Further Estimates for Certain Integrals of Six Bessel Functions

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Abstract

In this thesis we study integrals involving a sixfold product of Bessel functions of the first kind and integer order. We establish good asymptotic estimates with precise error bounds for a certain one-parameter subclass of integrals. In the general case of six arbitrary integer orders we conjecture a formula for the asymptotics of the integral that is consistent with the special case. Testing this formula on numerical reference data suggests that it approximates the integral very accurately if at least one of the integers is larger than 20.
The interest in these kinds of Bessel integrals stems from the study of a sharp Fourier restriction inequality on the circle and a program to characterize its extremizers by looking at those integrals. This program is proposed in [14] and [1] and also includes the works [12], [13] and [15]. This thesis lines up here and builds upon the work of Oliveira e Silva and Thiele in [1]. We improve and extend their methods and generalize the results to a much wider range of integrals.

The thesis consists of seven chapters.

In Chapter 1 we discuss the connection between the study of extremizers in the Fourier restriction theory on the circle and integrals of a sixfold product of Bessel functions. We review the existing results on asymptotic bounds for those integrals and state our main theorem. We then outline the four major parts of the proof and give a summary of the used techniques. Each of the Chapters 2, 3, 4 and 5 elaborates on one of the parts of the proof.

In Chapter 2 we expand a product of four Bessel functions into a power series of finite length. If all four functions are equal we provide an expression for the remainder term.

In Chapter 3 we replace four of the six Bessel functions in our integral with the power series we derived in Chapter 2 and deduce an alternative representation of the initial integral in terms of sums of quotients of gamma functions and hypergeometric functions. We identify those parts of this representation that carry the asymptotic information of the integral and those that contribute to the error. In the case of six different Bessel functions of arbitrary integer orders we only establish the main asymptotic term without a proof.

Chapters 4 and 5 are entirely dedicated to the one-parameter subclass of integrals such that the orders of the Bessel functions form the six vector $(n, n, n, n, 2n, 2n) \in \mathbb{Z}^6$. More precisely, in Chapter 4 we prove upper bounds on all components of the decomposition of the integral we deduced in Chapter 3. The analytical approach of this chapter fails for finitely many integrals. Those are estimated numerically in Chapter 5.

In Chapter 6 we take a closer look at the conjectured formula for the asymptotics of the general Bessel integral and test its quality on numerical data for selected subfamilies of integrals. Those numerical values have been calculated for the paper [13]. To demonstrate the excellent performance of our formula we use it to reproduce some of the findings of [13]. We then sketch open problems that have to be solved in order to turn the conjecture into a theorem and list some interesting questions for further research on this topic.

Chapter 7 provides results from the theory of gamma, Bessel and hypergeometric functions, as well as some other useful inequalities and identities, that are frequently used throughout this thesis.

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First and foremost, I would like to thank my advisor, Prof. Dr. Christoph Thiele, for giving me this invaluable chance and making this work possible. He always took his time and patiently worked his way through pages of awful calculations with me week after week. Even though it was certainly not always possible to follow along my highly technical descriptions of the problems I faced estimating huge quotients of gamma functions, I still left the office of Prof. Thiele happy and with new ideas almost every Friday because of his priceless experience and intuition. This and his lasting excitement and interest in the plots and graphics I showed him weekly also helped me to overcome periods of frustration and lack of motivation. Going beyond the professional relationship, Prof. Thiele also managed to be an excellent advisor on a human level. I always felt valued and comfortable in discussions with him and especially enjoyed that they were not limited to mathematics.

I would also like to thank the rest of my academic siblings at the University of Bonn and the entire Harmonic Analysis and PDE group for welcoming me and giving me a sense of belonging.

Special thanks go to Johann David Wochner and the Software *Wolfram Mathematica*. The Python code of Johann David Wochner saved me a lot of time in the analysis of the huge file that contained the numerical reference data I used for parts of my work. Without *Mathematica* the thesis in this form and many calculations inside it would not be possible. The software was not only essential to the brainstorming phase of the entire project, it was also used to execute the numerical part of this work and even strongly influenced the approaches we took in the analytical part.

Last but not least, I am grateful to my family and friends, who are always there for me with their unconditional support. In particular I would like to express my gratitude to my boyfriend, Jan Wirbatz, who had to face my moods, ups and downs and periods of frustration at the front line and always managed to cheer me up again.

Contents

CHAPTER 1

Introduction and Motivation

The main objective of this work is to get a better understanding of the behavior of the integral

$$I(\mathbf{n}) := \int_0^\infty \prod_{i=0}^{6} J_{n_i}(r) r dr \tag{1}$$

with a six vector $\mathbf{n} = (n_1, n_2, n_3, n_4, n_5, n_6) \in \mathbb{Z}^6$ and a product of Bessel functions J_{n_i} of the first kind.

The interest in these integrals arose from the study of extremizers in Fourier restriction theory. For a function $f \in L^2(0,1)$, that we view as a density on the unit circle on the plane, we consider the Fourier transform

$$g(\xi, \eta) := \int_0^1 f(\theta) e^{-i(\xi \cos(2\pi\theta) + \eta \sin(2\pi\theta))} d\theta \tag{2}$$

of the corresponding measure on the circle. The famous Tomas-Stein theorem asserts that for $||f||_2$ in the unit ball, $||f||_2 \leqslant 1$, the norm $||g||_6$ is finite and bounded by a universal constant. It is conjectured that $||g||_6$ is maximal among all f in the unit ball, if f is the constant function $f \equiv 1$. This conjecture, which we will refer to as the sharp Tomas-Stein conjecture for the circle, has partially been verified in [13] for the class of all real-valued functions with Fourier mode up to degree 120, that is real-valued functions of the form

$$f(\theta) = \sum_{n=-120}^{120} \hat{f}_n e^{2\pi i n\theta}. \tag{3}$$

Expressing g in terms of the $\hat{f}_n$ leads to the Bessel integral $I(\mathbf{n})$. The proof in [13] consists of reducing the problem to numerically showing positive semi-definiteness of certain matrices. The entries of the matrices are of the form

$$Q(\mathbf{x}, \mathbf{y}) := \frac{1}{6} \sum_{\sigma \in S_3} (R(\mathbf{x}, \sigma(\mathbf{y})) - L(\mathbf{x}, \sigma(\mathbf{y})), \tag{4}$$

$$R(\mathbf{x}, \mathbf{y}) := 2I(\mathbf{x} - \mathbf{y}, 0, 0, 0) + \sum_{\sigma \in S_3} I(\mathbf{x} - \mathbf{y}, \sigma(-1, 1, 0)), \tag{5}$$

$$L(\mathbf{x}, \mathbf{y}) := 2I(\mathbf{x}, -\mathbf{y}) + \sum_{\sigma \in S_3} I(\mathbf{x}, -\mathbf{y} + \sigma(-1, 1, 0)), \tag{6}$$

and $I(\mathbf{x}, \mathbf{y})$ is given by (1) with $\mathbf{n} = (\mathbf{x}, \mathbf{y})$.

The paper [13] is part of a program proposed in [14] and [1], to study the Fourier restriction theory of the circle by looking at the above matrix coefficients, use a computer for values of $\mathbf{n}$ in a large finite region and use good asymptotic estimates on $I(\mathbf{n})$ for $\mathbf{n}$ outside this finite region. The paper [1] establishes good asymptotic estimates for $I(\mathbf{n})$ with $n_4 = n_5 = n_6 = 0$, or $n_4 = n_5 = 1$ and $n_6 = 0$. It should be emphasized that the existence of asymptotic estimates follows from general abstract principles, the point of paper [1] is to obtain good estimates for the error between the asymptotic approximation and the absolute value of the integral. The paper [14] uses these asymptotic estimates together with a computer to make partial progress towards the sharp Tomas-Stein conjecture for the circle. Namely, the extension map from f to g is factored as the composition of two maps

and a sharp estimate is proven for one of the two maps.
It should be noted that the program from [14] and [1] stems from an approach of Foschi [15] for Fourier restriction on the sphere in three dimensions. Interestingly, this problem is much easier than the problem on the circle, since the exponent 6 is exchanged by an exponent of 4. This allows a purely analytical proof and does not lead to the distinction between a numerical and an asymptotic region.

The goal of this thesis is to extend the estimates in [1] to a larger range of parameters. We improve on the approach of [1] to obtain good asymptotic estimates for $n_1 = n_2 = n_3 = n_4 = n$ and $n_5 = n_6 = 2n$. In addition we conjecture an asymptotic formula for the general case that is motivated by this approach. We test this formula on numerical data that has been calculated on a modern computer cluster for [13].

We recall the main result of [1].

Theorem 1. *It is for* $n \geqslant 7$

$$\left| I(0,0,0,0,n,n) - \frac{3}{4\pi^2}\frac{1}{n} + \frac{3}{32\pi^2(n-1)n(n+1)} \right| \leqslant \frac{0.002}{n^4},$$

and for $n \geqslant 3$

$$\left| I(0,0,1,1,n,n) - \frac{3}{4\pi^2}\frac{1}{n} - \frac{3}{32\pi^2(n-1)n(n+1)} \right| \leqslant \frac{0.002}{n^4}.$$

Moreover, for $n \geqslant 2$ *and* $j = 0, 1$, *and with* $c_{2,0} = 15, c_{2,1} = 9$ *it is*

$$\left| I(0,j,j,2,n,n+2) - \frac{c_{2,j}}{64\pi^2 n(n+1)(n+2)} \right| \leqslant \frac{0.002}{n^4},$$

for $n \geqslant 4$ *and* $j = 0, 1$ *and with* $c_{4,0} = 1557, c_{4,1} = 855$ *it is*

$$\left| I(0,j,j,4,n,n+4) - \frac{c_{4,j}}{1024\pi^2 n(n+1)(n+2)(n+3)(n+4)} \right| \leqslant \frac{0.0015}{n^4},$$

and for $6 \leqslant m \leqslant n$, *even* m *and* $j = 0, 1$ *it is*

$$|I(0,j,j,m,n,n+m)| \leqslant \frac{0.0015}{n^4}.$$

The ultimate goal is to deduce similar bounds the ones above in the general case of six arbitrary integers $n_1, \ldots, n_6$. This would take us even one step closer to the sharp Tomas-Stein conjecture for the circle.

A particular focus in the application to Fourier restriction theory lies on those tuples of indices that add up to zero. Bessel functions of integer order obey the symmetry $J_{-n}(r) = (-1)^n J_n(r)$. This allows us to restrict ourselves to non-negative integers. Passing to non-negative indices, however, hides whether the tuple may add up to zero or not. Most of the tuples we consider stem from tuples summing to zero.

The method of [1], that we refine, suggests the existence of a main term $M(\mathbf{n})$, that describes the overall asymptotic behavior of the integral (1).

Conjecture 2. Let $\mathbf{n} \in \mathbb{Z}^6$ be an arbitrary nonnegative six vector such that n_5 and n_6 are the two largest indices. Moreover, define the set

$$D_{\mathbf{n}} = \left\{ k \in \mathbb{Z}, k \geqslant 0 \,\middle|\, \begin{matrix} \max\{|n_5 - n_6| - 1, 0\} \leqslant k \leqslant n_5 + n_6 - 1 \\ n_1 + n_2 + n_3 + n_4 + k \text{ even} \end{matrix} \right\}.$$

Then $I(\mathbf{n})$ is well approximated by $M(\mathbf{n})$, if at least one of the entries of $\mathbf{n}$ is large, where

$$\begin{aligned}
M(\mathbf{n}) &= \frac{1}{(2\pi)^2} \\
&\times \sum_{k\in D_{\mathbf{n}}} (-1)^{\frac{n_1+n_2-n_3-n_4+k}{2}} \frac{2^{-2k}\Gamma(k+1)\Gamma\left(\frac{n_5+n_6-k}{2}\right)}{\Gamma\left(\frac{n_5-n_6+k}{2}+1\right)\Gamma\left(-\frac{n_5-n_6+k}{2}+1\right)\Gamma\left(\frac{n_5+n_6+k}{2}+1\right)} \\
&\times \sum_{i=0}^{k} (-1)^i \frac{\Gamma\left(n_1+i+\frac{1}{2}\right)\Gamma\left(n_3+k-i+\frac{1}{2}\right)}{\Gamma(i+1)\Gamma(k-i+1)\Gamma\left(n_1-i+\frac{1}{2}\right)\Gamma\left(n_3-k+i+\frac{1}{2}\right)} \\
&\times \Bigg[{}_3F_2\left(\begin{matrix} -n_2+\frac{1}{2}, n_2+\frac{1}{2}, -i \\ -n_1-i+\frac{1}{2}, n_1-i+\frac{1}{2} \end{matrix} \,\Big|\, -1\right) {}_3F_2\left(\begin{matrix} -n_4+\frac{1}{2}, n_4+\frac{1}{2}, -k+i \\ -n_3-k+i+\frac{1}{2}, n_3-k+i+\frac{1}{2} \end{matrix} \,\Big|\, -1\right) \\
&\quad + {}_3F_2\left(\begin{matrix} -n_2+\frac{1}{2}, n_2+\frac{1}{2}, -i \\ -n_1-i+\frac{1}{2}, n_1-i+\frac{1}{2} \end{matrix} \,\Big|\, 1\right) {}_3F_2\left(\begin{matrix} -n_4+\frac{1}{2}, n_4+\frac{1}{2}, -k+i \\ -n_3-k+i+\frac{1}{2}, n_3-k+i+\frac{1}{2} \end{matrix} \,\Big|\, 1\right) \\
&\quad \times (-1)^{n_2+n_4}\left(1+(-1)^{n_1-n_2+i}\right)\Bigg].
\end{aligned} \tag{7}$$

In practice the approximation appears very useful, if the largest entry of the vector $\mathbf{n}$ is larger than 20. We want to note at this point that the asymmetry of $M(\mathbf{n})$ with respect to n_1, n_2, n_3, n_4 is caused artificially by some transformations we carry out in Chapter 2. The starting term is completely symmetric with respect to those variables. The *Wolfram Mathematica* program code we use in Chapter 6 to evaluate M for some specific values of $\mathbf{n}$ puts the entries of $\mathbf{n}$ in ascending order bevor inserting it into the formula. However, a runtime analysis of different permutations of the variables n_1, n_2, n_3, n_4, suggests that the ordering does not affect the performance of the calculation. At least not in the setting of *the Wolfram Language*.
Moreover, we want to point out that $M(\mathbf{0}) = 0$, since $D_{\mathbf{0}}$ is the empty set. Unfortunately, that is far off. However, we conjecture that the error $|I(\mathbf{n}) - M(\mathbf{n})|$ decreases rapidly with increasing size of $D_{\mathbf{n}}$. The verification and quantification of this assertion is another open problem.

The difficulty in our method is the rigorous quantitative estimation of the remaining terms that are small compared to $M(\mathbf{n})$.
To simplify the technicalities and display our ideas in an easier to digest form, we restrict our attention to the one-parameter subfamily $\mathbf{n} = (n, n, n, n, 2n, 2n)$. Our main result is the following.

THEOREM 3. *Let $n \in \mathbb{N}$ and $n \geqslant 2$. Then it is for $\mathbf{n} = (n, n, n, n, 2n, 2n)$*

$$\left| I(\mathbf{n}) - \frac{3^{\frac{1}{2}}}{4\pi^2}\frac{1}{n} + \frac{23}{2^8 3^{\frac{3}{2}}\pi^2}\frac{1}{n\left(n^2-\frac{1}{4}\right)} \right| \leqslant 8.002183 \cdot 10^{-4}\,\frac{1}{n(n^2-\frac{1}{4})(n^2-1)}. \tag{8}$$

This family is analytically more difficult than the ones in [1] and requires additional steps and arguments. These arguments should be strong enough to bound all terms that are small compared to the main term M also for a larger range of vectors $\mathbf{n} \in \mathbb{Z}^6$, namely, when the four smallest indices are sufficiently dominated by the remaining two. However, the full range of parameters, for example $(0, 0, n, n, n, n)$, and thus a proof of Conjecture 2 or an equivalent quantitative result to Theorem 3, may require additional ideas.

The motivation to study the one-parameter subfamily $\mathbf{n} = (n, n, n, n, 2n, 2n)$ stems from a numerical approximation of the two-parameter subfamily $\mathbf{n} = (m, m, n, n, m+n, m+n)$ of positive integrands. In Figure 1.1 the discrete data points of $I(\mathbf{n})$ are joined with piecewise polygonal surface elements. This way the obvious symmetry of the integral with respect to the main diagonal $m = n$ becomes visible.

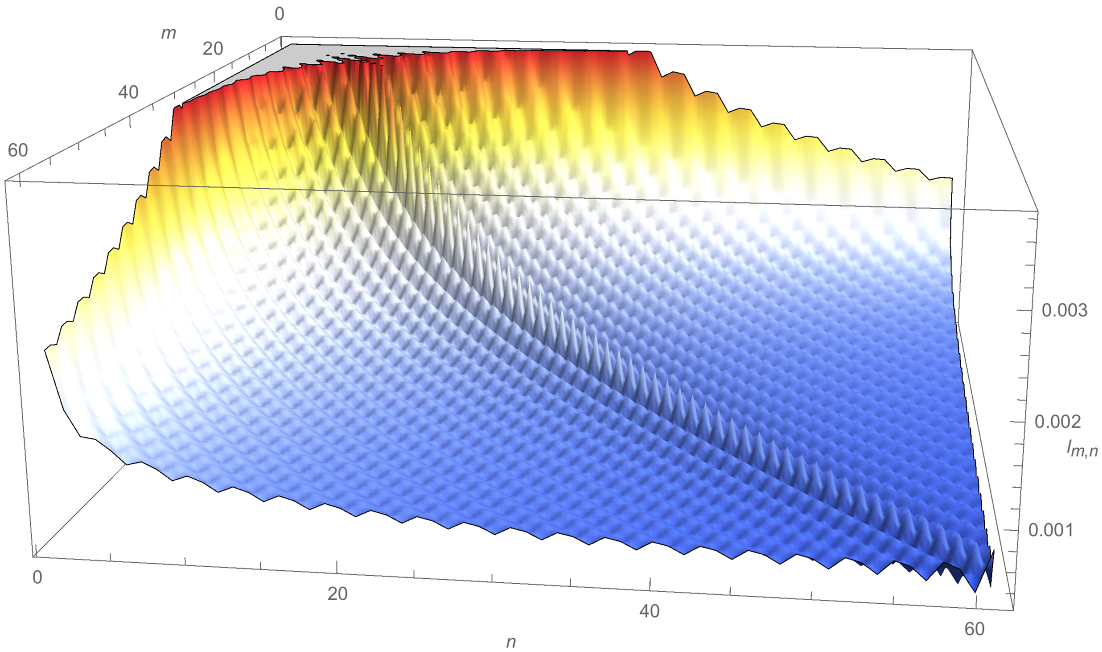

FIGURE 1.1. Interpolated listplot of $I(m, m, n, n, m + n, m + n)$.

The literally outstanding appearance of the diagonal itself piqued our interest and lead to the decision to further investigate this case.

As mentioned above our approach goes back to [1] and follows a program that consists of four major steps.

Step 1 We pick those four Bessel functions with the smallest indices and expand their product into a series with finite $\mathbf{n}$-dependent length and a remainder term.

Step 2 In $I(\mathbf{n})$ we replace the product of the four Bessel functions with the smallest indices by its series expansion and the remainder term. This decomposes $I(\mathbf{n})$ into a sum of integrals involving the remaining two Bessel functions. We express these integrals in terms of hypergeometric functions and quotients of products of gamma functions to make them accessible for upcoming estimations.

Step 3 We take the integral-free representation of $I(\mathbf{n})$ from Step 2 and provide upper bounds for all occurring terms.

Step 4 The analytical approach of Step 3 fails for finitely many $\mathbf{n}$. Those integrals are estimated numerically.

The majority of the work is hidden in Step 3. Although our techniques are quite elementary, the sheer amount of different cases we have to take into consideration makes this the largest and most difficult part of the proof of Theorem 3.
Step 2 is the one that provides us with a candidate for the main approximating term $M(\mathbf{n})$ of $I(\mathbf{n})$. The integrals we mentioned in the outline of the program can all be assigned to

one of the following classes

$$I_{0r}(k,\mathbf{n}) := \int_0^\infty J_{n_5}(r)J_{n_6}(r)r^{-k-1}dr,$$

$$I_{2r}(k,\mathbf{n}) := \int_0^\infty \sin\left(\frac{\sigma(\mathbf{n})+k}{2}\pi + 2r\right) J_{n_5}(r)J_{n_6}(r)r^{-k-1}dr,$$

$$I_{4r}(k,\mathbf{n}) := \int_0^\infty \cos\left(\frac{\sigma(\mathbf{n})+k}{2}\pi + 4r\right) J_{n_5}(r)J_{n_6}(r)r^{-k-1}dr,$$

with $\sigma(\mathbf{n}) = \sum_{i=1}^{i} c_i n_4$ and $c_i \in \{-1, 1\}$. Here we assume that the entries of the six vector $\mathbf{n}$ have ascending order. There exists a quite comprehensive theory of integrals of two and three Bessel functions, that includes results and methods we can use to analyze the integrals $I_{0r}(k,\mathbf{n})$, $I_{2r}(k,\mathbf{n})$ and $I_{4r}(k,\mathbf{n})$. Let us in particular mention Lemmata 56, 57 and 59, which are essential for our work. We will then see that the integrals $I_{0r}(k,\mathbf{n})$ make up the main term.

More precisely, our work is structured as follows. In Chapter 2 we carry out the series expansion described in Step 1. Chapter 3 then covers Step 2. Namely, the identification of the integrals I_{0r}, I_{2r} and I_{4r} and the application of the aforementioned lemmata to those integrals. At the end of Chapters 2 and 3 we are able to express $I(\mathbf{n})$ as a sum of three convolution-like sums of products and quotients of gamma functions for the subfamily $\mathbf{n} = (n, n, n, n, 2n, 2n)$. On our way to this representation, we also deduce the conjectured main term $M(\mathbf{n})$, that we stated in Conjecture 2 for general $\mathbf{n}$. Our goal throughout those two chapters is to present our approach in detail for the subfamily $\mathbf{n} = (n, n, n, n, 2n, 2n)$, while in the generalization for arbitrary $\mathbf{n} \in \mathbb{Z}^6$ we only consider those terms that we suppose to have significant contribution to $I(\mathbf{n})$.
Chapter 4 implements Step 3 for the subfamily $\mathbf{n} = (n, n, n, n, 2n, 2n)$ and is entirely devoted to the analytical proof of Theorem 3. An in-depth analysis, including precise error bounds of the three summands we deduce in Chapter 3, is done in Sections 4.2, 4.3 and 4.4.
Chapter 5 completes the proof of Theorem 3 by a numerical estimate of $I(\mathbf{n})$ for the values of n beyond the analytical scope of the proof in Chapter 4.
The purpose of Chapter 6 is twofold. We first provide a numerical validation of Conjecture 2. As reference data we use numerical values for a large class of $I(\mathbf{n})$, that have been calculated for [13] on a modern computer cluster with a prescribed accuracy of $0.73 \cdot 10^{-9}$. Afterwards we draw a deeper connection between our work and [13] by reproducing parts of the findings in [13] using our general main term $M(\mathbf{n})$. Since the numerical validation of the general main term is far from a rigorous proof, we conclude Chapter 6 with a list of open tasks and problems that have to be solved in order to prove Conjecture 2 and to quantify the error between $M(\mathbf{n})$ and $I(\mathbf{n})$.
To round off this work, Chapter 7 provides background on the theory of gamma, Bessel and hypergeometric functions to an extend that is sufficient for our purposes. We review and prove useful bounds and identities for different types of sums and integrals, including the Lemmata 56, 57 and 59 we mentioned earlier.

CHAPTER 2

Series Expansion of a Product of Four Bessel Functions

In this chapter we jump straight into Step 1 of our program on page 4 towards the analysis of the integral (1). In Lemma 7 we provide a formula for the series expansion of the product $J_n^4(r)$ on the basis of the expansion coefficients of a single Bessel function $J_n(r)$. The expression for the k-th expansion coefficient is then generalized for the case of a product of four Bessel functions with different indices in formula (24). We obtain a more precise expression for the k-th expansion coefficient of the product of four different Bessel functions in Theorem 8, by using information on the coefficient of a single Bessel function from Lemma 4. As a direct consequence of Theorem 8, we deduce in Corollary 9 the corresponding formula for the k-th expansion coefficient of the one-parameter case $J_n^4(r)$. The remaining Lemmata 10 and 11 provide closed form expressions for parts of the formulae in Theorem 8 and Corollary 9. This makes upcoming calculations less cumbersome.
To avoid ambiguity of indices, we will throughout this and the following chapter denote the general six-vector by $\mathbf{n} = (\alpha, \beta, \gamma, \delta, x, y)$ and assume that x and y are the two largest indices.

As mentioned above, at the heart of all calculations in this chapter lies the series expansion of a single Bessel function at infinity. It has first been used in the context of integrals of Bessel functions by Oliveira e Silva and Thiele in [1].

LEMMA 4 ([1]). *Let $n \in \mathbb{N}$ and $\mathrm{Re}(z) > 0$. Let further $l \in \mathbb{N}$ be such that $l \geqslant \max\{n - \frac{1}{2}, 1\}$. If l is even, then*

$$\left(\frac{\pi}{2}z\right)^{\frac{1}{2}} J_n(z) = \cos\left(z - \frac{\pi}{4} - \frac{\pi}{2}n\right) \sum_{k=0}^{\frac{l}{2}-1} (-1)^k a_{2k}(n)\, z^{-2k} - \sin\left(z - \frac{\pi}{4} - \frac{\pi}{2}n\right) \sum_{k=0}^{\frac{l}{2}-1} (-1)^k a_{2k+1}(n)\, z^{-(2k+1)} + R_{n,l}(z). \tag{9}$$

If l is odd, then

$$\left(\frac{\pi}{2}z\right)^{\frac{1}{2}} J_n(z) = \cos\left(z - \frac{\pi}{4} - \frac{\pi}{2}n\right) \sum_{k=0}^{\frac{l-1}{2}} (-1)^k a_{2k}(n)\, z^{-2k} - \sin\left(z - \frac{\pi}{4} - \frac{\pi}{2}n\right) \sum_{k=0}^{\frac{l-3}{2}} (-1)^k a_{2k+1}(n)\, z^{-(2k+1)} + R_{n,l}(z). \tag{10}$$

The expansion coefficients $a_j(n)$ are defined as

$$a_j(n) = \frac{\Gamma(n + j + \frac{1}{2})}{j! 2^j\, \Gamma(n - j + \frac{1}{2})}, \tag{11}$$

and the remainder $R_{n,l}(z)$ satisfies

$$|R_{n,l}(z)| \leqslant |a_l(n)| \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{l-n+\frac{1}{2}} \cosh\left(\mathrm{Im}(z)\right) |z|^{-l}. \tag{12}$$

For the proof of Lemma 4 Oliveira e Silva and Thiele fall back on the Poisson integral representation (377) of the Bessel function $J_n(z)$ and perform the change of variables $t = \cos(\theta)$ to turn it into

$$J_n(z) = \frac{z^n}{2^n \Gamma\left(\frac{1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)} \int_{-1}^{1} \cos(zt) \left(1-t^2\right)^{n-\frac{1}{2}} \, dt.$$

Then they split

$$J_n(z) = \frac{1}{2}\left(J_n^+(z) + J_n^-(z)\right),$$

with

$$J_n^+(z) = \frac{z^n}{2^n \Gamma\left(\frac{1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)} \int_{-1}^{1} e^{izt} \left(1-t^2\right)^{n-\frac{1}{2}} \, dt$$

and $J_n^-(z) = \overline{J_n^+(\overline{z})}$, and prove the claimed identities (9) and (10) via contour integration and the Taylor expansion of $(1+x)^{n-1}$ for both functions J_n^+ and J_n^-.
Their technique allows Oliveira e Silva and Thiele to immediately deduce explicit first and second order asymptotics from Lemma 4 that are valid for large z close to the positive real axis.

Corollary 5 ([1]). *Let $n \geqslant 2$ and $z \in \mathbb{C}$ such that $\mathrm{Im}(z) < \mathrm{Re}(z)$ and $\mathrm{Re}(z) > n^2$. Then*

$$\left| J_n^{\pm}(z) - \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{\pi}{4} - \frac{\pi}{2}n\right)\right| \leqslant \left(\frac{2}{\pi|z|}\right)^{\frac{1}{2}} \frac{n^2}{|z|} \cosh\left(\mathrm{Im}(z)\right) \left(\frac{|z|}{|\mathrm{Re}(z)|}\right)^{\frac{1}{2}}, \tag{13}$$

and

$$\begin{aligned} &\left| J_n^{\pm}(z) - \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{\pi}{4} - \frac{\pi}{2}n\right) + \frac{4n^2-1}{8z}\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin\left(z - \frac{\pi}{4} - \frac{\pi}{2}n\right)\right| \\ &\leqslant \left(\frac{2}{\pi|z|}\right)^{\frac{1}{2}} \frac{n^4}{4|z|^2} \cosh\left(\mathrm{Im}(z)\right) \left(\frac{|z|}{|\mathrm{Re}(z)|}\right)^{\frac{1}{2}}. \end{aligned} \tag{14}$$

For the sake of simplicity in notation we follow [1] and set from now on

$$\mathfrak{J}_n(r) := \left(\frac{\pi}{2} r\right)^{\frac{1}{2}} J_n(r)$$

and abbreviate the asymptotic expansion of $\mathfrak{J}_n(r)$ with length l by

$$\mathfrak{J}_n(r) = \sum_{k=0}^{l-1} c_k(n,r) r^{-k} + R_l(n) r^{-l}. \tag{15}$$

The exact formulas for the expansion coefficients $c_k(n)$ and the remainder term $R_l(n)$ can be read from Lemma 4. Taking absolute values, we find that

$$\begin{aligned} |c_k(n,r)| &\leqslant |a_k(n)|, \\ |R_l(n)| &\leqslant \begin{cases} \left(n+\frac{1}{2}\right)^{\frac{1}{4}} =: r_0(n), & l = 0, \\ |a_l(n)|, & l \geqslant 1 \end{cases} \end{aligned} \tag{16}$$

for $n \geqslant 1$. The estimate in the case $l = 0$ is due to Lemma 55 .

On the following pages, starting from (15) we first derive quite general expressions for the asymptotic expansion of the product of two and of four Bessel functions of the same index n. In a second step, we generalize the resulting expansion coefficient for the product of four different Bessel functions with nonnegative indices $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. Afterwards, we plug in the specific expression for the expansion coefficients and the remainder term of a single Bessel function and simplify the result algebraically as much as possible.
So, let us start with $\mathfrak{J}_n^2(r)$.

Lemma 6. *Let $n \in \mathbb{N}, n \geqslant 1$ and $r \geqslant 0$. Then the asymptotic expansion of $\mathfrak{J}_n^2(r)$ with length $l \geqslant 1$ is*

$$\mathfrak{J}_n^2(r) = \sum_{k=0}^{l-1} c_k^{(2)}(n,r)r^{-k} + R_l^{(2)}(n,r)r^{-l},$$

with

$$c_k^{(2)}(n,r) = \sum_{i=0}^{k} c_i(n,r)c_{k-i}(n,r) \tag{17}$$

and

$$R_l^{(2)}(n,r) = \sum_{k=0}^{l-1} c_k(n,r)R_{l-k}(n) + R_l(n)R_0(n). \tag{18}$$

Proof. Let $l \geqslant 1$ and

$$\mathfrak{J}_n(r) = \sum_{k=0}^{l-1} c_k(n,r)r^{-k} + R_l(n)r^{-l}.$$

Then it is by the definition of a remainder term

$$\begin{aligned}\sum_{m=j}^{l-1} c_m(n,r)r^{-m} + R_l(n)r^{-l} &= \mathfrak{J}_n(r) - \sum_{m=1}^{j-1} c_m(n,r)r^{-m} \\ &= R_j(n)r^{-j}\end{aligned}$$

for all $0 \leqslant j \leqslant l-1$, and the square of the expansion of $\mathfrak{J}_n(r)$ simplifies to

$$\begin{aligned}\mathfrak{J}_n(r)^2 &= \left(\sum_{k=0}^{l-1} c_k(n,r)r^{-k} + R_l(n)r^{-l}\right)^2 \\ &= \sum_{k=0}^{l-1}\left(\sum_{i=0}^{k} c_i(n,r)c_{k-i}(n,r)\right)r^{-k} + c_0(n,r)R_l(n)r^{-l} \\ &+ \sum_{k=1}^{l-1} c_k(n,r)\left(\sum_{i=l-k}^{l-1} c_i(n,r)r^{-i} + R_l(n)r^{-l}\right)r^{-k} \\ &+ R_l(n)\sum_{k=0}^{l-1}\left(c_k(n,r)r^{-k} + R_l(n)r^{-l}\right)r^{-l} \\ &= \sum_{k=0}^{l-1}\sum_{i=0}^{k} c_i(n,r)c_{k-i}(n,r)r^{-k} + \left(\sum_{k=0}^{l-1} c_k(n,r)R_{l-k}(n) + R_l(n)R_0(n)\right)r^{-l}.\end{aligned}$$

As we can see, the expansion coefficients $c_k^{(2)}(n)$ of the expansion of two Bessel functions are

$$c_k^{(2)}(n,r) = \sum_{i=0}^{k} c_i(n,r)c_{k-i}(n,r),$$

and the remainder term $R_l^{(2)}(n)$ is

$$R_l^{(2)}(n,r) = \sum_{k=0}^{l-1} c_k(n,r)R_{l-k}(n) + R_l(n)R_0(n).$$

∎

The trick of cutting the square of a sum into a triangular sum and a linear one makes all forthcoming calculations easier and less cumbersome since the number of terms we have to take into consideration reduces drastically. This technique was already used by Oliveira e Silva and Thiele in [1] for an expansion length of 6. Lemma 6 generalizes it for an arbitrary expansion length l. If we apply Lemma 6 twice and use the estimates (16) for $c_j(n,r)$ and $R_j(n)$, we immediately get

LEMMA 7. *Let $n \in \mathbb{N}, n \geqslant 1$ and $r \geqslant 0$. Then the asymptotic expansion of $\mathfrak{J}_n^4(r)$ with length $l \geqslant 1$ is*

$$\mathfrak{J}_n^4(r) = \sum_{k=0}^{l-1} c_k^{(4)}(n,r) r^{-k} + R_l^{(4)}(n,r) r^{-l}, \tag{19}$$

where the expansion coefficients $c_k^{(4)}(n)$ are defined by

$$c_k^{(4)}(n,r) = \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{m=0}^{j} c_m(n,r) c_{j-m}(n,r) c_{i-j}(n,r) c_{k-i}(n,r), \tag{20}$$

and the remainder term $R_l^{(4)}(n,r)$ satisfies $|R_l^{(4)}(n,r)| \leqslant r(n,l)$ with

$$\begin{aligned} r(n,l) := & \sum_{k=0}^{l-1} \sum_{i=0}^{k} \sum_{j=0}^{i} |a_j(n) a_{i-j}(n) a_{k-i}(n) a_{l-k}(n)| \\ & + r_0(n) \sum_{k=0}^{l-1} \sum_{i=0}^{k} |a_i(n) a_{k-i}(n) a_{l-k}(n)| \\ & + r_0^2(n) \sum_{k=0}^{l-1} |a_k(n) a_{l-k}(n)| + r_0^3(n) |a_l(n)|. \end{aligned} \tag{21}$$

The goal of the following calculations is to generalize formula (20) for the case of four different Bessel functions $\mathfrak{J}_\alpha(r)\mathfrak{J}_\beta(r)\mathfrak{J}_\gamma(r)\mathfrak{J}_\delta(r)$. We will then be able to formally express this product as a series of length l, but <u>without remainder term</u>. We refer to this representation as the approximated series expansion of $\mathfrak{J}_\alpha(r)\mathfrak{J}_\beta(r)\mathfrak{J}_\gamma(r)\mathfrak{J}_\delta(r)$ and denote it by

$$\mathfrak{J}_\alpha(r)\mathfrak{J}_\beta(r)\mathfrak{J}_\gamma(r)\mathfrak{J}_\delta(r) \approx \sum_{k=0}^{l-1} c_k^{(4)}(\alpha,\beta,\gamma,\delta,r) r^{-k}. \tag{22}$$

Now, let us have a closer look at $c_k^{(4)}(\alpha,\beta,\gamma,\delta,r)$. By replacing the two values n in the proof of Lemma 6 by α and β, we immediately see that (17) generalizes for the product of two Bessel functions with different indices in the following simple way

$$c_k^{(2)}(\alpha,\beta,r) := \sum_{i=0}^{k} c_{k-i}(\alpha,r) c_i(\beta,r). \tag{23}$$

We repeat the calculation that leads to (23) two more times and yield the expression

$$c_k^{(4)}(\alpha,\beta,\gamma,\delta,r) := \sum_{i+0}^{k} \sum_{j=0}^{i} \sum_{s=0}^{j} c_{k-i}(\alpha,r) c_{i-j}(\beta,r) c_{j-s}(\gamma,r) c_s(\delta,r) \tag{24}$$

for the k-th expansion coefficient $\mathfrak{J}_\alpha(r)\mathfrak{J}_\beta(r)\mathfrak{J}_\gamma(r)\mathfrak{J}_\delta(r)$.
Note that due to the associativity and commutativity of the product of the four Bessel functions, also the formula for the expansion coefficients must reflect these property. Meaning, the coefficients can expressed in the following equivalent way

$$c_k^{(4)}(\alpha,\beta,\gamma,\delta,r) = \sum_{i=0}^{k} \sum_{j=0}^{i} c_{i-j}(\alpha,r) c_j(\beta,r) \sum_{s=0}^{k-i} c_{k-i-s}(\gamma,r) c_s(\delta,r). \tag{25}$$

We now plug in the information we have from Lemma 4 on the single-function expansion coefficients $c_k(\alpha, r)$ into the expression (25) for the four-function expansion coefficient. Recall that depending on the parity of α and k the coefficient $c_k(\alpha, r)$ is

$$c_k(\alpha, r) = a_k(\alpha) \begin{cases} (-1)^{\frac{k}{2}} \cos\left(r - \frac{\pi}{4} - \frac{\pi}{2}n\right), & k \text{ even}, \\ (-1)^{\frac{k+1}{2}} \sin\left(r - \frac{\pi}{4} - \frac{\pi}{2}n\right), & k \text{ odd}. \end{cases}$$

This can also be expressed in the following, more convenient way

$$c_k(\alpha, r) = a_k(\alpha) \cos\left(r - \frac{\pi}{4} - (\alpha - k)\frac{\pi}{2}\right), \tag{26}$$

with a_k defined in (11) by

$$a_k(\alpha) = \frac{\Gamma(\alpha + k + \frac{1}{2})}{k! 2^k\, \Gamma(\alpha - k + \frac{1}{2})}.$$

Plugging (26) into (25) we yield

$$\begin{aligned} c_k^{(4)}(\alpha, \beta, \gamma, \delta) = &\sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{s=0}^{k-i} a_{i-j}(\alpha) a_j(\beta) a_{k-i-s}(\gamma) a_s(\delta) \\ &\times \cos\left[r - \frac{\pi}{4} - (\alpha - i + j)\frac{\pi}{2}\right] \cos\left[r - \frac{\pi}{4} - (\beta - j)\frac{\pi}{2}\right] \\ &\times \cos\left[r - \frac{\pi}{4} - (\gamma - k + i + s)\frac{\pi}{2}\right] \cos\left[r - \frac{\pi}{4} - (\delta - s)\frac{\pi}{2}\right]. \end{aligned} \tag{27}$$

Using the trigonometric identities

$$\begin{aligned} \cos(x)^2 &= \frac{1}{2}[\cos(2x) + 1], \\ \sin(x)^2 &= -\frac{1}{2}[\cos(2x) - 1], \\ \sin(x)\cos(x) &= \frac{1}{2}\sin(2x), \end{aligned} \tag{28}$$

we reduce the product of the four cosine factors to the sum

$$\begin{aligned} \frac{1}{8}\Bigg[&(-1)^i \cos\left[(\alpha + \beta - \gamma - \delta + k)\frac{\pi}{2}\right] + (-1)^{j+s} \cos\left[(-\alpha + \beta - \gamma + \delta + k)\frac{\pi}{2}\right] \\ &+ (-1)^{i+j+s} \cos\left[(\alpha - \beta - \gamma + \delta + k)\frac{\pi}{2}\right] - \cos\left[(\alpha + \beta + \gamma + \delta - k)\frac{\pi}{2} - 4r\right] \\ &+ (-1)^j \sin\left[(-\alpha + \beta - \gamma - \delta + k)\frac{\pi}{2} + 2r\right] + (-1)^s \sin\left[(-\alpha - \beta - \gamma + \delta + k)\frac{\pi}{2} + 2r\right] \\ &+ (-1)^{i+j} \sin\left[(\alpha - \beta - \gamma - \delta + k)\frac{\pi}{2} + 2r\right] + (-1)^{i+s} \sin\left[(-\alpha - \beta + \gamma - \delta - k)\frac{\pi}{2} + 2r\right]\Bigg]. \end{aligned} \tag{29}$$

For the sake of readability, we denote in the following

$$\begin{aligned} A_i(\alpha, \beta) &:= \sum_{j=0}^{i} a_{i-j}(\alpha) a_j(\beta), \\ B_i(\alpha, \beta) &:= \sum_{j=0}^{i} (-1)^j a_{i-j}(\alpha) a_j(\beta), \end{aligned} \tag{30}$$

and abbreviate

$$A_i(\alpha, \alpha) =: A_i(\alpha), \quad B_i(\alpha, \alpha) =: B_i(\alpha),$$

in the case that the parameters α and β are equal.

In terms of these A_i and B_i we obtain the following representation for the k-th expansion coefficient $c_k^{(4)}(\alpha,\beta,\gamma,\delta,r)$ of the series expansion of the product of four different Bessel functions.

THEOREM 8. *Let $\alpha,\beta,\gamma,\delta$ be non-negative integers and $k\geqslant 0$. Then the k-th expansion coefficient $c_k^{(4)}(\alpha,\beta,\gamma,\delta,r)$ of the series expansion of the product $J_\alpha(r)J_\beta(r)J_\gamma(r)J_\delta(r)$ obeys*

$$\begin{aligned}c_k^{(4)}(\alpha,\beta,\gamma,\delta,r) = \frac{1}{8}\sum_{i=0}^{k}\Bigg(&(-1)^i\cos\left((\alpha+\beta-\gamma-\delta+k)\frac{\pi}{2}\right)A_i(\alpha,\beta)A_{k-i}(\gamma,\delta)\\
&+(-1)^i\cos\left((\alpha-\beta-\gamma+\delta+k)\frac{\pi}{2}\right)B_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\\
&+\cos\left((-\alpha+\beta-\gamma+\delta+k)\frac{\pi}{2}\right)B_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\\
&-\cos\left((\alpha+\beta+\gamma+\delta-k)\frac{\pi}{2}-4r\right)A_i(\alpha,\beta)A_{k-i}(\gamma,\delta)\\
&+\sin\left((-\alpha+\beta-\gamma-\delta+k)\frac{\pi}{2}+2r\right)B_i(\alpha,\beta)A_{k-i}(\gamma,\delta)\\
&+\sin\left((-\alpha-\beta-\gamma+\delta+k)\frac{\pi}{2}+2r\right)A_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\\
&+(-1)^i\sin\left((\alpha-\beta-\gamma-\delta+k)\frac{\pi}{2}+2r\right)A_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\\
&+(-1)^i\sin\left((-\alpha-\beta+\gamma-\delta-k)\frac{\pi}{2}+2r\right)A_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\Bigg).\end{aligned}$$

PROOF. Plug in (29) into (27) and replace the sums over j and s by (30). ∎

If all indices $\alpha,\beta,\gamma,\delta$ are equal, Theorem 8 simplifies a lot and leads us directly to the following representation of the expansion coefficient (20) in the one-parameter case.

COROLLARY 9. *Let k,n be nonnegative integers, $r>0$ and $A_i(n),B_i(n)$ be the sums we defined in* (30)*, with $\alpha=\beta=n$. Then, the expansion coefficient $c_k^{(4)}(n,r)$ satisfies*

$$\begin{aligned}c_k^{(4)}(n,r) = &\frac{1}{8}\cos\left(\frac{k\pi}{2}\right)\left(\sum_{i=0}^{k}(-1)^iA_i(n)A_{k-i}(n)+2\sum_{\substack{i=0\\ even}}^{k}B_i(n)B_{k-i}(n)\right)\\
&+\frac{1}{4}(-1)^n\sin\left(\frac{k\pi}{2}+2r\right)\left(\sum_{\substack{i=0\\ even}}^{k}B_i(n)A_{k-i}(n)+\sum_{\substack{i=0\\ k+i\ even}}^{k}A_i(n)B_{k-i}(n)\right)\\
&-\frac{1}{8}\cos\left(\frac{k\pi}{2}+4r\right)\sum_{i=0}^{k}A_i(n)A_{k-i}(n).\end{aligned}\tag{31}$$

PROOF. Setting $\alpha=\beta=\gamma=\delta=n$ in (29) reduces it to

$$\begin{aligned}\frac{1}{8}\Bigg[&\cos\left(\frac{k\pi}{2}\right)\left[(-1)^i+(1+(-1)^i)(-1)^{j+s}\right]\\
&+(-1)^n\sin\left(\frac{k\pi}{2}+2r\right)\left[(1+(-1)^i)(-1)^j+(1+(-1)^{k+i})(-1)^s\right]-\cos\left(\frac{k\pi}{2}+4r\right)\Bigg].\end{aligned}\tag{32}$$

All that is left now ist to plug in (32) into (27) and identify the sums $A_i(n)$ and $B_i(n)$. ■

Next, we continue the discussion of the general four-functions expansion coefficient (27) with a closer analysis of the sums $A_i(\alpha,\beta)$ and $B_i(\alpha,\beta)$ and express them in terms of the generalized hypergeometric function ${}_3F_2$. This may seem like making it more complicated, but opens up the possibility of applying the machinery of hypergeometric functions to our problem.

LEMMA 10. *For all non-negative integers α,β and i it is*

$$A_i(\alpha,\beta) = 2^{-i}\frac{\Gamma\left(\alpha+i+\frac{1}{2}\right)}{\Gamma(i+1)\Gamma\left(\alpha-i+\frac{1}{2}\right)}{}_3F_2\left(\begin{matrix}-\beta+\frac{1}{2},\beta+\frac{1}{2},-i\\ -\alpha-i+\frac{1}{2},\alpha-i+\frac{1}{2}\end{matrix}\,\Big|\,-1\right),$$
$$B_i(\alpha,\beta) = 2^{-i}\frac{\Gamma\left(\alpha+i+\frac{1}{2}\right)}{\Gamma(i+1)\Gamma\left(\alpha-i+\frac{1}{2}\right)}{}_3F_2\left(\begin{matrix}-\beta+\frac{1}{2},\beta+\frac{1}{2},-i\\ -\alpha-i+\frac{1}{2},\alpha-i+\frac{1}{2}\end{matrix}\,\Big|\,1\right).$$

PROOF. We derive the expression for $A_i(\alpha,\beta)$. The one for $B_i(\alpha,\beta)$ is completely analogous. First we plug in the definition (11) of the expansion coefficients a into the definition of $A_i(\alpha,\beta)$ and apply the reflection formula (327) to the factors $\Gamma\left(\alpha+i-j+\frac{1}{2}\right)$ and $\Gamma\left(\beta-j+\frac{1}{2}\right)$. Then it is

$$A_i(\alpha,\beta) = 2^{-i}(-1)^{\alpha-\beta+i}\sum_{j=0}^{i}\frac{\Gamma\left(-\beta+j+\frac{1}{2}\right)\Gamma\left(\beta+j+\frac{1}{2}\right)}{\Gamma\left(\alpha-i+j+\frac{1}{2}\right)\Gamma\left(-\alpha-i+j+\frac{1}{2}\right)\Gamma(i-j+1)\Gamma(j+1)}.$$

Next, we note that

$$\frac{\Gamma(-i)\Gamma(i+1)}{\Gamma(-i+j)\Gamma(i-j+1)} = (-1)^j. \tag{33}$$

This can be seen by applying the reflection formula (327) for non-integer i and taking the limit $i\to\mathbb{Z}$ afterwards. Hence, we can further transform $A_i(\alpha,\beta)$ into

$$A_i(\alpha,\beta) = \frac{2^{-i}(-1)^{\alpha-\beta+i}}{\Gamma(i+1)\Gamma(-i)}\sum_{j=0}^{i}\frac{\Gamma\left(-\beta+j+\frac{1}{2}\right)\Gamma\left(\beta+j+\frac{1}{2}\right)\Gamma(-i+j)}{\Gamma\left(\alpha-i+j+\frac{1}{2}\right)\Gamma\left(-\alpha-i+j+\frac{1}{2}\right)\Gamma(j+1)}(-1)^j.$$

By definition (335) of ${}_3F_2$ this is equal to

$$\frac{2^{-i}(-1)^{\alpha-\beta+i}\Gamma\left(-\beta+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(i+1)\Gamma\left(-\alpha-i+\frac{1}{2}\right)\Gamma\left(\alpha-i+\frac{1}{2}\right)}{}_3F_2\left(\begin{matrix}-\beta+\frac{1}{2},\beta+\frac{1}{2},-i\\ -\alpha-i+\frac{1}{2},\alpha-i+\frac{1}{2}\end{matrix}\,\Big|\,-1\right),$$

which after another two applications of the reflection formula to the factors $\Gamma\left(-\beta+\frac{1}{2}\right)$ and $\Gamma\left(-\alpha-i+\frac{1}{2}\right)$ turns into the claimed formula. ■

In the special case of $\alpha=\beta=n$, we exploit some symmetry effects to simplify the first summand in (31) and deduce a closed form expression for $B_i(n)$.

LEMMA 11. *Let $i,n\in\mathbb{N}$ and $i,n\geqslant 0$. Then it is*

$$B_i(n) = \begin{cases} 0, & i \text{ odd},\\ (-1)^{\frac{i}{2}}\pi^{-\frac{1}{2}}\frac{\Gamma\left(\frac{i}{2}+\frac{1}{2}\right)\Gamma\left(n+\frac{i}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{i}{2}+1\right)\Gamma\left(n-\frac{i}{2}+\frac{1}{2}\right)}, & i \text{ even}.\end{cases} \tag{34}$$

Moreover, for even k we have

$$\sum_{i=0}^{k}(-1)^iA_i(n)A_{k-i}(n) = \sum_{i=0}^{\frac{k}{2}}B_{2i}(n)B_{k-2i}(n). \tag{35}$$

Proof. For the proof of (34) let $y_j(i,n) = (-1)^j a_j(n)a_{i-j}(n)$. Then $y_{i-j}(k,n) = -y_j(i,n)$ and therefore, for odd i

$$\sum_{j=0}^{i} y_j(i,n) = \sum_{j=0}^{\frac{i-1}{2}} (y_j(i,n) + y_{i-j}(i,n)) = 0.$$

So, let's assume that i is even and set $i = 2j$. By Lemma 10 it is

$$B_{2j}(n) = 2^{-2j} \frac{\Gamma\left(n+2j+\frac{1}{2}\right)}{\Gamma(2j+1)\,\Gamma\left(n-2j+\frac{1}{2}\right)} {}_3F_2\left(\begin{matrix} -n+\frac{1}{2}, n+\frac{1}{2}, -2j \\ -n-2j+\frac{1}{2}, n-2j+\frac{1}{2} \end{matrix} \,\Big|\, 1\right).$$

Note that the hypergeometric function above has exactly the form stated in Lemma 48, when we set $a = j, b = n - j - \frac{1}{2}$ and $c = -n - j - \frac{1}{2}$. Thus, an application of the lemma and the reflection formula (327) yields

$$\begin{aligned} B_{2j}(n) &= 2^{-2j} \frac{\Gamma\left(n+2j+\frac{1}{2}\right)}{\Gamma(2j+1)\,\Gamma\left(n-2j+\frac{1}{2}\right)} \\ &\quad\times \frac{\Gamma(-j+1)\,\Gamma\left(n-2j+\frac{1}{2}\right)\Gamma\left(-n-2j+\frac{1}{2}\right)\Gamma(-j)}{\Gamma(-2j+1)\Gamma\left(n-j+\frac{1}{2}\right)\Gamma\left(-n-j+\frac{1}{2}\right)\Gamma(-2j)} \\ &= 2^{-2j} \frac{\Gamma(-j)}{\Gamma(-2j)\Gamma(j+1)} \frac{\Gamma\left(n+j+\frac{1}{2}\right)}{\Gamma\left(n-j+\frac{1}{2}\right)}. \end{aligned}$$

We write the first quotient above as

$$\frac{\Gamma(-j)\Gamma(j+1)}{\Gamma(-2j)\Gamma(2j+1)} \frac{\Gamma(2j+1)}{\Gamma(j+2)^2}.$$

By relation (33) with $i = 2j$ and Legendre's duplication formula (328), this is equal to

$$(-1)^j 2^{2j} \pi^{-\frac{1}{2}} \frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma(j+1)}.$$

Combining the last three steps and switching back to $j = \frac{i}{2}$ completes the proof of (34). Let's turn to (35). Recall the definition of the sum $A_i(n)$ and plug in the formula for the factors $a_j(n)$ from Lemma 4.

$$\begin{aligned} A_i(n) &= \sum_{j=0}^{i} a_{i-j}(n)a_j(n) \\ &= 2^{-i} \sum_{j=0}^{i} \frac{\Gamma\left(n+i-j+\frac{1}{2}\right)\Gamma\left(n+j+\frac{1}{2}\right)}{\Gamma(i-j+1)\Gamma(j+1)\Gamma\left(n-i+j+\frac{1}{2}\right)\Gamma\left(n-j+\frac{1}{2}\right)}. \end{aligned}$$

Note that

$$\frac{1}{\Gamma(i-j+1)\Gamma(j+1)} = \frac{1}{\Gamma(i+1)} \binom{i}{j}.$$

Moreover, let

$$G(j) := \frac{\Gamma\left(n+j+\frac{1}{2}\right)}{\Gamma\left(n-j+\frac{1}{2}\right)}.$$

We omit the dependency on n, since it is not of relevance for the upcoming transformations. Then

$$A_i(n) = \frac{2^{-i}}{\Gamma(i+1)} \sum_{j=0}^{i} \binom{i}{j} G(i-j)(G(j)$$

and

$$
\begin{aligned}
&\sum_{i=0}^{k}(-1)^i A_{k-i}(n)A_i(n)\\
&= 2^{-k}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\sum_{s=0}^{i}\frac{(-1)^i}{\Gamma(k-i+1)\Gamma(i+1)}\binom{k-i}{j}\binom{i}{s}G(k-i-j)G(j)G(i-s)G(s)\\
&= \frac{2^{-k}}{\Gamma(k+1)}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\sum_{s=0}^{i}(-1)^i\binom{k}{i}\binom{k-i}{j}\binom{i}{s}G(k-i-j)G(j)G(i-s)G(s). \qquad (36)
\end{aligned}
$$

By symmetry of the convolution and property (34) of the B_i's, the asserted identity (35) is equivalent to

$$
\sum_{i=0}^{k}(-1)^i A_{k-i}(n)A_i(n) = \sum_{i=0}^{k} B_{k-i}(n)B_i(n).
$$

Repeating the transformations that lead to (36) for the B, we yield

$$
\begin{aligned}
&\sum_{i=0}^{k} B_{k-i}(n)B_i(n)\\
&= \frac{2^{-k}}{\Gamma(k+1)}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\sum_{s=0}^{i}(-1)^{j+s}\binom{k}{i}\binom{k-i}{j}\binom{i}{s}G(k-i-j)G(j)G(i-s)G(s).
\end{aligned} \qquad (37)
$$

In the following we show equality between (36) and (37). We start at (37) and use Fubini's theorem on the interchangeability of the summation order and the two binomial identities

$$
\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}, \qquad (38)
$$

$$
\binom{n}{m} = \binom{n}{n-m} \qquad (39)
$$

to end up at (36). In (37) we change the order of summation of i and s and shift $i = p + s$ afterwards to get

$$
\begin{aligned}
&2^k\Gamma(k+1)\sum_{i=0}^{k} B_{k-i}(n)B_i(n)\\
&= \sum_{s=0}^{k}\sum_{p=0}^{k-s}\sum_{j=0}^{k-s-p}(-1)^{j+s}\binom{k}{p+s}\binom{p+s}{s}\binom{k-s-p}{j}G(k-s-p-j)G(s)G(p)G(j).
\end{aligned}
$$

Next, we apply (38) to the first two binomial coefficients and reverse the summation over p and s by setting $q = k - s$ and $r = q - p$. Due to (39), this leads to

$$
\sum_{q=0}^{k}\sum_{r=0}^{q}\sum_{j=0}^{r}(-1)^{j+q}\binom{k}{q}\binom{q}{r}\binom{r}{j}G(k-q)(G(q-r)G(r-j)G(j).
$$

Now we interchange the summation over r and j and repeat the same steps as above. That is, we apply (38), shift $r = j + s$ and reverse $p = q - j$. The resulting sum looks the following

$$
\sum_{q=0}^{k}\sum_{p=0}^{q}\sum_{s=0}^{p}(-1)^p\binom{k}{p}\binom{q}{p}\binom{p}{s}G(k-q)G(q-p)G(p-s)G(s).
$$

We have to interchange the summation order once more. Now it is the turn for the sums over q and p. Again we apply (38) and shift $q = p + t$. This finally leads to the desired result

$$\sum_{p=0}^{k}\sum_{t=0}^{k-p}\sum_{s=0}^{p}(-1)^p\binom{k}{p}\binom{k-p}{t}\binom{p}{s}G(k-p-t)G(t)G(p-s)G(s)$$
$$= 2^k\Gamma(k+1)\sum_{p=0}^{k}(-1)^pA_{k-p}(n)A_p(n).$$

■

Note that the proof of identity (35) heavily relies on the fact that all Bessel indices are equal. Thus, we don't expect it to be true in the general case.

Remark 12. As a direct consequence of Lemma 11 we can simplify the expansion coefficient $c_k^{(4)}(n)$ for even k a little bit and get

$$c_k^{(4)}(n,r) = (-1)^{\frac{k}{2}}\Bigg(\frac{3}{8}\sum_{j=0}^{\frac{k}{2}}B_{2j}(n)B_{k-2j}(n) - \frac{1}{8}\cos(4r)\sum_{i=0}^{k}A_i(n)A_{k-i}(n)$$
$$+\frac{1}{4}(-1)^n\sin(2r)\sum_{j=0}^{\frac{k}{2}}\left(A_{2j}(n)B_{k-2j}(n)+A_{k-2j}(n)B_{2j}(n)\right)\Bigg). \tag{40}$$

CHAPTER 3

Integration of the Series Expansion and Derivation of the Main Term

The present Chapter implements Step 2 of our program on page 4. Its first half treats the one-parameter case and is devoted to the proof of Theorem 13, which proves a decomposition of the integral $I(\mathbf{n})$ into a sum of three terms $M(\mathbf{n}), S(\mathbf{n})$ and $R(\mathbf{n})$, the main term, the secondary term and the remainder term. We establish concrete formulae for the summands $M(\mathbf{n})$ and $S(\mathbf{n})$ in terms of sums of quotients of products of gamma functions with and without an additional hypergeometric factor, respectively. For the term $R(\mathbf{n})$ we derive an upper bound. The basis of the proof of Theorem 13 is the series expansion of J_n^4 that we developed in Lemma 7 and refined in Corollary 9 and Lemma 11. Another crucial ingredient are the Lemmata 56, 57 and 59, that provide us with the necessary tools to handle integrals of two Bessel functions.
In the second half of this chapter we turn to the case of a general six-vector $\mathbf{n}$. In formula (56) we produce the conjectured main term $M(\mathbf{n})$ of Conjecture 2. Its deduction does not have proof-character but is very similar to the procedure in the proof of Theorem 13.

THEOREM 13. *Let $n \geqslant 0$ and $\boldsymbol{n} = (n, n, n, n, 2n, 2n)$. Then the integral $\left(\frac{\pi}{2}\right)^2 I(\boldsymbol{n})$ can be splitted into a sum of three terms, a main term $M(\boldsymbol{n})$, a secondary term $S(\boldsymbol{n})$ and a remainder Term $R(\boldsymbol{n})$*

$$\left(\frac{\pi}{2}\right)^2 I(\boldsymbol{n}) = M(\boldsymbol{n}) + S(\boldsymbol{n}) + R(\boldsymbol{n}).$$

The individual terms are defined by

$$M(\boldsymbol{n}) = \frac{3}{16\pi^{\frac{3}{2}}} \times \sum_{p=0}^{2n-1} \sum_{j=0}^{p} \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)\Gamma\left(2n-p\right)\Gamma\left(n+j+\frac{1}{2}\right)\Gamma\left(n+p-j+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(j+1)\Gamma(p-j+1)\Gamma(2n+p+1)\Gamma\left(n-j+\frac{1}{2}\right)\Gamma\left(n-p+j+\frac{1}{2}\right)}, \tag{41}$$

and

$$S(\boldsymbol{n}) = \frac{1}{8}\frac{2^{-12n}}{\Gamma(2n+1)^2} \sum_{k=0}^{4n-2} (-1)^{k+1} 2^k \sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{m=0}^{j} {}_3F_2\left(\begin{matrix} 2n+\frac{1}{2}, 2n-\frac{k}{2}, 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1, 4n+1 \end{matrix} \,\Big|\, \frac{1}{4}\right)$$
$$\times \frac{\Gamma(4n-k)\Gamma\left(n+k-i+\frac{1}{2}\right)\Gamma\left(n+i-j+\frac{1}{2}\right)}{\Gamma(k-i+1)\Gamma(i-j+1)\Gamma\left(n-k+i+\frac{1}{2}\right)\Gamma\left(n-i+j+\frac{1}{2}\right)}$$
$$\times \frac{\Gamma\left(n+j-m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma(j-m+1)\Gamma(m+1)\Gamma\left(n-j+m+\frac{1}{2}\right)\Gamma\left(n-m+\frac{1}{2}\right)}, \tag{42}$$

and the remainder term satisfies

$$|R(\boldsymbol{n})| \leqslant r(n)\frac{\Gamma(2n)}{2\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}, \tag{43}$$

where $r(n) = r(n, 4n-1)$ is defined in (21).

Later in this work we will see that the main term $M(\mathbf{n})$ is the one, that carries all the information on the decay behavior of the Bessel integral $I(\mathbf{n})$. The secondary term $S(\mathbf{n})$ and the remainder term $R(\mathbf{n})$ on the other hand don't contribute substantially to the magnitude of $I(\mathbf{n})$ and will be subject of the error analysis.

PROOF. When we replace in $I(\mathbf{n})$ the product $J_n^4(r) = \left(\frac{2}{\pi r}\right)^2 \mathfrak{J}_n^4(n)$ by the series expansion we developed in Lemma 7, we get the sum

$$\left(\frac{\pi}{2}\right)^2 I(\mathbf{n}) = \sum_{k=0}^{l-1} \int_0^\infty J_{2n}^2(r) c_k^{(4)}(n,r) r^{-k-1} dr + \int_0^\infty J_{2n}^2(r) R_l^{(4)}(n,r) r^{-l-1} dr. \tag{44}$$

After plugging in the formula for the expansion coefficient $c_k^{(4)}(n,r)$ from Corollary 9 and (40), respectively, we encounter the three different types of integrals

$$\begin{aligned}
I_{0r}(k,n) &:= \int_0^\infty J_{2n}^2(r) r^{-k-1} dr, \\
I_{2r}(k,n) &:= \int_0^\infty \sin\left(\frac{k\pi}{2} + 2r\right) J_{2n}^2(r) r^{-k-1} dr, \\
I_{4r}(k,n) &:= \int_0^\infty \cos\left(\frac{k\pi}{2} + 4r\right) J_{2n}^2(r) r^{-k-1} dr.
\end{aligned}$$

In the following we address them by $0r$ - terms, $2r$ - terms and $4r$ - terms, respectively, referring to the r - dependency of the trigonometric factor under the integral. Using these abbreviations for the integrals, (44) turns into

$$\begin{aligned}
\left(\frac{\pi}{2}\right)^2 I(\mathbf{n}) = \sum_{k=0}^{l-1} \Bigg[& \frac{1}{8} \cos\left(\frac{k\pi}{2}\right) I_{0r}(k,n) \left(\sum_{i=0}^{k} A_i(n) A_{k-i}(n) + 2 \sum_{\substack{i=0 \\ \text{even}}}^{k} B_i(n) B_{k-i}(n) \right) \\
& + \frac{1}{4}(-1)^n I_{2r}(k,n) \left(\sum_{\substack{i=0 \\ \text{even}}}^{k} B_i(n) A_{k-i}(n) + \sum_{\substack{i=0 \\ k+i \text{ even}}}^{k} A_i(n) B_{k-i}(n) \right) \\
& - \frac{1}{8} I_{4r}(k,n) \sum_{i=0}^{k} A_i(n) A_{k-i}(n) \Bigg] + \int_0^\infty J_{2n}^2(r) R_l^{(4)}(n,r) r^{-l-1} dr.
\end{aligned} \tag{45}$$

The last integral in the above line will from now on be called remainder term of $I(\mathbf{n})$, and we denote it by

$$R(l,\mathbf{n}) := \int_0^\infty J_{2n}^2(r) R_l^{(4)}(n,r) r^{-l-1} dr. \tag{46}$$

Let us now further explore the single types of integrals starting with the $0r$-terms. Upon the condition that $k+1 \leqslant 4n$, we can apply Kaptain's formula Lemma 56 to express the integral in the closed form

$$\begin{aligned}
I_{0r}(k,n) &= \frac{2^{-k-1}\Gamma(k+1)\Gamma\left(2n - \frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)^2 \Gamma\left(2n + \frac{k}{2} + 1\right)} \\
&= \frac{1}{2} \frac{\Gamma\left(\frac{k}{2} + \frac{1}{2}\right)\Gamma\left(2n - \frac{k}{2}\right)}{\Gamma(k+1)\Gamma\left(2n + \frac{k}{2} + 1\right)}.
\end{aligned} \tag{47}$$

The second equality is due to the duplication formula (328).

Before we continue with the analysis of the $2r$- and $4r$-terms, we briefly discuss the conditions the expansion length l has to satisfy, and choose a suitable value for l.
Since we want to treat all $0r$-terms with Kapteyn's formula, our expansion length l has to satisfy $0 \leqslant k \leqslant l \leqslant 4n-1$. Moreover, we want our approximation of $\mathfrak{J}_n^4(r)$ by its expansion

to be as accurate as possible. Meaning we want to loose as few expansion terms as possible. Consequently, we choose the expansion length to be as large as possible. That is

$$l = 4n - 1. \tag{48}$$

Note that (46) leads to an integral of the type $I_{0r}(k,n)$, as well, when we take the absolute value and use the uniform bound on the expansion remainder from Lemma 7. By the above considerations regarding the expansion length, this $0r$-term related to the expansion remainder satisfies

$$I_{0r}(l,n) = I_{0r}(4n-1,n) = \frac{\Gamma(2n)}{2\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}, \tag{49}$$

and the claimed upper bound (43) on the remainder term $R(\mathbf{n}) := R(4n-1,\mathbf{n})$ follows directly from

$$|R(\mathbf{n})| \leqslant |R^{(4)}_{4n-1}(n,r)| I_{0r}(4n-1,n)$$

and Lemma 7.

Next, we consider the $2r$-terms. Note that depending on the parity of k, the trigonometric function underneath the integral is either a sine or a cosine of $2r$. For even k we have the sine, for odd k it is the cosine. Thus, all conditions of Lemma 57 are met and we can conclude that the $2r$-terms vanish.

In the case of the $4r$-terms we proceed similarly, with the intention to use Lemma 58. For even k the first formula (383) of the lemma applies, and for odd k it is the second one (384), leading to

$$I_{4r}(k,n) = (-1)^k 2^{2k-12n} \frac{\Gamma(4n-k)}{\Gamma(2n+1)^2} {}_3F_2\left(\begin{matrix} 2n+\frac{1}{2}, 2n-\frac{k}{2}, 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1, 4n+1 \end{matrix} \,\middle|\, \frac{1}{4}\right). \tag{50}$$

In the next step, we plug in our findings about $I_{0r}(k,n), I_{2r}(k,n)$ and $I_{4r}(k,n)$ into (45) and call the part related to the $0r$-terms the main term $M(\mathbf{n})$, and the part related to the $4r$-terms the secondary term $S(\mathbf{n})$. Then we yield from (47) and Lemma 11 for the main term

$$\begin{aligned} M(\mathbf{n}) &= \frac{1}{16} \sum_{k=0}^{4n-2} \cos\left(\frac{k\pi}{2}\right) \frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(2n-\frac{k}{2}\right)}{\Gamma(k+1)\Gamma\left(2n+\frac{k}{2}+1\right)} \\ &\quad \times \left[\sum_{i=0}^{k} A_i(n)A_{k-i}(n) + 2\sum_{\substack{i=0\\ \text{even}}}^{k} B_i(n)B_{k-i}(n)\right] \\ &= \frac{3}{16\pi^{\frac{1}{2}}} \sum_{p=0}^{2n-1} \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma(2n-p)}{\Gamma(p+1)\Gamma(2n+p+1)} (-1)^p \sum_{j=0}^{p} B_{2j}(n)B_{2p-2j}(n). \end{aligned}$$

For the secondary term we get from (50)

$$\begin{aligned} S(\mathbf{n}) &= \frac{1}{8} \sum_{k=0}^{4n-2} (-1)^{k+1} 2^{2k-12n} \frac{\Gamma(4n-k)}{\Gamma(2n+1)^2} {}_3F_2\left(\begin{matrix} 2n+\frac{1}{2}, 2n-\frac{k}{2}, 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1, 4n+1 \end{matrix} \,\middle|\, \frac{1}{4}\right) \\ &\quad \times \sum_{i=0}^{k} A_i(n)A_{k-i}(n). \end{aligned}$$

Now the claimed formulae (41) and (42) follow from the definition (30) of the sums A_i and B_i and formula (11) for the a_j. ∎

Since we will later only be interested in an upper bound on the absolute value of the secondary term (42), we want to take us a minute before we move on to place the following remark.

Remark 14. By definition (335) of a generalized hypergeometric function, we see that this function is positive, if all parameters $a_1, \dots, a_p$ and $b_1, \dots, b_q$ are positive. In the case of ${}_3F_2\left(\begin{smallmatrix} 2n+\frac12 & 2n-\frac{k}{2} & 2n-\frac{k}{2}+\frac12 \\ 2n+1 & 4n+1 \end{smallmatrix} \middle| \frac14\right)$ this condition is met, as we have $0 \leqslant k \leqslant l \leqslant 4n-1$. We thus can estimate the absolute value of $S(\mathbf{n})$ as follows

$$|S(\mathbf{n})| \leqslant \frac{1}{8}\frac{2^{-12n}}{\Gamma(2n+1)^2}\sum_{k=0}^{4n-2} 2^k \sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{m=0}^{j} {}_3F_2\left(\begin{matrix} 2n+\frac12, 2n-\frac{k}{2}, 2n-\frac{k}{2}+\frac12 \\ 2n+1, 4n+1 \end{matrix} \middle| \frac14\right)$$
$$\times \frac{\Gamma(4n-k)\Gamma\left(n+k-i+\frac12\right)\Gamma\left(n+i-j+\frac12\right)}{\Gamma(k-i+1)\Gamma(i-j+1)\left|\Gamma\left(n-k+i+\frac12\right)\Gamma\left(n-i+j+\frac12\right)\right|}$$
$$\times \frac{\Gamma\left(n+j-m+\frac12\right)\Gamma\left(n+m+\frac12\right)}{\Gamma(j-m+1)\Gamma(m+1)\left|\Gamma\left(n-j+m+\frac12\right)\Gamma\left(n-m+\frac12\right)\right|}.$$

The second half of this Chapter is devoted to the derivation of the main term $M(\mathbf{n})$ for the general integral $I(\mathbf{n})$ with $\mathbf{n} = (\alpha, \beta, \gamma, \delta, x, y)$ upon the assumption that x and y are the two largest indices. We follow the lines of the proof of Theorem 13, with the difference that we don't work with the series expansion of the product $J_\alpha(r)J_\beta(r)J_\gamma(r)J_\delta(r)$, but with the approximated series expansion

$$J_\alpha(r)J_\beta(r)J_\gamma(r)J_\delta(r) \approx \left(\frac{2}{\pi r}\right)^2 \sum_{k=0}^{l-1} c_k^{(4)}(\alpha, \beta, \gamma, \delta, r) r^{-k}$$

we deduced in Chapter 2. Meaning, we do not take into account the expansion remainder. The reason is, that on the basis of the analysis of the remainder term $R(\mathbf{n})$ of the integral $I(\mathbf{n})$ in Section 4.2 of the next Chapter 4, we suppose, that in the general case the integral of the expansion remainder also does not contribute to the overall asymptotic behavior of $I(\mathbf{n})$.

So, let us start by replacing the product $J_\alpha(r)J_\beta(r)J_\gamma(r)J_\delta(r)$ in $I(\mathbf{n})$ by its approximated series expansion. We still encounter similar $0r$-terms, $2r$-terms and $4r$-terms, this time in the more general form

$$\begin{aligned} I_{0r}(k,x,y) &:= \int_0^\infty J_x(r)J_y(r) r^{-k-1} dr, \\ I_{2r}(k,x,y) &:= \int_0^\infty \sin\left(\frac{\sigma+k}{2}\pi + 2r\right) J_x(r)J_y(r) r^{-k-1} dr, \\ I_{4r}(k,x,y) &:= \int_0^\infty \cos\left(\frac{\sigma+k}{2}\pi + 4r\right) J_x(r)J_y(r) r^{-k-1} dr, \end{aligned} \tag{51}$$

with σ being a sum of the kind $\pm\alpha \pm \beta \pm \gamma \pm \delta$. The precise cases for σ can be read from (29), but are not important at this point. Here, we only care about the parity of σ.

We now briefly discuss each of these integrals, starting with the $0r$-terms. Lemma 56 is still applicable to the $0r$-terms if $0 \leqslant k+1 \leqslant x+y$ and yields

$$I_{0r}(k,x,y) = \frac{2^{-k-1}\Gamma(k+1)\Gamma\left(\frac{x+y-k}{2}\right)}{\Gamma\left(\frac{x-y+k}{2}+1\right)\Gamma\left(-\frac{x-y+k}{2}+1\right)\Gamma\left(\frac{x+y+k}{2}+1\right)}. \tag{52}$$

With the same argument as in the proof of Theorem 13, we choose the expansion length to be as large as possible, such that the requirements of Lemma 56 are still met for all $0 \leqslant k \leqslant l-1$. That is

$$l = x + y - 1. \tag{53}$$

As in the special case, we define the main term $M(\mathbf{n})$ to be the sum of all terms that come with a $0r$-term factor after the replacement step above. In order to motivate and justify this choice, we now provide some arguments why we expect the $2r$- and the $4r$-terms to be negligible.
Having a closer look at $I_{2r}(k,x,y)$ and $I_{4r}(k,x,y)$, one notices that depending on the parity of x,y and σ always exactly one of those integrals meets the requirements of Lemma 57 or Lemma 58 and Remark 61, and thus vanishes for all k. To make this clearer, assume that $x+y$ is even. If σ is even, too, then we get in I_{2r} the trigonometric factor $\sin(2r)$ and an odd sum $x+y+k+1$ for even k, and $\cos(2r)$ with an even sum $x+y+k+1$ for odd k. Thus, the $2r$-terms vanish due to Lemma 57. If σ is odd, then the same chain of reasoning applies to the $4r$-terms, with Lemma 57 replaced by Lemma 58 and Remark 61.

Let us for example assume, that we are in the case that the $2r$-terms vanish. Then, the $4r$-terms have to be estimated. To do so, one may apply a contour integral argument similar to the one in the proof of Lemma 2.4 in [1]. This leads to the upper bound

$$|I_{4r}(k,x,y)| \leqslant 2^{k-2x-2y-1}\frac{\Gamma(x+y-k)}{\Gamma(x+1)\Gamma(y+1)}$$

and shows an exponential decay of the integrals $I_{4r}(k,x,y)$ in x and y. We therefore ignore them for our purposes.

After this excursion on the $2r$- and $4r$-terms, we continue our derivation of a formula for the main term $M(\mathbf{n})$. To this end, we have to take a closer look at the expression we get after replacing the product $J_\alpha(r)J_\beta(r)J_\gamma(r)J_\delta(r)$ in $I(\mathbf{n})$ by its approximated series expansion and need to identify the part, that contains the $0r$-terms. Recalling the formula for the expansion coefficient $c_k^{(4)}(\alpha,\beta,\gamma,\delta,r)$ from Theorem 8, we find that this is

$$\begin{aligned} M(\mathbf{n}) &= \left(\frac{2}{\pi}\right)^2 \frac{1}{8} \sum_{k=0}^{x+y-1} I_{0r}(k,x,y) \\ &\times \Bigg[\cos\left[(\alpha+\beta-\gamma-\delta+k)\frac{\pi}{2}\right] \sum_{i=0}^{k} (-1)^i A_i(\alpha,\beta)A_{k-i}(\gamma,\delta) \\ &+ \cos\left[(-\alpha+\beta-\gamma+\delta+k)\frac{\pi}{2}\right] \sum_{i=0}^{k} B_i(\alpha,\beta)B_{k-i}(\gamma,\delta) \\ &+ \cos\left[(\alpha-\beta-\gamma+\delta+k)\frac{\pi}{2}\right] \sum_{i=0}^{k} (-1)^i B_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\Bigg]. \end{aligned} \tag{54}$$

When we plug in (52), we see that the summands for $0 \leqslant k \leqslant |x-y|-2$ vanish in (54) due to the fact that the reciprocal gamma function $1/\Gamma(z)$ has zeros at $z=0,-1,-2,\ldots$. Moreover, it suffices to sum over those values of k such that the sum $\alpha+\beta+\gamma+\delta+k$ is even, since otherwise all cosine factors are zero. Thus, we can state the following, more compact version of M

$$M(\mathbf{n}) = \frac{1}{(2\pi)^2} \sum_{\substack{k=\max\{|x-y|-1,0\}\\ \alpha+\beta+\gamma+\delta+k \text{ even}}}^{x+y-1} (-1)^{\frac{\alpha+\beta-\gamma-\delta+k}{2}} \frac{2^{-k}\Gamma(k+1)\Gamma\left(\frac{x+y-k}{2}\right)}{\Gamma\left(\frac{x-y+k}{2}+1\right)\Gamma\left(-\frac{x-y+k}{2}+1\right)\Gamma\left(\frac{x+y+k}{2}+1\right)}$$
$$\times \left[\sum_{i=0}^{k}(-1)^i A_i(\alpha,\beta)A_{k-i}(\gamma,\delta) + (-1)^{\alpha+\delta}\sum_{i=0}^{k} B_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\right.$$
$$\left. + (-1)^{\beta+\delta}\sum_{i=0}^{k}(-1)^i B_i(\alpha,\beta)B_{k-i}(\gamma,\delta)\right]. \tag{55}$$

Note that for integers γ and δ the sum $\alpha+\beta-\gamma-\delta+k$ is even if and only if the sum $\alpha+\beta+\gamma+\delta+k$ is. Thus, the factor $(-1)^{\frac{\alpha+\beta-\gamma-\delta+k}{2}}$ is always real.
Finally, we express the sums A_j and B_j in terms of the hypergeometric functions, we found in Lemma 10 and end up with

$$M(\mathbf{n})$$
$$= \frac{1}{(2\pi)^2} \sum_{\substack{k=\max\{|x-y|-1,0\}\\ \alpha+\beta+\gamma+\delta+k \text{ even}}}^{x+y-1} (-1)^{\frac{\alpha+\beta-\gamma-\delta+k}{2}} \frac{2^{-2k}\Gamma(k+1)\Gamma\left(\frac{x+y-k}{2}\right)}{\Gamma\left(\frac{x-y+k}{2}+1\right)\Gamma\left(-\frac{x-y+k}{2}+1\right)\Gamma\left(\frac{x+y+k}{2}+1\right)}$$
$$\times \sum_{i=0}^{k}(-1)^i \frac{\Gamma\left(\alpha+i+\frac{1}{2}\right)\Gamma\left(\gamma+k-i+\frac{1}{2}\right)}{\Gamma(i+1)\Gamma\left(\alpha-i+\frac{1}{2}\right)\Gamma(k-i+1)\Gamma\left(\gamma-k+i+\frac{1}{2}\right)}$$
$$\times \left[{}_3F_2\left(\begin{matrix}-\beta+\frac{1}{2},\beta+\frac{1}{2},-i\\ -\alpha-i+\frac{1}{2},\alpha-i+\frac{1}{2}\end{matrix}\,\middle|\,-1\right) {}_3F_2\left(\begin{matrix}-\delta+\frac{1}{2},\delta+\frac{1}{2},-k+i\\ -\gamma-k+i+\frac{1}{2},\gamma-k+i+\frac{1}{2}\end{matrix}\,\middle|\,-1\right)\right.$$
$$+ {}_3F_2\left(\begin{matrix}-\beta+\frac{1}{2},\beta+\frac{1}{2},-i\\ -\alpha-i+\frac{1}{2},\alpha-i+\frac{1}{2}\end{matrix}\,\middle|\,1\right) {}_3F_2\left(\begin{matrix}-\delta+\frac{1}{2},\delta+\frac{1}{2},-k+i\\ -\gamma-k+i+\frac{1}{2},\gamma-k+i+\frac{1}{2}\end{matrix}\,\middle|\,1\right)$$
$$\left. \times (-1)^{\beta+\delta}\left(1+(-1)^{\alpha-\beta+i}\right)\right]. \tag{56}$$

CHAPTER 4

Precise Error Bound for the Integral $I(\mathbf{n})$ with $\mathbf{n}=(n,n,n,n,2n,2n)$

This Chapter is entirely devoted to the case $\mathbf{n}=(n,n,n,n,2n,2n)$ and the analytical part of the proof of Theorem 3. This corresponds to Step 3 in terms of the program on page 4. More precisely, we take the representation

$$\left(\frac{\pi}{2}\right)^2 I(\mathbf{n}) = M(\mathbf{n}) + S(\mathbf{n}) + R(\mathbf{n})$$

from Theorem 13 and analyze the right-hand side summand wise. The beginning makes the remainder term $R(\mathbf{n})$ which is estimated from above in Section 4.2, Theorem 15.
Section 4.3 then deals with the secondary term $S(\mathbf{n})$. Its main result, Theorem 26, provides an upper bound on $S(\mathbf{n})$.
In Section 4.4 we take a closer look at the main term $M(\mathbf{n})$. We first rearrange it in such a way that it reveals its asymptotic behavior. Theorem 36 then quantifies the error between the main term and its leading asymptotic terms.
Since it requires a fair amount of work to prove Theorems 15, 26 and 36, we want to start this chapter with a short section that puts together the results of the theorems into an analytical proof of Theorem 3 for $n \geqslant 20$.

4.1. Proof of Theorem 3 for $n \geqslant 20$

Before we start proving Theorems 15, 26 and 36, we use them to prove our main result Theorem 3 in the case that $n \geqslant 20$.

Proof of Theorem 3 for $n \geqslant 20$. Assume that $n \geqslant 20$. Recall the identity stated in Theorem 13

$$\left(\frac{\pi}{2}\right)^2 I(\mathbf{n}) = M(\mathbf{n}) + S(\mathbf{n}) + R(\mathbf{n}).$$

By Theorem 36, the right-hand side is equal to

$$\left(\frac{2}{\pi}\right)^2 \left[\frac{3^{\frac{1}{2}}}{2^4}\frac{1}{n} - \frac{23}{2^{10}3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)} + E_M(n) + S(\mathbf{n}) + R(\mathbf{n})\right].$$

Hence we estimate

$$\left| I(\mathbf{n}) - \frac{3^{\frac{1}{2}}}{4\pi^2}\frac{1}{n} + \frac{23}{2^8 3^{\frac{3}{2}}\pi^2}\frac{1}{n\left(n^2-\frac{1}{4}\right)} \right| \leqslant \left(\frac{2}{\pi}\right)^2 \left[|E_M(n)| + |S(\mathbf{n})| + |R(\mathbf{n})|\right]. \tag{57}$$

We use Theorem 36 once more, which states for the absolute value of E_M the upper bound

$$|E_M(n)| \leqslant \frac{2689}{2^{18}\cdot 3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)} + \frac{1}{2^5\pi^{\frac{1}{2}}}\frac{n+1}{n}2^{-2n} + \frac{3^{\frac{1}{2}}}{2^5\pi^{\frac{1}{2}}}e^{\frac{7}{36n}}\frac{n^{\frac{1}{2}}}{n-1}2^{-2n}.$$

Theorem 15 provides the estimate

$$|R(\mathbf{n})| \leqslant \frac{251}{250}\frac{3e}{\pi^{\frac{5}{2}}}n^{-\frac{1}{2}}\left(\frac{2}{3}\right)^{4n}$$

on the absolute value of the remainder term $R(\mathbf{n})$ and Theorem 26 gives

$$|S(\mathbf{n})| \leqslant e\left(\frac{3}{\pi}\right)^{\frac{5}{2}} n^{\frac{1}{2}}\left(\frac{2}{3}\right)^{4n}$$

for the absolute value of the secondary term $S(\mathbf{n})$. Combining these last three estimates allows us to bound the right-hand side of (57) from above by

$$\frac{2689}{2^{16}3^{\frac{3}{2}}\pi^2}\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)}+\frac{V(n)}{2^3\pi^{\frac{5}{2}}}2^{-2n}, \tag{58}$$

where the function $V(n)$ collects all the coefficients of $\frac{2^{-2n}}{2^3\pi^{\frac{5}{2}}}$ in the above estimates. More precisely, it reads

$$V(n) := 1+\frac{1}{n}+3^{\frac{1}{2}}e^{\frac{7}{36n}}\frac{n^{\frac{1}{2}}}{n-1}+\frac{251}{250}\frac{3e2^5}{\pi^2}\frac{1}{n^{\frac{1}{2}}}\left(\frac{2^6}{3^4}\right)^n+\frac{3^{\frac{5}{2}}e2^5}{\pi^2}n^{\frac{1}{2}}\left(\frac{2^6}{3^4}\right)^n. \tag{59}$$

This is a decreasing function in $n \geqslant 3$. For all summands despite the last one, this is immediately clear. In case of the last summand we take the derivative of $n^{\frac{1}{2}}\left(\frac{2^6}{3^4}\right)^n$, in order to determine the value of $n \geqslant 0$ where this summand attains its maximum. It is

$$\frac{d}{dn}n^{\frac{1}{2}}\left(\frac{2^6}{3^4}\right)^n=\frac{1}{2n^{\frac{1}{2}}}\left(\frac{2^6}{3^4}\right)\left(1-2n\log\left(\frac{3^4}{2^6}\right)\right),$$

which is negative for $n \geqslant 3 > \frac{1}{2\log\left(\frac{3^4}{2^6}\right)}$.

For the next step, we denote

$$b(n) := n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)2^{-2n}.$$

This function decreases as soon as $n \geqslant 4$, which can easily be seen by taking the derivative of its upper bound $n^5 2^{-2n}$.

Now we replace the factor 2^{-2n} in (58) by $b(n)\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)}$ and arrive at

$$\begin{aligned}\left|I(\mathbf{n})-\frac{3^{\frac{1}{2}}}{4\pi^2}\frac{1}{n}+\frac{23}{2^8 3^{\frac{3}{2}}\pi^2}\frac{1}{n\left(n^2-\frac{1}{4}\right)}\right| &\leqslant \left(\frac{2689}{2^{16}3^{\frac{3}{2}}\pi^2}+\frac{V(n)b(n)}{8\pi^{\frac{5}{2}}}\right)\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)}\\ &\leqslant \left(\frac{2689}{2^{16}3^{\frac{3}{2}}\pi^2}+\frac{V(20)b(20)}{8\pi^{\frac{5}{2}}}\right)\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)}\end{aligned} \tag{60}$$

for all $n \geqslant 20$. The values of $V(n)$ and $b(n)$ at the point $n = 20$ are

$$\begin{aligned}V(20) &\leqslant 7.040424,\\ b(20) &\leqslant 2.901293\cdot 10^{-6},\end{aligned}$$

which leads to

$$\frac{V(20)b(20)}{8\pi^{\frac{5}{2}}} \leqslant 1.459573\cdot 10^{-7}.$$

We thus arrive at the claimed constant

$$\begin{aligned}\frac{2689}{2^{16}3^{\frac{3}{2}}\pi^2}+\frac{V(20)b(20)}{8\pi^{\frac{5}{2}}} &\leqslant 8.000724\cdot 10^{-4}+1.459573\cdot 10^{-7}\\ &= 8.002183\cdot 10^{-4}.\end{aligned}$$

■

Now let's start the actual work with the proof of the estimate for the remainder term.

4.2. Estimate of the Remainder Term $R(\mathbf{n})$

The goal of this section is the proof of the following bound on the remainder term.

THEOREM 15. *Let $n \geqslant 20$. Then the remainder term $R(\boldsymbol{n})$ satisfies the bound*

$$\begin{aligned} |R(\boldsymbol{n})| \leqslant \frac{1}{\pi} n^{\frac{1}{4}} \left(\frac{3^3 5^5}{2^{20}}\right)^n + \frac{e^{\frac{3}{2}+\frac{2}{n}}}{\sqrt{3}\pi^{\frac{3}{2}}} n^{-1} \left(\frac{3^3}{2^8}\right)^n \\ + \frac{2e^{\frac{5}{2}+\frac{2}{n}}}{\pi^2} n^{-\frac{3}{4}} \left(\frac{\left(10+\sqrt{2}\right)^3 \left(2+3\sqrt{2}\right)}{2^{16}}\right)^n + \frac{3e}{\pi^{\frac{5}{2}}} n^{-\frac{1}{2}} \left(\frac{2}{3}\right)^{4n} \end{aligned} \tag{61}$$

$$\leqslant \frac{251}{250} \frac{3e}{\pi^{\frac{5}{2}}} n^{-\frac{1}{2}} \left(\frac{2}{3}\right)^{4n}. \tag{62}$$

Our starting point is estimate (43) of Theorem 13

$$|R(\mathbf{n})| \leqslant r(n, 4n-1) \frac{\Gamma(2n)}{2\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}.$$

Then we recall Lemma 7, estimate (21), which tells us that $r(n) = r_1(n) + r_2(n) + r_3(n) + r_4(n)$ is a sum, consisting of the four components

$$\begin{aligned} r_1(n) &:= r_0^3(n)|a_l(n)|, \\ r_2(n) &:= r_0^2(n) \sum_{k=0}^{l-1} |a_k(n) a_{l-k}(n)|, \\ r_3(n) &:= r_0(n) \sum_{k=0}^{l-1} \sum_{i=0}^{k} |a_i(n) a_{k-i}(n) a_{l-k}(n)|, \\ r_4(n) &:= \sum_{k=0}^{l-1} \sum_{i=0}^{k} \sum_{j=0}^{i} |a_j(n) a_{i-j}(n) a_{k-i}(n) a_{l-k}(n)|, \end{aligned} \tag{63}$$

with $l = 4n-1$, $a_k(n) = \frac{\Gamma\left(n+k+\frac{1}{2}\right)}{2^k k! \Gamma\left(n-k+\frac{1}{2}\right)}$ and $r_0(n) = \left(n+\frac{1}{2}\right)^{\frac{1}{4}}$. We thus also write our target as a sum

$$|R(\mathbf{n})| \leqslant \sum_{m=1}^{4} R_m(n)$$

with

$$R_m(n) := r_m(n) \frac{\Gamma(2n)}{2\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} \tag{64}$$

and cut the proof of Theorem 15 into smaller chunks by estimating each summand $R_m(n)$ separately. Those estimates are subject of the Subsections 4.2.1, 4.2.4, 4.2.5 and 4.2.6 for $R_1(n)$, $R_2(n)$, $R_3(n)$ and $R_4(n)$, respectively, and summarized in the corresponding Lemmata 16, 23, 24 and 25.

Before we dive deeper into the proofs of those Lemmata, let us first combine their results to a proof of the bound on the entire remainder term $R(\mathbf{n})$.

PROOF OF THEOREM 15. We assume that $n \geqslant 20$. Then estimate (61) is a direct consequence of Lemmata 16, 23, 24 and 25 below.

To prove estimate (62) we write $\tilde{R}_m(n)$ for the upper bound on $R_m(n)$ and consider the three quotients

$$\begin{aligned}
\frac{\tilde{R}_1(n)}{\tilde{R}_4(n)} &= \frac{\pi^{\frac{3}{2}}}{3e} n^{\frac{3}{4}} \left(\frac{3^7 5^5}{2^{24}}\right)^n, \\
\frac{\tilde{R}_2(n)}{\tilde{R}_4(n)} &= \frac{\pi e^{\frac{1}{2}+\frac{2}{n}}}{3^{\frac{3}{2}}} n^{-\frac{1}{2}} \left(\frac{3^7}{2^{12}}\right)^n, \\
\frac{\tilde{R}_3(n)}{\tilde{R}_4(n)} &= \frac{2\pi^{\frac{1}{2}} e^{\frac{3}{2}+\frac{2}{n}}}{3} n^{-\frac{1}{4}} \left(\frac{3^4 \left(10+\sqrt{2}\right)^3 \left(2+3\sqrt{2}\right)}{2^{20}}\right)^n.
\end{aligned}$$

All of them are monotonically decreasing in n, and thus, can be bounded from above by the value they take at $n_0 = 20$. Since $\tilde{R}_4(n) = \frac{3e}{\pi^{\frac{5}{2}}} n^{-\frac{1}{2}} \left(\frac{2}{3}\right)^{4n}$ by Lemma 25, this leads to

$$\begin{aligned}
\tilde{R}_1(n) + \tilde{R}_2(n) + \tilde{R}_3(n) + \tilde{R}_4(n) &= \left(\frac{\tilde{R}_1(n)}{\tilde{R}_4(n)} + \frac{\tilde{R}_2(n)}{\tilde{R}_4(n)} + \frac{\tilde{R}_3(n)}{\tilde{R}_4(n)} + 1\right) \tilde{R}_4(n) \\
&\leqslant \left(\frac{\tilde{R}_1(n_0)}{\tilde{R}_4(n_0)} + \frac{\tilde{R}_2(n_0)}{\tilde{R}_4(n_0)} + \frac{\tilde{R}_3(n_0)}{\tilde{R}_4(n_0)} + 1\right) \tilde{R}_4(n) \\
&\leqslant (0.00358133 + 1) \frac{3e}{\pi^{\frac{5}{2}}} n^{-\frac{1}{2}} \left(\frac{2}{3}\right)^{4n}.
\end{aligned}$$

Since $0.00358133 \leqslant \frac{1}{250}$ the claim follows. ∎

Now we turn to the analysis of the single components $R_m(n)$. Consider for example the summand of $r_2(n)$ in (63). When we plug in the definition of $a_k(n)$ and $a_{l-k}(n)$ we encounter quotients of products of gamma functions like

$$Q(n,k) = 2^{-l} \frac{\Gamma\left(n+k+\frac{1}{2}\right)\Gamma\left(n+l-k+\frac{1}{2}\right)}{\Gamma(k+1)\Gamma(l-k+1)\left|\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(n-l+k+\frac{1}{2}\right)\right|}, \tag{65}$$

depending on n and the summation index k. To save some time and space, we will from now on refer to such an object as gamma quotient.

Our approach in handling the $R_m(n)$ now simply consist of an ℓ^1-ℓ^∞-estimate. That is, we bound a sum of gamma quotients by the number of summands multiplied with the maximum of all summands

$$\sum_{k=0}^{l-1} Q(n,k) \leqslant l \max_k Q(n,k).$$

This results in another gamma quotient $\tilde{Q}(n)$, that then only depends on n, and thus can be estimated using Stirling's formula (332).

The difficulty in this approach is to actually identify the maximum of the gamma quotient Q viewed as functions of the summation indices k, i, j - a tasks, that becomes increasingly complex the more summation indices are involved.

As a little warmup we now jump into the estimate of $R_1(n)$, which is the easiest of the candidates, as we do not need to find the maximum of the corresponding gamma quotient first.

4.2.1. Upper Bound on $R_1(n)$. Since $R_1(n)$ already is in the form of a simple n-dependent gamma quotient without any summation involved, its estimation is straight forward.

LEMMA 16. *Let $n \geqslant 2$, then*

$$R_1(n) \leqslant \frac{n^{\frac{1}{4}}}{\pi} \left(\frac{3^3 5^5}{2^{20}}\right)^n.$$

To get an intuition on how $R(n)$ behaves in n and how it compares to the claimed upper bound, Figure 4.1 plots both quantities logarithmically.

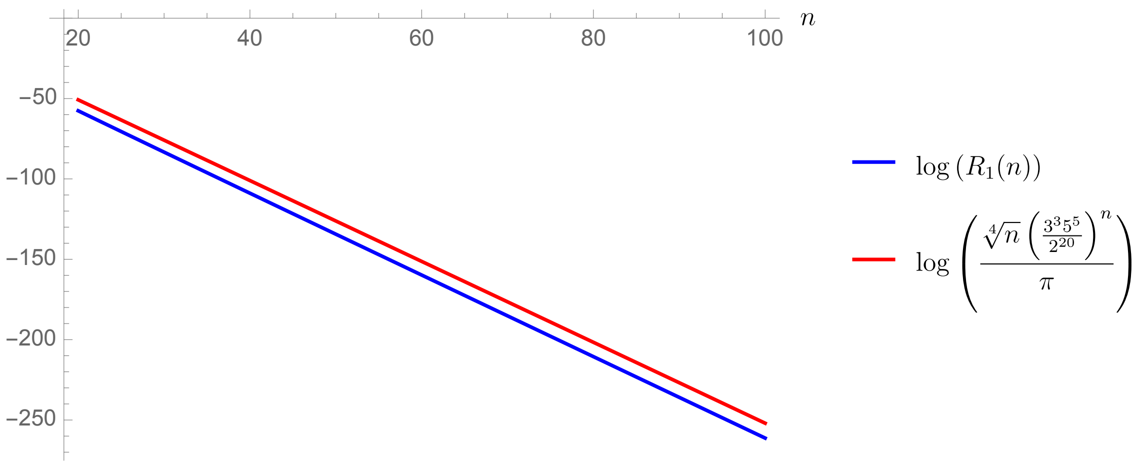

FIGURE 4.1. Logarithmic plot of $R_1(n)$ and its upper bound $\frac{n^{\frac{1}{4}}}{\pi}\left(\frac{3^3 5^5}{2^{20}}\right)^n$.

PROOF. By Euler's reflection formula (327) it is

$$|a_{4n-1}(n)| = 2^{-4n+1}\frac{\Gamma\left(5n-\frac{1}{2}\right)}{\Gamma(4n)\left|\Gamma\left(-3n+\frac{3}{2}\right)\right|} = 2^{-4n-1}\frac{\Gamma\left(5n-\frac{1}{2}\right)\Gamma\left(3n-\frac{1}{2}\right)}{\pi\Gamma(4n)}.$$

Consequently, the first term satisfies

$$\begin{aligned} R_1(n) &= \frac{\left(n+\frac{1}{2}\right)^{\frac{3}{4}}}{\pi}2^{-4n}\frac{\Gamma\left(5n-\frac{1}{2}\right)\Gamma\left(3n-\frac{1}{2}\right)\Gamma\left(2n\right)}{\Gamma\left(4n+\frac{1}{2}\right)\Gamma(4n)\Gamma\left(2n+\frac{1}{2}\right)} \\ &= \frac{\left(n+\frac{1}{2}\right)^{\frac{3}{4}}}{2\pi^{\frac{3}{2}}}2^{4n}\frac{\Gamma\left(5n-\frac{1}{2}\right)\Gamma\left(3n-\frac{1}{2}\right)\Gamma(2n)}{\Gamma(8n)\Gamma\left(2n+\frac{1}{2}\right)} \\ &\leqslant \frac{2^{4n}}{2\pi^{\frac{3}{2}}}\left(n+\frac{1}{2}\right)^{\frac{3}{4}}\frac{\Gamma(5n)\Gamma(3n)}{\Gamma(8n)}, \end{aligned}$$

where we used Legendre's duplication formula (328) in the second step. By an application of Stirling's formula (332) on the gamma quotient in the last line above we arrive at

$$R_1(n) \leqslant \frac{2}{\sqrt{15}\pi}e^{\frac{2}{45n}}\frac{\left(n+\frac{1}{2}\right)^{\frac{3}{4}}}{n^{\frac{1}{2}}}\left(\frac{3^3 5^5}{2^{20}}\right)^n.$$

Note that $e^{-\mu(n)} < e^{-\frac{1}{12n+1}} < 1$ for all $1 \leqslant n < \infty$. Moreover, since $\left(\frac{n+\frac{1}{2}}{n}\right)^{\frac{1}{2}} = \left(1+\frac{1}{2n}\right)^{\frac{1}{2}} \leqslant \frac{\sqrt{5}}{2}, e^{\frac{2}{45n}} < e^{\frac{1}{20}}$ and $\left(n+\frac{1}{2}\right)^{\frac{1}{4}} < \sqrt{\frac{3}{e^{\frac{1}{10}}}}n^{\frac{1}{4}}$ for all $n \geqslant 2$, we further estimate

$$R_1(n) \leqslant \frac{n^{\frac{1}{4}}}{\pi}\left(\frac{3^3 5^5}{2^{20}}\right)^n$$

as stated in the Lemma. ∎

For the ℓ^1-ℓ^∞-estimates of the other components $R_2(n), R_3(n)$, and $R_4(n)$ we have to work harder, since we now actually encounter sums of gamma quotients.

In the following two subsections we develop a strategy on how to maximize a gamma quotient like (65) on a subset of $\mathbb{Z}$. The first step is to reduce the complexity of the problem by narrowing down the relevant subset. To this end, we now take a closer look at symmetries in the definition of the functions $R_m(n)$.

4.2.2. Symmetry Considerations. A source of complexity in maximizing the summands of $R_2(n), R_3(n)$, and $R_4(n)$ are gamma function factors in the denominator of those summands that change sign depending on the value of the summation indices k, i, j. This leads to a large number of different cases that we have to consider for the summation indices. The goal of this subsection is to reduce the number of relevant cases by exploiting symmetry properties of the summands of $R_2(n), R_3(n)$, and $R_4(n)$. The result is summarized in Lemma 17. Corollary 18 then uses Lemma 17 to deduce upper bounds for the functions $r_2(n), r_3(n)$ and $r_4(n)$.

Let us start with a few comments on the notation. Although we have already set the expansion length $l = 4n - 1$ in (48), we stick to l here, since this makes the upcoming chain of reasoning easier to follow in our opinion. Moreover, from now on and throughout the entire chapter we denote the summands in the terms $r_2(n), r_3(n)$ and $r_4(n)$ by $2^{-l}F_2(n, k), 2^{-l}F_3(n, k, i)$ and $2^{-l}F_4(n, k, i, j)$, respectively. Those then read

$$
\begin{aligned}
F_2(n, k, l) &:= 2^l |a_k(n) a_{l-k}(n)|, \\
F_3(n, i, k, l) &:= 2^l |a_i(n) a_{k-i}(n) a_{l-k}(n)|, \\
F_4(n, j, i, k, l) &:= 2^l |a_j(n) a_{i-j}(n) a_{k-i}(n) a_{l-k}(n)|.
\end{aligned}
$$

Which function exactly hides behind the expansion coefficients $a_j(n)$ can be found in (11), but is not relevant at this point. Having a look at the structure of the sums in (63), we see that these functions live on the sets

$$
\begin{aligned}
\mathcal{D}_2 &:= \{k \in \mathbb{Z} \mid 0 \leqslant k \leqslant l - 1\}, \\
\mathcal{D}_3 &:= \{i, k \in \mathbb{Z} \mid 0 \leqslant i \leqslant k \leqslant l - 1\}, \\
\mathcal{D}_4 &:= \{j, i, k \in \mathbb{Z} \mid 0 \leqslant j \leqslant i \leqslant k \leqslant l - 1\}.
\end{aligned}
$$

Since we will at a later stage of this work encounter an object very similar to $F_4(n, k, i, j)$, we want to present our ideas and findings here in a generalized version. To this end, let $l = 4n - 1$ as above, $m \geqslant 2$ be an integer and $\mathbf{k} = (k_1, \ldots, k_{m-1}) \in \mathbb{Z}^{m-1}$. We consider the function

$$
F_m(n, \mathbf{k}, l) := 2^l \left| a_{k_1}(n) \prod_{i=1}^{m-2} a_{k_{i+1}-k_i}(n) a_{l-k_{m-1}}(n) \right|, \tag{66}
$$

defined on $\mathcal{D}_m \subset \mathbb{Z}^{m-1}$ with

$$
\mathcal{D}_m := \{k_1, \ldots, k_{m-1} \in \mathbb{Z} \mid 0 \leqslant k_1 \leqslant \cdots \leqslant k_{m-1} \leqslant l - 1\}. \tag{67}
$$

Upon the transformation $\tau : \mathbb{Z}^{m-1} \to \mathbb{Z}^m, \mathbf{k} \mapsto \mathbf{z}$ with

$$
z_i = \tau(\mathbf{k})_i = \begin{cases} k_1, & i = 1, \\ k_i - k_{i-1}, & i = 2, \ldots, m - 1, \\ l - k_{m-1}, & i = m, \end{cases} \tag{68}
$$

we obtain the following representation of F_m and its domain

$$
F_m(n, \mathbf{z}) = 2^l \left| \prod_{i=1}^{m} a_{z_i}(n) \right|, \tag{69}
$$

$$
\tau(\mathcal{D}_m) = \left\{ \mathbf{z} \in \mathbb{Z}^m \,\middle|\, \begin{matrix} 0 \leqslant z_1, \ldots, z_{m-1} \leqslant l - 1, 1 \leqslant z_m \leqslant l \\ \sum_{i=1}^{m} z_i = l \end{matrix} \right\}.
$$

Recall that we are interested in maximizing $F_m(n, \mathbf{z})$ on $\mathcal{D}_m$, and note that the maximum of $F_m(n, \mathbf{z})$ can only become larger when we consider a larger set of $\mathbf{z}$. Thus, we can safely pass to the symmetric superset

$$\left\{ \mathbf{z} \in \mathbb{N}_0^m \,\middle|\, \sum_{i=1}^{m} z_i = l \right\} \supset \mathcal{D}_m$$

when looking for the maximum of $F_m(n, \mathbf{z})$. Finally, we are able to exploit the symmetric structure of (69) under permutation of $z_1, \ldots, z_m$ to restrict our search for the maximum of $F_m(n, \mathbf{z})$ to those $\mathbf{z}$ such that $z_1 \leqslant \cdots \leqslant z_m$. Let us summarize this in the following Lemma.

Lemma 17. *Let $n \geqslant 1, l = 4n-1$, F_m be the function* (66) *with the domain $\mathcal{D}_m$ that we defined in* (67). *Moreover, let $\tau : \mathbb{Z}^{m-1} \to \mathbb{Z}^m, \boldsymbol{k} \mapsto \boldsymbol{z}$ be the transformation* (68). *Then*

$$\max_{\boldsymbol{k} \in \mathcal{D}_m} F_m(n, \boldsymbol{k}, l) \leqslant \max_{\boldsymbol{z} \in D_m} F_m(n, \boldsymbol{z}),$$

with

$$D_m := \left\{ \boldsymbol{z} \in \mathbb{Z}^m \,\middle|\, \begin{matrix} 0 \leqslant z_1 \leqslant \cdots \leqslant z_m \leqslant l \\ \sum_{i=1}^{m} z_i = l \end{matrix} \right\}.$$

In the case of $m = 2, 3, 4$ we pass to the notation

$$\begin{aligned} \tau(k) &= (c, d) \in \mathbb{Z}^2, \\ \tau(i, k) &= (b, c, d) \in \mathbb{Z}^3, \\ \tau(j, i, k) &= (a, b, c, d) \in \mathbb{Z}^4, \end{aligned} \tag{70}$$

respectively. As a direct consequence of Lemma 17, we obtain the following bounds on the functions $r_2(n), r_3(n)$ and $r_4(n)$.

Corollary 18. *Let $n \in \mathbb{Z}, n \geqslant 0$ and $l = 4n - 1$. Then the functions $r_2(n), r_3(n)$ and $r_4(n)$ defined in* (63) *satisfy the upper bounds*

$$\begin{aligned} r_2(n) &\leqslant r_0^2(n) l 2^{-l} \max_{D_2} F_2(n, c, d), \\ r_3(n) &\leqslant r_0(n) \frac{1}{2} l(l+1) 2^{-l} \max_{D_3} F_3(n, b, c, d), \\ r_4(n) &\leqslant \frac{1}{6} l(l+1)(l+2) 2^{-l} \max_{D_4} F_4(n, a, b, c, d), \end{aligned}$$

with

$$\begin{aligned} D_2 &= \left\{ c, d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant c \leqslant d \leqslant l \\ c + d = l \end{matrix} \right\} \\ D_3 &= \left\{ b, c, d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant b \leqslant c \leqslant d \leqslant l \\ b + c + d = l \end{matrix} \right\} \\ D_4 &= \left\{ a, b, c, d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant l \\ a + b + c + d = l \end{matrix} \right\}. \end{aligned}$$

Since we are working in a discrete setting, the structure of the sets D_2, D_3 and D_4 has the following implications on the magnitude of the parameters.

Corollary 19. *For $l = 4n - 1$ with $n \in \mathbb{Z}, n \geqslant 0$, we can infer that*

(i) If $c, d \in D_2$, then

$$\begin{aligned} c &\leqslant 2n - 1, \\ d &\geqslant 2n. \end{aligned}$$

(ii) If $b, d \in D_3$, then

$$b \leqslant \left\lfloor \frac{4}{3}n - \frac{1}{3} \right\rfloor,$$
$$d \geqslant \left\lceil \frac{4}{3}n - \frac{1}{3} \right\rceil.$$

(iii) If $a, d \in D_4$, then

$$a \leqslant n - 1,$$
$$d \geqslant n.$$

Note that in the case of $F_2(n, c, d)$ and $F_4(n, a, b, c, d)$ the smallest difference, i. e. c and a, respectively, is always strictly smaller than the largest one d. In the case of $F_3(n, b, c, d)$ equality can actually be obtained if $n \equiv 1 \mod 3$. For all other n it is $b < d$.

4.2.3. Monotonicity Properties. The purpose of this subsection is to describe the idea behind our approach of finding the maximum of a gamma quotient, and to set up the machinery that will accompany us throughout large parts of this entire work. More precisely, in Theorem 20 we first develop a general principle that helps us to narrow down the location of an extremum of a gamma quotient. It enables us to use a recurrence relation for our purposes instead of the actual, very involved derivative of a gamma quotient. This is then applied in Theorem 22 to prove a handy monotonicity property for a slightly more general version of the gamma quotient (65).

Let us start by taking a closer at the generalized gamma quotient consisting of $p \geqslant 1$ gamma factors in the numerator and $q \geqslant 1$ gamma factors in the denominator

$$Q(z) := \frac{\prod_{i=1}^{p} \Gamma(x_i + \alpha_i z)}{\prod_{i=1}^{q} \Gamma(y_i + \beta_i z)}, \tag{71}$$

defined on a subset $\mathcal{D}_Q \subset \mathbb{Z}$. We assume Q and its domain meet the following conditions

$$x_i, y_i \in \mathbb{R}, \tag{72}$$
$$\alpha_i, \beta_i \in \mathbb{Z}\backslash\{0\}, \tag{73}$$
$$\mathcal{D}_Q = [s, t] \cap \mathbb{Z}, \quad \text{for some } s, t \in \mathbb{Z}, \tag{74}$$
$$Q(z) \geqslant 0, \qquad \text{for all } z \in \mathcal{D}_Q. \tag{75}$$

Due to assumptions (73) and (74) we may apply the functional equation (326) for the gamma function factor-wise to (71) to deduce a recurrence relation for Q on $\mathcal{D}_Q$ of the form

$$Q(z) = \varphi(z)Q(z-1). \tag{76}$$

To make this clearer let for example $\alpha_i > 0$ for an $i \in \{1, \dots, p\}$. Then

$$\Gamma(x_i + \alpha_i z) = \prod_{j=1}^{\alpha_i} (x_i + \alpha_i z - j)\Gamma(x_i + \alpha_i(z-1)).$$

In the case of $\alpha_i < 0$ we yield

$$\Gamma(x_i - |\alpha_i| z) = \prod_{j=0}^{|\alpha_i|-1} (x_i - |\alpha_i| z + j)^{-1}\Gamma(x_i - |\alpha_i|(z-1)).$$

Due to the structure of Q, the recurrence function φ is a rational function in z. Note that by subtracting $Q(z-1)$ on both sides of (76), we obtain an expression for the finite backwards difference $Q(z) - Q(z-1)$ in terms of $\varphi(z)$, namely

$$Q(z) - Q(z-1) = (\varphi(z) - 1)Q(z-1). \tag{77}$$

Hence, the function $(\varphi - 1)Q$ may be interpreted as the discrete analogue of the derivative of Q, and similar to a derivative it carries valuable information on monotonicity properties and the location of extrema of Q. Since we additionally assume that Q is nonnegative, the aforementioned properties are directly linked to properties of φ in the following way.

THEOREM 20. *(i) If $\varphi(z) \geqslant 1$ for all $z \in \mathcal{D}_Q$, then Q increases monotonically on $\mathcal{D}_Q$.*

(ii) If $\varphi(z) \leqslant 1$ for all $z \in \mathcal{D}_Q$, then Q decreases monotonically on $\mathcal{D}_Q$.

(iii) If $\varphi(\zeta) = 1$ for $\zeta \in [z^*, z^* + 1)$ with $z^* \in \mathcal{D}_Q$, then Q has a maximum at z^*, if $\varphi(z) \geqslant 1$ for all $z \in \mathcal{D}_Q$ with $z \leqslant \zeta$ and $\varphi(z) \leqslant 1$ for all $z \in \mathcal{D}_Q$ with $z \geqslant \zeta$.

(iv) If $\varphi(\zeta) = 1$ for $\zeta \in [z_*, z_* + 1)$ with $z_* \in \mathcal{D}_Q$, then Q has a minimum at z_*, if $\varphi(z) \leqslant 1$ for all $z \in \mathcal{D}_Q$ with $z \leqslant \zeta$ and $\varphi(z) \geqslant 1$ for all $z \in \mathcal{D}_Q$ with $z \geqslant \zeta$.

PROOF. Assertions *(i)* and *(ii)* follow directly from (77) and the positivity assumption (75) on Q. Only *(iii)* and *(iv)* need a little more explanation. Upon the assumptions of *(iii)* we show that $Q(z^*) \geqslant Q(z)$ for all $z \in \mathcal{D}_Q$. Let us first assume that $z \in \mathcal{D}_Q$ and $z = z^* - m$ for a nonnegative integer m. Since $z^* \leqslant \zeta$ by assumption, we infer from the condition $\varphi(z) \geqslant 1$ for $z \leqslant \zeta$, that

$$\begin{aligned} Q(z^*) &= \sum_{i=0}^{m-1} (Q(z^* - i) - Q(z^* - (i+1))) + Q(z) \\ &= \sum_{i=0}^{m-1} (\varphi(z^* - i) - 1)Q(z^* - (i+1)) + Q(z) \\ &\geqslant Q(z). \end{aligned}$$

Next, let $z \in \mathcal{D}_Q$ with $z = z^* + n$ for an integer $n \geqslant 1$. Thus, $z \geqslant z^* + 1 \geqslant \zeta$ and since $\varphi(z) \leqslant 1$ for all $z \geqslant \zeta$, we obtain

$$\begin{aligned} Q(z) = Q(z^* + n) &= \sum_{i=0}^{n-1} (Q(z^* + n - i) - Q(z^* + n - (i+1))) + Q(z^*) \\ &= \sum_{i=0}^{n-1} (\varphi(z^* + n - i) - 1)Q(z^* + n - (i+1)) + Q(z^*) \\ &\leqslant Q(z^*). \end{aligned}$$

For the proof of *(iv)* we need to show that $Q(z^*) \leqslant Q(z)$ for all $z \in \mathcal{D}_Q$. Copying the proof of *(iii)* does the job. Whereas in this case the assumptions of *(iv)* imply that $\varphi(z^* - i) - 1 \leqslant 1$ and $\varphi(z^* + m - i) - 1 \geqslant 1$ and thus lead to the claim. ∎

To really drive this point home, we want to emphasize the following consequence of Theorem 20.

REMARK 21. From the location $\zeta \in [z^*, z^* + 1)$ or $\zeta \in [z_*, z_* + 1)$, respectively, of the point ζ with $\varphi(\zeta) = 1$, we conclude that the location z^* or z_*, respectively, of the extremum of Q on the discrete domain $\mathcal{D}_Q$ can be narrowed down to the interval $(\zeta - 1, \zeta]$.
Moreover, if $\zeta \in \mathcal{D}_Q$, then $Q(\zeta) = Q(\zeta - 1)$ and thus both, $Q(\zeta)$ and $Q(\zeta - 1)$, are extrema of Q on $\mathcal{D}_Q$.

In the next step we apply the above findings to the case of the gamma quotient (65) with two subsequent differences $z_1 \leqslant z_2$.

THEOREM 22. *Recall the factors*

$$a_z(n) = \frac{\Gamma\left(n+z+\frac{1}{2}\right)}{2^z\Gamma(z+1)\Gamma\left(n-z+\frac{1}{2}\right)}$$

and consider the gamma quotient

$$Q(n, z_1, z_2) = |a_{z_1}(n)a_{z_2}(n)|.$$

We distinguish the following three cases.

(i) If $0 < z_1 \leqslant z_2 < n$*, then*

$$Q(n, z_1, z_2) \geqslant Q(n, z_1 - 1, z_2 + 1).$$

(ii) If $n < z_1 \leqslant z_2$*, then*

$$Q(n, z_1, z_2) \leqslant Q(n, z_1 - 1, z_2 + 1).$$

(iii) If $0 < z_1 \leqslant n \leqslant z_2$*, then* $Q(n, z_1, z_2) \geqslant Q(n, z_1 - 1, z_2 + 1)$ *holds iff*

$$n^2(z_1 + z_2 + 1) \geqslant \frac{1}{2}\left(z_1 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(z_2 + \frac{1}{2}\right)^2 + (z_1 + z_2)\left(z_1 - \frac{1}{2}\right)\left(z_2 + \frac{1}{2}\right).$$

PROOF. We want to apply Theorem 20 to the gamma quotient Q in the two arguments z_1, z_2 simultaneously. In all three cases we have

$$\begin{aligned}\varphi(z_1, z_2) &= \frac{Q(n, z_1, z_2)}{Q(n, z_1 - 1, z_2 + 1)} \\ &= \left|\frac{n + z_1 - \frac{1}{2}}{n + z_2 + \frac{1}{2}} \cdot \frac{n - z_1 + \frac{1}{2}}{n - z_2 - \frac{1}{2}} \cdot \frac{z_2 + 1}{z_1}\right|.\end{aligned}$$

In *(i)* the enumerator and the denominator of φ are positive. Hence, $\varphi(z_1, z_2) \geqslant 1$ is equivalent to

$$\left(n^2 - \left(z_1 - \frac{1}{2}\right)^2\right)(z_2 + 1) \geqslant \left(n^2 - \left(z_2 + \frac{1}{2}\right)^2\right)z_1.$$

Expanding both sides of the above inequality yields

$$n^2(z_2 + 1) - (z_2 + 1)\left(z_1 - \frac{1}{2}\right)^2 \geqslant n^2 z_z - z_1\left(z_2 + \frac{1}{2}\right)^2$$

which is quickly simplified to

$$n^2(z_2 - z_1 + 1) \geqslant (z_2 + 1)\left(z_1 - \frac{1}{2}\right)^2 - z_1\left(z_2 + \frac{1}{2}\right)^2. \tag{78}$$

We add and subtract $z_1 - \frac{1}{2}$ in $\left(z_2 + \frac{1}{2}\right)^2$ and get

$$\left(z_2 - z_1 + 1 + \left(z_1 - \frac{1}{2}\right)\right)^2 = (z_2 - z_1 + 1)^2 + 2(z_2 - z_1 + 1)\left(z_1 - \frac{1}{2}\right) + \left(z_1 - \frac{1}{2}\right)^2.$$

Now we can simplify the right-hand side of (78) as follows

$$\begin{aligned}&(z_2 + 1)\left(z_1 - \frac{1}{2}\right)^2 - z_1\left(z_2 + \frac{1}{2}\right)^2 \\ &= (z_2 - z_1 + 1)\left(z_1 - \frac{1}{2}\right)^2 - z_1(z_2 - z_1 + 1)^2 - 2z_1(z_2 - z_1 + 1)\left(z_1 - \frac{1}{2}\right)^2 \\ &= (z_2 - z_1 + 1)\left(\left(z_1 - \frac{1}{2}\right)^2 - z_1(z_2 - z_1 + 1) - 2z_1\left(z_1 - \frac{1}{2}\right)\right) \\ &= (z_2 - z_1 + 1)\left(\frac{1}{4} - z_1 - z_1 z_2\right).\end{aligned}$$

This transforms (78) into

$$n^2\left(z_2 - z_1 + 1\right) \geqslant \left(z_2 - z_1 + 1\right)\left(\frac{1}{4} - z_1 - z_1 z_2\right).$$

Since we assumed that $0 < z_1 \leqslant z_2$, we obtain that $\varphi(z_1, z_2) \geqslant 1$ iff

$$n^2 \geqslant \frac{1}{4} - z_1 - z_1 z_2,$$

which is always true for positive integers n, z_1 and z_2.
In case *(ii)* the enumerator and denominator of φ are both negative. Hence, it is $\varphi(z_1, z_2) \geqslant 1$ iff

$$\left(n^2 - \left(z_1 - \frac{1}{2}\right)^2\right)(z_2 + 1) \leqslant \left(n^2 - \left(z_2 + \frac{1}{2}\right)^2\right) z_1.$$

All following inequalities in the proof of *(i)* are also reverted and we can immediately refer from the calculations in *(i)* that the reverted condition on φ, that is

$$\varphi(z_1, z_2) \leqslant 1, \tag{79}$$

is always true for $n < z_1 \leqslant z_2$. As by construction (79) is equivalent to $Q(n, z_1, z_2) \leqslant Q(n, z_1 - 1, z_2 + 1)$ assertion *(ii)* follows.
In case *(iii)* the enumerator of φ is positive and the denominator is negative. Consequently it is $\varphi(z_1, z_2) \geqslant 1$ iff

$$\left(n^2 - \left(z_1 - \frac{1}{2}\right)^2\right)(z_2 + 1) \geqslant -\left(n^2 - \left(z_2 + \frac{1}{2}\right)^2\right) z_1.$$

By copying the expansion and simplification steps that lead to (78) in the proof of *(i)* the above inequality turns into

$$n^2(z_1 + z_2 + 1) \geqslant (z_2 + 1)\left(z_1 - \frac{1}{2}\right)^2 + z_1\left(z_2 + \frac{1}{2}\right)^2. \tag{80}$$

In order to deduce the asserted inequality from (80), we write

$$\begin{aligned}(z_2 + 1)\left(z_1 - \frac{1}{2}\right)^2 &= \frac{1}{2}\left(z_1 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(z_1 - \frac{1}{2}\right)^2 + z_2\left(z_1 - \frac{1}{2}\right)^2, \\ z_1\left(z_2 + \frac{1}{2}\right)^2 &= \frac{1}{2}\left(z_2 + \frac{1}{2}\right)^2 - \frac{1}{2}\left(z_2 + \frac{1}{2}\right)^2 + z_1\left(z_2 + \frac{1}{2}\right)^2,\end{aligned}$$

and note that

$$\begin{aligned}& z_2\left(z_1 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(z_1 - \frac{1}{2}\right)^2 + z_1\left(z_2 + \frac{1}{2}\right)^2 - \frac{1}{2}\left(z_2 + \frac{1}{2}\right)^2 \\ &= \left(z_1 - \frac{1}{2}\right)\left(z_2 + \frac{1}{2}\right)\left(z_1 - \frac{1}{2} + z_2 + \frac{1}{2}\right).\end{aligned}$$

Thus the right-hand side of (80) is equal to

$$\frac{1}{2}\left(z_1 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(z_2 + \frac{1}{2}\right)^2 + \left(z_1 - \frac{1}{2}\right)\left(z_2 + \frac{1}{2}\right)\left(z_1 - \frac{1}{2} + z_2 + \frac{1}{2}\right).$$

∎

The upcoming Subsections 4.2.4, 4.2.5 and 4.2.6 are now entirely devoted to the proofs of upper bounds on the remainder summands $R_2(n), R_3(n)$ and $R_4(n)$, respectively. As mentioned previously this mainly consists of finding the maximum of the functions $F_2(n, c, d)$, $F_3(n, b, c, d)$ and $F_4(n, a, b, c, d)$ by applying the findings of this subsection.

4.2.4. Upper Bound on $R_2(n)$. In this subsection we will make heavy use of the machinery we developed throughout Subsection 4.2.3 to prove the following bound on $R_2(n)$.

LEMMA 23. *Let $n \geqslant 20$, then*

$$R_2(n) \leqslant \frac{e^{\frac{3}{2}+\frac{2}{n}}}{\sqrt{3}\pi^{\frac{3}{2}}} n^{-1} \left(\frac{3^3}{2^8}\right)^n.$$

Before we start proving the lemma, we visualize its statement graphically in Figure 4.2 in the same manner as for the bound of $R_1(n)$.

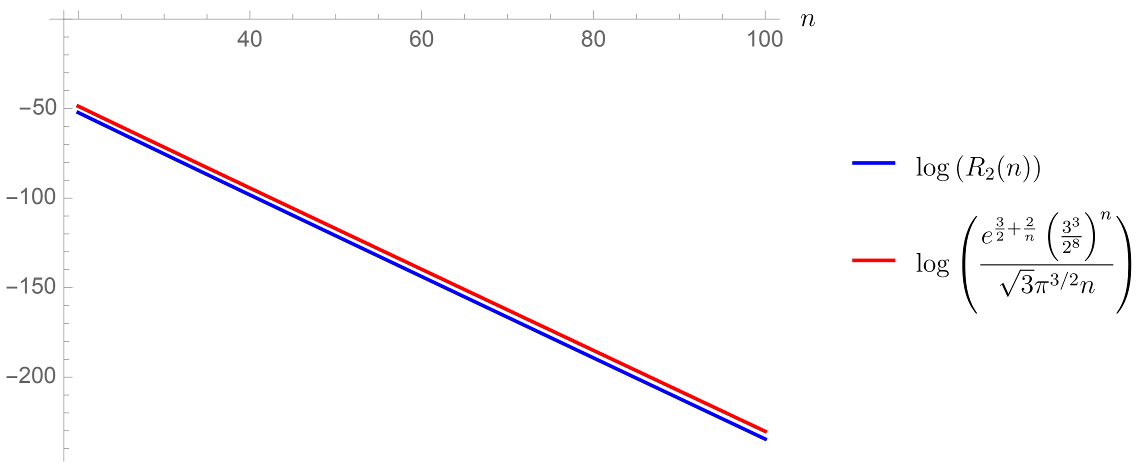

FIGURE 4.2. Logarithmic plot of $R_2(n)$ and its upper bound $\frac{e^{\frac{3}{2}+\frac{2}{n}}}{\sqrt{3}\pi^{\frac{3}{2}}} n^{-1} \left(\frac{3^3}{2^8}\right)^n$.

PROOF. By (64) and Corollary 18 it is

$$R_2(n) \leqslant (4n-1)\left(n+\frac{1}{2}\right)^{\frac{1}{2}} \frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} \max_{D_2} F_2(n,c,d),$$

with

$$F_2(n,c,d) = \frac{\Gamma\left(n+c+\frac{1}{2}\right)\Gamma\left(n+d+\frac{1}{2}\right)}{\Gamma(c+1)\Gamma(d+1)\left|\Gamma\left(n-c+\frac{1}{2}\right)\Gamma\left(n-d+\frac{1}{2}\right)\right|}$$

and

$$D_2 = \left\{c,d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant c \leqslant d \leqslant 4n-1 \\ c+d = 4n-1 \end{matrix}\right\}.$$

Thanks to the reduction in dimension by the constraint $c+d = 4n-1$, we can easily visualize $F_2(n,c,d)$ on its domain D_2. This is done in Figure 4.3, which plots $F_2(n,c,4n-1-c)$ in $0 \leqslant c \leqslant 2n-1$ for $n = 30$. The yellow and red vertical lines mark the lower and upper bound, respectively, we state in (81) for the location c^* of the maximum. Now, let's deduce these bounds and estimate the maximum.

By Corollary 19 we already know that $c \leqslant 2n-1$ and $d \geqslant 2n$. Let us for the moment assume that $c > n$. We then apply Theorem 22 *(ii)* repeatedly

$$F_2(n,c,d) \leqslant F_2(n,c-1,d+1) \leqslant \cdots \leqslant F_2(n,c-m,d+m)$$

with $m \geqslant 1$, until we reach $n = c-m \leqslant d+m$, in other words, until we pushed c down to n. This implies that $F_2(n,c,d)$ is maximized by (c,d) such that $0 < c \leqslant n < d$. Part *(iii)* of Theorem 22 and the reduction $c+d = 4n-1$ now allow to maximize in one of the two parameters. In fact, we set $d = 4n-c-1$ and introduce the recurrence factor $\varphi_2(c)$ as

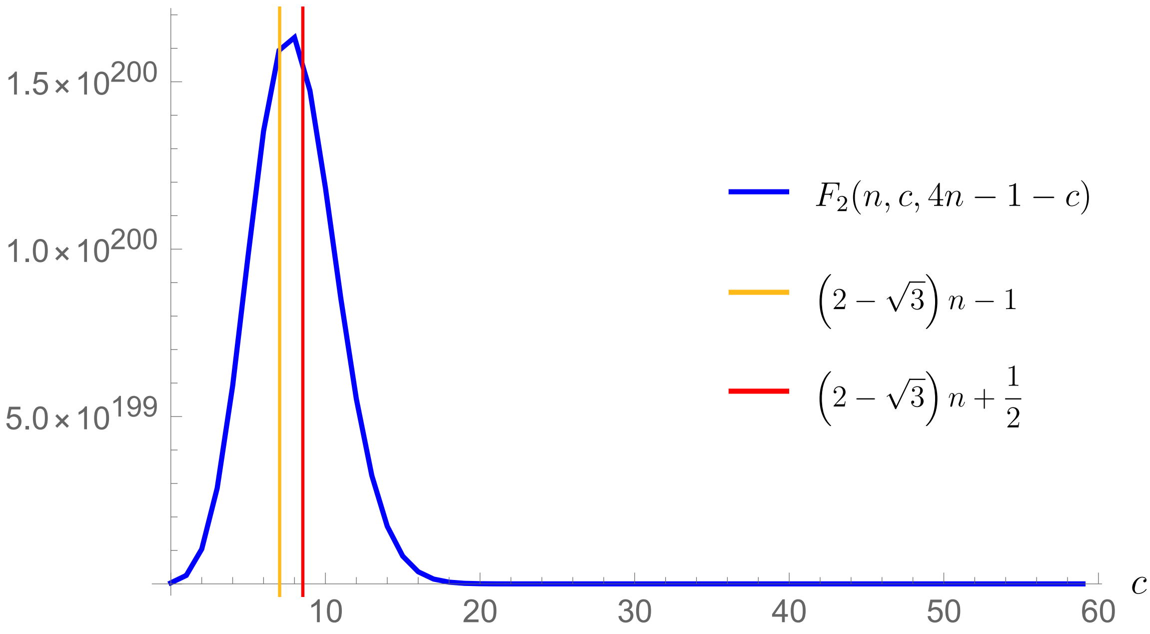

FIGURE 4.3. Plot of $F_2(n, c, 4n - 1 - c)$ for $n = 30$.

$F_2(n, c, 4n-c-1) = \varphi_2(c)F_2(n, c-1, 4n-c)$. We showed in the proof of *(iii)* that $\varphi(c) \geqslant 1$, i.e. that $F_2(n, c, 4n - c - 1)$ is increasing in c, iff

$$4n^3 \geqslant \frac{1}{2}\left(c - \frac{1}{2}\right)^2 + \frac{1}{2}\left(4n - c - \frac{1}{2}\right)^2 + (4n-1)\left(c - \frac{1}{2}\right)\left(4n - c - \frac{1}{2}\right).$$

As the right-hand side is equal to $-(2n-1)(2c^2 - 8cn) + n$ this simplifies to

$$(2n-1)(2c^2 - 8cn + 2n^2 + n) \geqslant 0.$$

Since $2n-1 > 0$, the above condition enables us to determine the value ζ such that $\varphi(\zeta) = 1$. The quadratic equation in c has the two solutions

$$c_{1,2} = 2n \mp \frac{1}{\sqrt{2}}\left(6n^2 - n\right)^{\frac{1}{2}}.$$

Due to the upper bound $c \leqslant 2n - 1$ on c we get $\zeta = c_1$, as well as $\varphi(c) \geqslant 1$ for $c \leqslant \zeta$ and $\varphi(c) \leqslant 1$ for $c \geqslant \zeta$.

According to Theorem 20 (iii), the point c^*, at which $F_2(n, c, 4n - c - 1)$ attains its maximum, satisfies $\zeta - 1 < c^* \leqslant \zeta$. In order to make the following calculations less laborious, we work with the two-sided bound

$$(2-\sqrt{3})n < \zeta < (2-\sqrt{3})n + \frac{1}{2}.$$

It can easily been seen, that the upper bound is true for $n \geqslant \frac{1}{6}$. The lower one is obvious for all $n \geqslant 0$.

This narrows down the location c^* of the maximum of $F_2(n, c, 4n - c - 1)$ to

$$(2-\sqrt{3})n - 1 < c^* < (2-\sqrt{3})n + \frac{1}{2}. \tag{81}$$

The following lines are now dedicated to the estimate of $F_2(n, c^*, 4n - c^* - 1)$. To simplify the notation we abbreviate $F_2(n, c^*) := F_2(n, c^*, 4n - c^* - 1)$ and $r := 2 - \sqrt{3}$. Moreover, we assume

$$c^* = rn - \frac{1}{2} + q$$

for $-\frac{1}{2} < q < 1$. Then we obtain by Stirling's formula (332)

$$
\begin{aligned}
F_2(n, c^*) &= \frac{\Gamma(n + rn + q)\Gamma(5n - rn - q)\Gamma(3n - rn - q)}{\pi\Gamma\left(rn + \frac{1}{2} + q\right)\Gamma\left(4n - rn + \frac{1}{2} - q\right)\Gamma(n - rn + 1 - q)} \\
&\leqslant \frac{1}{\pi}\exp\left[-4n + 2 + \frac{1}{12n}\left(\frac{1}{1 + r + \frac{q}{n}} + \frac{1}{5 - r - \frac{q}{n}} + \frac{1}{3 - r - \frac{q}{n}}\right)\right] \\
&\times \frac{[n(1 + r) + q]^{n(1+r)-\frac{1}{2}+q}[n(5 - r) - q]^{n(5-r)-\frac{1}{2}-q}[n(3 - r) - q]^{n(3-r)-\frac{1}{2}-q}}{\left[rn + \frac{1}{2} + q\right]^{rn+q}\left[n(4 - r) + \frac{1}{2} - q\right]^{n(4-r)-q}[n(1 - r) + 1 - q]^{n(1-r)+\frac{1}{2}-q}}.
\end{aligned}
\tag{82}
$$

Next, we estimate for $n \geqslant 1$ and $-\frac{1}{2} < q < 1$

$$
\begin{aligned}
\frac{1}{1 + r + \frac{q}{n}} + \frac{1}{5 - r - \frac{q}{n}} + \frac{1}{3 - r - \frac{q}{n}} &\leqslant \frac{1}{\frac{1}{2} + r} + \frac{1}{4 - r} + \frac{1}{2 - r} \\
&= \frac{108 - 14\sqrt{3}}{39},
\end{aligned}
\tag{83}
$$

and rewrite the second line of (82) as

$$
\begin{aligned}
&\left(\frac{1}{(1 + r)(1 - r)(5 - r)(3 - r)}\right)^{\frac{1}{2}} \frac{n^{4n}}{n^2} \\
&\left[\frac{(1 + r)(1 - r)(4 - r)}{r(5 - r)(3 - r)}\right]^{rn+q}\left[\frac{(1 + r)(5 - r)^5(3 - r)^3}{(4 - r)^4(1 - r)}\right]^n \\
&\times \frac{\left(1 + \frac{q}{n(1+r)}\right)^{n(1+r)+q-\frac{1}{2}}\left(1 - \frac{q}{n(5-r)}\right)^{n(5-r)-q-\frac{1}{2}}\left(1 - \frac{q}{n(3-r)}\right)^{n(3-r)-q-\frac{1}{2}}}{\left(1 + \frac{q+1}{2rn}\right)^{rn+q}\left(1 + \frac{1-2q}{2n(4-r)}\right)^{n(4-r)-q}\left(1 + \frac{1-q}{n(1-r)}\right)^{n(1-r)-q+\frac{1}{2}}}.
\end{aligned}
\tag{84}
$$

We want to emphasize at this point that

$$\frac{(1 + r)(1 - r)(4 - r)}{r(5 - r)(3 - r)} = 1,$$

and

$$\frac{(1 + r)(5 - r)^5(3 - r)^3}{(4 - r)^4(1 - r)} = 2^4 3^3.$$

Therefore, only the last line of (84) needs a little more attention. Our goal is to estimate the factors in the numerator by the standard inequality $1 + x \leqslant e^x$ and apply inequality (403) of Lemma 64 in the denominator. Since the latter is only applicable to positive x, we have to consider the factor in the denominator corresponding to $x = \frac{1-2q}{2(4-r)n}$ separately. This is

$$\left(1 + \frac{1 - 2q}{2n(4 - r)}\right)^{-n(4-r)+q}. \tag{85}$$

We plug in $r = 2 - \sqrt{3}$ and calculate the derivative in q, which is equal to

$$\frac{\left(2\left(\sqrt{3} + 2\right)n - 2q + 1\right)\log\left(\frac{1-2q}{2\sqrt{3}n+4n} + 1\right) + 2\left(\sqrt{3} + 2\right)n - 2q}{\left(2\left(\sqrt{3} + 2\right)n - 2q + 1\right)\left(\frac{1-2q}{2\sqrt{3}n+4n} + 1\right)^{\left(\sqrt{3}+2\right)n-q}}.$$

It's easily seen that the denominator is positive for all $n \geqslant 2$ and $-\frac{1}{2} < q < 1$. Moreover, since

$$\log\left(\frac{1-2q}{2\sqrt{3}n+4n}+1\right) > \log\left(1-\frac{1}{2\sqrt{3}n+4n}\right) > -\frac{1}{10}$$

for $n \geqslant 2$, also the numerator is positive and we estimate

$$\left(1+\frac{1-2q}{2n(4-r)}\right)^{n(4-r)-q} \geqslant \left(1-\frac{1}{2n(4-r)}\right)^{n(4-r)-1}.$$

Next, the application of Lemma 65 with $a = -\frac{1}{2}$ and $b = -1$ to the right-hand side above yields

$$\left(1-\frac{1}{2n(4-r)}\right)^{n(4-r)-1} \geqslant e^{-\frac{1}{2}} \tag{86}$$

as soon as $m \geqslant \frac{1}{2}$. Therefore, we can conclude that (85) is smaller than $e^{\frac{1}{2}}$ for all $n \geqslant 2$. Now, we can continue to estimate the last line of (84) and safely apply Lemma 64 and the basic inequality $1 + x \leqslant e^x$. This leads to the upper bound

$$\begin{aligned}\exp\Bigg[&-1-q\\ &+\frac{1}{n}\left(-\frac{4q^2-1}{8r}-\frac{(q-1)q}{2(1-r)}+\frac{\left(q+\frac{1}{2}\right)q}{3-r}+\frac{\left(q+\frac{1}{2}\right)q}{5-r}+\frac{\left(q-\frac{1}{2}\right)q}{r+1}\right)\\ &+\frac{1}{n^2}\left(\frac{q\left(q+\frac{1}{2}\right)^2}{2r^2}+\frac{\left(\frac{1}{2}-q\right)(1-q)^2}{(1-r)^2}\right)\Bigg].\end{aligned} \tag{87}$$

We plug in $r = 2-\sqrt{3}$ and obtain for the coefficient of $\frac{1}{n}$ in the second line above

$$\begin{aligned}&-\frac{4q^2-1}{8r}-\frac{(q-1)q}{2(1-r)}+\frac{\left(q+\frac{1}{2}\right)q}{3-r}+\frac{\left(q+\frac{1}{2}\right)q}{5-r}+\frac{\left(q-\frac{1}{2}\right)q}{r+1}\\ &=\frac{-6\left(3-\sqrt{3}\right)q^2+8\left(2\sqrt{3}-3\right)q+3}{24\left(2-\sqrt{3}\right)}\\ &\leqslant \frac{5}{72}\left(\sqrt{3}+6\right),\end{aligned}$$

since the quadratic function in q takes its maximal value for $q = \frac{2(2\sqrt{3}-3)}{3(3-\sqrt{3})}$. Moreover, in the third line of (87) we estimate the coefficient of $\frac{1}{n^2}$ by

$$\begin{aligned}\frac{q\left(q+\frac{1}{2}\right)^2}{2r^2}+\frac{\left(\frac{1}{2}-q\right)(1-q)^2}{(1-r)^2} &\leqslant \frac{\left(\frac{3}{2}\right)^2}{2r^2}+\frac{\left(\frac{3}{2}\right)^2}{(1-r)^2}\\ &=\frac{9}{8}\left(5\sqrt{3}+9\right),\end{aligned}$$

and bound (87) by

$$\exp\left[-\frac{1}{2}+\frac{5}{72n}\left(\sqrt{3}+6\right)+\frac{9}{8n^2}\left(5\sqrt{3}+9\right)\right]. \tag{88}$$

Now, we go back to (84), apply our recent findings and see that it is smaller than

$$\frac{n^{4n}}{2\sqrt{3}n^2}\exp\left[-\frac{1}{2}+\frac{5}{72n}\left(\sqrt{3}+6\right)+\frac{9}{8n^2}\left(5\sqrt{3}+9\right)\right]\left(2^4 3^3\right)^n. \tag{89}$$

We plug (89) into (82), use (83) and estimate

$$\frac{108-14\sqrt{3}}{468}+\frac{5}{72}\left(\sqrt{3}+6\right)+\frac{9}{8n}\left(5\sqrt{3}+9\right)\leqslant 2-\frac{13}{96}$$

for all $n\geqslant 20$. This way we eventually arrive at

$$F_2(n,c^*)\leqslant\frac{n^{4n}}{\pi 2\sqrt{3}n^2}\exp\left[-4n+\frac{3}{2}+\frac{2}{n}-\frac{13}{96n}\right]\left(2^4 3^3\right)^n. \tag{90}$$

Next, we estimate the factor $\frac{\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}$ in the definition of $R_2(n)$ using Stirling's formula (332) and Lemma 64.

$$\begin{aligned}\frac{\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}&\leqslant\frac{e^{1+4n+\frac{1}{24n}}}{(2\pi)^{\frac{1}{2}}}\frac{(2n)^{2n-\frac{1}{2}}}{\left(2n+\frac{1}{2}\right)^{2n}\left(4n+\frac{1}{2}\right)^{4n}}\\&=\frac{e^{1+4n+\frac{1}{24n}}}{2\pi^{\frac{1}{2}}n^{\frac{1}{2}}}n^{-4n}2^{-8n}\frac{1}{\left(1+\frac{1}{4n}\right)^{2n}}\frac{1}{\left(1+\frac{1}{8n}\right)^{4n}}\\&\leqslant\frac{e^{4n+\frac{13}{96n}}}{2\pi^{\frac{1}{2}}n^{\frac{1}{2}}}n^{-4n}2^{-8n}.\end{aligned} \tag{91}$$

In the final step we combine (90) and (91) and obtain

$$\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_2(n,c^*)\leqslant\frac{e^{\frac{3}{2}+\frac{2}{n}}}{4\sqrt{3}\pi^{\frac{3}{2}}}n^{-\frac{5}{2}}\left(\frac{3^3}{2^8}\right)^n. \tag{92}$$

Consequently, if $n\geqslant 20$ the second summand $R_2(n)$ of the remainder term satisfies the bound

$$\begin{aligned}R_2(n)&\leqslant(4n-1)\left(n+\frac{1}{2}\right)^{\frac{1}{2}}\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_2(n,c^*)\\&\leqslant\frac{e^{\frac{3}{2}+\frac{2}{n}}}{\sqrt{3}\pi^{\frac{3}{2}}}\frac{\left(n-\frac{1}{4}\right)\left(n+\frac{1}{2}\right)^{\frac{1}{2}}}{n^{\frac{5}{2}}}\left(\frac{3^3}{2^8}\right)^n.\end{aligned}$$

As

$$\left(n-\frac{1}{4}\right)^2\left(n+\frac{1}{2}\right)=n^3-\frac{3}{16}n+\frac{1}{32}<n^3$$

for $n\geqslant 1$, we end up with

$$R_2(n)\leqslant\frac{e^{\frac{3}{2}+\frac{2}{n}}}{\sqrt{3}\pi^{\frac{3}{2}}}n^{-1}\left(\frac{3^3}{2^8}\right)^n.$$

∎

4.2.5. Upper Bound on $R_3(n)$. The estimate of $R_3(n)$ is a bit more involved, since we now maximize a gamma quotient in three variables $b\leqslant c\leqslant d$ instead of two $c\leqslant d$. This increases the number of cases we have to consider in the application of Theorem 22. But apart from that almost all ideas are already known from the proof of Lemma 23. What we get is the following

LEMMA 24. *Let $n\geqslant 20$, then*

$$R_3(n)\leqslant\frac{2e^{\frac{5}{2}+\frac{2}{n}}}{\pi^2}n^{-\frac{3}{4}}\left(\frac{\left(10+\sqrt{2}\right)^3\left(2+3\sqrt{2}\right)}{2^{16}}\right)^n.$$

Figure 4.4 visualizes Lemma 24 by plotting the logarithm of $R_3(n)$ and the claimed upper bound, respectively.

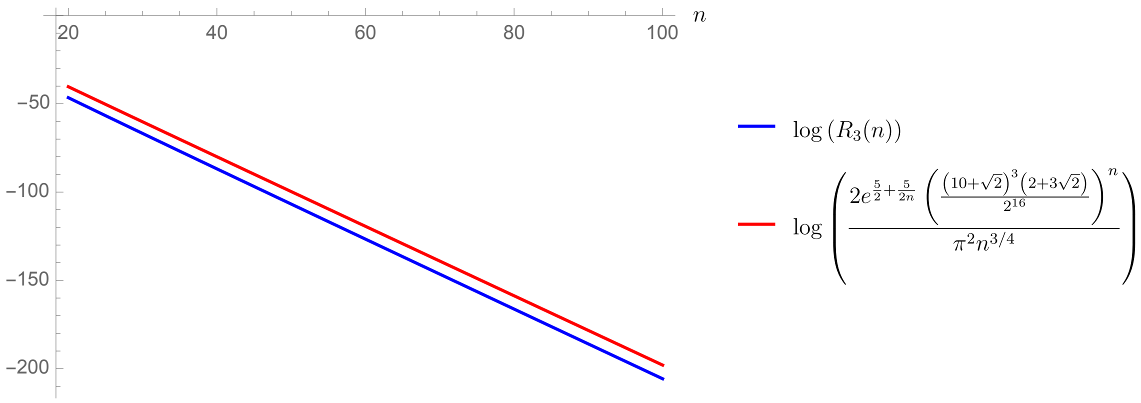

FIGURE 4.4. Logarithmic plot of $R_3(n)$ and its upper bound $\frac{2e^{\frac{5}{2}+\frac{2}{n}}}{\pi^2}n^{-\frac{3}{4}}\left(\frac{(10+\sqrt{2})^3(2+3\sqrt{2})}{2^{16}}\right)^n$.

Now, let's prove what we see above.

PROOF. By (64) and Corollary 18 it is

$$R_3(n) \leqslant 2n(4n-1)\left(n+\frac{1}{2}\right)^{\frac{1}{4}} \frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} \max_{D_3} F_3(n,b,c,d),$$

with

$$F_3(n,b,c,d) = \frac{\Gamma\left(n+b+\frac{1}{2}\right)\Gamma\left(n+c+\frac{1}{2}\right)\Gamma\left(n+d+\frac{1}{2}\right)}{\Gamma(b+1)\Gamma(c+1)\Gamma(d+1)\left|\Gamma\left(n-b+\frac{1}{2}\right)\Gamma\left(n-c+\frac{1}{2}\right)\Gamma\left(n-d+\frac{1}{2}\right)\right|}$$

and

$$D_3 = \left\{b,c,d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant b \leqslant c \leqslant d \leqslant 4n-1 \\ b+c+d = 4n-1 \end{matrix}\right\}.$$

Due to the increasing number of parameters, the function $F_3(n,b,c,d)$ is the last in the series F_2, F_3, F_4, that can easily be visualized on its domain. We do this in Figure 4.5 by plotting $F_3(n,b,c,4n-1-b-c)$ in $0 \leqslant b \leqslant \frac{4}{3}n$ and $0 \leqslant c \leqslant \frac{8}{3}n$ for $n = 30$. The black grid lines mark the lower and upper bound for the location of the maximum at $b = c = \left(1-\frac{1}{\sqrt{2}}\right)n - 1$ and $b = c = \left(1-\frac{1}{\sqrt{2}}\right)n + \frac{1}{4}$, respectively. We state them in (95). Since $F_3(n,b,c,d)$ takes ridiculously large values up to 1.67587×10^{203} for the chosen range of parameters, we had to scale it to one for technical reasons in order to get a meaningful plot. Similar to Figure 4.3 for $F_2(n,c,d)$ we note that the function has one clearly dominating global maximum. But now, let's track down the location of the maximum analytically and estimate it.

From Corollary 19 we already know, that $b \leqslant \left\lfloor \frac{4}{3}n - \frac{1}{3} \right\rfloor$ and $d \geqslant \left\lceil \frac{4}{3}n - \frac{1}{3} \right\rceil \geqslant n+1$ for all $n \geqslant 2$.
By the repeated application of Theorem 22 part *(ii)* to b and d, or c and d, respectively, under the assumption that $b > n$ or $c > n$, we deduce that $F_3(n,b,c,d)$ is maximized by tuples $b \leqslant c \leqslant n < d$. The argument follows the exact same chain of reasoning as in the beginning of the proof of Theorem 23 on page 34. Now we can apply Theorem 22 *(i)* to b and c

$$F_3(n,b,c,d) \leqslant F_3(n,b+1,c-1,d) \leqslant \cdots \leqslant F_3(n,b+m,c-m,d)$$

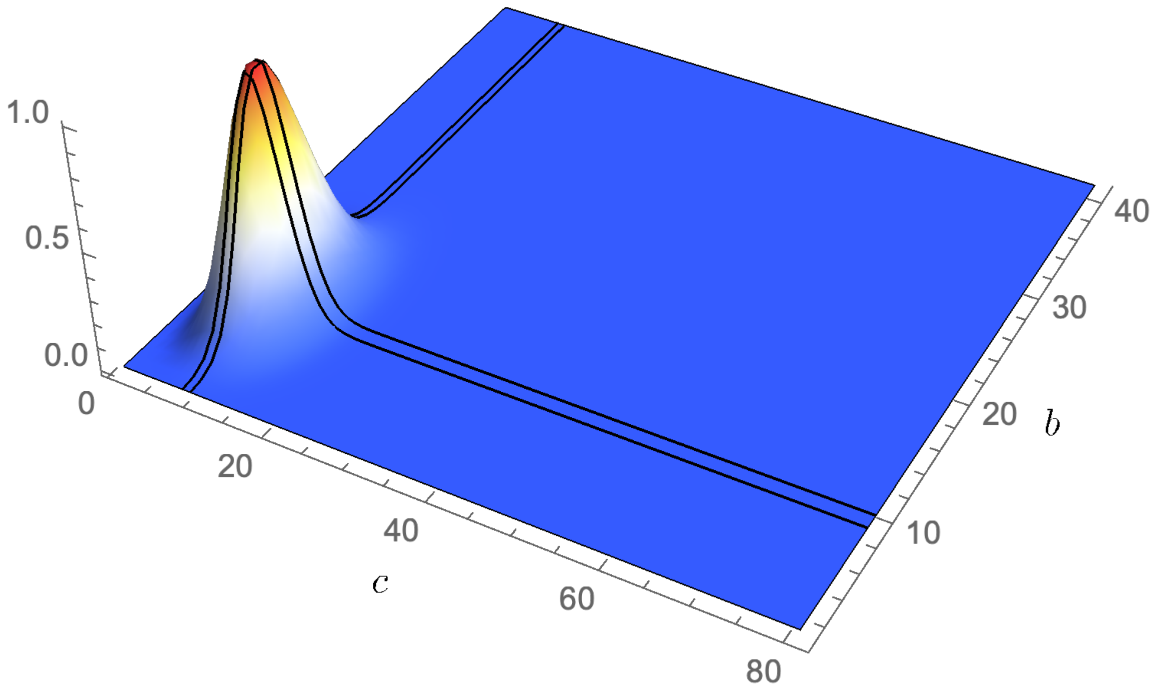

FIGURE 4.5. Scaled plot of $F_3(n, b, c, 4n - 1 - b - c)$ for $n = 30$.

$m \geqslant 0$ times, until it is either $b + m = c - m$ or $b + m + 1 = c - m$. Meaning, if $b \leqslant c \leqslant n$, we further increase $F_3(n, b, c, d)$ by pushing b and c together until they differ by at most 1. Since the sum of b, c, d is constant

$$b + c + d = 4n - 1$$

we get two different options for the maximizing tuple (b, c, d).

A: $F_3(n, b, c, d)$ is maximized by

$$\begin{aligned} b &= c \leqslant n, \\ d &= 4n - 2b - 1. \end{aligned}$$

B: $F_3(n, b, c, d)$ is maximized by

$$\begin{aligned} b + 1 &= c \leqslant n, \\ d &= 4n - 2b - 2. \end{aligned}$$

Let's have a closer look at **Case A**.

As shown above, we know that

$$\begin{aligned} F_3(n, b, c, d) &\leqslant F_3(n, b, b, 4n - 2b - 1) \\ &= \frac{\Gamma\left(n + b + \frac{1}{2}\right)^2 \Gamma\left(5n - 2b - \frac{1}{2}\right) \Gamma\left(3n - 2b - \frac{1}{2}\right)}{\pi \Gamma(b + 1)^2 \Gamma(4n - 2b) \Gamma\left(n - b + \frac{1}{2}\right)^2} \\ &=: F_{3,A}(n, b). \end{aligned}$$

So we were able to reduce the problem to one dimension. The task is now to find the maximum of $F_{3,A}(n,b)$ for $0 \leqslant b \leqslant n$.
We aim on applying Theorem 20, and thus determine the recurrence relation of $F_{3,A}(n,b)$ in b. This is $F_{3,A}(n,b) = \varphi_{3,A}(b) F_{3,A}(n,b-1)$ with

$$\varphi_{3,A}(b) = \frac{\left(n+b-\frac{1}{2}\right)^2 \left(n-b+\frac{1}{2}\right)^2 (4n-2b)(4n-2b+1)}{b^2 \left(5n-2b-\frac{1}{2}\right)\left(5n-2b+\frac{1}{2}\right)\left(3n-2b-\frac{1}{2}\right)\left(3n-2b+\frac{1}{2}\right)}, \tag{93}$$

for $1 \leqslant b \leqslant n$. Note that the function $\varphi_{3,A}$ obeys

$$\varphi_{3,A}(b) > \left[\frac{(n+b)(n-b)(4n-2b)}{b(5n-2b)(3n-2b)}\right]^2. \tag{94}$$

Solving for b such that the right-hand side of (94) is greater than or equal to one, yields the condition

$$(4n-b)(2b^2-4nb+n^2) \geqslant 0.$$

Since $b \leqslant n$, the first factor is always positive, and thus the above inequality is satisfied on $1 \leqslant b \leqslant n$ if $2b^2 - 4nb + n^2 \geqslant 0$. As our b is positive, we note that

$$\left[\frac{(n+b)(n-b)(4n-2b)}{b(5n-2b)(3n-2b)}\right]^2 \geqslant 1$$

for all $1 \leqslant b \leqslant \left(1-\frac{1}{\sqrt{2}}\right) n$ and denote in the following

$$\left(1-\frac{1}{\sqrt{2}}\right) n =: \underline{b}.$$

Due to (94), it is also $\varphi_{3,A}(b) \geqslant 1$ at least in the range $1 \leqslant b \leqslant \underline{b}$.
Moreover, since the inequality in (94) is strict, we can infer that the point ζ with $\varphi_{3,A}(\zeta) = 1$, satisfies $\underline{b} < \zeta$.

The following calculations aim at establishing an upper bound $\overline{b}$ on ζ. To this end, we first show that $\varphi_{3,A}$ decreases monotonically in b, and then we prove that $\varphi_{3,A}\left(\left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}\right) < 1$ for all $n \geqslant 2$. Then, $\overline{b} := \left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}$ is a possibility.
So let's rewrite $\varphi_{3,A}$ in the following way

$$\varphi_{3,A}(b) = \left(1+\frac{n-\frac{1}{2}}{b}\right)^2 \left(1-\frac{n-\frac{1}{2}}{5n-2b-\frac{1}{2}}\right)\left(1-\frac{n-\frac{1}{2}}{5n-2b+\frac{1}{2}}\right) \frac{\left(n-b+\frac{1}{2}\right)^2}{\left(3n-2b-\frac{1}{2}\right)\left(3n-2b+\frac{1}{2}\right)}.$$

Obviously, the first three factors are decreasing in b. The last one requires a bit more explanation. Its derivative with respect to b is

$$\begin{aligned}\frac{\partial}{\partial b}\frac{\left(n-b+\frac{1}{2}\right)^2}{\left(3n-2b-\frac{1}{2}\right)\left(3n-2b+\frac{1}{2}\right)} &= -\frac{4(2n-2b+1)(12n(n-1)-8b(n-1)-1)}{(4(3n-2b)^2-1)^2} \\ &\leqslant -\frac{12(4(n-1)(n+2)-1)}{(4(3n-2b)^2-1)^2}.\end{aligned}$$

To pass from the first to the second line, we use that $1 \leqslant b \leqslant n-1$ to bound the numerator. This last line is negative for all $n \geqslant 2$, meaning, this factor decreases in b as well, and so does the entire function $\varphi_{3,A}$.
Now we have a closer look at $\varphi_{3,A}\left(\left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}\right)$. Plugging in $b = \left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}$ into the definition of $\varphi_{3,A}$ in (93) ends up in

$$\varphi_{3,A}\left(\left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}\right) = \frac{\left(n+\frac{1}{2\sqrt{2}}\right)^2\left(n-\frac{1}{2(4-\sqrt{2})}\right)^2\left(n-\frac{1}{2(2+\sqrt{2})}\right)\left(n+\frac{1}{2(2+\sqrt{2})}\right)}{n^2\left(n+\frac{1}{2(2-\sqrt{2})}\right)^2\left(n-\frac{1}{1+\sqrt{2}}\right)\left(n-\frac{1}{3+\sqrt{2}}\right)}.$$

The plan is to apply Lemma 64. Since it requires all x_i to be positive, we first shift $\varphi_{3,A}$ by $y = \frac{1}{1+\sqrt{2}}$, meaning we set $n = m + y$. Then we call $\frac{1}{m} = x$ and Lemma 64 applied with $\delta = \frac{3}{10}$ yields

$$\begin{aligned}
&\varphi_{3,A}\left(\left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{x}+\frac{1}{1+\sqrt{2}}\right)+\frac{1}{4}\right)\\
&\leqslant \exp\Bigg[-\frac{26-11\sqrt{2}}{14}x-\frac{3}{10}x^2\\
&\times\left(2\left(y-\frac{1}{2\left(4-\sqrt{2}\right)}\right)^2+2\left(y+\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{2\left(\sqrt{2}+2\right)}+y\right)^2+\left(y-\frac{1}{2\left(\sqrt{2}+2\right)}\right)^2\right)\\
&+\frac{1}{2}\left(2y^2+2\left(\frac{1}{2(2-\sqrt{2})}+y\right)^2+\left(y-\frac{1}{3+\sqrt{2}}\right)\right)x^2\Bigg]\\
&=\exp\left[-\frac{26-11\sqrt{2}}{14}x+\frac{564-174\sqrt{2}}{245}x^2\right].
\end{aligned}$$

This is smaller than or equal to one, if

$$x < \frac{26-11\sqrt{2}}{14}\frac{245}{564-174\sqrt{2}}.$$

Moreover, since (404) is only valid for $0 \leqslant x_i \leqslant \frac{1}{\delta} - \left(\frac{2}{\delta}\right)^{\frac{1}{2}}$, we need to make sure that the largest x_i, which is $\left(y + \frac{1}{2(2-\sqrt{2})}\right)x$ in our case, satisfies this condition. It does if

$$x \leqslant \left(\frac{10}{3} - \left(\frac{20}{3}\right)^{\frac{1}{2}}\right)\frac{1}{\frac{1}{2(2-\sqrt{2})}+y}.$$

It is

$$0.57 < \frac{26-11\sqrt{2}}{14}\frac{245}{564-174\sqrt{2}} < 0.59 < \left(\frac{10}{3} - \left(\frac{20}{3}\right)^{\frac{1}{2}}\right)\frac{1}{\frac{1}{2(2-\sqrt{2})}+y},$$

and thus both conditions are satisfied if $x < 0.57$. Due to the relation $n = \frac{1}{x} + y$, we can conclude that $\varphi_{3,A}\left(\left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}\right) < 1$ as soon as $n \geqslant 3$.

Consequently, our upper bound $\overline{b}$ on ζ is $\overline{b} = \left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}$, and by Theorem 20 *(iii)*, we can bound the point b^*, for which $F_{3,A}$ attains its maximum, as follows

$$\left(1-\frac{1}{\sqrt{2}}\right)n - 1 < b^* < \left(1-\frac{1}{\sqrt{2}}\right)n+\frac{1}{4}. \tag{95}$$

Now it's time to estimate $F_{3,A}(n, b^*)$. We follow the same strategy as in the estimate of $F_2(n, c^*)$ on page 36. To this end we abbreviate $r = 1 - \frac{1}{\sqrt{2}}$ and assume

$$b^* = rn - \frac{1}{2} + q,$$

with $-\frac{1}{2} < q < \frac{3}{4}$. Then we have by Stirling's formula (332)

$$
\begin{aligned}
&F_{3,A}(n, b^*) \\
&= \frac{\Gamma((1+r)n+q)^2\Gamma\left((5-2r)n+\frac{1}{2}-2q\right)\Gamma\left((3-2r)n+\frac{1}{2}-2q\right)}{\pi\Gamma\left(rn+\frac{1}{2}+q\right)^2\Gamma((4-2r)n+1-2q)\Gamma((1-r)n+1-q)^2} \\
&\leqslant \frac{1}{\sqrt{2}\pi^{\frac{3}{2}}}\exp\left[-4n+3+\frac{1}{12n}\left(\frac{2}{1+r+\frac{q}{n}}+\frac{1}{5-2r+\frac{1-4q}{2n}}+\frac{1}{3-2r+\frac{1-4q}{2n}}\right)\right] \\
&\times \frac{[(1+r)n+q]^{2(1+r)n-1+2q}\left[(5-2r)n+\frac{1}{2}-2q\right]^{(5-2r)n-2q}\left[(3-2r)n+\frac{1}{2}-2q\right]^{(3-2r)n-2q}}{\left[rn+\frac{1}{2}+q\right]^{2rn+2q}[(4-2r)n+1-2q]^{(4-2r)n+\frac{1}{2}-2q}[(1-r)n+1-q]^{2(1-r)n+1-2q}}.
\end{aligned}
\tag{96}
$$

We estimate for $-\frac{1}{2} < q < \frac{3}{4}$ and $n \geqslant 1$

$$
\begin{aligned}
\frac{2}{1+r+\frac{q}{n}}+\frac{1}{5-2r+\frac{1-4q}{2n}}+\frac{1}{3-2r+\frac{1-4q}{2n}} &\leqslant \frac{2}{\frac{1}{2}+r}+\frac{1}{4-2r}+\frac{1}{2-2r} \\
&= \frac{1}{7}\left(4\sqrt{2}+19\right).
\end{aligned}
\tag{97}
$$

The last line is due to $r = 1-\frac{1}{\sqrt{2}}$. Next, we rewrite the second factor in (96) as

$$
\begin{aligned}
&\frac{n^{4n}}{(1-r^2)(4-2r)^{\frac{1}{2}}n^{\frac{5}{2}}}\left[\frac{(1+r)^2(5-2r)^5(3-2r)^3}{(4-2r)^4(1-r)^2}\right]^n\left[\frac{(1+r)(1-r)(4-2r)}{r(5-2r)(3-2r)}\right]^{2rn+2q} \\
&\times \frac{\left(1+\frac{q}{(1+r)n}\right)^{2(1+r)n+2q-1}\left(1+\frac{1-4q}{2(5-2r)n}\right)^{(5-2r)n-2q}\left(1+\frac{1-4q}{2(3-2r)n}\right)^{(3-2r)n-2q}}{\left(1+\frac{1+2q}{2rn}\right)^{2rn+2q}\left(1+\frac{1-2q}{(4-2r)n}\right)^{(4-2r)n+\frac{1}{2}-2q}\left(1+\frac{1-q}{(1-r)n}\right)^{2(1-r)n+1-2q}}.
\end{aligned}
\tag{98}
$$

Note that for $r = 1-\frac{1}{\sqrt{2}}$

$$
\frac{(1+r)(1-r)(4-2r)}{r(5-2r)(3-2r)} = 1.
$$

Hence, we only have to take care of the last line in (98). We aim on applying the estimate $1+x \leqslant e^x$ in the numerator and estimate (403) of Lemma 64 in the denominator. Since $1-2q$ is not positive for all admissible q, we handle the corresponding factor in the denominator separately in a first step. We obtain by almost exactly the same calculation as on page 37 and the application of Lemma 65 that

$$
\begin{aligned}
\left(1+\frac{1-2q}{(4-2r)n}\right)^{(4-2r)n+\frac{1}{2}-2q} &\geqslant \left(1-\frac{1}{2(4-2r)n}\right)^{(4-2r)n-1} \\
&\geqslant e^{-\frac{1}{2}}.
\end{aligned}
$$

Therefore, by Lemma 64, the last line in (98) is smaller than

$$\begin{aligned}&\exp\Bigg[-\frac{3}{2}-2q\\&+\frac{1}{n}\left(\frac{q(1-q)}{1-r}-\frac{q^2-\frac{1}{4}}{r}+\frac{(2q-1)q}{r+1}+\frac{2\left(2q-\frac{1}{2}\right)q}{3-2r}+\frac{2\left(2q-\frac{1}{2}\right)q}{5-2r}\right)\\&+\frac{1}{n^2}\left(\frac{q\left(q+\frac{1}{2}\right)^2}{r^2}+\frac{(1-2q)(1-q)^2}{(1-r)^2}\right)\Bigg].\end{aligned}\tag{99}$$

We plug in $r=1-\frac{1}{\sqrt{2}}$, and since

$$\begin{aligned}&\frac{q(1-q)}{1-r}-\frac{q^2-\frac{1}{4}}{r}+\frac{(2q-1)q}{r+1}+\frac{2\left(2q-\frac{1}{2}\right)q}{3-2r}+\frac{2\left(2q-\frac{1}{2}\right)q}{5-2r}\\&=\frac{2}{7}\left(6\sqrt{2}-11\right)q^2+\frac{1}{4}\left(\sqrt{2}+2\right)\\&\leqslant\frac{1}{4}\left(\sqrt{2}+2\right),\end{aligned}$$

as $6\sqrt{2}-11<0$, and

$$\begin{aligned}\frac{q\left(q+\frac{1}{2}\right)^2}{r^2}+\frac{(1-2q)(1-q)^2}{(1-r)^2}&=\left(\sqrt{2}+\frac{3}{2}\right)q(2q+1)^2-2(q-1)^2(2q-1)\\&\leqslant\frac{1}{32}\left(150\sqrt{2}+223\right),\end{aligned}$$

which takes its maximal value for $q=\frac{3}{4}$, we further estimate (99) by

$$\exp\left[-\frac{1}{2}+\frac{1}{4n}\left(\sqrt{2}+2\right)+\frac{1}{32n^2}\left(150\sqrt{2}+223\right)\right].\tag{100}$$

Now, we collect our findings from (96), (97), (98), (99) and (100), plug in $r=1-\frac{1}{\sqrt{2}}$ where we did not already do that, and obtain the bound

$$\begin{aligned}F_{3,A}(n,b^*)\leqslant{}&\frac{2n^{4n}}{\pi^{\frac{3}{2}}\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}}n^{-\frac{5}{2}}\left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^4}\right]^n\\&\times\exp\left[-4n+\frac{5}{2}+\frac{1}{84n}\left(4\sqrt{2}+19\right)+\frac{1}{4n}\left(\sqrt{2}+2\right)+\frac{1}{32n^2}\left(150\sqrt{2}+223\right)\right].\end{aligned}$$

We estimate

$$\frac{150\sqrt{2}+223}{32n}+\frac{1}{84}\left(4\sqrt{2}+19\right)+\frac{1}{4}\left(\sqrt{2}+2\right)\leqslant 2-\frac{13}{96}$$

for all $n\geqslant 20$ and arrive at

$$\begin{aligned}F_{3,A}(n,b^*)\leqslant{}&\frac{2n^{4n}}{\pi^{\frac{3}{2}}\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}}n^{-\frac{5}{2}}\left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^4}\right]^n\\&\times\exp\left[-4n+\frac{5}{2}+\frac{2}{n}-\frac{13}{96n}\right].\end{aligned}\tag{101}$$

Now, the combination of (91) and (101) eventually gives

$$\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_{3,A}(n,b^*) \leqslant \frac{e^{\frac{5}{2}+\frac{2}{n}}}{\pi^2\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}}n^{-3}\left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^{16}}\right]^n. \tag{102}$$

We now turn to **Case B**. In this case we have to maximize

$$\begin{aligned} F_3(n,b,c,d) &\leqslant F_3(n,b,b+1,4n-2b-2) \\ &= \frac{\Gamma\left(n+b+\frac{1}{2}\right)\Gamma\left(n+b+\frac{3}{2}\right)\Gamma\left(5n-2b-\frac{3}{2}\right)\Gamma\left(3n-2b-\frac{3}{2}\right)}{\pi\Gamma(b+1)\Gamma(b+2)\Gamma(4n-2b-1)\Gamma\left(n-b+\frac{1}{2}\right)\Gamma\left(n-b-\frac{1}{2}\right)} \\ &=: F_{3,B}(n,b) \end{aligned}$$

in $0 \leqslant b \leqslant n-1$. As in previous tasks of this kind our go-to tool is Theorem 20. In order to apply it here, we calculate

$$\begin{aligned} \varphi_{3,B}(b) &= \frac{F_{3,B}(n,b)}{F_{3,B}(n,b-1)} \\ &= \frac{\left(n+b-\frac{1}{2}\right)\left(n-b+\frac{1}{2}\right)(4n-2b)}{b\left(5n-2b-\frac{1}{2}\right)\left(3n-2b-\frac{1}{2}\right)} \cdot \frac{\left(n+b+\frac{1}{2}\right)\left(n-b-\frac{1}{2}\right)(4n-2b-1)}{(b+1)\left(5n-2b-\frac{3}{2}\right)\left(3n-2b-\frac{3}{2}\right)}. \end{aligned} \tag{103}$$

Our first interim goal is to narrow down the location of the value ζ with $\varphi_{3,B}(\zeta) = 1$ as closely as possible. It turns out that an old acquaintance is very helpful to us with this task. Recall the definition of $\varphi(c,d)$ in the proof of Theorem 22. In the case $c \leqslant n \leqslant d$ it is

$$\varphi(c,d) = \frac{(d+1)\left(n+c-\frac{1}{2}\right)\left(n-c+\frac{1}{2}\right)}{c\left(d+n+\frac{1}{2}\right)\left(d-n+\frac{1}{2}\right)}. \tag{104}$$

Now, note that $\varphi_{3,B}(b)$ can be expressed in terms of φ as follows

$$\varphi_{3,B}(b) = \varphi(b,4n-2b-1)\varphi(b+1,4n-2(b+1)). \tag{105}$$

In the following we derive two handy properties of (104). We first show that $\varphi(c,4n-2c-1)$ and $\varphi(c,4n-2c)$ are monotonously decreasing functions in $0 \leqslant c < n$ for all $n > \frac{3}{2}$ and $n > 0$, respectively. Afterwards we deduce a simplified condition on c and d in the spirit of Theorem 22 *(iii)* that already implies $\varphi(c,d) \geqslant 1$.

In order to prove the monotonicity assertions, we write

$$\begin{aligned} \varphi(c,4n-2c-1) &= \left(1+\frac{n-\frac{1}{2}}{c}\right)\frac{\left(n-c+\frac{1}{2}\right)(4n-2c)}{\left(3n-2c-\frac{1}{2}\right)\left(5n-2c-\frac{1}{2}\right)}, \\ \varphi(c,4n-2c) &= \left(1+\frac{n-\frac{1}{2}}{c}\right)\frac{\left(n-c+\frac{1}{2}\right)(4n-2c+1)}{\left(3n-2c+\frac{1}{2}\right)\left(5n-2c+\frac{1}{2}\right)}. \end{aligned}$$

Since the first factor in the above equations is obviously decreasing in c, it remains to show that the respective second factor is as well. We achieve this by proving that the derivative is negative in the claimed range of n.
In the case of $\varphi(c,4n-2c-1)$ we get

$$\begin{aligned} &\frac{d}{dc}\frac{\left(n-c+\frac{1}{2}\right)(4n-2c)}{\left(3n-2c-\frac{1}{2}\right)\left(5n-2c-\frac{1}{2}\right)} \\ &= -\frac{4\left(32c^2(n-1)-4c\left(28n^2-32n+1\right)+104n^3-132n^2+6n+1\right)}{(4c-10n+1)^2(4c-6n+1)^2}. \end{aligned}$$

As the denominator is always positive, the sign of the derivative is determined by the numerator. We calculate its roots

$$c_{1,2} = \frac{28n^2 \pm \sqrt{3}\sqrt{-(2n-1)^2\left(4n^2-4n-3\right)} - 32n + 1}{16(n-1)},$$

and find that $c_{1,2} \in \mathbb{C}\backslash\mathbb{R}$ as soon as $4n^2 - 4n - 3 > 0$. This inequality holds true for $n < -\frac{1}{2}$ and $n > \frac{3}{2}$, whereas only the second condition is of interest for us. In summary, we just showed that, if $n > \frac{3}{2}$, then the enumerator of the derivative in c of $\frac{\left(n-c+\frac{1}{2}\right)(4n-2c)}{\left(3n-2c-\frac{1}{2}\right)\left(5n-2c-\frac{1}{2}\right)}$ has no real root, implying that it is negative for all $0 \leqslant c < n$.
In the case of $\varphi(c, 4n-2c)$ the claimed monotonicity follows from

$$\frac{d}{dc}\frac{\left(n-c+\frac{1}{2}\right)(4n-2c+1)}{\left(3n-2c+\frac{1}{2}\right)\left(5n-2c+\frac{1}{2}\right)} = -\frac{8(2n-1)\left(8c^2-2c(14n+3)+26n^2+11n+1\right)}{(-4c+6n+1)^2(-4c+10n+1)^2},$$

which is always negativ if $c_{1,2} \in \mathbb{C}\backslash\mathbb{R}$. Here

$$c_{1,2} = \frac{1}{8}\left(\pm\sqrt{-12n^2-4n-239} + 14n + 3\right)$$

are the roots of the enumerator of the last derivativ above.

Next, note that

$$\varphi(c,d) > \frac{d\left(n+c-\frac{1}{2}\right)\left(n-c+\frac{1}{2}\right)}{c\left(d+n+\frac{1}{2}\right)\left(d-n+\frac{1}{2}\right)}.$$

The right-hand side of this inequality is greater than or equal to one, iff

$$-\frac{1}{4}(c+d)(4cd-4n^2+1) \geqslant 0.$$

Due to the positivity of $c+d$, we can can infer that

$$cd - n^2 + \frac{1}{4} \leqslant 0 \Rightarrow \varphi(c,d) > 1. \tag{106}$$

An immediate consequence of the monotonicity of $\varphi(c, 4n-2c-1)$ and $\varphi(c, 4n-2c)$ is that also $\varphi_{3,B}(b)$ is monotonously decreasing in $0 \leqslant b < n$ as soon as $n \geqslant 2$. Property (106) may be used to get a rough first idea of the location of ζ. In Fact, if $cd - n^2 + \frac{1}{4} \leqslant 0$ is satisfied for $c = b_0, d = 4n - 2b_0 - 1$ and for $c = b_0 + 1, d = 4n - 2(b_0+1)$, then we know that $\varphi_{3,B}(b_0) > 1$ and thus $\zeta > b_0$. The monotonicity of $\varphi_{3,B}(b)$ then allows us to find a finer two-sided bound on ζ. So, let's get to work.
It is

$$b(4n-2b-1) - n^2 + \frac{1}{4} = -2b^2 + b(4n-1) - n^2 + \frac{1}{4} \leqslant 0$$

iff $b \leqslant b_1$ or $b \geqslant b_2$, with

$$b_{1,2} = n \mp \frac{1}{4}\sqrt{8n^2-8n+3} - \frac{1}{4},$$

and

$$(b+1)(4n-2b-2) - n^2 + \frac{1}{4} = -2b^2 + b(4n-4) - n^2 + 4n - \frac{7}{4} \leqslant 0$$

iff $b \leqslant b_3$ or $b \geqslant b_4$, with

$$b_{3,4} = n \mp \frac{1}{2}\sqrt{2n^2+\frac{1}{2}} - 1.$$

Since $b \leqslant n-1$, we conclude that $\varphi_{3,B}(b) > 1$ if

$$b \leqslant \min\{b_1, b_3\} = b_3.$$

Note that $b_3 \approx \left(1 - \frac{1}{\sqrt{2}}\right) n - 1$. We use this value as a starting point for the location of ζ and continue by proving that for all $n \geqslant 4$

$$\left(1 - \frac{1}{\sqrt{2}}\right) n - \frac{1}{2} < \zeta < \left(1 - \frac{1}{\sqrt{2}}\right) n - \frac{1}{4}. \tag{107}$$

The strategy is to directly show

$$\varphi_{3,B}\left(\left(1 - \frac{1}{\sqrt{2}}\right) n - \frac{1}{2}\right) > 1 \tag{108}$$

for all $n \geqslant 2$, and

$$\varphi_{3,B}\left(\left(1 - \frac{1}{\sqrt{2}}\right) n - \frac{1}{4}\right) < 1 \tag{109}$$

for all $n \geqslant 4$. The bounds (107) then follow from the monotonicity of $\varphi_{3,B}(b)$.
We start with (108). Simply plugging in $b = \left(1 - \frac{1}{\sqrt{2}}\right) n - \frac{1}{2}$ into (103) yields

$$\varphi_{3,B}\left(\left(1 - \frac{1}{\sqrt{2}}\right) n - \frac{1}{2}\right) = \frac{n^3\left(n + \sqrt{2}\right)\left(n - \frac{2}{4-\sqrt{2}}\right)\left(n + \frac{1}{2+\sqrt{2}}\right)}{\left(n^2 - \left(\frac{1}{2-\sqrt{2}}\right)^2\right)\left(n^2 - \left(\frac{1}{2+2\sqrt{2}}\right)^2\right)\left(n^2 - \left(\frac{1}{6+2\sqrt{2}}\right)^2\right)}.$$

The right-hand side is greater than

$$\frac{\left(n + \sqrt{2}\right)\left(n - \frac{2}{4-\sqrt{2}}\right)\left(n + \frac{1}{2+\sqrt{2}}\right)}{n^3}.$$

This in turn is equal to

$$\frac{\left(n^2 + \frac{2}{7}\left(3\sqrt{2} - 2\right) n - \frac{2\sqrt{2}}{4-\sqrt{2}}\right)}{n^2} \cdot \frac{\left(n + \frac{1}{2+\sqrt{2}}\right)}{n}.$$

The first factor is greater than one if $n > \frac{1}{\sqrt{2}\left(\sqrt{2}-1\right)} \approx \frac{25}{14}$, and the second one for all positive n.
For the proof of (109) we note that

$$\begin{aligned}
\varphi_{3,B}\left(\left(1 - \frac{1}{\sqrt{2}}\right) n - \frac{1}{4}\right) &= \frac{\left(n^2 - \left(\frac{1}{4+2\sqrt{2}}\right)^2\right)\left(n - \frac{1}{2\sqrt{2}}\right)\left(n + \frac{3}{2\sqrt{2}}\right)\left(n + \frac{1}{8-2\sqrt{2}}\right)\left(n - \frac{3}{8-2\sqrt{2}}\right)}{n^2\left(n - \frac{1}{3+\sqrt{2}}\right)\left(n - \frac{1}{1+\sqrt{2}}\right)\left(n + \frac{3}{4-2\sqrt{2}}\right)\left(n - \frac{1}{4-2\sqrt{2}}\right)} \\
&< \frac{\left(n + \frac{3}{2\sqrt{2}}\right)\left(n + \frac{1}{8-2\sqrt{2}}\right)\left(n - \frac{3}{8-2\sqrt{2}}\right)}{\left(n - \frac{1}{1+\sqrt{2}}\right)\left(n + \frac{3}{4-2\sqrt{2}}\right)\left(n - \frac{1}{4-2\sqrt{2}}\right)}.
\end{aligned}$$

The last line above is smaller than or equal to one, iff

$$\frac{-8\left(199\sqrt{2} - 278\right) n^2 + 8\left(40 - 19\sqrt{2}\right) n + 57\sqrt{2} - 126}{416\sqrt{2} - 544} \leqslant 0,$$

which is satisfied for all

$$n \geqslant \frac{-38\sqrt{2} + \sqrt{2\left(62358 - 43960\sqrt{2}\right)} + 80}{4\left(199\sqrt{2} - 278\right)}$$

and

$$n \leqslant \frac{-38\sqrt{2} - \sqrt{2\left(62358 - 43960\sqrt{2}\right)} + 80}{4\left(199\sqrt{2} - 278\right)} < \frac{1}{2}.$$

Since $3 < \frac{-38\sqrt{2}+\sqrt{2\left(62358-43960\sqrt{2}\right)}+80}{4\left(199\sqrt{2}-278\right)} < 4$, the claim follows.

After finally reaching our interim goal of localizing ζ, we are now ready to apply Theorem 20 *(iii)* to bound the point b^*, for which $F_{3,B}$ attains its maximum. It satisfies

$$\left(1-\frac{1}{\sqrt{2}}\right)n-\frac{3}{2}<b^*<\left(1-\frac{1}{\sqrt{2}}\right)n-\frac{1}{4}. \tag{110}$$

We take the same approach to estimate $F_{3,B}(b^*)$ as in the case of $F_{3,A}(b^*)$ and set

$$b^*=rn+q-1,$$

with $-\frac{1}{2}<q<\frac{3}{4}$ and $r=1-\frac{1}{\sqrt{2}}$. Then we have by Stirling's formula (332)

$$\begin{aligned}
&F_{3,B}\left(n,b^*\right)\\
&=\frac{\Gamma\left((1+r)n+q-\frac{1}{2}\right)\Gamma\left((1+r)n+q+\frac{1}{2}\right)\Gamma\left((5-2r)n+\frac{1}{2}-2q\right)\Gamma\left((3-2r)n+\frac{1}{2}-2q\right)}{\pi\Gamma\left(rn+q\right)\Gamma\left(rn+q+1\right)\Gamma((4-2r)n+1-2q)\Gamma\left((1-r)n+\frac{3}{2}-q\right)\Gamma\left((1-r)n+\frac{1}{2}-q\right)}\\
&\leqslant\frac{1}{\sqrt{2}\pi^{\frac{3}{2}}}\exp\left[-4n+3+\frac{1}{12n}\left(\frac{1}{1+r+\frac{2q-1}{n}}+\frac{1}{1+r+\frac{2q+1}{n}}+\frac{1}{5-2r+\frac{1-4q}{2n}}+\frac{1}{3-2r+\frac{1-4q}{2n}}\right)\right]\\
&\times\frac{\left[(1+r)n+q-\frac{1}{2}\right]^{(1+r)n-1+q}\left[(1+r)n+q+\frac{1}{2}\right]^{(1+r)n+q}}{\left[rn+q\right]^{rn+q-\frac{1}{2}}\left[rn+q+1\right]^{rn+q+\frac{1}{2}}\left[(1-r)n+\frac{3}{2}-q\right]^{(1-r)n+1-q}\left[(1-r)n+\frac{1}{2}-q\right]^{(1-r)n-q}}\\
&\times\frac{\left[(5-2r)n+\frac{1}{2}-2q\right]^{(5-2r)n-2q}\left[(3-2r)n+\frac{1}{2}-2q\right]^{(3-2r)n-2q}}{\left[(4-2r)n+1-2q\right]^{(4-2r)n+\frac{1}{2}-2q}}.
\end{aligned} \tag{111}$$

We estimate for $-\frac{1}{2}<q<\frac{3}{4}$ and $n\geqslant 1$

$$\begin{aligned}
\frac{1}{1+r+\frac{2q-1}{n}}+\frac{1}{1+r+\frac{2q+1}{n}}+\frac{1}{5-2r+\frac{1-4q}{2n}}+\frac{1}{3-2r+\frac{1-4q}{2n}}&\leqslant\frac{1}{r+1}+\frac{1}{2-2r}+\frac{1}{4-2r}+\frac{1}{r}\\
&=\frac{1}{7}\left(8\sqrt{2}+25\right).
\end{aligned} \tag{112}$$

The last equality is due to $r=1-\frac{1}{\sqrt{2}}$. Next, we rewrite the the last two lines of (111) as

$$\begin{aligned}
&\frac{n^{4n}}{(1-r^2)(4-2r)^{\frac{1}{2}}n^{\frac{5}{2}}}\left[\frac{(1+r)^2(5-2r)^5(3-2r)^3}{(4-2r)^4(1-r)^2}\right]^n\left[\frac{(1+r)(1-r)(4-2r)}{r(5-2r)(3-2r)}\right]^{2rn+2q}\\
&\times\frac{\left(1+\frac{2q-1}{2(1+r)n}\right)^{(1+r)n+q-1}\left(1+\frac{2q+1}{2(1+r)n}\right)^{(1+r)n+q}}{\left(1+\frac{q}{rn}\right)^{rn+q-\frac{1}{2}}\left(1+\frac{q+1}{rn}\right)^{rn+q+\frac{1}{2}}\left(1+\frac{3-2q}{2(1-r)n}\right)^{(1-r)n+1-q}\left(1+\frac{1-2q}{2(1-r)n}\right)^{(1-r)n-q}}\\
&\times\frac{\left(1+\frac{1-4q}{2(5-2r)n}\right)^{(5-2r)n-2q}\left(1+\frac{1-4q}{2(3-2r)n}\right)^{(3-2r)n-2q}}{\left(1+\frac{1-2q}{(4-2r)n}\right)^{(4-2r)n+\frac{1}{2}-2q}}.
\end{aligned}$$

We have already seen in the estimate of the maximum of $F_{3,A}$ that the last formula can be simplified to

$$\frac{2\sqrt{2}n^{4n}}{(4-\sqrt{2})\sqrt{\sqrt{2}+2}}n^{-\frac{5}{2}}\left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^4}\right]^n$$

$$\times\frac{\left(1+\frac{2q-1}{2(1+r)n}\right)^{(1+r)n+q-1}\left(1+\frac{2q+1}{2(1+r)n}\right)^{(1+r)n+q}}{\left(1+\frac{q}{rn}\right)^{rn+q-\frac{1}{2}}\left(1+\frac{q+1}{rn}\right)^{rn+q+\frac{1}{2}}\left(1+\frac{3-2q}{2(1-r)n}\right)^{(1-r)n+1-q}\left(1+\frac{1-2q}{2(1-r)n}\right)^{(1-r)n-q}} \tag{113}$$

$$\times\frac{\left(1+\frac{1-4q}{2(5-2r)n}\right)^{(5-2r)n-2q}\left(1+\frac{1-4q}{2(3-2r)n}\right)^{(3-2r)n-2q}}{\left(1+\frac{1-2q}{(4-2r)n}\right)^{(4-2r)n+\frac{1}{2}-2q}}.$$

The two last lines in (113) are now taken care of in the same fashion as in the estimate of $F_{3,A}(b^*)$, that is by applying the estimate $1+x\leqslant e^x$ in the numerator and estimate (403) of Lemma 64 in the denominator to those factors that satisfy the conditions of the Lemma. Those are all except

$$\left(1+\frac{1-2q}{(4-2r)n}\right)^{(4-2r)n+\frac{1}{2}-2q}$$

and

$$\left(1+\frac{1-2q}{2(1-r)n}\right)^{(1-r)n-q}. \tag{114}$$

The former already appears in (98) and satisfies

$$\left(1+\frac{1-2q}{(4-2r)n}\right)^{(4-2r)n+\frac{1}{2}-2q}\geqslant e^{-\frac{1}{2}}. \tag{115}$$

The latter is estimated from below using the same strategy. We first show that it is decreasing in q and then apply Lemma 65. Calculating the q-derivative of (114) we get

$$\left(\frac{1-2q}{\sqrt{2}n}+1\right)^{\frac{n}{\sqrt{2}}-q}\left(\frac{1}{\sqrt{2}n-2q+1}-\log\left(\frac{1-2q}{\sqrt{2}n}+1\right)-1\right).$$

The first factor is positive all admissible q and $n>0$. The second one is increasing in q and thus smaller than

$$\frac{1}{\sqrt{2}n-\frac{1}{2}}-\log\left(1-\frac{1}{2\sqrt{2}n}\right)-1,$$

which in turn now is a decreasing function of n. For all $n\geqslant 2$ it is smaller than

$$\frac{1}{31}\left(8\sqrt{2}-29\right)+\log\left(\frac{4}{31}\left(\sqrt{2}+8\right)\right)<-\frac{1}{3}<0,$$

and we can infer that (114) is decreasing in $-\frac{1}{2}<q<\frac{3}{4}$.
Now we want to apply Lemma 65. In the case of (114), we have $a=\frac{1}{2}-q$ and $b=-q$. So it is $2b-a=-q-\frac{1}{2}<0$ as $q>-\frac{1}{2}$. Part *(ii)* of the Lemma tells us, that (114) is bounded from below by e^{1-2q}, if $a=\frac{1}{2}-q<0$, which is the case for $q>\frac{1}{2}$. Since we have seen above that the factor we want to bound from below is actually a decreasing function in q, we may assume that $q>\frac{1}{2}$ when we minimize in n. In summary, we showed

$$\left(1+\frac{1-2q}{2(1-r)n}\right)^{(1-r)n-q}\geqslant e^{\frac{1}{2}-q} \tag{116}$$

for all $\frac{1}{2} < q < \frac{3}{4}$ and $n \geqslant 2$.
Now we can continue the estimate of the last two lines in (113) and find that they are smaller than

$$\begin{aligned}
&\exp\Bigg[-\frac{3}{2} - 2q \\
&+ \frac{1}{n}\left(-\frac{q^2}{r} + \frac{4q^2-2q+1}{2(r+1)} + \frac{2\left(2q-\frac{1}{2}\right)q}{3-2r} + \frac{2\left(2q-\frac{1}{2}\right)q}{5-2r} + \frac{4(q-2)q+3}{8(r-1)} \right) \\
&+ \frac{1}{n^2}\left(\frac{\left(q+\frac{1}{4}\right)\left(2q^2+q+2\right)}{2r^2} + \frac{(1-q)(3-2q)^2}{4(1-r)^2} \right)\Bigg].
\end{aligned} \tag{117}$$

We plug in $r = 1 - \frac{1}{\sqrt{2}}$, and note that

$$\begin{aligned}
&-\frac{q^2}{r} + \frac{4q^2-2q+1}{2(r+1)} + \frac{2\left(2q-\frac{1}{2}\right)q}{3-2r} + \frac{2\left(2q-\frac{1}{2}\right)q}{5-2r} + \frac{4(q-2)q+3}{8(r-1)} \\
&= \frac{1}{56}\left(4\left(31\sqrt{2}-44\right)q^2 - 17\sqrt{2} + 16\right)
\end{aligned}$$

is maximized for $q = 0$, as $31\sqrt{2} - 44 < 0$, with the maximum being $\frac{1}{56}\left(16 - 17\sqrt{2}\right)$. Moreover it is

$$\begin{aligned}
&\frac{\left(q+\frac{1}{4}\right)\left(2q^2+q+2\right)}{2r^2} + \frac{(1-q)(3-2q)^2}{4(1-r)^2} \\
&= \left(4\sqrt{2}+5\right)q^3 + \left(3\sqrt{2}+\frac{17}{2}\right)q^2 + \frac{3}{2}\left(3\sqrt{2}+1\right)q + \sqrt{2} + \frac{15}{4}
\end{aligned}$$

obviously increasing in $-\frac{1}{2} < q < \frac{3}{4}$ and therefore bounded from above by $\frac{1}{64}\left(496\sqrt{2}+753\right)$. Hence, we further estimate (117) by

$$\exp\left[-\frac{1}{2} + \frac{1}{56n}\left(16-17\sqrt{2}\right) + \frac{1}{64n^2}\left(496\sqrt{2}+753\right)\right]. \tag{118}$$

We now finally collect our findings from (111), (112), (113) and (118) and obtain the bound

$$\begin{aligned}
F_{3,B}(n,b^*) \leqslant{}& \frac{2n^{4n}}{\pi^{\frac{3}{2}}\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}} n^{-\frac{5}{2}} \left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^4}\right]^n \\
&\times \exp\left[-4n + \frac{5}{2} + \frac{1}{84n}\left(8\sqrt{2}+25\right) + \frac{1}{56n}\left(16-17\sqrt{2}\right) + \frac{1}{64n^2}\left(496\sqrt{2}+753\right)\right].
\end{aligned}$$

Since it is

$$\frac{1}{84}\left(8\sqrt{2}+25\right) + \frac{1}{56}\left(16-17\sqrt{2}\right) + \frac{1}{64n}\left(496\sqrt{2}+753\right) \leqslant 2 - \frac{13}{96}$$

for all $n \geqslant 20$, we eventually see that $F_{3,B}(b^*)$ satisfies the same upper bound (101) as $F_{3,A}(b^*)$, that is

$$\begin{aligned}
F_{3,B}(n,b^*) \leqslant{}& \frac{2n^{4n}}{\pi^{\frac{3}{2}}\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}} n^{-\frac{5}{2}} \left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^4}\right]^n \\
&\times \exp\left[-4n + \frac{5}{2} + \frac{2}{n} - \frac{13}{96n}\right].
\end{aligned}$$

Consequently it is also

$$\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_{3,B}(n,b^*) \leqslant \frac{e^{\frac{5}{2}+\frac{2}{n}}}{\pi^2\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}}n^{-3}\left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^{16}}\right]^n.$$

by (91), and the third remainder summand satisfies

$$\begin{aligned} R_3(n) &\leqslant 2^3 n\left(n-\frac{1}{4}\right)\left(n+\frac{1}{2}\right)^{\frac{1}{4}} \\ &\times \frac{e^{\frac{5}{2}+\frac{2}{n}}}{\pi^2\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}}n^{-3}\left[\frac{\left(10+\sqrt{2}\right)^3\left(3\sqrt{2}+2\right)}{2^{16}}\right]^n \\ &\leqslant \frac{2e^{\frac{5}{2}+\frac{2}{n}}}{\pi^2}n^{-\frac{3}{4}}\left[\frac{\left(10+\sqrt{2}\right)^3\left(2+3\sqrt{2}\right)}{2^{16}}\right]^n \end{aligned}$$

for all $n \geqslant 20$. Here, we estimated $\frac{8}{\left(4-\sqrt{2}\right)\sqrt{\sqrt{2}+2}} < 2$ and

$$\left(n-\frac{1}{4}\right)^4\left(n+\frac{1}{2}\right) \leqslant n^5,$$

which becomes quite obvious by expanding the left-hand side above. This finishes the proof of Lemma 24. ■

4.2.6. Upper Bound on $R_4(n)$. Last but not least, we get to the last summand $R_4(n)$. The key task here is to locate and estimate the maximum of the function $F_4(n,a,b,c,d)$ in the four variables a,b,c,d. Very similar to our approach in Subsection 4.2.5, we apply the machinery we developed in Subsections 4.2.2 and 4.2.3, especially Theorem 20 and Theorem 22, to name the two most important. This time we get three different possibilities for the maximizing tuple (a,b,c,d). For each we derive an upper bound on the value of F_4 at this point. The result is the following bound on $R_4(n)$.

LEMMA 25. *Let $n \geqslant 17$, then*

$$R_4(n) \leqslant \frac{3e}{\pi^{\frac{5}{2}}}n^{-\frac{1}{2}}\left(\frac{2}{3}\right)^{4n}.$$

Let us again briefly explore Lemma 25 graphically, before we prove it. Figure 4.6 below shows our object of interest and the claimed upper bound. Both functions are plotted logarithmically. By comparing this plot to Figures 4.1, 4.2 and 4.4 for the terms $R_1(n), R_2(n)$ and $R_3(n)$, it becomes clear again that $R_4(n)$ is the dominating component of the remainder term. All these plots have in common that they illustrate in a nice way, that our approach of estimating expressions like the $R_m(n)$ is not completely off. Even though it might appear quite rough, we still get the correct rate of the exponential decay, which is the most important part here.

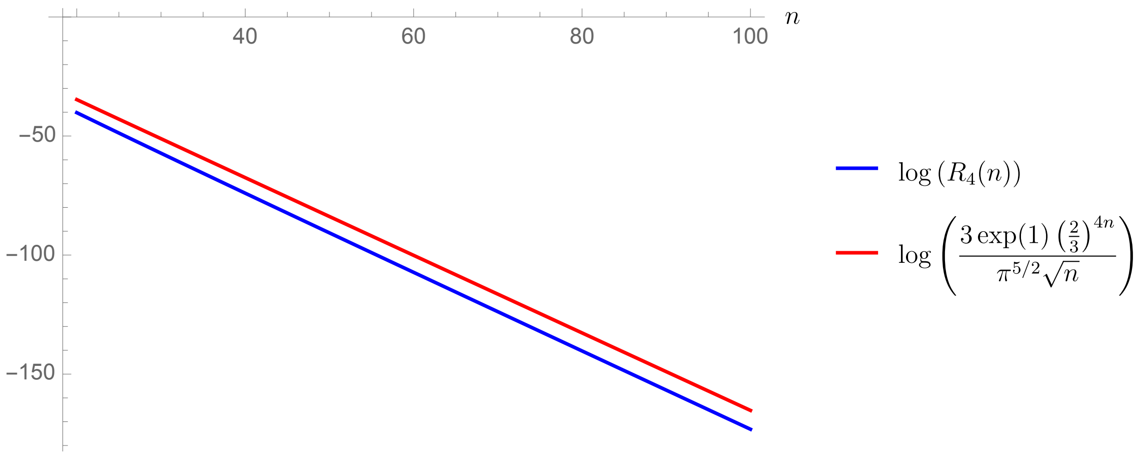

FIGURE 4.6. Logarithmic plot of $R_4(n)$ and its upper bound $\frac{3e}{\pi^{\frac{5}{2}}} n^{-\frac{1}{2}} \left(\frac{2}{3}\right)^{4n}$.

PROOF. By (64) and Corollary 18 it is

$$R_4(n) \leqslant \frac{2}{3} n(4n-1)(4n+1) \frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} \max_{D_4} F_4(n,a,b,c,d), \tag{119}$$

with

$$F_4(n,a,b,c,d) = \frac{\Gamma\left(n+a+\frac{1}{2}\right)\Gamma\left(n+b+\frac{1}{2}\right)\Gamma\left(n+c+\frac{1}{2}\right)\Gamma\left(n+d+\frac{1}{2}\right)}{a!b!c!d\left|\Gamma\left(n-a+\frac{1}{2}\right)\Gamma\left(n-b+\frac{1}{2}\right)\Gamma\left(n-c+\frac{1}{2}\right)\Gamma\left(n-d+\frac{1}{2}\right)\right|}$$

and

$$D_4 = \left\{ a,b,c,d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant 4n-1 \\ a+b+c+d = 4n-1 \end{matrix} \right\}.$$

From Corollary 19, we already know that a is always smaller than or equal to $n-1$, and that d is always greater than or equal to n. We apply Theorem 22 in the same manner as at the beginning of the proof of Lemma 24 and conclude that F_4 is maximized by tuples (a,b,c,d) such that $a \leqslant b \leqslant c \leqslant n$, and either $a = c$ or $a+1 = c$. This leads to the following three different options for the maximizing tuple (a,b,c,d).

A: $F_4(n,a,b,c,d)$ is maximized by

$$\begin{aligned} a = b = c &\leqslant n, \\ d &= 4n-3a-1. \end{aligned}$$

B: $F_4(n,a,b,c,d)$ is maximized by

$$\begin{aligned} a = b &\leqslant n-1, \\ c = a+1 &\leqslant n, \\ d &= 4n-2b-2. \end{aligned}$$

C: $F_4(n,a,b,c,d)$ is maximized by

$$\begin{aligned} a &\leqslant n-1, \\ b = c = a+1 &\leqslant n, \\ d &= 4n-2b-3. \end{aligned}$$

We start with **Case A**.

We set

$$F_4(n,a,a,a,4n-3a-1) = \frac{\Gamma\left(n+a+\frac{1}{2}\right)^3\Gamma\left(5n-3a-\frac{1}{2}\right)\Gamma\left(3n-3a-\frac{1}{2}\right)}{\pi\Gamma(a+1)^3\Gamma(4n-3a)\Gamma\left(n-a+\frac{1}{2}\right)^3} =: F_{4,A}(n,a). \tag{120}$$

In the following we maximize $F_{4,A}(n,a)$ in $0 \leqslant a \leqslant n$. As in the last subsections, the strategy is to apply Theorem 20. Therefore, we determine the recurrence relation of $F_{4,A}(n,a)$ in a. This is $F_{4,A}(n,a) = \varphi_{4,A}(a)F_{4,A}(n,a-1)$ with

$$\begin{aligned}\varphi_{4,A}(a) &= \frac{\left(n+a-\frac{1}{2}\right)^3\left(n-a+\frac{1}{2}\right)^3}{a^3\left(5n-3a-\frac{1}{2}\right)\left(5n-3a+\frac{1}{2}\right)\left(5n-3a+\frac{3}{2}\right)}\\ &\times\frac{(4n-3a)(4n-3a+1)(4n-3a+2)}{\left(3n-3a-\frac{1}{2}\right)\left(3n-3a+\frac{1}{2}\right)\left(3n-3a+\frac{3}{2}\right)}\\ &= \frac{\left(n+a-\frac{1}{2}\right)^3\left(n-a+\frac{1}{2}\right)^2(4n-3a)(4n-3a+1)(4n-3a+2)}{3a^3\left(5n-3a-\frac{1}{2}\right)\left(5n-3a+\frac{1}{2}\right)\left(5n-3a+\frac{3}{2}\right)\left(3n-3a-\frac{1}{2}\right)\left(3n-3a+\frac{1}{2}\right)}\end{aligned}$$

for $1 \leqslant a \leqslant n$.

The task now is to localize the point ζ such that $\varphi_{4,A}(\zeta) = 1, \varphi_{4,A}(a) > 1$ for $a < \zeta$ and $\varphi_{4,A}(a) < 1$ for $a > \zeta$. Then $F_{4,A}(n,a)$ has a maximum in a between $\zeta - 1$ and ζ according to Theorem 20 *(iii)*.

Our strategy is to prove a strict lower bound $\underline{\varphi}_{4,A}$ and a strict upper bound $\overline{\varphi}_{4,A}$ on $\varphi_{4,A}$, and then determine the two values $\underline{a}$ and $\overline{a}$, such that

$$\begin{aligned}\underline{\varphi}_{4,A}(a) &\geqslant 1,\\ \overline{\varphi}_{4,A}(a) &\leqslant 1\end{aligned}$$

for $a \leqslant \underline{a}$, and $a \geqslant \overline{a}$, respectively. Due to the continuity of $\varphi_{4,A}$ on $[1, n-1]$, we thus can infer that $\underline{a} < \zeta < \overline{a}$.

We start with the lower bound $\underline{\varphi}_{4,A}$ and its analysis. As a first step we use the trivially bounds $x(x+1) > x^2$ and $\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) < x^2$ for arbitrary $x \geqslant 0$, and find that

$$\varphi_{4,A}(a) > \frac{(n+a)^3(4n-3a)^2(4n-3a+2)}{(3a)^3(5n-3a)^2\left(5n-3a+\frac{3}{2}\right)}.$$

Moreover, it is easily seen that

$$\frac{p+2}{q+\frac{3}{2}} \geqslant \frac{p}{q},$$

if $q \geqslant \frac{3}{4}p$ and $p, q > 0$. For $p = 4n - 3a$ and $q = 5n - 3a$ this is obviously satisfied, and we get the lower bound

$$\underline{\varphi}_{4,A}(a) = \left[\frac{(n+a)(4n-3a)}{3a(5n-3a)}\right]^3.$$

We are interested in those values of a, for which the right-hand side is greater than or equal to one. The condition $\underline{\varphi}_{4,A}(a) \geqslant 1$ is equivalent to

$$(a-2n)(3a-n) \geqslant 0. \tag{121}$$

Since $a \leqslant n$, the first factor is always negative. Consequently (121) holds true for all $a \leqslant \frac{1}{3}n$, and we can set

$$\underline{a} := \frac{1}{3}n$$

as the lower bound for ζ.

Next we take care of the upper bound $\overline{\varphi}_{4,A}$ and claim that

$$\overline{\varphi}_{4,A}(a) := \left[\frac{\left(n+a-\frac{1}{2}\right)\left(n-a+\frac{1}{2}\right)(4n-3a-1)}{a\left(5n-3a-\frac{1}{2}\right)\left(3n-3a-\frac{1}{2}\right)}\right]^3$$

does the job for all $1 \leqslant a \leqslant n$ and all $n \geqslant 0$. This claim is equivalent to

$$\frac{(4n-3a-1)^3}{\left(5n-3a-\frac{1}{2}\right)^2\left(3n-3a-\frac{1}{2}\right)^2} > \frac{(4n-3a)(4n-3a+1)(4n-3a+2)}{\left(5n-3a+\frac{1}{2}\right)\left(5n-3a+\frac{3}{2}\right)\left(3n-3a+\frac{1}{2}\right)\left(3n-3a+\frac{3}{2}\right)}.$$

We call

$$p := 4n-3a+2.$$

Since a lives on $1 \leqslant a \leqslant n$, the value p satisfies $n+2 \leqslant p \leqslant 4n-1$. In terms of p we have to show that

$$\frac{(p-3)^3}{\left(p-n-\frac{5}{2}\right)^2\left(p+n-\frac{5}{2}\right)^2} > \frac{p(p-1)(p-2)}{\left(p-n-\frac{1}{2}\right)\left(p-n-\frac{3}{2}\right)\left(p+n-\frac{1}{2}\right)\left(p+n-\frac{3}{2}\right)}. \tag{122}$$

We do this step by step. The first step is the inequality

$$\frac{(p-3)^2}{\left(p-n-\frac{5}{2}\right)\left(p+n-\frac{5}{2}\right)} > \frac{(p-2)^2}{\left(p-n-\frac{3}{2}\right)\left(p+n-\frac{3}{2}\right)} \tag{123}$$

for all positive n. We multiply by the denominators and then subtract the right-hand side from the left. This transforms (123) into

$$-p^2 + \left(2n^2+\frac{9}{2}\right)p - 5n^2 - \frac{19}{4} > 0.$$

We determine the zeroes of the quadratic function in p and find that the above inequality is satisfied if

$$n^2+\frac{9}{4}-\frac{1}{4}\sqrt{16n^4-8n^2+5} < p < n^2+\frac{9}{4}+\frac{1}{4}\sqrt{16n^4-8n^2+5}.$$

Note that $16n^4-8n^2+5 > (4n^2-1)^2$. Therefore it is

$$n^2+\frac{9}{4}-\frac{1}{4}\sqrt{16n^4-8n^2+5} < \frac{5}{2} < n+5 \leqslant p,$$

and

$$p \leqslant 4n-1 < 2n^2+2 < n^2+\frac{9}{4}+\frac{1}{4}\sqrt{16n^4-8n^2+5},$$

for all $n \geqslant 0$. Consequently, inequality (123) holds true for all p in question, and we showed that

$$\frac{(p-3)^3}{\left(p-n-\frac{5}{2}\right)^2\left(p+n-\frac{5}{2}\right)^2} > \frac{(p-2)^2(p-3)}{\left(p-n-\frac{3}{2}\right)\left(p+n-\frac{3}{2}\right)\left(p-n-\frac{5}{2}\right)\left(p+n-\frac{5}{2}\right)}, \tag{124}$$

if n is non-negative.
In the next step we establish the bounds

$$\frac{p-3}{\left(p-n-\frac{5}{2}\right)\left(p+n-\frac{5}{2}\right)} > \frac{p-1}{\left(p-n-\frac{3}{2}\right)\left(p+n-\frac{3}{2}\right)}, \tag{125}$$

and

$$\frac{p-2}{\left(p-n-\frac{3}{2}\right)\left(p+n-\frac{3}{2}\right)} > \frac{p}{\left(p-n-\frac{1}{2}\right)\left(p+n-\frac{1}{2}\right)} \tag{126}$$

for $n \geqslant 1$. Both inequalities, (125), as well as (126), convert to $2n^2 - \frac{1}{2} > 0$, by standard algebraic methods. This is obviously true as soon as $n \geqslant 1$. Hence, we can continue (124) by the following chain of inequalities

$$\begin{aligned}
&\frac{(p-2)^2(p-3)}{\left(p-n-\frac{3}{2}\right)\left(p+n-\frac{3}{2}\right)\left(p-n-\frac{5}{2}\right)\left(p+n-\frac{5}{2}\right)} \\
&> \frac{(p-2)^2(p-1)}{\left(p-n-\frac{3}{2}\right)^2\left(p+n-\frac{3}{2}\right)^2} \\
&> \frac{p(p-2)(p-1)}{\left(p-n-\frac{1}{2}\right)\left(p+n-\frac{1}{2}\right)\left(p-n-\frac{3}{2}\right)\left(p+n-\frac{3}{2}\right)},
\end{aligned}$$

which finally proves (122).
Now we work with our newly gained insight and examine for which values of a the upper bound $\overline{\varphi}_{4,A}$ is smaller than or equal to one. The usual transformations of the condition $\overline{\varphi}_{4,A}(a) \leqslant 1$, like multiplication of both sides with the denominator and subsequent subtraction of the right-hand side of the inequality from the left one, lead to the requirement

$$\left(2n - \frac{1}{2} - a\right)(-n+1+a)(-2n-1+6a) \leqslant 0.$$

Since $1 \leqslant a \leqslant n-1$, the first factor is always positive and the second one is always negative. Thus, we need the last factor to be positive, in order for the above condition to be true. This is the case for all $a \geqslant \frac{1}{3}n + \frac{1}{6}$. Thus, we can set the value

$$\overline{a} := \frac{1}{3}n + \frac{1}{6}$$

as the wanted upper bound on ζ.

By Theorem 20 *(iii)* we now know that $F_{4,A}(n,a)$ attains its maximum at $a^* \in \mathbb{N}$ with

$$\frac{1}{3}n - 1 < a^* < \frac{1}{3}n + \frac{1}{6}. \tag{127}$$

Since $R_4(n)$ is the largest of the four remainder summands $R_1(n), R_2(n), R_3(n), R_4(n)$, we have to estimate it particularly careful. Therefore, we use the additional information that a^* is an integer, to find the exact location of the maximum depending on the divisibility of n by 3.
If n is a multiple of 3, let's say $n = 3p$, then $a^* \in \left(p-1, p+\frac{1}{6}\right)$. Since a^* is an integer, the only possibility is $a^* = p = \frac{1}{3}n$. We repeat the argument for the other cases of divisibility of n by 3 and end up with

$$\begin{aligned}
a^* &= \begin{cases} \frac{1}{3}n, & n \equiv 0 \mod 3, \\ \frac{1}{3}n - \frac{1}{3}, & n \equiv 1 \mod 3, \\ \frac{1}{3}n - \frac{2}{3}, & n \equiv 2 \mod 3, \end{cases} \\
&= \left\lfloor \frac{1}{3}n \right\rfloor.
\end{aligned} \tag{128}$$

Now we continue with the estimate of $\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} F_{4,A}(n,a^*)$ for each of the three cases of (128). Afterwards we prove a common upper bound for all of them.

We start with $\underline{n \equiv 0 \mod 3}$. We plug in $a^* = \frac{1}{3}n$ into (120) and estimate, using Stirling's formula (332)

$$\begin{aligned}
&\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_{4,A}\left(n,\frac{1}{3}n\right) \\
&= 2^{-4n}\frac{\Gamma(2n)\Gamma\left(2n-\frac{1}{2}\right)\Gamma\left(4n-\frac{1}{2}\right)}{\pi\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)\Gamma(3n)}\frac{\Gamma\left(\frac{4}{3}n+\frac{1}{2}\right)^3}{\Gamma\left(\frac{1}{3}n+1\right)^3\Gamma\left(\frac{2}{3}n+\frac{1}{2}\right)^3} \\
&\leqslant \frac{3^{\frac{1}{2}}}{2^{\frac{7}{2}}\pi}e^{n+\frac{1}{24n}}\frac{n^{-n}}{\left(n-\frac{1}{4}\right)\left(n-\frac{1}{8}\right)}2^{-2n}3^{-3n}\frac{\Gamma\left(\frac{4}{3}n+\frac{1}{2}\right)^3}{\Gamma\left(\frac{1}{3}n+1\right)^3\Gamma\left(\frac{2}{3}n+\frac{1}{2}\right)^3}.
\end{aligned} \tag{129}$$

Another application of Stirling's formula (332) to the remaining gamma quotient in (129) yields

$$\frac{\Gamma\left(\frac{4}{3}n+\frac{1}{2}\right)^3}{\Gamma\left(\frac{1}{3}n+1\right)^3\Gamma\left(\frac{2}{3}n+\frac{1}{2}\right)^3} \leqslant \frac{e^{-n+3+\frac{3}{16n}}}{(2\pi)^{\frac{3}{2}}}\frac{3^{\frac{3}{2}}n^n}{(n+3)^{\frac{3}{2}}}\frac{2^{6n}}{3^n}\frac{\left(1+\frac{3}{8n}\right)^{4n}}{\left(1+\frac{3}{n}\right)^n\left(1+\frac{3}{4n}\right)^{2n}}. \tag{130}$$

Next, we bound the last factor in (130) with the help of the inequality $1+x \leqslant e^x$, and Lemma 64, estimate (403)

$$\begin{aligned}
\frac{\left(1+\frac{3}{8n}\right)^{4n}}{\left(1+\frac{3}{n}\right)^n\left(1+\frac{3}{4n}\right)^{2n}} &\leqslant \exp\left[\frac{3}{2}-n\left(\frac{3}{n}-\frac{9}{2n^2}\right)-2n\left(\frac{3}{4n}-\frac{9}{32n^2}\right)\right] \\
&= e^{-3+\frac{81}{16n}}.
\end{aligned} \tag{131}$$

Putting everything together, we arrive at

$$\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_{4,A}\left(n,\frac{1}{3}n\right) \leqslant \frac{3^2e^{\frac{127}{24n}}}{2^5\pi^{\frac{5}{2}}}\frac{1}{(n+3)^{\frac{3}{2}}\left(n-\frac{1}{8}\right)\left(n-\frac{1}{4}\right)}\left(\frac{2}{3}\right)^{4n}. \tag{132}$$

The case $\underline{n \equiv 1 \mod 3}$ is next. By the same means as in the first case $n \equiv 0 \mod 3$, we estimate

$$\begin{aligned}
&\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_{4,A}\left(n,\frac{1}{3}n-\frac{1}{3}\right) \\
&= 2^{-4n}\frac{\Gamma(2n)}{\pi\Gamma(3n+1)}\frac{\Gamma\left(\frac{4}{3}n+\frac{1}{6}\right)^3}{\Gamma\left(\frac{1}{3}n+\frac{2}{3}\right)^3\Gamma\left(\frac{2}{3}n+\frac{5}{6}\right)^3} \\
&\leqslant \frac{3^2e^{\frac{91}{24n}}}{2^5\pi^{\frac{5}{2}}}\frac{1}{(n+2)^{\frac{1}{2}}n\left(n+\frac{1}{8}\right)\left(n+\frac{5}{4}\right)}\left(\frac{2}{3}\right)^{4n}.
\end{aligned} \tag{133}$$

In the case $\underline{n \equiv 2 \mod 3}$ the same techniques lead to

$$\begin{aligned}
&\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}F_{4,A}\left(n,\frac{1}{3}n-\frac{2}{3}\right) \\
&= 2^{-4n}\frac{\Gamma(2n)\Gamma\left(2n+\frac{3}{2}\right)\Gamma\left(4n+\frac{3}{2}\right)}{\pi\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)\Gamma(3n+2)}\frac{\Gamma\left(\frac{4}{3}n-\frac{1}{6}\right)^3}{\Gamma\left(\frac{1}{3}n+\frac{1}{3}\right)^3\Gamma\left(\frac{2}{3}n+\frac{7}{6}\right)^3} \\
&\leqslant \frac{3^2e^{\frac{4}{n}}}{2^5\pi^{\frac{5}{2}}}\frac{(n+1)^{\frac{1}{2}}\left(n+\frac{1}{4}\right)\left(n+\frac{1}{8}\right)}{n\left(n+\frac{1}{3}\right)\left(n-\frac{1}{8}\right)^2\left(n+\frac{7}{4}\right)^2}\left(\frac{2}{3}\right)^{4n}.
\end{aligned} \tag{134}$$

In the next step we show that for $n \geqslant 9$ all three expressions (132), (133), and (134) are bounded from above by

$$\frac{3^2 e}{2^5 \pi^{\frac{5}{2}}} \frac{1}{n^{\frac{3}{2}} \left(n+\frac{1}{4}\right)\left(n-\frac{1}{4}\right)} \left(\frac{2}{3}\right)^{4n}. \tag{135}$$

First, note that $e^{\frac{91}{24n}} < e^{\frac{4}{n}} < e^{\frac{199}{24n}} < e$ for all $n \geqslant 9$. Consequently, the first factor of each of the expressions (132), (133), and (134), respectively, is at most $\frac{3^2 e}{2^5 \pi^{\frac{5}{2}}}$, if $n \geqslant 9$. This marks the first step towards the common bound (135).
Now, to prove the claimed bound for (132) it suffices to show that

$$\frac{n^{\frac{3}{2}}\left(n+\frac{1}{4}\right)}{(n+3)^{\frac{3}{2}}\left(n-\frac{1}{8}\right)} \leqslant 1.$$

To this end, we estimate

$$\frac{n^{\frac{3}{2}}\left(n+\frac{1}{4}\right)}{(n+3)^{\frac{3}{2}}\left(n-\frac{1}{8}\right)} \leqslant \frac{n\left(n+\frac{1}{4}\right)}{(n+3)\left(n-\frac{1}{8}\right)}.$$

That this expression is smaller than or equal to one for $n \geqslant 1$ can easily be seen by solving $n\left(n+\frac{1}{4}\right) \leqslant (n+3)\left(n-\frac{1}{8}\right)$ for n.
In the case of (133) it is even simpler, since

$$\frac{n^{\frac{1}{2}}\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)}{(n+2)^{\frac{1}{2}}\left(n+\frac{1}{8}\right)\left(n+\frac{5}{4}\right)} < \frac{\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)}{\left(n+\frac{1}{8}\right)\left(n+\frac{5}{4}\right)} < 1.$$

For the proof that (134) is at most equal to (135), we again consider the quotient of both expressions, which is for $n \geqslant 9$ bounded from above by

$$\frac{n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}\left(n+\frac{1}{8}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)^2}{\left(n-\frac{1}{8}\right)^2\left(n+\frac{7}{4}\right)^2\left(n+\frac{1}{3}\right)} \leqslant \frac{n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}\left(n+\frac{1}{8}\right)\left(n+\frac{1}{4}\right)}{\left(n+\frac{3}{2}\right)^2\left(n+\frac{1}{3}\right)} < 1. \tag{136}$$

We obtain this result by estimating $\left(n-\frac{1}{8}\right)\left(n+\frac{7}{4}\right) \geqslant n\left(n+\frac{3}{2}\right)$ for $n \geqslant \frac{7}{4}$,
and $\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right) < n^2$.

To sum up the above findings, we note that our first candidate for the maximum of $F_4(n,a,b,c,d)$ is attained in the vicinity of $a=b=c=\frac{1}{3}n$ and $d=4n-3a-1$. Moreover, if $n \geqslant 9$ it satisfies

$$\begin{aligned} \frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} F_4(n,a,b,c,d) &\leqslant \frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} F_{4,A}(n,a^*) \\ &\leqslant \frac{3^2 e}{2^5 \pi^{\frac{5}{2}}} \frac{1}{n^{\frac{3}{2}}\left(n+\frac{1}{4}\right)\left(n-\frac{1}{4}\right)} \left(\frac{2}{3}\right)^{4n}. \end{aligned} \tag{137}$$

Let us now turn to **Case B**. We write

$$\begin{aligned} F_4(n,a,a,a+1,4n-3a-2) &= \frac{\Gamma\left(n+a+\frac{1}{2}\right)^2\Gamma\left(n+a+\frac{3}{2}\right)\Gamma\left(5n-3a-\frac{3}{2}\right)\Gamma\left(3n-3a-\frac{3}{2}\right)}{\pi\Gamma(a+1)^2\Gamma(a+2)\Gamma(4n-3a-1)\Gamma\left(n-a+\frac{1}{2}\right)^2\Gamma\left(n-a-\frac{1}{2}\right)} \quad (138) \\ &=: F_{4,B}(n,a), \end{aligned}$$

for $a \leqslant a \leqslant n-1$ and note that

$$\begin{aligned}
\varphi_{4,B}(a) &= \frac{F_{4,B}(n,a)}{F_{4,B}(n,a-1)} \\
&= \frac{\left(-a+n-\frac{1}{2}\right)\left(-a+n+\frac{1}{2}\right)^2\left(a+n-\frac{1}{2}\right)^2\left(a+n+\frac{1}{2}\right)}{a^2(a+1)\left(-3a+3n-\frac{3}{2}\right)\left(-3a+3n-\frac{1}{2}\right)\left(-3a+3n+\frac{1}{2}\right)} \\
&\times \frac{(-3a+4n-1)(4n-3a)(-3a+4n+1)}{\left(-3a+5n-\frac{3}{2}\right)\left(-3a+5n-\frac{1}{2}\right)\left(-3a+5n+\frac{1}{2}\right)} \\
&= \varphi(a,4n-3a)\varphi(a,4n-3a-1)\varphi(a+1,4n-3a-2),
\end{aligned} \tag{139}$$

with $\varphi(c,d)$ as in (104) for $c \leqslant n \leqslant d$, which was first defined in the proof of Theorem 22. Due to the three poles at $a=0$, $a=n-\frac{1}{6}$ and $a=n+\frac{1}{6}$, we expect the function $\varphi_{4,B}(a)$ to decrease on $0<a<\tilde{a}$ and increase on $\tilde{a}<a\leqslant n-1$. In the following we show that $\tilde{a}>\frac{1}{3}n+1$ for $n\geqslant 6$. To this end we inspect the monotonicity of each factor in (139) separately. The last one is the easiest one. Writing

$$\begin{aligned}
\varphi(a+1,4n-3a-2) &= \frac{\left(a+n+\frac{1}{2}\right)(-3a+4n-1)}{3(a+1)\left(-3a+5n-\frac{3}{2}\right)} \\
&= \frac{1}{3}\left(1+\frac{n-\frac{1}{2}}{a+1}\right)\left(1-\frac{n-\frac{1}{2}}{5n-3a-\frac{3}{2}}\right)
\end{aligned}$$

we immediately see that it is decreasing on the entire interval $0<a\leqslant n-1$. The remaining two make us work a little harder. We write

$$\varphi(a,4n-3a) = \frac{1}{3}\left(1+\frac{n-\frac{1}{2}}{a}\right)\frac{\left(n-a+\frac{1}{2}\right)(4n-3a+1)}{\left(n-a+\frac{1}{6}\right)\left(5n-3a+\frac{1}{2}\right)} \tag{140}$$

$$\varphi(a,4n-3a-1) = \frac{1}{3}\left(1+\frac{n-\frac{1}{2}}{a}\right)\frac{\left(n-a+\frac{1}{2}\right)(4n-3a)}{\left(n-a-\frac{1}{6}\right)\left(5n-3a-\frac{1}{2}\right)}. \tag{141}$$

As the first factor is decreasing in a, we only consider the respective second ones. Our approach is the same as usually with quotients of this kind: inspecting the sign of the derivative. Starting with (140) we get for the a-derivative of the second factor

$$-\frac{6\left(36a^2(2n-3)+a\left(-144n^2+240n+60\right)+72n^3-148n^2-74n-7\right)}{(-6a+6n+1)^2(-6a+10n+1)^2},$$

which is negative iff the enumerator is positive. This is the case for $a\leqslant a_1$ and $a\geqslant a_2$, with

$$a_{1,2} = \frac{12n^2-20n\mp(2n-1)\sqrt{2n+1}-5}{12n-18}$$

being the two zeros of the numerator. We estimate $1<\sqrt{2n+1}<2n-5$ for $n\geqslant 4$, and find that

$$\begin{aligned}
a_1 &> \frac{8n^2-8n-20}{12n-18} \\
&> \frac{2}{3}n-\frac{20}{12n-18} \\
&> \frac{2}{3}n-1,
\end{aligned}$$

and

$$\begin{aligned} a_2 &> \frac{12n^2 - 18n - 6}{12n - 18} \\ &= n - \frac{1}{2n - 3} \\ &> n - 1 \end{aligned}$$

as soon as $n \geqslant 2$. We thus conclude that (140) is decreasing until at least $a = \frac{2}{3}n - 1$.
The a-derivative of the second factor of (141) reads

$$-\frac{6\left(36a^2(2n-5) - 12a\left(12n^2 - 40n + 1\right) + 72n^3 - 332n^2 + 14n + 3\right)}{(6a - 10n + 1)^2(6a - 6n + 1)^2},$$

which again is negative iff the numerator is positive. This time this is the case for $a \leqslant a_1$ and $a \geqslant a_2$ with

$$a_{1,2} = \frac{12n^2 - 40n \mp 4\sqrt{(n-1)(2n-1)(2n+1)} + 1}{12n - 30}.$$

In this case we estimate $2n - 1 < \sqrt{(n-1)(2n-1)(2n+1)} < (n-6)(2n+1)$ for $n \geqslant 6$. Using this we yield

$$\begin{aligned} a_1 &> \frac{4n^2 + 4n + 25}{12n - 30} \\ &> \frac{4n^2 + 4n + 25}{12n - 24} \\ &= \frac{1}{3}n + \frac{12n + 25}{12n - 24} \\ &> \frac{1}{3}n + 1, \end{aligned}$$

and

$$\begin{aligned} a_2 &> \frac{12n^2 - 32n + 5}{12n - 30} \\ &= n - \frac{2n - 5}{12n - 30} \\ &> n - 1 \end{aligned}$$

for all $n \geqslant 3$. Hence (141) is decreasing until at least $a = \frac{1}{3}n + 1$ if $n \geqslant 6$. This proves the claimed bound $\tilde{a} > \frac{1}{3}n + 1$.
The next step is to show that despite of increasing for $a \to n - 1$, the function $\varphi_{4,B}(a)$ is still smaller than one at $a = n - 1$ if $n \geqslant 6$. In fact, it is

$$\begin{aligned} \varphi_{4,B}(n-1) &= \frac{3}{35} \frac{(4n-3)^2(4n-1)(n+2)(n+3)(n+4)}{(4n+3)(4n+5)(4n+7)n(n-1)^2} \\ &< \frac{3}{35} \frac{(n+2)(n+3)(n+4)}{n(n-1)^2}. \end{aligned}$$

Using that $\frac{3}{35} < \frac{1}{8}$, we further estimate the last line above by

$$\frac{n+4}{2n} \cdot \frac{n+2}{2n-2} \cdot \frac{n+3}{2n-2}.$$

If $n \geqslant 6$ each of those factors is smaller than one.
In summary, we now can conclude from the above calculations that if there is a value $\zeta \in (0, n-1]$ such that $\varphi_{4,B}(\zeta) = 1$ then it is unique, and moreover, $\varphi_{4,B}(a) > 1$ iff $a < \zeta$ and $\varphi_{4,B}(a) < 1$ iff $\zeta < a \leqslant n - 1$.

Next we prove that

$$\varphi_{4,B}\left(\frac{1}{3}n-\frac{1}{4}\right)>1, \tag{142}$$

$$\varphi_{4,B}\left(\frac{1}{3}n-\frac{1}{12}\right)<1 \tag{143}$$

for $n \geqslant 12$. This together with the lates insights on the monotonicity of $\varphi_{4,B}$ implies that

$$\frac{1}{3}n-\frac{1}{4}<\zeta<\frac{1}{3}n-\frac{1}{12}. \tag{144}$$

Then we know by Theorem 20 *(iii)* that $F_{4,B}(n,a)$ attains its maximum at $a^* \in \mathbb{N}$ with

$$\frac{1}{3}n-\frac{5}{4}<a^*<\frac{1}{3}n-\frac{1}{12}. \tag{145}$$

Before we exploit this information, we show the two bounds (142) and (143), starting with the former. It is

$$\varphi_{4,B}\left(\frac{1}{3}n-\frac{1}{4}\right)=\frac{\left(n-\frac{9}{16}\right)^2\left(n+\frac{1}{4}\right)\left(n+\frac{9}{8}\right)^2\left(n-\frac{1}{12}\right)\left(n+\frac{7}{12}\right)\left(n+\frac{3}{16}\right)}{\left(n-\frac{3}{4}\right)^2\left(n+\frac{9}{4}\right)\left(n+\frac{1}{8}\right)\left(n+\frac{5}{8}\right)\left(n-\frac{3}{16}\right)\left(n+\frac{1}{16}\right)\left(n+\frac{5}{16}\right)}.$$

Shifting each single factor using $n=m+\frac{9}{16}$ and then dividing it by m transforms the right-hand side above into

$$\frac{\left(\frac{23}{48m}+1\right)\left(\frac{3}{4m}+1\right)\left(\frac{13}{16m}+1\right)\left(\frac{55}{48m}+1\right)\left(\frac{27}{16m}+1\right)^2}{\left(1-\frac{3}{16m}\right)^2\left(\frac{3}{8m}+1\right)\left(\frac{5}{8m}+1\right)\left(\frac{11}{16m}+1\right)\left(\frac{7}{8m}+1\right)\left(\frac{19}{16m}+1\right)\left(\frac{45}{16m}+1\right)}.$$

Next we apply estimate (403) of Lemma 64 to the factors in the numerator and the simples estimate $1+x\leqslant e^x$ to the factors in the denominator. This yields the lower bound

$$\begin{aligned}
&\exp\Bigg[2\frac{27}{16m}+\frac{13}{16m}+\frac{55}{48m}+\frac{23}{48m}+\frac{3}{4m}\\
&\qquad-\left(\frac{45}{16m}+\frac{19}{16m}+\frac{11}{16m}-2\frac{3}{16m}+\frac{7}{8m}+\frac{5}{8m}+\frac{3}{8m}\right)\\
&\qquad-\frac{1}{2}\left(\left(\frac{3}{4m}\right)^2+\left(\frac{13}{16m}\right)^2+2\left(\frac{27}{16m}\right)^2+\left(\frac{23}{48m}\right)^2+\left(\frac{55}{48m}\right)^2\right)\Bigg]\\
&=\exp\left[\frac{3}{8m}-\frac{19493}{4608m^2}\right].
\end{aligned}$$

The last line is greater than one if $m>\frac{19493}{1728}$. This proves (142) for $n\geqslant 12>m+\frac{9}{16}$. We follow the same approach for the proof of (143). It is

$$\varphi_{4,B}\left(\frac{1}{3}n-\frac{1}{12}\right)=\frac{\left(n-\frac{7}{16}\right)^2\left(n+\frac{7}{8}\right)^2\left(n+\frac{1}{12}\right)\left(n+\frac{5}{12}\right)\left(n+\frac{5}{16}\right)}{\left(n-\frac{1}{4}\right)\left(n+\frac{11}{4}\right)\left(n-\frac{1}{8}\right)\left(n+\frac{3}{8}\right)\left(n-\frac{5}{16}\right)\left(n-\frac{1}{16}\right)\left(n+\frac{3}{16}\right)}.$$

We shift $n=m+\frac{5}{16}$, divide each linear factor by m and get

$$\frac{\left(1-\frac{1}{8m}\right)^2\left(\frac{19}{48m}+1\right)\left(\frac{5}{8m}+1\right)\left(\frac{35}{48m}+1\right)\left(\frac{19}{16m}+1\right)^2}{\left(\frac{1}{16m}+1\right)\left(\frac{3}{16m}+1\right)\left(\frac{1}{4m}+1\right)\left(\frac{1}{2m}+1\right)\left(\frac{11}{16m}+1\right)\left(\frac{49}{16m}+1\right)}.$$

This time we apply apply estimate (403) of Lemma 64 to the factors in the denominator and $1 + x \leqslant e^x$ in the numerator and arrive at the following lower bound

$$\begin{aligned}
&\exp\Bigg[-\frac{2}{8m} + \frac{19}{48m} + \frac{5}{8m} + \frac{35}{48m} + 2\frac{19}{16m} \\
&\quad - \left(\frac{1}{16m} + \frac{3}{16m} + \frac{1}{4m} + \frac{1}{2m} + \frac{11}{16m} + \frac{49}{16m} \right) \\
&\quad + \frac{1}{2}\left(2\left(\frac{1}{8m}\right)^2 + \left(\frac{19}{48m}\right)^2 + \left(\frac{5}{8m}\right)^2 + \left(\frac{35}{48m}\right)^2 + 2\left(\frac{19}{16m}\right)^2 \right) \Bigg] \\
&= \exp\left[\frac{283}{144m^2} - \frac{7}{8m} \right].
\end{aligned}$$

This is smaller than one as soon as $m > \frac{283}{126}$, which proves (143) for $n \geqslant 3 > m + \frac{5}{16}$.

Next on our to-do list is now to estimate $F_{4,B}(a^*)$. We try to safe some time and space and show that $F_{4,B}(a^*)$ must be smaller than the the maximum of $F_{4,A}(a)$. To this end, we first calculate the quotient of $F_{4,B}$ and $F_{4,A}$. This is an old friend, namely

$$\begin{aligned}
\frac{F_{4,B}(a)}{F_{4,A}(a)} &= \frac{\Gamma\left(n+a+\frac{3}{2}\right)\Gamma\left(5n-3a-\frac{3}{2}\right)\Gamma\left(3n-3a-\frac{3}{2}\right)}{\Gamma\left(a+2\right)\Gamma(4n-3a-1)\Gamma\left(n-a-\frac{1}{2}\right)} \\
&\quad \times \frac{\Gamma(a+1)\Gamma(4n-3a)\Gamma\left(n-a+\frac{1}{2}\right)}{\Gamma\left(n+a+\frac{1}{2}\right)\Gamma\left(5n-3a-\frac{1}{2}\right)\Gamma\left(3n-3a-\frac{1}{2}\right)} \\
&= \frac{\left(a+n+\frac{1}{2}\right)(-3a+4n-1)}{3(a+1)\left(-3a+5n-\frac{3}{2}\right)} \\
&= \varphi(a+1, 4n-3a-1).
\end{aligned}$$

We already know that $\varphi(a+1, 4n-3a-1)$ is monotonously decreasing on $0 < a \leqslant n-1$. We additionally determine the point ξ such that $\varphi(\xi+1, 4n-3\xi-1) = 1$. To do so we transform

$$\frac{\left(a+n+\frac{1}{2}\right)(-3a+4n-1)}{3(a+1)\left(-3a+5n-\frac{3}{2}\right)} = 1$$

into the quadratic equation

$$6a^2 + a(11-14n) + 4n^2 - 14n + 4 = 0$$

and calculate the roots $a_{1,2}$. They are

$$a_{1,2} = \frac{1}{12}\left(14n \mp \sqrt{100n^2+28n+25} - 11\right).$$

Since $a_2 > n-1$, it is $\xi = a_1$. Using that $100n^2 + 28n + 25 > (10n+1)^2$, as soon as $n > -3$ we bound

$$\begin{aligned}
\xi &= \frac{1}{12}\left(14n - \sqrt{100n^2+28n+25} - 11\right) \\
&< \frac{1}{3}n - 1. \qquad (146)
\end{aligned}$$

Moreover, we infer from the monotonicity of $\varphi(a+1, 4n-3a-1)$ that

$$F_{4,B}(a) > F_{4,A}(a) \quad \text{if } 0 \leqslant a < \xi,$$

and

$$F_{4,B}(a) < F_{4,A}(a) \quad \text{if } \xi < a \leqslant n-1. \qquad (147)$$

Recall (127) which states the two-sided bound $\frac{1}{3}n - 1 < a_A^* < \frac{1}{3}n + \frac{1}{6}$ on the location a_A^* of the maximum of $F_{4,A}(a)$. The function $F_{4,A}$ thus attains its maximum in the range of a where it is larger than $F_{4,B}$ anyway.

Next we have a closer look at the location a^* of the maximum of $F_{4,B}(a)$ and show that our bound (145) implies

$$a^* \geqslant \frac{1}{3}n - 1 > \xi. \tag{148}$$

Using (147) and (146) we thus conclude that

$$F_{4,B}(a^*) < F_{4,A}(a^*) \leqslant \max_{0 \leqslant a \leqslant n} F_{4,A}(a). \tag{149}$$

To complete the discussion of **Case B** it remains to prove (148). Recall that the function $F_{4,B}$ lives on integers. We therefore need to determine

$$a^* \in \left(\frac{1}{3}n - \frac{5}{4}, \frac{1}{3}n - \frac{1}{12}\right) \cap \mathbb{Z} \tag{150}$$

depending on the divisibility of n.
If $n \equiv 0 \mod 3$, we set $n = 3p$ and (150) becomes

$$\begin{aligned} a^* &\in \left(p - \frac{5}{4}, p - \frac{1}{12}\right) \cap \mathbb{Z} \\ &= \{p - 1\}. \end{aligned}$$

Consequently it is in this case

$$a^* = \frac{1}{3}n - 1.$$

If $n \equiv 1 \mod 3$, setting $n = 3p + 1$ leads to

$$\begin{aligned} a^* &\in \left(p - \frac{11}{12}, p + \frac{1}{4}\right) \cap \mathbb{Z} \\ &= \{p\} \end{aligned}$$

and therefore

$$a^* = \frac{1}{3}n - \frac{1}{3}.$$

In the case of $n \equiv 2 \mod 3$, we get with $n = 3p + 2$

$$\begin{aligned} a^* &\in \left(p - \frac{7}{12}, p + \frac{7}{12}\right) \cap \mathbb{Z} \\ &= \{p\} \end{aligned}$$

and

$$a^* = \frac{1}{3}n - \frac{2}{3}.$$

Last on our to-do list is **Case C**. We write

$$\begin{aligned} F_4(n, a, a+1, a+1, 4n-3a-3) &= \frac{\Gamma\left(n+a+\frac{1}{2}\right)\Gamma\left(n+a+\frac{3}{2}\right)^2\Gamma\left(5n-3a-\frac{5}{2}\right)\Gamma\left(3n-3a-\frac{5}{2}\right)}{\pi\Gamma(a+1)\Gamma\left(a+2\right)^2\Gamma(4n-3a-2)\Gamma\left(n-a+\frac{1}{2}\right)\Gamma\left(n-a-\frac{1}{2}\right)^2} \quad (151) \\ &=: F_{4,C}(n, a), \end{aligned}$$

for $a \leqslant a \leqslant n-1$. In this case we find that

$$\begin{aligned}
\varphi_{4,c}(a) &= \frac{F_{4,C}(n,a)}{F_{4,C}(n,a-1)} \\
&= \frac{\left(-a+n-\frac{1}{2}\right)^2\left(-a+n+\frac{1}{2}\right)\left(a+n-\frac{1}{2}\right)\left(a+n+\frac{1}{2}\right)^2}{a(a+1)^2\left(-3a+3n-\frac{5}{2}\right)\left(-3a+3n-\frac{3}{2}\right)\left(-3a+3n-\frac{1}{2}\right)} \\
&\times \frac{(-3a+4n-2)(4n-3a-1)(-3a+4n)}{\left(-3a+5n-\frac{5}{2}\right)\left(-3a+5n-\frac{3}{2}\right)\left(-3a+5n-\frac{1}{2}\right)} \\
&= \varphi(a,4n-3a-1)\varphi(a+1,4n-3a-2)\varphi(a+1,4n-3a-3). \qquad (152)
\end{aligned}$$

Recall (139) and set

$$\Phi(a) = \varphi(a, 4n-3a).$$

Then it is

$$\begin{aligned}
\varphi_{4,B}(a) &= \Phi(a)\varphi(a,4n-3a-1)\varphi(a+1,4n-3a-2), \\
\varphi_{4,C}(a) &= \Phi(a+1)\varphi(a,4n-3a-1)\varphi(a+1,4n-3a-2).
\end{aligned}$$

We have already seen on page 58 that $\varphi(a,4n-3a-1)\varphi(a+1,4n-3a-2)$ is monotonously decreasing in a for $0 < a \leqslant \tilde{a}$ with $\tilde{a} > \frac{1}{3}n+1$, and that $\Phi(a)$ decreases until at least $a = \frac{2}{3}n-1$. It thus follows that also $\Phi(a+1)\varphi(a,4n-3a-1)\varphi(a+1,4n-3a-2)$ is decreasing in $0 < a \leqslant \hat{a}$, where $\hat{a}$ and $\tilde{a}$ satisfy the same upper bound $\frac{1}{3}n+1$.
Next we show that

$$\varphi_{4,C}(n-1) < 1 \qquad (153)$$

for $n \geqslant 3$. We can then draw the same conclusion as for $\varphi_{4,B}$, namely that if there is a value $\zeta \in (0, n-1]$ such that $\varphi_{4,C}(\zeta) = 1$ then it is unique, and moreover, $\varphi_{4,C}(a) > 1$ iff $a < \zeta$ and $\varphi_{4,C}(a) < 1$ iff $\zeta < a \leqslant n-1$.
So let's prove (153). It is

$$\varphi_{4,C}(n-1) = \frac{(1-4n)^2(n+1)(n+2)(n+3)(4n-3)}{5(n-1)n^2(4n+1)(4n+3)(4n+5)}.$$

We split $5 = \frac{5}{4}\cdot 4$ and use that

$$(4n-3)(n+1) < (4n+5)\frac{5}{4}(n-1)$$

for $n \geqslant 2$ to bound the right-hand side from above by

$$\frac{(1-4n)^2}{(4n+1)(4n+3)} \cdot \frac{n+2}{2n} \cdot \frac{n+3}{2n}.$$

This is obviously smaller than one for all $n \geqslant 3$.

In the following we prove that

$$\varphi_{4,C}\left(\frac{1}{3}n - \frac{7}{12}\right) > 1, \qquad (154)$$

$$\varphi_{4,C}\left(\frac{1}{3}n - \frac{1}{2}\right) < 1 \qquad (155)$$

for $n \geqslant 17$. This and the monotonicity properties of $\varphi_{4,C}$ then imply

$$\frac{1}{3}n - \frac{7}{12} < \zeta < \frac{1}{3}n - \frac{1}{2}, \qquad (156)$$

and the application of Theorem 20 *(iii)* eventually yields that $F_{4,C}(n,a)$ attains its maximum at $a^* \in \mathbb{N}$ with

$$\frac{1}{3}n - \frac{19}{12} < a^* < \frac{1}{3}n - \frac{1}{2}. \tag{157}$$

We first show (154). It is

$$\varphi_{4,C}\left(\frac{1}{3}n - \frac{7}{12}\right) = \frac{\left(n-\frac{1}{16}\right)^2\left(n+\frac{1}{4}\right)\left(n+\frac{1}{8}\right)\left(n+\frac{13}{8}\right)\left(n-\frac{1}{12}\right)\left(n+\frac{7}{12}\right)\left(n-\frac{13}{16}\right)}{\left(n-\frac{7}{4}\right)\left(n+\frac{5}{4}\right)^2\left(n-\frac{3}{8}\right)\left(n+\frac{5}{8}\right)\left(n-\frac{3}{16}\right)\left(n+\frac{1}{16}\right)\left(n+\frac{5}{16}\right)}.$$

We have seen terms like this several times before and follow the same approach as usual. We shift $n = m + \frac{11}{16}$ and divide every linear factor by m to transform the right-hand side above into

$$\frac{\left(\frac{35}{48m}+1\right)\left(\frac{3}{4m}+1\right)^2\left(\frac{15}{16m}+1\right)\left(\frac{17}{16m}+1\right)\left(\frac{67}{48m}+1\right)\left(\frac{39}{16m}+1\right)}{\left(1-\frac{15}{16m}\right)\left(\frac{7}{16m}+1\right)\left(\frac{5}{8m}+1\right)\left(\frac{7}{8m}+1\right)\left(\frac{9}{8m}+1\right)\left(\frac{23}{16m}+1\right)\left(\frac{33}{16m}+1\right)^2}.$$

Now we apply $1 + x \leqslant e^x$ to the factors in the denominator and estimate (403) of Lemma 64 in the numerator and get the following lower bound on the last line

$$\begin{aligned}
&\exp\Bigg[\frac{39}{16m} + \frac{17}{16m} + \frac{15}{16m} + \frac{67}{48m} + \frac{35}{48m} + 2\frac{3}{4m} \\
&\quad - \frac{1}{2}\left(2\left(\frac{3}{4m}\right)^2 + \left(\frac{15}{16m}\right)^2 + \left(\frac{17}{16m}\right)^2 + \left(\frac{39}{16m}\right)^2 + \left(\frac{35}{48m}\right)^2 + \left(\frac{67}{48m}\right)^2\right) \\
&\quad - 2\frac{33}{16m} + \frac{23}{16m} - \frac{15}{16m} + \frac{7}{16m} + \frac{9}{8m} + \frac{7}{8m} + \frac{5}{8m}\Bigg] \\
&= \exp\left[\frac{3}{8m} - \frac{26621}{4608m^2}\right].
\end{aligned}$$

This is greater than one if the exponent is positive, which in turn is satisfied for $m > \frac{26621}{1728}$, or $n > \frac{28025}{1728} \approx 16.22$, respectively.
Next is (155). We have

$$\varphi_{4,C}\left(\frac{1}{3}n - \frac{1}{2}\right) = \frac{n^2\left(n-\frac{3}{4}\right)\left(n-\frac{1}{6}\right)\left(n+\frac{1}{6}\right)}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)}.$$

In this case we shift $n = m + \frac{3}{2}$ before dividing each factor by m and get

$$\frac{\left(\frac{3}{4m}+1\right)\left(\frac{4}{3m}+1\right)\left(\frac{3}{2m}+1\right)^2\left(\frac{5}{3m}+1\right)}{\left(\frac{1}{m}+1\right)\left(\frac{5}{4m}+1\right)\left(\frac{7}{4m}+1\right)\left(\frac{3}{m}+1\right)}. \tag{158}$$

Since this time the estimate is a bit tighter, we need to apply Lemma 64 to both the numerator and denominator. For the denominator we use estimate (403). To the numerator we apply estimate (404) with $\delta = \frac{3}{7}$. All conditions for this estimate are satisfied if

$$\frac{3}{2m} < \frac{1}{\frac{3}{7}} - \sqrt{\frac{2}{\frac{3}{7}}},$$

which leads to $m > -\frac{9}{2\sqrt{42}-14}$, or $n > \frac{3}{2} - \frac{9}{2\sqrt{42}-14} \approx 10.17$, respectively. Then we yield

$$\begin{aligned}
&\exp\Bigg[-\left(\frac{7}{4m}+\frac{5}{4m}+\frac{3}{m}+\frac{1}{m}\right) + 2\frac{3}{2m}+\frac{5}{3m}+\frac{4}{3m}+\frac{3}{4m} \\
&\qquad + \frac{1}{2}\left(\left(\frac{1}{m}\right)^2+\left(\frac{3}{m}\right)^2+\left(\frac{5}{4m}\right)^2+\left(\frac{7}{4m}\right)^2\right) \\
&\qquad - \frac{3}{7}\left(2\left(\frac{3}{2m}\right)^2+\left(\frac{4}{3m}\right)^2+\left(\frac{5}{3m}\right)^2+\left(\frac{3}{4m}\right)^2\right)\Bigg] \\
&= \exp\left[\frac{67}{21m^2} - \frac{1}{4m}\right]
\end{aligned}$$

as an upper bound for (158). The last line is smaller than one if $m > \frac{268}{21}$, or $n > \frac{599}{42} \approx 14.26$.

Our next interim goal is to determine the precise location a^* of the maximum of $F_{4,C}(a)$ with respect to the divisibility by 3 of n. To this end we use (157) and the same reasoning as on page 62.
If $n \equiv 0 \mod 3$, we set $n = 3p$ and get

$$\begin{aligned}
a^* &\in \left(p - \frac{19}{12}, p - \frac{1}{2}\right) \cap \mathbb{Z} \\
&= \{p-1\}.
\end{aligned}$$

Consequently it is in this case

$$a^* = \frac{1}{3}n - 1.$$

If $n \equiv 1 \mod 3$, setting $n = 3p+1$ leads to

$$\begin{aligned}
a^* &\in \left(p - \frac{15}{12}, p - \frac{1}{6}\right) \cap \mathbb{Z} \\
&= \{p-1\}
\end{aligned}$$

and therefore

$$a^* = \frac{1}{3}n - \frac{4}{3}.$$

In the case of $n \equiv 2 \mod 3$, we get with $n = 3p+2$

$$\begin{aligned}
a^* &\in \left(p - \frac{11}{12}, p + \frac{1}{6}\right) \cap \mathbb{Z} \\
&= \{p\}
\end{aligned}$$

and

$$a^* = \frac{1}{3}n - \frac{2}{3}.$$

In summary this is

$$a^* = \begin{cases} \frac{1}{3}n - 1, & n \equiv 0 \mod 3, \\ \frac{1}{3}n - \frac{4}{3}, & n \equiv 1 \mod 3, \\ \frac{1}{3}n - \frac{2}{3}, & n \equiv 2 \mod 3. \end{cases} \tag{159}$$

Now we estimate $F_{4,C}(a^*)$ in the three cases (159) and compare the result to the estimate of the maximum of $F_{4,A}$. But fortunately it is not necessary to activate the entire machinery of Stirling's formula and Co. in all three cases. Since $F_{4,A}$ and $F_{4,C}$ share the case $a^* = \frac{1}{3}n - \frac{2}{3}$ (compare (128) to (159)), we take a shortcut here and only look at the quotient

$$\begin{aligned}\frac{F_{4,C}(a)}{F_{4,A}(a)} &= \frac{\Gamma\left(n+a+\frac{5}{2}\right)\Gamma\left(5n-3a-\frac{5}{2}\right)\Gamma\left(3n-3a-\frac{5}{2}\right)}{\Gamma\left(a+2\right)\Gamma(4n-3a-2)\Gamma\left(n-a-\frac{1}{2}\right)} \\ &\times \frac{\Gamma(a+1)\Gamma(4n-3a-1)\Gamma\left(n-a+\frac{1}{2}\right)}{\Gamma\left(n+a+\frac{1}{2}\right)\Gamma\left(5n-3a-\frac{1}{2}\right)\Gamma\left(3n-3a-\frac{1}{2}\right)} \\ &\times \frac{\Gamma\left(n+a+\frac{3}{2}\right)\Gamma\left(5n-3a-\frac{3}{2}\right)\Gamma\left(3n-3a-\frac{3}{2}\right)}{\Gamma\left(a+2\right)\Gamma(4n-3a-1)\Gamma\left(n-a-\frac{1}{2}\right)} \\ &\times \frac{\Gamma(a+1)\Gamma(4n-3a)\Gamma\left(n-a+\frac{1}{2}\right)}{\Gamma\left(n+a+\frac{1}{2}\right)\Gamma\left(5n-3a-\frac{1}{2}\right)\Gamma\left(3n-3a-\frac{1}{2}\right)} \\ &= \varphi(a+1,4n-3a-3)\varphi(a+1,4n-3a-2).\end{aligned}$$

It is straight forward to show that $\varphi(a+1,4n-3a-3)\varphi(a+1,4n-3a-2) < 1$ for $a = \frac{1}{3}n - \frac{2}{3}$. In fact, it is

$$\begin{aligned}\varphi\left(\frac{1}{3}n+\frac{1}{3},3n-1\right)\varphi\left(\frac{1}{3}n+\frac{1}{3},3n\right) &= \frac{n(3n+1)(4n+1)(8n-1)}{3(n+1)^2(4n-1)(8n+1)} \\ &< \frac{n(4n+1)}{(n+1)(4n-1)}.\end{aligned}$$

The last line is smaller than one as soon as $n \geqslant 1$.
It thus remains to estimate $F_{4,C}(a^*)$ in the two cases $a^* = \frac{1}{3}n - 1$, and $a^* = \frac{1}{3}n - \frac{4}{3}$. We start with the former one and plug in $a = \frac{1}{3}n - 1$ into (152). Then it is

$$\frac{2^{-4n}\Gamma(2n)F_{4,C}\left(n,\frac{1}{3}n-1\right)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} = \frac{2^{-4n}\Gamma(2n)\Gamma\left(\frac{4n}{3}-\frac{1}{2}\right)\Gamma\left(\frac{4n}{3}+\frac{1}{2}\right)^2}{\pi\Gamma\left(\frac{n}{3}+1\right)^2\Gamma\left(\frac{2n}{3}+\frac{1}{2}\right)^2\Gamma\left(\frac{2n}{3}+\frac{3}{2}\right)\Gamma\left(\frac{n}{3}\right)\Gamma(3n+1)}. \quad (160)$$

The application of Stirling's formula (332) yields the upper bound

$$\begin{aligned}&\frac{2^{-4n}}{2^{\frac{3}{2}}\pi^{\frac{5}{2}}}\exp\left[5+\frac{1}{24n}+\frac{1}{16n-6}+\frac{1}{8n+3}\right] \\ &\times \frac{(2n)^{2n-\frac{1}{2}}\left(\frac{4n}{3}-\frac{1}{2}\right)^{\frac{4}{3}n-1}\left(\frac{4n}{3}+\frac{1}{2}\right)^{\frac{8}{3}n}}{\left(\frac{n}{3}+1\right)^{\frac{2}{3}n+1}\left(\frac{2n}{3}+\frac{1}{2}\right)^{\frac{4}{3}n}\left(\frac{2n}{3}+\frac{3}{2}\right)^{\frac{2}{3}n+1}\left(\frac{n}{3}\right)^{\frac{1}{3}n-\frac{1}{2}}(3n+1)^{3n+\frac{1}{2}}}.\end{aligned} \quad (161)$$

The last quotient is equal to

$$\frac{3^2}{2^{\frac{7}{2}}}\frac{2^{8n}}{3^{4n}}\frac{1}{\left(n+\frac{1}{3}\right)^{\frac{1}{2}}\left(n-\frac{3}{8}\right)(n+3)\left(n+\frac{9}{4}\right)}\frac{\left(1-\frac{3}{8n}\right)^{\frac{4}{3}n}\left(1+\frac{3}{8n}\right)^{\frac{8}{3}n}}{\left(1+\frac{1}{3n}\right)^{3n}\left(1+\frac{3}{n}\right)^{\frac{2}{3}n}\left(1+\frac{9}{4n}\right)^{\frac{2}{3}n}\left(1+\frac{3}{4n}\right)^{\frac{4}{3}n}}.$$

We further estimate it using $1+x<e^x$ in the numerator and estimate (403) of Lemma 64 in the denominator of the third factor and find that it is smaller than

$$\frac{3^2}{2^{\frac{7}{2}}}\frac{2^{8n}}{3^{4n}}\frac{1}{\left(n+\frac{1}{3}\right)^{\frac{1}{2}}\left(n-\frac{3}{8}\right)(n+3)\left(n+\frac{9}{4}\right)}\exp\left[-5+\frac{25}{6n}\right]. \quad (162)$$

By combining (160) with estimates (161) and (162) we arrive at

$$\begin{aligned}\frac{2^{-4n}\Gamma(2n)F_{4,C}\left(n,\frac{1}{3}n-1\right)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} &\leqslant \frac{3^2}{2^5\pi^{\frac{5}{2}}}\left(\frac{2}{3}\right)^{4n}\frac{1}{\left(n+\frac{1}{3}\right)^{\frac{1}{2}}\left(n-\frac{3}{8}\right)(n+3)\left(n+\frac{9}{4}\right)} \\ &\quad\times\exp\left[\frac{1}{24n}+\frac{1}{16n-6}+\frac{1}{8n+3}+\frac{25}{6n}\right] \\ &\leqslant \frac{3^2 e}{2^5\pi^{\frac{5}{2}}}\left(\frac{2}{3}\right)^{4n}\frac{1}{\left(n+\frac{1}{3}\right)^{\frac{1}{2}}\left(n-\frac{3}{8}\right)(n+3)\left(n+\frac{9}{4}\right)} \qquad (163)\end{aligned}$$

for all $n \geqslant 5$. Like in the discussion of **Case A** we show that (163) is smaller than (135) for $n \geqslant 17$. This is the case if

$$\frac{n^{\frac{3}{2}}\left(n+\frac{1}{4}\right)\left(n-\frac{1}{4}\right)}{\left(n+\frac{1}{3}\right)^{\frac{1}{2}}\left(n-\frac{3}{8}\right)(n+3)\left(n+\frac{9}{4}\right)} < 1.$$

Taking into account that $n\left(n-\frac{1}{4}\right) < \left(n-\frac{3}{8}\right)(n+3)$ as soon as $n \geqslant 1$ this becomes quite obvious.

Next we have to estimate $F_{4,C}$ at the point $a = \frac{1}{3}n-\frac{4}{3}$. Similar to the above calculations we get

$$\begin{aligned}\frac{2^{-4n}\Gamma(2n)F_{4,C}\left(n,\frac{1}{3}n-\frac{4}{3}\right)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} &= \frac{\Gamma\left(2n+\frac{3}{2}\right)\Gamma\left(4n+\frac{3}{2}\right)}{\pi\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)} \\ &\quad\times\frac{2^{-4n}\Gamma(2n)\Gamma\left(\frac{4n}{3}-\frac{5}{6}\right)\Gamma\left(\frac{4n}{3}+\frac{1}{6}\right)^2}{\Gamma\left(\frac{2n}{3}+\frac{5}{6}\right)^2\Gamma\left(\frac{2n}{3}+\frac{11}{6}\right)\Gamma\left(\frac{n-1}{3}\right)\Gamma\left(\frac{n+2}{3}\right)^2\Gamma(3n+2)}. \qquad (164)\end{aligned}$$

Stirling's formula (332) provides the following upper bound on (164)

$$\begin{aligned}&\frac{2^{-4n}}{(2\pi)^{\frac{3}{2}}}\frac{2^{-4n}(2n)^{2n-\frac{1}{2}}\left(\frac{4n}{3}-\frac{5}{6}\right)^{\frac{4}{3}n-\frac{4}{3}}\left(\frac{4n}{3}+\frac{1}{6}\right)^{\frac{8}{3}n-\frac{2}{3}}}{\left(\frac{2n}{3}+\frac{5}{6}\right)^{\frac{4}{3}n+\frac{2}{3}}\left(\frac{2n}{3}+\frac{11}{6}\right)^{\frac{2}{3}n+\frac{4}{3}}\left(\frac{n-1}{3}\right)^{\frac{1}{3}n-\frac{5}{6}}\left(\frac{n+2}{3}\right)^{\frac{2}{3}n+\frac{1}{3}}(3n+2)^{3n+\frac{3}{2}}} \\ &\times\exp\left[7+\frac{1}{24n}+\frac{1}{16n-10}+\frac{1}{8n+1}\right],\end{aligned}$$

which is equal to

$$\begin{aligned}&\frac{3^{2+\frac{5}{6}}}{2^8\pi^{\frac{3}{2}}}\left(\frac{2}{3}\right)^{4n}\frac{1}{n^{\frac{1}{2}}\left(n-\frac{5}{8}\right)^{\frac{4}{3}}\left(n+\frac{1}{8}\right)^{\frac{2}{3}}\left(n+\frac{2}{3}\right)^{\frac{3}{2}}(n+2)^{\frac{1}{3}}\left(n+\frac{11}{4}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{2}{3}}} \\ &\times\exp\left[7+\frac{1}{24n}+\frac{1}{16n-10}+\frac{1}{8n+1}\right] \\ &\times\frac{\left(1-\frac{5}{8n}\right)^{\frac{4}{3}n}\left(1+\frac{1}{8n}\right)^{\frac{8}{3}n}}{\left(1-\frac{1}{n}\right)^{\frac{1}{3}n-\frac{5}{6}}\left(1+\frac{5}{4n}\right)^{\frac{4}{3}n}\left(1+\frac{11}{4n}\right)^{\frac{2}{3}n}\left(1+\frac{2}{n}\right)^{\frac{2}{3}n}\left(1+\frac{2}{3n}\right)^{3n}}. \qquad (165)\end{aligned}$$

We use Lemma 65 *(ii)* to bound

$$\left(1-\frac{1}{n}\right)^{\frac{1}{3}n-\frac{5}{6}} > e^{-\frac{1}{3}}$$

and $1+x<e^x$ as well as estimate (403) of Lemma 64 for the remaining factors of the numerator and denominator, respectively, of (165). Then it is

$$\frac{\left(1-\frac{5}{8n}\right)^{\frac{4}{3}n}\left(1+\frac{1}{8n}\right)^{\frac{8}{3}n}}{\left(1-\frac{1}{n}\right)^{\frac{1}{3}n-\frac{5}{6}}\left(1+\frac{5}{4n}\right)^{\frac{4}{3}n}\left(1+\frac{11}{4n}\right)^{\frac{2}{3}n}\left(1+\frac{2}{n}\right)^{\frac{2}{3}n}\left(1+\frac{2}{3n}\right)^{3n}} \leqslant \exp\left[-7+\frac{89}{16n}\right].$$

Next we simplify

$$\frac{\Gamma\left(2n+\frac{3}{2}\right)\Gamma\left(4n+\frac{3}{2}\right)}{\pi\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}=\frac{2^3}{\pi}\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)$$

and put all estimates together to arrive at

$$\begin{aligned}
&\frac{2^{-4n}\Gamma(2n)F_{4,C}\left(n,\frac{1}{3}n-\frac{4}{3}\right)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}\\
&\leqslant \frac{3^{2+\frac{5}{6}}}{2^5\pi^{\frac{5}{2}}}\left(\frac{2}{3}\right)^{4n}\frac{\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)}{n^{\frac{1}{2}}\left(n-\frac{5}{8}\right)^{\frac{4}{3}}\left(n+\frac{1}{8}\right)^{\frac{2}{3}}\left(n+\frac{2}{3}\right)^{\frac{3}{2}}(n+2)^{\frac{1}{3}}\left(n+\frac{11}{4}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{2}{3}}}\\
&\times\exp\left[\frac{1}{24n}+\frac{1}{16n-10}+\frac{1}{8n+1}+\frac{89}{16n}\right]\\
&\leqslant \frac{3^{2+\frac{5}{6}}e}{2^5\pi^{\frac{5}{2}}}\left(\frac{2}{3}\right)^{4n}\frac{\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)}{n^{\frac{1}{2}}\left(n-\frac{5}{8}\right)^{\frac{4}{3}}\left(n+\frac{1}{8}\right)^{\frac{2}{3}}\left(n+\frac{2}{3}\right)^{\frac{3}{2}}(n+2)^{\frac{1}{3}}\left(n+\frac{11}{4}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{2}{3}}}
\end{aligned}\tag{166}$$

as soon as $n\geqslant 6$. Expression (166) is smaller than (135), if

$$3^{\frac{5}{6}}\frac{n\left(n+\frac{1}{4}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)}{\left(n-\frac{5}{8}\right)^{\frac{4}{3}}\left(n+\frac{1}{8}\right)^{\frac{2}{3}}\left(n+\frac{2}{3}\right)^{\frac{3}{2}}(n+2)^{\frac{1}{3}}\left(n+\frac{11}{4}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{2}{3}}}<1.$$

We write

$$\left(n-\frac{5}{8}\right)^{\frac{4}{3}}\left(n+\frac{2}{3}\right)^{\frac{3}{2}}=\left(n-\frac{5}{8}\right)^{\frac{1}{3}}\left(n+\frac{2}{3}\right)^{\frac{1}{2}}\cdot\left(n-\frac{5}{8}\right)\left(n+\frac{2}{3}\right)$$

and note that

$$n^2-\frac{1}{16}<\left(n-\frac{5}{8}\right)\left(n+\frac{2}{3}\right)$$

for all $n\geqslant 9$. Moreover it is obviously

$$\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)<\left(n+\frac{11}{4}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{2}{3}}$$

and

$$n<\left(n+\frac{1}{8}\right)^{\frac{2}{3}}(n+2)^{\frac{1}{3}}.$$

Thus we finally get that

$$3^{\frac{5}{6}}\frac{n\left(n+\frac{1}{4}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)}{\left(n-\frac{5}{8}\right)^{\frac{4}{3}}\left(n+\frac{1}{8}\right)^{\frac{2}{3}}\left(n+\frac{2}{3}\right)^{\frac{3}{2}}(n+2)^{\frac{1}{3}}\left(n+\frac{11}{4}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{2}{3}}}<\frac{3^{\frac{5}{6}}}{\left(n-\frac{5}{8}\right)^{\frac{1}{3}}\left(n+\frac{2}{3}\right)^{\frac{1}{2}}}<1$$

for all $n\geqslant 9$.

To finish off the proof of Lemma 25, we now jump back to (119) at the beginning of the proof and plug in estimate (137) for the maximum of $\frac{2^{-4n}\Gamma(2n)}{\Gamma(2n+\frac{1}{2})\Gamma(4n+\frac{1}{2})}F_4(n,a,b,c,d)$. This way we obtain

$$\begin{aligned} R_4(n) &\leqslant \frac{2}{3}n(4n-1)(4n+1)\frac{2^{-4n}\Gamma(2n)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma\left(4n+\frac{1}{2}\right)}\max_{D_4} F_4(n,a,b,c,d) \\ &\leqslant \frac{3e}{\pi^{\frac{5}{2}}}n^{-\frac{1}{2}}\left(\frac{2}{3}\right)^{4n} \end{aligned}$$

for all $n \geqslant 17$. ∎

4.3. Estimate of the Secondary Term $S(\mathbf{n})$

The purpose of this section is the proof of Theorem 26, which provides an upper bound on the absolute value of the secondary term $S(\mathbf{n})$.

THEOREM 26. *Let $n \in \mathbb{Z}, n \geqslant 17$. Then the secondary term is bounded from above by*

$$|S(\boldsymbol{n})| \leqslant e\left(\frac{3}{\pi}\right)^{\frac{5}{2}} n^{\frac{1}{2}}\left(\frac{2}{3}\right)^{4n}.$$

Before we turn to the derivation of the theorem, let us briefly look at its statement graphically in Figure 4.7, which plots the logarithms of $|S(\mathbf{n})|$ and the claimed upper bound. As in Figures 4.1, 4.2, 4.4 and 4.6 for the components $R_1(n), R_2(n), R_3(n)$ and $R_4(n)$ of the remainder term, we see that we got the rate of the exponential decay right. But it is also apparent, that there is still quite some space to improve the estimate.

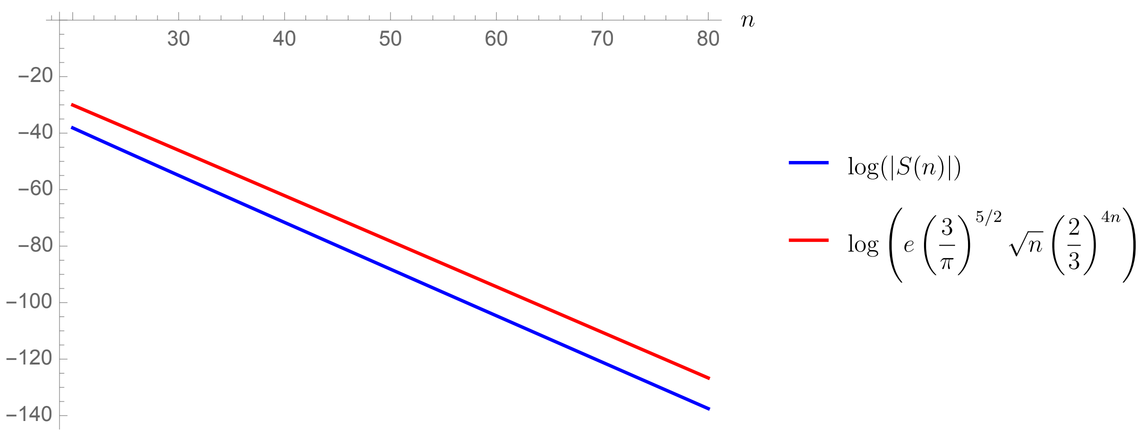

FIGURE 4.7. Logarithmic plot of $|S(\mathbf{n})|$ and its upper bound $e\left(\frac{3}{\pi}\right)^{\frac{5}{2}} n^{\frac{1}{2}}\left(\frac{2}{3}\right)^{4n}$.

To understand our pathway to the proof of Theorem 26, turn back to page 20 and recall the upper bound on $S(\mathbf{n})$ from Remark 14

$$\begin{aligned} |S(\mathbf{n})| \leqslant \frac{1}{8}\frac{2^{-12n}}{\Gamma(2n+1)^2} &\sum_{k=0}^{4n-2}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{m=0}^{j} 2^k {}_3F_2\left(\begin{matrix} 2n+\frac{1}{2}, 2n-\frac{k}{2}, 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1, 4n+1 \end{matrix} \,\middle|\, \frac{1}{4}\right) \\ &\times \frac{\Gamma(4n-k)\Gamma\left(n+k-i+\frac{1}{2}\right)\Gamma\left(n+i-j+\frac{1}{2}\right)}{\Gamma(k-i+1)\Gamma(i-j+1)\left|\Gamma\left(n-k+i+\frac{1}{2}\right)\Gamma\left(n-i+j+\frac{1}{2}\right)\right|} \\ &\times \frac{\Gamma\left(n+j-m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma(j-m+1)\Gamma(m+1)\left|\Gamma\left(n-j+m+\frac{1}{2}\right)\Gamma\left(n-m+\frac{1}{2}\right)\right|}. \end{aligned} \tag{167}$$

Note, that except for the factor $2^k{}_3F_2\left(\begin{smallmatrix} 2n+\frac{1}{2} & 2n-\frac{k}{2} & 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1 & 4n+1 \end{smallmatrix} \;\middle|\; \frac{1}{4}\right)$ inside the sum, the structure of $S(\mathbf{n})$ is very similar to that of the three terms $R_2(n), R_3(n)$ and $R_4(n)$ of the remainder term $R(\mathbf{n})$.

For that reason, we will in a first step trace back (167) to a term that resembles $R_4(n)$. This is mainly done by means of Lemma 27, that provides an upper bound on the hypergeometric function ${}_2F_3$. We are then able to identify the gamma quotient F_5, that we will have to maximize, when we eventually perform an l^1-l^∞ bound in order to estimate (167).

In a second step we modify the findings of Subsections 4.2.2 a bit, such they suit our purposes here. That way it becomes apparent that $F_5(n,a,b,c,d) = \Gamma(4n-a-b-c-d)F_4(n,a,b,c,d)$, with our well-known function $F_4(n,a,b,c,d)$ from Subsection 4.2.6. We then derive Lemma 28, that helps us to narrow down the set of values a,b,c,d, that are relevant in the search for the maximum of $F_5(n,a,b,c,d)$.

The final and most laborious part consist of locating the maximum of $F_5(n,a,b,c,d)$ in the four variables a,b,c,d more precisely and estimating it. This is subject of Lemma 29. Luckily, it can almost entirely be accomplished by means of the techniques from Subsection 4.2.3.

Let us start with the bound on the hypergeometric function in (167).

LEMMA 27. *For all $n \geqslant 1$ and $0 \leqslant k \leqslant 4n-2$ we have*

$$ {}_3F_2\left(\begin{matrix} 2n-\frac{k}{2}, 2n+\frac{1}{2}, 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1, 4n+1 \end{matrix} \;\middle|\; \frac{1}{4}\right) \leqslant \frac{1}{2} + 2^{2n-\frac{k}{2}-1}. \tag{168} $$

PROOF. We use Lemma 50 with $q=2, x=\frac{1}{4}$ and $\lambda = 2n-\frac{k}{2} > 0$. Moreover, in our case it is $\alpha_1 = 2n+\frac{1}{2}, \alpha_2 = 2n-\frac{k}{2}+\frac{1}{2}$ and $\beta_1 = 2n+1, \beta_2 = 4n+1$, and therefore

$$ e_1(\beta_1,\beta_2) = 6n+2 \geqslant 4n - \frac{k}{2} + 1 = e_1(\alpha_1,\alpha_2) > 0, $$
$$ e_2(\beta_1,\beta_2) = (4n+1)(2n+1) \geqslant \left(2n+\frac{1}{2}\right)\left(2n-\frac{k}{2}+\frac{1}{2}\right) = e_2(\alpha_1,\alpha_2) > 0 $$

for $0 \leqslant k \leqslant 4n-2$, and all requirements of Lemma 50 are fulfilled. Hence, it is by the Lemma

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} 2n-\frac{k}{2}, 2n+\frac{1}{2}, 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1, 4n+1 \end{matrix} \;\middle|\; \frac{1}{4}\right) &\leqslant 1 + \left(\frac{1}{(1-x)^\lambda} - 1\right)\frac{\alpha_1\alpha_2}{\beta_1\beta_2} \\ &= 1 + \left(\left(\frac{4}{3}\right)^{2n-\frac{k}{2}} - 1\right)\frac{\left(2n+\frac{1}{2}\right)\left(2n-\frac{k}{2}+\frac{1}{2}\right)}{(4n+1)(2n+1)}. \end{aligned} $$

We further estimate

$$ \left(\frac{4}{3}\right)^{2n-\frac{k}{2}} \leqslant 2^{2n-\frac{k}{2}} $$

and

$$ \frac{\left(2n+\frac{1}{2}\right)\left(2n-\frac{k}{2}+\frac{1}{2}\right)}{(4n+1)(2n+1)} \leqslant \frac{1}{2}, $$

for all $n \geqslant 1$ and $0 \leqslant k \leqslant 4n-2$, and thus end up with

$$ {}_3F_2\left(\begin{matrix} 2n-\frac{k}{2}, 2n+\frac{1}{2}, 2n-\frac{k}{2}+\frac{1}{2} \\ 2n+1, 4n+1 \end{matrix} \;\middle|\; \frac{1}{4}\right) \leqslant \frac{1}{2} + 2^{2n-\frac{k}{2}-1}. $$

∎

We proceed by plugging in estimate (168) into (167). Then

$$
\begin{aligned}
|S(\mathbf{n})| \leqslant & \frac{1}{8} \frac{2^{-12n}}{\Gamma(2n+1)^2} \sum_{k=0}^{4n-2} \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{m=0}^{j} \left(2^{k-1} + 2^{2n+\frac{k}{2}-1}\right) \\
& \times \frac{\Gamma(4n-k)\Gamma\left(n+k-i+\frac{1}{2}\right)\Gamma\left(n+i-j+\frac{1}{2}\right)}{\Gamma(k-i+1)\Gamma(i-j+1)\left|\Gamma\left(n-k+i+\frac{1}{2}\right)\Gamma\left(n-i+j+\frac{1}{2}\right)\right|} \\
& \times \frac{\Gamma\left(n+j-m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma(j-m+1)\Gamma(m+1)\left|\Gamma\left(n-j+m+\frac{1}{2}\right)\Gamma\left(n-m+\frac{1}{2}\right)\right|}.
\end{aligned}
$$

As $2^{k-1} + 2^{2n+\frac{k}{2}-1} \leqslant 3 \cdot 2^{4n-3}$ for $0 \leqslant k \leqslant 4n-2$, we further estimate

$$
\begin{aligned}
|S(\mathbf{n})| \leqslant & \, 3 \frac{2^{-8n}}{\Gamma(2n+1)^2} \\
& \times \sum_{k=0}^{4n-2} \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{m=0}^{j} \frac{\Gamma(4n-k)\Gamma\left(n+k-i+\frac{1}{2}\right)\Gamma\left(n+i-j+\frac{1}{2}\right)}{\Gamma(k-i+1)\Gamma(i-j+1)\left|\Gamma\left(n-k+i+\frac{1}{2}\right)\Gamma\left(n-i+j+\frac{1}{2}\right)\right|} \\
& \times \frac{\Gamma\left(n+j-m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma(j-m+1)\Gamma(m+1)\left|\Gamma\left(n-j+m+\frac{1}{2}\right)\Gamma\left(n-m+\frac{1}{2}\right)\right|}.
\end{aligned}
\tag{169}
$$

Now, we slowly see where the journey is going. Like in the case of the reminder term, the tactic is to bound $|S(\mathbf{n})|$ by an l^1-l^∞ bound. To this end, we have a closer look at the function

$$
\begin{aligned}
F_5(n,m,j,i,k) = & \frac{\Gamma\left(n+m+\frac{1}{2}\right)\Gamma\left(n+j-m+\frac{1}{2}\right)\Gamma\left(n+i-j+\frac{1}{2}\right)\Gamma\left(n+k-i+\frac{1}{2}\right)}{\left|\Gamma\left(n-m+\frac{1}{2}\right)\Gamma\left(n-j+m+\frac{1}{2}\right)\Gamma\left(n-i+j+\frac{1}{2}\right)\Gamma\left(n-k+i+\frac{1}{2}\right)\right|} \\
& \times \frac{\Gamma(4n-k)}{\Gamma(m+1)\Gamma(j-m+1)\Gamma(i-j+1)\Gamma(k-i+1)}
\end{aligned}
\tag{170}
$$

on the domain

$$
\mathcal{D}_5 = \{m,j,i,k \in \mathbb{N}_0 \mid 0 \leqslant m \leqslant j \leqslant i \leqslant k \leqslant 4n-2\}.
$$

Upon a slight variation $\tilde{\tau} : \mathbb{Z}^m \to \mathbb{Z}^m, \mathbf{k} \mapsto \mathbf{z}$ with

$$
z_i = \tilde{\tau}(\mathbf{k})_i = \begin{cases} k_1, & i = 1, \\ k_i - k_{i-1}, & i = 2, \ldots, m, \end{cases}
\tag{171}
$$

of the transformation τ on page 28, we obtain for $\mathbf{k} = (m,j,i,k) \in \mathbb{Z}^4$ and $\tilde{\tau}(\mathbf{k}) = (a,b,c,d) \in \mathbb{Z}^4$

$$
\begin{aligned}
F_5(n, \tilde{\tau}(m,j,i,k)) &= F_5(n,a,b,c,d) \\
&= \frac{\Gamma\left(n+a+\frac{1}{2}\right)\Gamma\left(n+b+\frac{1}{2}\right)}{\Gamma(a+1)\Gamma(b+1)\left|\Gamma\left(n-a+\frac{1}{2}\right)\Gamma\left(n-b+\frac{1}{2}\right)\right|} \\
&\quad \times \frac{\Gamma\left(n+c+\frac{1}{2}\right)\Gamma\left(n+d+\frac{1}{2}\right)\Gamma(4n-a-b-c-d)}{\Gamma(c+1)\Gamma(d+1)\left|\Gamma\left(n-c+\frac{1}{2}\right)\Gamma\left(n-d+\frac{1}{2}\right)\right|} \\
&= F_4(n,a,b,c,d)\Gamma(4n-a-b-c-d),
\end{aligned}
$$

and

$$
\tilde{\tau}(\mathcal{D}_5) = \bigcup_{k=0}^{4n-2} \left\{ a,b,c,d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant a,b,c,d \leqslant k \\ a+b+c+d = k \end{matrix} \right\}.
$$

Now, we reached a situation very similar to that in the derivation of Lemma 17 and can argue that it suffices to maximize $F_5(n,a,b,c,d)$ on the set

$$D_5 = \bigcup_{k=0}^{4n-2} \left\{ a,b,c,d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant k \\ a+b+c+d = k \end{matrix} \right\},$$

since the function $F_5(n,a,b,c,d)$ itself, as well as each layer of its domain $\tilde{\tau}(\mathcal{D}_5)$ are symmetric with respect to permutations of the variables a,b,c,d. This proves

LEMMA 28. *Let $\tilde{\tau}$ be the transformation defined in* (171) *and* $\tilde{\tau}(m,j,i,k) = (a,b,c,d)$. *Then it is*

$$\begin{aligned} \max_{\mathcal{D}_5} F_5(n,m,j,i,k) &= \max_{\tilde{\tau}(\mathcal{D}_5)} F_5(n,a,b,c,d) \\ &= \max_{D_5} F_5(n,a,b,c,d), \end{aligned}$$

where

$$\mathcal{D}_5 = \{m,j,i,k \in \mathbb{N}_0 \mid 0 \leqslant m \leqslant j \leqslant i \leqslant k \leqslant 4n-2\},$$

and

$$D_5 = \bigcup_{k=0}^{4n-2} \left\{ a,b,c,d \in \mathbb{Z} \,\middle|\, \begin{matrix} 0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant k \\ a+b+c+d = k \end{matrix} \right\}.$$

The location of the maximum of $F_5(n,a,b,c,d)$ on D_5 and an upper bound on it is given by the following lemma.

LEMMA 29. *Let $n \in \mathbb{Z}, n \geqslant 17$. Then the maximum of the function $F_5(n,a,b,c,d)$ on the set D_5 is attained for $a=b=c=d=\left\lceil \frac{1}{3}n - \frac{7}{6} \right\rceil$ and satisfies*

$$\max_{D_5} F_5(n,a,b,c,d) \leqslant \Gamma(2n+1)^2 \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} \frac{e}{n^{\frac{1}{2}}\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)} \left(\frac{2^{12}}{3^4}\right)^n.$$

Since the proof of Lemma 29 is quite lengthy and technical, we first prove the main result of this section.

PROOF OF THEOREM 26. Starting from (169), we estimate $|S(\mathbf{n})|$ by an l^1-l^∞ bound

$$\begin{aligned} |S(\mathbf{n})| &\leqslant 3\frac{2^{-8n}}{\Gamma(2n+1)^2} \sum_{k=0}^{4n-2} \sum_{i=0}^{k} \sum_{j=0}^{i} \sum_{m=0}^{j} F_5(n,m,j,i,k) \\ &\leqslant \frac{2^{-8n}}{\Gamma(2n+1)^2} n(4n-1)(4n+1)(2n+1) \max_{\mathcal{D}_5} F_5(n,m,j,i,k). \end{aligned}$$

Next, we apply Lemmata 28 and 29 and obtain for the second line above the upper bound

$$\frac{2^{-8n}}{\Gamma(2n+1)^2} n(4n-1)(4n+1)(2n+1) \max_{D_5} F_5(n,a,b,c,d) \leqslant e\left(\frac{3}{\pi}\right)^{\frac{5}{2}} n^{\frac{1}{2}} \left(\frac{2}{3}\right)^{4n},$$

if $n \geqslant 17$. ∎

The entire remaining pages of this section are dedicated to the proof of Lemma 29. The procedure remains largely the same as in the case of $F_4(n,a,b,c,d)$ in Subsection 4.2.6, namely using Theorem 22 to identify the (in this case four) possible tuples that might maximize F_5, determine their coordinates more accurately with the help of Theorem 20 and then finally estimate the value of the maximum. But since the structure of the two functions $F_5(n,a,b,c,d)$ and $F_4(n,a,b,c,d)$ is so similar, we can trace back several parts in the analysis of $F_5(n,a,b,c,d)$ to corresponding calculations in the study of $F_4(n,a,b,c,d)$, and thus benefit a lot from the hard work we did back in Subsection 4.2.6.

Before we jump into the proof of Lemma 29, we want to emphasize the following important fact.

Remark 30. Since the sum $a+b+c+d=k$ is constant, our method of proving Theorem 22 is not affected by the additional factor $\Gamma(4n-k)$. We thus can still apply it here.

Proof of Lemma 29. Due to the requirements

$$0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant k, \tag{172}$$

$$0 \leqslant a+b+c+d = k \leqslant 4n-2, \tag{173}$$

on the parameters, the smallest parameter a must satisfy

$$a \leqslant n-1. \tag{174}$$

Moreover, since $k \leqslant 4n-2$ is not fixed like in the case of the remainder terms, we now have the additional degree of freedom that d might be smaller or equal to n in the maximum. Let's for the moment assume the opposite, meaning that $d \geqslant n$ and $0 < a \leqslant b \leqslant c \leqslant n$. Then Theorem 22 *(iii)* applies and provides a condition on c and d upon which $F_5(n,a,b,c,d) \geqslant F_5(n,a,b,c-1,d+1)$. This condition is equivalent to

$$\varphi(c,d) = \frac{F_5(n,a,b,c,d)}{F_5(n,a,b,c-1,d+1)} \geqslant 1.$$

We have seen in (106) of Section 4.2.5 that this condition is already satisfied if

$$cd - n^2 + \frac{1}{4} \leqslant 0.$$

Hence for any $d \geqslant n$ there is an admissible $c \leqslant n$ such that Theorem 22 *(iii)* is satisfied and $F_5(n,a,b,c,d) \geqslant F_5(n,a,b,c-1,d+1)$. Consequently, we can push d down to n and infer that $F_5(n,a,b,c,d)$ is maximized by tuples $a \leqslant b \leqslant c \leqslant d \leqslant n$.
We therefore can omit the absolute values in the definition of $F_5(n,a,b,c,d)$ and consider from now on

$$\begin{aligned} F_5(n,a,b,c,d) &= \frac{\Gamma\left(n+a+\frac{1}{2}\right)\Gamma\left(n+b+\frac{1}{2}\right)}{\Gamma(a+1)\Gamma(b+1)\Gamma\left(n-a+\frac{1}{2}\right)\Gamma\left(n-b+\frac{1}{2}\right)} \\ &\quad \times \frac{\Gamma\left(n+c+\frac{1}{2}\right)\Gamma\left(n+d+\frac{1}{2}\right)\Gamma(4n-a-b-c-d)}{\Gamma(c+1)\Gamma(d+1)\Gamma\left(n-c+\frac{1}{2}\right)\Gamma\left(n-d+\frac{1}{2}\right)}. \end{aligned} \tag{175}$$

Now we are in a familiar setup and deduce from Theorem 22 *(i)* the following four options for the maximizing tuple (a,b,c,d).

A: $F_5(n,a,b,c,d)$ is maximized by

$$a=b=c=d \leqslant n-1,$$
$$4a \leqslant 4n-2.$$

B: $F_5(n,a,b,c,d)$ is maximized by

$$a=b=c \leqslant n-1,$$
$$d=a+1 \leqslant n,$$
$$4a+1 \leqslant 4n-2.$$

C: $F_5(n,a,b,c,d)$ is maximized by

$$a=b \leqslant n-1,$$
$$c=d=a+1 \leqslant n,$$
$$4a+2 \leqslant 4n-2.$$

D: $F_5(n,a,b,c,d)$ is maximized by

$$\begin{aligned} &a \leqslant n-2,\\ &b=c=d=a+1 \leqslant n-1,\\ &4a+3 \leqslant 4n-2. \end{aligned}$$

In what follows we first locate and estimate the local maximum of $F_5(n,a,b,c,d)$ in **Case A**. Afterwards we show that the maxima in the other three cases all have to be smaller than the one in A.

We start with **Case A**. Here we have to extremize

$$F_{5,A}(n,a) := \frac{\Gamma\left(n+a+\frac{1}{2}\right)^4 \Gamma\left(4n-4a\right)}{\Gamma\left(a+1\right)^4 \Gamma\left(n-a+\frac{1}{2}\right)^4} \tag{176}$$

in $0 \leqslant a \leqslant n-1$. The result is the following.

Lemma 31. *The function $F_{5,A}(n,a)$ attains its maximum for $a = a^* := \left\lceil \frac{1}{3}n - \frac{7}{6} \right\rceil$.*

Proof. We stick to our approach from Subsections 4.2.4, 4.2.5 and 4.2.6 and plan to apply Theorem 20. Therefore, we determine the recurrence relation for $F_{5,A}(n,a)$ in a. It is $F_{5,A}(n,a) = \varphi_{5,A}(a) F_{5,A}(n,a-1)$ with

$$\varphi_{5,A}(a) = \frac{\left(n+a-\frac{1}{2}\right)^4 \left(n-a+\frac{1}{2}\right)^4}{a^4(4n-4a)(4n-4a+1)(4n-4a+2)(4n-4a+3)}$$

for $1 \leqslant a \leqslant n-1$.

The task now is to localize the point ζ such that $\varphi_{5,A}(\zeta) = 1, \varphi_{5,A}(a) > 1$ for $a < \zeta$ and $\varphi_{5,A}(a) < 1$ for $a > \zeta$. Then $F_{5,A}(n,a)$ has a maximum in a between $\zeta - 1$ and ζ according to Theorem 20 *(iii)*. To solve this task, we proceed very similar to the estimate of the maximum of $F_{4,A}(n,a)$ on page 53. That is, we prove a strict lower bound $\underline{\varphi}_{5,A}$ and a strict upper bound $\overline{\varphi}_{5,A}$ on $\varphi_{5,A}$, and then determine the two values $\underline{a}$ and $\overline{a}$, such that

$$\begin{aligned} \underline{\varphi}_{5,A}(a) &\geqslant 1,\\ \overline{\varphi}_{5,A}(a) &\leqslant 1 \end{aligned}$$

for $a \leqslant \underline{a}$, and $a \geqslant \overline{a}$, respectively. Due to the continuity of $\varphi_{5,A}$ on $[1, n-1]$, we thus can infer that $\underline{a} < \zeta < \overline{a}$.

We use that $(x+1)(x+3) = x^2+4x+3 < (x+2)^2$, and therefore $x^4 < x(x+1)(x+2)(x+3) < (x+2)^2$. For $x = 4n-4a$ we then obtain

$$\varphi_{5,A}(a) \begin{cases} > \underline{\varphi}_{5,A}(a) & = \left[\frac{n+a-\frac{1}{2}}{4a}\right]^4, \\ < \overline{\varphi}_{5,A}(a) & = \left[\frac{\left(n+a-\frac{1}{2}\right)\left(n-a+\frac{1}{2}\right)}{4a(n-a)}\right]^4. \end{cases}$$

The function $\underline{\varphi}_{5,A}(a)$ is greater than or equal to one if $n+a-\frac{1}{2} \geqslant 4a$, and hence if

$$a \leqslant \frac{1}{3}n - \frac{1}{6} =: \underline{a}.$$

The upper bound $\overline{\varphi}_{5,A}(a)$ on $\varphi_{5,A}(a)$ on the other hand is smaller than or equal to one if

$$n^2 - \frac{1}{4} - a^2 + a \leqslant -4a^2 + 4an.$$

After subtracting of the right-hand side from the left, we find that the above inequality is equivalent to

$$\left(a-n+\frac{1}{2}\right)\left(a-\frac{1}{3}n-\frac{1}{6}\right)\leqslant 0.$$

This means that $\overline{\varphi}_{5,A}(a)\leqslant 1$ if

$$\frac{1}{3}n+\frac{1}{6}\leqslant a\leqslant n-\frac{1}{2}.$$

With $\overline{a}:=\frac{1}{3}n+\frac{1}{6}$, we thus obtain

$$\frac{1}{3}n-\frac{1}{6}<\zeta<\frac{1}{3}n+\frac{1}{6}.$$

By Theorem 20 *(iii)* and Remark 21 we now know that $F_{5,A}(n,a)$ attains its maximum at $a^*\in\mathbb{N}$ with $\zeta-1<a^*\leqslant\zeta$. With the above bounds on ζ this yields the range

$$\frac{1}{3}n-\frac{7}{6}<a^*<\frac{1}{3}n+\frac{1}{6}.$$

Since we are restricted to integers, we take a closer look at the interval

$$\left(\frac{1}{3}n-\frac{7}{6},\frac{1}{3}n+\frac{1}{6}\right)$$

for the three possibilities of divisibility of n modulo 3, and find that there are two candidates for a^*, namely

$$a_1^*:=\left\lceil\frac{1}{3}n-\frac{7}{6}\right\rceil,$$
$$a_2^*:=\left\lfloor\frac{1}{3}n+\frac{1}{6}\right\rfloor.$$

A simple calculation shows that for $n\equiv 1\mod 3$ and $n\equiv 2\mod 3$ the two options coincide, $a_1^*(n)=a_2^*(n)$. But, if n is an integer multiple of 3, then $a_1^*=\frac{1}{3}n-1$ and $a_2^*=\frac{1}{3}n$. In the following we show that in this case $F_{5,A}\left(n,\frac{1}{3}n-1\right)\geqslant F_{5,A}\left(n,\frac{1}{3}n\right)$ for $n\geqslant 3$.

Using the functional equation of the gamma function, we find that

$$\begin{aligned}\frac{F_{5,A}\left(\frac{1}{3}n-1\right)}{F_{5,A}\left(\frac{1}{3}n\right)}&=\frac{\Gamma\left(\frac{4}{3}n-\frac{1}{2}\right)^4\Gamma\left(\frac{2}{3}n+\frac{1}{2}\right)^4\Gamma\left(\frac{1}{3}n+1\right)^4\Gamma\left(\frac{8}{3}n+4\right)}{\Gamma\left(\frac{4}{3}n+\frac{1}{2}\right)^4\Gamma\left(\frac{2}{3}n+\frac{3}{2}\right)^4\Gamma\left(\frac{1}{3}n\right)^4\Gamma\left(\frac{8}{3}n\right)}\\&=\frac{n^5\left(n+\frac{3}{8}\right)\left(n+\frac{9}{8}\right)}{\left(n-\frac{3}{8}\right)^4\left(n+\frac{3}{4}\right)^3}.\end{aligned}\tag{177}$$

We set $\frac{1}{n}=x$ and show that $q(x)=\frac{\left(1+\frac{3}{8}x\right)\left(1+\frac{9}{8}x\right)}{\left(1-\frac{3}{8}x\right)^4\left(1+\frac{3}{4}x\right)^3}$ is greater than or equal to one for all $0\leqslant x\leqslant\frac{1}{3}$. To this end, we use the elementary inequalities

$$\frac{1+y}{1-y}\geqslant e^{2y},$$
$$1+y\leqslant e^y,$$

which are in particular valid for $0 \leqslant y \leqslant 1$, and obtain

$$
\begin{aligned}
q(x) &\geqslant \frac{e^{\frac{3}{4}x}}{1+\frac{3}{4}x} \frac{1+\frac{9}{8}x}{\left(1-\frac{3}{8}x\right)^3\left(1+\frac{3}{4}x\right)^2} \\
&\geqslant \frac{1+\frac{9}{8}x}{\left(1-\frac{3}{8}x\right)^3\left(1+\frac{3}{4}x\right)^2} \\
&\geqslant \frac{1}{\left(1-\frac{3}{8}x\right)^3\left(1+\frac{3}{4}x\right)}.
\end{aligned}
$$

Since the last fraction takes the value 1 for $x=0$, we are done after showing that $\left(1-\frac{3}{8}x\right)^3\left(1+\frac{3}{4}x\right)$ decreases monotonically for $0 \leqslant x \leqslant \frac{1}{3}$. Taking the derivative we get

$$
\frac{d}{dx}\left[\left(1-\frac{3}{8}x\right)^3\left(1+\frac{3}{4}x\right)\right] = -\frac{3}{8}\left(1-\frac{3}{8}x\right)^2(1+3x),
$$

which is negative for $0 \leqslant x \leqslant \frac{1}{3}$.

Consequently, we end up with

$$
a^* = \left\lceil \frac{1}{3}n - \frac{7}{6} \right\rceil.
$$

■

We now continue with the estimate of $\frac{F_{5,A}(n,a^*)}{\Gamma(2n+1)^2}$ and prove

LEMMA 32. *If $n \geqslant 8$, then*

$$
\frac{F_{5,A}(n,a^*)}{\Gamma(2n+1)^2} \leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} \frac{e}{n^{\frac{1}{2}}\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)} \left(\frac{2^{12}}{3^4}\right)^n.
$$

PROOF. Depending on the divisibility of n modulo 3, there are only three possible values a^* can take. Similar to the approach for the maximum of $F_{4,A.2}(n,a)$ on page 55, we consider these cases separately and establish a common upper bound in a second step. Those cases are

$$
a^* = \begin{cases} \frac{1}{3}n-1, & n \equiv 0 \mod 3, \\ \frac{1}{3}n-\frac{1}{3}, & n \equiv 1 \mod 3, \\ \frac{1}{3}n-\frac{2}{3}, & n \equiv 2 \mod 3. \end{cases} \tag{178}
$$

Let us start with $\underline{n \equiv 0 \mod 3}$. Plugging in $a^* = \frac{1}{3}n-1$ into (176) gives

$$
\begin{aligned}
\frac{F_{5,A}\left(n,\frac{1}{3}n-1\right)}{\Gamma(2n+1)^2} &= \frac{\Gamma\left(\frac{8}{3}n+4\right)}{\Gamma(2n+1)^2} \frac{\Gamma\left(\frac{4}{3}n-\frac{1}{2}\right)^4}{\Gamma\left(\frac{1}{3}n\right)^4\Gamma\left(\frac{2}{3}n+\frac{3}{2}\right)^4} \\
&= \frac{2^{10}\left(n+\frac{9}{8}\right)\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)}{3^4 n} \frac{\Gamma\left(\frac{8}{3}n\right)}{\Gamma(2n)} \frac{\Gamma\left(\frac{4}{3}n-\frac{1}{2}\right)^4}{\Gamma\left(\frac{1}{3}n\right)^4\Gamma\left(\frac{2}{3}n+\frac{3}{2}\right)^4}.
\end{aligned} \tag{179}
$$

By Stirling's formula (332) it is

$$
\frac{\Gamma\left(\frac{8}{3}n\right)}{\Gamma(2n)} \leqslant \frac{3^{\frac{1}{2}}e^{\frac{4}{3}n+\frac{1}{32n}}}{2\pi^{\frac{1}{2}}} n^{-\frac{4}{3}n} n^{\frac{1}{2}} 2^{4n} 3^{-\frac{8}{3n}}, \tag{180}
$$

and

$$
\frac{\Gamma\left(\frac{4}{3}n-\frac{1}{2}\right)^4}{\Gamma\left(\frac{1}{3}n\right)^4\Gamma\left(\frac{2}{3}n+\frac{3}{2}\right)^4} \leqslant \frac{3^6 e^{-\frac{4}{3}n+8+\frac{1}{4n-\frac{3}{2}}}}{2^{14}\pi^2} n^{\frac{4}{3}n} \frac{n^2}{\left(n-\frac{3}{8}\right)^4\left(n+\frac{9}{4}\right)^4} \frac{2^{8n}}{3^{\frac{4}{3}n}} \frac{\left(1-\frac{3}{8n}\right)^{\frac{16}{3}n}}{\left(1+\frac{9}{4n}\right)^{\frac{8}{3}n}}.
$$

Due to $1-x \leqslant e^{-x}$ and Lemma 64, estimate (403), we can bound

$$\frac{\left(1-\frac{3}{8n}\right)^{\frac{16}{3}n}}{\left(1+\frac{9}{4n}\right)^{\frac{8}{3}n}} \leqslant e^{-8+\frac{27}{4n}},$$

and thus get

$$\frac{\Gamma\left(\frac{4}{3}n-\frac{1}{2}\right)^4}{\Gamma\left(\frac{1}{3}n\right)^4\Gamma\left(\frac{2}{3}n+\frac{3}{2}\right)^4} \leqslant \frac{3^6 e^{-\frac{4}{3}n+\frac{27}{4n}+\frac{1}{4n-\frac{3}{2}}}}{2^{14}\pi^2} n^{\frac{4}{3}n} \frac{n^2}{\left(n-\frac{3}{8}\right)^4\left(n+\frac{9}{4}\right)^4} \frac{2^{8n}}{3^{\frac{4}{3}n}}. \tag{181}$$

Putting together (179), (180) and (181) we end up with

$$\frac{F_{5,A}\left(n,\frac{1}{3}n-1\right)}{\Gamma(2n+1)^2} \leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} e^{\frac{217}{32n}+\frac{1}{4n-\frac{3}{2}}} \frac{n^{\frac{3}{2}}\left(n+\frac{9}{8}\right)\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)}{\left(n-\frac{3}{8}\right)^4\left(n+\frac{9}{4}\right)^4} \left(\frac{2^{12}}{3^4}\right)^n. \tag{182}$$

Next, we repeat the above procedure in the case of $\underline{n \equiv 1 \mod 3}$. We plug in $a^* = \frac{1}{3}n-\frac{1}{3}$ into (176) and make heavy use of Stirling's formula (332) to get

$$\begin{aligned}
\frac{F_{5,A}\left(\frac{1}{3}n-\frac{1}{3}\right)}{\Gamma(2n+1)^2} &= \frac{\Gamma\left(\frac{4}{3}n+\frac{1}{6}\right)^4\Gamma\left(\frac{8}{3}n+\frac{4}{3}\right)}{\Gamma(2n+1)^2\Gamma\left(\frac{1}{3}n+\frac{2}{3}\right)^4\Gamma\left(\frac{2}{3}n+\frac{5}{6}\right)^4} \\
&\leqslant \frac{e^{4n}n^{-4n}}{4\pi n} 2^{-4n} \frac{\Gamma\left(\frac{4}{3}n+\frac{1}{6}\right)^4\Gamma\left(\frac{8}{3}n+\frac{4}{3}\right)}{\Gamma\left(\frac{1}{3}n+\frac{2}{3}\right)^4\Gamma\left(\frac{2}{3}n+\frac{5}{6}\right)^4} \\
&\leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} e^{4+\frac{9}{32n}} \frac{\left(n+\frac{1}{2}\right)^{\frac{5}{6}}}{n\left(n+2\right)^{\frac{2}{3}}\left(n+\frac{1}{8}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{4}{3}}} \left(\frac{2^{12}}{3^4}\right)^n \\
&\quad\times \frac{\left(1+\frac{1}{2n}\right)^{\frac{8}{3}n}\left(1+\frac{1}{8n}\right)^{\frac{16}{3}n}}{\left(1+\frac{2}{n}\right)^{\frac{4}{3}n}\left(1+\frac{5}{4n}\right)^{\frac{8}{3}n}}.
\end{aligned} \tag{183}$$

The first inequality is due to

$$\begin{aligned}
\Gamma(2n+1)^2 &= 4n^2\Gamma(2n)^2 \\
&\geqslant (4\pi n)n^{4n}2^{4n}e^{-4n}
\end{aligned} \tag{184}$$

by Stirling's Formula for all $n \geqslant 1$. To get the second inequality, Stirling is applied to the gamma quotient in the second line. With the help of estimate (403) from Lemma 64, and its first order counterpart $1+x \leqslant e^x$, we further estimate

$$\begin{aligned}
\frac{\left(1+\frac{1}{2n}\right)^{\frac{8}{3}n}\left(1+\frac{1}{8n}\right)^{\frac{16}{3}n}}{\left(1+\frac{2}{n}\right)^{\frac{4}{3}n}\left(1+\frac{5}{4n}\right)^{\frac{8}{3}n}} &\leqslant \exp\left[\frac{4}{3}+\frac{2}{3}-\frac{8}{3}-\frac{10}{3}+\frac{2}{3}\cdot\frac{4}{n}+\frac{4}{3}\cdot\frac{25}{16n}\right] \\
&= e^{-4+\frac{19}{4n}}.
\end{aligned} \tag{185}$$

Putting together (183) and (185) we arrive at

$$\frac{F_{5,A}\left(n,\frac{1}{3}n-\frac{1}{3}\right)}{\Gamma(2n+1)^2} \leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} e^{\frac{85}{16n}} \frac{\left(n+\frac{1}{2}\right)^{\frac{5}{6}}}{n\left(n+2\right)^{\frac{2}{3}}\left(n+\frac{1}{8}\right)^{\frac{4}{3}}\left(n+\frac{5}{4}\right)^{\frac{4}{3}}} \left(\frac{2^{12}}{3^4}\right)^n. \tag{186}$$

If $\underline{n \equiv 2 \mod 3}$, very similar calculations lead to

$$\begin{aligned}\frac{F_{5,A}\left(n, \frac{1}{3}n - \frac{2}{3}\right)}{\Gamma(2n+1)^2} &= \frac{\Gamma\left(\frac{4}{3}n - \frac{1}{6}\right)^4 \Gamma\left(\frac{8}{3}n + \frac{8}{3}\right)}{\Gamma(2n+1)^2 \Gamma\left(\frac{1}{3} + \frac{1}{3}\right)^4 \Gamma\left(\frac{2}{3}n + \frac{7}{6}\right)^4} \\ &\leqslant \frac{e^{4n} n^{-4n}}{4\pi n} 2^{-4n} \frac{\Gamma\left(\frac{4}{3}n - \frac{1}{6}\right)^4 \Gamma\left(\frac{8}{3}n + \frac{8}{3}\right)}{\Gamma\left(\frac{1}{3} + \frac{1}{3}\right)^4 \Gamma\left(\frac{2}{3}n + \frac{7}{6}\right)^4} \\ &\leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} e^{4 + \frac{1}{32n} + \frac{1}{4n - \frac{1}{2}}} \frac{(n+1)^{\frac{17}{6}}}{n\left(n - \frac{1}{8}\right)^{\frac{8}{3}} \left(n + \frac{7}{4}\right)^{\frac{8}{3}}} \left(\frac{2^{12}}{3^4}\right)^n \\ &\quad \times \frac{\left(1 + \frac{1}{n}\right)^{\frac{8}{3}n} \left(1 - \frac{1}{8n}\right)^{\frac{16}{3}n}}{\left(1 + \frac{1}{n}\right)^{\frac{4}{3}n} \left(1 + \frac{7}{4n}\right)^{\frac{8}{3}n}},\end{aligned} \tag{187}$$

by (184) in the first step and Stirling applied to the remaining gamma quotient in the second. Next, estimate (403) from Lemma 64, and $1 + x \leqslant e^x$ yield

$$\frac{\left(1 + \frac{1}{n}\right)^{\frac{8}{3}n} \left(1 - \frac{1}{8n}\right)^{\frac{16}{3}n}}{\left(1 + \frac{1}{n}\right)^{\frac{4}{3}n} \left(1 + \frac{7}{4n}\right)^{\frac{8}{3}n}} \leqslant e^{-4 + \frac{19}{4n}}. \tag{188}$$

Consequently, we estimate

$$\frac{F_{5,A}\left(n, \frac{1}{3}n - \frac{2}{3}\right)}{\Gamma(2n+1)^2} \leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} e^{\frac{153}{32n} + \frac{1}{4n - \frac{1}{2}}} \frac{(n+1)^{\frac{17}{6}}}{n\left(n - \frac{1}{8}\right)^{\frac{8}{3}} \left(n + \frac{7}{4}\right)^{\frac{8}{3}}} \left(\frac{2^{12}}{3^4}\right)^n. \tag{189}$$

The following lines serve the purpose of comparing the three estimates (181), (186), and (189). We show that all of them satisfy the same upper bound

$$\left(\frac{3}{4\pi}\right)^{\frac{5}{2}} \frac{e}{n^{\frac{1}{2}} \left(n - \frac{1}{4}\right) \left(n + \frac{1}{4}\right) \left(n + \frac{1}{2}\right)} \left(\frac{2^{12}}{3^4}\right)^n \tag{190}$$

if $n \geqslant 8$.

As a first step towards (190) we compare the exponents of e in (181), (186), and (189), respectively and find that

$$\frac{153}{32n} + \frac{1}{4n - \frac{1}{2}} < \frac{85}{16n} < \frac{217}{32n} + \frac{1}{4n - \frac{3}{2}} < 1, \tag{191}$$

as soon as $n \geqslant 8$. Hence, in order to establish the upper bound (190) for (181), (186), and (189), respectively, it suffices to show that the rational factor of all of the three expressions is at most $\left(n^{\frac{1}{2}} \left(n - \frac{1}{4}\right) \left(n + \frac{1}{4}\right) \left(n + \frac{1}{2}\right)\right)^{-1}$. To this end, we consider the quotients

$$q_{n \equiv 0}(n) := \frac{n^2 \left(n - \frac{1}{4}\right) \left(n + \frac{1}{4}\right) \left(n + \frac{3}{8}\right) \left(n + \frac{1}{2}\right) \left(n + \frac{3}{4}\right) \left(n + \frac{9}{8}\right)}{\left(n - \frac{3}{8}\right)^4 \left(n + \frac{9}{4}\right)^4}, \tag{192}$$

$$q_{n \equiv 1}(n) := \frac{\left(n + \frac{1}{2}\right)^{\frac{11}{6}} \left(n - \frac{1}{4}\right) \left(n + \frac{1}{4}\right)}{n^{\frac{1}{2}} (n+2)^{\frac{2}{3}} \left(n + \frac{1}{8}\right)^{\frac{4}{3}} \left(n + \frac{5}{4}\right)^{\frac{4}{3}}}, \tag{193}$$

$$q_{n \equiv 2}(n) := \frac{(n+1)^{\frac{17}{6}} \left(n - \frac{1}{4}\right) \left(n + \frac{1}{4}\right) \left(n + \frac{1}{2}\right)}{n^{\frac{1}{2}} \left(n - \frac{1}{8}\right)^{\frac{8}{3}} \left(n + \frac{7}{4}\right)^{\frac{8}{3}}}, \tag{194}$$

where $q_{\equiv i}(n)$ is constructed by dividing the rational factor corresponding to the estimate for $n \equiv i$ by $\left(n^{\frac{1}{2}} \left(n - \frac{1}{4}\right) \left(n + \frac{1}{4}\right) \left(n + \frac{1}{2}\right)\right)^{-1}$. By means of Lemma 64 we show that each of them is smaller than 1 for $n \geqslant 8$. In all three cases, we apply inequality (403) in the denominator and inequality (404) with $\delta = \frac{1}{3}$ in the numerator.

Let us consider $q_{n\equiv 0}$ first. We first shift all linear factors in (192) by the largest negative constant $\frac{3}{8}$ and set $n = m + \frac{3}{8}$. Next we take the reciprocal $\frac{1}{m} = x$. Then it is

$$q_{n\equiv 0}(x) = \frac{\left(1+\frac{3}{8}x\right)^2\left(1+\frac{1}{8}x\right)\left(1+\frac{5}{8}x\right)\left(1+\frac{3}{4}x\right)\left(1+\frac{7}{8}x\right)\left(1+\frac{9}{8}x\right)\left(1+\frac{3}{2}x\right)}{\left(1+\frac{21}{8}x\right)^4}.$$

We are allowed to apply estimate (404) to all linear factors in the numerator if the one with the largest slope, that is $\frac{3}{2}$, satisfies the requirements of inequality (404). This leads to

$$x \leqslant \frac{3-\sqrt{6}}{3/2}.$$

For those x we estimate

$$\begin{aligned} q_{n\equiv 0}(x) &\leqslant \exp\left[-\frac{19}{4}x - \frac{1}{3}\cdot\frac{177}{32}x^2 + 2\cdot\frac{441}{64}x^2\right] \\ &= \exp\left[-\frac{19}{4}x + \frac{191}{16}x^2\right], \end{aligned}$$

which is smaller than one, if $x < \frac{76}{191}$. Since $\frac{3-\sqrt{6}}{3/2} < \frac{76}{191}$, the condition $x \leqslant \frac{3-\sqrt{6}}{3/2}$ translates to $n \geqslant \frac{3}{2(3-\sqrt{6})} + \frac{3}{8}$, and we conclude that $q_{n\equiv 0}(n) < 1$ for the claimed range of n.

By repeating the above scheme for $q_{n\equiv 1}(n)$ with $x = \frac{1}{n-\frac{1}{4}}$, we find that

$$\begin{aligned} q_{n\equiv 1}(x) &= \frac{\left(1+\frac{3}{4}x\right)^{\frac{11}{6}}\left(1+\frac{1}{2}x\right)}{\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}\left(1+\frac{9}{4}x\right)^{\frac{2}{3}}\left(1+\frac{3}{8}x\right)^{\frac{4}{3}}\left(1+\frac{3}{2}\right)^{\frac{4}{3}}} \\ &\leqslant \exp\left[-\frac{9}{4}x + \frac{551}{192}x^2\right], \end{aligned}$$

which is smaller than one for $x < \min\left\{\frac{3-\sqrt{6}}{3/4}, \frac{432}{551}\right\} = \frac{3-\sqrt{6}}{3/4}$. This implies that $q_{n\equiv 1}(n) < 1$ if n satisfies $n > \frac{3}{4(3-\sqrt{6})} + \frac{1}{4}$. Since $\frac{3}{4(3-\sqrt{6})} + \frac{1}{4} < 8$, we are safe here as well.

Last but not least, we consider $q_{n\equiv 2}(n)$ and obtain by a similar calculation, again with $x = \frac{1}{n-\frac{1}{4}}$

$$\begin{aligned} q_{\equiv 2}(x) &= \frac{\left(1+\frac{5}{4}x\right)^{\frac{17}{6}}\left(1+\frac{1}{2}x\right)\left(1+\frac{3}{4}x\right)}{\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}\left(1+\frac{1}{8}x\right)^{\frac{8}{3}}\left(1+2x\right)^{\frac{8}{3}}} \\ &\leqslant \exp\left[-x + \frac{2087}{576}x^2\right]. \end{aligned}$$

This is smaller than one if $x < \min\left\{\frac{3-\sqrt{6}}{5/4}, \frac{576}{2087}\right\} = \frac{576}{2087}$. And thus, $q_{n\equiv 2}(n) < 1$ as soon as $n > \frac{2087}{576} + \frac{1}{4}$, that is, if $n \geqslant 4$.

So, eventually, we reached our goal

$$\frac{F_{5,A}(n,a^*)}{\Gamma(2n+1)^2} \leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} \frac{e}{n^{\frac{1}{2}}\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)}\left(\frac{2^{12}}{3^4}\right)^n \tag{195}$$

for all $n \geqslant 8$. ∎

In the following we take care of the remaining cases **B, C** and **D** and show that the maxima of the corresponding functions satisfy the same upper bound (195) as the one of $F_{5,A}$. Since the three cases basically share the same approach and a lot of techniques, we handle them all at once.

We introduce the function

$$g_n(a) := \frac{\Gamma\left(n+a+\frac{1}{2}\right)}{\Gamma(a+1)\Gamma\left(n-a+\frac{1}{2}\right)} \tag{196}$$

and write

$$\begin{aligned} F_{5,A}(n,a) &= g_n(a)^4\Gamma(4n-4a)\,, \\ F_{5,B}(n,a) &:= g_n(a)^3 g_n(a+1)\Gamma(4n-4a-1)\,, \\ F_{5,C}(n,a) &:= g_n(a)^2 g_n(a+1)^2\Gamma(4n-4a-2), \\ F_{5,D}(n,a) &:= g_n(a) g_n(a+1)^3\Gamma(4n-4a-3). \end{aligned} \tag{197}$$

We want to compare "subsequent" functions and therefore calculate the their quotients

$$\begin{aligned} \Phi_1(n,a) &:= \frac{F_{5,B}(n,a)}{F_{5,A}(n,a)} = \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-1)}{\Gamma(4n-4a)}, \\ \Phi_2(n,a) &:= \frac{F_{5,C}(n,a)}{F_{5,B}(n,a)} = \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-2)}{\Gamma(4n-4a-1)}, \\ \Phi_3(n,a) &:= \frac{F_{5,D}(n,a)}{F_{5,C}(n,a)} = \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-3)}{\Gamma(4n-4a-2)}. \end{aligned} \tag{198}$$

We will need one additional quotient later

$$\Phi_4(n,a) := \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-4)}{\Gamma(4n-4a-3)}.$$

Moreover, in order to narrow down the location a_X^* of the maximum of $F_{5,X}(n,a)$ with the help of Theorem 20 we need $\varphi_{5,X}(a) := \frac{F_{5,X}(n,a)}{F_{5,X}(n,a-1)}$ for $X = B, C, D$. Those are

$$\begin{aligned} \varphi_{5,B}(a) = &\frac{g_n(a)}{g_n(a-1)}\frac{\Gamma(4n-4a+2)}{\Gamma(4n-4a+3)} \cdot \frac{g_n(a)}{g_n(a-1)}\frac{\Gamma(4n-4a+1)}{\Gamma(4n-4a+2)} \\ &\times \frac{g_n(a)}{g_n(a-1)}\frac{\Gamma(4n-4a)}{\Gamma(4n-4a+1)} \cdot \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-1)}{\Gamma(4n-4a)}, \\ \varphi_{5,C}(a) = &\frac{g_n(a)}{g_n(a-1)}\frac{\Gamma(4n-4a+1)}{\Gamma(4n-4a+2)} \cdot \frac{g_n(a)}{g_n(a-1)}\frac{\Gamma(4n-4a)}{\Gamma(4n-4a+1)} \\ &\times \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-1)}{\Gamma(4n-4a)} \cdot \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-2)}{\Gamma(4n-4a-1)}, \end{aligned}$$

and

$$\begin{aligned} \varphi_{5,D}(a) = &\frac{g_n(a)}{g_n(a-1)}\frac{\Gamma(4n-4a)}{\Gamma(4n-4a+1)} \cdot \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-1)}{\Gamma(4n-4a)} \\ &\times \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-2)}{\Gamma(4n-4a-1)} \cdot \frac{g_n(a+1)}{g_n(a)}\frac{\Gamma(4n-4a-3)}{\Gamma(4n-4a-2)}. \end{aligned}$$

Those functions can all be expressed in terms of $\Phi_1, \ldots, \Phi_4$ as follows

$$\begin{aligned} \varphi_{5,B}(a) &= \Phi_2(n,a-1)\Phi_3(n,a-1)\Phi_4(n,a-1)\Phi_1(n,a), \\ \varphi_{5,C}(a) &= \Phi_3(n,a-1)\Phi_4(n,a-1)\Phi_1(n,a)\Phi_2(n,a), \\ \varphi_{5,D}(a) &= \Phi_4(n,a-1)\Phi_1(n,a)\Phi_2(n,a)\Phi_3(n,a). \end{aligned} \tag{199}$$

Thus, some properties of $\varphi_{5,X}$ can be deduced from the respective properties of Φ_1, Φ_2, Φ_3 to Φ_4. One property we need is the following

LEMMA 33. *Let $X = B, C, D$ and $n \geqslant 16$. Then the functions $\varphi_{5,X}$ have the property that if there is a value $\zeta \in (0, n-1]$ such that $\varphi_{5,X}(\zeta) = 1$ then it is unique. Moreover, $\varphi_{5,X}(a) > 1$ iff $a < \zeta$ and $\varphi_{5,X}(a) < 1$ iff $\zeta < a \leqslant n-1$.*

PROOF. The attentive reader might notice that we deduced the same property for $\varphi_{4,B}$ and $\varphi_{4,C}$ back in Subsection 4.2.6. In the following we copy the chain of reasoning we used there. That is, we prove for $j = 1, 2, 3, 4$ that there is an $\tilde{a}$ such that $\Phi_j(n, a)$ decrease monotonically on $0 < a \leqslant \tilde{a}$ and increase for $\tilde{a} < a \leqslant n-1$, as well as that $\varphi_{5,X}(n-1) < 1$. Before we jump into medias res let us rewrite Φ_j in a more easily digestible way. It is

$$\begin{aligned} \Phi_1(n,a) &= \frac{\left(-a+n-\frac{1}{2}\right)\left(a+n+\frac{1}{2}\right)}{4(a+1)\left(-a+n-\frac{1}{4}\right)}, \\ \Phi_2(n,a) &= \frac{a+n+\frac{1}{2}}{4a+4}, \\ \Phi_3(n,a) &= \frac{\left(-a+n-\frac{1}{2}\right)\left(a+n+\frac{1}{2}\right)}{4(a+1)\left(-a+n-\frac{3}{4}\right)}, \\ \Phi_4(n,a) &= \frac{\left(-a+n-\frac{1}{2}\right)\left(a+n+\frac{1}{2}\right)}{4(a+1)(-a+n-1)}. \end{aligned} \tag{200}$$

Next we show that $\tilde{a} = \frac{3}{4}n - 2$ does the job for all $n \geqslant 16$. In the case of Φ_1 and Φ_2 there is nothing to do, since

$$\Phi_1(n,a) = \frac{1}{4}\left(1+\frac{n-\frac{1}{2}}{a+1}\right)\left(1-\frac{\frac{1}{4}}{n-a-\frac{1}{4}}\right),$$

and

$$\Phi_2(n,a) = \frac{1}{4}\left(1+\frac{n-\frac{1}{2}}{a+1}\right)$$

are obviously monotonously decreasing for all $0 < a \leqslant n-1$ and $n \geqslant 1$. For the two remaining functions Φ_3 and Φ_4 we show that the a-derivative is negative for $0 < a \leqslant \tilde{a} = \frac{3}{4}n - 2$ if $n \geqslant 16$. It is

$$\frac{\partial \Phi_3(n,a)}{\partial a} = \frac{4a^2(3-4n)+16a(2(n-1)n+1)-4(n-1)n(4n-3)+5}{4(a+1)^2(4a-4n+3)^2} \tag{201}$$

which is negative for $a \leqslant a_1$, and $a \geqslant a_2$, with

$$a_{1,2} = \frac{8n^2 \mp \sqrt{32n^3-4n^2-8n+1}-8n+4}{2(4n-3)}$$

being the two roots of the numerator of (201). It is easy to see that $a_2 > n-1$. We therefore conclude that $\Phi_3(n,a)$ decreases in a if $0 < a \leqslant a_1$. We further estimate

$$\begin{aligned} a_1 &> \frac{3n^2-4n+2}{4n-3} \\ &> \frac{3}{4}n - \frac{4n-2}{4n-3} \\ &> \frac{3}{4}n - 2, \end{aligned}$$

using that

$$1-8n-4n^2+32n^3 < 4n^4$$

if $n \geqslant 8$.
We repeat this procedure for Φ_4 and find that

$$\frac{\partial \Phi_4(n,a)}{\partial a} = \frac{-4a^2(n-1) + a(8(n-1)n+6) + n(-4(n-2)n-3) + 2}{16(a+1)^2(a-n+1)^2} \tag{202}$$

is negative for $a \leqslant a_1$, and $a \geqslant a_2$, with

$$a_{1,2} = \frac{4n^2 \mp \sqrt{16n^3 - 4n^2 - 4n + 1} - 4n + 3}{4(n-1)}$$

being the two roots of the numerator of (202). Since again $a_2 > n$, we conclude that $\Phi_4(n,a)$ decreases in a if $0 < a \leqslant a_1$ and estimate

$$\begin{aligned} a_1 &> \frac{\frac{3}{4}n - n + 3}{n-1} \\ &> \frac{3}{4}n - \frac{n-3}{n-1} \\ &> \frac{3}{4}n - 1, \end{aligned}$$

using that

$$1 - 4n - 4n^2 + 16n^3 < n^4$$

if $n \geqslant 16$.
Finally note that for $X = B, C, D$

$$\begin{aligned} \varphi_{5,X}(n-1) &\leqslant \Phi_4(n-2) \prod_{j=1}^{3} \Phi_j(n-1) \\ &= \frac{3}{16}\left(\frac{1}{n-1} + 4\right)\left(\frac{1}{3} - \frac{1}{12n}\right)\left(\frac{1}{2} - \frac{1}{8n}\right)\left(1 - \frac{1}{4n}\right) \\ &< 1. \end{aligned}$$

This completes the proof of the above stated monotonicity property of the functions $\varphi_{5,B}, \varphi_{5,C}$ and $\varphi_{5,D}$. ■

In the next step we deduce two-sided bounds on the values ζ_X with $\varphi_{5,X}(\zeta_X) = 1$ for $X = B, C, D$.

LEMMA 34. *If $n \geqslant 17$, then the functions $\varphi_{5,B}, \varphi_{5,C}$ and $\varphi_{5,D}$ satisfy the bounds*

$$\varphi_{5,B}\left(\frac{1}{3}n - \frac{1}{3}\right) > 1 > \varphi_{5,B}\left(\frac{1}{3}n - \frac{1}{4}\right), \tag{203}$$

$$\varphi_{5,C}\left(\frac{1}{3}n - \frac{15}{24}\right) > 1 > \varphi_{5,C}\left(\frac{1}{3}n - \frac{1}{2}\right), \tag{204}$$

and

$$\varphi_{5,D}\left(\frac{1}{3}n - 1\right) > 1 > \varphi_{5,D}\left(\frac{1}{3}n - \frac{2}{3}\right). \tag{205}$$

Before we prove Lemma 34, we want to line out its implications. Primarily these are the following bounds on the ζ_X

$$\begin{aligned} \frac{1}{3}n - \frac{1}{3} < \zeta_B < \frac{1}{3}n - \frac{1}{4}, \\ \frac{1}{3}n - \frac{15}{24} < \zeta_C < \frac{1}{3}n - \frac{1}{2}, \end{aligned}$$

and

$$\frac{1}{3}n - 1 < \zeta_D < \frac{1}{3}n - \frac{2}{3}.$$

Together with the monotonicity of $\varphi_{5,X}(a)$ around $a = \zeta_X$ that follows from Lemma 33, Theorem 20 then postulates the following locations a_X^* of the maximum of $F_{5,X}(n,a)$

$$\frac{1}{3}n - \frac{4}{3} < a_B^* < \frac{1}{3}n - \frac{1}{4}, \tag{206}$$

$$\frac{1}{3}n - \frac{39}{24} < a_C^* < \frac{1}{3}n - \frac{1}{2}, \tag{207}$$

and

$$\frac{1}{3}n - 2 < a_D^* < \frac{1}{3}n - \frac{2}{3}. \tag{208}$$

Next, recall that we are looking for integers a_X^*, and thus can further cook down the ranges (206), (207) and (208) to three possibilities for each $a_X^*, X = B, C, D$ by a similar calculation as on page 62 in the case of the function $F_{4,B}$. For the sake of completeness we carry out the calculation for a_B^* but omit the details for a_C^* and a_D^*.
We need to determine

$$a_B^* \in \left(\frac{1}{3}n - \frac{4}{3}, \frac{1}{3}n - \frac{1}{4}\right) \cap \mathbb{Z}. \tag{209}$$

Depending on the divisibility of n by 3 we distinguish three cases.
If $n \equiv 0 \mod 3$, we set $n = 3p$ and (209) becomes

$$\begin{aligned} a_B^* &\in \left(p - \frac{4}{3}, p - \frac{1}{4}\right) \cap \mathbb{Z} \\ &= \{p - 1\}. \end{aligned}$$

Consequently it is in this case

$$a_B^* = \frac{1}{3}n - 1.$$

If $n \equiv 1 \mod 3$, setting $n = 3p + 1$ leads to

$$\begin{aligned} a_B^* &\in \left(p - 1, p + \frac{1}{12}\right) \cap \mathbb{Z} \\ &= \{p\} \end{aligned}$$

and therefore

$$a_B^* = \frac{1}{3}n - \frac{1}{3}.$$

In the case of $n \equiv 2 \mod 3$, we get with $n = 3p + 2$

$$\begin{aligned} a_B^* &\in \left(p - \frac{2}{3}, p + \frac{5}{12}\right) \cap \mathbb{Z} \\ &= \{p\} \end{aligned}$$

and

$$a_B^* = \frac{1}{3}n - \frac{2}{3}.$$

In summary we end up with

$$a_B^* = \begin{cases} \frac{1}{3}n - 1, & n \equiv 0 \mod 3, \\ \frac{1}{3}n - \frac{1}{3}, & n \equiv 1 \mod 3, \\ \frac{1}{3}n - \frac{2}{3}, & n \equiv 2 \mod 3. \end{cases} \tag{210}$$

The similar procedure results for a^*_C and a^*_D in

$$a^*_C = \begin{cases} \frac{1}{3}n - 1, & n \equiv 0 \mod 3, \\ \frac{1}{3}n - \frac{4}{3}, & n \equiv 1 \mod 3, \\ \frac{1}{3}n - \frac{2}{3}, & n \equiv 2 \mod 3, \end{cases} \tag{211}$$

and

$$a^*_D = \begin{cases} \frac{1}{3}n - 1, & n \equiv 0 \mod 3, \\ \frac{1}{3}n - \frac{4}{3}, & n \equiv 1 \mod 3, \\ \frac{1}{3}n - \frac{5}{3}, & n \equiv 2 \mod 3. \end{cases} \tag{212}$$

Before we carry on with estimating $F_{5,X}(n, a^*_X)$ for $X = B, C, D$ we now prove Lemma 34.

Proof of Lemma 34. All inequalities follow the same scheme. That is manipulation of $\varphi_{5,X}(a)$ until Lemma 64 is applicable and then showing that the Lemma proves the claimed inequality under the condition that n is at least 17.
We have done this several times before. For example in Subsection 4.2.6 on page 60, and therefore keep the calculations short here.

For the first inequality of (203) we get

$$\varphi_{5,B}\left(\frac{1}{3}n - \frac{1}{3}\right) = \frac{\left(n - \frac{1}{4}\right)\left(n + \frac{5}{4}\right)^2\left(n - \frac{5}{8}\right)^3}{(n-1)^3(n+2)\left(n + \frac{1}{2}\right)\left(n + \frac{7}{8}\right)}.$$

By shifting $m = n - 1$ the right-hand side turns into

$$\frac{\left(m + \frac{3}{8}\right)^3\left(m + \frac{3}{4}\right)\left(m + \frac{9}{4}\right)^2}{m^3\left(m + \frac{3}{2}\right)\left(m + \frac{15}{8}\right)(m+3)}.$$

Next we divide each linear factor in the numerator and denominator by m

$$\frac{\left(\frac{3}{8m} + 1\right)^3\left(\frac{3}{4m} + 1\right)\left(\frac{9}{4m} + 1\right)^2}{\left(\frac{3}{2m} + 1\right)\left(\frac{15}{8m} + 1\right)\left(\frac{3}{m} + 1\right)}$$

and apply estimate (403) to the factors in the numerator and estimate (404) with $\delta = \frac{2}{5}$ in the denominator to bound the last line from below by

$$\exp\left[\frac{945}{64m^2}\delta - \frac{711}{128m^2}\right] = \exp\left[\frac{45}{128m^2}\right] > 1$$

for all $m > 0$. The conditions of (404) are satisfied if $\frac{3}{m} \leqslant \frac{1}{\delta} - \sqrt{\frac{2}{\delta}}$ for $\delta = \frac{2}{5}$. This is true as soon as $m > -\frac{6}{2\sqrt{5}-5}$. Finally we go back to $n = m + 1$ and find that the claimed inequality holds for all $n > 1 - \frac{6}{2\sqrt{5}-5} \approx 12.4$.
For the proof of the second inequality of (203) we calculate

$$\varphi_{5,B}\left(\frac{1}{3}n - \frac{1}{4}\right) = \frac{\left(n - \frac{3}{8}\right)\left(n + \frac{9}{8}\right)^2\left(n - \frac{9}{16}\right)^3\left(n + \frac{3}{16}\right)}{n\left(n - \frac{3}{4}\right)^3\left(n + \frac{3}{4}\right)\left(n + \frac{9}{4}\right)\left(n + \frac{3}{8}\right)}.$$

The right-hand side is equivalent to

$$\frac{\left(m + \frac{3}{16}\right)^3\left(m + \frac{3}{8}\right)\left(m + \frac{15}{16}\right)\left(m + \frac{15}{8}\right)^2}{m^3\left(m + \frac{3}{4}\right)\left(m + \frac{9}{8}\right)\left(m + \frac{3}{2}\right)(m+3)}$$

$$= \frac{\left(\frac{3}{16m} + 1\right)^3\left(\frac{3}{8m} + 1\right)\left(\frac{15}{16m} + 1\right)\left(\frac{15}{8m} + 1\right)^2}{\left(\frac{3}{4m} + 1\right)\left(\frac{9}{8m} + 1\right)\left(\frac{3}{2m} + 1\right)\left(\frac{3}{m} + 1\right)}$$

for $m = n - \frac{3}{4}$. Now the last line above is smaller than

$$\exp\left[\frac{837}{128m^2} - \frac{3}{4m}\right]$$

by Lemma 64 with $\delta = 0$, and smaller than one as soon as $m > \frac{279}{32}$. This proves the claimed inequality for all $n > \frac{279}{32} + \frac{3}{4} = \frac{303}{320} \approx 9.5$.
We continue with the first inequality of (204). It is

$$\varphi_{5,C}\left(\frac{1}{3}n - \frac{15}{24}\right) = \frac{\left(n-\frac{3}{32}\right)^2\left(n-\frac{27}{32}\right)^2\left(n+\frac{3}{16}\right)\left(n+\frac{27}{16}\right)^2}{\left(n-\frac{15}{8}\right)^2\left(n+\frac{9}{8}\right)^2\left(n+\frac{9}{16}\right)\left(n+\frac{15}{16}\right)\left(n+\frac{21}{16}\right)}$$

equivalent to

$$\begin{aligned}&\frac{\left(m+\frac{33}{32}\right)^2\left(m+\frac{57}{32}\right)^2\left(m+\frac{33}{16}\right)\left(m+\frac{57}{16}\right)^2}{m^2\left(m+\frac{39}{16}\right)\left(m+\frac{45}{16}\right)(m+3)^2\left(m+\frac{51}{16}\right)}\\&=\frac{\left(\frac{33}{32m}+1\right)^2\left(\frac{57}{32m}+1\right)^2\left(\frac{33}{16m}+1\right)\left(\frac{57}{16m}+1\right)^2}{\left(\frac{39}{16m}+1\right)\left(\frac{45}{16m}+1\right)\left(\frac{3}{m}+1\right)^2\left(\frac{51}{16m}+1\right)}\end{aligned}$$

with $m = n - \frac{15}{8}$. Lemma 64 with $\delta = \frac{1}{3}$ yields the following lower bound on the last line above

$$\exp\left[\frac{10755}{256m^2}\delta - \frac{2439}{128m^2} + \frac{3}{8m}\right] = \exp\left[\frac{3}{8m} - \frac{1293}{256m^2}\right].$$

This bound lies above 1, if $m > \frac{431}{32}$, or $n = m + \frac{15}{8} > \frac{533}{32} \approx 16.7$, respectively. Moreover, the application of Lemma 64 with $\delta = \frac{1}{3}$ is justified if $m \geqslant -\frac{51}{16\sqrt{6}-48}$, or $n \geqslant \frac{15}{8} - \frac{51}{16\sqrt{6}-48} \approx 7.7$, respectively.
In the case of the second inequality of (204) we have

$$\varphi_{5,C}\left(\frac{1}{3}n - \frac{1}{2}\right) = \frac{\left(n-\frac{3}{4}\right)^2 n^3}{\left(n-\frac{3}{2}\right)^2\left(n+\frac{3}{4}\right)\left(n+\frac{3}{8}\right)\left(n+\frac{9}{8}\right)},$$

and do the usual shifting and dividing, this time with $m = n - \frac{3}{2}$, to get

$$\frac{\left(m+\frac{3}{4}\right)^2\left(m+\frac{3}{2}\right)^3}{m^2\left(m+\frac{15}{8}\right)\left(m+\frac{9}{4}\right)\left(m+\frac{21}{8}\right)} = \frac{\left(\frac{3}{4m}+1\right)^2\left(\frac{3}{2m}+1\right)^3}{\left(\frac{15}{8m}+1\right)\left(\frac{9}{4m}+1\right)\left(\frac{21}{8m}+1\right)}.$$

Now Lemma 64 with $\delta = 0$ provides the following upper bound on the last line

$$\exp\left[\frac{495}{64m^2} - \frac{3}{4m}\right],$$

which is smaller than one if $m > \frac{165}{16}$. Thus the claimed inequality holds for all $n = m + \frac{3}{2} > \frac{189}{16} \approx 11.8$.
Next is the first inequality of (205). We calculate

$$\varphi_{5,D}\left(\frac{1}{3}n - 1\right) = \frac{\left(n+\frac{3}{4}\right)^2\left(n+\frac{9}{4}\right)\left(n-\frac{9}{8}\right)\left(n-\frac{3}{8}\right)^3}{(n-3)n^3\left(n+\frac{3}{2}\right)\left(n+\frac{3}{8}\right)\left(n+\frac{9}{8}\right)}$$

and transform the right-hand side into

$$\frac{\left(m+\frac{15}{8}\right)\left(m+\frac{21}{8}\right)^3\left(m+\frac{15}{4}\right)^2\left(m+\frac{21}{4}\right)}{m(m+3)^3\left(m+\frac{27}{8}\right)\left(m+\frac{33}{8}\right)\left(m+\frac{9}{2}\right)} = \frac{\left(\frac{15}{8m}+1\right)\left(\frac{21}{8m}+1\right)^3\left(\frac{15}{4m}+1\right)^2\left(\frac{21}{4m}+1\right)}{\left(\frac{3}{m}+1\right)^3\left(\frac{27}{8m}+1\right)\left(\frac{33}{8m}+1\right)\left(\frac{9}{2m}+1\right)}$$

by shifting $m = n - 3$ in order to apply Lemma 64 with $\delta = \frac{1}{3}$. This then provides us with the lower bound

$$\exp\left[\frac{2421}{32m^2}\delta - \frac{639}{16m^2} + \frac{3}{2m}\right] = \exp\left[\frac{3}{2m} - \frac{471}{32m^2}\right],$$

which is greater than one if $m > \frac{157}{16}$ and thus $n > \frac{205}{16} \approx 12.8$. For the application of Lemma 64 we need that $\frac{9}{2m} < 3 - \sqrt{6}$. This leads to the additional condition $m > -\frac{9}{2\sqrt{6}-6}$ on m, or $n > 3 - \frac{9}{2\sqrt{6}-6} \approx 11.2$ on n, respectively.
Last on the list is the second inequality of (205). Here we get

$$\varphi_{5,D}\left(\frac{1}{3}n - \frac{2}{3}\right) = \frac{\left(n-\frac{1}{8}\right)^2\left(n+\frac{1}{4}\right)^2\left(n+\frac{7}{4}\right)\left(n-\frac{7}{8}\right)}{(n-2)(n+1)^4\left(n+\frac{5}{8}\right)},$$

shift the right-hand side by $m = n - 2$ and arrive at

$$\frac{\left(m+\frac{9}{8}\right)\left(m+\frac{15}{8}\right)^2\left(m+\frac{9}{4}\right)^2\left(m+\frac{15}{4}\right)}{m\left(m+\frac{21}{8}\right)(m+3)^4} = \frac{\left(\frac{9}{8m}+1\right)\left(\frac{15}{8m}+1\right)^2\left(\frac{9}{4m}+1\right)^2\left(\frac{15}{4m}+1\right)}{\left(\frac{21}{8m}+1\right)\left(\frac{3}{m}+1\right)^4},$$

after dividing each factor by m. Now Lemma 64 with $\delta = 0$ yields the upper bound

$$\exp\left[\frac{2745}{128m^2} - \frac{3}{2m}\right]$$

on the right-hand side above. This bound lies below one as soon as $m > \frac{915}{64}$ and the last remaining inequality follows for all $n = m + 2 > \frac{1043}{64} \approx 16.3$.

■

Now the way is clear for the estimate of the maxima $F_{5,X}(n, a_X^*)$. Luckily we don't have to go through all nine possible cases in (210), (211) and (212) "manually" with the help of Stirling's inequality and our other arsenal of estimates. Six of the cases can be excluded more elegantly by looking at the quotients (198). In fact, recall the three possible locations of the maximum of $F_{5,A}$ in (178)

$$a_A^* = \begin{cases} \frac{1}{3}n - 1, & n \equiv 0 \mod 3, \\ \frac{1}{3}n - \frac{1}{3}, & n \equiv 1 \mod 3, \\ \frac{1}{3}n - \frac{2}{3}, & n \equiv 2 \mod 3. \end{cases}$$

Those are exactly the same as for $F_{5,B}$. Moreover, we already know that the quotient $\Phi_1(n, a)$ of $F_{5,B}(n, a)$ and $F_{5,A}(n, a)$ is a decreasing function in a, that simplifies to

$$\Phi_1(n, a) = \frac{\left(-a+n-\frac{1}{2}\right)\left(a+n+\frac{1}{2}\right)}{4(a+1)\left(-a+n-\frac{1}{4}\right)}.$$

We determine the value of a such that Φ_1 equals one. Setting the right-hand side above equal to one leads to the quadratic equation

$$12a^2 - 16a(n-1) + 4n^2 - 16n + 3 = 0$$

with the two roots

$$a_{1,2} = \frac{1}{6}\left(4n - 4 \mp \sqrt{4n^2 + 16n + 7}\right).$$

Using that $4n^2 + 16n + 7 > (2n+3)^2$ for all $n \geqslant 1$, we see that a_2 is not an element of the domain of $\Phi_1(n, \cdot)$, and that

$$\begin{aligned} a_1 &< \frac{2n-7}{6} \\ &< \frac{1}{3}n - 1 \\ &< a_B^* = a_A^*. \end{aligned}$$

Consequently it is

$$F_{5,B}(n, a_B^*) < F_{5,A}(n, a_A^*). \tag{213}$$

We move on to $F_{5,C}$ and note that two of the possible locations of a_C^* coincide with the respective cases of a_B^*. Those are

$$a_{C,0}^* = a_{B,0}^* = \frac{1}{3}n - 1,$$
$$a_{C,2}^* = a_{B,2}^* = \frac{1}{3}n - \frac{2}{3}.$$

In order to compare $F_{5,B}$ and $F_{5,C}$ for those two values of a, we consider the quotient Φ_2 of the two functions. Here we are done quickly, as

$$\Phi_2(n,a) = \frac{a+n+\frac{1}{2}}{4a+4} = 1$$

has the unique solution

$$a = \frac{1}{3}n - \frac{7}{6} < a_{C,0}^* < a_{C,2}^*.$$

Hence,

$$\begin{aligned} F_{5,C}(n, a_{C,0}^*) &< F_{5,B}(n, a_{B,0}^*), \\ F_{5,C}(n, a_{C,2}^*) &< F_{5,B}(n, a_{B,2}^*). \end{aligned} \tag{214}$$

The last possible maximum we can exclude comparatively quickly is the one at

$$a_{D,0}^* = a_{C,0}^* = \frac{1}{3}n - 1.$$

We consult the quotient Φ_3 of $F_{5,D}$ and $F_{5,C}$ and solve

$$\Phi_3(n,a) = \frac{\left(-a+n-\frac{1}{2}\right)\left(a+n+\frac{1}{2}\right)}{4(a+1)\left(-a+n-\frac{3}{4}\right)} = 1.$$

The two solutions are

$$a_{1,2} = \frac{1}{6}\left(4n - 6 \mp \sqrt{4n^2+3}\right).$$

Obviously $a_2 > n-1$ is irrelevant and

$$a_1 < \frac{2n-6}{6} = a_{D,0}^*$$

and we can infer without further calculations that

$$F_{5,D}(n, a_{D,0}^*) < F_{5,C}(n, a_{C,0}^*). \tag{215}$$

Now there are still three candidates left for the global maximum of F_5. Those are

$$F_{5,C}\left(n, \frac{1}{3}n - \frac{4}{3}\right), \quad F_{5,D}\left(n, \frac{1}{3}n - \frac{4}{3}\right), \quad F_{5,D}\left(n, \frac{1}{3}n - \frac{5}{3}\right). \tag{216}$$

The remaining pages of this section are devoted to the proof of

Lemma 35. *If $n \geqslant 9$ then the three values* (216) *are bounded from above by*

$$\Gamma(2n+1)^2\left(\frac{3}{4\pi}\right)^{\frac{5}{2}} \frac{e}{n^{\frac{1}{2}}\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)}\left(\frac{2^{12}}{3^4}\right)^n.$$

Proof. We work the three candidates from left to right and thus start with $F_{5,C}\left(n, \frac{1}{3}n - \frac{4}{3}\right)$. It is

$$\frac{F_{5,C}\left(n, \frac{1}{3}n - \frac{4}{3}\right)}{\Gamma(2n+1)^2} = \frac{\Gamma\left(\frac{4n}{3} - \frac{5}{6}\right)^2 \Gamma\left(\frac{4n}{3} + \frac{1}{6}\right)^2 \Gamma\left(\frac{8}{3}n + \frac{10}{3}\right)}{\Gamma(2n+1)^2 \Gamma\left(\frac{2n}{3} + \frac{5}{6}\right)^2 \Gamma\left(\frac{2n}{3} + \frac{11}{6}\right)^2 \Gamma\left(\frac{n-1}{3}\right)^2 \Gamma\left(\frac{n+2}{3}\right)^2}.$$

By Stirling's Formula (332) the right-hand side satisfies the upper bound

$$\frac{1}{(2\pi)^{\frac{5}{2}}}\exp\left[6+\frac{1}{12}\left(\frac{2}{\frac{4}{3}n-\frac{5}{6}}+\frac{2}{\frac{4}{3}n+\frac{1}{6}}+\frac{1}{\frac{8}{3}n+\frac{10}{3}}\right)\right]$$

$$\times\frac{\left(\frac{4}{3}n-\frac{5}{6}\right)^{\frac{8}{3}n-\frac{8}{3}}\left(\frac{4}{3}n+\frac{1}{6}\right)^{\frac{8}{3}n-\frac{2}{3}}\left(\frac{8}{3}n+\frac{10}{3}\right)^{\frac{8}{3}n+\frac{17}{6}}}{(2n+1)^{4n+1}\left(\frac{2}{3}n+\frac{5}{6}\right)^{\frac{4}{3}n+\frac{2}{3}}\left(\frac{2}{3}n+\frac{11}{6}\right)^{\frac{4}{3}n+\frac{8}{3}}\left(\frac{1}{3}n-\frac{1}{3}\right)^{\frac{2}{3}n-\frac{5}{3}}\left(\frac{1}{3}n+\frac{2}{3}\right)^{\frac{2}{3}n+\frac{1}{3}}}. \tag{217}$$

Before we move on to the detailed breakdown of (217) into its polynomial and exponential parts note that

$$\begin{aligned}\left(\frac{1}{3}n-\frac{1}{3}\right)^{\frac{2}{3}n-\frac{5}{3}} &= \left(\frac{1}{3}\right)^{\frac{2}{3}n-\frac{5}{3}}\left(1-\frac{1}{n}\right)^{\frac{2}{3}n-\frac{5}{3}} \\ &> \left(\frac{1}{3}\right)^{\frac{2}{3}n-\frac{5}{3}} e^{-\frac{2}{3}}\end{aligned} \tag{218}$$

by Lemma 65 with $a=-\frac{2}{3}$ and $b=-\frac{5}{3}$. We now split (217) into exponentials and polynomials and additionally apply the last estimate to yield the upper bound

$$\left(\frac{3}{2}\right)^{\frac{5}{2}}\frac{n^{\frac{5}{3}}\left(n+\frac{5}{4}\right)^{\frac{13}{6}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{5}{8}\right)^{\frac{8}{3}}\left(n+\frac{11}{4}\right)^{\frac{8}{3}}\left(n+\frac{1}{8}\right)^{\frac{2}{3}}(n+2)^{\frac{1}{3}}}\left(\frac{2^{12}}{3^4}\right)^n \tag{219}$$

$$\times\frac{\left(1-\frac{5}{8n}\right)^{\frac{8}{3}n}\left(1+\frac{1}{8n}\right)^{\frac{8}{3}n}\left(1+\frac{5}{4n}\right)^{\frac{8}{3}n}}{e^{-\frac{2}{3}}\left(1+\frac{1}{2n}\right)^{4n}\left(1+\frac{5}{4n}\right)^{\frac{4}{3}n}\left(1+\frac{11}{4n}\right)^{\frac{4}{3}n}\left(1+\frac{2}{n}\right)^{\frac{2}{3}n}}. \tag{220}$$

Expression (220) is further treated with Lemma 64 in the denominator and estimate $1+x<e^x, x\in\mathbb{R}$ in the numerator and consequently smaller than

$$\exp\left[2-8+\frac{1}{2}\cdot\frac{95}{6n}\right].$$

Expression (219) on the other hand satisfies the upper bound

$$\begin{aligned}&\left(\frac{3}{2}\right)^{\frac{5}{2}}\frac{1}{\left(n+\frac{5}{4}\right)^{\frac{1}{2}}\left(n+\frac{1}{2}\right)n\left(n+\frac{1}{8}\right)^{\frac{2}{3}}(n+2)^{\frac{1}{3}}}\left(\frac{2^{12}}{3^4}\right)^n \\ \leqslant &\left(\frac{3}{2}\right)^{\frac{5}{2}}\frac{1}{n^{\frac{1}{2}}\left(n+\frac{1}{2}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)}\left(\frac{2^{12}}{3^4}\right)^n\end{aligned}$$

since

$$\left(n-\frac{5}{8}\right)^{\frac{8}{3}}\left(n+\frac{11}{4}\right)^{\frac{8}{3}}\geqslant n^{\frac{8}{3}}\left(n+\frac{5}{4}\right)^{\frac{8}{3}}$$

as soon as $n\geqslant\frac{55}{28}$. Now we are almost there. Collecting the recent estimates for (217) results in the inequality

$$\begin{aligned}\frac{F_{5,C}\left(n,\frac{1}{3}n-\frac{4}{3}\right)}{\Gamma(2n+1)^2}\leqslant&\left(\frac{3}{4\pi}\right)^{\frac{5}{2}}\frac{1}{n^{\frac{1}{2}}\left(n+\frac{1}{2}\right)\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)}\left(\frac{2^{12}}{3^4}\right)^n \\ &\times\exp\left[\frac{1}{8n+1}+\frac{1}{32n+40}+\frac{1}{8n-5}+\frac{95}{12n}\right].\end{aligned}$$

With

$$\left[\frac{1}{8n+1}+\frac{1}{32n+40}+\frac{1}{8n-5}+\frac{95}{12n}\right]_{n=9}=\frac{39472787}{43314696}<1$$

the claimed bound follows for $n \geqslant 9$.

We repeat this procedure for $F_{5,D}\left(n, \frac{1}{3}n - \frac{4}{3}\right)$ and get

$$\frac{F_{5,D}\left(n, \frac{1}{3}n - \frac{4}{3}\right)}{\Gamma(2n+1)^2} = \frac{\Gamma\left(\frac{4n}{3} - \frac{5}{6}\right)\Gamma\left(\frac{4n}{3} + \frac{1}{6}\right)^3\Gamma\left(\frac{8n}{3} + \frac{7}{3}\right)}{\Gamma(2n+1)^2\Gamma\left(\frac{2n}{3} + \frac{5}{6}\right)^3\Gamma\left(\frac{2n}{3} + \frac{11}{6}\right)\Gamma\left(\frac{n-1}{3}\right)\Gamma\left(\frac{n+2}{3}\right)^3},$$

which by Stirling's Formula (332) satisfies the upper bound

$$\begin{aligned} &\frac{1}{(2\pi)^{\frac{5}{2}}} \exp\left[6 + \frac{1}{12}\left(\frac{1}{\frac{4}{3}n - \frac{5}{6}} + \frac{3}{\frac{4}{3}n + \frac{1}{6}} + \frac{1}{\frac{8}{3}n + \frac{7}{3}}\right)\right] \\ &\times \frac{\left(\frac{4}{3}n - \frac{5}{6}\right)^{\frac{4}{3}n - \frac{4}{3}}\left(\frac{4}{3}n + \frac{1}{6}\right)^{4n-1}\left(\frac{8}{3}n + \frac{7}{3}\right)^{\frac{8}{3}n + \frac{11}{6}}}{(2n+1)^{4n+1}\left(\frac{2}{3}n + \frac{5}{6}\right)^{2n+1}\left(\frac{2}{3}n + \frac{11}{6}\right)^{\frac{2}{3}n + \frac{4}{3}}\left(\frac{1}{3}n - \frac{1}{3}\right)^{\frac{1}{3}n - \frac{5}{6}}\left(\frac{1}{3}n + \frac{2}{3}\right)^{n + \frac{1}{2}}}. \end{aligned} \tag{221}$$

The equivalent estimate to (218) in this case is

$$\begin{aligned} \left(\frac{1}{3}n - \frac{1}{3}\right)^{\frac{1}{3}n - \frac{5}{6}} &= \left(\frac{1}{3}\right)^{\frac{1}{3}n - \frac{5}{6}}\left(1 - \frac{1}{n}\right)^{\frac{1}{3}n - \frac{5}{6}} \\ &> \left(\frac{1}{3}\right)^{\frac{1}{3}n - \frac{5}{6}} e^{-\frac{1}{3}} \end{aligned} \tag{222}$$

by Lemma 65 with $a = -\frac{1}{3}$ and $b = -\frac{5}{6}$. We apply this estimate in the denominator of (221) and afterwards split the entire expression into the factors

$$\left(\frac{3}{2}\right)^{\frac{5}{2}} \frac{n^{\frac{5}{6}}\left(n + \frac{7}{8}\right)^{\frac{11}{6}}}{(n+2)^{\frac{1}{2}}\left(n + \frac{1}{2}\right)\left(n - \frac{5}{8}\right)^{\frac{4}{3}}\left(n + \frac{11}{4}\right)^{\frac{4}{3}}\left(n + \frac{1}{8}\right)\left(n + \frac{5}{4}\right)}\left(\frac{2^{12}}{3^4}\right)^n \tag{223}$$

$$\times \frac{\left(1 - \frac{5}{8n}\right)^{\frac{4}{3}n}\left(1 + \frac{1}{8n}\right)^{4n}\left(1 + \frac{7}{8n}\right)^{\frac{8}{3}n}}{e^{-\frac{1}{3}}\left(1 + \frac{1}{2n}\right)^{4n}\left(1 + \frac{5}{4n}\right)^{2n}\left(1 + \frac{11}{4n}\right)^{\frac{2}{3}n}\left(1 + \frac{2}{n}\right)^{n}}. \tag{224}$$

The last line is estimated by the same means as (220) and obeys the upper bound

$$\exp\left[2 - 8 + \frac{1}{2} \cdot \frac{79}{6n}\right].$$

For the further processing of (223) we note that

$$\left(n - \frac{5}{8}\right)^{\frac{4}{3}}\left(n + \frac{11}{4}\right)^{\frac{4}{3}} \geqslant n^{\frac{4}{3}}\left(n + \frac{7}{8}\right)^{\frac{4}{3}}$$

for $n \geqslant \frac{11}{8}$ and that therefore (223) is smaller than

$$\begin{aligned} &\left(\frac{3}{2}\right)^{\frac{5}{2}} \frac{1}{n^{\frac{1}{2}}\left(n + \frac{1}{2}\right)\left(n + \frac{1}{8}\right)\left(n + \frac{5}{4}\right)}\left(\frac{n + \frac{7}{8}}{n+2}\right)^{\frac{1}{2}}\left(\frac{2^{12}}{3^4}\right)^n \\ &\leqslant \left(\frac{3}{2}\right)^{\frac{5}{2}} \frac{1}{n^{\frac{1}{2}}\left(n + \frac{1}{2}\right)\left(n - \frac{1}{4}\right)\left(n + \frac{1}{4}\right)}\left(\frac{2^{12}}{3^4}\right)^n. \end{aligned}$$

Putting all estimates of (221) together results in

$$\begin{aligned} \frac{F_{5,D}\left(n, \frac{1}{3}n - \frac{4}{3}\right)}{\Gamma(2n+1)^2} \leqslant &\left(\frac{3}{4\pi}\right)^{\frac{5}{2}} \frac{1}{n^{\frac{1}{2}}\left(n + \frac{1}{2}\right)\left(n - \frac{1}{4}\right)\left(n + \frac{1}{4}\right)}\left(\frac{2^{12}}{3^4}\right)^n \\ &\times \exp\left[\frac{3}{16n+2} + \frac{1}{32n+28} + \frac{1}{16n-10} + \frac{79}{12n}\right], \end{aligned}$$

which obeys the claimed bound as soon as $n \geqslant 7$, since

$$\left[\frac{3}{16n+2} + \frac{1}{32n+28} + \frac{1}{16n-10} + \frac{79}{12n}\right]_{n=7} = \frac{22427291}{26139360} < 1.$$

We need one more iteration of those estimates for $F_{5,D}\left(n, \frac{1}{3}n - \frac{5}{3}\right)$. In this case we get

$$\frac{F_{5,D}\left(n, \frac{1}{3}n - \frac{5}{3}\right)}{\Gamma(2n+1)^2} = \frac{\Gamma\left(\frac{4n}{3} - \frac{7}{6}\right)\Gamma\left(\frac{4n}{3} - \frac{1}{6}\right)^3\Gamma\left(\frac{8n}{3} + \frac{11}{3}\right)}{\Gamma(2n+1)^2\Gamma\left(\frac{2n}{3} + \frac{7}{6}\right)^3\Gamma\left(\frac{2n}{3} + \frac{13}{6}\right)\Gamma\left(\frac{n-2}{3}\right)\Gamma\left(\frac{n+1}{3}\right)^3}.$$

Throwing Stirling's Formula (332) at the right-hand side yields the upper bound

$$\frac{1}{(2\pi)^{\frac{5}{2}}}\exp\left[6 + \frac{1}{12}\left(\frac{1}{\frac{4}{3}n - \frac{7}{6}} + \frac{3}{\frac{4}{3}n - \frac{1}{6}} + \frac{1}{\frac{8}{3}n + \frac{11}{3}}\right)\right]$$

$$\times \frac{\left(\frac{4}{3}n - \frac{7}{6}\right)^{\frac{4}{3}n - \frac{5}{3}}\left(\frac{4}{3}n - \frac{1}{6}\right)^{4n-2}\left(\frac{8}{3}n + \frac{11}{3}\right)^{\frac{8}{3}n + \frac{19}{6}}}{(2n+1)^{4n+1}\left(\frac{2}{3}n + \frac{7}{6}\right)^{2n+2}\left(\frac{2}{3}n + \frac{13}{6}\right)^{\frac{2}{3}n + \frac{5}{3}}\left(\frac{1}{3}n - \frac{2}{3}\right)^{\frac{1}{3}n - \frac{7}{6}}\left(\frac{1}{3}n + \frac{1}{3}\right)^{n - \frac{1}{2}}}. \quad (225)$$

Next we apply Lemma 65 with $a = -\frac{2}{3}$ and $b = -\frac{7}{6}$ on

$$\begin{aligned}\left(\frac{1}{3}n - \frac{2}{3}\right)^{\frac{1}{3}n - \frac{7}{6}} &= \left(\frac{1}{3}\right)^{\frac{1}{3}n - \frac{7}{6}}\left(1 - \frac{2}{n}\right)^{\frac{1}{3}n - \frac{7}{6}} \\ &> \left(\frac{1}{3}\right)^{\frac{1}{3}n - \frac{7}{6}} e^{-\frac{2}{3}}\end{aligned} \quad (226)$$

in the denominator of (225) and then do the usual extraction of the polynomial and exponential parts of each factor in the quotient. This leads to the following upper bound on (225)

$$\left(\frac{3}{2}\right)^{\frac{5}{2}} \frac{n^{\frac{7}{6}}(n+1)^{\frac{1}{2}}\left(n + \frac{11}{8}\right)^{\frac{19}{6}}}{\left(n + \frac{1}{2}\right)\left(n - \frac{7}{8}\right)^{\frac{5}{3}}\left(n + \frac{13}{4}\right)^{\frac{5}{3}}\left(n - \frac{1}{8}\right)^2\left(n + \frac{7}{4}\right)^2}\left(\frac{2^{12}}{3^4}\right)^n \quad (227)$$

$$\times \frac{\left(1 - \frac{7}{8n}\right)^{\frac{4}{3}n}\left(1 - \frac{1}{8n}\right)^{4n}\left(1 + \frac{11}{8n}\right)^{\frac{8}{3}n}}{e^{-\frac{2}{3}}\left(1 + \frac{1}{2n}\right)^{4n}\left(1 + \frac{7}{4n}\right)^{2n}\left(1 + \frac{13}{4n}\right)^{\frac{2}{3}n}\left(1 + \frac{1}{n}\right)^n}. \quad (228)$$

Next, estimate (403) of Lemma 64 and its simpler sibling $1 + x < e^x$ applied to the factors in the denominator and enumerator, respectively, of (228) provide us with the upper bound

$$\exp\left[2 - 8 + \frac{1}{2} \cdot \frac{91}{6n}\right]$$

for (228). This time we need to work a tiny little bit harder to show that the second factor of (227) is smaller than

$$\frac{1}{n^{\frac{1}{2}}\left(n - \frac{1}{4}\right)\left(n + \frac{1}{4}\right)\left(n + \frac{1}{2}\right)}$$

if n is larger than an $n_0 \leqslant 20$.

We start with the trivial bound $n + 1 < n + \frac{11}{8}$ and then use that

$$\left(n - \frac{7}{8}\right)^{\frac{5}{3}}\left(n + \frac{13}{4}\right)^{\frac{5}{3}} \geqslant \left(n + \frac{1}{4}\right)^{\frac{5}{3}}\left(n + \frac{11}{8}\right)^{\frac{5}{3}}$$

for all $n \geqslant \frac{17}{4}$, as well as

$$\left(n - \frac{1}{8}\right)^2\left(n + \frac{7}{4}\right)^2 \geqslant n^2\left(n + \frac{11}{8}\right)^2$$

if $n \geqslant \frac{7}{8}$. This way we get

$$\frac{n^{\frac{7}{6}}(n+1)^{\frac{1}{2}}\left(n+\frac{11}{8}\right)^{\frac{19}{6}}}{\left(n+\frac{1}{2}\right)\left(n-\frac{7}{8}\right)^{\frac{5}{3}}\left(n+\frac{13}{4}\right)^{\frac{5}{3}}\left(n-\frac{1}{8}\right)^{2}\left(n+\frac{7}{4}\right)^{2}} \leqslant \left(\frac{n}{n+\frac{1}{4}}\right)^{\frac{2}{3}} \frac{1}{n^{\frac{3}{2}}\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)}$$
$$\leqslant \frac{1}{n^{\frac{1}{2}}\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)}.$$

Now we put together all estimates of (225) and arrive at

$$\frac{F_{5,D}\left(n,\frac{1}{3}n-\frac{5}{3}\right)}{\Gamma(2n+1)^2} \leqslant \left(\frac{3}{4\pi}\right)^{\frac{5}{2}} \frac{1}{n^{\frac{1}{2}}\left(n-\frac{1}{4}\right)\left(n+\frac{1}{4}\right)\left(n+\frac{1}{2}\right)}\left(\frac{2^{12}}{3^4}\right)^{n}$$
$$\times \exp\left[\frac{3}{16n-2}+\frac{1}{32n+44}+\frac{1}{16n-14}+\frac{91}{12n}\right].$$

The fact that

$$\left[\frac{3}{16n-2}+\frac{1}{32n+44}+\frac{1}{16n-14}+\frac{91}{12n}\right]_{n=8} = \frac{314039}{319200} < 1$$

completes the proof of the lemma. ■

Now we can eventually finish off the proof of Lemma 29 by combining Lemmata 31 and 32 with the estimates (213), (214) and (215) as well as Lemma 35. Note that in order to deduce the respective locations (210), (211) and (212) of the functions $F_{5,B}, F_{5,C}$ and $F_{5,D}$, we need Lemmata 33 and 34, which require n to be larger than or equal to 17. ■

4.4. Analysis of the Main Term $M(\mathbf{n})$

As the name already suggests, the main term $M(\mathbf{n})$ of the series expansion of $\left(\frac{\pi}{2}\right)^2 I(\mathbf{n})$ is the one that governs the decay behavior of our Bessel integral. The goal of this section is to prove Theorem 36 below, which asserts that the main term obeys

$$M(\mathbf{n}) = \frac{3^{\frac{1}{2}}}{2^4}\frac{1}{n} - \frac{23}{2^{10}\cdot 3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)} + E_M(n), \tag{229}$$

with an error

$$|E_M(n)| \leqslant \frac{2689}{2^{18}\cdot 3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)} + \frac{2^{-2n}}{2^5\pi^{\frac{1}{2}}}\left[1+\frac{1}{n}+3^{\frac{1}{2}}e^{\frac{7}{36n}}\frac{n^{\frac{1}{2}}}{n-1}\right] \tag{230}$$

for $n \geqslant 20$.

Let us start with a quick graphical exploration of the formulae (229) and (230). Figure 4.8 shows the main term and its two leading asymptotic terms.

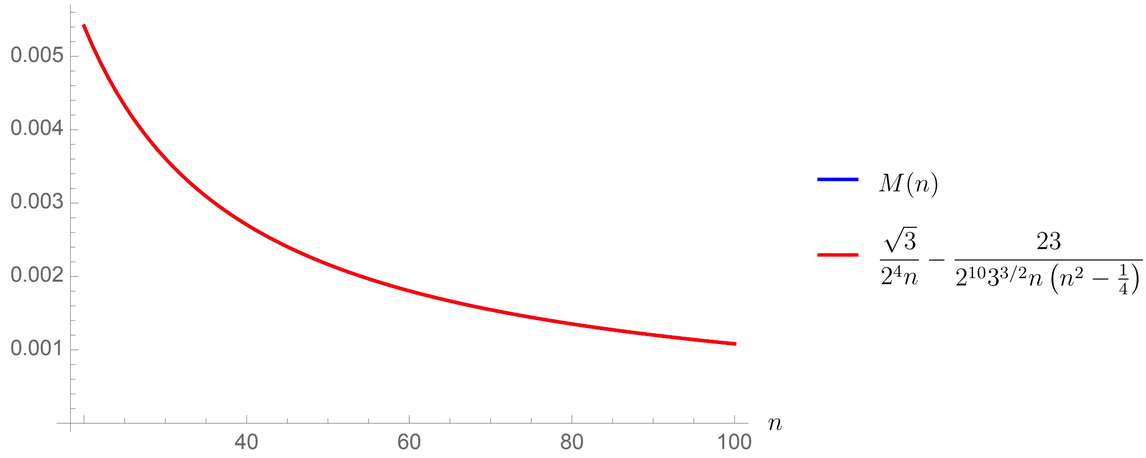

FIGURE 4.8. Plot of $M(\mathbf{n})$ and its two leading asymptotic terms.

Since it is impossible to distinguish the blue and the red curve with the naked eye, Figure 4.9 plots their difference.

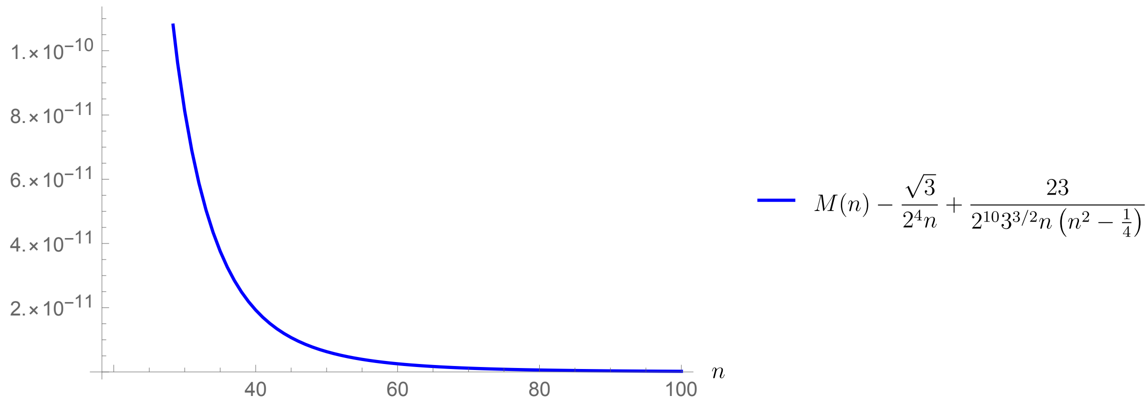

FIGURE 4.9. Plot of the difference of $M(\mathbf{n})$ and its claimed asymptotics.

This visualization suggests that the error term $E_M(n)$ is actually negative. Nevertheless, the upper bound (230) on its absolute value is still larger than the difference between $M(\mathbf{n})$ and its asymptotics. This is pictured in Figure 4.10.

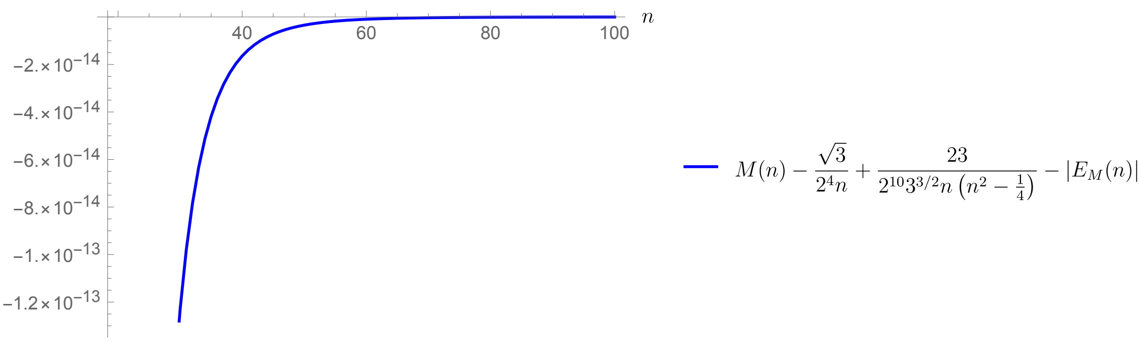

FIGURE 4.10. Plot of the difference of $M(\mathbf{n}) - \frac{3^{\frac{1}{2}}}{2^4}\frac{1}{n} + \frac{23}{2^{10}.3^{\frac{3}{2}}}\frac{1}{n\left(n^2 - \frac{1}{4}\right)}$ and the upper bound on $|E_M(n)|$.

Now back to an actual proof of the facts that are shown in the figures above. Before we state Theorem 36, we will first deduce the two leading terms of $M(\mathbf{n})$ in (229) and, as part of this, provide a formula for the error $E_M(n)$. This first step is done by expressing parts of $M(\mathbf{n})$ as a rational function, which is then written in terms of four telescoping sums. We plug in this new representation into $M(\mathbf{n})$ and reach our first intermediate goal.
This part, as well as the proof of estimate (230), that follows afterwards, make heavy use of the summation formulae from Lemma 66, as well as Lemma 69, that provides closed form expressions for sums of the kind

$$\sum_{j=0}^{p} q(j) \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma\left(j+1\right)\Gamma\left(p-j+1\right)},$$

where q is a polynomial in j. We also need Lemma 49 quite often, that proves identities for the sum

$$\sum_{p=0}^{\infty} p^a \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} x^p,$$

for $a \in \mathbb{Z}, 0 \leqslant a \leqslant 9$. Another crucial ingredient is our old friend Theorem 20. Moreover, at several points of this section we will face the task to calculate finite, but nevertheless large sums of fractions with increasingly large numerators and denominators. To avoid mistakes and master this task with the required diligence, we use the software *Wolfram Mathematica*. So far so good. Now, let us start with the derivation of (229) and the statement of Theorem 36.

Recall the definition (41) of the main term on page 17

$$\begin{aligned} M(\mathbf{n}) = {} & \frac{3}{16\pi^{\frac{3}{2}}} \sum_{p=0}^{2n-1} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} \\ & \times \sum_{j=0}^{p} \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)} \frac{\Gamma\left(n+j+\frac{1}{2}\right)\Gamma\left(n+p-j+\frac{1}{2}\right)\Gamma(2n-p)}{\Gamma\left(n-j+\frac{1}{2}\right)\Gamma\left(n-p+j+\frac{1}{2}\right)\Gamma(2n+p+1)}. \end{aligned} \tag{231}$$

This is a quite cryptical formula, that doesn't reveal its behaviour in n at the first glance. To make $M(\mathbf{n})$ do this, we have to work a little. One may easily guess that the crucial information about the decay in n is hidden in the quotient

$$Q(n,p,j) := \frac{\Gamma\left(n+j+\frac{1}{2}\right)\Gamma\left(n+p-j+\frac{1}{2}\right)\Gamma(2n-p)}{\Gamma\left(n-j+\frac{1}{2}\right)\Gamma\left(n-p+j+\frac{1}{2}\right)\Gamma(2n+p+1)}, \tag{232}$$

which is nothing but a camouflaged quotient of two polynomials: By the functional equation of the gamma function $z\Gamma(z) = \Gamma(z+1)$, it is

$$\begin{aligned} \frac{\Gamma(2n-p)}{\Gamma(2n+p+1)} &= \frac{1}{2n} \prod_{t=1}^{p} \frac{1}{(2n-t)(2n+t)} \\ \frac{\Gamma\left(n+j+\frac{1}{2}\right)}{\Gamma\left(n+j+\frac{1}{2}-p\right)} &= \prod_{t=1}^{p} \left(n+j-t+\frac{1}{2}\right), \\ \frac{\Gamma\left(n-j+\frac{1}{2}+p\right)}{\Gamma\left(n-j+\frac{1}{2}\right)} &= \prod_{t=1}^{p} \left(n-j+t-\frac{1}{2}\right), \end{aligned}$$

and thus,

$$Q(n,p,j) = \frac{2^{-2p-1}}{n} \prod_{t=1}^{p} \frac{n^2 - \left(t-j-\frac{1}{2}\right)^2}{n^2 - \frac{t^2}{4}}. \tag{233}$$

Now it is immediately apparent that $M(n)$ decays of order n^{-1}.

In order to increase the readability of the upcoming formulae, we abbreviate in the following

$$\begin{aligned} \alpha_t(j) &:= \left(t - j - \frac{1}{2}\right)^2, \\ \beta_t &:= \frac{t^2}{4}. \end{aligned} \tag{234}$$

Our goal is to extract the terms of order at most n^{-3} and to establish an upper bound of the form

$$\frac{const}{n\left(n^2 - \frac{1}{4}\right)\left(n^2 - 1\right)}$$

on all remaining orders. To this end we rewrite the product in (233) with the help of the telescoping sum

$$\prod_{t=1}^{p} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_t} = 1 + \sum_{i=1}^{p} \left[\prod_{t=1}^{i} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_t} - \prod_{t=1}^{i-1} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_t} \right],$$

which is equal to

$$\begin{aligned} &1 + \sum_{i=1}^{p} \frac{\prod_{t=1}^{i} \left(n^2 - \alpha_t(j)\right) - \left(n^2 - \beta_i\right) \prod_{t=1}^{i-1} (n^2 - \beta_t)}{\prod_{t=1}^{i} (n^2 - \beta_t)} \\ &= 1 + \sum_{i=1}^{p} \frac{\beta_i - \alpha_i(j)}{n^2 - \beta_1} \prod_{t=1}^{i-1} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_{t+1}}. \end{aligned}$$

We repeat this process two more times and obtain

$$\begin{aligned} Q(n,p,j) &= \frac{2^{-2p}}{2n} \prod_{t=1}^{p} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_t} \\ &= \frac{2^{-2p}}{2n} \left[1 + \sum_{i=1}^{p} \frac{\beta_i - \alpha_i(j)}{n^2 - \frac{1}{4}} + \sum_{i=1}^{p} \sum_{k=1}^{i-1} \frac{\left(\beta_i - \alpha_i(j)\right)\left(\beta_{k+1} - \alpha_k(j)\right)}{\left(n^2 - \frac{1}{4}\right)\left(n^2 - 1\right)} \right. \\ &\left. + \sum_{i=1}^{p} \sum_{k=1}^{i-1} \sum_{l=1}^{k-1} \frac{\left(\beta_i - \alpha_i(j)\right)\left(\beta_{k+1} - \alpha_k(j)\right)\left(\beta_{l+2} - \alpha_l(j)\right)}{\left(n^2 - \frac{1}{4}\right)\left(n^2 - 1\right)\left(n^2 - \frac{9}{4}\right)} \prod_{t=1}^{l-1} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_{t+3}} \right] \end{aligned} \tag{235}$$

In the next step we split

$$M(\mathbf{n}) = M_{<n}(n) + M_{>n}(n),$$

where

$$M_{<n}(n) := \frac{3}{16\pi^{\frac{3}{2}}} \sum_{p=0}^{n} \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma\left(p+1\right)} \sum_{j=0}^{p} \frac{\Gamma\left(j + \frac{1}{2}\right)\Gamma\left(p - j + \frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)} Q(n,p,j),$$

consists of the first $n+1$ summands of the main term, and

$$M_{>n}(n) := \frac{3}{16\pi^{\frac{3}{2}}} \sum_{p=n+1}^{2n-1} \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma\left(p+1\right)} \sum_{j=0}^{p} \frac{\Gamma\left(j + \frac{1}{2}\right)\Gamma\left(p - j + \frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)} Q(n,p,j) \tag{236}$$

includes the remaining ones.
Now we take $M_{<n}(n)$, plug in (235), and thus obtain the representation $M_{<n}(n) = M_1(n) +$

$M_2(n) + M_3(n)$ with

$$M_1(n) := \frac{3}{2^5\pi^{\frac{3}{2}}}\frac{1}{n}\sum_{p=0}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\sum_{j=0}^{p}\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)},$$

$$M_2(n) := \frac{3}{2^5\pi^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)}\sum_{p=1}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\sum_{j=0}^{p}\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}\sum_{i=1}^{p}(\beta_i-\alpha_i(j)),$$

and

$$\begin{aligned} M_3(n) := \frac{3}{2^5\pi^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)}&\left[\sum_{p=2}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\sum_{j=0}^{p}\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}\right.\\ &\times\sum_{i=1}^{p}\sum_{k=1}^{i-1}\left(\beta_i-\alpha_i(j)\right)\left(\beta_{k+1}-\alpha_k(j)\right)\\ &+\frac{1}{n^2-\frac{9}{4}}\sum_{p=3}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\sum_{j=0}^{p}\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}\\ &\left.\times\sum_{i=1}^{p}\sum_{k=1}^{i-1}\sum_{l=1}^{k-1}\left(\beta_i-\alpha_i(j)\right)\left(\beta_{k+1}-\alpha_k(j)\right)\left(\beta_{l+2}-\alpha_l(j)\right)\prod_{t=1}^{l-1}\frac{n^2-\alpha_t(j)}{n^2-\beta_{t+3}}\right]. \end{aligned} \tag{237}$$

By Lemma 66 the sum over i in the definition of $M_2(n)$ is equal to $\frac{1}{8}p\left(-2(p-2j)^2+p+1\right)$. Furthermore, we apply Lemma 69 equation (430) to the j-sum in $M_1(n)$ and obtain

$$M_1(n) = \frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{n}\sum_{p=0}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}.$$

Equations (430), (431) and (432) of the same lemma applied to $M_2(n)$ yield

$$\begin{aligned} M_2(n) &= \frac{3}{2^5\pi^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)}\\ &\quad\times\sum_{p=1}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\sum_{j=0}^{p}\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}\frac{1}{8}p\left(-2(p-2j)^2+p+1\right)\\ &= \frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)}\sum_{p=1}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\frac{1}{8}\left(-p^3+p\right). \end{aligned}$$

These expressions for $M_1(n)$ and $M_2(n)$ can be further evaluated with the help of Lemma 49. Equation (348) gives

$$M_1(n) = \frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{n}\left(\frac{2\pi^{\frac{1}{2}}}{3^{\frac{1}{2}}}-\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\right),$$

and by equations (349) and (351) we get

$$M_2(n) = \frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)}\left(-\frac{23\pi^{\frac{1}{2}}}{2^5\cdot 3^{\frac{5}{2}}}-\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\frac{1}{8}\left(-p^3+p\right)\right).$$

So we finally reached the point where we have introduced all necessary objects to state our main result of this section.

THEOREM 36. *The main term $M(\boldsymbol{n})$ of the series expansion of $\left(\frac{\pi}{2}\right)^2 I(\boldsymbol{n})$ satisfies*

$$M(\boldsymbol{n}) = \frac{3^{\frac{1}{2}}}{2^4}\frac{1}{n}-\frac{23}{2^{10}\cdot 3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)}+E_M(n),$$

where the error term $E_M(n)$ consists of

$$E_M(n) := M_3(n) + M_{>n}(n) - \frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{n}\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}$$
$$+\frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{8n\left(n^2-\frac{1}{4}\right)}\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\left(p^3-p\right),$$

and obeys for all $n \geqslant 20$ the bound

$$|E_M(n)| \leqslant \frac{2689}{2^{18}\cdot 3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)} + \frac{2^{-2n}}{2^5\pi^{\frac{1}{2}}}\left[1+\frac{1}{n}+3^{\frac{1}{2}}e^{\frac{7}{36n}}\frac{n^{\frac{1}{2}}}{n-1}\right].$$

All the remaining pages of this section are now devoted to the estimate of the absolute value of the error term

$$|E_M(n)| \leqslant |M_3(n)| + |M_{>n}(n)| + \frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{n}\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}$$
$$+\frac{3}{2^5\pi^{\frac{1}{2}}}\frac{1}{8n\left(n^2-\frac{1}{4}\right)}\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\left(p^3-p\right). \tag{238}$$

Our strategy is to bound it summand wise. The last two of them are handled in Lemma 37. The component $M_{>n}(n)$ is subject of Lemma 38, and $M_3(n)$ is estimated in Lemma 39. Since it takes some time and effort to establish those lemmata, we first make use of their results and finish the proof of Theorem 36.

Proof of Theorem 36. By Lemma 37 it is for $n \geqslant 2$

$$\frac{3}{2^5\pi^{\frac{1}{2}}n}\left[\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} + \frac{1}{8\left(n^2-\frac{1}{4}\right)}\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\left(p^3-p\right)\right]$$
$$\leqslant \frac{1}{2^5\pi^{\frac{1}{2}}}\frac{n+1}{n}2^{-2n}.$$

Lemma 38 yields that

$$|M_>(n)| \leqslant \frac{3^{\frac{1}{2}}}{2^5\pi^{\frac{1}{2}}}e^{\frac{7}{36n}}\frac{n^{\frac{1}{2}}}{n-1}2^{-2n}$$

if $n \geqslant 7$, and due to Lemma 39, we know that

$$M_3(n) \leqslant \frac{2689}{2^{18}\cdot 3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)}$$

for $n \geqslant 20$. Plugging those estimates into (238) proves Theorem 36 ∎

Now, we turn to the analysis of the single components that make up the error term (238). As mentioned a few lines above, we take care of the last two summands first. In principle, the following bound is a direct consequence of Lemma 67.

Lemma 37. *For all $n \geqslant 2$ it is*

$$\frac{3}{2^5\pi^{\frac{1}{2}}n}\left[\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} + \frac{1}{8\left(n^2-\frac{1}{4}\right)}\sum_{p=n+1}^{\infty} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\left(p^3-p\right)\right]$$
$$\leqslant \frac{1}{2^5\pi^{\frac{1}{2}}}\frac{n+1}{n}2^{-2n}.$$

Proof. We estimate $\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \leqslant 1$ and then apply Lemma 67 on the tail of the geometric series $\sum_{p=n+1}^{\infty} x^p$ and the related series $\sum_{p=n+1}^{\infty} p(p^2-1)x^p$ with $x = \frac{1}{4}$. This yields

$$\sum_{p=n+1}^{\infty} 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} \leqslant \frac{1}{3} 2^{-2n},$$

$$\sum_{p=n+1}^{\infty} p(p^2-1) 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} \leqslant \frac{1}{3}\left(n^3+4n^2+\frac{17}{3}n+\frac{32}{9}\right) 2^{-2n}.$$

Now the assertion follows from

$$n^3+4n^2+\frac{17}{3}n+\frac{32}{9} \leqslant n(n-2)(n-3)$$

and

$$\frac{(n-2)(n-3)}{8\left(n^2-\frac{1}{4}\right)} \leqslant 1$$

for all $n \geqslant 2$. ∎

Our next candidate for estimation is the term $M_>(n)$, that contains all the summands of the main term (231) for $n+1 \leqslant p \leqslant 2n-1$. Its contribution to the error term $E_M(n)$ is subject of the next lemma. The larger part of the proof of the lemma consists of estimating the maximum of a gamma quotient by means of the well-known techniques from Subsection 4.2.3.

Lemma 38. *The tail $M_>(n)$ of $M(\boldsymbol{n})$, defined in (236) satisfies for all $n \geqslant 7$*

$$|M_>(n)| \leqslant \frac{3^{\frac{1}{2}}}{2^5\pi^{\frac{1}{2}}} e^{\frac{7}{36n}} \frac{n^{\frac{1}{2}}}{n-1} 2^{-2n}.$$

Proof. By Lemma 69 equation (430) it is

$$\begin{aligned} |M_>(n)| &\leqslant \frac{3}{16\pi^{\frac{3}{2}}} \sum_{p=n+1}^{2n-1} \sum_{j=0}^{p} \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(j+1)\Gamma(p-j+1)} |Q(n,p,j)| \\ &\leqslant \frac{3}{16\pi^{\frac{1}{2}}} \max_{\substack{n+1\leqslant p\leqslant 2n-1 \\ 0\leqslant j\leqslant p}} |Q(n,p,j)| \sum_{p=n+1}^{2n-1} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}. \end{aligned}$$

Note that

$$\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} = 2\left(\frac{\Gamma\left(p+\frac{3}{2}\right)}{\Gamma(p+1)} - \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p)}\right).$$

Thus the last sum above is telescoping and evaluates to

$$\sum_{p=n+1}^{2n-1} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} = 2\left(\frac{\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n)} - \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+1)}\right),$$

which is at most

$$2\left((2n)^{\frac{1}{2}} - \left(n+\frac{1}{2}\right)^{\frac{1}{2}}\right) < 2n^{\frac{1}{2}},$$

due to Lemma 44. Consequently, our first bound on $M_>(n)$ is

$$M_>(n) \leqslant \frac{3}{8\pi^{\frac{1}{2}}} n^{\frac{1}{2}} \max_{\substack{n+1\leqslant p\leqslant 2n-1 \\ 0\leqslant j\leqslant p}} |Q(n,p,j)|. \tag{239}$$

In the remaining part of the proof we will locate and estimate the maximum of

$$|Q(n,p,j)| = \frac{\Gamma\left(n+j+\frac{1}{2}\right)\Gamma\left(n+p-j+\frac{1}{2}\right)\Gamma(2n-p)}{\left|\Gamma\left(n-j+\frac{1}{2}\right)\Gamma\left(n-p+j+\frac{1}{2}\right)\right|\Gamma(2n+p+1)}$$

for $n+1 \leqslant p \leqslant 2n-1$ and $0 \leqslant j \leqslant p$.

First of all note that by the same symmetry arguments as in the last two sections 4.2 and 4.3, we can assume an ascending ordering of the two differences

$$j \leqslant p-j$$

when looking for the maximum of $Q(n,p,j)$. Moreover, since $j+(p-j) = p \leqslant 2n-1$, we can immediately infer that

$$j \leqslant n-1.$$

Thus, it is

$$\max_{\substack{n+1\leqslant p\leqslant 2n-1\\ 0\leqslant j\leqslant p-j}} |Q(n,p,j)| = \max_{\substack{n+1\leqslant p\leqslant 2n-1\\ 0\leqslant j\leqslant p-j}} \frac{\Gamma\left(n+j+\frac{1}{2}\right)\Gamma\left(n+p-j+\frac{1}{2}\right)\Gamma(2n-p)}{\Gamma\left(n-j+\frac{1}{2}\right)\left|\Gamma\left(n-p+j+\frac{1}{2}\right)\right|\Gamma(2n+p+1)}. \quad (240)$$

In the following we distinguish between the two cases $p-j \leqslant n$ and $p-j \geqslant n+1$, starting with the former one. Note that, due to the missing factors $\Gamma(j+1)\Gamma(p-j+1)$ in the denominator, we can not simply apply Theorem 22 here.

Since $\Gamma\left(n-p+j+\frac{1}{2}\right) > 0$ in the case of $p-j \leqslant n$, we can omit the absolute value

$$|Q(n,p,j)| = Q(n,p,j).$$

Calculating the recurrence relation $Q(n,p,j) = \varphi_1(j)Q(n,p,j-1)$ for $1 \leqslant j \leqslant \min\{n-1, p-j\}$, we get

$$\begin{aligned}\varphi_1(j) &= \frac{\left(n+j-\frac{1}{2}\right)\left(n-j+\frac{1}{2}\right)}{\left(n-p+j-\frac{1}{2}\right)\left(n+p-j+\frac{1}{2}\right)}\\ &= \frac{n^2-\left(j-\frac{1}{2}\right)^2}{n^2-\left(p-j-\frac{1}{2}\right)^2}.\end{aligned}$$

Since the last line is greater than or equal to one for all $j \leqslant p-j$, we can conclude from Theorem 20 *(i)* that $Q(n,p,j)$ attains its maximum in j for $j = p-j$. Meaning in the maximum of $Q(n,p,j)$ the relation $p = 2j$ has to be satisfied. Thus our new object of interest is

$$Q(n,2j,j) = \frac{\Gamma\left(n+j+\frac{1}{2}\right)^2\Gamma(2n-2j)}{\Gamma\left(n-j+\frac{1}{2}\right)^2\Gamma(2n+2j+1)}$$

for j such that $n+1 \leqslant 2j \leqslant 2n-1$.
We continue by determining the recurrence $Q(n,2j,j) = \varphi_2(j)Q(n,2(j-1),j-1)$ in j, and yield

$$\varphi_2(j) = \frac{\left(n-j+\frac{1}{2}\right)\left(n+j-\frac{1}{2}\right)}{(4n-4j)(4n+4j)}.$$

The function $\varphi_2(j)$ is equal to one if

$$15j^2 + j - (15n^2+1) = 0,$$

and smaller than one between the two roots j_1, j_2 of this quadratic expression. Those are

$$j_{1,2} = -\frac{1}{30} \mp \left(n^2 - \frac{19}{225}\right)^{\frac{1}{2}}.$$

Since $j_1 < \frac{n+1}{2} \leqslant j \leqslant \frac{2n-1}{2} < j_2$ for $n \in \mathbb{R}$, Theorem 20 *(ii)* allows us to draw the conclusion that $Q(n, 2j, j)$ decreases monotonically in j and thus is maximal for $j = \frac{n+1}{2}$. Hence, the first applicant for the maximum is

$$\begin{aligned} Q\left(n, n+1, \frac{n+1}{2}\right) &= \frac{\Gamma\left(\frac{3}{2}n+1\right)^2 \Gamma(n-1)}{\Gamma\left(\frac{1}{2}n\right)^2 \Gamma(3n+2)} \\ &= \frac{n}{4\left(n+\frac{1}{3}\right)(n-1)} \frac{\Gamma\left(\frac{3}{2}n\right)^2 \Gamma(n)}{\Gamma\left(\frac{1}{2}n\right)^2 \Gamma(3n)}. \end{aligned} \tag{241}$$

We use that $n \leqslant n + \frac{1}{3}$ and estimate (241) using Stirling's formula (333). This way we obtain the upper bound

$$\frac{1}{4(n-1)} \frac{\Gamma\left(\frac{3}{2}n\right)^2 \Gamma(n)}{\Gamma\left(\frac{1}{2}n\right)^2 \Gamma(3n)} \leqslant \frac{1}{4\sqrt{3}} \frac{1}{n-1} e^{\frac{7}{36n}} 2^{-2n}. \tag{242}$$

Next we assume that $p - j \geqslant n+1$. This, together with the restriction $n+1 \leqslant p \leqslant 2n-1$ implies that either $j = n-2$ and $p = 2n-1$, or that $0 \leqslant j \leqslant n-3$ and $j+n+1 \leqslant p \leqslant 2n-1$. The first possibility immediately leads to the second candidate for the maximum

$$|Q(n, 2n-1, n-2)| = \frac{2}{3\pi} \frac{\Gamma\left(2n-\frac{3}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(4n)}. \tag{243}$$

Here, and in the following calculations for the case $p-j \geqslant n+1$, we use that by the reflection formula (327)

$$\left|\Gamma\left(n-p+j+\frac{1}{2}\right)\right| = \frac{\pi}{\Gamma\left(-n+p-j+\frac{1}{2}\right)}.$$

By the functional equation of the gamma function, the right-hand side of (243) is equal to

$$\frac{2}{3\pi} \frac{2n+\frac{1}{2}}{2n-\frac{3}{2}} \frac{\Gamma\left(2n-\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(4n)}. \tag{244}$$

Note that, due to

$$\begin{aligned} \max_{n+1 \leqslant 2j \leqslant 2n-1} Q(n, 2j, j) &\geqslant Q(n, 2n-1, n-1) \\ &= \frac{2}{\pi} \frac{\Gamma\left(2n-\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(4n)} \\ &\geqslant \frac{2}{3\pi} \frac{2n+\frac{1}{2}}{2n-\frac{3}{2}} \frac{\Gamma\left(2n-\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(4n)}, \end{aligned} \tag{245}$$

candidate (243) is smaller than the first candidate (241) for all $n \geqslant 2$.

For the other possibility $0 \leqslant j \leqslant n-3$ and $j+n+1 \leqslant p \leqslant 2n-1$ we calculate the recurrence $|Q(n,p,j)| = \varphi_3(p)|Q(n,p-1,j)|$ in p. This is

$$\begin{aligned} \varphi_3(p) &= \frac{\left(n+p-j-\frac{1}{2}\right)\left(-n+p-j-\frac{1}{2}\right)}{(2n-p)(2n+p)} \\ &= \frac{\left(p-j-\frac{1}{2}\right)^2 - n^2}{4n^2-p^2}. \end{aligned}$$

The function $\varphi_3(p)$ is obviously monotonically increasing in p. At the left boundary $p = j+n+1$ it satisfies

$$\varphi_3(j+n+1) = \frac{n+\frac{1}{4}}{4n^2-(j+n+1)^2},$$

which in turn is an increasing function in j with

$$\varphi_3(j+n+1)\Big|_{j=n-3} = \frac{n+\frac{1}{4}}{2(4n-2)} < 1.$$

At the right boundary $p = 2n-1$ the function φ_3 takes the value

$$\varphi_3(2n-1) = \frac{\left(3n-j-\frac{3}{2}\right)\left(n-j-\frac{3}{2}\right)}{4n+1}.$$

This is equal to one, if

$$j^2 - j(4n-3) + 3n^2 - 10n + \frac{5}{4} = 0.$$

Hence, $\varphi_3(2n-1) \leqslant 1$ for j between the two roots of the above quadratic equation, and $\varphi_3(2n-1) > 1$ else. Those roots are

$$j_{1,2} = 2n - \frac{3}{2} \mp \left(n^2+4n+1\right)^{\frac{1}{2}}.$$

Since our j lives in the range $0 \leqslant j \leqslant n-3$, the root j_2 is irrelevant for us. For the further analysis of the smaller root j_1, we estimate $n^2+4n+1 < (n+2)^2$ and thus find that

$$n-4 < j_1 < n-3,$$

meaning that $\varphi_3(2n-1) > 1$ for all $0 \leqslant j \leqslant n-4$, and $\varphi_3(2n-1) < 1$ for $j = n-3$. We consult Theorem 20 and conclude that the gamma quotient $|Q(n,p,n-3)|$ is monotonically decreasing in p, and thus maximal for $p = j+n+1 = 2n-2$. Moreover, for $0 \leqslant j \leqslant n-4$, the function $|Q(n,p,j)|$ has a minimum between $p = j+n+1$ and $p = 2n-1$, and thus has two local maxima, located at the left and the right boundary.

Before we continue with the determination of the j-maxima at the boundaries $p-j = n+1$ and $p = 2n-1$, we estimate our third candidate for the maximum at $j = n-3, p = 2n-2$. This is

$$\begin{aligned} |Q(n,2n-2,n-3)| &= \frac{4}{15\pi}\frac{\Gamma\left(2n-\frac{5}{3}\right)\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(4n-1)} \\ &= \frac{2}{\pi}\frac{(4n-1)(4n+1)}{\left(6n-\frac{15}{2}\right)\left(10n-\frac{15}{2}\right)}\frac{\Gamma\left(2n-\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(4n)} \\ &< Q(n,2n-1,n-1) \end{aligned} \tag{246}$$

by comparison with (244) for all $n \geqslant 2$.

Now we turn to the right boundary $p = 2n-1$. At this point we have

$$|Q(n,2n-1,j)| = \frac{\Gamma\left(n+j+\frac{1}{2}\right)\Gamma\left(3n-j-\frac{1}{2}\right)\Gamma\left(n-j-\frac{1}{2}\right)}{\pi\Gamma\left(n-j+\frac{1}{2}\right)\Gamma(4n)},$$

which satisfies in j the recurrence relation $|Q(n,2n-1,j)| = \varphi_4(j)|Q(n,2n-1,j-1)|$, with

$$\varphi_4(j) = \frac{\left(n-j+\frac{1}{2}\right)\left(n-j+\frac{1}{2}\right)}{\left(3n-j-\frac{1}{2}\right)\left(n-j-\frac{1}{2}\right)}$$

for $1 \leqslant j \leqslant n-4$.

By the usual procedure we find that $\varphi_4(j) < 1$ for $1 \leqslant n - \sqrt{n-\frac{1}{4}}$ and $\varphi_4(j) > 1$ for $n-\sqrt{n-\frac{1}{4}} < j < n-4$, and thus infer from Theorem 20 *(iv)* that $|Q(n,2n-1,j)|$ has a minimum at $j^* \in \left(n-\sqrt{n-\frac{1}{4}}-1, n-\sqrt{n-\frac{1}{4}}\right]$ and two endpoint maxima at $j = 0$ and $j = n-4$.

Consequently, in our search for the global maximum we have to take into consideration two more terms, namely

$$|Q(n, 2n-1, 0)| = \frac{\Gamma\left(3n - \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right)}{\pi\Gamma(4n)}, \tag{247}$$

and

$$|Q(n, 2n-1, n-4)| = \frac{2}{7\pi}\frac{\Gamma\left(2n - \frac{7}{2}\right)\Gamma\left(2n + \frac{7}{2}\right)}{\Gamma(4n)}. \tag{248}$$

By the functional equation of the gamma function, expression (248) is equal to

$$\frac{2}{\pi}\frac{1}{7}\frac{\left(2n + \frac{5}{2}\right)\left(2n + \frac{3}{2}\right)\left(2n + \frac{1}{2}\right)}{\left(2n - \frac{7}{2}\right)\left(2n - \frac{5}{2}\right)\left(2n - \frac{3}{2}\right)}\frac{\Gamma\left(2n - \frac{1}{2}\right)\Gamma\left(2n + \frac{1}{2}\right)}{\Gamma(4n)},$$

which is smaller than the first candidate (241) if $n \geqslant 2$. This can again be seen by comparison with $Q(n, 2n-1, n-1)$ as in (245).

The term (247) is treated with Stirling's formula (332). We then yield

$$\begin{aligned}\frac{\Gamma\left(3n - \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right)}{\pi\Gamma(4n)} &\leqslant \frac{\Gamma(3n)\Gamma(n)}{\pi\Gamma(4n)}, \\ &\leqslant \frac{2^{\frac{3}{2}}}{(3\pi n)^{\frac{1}{2}}} e^{\frac{1}{9n}} \left(\frac{3^3}{2^8}\right)^n. \end{aligned} \tag{249}$$

Next we handle the left boundary point $p - j = n + 1$, where the quotient $|Q(n, p, j)|$ takes the value

$$|Q(n, j+n+1, j)| = \frac{\Gamma\left(2n + \frac{3}{2}\right)\Gamma\left(n + j + \frac{1}{2}\right)\Gamma(n - j - 1)}{2\pi^{\frac{1}{2}}\Gamma\left(n - j + \frac{1}{2}\right)\Gamma(3n + j + 2)}.$$

It satisfies the recurrence relation $|Q(n, j+n+1, j)| = \varphi_5(j)|Q(n, j+n, j-1)|$ for $1 \leqslant j \leqslant n - 4$ with

$$\varphi_5(j) = \frac{\left(n + j - \frac{1}{2}\right)\left(n - j + \frac{1}{2}\right)}{(n - j - 1)(3n + j + 1)}.$$

We find that $\varphi_5(j) \leqslant 1$ for all $j \leqslant \frac{2n^2 - 2n - \frac{3}{4}}{2n+3}$. Since we are only interested in $j \leqslant n - 4 = \frac{(n-4)(2n+3)}{2n+3} < \frac{2n^2-2n-\frac{3}{4}}{2n+3}$, we can conclude by Theorem 20 *(ii)* that $|Q(n, j+n+1, j)|$ is monotonically decreasing in j and thus maximal for $j = 0$. Therefore, the last potential maximum is

$$\begin{aligned}|Q(n, n+1, 0)| &= \frac{\Gamma\left(2n + \frac{3}{2}\right)\Gamma(n-1)}{2\pi^{\frac{1}{2}}\Gamma(3n+2)} \\ &\leqslant \frac{1}{2\pi^{\frac{1}{2}}}\frac{\left(2n + \frac{1}{2}\right)\left(2n - \frac{1}{2}\right)}{3n(3n+1)}\frac{\Gamma(2n)\Gamma(n)}{\Gamma(3n)} \\ &\leqslant \frac{2}{9\pi^{\frac{1}{2}}}\frac{\Gamma(2n)\Gamma(n)}{\Gamma(3n)}. \end{aligned}$$

By Stirling's formula (332), the last line above is at most

$$\frac{2}{3^{\frac{3}{2}} n^{\frac{1}{2}}} e^{\frac{1}{8n}} \left(\frac{2^2}{3^3}\right)^n. \tag{250}$$

This is larger than (249). In fact, we divide (249) by (250) and get

$$3\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{72n}} \left(\frac{3^6}{2^{10}}\right)^n < 3\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{729}{1024}\right)^n,$$

which decreases monotonically in n, and falls below one as soon as $n \geqslant 3$.

Last but not least we compare (250) to the right-hand side of (242) to show that the value $Q\left(n, n+1, \frac{n+1}{2}\right)$ is actually the largest one the quotient $|Q(n,p,j)|$ takes for $n+1 \leqslant p \leqslant 2n-1$ and $0 \leqslant j \leqslant p$. To this end let's have a look at the quotient of (250) and the right-hand side of (242). It reads

$$\frac{8}{3}\frac{n-1}{n^{\frac{1}{2}}} e^{-\frac{5}{72n}} \left(\frac{16}{27}\right)^n,$$

and is obviously smaller than

$$\frac{8}{3} n^{\frac{1}{2}} \left(\frac{16}{27}\right)^n.$$

We determine the root of the derivative of the above line and find a maximum of this function at the point $n = \frac{1}{2\log\left(\frac{27}{16}\right)} < 1$. Hence, it is decreasing for $n \geqslant 1$ and satisfies

$$\frac{8}{3} n^{\frac{1}{2}} \left(\frac{16}{27}\right)^n \Big|_{n=3} < 1.$$

This proves that

$$\max_{\substack{n+1 \leqslant p \leqslant 2n-1 \\ 0 \leqslant j \leqslant p}} |Q(n,p,j)| \leqslant \frac{1}{4\sqrt{3}} \frac{1}{n-1} e^{\frac{7}{36n}} 2^{-2n}.$$

Plugging this into estimate (239), completes the proof of the Lemma. ∎

Next in the line is the largest and therefore most difficult component of $E_M(n)$, the term $M_3(n)$. The most important tools for the estimate of $M_3(n)$ are Lemma 66 on summation formulae for polynomials, Lemma 69, that provides closed form expressions for convolution like sums of the type

$$\sum_{j=0}^{p} j^a \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}$$

and the formulae from Lemma 49 for sums like

$$\sum_{p=0}^{\infty} p^a \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} x^p.$$

In both cases, the variable a is an integer. Then the main idea behind the proof of the following Lemma 39 is to first split $M_3(n)$ into two parts. One, that carries the important information, and a second one, that is small compared to the first one and negative. In order to show those properties of the second term, we split it once again into two partial sums. One of them consists of finitely many summands and can be calculated explicitly using *Wolfram Mathematica*, and the other one is treated by means of the aforementioned lemmata.

LEMMA 39. *For all $n \geqslant 20$ the term $M_3(n)$, defined in* (237)*, is positive and does not exceed*

$$\frac{2689}{2^{18} \cdot 3^{\frac{3}{2}}} \frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)}.$$

The claimed upper bound on $M_3(n)$ is quite sharp, as can be seen in Figure 4.11.

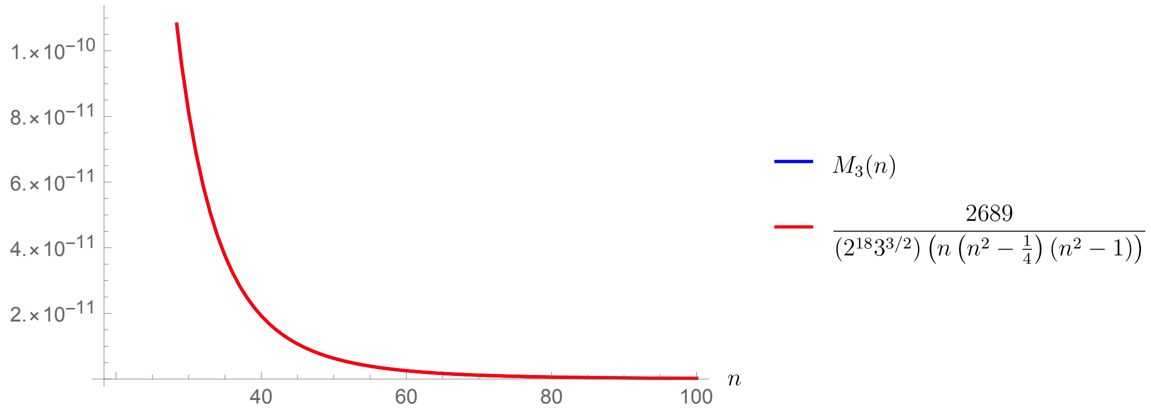

FIGURE 4.11. Plot of $M_3(n)$ and its upper bound.

Similar to Figure 4.8 it is not really possible to distinguish the two curves. For this reason, Figure 4.12 shows their difference.

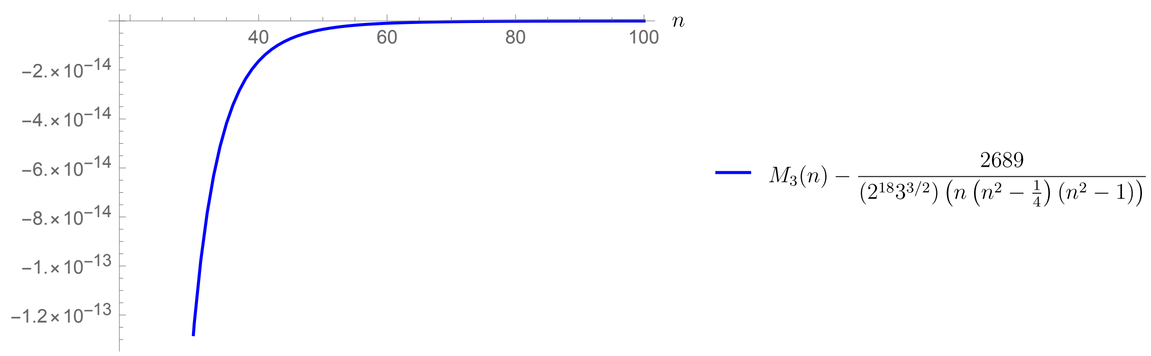

FIGURE 4.12. Plot of the difference of $M_3(n)$ and its upper bound.

Note that it is not surprising, that Figures 4.10 and 4.12 look so similar. Recall the total upper bound on the error

$$|E_M(n)| \leqslant \frac{2689}{2^{18}\cdot 3^{\frac{3}{2}}} \frac{1}{n\left(n^2-\frac{1}{4}\right)(n^2-1)} + \frac{2^{-2n}}{2^5\pi^{\frac{1}{2}}}\left[1+\frac{1}{n}+3^{\frac{1}{2}}e^{\frac{7}{36n}}\frac{n^{\frac{1}{2}}}{n-1}\right].$$

Since its second component is small compared to the first component, which is the upper bound on $M_3(n)$, Figure 4.10 essentially shows the error we make when estimating $M_3(n)$ by $\frac{2689}{2^{18}.3^{\frac{3}{2}}}\frac{1}{n\left(n^2-\frac{1}{4}\right)(n^2-1)}$.

After this quick graphical excursion, we now turn to the proof of Lemma 39.

PROOF OF LEMMA 39. In the following we denote

$$\hat{M}_3(n) := \sum_{p=2}^{n} 2^{-2p}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}\sum_{j=0}^{p}\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}\sum_{i=1}^{p}\sum_{k=1}^{i-1}\left(\beta_i-\alpha_i(j)\right)\left(\beta_{k+1}-\alpha_k(j)\right), \tag{251}$$

and

$$\tilde{M}_3(n) := \sum_{p=3}^{n} 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} \sum_{j=0}^{p} \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}$$
$$\times \sum_{i=1}^{p}\sum_{k=1}^{i-1}\sum_{l=1}^{k-1} \left(\beta_i - \alpha_i(j)\right)\left(\beta_{k+1} - \alpha_k(j)\right)\left(\beta_{l+2} - \alpha_l(j)\right) \prod_{t=1}^{l-1} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_{t+3}}. \tag{252}$$

So it is

$$M_3(n) = \frac{3}{2^5 \pi^{\frac{3}{2}}} \frac{1}{n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)} \left[\hat{M}_3(n) + \frac{\tilde{M}_3(n)}{n^2-\frac{9}{4}}\right]. \tag{253}$$

In a first step, we have a closer look at $\hat{M}_3(n)$. Recall the definition (234) of the quadratic expressions α and β

$$\alpha_t(j) = \left(t - j - \frac{1}{2}\right)^2, \quad \beta_t = \frac{t^2}{4}.$$

With the help of our summation formulae, Lemma 66, we evaluate the double sum over i and k and get

$$\sum_{i=1}^{p}\sum_{k=1}^{i-1} \left(\beta_i - \alpha_i(j)\right)\left(\beta_{k+1} - \alpha_k(j)\right) = \frac{1}{2^7} p(p-1)\Big(64j^4 - 128j^3 p + j^2\left(96p^2 - 48p - 64\right)$$
$$+ j\left(-32p^3 + 48p^2 + 64p\right) + 4p^4 - 12p^3 - 13p^2 + 9p + 6\Big).$$

Next, we apply Lemma 69 summand wise and arrive at

$$\frac{1}{2^7} p(p-1) \sum_{j=0}^{p} \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)} \Big[64j^4 - 128j^3 p + j^2\left(96p^2 - 48p - 64\right)$$
$$+ j\left(-32p^3 + 48p^2 + 64p\right) + 4p^4 - 12p^3 - 13p^2 + 9p + 6\Big] \tag{254}$$
$$= \frac{3}{2^8}\pi (p-1)p(p+1)\left(p^3 - 3p^2 - 4p + 4\right).$$

Thus, we just showed, that $\hat{M}_3(n)$ actually can be expressed in the following simpler form

$$\hat{M}_3(n) = \frac{3}{2^8}\pi \sum_{p=2}^{n} 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} (p-1)p(p+1)\left(p^3 - 3p^2 - 4p + 4\right). \tag{255}$$

Moreover, since the last line of (254) is positive for all $p \geqslant 4$, and

$$\hat{M}_3(6) = \frac{3}{2^8}\pi \sum_{p=2}^{6} 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} (p-1)p(p+1)\left(p^3 - 3p^2 - 4p + 4\right)$$
$$= \frac{151119\pi^{3/2}}{2^{26}} > 0.$$

we can infer that $\hat{M}_3(n)$ is positive and monotonically increasing as soon as $n \geqslant 6$. This leads us to the obvious upper and lower bounds for $n \geqslant 20$

$$0 < \hat{M}_3(20) \leqslant \hat{M}_3(n) \leqslant \hat{M}_3(\infty). \tag{256}$$

The upper bound can be further calculated as

$$\hat{M}_3(\infty) = \frac{3}{2^8}\pi \sum_{p=2}^{\infty} 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} (p-1)p(p+1)\left(p^3 - 3p^2 - 4p + 4\right)$$
$$= \frac{2689\pi^{3/2}}{2^{13} \cdot 3^{\frac{5}{2}}},$$

with the help of Lemma 49 for $x = \frac{1}{4}$. Consequently, for $n \geqslant 20$ we arrive at the bound

$$M_3(n) \leqslant \frac{3}{2^5 \pi^{\frac{3}{2}}} \frac{1}{n\left(n^2 - \frac{1}{4}\right)\left(n^2 - 1\right)} \left[\frac{2689\pi^{3/2}}{2^{13} \cdot 3^{\frac{5}{2}}} + \frac{\tilde{M}_3(n)}{n^2 - \frac{9}{4}}\right]. \tag{257}$$

Next, we take care of $\tilde{M}_3(n)$. In a minute we will see, that it has the following two properties.

Lemma 40. *For all $n \geqslant 20$ it is*

$$\frac{|\tilde{M}_3(n)|}{n^2 - \frac{9}{4}} < \hat{M}_3(20), \tag{258}$$

and

$$\tilde{M}_3(n) < 0. \tag{259}$$

Before we dive into the proof of Lemma 40, we use its statements to finish the proof of Lemma 39.
Since the factor $\frac{3}{2^5 \pi^{\frac{3}{2}}} \frac{1}{n\left(n^2 - \frac{1}{4}\right)\left(n^1 - 1\right)}$ in (253) is positive for $n \geqslant 20$, we obtain the positivity of $M_3(n)$ for all $n \geqslant 20$ from (259) and (258) above, paired with (256) as follows

$$\hat{M}_3(n) + \frac{\tilde{M}_3(n)}{n^2 - \frac{9}{4}} > \hat{M}_3(20) - \frac{|\tilde{M}_3(n)|}{n^2 - \frac{9}{4}} > 0.$$

The second assertion (259) of Lemma 40 together with the bound (257) provides the claimed inequality

$$M_3(n) \leqslant \frac{3}{2^5 \pi^{\frac{3}{2}}} \frac{1}{n\left(n^2 - \frac{1}{4}\right)\left(n^2 - 1\right)} \frac{2689\pi^{3/2}}{2^{13} \cdot 3^{\frac{5}{2}}},$$

and concludes the proof of Lemma 39. ∎

So let's turn to the

Proof of Lemma 40. Throughout the proof of the lemma we denote the summand of $\tilde{M}_3(n)$ by

$$\mathcal{S}(p) := 2^{-2p} \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma\left(p + 1\right)} \sum_{j=0}^{p} \frac{\Gamma\left(j + \frac{1}{2}\right)\Gamma\left(p - j + \frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}$$
$$\times \sum_{i=1}^{p} \sum_{k=1}^{i-1} \sum_{l=1}^{k-1} \left(\beta_i - \alpha_i(j)\right)\left(\beta_{k+1} - \alpha_k(j)\right)\left(\beta_{l+2} - \alpha_l(j)\right) P(n,j,l)$$

for $p \leqslant n$, with

$$P(n,j,l) := \prod_{t=1}^{l-1} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_{t+3}}. \tag{260}$$

Our strategy is to evaluate $\tilde{M}_3(20)$ exactly and then estimate the remaining sum over $p \geqslant 21$. This is, we split

$$\tilde{M}_3(n) = \tilde{M}_3(20) + \sum_{p=21}^{n} \mathcal{S}(p),$$

which leads to the first simple bounds

$$\tilde{M}_3(n) \leqslant \tilde{M}_3(20) + \sum_{p=21}^{n} |\mathcal{S}(p)|, \tag{261}$$

$$|\tilde{M}_3(n)| \leqslant |\tilde{M}_3(20)| + \sum_{p=21}^{n} |\mathcal{S}(p)|. \tag{262}$$

Hence, we have to find a suitable estimate on the absolute value of the expression $\mathcal{S}$.

We approach this goal from the rear end and establish an upper bound on the product $P(n, j, l)$.

Lemma 41. *Let* $\alpha_t(j) = \left(t - j - \frac{1}{2}\right)^2$ *and* $\beta_t = \frac{t^2}{4}$ *with* $0 \leqslant j \leqslant p$ *and*

$$P(n, j, l) = \prod_{t=1}^{l-1} \frac{n^2 - \alpha_t(j)}{n^2 - \beta_{t+3}}$$

be the product from (260)*. Then* $P(n, j, l)$ *satisfies for all* $n \geqslant 2$ *and* $l \leqslant p - 2 \leqslant n - 2$

$$\max_{0 \leqslant j \leqslant p} P(n, j, l) \leqslant \left(\frac{4}{3}\right)^2 e^{\frac{1}{2}}.$$

We postpone the proof of Lemma 41 until the end of the proof of Lemma 40 and instead immediately use the newly gained insight on the product $P(n, j, l)$ here to bound

$$\begin{aligned} |\mathcal{S}(p)| \leqslant & \left(\frac{4}{3}\right)^2 e^{\frac{1}{2}} 2^{-2p} \frac{\Gamma\left(p + \frac{1}{2}\right)}{\Gamma\left(p+1\right)} \sum_{j=0}^{p} \frac{\Gamma\left(j + \frac{1}{2}\right)\Gamma\left(p - j + \frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)} \\ & \times \sum_{i=1}^{p} \sum_{k=1}^{i-1} \sum_{l=1}^{k-1} \left|\left(\beta_i - \alpha_i(j)\right)\left(\beta_{k+1} - \alpha_k(j)\right)\left(\beta_{l+2} - \alpha_l(j)\right)\right|. \end{aligned} \tag{263}$$

Next, we determine the maximum of the three factors $|\beta_{i+m} - \alpha_i(j)|$ for $1 \leqslant i \leqslant p - m, 0 \leqslant j \leqslant p$ and $m = 0, 1, 2$. We start with $m = 0$ and recall

$$\beta_i - \alpha_i(j) = \frac{i^2}{4} - \left(i - j - \frac{1}{2}\right)^2.$$

Taking the derivative in i we find that $\beta_i - \alpha_i(j)$ has a maximum at $i = \frac{4}{3}j + \frac{2}{3}$. Moreover, it is

$$\frac{i^2}{4} - \left(i - j - \frac{1}{2}\right)^2 \begin{cases} \leqslant 0, & 1 \leqslant i \leqslant \frac{2}{3}j + \frac{1}{3}, \\ > 0, & \frac{2}{3}j + \frac{1}{3} < i < 2j + 1, \\ \leqslant 0, & 2j + 1 \leqslant i \leqslant p. \end{cases}$$

This means that the absolute value $|\beta_i - \alpha_i(j)|$ attains its maximum either at one of the two boundary points $i = 1$ and $i = p$, or at $i = \frac{4}{3}j + \frac{2}{3}$.
Note that due to the condition $j \leqslant p$, the right boundary $i = p$ is only relevant if $0 \leqslant j \leqslant \frac{p}{2} - \frac{1}{2}$. For the same reason we need that $j \leqslant \frac{3}{4}p - \frac{1}{2}$, in order for the apex $i = \frac{4}{3}j + \frac{2}{3}$ of the parabola to be a possible maximum. Thus, we yield

$$\begin{aligned} |\beta_i - \alpha_i(j)| &\leqslant \max\left\{j(j-1), \frac{1}{3}\left(j + \frac{1}{2}\right)^2, \left(p - j - \frac{1}{2}\right)^2 - \frac{p^2}{4}\right\} \\ &\leqslant \max\left\{p(p-1), \frac{3}{16}p^2, \frac{3}{4}p^2\right\} \\ &= p(p-1). \end{aligned}$$

Similar calculations for $m = 1$ and $m = 2$ lead to the same result in both cases

$$|\beta_{i+m} - \alpha_i(j)| \leqslant p(p-1).$$

Hence, we showed that

$$\sum_{i=1}^{p}\sum_{k=1}^{i-1}\sum_{l=1}^{k-1} |(\beta_i - \alpha_i(j))(\beta_{k+1} - \alpha_k(j))(\beta_{l+2} - \alpha_l(j))| \leqslant \sum_{i=1}^{p}\sum_{k=1}^{i-1}\sum_{l=1}^{k-1} p^3(p-1)^3.$$

With the help of our friends, the summation formulae of Lemma 66, this evaluates to

$$\frac{1}{6}p\left(p^2 - 3p + 2\right)p^3(p-1)^3,$$

and obeys for all $p \geqslant 1$ the upper bound

$$\frac{1}{6}p^4(p-1)^5.$$

Now, we plug in this estimate into (263) and apply another one of our good friends in this section, namely Lemma 69, to the remaining sum in j. This way we obtain

$$|\mathcal{S}(p)| \leqslant \left(\frac{2}{3}\right)^3 \pi e^{\frac{1}{2}} p^4(p-1)^5 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} \tag{264}$$

and our achievement so far is the bound

$$\sum_{p=21}^{n} |\mathcal{S}(p)| \leqslant \left(\frac{2}{3}\right)^3 \pi e^{\frac{1}{2}} \left[\sum_{p=0}^{\infty} p^4(p-1)^5 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} - \sum_{p=0}^{20} p^4(p-1)^5 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}\right]. \tag{265}$$

Now, we are almost there! The part containing the infinite series can be calculated using Lemma 49 with $x = 2^{-2}$ and evaluates to

$$\left(\frac{2}{3}\right)^3 e^{\frac{1}{2}} \pi \sum_{p=0}^{\infty} p^4(p-1)^5 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} = \frac{137297659}{2^5 3^{\frac{17}{2}}} e^{\frac{1}{2}} \pi^{\frac{3}{2}}. \tag{266}$$

Going back to (261) and (262), and plugging in (265) and (266) yields

$$\tilde{M}_3(n) \leqslant \tilde{M}_3(20) + \frac{137297659}{2^5 3^{\frac{17}{2}}} e^{\frac{1}{2}} \pi^{\frac{3}{2}} - \left(\frac{2}{3}\right)^3 e^{\frac{1}{2}} \pi \sum_{p=0}^{20} p^4(p-1)^5 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}, \tag{267}$$

$$|\tilde{M}_3(n)| \leqslant |\tilde{M}_3(20)| + \frac{137297659}{2^5 3^{\frac{17}{2}}} e^{\frac{1}{2}} \pi^{\frac{3}{2}} - \left(\frac{2}{3}\right)^3 e^{\frac{1}{2}} \pi \sum_{p=0}^{20} p^4(p-1)^5 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}. \tag{268}$$

Consequently, in order to finish the proof of inequalities (258) and (259) we have to calculate the exact values of $\hat{M}_3(20)$ and $\tilde{M}_3(20)$ and of the quantity

$$\tilde{\mathcal{S}} := \left(\frac{2}{3}\right)^3 e^{\frac{1}{2}} \pi \sum_{p=0}^{20} p^4(p-1)^5 2^{-2p} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}.$$

We apply the duplication formula (328) to $\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}$ and find that this quotient can also be written as

$$\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} = \pi^{\frac{1}{2}} 2^{-2p} \binom{2p}{p}.$$

Next, we plug this representation into the formulae for $\tilde{\mathcal{S}}, \hat{M}_3(20)$ and $\tilde{M}_3(20)$. This way, we get the representations

$$\tilde{\mathcal{S}} = \left(\frac{2}{3}\right)^3 e^{\frac{1}{2}} \pi^{\frac{3}{2}} \sum_{p=0}^{20} 2^{-4p} \binom{2p}{p} p^4 (p-1)^5, \tag{269}$$

$$\hat{M}_3(20) = \frac{3}{2^8} \pi^{\frac{3}{2}} \sum_{p=2}^{20} 2^{-4p} \binom{2p}{p} (p-1)p(p+1)\left(p^3 - 3p^2 - 4p + 4\right), \tag{270}$$

and

$$\begin{aligned} \tilde{M}_3(20) = \pi^{\frac{3}{2}} \sum_{p=3}^{20} 2^{-6p} \binom{2p}{p} \sum_{j=0}^{p} \binom{2j}{j} \binom{2(p-j)}{p-j} \\ \times \sum_{i=1}^{p} \sum_{k=1}^{i-1} \sum_{l=1}^{k-1} \left(\beta_i - \alpha_i(j)\right)\left(\beta_{k+1} - \alpha_k(j)\right)\left(\beta_{l+2} - \alpha_l(j)\right) \prod_{t=1}^{l-1} \frac{20^2 - \alpha_t(j)}{20^2 - \beta_{t+3}}. \end{aligned} \tag{271}$$

The point of this last transformation of $\tilde{\mathcal{S}}, \hat{M}_3(20)$ and $\tilde{M}_3(20)$ is to emphasize the fact, that all of the quantities (269), (270) and (271) are sums of rational numbers. Therefore, in theory, the evaluation of them could be done using any calculator, but can be quite tedious in practice and is very prone to mistakes.
We use *Wolfram Mathematica* for this task, since it is capable of symbolic calculations and evaluates fractions with arbitrary precision. The code is as follows for $\tilde{\mathcal{S}}$

```
(2/3)^3 Pi^(3/2) Exp[1/2]
Sum[
  2^(-4p) Binomial[2p, p] p^4 (p - 1)^5, {p, 0, 20}
]
```

for $\hat{M}_3(20)$

```
3/2^8 Pi^(3/2)
Sum[
  2^(-4p) Binomial[2p, p] (p - 1) p (p + 1)
  (p^3 - 3p^2 - 4p + 4),{p, 2, 20}
]
```

and for $\tilde{M}_3(20)$

```
Pi^(3/2)
Sum[
  2^(-6p) Binomial[2p, p]
  Sum[
    Binomial[2j, j] Binomial[2(p - j), p - j]
    Sum[
      Sum[
        Sum[
          (i^2/4 - (i - j - 1/2)^2) ((k + 1)^2/4 - (k - j - 1/2)^2)
          ((l + 2)^2/4 - (l - j - 1/2)^2)
          Product[
            (20^2 - (t - j - 1/2)^2)/(20^2 - (t + 3)^2/4), {t, 1, l - 1}
```

```
          ], {l, 1, k - 1}
        ], {k, 1, i - 1}
      ], {i, 1, p}
    ], {j, 0, p}
  ], {p, 3, 20}
]
```

The resulting values are

$$\tilde{S} = \frac{e^{\frac{1}{2}}}{3}\pi^{\frac{3}{2}}\frac{167149105866612550714859}{2^{67}}$$
$$\hat{M}_3(20) = \pi^{\frac{3}{2}}\frac{1018255216965759951933981}{2^{82}},$$

and

$$\tilde{M}_3(20) = -\frac{40715450484088683640584003236562927 8817}{5\cdot 11\cdot 17\cdot 23\cdot 2^{119}}\pi^{\frac{3}{2}}.$$

Since those exact values are definitely not very handy, we use *Mathematica* once more to express them as floating point values with ten digits precision. This yields

$$\hat{M}_3(20) > 0.1172524182, \tag{272}$$
$$\tilde{S} > 3466.132364, \tag{273}$$
$$\tilde{M}_3(20) < -0.1586262737, \tag{274}$$
$$\left|\tilde{M}_3(20)\right| < 0.1586262738, \tag{275}$$
$$\frac{137297659}{2^5 3^{\frac{17}{2}}}e^{\frac{1}{2}}\pi^{\frac{3}{2}} < 3466.207351. \tag{276}$$

Plugging in the respective values into (268) now gives for $n \geqslant 20$

$$\frac{\left|\tilde{M}_3(n)\right|}{n^2-\frac{9}{4}} < \frac{1}{397}(0.1586262738 + 3466.207351 - 3466.132364)$$
$$< \frac{0.2337}{397} < 0.0006,$$

which is obviously smaller than (272) and thus proves the first assertion (258) of the lemma. In order to prove the second assertion (259) we plug in (273), (274) and (276) into (267) and obtain

$$\tilde{M}_3(n) < -0.1586262737 + 3466.207351 - 3466.132364 < -0.08 < 0.$$

■

The careful reader will have noticed that we are not quite finished yet. We are still missing the proof of Lemma 41, which we will now catch up on.

Proof of Lemma 41. Since the variable j only appears in the enumerator of $P(n, j, l)$, the product attains its maximum in j for those values of j, such that

$$\prod_{t=1}^{l-1}\left[n^2 - \left(t - j - \frac{1}{2}\right)^2\right] \tag{277}$$

is maximal. This is the case if the terms $\left(t-j-\frac{1}{2}\right)^2$ are arranged around $\left(\pm\frac{1}{2}\right)^2$ "as symmetrically as possible".

If l is even, then (277) consists of an odd number of factors. Thus, the condition "as symmetrically as possible" is met, if either

$$1-j-\frac{1}{2}=-\left(l-1-j-\frac{1}{2}-1\right),$$

or

$$1-j-\frac{1}{2}=-\left(l-1-j-\frac{1}{2}+1\right).$$

That means, if either $j=\frac{l}{2}-1$ or $j=\frac{l}{2}$.
If l is odd, then the condition "as symmetrically as possible" implies $j=\frac{l-1}{2}$.

Moreover, we note that for even l

$$\prod_{t=1}^{l-1}\left[n^2-\left(t-\frac{l}{2}-\frac{1}{2}\right)^2\right]=\prod_{t=1}^{l-1}\left[n^2-\left(t-\frac{l}{2}+\frac{1}{2}\right)^2\right].$$

This can easily be seen by reversing the ordering of, for example, the left product above, and using that $\left(-t+\frac{l}{2}-\frac{1}{2}\right)^2=\left(t-\frac{l}{2}+\frac{1}{2}\right)^2$. So, in the case of even l it is

$$\begin{aligned}\prod_{t=1}^{l-1}\left[n^2-\left(t-j-\frac{1}{2}\right)^2\right]&\leqslant\prod_{t=1}^{l-1}\left[n^2-\left(t-\frac{l}{2}+\frac{1}{2}\right)^2\right]\\&=\left[n^2-\left(\frac{l}{2}-\frac{1}{2}\right)^2\right]\prod_{t=1}^{\frac{l}{2}-1}\left[n^2-\left(t-\frac{1}{2}\right)^2\right]^2,\end{aligned}\tag{278}$$

and for odd l, we obtain

$$\begin{aligned}\prod_{t=1}^{l-1}\left[n^2-\left(t-j-\frac{1}{2}\right)^2\right]&\leqslant\prod_{t=1}^{l-1}\left[n^2-\left(t-\frac{l}{2}\right)^2\right]\\&=\prod_{t=1}^{\frac{l-1}{2}}\left[n^2-\left(t-\frac{1}{2}\right)^2\right]^2.\end{aligned}\tag{279}$$

Next, we consider the denominator of $P(n,j,l)$ and rewrite it as follows

$$\prod_{t=1}^{l-1}\left[n^2-\left(\frac{t+3}{2}\right)^2\right]=\prod_{t=1}^{\lfloor\frac{l-1}{2}\rfloor}\left[n^2-\left(t+\frac{3}{2}\right)^2\right]\prod_{t=1}^{\lceil\frac{l-1}{2}\rceil}\left[n^2-(t+1)^2\right].$$

Consequently, lot's of factors cancel, when we divide the estimates for the numerator by the above expression for the denominator. For even l we get by (278)

$$P(n,j,l)\leqslant\frac{\left(n^2-\frac{1}{4}\right)\left(n^2-\frac{9}{4}\right)}{\left(n^2-\left(\frac{l}{2}+\frac{1}{2}\right)^2\right)\left(n^2-\left(\frac{l}{2}-\frac{1}{2}\right)^2\right)}\prod_{t=1}^{\frac{l}{2}}\frac{n^2-\left(t-\frac{1}{2}\right)^2}{n^2-(t+1)^2},$$

and by (279) we estimate for odd l

$$P(n,j,l)\leqslant\frac{\left(n^2-\frac{1}{4}\right)\left(n^2-\frac{9}{4}\right)}{\left(n^2-\left(\frac{l}{2}+1\right)^2\right)\left(n^2-\frac{l^2}{4}\right)}\prod_{t=1}^{\frac{l-1}{2}}\frac{n^2-\left(t-\frac{1}{2}\right)^2}{n^2-(t+1)^2}.$$

Since $l \leqslant p-2$ both of the above estimates of $P(n,j,l)$ are bounded from above by

$$\frac{n^4}{\left(n^2-\frac{p^2}{4}\right)^2}\prod_{t=1}^{\frac{l}{2}}\frac{n^2-\left(t-\frac{1}{2}\right)^2}{n^2-(t+1)^2},$$

which is smaller than or equal to

$$\left(\frac{4}{3}\right)^2\prod_{t=1}^{\frac{l}{2}}\frac{n^2-\left(t-\frac{1}{2}\right)^2}{n^2-(t+1)^2}, \tag{280}$$

due to the condition $p \leqslant n$.

Last but not least, we use the inequality $1+x \leqslant e^x$ and the bound $l \leqslant p-2$ to estimate

$$\begin{aligned}\prod_{t=1}^{\frac{l}{2}}\frac{n^2-\left(t-\frac{1}{2}\right)^2}{n^2-(t+1)^2} &= \prod_{t=1}^{\frac{l}{2}}\left(1+\frac{(t+1)^2-\left(t-\frac{1}{2}\right)^2}{n^2-(t+1)^2}\right)\\ &\leqslant \exp\left[\sum_{t=1}^{\frac{l}{2}}\frac{(t+1)^2-\left(t-\frac{1}{2}\right)^2}{n^2-\frac{p^2}{4}}\right].\end{aligned}$$

By Lemma 66, the last line is equal to

$$\exp\left[\frac{3}{8}\frac{l(l+3)}{n^2-\frac{p^2}{4}}\right]$$

and satisfies the upper bound

$$\exp\left[\frac{3}{8}\frac{(p+1)(p-2)}{n^2-\frac{p^2}{4}}\right] \leqslant e^{\frac{1}{2}}$$

for all $l \leqslant p-2$ and $p \leqslant n$. This together with (280) proves the claim. ∎

CHAPTER 5

Numerical Proof of Theorem 3 for $2 \leqslant n \leqslant 19$

The remaining values of n, for which our analytical machinery of Chapter 4 does not apply, are now taken care of numerically. More precisely, our goal is to show that for every $2 \leqslant n \leqslant 19$ there exists a constant $c_n \leqslant 8$, such that the estimate

$$\left| I(\mathbf{n}) - \frac{3^{\frac{1}{2}}}{4\pi^2}\frac{1}{n} + \frac{23}{2^8 3^{\frac{3}{2}}\pi^2}\frac{1}{n\left(n^2-\frac{1}{4}\right)} \right| \leqslant \frac{c_n \cdot 10^{-4}}{n(n^2-\frac{1}{4})(n^2-1)}$$

is true. Table 1 at the end of this chapter provides these constants c_n.
In order to achieve this goal and complete the proof of Theorem 3 for the missing values of n between 2 and 19, we calculate the integral $I(\mathbf{n})$ numerically for those n. The underlying idea is the same as in [1]. That is, we split the integral into a low and a high part

$$\begin{aligned} I(\mathbf{n}) &= I_{low}(n) + I_{high}(n) \\ &= \int_0^R J_{2n}^2(r) J_n^4(r) r\, dr + \int_R^\infty J_{2n}^2(r) J_n^4(r) r\, dr. \end{aligned}$$

We compute the low integral numerically using a composite 7 point Newton-Cotes quadrature rule. The high integral is estimated by means of the asymptotic expansion of the Bessel function from Corollary 5. The corollary allows for a decomposition of $I_{high}(n)$ into a main term and several error terms. This decomposition and the estimate of the error terms is subject of Theorem 42.
The value of the boundary R is determined in such a way, that we are allowed to apply the just mentioned Corollary 5, and such that the absolute error meets our accuracy goal of $4 \cdot 10^{-13}$. Moreover, its value should not be chosen too large, since otherwise the computation time for the low integral becomes unnecessarily long. We will see that $R = 600000$ does the job.
In the numerical computation of the low integral $I_{low}(n)$ we follow [1] quite closely, while we leave their path when it comes to the estimate of the high integral $I_{high}(n)$. The first major difference is that we don't rely on the built in functionalities of *Wolfram Mathematica* to evaluate trigonometric integrals and to transform them into trigonometric integral functions. We do this manually here with the help of some additional tools from the theory of trigonometric integrals, more precisely the sine and the cosine integrals. Lemmata 62 and 63 provide those tools. Moreover, we will encounter the task of estimating trigonometric integrals with an odd integrand. In [1] this task is solved by some cancellation argument. Another way is to use Lemmata 62 and 63. Since we need them anyway, we also apply them here to be consistent, even though the calculations might seem more laborious than in [1]. The second difference arises from the need for a higher accuracy in our case. We therefore pay more attention to the error terms we get from the aforementioned application of Corollary 5. In particular the error terms of second order are bounded more carefully than it is done in [1].

We start with the discussion of the high integral and prove the following theorem.

THEOREM 42. *The high integral satisfies for $R = 600000$ and all $n \leqslant 19$*

$$\left| I_{high}(n) - 1.343815198 \cdot 10^{-7} \right| \leqslant 2.101441586 \cdot 10^{-13},$$

if n is even, and

$$\left|I_{high}(n) - 2.687627594 \cdot 10^{-8}\right| \leqslant 2.101441586 \cdot 10^{-13}$$

if n is odd.

PROOF. As pointed out above we make use of the fact that $R \gg 38^2 \geqslant (2n)^2$, such that Corollary 5 is applicable. It provides us with helpful information on the asymptotic behavior of the Bessel functions on $[R, \infty)$, namely that

$$J_n(r) = \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \cos(\omega_n) + F_1(n,r), \tag{281}$$

$$J_n(r) = \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \left[\cos(\omega_n) - \frac{4n^2-1}{8r}\sin(\omega_n)\right] + F_2(n,r), \tag{282}$$

with

$$\omega_n := r - \frac{\pi}{4} - \frac{\pi}{2}n,$$

and error terms

$$|F_1(n,r)| \leqslant \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \frac{n^2}{r}, \tag{283}$$

$$|F_2(n,r)| \leqslant \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \frac{n^4}{(2r)^2}. \tag{284}$$

We now replace $J_n(r)$ and $J_{2n}(r)$ with its asymptotics from (281) and obtain

$$\begin{aligned} J_{2n}(r)^2 J_n(r)^4 &= \left[\left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \cos(\omega_{2n}) + F_1(2n,r)\right]^2 \left[\left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \cos(\omega_n) + F_1(n,r)\right]^4 \\ &= \sum_{j=0}^{6} t_j(n,r), \end{aligned}$$

where t_j is the sum of all products that contain j first order error terms $F_1(n,r)$ or $F_1(2n,r)$, respectively, and $6-j$ leading asymptotic terms from (281). More precisely, it is

$$t_0(n,r) = \left(\frac{2}{\pi}\right)^3 r^{-3} \cos(\omega_n)^4 \cos(\omega_{2n})^2 \tag{285}$$

the main term, and the error terms of first and second order read

$$\begin{aligned} t_1(n,r) &= \left(\frac{2}{\pi r}\right)^{\frac{5}{2}} \Big[2F_1(2n,r)\cos(\omega_n)^4\cos(\omega_{2n}) + 4F_1(n,r)\cos(\omega_n)^3\cos(\omega_{2n})^2\Big], \\ t_2(n,r) &= \left(\frac{2}{\pi r}\right)^{2} \Big[F_1(2n,r)^2\cos(\omega_n)^4 + 8F_1(2n,r)F_1(n,r)\cos(\omega_n)^3\cos(\omega_{2n}) \\ &\qquad + 6F_1(n,r)^2\cos(\omega_n)^2\cos(\omega_{2n})^2\Big]. \end{aligned}$$

Using (283) and $|\cos(\omega_n)| \leqslant 1$, we estimate the sum of the remaining terms of higher order by

$$\sum_{j=3}^{6} |t_j(n,r)| \leqslant \left(\frac{2}{\pi}\right)^3 \left[116\frac{n^6}{r^6} + 129\frac{n^8}{r^7} + 72\frac{n^{10}}{r^8} + 16\frac{n^{12}}{r^9}\right]. \tag{286}$$

The terms $t_1(n,r)$ and $t_2(n,r)$ need a more careful treatment. Recall the finer information from (282), which tells us that

$$F_1(n,r) = -\left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \frac{4n^2-1}{8r} \sin(\omega_n) + F_2(n,r). \tag{287}$$

Replacing all factors $F_1(n,r)$ and $F_1(2n,r)$ in $t_1(n,r)$ and $t_2(n,r)$ by (287) yields two more main terms $t_{1,0}(n,r)$ and $t_{2,0}(n,r)$, as well as two new error terms $t_{1,1}(n,r)$ and $t_{2,1}(n,r)$. Those consist of all products that contain one second order error $F_2(n,r)$ or $F_2(2n,r)$, respectively, from (282). The exact expressions for $t_{1,0}(n,r)$ and $t_{2,0}(n,r)$ are

$$\begin{aligned} t_{1,0}(n,r) = &-\left(\frac{2}{\pi}\right)^3 r^{-4} \\ &\times \left[\frac{16n^2-1}{4} \cos(\omega_n)^4 \sin(\omega_{2n}) \cos(\omega_{2n}) + \frac{4n^2-1}{2} \sin(\omega_n) \cos(\omega_n)^3 \cos(\omega_{2n})^2\right], \end{aligned}$$

and

$$\begin{aligned} t_{2,0}(n,r) = &\left(\frac{2}{\pi}\right)^3 r^{-5} \\ &\times \Bigg[\left(\frac{16n^2-1}{8}\right)^2 \cos(\omega_n)^4 \sin(\omega_{2n})^2 + 6\left(\frac{4n^2-1}{8}\right)^2 \sin(\omega_n)^2 \cos(\omega_n)^2 \cos(\omega_{2n})^2 \\ &+ \frac{(16n^2-1)(4n^2-1)}{8} \sin(\omega_n) \cos(\omega_n)^3 \sin(\omega_{2n}) \cos(\omega_{2n})\Bigg]. \end{aligned}$$

The error terms $t_{1,1}(n,r)$ and $t_{2,1}(n,r)$ are small enough, such that we don't need to consider them in such detail. Thus, we estimate them using (284) and obtain

$$|t_{1,1}(n,r)| \leqslant \left(\frac{2}{\pi}\right)^3 n^4 r^{-5} \Big[8\left|\cos(\omega_n)^4 \cos(\omega_{2n})\right| + \left|\cos(\omega_n)^3 \cos(\omega_{2n})^2\right|\Big], \tag{288}$$

and, since the absolute value of the sine and cosine are at most one,

$$|t_{2,1}(n,r)| \leqslant \left(\frac{2}{\pi}\right)^3 r^{-6} \frac{1}{16} n^4 \left(604n^2 - 91\right). \tag{289}$$

Moreover, from $t_2(n,r)$ we get another error term $t_{2,2}(n,r)$, containing the products with two factors of F_2. We also estimate it by means of (284) and bound all sine and cosine factors by one. This way we get

$$|t_{2,2}(n,r)| \leqslant \left(\frac{2}{\pi}\right)^3 r^{-7} \frac{391}{16} n^8. \tag{290}$$

In order to further simplify all the above terms that still contain trigonometric factors, we follow the lead of Oliveira e Silva and Thiele [1] and exploit the periodicity of $\sin(\omega_n)$

and $\cos(\omega_n)$. That is

$$\begin{aligned}
\cos(\omega_{2n}) &= (-1)^n \cos(\omega_0), \\
\sin(\omega_{2n}) &= (-1)^n \sin(\omega_0), \\
\cos(\omega_n) &= \begin{cases} (-1)^{\frac{n}{2}} \cos(\omega_0), & n \text{ even}, \\ (-1)^{\frac{n-1}{2}} \sin(\omega_0), & n \text{ odd}, \end{cases} \\
\sin(\omega_n) &= \begin{cases} (-1)^{\frac{n}{2}} \sin(\omega_0), & n \text{ even}, \\ (-1)^{\frac{n+1}{2}} \cos(\omega_0), & n \text{ odd}. \end{cases}
\end{aligned} \tag{291}$$

Using (291) we find that the main term $t_0(n,r)$ on page 114 equals

$$\begin{aligned}
t_0(n,r) &= \left(\frac{2}{\pi}\right)^3 r^{-3} \begin{cases} \cos(\omega_0)^6, & n \text{ even}, \\ \sin(\omega_0)^4 \cos(\omega_0)^2, & n \text{ odd} \end{cases} \\
&= \left(\frac{2}{\pi}\right)^3 2^{-5} r^{-3} \begin{cases} 15\sin(2r) - \sin(6r) - 6\cos(4r) + 10, & n \text{ even}, \\ -\sin(2r) - \sin(6r) + 2\cos(4r) + 2, & n \text{ odd}. \end{cases} \qquad (292)
\end{aligned}$$

The second equality is due to the standard trigonometric addition formulae

$$\cos(2r) = \cos(r)^2 - \sin(r)^2, \tag{293}$$

$$\sin(2r) = 2\sin(r)\cos(r), \tag{294}$$

as well as the well-known property $\sin(r)^2 + \cos(r)^2 = 1$.

For the term $t_{1,0}(n,r)$, we obtain by the same procedure

$$\begin{aligned}
t_{1,0}(n,r) = -\left(\frac{2}{\pi}\right)^3 r^{-4} \Bigg[&\frac{16n^2-1}{4} \begin{cases} \sin(\omega_0)\cos(\omega_0)^5, & n \text{ even}, \\ \sin(\omega_0)^5 \cos(\omega_0), & n \text{ odd} \end{cases} \\
&+ \frac{4n^2-1}{2} \begin{cases} \sin(\omega_0)\cos(\omega_0)^5, & n \text{ even}, \\ -\sin(\omega_0)^3 \cos(\omega_0)^3, & n \text{ odd} \end{cases} \Bigg],
\end{aligned}$$

which is equal to

$$-\left(\frac{2}{\pi}\right)^3 2^{-5} r^{-4} \begin{cases} \left(6n^2 - \frac{3}{4}\right)\left[-4\sin(4r) - 5\cos(2r) + \cos(6r)\right], & n \text{ even}, \\ (16n^2-1)\sin(4r) - \left(14n^2 + \frac{1}{4}\right)\cos(2r) + \left(6n^2 - \frac{3}{4}\right)\cos(6r), & \text{odd}. \end{cases} \tag{295}$$

And in the case of $t_{2,0}(n,r)$ this yields

$$\begin{aligned}
t_{2,0}(n,r) = \left(\frac{2}{\pi}\right)^3 r^{-5} \Bigg[&\left(\frac{16n^2-1}{8}\right)^2 \begin{cases} \sin(\omega_0)^2 \cos(\omega_0)^4, & n \text{ even}, \\ \sin(\omega_0)^6, & n \text{ odd} \end{cases} \\
&+ 6\left(\frac{4n^2-1}{8}\right)^2 \sin(\omega_0)^2 \cos(\omega_0)^4 \\
&+ \frac{(16n^2-1)(4n^2-1)}{8} \begin{cases} \sin(\omega_0)^2 \cos(\omega_0)^4, & n \text{ even}, \\ -\sin(\omega_0)^4 \cos(\omega_0)^2, & n \text{ odd}, \end{cases} \Bigg].
\end{aligned}$$

By means of the trigonometric addition formulae we transform the last expression into

$$\left(\frac{2}{\pi}\right)^3 2^{-11} r^{-5} \begin{cases} 3\left(288n^4 - 80n^2 + 5\right)\left[\sin(2r) + \sin(6r) + 2\cos(4r) + 2\right], & n \text{ even}, \\ -(3232n^4 - 272n^2 + 1)\sin(2r) + 3\left(288n^4 - 80n^2 + 5\right)\sin(6r) & \\ \qquad -(2368n^4 - 416n^2 + 10)\cos(4r) + 1728n^4 - 96n^2 + 6, & n \text{ odd}. \end{cases} \tag{296}$$

Next, we further estimate the absolute value of $t_{1,1}(n,r)$ with the help of the two bounds

$$\left|\cos(\omega_n)^4 \cos(\omega_{2n})\right| \leqslant \begin{cases} \cos(\omega_0)^4, & n \text{ even}, \\ \sin(\omega_0)^4, & n \text{ odd} \end{cases}$$

and

$$\left|\cos(\omega_n)^3 \cos(\omega_{2n})^2\right| \leqslant \begin{cases} \cos(\omega_0)^4, & n \text{ even}, \\ \frac{1}{4}\left((\sin(r)^2 - \cos(r)^2\right)^2, & n \text{ odd}. \end{cases}$$

The estimate for odd n in the above inequality is due to

$$\begin{aligned} \cos(\omega_n)^3 \cos(\omega_{2n})^2 &= \sin\left(r - \frac{\pi}{4}\right)^3 \cos\left(r - \frac{\pi}{4}\right)^2 \\ &= \frac{1}{4\sqrt{2}}\left(\sin(r)^2 - \cos(r)^2\right)^2 \left(\sin(r) - \cos(r)\right) \end{aligned}$$

and $\sin(r) - \cos(r) \leqslant \sqrt{2}$. Using this for the right-hand side of (288), we obtain

$$|t_{1,1}(n,r)| \leqslant \left(\frac{2}{\pi}\right)^3 n^4 r^{-5} \begin{cases} 9\cos(\omega_0)^4, & n \text{ even}, \\ 8\sin(\omega_0)^4 + \frac{1}{4}\left((\sin(r)^2 - \cos(r)^2\right)^2, & n \text{ odd}. \end{cases}$$

This is, once again by the trigonometric addition formulae (293), equal to

$$\left(\frac{2}{\pi}\right)^3 2^{-3} n^4 r^{-5} \begin{cases} 36\sin(2r) - 9\cos(4r) + 27, & n \text{ even}, \\ -32\sin(2r) - 7\cos(4r) + 25, & n \text{ odd}. \end{cases} \tag{297}$$

Thus, so far we achieved the following approximation of $I_{high}(n)$

$$I_{high}(n) = \int_R^\infty t_0(n,r) r dr + E_{high}, \tag{298}$$

with

$$|E_{high}| \leqslant E_{high}^{trig} + E_{high}^{rat}. \tag{299}$$

Here we collect all error terms that include trigonometric factors in the trigonometric error E_{high}^{trig}. This is

$$E_{high}^{trig} := \left|\int_R^\infty t_{1,0}(n,r) r dr\right| + \left|\int_R^\infty t_{2,0}(n,r) r dr\right| + \int_R^\infty |t_{1,1}(n,r)|\, r dr. \tag{300}$$

All remaining error terms we put in

$$E_{high}^{rat} := \int_R^\infty \left[|t_{2,1}(n,r)| + |t_{2,2}(n,r)| + \sum_{j=3}^{6} |t_j(n,r)|\right] r dr. \tag{301}$$

In the following we first evaluate E_{high}^{rat}. Then we take a closer look at $\int_R^\infty t_0(n,r) r dr$ in (298) and the trigonometric error E_{high}^{trig}. We establish upper bounds on both quantities by means of asymptotic estimates for generalized sine and the cosine integrals. The necessary tools are provided by Lemmata 62 and 63 in Section 7.4.

So, let's begin with E_{high}^{rat}. Since, it is merely a sum of simple integrals of the rational function r^{-m} for $m = 6, 7, 8, 9$, that is easily evaluated as

$$\int_R^\infty r^{-m} dr = \frac{R^{1-m}}{m-1},$$

we just plug in estimates (289), (290) and (286) into (301), sum up the resulting values and obtain

$$E_{high}^{rat} \leqslant \frac{n^4 \left(1024n^8 + 5376n^6 R + 14336n^4 R^2 + 7\left(2460n^2 - 91\right) R^3\right)}{56\pi^3 R^7}.$$

Now, we evaluate the above expression for $n = 19$ and $R = 600000$. This leaves us with

$$E_{high}^{rat} \leqslant \frac{170954917420150229492216161}{15309000000000000000000000000000000000\pi^3},$$

which is in floating-point representation

$$E_{high}^{rat} \leqslant 3.6015144071 \cdot 10^{-15}. \tag{302}$$

As outlined above, we now turn to the further analysis of the leading term

$$I_{high,0}(n) := \int_R^\infty t_0(n, r) r dr \tag{303}$$

in (298) and the trigonometric error E_{high}^{trig}. Our strategy is to trace those integrals back to some higher order sine and cosine integrals and then apply Lemma 63. To this end, we transform all occurring trigonometric integrals into sums of the functions

$$S(R, a, b) := \int_R^\infty \frac{\sin(ar)}{r^b} dr, \tag{304}$$

$$C(R, a, b) := \int_R^\infty \frac{\cos(ar)}{r^b}. \tag{305}$$

We first take care of the leading term (303) for even n. By (292) we get

$$I_{high,0}(n) = \frac{1}{4\pi^3}\left[15S(R, 2, 2) - S(R, 6, 2) - 6C(R, 4, 2) + \frac{10}{R}\right].$$

Now, Lemma 63 applies and we obtain for the right-hand side

$$\frac{5}{2\pi^3 R} + \frac{1}{4\pi^3 R^2}\left[\frac{15}{2}\cos(2R) - \frac{1}{6}\cos(6R) + \frac{3}{2}\sin(4R)\right]$$
$$+ \frac{1}{4\pi^3}\left[-30q_{c,2}(2R) + 6q_{c,2}(6R) - 24q_{s,2}(4R)\right].$$

We denote the first line above by I_{high}^{even} and the second line by $E_{0,even}$. Then we plug in $R = 600000$. This way we obtain

$$\begin{aligned} I_{high}^{even} &:= \frac{5}{2\pi^3 R} + \frac{1}{4\pi^3 R^2}\left[\frac{15}{2}\cos(2R) - \frac{1}{6}\cos(6R) + \frac{3}{2}\sin(4R)\right] \\ &= 1.3438151976 \cdot 10^{-7}, \end{aligned} \tag{306}$$

and

$$\begin{aligned} E_{0,even} &:= \frac{1}{4\pi^3}\left|\left[-30q_{c,2}(2R) + 6q_{c,2}(6R) - 24q_{s,2}(4R)\right]\right| \\ &\leqslant 6.2006229396 \cdot 10^{-19} \end{aligned} \tag{307}$$

where inequality (307) is due to the triangle inequality and the bounds $|q_{c,2}(x)|, |q_{s,2}(x)| \leqslant \frac{4}{x^3}$ of Lemma 62.

If n is odd, we get from (292)

$$I_{high,0}(n) = \frac{1}{4\pi^3}\left[-S(R,2,2) - S(R,6,2) + 2C(R,4,2) + \frac{2}{R}\right].$$

As in the case of even n, we plug in the asymptotics for $S(R,a,2)$ and $C(R,a,2)$ from Lemma 63 into the left-hand-side above and obtain

$$\frac{1}{2\pi^3 R} - \frac{1}{4\pi^3 R^2}\left[\frac{1}{2}\cos(2R) + \frac{1}{6}\cos(6R) + \frac{1}{2}\sin(4R)\right] + \frac{1}{4\pi^3}\left[2q_{c,2}(2R) + 6q_{c,2}(6R) + 8q_{s,2}(4R)\right].$$

Following an equivalent notation, we split the above expression into

$$I_{high}^{odd} := \frac{1}{2\pi^3 R} - \frac{1}{4\pi^3 R^2}\left[\frac{1}{2}\cos(2R) + \frac{1}{6}\cos(6R) + \frac{1}{2}\sin(4R)\right],$$

which evaluates to

$$I_{high}^{odd} = 2.6876275940 \cdot 10^{-8} \tag{308}$$

for $R = 600000$, and

$$\begin{aligned} E_{0,odd} &:= \frac{1}{4\pi^3}\left|2q_{c,2}(2R) + 6q_{c,2}(6R) + 8q_{s,2}(4R)\right| \\ &\leqslant E_{0,even}. \end{aligned} \tag{309}$$

The last inequality becomes obvious, when one first applies the triangle inequality to $E_{0,even}$ and $E_{0,odd}$, and the bound of Lemma 62 afterwards.

Next in the line is the trigonometric error E_{high}^{trig}. We handle it summand wise with the same strategy as (303). For each summand we consider even and odd n separately, and then choose the larger value as a common upper bound.

So, let us start with

$$\left|\int_R^\infty t_{1,0}(n,r) r dr\right|. \tag{310}$$

For even n, we get by (295)

$$\left|\int_R^\infty t_{1,0}(n,r) r dr\right| = \frac{6n^2 - \frac{3}{4}}{4\pi^3}\left|-4S(R,4,3) - 5C(R,2,3) + C(R,6,3)\right|.$$

By Lemma 63 this is equal to

$$\frac{6n^2 - \frac{3}{4}}{4\pi^3}\left|-32q_{s,2}(4R) - 10q_{c,2}(2R) + 18q_{c,2}(6R)\right|,$$

and by Lemma 62 not larger than

$$\frac{44n^2 - \frac{11}{2}}{4\pi^3}\frac{1}{R^3} \leqslant 5.3194501325 \cdot 10^{-16},$$

for $R = 600000$ and $n \leqslant 18$.

For odd n identity (295) leads to the expression

$$\frac{1}{4\pi^3}\left|\left(16n^2 - 1\right) S(R,4,3) - \left(14n^2 + \frac{1}{4}\right) C(R,2,3) + \left(6n^2 - \frac{3}{4}\right) C(R,6,3)\right|$$

for (310). This is equal to

$$\frac{1}{4\pi^3}\left|8\left(16n^2-1\right)q_{s,2}(4R)-2\left(14n^2+\frac{1}{4}\right)q_{c,2}(2R)+18\left(6n^2-\frac{3}{4}\right)q_{c,2}(6R)\right|$$

by Lemma 63, and bounded from above by

$$\frac{24n^2-\frac{1}{2}}{4\pi^3}\frac{1}{R^3}\leqslant 3.2339255621\cdot 10^{-16}$$

due to Lemma 62 for $R=600000$ and $n\leqslant 19$.

Hence, we note that

$$\left|\int_R^\infty t_{1,0}(n,r)rdr\right|\leqslant 5.3194501325\cdot 10^{-16}. \tag{311}$$

Next, we repeat the above procedure for $\left|\int_R^\infty t_{2,0}(n,r)rdr\right|$ and $\int_R^\infty |t_{1,1}(n,r)|\,rdr$. We start with the former one in the case of even n. Then identity (296) yields

$$\begin{aligned}\left|\int_R^\infty t_{2,0}(n,r)rdr\right| &= \frac{3\left(288n^4-80n^2+5\right)}{2^8\pi^3}\left|S(R,2,4)+S(R,6,4)+2C(R,4,4)+\frac{2}{3R^3}\right|\\ &\leqslant \frac{3\left(288n^4-80n^2+5\right)}{2^8\pi^3}\left(\frac{2}{3R^3}+\frac{7}{3R^4}\right)\\ &\leqslant 3.523702847\cdot 10^{-14},\end{aligned}$$

for even $n\leqslant 18$ and $R=600000$.

Likewise, for odd $n\leqslant 19$ and the same R we obtain by means of (296) and Lemma 63

$$\begin{aligned}\left|\int_R^\infty t_{2,0}(n,r)rdr\right| = \frac{1}{2^8\pi^3}\Bigg|&-(3232n^4-272n^2+1)S(R,2,4)+3\left(288n^4-80n^2+5\right)S(R,6,4)\\ &-(2368n^4-416n^2+10)C(R,4,4)+\frac{1728n^4-96n^2+6}{3R^3}\Bigg|\\ &\leqslant \frac{1}{2^8\pi^3}\left(\frac{1728n^4-96n^2+6}{3R^3}+\frac{64\left(9760n^4-2384n^2+121\right)}{R^4}\right)\\ &\leqslant 4.3854135383\cdot 10^{-14}.\end{aligned}$$

Thus, we arrive at

$$\left|\int_R^\infty t_{2,0}(n,r)rdr\right|\leqslant 4.3854135383\cdot 10^{-14}. \tag{312}$$

Last but not least in the analysis of the trigonometric error is $\int_R^\infty |t_{1,1}(n,r)|\,rdr$. Recalling (297) and invoking Lemma 63 one last time, we find for even $n\leqslant 18$

$$\begin{aligned}\int_R^\infty |t_{1,1}(n,r)|\,rdr &= \frac{n^4}{\pi^3}\left(36S(R,2,4)-9C(R,4,4)+\frac{9}{R^3}\right)\\ &\leqslant \frac{n^4}{\pi^3}\left(\frac{9}{R^3}+\frac{81}{2R^4}\right)\\ &\leqslant 1.4106926963\cdot 10^{-13},\end{aligned}$$

with $R = 600000$, and for odd $n \leqslant 19$

$$\begin{aligned}\int_R^\infty |t_{1,1}(n,r)|\, rdr &= \frac{n^4}{\pi^3}\left(-32S(R,2,4) - 7C(R,4,4) + \frac{25}{3R^3}\right)\\ &\leqslant \frac{n^4}{\pi^3}\left(\frac{25}{3R^3} + \frac{71}{2R^4}\right)\\ &\leqslant 1.6215594370 \cdot 10^{-13}.\end{aligned}$$

Consequently, we estimate

$$\int_R^\infty |t_{1,1}(n,r)|\, rdr \leqslant 1.6215594370 \cdot 10^{-13}. \tag{313}$$

After adding up estimates (311), (312) and (313), we end up with the upper bound on the trigonometric error

$$E_{high}^{trig} \leqslant 2.065420241 \cdot 10^{-13}. \tag{314}$$

Thus, (299) in combination with the estimates (302), (307), (309) and (314), eventually yields

$$\left| I_{high}(n) - I_{high}^{parity} \right| < 2.101441586 \cdot 10^{-13}, \tag{315}$$

where *parity* is either *even* or *odd*, and I_{high}^{even} and I_{high}^{odd} are the expressions in (306) and (308), respectively. This proves the claimed error estimates. ∎

Next we turn to the low integral $I_{low}(n) = \int_0^R J_{2n}^2(r) J_n^4(r) rdr$. For the sake of completeness we recall from [1] the definition and most important properties of the Newton-Cotes rule for real polynomials f with degree at most 7. It says that

$$\int_a^{a+6w} f(x)dx = F_{a,w}(f)$$

with

$$\begin{aligned}F_{a,w}(f) := \frac{w}{140}\big[&41f(a) + 216f(a+w) + 27f(a+2w) + 272f(a+3w)\\ &+ 27f(a+4w) + 216f(a+5w) + 41f(a+6w)\big].\end{aligned}$$

For any 8 times continuously differentiable function f on $[a, a+w]$, the function $F_{a,w}(f)$ approximates f with an error of

$$\left| \int_a^{a+6w} f(x)dx - F_{a,w}(f) \right| \leqslant \frac{6^4 w^9}{5 \cdot 8!} \sup_{a \leqslant \xi \leqslant 6w} \left| f^{(8)}(\xi) \right|.$$

In order for the Newton-Cotes rule to be accurate, we need the step size w to be small, meaning that the interval of integration $[a, a+w]$ must be small. Since in our case the total interval is $[0, 600000]$, this is definitely not the case. For this reason we split the interval in smaller subintervals, perform the Newton-Cotes rule on each subinterval and add up the results. This is where the name composite rule comes from. Let for example b be such that $b - a = N \cdot 6w$ for an integer N. Then the composite Newton-Cotes formula

$$\sum_{k=0}^{N-1} F_{a+kw,w}(f) =: F_{[a,b],w}(f)$$

approximates the integral of f on $[a,b]$ with an error

$$\left|\int_a^b f(x)dx - F_{[a,b],w}(f)\right| \leqslant (b-a)\frac{6^3 w^8}{5\cdot 8!}\sup_{a\leqslant\xi\leqslant b}\left|f^{(8)}(\xi)\right|. \tag{316}$$

Now, we return to our specific problem and split our interval of integration $[0,R]$ once again into $[0,S]\cup[S,R]$. The goal is to apply the composite Newton-Cotes rule on $[0,S]$ and $[S,R]$ with different step sizes w_1 and w_2. The parameters w_1, w_2 and S are chosen in such a way that the error of the numerical integration meets our accuracy goal of $4\cdot 10^{-13}$. Moreover, we want that the number of steps we have to perform on each interval is more or less the same. It turns out that the values

$$\begin{aligned} S &= 9843,\\ w_1 &= \frac{5}{10000},\\ w_2 &= \frac{3}{100} \end{aligned} \tag{317}$$

do the job.

Thus, it is left to actually estimate the error. This is equivalent to finding an upper bound on the eighth derivative of the function

$$f(r) := J_{2n}^2(r)J_n^4(r)r.$$

In [1] this is done via the Cauchy integral formula for the circle of radius one about r

$$f^{(m)}(r) = \frac{m!}{2\pi i}\int_{\partial B_1(r)}\frac{f(z)}{(z-r)^{m+1}}dz.$$

Luckily, the same technique applies in our case as well.

We first consider the interval $[0,S]$. Here we estimate the eighth derivative of f as follows

$$\begin{aligned} f^{(8)}(r) &\leqslant \frac{8!}{2\pi}\int_{\partial B_1(r)}\frac{\left|J_{2n}^2(z)J_n^4(z)\right||z|}{|z-r|^9}dz\\ &\leqslant e^6\frac{8!}{2\pi}\int_{\partial B_1(r)}|z|dz\\ &\leqslant 8!e^6(S+1). \end{aligned} \tag{318}$$

The second inequality above is due to the simple bound (380).

For the estimate of the derivative of f on the interval $[S,R]$ we need finer information on the behavior of the Bessel functions for large r. Corollary 5 provides us with the needed information. In fact, inequality (13) implies that

$$\left|J_n^{\pm}(z)\right| \leqslant \left(\frac{2}{\pi|z|}\right)^{\frac{1}{2}}\left[1+\frac{n^2}{|z|}\cosh(\mathrm{Im}(z))\left(\frac{|z|}{|\mathrm{Re}(z)|}\right)^{\frac{1}{2}}\right].$$

Hence, it is

$$\begin{aligned} |f(z)| &\leqslant \left(\frac{2}{\pi}\right)^3\frac{1}{|z|^2}\left[1+\frac{n^2\cosh(\mathrm{Im}(z))}{|z\mathrm{Re}(z)|^{\frac{1}{2}}}\right]^4\left[1+\frac{4n^2\cosh(\mathrm{Im}(z))}{|z\mathrm{Re}(z)|^{\frac{1}{2}}}\right]^2\\ &\leqslant \left(\frac{2}{\pi}\right)^3\frac{1}{|z|^2}\exp\left[\frac{12n^2\cosh(\mathrm{Im}(z))}{|\mathrm{Re}(z)|}\right], \end{aligned}$$

and thus for all $r \in [S, R]$ by Cauchy's Integral Formula

$$\begin{aligned} \left| f^{(8)}(r) \right| &\leqslant \frac{8!}{2\pi} \int_{\partial B_1(r)} \frac{|f(z)|}{|z-r|^9} dz \\ &\leqslant \left(\frac{2}{\pi}\right)^3 \frac{8!}{(S-1)^2} \exp\left[\frac{12n^2\cosh(1)}{S-1}\right]. \end{aligned} \tag{319}$$

Consequently, by combining inequalities (316), (318) and (319), we obtain for the low integral the following error bound

$$\begin{aligned} \left| I_{low}(n) - F_{[0,S],w_1}(f) - F_{[S,R],w_2}(f) \right| \leqslant \frac{6^3 e^6}{5} w_1^8 S(S+1) \\ + \left(\frac{2}{\pi}\right)^3 \frac{6^3}{5} w_2^8 \frac{R-S}{(S-1)^2} \exp\left[\frac{12n^2\cosh(1)}{S-1}\right] \end{aligned}$$

When we plug in $R = 600000, n \leqslant 19$ and the values we chose for w_1, w_2 and S from (317), we obtain

$$\left| I_{low}(n) - F_{[0,S],w_1}(f) - F_{[S,R],w_2}(f) \right| \leqslant 9.4471396343 \cdot 10^{-14}. \tag{320}$$

By Theorem 42 on the high integral and the just derived error bound (320) for the low integral, we end up with the following error bounds on our Bessel integral $I(\mathbf{n})$

$$\left| I(\mathbf{n}) - 1.343815198 \cdot 10^{-7} - F_{[0,S],w_1}(f) - F_{[S,R],w_2}(f) \right| \leqslant 3.0461555492 \cdot 10^{-13}$$

for even n, and

$$\left| I(\mathbf{n}) - 2.687627594 \cdot 10^{-8} - F_{[0,S],w_1}(f) - F_{[S,R],w_2}(f) \right| \leqslant 3.0461555492 \cdot 10^{-13}$$

for odd n.

Just like Oliveira e Silva and Thiele in [1], we use *Wolfram Mathematica* to carry out the numerical integration on $[0, S]$ and $[S, R]$. To meet our accuracy goal, we evaluate the product $J_{2n}^2(r)J_n^4(r)r$ at the grid points with a prescribed precision of 20 digits. This leads for all $2 \leqslant n \leqslant 19$ in the result for $F_{[0,S],w_1}(f) + F_{[S,R],w_2}(f)$ to a precision of at least 21 effective digits to the right of the decimal point. Thus, it is safe to estimate the rounding error by $1 \cdot 10^{-14}$. This now implies an overall accuracy of our approximation of $I(\mathbf{n})$ of $3.15 \cdot 10^{-13}$ for even, as well as odd n.

Since in our case the number of steps in the computation of $F_{[0,S],w_1}(f) + F_{[S,R],w_2}(f)$ is about 16.5 times the number of steps in [1], we have to put a little bit more effort into the code and make use of *Mathematica's* parallelization tools. This way we manage to keep the computation time within reasonable limits. The source code we used for the computation of $F_{[0,S],w_1}(f)$ is the following.

```
Jnn2n20P[n_,r_] := N[(BesselJ[n,r]BesselJ[n,r]BesselJ[2*n,r])^2 r, 20]
NewtonCotesnn2n0S20P[w1_,S_] :=
    Module[ {NewtonCotesnn2n0S20Ploc = ConstantArray[0,18],
        locw = w1, n1 = S/(6*w1), F, n, k},
        F[n_,k_] :=
            locw/140 (41*Jnn2n20P[n,(6*k + 0)*locw] +
               216*Jnn2n20P[n,(6*k + 1)*locw] +
               27*Jnn2n20P[n,(6*k + 2)*locw] +
               272*Jnn2n20P[n,(6*k + 3)*locw] +
               27*Jnn2n20P[n,(6*k + 4)*locw] +
               216*Jnn2n20P[n,(6*k + 5)*locw] +
```

```
                41*Jnn2n20P[n,(6*k + 6)*locw]);
        NewtonCotesnn2n0S20Ploc =
            Table[
                Total[
                    ParallelTable[
                        F[n,k], {k,Floor[(n1 - 1)/10000]*10000,n1 - 1}
                    ]
                ], {n,2,19}
            ];
        Do[
            NewtonCotesnn2n0S20Ploc[[n - 1]]
            = NewtonCotesnn2n0S20Ploc[[n - 1]] +
                Total[
                    ParallelTable[
                        F[n,k], {k,10000*j,10000*(j + 1) - 1}
                    ]
                ];
            ClearSystemCache[], {j,0,Floor[(n1 - 1)/10000] - 1}, {n,2,19}
        ];
        NewtonCotesnn2n0S20Ploc
    ]
```

Table 1 below lists upper bounds on the quantity

$$\left[\left|I_{num}(\mathbf{n}) - \frac{3^{\frac{1}{2}}}{4\pi^2}\frac{1}{n} + \frac{23}{2^8 3^{\frac{3}{2}}\pi^2}\frac{1}{n\left(n^2-\frac{1}{4}\right)}\right| + 3.15\cdot 10^{-13}\right] 10^4 n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)$$

with

$$I_{num}(\mathbf{n}) = F_{[0,S],w_1}(f) + F_{[S,R],w_2}(f) + \begin{cases} 1.343815198\cdot 10^{-7}, & n \text{ even,} \\ 2.687627594\cdot 10^{-8}, & n \text{ odd.} \end{cases}$$

It is immediately apparent that Theorem 3 is still valid for $2 \leqslant n \leqslant 19$.

TABLE 1. Asymptotics of $I(\mathbf{n})$ for small n.

n	$\left[\left\lvert I_{num}(\mathbf{n}) - \frac{\sqrt{3}}{4\pi^2}\frac{1}{n} + \frac{23}{2^8\sqrt{27}\pi^2}\frac{1}{n\left(n^2-\frac{1}{4}\right)}\right\rvert + 3.15\cdot 10^{-13}\right] 10^4 n\left(n^2-\frac{1}{4}\right)\left(n^2-1\right)$
2	5.5489
3	6.8505
4	7.3397
5	7.5731
6	7.7021
7	7.7806
8	7.8318
9	7.8672
10	7.8926
11	7.9115
12	7.9261
13	7.9375
14	7.9470
15	7.9547
16	7.9617
17	7.9673
18	7.9735
19	7.9776

CHAPTER 6

Discussion of the General Main Term

The intention of this chapter is to provide evidence that $M(\mathbf{n})$, defined in formula (7) is a good approximation for $I(\mathbf{n})$. This motivates looking for a rigorous proof of the qualitative statement of Conjecture 2 and a quantitative version of it. To this end, we compare the values of $M(\mathbf{n})$ to values of $I(\mathbf{n})$, that have been calculated numerically with a prescribed accuracy of $0.73 \cdot 10^{-9}$ for the paper [13]. We present our findings in various plots, and in particular replicate Figures 4 and 5 from [13] with our main term. We finish this chapter with a list of open problems that have to be solved in order to prove Conjecture 2.

6.1. Numerical Quality Check of the Main Term

Recall that our incentive to study the integral $I(\mathbf{n})$ originates from a computer aided program for a proof of the sharp Tomas-Stein conjecture on the circle, proposed in [14] and [1]. This program has partly been carried out in [13], where the conjecture is verified for the class of all real-valued functions with Fourier mode up to degree 120. This is done by numerically showing positive semi-definiteness of the matrices (4) for $\mathbf{x}, \mathbf{y} \in (2\mathbb{Z})^2$, both satisfying

$$\begin{aligned} x_1 + x_2 + x_3 &= D \leqslant 360, \\ |x_1|, |x_2|, |x_3| &\leqslant 120, \\ x_1 \leqslant x_2 &\leqslant x_3. \end{aligned}$$

This required the numerical calculation of the Bessel integrals $I(\mathbf{n})$ with an accuracy of $0.73 \cdot 10^{-9}$ for those vectors $\mathbf{n}$, such that either all entries are even, or exactly two of them are odd, and

$$\begin{aligned} n_3 &= D - n_1 - n_2, \\ n_6 &= D - n_4 - n_5, \\ |n_i| &\leqslant 120, \end{aligned} \tag{321}$$

for all $i = 1, \ldots, 6$. We denote those numerical values of $I(\mathbf{n})$ by $I_{num}(\mathbf{n})$. These numerical data are a valuable source for us to test our conjectured main term (7), at least for some subfamilies of $I(\mathbf{n})$.

We choose the subfamily corresponding to $D = 0$ and $n_1 = 20, n_2 = 70, n_3 = -90$, and calculate $M(\mathbf{n})$ for

$$\mathbf{n} = (20, 70, -90, m, n, -m-n) \tag{322}$$

in the range of $0 \leqslant m, n \leqslant 120$ with the help of *Wolfram Mathematica*. The code is the following.

```
Kapteyn[x_,y_,k_]:=
  Gamma[k+1]/2^(k+1)*Gamma[1/2(x+y)-1/2k]/Gamma[1/2(x+y)+1/2k+1]
  *1/(Gamma[1/2(x-y)+1/2k+1]Gamma[-1/2(x-y)+1/2k+1])

HypB[c_,d_,j_]:=
  HypergeometricPFQ[{-d+1/2,d+1/2,-j},{-c-j+1/2,c-j+1/2},1]

```

```
HypA[c_,d_,j_]:=
  HypergeometricPFQ[{-d+1/2,d+1/2,-j},{-c-j+1/2,c-j+1/2},-1]

HypFac[c_,d_,j_]:=
  2^-jGamma[c+j+1/2]/(Gamma[j+1]Gamma[c-j+1/2])

MainSummand[a_,b_,c_,d_,k_]:=
  If[
    Mod[a+b+c+d+k,2]==0,(-1)^((a+b-c-d+k)/2)/(2Pi^2)
    Sum[
      HypFac[a,b,i]HypFac[c,d,k-i]((-1)^iHypA[a,b,i]HypA[c,d,k-i]
      +(-1)^(a+d)(1+(-1)^(-a+b+i))HypB[a,b,i]HypB[c,d,k-i]),{i,0,k}
    ],0
  ]

VZ[a_,b_,c_,d_,x_,y_]:=
  Module[
    {ind=List[a,b,c,d,x,y]},
    For[
      i=1,i<=6,i++,
      If[
        ind[[i]]<0,ind[[i]]=ind[[i]],ind[[i]]=0
      ]
    ];
    Total[ind]
  ]

MainTerm[a_,b_,c_,d_,x_,y_]:=
  (-1)^VZ[a,b,c,d,x,y]*
  Module[
    {orderedParameters=Sort[Abs[{a,b,c,d,x,y}]]},
    If[
      a==b==c==d==x==y==0,0.3368279617664489`,
      Sum[
        Kapteyn[
          orderedParameters[[6]],orderedParameters[[5]],k
        ]
        MainSummand[
          orderedParameters[[1]],orderedParameters[[2]],
          orderedParameters[[3]],orderedParameters[[4]],k
        ],{k,Max[orderedParameters[[6]]-orderedParameters[[5]]-1,0],
        orderedParameters[[5]]+orderedParameters[[6]]-1}
      ]
    ]
  ]
```

Figure 6.1 reveals the interesting and beautiful structure of $M(20,70,-90,m,n,-m-n)$.

We compare the values of $M(\mathbf{n})$ to those for $I_{num}(\mathbf{n})$ from [13] and obtain for the difference

$$\max_{0\leqslant m+n\leqslant 120} |M(20,70,-90,m,n,-m-n) - I_{num}(20,70,-90,m,n,-m-n)| \leqslant 9\cdot 10^{-14}.$$

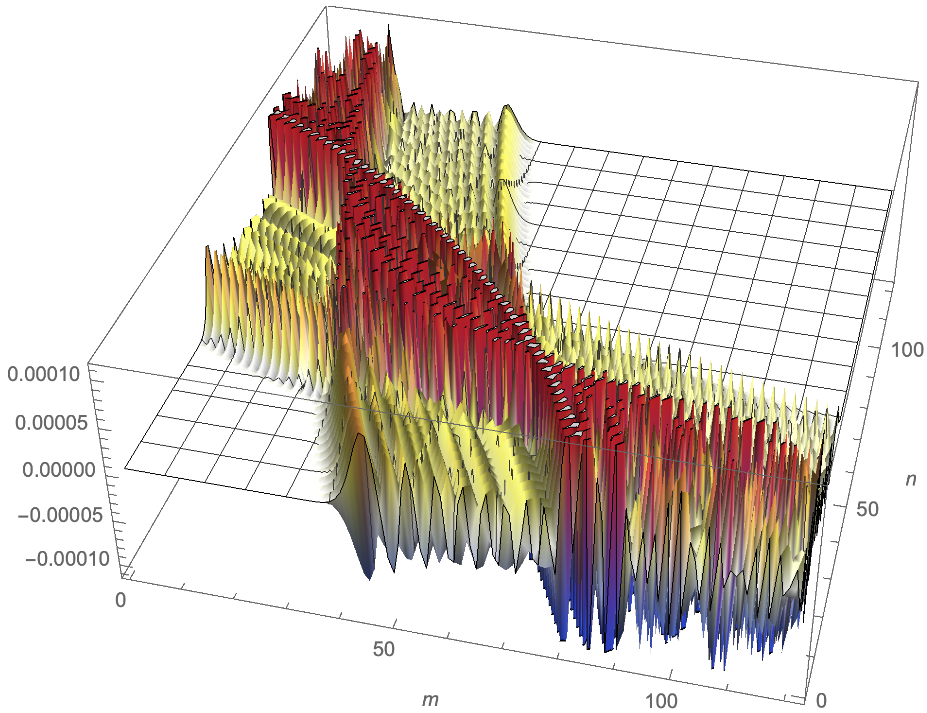

FIGURE 6.1. Plot of $M(20, 70, -90, m, n, -m-n)$.

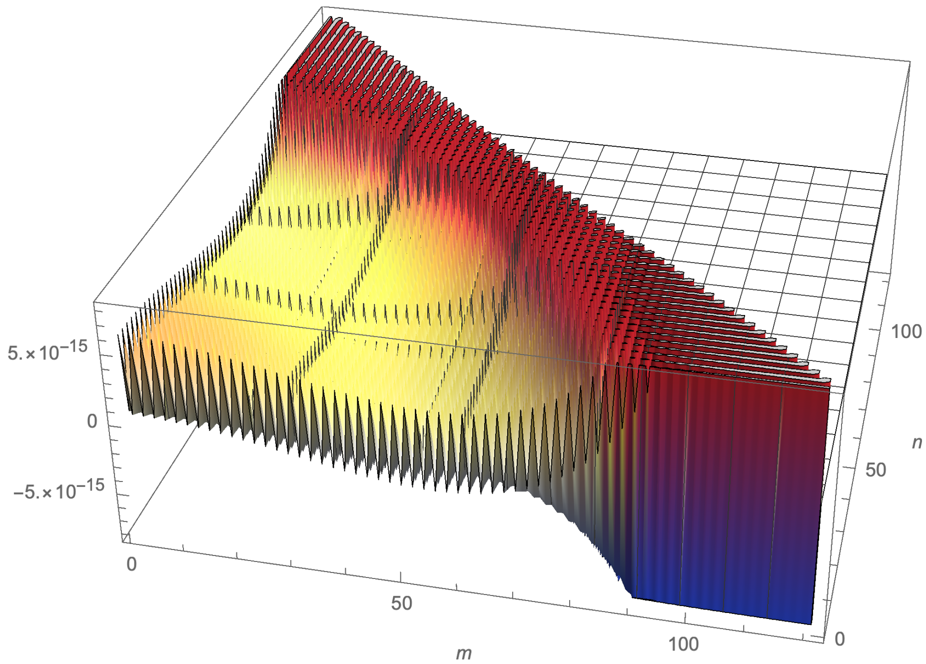

FIGURE 6.2. Plot of the difference
$M(20, 70, -90, m, n, -m-n) - I_{num}(20, 70, -90, m, n, -m-n)$.

As can be seen in Figure 6.2, the difference is uniformly small for all points (m, n) with $m + n \leqslant 120$, and increases in the vicinity of the diagonal $m + n = 120$. We suppose that this increase is due to the numerical methods used in [13], that produce slightly worse error bounds in this region. Note that, in contrast to Figure 6.1, the values for $m + n > 120$ are artificially set to zero in Figure 6.2, since we don't have any reference data $I_{num}(\mathbf{n})$ in this region. This is due to conditions (321). These pleasantly small differences encourage us to believe that our Conjecture 2 is correct and the main term (7) provides very good approximations of $I(\mathbf{n})$. At least, if the components of $\mathbf{n}$ are large enough. For $(20, 70, -90, m, n, -m-n)$ this seems to be the case. Of course, it has to be further investigated, what "large enough" exactly means. The next section might give some hints.

6.2. From $I(\mathbf{n})$ to $Q(\mathbf{n})$ and Interesting Questions

While our focus of attention lies on the integrals $I(\mathbf{n})$, the primary object of interest in [13] are the matrices (4). In order to further test our main term formula (7) and to gain some more insight in its behavior and performance, we use it to calculate $Q(\mathbf{x}, \mathbf{y})$ for test the vector (322).

By resolving some of the sums over the permutation group S_3 in the construction guide (4), (5) and (6) of the matrix $Q(\mathbf{x}, \mathbf{y})$ we obtain

$$Q(\mathbf{x}, \mathbf{y}) = \frac{1}{6} \sum_{\tau \in S_3} R(\mathbf{x}, \tau(\mathbf{y})) - L(\mathbf{x}, \mathbf{y}),$$

with

$$R(\mathbf{x}, \mathbf{y}) = 2I(\mathbf{x} - \mathbf{y}, 0, 0, 0) + 6I(\mathbf{x} - \mathbf{y}, -1, 1, 0),$$

$$L(\mathbf{x}, \mathbf{y}) = 2I(\mathbf{x}, -\mathbf{y}) + \sum_{\sigma \in S_3} I(\mathbf{x}, -\mathbf{y} + \sigma(-1, 1, 0)).$$

Now, we replace every instance of I with our main term M and denote the corresponding matrices by $L_M(\mathbf{x}, \mathbf{y}), R_M(\mathbf{x}, \mathbf{y})$ and $Q_M(\mathbf{x}, \mathbf{y})$. As above, we write $L_{num}(\mathbf{x}, \mathbf{y}), R_{num}(\mathbf{x}, \mathbf{y})$ and $Q_{num}(\mathbf{x}, \mathbf{y})$, respectively, when we use the numerical data from [13] to calculate those matrices. Furthermore, we split the test vector (322) as follows

$$(\mathbf{x}, \mathbf{y}) = (20, 70, -90, m, n, -m-n) = \mathbf{n}. \tag{323}$$

Note that since $J_{-n}(r) = (-1)^n J_n(r)$, it is

$$\begin{aligned} I(\mathbf{n}, -\mathbf{y}) &= I(\mathbf{x}, -m, -n, m+n) \\ &= I(\mathbf{x}, m, n, -m-n). \end{aligned}$$

Hence, we can use $M(20, 70, -90, m, n, -m-n)$ for the construction of the matrix $L_M(\mathbf{n})$. Its remaining components are calculated with the code above. Since the corresponding vectors are very close to (322), we expect the difference $L_M(\mathbf{n}) - L_{num}(\mathbf{n})$ to be comparable to that of $M(\mathbf{n})$ and $I_{num}(\mathbf{n})$. In fact, it is

$$\max_{0 \leqslant m+n \leqslant 120} |L_M(20, 70, -90, m, n, -m-n) - L_{num}(20, 70, -90, m, n, -m-n)| \leqslant 3 \cdot 10^{-14},$$

even smaller.

A three-dimensional plot of the difference can be seen in Figure 6.3. Similar to Figure 6.2, the area, where $m+n > 120$ is set to zero, since we lack reference data there. Note, that the difference does not show the same oscillatory structure as the difference $M(\mathbf{n}) - I_{num}(\mathbf{n})$ in Figure 6.2. This means that we are perfectly able to reproduce the structure of $L_{num}(\mathbf{n})$, even better than for a single Bessel integral $I_{num}(\mathbf{n})$. The reason might be cancellation effects due to the shifts $(-m, -n, m+n) + \sigma(-1, 1, 0)$.

The matrix $L_M(\mathbf{n})$ itself can be seen in Figures 6.4 and 6.5. In Figure 6.4, the outlying large values are dropped to emphasize the beautiful wavey structure of the function. It is not surprising, that $L_M(\mathbf{n})$ and $M(\mathbf{n})$ look very similar. To emphasize the fact that $L_M(\mathbf{n})$ is almost everywhere negligibly small except for a few values that lie on a very regular curve,

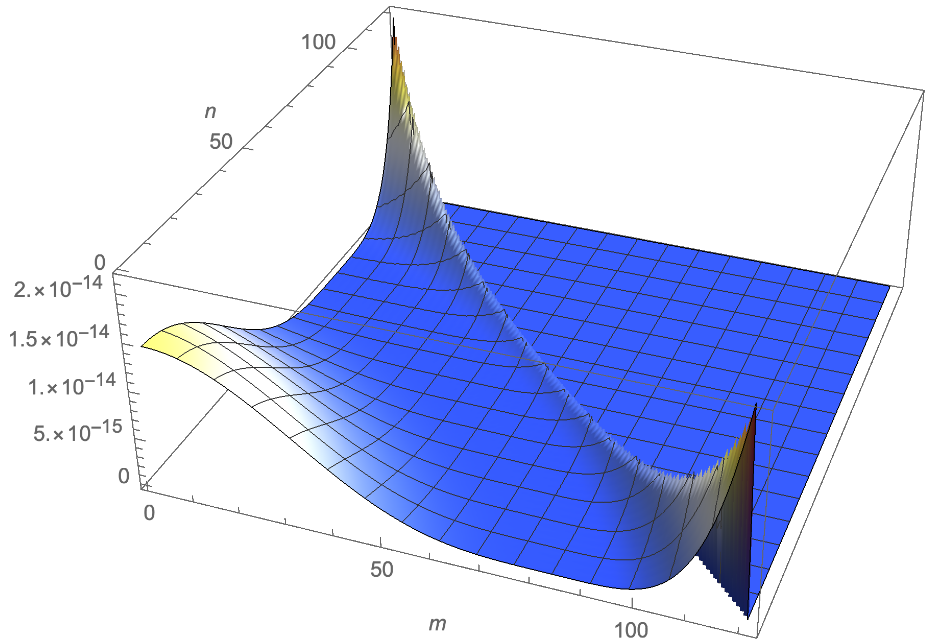

FIGURE 6.3. Plot of the difference
$L_M(20, 70, -90, m, n, -m-n) - L_{num}(20, 70, -90, m, n, -m-n)$.

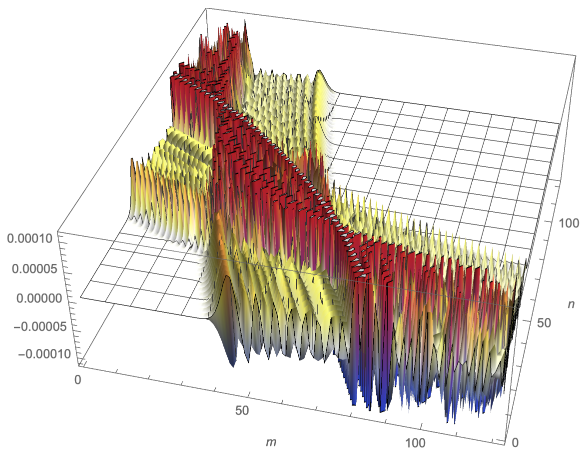

FIGURE 6.4. $L_M(\mathbf{n})$ without the large values.

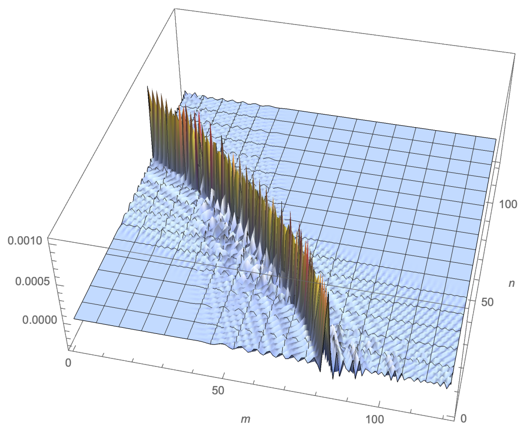

FIGURE 6.5. Full range plot of $L_M(\mathbf{n})$

Figure 6.5 shows the full range of the values. It is conjectured in [13] that this location of the large entries is actually a section of an ellipsis given by

$$m^2 + n^2 + (m+n)^2 = 20^2 + 70^2 + 90^2. \tag{324}$$

Next, we turn to the matrix $R_M(\mathbf{n})$. Recall that $M(\mathbf{0}) = 0$, since $D_\mathbf{0} = \varnothing$, whereas $I(\mathbf{0}) = 0.33682796$ is actually the largest of all Bessel integrals $I(\mathbf{z})$. This is definitely a weak point of the main term (7) and shows that it will most certainly only provide good approximations of $I(\mathbf{z})$, if $\mathbf{z}$ is sufficiently far from $\mathbf{0}$. Since the construction of $R_M(\mathbf{n})$ requires the calculation of $M(\mathbf{z})$ for $\mathbf{z}$ in the vicinity of $\mathbf{0}$, we expect to see larger deviations between $R_M(\mathbf{n})$ and $R_{num}(\mathbf{n})$ than in the case of $L_M(\mathbf{n})$. To soften the effects of $M(\mathbf{0}) = 0$ a little, we manually set the value $M(\mathbf{0}) = 0.3368279617664489$ in line 40 of the *Mathematica* code we use to calculate the main term.

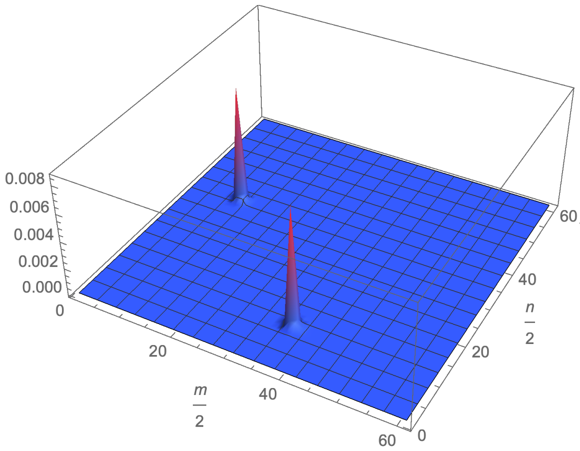

FIGURE 6.6. Full range plot of the difference $R_{num}(\mathbf{n}) - R_M(\mathbf{n})$ for even m, n.

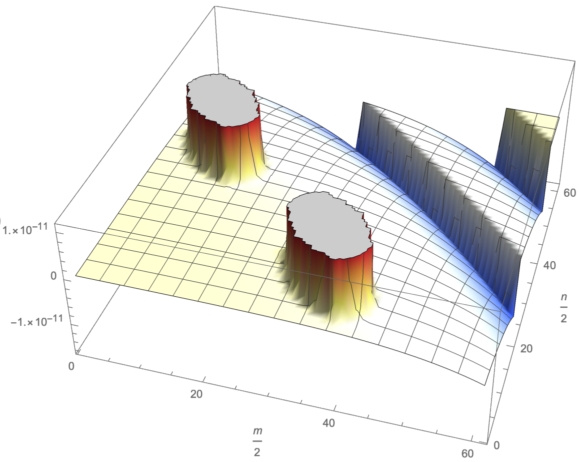

FIGURE 6.7. Plot of the difference $R_{num}(\mathbf{n}) - R_M(\mathbf{n})$ for even m, n.

For the calculation of the matrix $R_M(\mathbf{n})$ we then use the following little piece of extra *Wolfram Mathematica* code.

```
MatrixR[x_,y_,z_]:=
        2*MainTerm[20-x,70-y,-90-z,0,0,0]+
        6*MainTerm[20-x,70-y,-90-z,1,-1,0]
```

Unfortunately, in [13] the matrix $R_{num}(\mathbf{n})$ has only been calculated for even m and n, as this was sufficient for their purposes. We therefore have to restrict the comparison of $R_M(\mathbf{n})$ and $R_{num}(\mathbf{n})$ to even m, n. Nevertheless, that still provides valuable insights. As we expected, we see two huge spikes around $(m, n) = (20, 70)$ and $(m, n) = (70, 20)$ in the plot of the difference $R_{num}(\mathbf{n}) - R_M(\mathbf{n})$ in Figure 6.6. In Figure 6.7 we dropped those large outliers. This way it becomes easier to see for which vectors $\mathbf{z}$ one starts to obtain sufficiently good approximations for $I(\mathbf{z})$ from the main term $M(\mathbf{z})$. It looks like we are on the safe side, meaning the error is smaller than 10^{-11}, if the absolute value of the largest entry of $\mathbf{z}$ is larger than approximately 20. This corresponds to the two circles of radius 20 centered around $(20, 70)$ and $(70, 20)$ in Figure 6.7.

The matrix $R_M(\mathbf{n})$ itself has the quite interesting looking structure, that can be seen in Figure 6.8. However, with the exception of the two peaks around $(20, 70)$ and $(70, 20)$, its values are very small. This is shown in Figure 6.9.

Of course, the differences between $L_M(\mathbf{n})$ and $L_{num}(\mathbf{n})$ and between $R_M(\mathbf{n})$ and $R_{num}(\mathbf{n})$ carry over to $Q_M(\mathbf{n})$ and $Q_{num}(\mathbf{n})$. We therefore refrain from a comparison in the spirit of $L_M(\mathbf{n})$ and $R_M(\mathbf{n})$ and only provide the promised replication of Figures 4 and 5 from [13].

Figure 6.10 shows a logarithmic plot of the absolute value of $Q_M(\mathbf{n})$. In Figure 6.11 we see the same plot for $Q_{num}(\mathbf{n})$, which corresponds to the first quadrant of Figure 4 in [13]. The structure with the section of the ellipsis and the two large peaks can be recognized perfectly. Moreover, the differences between the two matrices $Q_M(\mathbf{n})$ and $Q_{num}(\mathbf{n})$ are so small that they cannot be seen by means of this type of visualization.

Next, we turn to Figure 5 from [13]. It plots for even m, n the values of $Q_{num}(\mathbf{n})$ as a function of the radial variable $r(m, n) := (m^2 + n^2 + (m + n)^2)^{\frac{1}{2}}$ near the radius of the ellipsis (324), that is, in the vicinity of $r_0 := 116$. The red dots in Figure 6.12 represent the

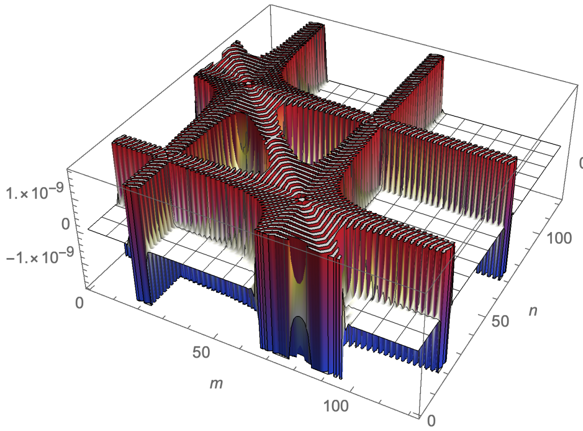

FIGURE 6.8. Plot of $R_M(\mathbf{n})$.

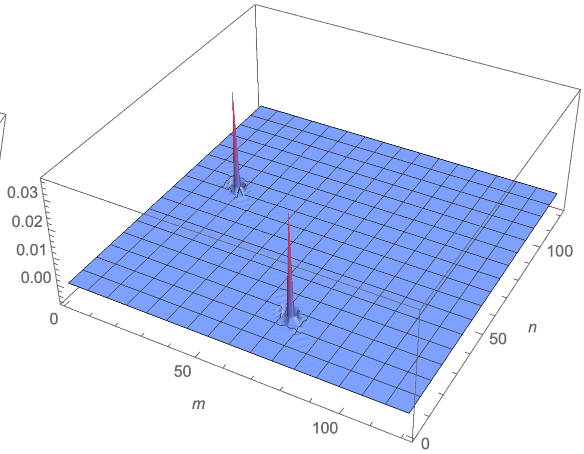

FIGURE 6.9. Full range plot of $R_M(\mathbf{n})$.

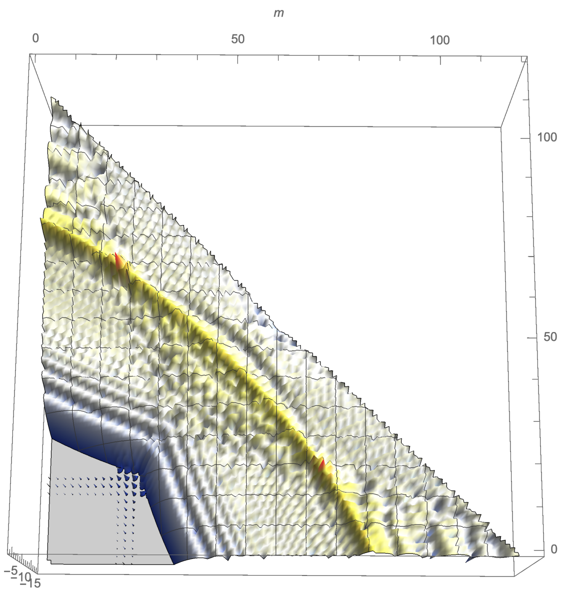

FIGURE 6.10. Logarithmic plot of $|Q_M(\mathbf{n})|$.

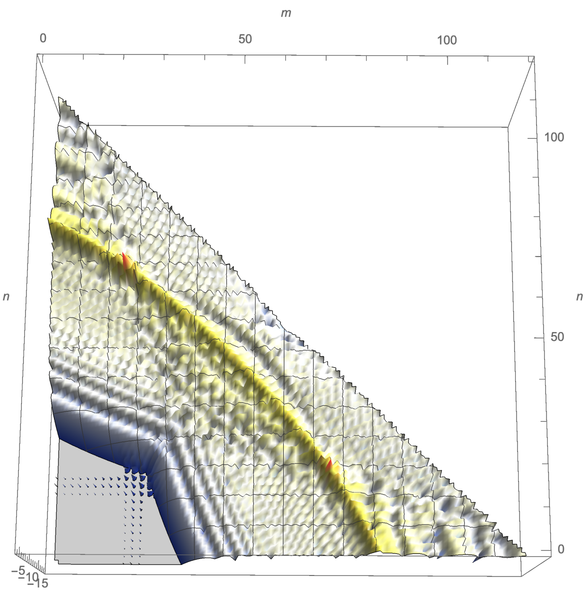

FIGURE 6.11. Logarithmic plot of $|Q_{num}(\mathbf{n})|$.

values of $Q_M(r(m,n))$, while the blue ones belong to $Q_{num}(r(m,n))$. We omitted the large outlier around r_0. Since most of the points are on top of each other, we almost only see blue ones that are slightly turning purplish. This is another pleasant confirmation of the good approximation of $I(\mathbf{n})$ by $M(\mathbf{n})$. This blue part of the plot is the counterpart of Figure 5 in [13]. One can perfectly see the very regular pattern, that has first been observed in [13]. We want to point out that this pattern of the radial plot entirely stems from the matrix $L(\mathbf{n})$, shown in Figure 6.13. This fact can be seen even more impressively in Figure 6.14, that plots the entire range of values of $Q(r(m,n))$ and in Figure 6.15, that only displays $R(r(m,n))$. While $L(\mathbf{n})$ is responsible for the wave-like structure, that is still recognizable in figure 6.14, $R(\mathbf{n})$ actually only contributes five dots, the major positive peak, and four secondary peaks of negative value. All other values are negligibly small compared to those of $L(\mathbf{n})$, as can be seen in Figure 6.15, where we omitted these five large values.

Note that apart from the interesting properties of the matrices $Q(\mathbf{n})$ and $L(\mathbf{n})$, the plots in Figures 6.12, 6.13 and 6.14 expose once again the great performance of our main term

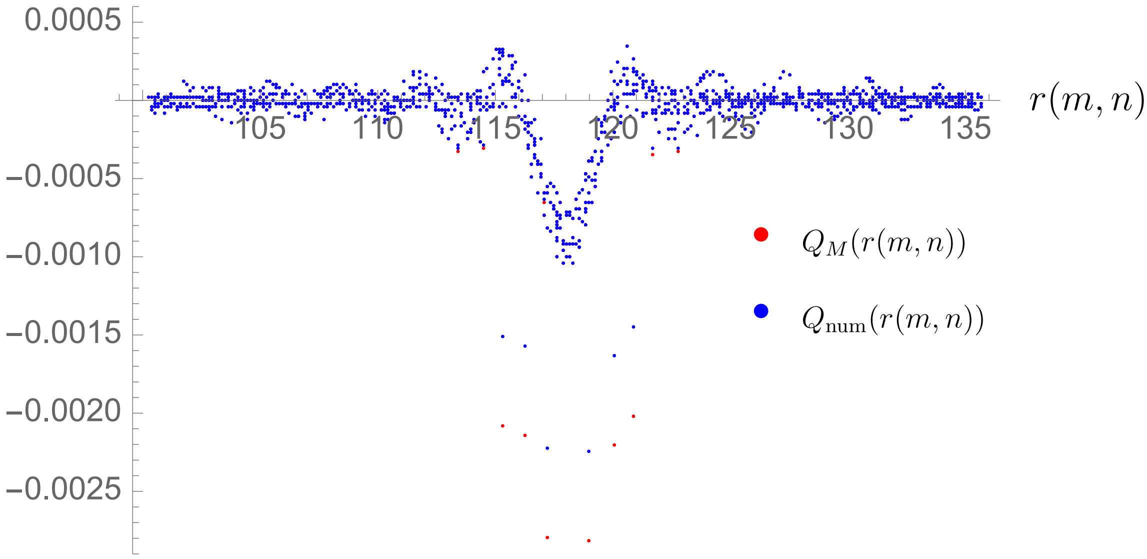

FIGURE 6.12. Truncated radial plot of $Q_M(\mathbf{n})$ and $Q_{num}(\mathbf{n})$.

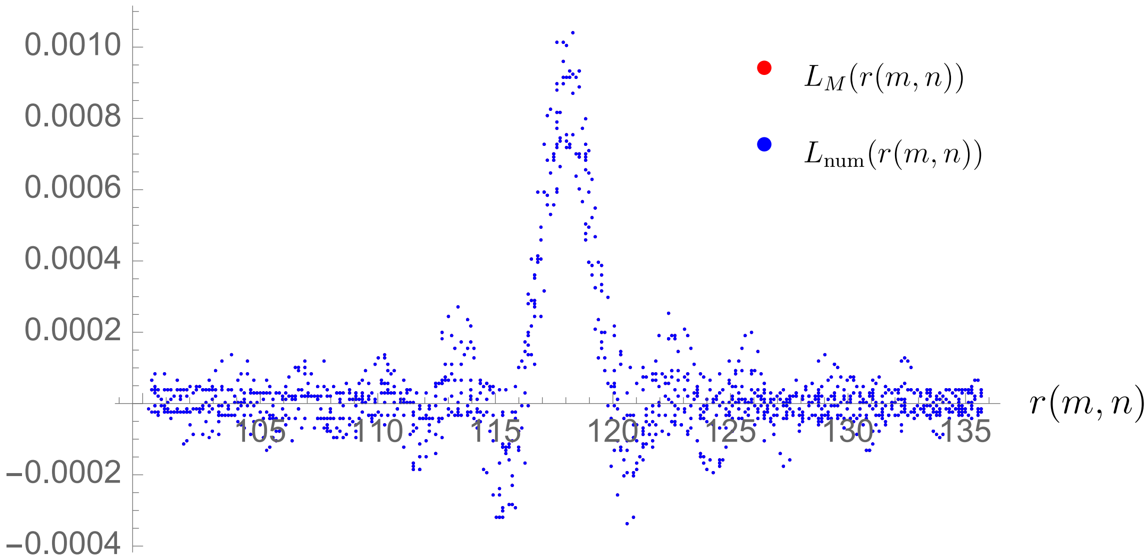

FIGURE 6.13. Full range radial plot of $L_M(\mathbf{n})$ and $L_{num}(\mathbf{n})$.

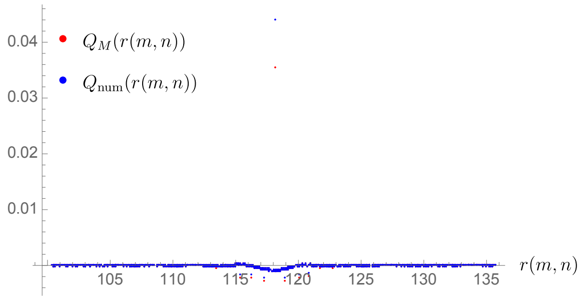

FIGURE 6.14. Full range radial plot of $Q_M(\mathbf{n})$ and $Q_{num}(\mathbf{n})$.

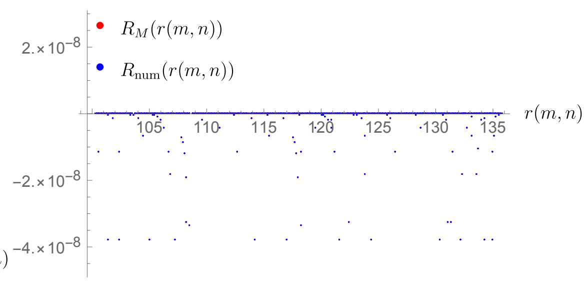

FIGURE 6.15. Truncated radial plot of $R_M(\mathbf{n})$ and $R_{num}(\mathbf{n})$.

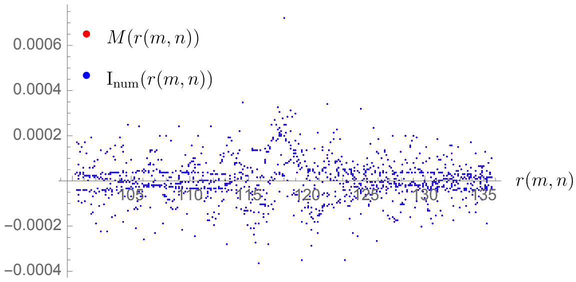

FIGURE 6.16. Full range radial plot of $M(\mathbf{n})$ and $I_{num}(\mathbf{n})$.

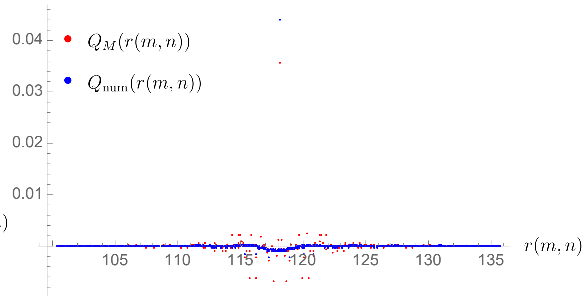

FIGURE 6.17. Full range radial plot of $Q_M(\mathbf{n})$ for even and odd m, n and $Q_{num}(\mathbf{n})$ only for even m, n.

$M(\mathbf{z})$ everywhere, except for vectors $\mathbf{z}$ close to $\mathbf{0}$. The discrepancies in the approximated values by means of M versus the numerical reference data for those vectors become brutally apparent by the quite large distance between the blue and red dots especially in Figures 6.12 and 6.14.

Two more noteworthy observations are that on the one hand the pattern we see in Figure 6.13 is actually due to the way the matrix $L(\mathbf{n})$ is constructed, and does not yet manifest itself so clearly in $I(\mathbf{n})$, that is displayed in Figure 6.16. The second observation is that taking into account odd m, n as well does not add any further substantial information to the pattern of $Q(\mathbf{n})$, as can be seen by the additional red dots in Figure 6.17 compared to Figure 6.14. The missing values for odd m, n in the case of $Q_{num}(\mathbf{n})$ are set to zero for this visualization.

6.3. Open Problems

We want to use this last section to summarize the open problems and questions concerning the main term $M(\mathbf{n})$ and Conjecture 2, that arose in the course of the discussion in this chapter and throughout the entire work.

In order to prove Conjecture 2 and to quantify the error

$$|I(\mathbf{n}) - M(\mathbf{n})|$$

we need a characterization of the expansion remainder $R_l^{(4)}(\alpha, \beta, \gamma, \delta, r)$ of the series expansion of the product of four different Bessel functions, similar to the one for $\alpha = \beta = \gamma = \delta = n$ in Lemma 7. This expansion remainder then needs to be included in the integration of the series expansion in Chapter 3. Moreover, in the integration step, we need to establish a more comprehensive characterization of the secondary term S, that contains not only the $4r$-terms from (51), as in the case of $\mathbf{n} = (n, n, n, n, 2n, 2n)$, but also the $2r$-terms. Then, we either suffer greatly and fight our way through the steps of Chapter 4, but with six different parameters instead of only one, or we find a smarter solution to estimate the remainder term and the secondary term in this general setting.

Beyond a qualitative and quantitative proof of Conjecture 2 it would be great to answer for example the question, whether the main term $M(\mathbf{n})$ can be further simplified in the spirit of Theorem 36 in Section 4.4. Meaning, is there a (simple) decomposition into leading asymptotic terms and an error? If so, how does it look like?
Another exciting question is, whether formula (7) enables us to explain the structures analytically we see for instance in Figures 6.1, 6.10 and 6.13.

CHAPTER 7

Background

In this chapter we provide some necessary tools from the theory of gamma functions, generalized hypergeometric functions and Bessel functions. Moreover, we introduce the sine and the cosine integral and give some insights on their asymptotics. The last section then collects some additional useful inequalities and summation formulae we use extensively throughout the entire thesis.

Most of the results are well-known. A few results, that are particularly useful in our setting, have not been taken from the literature, but are proven here. These include Lemma 55 on a uniform upper bound for $\left(\frac{\pi}{2}r\right)^{\frac{1}{2}} J_n(r)$, Lemma 63, that provides asymptotic estimates for integrals of the type

$$\int_x^\infty \frac{\cos(t)}{t^n} dt, \quad \int_x^\infty \frac{\sin(t)}{t^n} dt,$$

Lemma 64, which estimates $1+x$ in terms of $e^{x-\delta x^2}$ from above and below for suitable x and δ, and last but not least Lemmata 67 and 69, which provide closed form expressions for the tail of the geometric series and sums of the type

$$\sum_{j=0}^{p} j^a \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)},$$

respectively.

7.1. The Gamma Function

The purpose of this section is to introduce the gamma function and to display and deduce some of its basic properties as well as a few upper and lower bounds, that have proven to be extremely useful for our purposes, for example in estimating gamma quotients. Those are the reflection formula (327), the duplication formula (328), Stirling's formula (332) and Lemma 44, to name a few.

7.1.1. Definition and Some Properties. When you are working with Bessel functions, you won't be able to avoid the gamma function, defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{325}$$

for $\mathrm{Re}(z) > 0$. Using integration by parts, one sees that

$$\Gamma(z+1) = z\Gamma(z). \tag{326}$$

This functional equation can be used to uniquely extend the integral formula to a meromorphic function defined for all complex z, except negative integers and zero. Two important properties of the gamma function, we heavily use throughout this work, are Euler's reflection formula for $z \notin \mathbb{Z}$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{327}$$

and Legendre's duplication formula for $z \notin \{-\frac{n}{2} \mid n \in \mathbb{N}\}$

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \pi^{\frac{1}{2}} 2^{1-2z}\Gamma(2z). \tag{328}$$

The reflection formula (327) is a consequence of Liouville's Theorem. This can be seen as follows. We set

$$f(z) = \Gamma(z)\Gamma(1-z)\sin(\pi z).$$

Due to the multiplicative recursion of the gamma function, the function f is 1-periodic. By the triangle inequality applied to the defining integral (325), one finds that

$$|f(x+iy)| = O\left(e^{y}\right) \quad \text{as } y \to \infty$$

for $x \in [0,1]$ and positive y. Hence, it is $f(z) = F\left(e^{2\pi i z}\right)$ for a holomorphic function F on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, with $|F(\omega)| = O\left(|\omega|^{\frac{1}{2}}\right)$ as $\omega \to \infty$ and $|F(\omega)| = O\left(|\omega|^{-\frac{1}{2}}|\right)$ as $\omega \to 0$. Thus, the function F is constant. Plugging in $z = \frac{1}{2}$ evaluates the constant.

Legendre's duplication formula (328) follows from the identity

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{329}$$

for x and y with positive real part. By substituting $t = \frac{1+s}{2}$, the right-hand side above turns into

$$2^{-x-y+1}\Gamma(x+y)\int_{-1}^1 (1-s)^{x-1}(1+s)^{y-1}ds. \tag{330}$$

Now we set $x = y = z$ and obtain

$$2^{2z-1}\frac{\Gamma(z)^2}{\Gamma(2z)} = 2\int_0^1 \left(1-s^2\right)^{z-1} ds.$$

On the other hand we find that

$$2\int_0^1 \left(1-s^2\right)^{z-1} ds = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(z)}{\Gamma\left(z+\frac{1}{2}\right)},$$

when we substitute $t = s^2$ in (329) and let $x = \frac{1}{2}$ and $y = z$. Since $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$, identity (328) follows.

Identity (329) can be seen by writing the product $\Gamma(x)\Gamma(y)$ as the double integral

$$\int_0^\infty \int_0^\infty e^{-(t+s)}t^{x-1}s^{y-1}dtds.$$

Substituting $t = rv, s = (1-r)v$ turns the above line into

$$\int_0^\infty e^{-v}v^{x+y-1}dv \int_0^1 r^{x-1}(1-r)^{y-1}dr.$$

By definition, the first factor is equal to $\Gamma(x+y)$.

Remark 43. Another useful representation for $\Gamma(z)$ and $\mathrm{Re}(z) > 0$ is

$$\Gamma(z) = s^z \int_0^\infty e^{-st}t^{z-1}dt. \tag{331}$$

One deduces it from (325) by replacing t with st for $s > 0$.

7.1.2. Bounds. Throughout our work, we have to deal a lot with gamma quotients. Often it is not immediately apparent how large or small such a gamma quotient really is. Stirling's formula is a great tool to shed some light on this question. So let us introduce one of our best friends in this thesis: Stirling's formula for $x \geqslant 0$

$$\Gamma(x) = (2\pi)^{\frac{1}{2}} x^{x-\frac{1}{2}} e^{-x+\mu(x)}, \tag{332}$$

where the function μ satisfies

$$\frac{1}{12x+1} < \mu(x) < \frac{1}{12x}. \tag{333}$$

Here we use the same version of Stirling's formula as Oliveira e Silva and Thiele did in [1].

One more handy double sided inequality, we will heavily make use of, is the following.

LEMMA 44 ([1]). *Let $x \geqslant \frac{1}{2}$, then*

$$\left(x - \frac{1}{2}\right)^{\frac{1}{2}} \leqslant \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} \leqslant x^{\frac{1}{2}}. \tag{334}$$

Another useful convexity estimate is

LEMMA 45 ([1]). *Let $x \geqslant y > 0$ and $w \geqslant 0$. Then it is*

$$\frac{\Gamma(x)}{\Gamma(y)} \leqslant \frac{\Gamma(x+w)}{\Gamma(y+w)}.$$

7.2. Generalized Hypergeometric Functions

It lies in the nature of our approach to Bessel functions and to products and integrals of them that we encounter many sums of products and quotients of gamma functions. We refer to them as sums of gamma quotients. It is often convenient to express such a sum of gamma quotients as gamma quotients itself. For this we shall use some formulae from the theory of hypergeometric functions.
After a brief introduction of the generalized hypergeometric function, we will deduce these formulae, which are mainly hypergeometric identities. More precisely, in Lemma 47 we prove Gauss' identity, which expresses ${}_2F_1\left(\begin{smallmatrix} a & b \\ & c \end{smallmatrix} \,\middle|\, 1\right)$ as a gamma quotient, and in Lemma 48 we prove Dixon's identity, which expresses ${}_3F_2\left(\begin{smallmatrix} a & b & c \\ & d & e \end{smallmatrix} \,\middle|\, 1\right)$ as a gamma quotient. Afterwards, in Lemma 49 we prove a summation formula for the related series

$$\sum_{p=0}^{\infty} p^a \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} x^p,$$

for $a \in \mathbb{Z}$. A valuable tool for the proof of hypergeometric identities is Euler's integral representation of ${}_2F_1$. Therefore, we mention and prove it below in Lemma 46. We finish this section with Lemma 50, that provides an upper bound on generalized hypergeometric functions. We need this bound in our first step of the estimate of the secondary term in Section 4.3. Now, let us start with the definition.

The generalized hypergeometric function is defined as the series

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z\right) = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{m=0}^{\infty} \frac{\Gamma(a_1+m)\cdots\Gamma(a_p+m)}{\Gamma(b_1+m)\cdots\Gamma(b_q+m)} \frac{z^m}{m!}. \tag{335}$$

With $z = 1$ it is a sum of gamma quotients. Is one of the upper parameters a_i a negative integer, then the series in (335) is finite and the associated hypergeometric function is a polynomial in z, since the gamma function has simple poles at the non-positive integers. Is one of the lower parameters b_j a negative integer or zero, then ${}_pF_q\left(\begin{smallmatrix} a_1 & \dots & a_p \\ b_1 & \dots & b_q \end{smallmatrix} \,\middle|\, z\right)$ does not exist in general. In all other cases, the radius of convergence ρ of the hypergeometric series is given by

$$\rho = \begin{cases} \infty, & p < q+1, \\ 1, & p = q+1, \\ 0, & p > q+1. \end{cases} \tag{336}$$

This follows directly from the ratio test, since

$$\lim_{m\to\infty}\left|\frac{c_{m+1}}{c_m}\right| = \begin{cases} 0, & p < q+1, \\ |z|, & p = q+1, \\ \infty, & p > q+1, \end{cases}$$

with $c_m = \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m m!}$.

In the cases $p = q$ and $p = q+1$ the function ${}_pF_q$ is called generalized hypergeometric function of Kummer type and Gauss type, respectively. The name (Gaussian or ordinary) Hypergeometric function is historically reserved for the special case of $p = 2$ and $q = 1$ with principal value being the cut plane $\mathbb{C}\backslash[1,\infty)$. This function has the following integral representation due to Euler.

LEMMA 46. *Let* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. *Then*

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \,\middle|\, z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt.$$

PROOF. Let us first suppose $|z| < 1$. Then

$$(1-zt)^{-a} = \sum_{m=0}^{\infty}\binom{-a}{m}z^m t^m = \sum_{m=0}^{\infty}\frac{\Gamma(a+m)}{\Gamma(a)m!}z^m t^m. \tag{337}$$

The second equality above is due to Euler's reflection formula (327). In Fact, the reflection formula implies that

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} = \frac{\sin(\pi(\alpha-m+1))}{\sin(\pi(\alpha+1))}\frac{\Gamma(-\alpha+m)}{\Gamma(-\alpha)}.$$

Since

$$\frac{\sin(\pi(\alpha-m+1))}{\sin(\pi(\alpha+1))} = \frac{\sin(\pi(\alpha-m))}{\sin(\pi\alpha)} = \cos(\pi m) = (-1)^m$$

we find that

$$\binom{\alpha}{m} = \frac{\Gamma(-\alpha+m)}{\Gamma(-\alpha)}\frac{(-1)^m}{m!}. \tag{338}$$

Now, using representation (337) and identity (329) we obtain

$$\begin{aligned}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt &= \sum_{m=0}^{\infty}\frac{\Gamma(a+m)}{\Gamma(a)m!}z^m\int_0^1 t^{m+b-1}(1-t)^{c-b-1}dt \\ &= \sum_{m=0}^{\infty}\frac{\Gamma(a+m)}{\Gamma(a)m!}z^m\frac{\Gamma(b+m)\Gamma(c-b)}{\Gamma(c+m)} \\ &= \frac{\Gamma(c-b)}{\Gamma(a)}\sum_{m=0}^{\infty}\frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)}\frac{z^m}{m!} \\ &= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)}{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \,\middle|\, z\right).\end{aligned}$$

Since the integral is analytic in the cut plane $\mathbb{C}\backslash[1,\infty)$, the proof follows by analytic continuation for all $z \in \mathbb{C}\backslash[1,\infty)$. ∎

7.2.1. Hypergeometric Identities. A direct consequence of Euler's integral representation is Gauss' first hypergeometric identity. We need it in the following section for the proof of Lemma 59.

LEMMA 47. *Let* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ *and* $\mathrm{Re}(c-a-b) > 0$. *Then*

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\Big|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

PROOF. By setting $z = 1$ in Lemma 46 we get for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c-a-b) > 0$

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\Big|\, 1\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1}dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}, \end{aligned} $$

where we once again used (329). ∎

The ${}_3F_2$ counterpart of Gauss' identity is Dixon's hypergeometric identity. It is an important ingredient in the proof of Lemma 11.

LEMMA 48. *Let* a, b, c *be non-negative integers such that* $1 + \frac{a}{2} - b - c > 0$. *Then in its beautifully symmetric and easier-to-remember form the identity reads*

$$ \sum_{m=-a}^{a} (-1)^m \binom{a+b}{a+m}\binom{b+c}{b+m}\binom{c+a}{c+m} = \frac{(a+b+c)!}{a!b!c!}. \tag{339} $$

Moreover, the left-hand side of (339) *is equal to*

$$ \binom{b+c}{b-a}\binom{c+a}{c-a} {}_3F_2\left(\begin{matrix} -2a, -a-b, -a-c \\ 1+b-a, 1+c-a \end{matrix} \,\Big|\, 1\right). \tag{340} $$

This following very elegant and short proof of Dixon's identity for non-negative integers is due to Ekhad [4].

PROOF. We will prove (339) first. Set

$$ f(a) = \sum_{m=-a}^{a} (-1)^m \binom{a+b}{a+m}\binom{b+c}{b+m}\binom{c+a}{c+m} $$

and denote the summand by $F(a,m)$. Obviously, the statement is true for $a = 0$. Thus, it follows by induction if we show that

$$ (a+1)f(a+1) - (a+1+b+c)f(a) \equiv 0. \tag{341} $$

Our goal is to find a recurrence relation for $F(a,m)$ of the form

$$ (a+1)F(a+1,m) + (a+1+b+c)F(a,m) = G(a,m) - G(a,m-1). \tag{342} $$

It is easily verified that the function

$$ G(a,m) = \frac{(-1)^a(a+b)!(a+c)!(b+c)!}{2(a+m+1)!(a-m)!(b+m)!(b-m-1)!(c+m)!(c-m-1)!} $$

satisfies this recurrence. We have done calculations like this a hundred times in Sections 4.2 and 4.3. Summing (342) with respect to m yields (341), since the right-hand side telescopes to zero due to the simple poles of $\Gamma(a-m+1)$ for $m \geqslant a+1$ and $\Gamma(a+m+2)$ for $m \leqslant -a-2$.

The equivalence of (340) and (339) can be seen as follows. After shifting the summation on the left-hand side of (339) by a, we obtain by (338)

$$\frac{(-1)^{b+c}}{\Gamma(-a-b)\Gamma(-a-c)\Gamma(-b-c)}\sum_{m=0}^{2a}\frac{\Gamma(-2a+m)\Gamma(-a-b+m)\Gamma(-a-c+m)}{\Gamma(b-a+1+m)\Gamma(c-a+1+m)m!}. \tag{343}$$

On the other hand, it is by (338)

$$\binom{b+c}{b-a}\binom{c+a}{c-a}=\frac{(-1)^{b+c}\Gamma(-2a)}{\Gamma(b-a+1)\Gamma(c-a+1)\Gamma(-b-c)}, \tag{344}$$

and by definition (335)

$$\begin{aligned}{}_3F_2\left(\begin{matrix}-2a,-a-b,-a-c\\1+b-a,1+c-a\end{matrix}\,\middle|\,1\right)&=\frac{\Gamma(b-a+1)\Gamma(c-a+1)}{\Gamma(-2a)\Gamma(-a-b)\Gamma(-a-c)}\\&\quad\times\sum_{m=0}^{2a}\frac{\Gamma(-2a+m)\Gamma(-a-b+m)\Gamma(-a-c+m)}{\Gamma(b-a+1+m)\Gamma(c-a+1+m)m!}.\end{aligned} \tag{345}$$

Now, it's left to multiply (344) and (345) and to compare the result with (343). ∎

Another special case of a hypergeometric function is the (generalized) binomial series

$$(1+x)^{\alpha}=\sum_{m=0}^{\infty}\binom{\alpha}{m}x^m={}_1F_0\left(\begin{matrix}-\alpha\\0\end{matrix}\,\middle|\,-x\right), \tag{346}$$

which converges absolutely for all $|x|<1$ and complex α. The second equality in (346) is due to (338). In the next lemma we exploit the fact that we have a closed form expression for the binomial series and prove identities for the sum

$$C_a(x):=\sum_{p=0}^{\infty}p^a\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)}x^p, \tag{347}$$

for $a\in\mathbb{Z}, 0\leqslant a\leqslant 9$.

Lemma 49. *Let C_a be defined as in* (347). *Then it is for* $|x|<1$

$$C_0(x) = \frac{\sqrt{\pi}}{\sqrt{1-x}}, \tag{348}$$

$$C_1(x) = \frac{\sqrt{\pi}x}{2(1-x)^{3/2}}, \tag{349}$$

$$C_2(x) = \frac{\sqrt{\pi}x}{2^2(1-x)^{5/2}}(x+2), \tag{350}$$

$$C_3(x) = \frac{\sqrt{\pi}x}{2^3(1-x)^{7/2}}(x^2+10x+4), \tag{351}$$

$$C_4(x) = \frac{\sqrt{\pi}x}{2^4(1-x)^{9/2}}\left(x^3+36x^2+60x+8\right), \tag{352}$$

$$C_5(x) = \frac{\sqrt{\pi}x}{2^5(1-x)^{11/2}}\left(x^4+116x^3+516x^2+296x+16\right), \tag{353}$$

$$C_6(x) = \frac{\sqrt{\pi}x}{2^6(1-x)^{13/2}}\left(x^5+358x^4+3508x^3+5168x^2+1328x+32\right), \tag{354}$$

$$C_7(x) = \frac{\sqrt{\pi}x}{2^7(1-x)^{15/2}}\left(x^6+1086x^5+21120x^4+64240x^3+42960x^2+5664x+64\right), \tag{355}$$

$$C_8(x) = \frac{\sqrt{\pi}x}{2^8(1-x)^{17/2}}\Big(x^7+3272x^6+118632x^5+660880x^4+900560x^3+320064x^2 + 23488x+128\Big), \tag{356}$$

$$C_9(x) = \frac{\sqrt{\pi}x}{2^9(1-x)^{19/2}}\Big(x^8+9832x^7+638968x^6+6049744x^5+14713840x^4 + 10725184x^3+2225728x^2+95872x+256\Big). \tag{357}$$

Proof. By Euler's reflection formula (327) it is

$$\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)} = (-1)^p\pi\frac{1}{\Gamma\left(\frac{1}{2}-p\right)\Gamma(p+1)} = \pi^{\frac{1}{2}}(-1)^p\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-p\right)\Gamma(p+1)} = \pi^{\frac{1}{2}}(-1)^p\binom{-\frac{1}{2}}{p}.$$

Thus, we rewrite

$$C_a(x) = \pi^{\frac{1}{2}}\sum_{p=0}^{\infty} p^a\binom{-\frac{1}{2}}{p}(-x)^p \tag{358}$$

and immediately get for $a=0$

$$C_0(x) = \pi^{\frac{1}{2}}\sum_{p=0}^{\infty}\binom{-\frac{1}{2}}{p}(-x)^p = \frac{\pi^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$$

by (346). Inside the radius of convergence $|x|<1$ of C_0, we are allowed to interchange summation and differentiation, and thus obtain after differentiating k times

$$\tilde{C}_k(x) := \pi^{\frac{1}{2}}\sum_{p=0}^{\infty} p(p-1)\cdots(p-k+1)\binom{-\frac{1}{2}}{p}(-x)^p = \pi^{\frac{1}{2}}\frac{(2k-1)!!}{2^k}\frac{x^k}{(1-x)^{\frac{2k+1}{2}}}. \tag{359}$$

Hence, for $a = 1$ the two series $C_1(x)$ and $\tilde{C}_1(x)$ coincide and take the value

$$C_1(x) = \frac{\pi^{\frac{1}{2}} x}{2(1-x)^{\frac{3}{2}}}.$$

Note that $p(p-1)\cdots(p-k+1) = k!\binom{p}{k}$, and thus

$$\tilde{C}_k(x) = \pi^{\frac{1}{2}} k! \sum_{p=0}^{\infty} \binom{p}{k} \binom{-\frac{1}{2}}{p} (-x)^p. \tag{360}$$

Next, we express each power p^a as a linear combination of binomial coefficients for $1 \leqslant k \leqslant a$

$$p^a = \sum_{k=1}^{a-1} c_k k! \binom{p}{k} + a! \binom{p}{a}. \tag{361}$$

Then it is by (358), (360) and (361)

$$C_a(x) = \sum_{k=1}^{a-1} c_k \tilde{C}_k(x) + \tilde{C}_a(x). \tag{362}$$

The solutions of the systems of linear equations (361) for the coefficients c_k are

$a = 2 : c_1 = 1,$
$a = 3 : c_1 = 1, c_2 = 3,$
$a = 4 : c_1 = 1, c_2 = 7, c_3 = 6,$
$a = 5 : c_1 = 1, c_2 = 15, c_3 = 25, c_4 = 10,$
$a = 6 : c_1 = 1, c_2 = 31, c_3 = 90, c_4 = 65, c_5 = 15,$
$a = 7 : c_1 = 1, c_2 = 63, c_3 = 301, c_4 = 350, c_5 = 140, c_6 = 21,$
$a = 8 : c_1 = 1, c_2 = 127, c_3 = 966, c_4 = 1701, c_5 = 1050, c_6 = 266, c_7 = 28,$
$a = 9 : c_1 = 1, c_2 = 255, c_3 = 3025, c_4 = 7770, c_5 = 6951, c_6 = 2646, c_7 = 462, c_8 = 36.$

Now the expressions for $C_2(x)$ to $C_9(x)$ follow by plugging in these values into (362) and the knowledge about the value of $\tilde{C}_k(x)$ from (359). ∎

7.2.2. Upper Bounds. In the following we give an upper bound for Kummer type and Gauss type generalized hypergeometric functions, respectively. These bounds where first mentioned by Luke [6] without proof. Several years later, Karp [5] gave two different proves of them. We sketch the first one to be self-contained.

LEMMA 50 ([5]). *Define for each $k = 1, \ldots, q$ the elementary symmetric polynomial*

$$e_k(x_1, \ldots, x_q) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant q} x_{j_1} \cdots x_{j_k}. \tag{363}$$

Let $r \geqslant 0$ and assume for all $k = 1, \ldots, q$

$$e_k(\beta_1, \ldots, \beta_q) \geqslant e_k(\alpha_1, \ldots, \alpha_q), \tag{364}$$

and that every elementary polynomial is non-negative. Then

$${}_qF_q\left(\begin{matrix}\alpha_1, \ldots, \alpha_q \\ \beta_1, \ldots, \beta_q\end{matrix} \,\middle|\, r\right) \leqslant 1 + (e^r - 1) f_1, \tag{365}$$

with $f_1 = \prod_{k=1}^{q} \frac{\alpha_k}{\beta_k}$. Moreover, let $\lambda > 0$. Then it is for $0 \leqslant x < 1$

$${}_{q+1}F_q\left(\begin{matrix}\lambda, \alpha_1, \ldots, \alpha_q \\ \beta_1, \ldots, \beta_q\end{matrix} \,\middle|\, x\right) \leqslant 1 + \left(\frac{1}{(1-x)^{\lambda}} - 1\right) f_1. \tag{366}$$

Proof. Let

$$f_m = \prod_{k=1}^{q} \frac{\Gamma(\alpha_k + m)\Gamma(\beta_k)}{\Gamma(\alpha_k)\Gamma(\beta_k + m)}$$

be the coefficient of $\frac{r^m}{m!}$ in the definition of ${}_qF_q\left(\begin{smallmatrix}\alpha_1 & \dots & \alpha_q \\ \beta_1 & \dots & \beta_q\end{smallmatrix} \,\middle|\, r\right)$. Then it is by condition (364)

$$\frac{f_{m+1}}{f_m} = \prod_{k=1}^{q} \frac{\alpha_k + m}{\beta_k + m} = \frac{\sum_{k=1}^{q} e_k(\alpha_1, \dots, \alpha_q) m^k}{\sum_{k=1}^{q} e_k(\beta_1, \dots, \beta_q) m^k} \leqslant 1$$

Thus, the series of coefficients f_m is decreasing and $f_m \leqslant f_1$ for all $m \geqslant 1$. This implies for $r \geqslant 0$

$${}_qF_q\left(\begin{matrix}\alpha_1, \dots, \alpha_q \\ \beta_1, \dots, \beta_q\end{matrix} \,\middle|\, r\right) = 1 + \sum_{m=1}^{\infty} f_m \frac{r^m}{m!} \leqslant 1 + f_1 \sum_{m=1}^{\infty} \frac{r^m}{m!} = 1 + f_1(e^r - 1).$$

To pass from (365) to (366), we use that

$$\begin{aligned}
&\int_0^\infty {}_qF_q\left(\begin{matrix}\alpha_1, \dots, \alpha_q \\ \beta_1, \dots, \beta_q\end{matrix} \,\middle|\, r\right) e^{-yr} r^{\lambda-1} dr \\
&= \prod_{k=1}^{q} \frac{\Gamma(\beta_k)}{\Gamma(\alpha_k)} \sum_{m=0}^{\infty} \prod_{k=1}^{q} \frac{\Gamma(\alpha_k + m)}{\Gamma(\beta_k + m)} \frac{1}{m!} \int_0^\infty e^{-yr} r^{\lambda+m-1} dr \\
&= y^{-\lambda} \prod_{k=1}^{q} \frac{\Gamma(\beta_k)}{\Gamma(\alpha_k)} \sum_{m=0}^{\infty} \Gamma(\lambda + m) \prod_{k=1}^{q} \frac{\Gamma(\alpha_k + m)}{\Gamma(\beta_k + m)} \frac{y^{-m}}{m!} \\
&= y^{-\lambda} \Gamma(\lambda)\, {}_{q+1}F_q\left(\begin{matrix}\lambda, \alpha_1, \dots, \alpha_q \\ \beta_1, \dots, \beta_q\end{matrix} \,\middle|\, y^{-1}\right),
\end{aligned}$$

by (331), if $\lambda > 0$ and $0 \leqslant y^{-1} < 1$. Integration of the right-hand side of (365) as well, and setting $x = y^{-1}$ completes the proof. ∎

7.3. Bessel Functions

Next, we eventually reach our main object of interest, the Bessel function. Similar to the two previous sections, we first give an overview on how the Bessel function is defined and list its basic properties to an extend that is sufficient for our purposes, including for example its representation as the integral (373) and the Poisson Integral (377). In that part of the section, we closely follow [1]. Afterwards, we deduce upper bounds on J_n. Especially Lemma 55 is very important for the series expansion of products of Bessel functions we develop in Chapter 2. This result builds upon a work by Krasnikov [10]. In the last part of this section, we state some results from the theory of integrals of two and three Bessel functions, that are crucial for our work in Chapter 3. The most valuable ingredients there are Kapteyn's Lemma 56, as well as Lemmata 57 and 58.

7.3.1. Definition and Some Properties. Bessel functions are solutions $y(z)$ of Bessel's differential equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0 \tag{367}$$

for arbitrary $\nu \in \mathbb{C}$. By plugging in the ansatz function

$$y(z) = \sum_{m=0}^{\infty} c_m z^{\alpha+m}$$

into (367) upon the condition that $c_0 \neq 0$, we find the formal solution

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+1+m)} \left(\frac{z^2}{4}\right)^m, \tag{368}$$

called Bessel function of the first kind of order ν and argument z. For non-integers ν the second linearly independent solution is $J_{-\nu}(z)$. If $\nu = n$ is an integer, it is

$$J_{-n}(z) = (-1)^n J_n(z). \tag{369}$$

This can be seen by setting $m = l + n$ in (368). Then

$$J_{-n}(z) = \sum_{l=-n}^{-1} \frac{(-1)^{l+n} 2^{-2l-n}}{\Gamma(l+1)(l+n)!} z^{2l+n} + (-1)^n \sum_{l=0}^{\infty} \frac{(-1)^l 2^{-2l-n}}{l!\Gamma(n+1+l)} z^{2l+n}.$$

Since the gamma function has simple poles at the non-positive integers, the first sum vanishes.

Later in this work we need the Bessel functions $J_{-\frac{1}{2}}$ and $J_{\frac{1}{2}}$ of order $\pm\frac{1}{2}$. Setting $\nu = -\frac{1}{2}$ in (368) and using Legendre's duplication formula, we obtain

$$\begin{aligned} J_{-\frac{1}{2}}(z) &= \sum_{0}^{\infty} \frac{(-1)^m 2^{-2m+\frac{1}{2}}}{\Gamma(m+1)\Gamma\left(m+\frac{1}{2}\right)} z^{2m-\frac{1}{2}} \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{z^{\frac{1}{2}}} \sum_{0}^{\infty} \frac{(-1)^m z^{2m}}{\Gamma(2m+1)} \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\cos(z)}{z^{\frac{1}{2}}}. \end{aligned} \tag{370}$$

Likewise we yield for $\nu = \frac{1}{2}$

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin(z)}{z^{\frac{1}{2}}}. \tag{371}$$

In [1] Oliveira e Silva and Thiele define the Bessel function of the first kind of integer order $n \in \mathbb{Z}$ and non-negative argument $r \geqslant 0$ via the identity

$$\widehat{e^{in\cdot}\sigma}(r) = 2\pi(-i)^n J_n(|r|) e^{in\arg(r)}, \tag{372}$$

which is, by a simple coordinate transform, equivalent to the integral representation

$$J_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir\sin(\theta) - in\theta} d\theta. \tag{373}$$

Then they use (373) to deduce the recurrence relations

$$J_{n-1}(r) - J_{n+1}(r) = 2J_n'(r), \tag{374}$$

$$J_{n-1}(r) + J_{n+1}(r) = \frac{2n}{r} J_n(r) \tag{375}$$

in the sense of meromorphic functions. Adding and subtracting (374) and (375) results in

$$\begin{aligned} rJ_n'(r) + nJ_n(r) &= rJ_{n-1}(r), \\ rJ_n'(r) - nJ_n(r) &= -rJ_{n+1}(r). \end{aligned}$$

Alternatively one can write

$$\begin{aligned} \frac{d}{dr}\left(r^n J_n(r)\right) &= r^n J_{n-1}(r), \\ \frac{d}{dr}\left(r^{1-n} J_{n-1}(r)\right) &= -r^{1-n} J_n(r). \end{aligned} \tag{376}$$

Elimination of $J_{n-1}(r)$ in this last system then leads to

$$\frac{d}{dr}\left(r^{1-n}\frac{dJ_n(r)}{dr} + nr^{-n}J_n(r)\right) + r^{1-n}J_n(r) = 0,$$

which is equivalent to Bessel's differential equation (367). Thus, both representations (368) and (373) of $J_n(z)$ are a solution of (367) with initial values $J_0(0) = 1$ and $J_n(0) = 0$ for $n \in \mathbb{Z}\backslash\{0\}$. The equality of the initial values $J_0'(0)$ and $J_n'(0)$ for $n \in \mathbb{Z}\backslash\{0\}$ follows from (368) and the recurrence (374), respectively. By the Picard-Lindelöf theorem on the existence and uniqueness of the solution of an ordinary differential equation, (368) and (373) have to be identical for integer orders $n \in \mathbb{Z}$. By a very similar argument, Oliveira e Silva and Thiele show that (373) is equivalent to the Poisson integral

$$J_n(r) = \frac{\left(\frac{r}{2}\right)^n}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi \cos(r\cos(\theta))\sin(\theta)^{2n}d\theta. \tag{377}$$

Let us at this point for the sake of completeness at least mention the Bessel function of the second kind as well as briefly draw a connection between the Bessel function and hypergeometric functions.

REMARKS 51. (i) Since for integer-ν the two solutions J_ν and $J_{-\nu}$ are no longer linearly independent, we need a different second solution in this case. For general complex ν the second linearly independent solution of Bessel's differential equation (367) is

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}. \tag{378}$$

It is called Bessel function of the second kind of order ν and argument z. In the case of integer order $\nu = n$ the function is defined as the limit

$$\begin{aligned} Y_n(z) &= \lim_{\nu\to n} Y_\nu(z) \\ &= \frac{1}{\pi}\frac{\partial J_\nu(z)}{\partial\nu}\bigg|_{\nu=n} + \frac{(-1)^n}{\pi}\frac{\partial J_\nu(z)}{\partial\nu}\bigg|_{\nu=-n}. \end{aligned}$$

(ii) The Bessel function of the first kind (368) can be expressed in terms of the hypergeometric function ${}_0F_1$, the so-called confluent hypergeometric limit function. It is

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\begin{matrix} \\ \nu+1\end{matrix}\,\bigg|\, -\frac{z^2}{4}\right).$$

Now, we move on with a further exploration of Bessel functions of the first kind.

7.3.2. Upper Bounds. One of the simplest upper bounds on $J_n(r)$ for real r and $n \geqslant 0$ is

$$|J_n(r)| \leqslant \frac{|r|^n}{2^n\Gamma(n+1)}, \tag{379}$$

which follows from Poisson's integral (377). For complex values z, the integral representation (373) provides us with another simple and useful bound

$$|J_n(z)| \leqslant e^{\mathrm{Im}(z)}. \tag{380}$$

Next, we are interested in an upper bound on the function $\left(\frac{\pi}{2}r\right)^{\frac{1}{2}} J_n(r)$ for integer orders $n \geqslant 1$, which is uniform in $r \geqslant 0$. In order to achieve this goal, we combine two inequalities Krasnikov proves in [10]. The first one is valid in the monotonicity region $0 < r \leqslant n + \frac{1}{2}$ of the Bessel function.

Lemma 52 ([10]). *Let $n > 0$ and $0 < r \leqslant n + \frac{1}{2}$, then*

$$J_n(r) < \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)} \frac{r^n}{n^{n+\frac{1}{3}}} e^{\frac{n^2-r^2}{2n+1}}.$$

The second one provides a bound for the oscillatory part of J_n.

Lemma 53 ([10]). *Let $n \geqslant \frac{1}{2}$ and $r \geqslant \left(n^2 - \frac{1}{4}\right)^{\frac{1}{2}}$, then*

$$\left(r^2 - n^2 + \frac{1}{4}\right)^{\frac{1}{4}} |J_n(r)| < \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.$$

For the sake of self-completeness we sketch the proofs of Lemmata 52 and 53.

Proof of Lemma 52. Set $u_n(r) = r^{-n}J_n(r)$ for $n \geqslant 0$. Then Bessel's differential equation (367) implies that

$$ru_n''(r) + (2n+1)u_n'(r) + ru_n(r) = 0,$$

and therefore, since $0 \leqslant r \leqslant n + \frac{1}{2}$

$$\begin{aligned} 0 \leqslant \left(u_n(r) + u_n'(r)\right)^2 &\leqslant u_n(r)^2 + \frac{2n+1}{r} u_n(r)u_n'(r) + u_n'(r)^2 \\ &= u_n'(r)^2 - u_n(r)u_n''(r), \end{aligned}$$

or, equivalently,

$$rt_n(r)^2 + (2n+1)t_n(r) + r \geqslant 0.$$

Here we put $t_n(r) = \frac{u_n'(r)}{u_n(r)}$. It follows that

$$t_n(r) \notin \left(-\frac{2n+1+\sqrt{(2n+1)^2-4r^2}}{2r}, -\frac{2r}{2n+1+\sqrt{(2n+1)^2-4r^2}}\right).$$

It is $\lim_{r\to 0+} -\frac{2n+1+\sqrt{(2n+1)^2-4r^2}}{2r} = -\infty$, but (376) and the Poisson integral (377) for $J_n(r)$ imply that $\lim_{r\to 0+} t(r) = 0$. Thus, it is

$$t(r) \geqslant -\frac{2r}{2n+1+\sqrt{(2n+1)^2-4r^2}} \geqslant -\frac{2r}{2n+1}.$$

Integration of the above inequality yields

$$\ln \frac{u_n(n)}{u_n(r)} \geqslant -\int_r^n \frac{2s}{2n+1} ds = -\frac{n^2-r^2}{2n+1}.$$

This, together with the inequality $J_n(n) \leqslant \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})n^{\frac{1}{3}}}$ by Lorch from [11], proves the claim. ∎

The proof of Lemma 53 is based on the theory of Sonin's functions. It would go beyond the scope of this work to go deeper into it. As a brief explanation, let us at least state the definition. Let $y(x)$ be a solution of

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0, \quad b(x) > 0.$$

Then the function

$$S(x) = y^2(x) + \frac{(y')^2(x)}{b(x)}$$

is called Sonin's function. It is an envelope of y^2 and coincides with it in all of its maxima.

Proof of Lemma 53. Set $v_n(r) = \left(r^2 - n^2 + \frac{1}{4}\right)^{\frac{1}{4}} J_n(r)$ and $\eta = \left(n^2 - \frac{1}{4}\right)^{\frac{1}{2}} \geqslant 0$ for $n \geqslant \frac{1}{2}$. For $r = \eta$ the result is trivial.
Now let $r > \eta$. By Bessel's differential equation (367), one easily verifies that the function $v_n(r)$ satisfies the differential equation

$$v_n''(r) - a(r)v_n'(r) + b(r)v_n(r) = 0, \tag{381}$$

with $a(r) = \frac{\eta^2}{r(r^2-\eta^2)}$ and

$$b(r) = \frac{4(r^2-\eta^2)^3 + \eta^2(6r^2-\eta^2)}{4r^2(r^2-\eta^2)^2} > 0.$$

Then, th Sonin's function is

$$S(r) = v_n^2(r) + \frac{1}{b(r)} v_n'^2(r),$$

which is an envelope of $v_n^2(r)$, as stated above, and coincides with $v_n^2(r)$ in all the maxima. Due to (381) the derivative of S can be simplified to

$$S'(r) = \frac{24\eta^2 r^3(r^2-\eta^2)(4r^2+\eta^2)}{((r^2-\eta^2)^3 + \eta^2(6r^2-\eta^2))^2} v_n'^2(r) \geqslant 0.$$

Thus,

$$|v_n(r)| < \left(\lim_{r\to\infty} S(r)\right)^{\frac{1}{2}}.$$

Using the recurrence relation (374), the asymptotics $J_n(r) \sim \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \cos\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right)$, that can be deduced from Corollary 5, and some algebra we arrive at the desired bound $|v_n(r)| < \left(\frac{2}{\pi}\right)^{\frac{1}{2}}$. ∎

Remark 54. Actually, the inequality we stated in Lemma 53 is true for all positive arguments $r \geqslant 0$ and the constant $\left(\frac{2}{\pi}\right)^{\frac{1}{2}}$ is sharp. The interested reader is referred to [10].

Finally, we have everything we need to deduce a uniform upper bound on $\left(\frac{\pi}{2}r\right)^{\frac{1}{2}} J_n(r)$.

Lemma 55. *Let $n \in \mathbb{N}, n \geqslant 1$. Then it is*

$$\left(\frac{\pi}{2}r\right)^{\frac{1}{2}} |J_n(r)| \leqslant \left(n + \frac{1}{2}\right)^{\frac{1}{4}}$$

for all $r \geqslant 0$.

Proof. Lemma 52 implies that for all $0 < r \leqslant n + \frac{1}{2}$

$$\begin{aligned} \left(\frac{\pi}{2}r\right)^{\frac{1}{2}} J_n(r) &< \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{6}} 3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)} \frac{r^{n+\frac{1}{2}}}{n^{n+\frac{1}{3}}} e^{\frac{n^2-r^2}{2n+1}} \\ &\leqslant \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{6}} 3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)} \frac{\left(n+\frac{1}{2}\right)^{\frac{1}{2}}}{n^{\frac{1}{3}}} \left(1 + \frac{1}{2n}\right)^n e^{-\frac{n+\frac{1}{4}}{2n+1}} \\ &< \frac{\pi^{\frac{1}{2}} e^{\frac{1}{8n+4}}}{2^{\frac{1}{6}} 3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)} \frac{\left(n+\frac{1}{2}\right)^{\frac{1}{2}}}{n^{\frac{1}{3}}}. \end{aligned}$$

The second inequality above can be seen by taking a closer look at the function

$$r^{n+\frac{1}{2}} e^{\frac{n^2-r^2}{2n+1}} \tag{382}$$

for $0 < r \leqslant n + \frac{1}{2}$. Its derivative in r is

$$\frac{e^{\frac{n^2-r^2}{2n+1}} r^{n-\frac{1}{2}} \left(4n^2 + 4n - 4r^2 + 1\right)}{2(2n+1)},$$

which is positive for all the values of r, we are interested in. Thus, we take the right boundary value of (382) as an upper bound. In the step following this, we estimated $\left(1 + \frac{1}{2n}\right)^n \leqslant e^{\frac{1}{2}}$.

Beyond the monotonicity region, for $r \geqslant n + \frac{1}{2} > \left(n^2 - \frac{1}{4}\right)^{\frac{1}{2}}$, Lemma 53 gives

$$\begin{aligned} \left(\frac{\pi}{2} r\right)^{\frac{1}{2}} |J_n(r)| &< \left(\frac{r^2}{r^2 - n^2 + \frac{1}{4}}\right)^{\frac{1}{4}} \\ &\leqslant \left(n + \frac{1}{2}\right)^{\frac{1}{4}}. \end{aligned}$$

Thus, we find that

$$\begin{aligned} \left(\frac{\pi}{2} r\right)^{\frac{1}{2}} |J_n(r)| &\leqslant \max \left\{ \left(n + \frac{1}{2}\right)^{\frac{1}{4}}, \frac{\pi^{\frac{1}{2}} e^{\frac{1}{8n+4}}}{2^{\frac{1}{6}} 3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)} \frac{\left(n + \frac{1}{2}\right)^{\frac{1}{2}}}{n^{\frac{1}{3}}} \right\} \\ &= \left(n + \frac{1}{2}\right)^{\frac{1}{4}} \end{aligned}$$

for all $n \geqslant 1$. ∎

7.3.3. Integrals Involving Bessel Functions. Infinite integrals involving Bessel functions have been of great interest around 1900. A vast amount of identities and formulae have been produced in this era. A great, but by far not comprehensive, overview provide the book by Watson [2] and the series by Erdélyi et al. [3]. In our analysis of $I(\mathbf{n})$ we fall back on some known results about Bessel integrals. The first one is a generalization of Kapteyn's identity [7], which has already been a crucial ingredient for the results of our companion paper [1].

Lemma 56 ([1]). *Let $m, n \geqslant 0$ and $1 \leqslant k \leqslant m + n$. Then*

$$\int_0^\infty J_m(r) J_n(r) r^{-k} dr = \frac{2^{-k} \Gamma(k) \Gamma(\frac{m+n+1-k}{2})}{\Gamma(\frac{m+n+k+1}{2}) \Gamma(\frac{m-n+k+1}{2}) \Gamma(\frac{n-m+k+1}{2})}.$$

The second one is due to Oliveira e Silva and Thiele and investigates the case of an additional trigonometric factor $\cos(2r)$ or $\sin(2r)$ under the integral.

Lemma 57 ([1]). *Let $m, n \geqslant 0$, and $1 \leqslant k \leqslant m + n$.*
If $m + n + k$ is even, then

$$\int_0^\infty J_m(r) J_n(r) \cos(2r) r^{-k} dr = 0.$$

If $m + n + k$ is odd, then

$$\int_0^\infty J_m(r) J_n(r) \sin(2r) r^{-k} dr = 0.$$

Luckily, there are still closed form expressions when we replace $\cos(2r)$ and $\sin(2r)$ by $\cos(4r)$ and $\sin(4r)$. However, they are a bit more complicated and involve the hypergeometric function ${}_3F_2$.

LEMMA 58. *Let $n \geqslant 0$ and $0 \leqslant k \leqslant 4n-1$. Then it is*

$$\begin{aligned}
&\int_0^\infty J_{2n}^2(r)\,\cos(4r)\,r^{-k-1}dr \\
&= \cos\left(\frac{\pi}{2}(4n-k)\right) 2^{2k-12n}\frac{\Gamma(4n-k)}{\Gamma(2n+1)^2}{}_3F_2\left(\begin{matrix}2n+\frac{1}{2}, 2n-\frac{k}{2}, 2n-\frac{k}{2}+\frac{1}{2}\\ 2n+1, 4n+1\end{matrix}\;\middle|\;\frac{1}{4}\right),
\end{aligned} \tag{383}$$

and

$$\begin{aligned}
&\int_0^\infty J_{2n}^2(r)\,\sin(4r)\,r^{-k-1}dr \\
&= -\sin\left(\frac{\pi}{2}(4n-k)\right) 2^{2k-12n}\frac{\Gamma(4n-k)}{\Gamma(2n+1)^2}{}_3F_2\left(\begin{matrix}2n+\frac{1}{2}, 2n-\frac{k}{2}, 2n-\frac{k}{2}+\frac{1}{2}\\ 2n+1, 4n+1\end{matrix}\;\middle|\;\frac{1}{4}\right).
\end{aligned} \tag{384}$$

This is a special case of the following lemma.

LEMMA 59. *Let $\mathrm{Re}(\lambda+\mu+\nu+\rho) > 0$, $\mathrm{Re}(\lambda) < \frac{3}{2}$ and b real and positive with $b > 2$. Then*

$$\begin{aligned}
&\int_0^\infty J_\mu(r)J_\nu(r)J_\rho(br)r^{\lambda-1}dr = \frac{1}{2\pi}\left(\frac{2}{b}\right)^\lambda b^{-\mu-\nu}\sin\left(\frac{\pi}{2}(\lambda+\mu+\nu-\rho)\right) \\
&\times \frac{\Gamma\left(\frac{1}{2}(\lambda+\mu+\nu+\rho)\right)}{\Gamma(\mu+1)}\frac{\Gamma\left(\frac{1}{2}(\lambda+\mu+\nu-\rho)\right)}{\Gamma(\nu+1)} \\
&\times {}_4F_3\left(\begin{matrix}\frac{1}{2}(\mu+\nu)+\frac{1}{2}, \frac{1}{2}(\mu+\nu)+1, \frac{1}{2}(\lambda+\mu+\nu+\rho), \frac{1}{2}(\lambda+\mu+\nu-\rho)\\ \mu+1, \nu+1, \mu+\nu+1\end{matrix}\;\middle|\;\left(\frac{2}{b}\right)^2\right)
\end{aligned}$$

This result was first proven by Bailey [8] and Rice [9] almost simultaneously. We vaguely follow the proof in [3] vol. 2. It relies on the following identity.

LEMMA 60. *Let $-\mathrm{Re}(\mu) < \mathrm{Re}(\lambda) < \frac{1}{2}$. Then*

$$\int_0^\infty J_\mu(r)r^{\lambda-1}dr = 2^{\lambda-1}\frac{\Gamma\left(\frac{1}{2}(\mu+\lambda)\right)}{\Gamma\left(\frac{1}{2}(\mu-\lambda)+1\right)}.$$

PROOF. Note that the assumptions on λ and μ make the integral absolutely integrable. We consider the integral

$$\int_0^\infty J_\mu(r)r^{\lambda-1}e^{-4\gamma r^2}dr$$

with the aim of letting γ tend to zero later. We replace the Bessel function by its Taylor series at 0 (see (368))

$$\int_0^\infty J_\mu(r)r^{\lambda-1}e^{-4\gamma r^2}dr = \sum_{m=0}^\infty \frac{(-1)^m 2^{-2m-\mu}}{\Gamma(\mu+1+m)m!}\int_0^\infty r^{\lambda+\mu+2m-1}e^{-4\gamma r^2}\,dr. \tag{385}$$

Substituting $4\gamma r^2$ by s and identifying the gamma integral in s gives

$$\frac{2^{\lambda-1}}{\gamma^{\frac{1}{2}(\mu+\lambda)}}\sum_{m=0}^\infty \frac{\Gamma\left(\frac{1}{2}(\mu+\lambda)+m\right)}{\Gamma(\mu+1+m)m!}(-\gamma)^{-m}. \tag{386}$$

The process of interchanging integration and summation in (385) is justified by the theorem of Fubini-Tonelli. In fact, we estimate

$$\left|\frac{(-1)^m 2^{-2m-\mu}}{\Gamma(\mu+1+m)m!}r^{\lambda+\mu+2m-1}e^{-4\gamma r^2}\right| \leqslant 2^{-\mu}r^{\lambda+\mu-1}e^{-4\gamma r^2}\left(\frac{\left(\frac{r}{2}\right)^m}{m!}\right)^2,$$

and see that

$$\int_0^\infty \sum_{m=0}^\infty 2^{-\mu}r^{\lambda+\mu-1}e^{-4\gamma r^2}\left(\frac{\left(\frac{r}{2}\right)^m}{m!}\right)^2\,dr \leqslant \int_0^\infty 2^{-\mu}r^{\lambda+\mu-1}e^{r-4\gamma r^2}\,dr < \infty$$

provided that $\mathrm{Re}(\lambda+\mu) > 0$. Now we multiply and divide (386) by $\Gamma\left(\frac{1}{2}(\mu-\lambda)+1\right)$ and identify the Beta integral (329) to obtain

$$\frac{2^{\lambda-1}}{\gamma^{\frac{1}{2}(\mu+\lambda)}} \sum_{m=0}^{\infty} \frac{1}{\Gamma\left(\frac{1}{2}(\mu-\lambda)+1\right) m!} (-\gamma)^{-m} \int_0^1 t^{\frac{1}{2}(\mu+\lambda)+m-1} (1-t)^{\frac{1}{2}(\mu-\lambda)} \, dt. \tag{387}$$

Looking carefully at the last line reveals the exponential series, which is uniformly convergent on any compact interval. We thus can drag the sum into the integral and (387) turns into

$$\frac{2^{\lambda-1}}{\gamma^{\frac{1}{2}(\mu+\lambda)}} \frac{1}{\Gamma\left(\frac{1}{2}(\mu-\lambda)+1\right)} \int_0^1 t^{\frac{1}{2}(\mu+\lambda)-1} (1-t)^{\frac{1}{2}(\mu-\lambda)} e^{-\frac{t}{\gamma}} \, dt,$$

where we now substitute $s = \frac{t}{\gamma}$ and get

$$\frac{2^{\lambda-1}}{\Gamma\left(\frac{1}{2}(\mu-\lambda)+1\right)} \int_0^{\frac{1}{\gamma}} s^{\frac{1}{2}(\mu+\lambda)-1} (1-\gamma s)^{\frac{1}{2}(\mu-\lambda)} e^{-s} \, ds.$$

Finally, we let γ tend to zero, and end up with

$$\begin{aligned} \int_0^\infty J_\mu(r) r^{\lambda-1} dr &= \frac{2^{\lambda-1}}{\Gamma\left(\frac{1}{2}(\mu-\lambda)+1\right)} \int_0^\infty s^{\frac{1}{2}(\mu+\lambda)-1} e^{-s} \, ds \\ &= \frac{\Gamma\left(\frac{1}{2}(\mu+\lambda)\right)}{\Gamma\left(\frac{1}{2}(\mu-\lambda)+1\right)}, \end{aligned}$$

by the gamma integral (325). ∎

Now, let's turn to the proof of Lemma 59.

Proof of Lemma 59. In a first step, we consider the product of two Bessel functions of different orders μ and ν but the same argument z. We expand the Bessel functions into their exponential series (368) and collect the coefficients of $\left(-\frac{z^2}{4}\right)^m$. Then

$$J_\mu(z) J_\nu(z) = \left(\frac{z}{2}\right)^{\mu+\nu} \sum_{m=0}^{\infty} \left(-\frac{z^2}{4}\right)^m \sum_{l=0}^{m} \frac{1}{\Gamma(\mu+1+m-l)\Gamma(\nu+1+l)\Gamma(m+1-l) l!}.$$

Note that a quick rearrangement of relation (338) implies that

$$\Gamma(m-l) = (-1)^l \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1-m+l)}.$$

Using this for the first and third factor in the denominator above, we identify a hypergeometric function and the right-hand side of the second last line turns into

$$\left(\frac{z}{2}\right)^{\mu+\nu} \sum_{m=0}^{\infty} \left(-\frac{z^2}{4}\right)^m \frac{1}{\Gamma(\nu+1)\Gamma(\mu+1+m) m!} {}_2F_1\left(\begin{matrix} -\mu-m, -m \\ \nu+1 \end{matrix} \,\middle|\, 1 \right).$$

Next, Gauss' hypergeometric identity Lemma 47 provides us with a closed expression for this hypergeometric function. This and Legendre's duplication formula (328) leads to

$$J_\mu(z) J_\nu(z) = z^{\mu+\nu} \pi^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(\mu+\nu)+\frac{1}{2}+m\right)\Gamma\left(\frac{1}{2}(\mu+\nu)+1+m\right)}{\Gamma(\mu+1+m)\Gamma(\nu+1+m)\Gamma(\mu+\nu+1+m) m!} \left(-z^2\right)^m. \tag{388}$$

Now, we replace $J_\mu(r)J_\nu(r)$ in $\int_0^\infty J_\mu(r)J_\nu(r)J_\rho(br)r^{\lambda-1}dr$ by (388) and integrate term by term

$$\int_0^\infty J_\mu(r)J_\nu(r)J_\rho(br)r^{\lambda-1}dr$$
$$= \frac{1}{\pi^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma\left(\frac{1}{2}(\mu+\nu)+\frac{1}{2}+m\right)\Gamma\left(\frac{1}{2}(\mu+\nu)+1+m\right)}{\Gamma(\mu+1+m)\Gamma(\nu+1+m)\Gamma(\mu+\nu+1+m)m!}\int_0^\infty J_\rho(br)r^{\lambda+\mu+\nu+2m-1}dr.$$

We treat the remaining integral with Lemma 60 and apply the reflection formula (327). This way we obtain

$$\frac{1}{\pi^{\frac{3}{2}}}2^{\lambda+\mu+\nu+1}\left(\frac{1}{b}\right)^{\lambda+\mu+\nu}\sin\left(\tfrac{\pi(\lambda+\mu+\nu-\rho)}{2}\right)\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}(\mu+\nu)+\frac{1}{2}+m\right)\Gamma\left(\frac{1}{2}(\mu+\nu)+1+m\right)}{\Gamma(\mu+1+m)\Gamma(\nu+1+m)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}(\lambda+\mu+\nu+\rho)+m\right)\Gamma\left(\frac{1}{2}(\lambda+\mu+\nu-\rho)+m\right)}{\Gamma(\mu+\nu+1+m)m!}\left(\frac{2}{b}\right)^m.$$

We take a closer look at the series above and eventually identify it as the claimed hypergeometric function.

$$\frac{\Gamma\left(\frac{1}{2}(\mu+\nu)+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\mu+\nu)+1\right)\Gamma\left(\frac{1}{2}(\lambda+\mu+\nu+\rho)\right)}{\Gamma(\mu+1)\Gamma(\mu+\nu+1)}\frac{\Gamma\left(\frac{1}{2}(\lambda+\mu+\nu-\rho)\right)}{\Gamma(\nu+1)}$$
$$\times\,{}_4F_3\left(\begin{matrix}\frac{1}{2}(\mu+\nu)+\frac{1}{2},\frac{1}{2}(\mu+\nu)+1,\frac{1}{2}(\lambda+\mu+\nu+\rho),\frac{1}{2}(\lambda+\mu+\nu-\rho)\\ \mu+1,\nu+1,\mu+\nu+1\end{matrix}\;\middle|\;\left(\frac{2}{b}\right)^2\right).$$

After one application of Legendre's duplication formula (328) the proof is finished. ∎

Now we are ready for the

Proof of Lemma 58. The claim follows by (370) and (371), respectively, and Lemma 59. Note that

$${}_4F_3\left(\begin{matrix}2n+\frac{1}{2},2n+1,2n-\frac{k}{2},2n-\frac{k}{2}+\frac{1}{2}\\ 2n+1,2n+1,4n+1\end{matrix}\;\middle|\;\frac{1}{4}\right)={}_3F_2\left(\begin{matrix}2n+\frac{1}{2},2n-\frac{k}{2},2n-\frac{k}{2}+\frac{1}{2}\\ 2n+1,4n+1\end{matrix}\;\middle|\;\frac{1}{4}\right).$$

∎

Remark 61. Another consequence of Lemma 59, in combination with (370) and (371), is that Lemma 57 remains true when the argument $2r$ of the trigonometric functions is replaced by $4r$.

7.4. Sine and Cosine Integrals

In this section we consider integrals of the type

$$\int_x^\infty\frac{\cos(t)}{t^n}dt,\quad\int_x^\infty\frac{\sin(t)}{t^n}dt,\tag{389}$$

which play an important role in the numerical part of the proof of our main Theorem 3 in Chapter 5. More precisely, we proof two asymptotic expansions with error bounds. Lemma 62 is for $n=1$, while Lemma 63 deals with $n=2,3,4$. In the literature some of these integrals have particular names

$$\mathrm{Ci}(x)=-\int_x^\infty\frac{\cos(t)}{t}dt,\tag{390}$$

and

$$\mathrm{Si}(x)=\int_0^x\frac{\sin(t)}{t}dt,\tag{391}$$

are the so called cosine and sine integral, respectively. Actually, more than in the sine integral, we are interested in its complement with respect to integration along the positive real axis. That is

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin(t)}{t} dt. \tag{392}$$

Here we stick to the established notation, which calls for the minus signs in front of the integrals in (390) and (392).
The following lemma gives pleasantly simple and increasingly accurate approximations for $\mathrm{Ci}(x)$ and $\mathrm{si}(x)$ for large x. As a byproduct it also shows that the integrals (390) and (392) converge in the first place.

Lemma 62. *We have that*

$$\mathrm{Ci}(x) = \frac{\sin(x)}{x} - \frac{\cos(x)}{x^2} + q_{c,2}(x), \tag{393}$$

$$\mathrm{Ci}(x) = \frac{\sin(x)}{x} - \frac{\cos(x)}{x^2} - 2\frac{\sin(x)}{x^3} + q_{c,3}(x), \tag{394}$$

and

$$\mathrm{si}(x) = -\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} + q_{s,2}(x), \tag{395}$$

$$\mathrm{si}(x) = -\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} + 2\frac{\cos(x)}{x^3} + q_{s,3}(x), \tag{396}$$

where $|q_{c,2}(x)|$ *and* $|q_{s,2}(x)|$ *are at most* $\frac{4}{x^3}$ *and* $|q_{c,3}(x)|$ *and* $|q_{s,3}(x)|$ *are not greater than* $\frac{12}{x^4}$.

Proof. For the proof of (393) we integrate $\mathrm{Ci}(x)$ three times by parts and obtain

$$\mathrm{Ci}(x) = \frac{\sin(x)}{x} - \frac{\cos(x)}{x^2} - \frac{2\sin(x)}{x^3} + \int_x^\infty \frac{6\sin(t)}{t^4} dt.$$

Thus, it is

$$\begin{aligned} |q_{c,2}(x)| &= \left| -\frac{2\sin(x)}{x^3} + \int_x^\infty \frac{6\sin(t)}{t^4} dt \right| \\ &\leqslant \frac{4}{x^3}. \end{aligned}$$

The proof for each of the three remaining asymptotics (394), (395) and (396) follows the same lines and is left to the reader. ∎

Note that we can obtain arbitrarily accurate asymptotics and inequalities for $\mathrm{Ci}(x)$ and $\mathrm{si}(x)$ this way. Moreover, the proof of Lemma 62 shows that via partial integration we can link the standard cosine and sine integrals to the integrals (389), where the factor $1/t$ occurs in higher powers. For example for arbitrary $a \in \mathbb{R}$ the substitution $at = s$ yields

$$\begin{aligned} \int_x^\infty \frac{\sin(at)}{t^2} dt &= a \int_{ax}^\infty \frac{\sin(s)}{s^2} ds \\ &= a \left[-\frac{\sin(s)}{s} \Big|_{ax}^\infty + \int_{ax}^\infty \frac{\cos(s)}{s} ds \right] \\ &= \frac{\sin(ax)}{x} - a\,\mathrm{Ci}(ax). \end{aligned} \tag{397}$$

A similar calculation leads to

$$\int_x^\infty \frac{\cos(at)}{t^2} dt = \frac{\cos(ax)}{x} + a\,\mathrm{si}(ax), \tag{398}$$

as well as

$$\int_x^\infty \frac{\sin(at)}{t^3}dt = \frac{\sin(ax)}{2x^2} + a\frac{\cos(ax)}{2x} + \frac{a^2}{2}\,\mathrm{si}(ax), \tag{399}$$

$$\int_x^\infty \frac{\cos(at)}{t^3}dt = \frac{\cos(ax)}{2x^2} - a\frac{\sin(ax)}{2x} + \frac{a^2}{2}\,\mathrm{Ci}(ax), \tag{400}$$

$$\int_x^\infty \frac{\sin(at)}{t^4}dt = \frac{\sin(ax)}{3x^3} + a\frac{\cos(ax)}{6x^2} - a^2\frac{\sin(ax)}{6x} + \frac{a^3}{6}\,\mathrm{Ci}(ax), \tag{401}$$

$$\int_x^\infty \frac{\cos(at)}{t^4}dt = \frac{\cos(ax)}{3x^3} - a\frac{\sin(ax)}{6x^2} - a^2\frac{\cos(ax)}{6x} - \frac{a^3}{6}\,\mathrm{si}(ax). \tag{402}$$

Combining the above identities (397) to (402) with Lemma 62 we obtain nice asymptotics for the integrals (389) for $n = 2, 3, 4$.

Lemma 63. *Let $q_{c,2}(x), q_{s,2}(x)$ and $q_{c,3}(x), q_{s,3}(x)$ be as in Lemma 62. Then we have for large x*

$$\int_x^\infty \frac{\sin(at)}{t^2}dt = \frac{\cos(ax)}{ax^2} - aq_{c,2}(ax),$$

$$\int_x^\infty \frac{\cos(at)}{t^2}dt = -\frac{\sin(ax)}{ax^2} + aq_{s,2}(ax),$$

$$\int_x^\infty \frac{\sin(at)}{t^3}dt = \frac{a^2}{2}q_{s,2}(ax),$$

$$\int_x^\infty \frac{\cos(at)}{t^3}dt = \frac{a^2}{2}q_{c,2}(ax),$$

$$\int_x^\infty \frac{\sin(at)}{t^4}dt = \frac{a^3}{6}q_{c,3}(ax),$$

$$\int_x^\infty \frac{\cos(at)}{t^4}dt = -\frac{a^3}{6}q_{s,3}(ax).$$

7.5. Other Useful Inequalities and Identities

This last section gives a home to all formulae and estimates that don't fit anywhere else, but nevertheless have proven to be extremely helpful and important during our journey through the proof of Theorem 3. Lemma 64, for example, is used very often to estimate rational functions with factorizing enumerator and denominator like

$$q(n) = \frac{(n+x_1)(n+x_2)(n+x_3)}{n(n+x_4)(n+x_5)}$$

with $x_i \in \mathbb{R}, n \neq x_i$ for $i = 1, \ldots, 5$. Moreover, in the analysis of the main term in Section 4.4, we are very grateful for the summation identities Lemmata 66, 67 and 69 provide for the objects

$$\sum_{t=0}^{n} t^q, \quad \sum_{p=n+1}^{\infty} x^p \quad \text{and} \quad \sum_{j=0}^{p} j^a \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)},$$

respectively.

We first present Lemma 64, that proves the following second order exponential function estimates, that are useful to bound functions like q.

LEMMA 64. *It is*

$$1 + x \geqslant e^{x - \frac{1}{2}x^2} \tag{403}$$

for all $0 \leqslant x \leqslant 2$, *and*

$$1 + x \leqslant e^{x - \delta x^2} \tag{404}$$

for all $0 < \delta < \frac{1}{2}$ *and* $0 \leqslant x \leqslant \frac{1}{\delta} - \left(\frac{2}{\delta}\right)^{\frac{1}{2}}$.

PROOF. Let's call $1 + x =: f(x)$ and $e^{x-\delta x^2} =: g_\delta(x)$. It is

$$\begin{aligned}
\frac{d}{dx}(f - g_\delta)(x) &= 1 - (1 - 2\delta x) g_\delta(x), \\
\frac{d^2}{dx^2}(f - g_\delta)(x) &= (2\delta - (1 - 2\delta x)^2) g_\delta(x).
\end{aligned}$$

It is immediately apparent that $f - g_\delta$, as well as its first derivative, vanish for $x = 0$. Furthermore, the second derivative is zero at the two points

$$\begin{aligned}
x_1(\delta) &= \frac{1}{2\delta} - \frac{1}{\sqrt{2\delta}}, \\
x_2(\delta) &= \frac{1}{2\delta} + \frac{1}{\sqrt{2\delta}}.
\end{aligned}$$

This means that the first derivative is extremal for $x_1(\delta)$ and $x_2(\delta)$. Moreover, we find that

$$(f - g_\delta)'(x_1(\delta)) = 1 - \sqrt{2\delta} e^{\frac{1}{4}\left(\frac{1}{\delta} - 2\right)},$$

which vanishes for $\delta = \frac{1}{2}$ and tends to $-\infty$ monotonically for $\delta < \frac{1}{2}, \delta \to 0$ by l'Hospital's rule. The other extremum satisfies

$$(f - g_\delta)'(x_2(\delta)) = 1 + \sqrt{2\delta} e^{\frac{1}{4}\left(\frac{1}{\delta} - 2\right)} \geqslant 2$$

for all $0 < \delta \leqslant \frac{1}{2}$. Consequently, the derivative $(f - g_\delta)'$ is minimal at $x_1(\delta)$.

In the case of $\delta = \frac{1}{2}$, this implies that the derivative is always nonnegative for all $x_1\left(\frac{1}{2}\right) = 0 \leqslant x \leqslant 2 = x_2\left(\frac{1}{2}\right)$. This, together with the fact that $(f - g_{\frac{1}{2}})(0) = 0$, proves (403).

Next, we consider the case $0 < \delta < \frac{1}{2}$. Note that in this case $x_1(\delta) = \frac{1}{\sqrt{2\delta}}\left(\frac{1}{\sqrt{2\delta}} - 1\right) > 0$, and thus that $(f - g_\delta)'(x) \leqslant 0$ for at least $0 \leqslant x \leqslant x_1(\delta)$. Since $(f - g_\delta)(0) = 0$, we also have $(f - g_\delta)(x) \leqslant 0$ in this range of x. We now show that we can actually at least double the size of the interval where $f - g_\delta$ is negative. To this end consider

$$(f - g_\delta)(2x_1(\delta)) = 1 + \frac{1}{\delta} - \sqrt{\frac{2}{\delta}} - e^{\sqrt{\frac{2}{\delta}} - 2}.$$

Since $\delta < \frac{1}{2}$ we have that $\sqrt{\frac{2}{\delta}} - 2 > 0$ and hence it is

$$\begin{aligned}
e^{\sqrt{\frac{2}{\delta}} - 2} &> 1 + \sqrt{\frac{2}{\delta}} - 2 + \frac{1}{2}\left(\sqrt{\frac{2}{\delta}} - 2\right)^2 \\
&= 1 + \frac{1}{\delta} - \sqrt{\frac{2}{\delta}}.
\end{aligned}$$

Thus,

$$(f - g_\delta)(x) < 0$$

for $0 \leqslant x \leqslant 2x_1(\delta) = \frac{1}{\delta} - \left(\frac{2}{\delta}\right)^{\frac{1}{2}}$. ∎

Next, we take a closer look at the function

$$e(m,a,b) := \left(1+\frac{a}{m}\right)^{m+b} \tag{405}$$

for positive integers m and $a, b \in \mathbb{R}$.

LEMMA 65. *The function $e(m,a,b)$ satisfies the following bounds.*

(i) If $a < 0$ and $2b - a > 0$ or $a > 0$ and $2b - a < 0$ then

$$e(m,a,b) < e^a$$

for all $m > \frac{-ab}{2b-a}$.

(ii) If both $a > 0$ and $2b - a > 0$ or both $a < 0$ and $2b - a < 0$, then

$$e(m,a,b) > e^a$$

for all $m > \frac{-ab}{2b-a}$.

PROOF. It is by the definition of the exponential function

$$\lim_{m\to\infty} e(m,a,b) = e^a.$$

To prove the assertion we show that $e(m,a,b)$ is monotonously increasing in m if a, b satisfy the conditions in *(i)*, and that it decreases in the case of *(ii)*. To this end we calculate

$$\frac{\partial e(m,a,b)}{\partial m} = \left(\frac{a}{m}+1\right)^{b+m}\left(\log\left(\frac{a}{m}+1\right) - \frac{a(b+m)}{m^2\left(\frac{a}{m}+1\right)}\right) \tag{406}$$

and set

$$\tilde{e}(m) = \log\left(\frac{a}{m}+1\right) - \frac{a(b+m)}{m^2\left(\frac{a}{m}+1\right)}.$$

As the first factor in (406) is always positive, the sign of the m-derivative of e is determined by the sign of $\tilde{e}$. It is obviously $\lim_{m\to\infty} \tilde{e}(m) = 0$. In the following we show that $\tilde{e}$ is decreasing under the conditions in *(i)* and increasing if the conditions in *(ii)* are satisfied. We then can infer that $\tilde{e}(m) > 0$, or $\tilde{e}(m) < 0$, respectively, in the claimed range of m, which immediately implies that $e(m,a,b)$ is increasing in m or decreasing, respectively. We determine the derivative of $\tilde{e}$

$$\frac{d\tilde{e}(m)}{dm} = \frac{a(a(b-m)+2bm)}{m^2(a+m)^2}. \tag{407}$$

Since the denominator is always positive, we only have to check the sign of the numerator. If $a < 0$ and $2b - a > 0$ it is negative iff

$$ab + (2b-a)m > 0,$$

which is satisfied for all $m > \frac{-ab}{2b-a}$. If $a > 0$ and $2b - a < 0$ it is negative iff

$$ab + (2b-a)m < 0,$$

which is again satisfied for all $m > \frac{-ab}{2b-a}$. In the cases $a, 2b - a < 0$ and $a, 2b - a > 0$, the right-hand side of (407) is positive if $ab+(2b-a)m < 0$ and $ab+(2b-a)m > 0$, respectively. They both are satisfied for $m > \frac{-ab}{2b-a}$. ∎

Now, we turn to formulae for the q-th powers of the first n positive integers $\sum_{t=0}^{n} t^q$. For the sake of completeness we give a general proof of how to derive such identities and state the result for $q = 2, \ldots, 8$.

Lemma 66. *Let n and q be nonnegative integers, and set $S_q(n) = \sum_{t=0}^n t^q$. Then it is*

$$S_2(n) = \frac{1}{6}n(n+1)(2n+1), \tag{408}$$

$$S_3(n) = \frac{1}{4}n^2(n+1)^2, \tag{409}$$

$$S_4(n) = \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right), \tag{410}$$

$$S_5(n) = \frac{1}{12}n^2(n+1)^2\left(2n^2+2n-1\right), \tag{411}$$

$$S_6(n) = \frac{1}{42}n(n+1)(2n+1)\left(3n^4+6n^3-3n+1\right), \tag{412}$$

$$S_7(n) = \frac{1}{24}n^2(n+1)^2\left(3n^4+6n^3-n^2-4n+2\right), \tag{413}$$

$$S_8(n) = \frac{1}{90}n(n+1)(2n+1)\left(5n^6+15n^5+5n^4-15n^3-n^2+9n-3\right). \tag{414}$$

The proof consists of two steps. The first is to show that the sums $S_q(n)$ are polynomials in n. The second one is to verify the stated coefficients.

Proof. One of the most general ways to approach such summation identities, if you don't want to find yourself stuck in tedious induction steps, is via binomial coefficients. These have the nice property that

$$\binom{t}{k} + \binom{t}{k+1} = \binom{t+1}{k+1}, \tag{415}$$

which makes $\sum_{t=0}^n \binom{t}{k}$ an easy to evaluate, telescoping sum

$$\sum_{t=0}^n \binom{t}{k} = \sum_{t=0}^n \left[\binom{t+1}{k+1} - \binom{t}{k+1}\right] = \binom{n+1}{k+1} \tag{416}$$

for arbitrary integers $0 \leqslant k \leqslant t$. Since the binomial coefficient $\binom{t}{k}$ is a polynomial in t of degree k

$$\binom{t}{k} = \frac{t(t-1)\cdot\dots\cdot(t-k+1)}{k!},$$

we can express t^q as a linear combination of binomial coefficients

$$t^q = \sum_{k=1}^{q-1} a_k \binom{t}{k} + q!\binom{t}{q}.$$

Now, we replace t^q in the definition of $S_q(n)$ by the above relation and evaluate the sum in t with the help of (416). This way, we obtain

$$S_q(n) = \sum_{k=1}^{q-1} a_k \binom{n+1}{k+1} + q!\binom{n+1}{q+1} \tag{417}$$

and it is immediately apparent that $S_q(n)$ is a polynomial in n of degree $n+1$.

To verify the coefficients stated in the lemma, one can now either validate them for example by testing, or determine the coefficients a_k for each q by solving a system of linear equations.

In the case of $q=2$ this system consists of one equation that reads $a_1 - 1 = 0$, coming from the condition $a_1 t + t(t-1) = t^2$ and hence,

$$S_2(n) = \binom{n+1}{2} + 2\binom{n+1}{3} = \frac{1}{6}n(n+1)(2n+1).$$

The coefficients for $3 \leqslant q \leqslant 8$ are

$q = 3 : a_1 = 1, a_2 = 6,$

$q = 4 : a_1 = 1, a_2 = 14, a_3 = 36,$

$q = 5 : a_1 = 1, a_2 = 30, a_3 = 150, a_4 = 240,$

$q = 6 : a_1 = 1, a_2 = 62, a_3 = 540, a_4 = 1560, a_5 = 1800,$

$q = 7 : a_1 = 1, a_2 = 126, a_3 = 1806, a_4 = 8400, a_5 = 16800, a_6 = 15120,$

$q = 8 : a_1 = 1, a_2 = 254, a_3 = 5796, a_4 = 40824, a_5 = 126000, a_6 = 191520, a_7 = 141120.$

■

Next in the line of useful identities is the following one for the tail of the geometric series and a related series.

LEMMA 67. *Let $|x| < 1$ and n be a non-negative integer. Then it is*

$$\sum_{p=n+1}^{\infty} x^p = \frac{x^{n+1}}{1-x},$$

$$\sum_{p=n+1}^{\infty} p(p^2-1)x^p = \left(n^3(1-x)^3 + 3n^2(1-x)^2 + n\left(x^3 - 6x^2 + 3x + 2\right) + 6x\right)\frac{x^{n+1}}{(1-x)^4}.$$

PROOF. The first assertion follows directly from the well-known identity

$$\sum_{p=0}^{n} x^p = \frac{1-x^{n+1}}{1-x} \tag{418}$$

and its limiting case for $n \to \infty$

$$\sum_{p=0}^{\infty} x^p = \frac{1}{1-x}. \tag{419}$$

For the proof of the second assertion of the lemma we take the second and the third derivative on both sides of (418), and obtain the identities

$$\sum_{p=0}^{n} p(p-1)x^{p-2} = \frac{x^n\left(nx^2 - n^2(1-x)^2 - n - 2x\right)}{(1-x)^3 x} + \frac{2}{(1-x)^3} \tag{420}$$

for the second derivative, and

$$\sum_{p=0}^{n} p(p-1)(p-2)x^{p-3} = \frac{x^n\left(-n^3(1-x)^3 - 3n^2x(1-x)^2 - n(x(2x+5)-1)(1-x) - 6x^2\right)}{(1-x)^4x^2} + \frac{6}{(1-x)^4} \tag{421}$$

for the third one. Since by the same technique applied to (419)

$$\sum_{p=0}^{\infty} p(p-1)x^{p-2} = \frac{2}{(1-x)^3}, \tag{422}$$

and

$$\sum_{p=0}^{\infty} p(p-1)(p-2)x^{p-3} = \frac{6}{(1-x)^4}, \tag{423}$$

we deduce from (420) and (422) that

$$\sum_{p=n+1}^{\infty} p(p-1)x^p = -\frac{x^{n+1}\left(nx^2 - n^2(1-x)^2 - n - 2x\right)}{(1-x)^3}, \tag{424}$$

and from (421) and (423) that

$$\sum_{p=n+1}^{\infty} p(p-1)(p-2)x^p$$
$$= -\frac{x^{n+1}\left(-n^3(1-x)^3 - 3n^2x(1-x)^2 - n(x(2x+5)-1)(1-x) - 6x^2\right)}{(1-x)^4}. \tag{425}$$

Now we write $p(p^2-1) = p(p-1)(p-2) + 3p(p-1)$ and the claimed identity follows from (424) and (425). ∎

Last but not least, we present a way how to establish closed form expressions for sums of the kind

$$\sum_{j=0}^{p} q(j)\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma\left(j+1\right)\Gamma\left(p-j+1\right)},$$

that occur in Section 4.4. Here, q is a polynomial in j. The key ingredient in this task is the following central binomial convolution identity.

LEMMA 68. *Let p and k be non-negative integers such that $k \leqslant \frac{p}{2}$. Then it is*

$$\sum_{j=0}^{p} \frac{j!}{(j-k)!}\frac{(p-j)!}{(p-j-k)!}\binom{2j}{j}\binom{2(p-j)}{p-j} = 2^{2p-3k}\frac{(2k-1)!!}{k!}\frac{p!}{(p-2k)!},$$

where the double factorial $n!!$ is for odd n defined by

$$n!! = \begin{cases} n(n-2)\cdot\ldots\cdot 3\cdot 1, & n>0 \text{ odd},\\ 1, & n=-1.\end{cases}$$

PROOF. By the duplication formula (328) it is

$$\binom{2p}{p} = 2^{2p}\pi^{-\frac{1}{2}}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)}.$$

Moreover, the reflection formula (327) implies

$$\begin{aligned}\frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma\left(p+1\right)} &= \frac{(-1)^p\pi}{\Gamma\left(\frac{1}{2}-p\right)\Gamma(p+1)}\\ &= \frac{(-1)^p\pi}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-p\right)\Gamma(p+1)}\\ &= \pi^{\frac{1}{2}}(-1)^p\binom{-\frac{1}{2}}{p}.\end{aligned}$$

Thus, the central binomial coefficient can be expressed as

$$\binom{2p}{p} = (-1)^p 2^{2p}\binom{-\frac{1}{2}}{p}.$$

Consequently, the generalized binomial theorem tells us that

$$\sum_{p=0}^{\infty}\binom{2p}{p}2^{-2p}x^p = \sum_{p=0}^{\infty}\binom{-\frac{1}{2}}{p}(-x)^p = (1-x)^{-\frac{1}{2}}. \tag{426}$$

For x inside the radius of convergence of the series in (426), we are allowed to interchange summation and differentiation. Differentiating the left-hand and the right-hand side k times yields

$$\sum_{p=0}^{\infty}\frac{p!}{(p-k)!}\binom{2p}{p}2^{-2p+k}x^p = (2k-1)!!\frac{x^k}{(1-x)^{\frac{2k+1}{2}}}. \tag{427}$$

Next, we square both sides of (427) and obtain the identity

$$\sum_{p=0}^{\infty}\left[\sum_{j=0}^{p}\frac{j!}{(j-k)!}\frac{(p-j)!}{(p-j-k)!}\binom{2j}{j}\binom{2(p-j)}{p-j}2^{-2p+2k}\right]x^p=[(2k-1)!!]^2\,\frac{x^{2k}}{(1-x)^{2k+1}}. \tag{428}$$

On the other hand, from differentiating the geometric series $2k$ times we know that

$$\sum_{p=0}^{\infty}\frac{p!}{(p-2k)!}x^p=(2k)!\frac{x^{2k}}{(1-x)^{2k+1}}. \tag{429}$$

By comparing the terms of (428) and (429) we can thus infer that

$$\sum_{j=0}^{p}\frac{j!}{(j-k)!}\frac{(p-j)!}{(p-j-k)!}\binom{2j}{j}\binom{2(p-j)}{p-j}2^{-2p+2k}=\frac{[(2k-1)!!]^2}{(2k)!}\frac{p!}{(p-2k)!}.$$

Since

$$\frac{[(2k-1)!!]^2}{(2k)!}=2^{-k}\frac{(2k-1)!!}{k!},$$

the claimed identity follows. ∎

With the just proven lemma at hand, we have all we need to prove the following identities.

LEMMA 69. *Let p be a nonnegative integer and*

$$B(a):=\sum_{j=0}^{p}j^a\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}.$$

Then

$$B(0)=\pi, \tag{430}$$

$$B(1)=\frac{\pi}{2}p, \tag{431}$$

$$B(2)=\frac{\pi}{2^3}p(3p+1), \tag{432}$$

$$B(3)=\frac{\pi}{2^4}p^2(5p+3), \tag{433}$$

$$B(4)=\frac{\pi}{2^7}p\left(35p^3+30p^2+p-2\right), \tag{434}$$

$$B(5)=\frac{\pi}{2^8}p^2\left(63p^3+70p^2+5p-10\right), \tag{435}$$

$$B(6)=\frac{\pi}{2^{10}}p\left(231p^5+315p^4+35p^3-75p^2-2p+8\right). \tag{436}$$

PROOF. Using the duplication formula (328), we find that

$$\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)}=\pi 2^{-2p}\binom{2j}{j}\binom{2(p-j)}{p-j}.$$

Thus, (430) follows directly from Lemma 68 for $k=0$.

For the proof of (431) we make use of the fact that

$$\begin{aligned}
B(1) &= \sum_{j=0}^{p} j \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)} \\
&= \sum_{j=0}^{p} (p-j) \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(p-j+\frac{1}{2}\right)}{\Gamma(j+1)\Gamma(p-j+1)} \\
&= pB(0) - B(1),
\end{aligned}$$

by symmetry of the summand. Hence, it is $B(1) = \frac{p}{2}B(0) = \frac{\pi}{2}p$. The same trick applies for $a = 3$ and $a = 5$ as well, and lets us determine the value of $B(a)$ recursively from the values of $B(0), B(1), \ldots, B(a-1)$.
For the even powers of j, that is (432), (434) and (436), we have to work a little bit harder. We write

$$\begin{aligned}
j^2 &= pj - j(p-j) \\
&= \frac{j!}{(j-1)!} \frac{(p-j)!}{(p-j-1)!}, \\
j^4 &= 2j^3 p - j^2\left(p^2+p-1\right) + j\left(p^2-p\right) + j(j-1)(p-j)(p-j-1) \\
&= \frac{j!}{(j-2)!} \frac{(p-j)!}{(p-j-2)!},
\end{aligned}$$

and

$$\begin{aligned}
j^6 &= 3j^5 p - j^4\left(3p^2+3p-5\right) + j^3\left(p^3+6p^2-10p\right) - j^2\left(3p^3-3p^2-6p+4\right) \\
&\quad + j\left(2p^3-6p^2+4p\right) - j(j-1)(j-2)(p-j)(p-j-1)(p-j-2) \\
&= \frac{j!}{(j-3)!} \frac{(p-j)!}{(p-j-3)!}.
\end{aligned}$$

Now, it is apparent, that the expressions for $B(2), B(4)$ and $B(6)$ follow from Lemma 68 with $k = 1, k = 2$ and $k = 3$, respectively, together with the results for all lower orders of j. ∎

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