y} \to \mathcal{M}^G$. Conversely, giving an equivariant morphism $h\colon T \to M \times B\mathbb{G}_m$, where the scheme T carries the trivial G-action, is equivalent to a line bundle L on T, a morphism $h' = p_1 \circ h\colon T \to M$ and maps $h'^*\mathcal{L}_g \to \mathcal{O}_T$ satisfying the cocycle condition. The cocycle condition implies that the induced map

$$h'^*(\oplus_{g \in G_{\mathrm{ab}}} \mathcal{L}_g) \to \mathcal{O}_T$$

is an algebra homomorphism with respect to the algebra structure on $\oplus_g \mathcal{L}_g$ defined by θ. Hence the map $T \to M$ factors through Y and thus h factors through $Y \times B\mathbb{G}_m$. This yields the inverse $\mathcal{M}^G \to \mathcal{Y}$. ☐

Remark E.3.7. Parallel results hold for a non-trivial $\mathbb{G}_m$-gerbe $\pi\colon \mathcal{M} \to M$ with Brauer class $\alpha \in \mathrm{Br}(M)$: There exists a $\pi^*(\alpha)$-twisted line bundle L_{univ} on $\mathcal{M}$ (playing the role of $\widetilde{L}_{\mathrm{univ}}$ as before) such that for every morphism $f\colon X \to M$ and every $f^*(\alpha)$-twisted line bundle $\mathcal{L}$ on X there exists a unique map $F\colon X \to \mathcal{M}$ such that $F^*(L_{\mathrm{univ}}) = \mathcal{L}$ and $f = \pi \circ F$. A morphism $F\colon \mathcal{M} \to \mathcal{M}$ of $\mathbb{G}_m$-gerbes is then equivalent to the pair of an (untwisted) line bundle L on M and a morphism $f\colon M \to M$ such that $f^*(\alpha) = \alpha$. See also [89, Sec. 5]. The formulation of the analogue of Proposition E.3.6 is similar.

E.3.3. Moduli spaces of equivariant objects

Let X be a smooth projective variety over $\mathbb{C}$. Recall from [129] the stack

$$\mathfrak{M}\colon \mathrm{Sch}/\mathbb{C} \to \mathfrak{Grpds}$$

which associates to each scheme T the groupoid of T-perfect universally gluable objects in $D(X \times T)$. As proven in *loc. cit.* $\mathfrak{M}$ is a quasi-separated algebraic stack locally of finite type over $\mathbb{C}$ with affine diagonal, see also [200, 0DPV] and [16, Sec. 8].

Let G be a finite group which acts on $D^b(X)$. By Lemma E.2.12 the action is given by Fourier–Mukai transforms. The pullback of the Fourier–Mukai kernels define a Fourier–Mukai action $D(X \times T)$ such that the pullback morphisms are G-equivariant. This defines an action of G on $\mathfrak{M}$ in the sense of Section E.3.1,

$$(\rho, \theta)\colon G \times \mathfrak{M} \to \mathfrak{M}.$$

Remark E.3.2 yields the following description of the fixed stack:

Proposition E.3.8. *The fixed stack $\mathfrak{M}^G$ is the stack of G-equivariant universally gluable perfect complexes in $D(X)$, i.e. for every scheme T we have*

$$\mathfrak{M}^G(T) = \{(\mathcal{E}, \phi) \in D(X \times T)_{G\times 1} \mid \mathcal{E} \textit{ is universally gluable, } T\textit{-perfect}\}.$$

The isomorphisms in $\mathfrak{M}^G(T)$ are the isomorphisms of objects in $D(X \times T)_{G\times 1}$. The pullback is the equivariant pullback. The morphism $\epsilon\colon \mathfrak{M}^G \to \mathfrak{M}$ is the map that forgets the G-linearization.

From now on let σ be a stability condition on $D^b(X)$ which is preserved by the G-action. Let $\mathcal{M}_\sigma(v)$ be the moduli stack of σ-semistable objects of class $v \in K(\mathcal{A})$, i.e. for any scheme T we let

$$\mathcal{M}_\sigma(v)(T) = \{\mathcal{E} \in D(X \times T) \mid \forall t \in T\colon \mathcal{E}_t \text{ is } \sigma\text{-semistable with } [E_t] = v\}.$$

Since G preserves σ-semistability, for any G-invariant $v \in K(\mathcal{A})$ we have an action

$$G \times \mathcal{M}_\sigma(v) \to \mathcal{M}_\sigma(v).$$

The following result follows immediately from Proposition E.3.8 and Lemma E.2.8.

Proposition E.3.9. *We have*

$$\mathcal{M}_\sigma(v)^G = \bigsqcup_{\substack{v' \in K(\mathcal{A}_G) \\ p_*(v') = v}} \mathcal{M}_{\sigma_G}(v'),$$

where $\mathcal{M}_{\sigma_G}(v')$ is the substack of $\mathfrak{M}^G$ defined by

$$\mathcal{M}_{\sigma_G}(v')(T) = \{\mathcal{E} \in D(X \times T)_{G\times 1} \mid \forall t \in T\colon \mathcal{E}_t \textit{ is } \sigma_G\textit{-semistable, } [\mathcal{E}_t] = v'\}.$$

E.3.4. The fixed stack of a fine moduli space

In the setting of Section E.3.3, let $v \in K(D^b(X))$ be a G-invariant class such that $\mathcal{M}_\sigma(v)$ has a fine moduli space $M_\sigma(v)$ which is smooth. The goal of this section is to determine the fixed stack $\mathcal{M}_\sigma(v)^G$.

Write $\mathcal{M} = \mathcal{M}_\sigma(v)$ and $M = M_\sigma(v)$. By assumption there is a universal family

$$\mathcal{E} \in D(M \times X),$$

unique up to tensoring with a line bundle pulled back from the first factor. By the universal property of $\mathcal{M}$ this yields a section $s_\mathcal{E} \colon M \to \mathcal{M}$ of the $\mathbb{G}_m$-gerbe $\mathcal{M} \to M$. Hence $s_\mathcal{E}$ defines a trivialization

$$\mathcal{M} \cong M \times B\mathbb{G}_m. \tag{E.3.3}$$

The universal family $\mathcal{E}_\mathcal{M} \in D(\mathcal{M} \times X)$ is identified under (E.3.3) with

$$(p_1 \times \mathrm{id}_X)^*(\mathcal{E}) \otimes p_2^*(L_{\mathrm{univ}})$$

where p_1, p_2 are the projections to the factors.

Let $f \colon \mathcal{M} \to \mathcal{M}$ be a morphism of $\mathbb{G}_m$-gerbes and let

$$F = p_1 \circ f \circ s_\mathcal{E}, \quad \mathcal{L} = (p_2 \circ f \circ s_\mathcal{E})^* L_{\mathrm{univ}}$$

be the associated automorphism and line bundle as in Lemma E.3.5. We consider the difference of the pullbacks of the universal families under F and f.

Lemma E.3.10. *In the situation above, we have*

$$((f \times \mathrm{id}_X)^* \mathcal{E}_\mathcal{M})|_M = (F \times \mathrm{id}_X)^*(\mathcal{E}) \otimes \mathcal{L}.$$

Proof. Under the identification (E.3.3) we have $\mathcal{E}_\mathcal{M} = (p_1 \times \mathrm{id}_X)^*(\mathcal{E}) \otimes p_2^*(L_{\mathrm{univ}})$. Hence

$$\begin{aligned}(f \times \mathrm{id}_X)^*(\mathcal{E}_\mathcal{M}) &= (f \times \mathrm{id}_X)^*((p_1 \times \mathrm{id}_X)^*(\mathcal{E})) \otimes (f \times \mathrm{id}_X)^* p_2^*(L_{\mathrm{univ}}) \\ &= (p_1 \times \mathrm{id}_X)^*((F \times \mathrm{id}_X)^*(\mathcal{E})) \otimes ((p_1 \times \mathrm{id}_X)^*(\mathcal{L}) \otimes p_2^*(L_{\mathrm{univ}})) \\ &= (p_1 \times \mathrm{id}_X)^*((F \times \mathrm{id}_X)^*(\mathcal{E}) \otimes \mathcal{L}) \otimes p_2^*(L_{\mathrm{univ}}).\end{aligned}$$

Restricting to M completes the claim. □

Consider the action of G on $\mathcal{M}$. For every $g \in G$ the morphism $\rho_g \colon \mathcal{M} \to \mathcal{M}$ commutes with the inclusion of the automorphism groups (in the derived category, we have $g(\lambda \mathrm{id}) = \lambda g(\mathrm{id}) = \lambda \mathrm{id}$) and hence is a morphism of $\mathbb{G}_m$-gerbes. Let

$$F_g \colon M \to M, \quad \mathcal{L}_g \in \mathrm{Pic}(M)$$

be the associated pair constructed in Lemma E.3.5. By Lemma E.3.10 the line bundle $\mathcal{L}_g$ can also be described by

$$(1 \times g)(\mathcal{E}) = ((1 \times g)\mathcal{E}_\mathcal{M})|_M = ((\rho_g \times \mathrm{id}_X)^* \mathcal{E}_\mathcal{M})|_M = (F_g \times \mathrm{id}_X)^*(\mathcal{E}) \otimes \mathcal{L}_g. \tag{E.3.4}$$

Let F be a connected component of the fixed locus $M^G \subset M$ and let $L_g = \mathcal{L}_g|_F$ which only depends on the conjugacy class of g, see the discussion in Section E.3.1. Consider further the associated étale cover

$$Y = \operatorname{Spec}\left(\bigoplus_{g \in G_{ab}} L_g\right), \quad \pi\colon Y \to F \tag{E.3.5}$$

and define

$$\mathcal{Y} = Y \times B\mathbb{G}_m, \quad \epsilon\colon \mathcal{Y} \xrightarrow{\pi \times \mathrm{id}_{B\mathbb{G}_m}} F \times B\mathbb{G}_m \to \mathcal{M}.$$

Proposition E.3.11. *In the setting above, if F contains a G-linearizable point, then $\mathcal{Y}$ is the union of the connected components of $\mathcal{M}^G$ which map to F and $\epsilon\colon \mathcal{Y} \to \mathcal{M}$ is the restriction of the classifying map $\mathcal{M}^G \to \mathcal{M}$ to $\mathcal{Y}$. The universal linearization of $\epsilon^*(\mathcal{E}_{\mathcal{M}})$ is pulled back from the canonical linearization of $(\pi \times \mathrm{id}_X)^*(\mathcal{E}|_{F \times X})$.*

By Proposition E.3.8, a point $p \in F$ is G-linearizable if and only if the corresponding G-invariant object $\mathcal{E}_p$ is G-linearizable. Using Proposition E.3.11 we see that there exists a G-linearizable point $p \in F$ if and only if every point on F is G-linearizable. In this case we say that the connected component F of M^G is G-linearizable.

Proof. The first statement is Proposition E.3.6. The second part follows since the linearization on $\mathcal{Y}$ is the pullback of the linearization on Y given by (E.3.2). ☐

Remark E.3.12. The action of $G^\vee$ on $D^b(X)_G$ by twisting the linearization preserves the stability condition σ_G. Moreover, for every $\chi \in G^\vee$ we have $p_*\chi v' = p_*v'$. Hence we have an induced action of $G^\vee$ on

$$\mathcal{M}_\sigma(v)^G = \bigsqcup_{p_*(v')=v} \mathcal{M}_{\sigma_G}(v').$$

By Lemma E.2.6 the action is free if $\mathcal{M}_\sigma(v)$ is a moduli space of stable objects.

In the setting of Proposition E.3.11, the $G^\vee$-action can be described by letting a character $\chi \in G^\vee$ act on the line bundle $\mathcal{L}_g$ by multiplication by $\chi(g)$. In particular, $Y/G^\vee = F$ and the projection $\pi\colon Y \to F$ is a $G^\vee$-torsor (in the étale topology).

Remark E.3.13. The results of this section can be generalized to the case when $\pi\colon \mathcal{M}_\sigma(v) \to M_\sigma(v)$ is a non-trivial $\mathbb{G}_m$-gerbe (if $\mathcal{E} \in D(M \times X, -\alpha)$ is the twisted universal object, then the universal family $\mathcal{E}_{\mathcal{M}}$ on the stack $\mathcal{M} \times X$ is given by $\pi^*(\mathcal{E}) \otimes L_{\mathrm{univ}}$, see also Remark E.3.7).

Example E.3.14. Let E be an elliptic curve and let $t_a\colon E \to E$ be the translation by a 2-torsion point $a \in E$. The group $G = \mathbb{Z}_2$ acts on $\mathrm{Coh}(E)$ by t_a^*. Let $E' = E/t_a$. The equivariant category is $\mathrm{Coh}(E)_G = \mathrm{Coh}(E')$. Consider the moduli stack $\mathcal{M} = \mathcal{M}(1,0)$ of Gieseker stable sheaves with Chern characters $v = (1,0) \in H^{2*}(E)$ or equivalently the moduli stack of degree 0 line bundles. It admits the fine moduli space $M \cong E$ with universal family the Poincaré bundle $\mathcal{P}$ on $E \times E$. Hence $\mathcal{M} \cong E \times B\mathbb{G}_m$. Since every degree 0 line bundle is translation invariant, the group G induces the trivial action on M. However, because of

$$(1 \times t_a^*)(\mathcal{P}) = (\mathrm{id} \times t_a)^*\mathcal{P} = \mathcal{P} \otimes p_1^*\mathcal{P}_a,$$

the bundle $\mathcal{P}$ can not be linearized over M. Indeed by Proposition E.3.11 (with $L_g = \mathcal{P}_a$) one has $\mathcal{M}^G = \tilde{E} \times B\mathbb{G}_m$ where $\tilde{E}$ is the cover of E defined by $\mathcal{P}_a$.

An alternative description of the fixed stack is also given by Proposition E.3.9 as follows:

$$\mathcal{M}^G = \mathcal{M}_{E'}(1,0) \cong E' \times B\mathbb{G}_m.$$

Since $E' \cong \tilde{E}$ these two presentations agree with each other.

Example E.3.15. We give an example of a component which is not G-linearizable.

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the subgroup of 2-torsion points of E acting by translation. Let $\mathcal{M} = \mathcal{M}(1,2)$ be the moduli stack of degree 2 line bundles and let $M \cong E$ be its fine moduli space. Then $M^G = M$ but $\mathcal{M}^G = \varnothing$, so M is not G-linearizable. Indeed, any G-linearization of a degree 2 line bundle L is a descent datum for the quotient map $\pi\colon E \to E/G$. Hence there would exists a line bundle L' on E/G with $\pi^* L' = L$ which would imply that the degree of L is divisible by 4.

E.3.5. The Artin–Zhang functor

As before we consider an action of a finite group G on $D^b(X)$ which preserves a stability condition $\sigma = (\mathcal{A}, Z)$. In this section we further assume the following properties:

- $\mathcal{A}$ is Noetherian,
- $\mathcal{A}$ satisfies the generic flatness property of [1, Prop. 3.5.1].

The second condition implies that the subfunctor $\mathfrak{M}_{\mathcal{A}} \subset \mathfrak{M}$ of objects, such that every geometric fiber lies in $\mathcal{A}$, is open. By Remark E.3.3 the open immersion $\mathfrak{M}_{\mathcal{A}} \subset \mathfrak{M}$ yields the fiber diagram

$$\begin{tikzcd}
(\mathfrak{M}_{\mathcal{A}})^G \arrow[r, hook] \arrow[d, "\epsilon"] & \mathfrak{M}^G \arrow[d] \\
\mathfrak{M}_{\mathcal{A}} \arrow[r, hook] & \mathfrak{M}.
\end{tikzcd} \tag{E.3.6}$$

By base change this shows that also $(\mathfrak{M}_{\mathcal{A}})^G \subset \mathfrak{M}^G$ is an open immersion.

Given a cocomplete[6], locally noetherian, k-linear abelian category $\mathcal{C}$, let $\mathcal{N}_{\mathcal{C}}$ be the stack of finitely presented objects in $\mathcal{C}$ as introduced by Artin and Zhang [11], see also [6, Def. 7.8]. Concretely, for a commutative ring R let $\mathcal{C}_R$ be the category of pairs (E, ϕ) with E an object in $\mathcal{C}$ and $\phi\colon R \to \mathrm{End}_{\mathcal{C}}(E)$ a morphism of k-algebras. Then $\mathcal{N}_{\mathcal{C}}(\operatorname{Spec} R)$ is the groupoid of flat and finitely presented objects in $\mathcal{C}_R$,

As discussed in [6, Ex. 7.20] our assumptions on $\mathcal{A}$ imply that the stacks $\mathfrak{M}_{\mathcal{A}}$ and $\mathcal{N}_{\mathrm{Ind}(\mathcal{A})}$ are equivalent, where $\mathrm{Ind}(\mathcal{A})$ is the Ind-completion of $\mathcal{A}$. Our first goal is to prove the parallel result for the equivariant abelian category $\mathcal{A}_G$:

Proposition E.3.16. $(\mathfrak{M}_{\mathcal{A}})^G \cong \mathcal{N}_{\mathrm{Ind}(\mathcal{A}_G)}$.

We begin with two technical lemmata.

Lemma E.3.17. *If $\mathcal{A}$ is a Noetherian abelian $\mathbb{C}$-linear category, then every object in* $\mathrm{Ind}(\mathcal{A})$ *can be written as a union of objects in $\mathcal{A}$.*

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[6] i.e. $\mathcal{C}$ has all small filtered colimits

Proof. [7] Given objects $E \in \mathcal{A}$ and $F \in \operatorname{Ind}(\mathcal{A})$ and an inclusion $F \subset E$ in $\operatorname{Ind}(\mathcal{A})$ we first claim that $F \in \mathcal{A}$. Indeed, write $F = \lim_i F_i$ where the F_i lie in $\mathcal{A}$. Then since $F \to E$ is a monomorphism we have $F_i' := \operatorname{Im}(F_i \to F) = \operatorname{Im}(F_i \to E)$ and thus this image lies in $\mathcal{A}$. Therefore, F is a union of objects in $\mathcal{A}$ (namely the F_i') which are subjects of E. Since E is Noetherian, this union has to stabilize and since abelian categories contain finite colomits, $F \in \mathcal{A}$ as desired. Now, if $E \to F$ is a quotient in $\operatorname{Ind}(\mathcal{A})$ with $E \in \mathcal{A}$ and $F \in \operatorname{Ind}(\mathcal{A})$ then by the above the kernel lies in $\mathcal{A}$ and hence so does F. Therefore $\mathcal{A}$ is closed under quotients in $\operatorname{Ind}(\mathcal{A})$. We conclude, that if $E = \lim_i E_i$ with $E_i \in \mathcal{A}$, then E is the union of the $F_i = \operatorname{Im}(E_i \to E)$. □

Lemma E.3.18. *Let $\mathcal{A}$ be a Noetherian abelian $\mathbb{C}$-linear category and G a finite group. Then there exists a canonical isomorphism $\operatorname{Ind}(\mathcal{A}_G) \cong \operatorname{Ind}(\mathcal{A})_G$.*

We refer to [179, Lem. 3.6] for a parallel result for ∞-categories.

Proof. If $\mathcal{A}$ is cocomplete and (E_i, ϕ_i) is a direct system in $\mathcal{A}_G$, then the ϕ_i define a canonical G-linearization on $E = \lim E_i$. Hence $\mathcal{A}_G$ is also cocomplete.

Let $\mathcal{A}$ now be Noetherian. Applying the above argument to $\operatorname{Ind}(\mathcal{A})$ we see that $\operatorname{Ind}(\mathcal{A})_G$ is cocomplete. Hence by the universal property of Ind-completion, the inclusion $\mathcal{A}_G \to \operatorname{Ind}(\mathcal{A})_G$ lifts to a functor $\operatorname{Ind}(\mathcal{A}_G) \to \operatorname{Ind}(\mathcal{A})_G$. By composing with the forgetful functor $\operatorname{Ind}(\mathcal{A})_G \to \operatorname{Ind}(\mathcal{A})$ one sees the functor is faithful. We check that the functor is essentially surjective and full: Let $(E, \phi) \in \operatorname{Ind}(\mathcal{A})_G$ where $E = \bigcup_i E_i$ is a union of objects E_i in $\mathcal{A}$. By replacing E_i by $\bigcup_{g \in G} \phi_g^{-1}(gE_i)$ if necessary we get that the restrictions $\phi_g|_{E_i} \colon E_i \to gE_i$ define G-linearizations on E_i. Moreover, after replacing the E_i and F_i suitably, any morphism $(E, \phi) \to (F, \psi)$ is the limit of a morphism $(E_i, \phi_i) \to (F_i, \psi_i)$. □

Proof of Proposition E.3.16. Since $\mathfrak{M}_{\mathcal{A}} = \mathcal{N}_{\operatorname{Ind}(\mathcal{A})}$ we have that $\mathfrak{M}_{\mathcal{A}}^G(\operatorname{Spec} R)$ is the groupoid of pairs of $x \in \mathcal{N}_{\mathcal{A}}(R)$ together with linearizations $\phi_g \colon x \to gx$ satisfying the cocycle condition. Spelling this out this is the groupoid of triples of objects $E \in \operatorname{Ind}(\mathcal{A})$, homomorphisms $\sigma \colon R \to \operatorname{End}(E)$ and linearizations $\phi_g \colon E \to gE$ satisfying

$$\phi_g \circ \sigma_r = g\sigma_r \circ \phi_g,$$

or equivalently, the groupoid of pairs $(E, \phi) \in \operatorname{Ind}(\mathcal{A})_G$ and $\sigma \colon R \to \operatorname{End}_{\operatorname{Ind}(\mathcal{A})_G}(E, \phi)$. However, G finite implies that $\operatorname{Ind}(\mathcal{A})_G = \operatorname{Ind}(\mathcal{A}_G)$ (see Lemma E.3.18) and hence this is precisely the groupoid $\mathcal{N}_{\operatorname{Ind}(\mathcal{A}_G)}(\operatorname{Spec} R)$. □

A stability condition $\sigma = (\mathcal{A}, Z)$ is called *algebraic* if $Z(K(\mathcal{A})) \subset \mathbb{Q} + i\mathbb{Q}$.

Theorem E.3.19. *In the above situation assume moreover that σ is algebraic and that $\mathcal{M}_\sigma(v)$ is bounded for every $v \in K(D(X))$. Then for every $v' \in K(D^b(X)_G)$ the moduli stack $\mathcal{M}_{\sigma_G}(v')$ is an universally closed Artin stack of finite type over $\mathbb{C}$ which has a proper good moduli space. The inclusion $\mathcal{M}_{\sigma_G}(v') \to \mathfrak{M}^G$ is an open embedding.*

Proof. Let $v = p_* v'$ and let $\mathfrak{M}_{\mathcal{A},v} \subset \mathfrak{M}_{\mathcal{A}}$ be the open and closed substack parametrizing objects of class v. Invoking [6, Ex. 7.27], the stack $\mathfrak{M}_{\mathcal{A},v}$ has a Θ-stratification whose open

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[7] We thank Eugen Hellman for providing this argument.

piece is $\mathcal{M}_\sigma(v)$. This yields the fiber diagram

$$\begin{tikzcd}
\mathcal{M}_\sigma(v)^G \arrow[r, hook] \arrow[d, "\epsilon"] & (\mathfrak{M}_{\mathcal{A},v})^G \arrow[d, "\epsilon"] \\
\mathcal{M}_\sigma(v) \arrow[r, hook] & \mathfrak{M}_{\mathcal{A},v},
\end{tikzcd}$$

where the horizontal maps are open immersions. Since $\mathfrak{M}_{\mathcal{A},v} \subset \mathfrak{M}$ is open and $\mathfrak{M}$ is an Artin stack locally of finite type with affine diagonal over $\mathbb{C}$, applying Proposition E.3.4 the same holds for $(\mathfrak{M}_{\mathcal{A},v})^G$. Moreover, by Proposition E.3.4 again both vertical morphisms ϵ are affine. Since $\mathcal{M}_\sigma(v)$ is of finite type, so is $\mathcal{M}_\sigma(v)^G$.

By [6, Sec. 7] the stack $\mathcal{M}_\sigma(v)$ is Θ-reductive and S-complete. By [6, Prop. 3.20(1)] affine morphisms are Θ-reductive and by [6, Prop. 3.42(1)] they are S-complete. Since both these properties are stable under composition, $\mathcal{M}_\sigma(v)^G$ is Θ-reductive and S-complete and hence by [6, Thm. A] admits a separated good moduli space.

It remains to show that $\mathcal{M}_\sigma(v)^G$ is universally closed.[8] For this recall from Proposition E.3.16 the isomorphism $(\mathfrak{M}_{\mathcal{A}})^G \cong \mathcal{N}_{\mathrm{Ind}(\mathcal{A}_G)}$. It follows from [6, Lem. 7.17] that $\mathfrak{M}_{\mathcal{A}}^G$ satisfies the existence part of the valuative criterion of properness. Since $\epsilon\colon (\mathfrak{M}_{\mathcal{A},v})^G \to \mathfrak{M}_{\mathcal{A},v}$ is affine, by [85, Prop. 1.19] the preimage of the Θ-stratification of $\mathfrak{M}_{\mathcal{A},v}$ defines a Θ-stratification of $(\mathfrak{M}_{\mathcal{A},v})^G$. By definition its open piece is the preimage of the stack of σ-semistable objets, which, is precisely the stack of σ_G-semistable objects.[9] By semistable reduction [6, Thm. B/C] we conclude that $\mathcal{M}_\sigma(v)^G$ is universally closed and therefore that its good moduli space is proper. By Proposition E.3.9 the stack $\mathcal{M}_{\sigma_G}(v')$ is a closed and open substack of $\mathcal{M}_\sigma(v)^G$, hence it satisfies the same conclusion. □

We consider the deformation-obstruction theory of the functor $\mathfrak{M}_{\mathcal{A}}^G$.

Proposition E.3.20. *Suppose that $\mathcal{A}$ is Noetherian, satisfies the generic flatness property and we have $D^b(\mathcal{A}) \cong D^b(X)$.*

Let $0 \to I \to A' \to A \to 0$ be a square zero extension of rings and let $\iota\colon X \times \operatorname{Spec} A \to X \times \operatorname{Spec} A'$ be the natural inclusion. Let $(E, \phi) \in \mathfrak{M}_{\mathcal{A}}^G(\operatorname{Spec} A)$. Then there exists an obstruction class

$$\omega(E, \phi) \in \operatorname{Ext}^2(E, E \otimes I)_0^G$$

which vanishes if and only if there exists a complex $(E', \phi') \in \mathfrak{M}_{\mathcal{A}}^G(A')$ such that $\iota^*(E', \phi') \cong (E, \phi)$. Moreover, in this case the set of extensions is a torsor over $\operatorname{Ext}^1(E, E \otimes I)^G$.

Here the subscript 0 stands for the traceless part defined by

$$\operatorname{Ext}^2(E, E)_0 = \operatorname{Ker}\left(\operatorname{Tr}\colon \operatorname{Ext}^2(E, E) \to H^2(X, \mathcal{O}_X)\right).$$

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[8] Since ϵ is not proper in general (see Section E.7.2 for an example where this fails) this does not follow directly from the fact that $\mathcal{M}_\sigma(v)$ is universally closed. Instead we have to use the alternative description of the bigger stack $(\mathfrak{M}_{\mathcal{A}})^G$.

[9] The Θ-stratification of $\mathfrak{M}_{\mathcal{A},v}$ corresponds to the Harder–Narasimhan filtration in $\mathcal{A}$. Given an equivariant object (E, ϕ) and a Harder–Narasimhan filtration E_i of E with respect to σ the restrictions $(E_i, \phi|_{E_i})$ define a Harder–Narasimhan filtration of (E, ϕ) which corresponds to the 'preimage' Θ-stratification of $(\mathfrak{M}_{\mathcal{A}})^G$.

Proof. By Proposition E.3.16 we can use the deformation theory of the Artin–Zhang functor $\mathcal{N}_{\mathrm{Ind}(\mathcal{A})}$. Since $D^b(\mathcal{A}) = D^b(X)$ for any $(E, \phi) \in \mathcal{A}_G$ we have

$$\mathrm{Ext}^i_{D^b(\mathcal{A}_G)}((E,\phi),(E,\phi)) = \mathrm{Ext}^i_{D^b(X)_G}((E,\phi),(E,\phi)) = \mathrm{Ext}^i_{D^b(X)}(E,E)^G.$$

Hence the existence of the obstruction class $\omega(E,\phi) \in \mathrm{Ext}^2(E, E \otimes I)^G$ follows from [131]. The (G-invariant) trace map is the derivative to the determinant map on S. Since the Picard stack is smooth, all obstructions to deforming $\det(E)$ vanishes. This shows that the obstruction class lies in the kernel of

$$\mathrm{Ext}^2(E,E)^G \xrightarrow{p_*} \mathrm{Ext}^2(E,E) \xrightarrow{\mathrm{Tr}} \mathbb{C}. \qquad \square$$

E.3.6. Conclusion

Let X be a smooth projective variety and let $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ be a connected component of the stability manifold satisfying the following condition:

(†) There exists an algebraic stability conditions $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}^*(X)$ such that
  - $\mathcal{A}$ satisfies the generic flatness property and
  - for all $v \in K(\mathcal{A})$ the stack $\mathcal{M}_\sigma(v)$ is bounded.

Then by [182, Prop. 4.12] the same holds for all algebraic stability conditions in $\mathrm{Stab}^*(X)$. Moreover, as explained in [6, Ex. 7.27], for any $v \in K(D^b(X))$ and stability condition $\sigma \in \mathrm{Stab}^*(X)$ one can find an algebraic stability condition σ' such that $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_{\sigma'}(v)$ define the same moduli functor.

Assume as before that we have a G-action on $D^b(X)$. We will need the following G-invariant version of the argument in [6, Ex. 7.27].

Lemma E.3.21. *Let $v \in K(D^b(X))^G$ and $\sigma \in \mathrm{Stab}^*(X)^G$. Then there exists an algebraic stability condition $\sigma' \in \mathrm{Stab}^*(X)^G$, such that $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_{\sigma'}(v)$ define the same moduli functor.*

Proof. We follow the arguments and notations from [6, Ex. 7.27]. Note also that the arguments from [133, Lem. 2.15] apply in our setting. We restrict the decomposition of [6]

$$\mathcal{C}_{S'} = \left(\bigcup_{\gamma' \in S'} \mathcal{W}_{\gamma'} \right) \setminus \bigcup_{\gamma' \notin S'} \mathcal{W}_{\gamma'}$$

associated to v and σ to the set of invariant stability conditions $\mathrm{Stab}^*(X)^G$. Since we have $\sigma \in \mathcal{C}_{S'}$, we conclude for all $\gamma' \notin S'$ that the connected component of the submanifold $\mathrm{Stab}^*(X)^G$ containing σ is not entirely contained in $\mathcal{W}_{\gamma'}$. Then arguing as in [6, Ex. 7.27] for $\mathcal{C}_{S'} \cap \mathrm{Stab}^*(X)^G$ completes the proof. $\square$

This yields the following existence result.

Theorem E.3.22. *Let $\sigma \in \mathrm{Stab}^*(X)$ be a G-fixed stability condition. Then for every $v' \in K(D^b(X)_G)$ the stack $\mathcal{M}_{\sigma_G}(v')$ is a universally closed Artin stack of finite type over $\mathbb{C}$ which has a proper good moduli space.*

Proof. By Lemma E.3.21 we may assume that σ is algebraic and apply Theorem E.3.19. $\square$

We are ready to give a proof of Theorem E.1.2.

Proof of Theorem E.1.2. We will assume for simplicity that M is a fine moduli space. The case of a coarse moduli space of stable objects works parallel by using a twisted universal object instead, see Remark E.3.13. By Proposition E.3.9 we have the decomposition

$$\mathcal{M}_{\sigma}(v)^{G} = \bigsqcup_{p_* v' = v} \mathcal{M}_{\sigma_G}(v'). \tag{E.3.7}$$

The map (E.1.1) is induced from $\epsilon\colon \mathcal{M}_{\sigma}(v)^G \to \mathcal{M}_{\sigma}(v)$ by passing to good moduli spaces. For every G-linearizable connected component $F \subset M^G$, the scheme $Y = \mathrm{Spec}\,(\oplus_{g \in G_{\mathrm{ab}}} \mathcal{L}_g)$ as defined in (E.3.5) is a $G^{\vee}$-torsor over F, see Remark E.3.12. By Proposition E.3.11 the gerbe $Y \times B\mathbb{G}_m$ is the union of all connected components of (E.3.7) mapping to F. Since every connected component maps to some F this shows the first claim.

If G factors through a Schur cover $G \to Q$, then we have $M^G = M^Q$. Moreover for every connected component F and point $p \in F$ the obstruction of being G-linearizable (as given by Lemma E.2.6) is the pullback of a class in $H^2(Q, \mathbb{C}^*)$ and hence vanishes. This shows that every connected component of M^G is G-linearizable and so (E.1.1) is surjective. $\square$

Part 2. Equivariant categories of symplectic surfaces

E.4. More on equivariant categories

E.4.1. Calabi–Yau categories

The main reference for this section is [29].

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category with finite-dimensional Hom spaces. A *Serre functor* for $\mathcal{D}$ is an equivalence $S\colon \mathcal{D} \to \mathcal{D}$ together with bifunctorial isomorphisms

$$\eta_{A,B}\colon\ \mathrm{Hom}(A,B) \xrightarrow{\cong} \mathrm{Hom}(B, SA)^{\vee}$$

for all objects $A, B \in \mathcal{D}$. By [29, Sec. 5] if we are given an action by a finite group G on $\mathcal{D}$ the Serre functor S lifts to a Serre functor

$$\tilde{S}\colon \mathcal{D}_G \to \mathcal{D}_G$$

which is of the form $\tilde{S}(A, \phi) = (SA, \phi')$ for a certain linearization ϕ'. Moreover, for any objects (A, ϕ) and (B, ψ) in $\mathcal{D}_G$ the restriction of $\eta_{A,B}$ to the G-invariant part defines bifunctorial isomorphisms

$$\eta_{A,B}\colon\ \mathrm{Hom}(A,B)^G \xrightarrow{\cong} (\mathrm{Hom}(B, SA)^G)^{\vee}$$

where the G-action on the left is defined by the linearizations ϕ, ψ and the G-action on the right is defined by the linearizations ψ and ϕ'

We say that the category $\mathcal{D}$ is Calabi–Yau if there exists a 2-isomorphism

$$\mathrm{id}_{\mathcal{D}} \xrightarrow{\cong} S[-n]$$

for some integer n, called the dimension of $\mathcal{D}$.

Remark E.4.1. The derived category $D^b(X)$ of a smooth projective n-dimensional variety X has the Serre functor $S = (-) \otimes \omega_X[n]$. In this case we will usually denote the lifted functor $\tilde{S}$ also by $(-) \otimes \omega_X[n]$ where the action on the linearization is implicitly understood. So

$$(A, \phi) \otimes \omega_X[n]$$

will stand for $\tilde{S}(A, \phi) = (A \otimes \omega_X[n], \phi')$.

Remark E.4.2. The results discussed above work also in the relative case of a smooth projective morphism $\pi\colon X \to T$ with geometrically connected fibers as in Section E.2.3. Given a Fourier–Mukai G-action on $D(X)$, the π-relative Serre functor lifts to a π-relative Serre functor of the equivariant category $D(X)_G$.

We have the following criterion for the equivariant category of a Calabi–Yau variety to be Calabi–Yau.

Proposition E.4.3. ([29, Sec. 6.3, 6.4]) *Let X be a smooth projective variety which is Calabi–Yau, i.e. $\omega_X \cong \mathcal{O}_X$. Consider the action of a finite group G on $D^b(X)$ which lifts to an action on the dg-enhancement $D_{\mathrm{dg}}(X)$.*

- *(i) If the induced action of G on singular cohomology preserves the class of the Calabi–Yau form $[\omega_X] \in H^0(X, \Omega_X^n)$, then $D^b(X)_G$ is Calabi–Yau of dimension n.*
- *(ii) Suppose that, moreover, we have an equivalence $D^b(X)_G \cong D^b(X')$ for a variety X'. The induced action of $G^\vee$ on $H^*(X', \mathbb{C})$ preserves the class of $\omega_{X'}$.*

E.4.2. Equivariant Fourier–Mukai transforms

Let X and Y be smooth projective varieties and let G be a finite group which acts on $D^b(X)$. By Lemma E.2.12 this action is given by Fourier–Mukai transforms and hence defines an action by Fourier–Mukai transforms on $D^b(X \times Y)$, see Section E.2.3.1.[10] Since this action is pulled back from X, we often write $G \times 1$ for the group which acts on $D^b(X \times Y)$.

Consider the projections $X \xleftarrow{\rho} X \times Y \xrightarrow{\pi} Y$. The (equivariant) Fourier–Mukai transform $\mathsf{F}_{\mathcal{E}}\colon D^b(Y) \to D^b(X)_G$ with kernel $\mathcal{E} \in D^b(X \times Y)_{G\times 1}$ is defined by

$$\mathsf{F}_{\mathcal{E}} A = \rho_*(\pi^*(A) \otimes \mathcal{E})$$

where the tensor product takes values in $D^b(X \times Y)_{G\times 1}$ and ρ_* is the equivariant pushforward. Similarly, the (reverse) equivariant Fourier–Mukai transform $\mathsf{G}_{\mathcal{E}}\colon D^b(X)_G \to D^b(Y)$ is defined by

$$\mathsf{G}_{\mathcal{E}}(E, \phi) = \mathscr{H}om_\pi\left(\mathcal{E}, \rho^*(E, \phi)\right)^G$$

where we used equivariant pullback and the π-relative Hom of Section E.2.3.2.

---

[10] Take β to be $Y \to \mathrm{Spec}(\mathbb{C})$.

Lemma E.4.4. *For any $\mathcal{E} \in D^b(X \times Y)_{G\times 1}$ let*

$$\mathcal{E}_L = \mathcal{E} \otimes \rho^* \omega_X^\vee[-\dim X], \qquad \mathcal{E}_R = \mathcal{E} \otimes \pi^* \omega_Y^\vee[-\dim Y].$$

Then $\mathsf{G}_{\mathcal{E}_L}$ and $\mathsf{G}_{\mathcal{E}_R}$ is the left and right adjoint of $\mathsf{F}_{\mathcal{E}}$ respectively.

Here we followed Remark E.4.1 and have written $\mathcal{E} \otimes \rho^* \omega_X^\vee[-\dim X]$ for the application of the inverse of the π-relative Serre functor of $D^b(X \times Y)_{G\times 1}$.

Proof of Lemma E.4.4. For any $(A, \phi) \in D^b(X)$ and $B \in D^b(Y)$ we have

$$\begin{aligned}
&\mathrm{Hom}_{D^b(X)_G}((A,\phi), \mathsf{F}_{\mathcal{E}} B) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)_{G\times 1}}(\rho^*(A,\phi), \pi^*(B) \otimes \mathcal{E}) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}(\rho^* A, \pi^*(B) \otimes \mathcal{E})^G \\
&\cong \Big(\mathrm{Hom}_{D^b(X\times Y)}(\pi^*(B) \otimes \mathcal{E}, \rho^*(A) \otimes \omega_{X\times Y}[\dim X + \dim Y])^\vee\Big)^G \\
&\cong \Big(\mathrm{Hom}_{D^b(Y)}(B, \mathscr{H}om_\pi(\mathcal{E}, \rho^*(A) \otimes \omega_{X\times Y}[\dim X + \dim Y]))^\vee\Big)^G \\
&\cong \mathrm{Hom}_{D^b(Y)}(\mathscr{H}om_\pi(\mathcal{E}, \rho^*(A) \otimes \rho^*\omega_X[\dim X]), B)^G \\
&\cong \mathrm{Hom}_{D^b(Y)}(\mathsf{G}_{\mathcal{E}\otimes\rho^*\omega_X^\vee[-\dim X]}(A), B).
\end{aligned}$$

The other case is similar. ☐

We have the following criterion when a Fourier–Mukai transform $\mathsf{F}_{\mathcal{E}} \colon D^b(Y) \to D^b(X)_G$ is an equivalence.

Proposition E.4.5. *Let $\mathcal{E} \in D^b(X \times Y)_{G\times 1}$. Assume that*

(i) $\mathrm{Hom}_{D^b(X)_G}(\mathcal{E}_x, \mathcal{E}_y[i]) = \mathrm{Hom}_{D^b(Y)}(\mathbb{C}_x, \mathbb{C}_y[i])$ *for all $x, y \in Y$.*

(ii) $D^b(X)_G$ is indecomposable.

(iii) The functor $\mathsf{F}_{\mathcal{E}}$ commutes on objects with Serre functors, i.e. $\tilde{S}\mathsf{F}_{\mathcal{E}}(A) \cong F_{\mathcal{E}} S(A)$ for all $A \in D^b(Y)$.

Then $\mathsf{F}_{\mathcal{E}}$ is an equivalence.

Proof. By Lemma E.4.4 the functor $\mathsf{F}_{\mathcal{E}} \colon D^b(Y) \to D^b(X)_G$ has both right and left adjoints. The assertion then follows from [42, Thm. 2.3]. ☐

E.5. Proof of results

Let S be a symplectic surface with a G-action on $D^b(S)$ satisfying conditions (i)-(iii) of Section E.1.1 and let $\sigma \in \mathrm{Stab}^\dagger(S)$ be a G-fixed stability condition.

E.5.1. Preliminaries

We have the following structure result.

Proposition E.5.1. *The equivariant category $D^b(S)_G$ is triangulated, indecomposable and Calabi–Yau of dimension* 2.

Proof. Write $\sigma = (\mathcal{A}, Z)$. Since the actions of $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$ and G on the stability manifold commute, by Proposition E.A.1 we may assume that $D^b(\mathcal{A}) \cong D^b(S)$. Applying Proposition E.2.4 we see that $D^b(S)_G$ is triangulated and that the G-action on $D^b(S)$ lifts to an action on the dg-enhancement. Hence by Proposition E.4.3 and assumption (i) the category $D^b(S)_G$ is Calabi–Yau. Since G acts faithfully, the indecomposability of $D^b(S)_G$ holds by definition. $\square$

E.5.2. Moduli spaces

By work of Toda [203] the distinguished component $\mathrm{Stab}^\dagger(S)$ satisfies condition (†) of Section E.3.6. Hence by Theorem E.3.22 we have the following.

Proposition E.5.2. *Let $v' \in K(D^b(S)_G)$. Then $\mathcal{M}_{\sigma_G}(v')$ is a universally closed Artin stack of finite type over $\mathbb{C}$ which admits a proper good moduli space.*

Recall the notion of a (G, σ)-generic class from Definition E.2.9.

Proposition E.5.3. *If $v \in \Lambda^G$ is (G, σ)-generic, then $\mathcal{M}_\sigma(v)^G$ has a good moduli space N which is smooth, symplectic and proper. The map $\pi \colon \mathcal{M}_\sigma(v)^G \to N$ is a $\mathbb{G}_m$-gerbe.*

Proof. By arguing as in the proof of Lemma E.3.21 we can deform σ inside $\mathrm{Stab}^\dagger(S)^G$ to an algebraic stability condition, without modifying the moduli functor $\mathcal{M}_\sigma(v)$. Together with Remark E.A.5 we hence can assume that σ is algebraic and that $D^b(\mathcal{A}) \cong D^b(S)$.

Let $\pi \colon \mathcal{M}_\sigma(v)^G \to N$ be the good moduli space of $\mathcal{M}_\sigma(v)^G$. For every $x \in \mathcal{M}_\sigma(v)^G(T)$ over a scheme T corresponding to an equivariant object (E, ϕ) we have an inclusion $\mathbb{G}_m(T) \hookrightarrow \mathrm{Aut}(x)$ by sending $f \in \mathbb{G}_m(T)$ to $f \cdot \mathrm{id}_E$. Moreover, for every $\mathbb{C}$-point $p \in \mathcal{M}_\sigma(v)^G$ by Lemma E.2.10 we have

$$\mathrm{Aut}_{\mathcal{M}_\sigma(v)^G}(p) = \mathrm{Aut}_{\mathcal{M}_{\sigma_G}(v')}(p) = \mathrm{Aut}_{\mathcal{A}_G}(E, \phi) = \mathbb{C}^* \cdot \mathrm{id}.$$

This shows that π is a $\mathbb{G}_m$-gerbe.

Let $p \in \mathcal{M}_\sigma(v)^G$ be a $\mathbb{C}$-valued point corresponding to some object $(E, \phi) \in \mathcal{A}_G$. Let $v' \in K(\mathcal{A}_G)$ be the class of (E, ϕ). Applying Lemma E.2.10 again we have

$$\mathrm{Hom}_{\mathcal{A}_G}((E, \phi), (E, \phi)) = \mathbb{C}.$$

Since $D^b(S)_G$ is Calabi–Yau of dimension 2, we find that

$$\mathrm{Ext}^2_{\mathcal{A}_G}((E, \phi), (E, \phi)) = \mathrm{Hom}_{\mathcal{A}_G}((E, \phi), (E, \phi))^\vee \cong \mathbb{C}.$$

By Lemma E.2.13 the Euler characteristic $\chi((E, \phi), (E, \phi))$ is locally constant and hence depends only on v'. We write $\chi(v', v')$ for its value. By Proposition E.3.20 we conclude that the dimension of the tangent space of N at p is

$$\dim T_{N,p} = \dim \mathrm{Ext}^1_{\mathcal{A}_G}((E, \phi), (E, \phi)) = -\chi(v', v') + 2.$$

In particular, the dimension is locally constant in p. Moreover, from the G-invariant inclusion $\mathbb{C}\cdot \mathrm{id} \subset \mathrm{Hom}(E,E)$ we obtain via Serre duality a G-invariant surjection $\mathrm{Ext}^2(E,E) \to \mathbb{C}$ which is precisely the trace map. This shows that the trace map is surjective on the G-invariant part and thus that the trace-free part vanishes:

$$\mathrm{Ext}^2(E,E)_0^G = 0.$$

Using Proposition E.3.20 again we find that all obstructions vanish and N is smooth.

The symplectic form on N can be constructed from the fact that it is a moduli space of stable objects in a 2-CY category. It can be seen also directly:

Recall from [103, Sec. 10] the anti-symmetric Yoneda pairing on $\mathcal{M}_\sigma(v)$,

$$\mathscr{E}xt^1_\rho(\mathcal{E},\mathcal{E}) \times \mathscr{E}xt^1_\rho(\mathcal{E},\mathcal{E}) \to \mathscr{E}xt^2_\rho(\mathcal{E},\mathcal{E}), \tag{E.5.1}$$

where $\mathcal{E}$ is the universal family on $S\times\mathcal{M}_\sigma(v)$ and $\rho\colon S\times\mathcal{M}_\sigma(v)\to\mathcal{M}_\sigma(v)$ is the projection to the second factor. Restricting to the G-invariant part and pulling back (E.5.1) via $\epsilon\colon \mathcal{M}_{\sigma_G}(v')\to\mathcal{M}_\sigma(v)$ yields a pairing

$$\epsilon^*\mathscr{E}xt^1_\rho(\mathcal{E},\mathcal{E})^G \times \epsilon^*\mathscr{E}xt^1_\rho(\mathcal{E},\mathcal{E})^G \to \epsilon^*\mathscr{E}xt^2_\rho(\mathcal{E},\mathcal{E}). \tag{E.5.2}$$

By Proposition E.3.20 the sheaf $\epsilon^*\mathscr{E}xt^1_\rho(\mathcal{E},\mathcal{E})^G$ is the tangent bundle of N. Since the symplectic form is G-invariant, the image of (E.5.2) is the G-invariant part $\epsilon^*_\rho\mathscr{E}xt^2(\mathcal{E},\mathcal{E})^G = \mathcal{O}_N$. Equivariant Serre duality implies that the pairing (E.5.2) is non-degenerate and hence a symplectic form. □

E.5.3. Proof of Theorem E.1.1

Consider the $G^\vee$-torsor given in (E.1.1),

$$\bigsqcup_{p_*v'=v} M_{\sigma_G}(v') \to M^G. \tag{E.5.3}$$

Let $F\subset M^G$ be a G-linearizable 2-dimensional component and let

$$S'\subset M_{\sigma_G}(v')$$

be a connected component which maps to F. The map $S'\to F$ is a torsor for the subgroup of $G^\vee$ that preserves this component.

By the second part of Proposition E.3.11 the moduli space $M_{\sigma_G}(v')$ is fine, i.e. there is a universal equivariant object on $M_{\sigma_G}(v')\times S$. Let

$$\mathcal{E} = (E,\phi)\in D^b(S'\times S)_{1\times G}.$$

be its restriction to $S'\times S$. We will check that the induced Fourier–Mukai transform

$$\mathsf{F}_{\mathcal{E}}\colon D^b(S')\to D^b(S)_G$$

is an equivalence.

For any $x \in S'$ we have

$$\begin{aligned}
\mathrm{Hom}_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x) &= \mathrm{Hom}_{D^b(S)}(\mathcal{E}_x, \mathcal{E}_x)^G = \mathbb{C} \\
\mathrm{Ext}^1_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x) &= \mathrm{Ext}^1_{D^b(S)}(\mathcal{E}_x, \mathcal{E}_x)^G = T_{S',x} \cong \mathbb{C}^2 \\
\mathrm{Ext}^2_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x) &= \mathrm{Hom}_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x)^\vee \cong \mathbb{C}.
\end{aligned}$$

The first line follows from the stability of $\mathcal{E}_x$. The second line follows from Proposition E.3.20, the smoothness of S', and since F and hence S' are 2-dimensional. The third line follows since the equivariant category is Calabi–Yau. In particular, we have $\chi(\mathcal{E}_x, \mathcal{E}_x) = 0$, and using Lemma E.2.13 this yields

$$\chi(\mathcal{E}_x, \mathcal{E}_y) = 0 \quad \text{for all } x, y \in S'.$$

Further for all distinct $x, y \in S'$ by the stability of $\mathcal{E}_x$ and $\mathcal{E}_y$ we have

$$\begin{aligned}
\mathrm{Hom}_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_y) &= 0 \\
\mathrm{Ext}^2_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_y) &= \mathrm{Hom}_{D^b(S)_G}(\mathcal{E}_y, \mathcal{E}_x)^\vee = 0.
\end{aligned}$$

Hence from the Euler characteristic calculation we also get $\mathrm{Ext}^1(\mathcal{E}_x, \mathcal{E}_y) = 0$. We have therefore proven that for all $x, y \in S'$ we have

$$\mathrm{Hom}_{D^b(S')}(\mathbb{C}_x, \mathbb{C}_y[i]) = \mathrm{Hom}_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_y[i]).$$

By Proposition E.5.1 the category $D^b(S)_G$ is indecomposable and Calabi–Yau of dimension 2. Applying Proposition E.4.5 we conclude that $\mathsf{F}_{\mathcal{E}}$ is an equivalence. ☐

E.5.4. A stronger version of Theorem E.1.1

We state a version of Theorem E.1.1 where we drop the condition on the moduli space to parametrize only stable objects. This is useful since not every group action on $D^b(S)$ induces an action on such a moduli space.

Theorem E.5.4. *Let $v \in \Lambda^G_{\mathrm{alg}}$ be (G, σ)-generic and let N be the good moduli space of $\mathcal{M}_\sigma(v)^G$. If N has a 2-dimensional connected component S', then we have an equivalence*

$$D^b(S', \alpha) \xrightarrow{\cong} D^b(S)_G$$

where $\alpha \in \mathrm{Br}(S')$ is the Brauer class of the gerbe $\pi \colon \mathcal{M}_\sigma(v)^G \to N$ restricted to S'.

Proof. Since π (restricted to $\pi^{-1}(S')$) is a $\mathbb{G}_m$-gerbe with Brauer class α, the universal equivariant object on $\mathcal{M}_{\sigma_G}(v)^G \times S$ restricted to $\pi^{-1}(S') \times S$ descends to an $\alpha \times 1$-twisted $1 \times G$-equivariant universal family $\mathcal{E}$ on $S' \times S$. Arguing as in Theorem E.1.1 shows that the associated Fourier–Mukai transform $\mathsf{F}_{\mathcal{E}} \colon D^b(S', \alpha) \to D^b(S)_G$ is an equivalence. ☐

E.5.5. Proof of Theorem E.1.3

For every $v' \in R_v$ consider the natural morphism

$$M_{\sigma_G}(v') \to M^G. \tag{E.5.4}$$

By Theorem E.1.2 this is a H-torsor over a connected component of M^G, where H is the stabilizer of v' under the $G^\vee$-action on $\Lambda_{(S',\alpha)}$. In particular, H acts freely on $M_{\sigma_G}(v')$.

Assume first that the induced stability condition σ_G lies in the distinguished component $\mathrm{Stab}^\dagger(S')$. Since S' is a K3 surface, this implies that $M_{\sigma_G}(v')$ is an irreducible holomorphic symplectic variety. By the second part of Proposition E.4.3 the group H acts symplectically on $M_{\sigma_G}(v')$ and thus by the holomorphic Lefschetz fixed point formula every non-trivial element must have a fixed point. This shows that $H = 1$ and that (E.5.4) is an isomorphism onto its image. In the general case, the main result of [135] implies that $\oplus_i H^{0,i}(M_{\sigma_G}(v'))$ is generated by (the conjugate of) the class of a symplectic form, so by the holomorphic Lefschetz fixed point formula we again obtain $H = 1$. In any case, the morphism (E.5.3) is a trivial $G^\vee$-torsor over its image. Since G is cyclic, every point of M^G is G-linearizable hence (E.5.3) is also surjective. This shows the claim. ☐

E.6. Existence and properties of auto-equivalences

Let S be a symplectic surface. In this section we tie up some loose ends in order to make the theorems we proved in the last section effective in practice. After some preliminary notation, we will consider the following topics:

(i) Given a G-fixed distinguished stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ we will show that the induced stability condition is distinguished, at least if the equivalence arises from a universal family. This is useful, because for distinguished stability conditions the moduli spaces of objects are well-understood.

(ii) We will prove that any symplectic action on a moduli space of stable objects on a K3 surface is induced by an action on the derived category (Proposition E.1.4).

E.6.1. Mukai lattice

The even cohomology of the symplectic surface S,

$$\Lambda = H^{2*}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}),$$

admits a non-degenerate pairing, called the *Mukai pairing*, defined by

$$\langle (r_1, D_1, n_1), (r_2, D_2, n_2) \rangle = -r_1 n_2 - r_2 n_1 + \int_S D_1 \cup D_2.$$

We will also write $\alpha \cdot \beta$ for $\langle \alpha, \beta \rangle$. For any $E, F \in D^b(S)$ we have

$$v(E) \cdot v(F) = -\chi(E,F)$$

where $v(E) = \mathrm{ch}(E)\sqrt{\mathtt{td}(S)}$ is the *Mukai vector* of E.

E.6.2. Stability conditions

Given a stability condition $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}^\dagger(S)$ in the distinguished component we will identify the stability function

$$Z \colon \Lambda_{\mathrm{alg}} \to \mathbb{C}$$

with the corresponding element in $\Lambda_{\mathrm{alg}} \otimes \mathbb{C}$ under the Mukai pairing.

Let $\mathcal{P}(S) \subset \Lambda_{\mathrm{alg}} \otimes \mathbb{C}$ be the open subset of elements whose real and imaginary part span a positive-definite 2-plan, let $\mathcal{P}^+(S) \subset \mathcal{P}(S)$ be the connected component which contains $e^{i\omega}$ for an ample class ω, and let

$$\mathcal{P}_0^+(S) = \mathcal{P}^+(S) \setminus \bigcup_{\substack{\delta \in \Lambda_{\mathrm{alg}} \\ \delta \cdot \delta = -2}} \delta^\perp.$$

Bridgeland [41] proved that

$$\pi \colon \mathrm{Stab}^\dagger(S) \to \mathcal{P}_0^+(S), \ \sigma = (\mathcal{A}, Z) \mapsto Z \tag{E.6.1}$$

is a covering map. His results were generalized to the twisted case in [104].

E.6.3. Induced stability conditions

Let $\sigma \in \mathrm{Stab}^\dagger(S)$ be a stability condition and let G be a finite group which acts on $D^b(S)$. We assume the conditions (i), (ii) and (iii) of Section E.1.1 are satisfied. Suppose we are given an equivalence

$$\mathsf{F}_{\mathcal{E}} \colon D^b(S', \alpha) \to D^b(S)_G$$

induced from a universal family $\mathcal{E}$ as in Theorem E.1.1 or Theorem E.5.4.

Proposition E.6.1. *We have* $\mathsf{F}_{\mathcal{E}}^{-1}(\sigma_G) \in \mathrm{Stab}^\dagger(S')$.

We begin with a description how the Mukai lattices Λ and Λ' of the surfaces S and S' interact. Consider the composition of the forgetful and linearization functors with the equivalence $\mathsf{F}_{\mathcal{E}}$:

$$\mathsf{FM}_{p(\mathcal{E})} = p \circ \mathsf{F}_{\mathcal{E}}, \quad \mathsf{FM}_{p(\mathcal{E})^\vee[2]} = \mathsf{F}_{\mathcal{E}}^{-1} \circ q,$$

where we have also written p for the forgetful functor of $D^b(S' \times S)_{1 \times G}$. Passing to cohomology this yields morphisms

$$p \colon \Lambda' \to \Lambda, \quad q \colon \Lambda \to \Lambda'$$

which are both left and right adjoints of each other. The composition is $pq = \oplus_g g$. Let

$$L \subset \Lambda'$$

denote the saturation of the sublattice $q(\Lambda)$.

Given a lattice M we write $M(n)$ for the lattice obtained by multiplying the intersection form with the integer n.

Lemma E.6.2. *We have the finite-index sublattices*

$$\Lambda^G \oplus (\Lambda^G)^\perp \subset \Lambda, \quad L \oplus L^\perp \subset \Lambda'.$$

The map p vanishes on $L^\perp$ and defines an embedding of lattices $p \colon L(|G|) \hookrightarrow \Lambda^G$. The map q vanishes on $(\Lambda^G)^\perp$ and defines an embedding of lattices $q \colon \Lambda^G(|G|) \hookrightarrow L$.

Proof. The isomorphism of correspondences

$$\rho_g \circ p(\mathcal{E}) = (\mathrm{id} \times \rho_g)(p(\mathcal{E})) \cong p(\mathcal{E}),$$

shows that the image of $p\colon \Lambda' \to \Lambda$ lies in the invariant lattice Λ^G. By adjunction it follows that q vanishes on $(\Lambda^G)^\perp$. In particular, for all $v', w' \in L$ we can write $v' = q(v)$ and $w' = q(w)$ where $v, w \in \Lambda^G \otimes \mathbb{Q}$. We obtain

$$\langle v', w' \rangle_{\Lambda'} = \langle qv, qw \rangle_{\Lambda'} = \langle v, pqw \rangle_\Lambda = |G| \langle v, w \rangle_\Lambda.$$

Since Λ^G is non-degenerate, this shows that L is non-degenerate and we have the finite-index sublattice $L \oplus L^\perp \subset \Lambda'$. It also shows that q defines an embedding $\Lambda^G(|G|) \hookrightarrow L$. Moreover, with the same notation as above we have

$$\langle pv', pw' \rangle_\Lambda = \langle pqv, pqw \rangle_\Lambda = |G| \langle v, pqw \rangle_\Lambda = |G| \langle qv, qw \rangle_{\Lambda'} = |G| \langle v', w' \rangle_{\Lambda'}.$$

We find that p defines an embedding $L(|G|) \hookrightarrow \Lambda^G$. For every $w' \in L^\perp$ we have $\langle pw', v \rangle_\Lambda = \langle w', qv \rangle_{\Lambda'} = 0$ for all $v \in \Lambda$, which shows that $pw' = 0$. ☐

If G is abelian, then one can show that L is the invariant lattice for the action of the dual group on $D^b(S')$, that is $L = (\Lambda')^{G^\vee}$.

Proof of Proposition E.6.1. To ease the notation we assume that the Brauer class α vanishes and hence that we work with the usual derived category $D^b(S')$. The case with non-trivial Brauer class works parallel.

Let $\tau = \mathsf{F}_{\mathcal{E}}^{-1}(\sigma_G)$. By construction the functor $\mathsf{F}_{\mathcal{E}}$ is induced from a universal family $\mathcal{E} \in D^b(S' \times S)_{1 \times G}$ of σ_G-stable objects. Since $\mathcal{E}_x$ is σ_G-stable for all $x \in S'$, the skyscraper sheaves $\mathbb{C}_x$ are τ-stable for all $x \in S'$.

Let us consider the central charge Z_τ of the stability condition τ. By definition, it is given by the composition

$$Z_\tau \colon \Lambda' \xrightarrow{p} \Lambda^G_{\mathrm{alg}} \subset \Lambda_{\mathrm{alg}} \xrightarrow{Z} \mathbb{C}.$$

By Lemma E.6.2 the central charge Z_τ factors over L and the real and imaginary part of Z_τ span a positive-definite 2-plane, because $\Re(Z)$ and $\Im(Z)$ do so.

We want to apply now the reasoning of the proof of [41, Prop. 10.3]. As in [41, Sec. 10], there is a unique $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that the central charge of $g\tau$ is of the form $\exp(\beta + i\omega)$ for some $\beta, \omega \in \mathrm{NS}(S')$ with $\omega^2 > 0$, and such that the sheaves $\mathbb{C}_x$ have phase 1. Then as in the first step in [41, Prop. 10.3] we apply [41, Lem. 10.1] to conclude that for any curve $C \subset S'$ and torsion sheaf $\mathcal{E}$ supported on C we have $\Im Z_\tau(\mathcal{E}) > 0$ which implies $\omega \cdot [C] > 0$. Combining this with $\omega^2 > 0$ we find that the class ω is ample.

Invoking again [41, Lem. 10.1] we find further that the heart $\mathcal{B}$ of $g\tau$ is the tilt of the torsion pair $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \mathrm{Coh}(S') \cap \mathcal{P}(0, 1]$ and $\mathcal{F} = \mathrm{Coh}(S') \cap \mathcal{P}(-1, 0]$ and $\mathcal{P}$ is the slicing corresponding to $g\tau$ (for more on tilting we refer to Appendix E.A or [87]). Arguing as in the second step of the proof of [41, Prop. 10.3] we deduce that the torsion pair $(\mathcal{T}, \mathcal{F})$ coincides with the torsion pair $(\mathcal{T}_{\omega,\beta}, \mathcal{F}_{\omega,\beta})$ associated with the classes ω, β which is constructed in [41, Sec. 6]. With the notation of *loc. cit.* this yields that $\mathcal{B} = \mathcal{A}(\omega, \beta)$ and therefore $g\tau = \sigma_{\omega,\beta}$. In particular, $\tau \in \mathrm{Stab}^\dagger(S')$ and the proof is finished. ☐

E.6.4. Proof of Proposition E.1.4

Let S be a K3 surface with a stability condition $\sigma' = (\mathcal{A}', Z') \in \mathrm{Stab}^\dagger(S)$. Let M be a fine[11] moduli space of σ'-stable objects of Mukai vector $v \in \Lambda$ and let G be a finite group which acts symplectically on M. Consider the Hodge isometry

$$\Lambda \supset v^\perp \cong H^2(M, \mathbb{Z}).$$

By [151, Thm. 26] the induced action of G on $H^2(M, \mathbb{Z})$ acts trivially on the discriminant lattice. Hence, the action lifts to an action on Λ which fixes the vector v and acts by Hodge isometries. Since G acts symplectically on M, the action on Λ preserves the class of the symplectic form.

Let $H \in H^2(M, \mathbb{Z})$ be a G-invariant ample class (obtained for example by averaging any ample class over its images under G). Recall the wall and chamber decomposition of $\mathrm{Stab}^\dagger(S)$ associated to v [41, Sec. 9] and denote by $\mathcal{C}$ the chamber which contains σ'. From [17, Thm. 1.2] we infer that there exists a stability condition $\sigma = (\mathcal{A}, Z) \in \mathcal{C}$ such that the associated divisor class ℓ_σ equals the class H (for the construction and properties of the divisor classes ℓ_σ we refer to [18]). By definition the central charge Z is contained in the $\mathbb{C}$-vector space $\mathrm{Span}_{\mathbb{C}}\langle H, v\rangle \subset \Lambda \otimes \mathbb{C}$ and hence fixed by G. Moreover, since σ and σ' lie in the same chamber, the moduli functors $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_{\sigma'}(v)$ agree. This proves $M = M_\sigma(v)$.

Hence we have obtained a subgroup $G \subset O(\Lambda)$ which acts by Hodge isometries, preserves the class of the symplectic form and Z. An application of [101, Prop. 1.4] shows that this action on Λ is induced by a subgroup $G \subset \mathrm{Aut}\, D^b(S)$ which preserves σ and acts symplectically. Using part (b) of Lemma E.2.7 there is a surjection $\tilde{G} \to G$ from a finite group $\tilde{G}$ which acts on $D^b(S)$ with image G in $\mathrm{Aut}\, D^b(S)$. By construction the action of $\tilde{G}$ preserves σ and v and hence induces an action on $M = M_\sigma(v)$. Since the restriction map $\mathrm{Aut}(M) \to \mathrm{O}(H^2(M, \mathbb{Z}))$ is injective [150, Lem. 7.1.3], the action of $\tilde{G}$ on M factors through the given action by G. This proves the first part.

For the second part, assume that $G \subset \mathrm{Aut}\, M$ is cyclic. Then the action of $\mathbb{Z}_n$ on M has at least one fixed point which corresponds to a $\mathbb{Z}_n$-invariant simple object F. Hence the claim follows from [29, Sec. 4.8]. ☐

E.7. Examples

We consider a series of examples to illustrate our methods. For simplicity we restrict ourselves mostly to cyclic groups acting on the derived category of a K3 surface.

E.7.1. Classification

Given a variety X and an element $g \in \mathrm{Aut}\, H^*(X, \mathbb{C})$ of finite order n we define the *frameshape* of g as the formal symbol

$$\pi_g = \prod_{a|n} a^{m(a)}$$

that encodes the characteristic polynomial of g via

$$\det(t \cdot \mathrm{id} - g) = \prod_{a|n} (t^a - 1)^{m(a)}.$$

---

[11] The case of a coarse moduli space works similarly.

Symplectic auto-equivalences of K3 surfaces of finite order preserving a stability condition are neatly classified in terms of the frameshape of their action on cohomology. There are 42 frameshapes and at most 82 $\mathrm{O}_+(\Lambda)$-conjugacy classes which can occur [56]. The invariant lattices can be found in [181, App. C]. For example, in order 2 there are three cases: $1^8 2^8$, $1^{-8} 2^{16}$, and 2^{12}, each in a unique conjugacy class. Symplectic involutions of K3 surfaces have frameshape $1^8 2^8$, while the others are strictly of derived nature.

E.7.2. The dual action of a geometric involution

Let $\iota\colon S \to S$ be a symplectic involution of a symplectic surface with at least one fixed point and let $G = \mathbb{Z}_2$ be the group generated by ι. Hence we are in one of the following two cases:

(i) S is an abelian surface and ι is multiplication by (-1), or

(ii) S is a K3 surface and ι is a *Nikulin involution* [192].

The number r of fixed points of G is 16 and 8 respectively, and in both cases the minimal resolution S' of $S/\mathbb{Z}_2$ is a K3 surface. In the fiber diagram

$$\begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d, "\beta"] & S' \arrow[d] \\ S \arrow[r] & S/\mathbb{Z}_2 \end{tikzcd}$$

the map β is the blowup at the fixed points and α identifies S' with the fixed locus $\mathsf{Hilb}^2(S)^G$. By [42] (or Theorem E.1.1) we have the equivalence $\Phi = \beta_* \alpha^* \colon D^b(S') \to D^b(S)_G$.

Let $\mathsf{Q}\colon D^b(S') \to D^b(S')$ be the involution given by the action of the dual group $G^\vee$. By applying both sides to skyscraper sheaves one finds[12]

$$\mathsf{Q} = \mathsf{T}_{\mathcal{O}_S(-\delta)} \circ \prod_{i=1}^{r} \mathsf{ST}_{\mathcal{O}_{E_i}(-2)}$$

where we let $\mathsf{ST}_E(F) = \mathrm{Cone}(\mathrm{Hom}^\bullet(E,F) \otimes E \to F)$ denote the spherical twist by the spherical object E, and $\mathsf{T}_{\mathcal{L}}(E) = E \otimes \mathcal{L}$ is the twist by a line bundle $\mathcal{L}$. The E_i are the exceptional divisors of the resolution S' and $\delta = \frac{1}{2}\sum_{i=1}^{r} E_i$.

The involution Q fixes skyscraper sheaves of points not on the exceptional divisor and sends $\mathcal{O}_{S'}$ to $\mathcal{O}_{S'}(\delta)$ as well as $\mathcal{O}_{E_i}(-1)$ to $\mathcal{O}_{E_i}(-2)[1]$. For $x \in E_i$ the action exchanges the two distinguished triangles

$$\begin{gathered} \mathcal{O}_{E_i}(-1) \to \mathbb{C}_x \to \mathcal{O}_{E_i}(-2)[1] \\ \mathcal{O}_{E_i}(-2)[1] \to \mathsf{Q}(\mathbb{C}_x) \to \mathcal{O}_{E_i}(-1). \end{gathered} \tag{E.7.1}$$

The frameshape of Q is $1^{-8} 2^{16}$ if S is an abelian surface, and $1^8 2^8$ if S is a K3 surface.[13]

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[12] See also [125] for a related discussion of this involution.

[13] On the Mukai lattice the involution Q acts by

$$(1,0,0) \mapsto (1,\delta,-r/4), \quad (0,E_i,0) \mapsto (0,-E_i,1), \quad (0,0,1) \mapsto (0,0,1).$$

As an example of a fixed stack computation, consider the moduli space

$$\mathcal{M} = \mathcal{M}_{\sigma_G}(0,0,1)$$

where σ_G is induced by a G-fixed stability condition on $D^b(S)$ which is equivalent to Gieseker stability for the Mukai vector $v = (0,0,1)$. The $\mathbb{C}$-points of $\mathcal{M}$ correspond to the objects

$$\mathbb{C}_x \text{ for all } x \in S', \quad \mathsf{Q}(\mathbb{C}_x) \text{ for all } x \in E_i, \quad \mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(-2)[1].$$

In this list the $\mathbb{C}_x$ for all $x \notin E_i$ and the $\mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(-2)[1]$ are invariant under Q. Every $\mathbb{C}_x$ for $x \notin E_i$ admits two distinct $G^\vee$-linearizations, while $\mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(-2)[1]$ admits only one. We find that the good moduli space of $\mathcal{M}$ is the quotient $S/\mathbb{Z}_2$, and that the good moduli space of the fixed stack $\mathcal{M}^{G^\vee}$ is S. Moreover, the forgetful map $\epsilon\colon \mathcal{M}^{G^\vee} \to \mathcal{M}$ induces the quotient map $S \to S/\mathbb{Z}_2$ on good moduli spaces. Applying Theorem E.5.4 we obtain the equivalence

$$D^b(S) \xrightarrow{\cong} D^b(S')_{G^\vee} \tag{E.7.2}$$

where the Brauer class α is trivial since $S/\mathbb{Z}_2$ is a fine moduli space away from the singularities. (The equivalence (E.7.2) also follows by a result of Elagin [67, Thm. 1.3].)

Among other things this example shows that while the good moduli space of $\mathcal{M}$ may be singular, its fixed stack has a smooth proper good moduli space (as guaranteed by Proposition E.5.3). We also see that ϵ is not proper, because it does not satisfy the valuative criterion of properness.

E.7.3. Involutions on a genus 2 K3 surface

Let $\pi\colon S \to \mathbb{P}^2$ be a K3 surface obtained as the double cover of a sextic plane curve, and let $g\colon S \to S$ be a symplectic involution which fixes the hyperplane class $H \in \operatorname{Pic}(S)$. In this section we will determine the fixed locus of the moduli spaces of Gieseker semistable sheaves with Mukai vector $(0,H,0)$ and $(0,2H,0)$. As an application we describe the fixed locus of the induced symplectic birational involution of the resolution of $M_\sigma(0,2H,0)$ of O'Grady 10 type.

Recall that the involution g descends to an involution $g_{\mathbb{P}^2}$ of $\mathbb{P}^2$ which can be choosen to act by $(x,y,z) \mapsto (-x,y,z)$, see [192, Sec. 3.2]. The fixed locus of $g_{\mathbb{P}^2}$ is $p = (1,0,0)$ and the line $x = 0$. Let C_0 be the preimage under π of the line $x = 0$ and let C_1 be the preimage of a generic line of the form $\lambda y + \mu z$. Let also $C \in |\mathcal{O}(2H)|$ be a curve that is preserved under g but disjoint from the fixed points p_i. These curves are preserved by g and contain 6, 2 and 0 fixed points respectively. Consider the quotients

$$C_0' = C_0/\mathbb{Z}_2, \quad C_1' = C_1/\mathbb{Z}_2 \quad \text{and} \quad C' = C/\mathbb{Z}_2$$

which are rational, elliptic, and of genus 3 respectively. After reordering the exceptional divisors one has in $\operatorname{Pic}(S')$ the relations[14]

$$C_0' = \frac{1}{2}C' - \frac{1}{2}(E_3 + \ldots + E_8), \quad C_1' = \frac{1}{2}C' - \frac{1}{2}(E_1 + E_2).$$

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[14] We denote the class in the Picard group with the same symbol as the underlying curve.

Suppose that S is of minimal Picard rank 9. Then by [192, Lem. 1.10] the Picard group of S' has the $\mathbb{Z}$-basis $C_1', \delta, E_2, \dots, E_8$. The map on cohomology $H^*(S', \mathbb{Z}) \to H^*(S, \mathbb{Z})$ induced by the composition $D^b(S') \xrightarrow{\Phi} D^b(S)_G \to D^b(S)$ is given by

$$1 \mapsto 1 - \mathsf{p}, \quad \mathsf{p} \mapsto 2\mathsf{p}, \quad E_i \mapsto \mathsf{p}, \quad \delta \mapsto 4\mathsf{p}, \quad C' \mapsto 2H, \quad C_1' \mapsto H - \mathsf{p}$$

where we let p denote the class of a point on both S and S'.

Let σ be a generic G-fixed stability condition on S which for vectors $(0, kH, 0)$ is equivalent to Gieseker stability. We consider the moduli spaces $M_\sigma(0, kH, 0)$ for $k = 1, 2$ and their fixed loci: Since H is irreducible on S, the coarse moduli space $M_\sigma(0, H, 0)$ is smooth. Hence by Theorem E.1.3 (and using the notation given there) we have

$$M_\sigma(0, H, 0)^G = \bigsqcup_{v' \in \overline{R}_H} M_{\sigma_G}(v').$$

A direct calculation shows that there is a unique vector in $\overline{R}_H$ of square 0 given by $C_1' + E_1$, and 28 vectors of square -2. Therefore,

$$M_\sigma(0, H, 0)^G = \widetilde{S} \sqcup (28 \text{ points})$$

where $\widetilde{S} = M_{\sigma_G}(0, C_1' + E_1, 0)$ is a smooth K3 surface. This matches the results of [118].

We turn to $M_\sigma(0, 2H, 0)$. Since the moduli space contains strictly semistable objects, we can not apply Theorem E.1.2 directly, but have to account for the semistable locus. We begin by describing the set R_{2H}. It is given by vectors of the form

$$v' = C' + \sum_{i=1}^{8} a_i E_i + c\mathsf{p}$$

where all the a_i are either integers or half-integers, $\sum_i a_i$ is even and $c = -\sum_i a_i/2$. Moreover, only vectors satisfying

- $(v')^2 \geq -2$ (equivalently $\sum_i a_i^2 \leq 3$), or
- $v' = v_1 + v_2$ with $v_i \in R_H$

contribute to R_{2H}. One finds that $\overline{R}_{2H}$ (i.e. modulo Q) consists of the following:

(i) The vector C' of square 4. It can be decomposed in 28 different ways as a sum $v_1 + v_2$ with $v_1, v_2 \in R_H$ both of square -2, and in a unique way as $v_1 + v_2$ with $v_1, v_2 \in R_H$ both of square 0 (given as $C_1' + E_i$). The moduli space $M_{\sigma_G}(C')$ is of dimension 6. Its singular locus is the disjoint union of the product variety $\widetilde{S} \times \widetilde{S}$ and 28 isolated points.

(ii) 63 vectors of square 0. Each vector can be written in 6 different ways as a sum of two (-2)-vectors in R_H. The moduli space in each case is a K3 surface with 6 singularities of type A_1.

(iii) 56 vectors of square 0, each written uniquely as $v_1 + v_2$ where v_1 is of square 0 (equal to $C_1' + E_1$) and v_2 is of square -2. In each case we have $M_{\sigma_G}(v') = M_{\sigma_G}(v_1) = \widetilde{S}$.

(iv) 1 vector of square 0 obtained as $2v_1$, where $v_1 = C_1' + E_1 \in R_H$ is of square 0. The good moduli space $M_{\sigma_G}(2v_1)$ is $\mathrm{Sym}^2 M_{\sigma_G}(v_1) = \mathrm{Sym}^2 \tilde{S}$.

(v) 378 vectors of square -4 written uniquely as $v_1 + v_2$ where $v_1, v_2 \in R_H$ are both of square -2. The good moduli space is a point.

(vi) 28 vectors of square -8 obtained as $2v$, where $v \in R_H$ is of square -2. The good moduli space is a point.

By considering the possible types of semistable points in $M_\sigma(0, 2H, 0)$ and using that G is cyclic one finds that the image of $\bigsqcup_{v' \in \overline{R}_{2H}} \mathcal{M}_{\sigma_G}(v')$ in $M_\sigma(0, 2H, 0)$ is precisely the fixed locus we are interested in. A basic sublocus of the fixed locus is

$$\mathrm{Sym}^2 \left(M_\sigma(0, H, 0)^G \right) \subset M_\sigma(0, 2H, 0)^G.$$

The scheme $\mathrm{Sym}^2 \left(M_\sigma(0, H, 0)^G \right)$ consists of

(a) 1 copy of $\mathrm{Sym}^2(\tilde{S})$,

(b) 28 copies of $\tilde{S}$ corresponding to sheaves $E \oplus F$ with $E \in \tilde{S}$ and F corresponding to one of the 28 fixed points and

(c) Sym^2(28 points) consisting of 378+28 points corresponding to the direct sum of distinct and identical stable sheaves respectively.

Given distinct G-invariant stable sheaves E, F of the same slope, the direct sum $E \oplus F$ admits precisely $|G^\vee|^2$ many G-linearizations. Moreover, if distinct $E, F \in M_\sigma(0, H, 0)$ are isolated points of the fixed locus, then no equivariant lift of $E \oplus F$ has class C' (since otherwise $(E, \phi) = \mathsf{Q}(F, \phi)$, so $E = F$). We see that the 378 points in (c) are the image of the points (v), but also of the $6 \cdot 63$ singular points on the K3 surfaces in (ii).

Similarly, the 28 K3 surfaces in (b) are the image of the 56 K3 surfaces in (iii). Since there are precisely 4 linearizations, these K3 surfaces can not appear in the image of other components, and so yield connected components of $M_\sigma(0, 2H, 0)^G$. A direct sum $E \oplus E$ of a stable object E admits precisely $|\mathrm{Sym}^2(G^\vee)| = \binom{|G^\vee|+1}{2}$ many linearizations (here 3). Hence the 28 remaining points in (c) are the image of the 28 points in (vi) and the 28 isolated singularities in (i). Moreover, if $v_1 \in R_H$ of square 0, then $M_{\sigma_G}(2v_1) = \mathrm{Sym}^2 M_{\sigma_G}(v_1)$ maps to the same locus as the inclusion

$$M_{\sigma_G}(v_1) \times M_{\sigma_G}(\mathsf{Q}v_1) \subset M_{\sigma_G}(0, C', 0). \tag{E.7.3}$$

Hence the image of $M_{\sigma_G}(2v_1)$ lies in the image of the main component $M_{\sigma_G}(0, C', 0)$. The 63 moduli spaces in (ii) contain stable points and since we have already taken the coset modulo Q, they must embed into $M_\sigma(0, 2H, 0)^G$ as isolated components. We conclude that

$$M_\sigma(0, 2H, 0)^G = Y \sqcup (\text{28 smooth K3s}) \sqcup (\text{63 K3s with 6 nodes})$$

where Y is the image of $\mathcal{M}_{\sigma_G}(0, C', 0)$ and hence 6-dimensional.

Rcall that the singular moduli space $M(0, 2H, 0)$ admits an irreducible holomorphic symplectic resolution $X \to M_\sigma(0, 2H, 0)$ of O'Grady 10 type [8, 170]. Recall from [192] that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus E_8(-2)$. Hence there exists 240 vectors $\alpha \in E_8(-2)$ of square -4. The involution g acts on these vectors by $g\alpha = -\alpha$. Let $A \subset E_8(-2)$ be a list of representatives of the orbits of the (-4)-vectors under this action. The singular locus of $M_\sigma(0,2H,0)$ is the locus of polystable sheaves, and therefore given by

$$M_\sigma(0,2H,0)^{\mathrm{sing}} = \mathrm{Sym}^2 M_\sigma(0,H,0) \sqcup \bigsqcup_{\alpha\in A} \left(M_\sigma(H+\alpha)\times M_\sigma(H-\alpha)\right).$$

The resolution X is obtained by a blowup of $M_\sigma(0,2H,0)$ along $\mathrm{Sym}^2 M_\sigma(0,H,0)$, followed by a resolution of the 120 isolated points. The fiber of X over each of these 120 points is a $\mathbb{P}^5$. The automorphism $g\colon M_\sigma(0,2H,0)\to M_\sigma(0,2H,0)$ natural lifts to the blowup (by universal property), but it is not clear a priori whether it lifts along the resolution of the 120 points. Hence we only obtain a birational involution $g'\colon X \dashrightarrow X$ defined away from 120 disjoint copies of $\mathbb{P}^5$. We will show the following:

Proposition E.7.1. *The closure of the fixed locus of the birational symplectic involution $g\colon X \dashrightarrow X$ is smooth and the disjoint union of one connected component of dimension* 6 *containing* 120 *copies of* $\mathbb{P}^5$, *and* 119 *K3 surfaces of which* 88 *are derived equivalent to* S'.

Proof. The claim follows from our discussion above and a local analysis of g along

$$M_\sigma(0,2H,0)^{\mathrm{sing}} \cap M_\sigma(0,2H,0)^G$$

using the local description of the moduli spaces given in [116, Sec. 2] and [8, Sec. 3]. This is straightforward and we just highlight the main points:

- The 120 isolated singular points of $M_\sigma(0,2H,0)$ lie in Y. They are the images of the stable points of $M_{\sigma_G}(C')$ corresponding to $q(E_\alpha)$ where E_α is the unique stable object in class $H+\alpha$. The map g' does not extend to the resolution and the closure of the fixed locus of g' contains the whole exceptional $\mathbb{P}^5$.
- The 63 K3 surfaces with 6 nodes described in (ii) meet the singular locus of $M_\sigma(0,2H,0)$ at the singularities. The corresponding component in the fixed locus of g' is the proper transform and smooth.
- The 28 smooth K3 surfaces in $M_\sigma(0,2H,0)^G$ corresponding to (iii) lie completely in the singular locus $M_\sigma(0,2H,0)^{\mathrm{sing}}$. The corresponding component in the fixed locus of g' is a trivial $2:1$ cover of this locus and hence given by 56 K3 surfaces.
- The K3 surfaces in (iii) and precisely 32 of the K3 surfaces in (ii) arise as moduli spaces of semistable objects on S' for a Mukai vector w which satisfies $\langle w,\Lambda'\rangle=\mathbb{Z}$. Hence all of them are derived equivalent to S'. ☐

E.7.4. An order 3 equivalence

Let E, F be elliptic curves defined by cubic equations f, g respectively and consider the cubic fourfold $X\subset\mathbb{P}^5$ defined by the equation $f(x_0,x_1,x_2)+g(x_3,x_4,x_5)=0$. Let ζ be a non-trivial third root of unity. As in [158, Ex. 1.7(iv)] we define a $G=\mathbb{Z}_3$-action on X by letting the generator act by

$$(x_0,\ldots,x_5)\mapsto(x_0,x_1,x_2,\zeta x_3,\zeta x_4,\zeta x_5).$$

The induced action of G on the Fano variety of lines on X has fixed locus $F(X)^G=E\times F$. Since $F(X)$ is a moduli space of stable objects in the Kuznetsov component $\mathfrak{A}$ of $D^b(X)$, and the Kuznetsov component $\mathfrak{A}$ is equivalent to the derived category of a K3 surface by a result of Ouchi [177], Theorem E.1.1 shows that $\mathfrak{A}_G \cong D^b(A)$ for some connected étale cover $A \to E \times F$ of degree 1 or 2. In particular, A is an abelian surface. Theorem E.1.2 then determines the fixed loci of the induced action on any smooth $M_\sigma(v)$ (with $v \in K(\mathfrak{A})^{\mathbb{Z}_3}$).

E.7.5. Frameshape 2^{12}

We give an example which shows that the equivariant category can behave rather strange. Consider a symplectic automorphism $\tau\colon S \to S$ of a K3 surface of order 4, and let S' be the resolution of the quotient $S/\langle\tau^2\rangle$. Since we have taken the quotient only by τ^2, we have a residual involution $\bar{\tau}\colon S' \to S'$. The equivalences $\bar{\tau}^*$ and the dual action Q of Section E.7.2 commute and are symplectic. One checks that the composition $g = \bar{\tau}^* \circ \mathsf{Q}$ is an involution of $D^b(S')$ of frameshape 2^{12}. Then, as a special case of [29, Sec. 4.9] the involution g does *not* define an action of $\mathbb{Z}_2$ on the category, but defines instead a faithful(!) action of $\mathbb{Z}_4$. Moreover one has the equivalence:

$$D^b(S')_{\mathbb{Z}_4} \cong D^b(S').$$

In other words, the equivariant category under this action is equivalent to the category we started with. In particular, there does not exist a stable object which is G-invariant and G does not act on any fine moduli space of S.[15]

E.7.6. Order 11 equivalences

Let $g\colon D^b(S) \to D^b(S)$ be a symplectic auto-equivalence of a K3 surface S of order 11 fixing a stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$. The associated action on cohomology is one of three possible conjugacy classes, each with invariant lattice of rank 4 [181, App. C]. This implies that the pairs (S, g) are isolated points in their moduli space. Using the Huybrechts–Mongardi criterion [101, 151] each such g induces an automorphism of a moduli spaces M of stable objects in $D^b(S)$. If we want to determine the equivariant category $D^b(S)_{\mathbb{Z}_{11}}$ through Theorem E.1.1, we would need to find a 2-dimensional component of the fixed locus in some M. This seems difficult in this case without studying the concrete geometry. By Appendix E.B we can at least read off the Euler characteristic of the fixed locus: If M is of dimension $2n$, then $e(M^g)$ is the coefficient of q^{n-1} of the series

$$\frac{1}{\eta(q)^2\eta(q^{11})^2} = \frac{1}{q} + 2 + 5q + 10q^2 + 20q^3 + 36q^4 + 65q^5 + 110q^6 + O(q^7).$$

We hence should expect 2-dimensional fixed components only in cases where $\dim M \geq 10$.

E.A. Hearts on symplectic surfaces

Let S be a smooth projective symplectic surface and recall the notation from Section E.6.2. The goal of this section is to prove the following result:

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[15] This example first appeared in [56, Sec. 4.2] as a symmetry of K3 non-linear sigma models. We expect that the behaviour $D^b(S)_G \cong D^b(S)$ is typical of the case where we have a 'failure of the level-matching condition', i.e. $\lambda > 1$ in [181, App. C].

Proposition E.A.1. *Let $\sigma \in \operatorname{Stab}^\dagger(S)$ be a stability condition. Then there exists an element $g \in \widetilde{\mathrm{GL}^+}(2,\mathbb{R})$ such that $g\sigma = (\mathcal{A}, Z)$ satisfies*

$$D^b(\mathcal{A}) \cong D^b(S).$$

Let us first recall from [41] how the component $\operatorname{Stab}^\dagger(S)$ is built up. First one considers the set $V(S)$ of stability conditions $\sigma_{\omega,\beta} = (\mathcal{A}_{\omega,\beta}, Z_{\omega,\beta})$ with central charge $Z_{\omega,\beta} = \langle \exp(\beta + i\omega), \_ \rangle$ where $\beta, \omega \in \mathrm{NS}(S) \otimes \mathbb{R}$ with ω ample. The heart $\mathcal{A}_{\omega,\beta}$ is obtained from the torsion pair $(\mathcal{T}_{\omega,\beta}, \mathcal{F}_{\omega,\beta})$ of $\operatorname{Coh}(S)$ by tilting, see [41, Sec. 6]. Next, let $U(S)$ be the orbit of $V(S)$ under the free action of $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$ on $\operatorname{Stab}^\dagger(S)$. Elements in $U(S)$ are characterized as those stability conditions in $\operatorname{Stab}^\dagger(S)$ such that all skyscraper sheaves are stable of the same phase. Finally, a detailed analysis of the boundary $\partial U(S)$ [41, Thm. 12.1] yields that any $\sigma \in \operatorname{Stab}^\dagger(S)$ can be mapped into $\overline{U(S)}$ using (squares of) spherical twists. If S is an abelian surface, then we even have $U(S) = \operatorname{Stab}^\dagger(S)$ [41, Thm. 15.2].

We start the proof by considering the set of geometric stability conditions $V(S)$.

Lemma E.A.2. *For all $\sigma = (\mathcal{A}, Z) \in V(S)$ we have $D^b(\mathcal{A}) \cong D^b(S)$.*

Proof. Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ of an abelian category $\mathcal{C}$ is called cotilting, if for all $E \in \mathcal{C}$ there is a surjection $F \twoheadrightarrow E$ with $F \in \mathcal{F}$. By [36, Prop. 5.4.3], which is a refined version of [87], for any cotilting torsion pair $(\mathcal{T}, \mathcal{F})$ one has $D^b(\mathcal{C}') \cong D^b(\mathcal{C})$, where $\mathcal{C}'$ is the tilt along $(\mathcal{T}, \mathcal{F})$.

If $\sigma_{\omega,\beta} \in V(S)$, then its heart $\mathcal{A}_{\omega,\beta}$ is obtained from $\operatorname{Coh}(S)$ by tilting along the torsion pair $(\mathcal{T}_{\omega,\beta}, \mathcal{F}_{\omega,\beta})$. Huybrechts proved in [98, Prop. 1.2] that this torsion pair is cotilting. ☐

Proposition E.A.3. *Let $\sigma \in V(S)$ and let $\mathcal{P}$ be the associated slicing. Then for all $a \in \mathbb{R}$ there is a natural derived equivalence $D^b(\mathcal{P}(a, a+1]) \cong D^b(S)$.*

Since Lemma E.A.2 proves the assertion for $a = 0$ and the property is preserved by shifts, we only need to consider the case $a \in (0,1)$. Write $\sigma = (\mathcal{A}_{\omega,\beta}, Z_{\omega,\beta})$ and $\mathcal{A} := \mathcal{P}(a, a+1]$. Then

$$\mathcal{A} \subset \langle \mathcal{A}_{\omega,\beta}, \mathcal{A}_{\omega,\beta}[1] \rangle$$

and $\mathcal{A}$ is a tilt of $\mathcal{A}_{\omega,\beta}$ for the torsion pair $\mathcal{T} = \mathcal{A}_{\omega,\beta} \cap \mathcal{A} = \mathcal{P}(a,1]$ and $\mathcal{F} = \mathcal{A}_{\omega,\beta} \cap \mathcal{A}[-1] = \mathcal{P}(0,a]$. There is a natural exact functor

$$\Phi \colon D^b(\mathcal{A}) \to D^b(\mathcal{A}_{\omega,\beta}) \cong D^b(S)$$

of triangulated categories [164, Sec. 7.3]. The proof given below shows that this functor defines a derived equivalence.

Proof of Proposition E.A.3. The main idea in the proof is to show that Φ is essentially surjective. For this we make first some observations.

Take a very ample line bundle $\mathcal{O}(1)$. The line bundle $\mathcal{O}(-i)$ will lie in $\mathcal{F}_{\omega,\beta}$ for $i \gg 0$. Recall from [41, Sec. 6] that the central charge $Z_{\omega,\beta}$ of the stability condition $\sigma_{\omega,\beta}$ sends an object $E \in D^b(S)$ with Mukai vector $v(E) = (r, l, s)$ to

$$Z_{\omega,\beta}(E) = -s + \frac{r}{2}(\omega^2 - \beta^2) + l\beta + i(l\omega - r\omega\beta). \tag{E.A.1}$$

Thus there exists an i_0 such that for all $i \geq i_0$ the object $\mathcal{O}(-i)[1]$ lies in $\mathcal{P}(0,a]$. Let us assume (after relabelling) that already $i_0 = 1$ is sufficient.

Consider a morphism of sheaves

$$\mathcal{O}(-i)^{\oplus m} \xrightarrow{\alpha} \mathcal{O}(-j)^{\oplus n}.$$

Since $\mathcal{F}_{\omega,\beta}$ is the free part of a torsion pair and hence closed under subobjects, the kernel $K = \mathrm{Ker}(\alpha)$ lies in $\mathcal{F}_{\omega,\beta}$. Similarly, $R = \mathrm{Image}(\alpha)$ is a subsheaf of $\mathcal{O}(-j)^{\oplus n}$ and lies in $\mathcal{F}_{\omega,\beta}$. Therefore the distinguished triangle

$$K[1] \to \mathcal{O}(-i)^{\oplus m}[1] \to R[1]$$

in $D^b(S)$ yields a short exact sequence in $\mathcal{P}(0,1]$. In particular, $K[1] \in \mathcal{P}(0,a]$.

Let $E \in D^b(S)$ be an object. Using the line bundles $\mathcal{O}(-i)$ we can find a quasi-isomorphism $O_E \xrightarrow{\simeq} E$ in the homotopy category $K(S) = K(\mathrm{Coh}(S))$, where $O_E = (\dots O_E^{i-1} \to O_E^i \to \dots)$ is a (possibly only bounded above) complex whose components are all direct sums of the line bundles $\mathcal{O}(-i)$ for $i > 0$. Let c be the smallest integer such that the cohomology $\mathcal{H}^c(E) \in \mathrm{Coh}(S)$ is not isomorphic to zero. Define a new complex

$$F_E = (\dots 0 \to \mathrm{Ker}(\partial^{c-1}) \to O_E^c \to O_E^{c+1} \to \dots).$$

This is a subcomplex of O_E which is bounded and the composition yields a quasi-isomorphism $F_E \xrightarrow{\simeq} E$.

From the above discussion we infer that $F_E[1]$ is a bounded complex whose components all lie inside $\mathcal{P}(0,a]$. In particular, the complex $F_E[2]$ viewed inside $K^b(\mathcal{P}(1,1+a])$ is an element in $D^b(\mathcal{A})$. This shows that the realization functor

$$\Phi\colon D^b(\mathcal{A}) \to D^b(\mathcal{P}(0,1]) \cong D^b(S)$$

is essentially surjective. Invoking [55, Thm. A] finishes the proof. ☐

Corollary E.A.4. *For all $\sigma = (\mathcal{A}, Z) \in U(S)$ we have $D^b(\mathcal{A}) \cong D^b(S)$.*

Proof. Any $\sigma \in U(S)$ is a $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$-translate of a unique $\tau \in V(S)$. Thus we have $\mathcal{A} = \mathcal{P}(a, a+1]$ for some $a \in \mathbb{R}$, where $\mathcal{P}$ is the slicing corresponding to τ. The assertion follows from Proposition E.A.3. ☐

Proof of Proposition E.A.1. Corollary E.A.4 proves the assertion for abelian surfaces. Hence we can assume that S is a K3 surface.

If $\Phi\colon D^b(S) \to D^b(S)$ is a derived auto-equivalence and $\mathcal{A} \subset D^b(S)$ is a heart, then the restriction $\Phi|_{\mathcal{A}}\colon \mathcal{A} \to \Phi(\mathcal{A})$ induces an equivalence $D^b(\mathcal{A}) \cong D^b(\Phi(\mathcal{A}))$. Hence $D^b(\mathcal{A}) \cong D^b(S)$ if and only of $D^b(\Phi(\mathcal{A})) \cong D^b(S)$. Moreover any auto-equivalence commutes with the $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$-action. Since, as discussed earlier, any stability condition in $\mathrm{Stab}^\dagger(S)$ can be mapped by an auto-equivalence into the closure of $U(S)$, and we know the claim for elements in the interior of $U(S)$ by Corollary E.A.4, we may therefore assume that σ lies on the boundary of $U(S)$.

As σ is contained in $\overline{U(S)}$, all skyscraper sheaves $\mathbb{C}_x$ are semistable. After applying an element of $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$ we may further assume that all skyscraper sheaves have phase 1 with respect to σ.

Following ideas of [19] we will consider a stability condition $\sigma' = (\mathcal{A}', Z') \in U(S)$ such that skyscraper sheaves have slope 1 and approach $\sigma = (\mathcal{A}, Z) \in \partial U(S)$ by first deforming only the real part of Z' and afterwards the imaginary part of the central charge

Concretely, consider the covering map $\pi\colon \mathrm{Stab}^\dagger(S) \to \mathcal{P}_0^+(S) \subset \Lambda_{\mathrm{alg}}^G \otimes \mathbb{C}$ and choose an open ball $B \subset \mathcal{P}_0^+(S)$ of small radius containing Z. Choose a stability condition $\sigma' = (\mathcal{A}', Z') \in U(S)$ such that skyscraper sheaves have slope 1 and such that the line from Z' to $\Re Z + \Im Z'$ and the line from $\Re Z + \Im Z'$ to Z viewed in the vector space $\Lambda_{\mathrm{alg}}^G \otimes \mathbb{C}$ are contained inside B. Let $\tilde{Z}$ be the stability function $\Re Z + \Im Z'$ and let $\tilde{\sigma} = (\tilde{\mathcal{A}}, \tilde{Z})$ be the stability condition obtained from the covering property of π. By construction all skyscraper sheaves remain of phase 1 along this deformation from σ to σ'.

The crucial observation now is that the stability condition $\tilde{\sigma}$ is still contained in the open subset $U(S)$. Indeed, recall that the set $U(S)$ can be characterized as the set of all stability conditions for which all skyscraper sheaves $\mathbb{C}_x$ are stable of the same phase. Assume that a skyscraper sheaf $\mathbb{C}_x$ becomes unstable along the line segment from Z' to $\tilde{Z}$. Since semistablity is a closed property, there would have to exist a τ on this line segment where $\mathbb{C}_x$ becomes semistable. Since the imaginary part of the central charges stays constant along the path, $\mathbb{C}_x$ is still contained in the abelian category $\mathcal{P}(1)$, where $\mathcal{P}$ is the slicing associated to τ. As $\mathbb{C}_x$ is semistable, there exists a stable object $F \in \mathcal{P}(1)$ and a non-zero morphism $F \to \mathbb{C}_x$ which is not an isomorphism. Since being stable is an open property [15, Prop. 2.10], the object F was also stable for a stability condition on the line segment where $\mathbb{C}_x$ is stable. However, a morphism between stable objects of the same phase is either an isomorphism or 0, yielding a contradiction. We conclude that $\tilde{\sigma} \in U(S)$.

Let $\tilde{\mathcal{P}}$ be the the slicing associated to $\tilde{\sigma}$. Then as argued in [19, Lem. 5.2] the abelian category $\tilde{\mathcal{A}} = \tilde{\mathcal{P}}(1/2, 3/2]$ is constant along a deformation that only changes the imaginary part of the stability condition. This yields $\mathcal{P}(1/2, 3/2] = \tilde{\mathcal{A}}$, where $\mathcal{P}$ is the slicing associated to σ. Let $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ denote the rotation by $\pi/2$. Then $\tilde{\mathcal{A}}$ is the heart of both $g\tilde{\sigma}$ and $g\sigma$. Since $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ preserves $U(S)$, we have $g\tilde{\sigma} \in U(S)$ and therefore by Corollary E.A.4 we conclude that $D^b(\tilde{\mathcal{A}}) \cong D^b(S)$. ☐

Remark E.A.5. Given an algebraic stability condition $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}^\dagger(S)$, the proof above shows that in Proposition E.A.1 one can choose the element g such that $g\sigma$ is algebraic as well. Indeed, this is immediate for stability conditions which are mapped by some auto-equivalence into $U(S)$. For $\sigma \in \partial U(S)$, we first applied an element from $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ so that skyscraper sheaves get mapped to -1 and then applied the rotation by $\pi/2$. If σ is algebraic, both steps can be achieved by multiplying Z with elements from $\mathbb{Q} + i\mathbb{Q}$.

E.B. The Euler characteristic of fixed loci

We state a result which may be viewed as a numerical version of Theorems E.1.3:

Let $M = M_\sigma(v)$ be a moduli space of stable objects of Mukai vector v on a K3 surface S and let $g\colon M \to M$ be a symplectic automorphism of finite order. Let $\pi_g = \prod_a a^{m(a)}$ be the frameshape of the induced action on the Mukai lattice Λ (obtained from lifting the action on $H^2(M, \mathbb{Z})$ to Λ, see Section E.6.4). We define the modular form

$$f_g(q) = \prod_a \eta(q^a)^{m(a)} = q + O(q^2),$$

where $\eta(q) = q^{1/24} \prod_{m \geq 1}(1-q^m)$ is the Dedekind elliptic function.

Proposition E.B.1. *$e(M_\sigma(v)^g) =$ Coefficient of $q^{v \cdot v/2}$ of $f_g(q)^{-1}$.*

Here $e(Z)$ denotes the topological Euler characteristic of a finite type scheme Z. If M is the Hilbert scheme of points and the automorphism is induced by an automorphism of the underlying surface, this follows by a local analysis, see [45] and also [44] for the extension to non-cyclic groups. The general case is evidence for an affirmative answer to Question E.1.5.

Proof. We prove the claim by computing the trace of the induced automorphism

$$g_* \colon H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{Z})$$

which by definition is a monodromy operator. Recall that the Zariski closure of the monodromy group in $O(H^*(M, \mathbb{C}))$ is canonically isomorphic to $O(H^2(M, \mathbb{C})) \times \mathbb{Z}_2$, see [136, Lem. 4.11] and also [167, Sec. 1.2]. Let $\psi \colon \Lambda \to \Lambda$ denote the unique lift of $g^*|_{H^2(M,\mathbb{Z})}$ to an automorphism of the Mukai lattice. By a result of Mongardi [151, Thm. 26] the lift ψ fixes $v \in \Lambda$. Hence g^* lies in fact in $O(H^2(M, \mathbb{C})) \times 1$ under the above isomorphism.

It hence remains to prove that, given an element $\varphi \in \mathrm{O}(H^2(M, \mathbb{C})) \times 1$ of finite order whose extension $\widetilde{\varphi} : \Lambda \otimes \mathbb{C} \to \Lambda \otimes \mathbb{C}$ has frameshape $\prod_a a^{m(a)}$ (where we let $\widetilde{\varphi}$ act by the identity on $H^2(M, \mathbb{C})^\perp$), the trace of φ on $H^*(M, \mathbb{C})$ has the desired form. Since this is a purely topological question, we may assume $M = \mathsf{Hilb}_n(S)$ where $n = (v \cdot v)/2 + 1$. Moreover, after conjugation by an element of $\mathrm{SO}(H^2(M, \mathbb{C}))$ we may assume that $\widetilde{\varphi}$ preserves the decomposition by degree and acts as the identity on $H^0(S, \mathbb{C}) \oplus H^4(S, \mathbb{C})$. In particular, $\widetilde{\varphi}$ induces an action on $H^*(\mathsf{Hilb}_k(S))$ for all k. As explained in [167, Sec. 1.3], the Nakajima operators are equivariant with respect to the action of $\widetilde{\varphi}$ on $\Lambda \otimes \mathbb{C}$ and $H^*(\mathsf{Hilb}_k(S))$. If V_i are the eigenspace of $\widetilde{\varphi}$ on $\Lambda \otimes \mathbb{C}$ with eigenvalue λ_i, this yields the equivariant decomposition $\oplus_{k \geq 0} H^*(\mathsf{Hilb}_k(S)) \cong \otimes_{i=1}^{24} \mathrm{Sym}^\bullet(V_i)$ and thus

$$\sum_{k \geq 0} \mathrm{Tr}\left(\widetilde{\varphi}|_{H^*(\mathsf{Hilb}_k(S))}\right) q^n = \prod_{m \geq 1} \prod_{i=1}^{24} \frac{1}{(1-\lambda_i q^m)} = \prod_{n \geq 1} \prod_{a \geq 1} \left(\frac{1}{1-q^{an}}\right)^{m(a)}$$

where the last equality follows from a direct computation. ☐

F. Integral Fourier transforms and the integral Hodge conjecture for one-cycles on abelian varieties

Abstract. We prove the integral Hodge conjecture for one-cycles on a principally polarized complex abelian variety whose minimal class is algebraic. In particular, the Jacobian of a smooth projective curve over the complex numbers satisfies the integral Hodge conjecture for one-cycles. The main ingredient is a lift of the Fourier transform to integral Chow groups. Similarly, we prove the integral Tate conjecture for one-cycles on the Jacobian of a smooth projective curve over the separable closure of a finitely generated field. Furthermore, abelian varieties satisfying such a conjecture are dense in their moduli space.

F.1. Introduction

Let g be a positive integer and let A be an abelian variety of dimension g over a field k with dual abelian variety $\widehat{A}$. The correspondence attached to the Poincaré bundle $\mathcal{P}_A$ on $A \times \widehat{A}$ defines a powerful duality between the derived categories, rational Chow groups and cohomology of A and $\widehat{A}$ [20, 97, 154]. We shall refer to such morphisms as *Fourier transforms*.

On the level of cohomology, the Fourier transform preserves integral ℓ-adic étale cohomology when $k = k_s$ and integral Betti cohomology when $k = \mathbb{C}$. It is thus natural to ask whether the Fourier transform on rational Chow groups preserves integral cycles modulo torsion or, more generally, lifts to a homomorphism between integral Chow groups. This question was raised by Moonen–Polishchuk [153] and Totaro [206]. More precisely, Moonen and Polishchuk gave a counterexample for abelian varieties over non-closed fields and asked about the case of algebraically closed fields.

In this paper we further investigate this question with a view towards applications concerning the integral Hodge conjecture for one-cycles when A is defined over $\mathbb{C}$. To state our main result, we recall that whenever $\iota\colon C \hookrightarrow A$ is a smooth curve, the image of the fundamental class under the pushforward map $\iota_*\colon \mathrm{H}_2(C,\mathbb{Z}) \to \mathrm{H}_2(A,\mathbb{Z}) \cong \mathrm{H}^{2g-2}(A,\mathbb{Z})$ defines a cohomology class $[C] \in \mathrm{H}^{2g-2}(A,\mathbb{Z})$. This construction extends to one-cycles and factors modulo rational equivalence. As such, it induces a canonical homomorphism, called the *cycle class map*,

$$cl\colon \mathrm{CH}_1(A) \to \mathrm{Hdg}^{2g-2}(A,\mathbb{Z}),$$

which is a direct summand of a natural graded ring homomorphism $cl\colon \mathrm{CH}(A) \to \mathrm{H}^\bullet(A,\mathbb{Z})$.

The liftability of the Fourier transform turns out to have important consequences for the image of the cycle class map. Recall that an element $\alpha \in \mathrm{H}^\bullet(A,\mathbb{Z})$ is called *algebraic* if it is in the image of cl, and that A satisfies the *integral Hodge conjecture for k-cycles* if all Hodge classes in $\mathrm{H}^{2g-2k}(A, \mathbb{Z})$ are algebraic. Although the integral Hodge conjecture fails in general [12, 14, 205], it is an open question for abelian varieties. Our main result is as follows.

Theorem F.1.1. *Let A be a complex abelian variety of dimension g with Poincaré bundle $\mathcal{P}_A$. The following three statements are equivalent:*

(i) *The cohomology class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic.*

(ii) *The Chern character $\mathrm{ch}(\mathcal{P}_A) = \exp(c_1(\mathcal{P}_A)) \in \mathrm{H}^\bullet(A \times \widehat{A}, \mathbb{Z})$ is algebraic.*

(iii) *The integral Hodge conjecture for one-cycles holds for $A \times \widehat{A}$.*

Any of these statements implies that

(iv) *The integral Hodge conjecture for one-cycles holds for A and $\widehat{A}$.*

Suppose that A is principally polarized by $\theta \in \mathrm{Hdg}^2(A, \mathbb{Z})$ and consider the following statements:

(v) *The minimal cohomology class $\gamma_\theta := \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ is algebraic.*

(vi) *The cohomology class $c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}(A \times \widehat{A}, \mathbb{Z})$ is algebraic.*

Then statements (i) – (vi) *are equivalent. If they hold, then $\theta^i/i! \in \mathrm{H}^{2i}(A, \mathbb{Z})$ is algebraic for $i \geq 1$.*

Remark that Condition (v) is stable under products, so a product of principally polarized abelian varieties satisfies the integral Hodge conjecture for one-cycles if and only if each of the factors does. More importantly, if $J(C)$ is the Jacobian of a smooth projective curve C of genus g, then every integral Hodge class of degree $2g-2$ on $J(C)$ is a $\mathbb{Z}$-linear combination of curves classes:

Theorem F.1.2. *Let $C_1, \ldots, C_n$ be smooth projective curves over $\mathbb{C}$. Then the integral Hodge conjecture for one-cycles holds for the product of Jacobians $J(C_1) \times \cdots \times J(C_n)$.*

See Remark F.4.2.(i) for another approach towards Theorem F.1.2 in the case $n = 1$. A second consequence of Theorem F.1.1 is that the integral Hodge conjecture for one-cycles on principally polarized complex abelian varieties is stable under specialization, see Corollary F.4.3. An application of somewhat different nature is the following density result, proven in Section F.4.2:

Theorem F.1.3. *Let $\delta = (\delta_1, \ldots, \delta_g)$ be positive integers such that $\delta_i | \delta_{i+1}$ and let $\mathsf{A}_{g,\delta}(\mathbb{C})$ be the coarse moduli space of polarized abelian varieties over $\mathbb{C}$ with polarization type δ. There is a countable union $X \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ of closed algebraic subvarieties of dimension at least g, that satisfies the following property: X is dense in the analytic topology and the integral Hodge conjecture for one-cycles holds for those polarized abelian varieties whose isomorphism class lies in X.*

Remark F.1.4. The lower bound that we obtain on the dimension of the components of X actually depends on δ and is often greater than g. For instance, when $\delta = 1$ and $g \geq 2$, there is a set X as in the theorem, whose elements are prime-power isogenous to products of Jacobians of curves. Therefore, the components of X have dimension $3g-3$ in this case, c.f. Remark F.4.7.

One could compare Theorem F.1.1 with the following statement, proven by Grabowski [80]: if g is a positive integer such that the minimal cohomology class $\gamma_\theta = \theta^{g-1}/(g-1)!$ of every principally polarized abelian variety of dimension g is algebraic, then every abelian variety of dimension g satisfies the integral Hodge conjecture for one-cycles. In this way, he proved the integral Hodge conjecture for abelian threefolds, a result which has been extended to smooth projective threefolds X with $K_X = 0$ by Voisin and Totaro [206, 213]. For abelian varieties of dimension greater than three, not many unconditional statements seem to have been known.

The idea behind the proof of Theorem F.1.1 is the following. Let A be a complex abelian variety of dimension g and let $i \geq 0$ be an integer. Then Poincaré duality induces a canonical isomorphism $\varphi \colon \mathrm{H}^{2i}(A, \mathbb{Z}) \cong \mathrm{H}^{2g-2i}(A, \mathbb{Z})^\vee \cong \mathrm{H}^{2g-2i}(\widehat{A}, \mathbb{Z})$. The map φ respects the Hodge structures and thus induces an isomorphism $\mathrm{Hdg}^{2i}(A, \mathbb{Z}) \cong \mathrm{Hdg}^{2g-2i}(\widehat{A}, \mathbb{Z})$. However, it is unclear a priori whether φ sends algebraic classes to algebraic classes. We prove that the algebraicity of $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ forces φ to be algebraic, i.e. to be induced by a correspondence $\Gamma \in \mathrm{CH}(A \times \widehat{A})$. In particular, one then has $\mathrm{Z}^{2i}(A) := \mathrm{Hdg}^{2i}(A, \mathbb{Z})/\mathrm{H}^{2i}(A, \mathbb{Z})_{\mathrm{alg}} \cong \mathrm{Z}^{2g-2i}(\widehat{A})$. To prove this, we lift the cohomological Fourier transform to a homomorphism between integral Chow groups whenever $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is algebraic. For this we use a theorem of Moonen–Polishchuk saying that the ideal of positive dimensional cycles in the Chow ring with Pontryagin product of an abelian variety admits a divided power structure [153, Theorem 1.6].

In Section F.5, we consider an abelian variety $A_{/\mathbb{C}}$ of dimension g and ask: if $n \in \mathbb{Z}_{\geq 1}$ is such that $n \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})_{\mathrm{alg}}$, can we bound the order of $\mathrm{Z}^{2g-2}(A)$ in terms of g and n? For a smooth complex projective d-dimensional variety X, $\mathrm{Z}^{2d-2}(X)$ is called the degree $2d-2$ *Voisin group* of X [180], is a stably birational invariant [215, Lemma 2.20], and related to the unramified cohomology groups by Colliot-Thélène–Voisin and Schreieder [57, 195]. We prove that if $n \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is algebraic, then $\gcd(n^2, (2g-2)!) \cdot \mathrm{Z}^{2g-2}(A) = (0)$. In particular, $(2g-2)! \cdot \mathrm{Z}^{2g-2}(A) = (0)$ for any g-dimensional complex abelian variety A. Moreover, if A is principally polarized by $\theta \in \mathrm{NS}(A)$ and if $n \cdot \gamma_\theta \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ is algebraic, then $n \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is algebraic. Since it is well known that for Prym varieties, the Hodge class $2 \cdot \gamma_\theta$ is algebraic, these observations lead to the following result (see also Theorem F.5.3).

Theorem F.1.5. *Let A be a g-dimensional Prym variety over $\mathbb{C}$. Then $4 \cdot \mathrm{Z}^{2g-2}(A) = (0)$.*

For the study of the liftability of the Fourier transform, which was initiated by Moonen and Polishchuk in [153], it is more natural to consider abelian varieties defined over arbitrary fields. For this reason we define and study integral Fourier transforms in this generality, see Section F.3. We provide, for an abelian variety principally polarized by a symmetric ample line bundle, necessary and sufficient conditions for an integral Fourier transform to exist, see Theorem F.3.8.

This generality also allows to obtain the analogue of Theorem F.1.1 over the separable closure k of a finitely generated field. Recall that a smooth projective variety X of dimension d over k satisfies the *integral Tate conjecture for one-cycles over k* if, for every prime number ℓ different from $\mathrm{char}(k)$ and for some finitely generated field of definition $k_0 \subset k$ of X, the cycle class map

$$cl\colon \mathrm{CH}_1(X)_{\mathbb{Z}_\ell} = \mathrm{CH}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \bigcup_U \mathrm{H}^{2d-2}_{\text{ét}}(X, \mathbb{Z}_\ell(d-1))^U \tag{F.1.1}$$

is surjective, where U ranges over the open subgroups of $\mathrm{Gal}(k/k_0)$.

Theorem F.1.6. *Let A be an abelian variety of dimension g over the separable closure k of a finitely generated field.*

- *The abelian variety A satisfies the integral Tate conjecture for one-cycles over k if the cohomology class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}_{\text{ét}}(A \times \widehat{A}, \mathbb{Z}_\ell(2g-1))$ is the class of a one-cycle with $\mathbb{Z}_\ell$-coefficients for every prime number $\ell < (2g-1)!$ unequal to* $\mathrm{char}(k)$.

- *Suppose that A is principally polarized and let $\theta_\ell \in \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1))$ be the class of the polarization. The map (F.1.1) is surjective if $\gamma_{\theta_\ell} := \theta_\ell^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A, \mathbb{Z}_\ell(g-1))$ is in its image. In particular, if $\ell > (g-1)!$, then this always holds. Thus A satisfies the integral Tate conjecture for one-cycles if γ_{θ_ℓ} is in the image of (F.1.1) for every prime number $\ell < (g-1)!$ unequal to* $\mathrm{char}(k)$.

Theorem F.1.6 implies in particular that products of Jacobians of smooth projective curves over k satisfy the integral Tate conjecture for one-cycles over k. Moreover, for an abelian variety A_K over a number field $K \subset \mathbb{C}$, the integral Hodge conjecture for one-cycles on $A_{\mathbb{C}}$ is equivalent to the integral Tate conjecture for one-cycles on $A_{\bar{K}}$ (Corollary F.6.2), which in turn implies the integral Tate conjecture for one-cycles on the geometric special fiber $A_{\overline{k(\mathfrak{p})}}$ of the Néron model of A_K over $\mathcal{O}_K$ for any prime $\mathfrak{p} \subset \mathcal{O}_K$ at which A_K has good reduction (Corollary F.6.3).

Finally, Theorem F.1.3 has an analogue in positive characteristic. The definition for a smooth projective variety over the algebraic closure k of a finitely generated field to satisfy the *integral Tate conjecture for one-cycles over k* is analogous to the definition above (see e.g. [53]).

Theorem F.1.7. *Let k be the algebraic closure of a finitely generated field of characteristic $p > 0$. Let A_g be the coarse moduli space over k of principally polarized abelian varieties of dimension g over k. The subset of $\mathsf{A}_g(k)$ of isomorphism classes of principally polarized abelian varieties over k that satisfy the integral Tate conjecture for one-cycles over k is Zariski dense in A_g.*

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We stress that work of Moonen and Polishchuk [153] has been essential for our results.

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F.2. Notation

- We let k be a field with separable closure k_s and ℓ a prime number different from the characteristic of k. For a smooth projective variety X over k, we let $\mathrm{CH}(X)$ be the Chow group of X and define $\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X) \otimes \mathbb{Q}$, $\mathrm{CH}(X)_{\mathbb{Q}_\ell} = \mathrm{CH}(X) \otimes \mathbb{Q}_\ell$ and $\mathrm{CH}(X)_{\mathbb{Z}_\ell} = \mathrm{CH}(X) \otimes \mathbb{Z}_\ell$. We let $\mathrm{H}^i_{\text{ét}}(X_{k_s}, \mathbb{Z}_\ell(a))$ be the i-th degree étale cohomology group with coeffients in $\mathbb{Z}_\ell(a)$, $a \in \mathbb{Z}$.

- Often, A will denote an abelian variety of dimension g over k, with dual abelian variety $\widehat{A}$ and (normalized) Poincaré bundle $\mathcal{P}_A$ on $A \times \widehat{A}$. The abelian group $\mathrm{CH}(A)$ will in that case be equipped with two ring structures: the usual intersection product $\cdot$ as well as the Pontryagin product $\star$. Recall that the latter is defined as follows:

$$\star \colon \mathrm{CH}(A) \times \mathrm{CH}(A) \to \mathrm{CH}(A), \quad x \star y = m_*(\pi_1^*(x) \cdot \pi_2^*(y)).$$

Here, as well as in the rest of the paper, π_i denotes the projection onto the i-th factor, $\Delta \colon A \to A \times A$ the diagonal morphism, and $m \colon A \times A \to A$ the group law morphism of A.

- For any abelian group M and any element $x \in M$, we will denote by $x_{\mathbb{Q}} \in M \otimes_{\mathbb{Z}} \mathbb{Q}$ the image of x in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ under the canonical homomorphism $M \to M \otimes_{\mathbb{Z}} \mathbb{Q}$.

F.3. Integral Fourier transforms and one-cycles on abelian varieties

Our goal in this section is to provide necessary and sufficient conditions for the Fourier transform on rational Chow groups or cohomology to lift to a motivic homomorphism between integral Chow groups. We will relate such lifts to the integral Hodge conjecture when $k = \mathbb{C}$. In subsequent Section F.4 we will use the theory developed in this section to prove Theorem F.1.1.

F.3.1. Integral Fourier transforms and integral Hodge classes

For abelian varieties A over $k = k_s$, it is unknown whether the Fourier transform

$$\mathcal{F}_A \colon \mathrm{CH}(A)_{\mathbb{Q}} \to \mathrm{CH}(\widehat{A})_{\mathbb{Q}}$$

preserves the subgroups of integral cycles modulo torsion. A sufficient condition for this to hold is that $\mathcal{F}_A$ lifts to a homomorphism $\mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$. In this section we outline a second consequence of such a lift $\mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ when A is defined over the complex numbers: the existence of an integral lift of $\mathcal{F}_A$ implies the integral Hodge conjecture for one-cycles on $\widehat{A}$.

Let A be an abelian variety over k. The Fourier transform on the level of Chow groups is the group homomorphism

$$\mathcal{F}_A \colon \mathrm{CH}(A)_{\mathbb{Q}} \to \mathrm{CH}(\widehat{A})_{\mathbb{Q}}$$

induced by the correspondence $\mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$, where $\mathrm{ch}(\mathcal{P}_A)$ is the Chern character of $\mathcal{P}_A$. Similarly, one defines the Fourier transform on the level of étale cohomology:

$$\mathcal{F}_A \colon \mathrm{H}^{\bullet}_{\text{ét}}(A_{k_s}, \mathbb{Q}_\ell(\bullet)) \to \mathrm{H}^{\bullet}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Q}_\ell(\bullet)).$$

In fact, $\mathcal{F}_A$ preserves the integral cohomology classes and induces, for each integer j with $0 \leq j \leq 2g$, an isomorphism [20, Proposition 1], [206, page 18]:

$$\mathcal{F}_A \colon \mathrm{H}^{j}_{\text{ét}}(A_{k_s}, \mathbb{Z}_\ell(a)) \to \mathrm{H}^{2g-j}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Z}_\ell(a+g-j)).$$

Similarly, if $k = \mathbb{C}$, then $\mathrm{ch}(\mathcal{P}_A)$ induces, for each integer i with $0 \leq i \leq 2g$, an isomorphism of Hodge structures

$$\mathcal{F}_A \colon \mathrm{H}^i(A, \mathbb{Z}) \to \mathrm{H}^{2g-i}(\widehat{A}, \mathbb{Z})(g-i). \tag{F.3.1}$$

In [153], Moonen and Polishchuk consider an isomorphism $\phi \colon A \xrightarrow{\sim} \widehat{A}$, a positive integer d, and define the notion of motivic integral Fourier transform of (A, ϕ) up to factor d. The definition goes as follows. Let $\mathcal{M}(k)$ be the category of effective Chow motives over k with respect to ungraded correspondences, and let $h(A)$ be the motive of A. Then a morphism $\mathcal{F} \colon h(A) \to h(A)$ in $\mathcal{M}(k)$ is a *motivic integral Fourier transform of* (A, ϕ) *up to factor* d if the following three conditions are satisfied: (i) the induced morphism $h(A)_{\mathbb{Q}} \to h(A)_{\mathbb{Q}}$ is the composition of the usual Fourier transform with the isomorphism $\phi^* \colon h(\widehat{A})_{\mathbb{Q}} \xrightarrow{\sim} h(A)_{\mathbb{Q}}$, (ii) one has $d \cdot \mathcal{F} \circ \mathcal{F} = d \cdot (-1)^g \cdot [-1]_*$ as morphisms from $h(A)$ to $h(A)$, and (iii) $d \cdot \mathcal{F} \circ m_* = d \cdot \Delta^* \circ \mathcal{F} \otimes \mathcal{F} \colon h(A) \otimes h(A) \to h(A)$.

For our purposes, we will consider similar homomorphisms $\mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$. However, to make their existence easier to verify (c.f. Theorem F.3.8) we relax some of the above conditions:

Definition F.3.1. Let $A_{/k}$ be an abelian variety and let $\mathcal{F} \colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ be a group homomorphism. We call $\mathcal{F}$ a *weak integral Fourier transform* if the following diagram commutes:

$$\begin{tikzcd} \mathrm{CH}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}(\widehat{A}) \arrow[d] \\ \mathrm{CH}(A)_{\mathbb{Q}} \arrow[r, "\mathcal{F}_A"] & \mathrm{CH}(\widehat{A})_{\mathbb{Q}}. \end{tikzcd} \tag{F.3.2}$$

We call a weak integral Fourier transform $\mathcal{F}$ *motivic* if it is induced by a cycle $\Gamma \in \mathrm{CH}(A \times \widehat{A})$ that satisfies $\Gamma_{\mathbb{Q}} = \mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$. A group homomorphism $\mathcal{F} \colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ is an *integral Fourier transform up to homology* if the following diagram commutes:

$$\begin{tikzcd} \mathrm{CH}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}(\widehat{A}) \arrow[d] \\ \oplus_{r \geq 0} \mathrm{H}^{2r}_{\text{ét}}(A_{k_s}, \mathbb{Z}_\ell(r)) \arrow[r, "\mathcal{F}_A"] & \oplus_{r \geq 0} \mathrm{H}^{2r}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Z}_\ell(r)). \end{tikzcd} \tag{F.3.3}$$

Similarly, a $\mathbb{Z}_\ell$-module homomorphism $\mathcal{F}_\ell \colon \mathrm{CH}(A)_{\mathbb{Z}_\ell} \to \mathrm{CH}(\widehat{A})_{\mathbb{Z}_\ell}$ is called an *ℓ-adic integral Fourier transform up to homology* if $\mathcal{F}_\ell$ is compatible with $\mathcal{F}_A$ and the ℓ-adic cycle class maps. Finally, an integral Fourier transform up to homology $\mathcal{F}$ (resp. an ℓ-adic integral Fourier transform up to homology $\mathcal{F}_\ell$) is called *motivic* if it is induced by a cycle $\Gamma \in \mathrm{CH}(A \times \widehat{A})$ (resp. $\Gamma_\ell \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Z}_\ell}$) such that $cl(\Gamma)$ (resp. $cl(\Gamma_\ell)$) equals $\mathrm{ch}(\mathcal{P}_A) \in \oplus_{r \geq 0} \mathrm{H}^{2r}_{\text{ét}}((A \times \widehat{A})_{k_s}, \mathbb{Z}_\ell(r))$.

Remark F.3.2. If $\mathcal{F} \colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ is a weak integral Fourier transform, then $\mathcal{F}$ is an integral Fourier transform up to homology, the $\mathbb{Z}_\ell$-module $\oplus_{r \geq 0} \mathrm{H}^{2r}_{\text{ét}}(\widehat{A}_{k_s}, \mathbb{Z}_\ell(r))$ being torsion-free. If $k = \mathbb{C}$, then $\mathcal{F} \colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ is an integral Fourier transform up to homology if and only if $\mathcal{F}$ is compatible with the Fourier transform $\mathcal{F}_A \colon \mathrm{H}^\bullet(A, \mathbb{Z}) \to \mathrm{H}^\bullet(\widehat{A}, \mathbb{Z})$ on Betti cohomology.

The relation between integral Fourier transforms and integral Hodge classes is as follows:

Lemma F.3.3. *Let A be a complex abelian variety and $\mathcal{F} \colon \mathrm{CH}(A) \to \mathrm{CH}(\widehat{A})$ an integral Fourier transform up to homology. For each $i \in \mathbb{Z}_{\geq 0}$, the integral Hodge conjecture for degree $2i$ classes on A implies the integral Hodge conjecture for degree $2g - 2i$ classes on $\widehat{A}$. If $\mathcal{F}$ is motivic, then $\mathcal{F}_A$ induces a group isomorphism $\mathrm{Z}^{2i}(A) \xrightarrow{\sim} \mathrm{Z}^{2g-2i}(\widehat{A})$ and, therefore, the integral Hodge conjectures for degree $2i$ classes on A and degree $2g - 2i$ classes on $\widehat{A}$ are equivalent for all i.*

Proof. We can extend Diagram (F.3.3) to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^i(A) \arrow[r] \arrow[d, "cl^i"] & \mathrm{CH}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}(\widehat{A}) \arrow[r] \arrow[d] & \mathrm{CH}_i(\widehat{A}) \arrow[d, "cl_i"] \\
\mathrm{H}^{2i}(A, \mathbb{Z}) \arrow[r] & \mathrm{H}^\bullet(A, \mathbb{Z}) \arrow[r, "\mathcal{F}_A"] & \mathrm{H}^\bullet(\widehat{A}, \mathbb{Z}) \arrow[r] & \mathrm{H}^{2g-2i}(\widehat{A}, \mathbb{Z}).
\end{tikzcd}$$

The composition $\mathrm{H}^{2i}(A, \mathbb{Z}) \to \mathrm{H}^{2g-2i}(\widehat{A}, \mathbb{Z})$ appearing on the bottom line agrees up to a suitable Tate twist with the map $\mathcal{F}_A$ of Equation (F.3.1). Therefore, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^i(A) \arrow[r] \arrow[d, "cl^i"] & \mathrm{CH}_i(\widehat{A}) \arrow[d, "cl_i"] \\
\mathrm{Hdg}^{2i}(A, \mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Hdg}^{2g-2i}(\widehat{A}, \mathbb{Z}).
\end{tikzcd} \tag{F.3.4}$$

Thus the surjectivity of cl^i implies the surjectivity of cl_i. Moreover, if $\mathcal{F}$ is motivic, then replacing A by $\widehat{A}$ and $\widehat{A}$ by $\widehat{\widehat{A}}$ in the argument above shows that the images of cl^i and cl_i are identified under the isomorphism $\mathcal{F}_A \colon \mathrm{Hdg}^{2i}(A, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hdg}^{2g-2i}(\widehat{A}, \mathbb{Z})$ in Diagram (F.3.4). ☐

F.3.2. Properties of the Fourier transform on rational Chow groups

The above suggests that to prove Theorem F.1.1, one would need to show that for a complex abelian variety of dimension g whose minimal Poincaré class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic, all classes of the form $c_1(\mathcal{P}_A)^i/i! \in \mathrm{H}^{2i}(A \times \widehat{A}, \mathbb{Z})$ are algebraic. With this goal in mind we shall study Fourier transforms on rational Chow groups in Section

F.3.2, and investigate how these relate to $\mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$. In turns out that the cycles $c_1(\mathcal{P}_A)^i/i! \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$ satisfy several relations that are very similar to those proved by Beauville for the cycles $\theta^i/i! \in \mathrm{CH}(A)_{\mathbb{Q}}$ in case A is principally polarized, see [20]. Since we will need these results in any characteristic in order to prove Theorem F.1.6, we will work over our general field k, see Section F.2.

Let A be an abelian variety over k. Define cycles $\ell = c_1(\mathcal{P}_A) \in \mathrm{CH}^1(A \times \widehat{A})_{\mathbb{Q}}$ and $\mathcal{R}_A = c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}$. Dually, define $\widehat{\ell} = c_1(\mathcal{P}_{\widehat{A}}) \in \mathrm{CH}^1(\widehat{A} \times A)_{\mathbb{Q}}$ and $\mathcal{R}_{\widehat{A}} = c_1(\mathcal{P}_{\widehat{A}})^{2g-1}/(2g-1)! \in \mathrm{CH}_1(\widehat{A} \times A)_{\mathbb{Q}}$. For $a \in \mathrm{CH}(A)_{\mathbb{Q}}$, define $\mathrm{E}(a) \in \mathrm{CH}(A)_{\mathbb{Q}}$ as the $\star$-exponential of a:

$$\mathrm{E}(a) := \sum_{n \geq 0} \frac{a^{\star n}}{n!} \in \mathrm{CH}(A)_{\mathbb{Q}}.$$

The key to Theorem F.1.1 will be the following:

Lemma F.3.4. *We have* $\mathrm{ch}(\mathcal{P}_A) = e^{\ell} = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathcal{R}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$.

Proof. The most important ingredient in the proof is the following:

Claim $(*)$: With respect to the Fourier transform $\mathcal{F}_{A\times\widehat{A}} \colon \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}} \to \mathrm{CH}(\widehat{A} \times A)_{\mathbb{Q}}$, one has $\mathcal{F}_{A\times\widehat{A}}(e^{\ell}) = (-1)^g \cdot e^{-\widehat{\ell}} \in \mathrm{CH}(\widehat{A} \times A)_{\mathbb{Q}}$.

To prove Claim $(*)$, we lift the desired equality in the rational Chow group of $\widehat{A} \times A$ to an isomorphism in the derived category $\mathrm{D}^b(\widehat{A} \times A)$ of $\widehat{A} \times A$. For $X = A \times \widehat{A}$ the Poincaré line bundle $\mathcal{P}_X$ on $X \times \widehat{X} \cong A \times \widehat{A} \times \widehat{A} \times A$ is isomorphic to $\pi_{13}^*\mathcal{P}_A \otimes \pi_{24}^*\mathcal{P}_{\widehat{A}}$. Consider

$$\Phi_{\mathcal{P}_X}(\mathcal{P}_A) \cong \pi_{34,*}\left(\pi_{13}^*\mathcal{P}_A \otimes \pi_{24}^*\mathcal{P}_{\widehat{A}} \otimes \pi_{12}^*\mathcal{P}_A\right) \in \mathrm{D}^b(\widehat{A} \times A) \tag{F.3.5}$$

whose Chern character is exactly $\mathcal{F}_X(e^{\ell})$. Applying the pushforward along the permutation map

$$(123) \colon A \times \widehat{A} \times \widehat{A} \times A \cong \widehat{A} \times A \times \widehat{A} \times A$$

the object (F.3.5) becomes

$$\pi_{14,*}\left(\pi_{12}^*\mathcal{P}_{\widehat{A}} \otimes \pi_{23}^*\mathcal{P}_A \otimes \pi_{34}^*\mathcal{P}_{\widehat{A}}\right)$$

which is isomorphic to the Fourier–Mukai kernel of the composition

$$\Phi_{\mathcal{P}_{\widehat{A}}} \circ \Phi_{\mathcal{P}_A} \circ \Phi_{\mathcal{P}_{\widehat{A}}}.$$

Since $\Phi_{\mathcal{P}_A} \circ \Phi_{\mathcal{P}_{\widehat{A}}}$ is isomorphic to $[-1_{\widehat{A}}]^* \circ [-g]$ by [154, Theorem 2.2], we have

$$\Phi_{\mathcal{P}_{\widehat{A}}} \circ \Phi_{\mathcal{P}_A} \circ \Phi_{\mathcal{P}_{\widehat{A}}} \cong \Phi_{\mathcal{P}_{\widehat{A}}} \circ [-1_{\widehat{A}}]^* \circ [-g].$$

This is the Fourier–Mukai transform with kernel the object $\mathcal{E} = \mathcal{P}_{\widehat{A}}^{\vee}[-g] \in \mathrm{D}^b(\widehat{A} \times A)$. By uniqueness of the Fourier–Mukai kernel of an equivalence [175, Theorem 2.2] and the fact that the Chern character of $\mathcal{E}$ equals $(-1)^g \cdot e^{-\widehat{\ell}} \in \mathrm{CH}(\widehat{A} \times A)_{\mathbb{Q}}$, this finishes the proof of Claim $(*)$.

Next, we claim that $(-1)^g \cdot \mathcal{F}_{\widehat{A}\times A}(-\widehat{\ell}) = \mathcal{R}_A$. To see this, recall that for each integer i with $0 \leq i \leq g$, there is a canonical *Beauville decomposition* $\mathrm{CH}^i(A)_{\mathbb{Q}} = \oplus_{j=i-g}^{i} \mathrm{CH}^{i,j}(A)_{\mathbb{Q}}$ [22]. Since the Poincaré bundle $\mathcal{P}_A$ is symmetric, we have $\ell \in \mathrm{CH}^{1,0}(A \times \widehat{A})_{\mathbb{Q}}$ and hence $\ell^i \in \mathrm{CH}^{i,0}(A \times \widehat{A})_{\mathbb{Q}}$. In particular, we have $\mathcal{R}_A \in \mathrm{CH}^{2g-1,0}(A \times \widehat{A})_{\mathbb{Q}}$. The fact that $\mathcal{P}_A$ is symmetric also implies - via Claim $(*)$ - that we have $\mathcal{F}_{\widehat{A}\times A}((-1)^g \cdot e^{-\widehat{\ell}}) = e^{\ell}$. Indeed, $\mathcal{F}_{\widehat{A}\times A} \circ \mathcal{F}_{A\times\widehat{A}} = [-1]^* \cdot (-1)^{2g} = [-1]^*$, see [66, Corollary 2.22]. Since $\mathcal{F}_{\widehat{A}\times A}$ identifies $\mathrm{CH}^{i,0}(\widehat{A} \times A)_{\mathbb{Q}}$ with $\mathrm{CH}^{g-i,0}(A \times \widehat{A})$ (see [66, Lemma 2.18]), we must indeed have

$$(-1)^g \cdot \mathcal{F}_{\widehat{A}\times A}(-\widehat{\ell}) = \mathcal{F}_{\widehat{A}\times A}((-1)^{g+1} \cdot \widehat{\ell}) = \frac{\ell^{2g-1}}{(2g-1)!} = \mathcal{R}_A. \tag{F.3.6}$$

For a g-dimensional abelian variety X and any $x, y \in \mathrm{CH}(X)_{\mathbb{Q}}$, one has $\mathcal{F}_X(x \cdot y) = (-1)^g \cdot \mathcal{F}_X(x) \star \mathcal{F}_X(y) \in \mathrm{CH}(\widehat{X})_{\mathbb{Q}}$, see [20, Proposition 3]. This implies (see also [153, §3.7]) that if a is a cycle on X such that $\mathcal{F}_X(a) \in \mathrm{CH}_{>0}(\widehat{X})_{\mathbb{Q}}$, then $\mathcal{F}_X(e^a) = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathcal{F}_X(a))$. This allows us to conclude that

$$e^{\ell} = \mathcal{F}_{\widehat{A}\times A}((-1)^g \cdot e^{-\widehat{\ell}}) = (-1)^g \cdot \mathcal{F}_{\widehat{A}\times A}(e^{-\widehat{\ell}}) = (-1)^g \cdot \mathrm{E}(\mathcal{F}_{\widehat{A}\times A}(-\widehat{\ell})) = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathcal{R}_A),$$

which finishes the proof. □

Next, assume that A is equipped with a *principal* polarization $\lambda \colon A \xrightarrow{\sim} \widehat{A}$, define $\ell = c_1(\mathcal{P}_A)$, and let

$$\Theta = \frac{1}{2} \cdot (\mathrm{id}, \lambda)^* \ell \in \mathrm{CH}^1(A)_{\mathbb{Q}} \tag{F.3.7}$$

be the symmetric ample class corresponding to the polarization. Here (id, λ) is the morphism $(\mathrm{id}, \lambda) \colon A \to A \times \widehat{A}$. One can understand the relation between

$$\Gamma_\Theta \coloneqq \Theta^{g-1}/(g-1)! \in \mathrm{CH}_1(A)_{\mathbb{Q}}$$

and $\mathcal{R}_A = \ell^{2g-1}/(2g-1)! \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}$ in the following way. Define $j_1 \colon A \to A \times \widehat{A}$ and $j_2 \colon \widehat{A} \to A \times \widehat{A}$ by $x \mapsto (x, 0)$ and $y \mapsto (0, y)$ respectively. Let $\widehat{\Theta} \in \mathrm{CH}^1(\widehat{A})_{\mathbb{Q}}$ be the dual of Θ, and define a one-cycle τ on $A \times \widehat{A}$ as follows:

$$\tau \coloneqq j_{1,*}(\Gamma_\Theta) + j_{2,*}(\Gamma_{\widehat{\Theta}}) - (\mathrm{id}, \lambda)_*(\Gamma_\Theta) \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}.$$

Lemma F.3.5. *One has* $\tau = (-1)^{g+1} \cdot \mathcal{R}_A \in \mathrm{CH}_1(A \times \widehat{A})_{\mathbb{Q}}$.

Proof. Identify A and $\widehat{A}$ via λ. This gives $\ell = m^*(\Theta) - \pi_1^*(\Theta) - \pi_2^*(\Theta)$, and the Fourier transform becomes an endomorphism $\mathcal{F}_A \colon \mathrm{CH}(A)_{\mathbb{Q}} \to \mathrm{CH}(A)_{\mathbb{Q}}$. We claim that $\tau = (-1)^g \cdot (\Delta_* \mathcal{F}_A(\Theta) - j_{1,*}\mathcal{F}_A(\Theta) - j_{2,*}\mathcal{F}_A(\Theta))$. For this, it suffices to show that $\mathcal{F}_A(\Theta) = (-1)^{g-1} \cdot \Theta^{g-1}/(g-1)! \in \mathrm{CH}_1(A)_{\mathbb{Q}}$. Now $\mathcal{F}_A(e^{\Theta}) = e^{-\Theta}$ by Lemma F.3.6 below. Moreover, since Θ is symmetric, we have $\Theta \in \mathrm{CH}^{1,0}(A)_{\mathbb{Q}}$, hence $\Theta^i/i! \in \mathrm{CH}^{i,0}(A)_{\mathbb{Q}}$ for each $i \geq 0$. Therefore, $\mathcal{F}_A\left(\Theta^i/i!\right) \in \mathrm{CH}^{g-i,0}(A)_{\mathbb{Q}}$ by [66, Lemma 2.18]. This implies that in fact, $\mathcal{F}_A\left(\Theta^i/i!\right) = (-1)^{g-i} \cdot \Theta^{g-i}/(g-i)! \in \mathrm{CH}^{g-i,0}(A)_{\mathbb{Q}}$ for every i. In particular, the claim follows.

Next, recall that $\mathcal{F}_{A\times A}(\ell) = (-1)^{g+1} \cdot \mathcal{R}_A$, see Claim $(*)$. So at this point, it suffices to prove the identity $\mathcal{F}_{A\times A}(\ell) = (-1)^g \cdot (\Delta_* \mathcal{F}_A(\Theta) - j_{1,*}\mathcal{F}_A(\Theta) - j_{2,*}\mathcal{F}_A(\Theta))$. To prove this, we use the following functoriality properties of the Fourier transform on the level of rational Chow groups. Let X and Y be abelian varieties and let $f\colon X \to Y$ be a homomorphism with dual homomorphism $\widehat{f}\colon \widehat{Y} \to \widehat{X}$. We then have the following equalities [153, (3.7.1)]:

$$(\widehat{f})^* \circ \mathcal{F}_X = \mathcal{F}_Y \circ f_*, \quad \mathcal{F}_X \circ f^* = (-1)^{\dim X - \dim Y} \cdot (\widehat{f})_* \circ \mathcal{F}_Y. \tag{F.3.8}$$

Since $\ell = m^*\Theta - \pi_1^*\Theta - \pi_2^*\Theta$, it follows from Equation (F.3.8) that

$$\begin{aligned}\mathcal{F}_{A\times A}(\ell) &= \mathcal{F}_{A\times A}(m^*\Theta) - \mathcal{F}_{A\times A}(\pi_1^*\Theta) - \mathcal{F}_{A\times A}(\pi_2^*\Theta) \\ &= (-1)^g \cdot (\Delta_* \mathcal{F}_A(\Theta) - j_{1,*}\mathcal{F}_A(\Theta) - j_{2,*}\mathcal{F}_A(\Theta)).\end{aligned}$$

□

Lemma F.3.6 (Beauville)**.** *Let A be an abelian variety over k, principally polarized by $\lambda\colon A \xrightarrow{\sim} \widehat{A}$, and define $\Theta = \frac{1}{2} \cdot (\mathrm{id}, \lambda)^* c_1(\mathcal{P}_A) \in \mathrm{CH}^1(A)_{\mathbb{Q}}$. Identify A and $\widehat{A}$ via λ. With respect to the Fourier transform $\mathcal{F}_A\colon \mathrm{CH}(A)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}(A)_{\mathbb{Q}}$, one has $\mathcal{F}_A(e^{\Theta}) = e^{-\Theta}$.*

Proof. Our proof follows the proof of [20, Lemme 1], but has to be adapted, since Θ does not necessarily come from a symmetric ample line bundle on A. Since one still has $\ell = m^*\Theta - \pi_1^*\Theta - \pi_2^*\Theta$, the argument can be made to work: one has

$$\mathcal{F}_A(e^{\Theta}) = \pi_{2,*}\left(e^{\ell} \cdot \pi_1^* e^{\Theta}\right) = \pi_{2,*}\left(e^{m^*\Theta - \pi_2^*\Theta}\right) = e^{-\Theta}\pi_{2,*}(m^* e^{\Theta}) \in \mathrm{CH}(A)_{\mathbb{Q}}.$$

For codimension reasons, one has $\pi_{2,*}(m^*e^{\Theta}) = \pi_{2,*}m^*(\Theta^g/g!) = \deg(\Theta^g/g!) \in \mathrm{CH}^0(A)_{\mathbb{Q}} \cong \mathbb{Q}$. Pull back $\Theta^g/g!$ along $A_{k_s} \to A$ to see that $\deg(\Theta^g/g!) = 1 \in \mathrm{CH}^0(A)_{\mathbb{Q}} \cong \mathrm{CH}^0(A_{k_s})_{\mathbb{Q}}$, since over k_s the cycle Θ becomes the cycle class attached to a symmetric ample line bundle. □

F.3.3. Divided powers and integral Fourier transforms

It was asked by Bruno Kahn whether there exists a PD-structure on the Chow ring of an abelian variety over any field with respect to its usual (intersection) product. There are counterexamples over non-closed fields: see [69], where Esnault constructs an abelian surface X and a line bundle $\mathcal{L}$ on X such that $c_1(\mathcal{L}) \cdot c_1(\mathcal{L})$ is not divisible by 2 in $\mathrm{CH}_0(X)$. However, the case of algebraically closed fields remains open [153, Section 3.2]. What we do know, is the following:

Theorem F.3.7 (Moonen–Polishchuk)**.** *Let A be an abelian variety over k. The ring $(\mathrm{CH}(A), \star)$ admits a canonical PD-structure γ on the ideal $\mathrm{CH}_{>0}(A) \subset \mathrm{CH}(A)$. If $k = \bar{k}$, then γ extends to a PD-structure on the ideal generated by $\mathrm{CH}_{>0}(A)$ and the zero cycles of degree zero.*

In particular, for each element $x \in \mathrm{CH}_{>0}(A)$ and each $n \in \mathbb{Z}_{\geq 1}$, there is a canonical element $x^{[n]} \in \mathrm{CH}_{>0}(A)$ such that $n! x^{[n]} = x^{\star n}$, see [200, Tag 07GM]. For $x \in \mathrm{CH}_{>0}(A)$, we may then define $\mathrm{E}(x) = \sum_{n\geq 0} x^{[n]} \in \mathrm{CH}(A)$ as the $\star$-exponential of x in terms of its divided powers.

Together with the results of Section F.3.2, Theorem F.3.7 enables us to provide several criteria for the existence of a motivic weak integral Fourier transform. We recall that for an abelian variety A over k, principally polarized by $\lambda \colon A \xrightarrow{\sim} \widehat{A}$, we defined $\Theta \in \mathrm{CH}^1(A)_{\mathbb{Q}}$ to be the symmetric ample class attached to the polarization λ, see Equation (F.3.7).

Theorem F.3.8. *Let $A_{/k}$ be an abelian variety of dimension g. The following are equivalent:*

(i) *The one-cycle $\mathcal{R}_A = c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$ lifts to a one-cycle in $\mathrm{CH}(A \times \widehat{A})$.*

(ii) *The abelian variety A admits a motivic weak integral Fourier transform.*

(iii) *The abelian variety $A \times \widehat{A}$ admits a motivic weak integral Fourier transform.*

Moreover, if A carries a symmetric ample line bundle that induces a principal polarization $\lambda \colon A \xrightarrow{\sim} \widehat{A}$, then the above statements are equivalent to the following equivalent statements:

(iv) *The two-cycle $\mathcal{S}_A = c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}$ lifts to a two-cycle in $\mathrm{CH}(A \times \widehat{A})$.*

(v) *The one-cycle $\Gamma_\Theta = \Theta^{g-1}/(g-1)! \in \mathrm{CH}(A)_{\mathbb{Q}}$ lifts to a one-cycle in $\mathrm{CH}(A)$.*

(vi) *The abelian variety A admits a weak integral Fourier transform.*

(vii) *The Fourier transform $\mathcal{F}_A$ satisfies $\mathcal{F}_A\left(\mathrm{CH}(A)/\mathrm{torsion}\right) \subset \mathrm{CH}(\widehat{A})/\mathrm{torsion}$.*

(viii) *There exists a PD-structure on the ideal $\mathrm{CH}^{>0}(A)/\mathrm{torsion} \subset \mathrm{CH}(A)/\mathrm{torsion}$.*

Proof. Suppose that (i) holds, and let $\Gamma \in \mathrm{CH}_1(A \times \widehat{A})$ be a cycle such that $\Gamma_{\mathbb{Q}} = \mathcal{R}_A$. Then consider the cycle $(-1)^g \cdot \mathrm{E}((-1)^g \cdot \Gamma) \in \mathrm{CH}(A \times \widehat{A})$. By Lemma F.3.4, we have

$$(-1)^g \cdot \mathrm{E}((-1)^g \cdot \Gamma)_{\mathbb{Q}} = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \Gamma_{\mathbb{Q}}) = (-1)^g \cdot \mathrm{E}((-1)^g \cdot \mathcal{R}_A) = \mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times \widehat{A})_{\mathbb{Q}}.$$

Thus (ii) holds. We claim that (iii) holds as well. Indeed, consider the line bundle $\mathcal{P}_{A \times \widehat{A}}$ on the abelian variety $X = A \times \widehat{A} \times \widehat{A} \times A$; one has that $\mathcal{P}_{A \times \widehat{A}} \cong \pi_{13}^* \mathcal{P}_A \otimes \pi_{24}^* \mathcal{P}_{\widehat{A}}$, which implies that

$$\begin{aligned} \mathcal{R}_{A \times \widehat{A}} &= \frac{1}{(4g-1)!} \cdot \left(\pi_{13}^* c_1(\mathcal{P}_A) + \pi_{24}^* c_1(\mathcal{P}_{\widehat{A}})\right)^{4g-1} \\ &= \frac{1}{(2g)!(2g-1)!} \cdot \left(\pi_{13}^* c_1(\mathcal{P}_A)^{2g-1} \cdot \pi_{24}^* c_1(\mathcal{P}_{\widehat{A}})^{2g} + \pi_{13}^* c_1(\mathcal{P}_A)^{2g} \cdot \pi_{24}^* c_1(\mathcal{P}_{\widehat{A}})^{2g-1}\right) \\ &= \frac{1}{(2g)!(2g-1)!} \cdot \left(\pi_{13}^* c_1(\mathcal{P}_A)^{2g-1} \cdot \pi_{24}^* \left((2g)! \cdot [0]_{A \times \widehat{A}}\right) + \pi_{13}^* \left((2g)! \cdot [0]_{\widehat{A} \times A}\right) \cdot \pi_{24}^* c_1(\mathcal{P}_{\widehat{A}})^{2g-1}\right) \\ &= \pi_{13}^* \left(\frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!}\right) \cdot \pi_{24}^*([0]_{A \times \widehat{A}}) + \pi_{13}^*([0]_{\widehat{A} \times A}) \cdot \pi_{24}^* \left(\frac{c_1(\mathcal{P}_{\widehat{A}})^{2g-1}}{(2g-1)!}\right) \in \mathrm{CH}_1(X)_{\mathbb{Q}}. \end{aligned} \tag{F.3.9}$$

We conclude that $\mathcal{R}_{A \times \widehat{A}}$ lifts to $\mathrm{CH}_1(X)$ which, by the implication [(i) $\Longrightarrow$ (ii)] (that has already been proved), implies that $A \times \widehat{A}$ admits a motivic weak integral Fourier transform. On the other hand, the implication [(iii) $\Longrightarrow$ (i)] follows from the fact that $(-1)^g \cdot \mathcal{F}_{\widehat{A} \times A}(-\widehat{\ell}) = \mathcal{R}_A$ (see Equation (F.3.6)) and the fact that an abelian variety admits a motivic weak integral

Fourier transform if and only if its dual abelian variety does. Therefore, we have [(i) $\iff$ (ii) $\iff$ (iii)].

Let us from now on assume that A is principally polarized by $\lambda\colon A \xrightarrow{\sim} A$, where λ is the polarization attached to a symmetric ample line bundle $\mathcal{L}$ on A. Moreover, in what follows we shall identify $\widehat{A}$ and A via λ.

Suppose that (iv) holds and let $S_A \in \mathrm{CH}_2(A \times A) = \mathrm{CH}^{2g-2}(A \times A)$ be such that $(S_A)_{\mathbb{Q}} = \mathcal{S}_A \in \mathrm{CH}_2(A \times A)_{\mathbb{Q}}$. Define $\mathrm{CH}^{1,0}(A) := \mathrm{Pic}^{\mathrm{sym}}(A)$ to be the group of isomorphism classes of symmetric line bundes on A. Then S_A induces a homomorphism $\mathcal{F}\colon \mathrm{CH}^{1,0}(A) \to \mathrm{CH}_1(A)$ defined as the composition

$$\mathcal{F}\colon \mathrm{CH}^{1,0}(A) \xrightarrow{\pi_1^*} \mathrm{CH}^1(A \times A) \xrightarrow{\cdot S_A} \mathrm{CH}^{2g-1}(A \times A) = \mathrm{CH}_1(A \times A) \xrightarrow{\pi_{2,*}} \mathrm{CH}_1(A).$$

Since $\mathcal{F}_A\left(\mathrm{CH}^{1,0}(A)_{\mathbb{Q}}\right) \subset \mathrm{CH}_1(A)_{\mathbb{Q}}$ (see [66, Lemma 2.18]) we see that the diagram

$$\begin{tikzcd}
\mathrm{CH}^{1,0}(A) \arrow[r, "\mathcal{F}"] \arrow[d] & \mathrm{CH}_1(A) \arrow[d] \\
\mathrm{CH}^{1,0}(A)_{\mathbb{Q}} \arrow[r, "\mathcal{F}_A"] & \mathrm{CH}_1(A)_{\mathbb{Q}}
\end{tikzcd} \tag{F.3.10}$$

commutes. On the other hand, since the line bundle $\mathcal{L}$ is symmetric, we have

$$\Theta = \frac{1}{2} \cdot (\mathrm{id}, \lambda)^* c_1(\mathcal{P}_A) = \frac{1}{2} \cdot c_1\left((\mathrm{id}, \lambda)^* \mathcal{P}_A\right) = \frac{1}{2} \cdot c_1(\mathcal{L} \otimes \mathcal{L}) = c_1(\mathcal{L}) \in \mathrm{CH}^1(A)_{\mathbb{Q}}. \tag{F.3.11}$$

The class $c_1(\mathcal{L}) \in \mathrm{CH}^{1,0}(A)$ of the line bundle $\mathcal{L}$ thus lies above $\Theta \in \mathrm{CH}^1(A)_{\mathbb{Q}}$. Therefore, $\mathcal{F}(c_1(\mathcal{L})) \in \mathrm{CH}_1(A)$ lies above $\Gamma_\Theta = (-1)^{g-1}\mathcal{F}_A(\Theta)$ by the commutativity of (F.3.10), and (v) holds.

Suppose that (v) holds. Then (i) follows readily from Lemma F.3.5. Moreover, if (ii) holds, then $\mathrm{ch}(\mathcal{P}_A) \in \mathrm{CH}(A \times A)_{\mathbb{Q}}$ lifts to $\mathrm{CH}(A \times A)$, hence in particular (iv) holds. Since we have already proved that (i) implies (ii), we conclude that [(iv) $\implies$ (v) $\implies$ (i) $\implies$ (ii) $\implies$ (iv)].

The implications [(ii) $\implies$ (vi) $\implies$ (vii)] are trivial. Assume that (vii) holds. By Equation (F.3.11), $\Theta \in \mathrm{CH}^1(A)_{\mathbb{Q}}$ lifts to $\mathrm{CH}^1(A)$, hence $\mathcal{F}_A(\Theta) = (-1)^{g-1} \cdot \Gamma_\Theta$ lifts to $\mathrm{CH}_1(A)$, i.e. (v) holds.

Assume that (vii) holds. The fact that $\mathcal{F}_A\left(\mathrm{CH}(A)/\mathrm{torsion}\right) \subset \mathrm{CH}(A)/\mathrm{torsion}$ implies that

$$\mathrm{CH}(A)/\mathrm{torsion} = \mathcal{F}_A\left(\mathcal{F}_A\left(\mathrm{CH}(A)/\mathrm{torsion}\right)\right) \subset \mathcal{F}_A\left(\mathrm{CH}(A)/\mathrm{torsion}\right) \subset \mathrm{CH}(A)/\mathrm{torsion}.$$

Thus the restriction of the Fourier transform $\mathcal{F}_A$ to $\mathrm{CH}(A)/\mathrm{torsion}$ defines an isomorphism $\mathcal{F}_A\colon \mathrm{CH}(A)/\mathrm{torsion} \xrightarrow{\sim} \mathrm{CH}(A)/\mathrm{torsion}$. Now if R is a ring and γ is a PD-structure on an ideal $I \subset R$, then γ extends to a PD-structure on $I/\mathrm{torsion} \subset R/\mathrm{torsion}$. Consequently, the ideal $\mathrm{CH}_{>0}(A)/\mathrm{torsion} \subset \mathrm{CH}(A)/\mathrm{torsion}$ admits a PD-structure for the Pontryagin product $\star$ by Theorem F.3.7. Since $\mathcal{F}_A$ exchanges the Pontryagin and intersection product (up to a sign, see [20, Proposition 3(ii)]), it follows that (viii) holds. Since (viii) trivially implies (v), we are done. □

Question F.3.9 (Moonen–Polishchuk [153], Totaro [206])**.** Let A be any principally polarized abelian variety over $k = \bar{k}$. Are the equivalent conditions in Theorem F.3.8 satisfied for A?

Remark F.3.10. For Jacobians of hyperelliptic curves the answer to Question F.3.9 is "yes" [153].

Similarly, there is a relation between integral Fourier transforms up to homology and the algebraicity of minimal cohomology classes induced by Poincaré line bundles and theta divisors.

Proposition F.3.11. *Let $A_{/k}$ be an abelian variety of dimension g. The following are equivalent:*

(i) *The class* $\rho_A \coloneqq c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}_{\text{ét}}((A\times\widehat{A})_{k_s}, \mathbb{Z}_\ell(2g-1))$ *lifts to* $\mathrm{CH}_1(A\times\widehat{A})$.

(ii) *The abelian variety A admits a motivic integral Fourier transform up to homology.*

(iii) *The abelian variety $A\times\widehat{A}$ admits a motivic integral Fourier transform up to homology.*

Moreover, if A carries a symmetric ample line bundle that induces a principal polarization $\lambda\colon A \xrightarrow{\sim} \widehat{A}$, then the above statements are equivalent to the following equivalent statements:

(iv) *The class* $\sigma_A \coloneqq c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}_{\text{ét}}((A\times\widehat{A})_{k_s}, \mathbb{Z}_\ell(2g-2))$ *lifts to* $\mathrm{CH}_2(A\times\widehat{A})$.

(v) *The class* $\gamma_\theta = \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{k_s}, \mathbb{Z}_\ell(g-1))$ *lifts to a cycle in* $\mathrm{CH}_1(A)$.

(vi) *The abelian variety A admits an integral Fourier transform up to homology.*

Proof. The proof of Theorem F.3.8 can easily be adapted to this situation. □

Proposition F.3.12. (i) *If $k = \mathbb{C}$, then each of the statements* (i) – (vi) *in Proposition F.3.11 is equivalent to the same statement with étale cohomology replaced by Betti cohomology.*

(ii) *Proposition F.3.11 remains valid if one replaces integral Chow groups in statements* (i), (iv) *and* (v) *by their tensor product with $\mathbb{Z}_\ell$ as well as 'integral Fourier transform up to homology' by 'ℓ-adic integral Fourier transform up to homology' in statements* (ii), (iii) *and* (vi).

Proof. (i) In this case $\mathbb{Z}_\ell(i) = \mathbb{Z}_\ell$ and the Artin comparison isomorphism

$$\mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell) \xrightarrow{\sim} \mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z}_\ell)$$

[10, III, Exposé XI] is compatible with the cycle class map. Since the map $\mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z}) \to \mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell)$ is injective, a class $\beta \in \mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z})$ is in the image of $cl\colon \mathrm{CH}^i(A) \to \mathrm{H}^{2i}(A(\mathbb{C}), \mathbb{Z})$ if and only if its image $\beta_\ell \in \mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell)$ is in the image of $cl\colon \mathrm{CH}^i(A) \to \mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell)$.

(ii) Indeed, for an abelian variety A over k, the PD-structure on $\mathrm{CH}_{>0}(A) \subset (\mathrm{CH}(A), \star)$ induces a PD-structure on $\mathrm{CH}_{>0}(A) \otimes \mathbb{Z}_\ell \subset (\mathrm{CH}(A)_{\mathbb{Z}_\ell}, \star)$ by [200, Tag 07H1], because the ring map $(\mathrm{CH}(A), \star) \to (\mathrm{CH}(A)_{\mathbb{Z}_\ell}, \star)$ is flat. The latter follows from the flatness of $\mathbb{Z} \to \mathbb{Z}_\ell$. □

F.4. The integral Hodge conjecture for one-cycles on complex abelian varieties

In this section we use the theory developed in Section F.3 to prove Theorem F.1.1. We also prove some applications of Theorem F.1.1: the integral Hodge conjecture for one-cycles on products of Jacobians (Theorem F.1.2), the fact that the integral Hodge conjecture for one-cycles on principally polarized complex abelian varieties is stable under specialization (Corollary F.4.3) and density of polarized abelian varieties satisfying the integral Hodge conjecture for one-cycles (Theorem F.1.3).

F.4.1. Proof of the main theorem

Let us prove Theorem F.1.1.

Proof of Theorem F.1.1. Suppose that (i) holds. Then (ii) holds by Propositions F.3.11 and F.3.12.(i). Suppose that (ii) holds. Then (iv) follows from Lemma F.3.3. So we have [(i) $\iff$ (ii) $\implies$ (iv)]. If (i) holds, then $\rho_A = c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic, which implies that $\rho_{\widehat{A}} \in \mathrm{H}^{4g-2}(\widehat{A} \times A, \mathbb{Z})$ is algebraic. Therefore, $\rho_{A \times \widehat{A}} \in \mathrm{H}^{8g-2}(A \times \widehat{A} \times \widehat{A} \times A, \mathbb{Z})$ is algebraic by Equation (F.3.9). We then apply the implication [(i) $\implies$ (iv)] to the abelian variety $A \times \widehat{A}$, which shows that (iii) holds. Since [(iii) $\implies$ (i)] is trivial, we have proven [(i) $\iff$ (ii) $\iff$ (iii) $\implies$ (iv)].

Next, assume that A is principally polarized by $\theta \in \mathrm{NS}(A) \subset \mathrm{H}^2(A, \mathbb{Z})$. The directions [(iv) $\implies$ (v)] and [(ii) $\implies$ (vi)] are trivial and [(v) $\implies$ (i)] follows from Propositions F.3.11 and F.3.12.(i). We claim that (vi) implies (iv). Define $\sigma_A = c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}(A \times \widehat{A}, \mathbb{Z})$ and let $S \in \mathrm{CH}_2(A \times \widehat{A})$ be such that $cl(S) = \sigma_A$. The squares in the following diagram commute:

$$\begin{tikzcd}
\mathrm{CH}^1(A) \arrow[r, "\pi_1^*"] \arrow[d, "cl"] & \mathrm{CH}^1(A \times \widehat{A}) \arrow[r, "\cdot S"] \arrow[d, "cl"] & \mathrm{CH}^{2g-1}(A \times \widehat{A}) \arrow[r, "\pi_{2,*}"] \arrow[d, "cl"] & \mathrm{CH}_1(\widehat{A}) \arrow[d, "cl"] \\
\mathrm{H}^2(A, \mathbb{Z}) \arrow[r, "\pi_1^*"] & \mathrm{H}^2(A \times \widehat{A}, \mathbb{Z}) \arrow[r, "\cdot \sigma_A"] & \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z}) \arrow[r, "\pi_{2,*}"] & \mathrm{H}^{2g-2}(\widehat{A}, \mathbb{Z}).
\end{tikzcd} \tag{F.4.1}$$

Since $\mathcal{F}_A = \pi_{2,*}\left(\mathrm{ch}(\mathcal{P}_A) \cdot \pi_1^*(-)\right)$ restricts to an isomorphism $\mathcal{F}_A \colon \mathrm{H}^2(A, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{2g-2}(\widehat{A}, \mathbb{Z})$ by [20, Proposition 1], the composition $\pi_{2,*} \circ (- \cdot \sigma_A) \circ \pi_1^*$ on the bottom row of (F.4.1) is an isomorphism. Thus, by Lefschetz $(1,1)$, $cl \colon \mathrm{CH}_1(\widehat{A}) \to \mathrm{Hdg}^{2g-2}(\widehat{A}, \mathbb{Z})$ is surjective.

It remains to prove the algebraicity of the classes $\theta^i/i! \in \mathrm{H}^{2i}(A, \mathbb{Z})$. This follows from Theorem F.3.7 together with the following equality, proven by Beauville [20, Corollaire 2]):

$$\frac{\theta^i}{i!} = \frac{\gamma_\theta^{\star j}}{j!}, \quad \gamma_\theta = \frac{\theta^{g-1}}{(g-1)!} \in \mathrm{H}^{2g-2}(A, \mathbb{Z}), \quad i + j = g.$$

Therefore, the proof is finished. □

Corollary F.4.1. *Let A and B be complex abelian varieties of respective dimensions g_A and g_B.*

- *The Hodge classes* $\rho_A \in \mathrm{H}^{4g_A-2}(A \times \widehat{A}, \mathbb{Z})$ *and* $\rho_B \in \mathrm{H}^{4g_B-2}(B \times \widehat{B}, \mathbb{Z})$ *are algebraic if and only if* $A \times \widehat{A}$, $B \times \widehat{B}$, $A \times B$ *and* $\widehat{A} \times \widehat{B}$ *satisfy the integral Hodge conjecture for one-cycles.*
- *If* A *and* B *are principally polarized, then the integral Hodge conjecture for one-cycles holds for* $A \times B$ *if and only if it holds for* A *and* B*.*

Proof. The first statement follows from Theorem F.1.1 and Equation (F.3.9). The second statement follows from the fact that the minimal cohomology class of the product $A \times B$ is algebraic if and only if the minimal cohomology classes of the factors A and B are both algebraic. ☐

Proof of Theorem F.1.2. By Corollary F.4.1 we may assume $n = 1$, so let C be a smooth projective curve. Let $p \in C$ and consider the morphism $\iota \colon C \to J(C)$ defined by sending a point q to the isomorphism class of the degree zero line bundle $\mathcal{O}(p-q)$. Then $cl(\iota(C)) = \gamma_\theta \in \mathrm{H}^{2g-2}(J(C), \mathbb{Z})$ by Poincaré's formula [7], so γ_θ is algebraic and the result follows from Theorem F.1.1. ☐

Remark F.4.2. (i) Let us give another proof of Theorem F.1.2 in the case $n = 1$, i.e. let C be a smooth projective curve of genus g and let us prove the integral Hodge conjecture for one-cycles on $J(C)$ in a way that does not use Fourier transforms. It is classical that any Abel-Jacobi map $C^{(g)} \to J(C)$ is birational. On the other hand, the integral Hodge conjecture for one-cycles is a birational invariant, see [214, Lemma 15]. Therefore, to prove it for $J(C)$ it suffices to prove it for $C^{(g)}$. One then uses [65, Corollary 5] which says that for each $n \in \mathbb{Z}_{\geq 1}$, there is a natural polarization η on the n-fold symmetric product $C^{(n)}$ such that for any $i \in \mathbb{Z}_{\geq 0}$, the map $\eta^{n-i} \cup (-) \colon \mathrm{H}^i(C^{(n)}, \mathbb{Z}) \to \mathrm{H}^{2n-i}(C^{(n)}, \mathbb{Z})$ is an isomorphism. In particular, the variety $C^{(n)}$ satisfies the integral Hodge conjecture for one-cycles for any positive integer n.

(ii) Along these lines, observe that the integral Hodge conjecture for one-cycles holds not only for symmetric products of smooth projective complex curves but also for any product $C_1 \times \cdots \times C_n$ of smooth projective curves C_i over $\mathbb{C}$. Indeed, this follows readily from the Künneth formula.

(iii) Let C be a smooth projective complex curve of genus g. Our proof of Theorem F.1.1 provides an explicit description of $\mathrm{Hdg}^{2g-2}(J(C), \mathbb{Z})$ depending on $\mathrm{Hdg}^2(J(C), \mathbb{Z})$. More generally, let (A, θ) be a principally polarized abelian variety of dimension g, identify A and $\widehat{A}$ via the polarization, and let $\ell = c_1(\mathcal{P}_A) \in \mathrm{H}^2(A \times \widehat{A}, \mathbb{Z})$. Then $\ell = m^*(\theta) - \pi_1^*(\theta) - \pi_2^*(\theta)$, which implies that

$$\sigma_A = \frac{\ell^{2g-2}}{(2g-2)!} = \sum_{\substack{i,j,k \geq 0 \\ i+j+k = 2g-2}}^{2g-2} (-1)^{j+k} \cdot m^* \left(\frac{\theta^i}{i!} \right) \cdot \pi_1^* \left(\frac{\theta^j}{j!} \right) \cdot \pi_2^* \left(\frac{\theta^k}{k!} \right).$$

On the other hand, any $\beta \in \mathrm{Hdg}^{2g-2}(A, \mathbb{Z})$ is of the form $\pi_{2,*}\left(\sigma_A \cdot \pi_1^*[D]\right)$, where $[D] = cl(D)$ for a divisor D on A, as follows from (F.4.1). Therefore, any $\beta \in \mathrm{Hdg}^{2g-2}(A, \mathbb{Z})$ may be written as

$$\beta = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=2g-2}}^{2g-2} (-1)^{j+k} \cdot \pi_{2,*}\left(m^*\left(\frac{\theta^i}{i!}\right) \cdot \pi_1^*\left(\frac{\theta^j}{j!}\right) \cdot \pi_1^*[D] \right) \cdot \frac{\theta^k}{k!}. \tag{F.4.2}$$

Returning to the case of a Jacobian $J(C)$ of a smooth projective curve C of genus g, the classes $\theta^i/i!$ appearing in (F.4.2) are effective algebraic cycle classes. Indeed, for $p \in C$ and $d \in \mathbb{Z}_{\geq 1}$, the image of the morphism $C^d \to J(C)$, $(x_i) \mapsto \mathcal{O}(\sum_i x_i - d \cdot p)$ defines a subvariety $W_d(C) \subset J(C)$ and by Poincaré's formula [7, §I.5] one has $cl(W_d(C)) = \theta^{g-d}/(g-d)! \in \mathrm{H}^{2g-2d}(J(C), \mathbb{Z})$.

Besides Theorem F.1.2, we obtain the following corollary of Theorem F.1.1:

Corollary F.4.3. *Let $A \to S$ be a principally polarized abelian scheme over a proper, smooth and connected variety S over $\mathbb{C}$. Let $X \subset S(\mathbb{C})$ be the set of $x \in S(\mathbb{C})$ such that the abelian variety A_x satisfies the integral Hodge conjecture for one-cycles. Then $X = \cup_i Z_i(\mathbb{C})$ for some countable union of closed algebraic subvarieties $Z_i \subset S$. In particular, if the integral Hodge conjecture for one-cycles holds on $U(\mathbb{C})$ for a non-empty open subscheme U of S, then it holds on all of $S(\mathbb{C})$.*

Proof. Write $\mathcal{A} = A(\mathbb{C})$ and $B = S(\mathbb{C})$ and let $\pi\colon \mathcal{A} \to B$ be the induced family of complex abelian varieties. Let $g \in \mathbb{Z}_{\geq 0}$ be the relative dimension of π and define, for $t \in S(\mathbb{C})$, $\theta_t \in \mathrm{NS}(\mathcal{A}_t) \subset \mathrm{H}^2(\mathcal{A}_t, \mathbb{Z})$ to be the polarization of $\mathcal{A}_t$. There is a global section $\gamma_\theta \in \mathrm{R}^{2g-2}\pi_*\mathbb{Z}$ such that for each $t \in B$, $\gamma_{\theta_t} = \theta_t^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(\mathcal{A}_t, \mathbb{Z})$. Note that γ_θ is Hodge everywhere on B. For those $t \in B$ for which γ_{θ_t} is algebraic, write γ_{θ_t} as the difference of effective algebraic cycle classes on $\mathcal{A}_t$. This gives a countable disjoint union $\phi\colon \sqcup_{ij} H_i \times_S H_j \to S$ of products of relative Hilbert schemes $H_i \to S$. By Lemma F.4.4 below, γ_{θ_t} is algebraic precisely for closed points t in the image $Y \subset S$ of ϕ. Theorem F.1.1 implies that $X = Y$ and the assertion is proven. □

Lemma F.4.4. *Let S be an integral variety over $\mathbb{C}$, let $\mathcal{A} \to S$ be a principally polarized abelian scheme of relative dimension g over S and let $\mathcal{C}_i \subset \mathcal{A}$ for $i = 1, \ldots, k$ be relative curves in $\mathcal{A}$ over S. Let $n_1, \ldots, n_k$ be integers and let $y \in S(\mathbb{C})$ be a point that satisfies $\sum_{i=1}^k n_i \cdot cl(C_{i,y}) = \gamma_{\theta_y} \in \mathrm{H}^{2g-2}(A_y, \mathbb{Z})$. Then for every $x \in S(\mathbb{C})$, one has $\sum_{i=1}^k n_i \cdot cl(C_{i,x}) = \gamma_{\theta_x} \in \mathrm{H}^{2g-2}(A_x, \mathbb{Z})$.*

Proof. Since it suffices to prove the lemma for any open affine $U \subset S$ that contains y, we may assume that S is quasi-projective. Fix $x \in S(\mathbb{C})$. After replacing S by a suitable base change containing x and y, we may assume that S is an open subscheme of a smooth connected curve. For $t \in S$, denote by $\theta_{\bar{t}} \in \mathrm{H}^2_{\text{ét}}(A_{\bar{t}}, \mathbb{Z}_\ell)$ the class of the polarization and $\gamma_{\theta_{\bar{t}}} = \theta_{\bar{t}}^{g-1}/(g-1)!$. Let $\eta = \operatorname{Spec} K$ be the generic point of S. The elements $\sum_i n_i \cdot cl(C_{i,\bar{\eta}})$ and $\gamma_{\theta_{\bar{\eta}}}$ in $\mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Z}_\ell)$ both map to $\sum_i n_i \cdot cl(C_{i,y}) = \gamma_{\theta_y} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_y, \mathbb{Z}_\ell)$ under the specialization homomorphism $s\colon \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\eta}}, \mathbb{Z}_\ell) \to \mathrm{H}^{2g-2}_{\text{ét}}(A_y, \mathbb{Z}_\ell)$ by [76, Example 20.3.5]. Since s is an isomorphism, we have $\sum_i n_i \cdot cl(C_{i,\bar{\eta}}) = \gamma_{\theta_{\bar{\eta}}}$, which implies that $\sum_i n_i \cdot cl(C_{x,i}) = \gamma_{\theta_x} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_x, \mathbb{Z}_\ell)$. □

F.4.2. Density of abelian varieties satisfying IHC$_1$

The goal of this section is to prove that Conditions (i) – (iii) in Theorem F.1.1 are satisfied on a dense subset of the moduli space of complex abelian varieties. To do so, will we state yet another criterion that a complex abelian variety may satisfy. In some sense this criterion provides a bridge between abelian varieties outside the Torelli locus and those lying within, thereby implying the integral Hodge conjecture for one-cycles for the abelian variety under consideration.

Definition F.4.5. Let A and B be a complex abelian varieties and let p a prime number. We say that A is *prime-to-p isogenous to a B* if there exists an isogeny $\alpha\colon A \to B$ whose degree $\deg(\alpha)$ is not divisible by p. We say that A is *p-power isogenous to B* if A is isogenous to B for some isogeny α whose degree is a power of p.

The following proposition shows in particular that to prove the density part of the statement in Theorem F.1.3, it suffices to prove that for any prime number ℓ, those abelian varieties that are ℓ-power isogenous to a product of elliptic curves are dense in their moduli space.

Proposition F.4.6. *Let A be a complex abelian variety of dimension g. Let $\widehat{A}$ be the dual abelian variety and let $\mathcal{P}_A$ be the Poincaré bundle. Let κ be a non-zero integer such that the cohomology class $\kappa \cdot c_1(\mathcal{P}_A)/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic. Consider the following statements:*

(i) *The abelian variety A satisfies the integral Hodge conjecture for one-cycles.*

(ii) *For every prime number p, there exists an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to the Jacobian of a smooth projective curve.*

(iii) *For every prime number p that divides κ, there exists an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to a Jacobian of a smooth projective curve.*

(iv) *For every prime number p, there exists an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to a product of Jacobians of smooth projective curves.*

(v) *For every prime number p dividing κ, there exists an abelian variety B such that the abelian variety $A \times B$ is prime-to-p isogenous to a product of Jacobians of smooth projective curves.*

Then [(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) $\Longrightarrow$ (i)] *and* [(ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v)]*. Moreover, if A is principally polarized by $\theta_A \in \mathrm{NS}(A)$, then* (i) *is implied by*

(vi) *For any prime number $p|(g-1)!$ there exists a smooth projective curve C and a morphism of abelian varieties $\phi\colon A \to J(C)$ such that $\phi^*\theta_{J(C)} = m \cdot \theta_A$ for $m \in \mathbb{Z}_{\geq 1}$ with* $\gcd(m,p) = 1$.

Finally, if A is principally polarized of Picard rank one, then the statements (i) – (vi) *are equivalent.*

Proof. **Step one**: [(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (v)] *and* [(ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v)]. These implications are trivial.

Step two: [(v) $\Longrightarrow$ (i)]. Let g be the dimension of A. We want to prove that the class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic. Let p be any prime number that divides κ. Then by Condition (v), there exists an abelian variety B and an isogeny $\alpha\colon A \times B \to Y$ to the product $Y = \prod_i J(C_i)$ of Jacobians $J(C_i)$ of smooth projective curves C_i such that $\gcd(\deg(\alpha), p) = 1$. Define $X = A \times B$. Let g_B be the dimension of B, let $h = g + g_B = \dim(X) = \dim(Y)$, and let $m_p = \deg(\alpha)$. There exists an isogeny $\beta\colon Y \to X$ such that $\beta \circ \alpha = [m_p]_X$. If we define $n_p = \deg(\beta)$ then $m_p \cdot n_p = \deg(\alpha) \cdot \deg(\beta) = \deg(\alpha \circ \beta) = m_p^{2h}$. Therefore, $(\beta \circ \alpha) \times (\widehat{\alpha} \circ \widehat{\beta}) = [m_p]_{X \times \widehat{X}}$. Consequently, if $N_p = 2h \cdot (4h-2)$, then the homomorphism $[m_p^{2h}]^* = (m_p^{N_p} \cdot (-))\colon \mathrm{H}^{4h-2}(X \times \widehat{X}, \mathbb{Z}) \to \mathrm{H}^{4h-2}(X \times \widehat{X}, \mathbb{Z})$ factors through $\mathrm{H}^{4h-2}(Y \times \widehat{Y}, \mathbb{Z})$. Since $Y \times \widehat{Y}$ satisfies the integral Hodge conjecture by Theorem F.1.2, the Hodge class $m_p^{N_p} \cdot c_1(P_X)^{2h-1}/(2h-1)! \in \mathrm{H}^{4h-2}(X \times \widehat{X}, \mathbb{Z})$ is algebraic. Let $f\colon A \times B \times \widehat{A} \times \widehat{B} \to A \times \widehat{A}$ and $g\colon A \times B \times \widehat{A} \times \widehat{B} \to B \times \widehat{B}$ be the canonical projections. Then $\mathcal{P}_X \cong f^*\mathcal{P}_A \otimes g^*\mathcal{P}_B$. Using this and denoting $\mu = c_1(\mathcal{P}_A)$ and $\nu = c_1(\mathcal{P}_B)$ we have

$$\frac{c_1(\mathcal{P}_X)^{2h-1}}{(2h-1)!} = f^*\left(\frac{\mu^{2g-1}}{(2g-1)!}\right) \cdot g^*\left(\frac{\nu^{2g_B}}{(2g_B)!}\right) + f^*\left(\frac{\mu^{2g}}{(2g)!}\right) \cdot g^*\left(\frac{\nu^{2g_B-1}}{(2g_B-1)!}\right).$$

This implies that $f_*\left(c_1(\mathcal{P}_X)^{2h-1}/(2h-1)!\right) = (-1)^{g_b}\mu^{2g-1}/(2g-1)!$. In particular, the class $m_p^{N_p} \cdot c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic.

Let $p_1, \ldots, p_n$ be all prime divisors of κ and observe that $\gcd(\kappa, m_{p_1}^{N_{p_1}}, m_{p_2}^{N_{p_2}}, \ldots, m_{p_n}^{N_{p_n}}) = 1$. Therefore, there are integers $a, b_1, \ldots, b_n$ such that $a \cdot \kappa + \sum_{i=1}^n b_i \cdot m_{p_i}^{N_{p_i}} = 1$. One obtains

$$\frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} = a \cdot \kappa \cdot \frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} + \sum_{i=1}^n b_i \cdot m_{p_i}^{N_{p_i}} \cdot \frac{c_1(\mathcal{P}_A)^{2g-1}}{(2g-1)!} \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z}).$$

This proves that $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is a $\mathbb{Z}$-linear combination of algebraic classes, hence algebraic. Condition (i) follows then from Theorem F.1.1.

Step three: [(vi) $\Longrightarrow$ (i)] *for A principally polarized by $\theta_A \in \mathrm{NS}(A)$.* Let $p_1, \ldots, p_k$ be the prime factors of $(g-1)!$ and let $C_1, \ldots, C_k$ be smooth proper curves for which there exist homomorphisms $\phi_i\colon A \to J(C_i)$ such that $\phi^*\theta_{J(C_i)} = m_i \cdot \theta_A$ for some $m_i \in \mathbb{Z}_{\geq 1}$ with $p_i \nmid m$. Since $\theta_{J(C_i)}^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(J(C_i), \mathbb{Z})$ is algebraic for each i, the classes $\phi_i^*(\theta_{J(C_i)}^{g-1}/(g-1)!) = m_i^{g-1} \cdot \theta_A^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ are algebraic. Since $\gcd((g-1)!, m_1, \ldots, m_k) = 1$, this implies that $\theta_A^{g-1}/(g-1)!$ is algebraic. Condition (i) follows then from Theorem F.1.1.

Step four: [(vi) $\Longleftarrow$ (i) $\Longrightarrow$ (ii)] *for (A, θ_A) principally polarized with $\rho(A) = 1$.* Write $\theta = \theta_A$. Let $Z_1, \ldots, Z_n$ be integral curves $Z_i \subset A$ and let $e_1, \ldots, e_n \in \mathbb{Z}$ with $e_i \neq 0$ for all i be such that $\theta^{g-1}/(g-1)! = \sum_{i=1}^n e_i \cdot [Z_i] \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$. Since $\rho(A) = 1$, the group $\mathrm{Hdg}^{2g-2}(A, \mathbb{Z})$ is generated by $\theta^{g-1}/(g-1)!$. Consequently, we have $[Z_i] = f_i \cdot (\theta^{g-1}/(g-1)!)$ for some non-zero $f_i \in \mathbb{Z}$. Hence we can write $\theta^{g-1}/(g-1)! = \sum_{i=1}^n e_i \cdot [Z_i] = \sum_{i=1}^n e_i \cdot f_i \cdot \theta^{g-1}/(g-1)!$, which implies $\sum_{i=1}^n e_i \cdot f_i = 1$. Now let p be any prime number. Then there exists an integer

i with $1 \leq i \leq n$ such that p does not divide f_i. Let $C_i \to Z_i$ be the normalization of Z_i and let $\lambda_A = \varphi_\theta \colon A \to \widehat{A}$ be the polarization corresponding to θ. This gives a diagram

$$\begin{tikzcd}
C_i \arrow[rr, "\varphi"] \arrow[dr, "\iota"'] & & A \arrow[r, "\lambda_A", "\sim"'] \arrow[rrr, bend right, "\phi"'] & \widehat{A} \arrow[r, "\varphi^*"] & \mathrm{Pic}^0(C_i) \arrow[r, "a", "\sim"'] & J(C_i), \\
& J(C_i) \arrow[ur, "\psi"'] & & & &
\end{tikzcd}$$

(F.4.3)

where $\iota \colon C_i \to J(C_i) = H^0(C, \Omega_C)^*/H_1(C, \mathbb{Z})$ is the Abel–Jacobi map (for some $p \in C$), and $\varphi^* \colon \widehat{A} = \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(C_i)$ is the pullback of line bundles along $\varphi \colon C_i \to A$. The natural homomorphism $a \colon \mathrm{Pic}^0(C_i) \to J(C_i)$ is an isomorphism by the Abel–Jacobi theorem. Since the triangle on the left in Diagram (F.4.3) commutes and $[Z_i] \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$ is non-zero, the morphism $\psi \colon J(C_i) \to A$ is non-zero. As $\rho(A) = 1$, the map $\psi \colon J(C_i) \to A$ must be surjective, the Picard rank of a non-simple abelian variety being greater than one. Dually, ψ gives rise to a non-zero homomorphism $\widehat{\psi} \colon \widehat{A} \to \widehat{J(C_i)}$, and the simpleness of $\widehat{A}$ implies that $\widehat{\psi}$ is finite onto its image. We claim that the same is true for ϕ. To prove this, it suffices to show that the kernel of $\varphi^* \colon \widehat{A} \to \mathrm{Pic}^0(C_i)$ is finite. Since the homomorphism $\iota^* \colon \widehat{J(C_i)} \to \mathrm{Pic}^0(C_i)$ induced by the embedding $\iota \colon C_i \to J(C_i)$ is an isomorphism, dualizing the triangle on the left in Diagram (F.4.3) proves our claim. By construction, we have $\varphi_*[C_i] = [Z_i] = f_i \cdot \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$. By a version of Welters' Criterion (see [31, Lemma 12.2.3]), this implies that $\phi^* \left(\theta_{J(C_i)} \right) = f_i \cdot \theta \in \mathrm{H}^2(A, \mathbb{Z})$, where $\theta_{J(C_i)} \in \mathrm{H}^2(J(C_i), \mathbb{Z})$ is the canonical principal polarization. In particular, (vi) holds.

We claim that also (ii) holds. Let $j \colon A_0 \hookrightarrow J(C_i)$ be the embedding of $A_0 = \phi(A)$ into $J(C_i)$ and let $\lambda_0 \colon A_0 \to \widehat{A}_0$ be the polarization on A_0 induced by j. We have $\phi^*(\lambda) = \varphi_{f_i \cdot \theta} = f_i \cdot \varphi_\theta = f_i \cdot \lambda_A$. We obtain a commutative diagram

$$\begin{tikzcd}
& A \arrow[r, "\pi"] \arrow[dl, "{[f_i]_A}"'] \arrow[d, "f_i \cdot \lambda_A"] & A_0 \arrow[r, "j"] \arrow[d, "\lambda_0"] & J(C_i) \arrow[d, "\lambda"] \\
A & \widehat{A} \arrow[l, "\lambda_{\widehat{A}}"] & \widehat{A}_0 \arrow[l, "\widehat{\pi}"] & \widehat{J(C_i)}. \arrow[l, "\widehat{j}"]
\end{tikzcd}$$

Let G be the kernel of π. Define $K = \mathrm{Ker}([f_i]_A) = \mathrm{Ker}(f_i \cdot \lambda_A) \cong (\mathbb{Z}/f_i)^{2g} \subset A$, and $U = \mathrm{Ker}(\widehat{\pi} \circ \lambda_0) \subset A_0$. Also define $H = \mathrm{Ker}(\lambda_0)$, and observe that $H \subset U$. The exact sequence $0 \to G \to K \to U \to 0$ shows that if a, k, u and h are the respective orders of G, K, U and H, then one has

$$h|u|k|f_i \quad \text{and} \quad a|k|f_i. \tag{F.4.4}$$

Then define $B = \mathrm{Ker}(\widehat{j} \circ \lambda) \subset J(C_i)$ with inclusion $i \colon B \hookrightarrow J(C_i)$. It is easy to see that B is connected. Moreover, we have $A_0 \cap B = H$ and, therefore, an exact sequence of commutative group schemes

$$0 \to H \to A_0 \times B \xrightarrow{\psi} J(C_i) \to 0.$$

The morphism $\alpha \colon A \times B \to J(C_i)$, defined as the composition

$$A \times B \xrightarrow{\pi \times \mathrm{id}} A_0 \times B \xrightarrow{\psi} J(C_i),$$

is an isogeny. Since the degree of an isogeny is multiplicative in compositions, we have $\deg(\alpha) = \deg(\psi \circ (\pi \times \mathrm{id})) = \deg(\psi) \cdot \deg(\pi \times \mathrm{id}) = h \cdot \deg(\pi) = h \cdot a$. In particular, p does not divide $\deg(\alpha)$ because h and a divide f_i by Equation (F.4.4). $\square$

Proof of Theorem F.1.3. According to Theorem F.1.1, it suffices to show that the cohomology class $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic for $[(A, \lambda)]$ in a dense subset X of $\mathsf{A}_{g,\delta}(\mathbb{C})$ as in the statement. Define $D = \mathrm{diag}(\delta_1, \ldots, \delta_g)$ and define, for each subring R of $\mathbb{C}$, a group

$$\mathrm{Sp}_{2g}^{\delta}(R) = \left\{ M \in \mathrm{GL}_{2g}(R) \mid M \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} M^t = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \right\}.$$

The isomorphism

$$\mathrm{Sp}_{2g}^{\delta}(\mathbb{R}) \to \mathrm{Sp}_{2g}(\mathbb{R}), \quad M \mapsto \begin{pmatrix} 1_g & 0 \\ 0 & D \end{pmatrix}^{-1} M \begin{pmatrix} 1_g & 0 \\ 0 & D \end{pmatrix}$$

induces an action of $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z})$ on the genus g Siegel space $\mathbb{H}_g$, and the period map defines an isomorphism of complex analytic spaces $\mathsf{A}_{g,\delta}(\mathbb{C}) \cong \mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}) \setminus \mathbb{H}_g$ [31, Theorem 8.2.6]. Pick any prime number $\ell > (2g-1)!$ and consider, for a period matrix $x \in \mathbb{H}_g$, the orbit $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell]) \cdot x \subset \mathbb{H}_g$. Let (A, λ) be a polarized abelian variety admitting a period matrix equal to x. The image of $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell]) \cdot x$ in $\mathsf{A}_{g,\delta}(\mathbb{C})$ is the *Hecke-ℓ-orbit* of $[(A, \lambda)] \in \mathsf{A}_{g,\delta}(\mathbb{C})$, i.e. the set of isomorphism classes of polarized abelian varieties $[(B, \mu)] \in \mathsf{A}_{g,\delta}(\mathbb{C})$ for which there exists integers $n, m \in \mathbb{Z}_{\geq 0}$ and an isomorphism of polarized rational Hodge structures $\phi \colon \mathrm{H}_1(B, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_1(A, \mathbb{Q})$ such that $\ell^n \cdot \phi$ and $\ell^m \cdot \phi^{-1}$ are morphisms of integral Hodge structures (Hecke orbits were studied in positive characteristic in e.g. [51, 52]). The degree of the isogeny $\alpha = \ell^n \phi$ must be ℓ^k for some nonnegative integer k. In particular, if one abelian variety in a Hecke-ℓ-orbit happens to be isomorphic to a Jacobian, then every abelian variety in that orbit is ℓ-power isogenous to a Jacobian, see Definition F.4.5.

The decomposition of a polarized abelian variety into non-decomposable polarized abelian subvarieties is unique [63, Corollaire 2], which implies that the following morphism

$$\pi \colon \prod_{i=1}^{g} \mathsf{A}_{1,1} \to \mathsf{A}_{g,\delta}, \quad ([(E_1, \lambda_1)], \ldots, [(E_g, \lambda_g)]) \mapsto ([E_1 \times \cdots \times E_g, \delta_1 \cdot \lambda_1 \times \cdots \times \delta_g \cdot \lambda_g)]$$

is finite onto its image. Thus $\mathsf{A}_{g,\delta}$ contains a g-dimensional subvariety on which the integral Hodge conjecture for one-cycles holds. We claim that $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell])$ is dense in $\mathrm{Sp}_{2g}(\mathbb{R})$. Since $\mathrm{Sp}_{2g}^{\delta}(\mathbb{Q})$ arises as the group of rational points of an algebraic subgroup $\mathrm{Sp}_{2g}^{\delta}$ of GL_{2g} over $\mathbb{Q}$ [183, Chapter 2, §2.3.2], which is isomorphic to Sp_{2g} over $\mathbb{Q}$, this claim follows from the well-known fact that for $S = \{\ell\} \subset \operatorname{Spec} \mathbb{Z}$, the algebraic group Sp_{2g} satisfies the strong approximation property with respect to S [183, Chapter 7, §7.1] (indeed, this is classical and follows from the non-compactness of $\mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$, see [183, Theorem 7.12]).

Let $V = \pi\left(\prod_{i=1}^{g} \mathsf{A}_{1,1}\right) \subset \mathsf{A}_{g,\delta}$. Then $X' := \mathrm{Sp}_{2g}^{\delta}(\mathbb{Z}[1/\ell]) \cdot V = \cup_i Z_i \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ is a countable union of closed analytic subsets $Z_i \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ of dimension $\dim Z_i \geq g$ such that $X' \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ is dense in the analytic topology and $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})$ is algebraic for every polarized abelian variety (A, λ) of polarization type δ whose isomorphism class lies in X'. To prove the theorem, we are reduced to proving that there exists a similar countable union $X \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ whose components are algebraic. For this, it suffices to prove the following *claim:* the locus of $[(A,\lambda)] \in \mathsf{A}_{g,\delta}(\mathbb{C})$ such that $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A}, \mathbb{Z})_{\mathrm{alg}}$ is a countable union $W = \cup_j Y_j \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ of closed algebraic subsets $Y_j \subset \mathsf{A}_{g,\delta}(\mathbb{C})$. Indeed, if this holds, then $X' \subset W$ and since each $Z_i \subset X$ is irreducible, each Z_i is contained in an irreducible component $Y_j \subset W$. We may then define X as the union of those $Y_j \subset W$ that contain some Z_i.

To prove the claim, let $U \to \mathcal{A}_{g,\delta}$ be a finite étale cover of the moduli stack $\mathcal{A}_{g,\delta}$ and let $\mathcal{X} \to U$ be the pullback of the universal family of abelian varieties along $U \to \mathcal{A}_{g,\delta}$. This gives an abelian scheme $\mathcal{X} \times \widehat{\mathcal{X}} \to U$ carrying a relative Poincaré line bundle $\mathcal{P}_{\mathcal{X}/U}$ and arguments similar to those used to prove Lemma F.4.4 show that indeed, for each irreducible component $U' \subset U$, the locus in $U'(\mathbb{C})$ where $c_1(\mathcal{P}_A)^{2g-1}/(2g-1)!$ is algebraic is a countable union of closed algebraic subvarieties of $U'(\mathbb{C})$.

Finally, Theorem F.1.1 implies that for each $[(A,\lambda)] \in X$, the integral Hodge conjecture for one-cycles holds for the abelian variety A, so we are done. ☐

Remark F.4.7. Using level structures one can show that whenever $\gcd(\prod_i \delta_i, (2g-1)!) = 1$ (or, more generally, $\gcd(\prod_i \delta_i, (2g-2)!) = 1$, see Section F.5 below), there is a countable union $X = \cup_i Z_i \subset \mathsf{A}_{g,\delta}(\mathbb{C})$ as in Theorem F.1.3 such that $\dim Z_i \geq 3g-3$. Indeed, let $\mathsf{A}^*_{g,\delta_g}$ be the moduli space of principally polarized abelian varieties of dimension g with δ_g-level structure. Then there is a natural morphism $\phi\colon \mathsf{A}^*_{g,\delta_g} \to \mathsf{A}_{g,\delta}$ such that for any $x = [(A,\lambda)] \in \mathsf{A}^*_{g,\delta_g}(\mathbb{C})$ with $[(B,\mu)] = \phi(x) \in \mathsf{A}_{g,\delta}(\mathbb{C})$, there exists an isogeny $\alpha\colon A \to B$ of degree $\prod_{i=1}^g \delta_i$, see [157].

Remark F.4.8. In the principally polarized case, the density in the moduli space of those abelian varieties that satisfy the integral Hodge conjecture for one-cycles admits another proof which might be interesting for comparison. Let A_g be the coarse moduli space of principally polarized complex abelian varieties of dimension g and let $[(A,\theta)]$ be a closed point of A_g. Then by [31, Exercise 5.6.(10)], the following are equivalent: (i) A is isogenous to the g-fold self-product E^g for an elliptic curve E with complex multiplication, (ii) A has maximal Picard rank $\rho(A) = g^2$, (iii) A is *isomorphic* to the product $E_1 \times \cdots \times E_g$ of pairwise isogenous elliptic curves E_i with complex multiplication. If any of these conditions is satisfied, then A satisfies the integral Hodge conjecture for one-cycles by Theorem F.1.2. Moreover, the set of isomorphism classes of principally polarized abelian varieties (A,θ) for which this holds is dense in A_g by [127]. For an explicit example in dimension $g = 4$ of a principally polarized abelian variety (A,θ) that satisfies one of the equivalent conditions above, but which is not isomorphic to a Jacobian, see [62, §5].

F.5. The integral Hodge conjecture for one-cycles up to factor n

In this section, we study a property of a smooth projective complex variety that lies somewhere in between the integral Hodge conjecture and the usual (i.e. rational) Hodge conjecture. The key will be the following:

Definition F.5.1. Let $d, k, n \in \mathbb{Z}_{\geq 1}$ and let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Recall the definition of the degree $2d-2k$ *Voisin group* of X [180, 215]:

$$\mathrm{Z}^{2d-2k}(X) := \mathrm{Hdg}^{2d-2k}(X,\mathbb{Z})/\mathrm{H}^{2d-2k}(X,\mathbb{Z})_{\mathrm{alg}} = \mathrm{Coker}\left(\mathrm{CH}_k(X) \to \mathrm{Hdg}^{2d-2k}(X,\mathbb{Z})\right).$$

We say that X *satisfies the integral Hodge conjecture for k-cycles up to factor n* if $\mathrm{Z}^{2d-2k}(X)$ is annihilated by n (in other words, if $n \cdot x \in \mathrm{H}^{2d-2k}(X,\mathbb{Z})_{\mathrm{alg}}$ for every $x \in \mathrm{Hdg}^{2d-2k}(X,\mathbb{Z})$).

Lemma F.5.2. *Let A be a complex abelian variety of dimension g. Define $\sigma_A \in \mathrm{H}^{4g-4}(A \times \widehat{A},\mathbb{Z})$ to be the class $c_1(\mathcal{P}_A)^{2g-2}/(2g-2)!$.*

(i) *Let n be a positive integer and let $\mathcal{F}_n \colon \mathrm{CH}^1(\widehat{A}) \to \mathrm{CH}_1(A)$ be a group homomorphism such that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{CH}^1(\widehat{A}) \arrow[r, "\mathcal{F}_n"] \arrow[d] & \mathrm{CH}_1(A) \arrow[d] \\
\mathrm{H}^2(\widehat{A},\mathbb{Z}) \arrow[r, "n\cdot\mathcal{F}_{\widehat{A}}"] & \mathrm{H}^{2g-2}(A,\mathbb{Z}).
\end{tikzcd}$$

Then A satisfies the integral Hodge conjecture for one-cycles up to factor n.

(ii) *Let $n \in \mathbb{Z}_{\geq 1}$ be such that $n \cdot \sigma_A$ is algebraic. Then a homomorphism $\mathcal{F}_n$ as in* (i) *exists.*

Proof. Statement (i) follows immediately from the fact that $\mathrm{CH}^1(\widehat{A}) \to \mathrm{Hdg}^2(\widehat{A},\mathbb{Z})$ is surjective by Lefschetz $(1,1)$. To prove (ii), first observe that if $\sigma_{\widehat{A}} := c_1(\mathcal{P}_{\widehat{A}})^{2g-2}/(2g-2)! \in \mathrm{H}^{4g-4}(\widehat{A} \times A,\mathbb{Z})$, then $n \cdot \sigma_{\widehat{A}}$ is algebraic since $n \cdot \sigma_A$ is. Let $\Sigma_n \in \mathrm{CH}_2(\widehat{A} \times A)$ be such that $cl(\Sigma_n) = n \cdot \sigma_{\widehat{A}}$. This gives a commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^1(\widehat{A}) \arrow[r, "\pi_1^*"] \arrow[d, "cl"] & \mathrm{CH}^1(\widehat{A} \times A) \arrow[r, "\cdot\Sigma_n"] \arrow[d, "cl"] & \mathrm{CH}^{2g-1}(\widehat{A} \times A) \arrow[r, "\pi_{2,*}"] \arrow[d, "cl"] & \mathrm{CH}_1(A) \arrow[d, "cl"] \\
\mathrm{H}^2(\widehat{A},\mathbb{Z}) \arrow[r, "\pi_1^*"] & \mathrm{H}^2(\widehat{A} \times A,\mathbb{Z}) \arrow[r, "\cdot n\cdot\sigma_{\widehat{A}}"] & \mathrm{H}^{4g-2}(\widehat{A} \times A,\mathbb{Z}) \arrow[r, "\pi_{2,*}"] & \mathrm{H}^{2g-2}(A,\mathbb{Z}).
\end{tikzcd}$$

Since $\pi_{2,*} \circ \left((-) \cdot n \cdot \sigma_{\widehat{A}}\right) \circ \pi_1^* = n \cdot \mathcal{F}_{\widehat{A}}$, the homomorphism $\mathcal{F}_n := \pi_{2,*} \circ ((-) \cdot \Sigma_n) \cdot \pi_1^*$ has the required property. ☐

Theorem F.5.3. *Consider a complex abelian variety A of dimension g. Define the cycle $\sigma_A \in \mathrm{H}^{4g-4}(A \times \widehat{A},\mathbb{Z})$ as before and define $\rho_A = c_1(\mathcal{P}_A)^{2g-1}/(2g-1)! \in \mathrm{H}^{4g-2}(A \times \widehat{A},\mathbb{Z})$.*

(i) *Let $n \in \mathbb{Z}_{\geq 1}$ be such that $n \cdot \rho_A$ is algebraic. Then $n^2 \cdot \sigma_A$ is algebraic. In particular, A satisfies the integral Hodge conjecture up to factor* $\gcd(n^2,(2g-2)!)$ *in this case.*

(ii) *If A is principally polarized, and $n \in \mathbb{Z}_{\geq 1}$ is such that $n \cdot \gamma_\theta \in \mathrm{Hdg}^{2g-2}(A,\mathbb{Z})$ is algebraic, then $n \cdot \rho_A \in \mathrm{Hdg}^{4g-2}(A \times \widehat{A},\mathbb{Z})$ is algebraic.*

(iii) *We have that A satisfies the integral Hodge conjecture for one-cycles up to factor* $(2g-2)!$*, and Prym varieties satisfy the integral Hodge conjecture for one-cycles up to factor* 4.

Proof. (i). By Lemma F.3.4, one has

$$\sigma_A = c_1(\mathcal{P}_A)^{2g-2}/(2g-2)! = (-1)^g \cdot (\rho_A)^{\star 2}/2! \in \mathrm{H}^{4g-4}(A \times \widehat{A},\mathbb{Z}).$$

By Theorem F.3.7, this implies that if $n \cdot \rho_A$ is algebraic, then $n^2 \cdot \sigma_A$ is algebraic. Since $(2g-2)! \cdot \sigma_A$ is algebraic, it follows that $\gcd(n^2, (2g-2)!) \cdot \sigma_A$ is algebraic. Thus we are done by Lemma F.5.2.

(ii). This follows from Lemma F.3.5.

(iii). This follows from Lemma F.5.2, parts (i) and (ii) and the fact that if A is a g-dimensional Prym variety with principal polarization $\theta \in \mathrm{Hdg}^2(A, \mathbb{Z})$, then

$$2 \cdot \gamma_\theta \in \mathrm{H}^{2g-2}(A, \mathbb{Z})$$

is algebraic. □

F.6. The integral Tate conjecture for one-cycles on abelian varieties over the separable closure of a finitely generated field

Let X be a smooth projective variety over the separable closure k of a finitely generated field. Let k_0 be a finitely generated field of definition of X. A class $u \in \mathrm{H}^{2i}_{\text{ét}}(X, \mathbb{Z}_\ell(i))$ is an *integral Tate class* if it is fixed by some open subgroup of $\mathrm{Gal}(k/k_0)$. Totaro has shown that for codimension-one cycles on X, the Tate conjecture over k implies the integral Tate conjecture over k [206, Lemma 6.2]. This means that every integral Tate class is the class of an algebraic cycle over k with $\mathbb{Z}_\ell$-coefficients.

Suppose that $A_{/k}$ is an abelian variety, defined over a finitely generated field $k_0 \subset k$ such that k is the separable closure of k_0. Then the Tate conjecture for codimension-one cycles holds for A over k by results of Tate [202], Faltings [70, 71], and Zarhin [218, 219]. By the above, A satisfies the integral Tate conjecture for codimension-one cycles over k. On the other hand, the Fourier transform defines an isomorphism

$$\mathcal{F}_A \colon \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} \mathrm{H}^{2g-2}_{\text{ét}}(\widehat{A}, \mathbb{Z}_\ell(g-1)), \tag{F.6.1}$$

see [206, Section 7]. Since (F.6.1) is Galois-equivariant (the Poincaré bundle being defined over k_0) it sends integral Tate classes to integral Tate classes. Therefore, to prove the integral Tate conjecture for one-cycles on A, it suffices to lift (F.6.1) to a homomorphism $\mathrm{CH}^1(A)_{\mathbb{Z}_\ell} \to \mathrm{CH}_1(\widehat{A})_{\mathbb{Z}_\ell}$.

Proof of Theorem F.1.6. This follows from the above together with Proposition F.3.12.(ii). □

Corollary F.6.1. *Let A and B be abelian varieties defined over the separable closure k of a finitely generated field, of respective dimensions g_A and g_B.*

(i) *The classes $\rho_A \in \mathrm{H}^{4g_A-2}_{\text{ét}}(A \times \widehat{A}, \mathbb{Z}_\ell(2g_A - 1))$ and $\rho_B \in \mathrm{H}^{4g_B-2}_{\text{ét}}(B \times \widehat{B}, \mathbb{Z}_\ell(2g_A - 1))$ are algebraic if and only if $A \times \widehat{A}$, $B \times \widehat{B}$, $A \times B$ and $\widehat{A} \times \widehat{B}$ satisfy the integral Tate conjecture for one-cycles.*

(ii) *If A and B are principally polarized, then the integral Tate conjecture for one-cycles holds for $A \times B$ if and only if it holds for both A and B.*

(iii) *If $\theta \in \mathrm{H}^2_{\text{ét}}(A, \mathbb{Z}_\ell(1))$ is the first Chern class of an ample line bundle that induces a principal polarization on A, and if the minimal class $\gamma_\theta = \theta^{g-1}/(g-1)! \in \mathrm{H}^{2g-2}_{\text{ét}}(A, \mathbb{Z}_\ell(g-1))$ is algebraic, then $\theta^i/i!$ in $\mathrm{H}^{2i}_{\text{ét}}(A, \mathbb{Z}_\ell(i))$ is algebraic for each $i \in \mathbb{Z}_{\geq 1}$.*

Proof. (i). See Equation (F.3.9).

(ii). This is true because the minimal cohomology class of the product is algebraic if and only if the minimal cohomology classes of the factors are algebraic.

(iii). One has $\theta^i/i! = \gamma_\theta^{\star(g-i)}/(g-i)!$ by [20, Corollaire 2]. □

Combining Theorems F.1.1 and F.1.6, we obtain:

Corollary F.6.2. *Let A_K be a principally polarized abelian variety over a number field $K \subset \mathbb{C}$ and let $A_\mathbb{C}$ be its base change to $\mathbb{C}$. Then $A_\mathbb{C}$ satisfies the integral Hodge conjecture for one-cycles if and only if $A_{\bar{K}}$ satisfies the integral Tate conjecture for one-cycles over $\bar{K} = \bar{\mathbb{Q}}$.*

Proof. We view $\bar{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ in a way compatible with the inclusion $K \hookrightarrow \mathbb{C}$. For a prime number ℓ, let $\theta_\ell = c_1(\mathcal{L}) \in \mathrm{H}^2_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(1))$ be the ℓ-adic étale cohomology class of $\mathcal{L}$. On the other hand, define $\theta_\mathbb{C} \in \mathrm{NS}(A) \subset \mathrm{H}^2(A_\mathbb{C}, \mathbb{Z})$ to be the polarization of the complex abelian variety $A_\mathbb{C}$. By Theorems F.1.1 and F.1.6, it suffices to show that $\gamma_{\theta_\mathbb{C}} \in \mathrm{H}^{2g-2}(A_\mathbb{C}, \mathbb{Z})$ is algebraic if and only if $\gamma_{\theta_\ell} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(g-1))$ is in the image of (F.1.1) for each prime number ℓ. We have, by definition, that $\mathrm{Z}^{2g-2}(A) = \mathrm{Coker}\left(\mathrm{CH}_1(A_\mathbb{C}) \to \mathrm{H}^{2g-2}(A_\mathbb{C}, \mathbb{Z})\right)$. This implies that $\mathrm{Z}^{2g-2}(A) \otimes \mathbb{Z}_\ell = \mathrm{Coker}\left(\mathrm{CH}_1(A_\mathbb{C})_{\mathbb{Z}_\ell} \to \mathrm{H}^{2g-2}(A_\mathbb{C}, \mathbb{Z}_\ell)\right)$.

Suppose that γ_{θ_ℓ} is in the image of (F.1.1) for every prime number ℓ. Then $\mathrm{Z}^{2g-2}(A) \otimes \mathbb{Z}_\ell = 0$ for each prime number ℓ by Theorem F.1.6, which means that $\mathrm{Z}^{2g-2}(A) = 0$. Conversely, suppose that $\gamma_\theta = \sum_{i=1}^k n_i \cdot cl(C_i)$ for some smooth projective curves C_i over $\mathbb{C}$. The Hilbert scheme $\mathcal{H} = \mathrm{Hilb}_{A_K/K}$ is defined over K; for each $i = 1, \ldots, k$ we pick a $\bar{\mathbb{Q}}$-point in the connected component of $\mathcal{H}$ containing $[C_i \subset A]$. This gives smooth projective curves $C_i' \subset A_{\bar{\mathbb{Q}}}$ over $\bar{\mathbb{Q}}$ and we define $\Gamma = \sum_i n_i \cdot [C_i'] \in \mathrm{CH}_1(A_{\bar{\mathbb{Q}}})$. On the one hand, we have $cl(\Gamma_\mathbb{C}) = \gamma_{\theta_\mathbb{C}}$ by Lemma F.4.4. On the other hand, the Artin comparison theorem gives an isomorphism of $\mathbb{Z}_\ell$-algebras

$$\phi\colon \mathrm{H}^\bullet_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell) = \mathrm{H}^\bullet_{\text{ét}}(A_\mathbb{C}, \mathbb{Z}_\ell) \cong \mathrm{H}^\bullet(A_\mathbb{C}, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Z}_\ell.$$

Since ϕ is compatible with the cycle class maps $\mathrm{CH}(A_{\bar{\mathbb{Q}}}) \to \mathrm{H}^\bullet_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell)$ and $\mathrm{CH}(A_\mathbb{C}) \to \mathrm{H}^\bullet(A_\mathbb{C}, \mathbb{Z})$, we have $\phi(\gamma_{\theta_\ell}) = \gamma_{\theta_\mathbb{C}}$ and $\phi(cl(\Gamma)) = cl(\Gamma_\mathbb{C}) = \gamma_{\theta_\mathbb{C}}$. Therefore, $cl(\Gamma) = \gamma_{\theta_\ell}$. □

Another corollary of Theorem F.1.6 is that the integral Tate conjecture for one-cycles on abelian varieties is stable under specialization. For example, one has the following (c.f. Corollary F.4.3):

Corollary F.6.3. *Let A_K be a principally polarized abelian variety over a number field K and suppose that $A_{\bar{K}}$ satisfies the integral Tate conjecture for one-cycles over $\bar{K}$. Let $\mathfrak{p}$ be a prime ideal of the ring of integers $\mathcal{O}_K$ of K at which A_K has good reduction and write $\kappa = \mathcal{O}_K/\mathfrak{p}$. Then the abelian variety $A_{\bar{\kappa}}$ over $\bar{\kappa}$ satisfies the integral Tate conjecture for one-cycles over $\bar{\kappa}$.*

Proof. Write $S = \mathrm{Spec}\,\mathcal{O}_K$ and let $A \to S$ be the Néron model of A_K. Let R (resp. $K_\mathfrak{p}$) be the completion of $\mathcal{O}_K$ (resp. K) at the prime $\mathfrak{p}$. The natural composition $K \to K_\mathfrak{p} \to \bar{K}_\mathfrak{p}$ induces an embedding $\bar{K} \to \bar{K}_{\mathfrak{p}}$, where $\bar{K}_{\mathfrak{p}}$ is an algebraic closure of $K_{\mathfrak{p}}$. This gives a commutative diagram, where the square on the right is provided in [76, Example 20.3.5]:

$$\begin{tikzcd}
\mathrm{CH}(A_{\bar{K}})_{\mathbb{Z}_\ell} \arrow[r] \arrow[d] & \mathrm{CH}(A_{\bar{K}_{\mathfrak{p}}})_{\mathbb{Z}_\ell} \arrow[r] \arrow[d] & \mathrm{CH}(A_{\bar{\kappa}})_{\mathbb{Z}_\ell} \arrow[d] \\
\oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_\ell(r)) \arrow[r, "\sim"] & \oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(A_{\bar{K}_{\mathfrak{p}}}, \mathbb{Z}_\ell(r)) \arrow[r, "\sim"] & \oplus_{r\geq 0}\mathrm{H}^{2r}_{\text{ét}}(A_{\bar{\kappa}}, \mathbb{Z}_\ell(r)).
\end{tikzcd} \tag{F.6.2}$$

Now the principal polarization $\lambda_K \colon A_K \xrightarrow{\sim} \widehat{A}_K$ extends uniquely to a homomorphism $\lambda \colon A \to \widehat{A}$ by the Néron mapping property [37, Section 1.2, Definition 1] and since the same is true for the inverse $\lambda_K^{-1} \colon \widehat{A}_K \xrightarrow{\sim} A_K$ we find that λ is an isomorphism. In particular, we see that $A_{\bar{\kappa}}$ is principally polarized and that the class in $\mathrm{CH}^1(A_{\bar{K}})_{\mathbb{Z}_\ell}$ of a theta divisor on $A_{\bar{K}}$ is sent to the class in $\mathrm{CH}^1(A_{\bar{\kappa}})_{\mathbb{Z}_\ell}$ of a theta divisor on $A_{\bar{\kappa}}$. Thus, the minimal class $\gamma_{\theta_{\bar{K}}} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_\ell(g-1))$ is sent to the minimal class $\gamma_{\theta_{\bar{\kappa}}} \in \mathrm{H}^{2g-2}_{\text{ét}}(A_{\bar{\kappa}}, \mathbb{Z}_\ell(g-1))$ by the isomorphism on the bottom of Diagram (F.6.2). It follows that $\gamma_{\theta_{\bar{\kappa}}}$ is algebraic which by Theorem F.1.6 means that we are done. $\square$

Finally, let us prove Theorem F.1.7. The theorem follows from Theorem F.1.6 together with a result of Chai on the density of an ordinary isogeny class in positive characteristic [51].

Proof of Theorem F.1.7. For any $t \in A_g(k)$, let (A_t, λ_t) be a principally polarized abelian variety such that $[(A_t, \lambda_t)] = t$. Let $A = E_1 \times \cdots \times E_g$ be the product of g ordinary elliptic curves E_i over k and provide A with its natural principal polarization. Let $x \in \mathsf{A}_g(k)$ be the point corresponding to the isomorphism class of A. Let $q > (g-1)!$ be a prime number different from p and let $\mathcal{G}_q(x) \subset \mathsf{A}_g(k)$ be the set of isomorphism classes $y = [(A_y, \lambda_y)]$ that admit an isogeny $\phi \colon A_y \to A_x$ with $\phi^*\lambda_x = q^N \cdot \lambda_y$ for some nonnegative integer N. We claim that A_y satisfies the integral Tate conjecture for one-cycles over k for any $y \in \mathcal{G}_q(x)$. Indeed, for such y there exists a nonnegative integer N such that the isogeny $[q^N] \colon A_y \to A_y$ factors through A_x. Consequently, $q^{(2g-2)\cdot N} \cdot \gamma_\theta$ is algebraic for the first Chern class θ of the principal polarization on A_y, which implies that γ_θ is algebraic (as $q > (g-1)!$). Thus, the claim follows from Theorem F.1.6. Now $\mathcal{G}_q(z)$ is dense in A_g for any ordinary principally polarized abelian variety (A_z, λ_z) by a result of Chai [51, Theorem 2]. Therefore, $\mathcal{G}_q(x)$ is dense in A_g and the proof is finished. $\square$

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