Modular q–difference equations and quantum invariants of hyperbolic three–manifolds

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Introduction

There are three sources where we can find wonder; the world, people, and ourselves. The study of mathematics is a perfect union of all three. Firstly, our understanding of the universe and our ability to shape it all stem from mathematics. Whether it be the analysis of statistics used to determine patterns in the natural world, understanding the shape of a snowflake, or in the harnessing of power from the sun, mathematics allows us to understand and probe deeper the makeup of the universe. More abstractly, the study of mathematics provides glimpses of the vast shadows of structures that can stem from seemingly simple and basic principles. This leads to the ability to ask simply phrased questions that admit only profoundly deep answers. Secondly, there is great awe that can be found in the clarity of others. The ability to step through the doors left open by our predecessors or see doors opened by our contemporaries is a privilege and of endless encouragement. This collective knowledge, consisting of thousands of years of work, is important and must be made accessible to anyone who shares a passion for its preservation and expansion. Finally, in mathematics, there is an indescribable joy and excitement in ones own understanding of a problem or its solution.

> *"Mathematics directs the flow of the universe, lurks behind its shapes and curves, holds the reins of everything from tiny atoms to the biggest stars."*
>
> Edward Frenkel, Love and Math, 2013.

> *"Pure mathematics is an immense organism built entirely and elusively of ideas that emerge in the minds of mathematicians and live within these minds."*
>
> Yuri Manin, Mathematics as Metaphor, 2007.

> *"Of course, the most rewarding part is the 'Aha' moment, the excitement of discovery and enjoyment of understanding something new — the feeling of being on top of a hill and having a clear view. But most of the time, doing mathematics for me is like being on a long hike with no trail and no end in sight."*
>
> Maryam Mirzakhani, 2014.

Understanding shapes with numbers and physics

Imagine yourself hundreds of year ago, living on the surface of the earth. No matter where you travel, everything looks roughly the same. There may be mountains and valleys but, besides these bumps and dents, everything looks like a flat surface. Given this, how could one hope to determine the shape of the earth? This was of course done by explorers travelling outwards from their origin, creating maps as they went. They would then collect these maps into an atlas, whose pages contained maps with instructions on which page to go to when you reach the edge. While each page of the atlas contains a map that is the shape of a flat rectangle, if you tear out the pages and line them up side by side according to the instructions, you eventually have a map of the entire globe and see that it is indeed a globe[1]. Of course one can imagine that, if the laws of physics were different, perhaps the explorers could have returned with their maps and, putting them together, found the earth was shaped like an enormous doughnut[2]. Topology is a field concerned with large scale properties of spaces and questions like determining the shape of the earth from an atlas.

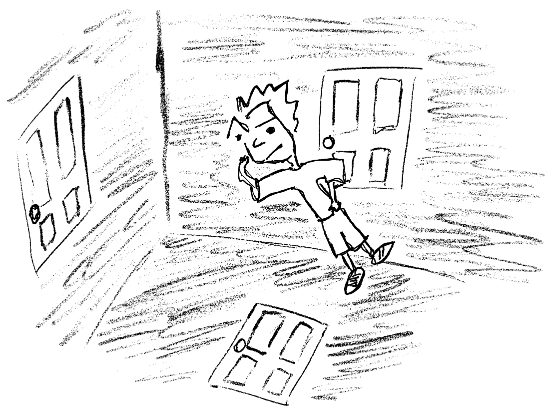
Surfaces are two-dimensional and so, given this dimension is small enough, our brains have the ability to simply see the difference between spaces. For example, the difference between a sphere and a doughnut is clear, as there is a hole in one and not the other. However, if we consider three–dimensional spaces, things becomes much harder to picture. What is the shape of a three–dimensional doughnut hole? Surprisingly, work throughout the last century has shown that understanding these questions very much depends on the dimension you are studying. Although we (or at least I) cannot picture it, we can consider shapes in dimensions of any whole number[3]. Remarkably, due to work of Smale, certain questions are better understood in five or more dimensions. The essential point of difference is that there is "enough space" for certain tricks to work. Then for dimensions zero, one, and

[1] Contrary to what you may find on the internet.

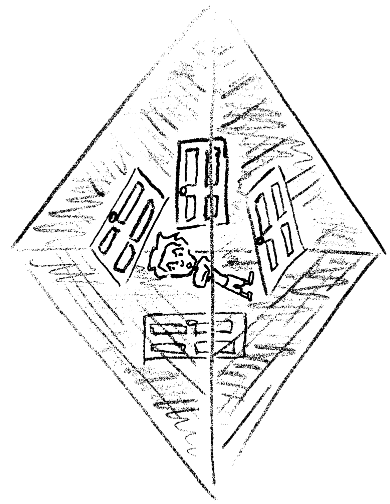
[2] A potentially much more exciting theory to be investigated that could similarly be called flat earth.

[3] Even wilder numbers if we consider fractals.

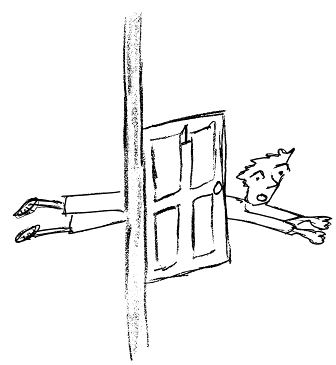
two, things can be understood with similar ease, although they behave differently. In a somewhat conspiratorial outcome[4], four dimensions — exactly the dimension of spacetime we seem to live in — is one of the most difficult to understand.



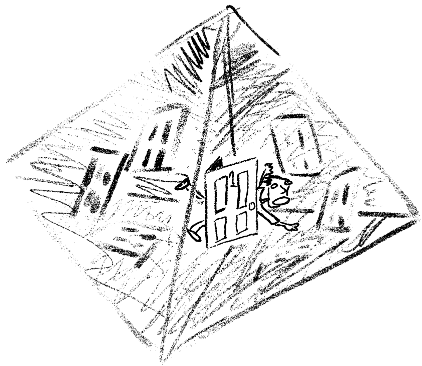
The main spaces of concern to this thesis are three dimensional. Think of spaces that locally look like the space around you with up and down, left and right, and backwards and forwards. However, in these kinds of spaces, if you travelled in a straight line far off into the universe, you could end up back where you started. This is just like the old arcade game asteroids. From the work of Thurston, culminating in the work of Perelman, many important questions about three dimensional spaces are now understood. To get a better idea of such spaces, picture yourself floating inside a triangular based pyramid. As you are floating, you see four triangular walls — three around you and one below. In each of these walls you see a door.

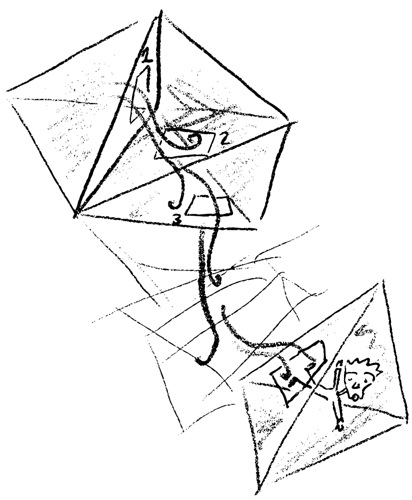


You float towards one of the doors and, before going through, you mark the door with a large "1". As you pass through the door, you find yourself floating inside another triangular based pyramid, again with four triangular walls and doors. You leave a mark on the door you entered again with a "1".



[4]Name of which is still pending.

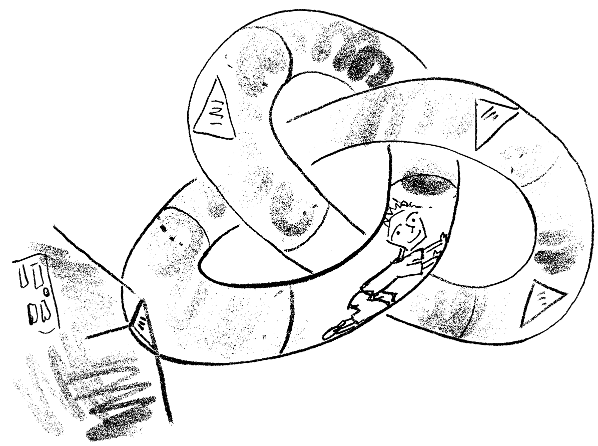


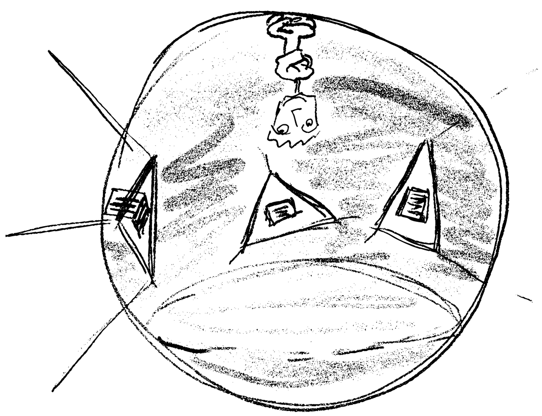


You continue to explore leaving marks “2”, “3”, “4”, *etc.* as you go. At some point you go through a door that takes you into a pyramid you’ve been in before. You are back to where you started and you see a large “1” on one of the other doors. Now you start going through the other doors systematically until you find that you have left a mark on every possible door. You have essentially mapped out this strange layout of tetrahedral rooms. The shape of the labyrinth you have stumbled into is a called a three–manifold.

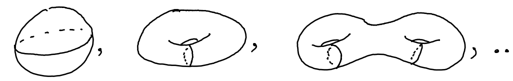
After your mapping of this labyrinth, you look towards one of the corners. You realise that there is a small vent leading to one of the points of the pyramid. You open the vent and find a small tunnel.

You find that there are more vents to the rooms you have previously been in. You crawl through the tunnel and realise that it is a loop. The loop you have crawled through went up and down and around and you realise that this loop is all twisted around itself into a knot.





You realise that this loop does not have vents to all the rooms. You find a vent that does not go to the tunnel you have discovered and find a new space, this time a small spherical area. You discover all the areas that the vents lead to and complete your mapping of the labyrinth and its ventilation system. You are led to think: what are the kinds of labyrinths that could be constructed in a similar way, that is, what are the possible three–manifolds?

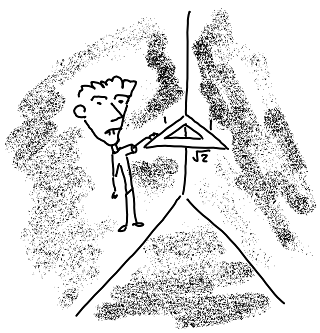


Surfaces without boundary, or two–manifolds, are completely determined by the number of doughnut holes they have. Whereas for three–manifolds, the situation is much more complex and there is no such classification. Three dimensional labyrinths can be understood using the insights of Thurston. In topology, we still consider a labyrinth to have the same shape if all the walls are slightly shifted, much the same way that we ignored the details of the mountains and valleys when discussing the shape of the earth. In geometry, however, there is less freedom as measuring the distance between two points depends on whether there is a mountain in the way. Thurston showed that, for a given labyrinth, there is a way to break it into pieces that have a favourite determined geometry. One could imagine that the architects of these strange labyrinths decided that there is a best way to construct certain sub–labyrinths and that every possibility is then constructed by putting these special sub–labyrinths together.

The structure of these special geometric labyrinths goes deeper. If one studies the geometries these eccentric architects have designed, such as the angles between the various walls, one finds that these numbers satisfy polynomial equations such as

$$x^2 - x + 1 = 0\,.$$

Another example is the length of the vents, which can similarly be written in terms of solutions to polynomial equations. This has turned the study of possible labyrinths into the study of polynomials and, in particular, number theory.



As the name suggests, number theory involves the study of numbers. This can be related to whole numbers or any kind of solutions to polynomial equations, such as $\sqrt{2} = 1.41421\cdots$, which solves the equation

$$x^2 - 2 \;=\; 0\,.$$

More generally, it can involve interesting transcendental numbers like $\pi = 3.14159\cdots$. Number theory has a tendency to pose extremely simple questions that defy simple answers or solutions. Famously, Fermat's last theorem evaded solution for hundreds of years, only to be solved by Wiles in the mid 1990s with the full use of modern mathematical methods. The statement says that there are no whole number solutions for (x, y, z) to the equation

$$x^n + y^n \;=\; z^n\,,$$

when $n > 2$ is a whole number. The proof of this theorem uses something called modular forms. In particular, to certain polynomial equations, there is an associated function called a modular form. These modular forms show up everywhere in number theory and provide powerful tools in understanding the geometry and symmetries of polynomials. Modular forms also show up in relation to three–manifolds. However, this will be for somewhat trivial examples, where the whole space is squashed so that the volume is zero. However, there is an extension of these functions that arise conjecturally for all three–manifolds, which will be studied in this thesis.

Back to the labyrinth or three–manifold, the equations that are associated to a three–manifold are determined by the numbering of the doors and rooms of the labyrinth that we performed previously. From this, we can determine whether two labyrinths are different by checking whether the equations lead to the same kind of solutions. The connection to numbers does not stop at polynomials. In particular, each of these labyrinths has some finite volume, which we can compute.

To compute this volume, we calculate the volume of each room and add them together. There is a function that can be used to calculate the volume of one of the rooms; it is called the dilogarithm. This dilogarithm is a deformation of Euler's beautiful identity[5],

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots \;=\; \frac{\pi^2}{6}\,.$$

The $+\cdots$ means that the sum goes on forever — or more specifically, taking a large enough number of terms, you can get as close to $\pi^2/6$ as you like. We define the dilogarithm as

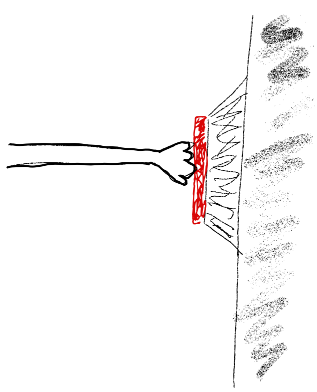
$$\mathrm{Li}_2(x) \;=\; x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \frac{x^5}{25} + \cdots\,.$$

[5]Solving the Basel problem.

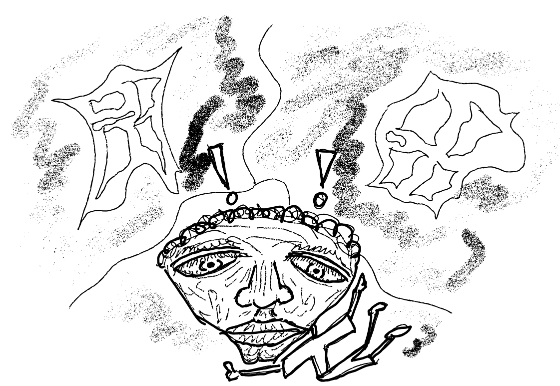
Taking the solutions to our polynomial equations and using this dilogarithm function, we can compute the volume of the rooms and therefore of the entire labyrinth.

After understanding the structures that underpin the topology, geometry, and number theory of this labyrinth, you find a peculiar button on a wall in a room that you missed. Above the large red button, there is a sign that reads:

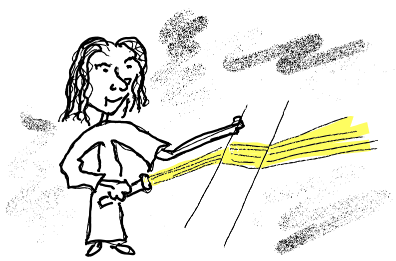
Achtung! Plancksches Wirkungsquantum!



Unfortunately, both your German and sense are momentarily lacking. After having spent some time at the Max Planck institute, the pull of the big red button is irresistible. You push the button. It starts slowly, but then the room you are in becomes weird and seems to be constantly changing. You see strange distortions out of the corners of your eyes. Everywhere you focus and observe remains almost as it had done previously but, when you close your eyes, you get the feeling that you are floating in an ever–changing chaos as the geometry of the whole labyrinth constantly shifts around you like a kaleidoscope.

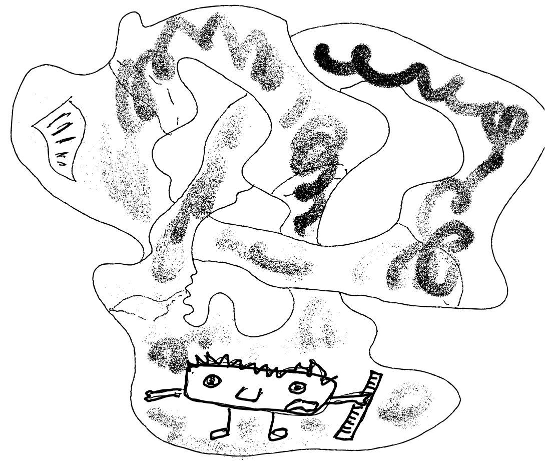


You unwittingly just increased Planck's constant — which determines the strength of quantum mechanical effects — to such a degree that the quantum effects have started to overpower usual physics. You instantly gain sympathy for Schrödinger's poor cat.



Classical physics is, in a sense, very much related to geometry. While we are often taught to think locally in terms of forces pushing and pulling on objects, there exists a different approach. In trying to understand the behaviour of light passing through a lens, Fermat postulated that light will always travel along the path that reaches its destination in the shortest time. In a lens light travels slower than in air and it therefore bends to compensate. The principle of least action generalises this to more complicated systems. These physical system behaves in a way that minimises something called *the action*. Finding the evolution of a system that minimises the action, and measuring quantities like energy, is then similar to finding the shortest line connecting two points and measuring its length.

Quantum mechanics takes yet another approach. In quantum mechanics, all possible evolutions of a system contribute to a probability of a value of a particular measurement. This means that until a measurement of some quantity is made, the system is in a kind of superposition of all possible states. The most likely state in which you will find the system, where quantum fluctuations are more stable, is around the solutions to the classical problem of where the action is minimal. Therefore, when Planck's constant — the constant that determined the strength of quantum effects — tends to zero, all other possible states become less and less likely and we return to the classical picture. For the labyrinth or three–manifold, the possible states are given by the possible geometries of the labyrinth. The geometry that minimises the action is given by the perfect geometry designed by the architects.



Back in the labyrinth, you stumble around these ever–changing geometries and make your way to one of the vents. There, things are not much better. To calm yourself down, you try to measure the length of the vents as you crawl around and around. You find a different length each time. There seems to be no order in this chaos until, after measuring the length of this vent again and again, you find that it seems to behave randomly but certain lengths are more common. In particular, you find that the lengths vary with each measurement, however, they are most likely to be close to your original measurement before you clicked the button. You realise that this random geometry is in fact related to the shape of the labyrinth. This again provides a method for comparing various labyrinths. The space is not rigid and any possible geometry, not just the perfect geometry of the architects, can occur. However, when one measures the geometry, certain lengths and angles are more likely and the likelihood is determined by the shape of the whole labyrinth or three–manifold. This quantum world you have stumbled into, depending just on the topology of the universe, is an example of a topological quantum field theory.

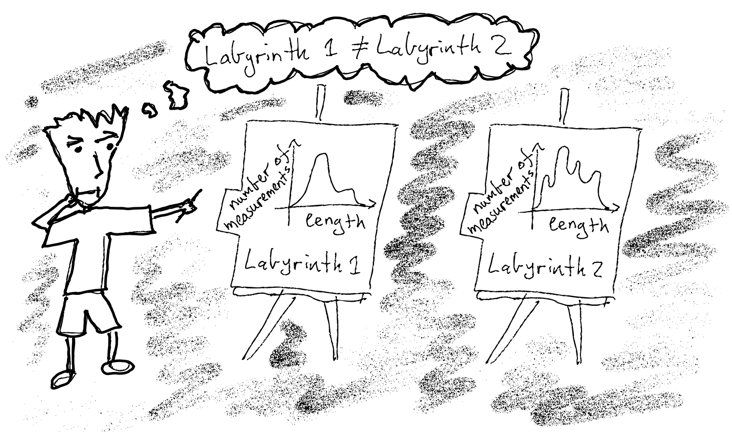
Measuring these quantum mechanical probabilities, you find that the number theory goes deeper. Not only does the most likely geometry, constructed by the architects, give measurements coming from solutions to polynomials, but the quantum corrections are also determined by polynomials. You make your way back to the button and manage to lower Planck's constant back down to something more reasonable. You start to ponder your experience and try to come up with a theory that describes the marvellous things you have observed. You try and try but the best you can do is find strange infinite sums like

$$1+2+3+4+5+6+\cdots\,.$$

How could one possibly make sense of this? You think back to the Euler's identity and recall that he had another

$$1+2+3+4+5+6+\cdots \;=\; -\frac{1}{12}\,.$$

Remembering that Riemann made sense of this kind of identity through something called analytic continuation, you try to apply the same ideas to your infinite sums. Computing these quantities for a fixed value of Planck's constant ends up requiring the use of some new and beautiful functions generalising the modular forms from number theory. These are called quantum modular forms and this thesis will develop some of the basic structures that surround these fascinating functions.



This thesis

"On the whole, divergent series are the work of the Devil and it's a shame that one dares base any demonstration on them. You can get whatever result you want when you use them, and they have given rise to so many disasters and so many paradoxes. Can anything more horrible be conceived than to have the following oozing out of you:

$$0 = 1 - 2^n + 3^n - 4^n + \cdots ,$$

where n is a whole number? Risum teneatis amici.

Niels Abel, in a letter to Bernt Holmboe, 1826.

Divergent series and Riemann surfaces

While divergent series may be the work of the Devil, they are an essential part of our current theories of physics. Asymptotic series where introduced by Poincaré and Stieltjes to give approximations to smooth functions via polynomials to any order. What distinguishes them from power series is that these infinite sums can have a zero radius of convergence. Using perturbation theory, physicists can give asymptotic series for quantities in certain parameters of their theories. However, in the real world these parameters are finite numbers and one needs to make sense of these divergent sums. One of the main issues with quantum chromodynamics is that this parameter is quite large and the approximation immediately has a large error. To understand this in physical theories associated to the real world would be extremely interesting, however not an aim of this thesis. Instead, much of the motivation of this thesis comes from trying to understand the behaviour of a physical theory of a much simpler form. In particular, this thesis will be interested in a family of topological quantum field theories associated to three–manifolds.

Before describing such a theory, lets consider how these divergent series were understood historically. One of the most powerful results in complex analysis is that if a holomorphic function has an accumulation point of zeros in its domain, then it is the zero function. This captures exactly the kind of rigidity that comes from complex analytic functions. This also naturally leads to the idea of analytic continuation. Once a holomorphic function is defined on some open set, then we can try and extend the function in patches to larger and larger domains obtained by gluing together little disks. One of the most important examples of this was given by Riemann who considered the ζ–function studied by Euler

$$\zeta(s) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^s} \, .$$

This equation makes sense when $\Re(s) > 1$ where it is absolutely convergent. Euler had conjectured that this function should be defined for $s \in \mathbb{R} - \{1\}$ and satisfy a functional equation

$$\zeta(s) \;=\; 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\,.$$

Riemann then proved this for $s \in \mathbb{C}$ using explicitly the idea of analytic continuation. Moreover, he showed that the function is meromorphic with a simple pole at $s = 1$. In this way, we can make sense of the equation

$$1+2+3+4+5+\cdots \text{ ``}=\text{'' } \zeta(-1) \;=\; -\frac{1}{12}\,.$$

Analytic continuation led Riemann to define Riemann surfaces, which come from the local structure of an analytic function that is continued to the largest possible domain. This domain will be a universal cover. Such a function and domain, has symmetries given by Deck transformations. When we quotient by these symmetries this leads to a Riemann surfaces with potentially interesting topology[6]. Remarkably, for compact Riemann surfaces, it was then understood that these all came from algebraic curves giving rise the Serre's GAGA principle. Riemann also described the complex structure of any simply connected and bounded domain in the complex plane. This was done via the Riemann mapping theorem, which shows that these domains are all equivalent to an open unit disk with the standard complex structure. Poincaré and Koebe then went onto show the famous trichotomy of uniformisation. This states that any simply connected Riemann surface is isomorphic to the Riemann sphere $\mathbb{CP}^1$, the complex plane $\mathbb{C}$, or the complex unit disk. Uniformisation also implies geometric content. In particular, for some Riemann surface, one of these spaces will appear as the universal cover. These three spaces then have natural geometries the first spherical, the second Euclidean, and the third hyperbolic respectfully. Taking quotients of these geometries then give the Riemann surfaces natural geometric structures. The only Riemann surface with spherical geometry is $\mathbb{CP}^1$ and the only Riemann surfaces with Euclidean geometries are tori. All the others are hyperbolic and come with a moduli space of such structures of real dimension $6g - 6$ where g is the genus of the underlying topological surface.

From three–manifolds to geometry to number theory

Based on this beautiful theory of surfaces, one can ask what happens in one dimension up. Topologically, these spaces immediately become much more complicated. However, there are still many descriptions of three–manifolds as combinatorial objects with a known set of equivalences. Unfortunately, determining whether two sets of this combinatorial data gives the same three–manifold is not a solvable problem. Also, these spaces are now three real dimensional and therefore cannot carry something like a complex structure. However, as

[6]Of course this could lead to singular spaces. One should then not quotient by the whole set of automorphisms. This will then lead to a Riemann surface with automorphisms, which is generically not the case.

is often the case, some form of uniformity in mathematics prevails. Indeed while complex structures can no longer be defined, we can still define geometric structures. For real surfaces we had three geometries while for three dimensional manifolds there are eight. These different geometries were introduced by Thurston [184]. He argued that every three–manifold can be broken up into pieces where each piece has a canonical geometric structure. This became know the geometrisation conjecture. As in the case of surfaces, all but one of the geometries appear as somewhat special cases. It is again hyperbolic geometry that provides the most general example of geometric structures. This program, initiated by Thurston, led ultimately to the work of Perelman [155, 157, 156], which as a by–product proved that every compact simply connected three–manifold is isomorphic to the three–sphere, a theorem previously known as the Poincaré conjecture.

Importantly, for three dimensional complete hyperbolic structures, there are no moduli in contrast to surfaces. This follows from Mostow–Prasad rigidity [135, 159]. Combining this with the fact that the isometries of hyperbolic three space are given by $\mathrm{PSL}_2(\mathbb{C})$, we see that these unique geometric structures associated to three–manifolds give rise to solutions to polynomials and therefore to number fields. This has led to the following diagram

$$\text{Topology} \rightsquigarrow \text{Geometry} \rightsquigarrow \text{Number theory}\,.$$

The connection to number theory goes quite deep. For any three–manifold and a representation of the fundamental group into $\mathrm{PSL}_2(\mathbb{C})$, we can define an element of the Bloch group [180, 145, 210] or the third algebraic K–group of the algebraic numbers, $K_3(\overline{\mathbb{Q}})$. The so called regulator gives a map from $K_3(\overline{\mathbb{Q}})$ to $\mathbb{R}$, which for the three–manifold corresponds to the volume of the associated geometric structure. Here one sees a remarkable connection. If the volume of the three–manifold[7] is not zero for some representation, then the manifold is hyperbolic and if it is zero, it is not. Similarly, if the regulator vanishes for every Galois conjugate of an element of $K_3(\overline{\mathbb{Q}})$, then the element is torsion. This shows that there is a connection between being torsion in algebraic K–theory and being hyperbolic as a three–manifold. One could think that was the end of it. However, in work on conformal field theory, Nahm [142] introduced a striking conjecture relating K–theory to modular forms.

Modular forms and asymptototics

Modular forms [32, 176] are functions with many arithmetic symmetries. This means they store an enormous amount of information about numbers and varieties. The amount of symmetries they satisfy often leads to non–trivial finite dimensional vector spaces of such functions. Having a finite dimension space of course means that, upon checking some small set of information, we can completely determine a function in this space. Modular forms first

[7]This manifold we are considering should of course be one of the building blocks of Thurston and not a collection of these, which may contain both hyperbolic and non–hyperbolic pieces.

arose as θ–functions. These θ–functions can be described as generating functions counting the number of ways of representing a number via some quadratic form. The θ–function

$$\vartheta_{00}(q) \;=\; \sum_{k\in\mathbb{Z}} q^{k^2/2}$$

was crucial in Riemann's proof of the functional equation of the ζ–function. Here the important property of the θ–function making it a modular form is that for $q = \mathbf{e}(\tau) = \exp(2\pi i\tau)$

$$\vartheta_{00}(-1/\tau) \;=\; \mathbf{e}(-1/8)\sqrt{\tau}\vartheta_{00}(\tau)\,.$$

Taking the Mellin transform of both sides of this equality gives rise to the functional equation for the ζ–function.

Let $\mathfrak{h} = \{z \in \mathbb{C} : \Im(z) > 0\}$. A function $f : \mathfrak{h} \to \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$ on some discrete subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$, if for $\gamma = [a, b; c, d] \in \Gamma$ we have

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) \;=\; (c\tau + d)^k f(\tau)\,,$$

and some condition of the growth as $\tau \to \infty$. Already we see that ϑ_{00} is not quite of this form as $k = 1/2$ and there is an eight root of unity appearing. These are of course related and the main point is that we can replace $(c\tau + d)^k$ by something called an automorphy factor. There are many details like this that one can specify, which — while leading to infinitely many such functions — when specified give rise to finite dimensional spaces. Once modularity is proved, many identities can then be checked for some finite set of data, which lead to identities between functions like an identity due originally to Jacobi,

$$\Big(\sum_{k\in\mathbb{Z}} q^{k^2/2}\Big)^4 \;=\; \Big(\sum_{k\in\mathbb{Z}}(-1)^k q^{k^2/2}\Big)^4 + \Big(\sum_{k\in\mathbb{Z}+1/2} q^{k^2/2}\Big)^4.$$

In this thesis, the examples we will be interested in are related to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This is generated by the matrices

$$T \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad S \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Importantly, the affect of the T matrix on modular forms is that

$$f(\tau + 1) \;=\; f(\tau)\,.$$

Therefore, modular forms on $\mathrm{SL}_2(\mathbb{Z})$ have Fourier series and we can write them in the variable $q = \exp(2\pi i\tau)$, which is a variable on the unit disk. For modular forms, the transformation under S determines the asymptotic behaviour as $q \to 1$. Thinking in the variable q we see that if $\mathbf{e}(-1/\tau)$ is close to 1 so the boundary of the unit disk then $\mathbf{e}(\tau)$ is close to 0. Therefore, the asymptotics near 1 are completely determined by the behaviour around $q = 0$. This similarly works for other roots of unity. For example, taking $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$ we see that when $\mathbf{e}((a\tau + b)/(c\tau + d))$ is close to $\mathbf{e}(a/c)$ that $\mathbf{e}(\tau)$ is close to 0. This means that around each root of unity the behaviour is completely determined by the behaviour near $q = 0$.

One of the fathers of the theory of modular forms was Ramanujan. While he thought very differently from the modern perspective, his intuitive approach led to many deep and remarkable insights. Another set of objects that Ramanujan had great intuition for was q–hypergeometric functions. These classes of functions rarely overlap but a beautiful example where they do are the Rogers–Ramanujan functions [164]. Nahm [142] studied certain q–hypergeometric sums generalising the examples of Rogers–Ramanujan. To these functions, one can find associated elements of $K_3(\overline{\mathbb{Q}})$. Nahm showed, in examples, that these elements being torsion led to the q–hypergeometric functions that happened to be modular forms. This beautiful connection has been partially understood but even the full formulation of the conjecture has not been precisely given [206, 191]. Regardless, the idea that one could relate modular forms to three–manifolds is enticing. Indeed, the first evidence that such a connection could exist was given in the work of Lawrence–Zagier [118]. They showed that, for Poincaré's famous homology sphere, certain quantum invariants are related to modular forms. What became clear after the work of Zwegers [211] is that, more specifically, they were related to the mock modular forms of Ramanujan [163]. They also outlined how similar statements should hold more generally, which was later studied in detail by Hikami [99] and [39].

Quantum field theory and topology

To understand how Lawrence–Zagier came to these modular objects, we need to investigate a different thread of ideas that came to fruition in the 1980s. Firstly, in the first half of the 20th century, Alexander [2] defined certain invariants of knots and links. These invariants took the form of polynomials. These polynomials satisfied certain *skein relations*, which were forgotten until they were rediscovered by Conway [41] at the beginning of the 1970s. These skein relations gave a simple combinatorial definition of the Alexander polynomial from link diagrams and could be used to effectively compute it[8]. Jones, in the study of representations of von Neumann algebras, came to an almost identical set of relations that gave rise to a new polynomial invariant [103, 104]. This polynomial invariant was a completely unexpected development and spurred many similar constructions [63, 161]. All of these descriptions use a link diagram, an inherently two dimension object. This contrasted the original definition of Alexander, which — while being computable using link diagrams — had an inherently

[8]See Figure 2.2 and equation (2.1).

three–dimensional definition. This led Atiyah to ask whether the Jones polynomial could similarly be given a three dimensional interpretation.

Witten [195] then used ideas from physics, which involve infinite dimensional integrals that remain undefined, to construct the framework of a theory that should give rise to the Jones polynomial and much more. This theory used the classical geometric ideas similar to Thurston but instead of choosing a specific canonical geometry to study the manifold one integrates over all possible geometries. This integral is done against a special function called the Chern–Simons invariant related to the classical geometry [40, 37]. As this is not mathematically defined, one could argue as to what this theory could be used for. However, Witten's insights gave such a rich array of relations which, essentially uniquely determined such a theory — if it were to exist. Soon after Witten's proposal, Reshetikhin–Turaev [166, 167] gave a completely mathematical construction of Witten's theory using quantum groups and their representations. This construction not only proved invariance of such objects, but gave effective ways to compute these invariants. Invariants of this kind became known as topological quantum field theories and were axiomatised by Atiyah [10, 11] following the ideas of Segal [175].

From Witten's perspective — using functional integrals — there should be some asymptotic expansion that should come associated to these invariants as we vary the analogue of Planck's constant towards zero. While these asymptotic series, coming from some infinite dimensional analogue of stationary phase approximation, are completely natural from the physical and more geometric perspective, from the mathematical construction of these invariants, it is not in the least bit clear. This leads to some remarkable mathematical conjectures that remain open for all but some small collection of three–manifolds. The first case that I'm aware of a proof of such a statement comes in the work of Lawrence–Zagier for the Poincaré homology sphere.

Therefore, coming back to Lawrence–Zagier, they considered[9] the q–series

$$\sum_{k=0}^{\infty} q^k (q^k;q)_k \;=\; 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - q^{29} + \cdots$$

where $(a;q) = \prod_{j=0}^{n-1}(1-aq^j)$. While this sum converges as a q–series it also makes sense at root of unity, where the sum becomes finite as the number of non-zero terms is given by the order of the root of unity. Evaluating at roots of unity gives precisely the Witten–Reshetikhin–Turaev invariant of the Poincaré homology sphere $\Sigma(2,3,5)$. This q–series on the other hand can be described as the Eichler integral of a unary θ–series. This then inherits certain modular behaviour. In particular, the additive failure of modularity of this q–series has an analytic continuation to the cut plane. These functions have asymptotics at each root

[9] This q–hypergeometric formula was not known at the time and they present this in a different fashion using Eichler integrals.

of unity that are combinations of three asymptotic series. In particular, for $q = \exp(2\pi i/n)$ where $n \in \mathbb{Z}$ as $n \to \infty$

$$\begin{aligned}\sum_{k=0}^{\infty} q^k (q^k;q)_k \;\sim\; &\sqrt{-in}\exp\Big(-\frac{\pi i}{60n}\Big)\sqrt{\frac{1}{2}\Big(1-\frac{1}{\sqrt{5}}\Big)}\exp\Big(-\frac{\pi i n}{60}\Big)\\ &+\sqrt{-in}\exp\Big(-\frac{\pi i}{60n}\Big)\sqrt{\frac{1}{2}\Big(1+\frac{1}{\sqrt{5}}\Big)}\exp\Big(-\frac{49\pi i n}{60}\Big)\\ &+1-\frac{2\pi i}{n}+\frac{9}{2}\frac{(2\pi i)^2}{n^2}-\frac{205}{6}\frac{(2\pi i)^3}{n^3}+\cdots\,.\end{aligned}$$

The remarkable observation is that if $\tilde{q} = \exp(-2\pi i n)$ and $n \in \mathbb{Q}$ with bounded denominator but still tending to ∞, then we find that

$$\begin{aligned}\sum_{k=0}^{\infty} q^k (q^k;q)_k \;\sim\; &\sqrt{-in}\, q^{\frac{1}{120}}\sqrt{\frac{1}{2}\Big(1-\frac{1}{\sqrt{5}}\Big)}\,\tilde{q}^{-\frac{1}{120}}\sum_{k=0}^{\infty}\tilde{q}^k(\tilde{q}^k;\tilde{q})_k\\ &+\sqrt{-in}\, q^{\frac{1}{120}}\sqrt{\frac{1}{2}\Big(1+\frac{1}{\sqrt{5}}\Big)}\,\tilde{q}^{\frac{49}{120}}\sum_{k=0}^{\infty}\tilde{q}^k(\tilde{q}^{k+1};\tilde{q})_k\\ &+1-\frac{2\pi i}{n}+\frac{9}{2}\frac{(2\pi i)^2}{n^2}-\frac{205}{6}\frac{(2\pi i)^3}{n^3}+\cdots\,.\end{aligned}$$

This is just the asymptotic series is associated to

$$S \;=\; \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,.$$

Replacing this by another element $[a,b;c,d] \in \mathrm{SL}_2(\mathbb{Z})$, we take $\tilde{q}_\gamma = \exp(2\pi i(an+b)/(cn+d))$ and again find three asymptotic series associated to the root of unity $\exp(2\pi i a/c)$. This modular property is some kind of generalisation to the usual definition of a modular form. In fact, this modularity can be vastly extended. Using Zwegers's theory of mock modular forms, one can find an analytic function Ω_γ such that

$$\begin{pmatrix}1\\ \sum_{k=0}^{\infty}\tilde{q}_\gamma^k(\tilde{q}_\gamma^k;\tilde{q}_\gamma)_k\\ \sum_{k=0}^{\infty}\tilde{q}_\gamma^k(\tilde{q}_\gamma^{k+1};\tilde{q}_\gamma)_k\end{pmatrix} \;=\; \Omega_\gamma(\tau)\begin{pmatrix}1\\ \sum_{k=0}^{\infty}q^k(q^k;q)_k\\ \sum_{k=0}^{\infty}q^k(q^{k+1};q)_k\end{pmatrix}\,.$$

This extremely simple example lays out the properties that will underlie all the examples we will consider in this thesis. The main points are as follows:

- to each root of unity we can associate a vector of formal series[10],
- in the upper half plane (resp. in the lower half plane) we have a vector of q–series (resp. q^{-1} series) that have asymptotics as we approach a root of unity agreeing with the asymptotic series,
- this vector $f(q)$ of asymptotic series and $q^{\pm}$–series satisfies a modular transformation

$$f(\tilde{q}_\gamma) \;=\; \Omega_\gamma(\tau) f(q)$$

  where Ω_γ has analytic continuation to a cut plane $\mathbb{C}_\gamma = \mathbb{C} - \mathrm{sign}(c)\mathbb{R}_{\leq -\mathrm{sign}(c)d/c}$ when $c \neq 0$.

We see that Ω has the same asymptotics as the prescribed formal series and it is analytic on a dense, open, and connected set in $\mathbb{C}$.

Quantum modular forms: from experiments to proofs

These remarkable properties of quantum invariants, computed in [106], were then studied for some simple knots by Zagier. The first [204] was the simplest knot called the trefoil. This was then extended to the next simplest knot, the figure eight knot in [208]. The trefoil behaves much in the same way to the Poincaré homology sphere and this follows from the fact they are both not hyperbolic and therefore their invariants behave like mock modular forms. The figure eight knot is the simplest hyperbolic three–manifold and its quantum invariants already behave in much more interesting ways. To understand this leads us to the work of Kashaev.

In somewhat parallel to much of the development, Kashaev [106, 107] defined invariants of knots using a quantisation of the dilogarithm called the Faddeev quantum dilogarithm [58, 59] denoted here[11] $\Phi_S(z;\tau)$. Kashaev [108] then noted that taking the asymptotics of his invariant took the quantum dilogarithm to the classical dilogarithm for each tetrahedron. He conjectured that his invariant should then have asymptotics given by the hyperbolic volume of the knot's complement. It was then shown by Murakami–Murakami [138] that Kashaev's invariant reproduced Jones's invariant at roots of unity. This then gives the conjecture that the asymptotics of the Jones invariants should grow exponentially like the hyperbolic volume of the knot complement. This conjecture became known as the volume conjecture. This conjecture is one of the outstanding problems in quantum topology.

[10] These formal series are also defined over some Kummer extensions to one of the fields associated to the three–manifold.

[11] See Section 8.10 for the comparison with the more standard notation.

This conjecture shows that the next simplest example of a knot after the trefoil, the figure eight knot, will already have to be much more complicated behaviour. The study of this example in full was taken up by Garoufalidis and Zagier [86, 85] in work over the last decade. The first step was taken by Zagier in [208] where remarkable modular properties of the Kashaev invariant

$$\sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2\,,$$

were discovered experimentally. The main observation was that for the complexified volume of the figure eight knot, $\mathrm{VC} = 2.0299\cdots i$, computing the assymptotics of the Kashaev invariant as $n \to \infty$ with $n \in \mathbb{Q}$ where $\tilde{q} = \mathbf{e}(-1/n)$ and $q = \mathbf{e}(n)$

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k \tilde{q}^{-k(k+1)/2}(\tilde{q};\tilde{q})_k^2 \;\sim\; &\sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2 \\ &\times \mathbf{e}\Big(\frac{\mathrm{VC}}{(2\pi i)^2}n\Big)\frac{\mathbf{e}(1/8)}{\sqrt{\sqrt{-3}}}\Big(1-\frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{n}+\frac{697}{1152\sqrt{-3}^6}\Big(\frac{2\pi i}{n}\Big)^2+\cdots\Big)\,.\end{aligned}$$

The next surprise came from a seemingly unrelated problem to do with quantum spin networks [81, 85]. There, the q–series

$$g(q) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2}}{(q;q)_k^2}$$

appeared with seemingly nothing to do with the figure eight knot. When computing the asymptotics as $q \to 1$, Garoufalidis and Zagier realised that this has exactly the same asymptotic series as the Kashaev invariant of the figure eight knot but missing an asymptotic series associated to substituting formally $q = e^h$ in the Kashaev invariant. From the physical perspective, this missing asymptotic series is associated to a trivial connection, which corresponds to some completely degenerate geometry.

Over the proceeding years many beautiful numerical observations were made by Garoufalidis and Zagier. For example, using optimal truncation and smooth optimal truncation [87] of the asymptotic series arising for the Kashaev invariant they found that numerically

$$\begin{aligned}&\sum_{k=0}^{\infty}(-1)^k \tilde{q}^{-k(k+1)/2}(\tilde{q};\tilde{q})_k^2 \\ &\sim \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2\,\mathbf{e}\Big(\frac{\mathrm{VC}}{(2\pi i)^2}n\Big)\frac{\mathbf{e}(1/8)}{\sqrt{\sqrt{-3}}}\mathrm{SmOpTrunc}\Big(1-\frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{n}+\frac{697}{1152\sqrt{-3}^6}\Big(\frac{2\pi i}{n}\Big)^2+\cdots\Big) \\ &\quad+\mathrm{SmOpTrunc}\Big(1-\Big(\frac{2\pi i}{n}\Big)^2+\frac{47}{12}\Big(\frac{2\pi i}{n}\Big)^4+\cdots\Big) \\ &\quad+\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k(q^{k+1}-q^{-k-1})q^{-k(k+1)/2}(q;q)_k^2\,\mathbf{e}\Big(\frac{-\mathrm{VC}}{(2\pi i)^2}n\Big)\frac{\mathbf{e}(1/8)}{\sqrt{-\sqrt{-3}}}\mathrm{SmOpTrunc}\Big(1+\frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{n}+\frac{697}{1152\sqrt{-3}^6}\Big(\frac{2\pi i}{n}\Big)^2+\cdots\Big)\,.\end{aligned}$$

This is of the same form of the asymptotics as the quantum invariant of the Poincaré homology sphere. The main difference is the appearance of factorially divergent series at exponentially large order.

Many of the observations that Garoufalidis and Zagier had made were then understood with new invariants defined by Andersen–Kashaev [8, 7, 9]. These invariants again used the Faddeev quantum dilogarithm and were called state integrals. They were introduced originally by Hikami [98] and subsequently studied in [47, 49, 45]. All these previous attempts lacked a precise contour of integration, which was supplied by Andersen–Kashaev [8]. The key to understanding many of the observations related to the modular behaviour of these quantum invariants was laid out in two papers of Garoufalidis–Kashaev [72, 73]. In these two papers, it was shown how the state integrals of Andersen–Kashaev could be written as bilinear combinations of q and $\tilde{q}$ functions. These functions are defined on $\mathbb{C} - (\mathbb{R} - \mathbb{Q})$. There was one point that was still missing. In particular, these state integrals factorised into two sets of seemingly different q and $\tilde{q}$ functions. Garoufalidis and Zagier realised that these bilinear combinations could be described as entries of a matrix product. The final step needed to give the modular properties was then understood as certain quadratic relations between the q–series or the asymptotic series in [85, 86]. This was interpreted in [82] as dualities between associated q–holonomic modules. Essentially, this writes one of the matrices that appears in the factorisation as the inverse of the other. Together the factorisation and quadratic relations give a method to prove these modular properties for essentially all simple examples. This leads to matrix equations

$$U(\tilde{q}_\gamma) \;=\; \Omega_\gamma(\tau)U(q)\Delta(\tau)$$

where Δ is some diagonal weight matrix. The matrix Ω appears in the place of the asymptotic series. Of course taking the quotient of the matrices writes this Ω as an analytic function the interior of $\mathbb{C} - (\mathbb{R} - \mathbb{Q})$. The remarkable property of Ω is that it has analytic continuation to a cut plane. This gives an analytic function on some open subset of $\mathbb{R}$. The main method of proof is to write Ω in as an integral that clearly extends holomorphically. Functions U satisfying this kind of equation are called *quantum modular forms* and the associated Ω is called its *cocycle*. This was understood [73, 72, 85, 86] for q–series and functions similar to the Kashaev invariant. However, for the figure eight knot these functions do not see the asymptotic series

$$1 - \hbar^2 + \frac{47}{12}\hbar^4 + \cdots .$$

In other words, the matrix valued quantum modular form constructed[12] in [85, 86] for the figure eight knot was only a 2×2 and we want a 3×3.

[12] In fact they had a 3×3 matrix at $\mathbb{Q}$. However, the analytic extension property of Ω was conjectural.

The missing series and q–difference equations

While the Andersen–Kashaev invariant is a bona fide invariant of knots, it has long been known that it is missing important information needed to define a fully fledged topological quantum field theory in a similar sense to the WRT theory. An important step towards understanding how this could be made possible involves finding a q–series, like the series $g(q)$, that has asymptotics that include the missing asymptotic series associated to the trivial geometry or $\mathrm{SL}_2(\mathbb{C})$–connection. The key to the construction of such a series relies on a different structure that these quantum invariants satisfy. It was shown by Garoufalidis and Lê [77], that the Jones invariant satisfies a system of q–difference equations and is in fact holonomic. Garoufalidis then conjectured that the characteristic varieties associated to these q–difference equations agree with a classical moduli of certain geometric structures associated to the knot called the A–polynomial [42, 66]. In the physical literature, Gukov argues that there could be a theory like Witten's associated to three–dimensional gravity, which should give an associated quantisation of the A–polynomial [91]. This link between the q–difference equations satisfied by the Jones invariant and the A–polynomial has become known as the AJ conjecture.

It became clear that to find a q–series that sees the missing asymptotic series, we need a basis of solutions to the q–difference equations. This was also taken up recently in the physics literature [92, 153]. However, a clear downside to this approach is that these q–difference equations are almost impossible to compute for complicated knots. However, any theory needs to be understood first for simple knots. For the Kashaev invariant, there was no such q–difference equation and this approach will then not work. In trying to understand some of their observations associated to Kashaev's invariant, Garoufalidis and Zagier noticed that there in fact seemed to be a naturally appearing family of invariants associated to the Kashaev invariant. Garoufalidis and Kashaev [71], then introduced an additional parameter in the definition of the Jones invariant, which gives rise to a sequence of invariants. This importantly gives rise to a sequence of invariants specialising to the Kashaev invariant.

These q–difference equations satisfied by knot invariants can often be made inhomogeneous. The q–series $g(q)$ naturally satisfies a homogenous version of this equation. Therefore, to find a q–series seeing all asymptotic series, we need the q–series to satisfy the inhomogenous recursion. If one considers mock modular forms exactly the same thing happens. In particular, mock modular forms satisfy inhomogeneous versions of q–difference equations satisfied by modular forms. This appears for indefinite and partial θ–functions as well, where we only sum over part of the lattice, which leads to boundary terms. Remarkably, the initial observations of Lawrence–Zagier that, with the work of Zwegers, mock modular forms describe quantum invariants of non–hyperbolic manifolds, also applies to finding the missing q–series associated to hyperbolic examples. This is summarised for the figure eight knot in the following theorem, which closes this aspect of the study of various works [85, 86, 69, 68, 70].

Theorem. *(Theorem 7) The Kashaev invariant of 4_1, given by*

$$\widetilde{J}_0(q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2\,,$$

and the q–series

$$\mathfrak{G}(q) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_k^2}\left(\frac{1}{8}\Big(-4G_1(q)+2\sum_{\ell=1}^{k}\frac{1+q^j}{1-q^j}\Big)^2-\frac{1}{24}+\sum_{\ell=1}^{k}\frac{q^\ell}{(1-q^\ell)^2}\right),$$

are part of a vector–valued quantum modular form whose cocycle is given by a matrix of state integrals one of which is given by

$$\int_{\mathcal{C}_\tau}\frac{\mathbf{e}(z(z+1+\tau)/2\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2}dz\,.$$

What about closed manifolds?

Much of this story was carried out for knots and in particular their compliments. How to make sense of this for closed manifolds remained somewhat elusive. However, we have the invariant defined by Reshetikhin–Turaev that makes sense at roots of unity. Therefore, we can carry out exactly the same experiments and proof for the WRT invariant as we did for the Kashaev invariant. The question then becomes how to extend this to the case of q–series. Gukov–Manolescu [92] took the solutions to the difference equations of the Jones invariant as q–series and gave a surgery procedure to give conjectural q–series invariants of closed three–manifolds called $\widehat{Z}$[13]. The passage from q–difference equation to solutions as q–series requires a q–difference equation. Again this was lacking for closed manifolds. Moreover, the difference equations for knots are associated to the boundaries and if we have none how could we expect a canonical difference equation?

The key to constructing difference equations for closed three–manifolds is to forget about requiring something canonical for just the closed three–manifold and instead associated something for the closed manifold with additional information. In particular, if we take a closed three–manifold and a link inside the three–manifold, then there is a canonical set of discrete q–difference equations whose span gives rise to a vector space over $\mathbb{Q}(q)$ that is finite dimensional. This finite dimensional space is an invariant of the closed three–manifold.

[13] I will often write as if $\widehat{Z}$ is an invariant of three–manifolds but I want to be clear that currently I don't know a proof of this or even a proper definition. Potentially the methods used for computations could be made into a more precise definition but unfortunately the current attempts to construct R–matrices [154] require restriction on the braids which could lead to disconnected set of combinatorial data where it can be defined and give rise to different q–series for knots. Then surgery giving rise to invariants is another question.

Theorem. *(Theorem 1) If M is an closed manifold obtained by surgery on the framed link L then for the coloured Jones polynomial $\widetilde{J}$, σ_L the signature of the link and $q=\mathbf{e}(a/c)$ taking*

$$\begin{aligned} Ш(M,L,b;q) \;&=\; \left(\frac{2\mathbf{e}(-1/8)(q^{1/2}-q^{-1/2})}{\sum_{k=1}^{4c} q^{k^2/4}}\right)^{\ell} \mathbf{e}\Big(\frac{-3\sigma_L}{8}\Big)\, q^{\frac{3\sigma_L}{4}} \\ &\qquad \times \sum_{N_1,\cdots,N_\ell=1}^{c-1} \widetilde{J}_{N_1,\cdots,N_\ell}(L;q) \prod_{j=1}^{\ell} \frac{q^{b_jN_j/2}-q^{-b_jN_j/2}}{q^{1/2}-q^{-1/2}}\,, \end{aligned}$$

the space

$$\Upsilon_M \;=\; \mathrm{Span}_{\mathbb{Q}(q)}\{Ш(M,L,b;q) : b\in\mathbb{Z}^{\ell}\}$$

is an invariant of M. Moreover, Υ_M is finite dimensional.

Finding a solution to one of these sets of q–difference equations in q–series should then give something related to the three–manifold in a similar way to the construction of Gukov–Manulescu for knots and their two variable series. In examples, solving these q–difference equations gives rise to families of q–series invariants that specialise to $\widehat{Z}$ much in the same way the descendant Kashaev invariants specialise to the Kashaev. This search for solutions to q–difference equations is related to the belief that q–difference equations from three–manifolds are modular in the sense of [82].

It is natural to ask how the WRT invariant and the $\widehat{Z}$ invariant are related more explicitly. Previously [92], it was thought that the $\widehat{Z}$ series had radial limits giving the WRT invariant. This seems not to be the case. However, as a definition is currently not available, this is a slightly tricky statement to make. Regardless, for q–series associated to simple manifolds, we can find q–hypergeometric formulae for these conjectured invariants. These series can then be proved to grow exponentially as q approaches a root of unity. These formulae allow us to take a state integral that factorises at the rational numbers in terms of the WRT invariant and in the upper half plane as the $\widehat{Z}$ invariant in a completely analogous way to the situation for knots discussed in [70]. For half surgery on the figure eight knot, we have the following theorem.

Theorem. *(Theorem 8) The WRT invariant of $4_1(1,2)$, given by*

$$Ш(q) \;=\; \sum_{0\le\ell\le k} (-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)} \frac{(q;q)_{2k+1}}{(q;q)_\ell (q;q)_{k-\ell}}\,,$$

and the $\widehat{Z}$ invariant

$$\widehat{Z}(q) \;=\; \sum_{0\le k\le\ell} (-1)^{k+\ell} q^{\frac{1}{2}3k(k+1)+\frac{1}{2}\ell(\ell+1)-k} \frac{(q;q)_\ell}{(q;q)_{2k}(q;q)_{\ell-k}}\,,$$

are part of a vector–valued quantum modular form whose cocycle is given by a matrix of state integrals one of which is given by

$$\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(3z_1(z_1+1+\tau)/2\tau + z_3(z_3+1+\tau)/2\tau - z_1(m+1+(m'+1)/\tau))\Phi_S(z_3+1+\tau;\tau)}{(1-\mathbf{e}(z_1/\tau))(1-\mathbf{e}(z_3/\tau))\Phi_S(z_3-z_1+\tau+1)\Phi_S(2z_1+1+\tau;\tau)}\, dz_1 dz_3\,.$$

What about three–manifolds?

Many of the observations about the behaviour of quantum invariants of three–manifolds seem to stem from properties of a, most likely, larger class of functions. These functions are proper q–hypergeometric functions. These functions can be expressed in terms of q–Pochhammer symbols and have the form

$$\sum_{k} \mathbf{e}(\nu(k)) q^{Q(k)+\mu(k)} \prod_{j} (q;q)^{\pm}_{\lambda_j(k)}\,,$$

where Q is a quadratic form and μ, ν, λ are linear forms. Indeed, all the proofs of quantum modularity follow from properties of the q–Pochhammer symbol all essentially coming back to the Faddeev quantum dilogarithm[14]. For example, the sums studied by Nahm give rise to quantum modular forms, though there is no reason that they should be related to a three–manifold.

Theorem. *(Theorems 5 and 6) For $A \in 2\mathbb{Z}$, the q–series*

$$f_A(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{\frac{A}{2}k(k+1)}}{(q;q)_k}\,,$$

and, for

$$1 - X_i \;=\; X_i^A\,,$$

the functions such that for $q = \mathbf{e}(a/c)$

$$f_{A,i}(q) \;=\; \frac{\prod_{\ell=1}^{|c|-1} \left(1-q^{\ell}\right)^{\frac{\ell}{|c|}-\frac{1}{2}}}{\sqrt{|c|X_i/(1-X_i) + A|c|}} \sum_{r=0}^{|c|-1} \frac{q^{Ar(r+1)/2} X_i^{Ar/|c|} X_i^{A/2|c|}}{\prod_{s=0}^{|c|-1} (1 - q^{r+s+1} X_i^{1/|c|})^{\frac{r+s+1}{|c|}-\frac{1}{2}}}\,,$$

are parts of vector–valued quantum modular forms who's cocycle is given by a matrix of state integrals one of which is given by

$$\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(Az(z+1+\tau)/2\tau)}{\Phi_S(z+1+\tau;\tau)}\,.$$

I expect that similar properties will hold for a large class of these proper q–hypergeometric functions. These Nahm sums are the second set of non–trivial infinite families of quantum modular forms I know. The other family is Heine's q–hypergeometric functions studied in [82]. I also expect that many of the properties observed will hold for deformations of q–hypergeometric functions. Indeed, the functions introduced for the figure eight knot $\mathfrak{G}(q)$ come from a deformation of $g(q)$ however on some cone as opposed to the full lattice[15]. To illustrate how this works I provide a proof that the deformation of the Roger–Ramanujan functions gives rise to a usual Jacobi form.

[14]To quote Kashaev: "The most beautiful function in the world". Les Diablerets, 2023.
[15]Recall the comments on mock modularity.

Theorem. *(Theorems 2 and 3) The function*

$$g(x;q) \;=\; \begin{pmatrix} q^{-\frac{1}{60}}G(x;q) \\ q^{\frac{11}{60}}H(x;q) \end{pmatrix} .$$

where

$$G(x;q) \;=\; \frac{(qx;q)_\infty}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}\frac{q^{k^2}x^{2k}}{(qx;q)_k}\,, \quad \textit{and} \quad H(x;q) \;=\; \frac{(qx;q)_\infty}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}\frac{q^{k^2+k}x^{2k+1}}{(qx;q)_k}\,,$$

is a vector–valued Jacobi form of weight 0 *and index* 2 *so that,*

$$\begin{aligned}
g(z+1;\tau) &= g(z;\tau)\\
g(z+\tau;\tau) &= q^{-1}x^{-2}g(z;\tau)\\
g(z;\tau+1) &= \begin{pmatrix} \mathbf{e}\big(-\frac{1}{60}\big) & 0 \\ 0 & \mathbf{e}\big(\frac{11}{60}\big) \end{pmatrix} g(z;\tau)\\
g\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big) &= \mathbf{e}\Big(\frac{z^2}{\tau}\Big)\frac{2}{\sqrt{5}}\begin{pmatrix} \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} g(z;\tau)\,.
\end{aligned}$$

The recipe for proving quantum modularity of q–hypergeometric functions is as follows:

(1) take your q–hypergeometric function

$$\sum_k q^{Q(k)+\mu(k)}\prod_j (q;q)^{\pm}_{\lambda_j(k)}\,,$$

and, using some q–series identities, write this in terms of functions that make sense when $|q|\neq 1$ multiplied by modular forms,

(2) Take a state integral of the same form as your sum where you replace the sum by an integral and the Pochhammer symbols by Faddeev quantum dilogarithms, which — if the expression is as above — is

$$\int \mathbf{e}(Q(z)/2\tau+\mu(z)/\tau)\prod_j \Phi_S(\lambda_j(z);\tau)^{\pm}dz\,,$$

(3) factorise the state integral using the residue Theorem [73] of the fundamental lemma of [72],

(4) prove the quadratic relations using Wilf–Zeilburger theory [194],

(5) combine this all together as a vector or matrix valued equation using the various identities along the way.

Every step is completely clear besides the first. In this thesis, I do not tackle the question of when such a procedure can be carried out, however in all of our examples will illustrate how such a procedure works.

Resurgence and quantum modularity

Before closing the introduction for a more detailed description of the contents of each section, an important set of conjectures relating to resummation of asymptotic series of q–hypergeometric sums has emerged in recent years and should be discussed. If you take a random sequence of elements on the unit disk a_n, then the series

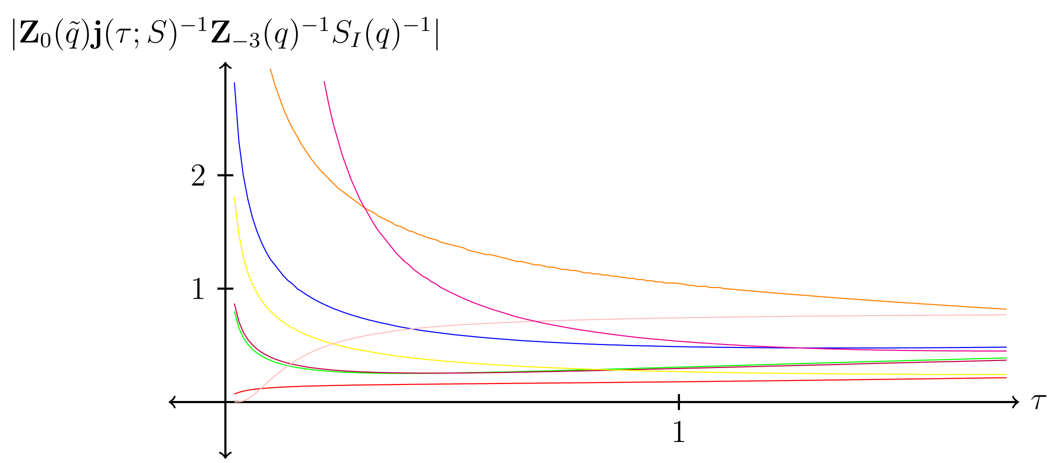
$$f(x) \;=\; \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

will converge to an analytic function on the unit disk but, by a theorem of Borel, will almost always have no analytic continuation for $|x| \geq 1$. Often we call $|x| = 1$ the natural boundary of such a function. If one instead assumes that f is analytic on almost all rays centred on the origin with sub–exponential growth, then we can take the Laplace transforms along any ray where the function is analytic and find an asymptotic expansion

$$\int_0^{\infty} \exp(-\zeta) f(\zeta\xi) d\zeta \;\sim\; \sum_{n=0}^{\infty} n! a_n \xi^n \,.$$

Assuming that f is nice enough, the function on the left will have analytic continuation to at least some half plane. Resurgence is related to an inverse problem of finding an analytic function with prescribed asymptotics. The main point is that after applying the Borel transform, which divides each term in the expansion by a factorial, we look for endless analytic continuation. If this exists with sufficient bounds at infinity we can take the Laplace transform and find the desired function. This process is called Borel resummation. If this works, it of course depends on the argument of ξ. If we cross an isolated singularity of the integrand of the Laplace transform, the integral will jump by the keyhole integral around the ray emanating from the singularity (or branch point). These keyhole integrals can then be described in terms of other functions with their own asymptotic expansions. We can apply the same analysis to these functions and find new functions. If this process of constructing functions and asymptotic series closes up with some finite set of asymptotic series, then our original asymptotic series is called resurgent. For resurgent series, at each of the singularities in the integrand of the Laplace transform, we find that the keyhole integrals are related to this finite set of functions constructed with some additional constant. These constants are called Stokes constants. The theory of resurgence is interested in computing these sets of asymptotic series, all related each other, and these Stokes constants. Fundamentally, this theory relies on analytic continuation.

In relation to q–hypergeometric functions, Garoufalidis [67] conjectured that generating series of WRT and Kashaev invariants should have endless analytic continuation with explicit singularities. This is equivalent to the more usual description of resurgence we have just described. Then, in recent work [69, 68, 70], it was understood how to conjecturally compute both the Borel resummation of asymptotic series associated to simple knots and also all of the Stokes behaviour. While both of these conjectures are related to knots, it appears that they hold more generally for asymptotic series coming from q–hypergeometric functions. To compute the Borel resummation and Stokes phenomenon associated to asymptotic series coming from q–hypergeometric functions, one can use quantum modularity. Instead of using the cocycle Ω , you can use the Borel resummation of the asymptotic series and conjecturally find an expression of quantum modularity giving a similar equality to the one with Ω. This then gives an expressions for the Borel resummation in terms of Ω. Then using $\Omega_{\pm\gamma}$ just above and below the reals, we can compute the Stokes phenomenon by taking a quotient of the two cocycles. The quantum modularity shows that this Stokes automorphism will be a q–series. Then conjectures on the behaviour of the Stokes constants and the series arising from resurgence imply that these q–series are generating functions for the Stokes constants. This is extremely practical and easy to compute in simple examples. However, this remains conjectural besides simple examples like the Faddeev quantum dilogarithm [75]. For knots these Stokes constants are then conjectured to coincide with a particular value of the $3d$–index of a knot. This carries its own definition from the combinatorial data of a triangulation. It would be interesting if the definition could be extended to include the Stokes constants coming from closed three–manifolds and the full set of Stokes constants studied in [70]. I include this conjectural computation for half surgery on the figure eight knot. As far as I know, this is the first closed hyperbolic three–manifold whose Stokes constants associated to quantum $\mathrm{SL}_2(\mathbb{C})$ Chern–Simons invariants have been computed. We close the introduction with a picture of the absolute value of the resummation associated to the WRT invariant of $4_1(1,2)$. In fact, the plot is of its cocycle from Theorem 8, which agrees with the resummation to all computable accuracy, which is much more than the naked eye can see.



Outline

The first part gives background in the topology and geometry of three–manifolds. It also includes an introduction to quantum invariants of three–manifolds with a focus of $\mathfrak{sl}_2$ invariants.

The part starts by defining knots and their diagrams. Examples, notation and equivalences between diagrams (Reidemeister moves) is discussed along with a definition of linking matrix. A description of some of the algebraic structures associated to knots are then discussed, which will be used later in defining quantum invariants. In particular the braid groups, Alexander's theorem (that all links come from braid closure) and Markov's equivalence theorem are given. Then the construction of three–manifolds via gluing is discussed explaining that all three–manifolds arise from gluing handlebodies or from surgery on links. Then Kirby's calculus used from describing when two manifolds obtained from surgery of a link are the same is described. Then the fundamental group of link complements is calculated and also their representations, which give rise to an algebraic curve associated to knots called the A–polynomial.

Next a thorough description of ideal triangulations is given. This gives the basic computational tools for hyperbolic geometry in three–dimensions. Firstly, topological considerations are given. Then we describe an algorithm that can be used to triangulate any knot complement. Next a brief description of the geometry of three–manifolds following Thurston's program of geometrisation, with the inclusion of the rigidity theorem for complete hyperbolic structures. The geometric structures of ideal tetrahedra are then discussed including their moduli, symmetries and volumes. Then Thurston's gluing equations are discussed, which gives a way to construct hyperbolic structures from ideal triangulations. The volume of the figure eight knot complement is given as a full example.

Then the study of the geometry of three–manifolds is taken up in full. Firstly, the combinatorial data of Thurston's gluing equations are considered and put into matrices called the Neumann–Zagier matrices. The theorem that these are half symplectic is quoted and the example of their computation for the figure eight knot is given. Then computer computations for some other simple knots is also given. Then Pachner moves are discussed and the various equivalences between Neumann–Zagier data. Then less combinatorial things are considered as we investigate the Chern–Simons functional, which requires a discussion on connections on three–manifolds. Then to understand this more computationally we go on to the Bloch group, which gives an algebraic way to consider triangulations and relates to algebraic K–theory. Various examples of explicit elements are constructed including in Neumann's description of the extended Bloch group. Next, using the Bloch group, a description of volumes and the Chern–Simons invariant is given and computations included for some simple closed manifolds and knots.

The next section discusses quantum invariants. A historical description of the Alexander polynomial until the discovery of the Jones polynomial is given. Then a vague discussion of Witten's physical theory is used to outline a hopeful mathematical theory extending the Jones polynomial to a much larger set of invariants. Then the coloured Jones polynomial will be defined by representations of quantum $\mathfrak{sl}_2$ with emphasis given on explicit formulae that can be used to compute. Various basic properties will be shown such as framing shifts and the computation of the coloured Jones polynomial of the figure eight knot will be given. Then the structural properties observed by Habiro will be discussed including the cyclotomic expansion and formulae for twist knots will be given. Then the recursions for the coloured Jones polynomial or $\hat{A}$–polynomial of a knot will be discussed and given for the figure eight knot along with a brief discussion of the AJ conjecture relating this to the A–polynomial. With all that, we can then define the WRT invariants of closed manifolds via surgery using the coloured Jones polynomials. We give examples for surgery on twist knots. Then we discuss operators in TQFT and use this to justify the construction of modules associated to links in closed three–manifolds. We then show that while these modules are not invariants, their span is. This then specialises to the proof that the WRT invariant is unchanged under Kirby moves. We close with some questions related to this seemingly new vector space associated to closed three–manifolds.

While the previous section describes (with the addition of some ideas of Witten) the mathematical theory and construction of invariants, this section gives the semiclassical story, which while clear from the physical perspective, is not so from the mathematical. The justification of the expectations from physics using infinite dimensional integrals is initially given along with Witten's asymptotic expansion conjecture, which relates the asymptotics of the WRT invariant at $\exp(2\pi i/N)$ for $N \in \mathbb{Z}$ to the values of the Chern–Simons invariant for SU(2) connections. An example is discussed and some pictures given, which give some vague feeling for these conjectures. Next Kashaev's volume conjecture on the asymptotics of knot invariants is discussed, which takes the story from SU(2) to $\mathrm{SL}_2(\mathbb{C})$. Then the Chen–Yang volume conjecture, which discusses the asymptotics of WRT invariants at other roots of unity. Then Vassilev invariants are discussed and the Melvin–Morton–Rozansky conjecture (Bar Natan–Garoufalidis theorem). Then after discussing the trivial connection, we move to the non–abelian connections and in particular the geometric. We describe the conjectured invariants defined by Dimofte–Garoufalidis and discuss their relation to the asymptotics of the Kashaev invariant. We close the part discussing the analogous series for closed manifolds, which were conjectured by Gang–Romo–Yamazaki.

The second part takes up the study of asymptotic series and starts by considering numerical methods. Then it considers some relations to differential equations and resummation. We then discuss the asymptotics of q–hypergeometric functions and particular depth is given to the Pochhammer symbol.

First extrapolation methods are discussed. Some brief recollections on asymptotics is given defining the o, O notation and the notion of asymptotic series given. Next the Richardson–

Zagier methods are discussed and compared. Variants on these methods are given, illustrating how to numerically find the asymptotic behaviour of sequences. Then a new variant that can be used to detect oscillating asymptotics is given and applied in the example of the Airy function.

Then we explore divergent series. Firstly, we describe linear differential equations, describing them as modules of the Weyl algebra. Then their Wronskians and companion matrices are given with a description of constants. The Apéry numbers are used to describe hypergeometric sums. Next the Frobenius algorithm for finding a basis of solutions to linear differential equations is given and a discussion on divergent series and exponential singularities is given. The exponential integral is used as an example of these methods. Secondly, we consider optimal truncation of divergent series. This is discussed in the more classical set up and then using the framework of smooth optimal truncation we discuss the methods of refining the truncation. Thirdly, we describe Padé approximation. The extension of Taylor series is discussed. Then using Gauss's hypergeometric function ${}_2F_1$ and his continued fraction we give the example of the Padé approximation of the logarithm. Finally, we describe Borel resummation. This involves a discussion of the Laplace transform and Watson's lemma is given. This is applied to the exponential integral example. Then the singularities in Borel plane and their leading to Stokes discontinuities is described and the Stokes matrix of the exponential integral is given.

The study then turns to q–hypergeometric functions. To start, we go through various methods that can be used to study asymptotics of series such as Euler–Maclaurin formula, Poisson summation, Abel–Plana summation and Laplace's method for the asymptotic of exponential integrals. Next, the dilogarithm is discussed. More generally, polylogarithms are introduced and the various identities satisfied by the dilogarithm are given. Next, we apply the methods of the previous sections to the Pochhammer symbol. To do this, we use a lemma for the logarithm of the Pochhammer symbol. The first version of the asymptotics is for generic parameters and the proof from Euler–Maclaurin is given. This is then used to describe the asymptotics of the Pochhammer with a Möbius transformation. This involves some kind of cyclic dilogarithm. Then applying Abel–Plana, we get an integral formula for the Pochhammer symbol, which will be crucial for quantum modularity later. Again the Möbius transformed version is given. Then using the integral formula, the asymptotics in the non–generic case is proved, which while clear from previous work has seemingly an explicit formula of which seemingly isn't given in previous literature. A corollary is a computation of the asymptotics of the η function. This is also considered for the Möbius transformed versions. This gives a formula for the multiplier system of the η function.

The next section turns to the study of q–hypergeometric functions as they approach roots of unity. Firstly, the general behaviour of such asymptotics is discussed with important emphasis given on number theoretic aspects. Next, the behaviour of the Borel transforms is described and the asymptotics of the coefficients of such series. The first examples are then given by simple Nahm sums. The Rogers–Ramanujan functions are discussed in this perspective. Then a similar Nahm sum is given and discussed in detail as $q \to 1$. Then the behaviour as q tends to other roots of unity is given. Next, we turn to simple knots and describe the behaviour of the trefoil and figure eight knot. The figure eight knot includes a discussion on the missing q–series that was described in the introduction and discusses its asymptotics. Next, we turn to a closed hyperbolic three–manifold, half surgery on the figure eight knot. The asymptotics of the WRT invariant and $\widehat{Z}$ invariant are discussed along with the behaviour of the various asymptotic series.

The third part concerns q–difference equations. These will provide some of the algebraic structures underlying the theory.

We start by defining linear q–difference equations in parallel to the discussion on differential equations. Wronskians and companion matrices are discussed and also the constants, which turn out to be elliptic functions. Then duals of various modules are discussed, which are implicit in the quadratic relations used in the proofs of quantum modularity. Next, the Pochhammer symbol and the θ–functions are discussed. Some of the most important identities for q–series manipulations are given in the infinite form of the q–binomial theorem. This is used to give a short proof of the Jacobi triple product identity for the θ–function. Then the asymptotics of the Pochhammer symbol are considered and the analogous results for these functions at roots of unity are given.

Then we describe the Frobenius method for solving q–difference equations as q–series or when q is close to 0. This involves describing the Frobenius Ansatz and a discussion on conventions, which arise from the fact constants are elliptic functions. An example from the figure eight knot is considered. The algorithm can produce divergent q–series and so q–Borel resummation is discussed. Then the theorem proving the existence of bases of solutions is given for q–difference equations. A brief discussion of monodromy is then given. We close with a small discussion on proving q–series identities.

Next, we solve q–difference equations when $q \sim 1$. This involves the WKB Ansatz. Firstly, we solve with general parameter, which involves towers of differential equations. Next we consider we the parameter is also near 1, which involves simple linear algebra. Secondly, we discuss the Habiro ring. This gives a ring that potential solutions to q–difference equations can live. The various rigidity results are given and the Kontsevich–Zagier function is studied as an example. Thirdly, we describe new types of functions that behave in a similar way to elements of the Habiro ring. These led to different kinds of solutions to q–difference equations that are homogeneous while the Habiro ring is naturally inhomogeneous. Finally, simple identities for elements of the Habiro ring are discussed that are crucial for quantum modularity of elements of the Habiro ring.

Next, we study q–hypergeometric functions and their q–difference equations. Starting with rank one Nahm sums, we consider their difference equations. We then prove various duality results for these sums. Using these we discuss the various identities that these imply between functions. Then we discuss various equalities between Nahm sums related to the equivalences between Neumann–Zagier matrices. Next we turn to a discussion on deformations of q–hypergeometric functions. Then deformations of the Rogers–Ramanujan identities are given in great detail.

Then we turn to knots. The descendant Kashaev is defined and a discussion of the various associated q–difference equations is given. We discuss solutions at roots of unity. Moreover, the missing q–series is put into this framework. The we go on to holomorphic blocks. These are q–series satisfying the $\hat{A}$–polynomial. These satisfy a submodule of the full module and this is then used to introduce the two variable series of Gukov–Manolescu. Next, a brief discussion on dualities associated to knots and lack of in come cases.

Next, we turn to closed manifolds. The $\widehat{Z}$ invariant is introduced from the two variable series by applying a Laplace transform. Then half surgery on the figure eight is considered in detail and its WRT invariant is studied. Then the $\widehat{Z}$ invariant is shown to come in a family satisfying the same q–difference equations and a q–hypergeometric formula is given that is related to the formula for the WRT invariant. We close by giving the self duality explicitly in this example.

The fourth part involves modularity. We start from classical theory and end in the quantum.

Classical elliptic modular forms are discussed first. The modular group and its various actions are discussed. Modular forms are defined and their finite dimensionality is discussed. Next, the Eisenstein series are defined. The expansion of the infinite Pochhammer symbol is given and related to the Eisenstein series. Then an efficient computation of the q–series expansions of the Eisenstein series is given. This is most helpful for odd Eisenstein series. The transformation of the second Eisenstein series is also given. Then we move to the infinite Pochhammer symbol or the Dedekind η–function. Its multiplier system is discussed and Ramanujan's Δ function is also discussed. Then the θ–function is discussed and its modularity is proved with Poisson summation. Next, vector–valued modular forms are studied and the θ–function is given as a simple example. Automorphy factors are defined and then the Rogers–Ramanujan functions are described in this set up. Then a discussion on modularity of q–hypergeometric functions is given and Nahm's conjecture described along with an analogue for three–manifolds.

Jacobi forms are discussed next. This starts with a discussion on elliptic functions from the θ–function and the Weierstrass $\wp$–function. The modularity of the Weierstrass $\wp$–function is given. This leads to the definition of Jacobi forms following [56]. The proof that the θ–function is a Jacobi form is then given again using Poisson summation. Then vector–valued θ–functions are considered and their modularity given. The construction of modular forms from a Jacobi form and using the second Eisenstein series to fix quasi–modularity is then discussed. The deformation of the Rogers–Ramanujan functions is shown to be a Jacobi form. Deformation of q–hypergeometric modular forms is then discussed along with q–Borel resummation of functions expected to be modular.

After this, we leave the more classical aspects and go to mock modular forms. Asymptotics of Ramanujan's original examples are considered. Then Appell–Lerch sums are studied and their asymptotics. The modular properties of these sums are then given and the description of the Appell–Lerch sums as the q–Borel resummation. Next, the inhomogeneity of the q–difference equations of mock modular forms is discussed and we give a brief method of constructing the $O(1)$ series of Ramanujan from nothing but the inhomogeneous q–difference equation. We close the discussion on a conjectural new order seven mock modular form that arose in studies with Matthias Storzer.

Then, we finally come to quantum modular forms. We start by describing modular forms at roots of unity. We extend the Eisenstein series to $\mathfrak{h}\cup\mathbb{Q}\cup\overline{\mathfrak{h}}$. Then, we consider the multiplier system of the η function and its modularity properties. Using this we give an extension of Ramanujan's Δ function to $\mathfrak{h}\cup\mathbb{Q}\cup\overline{\mathfrak{h}}$ and describe its automorphy factor. Then, we consider the Rogers–Ramanujan functions at roots of unity. After this, we consider the original description of quantum modular forms and the relation to improved analyticity properties. Using this, we give a description for q–series to be quantum modular along the same lines and prove this for rank one Nahm sums. Then, we turn to the study of the WRT invariant of half surgery on the figure eight knot. We describe how the various conjectures on asymptotics appear and how modularity similar to the original observations of Zagier for the Kashaev invariant also holds here. Importantly, we describe how Witten's asymptotic expansion conjecture and Chen–Yang's volume conjecture are unified by quantum modularity. Then, we prove this form of quantum modularity for this example. We then go on to refine this form of quantum modularity along the lines of Garoufalidis and Zagier [86].

The refinements of quantum modularity lead to discussions of analytic cocycles. We describe non–commutative first group cohomology and how this is used to describe $\mathrm{SL}_2(\mathbb{Z})$ cocycles valued in analytic functions. Various examples are given and some rigidity results are also discussed. Then we turn to some kind of Jacobi version that incorporates q–difference equations. Rigidity results are also given in this context. Using these analytic cocycles we give the working definition of quantum modular forms for this thesis following roughly the ideas of [201, 202]. Similarly, Jacobi versions are also described compatibly with the modular q–holonomic modules of [82]. Then, we go through the various theorems on the quantum modularity. First, for rank one Nahm sums, then the figure eight knot, and then for half surgery on the figure eight knot.

Quantum modularity of the Pochhammer symbol is then studied in detail. The main tool is the Faddeev quantum dilogarithm and we start with its definition. Then, we go onto to the difference equations it satisfies and its modular properties. Then the analytic properties of the quantum dilogarithm are given and the asymptotic behaviour. Then the various formulae for the dilogarithm are described. The symmetry is given and then finally the descriptions of the Fourier transforms are given. These are the analogues of the q–binomial theorem. Next, we briefly discuss the cocycle of the Pochhammer symbol related to the modular quantum dilogarithm of [76]. The dilogarithm is then discussed at rational numbers, which uses the cyclic dilogarithm.

State integrals are the main technical tool for proving quantum modularity. These are discussed in the last part. These are introduced with comparisons to Gaussian integrals and then the Mordell integral.

Firstly, we describe how to factorise state integrals at rationals. This relies on the fundamental lemma in [72]. This is extended to any dimension and the Mordell integral is considered at rationals. Then, we go onto to factorise the state integrals at rationals of the rank one Nahm sums, the trefoil knot, the figure eight, and then a toy example used to illustrate the final example of half surgery on the figure eight knot.

Then, we go onto to show how to factorise state integrals in terms of q–series. This is illustrated in terms of a proof of the analogue of the q–binomial theorem. We then discuss untrapping and run parallel computations between q–series and state integrals. Then we go onto factorise the state integrals associated to rank one Nahm sums, the trefoil and the figure eight knot. We use the figure eight knot, as an example of computing q–difference equations associated to state integrals. Then, we go onto explain how to factorise the state integrals associated to half surgery on the figure eight knot.

The final section discusses various conjectures relating to Borel resummation of the asymptotic series that come from state integrals and q–hypergeometric functions. This starts with computing the quantum modular properties with Borel resummation of a Nahm sum and the $\widehat{Z}$ series associated to half surgery on the figure eight knot. Next, we describe the general conjectures related to resurgence and the Stokes phenomenon coming from [69, 68]. Along with the quantum modularity, these conjectures allow conjectural computation of all of the Stokes constants. This is illustrated in the two examples and importantly the example of half surgery on the figure eight knot.

Some questions for the future are then given. We close with a large collection of mostly PARI/GP [20] programs of various clarity associated to the various sections of the thesis.

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So secondly, Ale, Dani and Séverin, you all gave me the great sense of excitement and drive in maths. I remember that Gaëtan had introduced a problem related to geometric recursion and outlined roughly how a solution could be given. I remember having absolutely no idea what was going on and all four of us in 404 (even if it wasn't the page we were looking for) standing at the board with you all seemingly having perfect grasp of the problem and me absolutely in the dark. I remember asking simple after simple questions until I finally understood the problem. I then remember working through examples with Ale until we understood the pattern and once this was done the proof. Ale I still miss being able to jump up in my desk and say do you wanna go to the blackboards. Séverin I think you were always immediately on the same wavelength and always explained things in a way that I understood. Dani I think you shared the most of the many things you know and taught me a lot. To all of you, while much of this thesis doesn't overlap with our work, your contribution is clear to me. I'm happy that we got a whole year to draw an uncountable number of graphs in surfaces. Also to Elba, Jørgen, Nezhla, Nitin, Reinier, Remi, Vincent. All of you have been great sources of discussion, whether it be about a practical problem we wanted to get to the bottom of or just for the fun of discussing some new or old mathematics. You each have my thanks.

Each year of this doctoral project was quite different. After the first year we of course landed in a global pandemic. So thirdly, Stavros. While we discussed somewhat in the first year we got deep into discussions in the second year. Your misfortune of being stuck in Bonn for months ended up being, for me, a great fortune. I remember your patience in organising

the chaos of my initial computations. You have shared and taught me so much over the following years and we have understood many interesting things. You have showed me how to really do maths and keep your head down in a problem but also to come up for air and look to the horizon. You have so many fantastic ideas and I hope to get to the bottom of many more of them together. Much of this thesis is based on ideas that you have developed over the last ten years and I have been glad to learn this beautiful collection of remarkable results and conjectures. You also introduced me to the collaborators who, besides you and Don, have had the most direct impact on this thesis.

So fourthly, Marcos and Jie. It is of course a sign of the times that Marcos we only met in person last week and that Jie we are still to meet in person. Your generous sharing of ideas and perspectives have been very important in the mathematical content of the theory developed in this thesis. I can't recall if you were quoting someone else or whether it was your own Marcos but I still keep the phase "shut up and compute" constantly in my mind. Indeed, the final steps in some of the proofs involved me thinking and thinking that things wouldn't work and I asked Matthias should I think more or as Marcos would say shut up and compute and after his suggestions of following Marcos I of course found the computations turned out better than I imagined and furnished the proof I needed.

So as already mentioned fifth, Matthias. I hope I have been somewhat of a good big brother! I'm sad that we didn't get to spend the majority of our studies in a normal situation but have been glad for our discussions throughout the pandemic and beyond. Thank you for getting up in the dark and discussing while I was in Australia. The constant sharing of ideas and thoughts has been invaluable and I have been glad to actually spend time in the office together after two years. Looking forward to writing up the countless the observations we have made together. Also to Michael, thank you for the discussions and sharing your ideas when I first arrived in Bonn.

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Last but not least of the mathematicians, Don. Many of these amazing people have been brought together around you and your guidance. I still have my notes from our first meeting in person where we discussed the literature, which was split into two. Both had a heading Borot and Garoufalidis. The next weeks I spent deep diving into literature and I recall we discussed many more times about problems initially a lot to do with the Bloch group. I then recall going for coffee where you asked if I was interested in checking for some of the same things you had been observing with Stavros for a closed manifold. I said I was interested and the result of that conversation has led to this thesis. Since then, I think we have clarified many aspects of the theory. It has been fun and exciting to be there alongside these developments and make contributions. At the very least, the outcome of this thesis answers your question in the affirmative. From you, I have learnt not only amazing mathematical

ideas but similarly how to think about maths in a larger context. Your generosity with your ideas and problems has given me so many wonderful things to think about. I am also glad for the various discussions on typesetting, coding, language, or the history of mathematics and I find these an absolute joy. I look forward to many more discussions to come.

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Notation and terminology

This section contains some notation but maybe not all. Use it as a guide but not as gospel. The most basic and important notation that will be used through the entire thesis is

$$\mathbf{e}(x) \;=\; \exp(2\pi i x)\,, \quad q \;=\; \mathbf{e}(\tau)\,, \quad \tilde{q} \;=\; \mathbf{e}(-1/\tau)\,, \quad q^{a/c} \;=\; \mathbf{e}(a\tau/c)\,.$$

We will also use the notation

$$\tilde{q}_\gamma \;=\; \mathbf{e}\Big(\frac{a\tau+b}{c\tau+d}\Big)\,, \quad \tilde{\tau}_\gamma \;=\; -\frac{|c|}{c^2\tau+cd}\,.$$

Of similar importance are

$$(a;q)_n \;=\; \prod_{j=0}^{n-1}(1-aq^j)\,, \quad \binom{n}{k}_q \;=\; \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\,.$$

We take $\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$ as the integers, the rationals, the reals and the complex numbers as usual. For all but the complex numbers we take $\mathbb{K}_{\leq a}$ and similar combinations to mean the elements less than or equal to a *e.g.* $\mathbb{Z}_{\leq 0}$ is the non–positive integers. If $\mathbb{K}$ is a field then $\mathbb{K}(x_1,\dots,x_N)$ is the field of rational functions in x_i over $\mathbb{K}$. The sets of matrices that appear are

- $\mathrm{SL}_n(\mathbb{K})$ the special linear group of matrices with determinant 1.
- $\mathrm{PSL}_n(\mathbb{K})$ the projective special linear group consisting of matrices with determinant 1 over $\mathbb{K}$ modulo the centre.
- $\mathrm{Sp}_{2N}(\mathbb{K})$ the group of symplectic matrices over $\mathbb{K}$, that is the group of matrices preserving the standard symplectic form on $\mathbb{K}^{2N}$.
- $M_{N\times M}(\mathbb{K})$ the $N\times M$ matrices over $\mathbb{K}$.
- $\mathrm{GL}_N(\mathbb{K})$ the general linear group of invertible $N\times N$ matrices over $\mathbb{K}$.

Besides this, the notation in order of first appearance is now listed.

- S^n the n-sphere.
- $3_1, 4_1, 5_1, 5_2, \cdots$, where a_b is the b-th knot with some purely historical numbering with at least a crossings when expressed as a diagram.
- $\mathrm{lk}(L_1,L_2)$ the linking number between two components L_1, L_2 of a link L. For a framed link $\mathrm{lk}(L)$, is the linking matrix.

- B_n braid group on n strands, $\iota_n : B_n \to B_{n+1}$ the canonical embedding.
- $\sigma_i \in B_n$ the generators of the braid group givin one half twist.
- π_1 is the fundamental group.
- G_L the fundamental group of the complement of the link L.
- A^K an A–polynomial of the knot K.
- $\mathbb{H}^n$ hyperbolic n–space.
- Isom the group of isometries.
- Δ_z ideal tetrahedron with shape parameter $z \in \mathbb{C}$.
- $z' = (1-z)^{-1}$ and $z'' = 1 - z^{-1}$.
- $\Lambda(x) = -\int_0^x \log|2\sin t| dt$ is Lobachevsky's function.
- $\mathrm{Vol}(M)$ is the volume of M.
- $D(z) = \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$ the Bloch–Wigner dilogarithm.
- $\mathrm{Li}_2(z) = -\int_0^z \log(1-w)\frac{dw}{w}$ the dilogarithm.
- $\Omega^k(M,V)$ the space of k–forms on M valued in V.
- $\mathcal{G}_P$ the gauge group of a principle bundle P, *i.e.* the group of automorphisms of P.
- CS the Chern–Simons functional.
- Tr the trace of a matrix and a representative of the Killing form.
- F_A the curvature of A.
- $\mathcal{R}(M,G)$ the representation variety of M in G, *i.e.* representations of the fundamental group of M into G up to conjugation.
- $c(x,y) = [x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)]$.
- $\mathcal{F}$ five term relations.
- $\mathcal{B}(\mathbb{K})$ the Bloch group of $\mathbb{K}$.
- μ the roots of unity and $\mu_{\mathbb{K}}$ the roots of unity in $\mathbb{K}$.

- $\widehat{C}$ the Riemann–surface of $(\log(z), \log(1-z))$
- $\ell(z;p,q) = (\log(z) + p\pi i, -\log(1-z) + q\pi i, -\log(z) + \log(1-z) - p\pi i - q\pi i)$
- $\widehat{\mathcal{B}}(\mathbb{K})$ the extended Bloch group of $\mathbb{K}$.
- $R(z;p,q) = \frac{1}{2}\log(z)\log(1-z) - \int_0^z \log(1-t)\frac{dt}{t} + \frac{\pi i}{2}(p\log(1-z) + q\log(z)) - \frac{\pi^2}{6}$
- $\Delta_L(x)$ the Alexander polynomial of a link L in variable x.
- $J_2(L;q)$ the Jones polynomial of L in variable q.
- $Z(M;\hbar)$ the partition function of Chern–Simons theory.
- $R : V \otimes W \cong W \otimes V$ a quantum R matrix.
- $(R, \mu, E, E_\mu, F, F_{\mu^{-1}})$ the local pieces associated to knots for quantum invariants.
- $\widetilde{J}_N(K;q)$ the coloured Jones polynomials of a knot.
- $(q^{N/2} - q^{-N/2})J_N(K;q) = (q^{1/2} - q^{-1/2})\widetilde{J}_N(K;q)$.
- $C_k(K;q)$ the cyclotomic coefficients of a knot K.
- $\gamma_{k,N}$ the change of basis taking the cyclotomic coefficients and the coloured Jones polynomial.
- σ_L the signature of the linking matrix, *i.e.* the number of positive eigenvalues minus the negatives.
- $\text{Ш}(M;q)$ the WRT invariant and the generators of the WRT module.
- $K(a,b)$ the a/b–Dehn surgery on the knot K.
- Z a TQFT functor.
- Cob_{n+1} the category of cobordisms.
- Vect the category of vector spaces.
- Υ_M the WRT module.
- VC complexified volume or Chern–Simons invariant.
- δ the one–loop invariant.
- $\langle L \rangle_N$ the Kashaev invariant.

- B_n the Bernoulli numbers.
- $\mathrm{Li}_k(z)$ the k-th polylogarithm.
- $\psi_\hbar$ asymptotics of the quantum dilogarithm.
- S the loop expansions of the formal series.
- Π the propagator.
- Γ vertex weights.
- Φ asymptotic series.
- $\sim, o, O$ asymptotic notations.
- $\mathbf{n}$ multiplication by n map of sequence in n.
- Δ the difference of a sequence.
- extrap_k the k-th order extrapolation operator.
- $S(s,k)$ the Stirling numbers of the second kind.
- $D_x = x\frac{\partial}{\partial x}$.
- $\mathcal{E}_b(x) = \int_0^\infty t^{|x|-b}\exp(-t|x|)\frac{dt}{t-\frac{x}{|x|}}$.
- $f[m/n](z)$ the Padé approximate of f with numerator of degree m and denominator of degree n.
- $\mathcal{B}_1$ the Borel transform of weight one.
- $\mathcal{L}$ the Laplace transform of weight one.
- $B_n(x)$ the Bernoulli polynomial.
- $f^{(n)}$ the n–th derivative of f.
- $\hat{f}$ the Fourier transform of f.
- Analytic continuation of $\zeta(s) = \sum_{k=1}^\infty k^{-s}$.
- $\Delta(m,x;q)$ and asymptotic cyclic dilogarithm.
- $\psi(\tau,m,z,M)$ the integral expression for the Pochhammer symbol.

- $\Phi_{a/c}(\hbar)$ the asymptotics of a q–hypergeometric function at $\mathbf{e}(a/c)$.
- D_n universal denominators.
- λ_γ the tweaking factor.
- $f_{A,m,n}(q)$ a Nahm sum.
- M a matrix of Stokes constants near the origin.
- $\mathrm{denom}(x), \mathrm{numer}(x)$ the denominator and numerator of $x \in \mathbb{Q}$ respectfully.
- $\widehat{Z}$ the expected invariant predicted for example in [92].
- σ_x a generator of the q–Weyl algebra or the operator $(\sigma_x f)(x) = f(qx)$.
- $W(f^{(1)}, \cdots, f^{(N)})$ the Wronskian.
- $M^\vee$ the dual of M in the usual sense.
- $M^\wedge$ the dual of M induced by the map $q \mapsto q^{-1}$.
- $\theta(x;q) = \sum_{k\in\mathbb{Z}} (-1)^k q^{k(k+1)/2} x^k$.
- $\varepsilon(q) = \sqrt{-i} \prod_{\ell=1}^{\mathrm{ord}(q)-1} \left(1 - q^\ell\right)^{\frac{1}{2} - \frac{\ell}{\mathrm{ord}(q)}}$ the multiplier system of the η function.
- $\mathrm{ord}(q)$ the smallest positive number n such that $q^n = 1$.
- $$\theta_\kappa(x;q) \;=\; \begin{cases} \theta(x^\kappa; q^\kappa) & \text{if } \kappa > 0\,, \\ 1 & \text{if } \kappa = 0\,, \\ \theta(q^\kappa x^\kappa; q^{-\kappa})^{-1} & \text{if } \kappa > 0\,, \end{cases}$$
- $\mathcal{B}_\kappa$ the q–Borel transform of weight κ.
- $\mathcal{L}_\kappa$ the q–Laplace transform of weight κ.
- $L(t,\lambda;q)$ the Appell–Lerch sum.
- M monodromy matrix.
- $\widehat{\mathbb{Z}[q]}$ the Habiro ring.
- $G_{A,B,r}$ a generalised Nahm sum.
- the Descendant coloured Jones polynomial $J_{N,m}$ with $J_{0,m}$ the descendant Kashaev invariant.

- $G_n(q)$ the n-th Eisenstein series with constant $\zeta(1-n)/2$.
- Ξ_K and F_K the two variable series associated to knots symmetrised or not.
- $|_k\gamma$ the slash action of weight k where $f|_k\gamma(\tau) = (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$
- $E_{2k}(q) = 2G_{2k}(q)/\zeta(1-2k)$
- $\eta(q) = q^{1/24}(q;q)_\infty$ the Dedekind η–function.
- $\vartheta_{ij}(q)$ classical modular θ–functions.
- $\mathbf{j}$ a multiplier system.
- $\wp$ the Weierstrass $\wp$–function.
- ϑ_κ vector–valued θ–function.
- Ω, Ξ analytic cocycles.
- $\mathbb{C}_\gamma$ the cut plane associated to $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
- $\Phi_{\mathsf{b}}(x)$ the Faddeev quantum dilogarithm in notation of [8].
- $\Phi_\gamma(z;\tau)$ cocycle of the Pochhammer symbol.
- Cyclic dilogarithm $D_M(x;q)$.
- $\mathcal{C}_\tau = i\sqrt{\tau}\mathbb{R}$ the contour used for state integrals
- $S_{\rho,\rho',k}$ a Stokes constant.

Remark 1. *We will consider certain q–hypergeometric sums known to me as Nahm sums. Recently, I became aware — thanks to remarks of Sergei Gukov — that these sums appeared earlier under the name fermionic sums [110]. In honour of Werner Nahm's contribution to the study of these sums, I will continue to use the name Nahm sums throughout the thesis, hoping that this leads to no confusion.*

Part I

Three–dimensional topology

Chapter 1

Combinatorics and geometry of three–manifolds

By the end of the 19–th century the topological aspects of surfaces were, at some level, well understood. Three–manifolds were a natural next step. Around the turn of the 19–th century, Poincaré famously made an initial guess that the homology of three–manifolds would determine them as for surfaces. None other than Poincaré himself constructed a counter example to this guess, now know as the Poincaré homology sphere. This led Poincaré to introduce the fundamental group, which is a non-abelian version of the first homology. This led to the question, more than a conjecture, of whether a three–manifold having a trivial fundamental group implies that it was the 3–sphere. This question became know as the Poincaré conjecture and over the 20–th century many proofs were given and all turned out to be flawed. The generalisation to dimensions higher than three was in fact done first. Smale and Freedman showed this is true in the topological category and Milnor showed that in the smooth category this was no longer true. Donaldson proved that there are infinitely many smooth structures on $\mathbb{R}^4$ but it is still unknown how many smooth structures exist on the 4–sphere. The first major step towards understanding this in three dimensions, as often happens in mathematics, was a vast and more detailed generalisation of this conjecture. This was introduced in the work of Thurston, which gave a picture of the structure of three–manifolds through geometrisation. Not only did Thurston describe generally how these structures should arise, but he gave concrete means to calculate them. Ultimately, Thurston's vision was proved a century after Poincaré's question by Perelman.

While we now have a picture of three–manifolds through geometrisation, what a three–manifold means in practice depends on the person you ask and in what context. Some will think of this purely abstractly as a space locally the same as $\mathbb{R}^3$ with some additional structure. Others may think of triangulations in terms of tetrahedra, the various combinatorics of how they can be glued, and which alterations can be made to get back the same three–manifold. Similarly, some may think of framed link via their diagrams and the various alterations one can make to get back the same three–manifold. Each way has its advantages when thinking about a particular problem. For example, calculating volumes of manifolds is most easy done with triangulations while proving invariance of some quantum objects can be most easily tackled using link diagrams.

In this section, I will outline the important results that are needed to define, calculate and prove invariance of the quantities we will study later. Technical aspects will be avoided and minimal proofs given. However, most of the results are well know and many have references to textbooks. Emphasis will be given on the tools needed to compute examples. I will firstly consider the algebraic aspects of three–manifolds given by link diagrams, braids and surgery. Then I will go onto study triangulations both topologically and geometrically. Next I will consider the combinatorics of geometric triangulations via Thurston's gluing equations encoded in Neumann–Zagier matrices. Finally, I will discuss geometric invariants and in particular the Cheeger–Chern–Simons invariant of a flat $\mathrm{PSL}_2(\mathbb{C})$–connection. In some sense, this section covers the classical aspect of the theory before we go onto the quantum aspects in the next section.

1.1 Knot and link diagrams

A *knot* is a smooth embedding of the circle into S^3 up to isotopy. Some care needs to be made with the topological aspects as discussed, for example, in [33, 121]. However, we can simply represent knots combinatorially in a *knot diagram*. This takes a projection of the knot onto some S^2 that is an embedding away from a finite set of points, where locally we have a transverse intersection of two arcs of S^1. This gives a four–valent graph in S^2 and to each vertex we keep track of which arc was higher in the original embedding into S^3. These vertices are called crossings. This just describes how one would simply draw a knot on a piece of paper. For example, the simplest non-trivial knot is the *trefoil* and its knot diagram is shown in Figure 1.1.

The trivial example is the *unknot*. This simply embeds S^1 with the usual embedding into a sphere S^2 inside S^3. It has a knot diagram which is just a circle with no crossings. Determining whether a knot is the unknot is an extremely difficult question. For a more complicated illustration of this, see [183, Ch. 1] but we give another example in Figure 1.2 for fun.

We will use knot diagrams to describe knots for most of this thesis. It then becomes important to understand how different diagrams for the same knot can be related. One can always take isotopies that introduce finitely many points crossings in the diagram. As one applies one of these isotopies, there will be a critical point where some crossings come together in the knot diagram, or new crossings are created. There are three fundamental cases. Firstly, there are two fundamental ways a crossing can be created (and conversely removed); by an edge

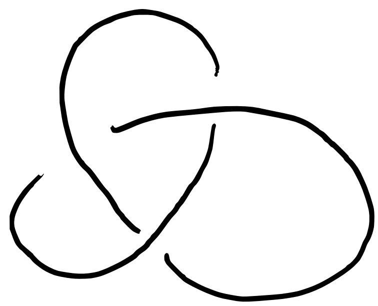

Figure 1.1: Left handed trefoil knot.

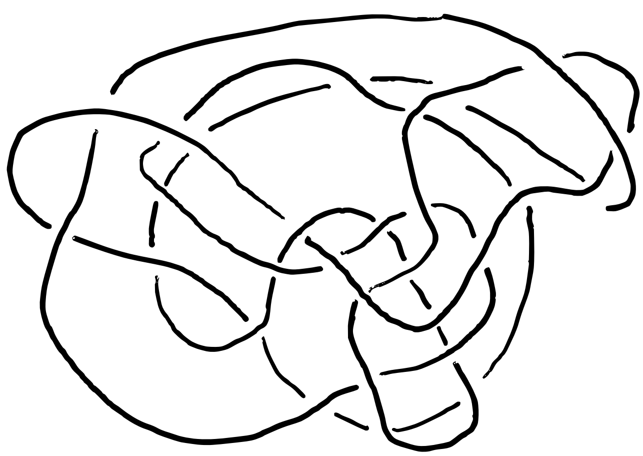

Figure 1.2: Is this the unknot?

Figure 1.3: The Reidemeister moves.

twisting around itself into a kink, and by two edges coming together creating two crossings. Finally, if more than two crossings come together this can always be decomposed into multiple steps where three crossings come together or one of the previous two cases. Then, if three edges come together, there is a face with three edges that shrinks. Analysing how this can happen, we find it always reduces to moving an edge under a crossing. All possible relations between knot diagrams are then combinations of these moves called Reidemeister moves. This is discussed in more detail in, for example, [33, 121]. These moves are shown in terms of the diagram in Figure 1.3.

Historically, tables of knots were made in order of the number of crossings. This historical notation is still somewhat used for simple knots. The notation is of the form a_b, where a is the minimal number of crossings in a knot diagram of the knot, and b indicates[1] which knot of the finite list it is. The minimal number of crossings is often simply referred to as the number of crossings of the knot. In Figure 1.4, we give a table of knots up to seven crossings up to mirror images. A knot is called *amphichiral* if it is the same as its mirror image. The only examples of amphichiral knots on the table in Figure 1.4 are 4_1 and 6_3. So all others differ from their mirror image. The number of knots with a given number of crossings increases rapidly, which is an obvious defect of this historical notation. The table of the numbers up to fifteen can be found in [121, Table. 1.2] and is reproduced here.

Number of crossings	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of knots	1	1	2	3	7	21	49	165	552	2176	9988	46972	253293

A natural and important generalisation of a knot is to a *link*. This is an embedding of a compact one-manifold in S^3. All compact one-manifolds are disjoint unions of circles S^1. We can generalise the previous discussion to define link diagrams. Similarly, link diagrams will be related by the Reidemeister moves from Figure 1.3. An example of a link is given in Figure 1.5. If a link is an embedding of $\sqcup_{i=1}^n S^1$ then we call it an n-component link. The components of the link are the restrictions of the embeddings to one of the copies of S^1.

[1] b is not canonical and purely historical.

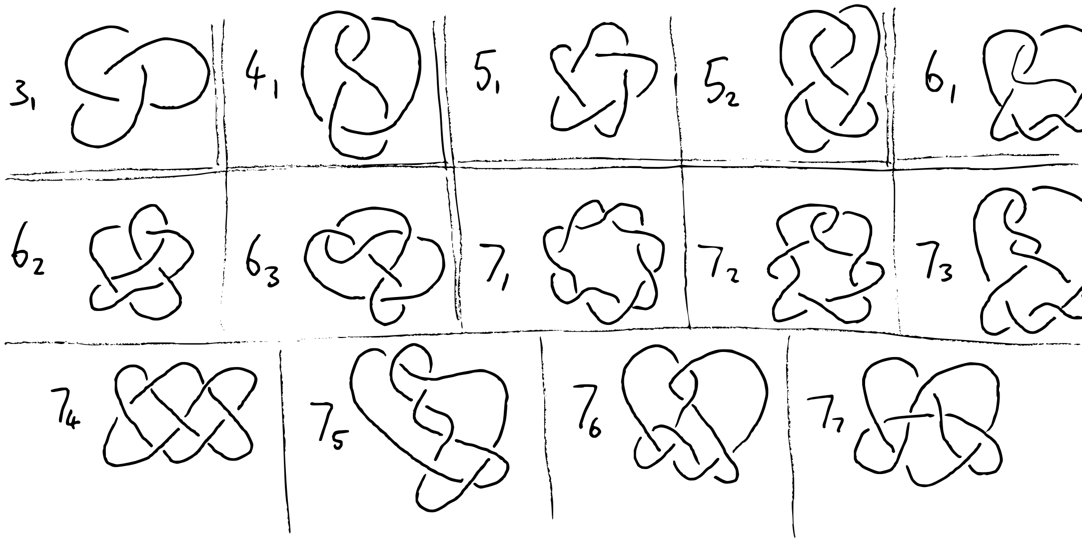

Figure 1.4: Table of knots up to seven crossings.

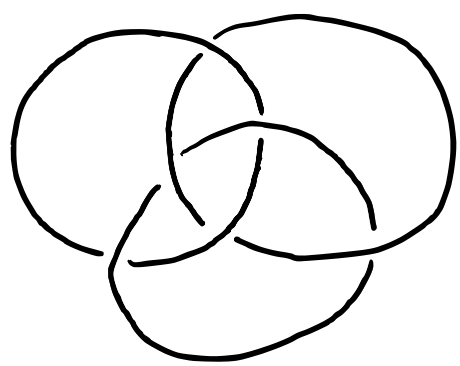

Figure 1.5: The Borromean rings. All components are unknotted and no two components are linked.

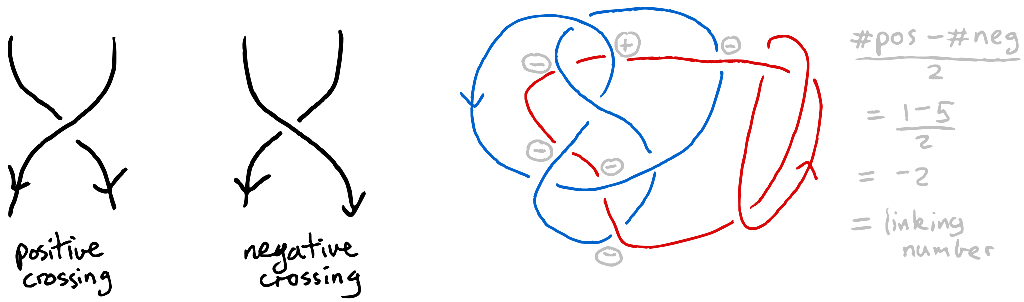

Figure 1.6: Positive and negative crossings and an example of linking number.

Often we will be interested in *oriented links*. For the diagram, this means putting an arrow on an arc in each component. There are at most two orientations of each component. For example, the two orientations of the trefoil are equivalent. Using orientations we can define positive and negative crossings as shown in Figure 1.6 Finally, there is a natural measure of how linked two components of a link are given in the following definition, which can be checked to be well defined under the equivalence induced by the Reidemeister moves in Figure 1.3.

Definition 1 (Linking number)**.** *[121, Def. 1.4] If L is an oriented link with components $L_1, \cdots, L_n$ then the* linking number $\mathrm{lk}(L_i, L_j)$ *between L_i and L_j is defined to be the number of positive crossings between the two components minus the number of negative crossings between the two components in a link diagram.*

An example of the computation of the linking number between the two components of a two component link is given in Figure 1.6. Choosing an orientation one can check that the linking number between any two components of the Borromean rings in Figure 1.5 is zero.

Knots and links can be described entirely in terms of algebra. This will be described in the next section.

1.2 Braids and the algebraic structures of links

Braid groups are algebraic objects that can be used to decompose knots and links into some simple building blocks corresponding to crossings.

Definition 2 (Braid groups)**.** *The group*

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} : \begin{array}{llll} \textit{if} & |i-j| > 1, & \sigma_i\sigma_j & = \sigma_j\sigma_j \\ \textit{if} & i = 1, \ldots, n-2 & \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle, \tag{1.1}$$

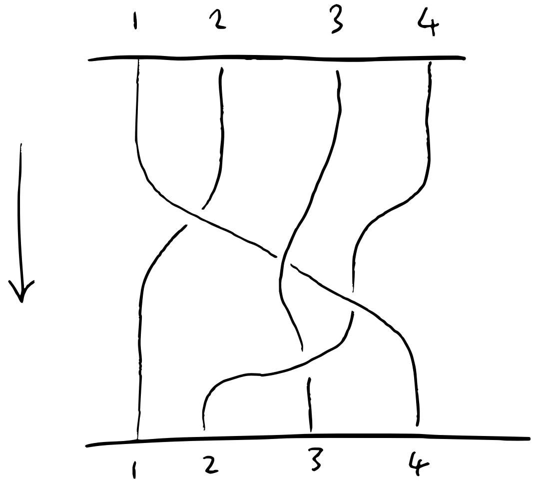

Figure 1.7: A braid on four strands.

is called the braid group on n strands.

The braid group on n strands can be described visually as the group generated by isotopty classes of embeddings of arcs connecting n points on a base line to n to the same points on a translation of the base line. This can be made into a braid diagram in the same way as for knots. For example, a typical element looks something like that shown in Figure 1.7 Composition is then done by stacking braids on top of each other as shown in Figure 1.8 Now the elements σ_i are given by the diagrams in Figure 1.9, which clearly generate the group. With these pictorial representations, the relations become clear and are describe in figures 1.10 and 1.11. We can include braid groups in each other by

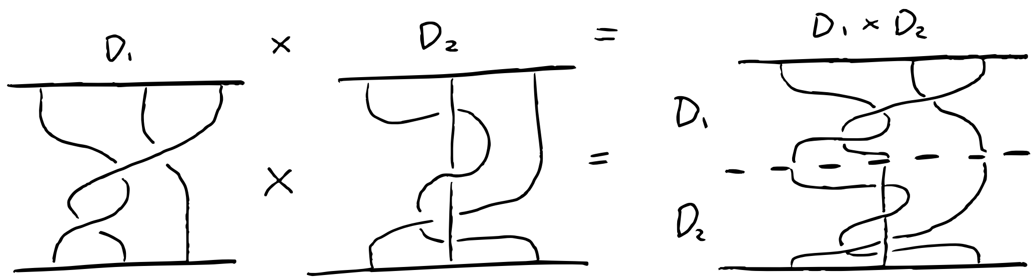

Figure 1.8: Composition of braid diagrams.

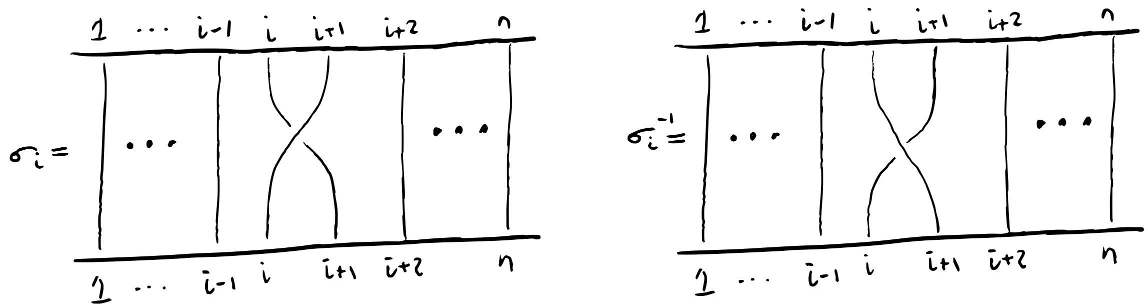

Figure 1.9: Generators of the braid group from crossings.

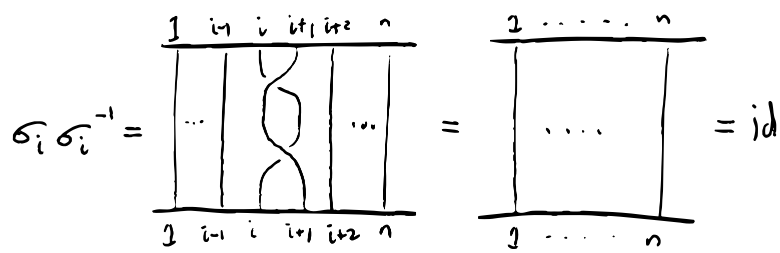

Figure 1.10: Inverse in the braid group corresponds to Reidemeister II in Figure 1.3.

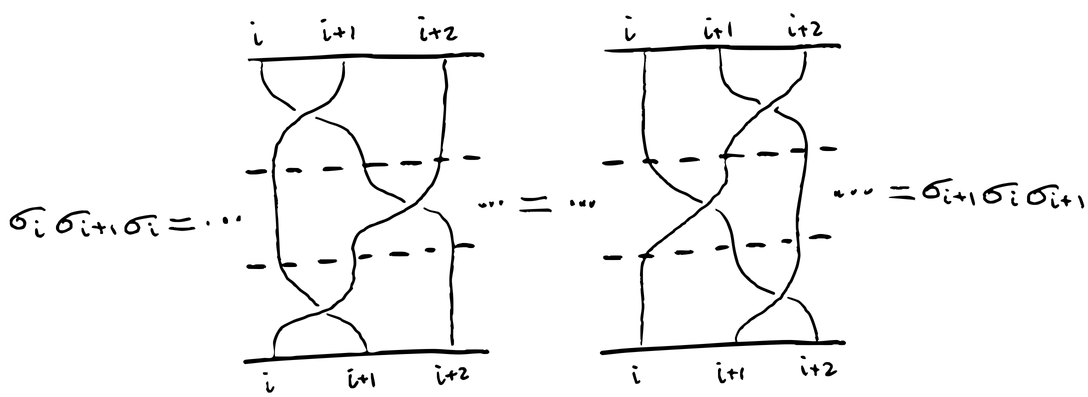

Figure 1.11: Relations in the braid group corresponds to Reidemeister III in Figure 1.3. The top left crossing is passed under.

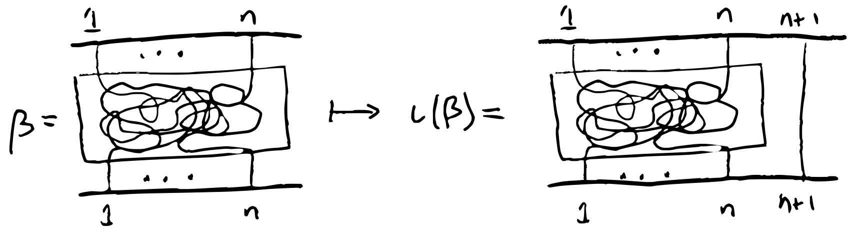

Figure 1.12: Embeddings of the braid groups in each other.

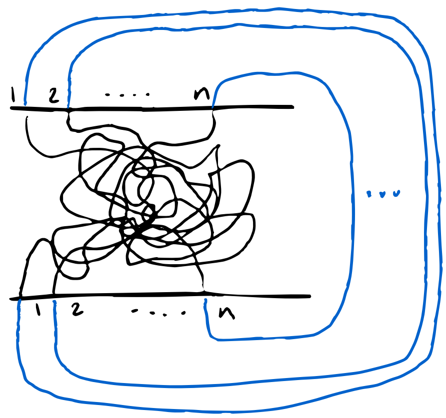

Figure 1.13: The closure of a braid.

$$\iota_n : B_n \to B_{n+1} \quad s.t. \quad \iota(\sigma_i) \ = \ \sigma_i \, . \tag{1.2}$$

This is given diagrammatically in Figure 1.12. To any braid we can construct a link by taking its closure. This involves connecting the points on the base lines above the braid. This is pictured in a link diagram in Figure 1.13. Not only does this give a link but an oriented link. Now a natural question is whether the links that arise from braids give rise to all links. This is the content of the next theorem due to Alexander [1].

Theorem A–1. *[109, Thm. 2.3]. Every oriented link in S^3 is isotopic to a braid closure.*

This shows that we have a surjection from the set of braids onto the set of links. The question is then of course what equivalence relation this surjection gives *i.e.* how are braids in the preimage of a link related. To describe this equivalence relation, we need to understand how the closure can swap a braid element from the top to the bottom, and how Reidemeister I in Figure 1.3 can be realised with braids.

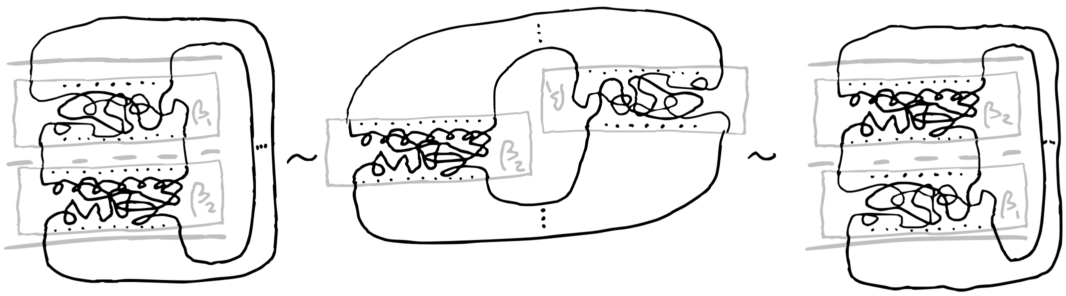

Figure 1.14: The first Markov move implied by isotoping along the closure.

We can construct a relation by taking the composition of two braid elements, β_1, β_2, given by $\beta_1\beta_2$ as seen in Figure 1.14. Then, taking the closure, we see that we can isotope β_1 along the closure to be below β_2. Therefore, we see that $\beta_1\beta_2$ and $\beta_2\beta_1$ give the same link. This implies that conjugation preserves the link. Therefore, under the equivalence relation coming from Theorem 1 we have

$$\beta_1 \sim \beta_2\beta_1\beta_2^{-1}\,. \tag{1.3}$$

A second relation can be constructed by understanding how braids interact with Reidemeister I in Figure 1.3. This can be done by considering a braid $\beta \in B_n$ and taking its embedding into the braid group with an extra strand, as shown in Figure 1.12, and composing with $\sigma_n^{\pm}$. This is depicted in Figure 1.15 and is the equivalence is then implied by Reidemeister I. Therefore, we have

$$\beta \sim \iota_n(\beta)\sigma_n^{\pm}\,. \tag{1.4}$$

These relations were introduced by Markov who proved the following Theorem [128].

Theorem A–2. *[109, Thm. 2.3] The closure of two braids give isotopic links if and only if they are related by equivalences generated by equation* (1.3) *and equation* (1.4)*.*

We can use this theorem to construct invariants of links. In particular, we must construct a function from the braid group that is invariant under the two equivalences in equation (1.3) and equation (1.4). Already, if we have some kind of representation of the braid group, taking some kind of trace should give invariance under equation (1.3). We will return to this is Section 2.3.

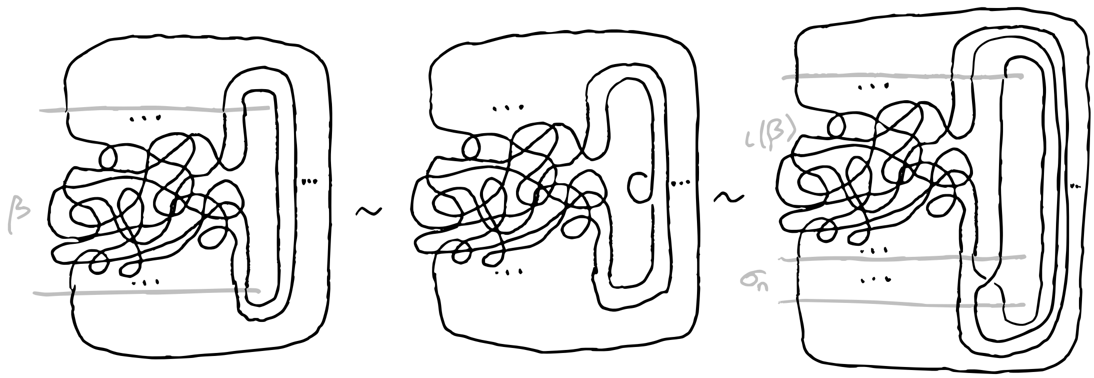

Figure 1.15: The second Markov move implied by Reidemeister I in Figure 1.3.

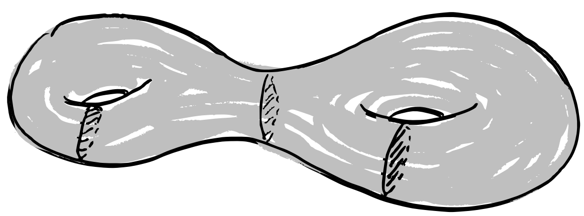

Figure 1.16: Genus two handlebody.

1.3 How to glue three–manifolds

Gluing topological manifolds with boundary is done by homeomorphisms of the boundaries to be glued. With additional structures one must be careful in general, however, in dimension 3 these issues cause us no problem due to work of Moise. The gluing will only depend on the isotopy class of the diffeomorphism and therefore elements of the mapping class group. The boundary of a three–manifold is a surface and therefore we find that we need to understand mapping class groups of surfaces. These groups are somewhat understood but aspects remain very mysterious. However mysterious, elements of the mapping class group can be used to give rise to all three–manifolds. A genus g *handlebody* is the closure of a small neighbourhood of a graph of genus g embedded in $\mathbb{R}^3$. Alternatively, take a standard embedding of a genus g surface, which splits $\mathbb{R}^3$ into two regions and the handlebody is the bounded region and the surface. For a graphical representation see Figure 1.16. Then we have the following theorem.

Theorem A–3. *[121, Lem. 12.12][173, Thm. 1.1] Every three–manifold can be obtained by*

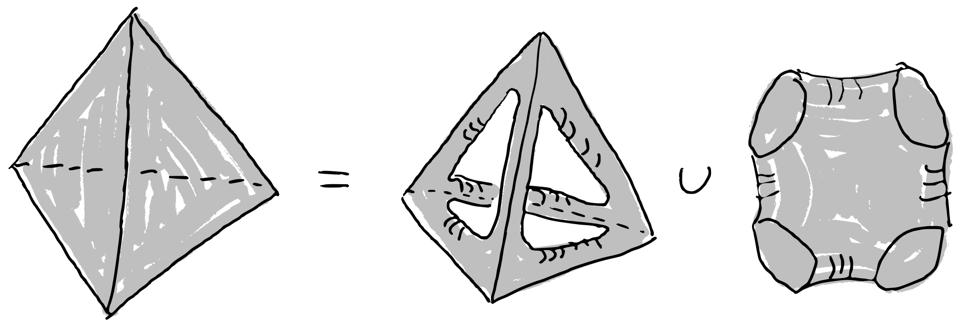

Figure 1.17: Heegaard splitting from a triangulation in a tetrahedron.

gluing two handle bodies together.

Proof. Triangulate the manifold and take the closure of a small neighbourhood of the edges of the triangulation, which will be a handlebody. Then in each tetrahedron what is left is a "solid sphere with four boundaries" as depicted in Figure 1.17, which will also glue together along the faces into a handlebody. The genus of this splitting will be one more than half the number of tetrahedra. □

This splitting is called a *Heegaard splitting* after its inventor. We will take a different approach in constructing closed manifolds. Consider a link with n components. Taking the complement of the link inside S^3 gives a manifold with n toroidal boundaries. We can glue a solid torus into each of these boundaries corresponding to the components of the link to construct a closed three–manifold. Notice that gluing in a solid torus is completely determined by where it sends the curve on the boundary that bounds a disk. Choose two dual curves, m_i, ℓ_i, generating the first homology of the torus corresponding to the i-th boundary component such that m_i is contained in a ball not containing the whole i-th component of the link. These curves are given the following names: m_i is a *meridian*, and ℓ_i is a *longitude*. Then for every component i of the link, we can glue so that the simple curve corresponding to the class $[a_i m_i + b_i \ell_i]$ is identified with the curve on the boundary bounding a disk in the solid torus. As a_i, b_i are coprime, as they represent a simple closed curve, this surgery is determined by the number called the *slope* $a_i/b_i \in \mathbb{Q} \cup \{\infty\}$. The way we have defined m_i, the surgery when the slope is infinity simply removes that component of the link. For example, infinite surgery on all components of a link will return S^3. This procedure is called *Dehn surgery*. The following theorem is due to Lickorish [120] and Wallace [192].

Theorem A–4. *[121, Thm. 12.13][173, Thm. 2.1] Every closed three–manifold is obtained by integral Dehn surgery on links in S^3.*

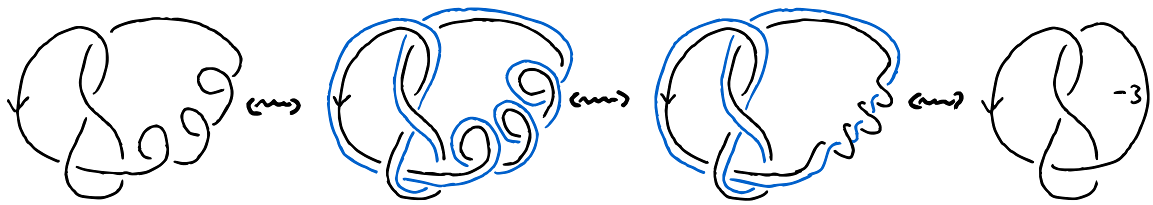

Figure 1.18: Blackboard framing.

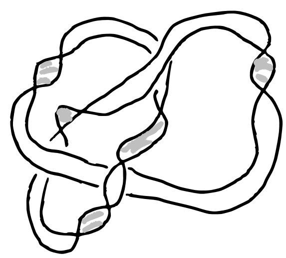

Figure 1.19: Example of a ribbon graph giving a framed trefoil.

Therefore, we can completely describe closed three–manifolds by a link with peripheral curves, and an integer for each component. A convenient way to choose the curves is the so called *blackboard framing*. This takes a link diagram and chooses ℓ_i to be the curve given by a parallel to the i-th component, which is always say on the right of the link of the i-th component in a small neighbourhood of the link. Then if the i-th component is framed by an integer a_i we take a portion of the i-th component and its parallel and twist them a_i times. To recover a_i, one can take the linking number, defined in definition 1, between the component and its parallel. There is a canonical 0–*framing* such that each component has 0–linking number between the two parallel curves. An example of blackboard framing is given in Figure 1.18. This is conveniently described by *ribbon graphs*, which are embeddings of $[0, 1] \times S^1$ into S^3 as opposed to S^1. Their projections keep track of the framing and can therefore be used to describe closed three–manifolds. These projections are often called *framed link diagrams*. For an example of a ribbon graph on the trefoil see Figure 1.19. The main difference is that Reidemeister I moves now alter the framing, as can be seen in Figure 1.20.

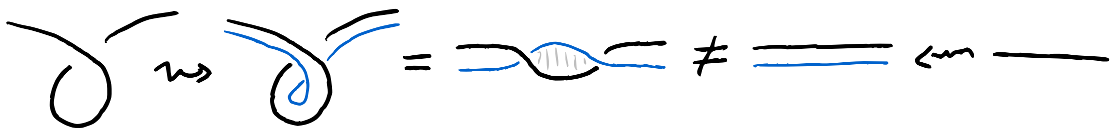

Figure 1.20: Reidemeister I does not hold for framed links or ribbon graphs.

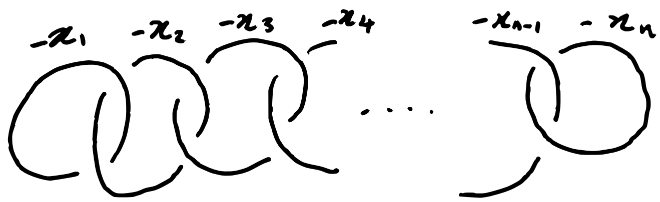

Figure 1.21: Lens spaces as a framed link diagram.

Example 1 (Lens spaces)**.** *[173, Thm. 2.3] Consider the continued fraction*

$$\frac{p}{q} = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}} \tag{1.5}$$

Then the lens space $L(p, q)$ is given by the framed link diagram in Figure 1.21.

With oriented framed links we have a natural notion of self linking given by the framing. Using this we have the following definition.

Definition 3 (Linking matrix)**.** *[121, Sec. 13] Let L be a framed link with components $L_1, \ldots, L_n$ and framing of component L_i given by a_i. Then define the* linking matrix

$$\mathrm{lk}(L) = \begin{pmatrix} a_1 & \mathrm{lk}(L_1, L_2) & \cdots & \mathrm{lk}(L_1, L_n) \\ \mathrm{lk}(L_2, L_1) & a_2 & \cdots & \mathrm{lk}(L_2, L_n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{lk}(L_n, L_1) & \mathrm{lk}(L_n, L_2) & \cdots & a_n \end{pmatrix}. \tag{1.6}$$

The linking matrix stores various information about the manifold obtained by surgery. For example, the first Betti number of the manifold obtained by surgery is given by the dimension of the kernel of the matrix.

As usual, it is important to understand the relations between various framed link diagrams giving the same manifold. This is described in the next section.

Figure 1.22: The Kirby move.

1.4 A calculus for the topology of three-manifolds

Treating three–manifolds given by a frame link diagram as the boundary of a four manifold, Kirby [111] studied the effect of various operations on four manifolds, which preserve the three–manifold at the boundary. Although these operations preserve the three–manifold, they alter the framed link diagram. This of course gives equivalence relations between different framed link diagrams that give the same three–manifold. Kirby's original version had some more global aspects but this was made entirely local by Fenn–Rourke [60]. Moves of this type are called *Kirby moves*. The most important Kirby moves are given by adding or removing an unknotted component with framing ± 1. If the unknotted component is linked around some other components which intersect a disk bounded by the unknotted component, then the arcs crossing the disk a twisted around once clockwise or anticlockwise depending on the sign. If the component bounds a disk not intersection any other component of the link, this is called a special Kirby move. The Kirby move is shown in Figure 1.22. Notice that it changes the framing of a component by the square of the linking number. This can be seen by considering Figure 1.23. It is not too hard to see that these moves give the same three–manifolds. However, all isomorphic three–manifolds can be related by them. This is the content of the following theorem, which in its original form is due to Kirby.

Theorem A–5. *[167, Thm. 6.2, 6.3][173, Thm. 3.1] If M_1 and M_2 are closed three–manifolds associated to framed links L_1 and L_2 then they are homeomorphic if and only if L_1 and L_2 are related by (± 1)–Kirby moves.*

As discussed for links, we can use this theorem to construct invariants of closed three–manifolds. To do this we must construct a function from the set of framed links that is invariant under the Kirby moves. This then implies that it is a topological invariant. Examples of such invariants will be given in Section 2.4.

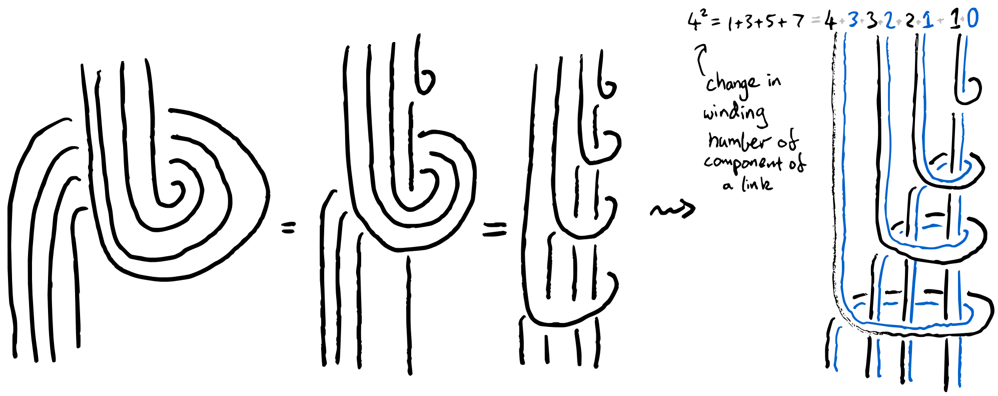

Figure 1.23: The affect of the Kirby move on the framing.

1.5 Representing fundamental groups and A–polynomial

With a surgery description of three–manifolds in terms of links in S^3, we see that to understand the fundamental group of a three–manifold we need to understand the fundamental group of a link complement. Doing this we can use the Seifert–Van Kampen theorem to describe the fundamental group of all three–manifolds. Given a link diagram one can explicitly compute a presentation for the fundamental group. This is called the *Wirtinger presentation*. Given an oriented link L it is described as follows:

- for every arc in the link take a generator g_i, which correspond to an arc coming from a base point high above the projection as wrapping once around the component following the right hand rule,
- at every crossing take the r_j relations given in Figure 1.24,
- and then take

$$G_L \;=\; \langle g_i : r_j \rangle \,. \tag{1.7}$$

Theorem A–6. *[169, Sec. 3, Thm. 2] The Wirtinger presentation is a presentation of the fundamental group of the link complement* i.e.

$$\pi_1(S^3 - L) \;=\; G_L \,. \tag{1.8}$$

There are similar algorithmic presentation from a link diagram such as the Dehn presentation. Given a manifold SnapPy [43] has algorithms to compute a presentation of the fundamental group. For example, we can compute a presentation of the fundamental group of the figure eight knot complement using SnapPy as follows.

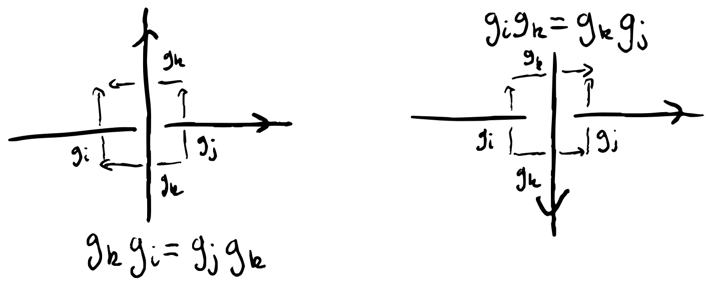

Figure 1.24: Relation for the Wirtinger presentation.

```
In[1]: M=Manifold("4_1")
In[2]: M.fundamental_group(simplify_presentation = False)
Out[2]:
Generators:
   a,b,c
Relators:
   BabC
   CaBcA
```

This says that the fundamental group of $S^3 - 4_1$ has presentation

$$\langle a, b, c : b^{-1}abc^{-1}, c^{-1}ab^{-1}ca^{-1} \rangle \,. \tag{1.9}$$

Later we will be very interested in representation of the fundamental group into $\mathrm{SL}_2(\mathbb{C})$ up to conjugation. These representations can also be computed by SnapPy [43] in Sage [182]. For example, boundary parabolic representations are computed in Code 2. The output there indicates that we have a representation defined on the generators by

$$\rho(a) = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -e^{\frac{2\pi i}{6}} + 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} -e^{\frac{2\pi i}{6}} + 2 & 1 \\ e^{\frac{2\pi i}{6}} & e^{\frac{2\pi i}{6}} \end{pmatrix}. \tag{1.10}$$

Also we can see that the boundary curves $a^{-1}b$ and $ca^{-1}cb^{-1}ac^{-1}ba^{-1}b$ are indeed parabolic *i.e.* conjugate to $\pm(1,1;0,1)$. These representations are related to flat connections and will be discussed again in Section 1.13. Finally, using these representations into $\mathrm{SL}_2(\mathbb{C})$ we can define the A–polynomial of a knot. Let K is a knot with a basis m, ℓ for the boundary (or peripheral) curves in $S^3 - K$. Then we can take a map defined almost everywhere locally $\iota : \mathrm{Hom}(G_K, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^\times \times \mathbb{C}^\times$ such that for a representation $\rho : G_K \to \mathrm{SL}_2(\mathbb{C})$ we take an eigenvalue of $\rho(m)$ and an eigenvalue of $\rho(\ell)$. This map descends to the quotient $\mathrm{Hom}(G_K, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ where the action is given by conjugation.

Definition 4 (A–polynomial)**.** *[42] The A–polynomial of the knot K with peripheral curves m, ℓ is defined to be the polynomial that defines the variety $\overline{\mathrm{Imag}(\iota)}$.*

Of course there is an ambiguity up to multiplication by constants. In fact these polynomials can be made to have integer coefficients [42, Prop. 2.3]

Example 2. *For the trefoil* 3_1 *the* A*–polynomial is*

$$A^{3_1}(m,\ell) \;=\; (\ell-1)(\ell+m^6)\,, \tag{1.11}$$

and for the figure eight knot 4_1 *the* A*–polynomial is*

$$A^{4_1}(m,\ell) \;=\; (\ell-1)(\ell^{-1}-(m^{-4}-m^{-2}-2-m^2+m^4)+\ell)\,. \tag{1.12}$$

Finally, representations on closed manifolds that come from surgery on links then come from representations on link complements with some addition conditions at each boundary, which corresponds to where the disk in the solid torus gets glued after surgery. Therefore, representaions on closed manifolds correspond to special points on the A–polynomial that satisfy an additional equation such as $m^a\ell^b = 1$ where $a, b \in \mathbb{Z}$ are determined by the slope of the surgery.

1.6 Topological aspects of triangulations

We have described various combinatorial descriptions of three–manifolds using diagrams. For a more explicitly three dimensional description, we will use triangulations. Firstly, we need to understand how this works topologically. We want to understand which gluings are allowed and lead to manifolds. If we have a finite number of triangles and glue all their edges to another edge we will always find a surface. The only place that the gluing might cause issues is a vertex. However, going around a vertex we see that we can pass through at most a finite number of vertices of the triangles, which leads to a picture reminiscent of a pizza as seen in Figure 1.25. Essentially, it always works here as there is only one connected compact one dimensional manifold; the circle or one–sphere. If we consider gluing tetrahedra, we find a more complicated situation. Not only could the gluing go wrong in codimension two, but it could wrong in codimension three. In codimension two we are considering gluing around an edge. Locally this is done gluing wedges which reduces to the previous analysis for triangles as seen in Figure 1.25. Around the vertices, we find a new phenomenon and an obstruction to gluing. If we remove a neighbourhood of the vertices of a tetrahedron, we find a truncated tetrahedron as shown in Figure 1.26. This can always be glued, now along the hexagonal faces of the truncated tetrahedra, and we find a manifold with boundary coming from the truncation. The little triangles of the truncated tetrahedra, triangulate this boundary. If the boundary component is a sphere, then one can glue in a ball, which would imply that we could have glued using the full tetrahedra as depicted in Figure 1.27. However, if the boundary is not a sphere then one will not obtain a manifold by gluing the full tetrahedra. This is because you would be gluing a ball into a higher genus hole. We can also just remove

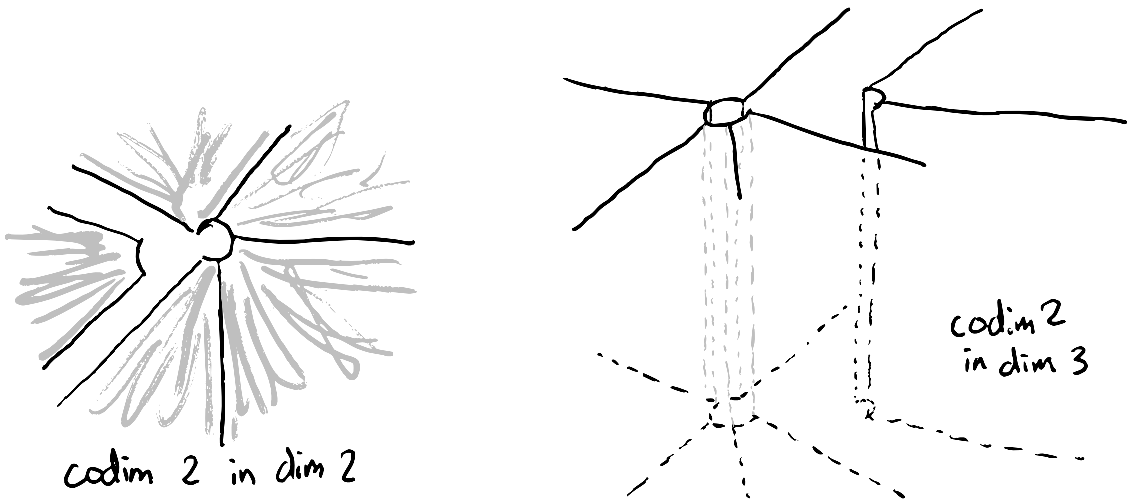

Figure 1.25: Gluing in codimension 2 for surfaces and three–manifolds. Pizza and tinned pineapple.

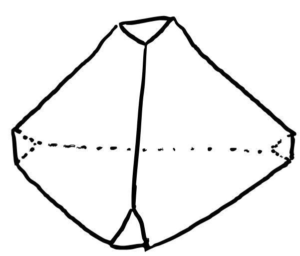

Figure 1.26: Truncated tetrahedron.

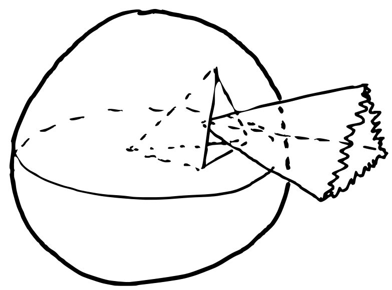

Figure 1.27: The extension of a triangulation to the vertices can only happen when the boundary of the truncated triangulation are disjoint unions of sphere. The picture would then correspond to pineapple segments coming together to form the ball.

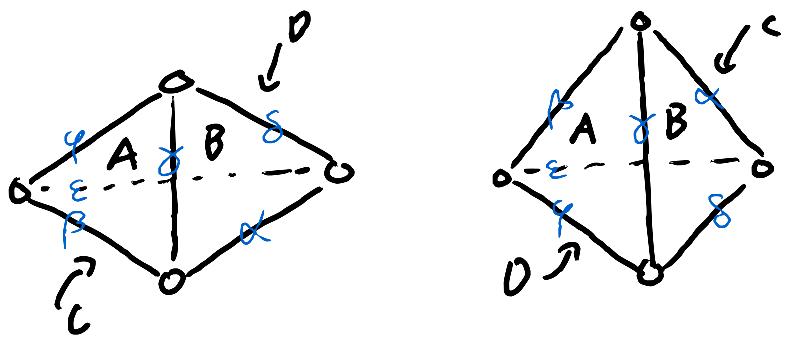

Figure 1.28: Triangulation of S^3.

the vertices to get an *ideal triangulation*, which is similar to the triangulation with truncated tetrahedra but without boundary and therefore non-compact.

Example 3. *Take the triangulation given by identifying edges and faces given in Figure 1.28. Checking the behaviour around the vertices, we see that there are four boundary components given by spheres triangulated by two triangles. Therefore, this triangulation extends to the vertices to give a manifold. In fact, this manifold is nothing but the three sphere. Now take the identification of two tetrahedra given in Figure 1.29. Analysing the vertices, one finds that the boundary of the triangulation, using the truncated tetrahedra, has one connected component given by a torus. Therefore, this gluing only gives an ideal triangulation. In fact, this is an ideal triangulation of the figure eight knot complement as described in Example 4.*

One may think that this obstruction could lead to problems in trying to understand three–manifolds. However, it turns out to be a useful tool. Instead of taking triangulations using

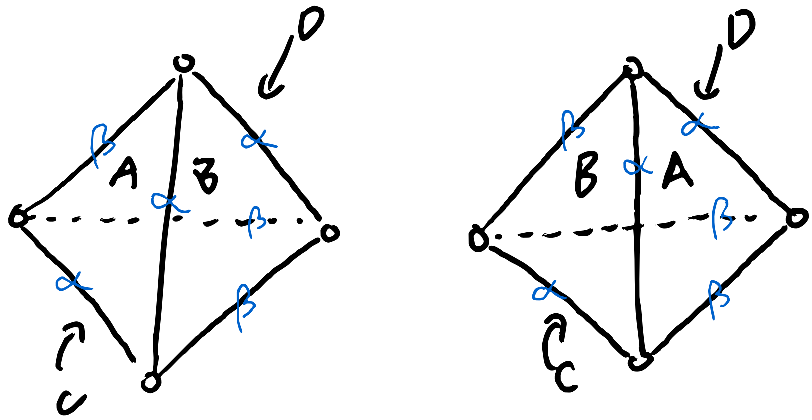

Figure 1.29: Triangulation of complement of the figure eight knot complement.

full tetrahedra, we can use ideal tetrahedra where we can always glue. This gives a way to ideally triangulate knot complements.

1.7 Triangulation algorithm from a knot diagram

Given a knot diagram, there is an explicit algorithm to decompose the complement of the knot into two polyhedra. Once this is done, one can choose a decomposition of these polyhedra into tetrahedra. The decomposition into the two polyhedra is constructed by putting a thickening of the knot diagram on a plane in $\mathbb{R}^3$, except at the crossing, where it goes above and below. Then, away from the crossings, take the plane outside of the thickened knot complement as the faces to be glued, as depicted in Figure 1.30. Finally, at a crossing, let the four incident faces twist and come together in an edge linking the strand of the knot in the over-crossing, to the stand in the under-crossing, as depicted in Figure 1.31. This will give a polyhedral decomposition with two type of faces; the faces that glue, and the faces on the boundary of the knot that are not glued. After cutting, the edge associated to a crossing become four edges. An intermediate picture is given in Figure 1.32. This decomposition can be given by the following algorithm:

- Take the knot diagram with at least one crossing and make two versions corresponding to the top, and the bottom polyhedra.
- For the bottom, draw two vertices on either side of each over-crossing and connect these along the knot diagram by some thickened shaded regions. Do the same for the top but with the under-crossings.

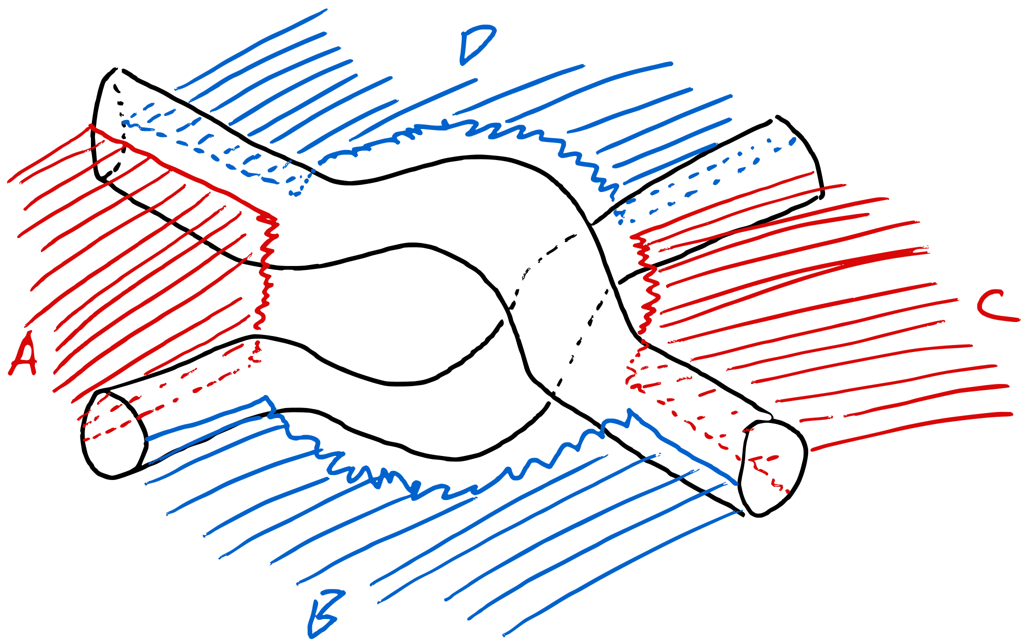

Figure 1.30: Polyhedral decomposition of a knot complement away from the crossings.

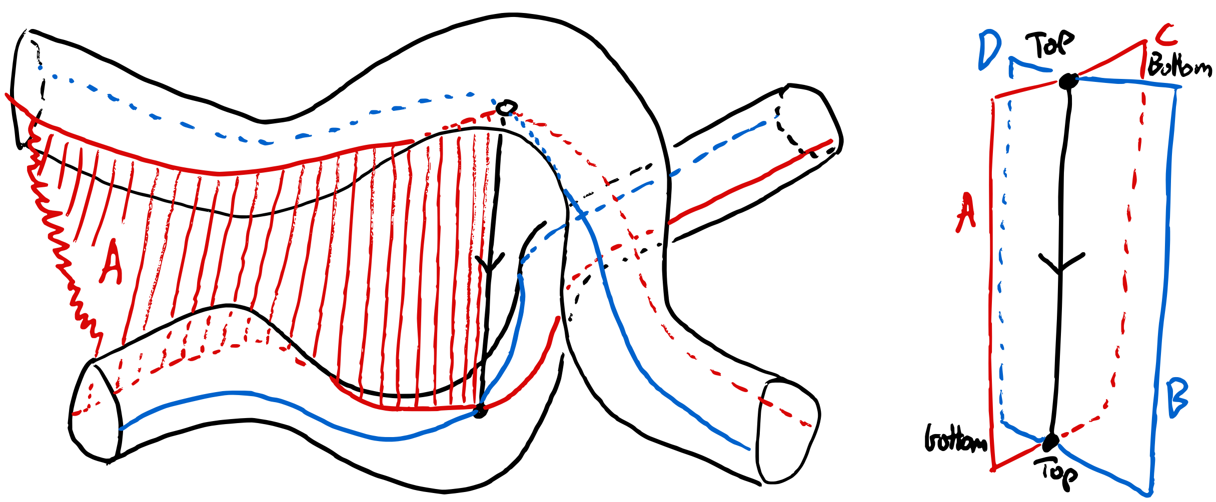

Figure 1.31: Polyhedral decomposition of a knot complement at the crossings.

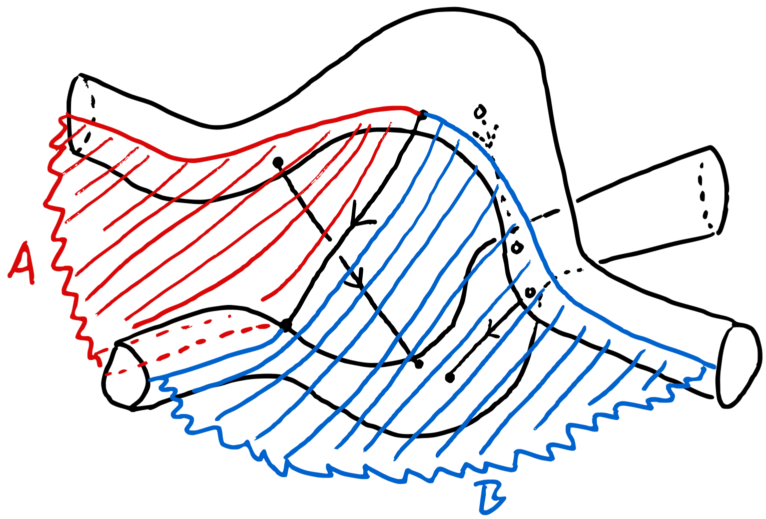

Figure 1.32: Two faces and an edge of the top polyhedron in the decomposition of a knot complement.

- Then, draw an edge connecting the vertices to a point on the boundary of the thickened shade regions, where the crossing was, with an arrow pointing towards the crossing if on the bottom, and away from the crossing if on the top (see figures 1.31 and 1.32). Label all edges associated to a crossing by the same label.
- This will decompose the plane into various simply connected regions, which will be the faces of the polyhedron. Label the various faces consistently between the top and the bottom pictures.
- The thicken shaded regions lie on the boundary of the knot, and these can be contracted to a point to find the ideal polyhedral decomposition. Alternatively, these can simply not be glued and they give a decomposition of the boundary into polygons.
- To finish, note that this picture of the two polyhedra is from outside of the bottom and inside of the top. Therefore, flipping the picture corresponding to the top we find the description of the two polyhedra.

This algorithm around a crossing is depicted in Figure 1.33. This algorithm will produce unigons and bigons. Additional steps can be added to remove these. Unigons can only appear from a twist as appears in Reidemeister I in Figure 1.3 and can therefore be removed before the algorithm. There are two ways a bigon can appear. One comes from bigons of the form of Reidemeister II in Figure 1.3. These can be removed before the algorithm. Then

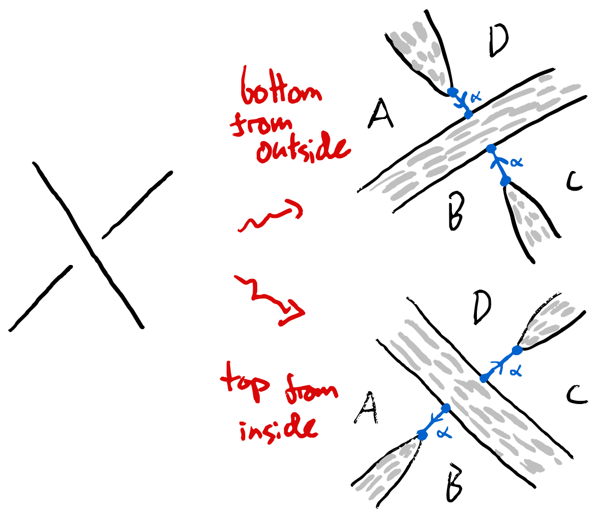

Figure 1.33: The algorithm to triangulate around a crossing. Imagine a worm in an apple eating out the shape of a knot. Then cutting the apple in half where it has eaten gives the thicken shaded regions. Compare the right part of this figure with the three dimensional picture in Figure 1.32

the other option, where they are linked locally, will always have a picture of the form in Figure 1.34, which can always be removed by identifying the edges bounding the bigon as shown in the Figure 1.34.

Example 4 (Triangulating the figure eight knot complement)**.** *This method, applied in the case of the figure eight knot, is shown in Figure 1.35. This example is also discussed in [183, Ch. 1]. Here we leave the bigons, but after their removal we find the picture in Figure 1.29.*

This algorithm is implemented in SnapPy [43]. This will be used later to implement various computations. Before finishing we remark that ideal triangulations exist more generally for cusped hyperbolic manifolds [35]. For links in S^3, essentially the same arguments and algorithms will construct a decomposition of the link complement into ideal tetrahedra.

1.8 Geometrisation

One of the triumphs of 3-dimensional topology is Thurston's geometrisation conjecture [184]. This is now a theorem thanks to the work of Perelman [155, 157, 156]. Thurston's geometrisation conjecture can be thought of as an analogue of the uniformisation conjecture for surfaces. For surfaces, the uniformisation conjecture states that every conformal equivalence

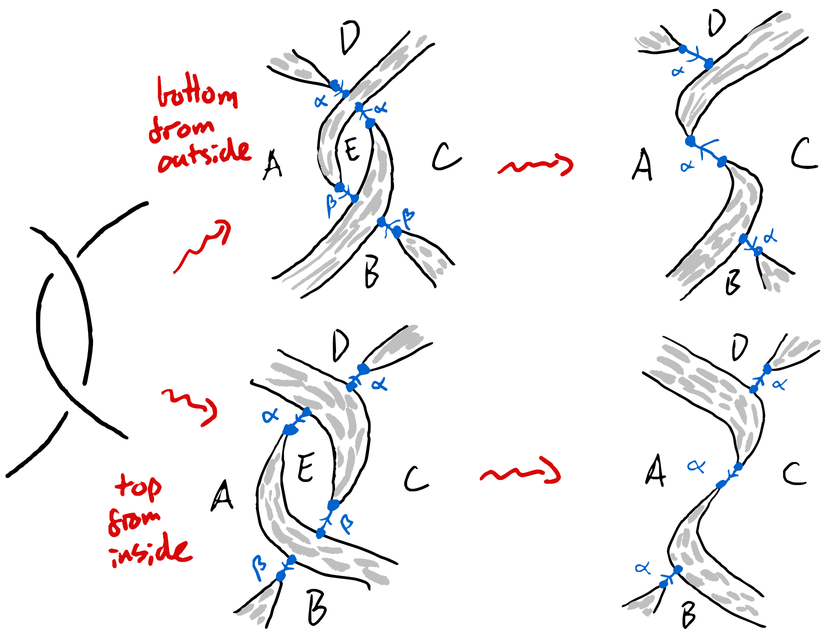

Figure 1.34: Removing non-trivial bigons.

class of metric on a surface has a representative of constant curvature. This then splits surfaces into the famous trichotomy: constant curvature 1, constant curvature 0, constant curvature −1. Pushing further, one can show that the only connected compact surface of positive curvature corresponds to the two sphere, while the only one with vanishing curvature is the torus. All higher genus curves are then hyperbolic, which shows that this is the most common geometry. Although the situation in three dimensions is much more complicated, it turns out to be very similar in spirit.

Firstly, one must decompose into basic building blocks. This is done by taking a so called prime decomposition of the three–manifold. This expresses the manifold as the gluing of a finite collection of manifolds with spherical boundaries that has no further non-trivial decompositions. These building blocks are then cut further along non-trivially embedded tori. What is left are some manifolds with torus boundaries. These are then given one of eight geometries [184]. Hyperbolic geometry is the most common and important. As with spherical and Euclidean geometries in two dimensions, the other geometries appear as special cases. We will be mainly interested in the hyperbolic manifolds.

Hyperbolic structures come from quotients of hyperbolic space by a discrete group of isometries. Complete hyperbolic surfaces have moduli. Indeed, for genus g surfaces there is a $6g-6$ (real) dimensional space of hyperbolic structures. This is not the case for higher

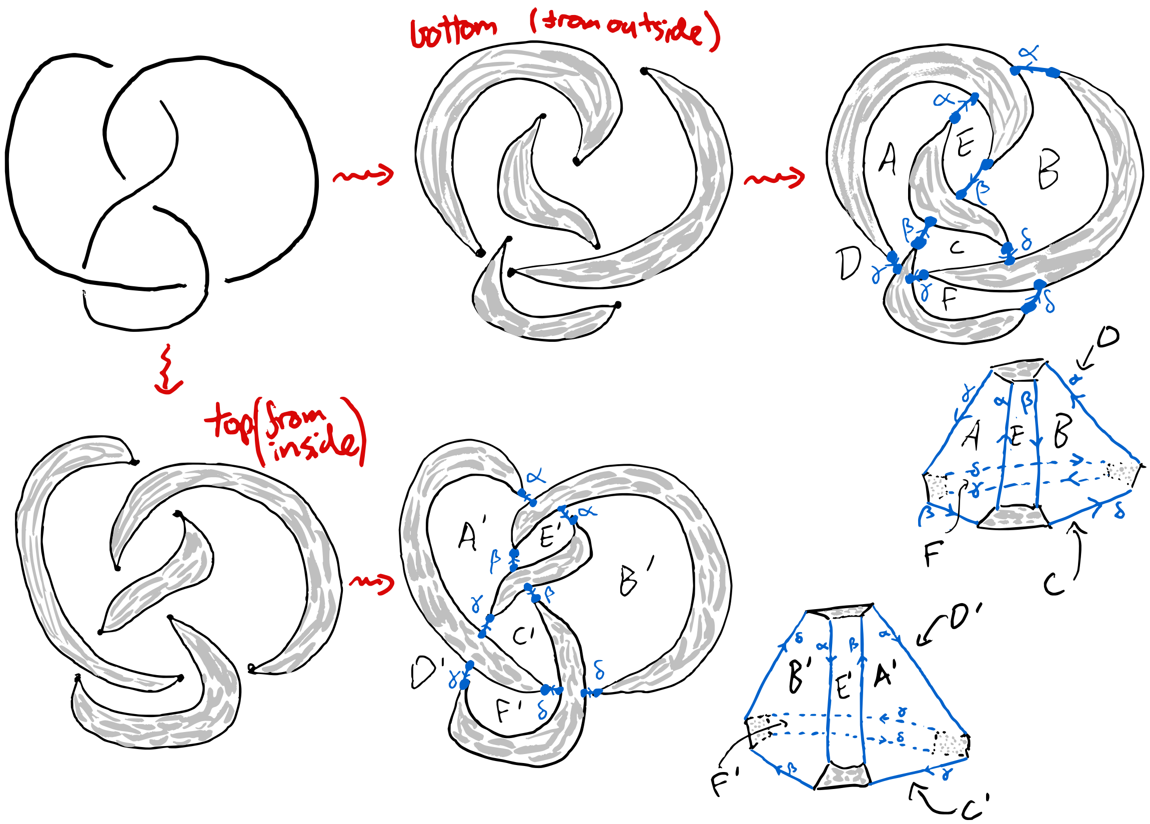

Figure 1.35: Algorithm applied to the figure eight knot.

dimensional manifolds. For higher dimensional hyperbolic manifolds the structure is unqiue. This follows from the following theorem due to Mostow [135] and Prasad [159].

Theorem A–7 (Mostow–Prasad rigidity)**.** *Suppose M_1 and M_2 are connected, complete, hyperbolic manifolds of dimension at least 3, and there exists an isomorphism*

$$\phi : \pi_1(M_1) \cong \pi_1(M_2)\,. \tag{1.13}$$

Then, ϕ is induced by an isometry of M_1 and M_2.

One of the beautiful outcomes of this is that, from topology, we get unique geometries. This allows us to construct topological invariants from geometric invariants. For example, the volume of a hyperbolic manifold will be a topological invariant. Isometries are also algebraic, which turns topology, not only into geometry, but number theory. To construct geometric structures we will decompose our manifolds, using triangulations, into geometric tetrahedra.

1.9 Ideal tetrahedra

Before considering tetrahedra, we will consider ideal triangles. An *ideal triangle* is a hyperbolic (constant curvature negative one) triangle with geodesic boundaries with vertices infinitely far from any other point. They can be embedded into $\mathbb{H}^2 = \{x + iy = z \in \mathbb{C} : \Im(z) > 0\}$ the hyperbolic upper half space (which has metric $(dx^2 + dy^2)/y^2$). Recall that the geodesics of $\mathbb{H}^2$ are given by semicircles perpendicular to $\mathbb{R}$ and vertical lines. Then an example of an embedding is shown in Figure 1.36. Given a geometric object like this, it is natural to consider its moduli space. Here we can parametrise embeddings by three distinct points on $\partial\mathbb{H}^2 = \mathbb{RP}^1$. However, we must consider the orbits under the automorphism group of $\mathbb{H}^2$. The group of isometries of $\mathbb{H}^2$ is given by

$$\mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})\,, \tag{1.14}$$

with action defined by real *Möbius transformations* such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \;=\; \frac{az + b}{cz + d}\,. \tag{1.15}$$

Considering our parametrisation of the embedding by x_1, x_2, x_3, as shown in Figure 1.36, we can take the following isometry

$$\begin{pmatrix} x_3 - x_1 & x_2(x_1 - x_3) \\ x_3 - x_2 & x_1(x_2 - x_3) \end{pmatrix}\,, \tag{1.16}$$

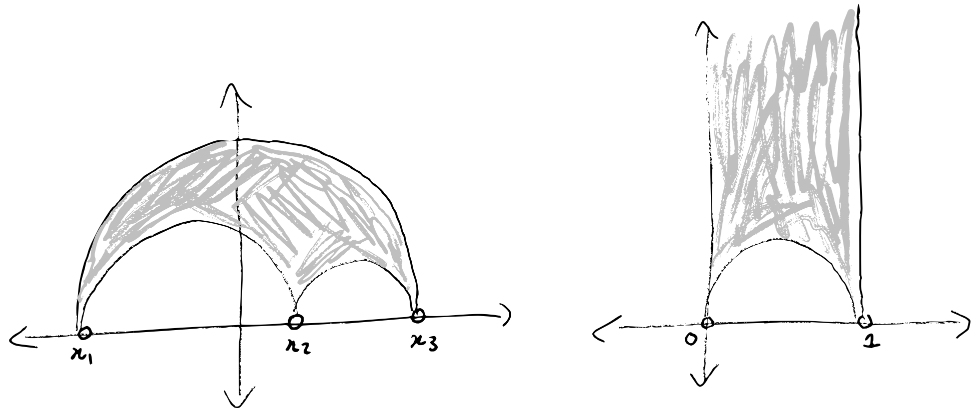

Figure 1.36: Embeddings of an ideal hyperbolic triangle.

which sends

$$x_1 \mapsto \infty\,, \quad x_2 \mapsto 0\,, \quad \text{and} \quad x_3 \mapsto 1\,. \tag{1.17}$$

Therefore, all ideal triangles are isometric to the embedding associated to $x_1 = \infty, x_2 = 0, x_3 = 1$, which is depicted in Figure 1.36. So we have found that the moduli space of hyperbolic triangles consists of a point. The volume of the ideal triangle is π, which of course agrees with the Gauss–Bonnet theorem.

We can try the same thing for ideal tetrahedra. An *ideal tetrahedron* is a hyperbolic tetrahedron with geodesic boundaries whose vertices are infinitely far away from every other point. To describe the moduli space of these objects we must understand upper half space $\mathbb{H}^3 = \{(x+iy, h) \in \mathbb{C} \times \mathbb{R}_{>0}\}$ (which has metric $(dx^2+dy^2+dh^2)/h^2$). Firstly, $\mathbb{H}^3$ has geodesics given by vertical lines and semi-circles perpendicular to $\mathbb{C}$. The boundary is now $\partial\mathbb{H}^3 = \mathbb{CP}^1$. The biholomorphisms of $\mathbb{CP}^1$ uniquely extend to isometries of $\mathbb{H}^3$. See for Example [57, Ch. 1]. On the boundary these are given by complex Möbius tranforms and we find that

$$\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})\,. \tag{1.18}$$

They act on $\mathbb{H}^3$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,h) \;=\; \left(\frac{(az+b)(\overline{cz}+\overline{d}) + a\bar{c}h^2}{|cz+d|^2+|c|^2h^2}, \frac{h}{|cz+d|^2+|c|^2h^2} \right). \tag{1.19}$$

We can embed ideal tetrahedra into $\mathbb{H}^3$ by specifying four points on the boundary that don't lie in a single line, $z_1, z_2, z_3, z_4 \in \mathbb{CP}^1$. An example is depicted in Figure 1.37. We can act on these embeddings via isometries and in particular we can take the isometry

$$\begin{pmatrix} z_3 - z_1 & z_2(z_1 - z_3) \\ z_3 - z_2 & z_1(z_2 - z_3) \end{pmatrix}\,, \tag{1.20}$$

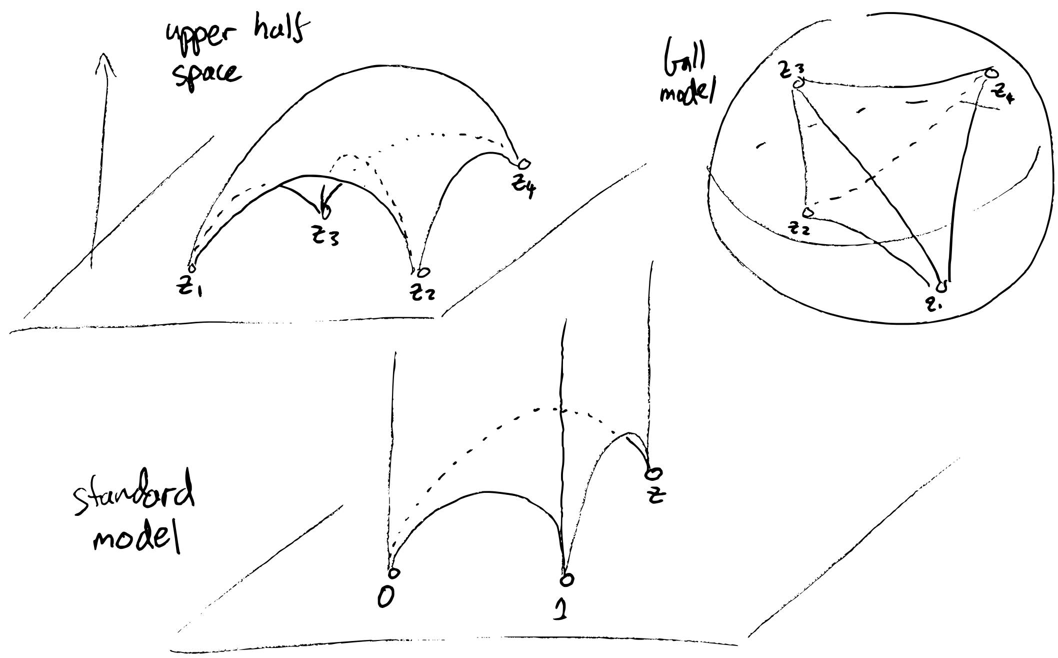

Figure 1.37: Embeddings of an ideal hyperbolic tetrahedra.

which sends

$$z_1 \mapsto \infty, \quad z_2 \mapsto 0\,, \quad z_3 \mapsto 1\,, \quad \text{and} \quad z_4 \mapsto z \;=\; \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}\,. \tag{1.21}$$

Therefore, we see that all ideal tetrahedra are isometric to one with vertices at $\infty, 0, 1, z$ where $z \in \mathbb{C}\backslash\mathbb{R}$ is called the *shape parameter*. Denote this tetrahedron by Δ_z. This is depicted in Figure 1.37. However, there are more relations. Indeed, the following Möbius transformations fix the set $\{\infty, 0, 1\}$ but give new values of z:

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.22}$$

Letting $z' = 1/(1-z)$ and $z'' = 1 - z^{-1}$ we see these isometries identify the tetrahedra

$$\Delta_z \cong \Delta_{z'} \cong \Delta_{z''}\,. \tag{1.23}$$

These preserve the orientation of the tetrahedron. There are still additional isometries, which invert the orientation that are given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \text{sending} \quad z \mapsto z^{-1}, \quad z \mapsto 1 - z\,, \quad z \mapsto \frac{z}{z-1}\,. \tag{1.24}$$

This gives the full permutation group action on the vertices $\infty, 0, 1$ and therefore all remaining isometries. Considering these actions, one can show that a fundamental domain is given by Figure 1.38. The parameters z can be directly related to aspects of the geometry of the tetrahedron. Imagine looking down on the tetrahedron from infinity in a standard model in the upper half space. Then, one would find an equilateral triangle with points at $0, 1, z$ as seen in Figure 1.39. There, we see see that angles between faces can be computed from the parameter z. In fact, using the angle formula there, one can see that the angles and an oriented edge reconstruct z. To each edge of the tetrahedra we can associated one of the parameters z, z', z'', which come in pairs of opposite edges as seen in Figure 1.39. The edge from 0 to ∞ is given parameter z. All other edges can be determined using the five automorphisms from equations (1.22) (1.24). The volume of the ideal tetrahedra can be computed explicitly in terms of some special functions. Consider the *Lobachevsky function*

$$\Lambda(x) \;=\; -\int_0^x \log|2\sin t| dt\,, \tag{1.25}$$

and the *Bloch-Wigner dilogarithm*

$$D(z) \;=\; \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|\,, \quad \text{where} \quad \mathrm{Li}_2(z) \;=\; -\int_0^z \log(1-w)\frac{dw}{w}\,, \tag{1.26}$$

is the *dilogarithm function* discussed in more detail in Section 4.2.

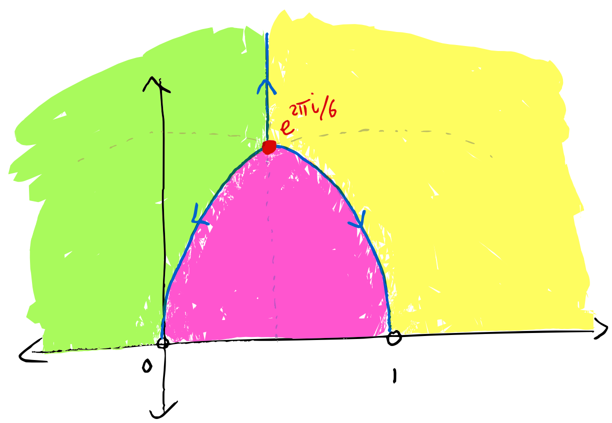

Figure 1.38: Moduli space of ideal tetrahedra where the edges with arrows are identified by $z \mapsto z' \mapsto z''$ and the coloured regions are identified with the same maps. The fixed point is given by $e^{\frac{2\pi i}{6}}$, which is like an equilateral ideal tetrahedron (see Figure 1.39).

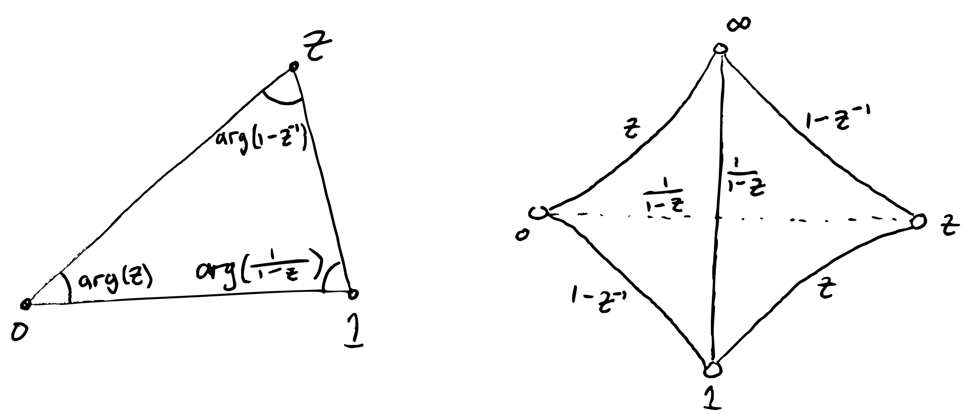

Figure 1.39: On the left, an ideal tetrahedron from infinity. The angles are computed by $\arg(z), \arg(1/(1-z)), \arg(1-1/z)$ as depicted. Note that when $z = e^{\frac{2\pi i}{6}}$ the triangle is equilateral. On the right, the labellings of an ideal tetrahedron

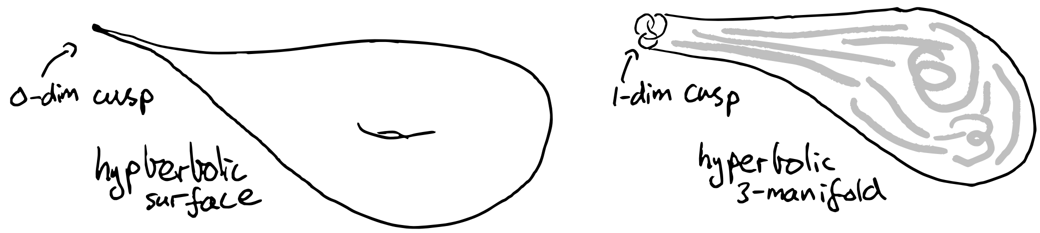

Figure 1.40: Cusps of three–manifolds are one–manifolds.

Theorem A–8. *[183, Thm. 7.2.1][206, Eq. 8] The volume of an ideal tetrahedron associated to $z \in \mathfrak{h}$ is given by*

$$\mathrm{Vol}(\Delta_z) \;=\; \Lambda(\arg(z)) + \Lambda(-\arg(1-z)) + \Lambda(\arg(1-1/z)) \;=\; D(z)\,. \tag{1.27}$$

For example, the volume of the equilateral tetrahedron, which is the maximum volume of any ideal hyperbolic tetrahedron, is given by

$$D(e^{2\pi i/6}) \;=\; 1.0149416064096536250\cdots\,. \tag{1.28}$$

1.10 Gluing equations

Topologically, we can construct ideal triangulations of knot complements and more general three–manifolds. It now becomes of interest as to whether these ideal tetrahedra can be glued together geometrically. The complete hyperbolic structure on the complement of a knot has the knot has a cusp, as it is an open manifold. In two dimensions, cusps correspond to points, while in three dimensions this increases to a one manifold. This is depicted in Figure 1.40. The boundaries of the hyperbolic ideal tetrahedron are hyperbolic ideal triangles. As we saw in Section 1.9, there is only one hyperbolic ideal triangle and it has no automorphisms that don't swap edges. Therefore, the topological identification of the faces has a unique geometric gluing. Therefore we need to check whether the gluing around an edge is geometric. To do this, consider an edge in an ideal triangulation. This edge has, say, n edges of tetrahedra incident to it[2]. Suppose that this edge, in these tetrahedra, is associated to the shape parameters $z_1, \dots z_n$ as we go around the edge. Consider the picture in the universal cover in upper half space looking down from the point at infinity. Moreover, suppose that the sequence of tetrahedral edges goes counter clockwise. The isometries that glue the tetrahedra together fix the points at infinity and zero and therefore are of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\,. \tag{1.29}$$

[2]The same tetrahedra can contribute more than once.

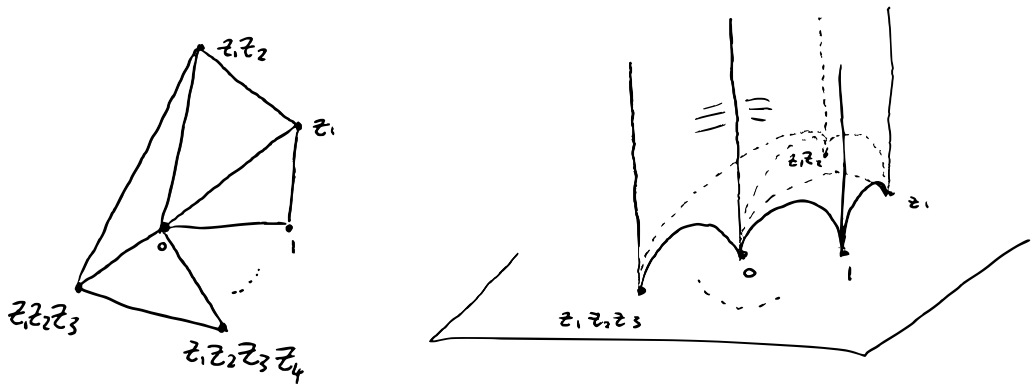

Figure 1.41: Gluing ideal tetrahedra around an edge.

Importantly, this means they all commute. Using the isometries that send $z \mapsto z \prod_{i=1}^{j} z_i$ on the standard model of the sequence of tetrahedra, we find the picture in Figure 1.41. The monodromy is then given by

$$\begin{pmatrix} \prod_{i=1}^{n} \sqrt{z_i} & 0 \\ 0 & \prod_{i=1}^{n} \sqrt{z_i}^{-1} \end{pmatrix} . \tag{1.30}$$

The vanishing of the monodromy around the edge, or equivalently the picture fitting together[3], immediately indicates that if this triangulation is geometric with parameters z_i, we must have

$$\prod_{i=1}^{n} z_i \;=\; 1 . \tag{1.31}$$

This equation is called an *edge equation*. If this equation holds for every edge, then we see that gluing these geometric tetrahedra gives a geometric structure on the three–manifold. This has reduced the construction of a hyperbolic metric to finding an ideal triangulation and a point in an associated algebraic variety. This description of the variety has some redundancy and in general the variety will have positive dimension. To get a complete hyperbolic metric, which should have no moduli from Theorem 7, we need to add an additional equation that deals with the behaviour at the boundaries. To understand this equation we need to introduce *horospheres*.

In the standard model of the tetrahedron, the horospheres around the point at infinity are just given by equilateral triangles parallel[4] to $\mathbb{C}$. They are defined more generally as the subspaces around a point such that all geodesics through that point are orthogonal to the

[3] This means that the final tetrahedron glues to the first by an isometry.
[4] All in the Euclidean metric.

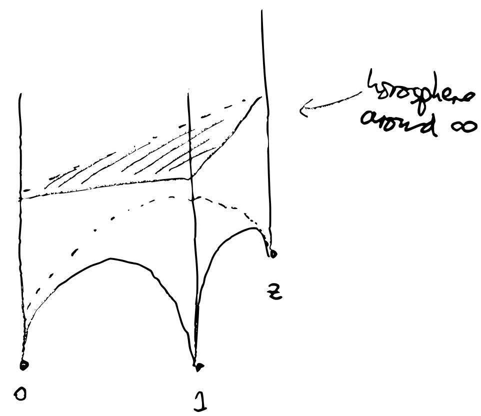

Figure 1.42: A horosphere around ∞.

space. These triangles are flat with the induced metric and are depicted in Figure 1.42. Importantly, their metrics, regardless of all being flat, are scaled by the inverse of their height squared.

Take the union of all cells of the triangulation incident to an ideal vertex. This is some open submanifold and we can take its universal cover. The ideal vertex of a knot complement corresponds to a toroidal boundary component. Therefore, looking from the vertex, which we put at infinity, we see locally tessellations of $\mathbb{C}$ by triangles in the universal cover. Consider a closed curve in that boundary. Take a representative in the universal cover of the manifold, starting in some tetrahedra, and continue the curve in horospheres. As we continue a curve through the various tetrahedra in the universal cover, the tetrahedra are then all related to the standard models by isometries. This time, the isometries won't preserve zero as the curve will pass multiple edges. However, they will preserve infinity as we have continued the curve in horospheres and it therefore never leaves through a bottom face[5]. Therefore, the isometries will be of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} . \tag{1.32}$$

These matrices do not in general commute. However, the map to $\mathbb{C}^\times$ taking the $(1,1)$ entry is a homomorphism. Take the shapes z_i of the edges of tetrahedra that the path travels around. We see that the gluing is then done by isometries of the form $z \mapsto z \prod_{i=1}^{j} z_i + \beta_j$ for some β_j as pictured in Figure 1.43. Then we see that the monodromy around the curve in

[5]We can always choose higher horospheres to avoid hitting a bottom face, as we will only pass through finitely many tetrahedra for one loop.

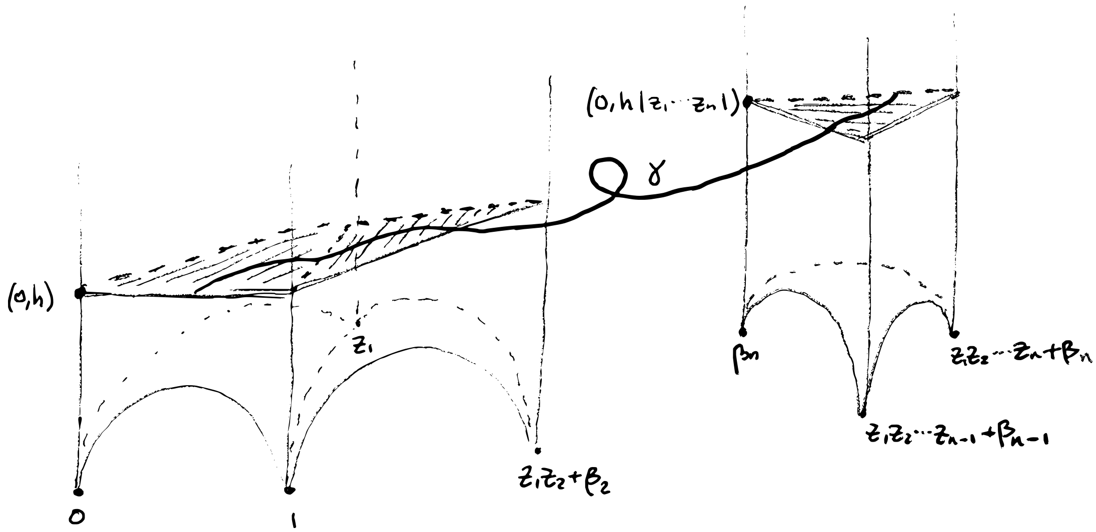

Figure 1.43: A boundary curve in the universal cover continued in horospheres. The tetrahedra it passes through are glued from their standard models via the automorphisms $z \mapsto z \prod_{i=1}^{j} z_i + \beta_j$.

the boundary is given by

$$\begin{pmatrix} \prod_{i=1}^{n} \sqrt{z_i} & \beta_n \\ 0 & \prod_{i=1}^{n} \sqrt{z_i} \end{pmatrix} . \tag{1.33}$$

We claim that a complete hyperbolic structure implies that

$$\prod_{i=1}^{n} z_i = 1 . \tag{1.34}$$

To show this, we will first show that the holonomy around the boundary curves must act by isometries of two–dimensional Euclidean space, which is equivalent to $\prod_{i=1}^{n} |z_i| = 1$. If not, then one curve must induce an element that is not an isometry and therefore must expand or contract. For some curve on the boundary, we must then have $\prod_{i=1}^{n} |z_i| > 1$, as we can always take the inverse of a curve. Consider a base point $(z, h) \in \mathbb{H}^3$ for this curve in the initial tetrahedra with vertices places at $(\infty, 0, 1, z_1)$. Then continue the curve in horospheres from one tetrahedra to another. After one loop we find that we land back in the a shift of the original tetrahedron by the isometry (1.33). This shift of the original tetrahedron has vertices at $(\infty, \beta_n, \prod_{i=1}^{n} z_i + \beta_n, z_1 \prod_{i=1}^{n} z_i + \beta_n)$ and we end up in the horocycle of height[6] $h \prod_{i=1}^{n} |z_i|$. Therefore, the projection[7] of the horosphere onto $\mathbb{C}$ appears to be $\prod_{i=1}^{n} |z_i| > 1$

[6] Euclidean.
[7] Euclidean.

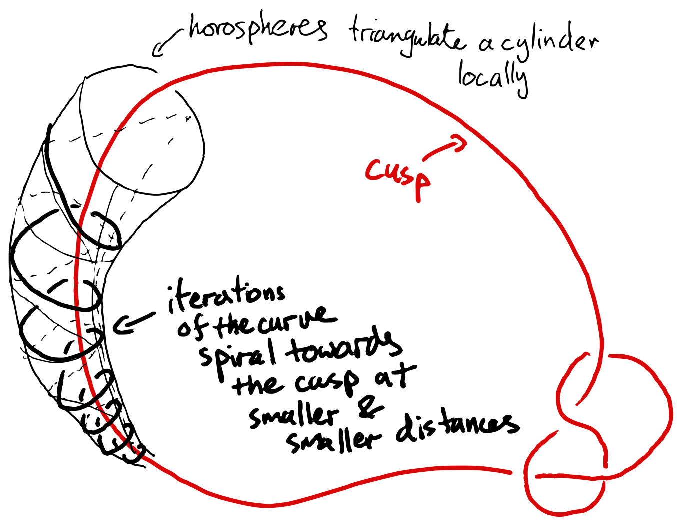

Figure 1.44: The lift of a curve in horospheres spiralling towards the cusp of an incomplete hyperbolic structure.

times as big, however, the metric is quadratically decaying in the height and therefore the length of the arc in the horosphere in that tetrahedron decreases by a factor of $\prod_{i=1}^{n} |z_i|$ at each iteration. This implies that the lift of the all the iterations of the curve has finite length[8]. Choosing the lifts of the initial point in this curve, therefore, gives a Cauchy sequence that is not convergent. This argument is discussed more generally in [183] and depicted here in Figure 1.44

Finally, the only way to find a manifold as the quotient under Euclidean isometries is to have no fixed points. This implies we must have a Euclidean torus and therefore that $\prod_{i=1}^{n} z_i = 1$ for all boundary curves. Equivalently, this means the holonomy at the boundary is parabolic, *i.e.* conjugate to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{1.35}$$

Example 5 (The figure eight knot)**.** *Consider the triangulation of the figure eight knot in Figure 1.29, which appears again in the top of Figure 1.45. We can compute the gluing equations as shown in Figure 1.45. This shows that the two tetrahedra that give the complete hyperbolic structure are given by the equilateral tetrahedra with shapes $e^{\frac{2\pi i}{6}}$. This already indicates that the hyperbolic volume of the figure eight knot complement is given by*

$$\mathrm{Vol}(S^3 - 4_1) \;=\; 2\mathrm{Vol}(\Delta_{e^{2\pi i/6}}) \;=\; 2D(e^{2\pi i/6}) \;=\; 2.0298832128193072500\cdots . \tag{1.36}$$

[8]This is because it converges like a geometric series with each iteration of the loop.

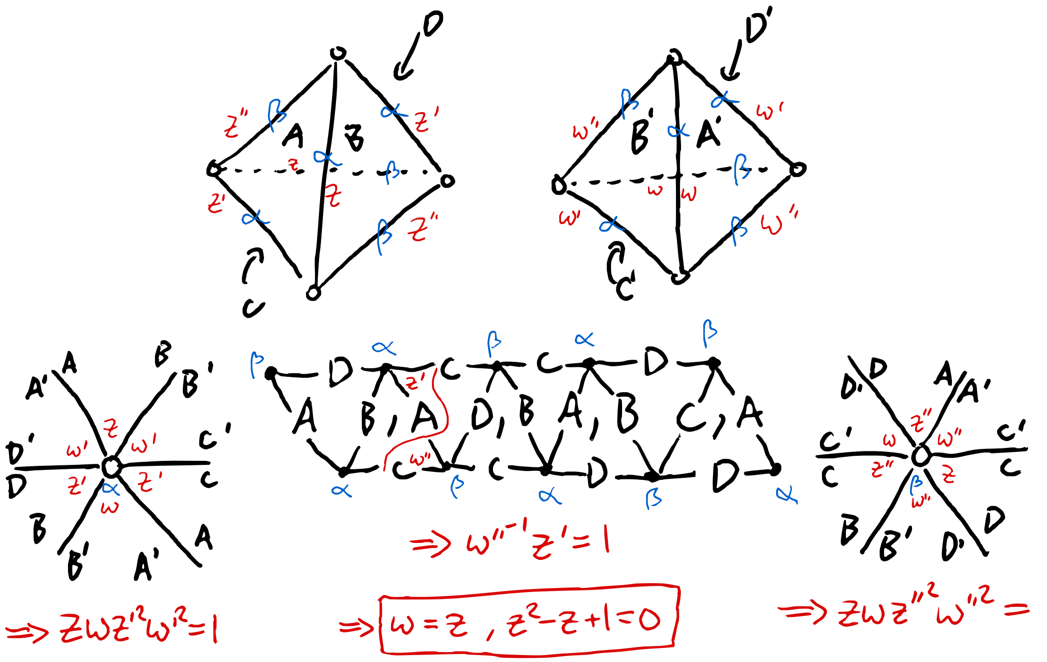

Figure 1.45: Gluing equations of the figure eight knot.

Although for knots we are interested mostly in the complete solution, when we perform surgery it becomes natural to take deformations of the hyperbolic structure on the knot complement, which will of course no longer be complete. When one performs surgery one of the peripheral curves is glued to a disk in the solid torus. This curve must then have trivial monodromy. Therefore, if we find a solution that makes this curve have trivial monodromy the non-complete structure on the knot complement will in fact give rise to a hyperbolic structure on the manifold obtained by surgery. The structure on the ideal triangulation is given by the hyperbolic structure on the manifold obtained by surgery with a geodesic removed (which corresponds to the solid torus we have filled in). The length of this geodesic can be computed from the cusp equations. Indeed, if M and L represent the holonomy of the longitude and the combination $\gamma m + \delta\ell$ is dual to the curve that is glued to the disk in the solid torus then the length is

$$\mathrm{length}(\gamma m + \delta\ell) \;=\; -\gamma \log|M| - \delta \log|L| \,. \tag{1.37}$$

More generally, we can compute the complex length by removing the absolute values. This captures some torsion of the curve as an imaginary part [146, 145].

With all of these gluing equations, we want a way to store this data and understand how the various choices effect it. This will done via Neumann-Zagier matrices and will be discussed in the next section.

1.11 Neumann–Zagier matrices of a knot

We want to completely describe the variety that gives the complete hyperbolic structure. Firstly, we need to understand how many edges and tetrehedra there are in the polyhedral complex. For a knot K, from for example Mayer-Vietoris, we have

$$H_k(S^3 - K) \;=\; \begin{cases} \mathbb{Z} & \text{if } k \;=\; 0,1 \\ 0 & \text{else} \end{cases} \,. \tag{1.38}$$

Therefore, the Euler characteristic vanishes. If we have an ideal triangulation of a knot complement with tetrahedra T, faces F and edges E, noting $\#F = 2\#T$, we have that

$$\#E \;=\; \#T \,. \tag{1.39}$$

Therefore, we have non–canonical bijections between the tetrahedra and the edges. Let this number be denoted by $\#E = \#T = N$. To store the data of the triangulation choose for each tetrahedra an edge, which will correspond to the edge connecting 0 and ∞ in the standard model. Let this edge for $t \in T$ have shape parameter z_t. Then every edge of the tetrahedron t are labeled by z_t, z'_t, z''_t following Figure 1.39. To each edge $e \in E$ in the triangulation we can store how many times the edges associated with z_t, z'_t, z''_t of a tetrahedron t appears. For the three pairs of edges this will give us a number in the set $\{0,1,2\}$. Denote these numbers by

$$G_{e,t}, \quad G'_{e,t}, \quad \text{and} \quad G''_{e,t}\,, \tag{1.40}$$

and note that

$$\sum_{e\in E} G_{e,t} \;=\; \sum_{e\in E} G'_{e,t} \;=\; \sum_{e\in E} G''_{e,t} \;=\; 2\,. \tag{1.41}$$

With this definition, the edge equation associated to e is given by

$$\prod_{t\in T} z_t^{G_{e,t}}(z'_t)^{G'_{e,t}}(z''_t)^{G''_{e,t}} \;=\; 1\,. \tag{1.42}$$

These equations have one redundancy associated the fact the product of all the equations (1.42) over e is one, which follows from equation (1.41) and the equation

$$zz'z'' \;=\; \frac{z(1-z^{-1})}{1-z} \;=\; -1\,. \tag{1.43}$$

To the edge equations, if we want the complete hyperbolic structure, we add the cusp equations. Choose curves generating the homology of the boundary and represent them as a path in the boundary using the truncated tetrahedra avoiding the edges, which are vertices on the triangulation of the boundary surface. Let $G_{c,t}, G'_{c,t}, G''_{c,t} \in \mathbb{Z}$ be a signed count of the number of times the cycle around the cusp passes an edge of the tetrahedron t, keeping track of the orientation. The cusp equations are then of exactly the same form as equation (1.42),

$$\prod_{t\in T} z_t^{G_{c,t}}(z'_t)^{G'_{c,t}}(z''_t)^{G''_{c,t}} \;=\; 1\,. \tag{1.44}$$

Choose a meridian for the cusp and denote it m. We can use equation (1.43) to reduce these equations further. For $d \in E \cup \{m\}$ let

$$A_{d,t} \;=\; G_{d,t} - G'_{d,t}\,, \quad B_{d,t} \;=\; G''_{d,t} - G'_{d,t}\,, \text{and} \quad \nu_d \;=\; 2 - 2\delta_{c,d} - \sum_{t\in T} G'_{d,t}\,. \tag{1.45}$$

The gluing equations equations now become

$$\prod_{t\in T} z_t^{A_{d,t}}(z''_t)^{B''_{d,t}} \;=\; (-1)^{\nu_d}\,. \tag{1.46}$$

Then, by dropping an edge $e \in E$ and labeling the remaining edges $1,\dots,N-1$, the cusp N, and the tetrahedra from $1,\dots N$ we can put this into a $N \times 2N$ matrix,

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,N} & B_{1,1} & \dots B_{1,N} \\ \vdots & \dots & \vdots & \vdots & \cdots \vdots \\ A_{N,1} & \dots & A_{N,N} & B_{N,1} & \dots B_{N,N} \end{pmatrix} \tag{1.47}$$

called a *Neumann–Zagier matrix*. We can add half of the additional cusp equation, corresponding to the longitude ℓ, as an additional row to get an *extended Neumann–Zagier matrix* .

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,N} & B_{1,1} & \dots B_{1,N} \\ \vdots & \dots & \vdots & \vdots & \dots \vdots \\ A_{N,1} & \dots & A_{N,N} & B_{N,1} & \dots B_{N,N} \\ C_{1,1} & \dots & C_{1,N} & D_{1,1} & \dots D_{1,N} \end{pmatrix} \tag{1.48}$$

This matrix satisfies an important symplectic property summarised in the following theorem.

Theorem A–9. *[144, Thm.4.1][146, Thm. 2.2] The matrix in equation (1.47) is half symplectic, meaning that AB^{T} is symmetric and that the $2N$ columns span $\mathbb{Z}^N$. Moreover, the matrix in equation (1.48) can be extended to a symplectic matrix in $\mathrm{Sp}_{2N}(\mathbb{Q})$[9].*

This result is presented here for manifolds with one torus boundary. However, it follows more generally for multiple torus boundaries. The matrices are then constructed by removing additional redundant edge equations and replacing them with meridian equations and adding the longitudes in the bottom half [146, 144]. For another introduction to Neumann–Zagier matrices, see [45]. A nice discussion of the work related to this result is given can be found in [84]. Also see [48] for another approach using abelianisation.

Example 6 (Figure eight knot)**.** *The previous example of the figure eight knot can now be put into the form of Neumann–Zagier. In particular, considering Figure 1.35 we find matrices*

$$A = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix} . \tag{1.49}$$

One can check that $(A\ B)$ is half symplectic.

Computation of the Neumann–Zagier matrices is implemented in SnapPy [43]. For example, we can compute for the figure eight knot again.

```
In[1]: M=Manifold("4_1")
In[2]: M.gluing_equations()
Out[2]:
matrix([[ 2,  1,  0,  2,  1,  0],
        [ 0,  1,  2,  0,  1,  2],
        [ 1,  0,  0,  0,  0, -1],
        [ 1,  1,  1,  1, -1, -3]])
```

The first two entries are the edge equations, the third represents the meridian, and the last is a longitude equation. Therefore, the first and third equations give

$$z^2 z' w^2 w' = 1 \quad \text{and} \quad z w''^{-1} = 1 \,. \tag{1.50}$$

[9] The denominators are at most 2

This gives Neumann–Zagier matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} . \tag{1.51}$$

Solving, we again find that $z = w = \exp(2\pi i/6)$. We can also read off the extended Neumann–Zagier matrices by considering the last row, which gives the cusp equation for the longitude. The equation is

$$w^2 w''^{-2} = 1 . \tag{1.52}$$

Therefore, the first row of the bottom of the extended Neumann–Zagier matrix is given by

$$C = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & -1 \end{pmatrix} . \tag{1.53}$$

SnapPy has the ability to draw in knots, but has stored a few simple knot. We can try with a few knots on the table 1.4. Next we will take 5_2 in Code 3. This shows that 5_2 has Neumann–Zagier matrices

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \tag{1.54}$$

Solving these equations gives

$$z_1^3 - 2z_1^2 + 3z_1 - 1 = 0 , \quad z_2 = z_1^2 - z_1 + 2 , \quad \text{and} \quad z_3 = z_1 . \tag{1.55}$$

The field defined by this variety is the cubic field of discriminant -23. Finally, we will compute for 7_4 in Code 4. This gives Neumann–Zagier matrices (in our convention as the computer uses z, z'^{-1})

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & -1 \end{pmatrix} \tag{1.56}$$

To compute the solutions this is best done with SnapPy [43] in Sage [182]. Here we find two Galois orbits of solutions. One is defined over the field generated by roots of the polynomial

$$x^4 + 3x^3 + 2x^2 + 1 , \tag{1.57}$$

while the other is defined over the field generated by roots of the polynomial

$$x^3 - 2x^2 - x - 2 . \tag{1.58}$$

This is done in Code 5.

1.12 Pachner moves and Neumann–Zagier equivalences

There are various choices that we have made in defining Neumann–Zagier matrices. These led to equivalence relations between half symplectic matrices. As usual, constructing invariants of half symplectic matrices under these moves leads to invariants of the manifolds. These equivalences are discussed in detail in [45]. We will recall them here. The choices we made in defining Neumann–Zagier matrices were as follows:

- choosing an ordering of a tetrahedra,
- choosing an ordering of the edges, an edge to discard, and a meridian,
- choosing an edge to label z_i for each tetrahedra (also know as a quad type).

We will now see how changing these choices alters the matrices. Suppose we alter the order of the tetrahedra. Then this simply permutes the columns of A and B. Let σ be the permutation matrix associated to the permutation of the columns. Then this says in matrix form that

$$\begin{pmatrix} A & B \end{pmatrix} \sim \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} . \tag{1.59}$$

Notice that $\sigma^{-1} = \sigma^T$ and therefore,

$$\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \in \mathrm{Sp}_{2N}(\mathbb{Z}) , \tag{1.60}$$

where

$$\mathrm{Sp}_{2N}(\mathbb{Z}) \;=\; \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2N}(\mathbb{Z}) : A^{\mathrm{T}}C = C^{\mathrm{T}}A\,, \quad B^{\mathrm{T}}D = D^{\mathrm{T}}B\,, \quad A^{\mathrm{T}}D - C^{\mathrm{T}}B = I_N \right\} . \tag{1.61}$$

If we alter the order of the edges, the edge we remove, or the meridian's path, this will alter the Neumann-Zagier matrices by multiplication on the left by some special elements $P \in \mathrm{GL}_N(\mathbb{Z})$. See [45, Sec. 3.3 and 3.4.]. However, more generally, it won't affect solutions if we multiply by any $P \in \mathrm{GL}_N(\mathbb{Z})$. Therefore, for any $P \in \mathrm{GL}_N(\mathbb{Z})$ we have

$$\begin{pmatrix} A & B \end{pmatrix} \sim P \begin{pmatrix} A & B \end{pmatrix} . \tag{1.62}$$

Notice that

$$\begin{pmatrix} P & 0 \\ 0 & P^{-\mathrm{T}} \end{pmatrix} \in \mathrm{Sp}_{2N}(\mathbb{Z}) . \tag{1.63}$$

Suppose we alter the edge we chose for the first tetrahedra. That is, we rewrite the equations in z_1' and z_1 as opposed to z_1 and z_1''. Then as $z_1'' = -z_1^{-1} z_1'^{-1}$ we find that the first column

$A_{-,1}$ becomes $-B_{-,1}$ while the first column $B_{-,1}$ becomes $A_{-,1} - B_{-,1}$. This can be written in matrix form

$$\begin{pmatrix} A & B \end{pmatrix} \sim \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \tag{1.64}$$

where

$$\begin{aligned} Q_{11} &= \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad Q_{12} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \\ Q_{21} &= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad Q_{22} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \end{aligned} \tag{1.65}$$

Notice that

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \in \mathrm{Sp}_{2N}(\mathbb{Z}) . \tag{1.66}$$

These equivalences generate a relation that makes all of the different choices lead to the same equivalence class. However, this says nothing about the relations between different triangulations. To understand how to relate ideal triangulations we need to understand *Pachner moves*. The only Pachner move we will need is the 2–3 Pachner move. This takes two geometric tetrahedra glued along a face and adds an edge through this face from the vertices not bounding the glued faces in the tetrahedra. This gives three tetrahedra and can be seen in Figure 1.46. The shapes completely determine each other. This move generates the equivalence relation on triangulations of cusped manifolds.

Theorem A–10. *[130, 131, 158] All ideal triangulations of a three–manifold are related by* 2–3 *Pachner moves.*

From Figure 1.46 we get the following equations for the new variables

$$w_1' = z_1 z_2 , \quad w_2' = z_1' z_2'' , \quad \text{and} \quad w_3' = z_1'' z_2' , \tag{1.67}$$

and inversely

$$\begin{aligned} z_1 &= w_2 w_3'' , \quad & z_1' &= w_3 w_1'' , \quad & z_1'' &= w_1 w_2'' , \\ z_2 &= w_2'' w_3 , \quad & z_2' &= w_1'' w_2 , \quad & z_2'' &= w_3'' w_1 . \end{aligned} \tag{1.68}$$

This implies that going from the matrices $(A\ B)$ associated to the z variables we find that

$$\begin{pmatrix} A & B \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{1.69}$$

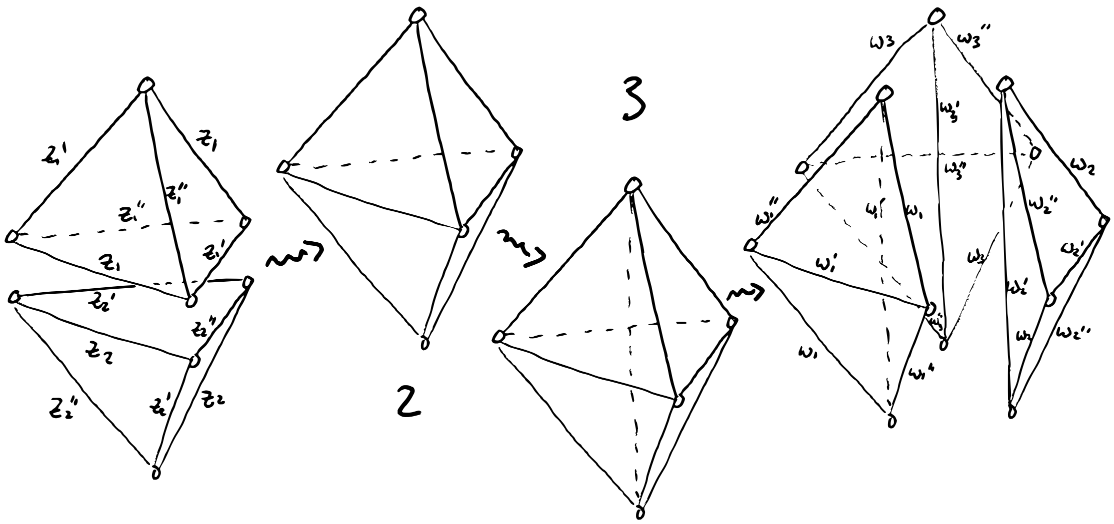

Figure 1.46: 2-3 Pachner move with shapes labelled. See equations (1.67) (1.68) for the relations between the shapes.

where

$$
\begin{aligned}
T_{11} &= \begin{pmatrix} -1 & -1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad T_{12} = \begin{pmatrix} -1 & -1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \\
T_{21} &= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad T_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.
\end{aligned}
\tag{1.70}
$$

Notice that

$$
\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathrm{Sp}_{2N+2}(\mathbb{Z})\,. \tag{1.71}
$$

Therefore, we see that all Neumann–Zagier matrices for a given manifold are related by the equivalences in equations (1.59) (1.62) (1.64) (1.69).

Remark 2. *Notice that these equivalences can be lifted from half symplectic matrices to full symplectic matrices. Then we can add additional equivalence given by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.72}$$

where $S = S^{\mathrm{T}} \in M_{N\times N}(\mathbb{Z})$. The set of equivalence classes will then be the same as this fixes the $\begin{pmatrix} A & B \end{pmatrix}$ but gives all completions to the lower half of the symplectic matrix.

1.13 The Chern–Simons functional

The Chern–Simons functional gives an invariant of connections on a three–manifold [61]. More generally, Chern–Simons forms [40, 37], are antiderivatives of characteristic classes on a manifold M formed by taking polynomials in the curvature form on $M \times [0,1]$. Recall, that for a Lie group G, with Lie algebra $\mathfrak{g}$, a connection on a principle G bundle $\pi : P \to M$ is equivalently defined to be either: a splitting of the tangent bundle TP into vertical and horizontal sub-bundles, an equivariant projection onto the vertical subbundle of TP, or a connection 1–form (a form that looks like the Maurer–Cartan on the fibres). The space of connections $\mathcal{A}_P$ is an affine space modelled on $\Omega^1(M, \mathfrak{g}_P)$, the 1–forms on M valued in the adjoint bundle $\mathfrak{g}_P$. Locally in some neighbourhood $U \subseteq M$ (or globally when the bundle is trivial $P = M \times G$) there is a canonical connection associated to a trivialisation of the bundle. Moreover, $\Omega^1(M, \mathfrak{g}_P)$ are locally given by elements of $\Omega^1(U, \mathfrak{g})$, Lie algebra valued one-forms. The automorphisms of the bundle, $\mathcal{G}_P$, act on the space of connections. These automorphisms are called gauge transformations. They are locally given by $g : U \to G$ and act locally via

$$g \cdot A \;=\; g^{-1}Ag - g^{-1}dg\,. \tag{1.73}$$

The Chern–Simons invariant is almost independent of gauge in $\mathbb{C}$. It is defined for closed three–manifolds with trivial bundle as

$$\mathrm{CS} : \mathcal{A}_{M\times G}/\mathcal{G}_{M\times G} \to \mathbb{C}/4\pi^2\mathbb{Z} \quad s.t. \quad \mathrm{CS}[A] \;=\; \int_M \mathrm{Tr}\Big(dA \wedge A + \frac{2}{3} A \wedge A \wedge A\Big)\,, \tag{1.74}$$

where Tr is an invariant bilinear form normalised so that, for the Maurer–Cartan form on G given by $g^{-1}dg$, the class

$$\frac{1}{6\pi^2}\mathrm{Tr}\big(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\big) \tag{1.75}$$

is an integral cohomology class. For the case that $G = \mathrm{SL}_2(\mathbb{C})$ taking the standard $\mathrm{Tr}(a,b) = \mathrm{Tr}(ab)$ gives such an example. The reason this is only valued in $\mathbb{C}/4\pi^2\mathbb{Z}$ is that

$$\mathrm{CS}[g\cdot A] \;=\; \mathrm{CS}[A] - \frac{2}{3}\int_M \mathrm{Tr}\big(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\big) \in \mathrm{CS}[A] + 4\pi^2\mathbb{Z}\,, \tag{1.76}$$

as the second integral is the integral of a pullback under the gauge transformation g of $4\pi^2$ times the integral cohomology class in equation (1.75). When M has boundary, the Chern–Simons invariant has an additional boundary term under gauge transformations given by

$$\mathrm{CS}[g\cdot A]-\mathrm{CS}[A] \;=\; \int_M d\,\mathrm{Tr}\big(g^{-1}Ag\wedge g^{-1}dg\big)-\frac{2}{3}\int_M \mathrm{Tr}\big(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\big)\,. \quad (1.77)$$

Taking $\exp((\mathrm{CS}[g\cdot A]-\mathrm{CS}[A])/2\pi i)$ then gives a cocycle on the boundary ∂M. This implies that, generally the Chern–Simons invariant naturally lives on a complex line bundle on the boundary called the Chern–Simons lines, discussed for example in [61, 162]. Now we will almost exclusively be interested in this invariant at special connections called *flat connections*. A connection is called flat when its curvature,

$$F_A \;=\; dA+\frac{1}{2}A\wedge A\in\Omega^2(M,\mathfrak{g}_P)\,, \quad (1.78)$$

vanishes. The flat connections up to gauge equivalence on a connected manifold are equivalent to the representations of the fundamental group up to conjugation *i.e.*

$$\bigsqcup_{P/\sim}\mathcal{A}_P^{\mathrm{flat}}/\mathcal{G}_P\cong\mathrm{Hom}(\pi_1(M),G)/G \;=:\; \mathcal{R}(M,G)\,. \quad (1.79)$$

This equivalence is constructed via holonomy representations. The Chern–Simons lines can be used to define a line bundle on the analogue of this space on the boundary [61, 115]. For $G=\mathrm{SL}_2(\mathbb{C})$, we will often be interested in finitely many flat connections up to gauge. Then, the line bundles trivialise at these points and we will just get complex numbers.

A complete hyperbolic three–manifold M can be expressed as the quotient of hyperbolic three–space $\mathbb{H}^3$ via some subgroup $\Gamma\subseteq\mathrm{SL}_2(\mathbb{C})$. This gives an isomorphism $\Gamma\cong\pi_1(M)$ and therefore a flat $\mathrm{SL}_2(\mathbb{C})$–connection on M. We call this connection the *geometric connection*. Noting that $\mathrm{PSL}_2/\mathrm{SO}(3)=\mathbb{H}^3$, this manifold also has an associated flat $\mathrm{SO}(3)$–connection. The $\mathrm{SL}_2(\mathbb{C})$ Chern–Simons invariant of the geometric connection splits into a real part, given by the Chern–Simons invariant of the natural $\mathrm{SO}(3)$ connection, and an imaginary part, given by the volume of the manifold. This was first suspected by Thurston [184], extended in [146] and proved in [200]. This is one indication that the calculation of the Chern–Simons invariant could be abstractly made with simplices. Summing over contributions from simplices is one of the most natural ways to compute volume. This indeed extends to the complexified volumes or Cheeger–Chern–Simons invariant and will be discussed in the next section. Alternatively, to compute the values at flat connections using the connections more explicitly see [114, 115, 127]. However, even there simplices play an important role.

Remark 3. *The geometric connection was implicitly constructed in Section 1.10 via tetrahedra. Tetrahedra can also be used to construct other* $\mathrm{SL}_2(\mathbb{C})$*–connections as well. This is done by gluing the flat connections associated to the to the shape parameters of the tetrahedra. These all correspond to special points on the* A*–polynomial.*

1.14 The Bloch group

The 2–3 Pachner move completely describes the equivalence relation between triangulations. This relation, at the geometric level, can be used to define an algebraic relation. This kind of construction was introduced by Dehn in his solution to Hilbert's third problem. There, he constructs an algebraic invariant of polyhedra in Euclidean three space that is invariant under cutting and pasting along planes. Two polyhedra obtained by cutting and gluing in this way are called *scissor congruent*. Dehn showed that the cube and the tetrahedron have different algebraic invariants, which solved the long standing question, in the negative, of whether the formula for the volume of a pyramid could be proved without methods of exhaustion. We can construct a similar invariant of ideal hyperbolic polyhedra [52, 53]. This is done by decomposing into tetrahedra and taking formal sums while keeping track of the shape parameters associated to each tetrahedra. This captures part of the information of the 2–3 Pachner move, as any triangulations related by this move are scissor congruent. This algebraic construction can be defined more generally over any field and relates to algebraic K–theory of that field.

Let $\mathbb{K}$ be a field. Then consider the linear map

$$d : \mathbb{Z}[\mathbb{KP}^1] \to \mathbb{K}^\times \wedge_{\mathbb{Z}} \mathbb{K}^\times \quad s.t. \quad d([z]) \;=\; 2(z \wedge (1-z))\,, \tag{1.80}$$

where $d([0]) = d([1]) = d([\infty]) = 0$. Notice that letting

$$c(x,y) \;=\; [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right], \tag{1.81}$$

we have

$$d(c(x,y)) \;=\; 0\,. \tag{1.82}$$

Notice that $c(x,y)$ is very closely related to the behaviour of the shapes in the 2–3 move 1.46. In fact, setting $x = x_1, y = 1/x_2$ and using the relations below in equation (1.84) gives the equivalence. Let $\mathcal{F}$ be the module generated by $c(x,y)$ where we avoid any x, y that lead to terms of the form $0/0, \infty/\infty$.

Definition 5 (Bloch group)**.** *The Bloch group is defined to be the quotient of the kernel of d by the five term relation* (1.81)*,* i.e.

$$\mathcal{B}(\mathbb{K}) \;=\; \ker(d)/\mathcal{F}\,. \tag{1.83}$$

Considering combinations of $c(x,y)$ for some $x, y \in \{0, 1, \infty\}$ gives some immediate relations

$$[x] \;=\; \left[\frac{1}{1-x}\right] \;=\; \left[1-x^{-1}\right] \;=\; -\left[x^{-1}\right] \;=\; -[1-x] \;=\; -\left[\frac{1}{1-x^{-1}}\right] \tag{1.84}$$

and

$$[0] \;=\; [1] \;=\; [\infty] \;=\; 0\,. \tag{1.85}$$

Compare these relations with the properties of the ideal hyperbolic tetrahedron discussed in Section (1.9). We will mainly be interested in $\mathcal{B}(\mathbb{C})$. This Bloch group fits into a short exact sequence [145]

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3^{\text{grp}}(\text{PSL}_2(\mathbb{C}), \mathbb{Z}) \to \mathcal{B}(\mathbb{C}) \to 0\,. \tag{1.86}$$

This can be described more generally for infinite fields using the third indecomposable algebraic K group [180, 210] as

$$0 \to \widetilde{\mu_{\mathbb{K}}} \to K_3^{\text{ind}}(\mathbb{K}) \to \mathcal{B}(\mathbb{K}) \to 0\,, \tag{1.87}$$

where $\widetilde{\mu_{\mathbb{K}}}$ is the unique non-trivial extension of the roots of unity by $\mathbb{Z}/2\mathbb{Z}$ (and in characteristic 2 it is just the roots of unity $\mu_{\mathbb{K}}$). The description of the Bloch group, at least for number fields, can be extended to include torsion giving a presentation of $K_3^{\text{ind}}(\mathbb{K})$ [210]. We will, however, mainly be interested in the case of $\mathbb{K} = \mathbb{C}$ covered in [145]. This will be discussed below in definition 6.

Firstly, an obvious question is whether we can find elements of $\mathcal{B}(\mathbb{C})$. In fact, three–manifolds provide the ability to construct elements using Neumann–Zagier matrices. However, more generally half symplectic matrices lead to elements. Let $\begin{pmatrix} A & B \end{pmatrix}$ be an $N \times 2N$ half symplectic matrix with completion

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2N}(\mathbb{Z}) \tag{1.88}$$

and let $z_i \in \mathbb{CP}^1 - \{0, 1, \infty\}$ such that

$$\prod_{j=1}^{N} z_j^{A_{i,j}} (1 - z_j)^{B_{i,j}} \;=\; 1\,. \tag{1.89}$$

Then we find that

$$\begin{aligned} d\Big(\sum_{i=1}^{N} z_i\Big) \;&=\; \sum_{i=1}^{N} z_i \wedge (1 - z_i) \;=\; \sum_{i,j,k=1}^{N} (A_{j,i}D_{j,k} - C_{j,i}B_{j,k}) z_i \wedge (1 - z_k) \\ &=\; \sum_{j,k=1}^{N} D_{j,k}\Big(\prod_{i=1}^{N} z_i^{A_{j,i}}\Big) \wedge (1 - z_k) - \sum_{i,j=1}^{N} C_{j,i} z_i \wedge \Big(\prod_{k=1}^{N} (1 - z_k)^{B_{j,k}}\Big) \\ &=\; \sum_{j,k=1}^{N} D_{j,k}\Big(\prod_{i=1}^{N} (1 - z_i)^{-B_{j,i}}\Big) \wedge (1 - z_k) - \sum_{i,j=1}^{N} C_{j,i} z_i \wedge \Big(\prod_{k=1}^{N} z_k^{-A_{j,k}}\Big) \\ &=\; -\sum_{i,j,k=1}^{N} B_{j,i}D_{j,k}(1 - z_i) \wedge (1 - z_k) + \sum_{i,j,k=1}^{N} C_{j,i}A_{j,k} z_i \wedge z_k \;=\; 0\,, \end{aligned} \tag{1.90}$$

where the second equality follows from the fact that $A^{\mathrm{T}}D - C^{\mathrm{T}}B = I$, and the last equality follows form the fact $B^{\mathrm{T}}D$ and $C^{\mathrm{T}}A$ are symmetric, exactly the conditions given by equation (1.88). Now we can construct large numbers of elements, however an obvious question is whether these are non-trivial. The easiest way to check this is using a homomorphism to $\mathbb{R}$. This will be discussed further in the next Section 1.15. For now, we will desribe how to deal with the torsion elements in $H_3^{\mathrm{grp}}(\mathrm{PSL}_2(\mathbb{C}), \mathbb{Z})$.

To deal with the torsion we need to introduce *flattenings*. These are logarithms of the shapes such that the sum over the three logarithms vanishes. A combinatorial flattening of an ideal tetrahedra with shape z is given by

$$\ell(z;p,q) \;=\; (\log(z) + p\pi i, -\log(1-z) + q\pi i, -\log(z) + \log(1-z) - p\pi i - q\pi i)\,. \tag{1.91}$$

This can be thought of as a map from $\widehat{\mathbb{C}}$, the Riemann–surface for the multivalued function $(\log(z) + p\pi i, -\log(1-z) + q\pi i)$ [145, Sec.2], to $\mathbb{C}^3$. Then the lift of d in equation (1.80) is given by the linear map

$$\widehat{d} : \mathbb{Z}[\widehat{\mathbb{C}}] \to \mathbb{C} \wedge_{\mathbb{Z}} \mathbb{C} \quad s.t. \quad \widehat{d}([z;p,q]) \;=\; (\log(z) + p\pi i) \wedge (-\log(1-z) + q\pi i)\,. \tag{1.92}$$

The analogue of the 5–term relation must also be lifted. Consider again the 2–3 move in Figure 1.46. This should give a five term relation where we insist that the additional edge, in the middle, has the sum of the three contributions vanish. Checking the implications this has on the flattenings [145, Lem. 3.4], we find that the correct five term relation lifting equation (1.81) is given by

$$\begin{aligned} c(x,y;p_0,p_1,q_0,q_1,q_2) \;&=\; [x;p_0,q_0] - [y;p_1,q_1] + \left[\frac{y}{x};p_1-p_0,q_2\right] \\ -\left[\frac{1-x^{-1}}{1-y^{-1}};p_1-p_0+q_1-q_0,q_2-q_1\right] &+ \left[\frac{1-x}{1-y};q_1-q_0,q_2-q_1-p_0\right]. \end{aligned} \tag{1.93}$$

We include the additional *transfer relation*

$$[z;p,q] + [z;p',q'] \;=\; [z;p,q'] + [z;p',q]\,, \tag{1.94}$$

which, when multiplied by two, is a combination of 5–term relations. Again, we can explicitly compute that, for x, y such that all the terms in equation (1.81) are in $\mathfrak{h}$,

$$\widehat{d}(c(x,y;p_0,p_1,q_0,q_1,q_2)) \;=\; 0\,. \tag{1.95}$$

Let $\widehat{\mathcal{F}}$ be the module generated by $c(x,y;p_0,p_1,q_0,q_1,q_2)$ with x, y such that all terms in equation (1.81) are in $\mathfrak{h}$ and the elements $[z;p,q] + [z;p',q'] - [z;p,q'] - [z;p',q]$.

Definition 6 (The extended Bloch group)**.** *The extended Bloch group is defined to be the quotient of the kernel of* $\widehat{d}$ *by the extended five term relation and the transfer relation,* i.e.

$$\widehat{\mathcal{B}}(\mathbb{K}) \;=\; \ker(\widehat{d})/\widehat{\mathcal{F}}\,. \tag{1.96}$$

The extended Bloch group exactly describes the addition of torsion to the group homology of $H_3^{\mathrm{grp}}(\mathrm{PSL}_2(\mathbb{C}),\mathbb{Z})$. Neumann [145, Thm. 2.6] proves the following theorem with an explicit isomorphism.

Theorem A–11. *There is an isomorphism*

$$H_3^{\mathrm{grp}}(\mathrm{PSL}_2(\mathbb{C}),\mathbb{Z}) \cong \widehat{\mathcal{B}}(\mathbb{C})\,. \tag{1.97}$$

Neumann [145, Sec. 14] goes on to show that for ideal triangulations of three–manifolds and surgery on them, we can construct elements of the extended Bloch group for any flat $\mathrm{SL}_2(\mathbb{C})$ connection. This is done by solving the logarithmic gluing equations (1.42)

$$\begin{aligned}\sum_{t\in T} G_{d,t}(\log(z_t)+p_t\pi i) + G'_{d,t}(-\log(1-z_t)+q_t\pi i)&\\ + G''_{d,t}(-\log(z_t)+\log(1-z_t)-p_t\pi i - q_t\pi i) \;=\; 0\,,&\end{aligned} \tag{1.98}$$

for all $d\in E\cup C$ where E is the set of edges and C is the set of peripheral curves (including the longitude). Importantly, not all connections will appear as solutions to the gluing equations for a given triangulation. However, the connection corresponding to the geometric structure will always appear. The reasons other solutions may not appear is that that may correspond to solutions where $z\in\{0,1,\infty\}$. The class corresponding to the geometric connection will be a solution with all shapes in $\mathfrak{h}$. This will give the complete hyperbolic structure and all solutions with this property will give the same class in the Bloch group by Mostow–Prasad rigidity, given in Theorem 7.

Remark 4. *There is another notation of flattening which is an integral solution to gluing equations. This is used for example in [45, 144] and it constructs elements in a quotient of the extended Bloch group, which removes some torsion.*

Example 7 (Elements of the extended Bloch group from 5_2)**.** *We can represent the element of the extended Bloch group corresponding to a solution* $z_1^{(j)}, z_2^{(j)}, z_3^{(j)}$ *for the gluing equations* (1.55) *for* 5_2 *as*

$$[z_1^{(j)};p_1^{(j)},q_1^{(j)}] + [z_2^{(j)};p_2^{(j)},q_2^{(j)}] + [z_3^{(j)};p_3^{(j)},q_3^{(j)}]\,. \tag{1.99}$$

Then we must check the logarithmic gluing equations around the edges and the peripheral curves. This means we need the extended Neumann–Zagier matrix, which includes the longitude 1.48. Considering, the solution to the gluing equations with $z_1^{(3)} = 0.78492\dots + 1.3071\dots i$ *(the third solution in PARI/GP indexing* `z1[3]`*) we get*

$$p_1^{(3)} \;=\; p_3^{(3)}\,,\quad q_1^{(3)} \;=\; q_3^{(3)}\,,\quad p_2^{(3)} \;=\; -2p_3^{(3)} - 2q_3^{(3)} + 2\,,\quad q_2^{(3)} \;=\; p_3^{(3)} + 2q_3^{(3)} + 1\,. \tag{1.100}$$

If we consider the real embedding $z_1^{(1)} = 0.43015\cdots$ *(the first solution in PARI/GP indexing* z1[1]*), then from a similar computation we find that*

$$p_1^{(1)} = p_3^{(1)}\,, \quad q_1^{(1)} = q_3^{(1)}\,, \quad p_2^{(1)} = -2p_3^{(1)} - 2q_3^{(1)}\,, \quad q_2^{(1)} = p_3^{(1)} + 2q_3^{(1)} + 1\,. \tag{1.101}$$

This is illustrated for the first example in the PARI/GP [20] Code 6.

Example 8 (Elements of the extended Bloch group from surgery on 5_2)**.** *Now the equations for the holonomy of the meridian and longitude are given by*

$$M = z_1^{-1}(1-z_2^{-1})\,, \quad \text{and} \quad L = z_1^2(1-z_1^{-1})^{-3}z_2(1-z_2^{-1})^{-2}(1-z_3^{-1})\,. \tag{1.102}$$

Therefore, the equations for the hyperbolic structure on $5_2(1,2)$ *are given by*

$$\begin{aligned} z_1z_2^{-1}z_3(1-z_1^{-1})(1-z_2^{-1})^{-2}(1-z_3^{-1}) &= 1 \\ z_1^{-1}z_3^{-1}(1-z_2^{-1})^2 &= 1 \\ ML^2 = (z_1^{-1}(1-z_2^{-1}))(z_1^2(1-z_1^{-1})^{-3}z_2(1-z_2^{-1})^{-2}(1-z_3^{-1}))^2 &= 1 \end{aligned} \tag{1.103}$$

This has solutions, (which can be computed for example with Mathematica [101])

$$\begin{aligned} 0 &= 1-7z_1+21z_1^2-29z_1^3+4z_1^4+41z_1^5-36z_1^6-40z_1^7+101z_1^8-81z_1^9+29z_1^{10}-4z_1^{11}+z_1^{12} \\ z_2 &= \frac{1}{809}(-8072+44732z_1-81810z_1^2+22655z_1^3+115415z_1^4-112614z_1^5-100341z_1^6 \\ &\qquad +254285z_1^7-185366z_1^8+56695z_1^9-8819z_1^{10}+1811z_1^{11}) \\ z_3 &= \frac{1}{809}(-3693+25399z_1-62740z_1^2+47520z_1^3+65783z_1^4-129841z_1^5-25478z_1^6 \\ &\qquad +206416z_1^7-179494z_1^8+48283z_1^9-8593z_1^{10}+1527z_1^{11}) \end{aligned} \tag{1.104}$$

Then taking the solution (the third in PARI/GP indexing) with $z_1 = -0.72561\cdots - 0.32484\cdots i$ *we find that*

$$\begin{aligned} p_1 &= \frac{15}{7}p_3 - \frac{8}{7}q_2 + \frac{16}{7}q_3 + \frac{12}{7}\,, \quad q_1 = -\frac{11}{7}p_3 + \frac{11}{7}q_2 - \frac{15}{7}q_3 + \frac{1}{7}\,, \\ p_2 &= -\frac{11}{7}p_3 - \frac{3}{7}q_2 - \frac{8}{7}q_3 - \frac{13}{7}\,. \end{aligned} \tag{1.105}$$

These have the potential to not be integers. However, there are additional parity parameters [145, Def. 4.3], which restrict the allowed values of p_j, q_j*. However, ignoring this we find the same elements up to two torsion [145, Lem. 11.3], so we can choose integral solutions to get the element up to two torsion. For example,*

$$p_1 = 4\,, \quad q_1 = -3\,, \quad p_2 = -1\,, \quad q_2 = -2\,, \quad p_3 = 0\,, \quad q_3 = 0\,. \tag{1.106}$$

1.15 Complexified volumes using scissors

We have been discussing abstractly some of the structures behind the geometry of three–manifolds but lets take a step back. Given a geometric manifold it is interesting to see whether you can compute its volume. A natural way to do this is by breaking the manifold into simpler pieces, calculating their volumes, and summing them up. Of course, calculating volume is a local calculation. The theory we have been building up over the last sections gives us the tools to do this. Given an ideal triangulation and the unique solution to the gluing equations giving the complete hyperbolic structure, coming from the Mostow-Prasad Theorem 7, we find that we simply sum the contributions from each tetrahedra using Theorem 8. We already did this in the Example 5. We can go one step further and use the extended Bloch group to not only compute volumes but also to compute complexified volumes *i.e.* the Cheeger–Chern–Simons invariant.

The first thing to note is that the 2–3 Pachner move indicates a consequence for the volumes. In particular, we get the following theorem due originally due to Spence.

Theorem A–12 (5–term relation)**.** *Let $x, y \in \mathbb{C}$ then*

$$\mathrm{Vol}(\Delta_x) + \mathrm{Vol}\big(\Delta_{\frac{y}{x}}\big) + \mathrm{Vol}\big(\Delta_{\frac{1-x}{1-y}}\big) \;=\; \mathrm{Vol}(\Delta_y) + \mathrm{Vol}\Big(\Delta_{\frac{1-x^{-1}}{1-y^{-1}}}\Big) \tag{1.107}$$

or for the Bloch–Wigner dilogarithm in equation (1.26)

$$D(x) - D(y) + D\left(\frac{y}{x}\right) - D\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + D\left(\frac{1-x}{1-y}\right) \;=\; 0\,. \tag{1.108}$$

The relations in the Bloch group, and its extended version, exactly fit the geometry of ideal tetrahedra. This theorem of course has the following consequence for the Bloch group.

Corollary 1. *The Bloch–Wigner dilogarithm given in equation* (1.26) *gives a homomorphism from the Bloch group to $\mathbb{R}$ via*

$$D : \mathcal{B}(\mathbb{C}) \to \mathbb{R} \quad s.t. \quad D([z]) \;=\; D(z)\,. \tag{1.109}$$

We can use this to compute volumes of three–manifolds and check non-triviality of elements of the Bloch group.

Example 9 (Bloch–Wigner dilogarithms for calculating volumes and non-triviality)**.** *Consider the elements we constructed for the knots $4_1, 5_2$ and the closed manifold $5_2(1,2)$ in equations* (1.50) (1.55) (1.104)*. In particular, we find that for 4_1*

$$2[e^{2\pi i/6}] \mapsto 2D(e^{2\pi i/6}) \;=\; 2.0299\cdots \;=\; \mathrm{Vol}(S^3 - 4_1)\,, \tag{1.110}$$

for 5_2 *with* $z_1^3 - 2z_1^2 + 3z_1 - 1$ *with* $z_1 = 0.78492\cdots + 1.3071\cdots i$ *we find*

$$2[z_1] + [z_1^2 - z_1 + 2] \mapsto 2D(z_1) + D(z_1^2 - z_1 + 2) \;=\; 2.8281\cdots \;=\; \mathrm{Vol}(S^3 - 5_2)\,, \tag{1.111}$$

and for $5_2(1,2)$ *with* z_1, z_2, z_3 *given in equation* (1.104) *with embeddings*

$$\begin{aligned} z_1 &= 0.29858\cdots + 4.7288\cdots i\,, \\ z_2 &= -0.031108\cdots + 0.55397\cdots i\,, \\ z_3 &= 0.80772\cdots + 0.47941\cdots i\,, \end{aligned} \tag{1.112}$$

we find that

$$[z_1] + [z_2] + [z_3] \mapsto D(z_1) + D(z_2) + D(z_3) \;=\; 2.2267\cdots \;=\; \mathrm{Vol}(5_2(1,2))\,. \tag{1.113}$$

Therefore, all of these elements are non–trivial in the Bloch group as they don't vanish under this homomorphism. Consequentially, this implies they are non–torsion in K*–theory.*

As mentioned, we can use the extended Bloch group to calculate the complexified volume. To do this, let

$$R(z;p,q) \;=\; \frac{1}{2}\log(z)\log(1-z) - \int_0^z \log(1-t)\frac{dt}{t} + \frac{\pi i}{2}(p\log(1-z) + q\log(z)) - \frac{\pi^2}{6}\,. \tag{1.114}$$

Then, R is a well defined map from $\widehat{\mathbb{C}} \to \mathbb{C}/\pi^2\mathbb{Z}$. Moreover, R satisfies the relations given by the generators of $\widehat{\mathcal{F}}$. Therefore, this induces a homomorphism

$$R : \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z} \quad s.t. \quad R([z;p,q]) \;=\; R(z;p,q)\,. \tag{1.115}$$

The Cheeger–Chern–Simons class gives a map

$$CS : H_3^{\mathrm{grp}}(\mathrm{PSL}_2(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}/\pi^2\,. \tag{1.116}$$

Then Neumann [145, Thm. 2.6] proves the following theorem.

Theorem A–13. *There is an isomorphism*

$$\lambda : H_3^{\mathrm{grp}}(\mathrm{PSL}_2(\mathbb{C}), \mathbb{Z}) \cong \widehat{\mathcal{B}}(\mathbb{C}) \tag{1.117}$$

such that

$$CS(\gamma) \;=\; R(\lambda(\gamma))\,. \tag{1.118}$$

See Neumann [145, Sec. 15] for an example with the figure eight knot. Also see [80] for a discussion of how this generalises to $\mathrm{SL}_N(\mathbb{C})$. We will continue our other examples.

Example 10 (Complexified volumes for 5_2 and $5_2(1,2)$)**.** *Now with our elements of the extended Bloch group given in Example 7 and Example 8 we can calculate their associated complexified volumes. So taking the elements $z_j^{(k)}$ and their flattening $p_j^{(k)}$ from Example 7 we have*

$$\begin{aligned} &R(z_1^{(3)};p_3^{(3)},q_3^{(3)}) + R(z_2^{(3)};-2p_3^{(3)}-2q_3^{(3)}-2,p_3^{(3)}+2q_3^{(3)}+1) + R(z_3^{(3)};p_3^{(3)},q_3^{(3)}) \\ &\qquad = -\pi^2 q_3^{(3)} - 6.8455\cdots + 2.8281\cdots i = \mathrm{VC}_{\rho_3}(5_2) \\ &R(z_1^{(1)};p_3^{(3)},q_3^{(3)}) + R(z_2^{(1)};-2p_3^{(3)}-2q_3^{(3)},p_3^{(3)}+2q_3^{(3)}+1,-1) + R(z_3^{(1)};p_3^{(3)},q_3^{(3)}) \\ &\qquad = \pi^2 p_3^{(3)} + \pi^2 q_3^{(3)} - 1.1135\cdots = \mathrm{VC}_{\rho_1}(5_2) \end{aligned} \tag{1.119}$$

We find, for the element given in Example 8 by $z_1 = -0.72561\cdots - 0.32484\cdots i$, with its flattening, that

$$\begin{aligned} &R\Big(z_1, \frac{15}{7}p_3 - \frac{8}{7}q_2 + \frac{16}{7}q_3 + \frac{12}{7}, -\frac{11}{7}p_3 + \frac{11}{7}q_2 - \frac{15}{7}q_3 + \frac{1}{7}\Big) \\ &\qquad + R\Big(z_2, -\frac{11}{7}p_3 - \frac{3}{7}q_2 - \frac{8}{7}q_3 - \frac{13}{7}, q_2\Big) + R(z_3,p_3,q_3) \\ &\qquad = \frac{\pi^2}{2}q_2 - \frac{3\pi^2}{2}p_3 - 2\pi^2 q_3 - 11.995\cdots = \mathrm{VC}_{\rho_1}(5_2(1,2)) \end{aligned} \tag{1.120}$$

Notice that without keeping track of the parity, as mentioned, this only gives an element up to $\pi^2/2$. The example for 5_2 can be computed using the PARI/GP [20] Code 7.

Now for cusped manifolds these computations are in fact implemented in SnapPy [43] in Sage [182]. However, these can differ by multiples of $\pi^2/6$. This is given in Code 8.

Chapter 2

Quantum invariants of three manifolds

In the previous sections, I have given various descriptions of three–manifolds and their associated geometries. In particular, we studied flat $\mathrm{SL}_2(\mathbb{C})$–connections and how they relate to hyperbolic structures. This section should correspond to a quantisation of this theory. While this is a well defined procedure in more finite dimensional settings, here we want to make some sense of some kind of gauge theory associated to $\mathrm{SL}_2(\mathbb{C})$ connections. Originally, after the discovery by Jones [103, 104] of a new invariant of knots, Witten introduced an SU(2) version of this theory [195]. From Witten's path integral, he could derive enough relations to determine the theory from some small initial data. This was then mathematically constructed and shown to be truly invariant by Reshetikhin–Turaev soon after [166, 167]. From the physical side the extension to $\mathrm{SL}_2(\mathbb{C})$ was studied by Witten [196, 197] and Gukov [91] however there is still no definition of this theory or enough conditions to compute it. There have been many constructions, which hope to give a definition of this theory and prove invariance à la Reshetikhin–Turaev. For example, Hikhami [98] introduced state integrals, which were then studied by Dimofte, Garoufalidis, Gukov, Lennels, and Zagier in [49, 45, 47]. These integrals lacked a precise contour, so could only give rise to asymptotics. This contour was then specified by Andersen–Kashaev [8, 7, 9] giving rise to invariants of cusped manifolds with certain triangulations.

While mathematical theories were constructed, there are certain properties that one would expect from physics that are not clear from their mathematical definition. In particular, these theories are expected to have certain asymptotic behaviour as we vary an analogue of $\hbar$. These properties then become conjectures and apart from a handful of cases these conjecture remain mostly open. There is substantial numerical evidence to support them on top of the physical intuition. The first is know as Witten's asymptotic expansion conjecture [195], which has to do with invariants of closed manifolds and SU(2) connections. Next came Kashaev's volume conjecture [108, 138], which implies that certain quantum invariants can compute the hyperbolic volume of hyperbolic links. More recently, Chan–Yang introduced a volume conjecture for closed manifolds [38]. These conjectures are all of one class, which will be somewhat unified in the later Section 8.3.

These asymptotic properties have driven much of the research over he past decade in hopes of constructing a mathematical theory. The closest construction we have is that of Andersen–Kashaev. Unfortunately, their theory does not encode any information about the trivial connection in the asymptotics, which from physics, would be required to make a fully fledged TQFT. Over the course of the work on this thesis there has been hints at extensions of this theory will, which be discussed in later sections 6.5 and 10.2 using the work of [70].

2.1 From the Alexander polynomial to the Jones polynomial

In the 1920s Alexander [2] constructed polynomial invariants of knots in the three sphere. Alexander considered the infinite cyclic cover, which can be explicitly constructed by cutting along a Seifert surface and gluing $\mathbb{Z}$ copies along two boundaries of the surface. One can think of $\mathbb{Z}$ copies of the knot complement with two way portals connecting the knot complement indexed by k to the copies indexed by $k \pm 1$. The generator of the covering transformations, say x, then acts on the homology of this space giving a $\mathbb{Z}[x^{\pm}]$–module. This module is finitely presented and its order ideal[1] is principle. Taking this generator, appropriately normalised, gives the *Alexander polynomial*[2] denoted for a knot K by $\Delta_K(x)$.

This definition gives a completely three–dimensional description of the Alexander polynomial but is computationally a little difficult to work with. Around 1970 Conway [41] introduced a new way to calculate the Alexander polynomial. Consider three oriented links $L_\pm$ and L_0 that have the same link diagram away from one crossing where they differ as shown in Figure 2.1. Alexander proved the following relation for his polynomial called a *Skein relation.*

$$\Delta_{L_+}(x) - \Delta_{L_-}(x) \;=\; (x^{1/2} - x^{-1/2})\Delta_{L_0}(x)\,. \tag{2.1}$$

Conway showed that fixing the value of the Alexander polynomial on the unknot $\Delta_O(x) = 1$, and imposing the Skein relation, uniquely determined the Alexander polynomial. We give an example of the computation of the Alexander polynomial for unlinked unions of unknots in Figure 2.3 and of the trefoil and figure eight knot in Figure 2.2. If L is an oriented link and $\overline{L}$ represents the mirror, then

$$\Delta_L(x) \;=\; \Delta_L(x^{-1}) \;=\; \Delta_{\overline{L}}(x)\,. \tag{2.2}$$

[1] This ideal is defined by taking the presentation in matrix form over $\mathbb{Z}[x^{\pm}]$ and taking the ideal generated by the minors of this matrix.

[2] See [123, 174] for a more detailed introduction and [33, Ch. 9][134, Ch. 12] for a generalisation to links where we have multivariable polynomials, one variable for each component.

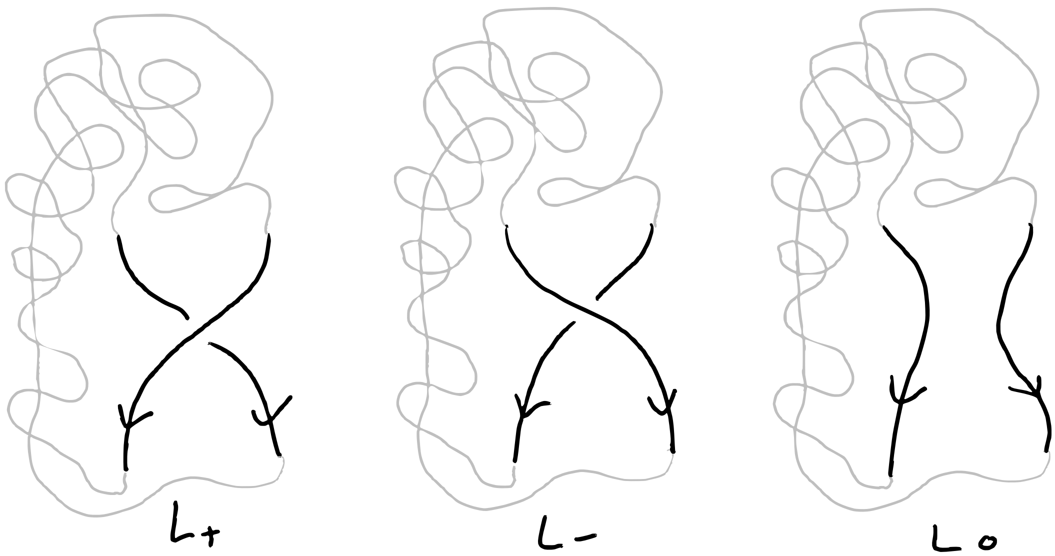

Figure 2.1: Three links L_+, L_-, L_0 with the same diagram away from one crossing. These local difference are related to skein relations.

Therefore, the Alexander polynomial does not detect the difference between mirrors. After Alexander's polynomial no new polynomial invariants of knots were introduced until the 1980s. In [103, 104], Jones discovered a surprising new invariant of oriented links. One important difference is that this polynomial invariant can detect the difference between mirror knots. Jones discovered the construction when studying von Neumann algebras that led to braid group representations. Taking the trace of these representations leads to these new oriented link invariants. We hinted at similar constructions art the end of Section 1.2.

Jones showed that his polynomial can also be defined via a recursion almost identical to Conway's contruction of the Alexander polynomial. Let L_+, L_- and L_0 three link diagrams the same everywhere except one crossing and at this crossing their differences are depicted in Figure 2.1 as before. Then, to define the Jones polynomial, we take as initial condition $J_2(O; q) = 1$, where O is the unknot, and insist the following relation

$$q\ J_2(L_+; q) - q^{-1} J_2(L_-; q) \ = \ (q^{1/2} - q^{-1/2})\ J_2(L_0; q)\,. \tag{2.3}$$

This is almost identical with the relation in equation (2.1) but surprisingly also leads to an invariant. Perhaps more surprising is that this invariant is of a very different nature.

Example 11. *We can take a disjoint union of unlinked unknots as shown in Figure 2.3 and calculate its Jones invariant. We can use this recursion to calculate the Jones polynomial of the trefoil and figure eight knot as shown in Figure 2.4. We find that*

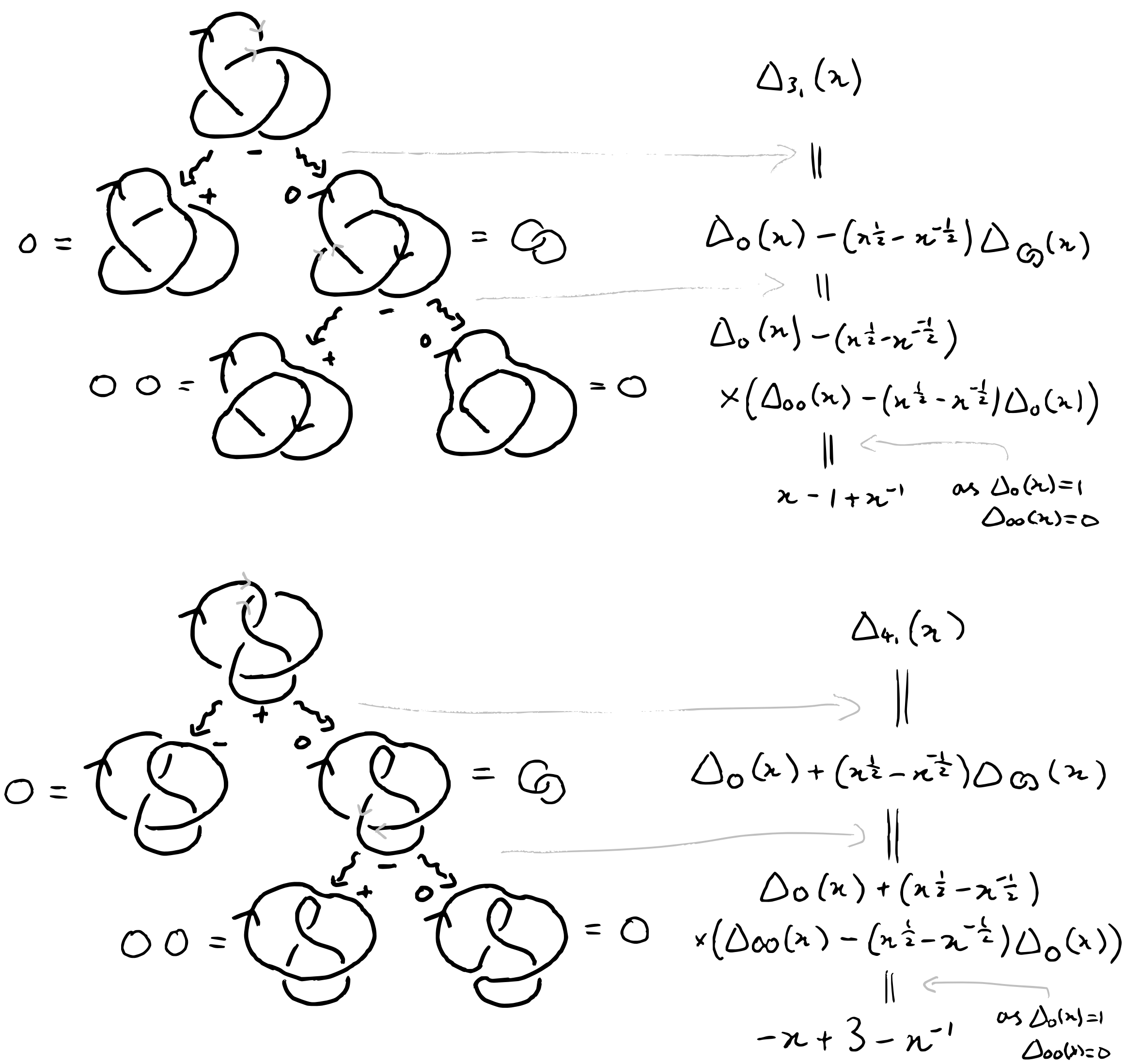

Figure 2.2: Calculation of the Alexander polynomial of the trefoil and the figure eight knot via skein relations where we also use Figure 2.4.

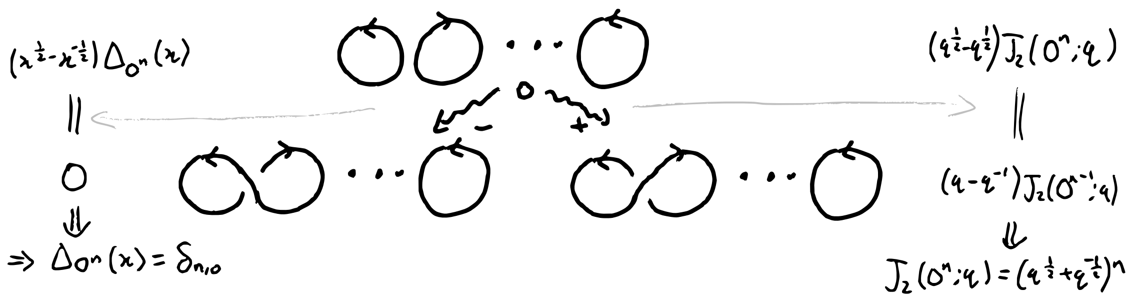

Figure 2.3: Calculation of the Alexander polynomial and Jones polynomial of disjoint union of unlinked unknots.

$$J_2(3_1;q) \;=\; -q^4+q^3+q\,, \quad \textit{and} \quad J_2(4_1;q) \;=\; q^2-q+1-q^{-1}+q^{-2}. \tag{2.4}$$

Notice that $J_2(4_1;q)=J_2(4_1;q^{-1})$, which follows from the fact that 4_1 is amphichiral while $J_2(3_1;q)\neq J_2(3_1;q^{-1})$ proving that it is not.

This seemingly simple definition, that could be explained to a high school student, contains surprisingly deep consequences. Currently, we only described the Jones polynomial by diagrams. In the late 1980s, Atiyah asked whether one could give a three dimensional construction of this invariant. The answer was famously given by Witten using quantum field theory [195]. See also [11]. To give his answer requires a detour into physics.

2.2 Some physical intuition

In physics everything is determined from the Lagrangian. Witten [195] considers the Chern–Simons functional described in Section 1.13 as a Lagrangian and takes a compact gauge group. We will mainly consider gauge group SU(N) and most importantly SU(2). Then, to understand the quantum theory associated to this Lagrangian he integrates over the space of states, in this case the space of connections. Unfortunately, making sense of this integral is quite a task. However, one can use it as a source of intuition, and, importantly, a tool that can be used to construct invariants mathematically. Let M be a closed three–manifold, then Witten takes

$$Z(M;\hbar) \text{ “ = ” } \int_{\mathcal{A}_{M\times\mathrm{SU}(N)}/\mathcal{G}_{M\times\mathrm{SU}(N)}} \exp\Big(\frac{\mathrm{CS}(A)}{2\pi i\hbar}\Big) DA\,. \tag{2.5}$$

Even physically this only makes sense when $1/\hbar\in\mathbb{Z}$ as CS is only defined up to gauge by elements of $4\pi^2\mathbb{Z}$. Therefore, Witten's interpretation here is restricted to this case. The

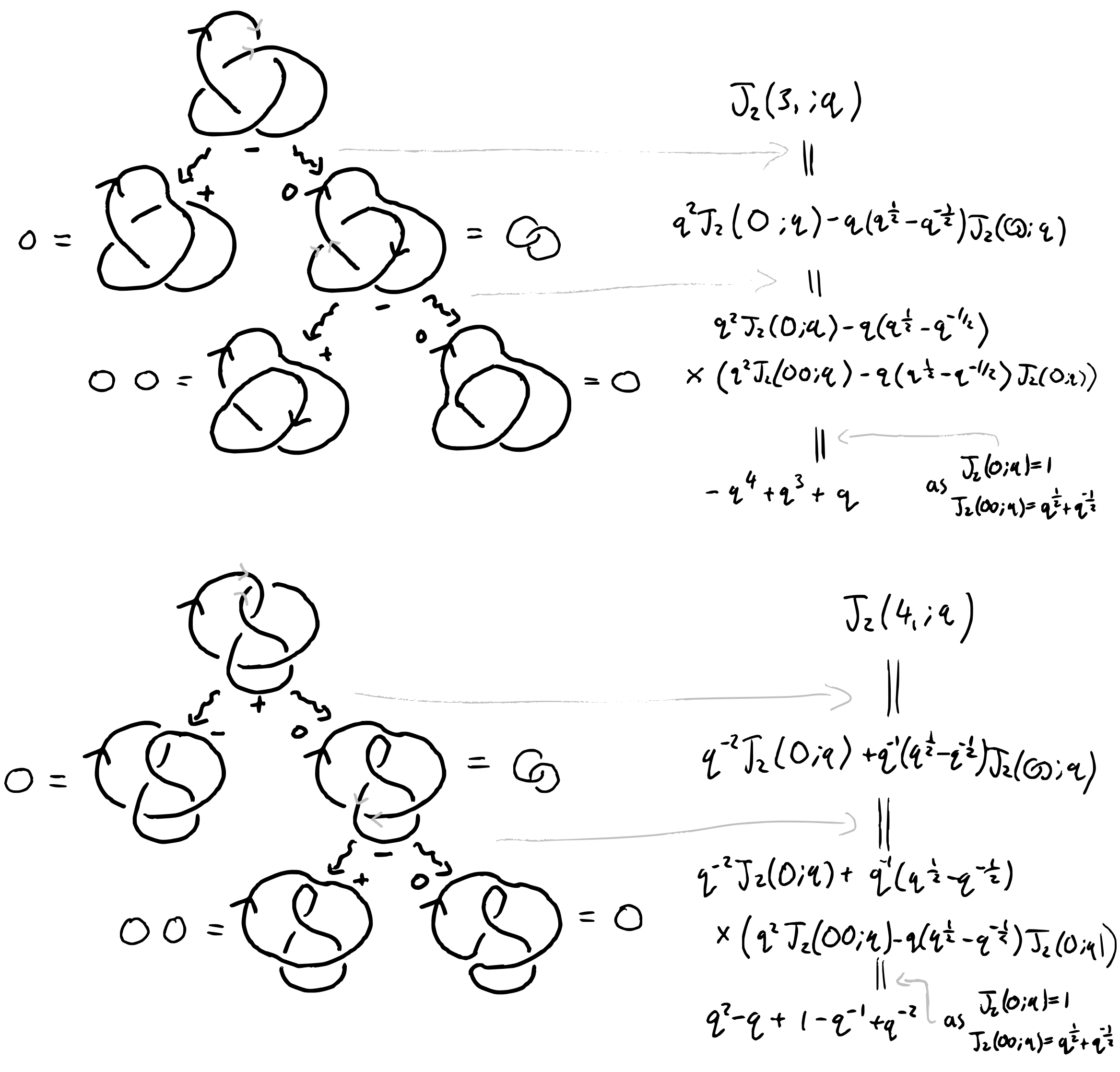

Figure 2.4: Calculation of the Jones polynomial of the trefoil and the figure eight knot via skein relations.

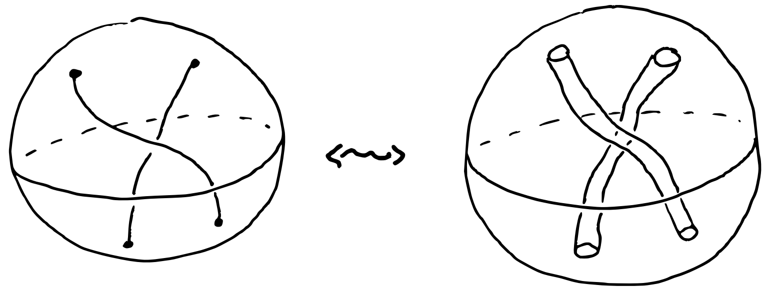

Figure 2.5: Manifold with corners given by the four circles on the sphere.

way one could try to define such an integral is decompose the manifold into small pieces and consider only classical solutions on these pieces with some boundary conditions. The integral may then be over a finite dimensional space. Performing this over finer and finer decompositions should give back the desired integral in the limit. This indicates that understanding the spaces of classical solutions on the boundary is important. Given a manifold with boundary and taking a vector space generated by appropriate boundary conditions turns these integrals into linear maps between these vector spaces of boundary conditions. Moreover, these spaces are representations under the action of their mapping class group. Before describing the theory on the boundary it is important to include links inside our three–manifolds. This can also be interpreted as allowing manifolds with corners as depicted in Figure 2.5. Then our boundaries will correspond to curves with marked points or with boundary themselves.

In Chern–Simons theory, the space of classical solutions on the boundary are flat $\mathrm{SU}(N)$ connections, which corresponds to the character variety as shown in equation (1.79). The moduli space with marked points or boundaries is the same with some choice of holonomy around the marked points. The moduli space of flat $\mathrm{SU}(N)$ connections on a surface has a canonical symplectic form [12]. This moduli space is compact so the classical quantisation of the cotanget bundle, which is of course non-compact, won't work. To quantise this space Witten introduces an auxiliary complex structure. This gives the moduli space a Kähler structure. One can then perform geometric quantisation. This gives a vector space for each complex structure, which varies holomorphically and therefore gives rise to bundle on the moduli space of complex structures. For this to be independent of the complex structure requires the bundle to be flat. This description also came up in Segal's work on conformal field theory [175], which leads to the spaces of conformal blocks. Therefore, Witten argues that this quantisation of the three dimensional Chern–Simons theory on the boundary should be given by a space of conformal blocks. This space is algebro–geometrically described as non–abelian θ–functions associated to the group, which for us is $\mathrm{SU}(N)$. The dimension is given by the Verlinde formula [190], which Witten re-derives from his methods. Around the

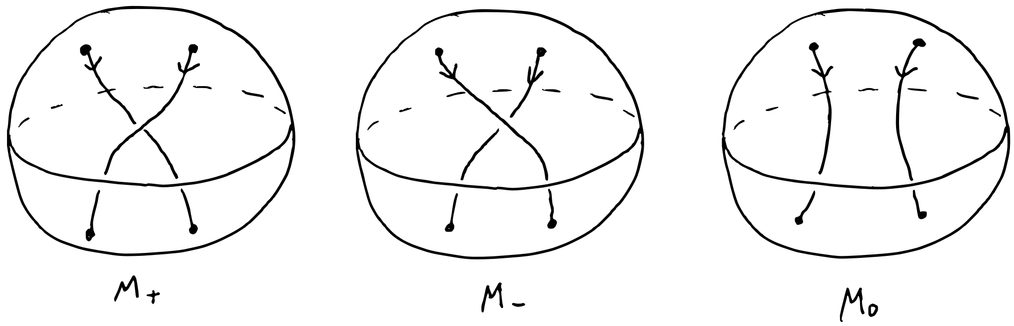

Figure 2.6: Three dimensional version of the skein relations as a ball with marked points on the boundary connected by embedded arcs.

marked points the quantisation leads to spaces of representations. In particular, choosing representations of SU(N) at each marked point gives rise a space of conformal blocks[3]. Importantly, the space associated to the sphere with four marked points labelled by two fundamental representation and two of its conjugate is two dimensional.

With this framework Witten studies the three–manifolds that correspond to the local moves in the Skein relation. These are given by the manifolds M_+, M_-, M_0 as shown in Figure 2.6. Labelling the incoming points via the fundamental representation ρ of SU(N) and the outgoing by its conjugate, these manifolds give elements, $Z(M_+;\hbar), Z(M_-;\hbar), Z(M_0;\hbar) \in Z(S^2, \rho, \rho, \overline{\rho}, \overline{\rho}; \hbar)$, of the vector space associated to the sphere with four marked points labelled the representations and their conjugates. As we mentioned this space is two dimensional and therefore there must be a relation

$$\alpha Z(M_+;\hbar) + \beta Z(M_-;\hbar) + \gamma Z(M_0;\hbar) \ = \ 0\,. \tag{2.6}$$

Witten goes on to use some physical arguments from work of Moore and Seiberg [133] to state that for the fundamental representation ρ_N of SU(N) we should have

$$q^{N/2} Z(M_+;\hbar) - q^{-N/2} Z(M_-;\hbar) \ = \ (q^{1/2} - q^{-1/2}) Z(M_0;\hbar)\,, \tag{2.7}$$

where $q = \exp(2\pi i/(N + 1/\hbar))$. Therefore, we see that taking the standard representation of SU(2) leads to the same skein relation that determined the Jones polynomial (2.3) at roots of unity. The larger family of invariants can be stored in the HOMFLY-PT polynomial [63, 161]. This example was also discussed in [185].

[3] This structure linear maps valued in the category of vector spaces leads naturally to higher categories and this field theory can be described as an extended field theory when using these structures and manifolds with corners.

Assuming Witten's ideas make sense, there should be some invariants associated to closed 3–manifolds and that these should be computable by applying surgery on links as discussed in Section 1.3. The vector space associated to the torus is generated by the irreducible representations of the loop group $L\mathrm{SU}(N)$ at level $1/\hbar$, which are in canonical bijection with some subset of representations of $\mathrm{SU}(N)$. In the case of $\mathrm{SU}(2)$ there are exactly $1/\hbar$ many representations. Taking contractions of vectors in the vector space should then give rise to invariants of closed three–manifolds.

Finally, it is important to note that Witten suggested [197] a method to deform the parameter $\hbar$ to be any number in $\mathbb{C}$. This involves integrating over a "half dimensional" cycle in $\mathcal{A}_{M\times \mathrm{SL}_N(\mathbb{C})}/\mathcal{G}_{M\times \mathrm{SL}_N(\mathbb{C})}$ of the same kind as in equation (2.5). This extension has still not been fully mathematically defined when $\hbar \notin \mathbb{Q}$. This is also related to the study of Chern–Simons theory with gauge group $\mathrm{SL}_2(\mathbb{C})$ given for example in [91, 196, 47].

2.3 The Coloured Jones polynomial

The coloured Jones polynomial is defined as a sequence of polynomials associated to oriented framed links. They are defined most naturally through representations of the braid group [185]. Many interesting examples come from quantum groups associated to simple Lie algebras as introduced by Drinfel'd. More generally, to any ribbon Hopf algebra one gets a functor from the cobordism category of ribbon graphs to the category of representations of the algebra [166]. This is similar to the kind of result discussed in [116]. In [185], Turaev introduced sufficient algebraic data needed to define link invariants via braid group representations.

Definition 7 (Enhanced Yang-Baxter operator [185])**.** *Let V be an N dimensional vector space over $\mathbb{C}$, $R : V\otimes V \cong V\otimes V$, $\mu : V \cong V$, and $a, b \in \mathbb{C}$. Then (R, μ, a, b) is an* enhanced Yang-Baxter operator *if it satisfies*

$$(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R\otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes R)(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R) \tag{2.8a}$$

$$R(\mu\otimes\mu) = (\mu\otimes\mu)R \tag{2.8b}$$

$$\mathrm{Tr}_2(R^{\pm}(\mathrm{Id}_V\otimes\mu)) = a^{\pm} b\,\mathrm{Id}_V \tag{2.8c}$$

where $\mathrm{Tr}_k : \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes(k-1)})$ such that for $f \in \mathrm{End}(V^{\otimes k})$ and e_i a basis of V with $f(e_I) = \sum_J f_I^J e_J$ we have $\mathrm{Tr}_k(f)(e_I) = \sum_{J,j} f_{I,j}^{J,j} e_J$.

The main example of an enhanced Yang–Baxter operator we will take comes from $U_q(\mathfrak{sl}_2)$. This ribbon Hopf algebra gives an enhanced Yang–Baxter operator, which comes from the associated universal R–matrix and its irreducible representations. If q is an r-th root of unity, then the irreducible representations of dimension $1, \dots, r-1$ of $\mathfrak{sl}_2$ have a corresponding irreducible representation for $U_q(\mathfrak{sl}_2)$. See [113] for a discussion on $U_q(\mathfrak{sl}_2)$ and for the following explicit expression[4] see [112, Cor. 2.32, Def. 2.35, Lem. 2.36, Thm. 3.24].

Theorem A–14 (R-matrix for $\mathfrak{sl}_2$)**.** *Let $m, n \in \frac{1}{2}\mathbb{Z}_{>0}$, $R : V_n \otimes V_m \to V_m \otimes V_n$ where $V_m = \mathrm{Span}\{e_{-m}, e_{-m+1}, \cdots, e_{m-1}, e_m\}$, $V_n = \mathrm{Span}\{e_{-n}, e_{-n+1}, \cdots, e_{n-1}, e_n\}$, such that*

$$\begin{aligned} R(e_k \otimes e_\ell) &= \sum_{i=-m}^{m} \sum_{j=-n}^{n} \sum_{p=0}^{\min(m-i,j+n)} \delta_{\ell,i+p}\delta_{k+p,j} \\ &\quad \times (-1)^p q^{ij-\frac{p}{2}(m+n)-(i-j)p-p(p+1)/2} \frac{(q;q)_{m+\ell}(q;q)_{n-k}}{(q;q)_{m+i}(q;q)_p(q;q)_{n-j}} e_i \otimes e_j \,. \end{aligned} \tag{2.9}$$

and $\mu : V_n \to V_n$ such that $\mu(e_j) = q^j e_j$. Then $(R, \mu, q^{n(n+1)}, 1)$ is an enhanced Yang–Baxter operator when restricted to V_n for some n. More generally, this gives a representation of a ribbon Hopf algebra associated to $U_q(\mathfrak{sl}_2)$.

Lemma 1. *The inverse of the R matrix in Theorem 14 is given explicitly by*

$$\begin{aligned} R^{-1}(e_k \otimes e_\ell) &= \sum_{i=-m}^{m} \sum_{j=-n}^{n} \sum_{p=0}^{\min(m-i,j+n)} \delta_{\ell,i-p}\delta_{k-p,j} \\ &\quad \times q^{-ij-\frac{p}{2}(m+n)} \frac{(q;q)_{m-\ell}(q;q)_{n+k}}{(q;q)_{m-i}(q;q)_p(q;q)_{n+j}} e_i \otimes e_j \,. \end{aligned} \tag{2.10}$$

Proof. This can be explicitly proved using the q–binomial theorem

$$(t;q)_n = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \binom{n}{k}_q t^k \,. \tag{2.11}$$

We have some function f such that

$$R^{-1}R(e_k \otimes e_\ell) = \sum_{p,r} \frac{(-1)^p q^{p(p-1)/2}}{(q;q)_p (q;q)_r} f(p+r) \,. = \sum_{s} \sum_{p=0}^{s} (-1)^p q^{p(p-1)/2} \binom{s}{p}_q \frac{f(s)}{(q;q)_s} \,. \tag{2.12}$$

Therefore, this gives a non-zero contribution to the sum only when $s = p + r = 0$. Therefore,

$$R^{-1}R(e_k \otimes e_\ell) = q^{\ell k} \frac{(q;q)_{m+\ell}(q;q)_{n-k}}{(q;q)_{m+\ell}(q;q)_{n-k}} q^{-k\ell} \frac{(q;q)_{n+k}(q;q)_{m-\ell}}{(q;q)_{n+k}(q;q)_{m-\ell}} e_k \otimes e_\ell = e_k \otimes e_\ell \tag{2.13}$$

□

Then with this example in mind, we have the following theorem.

[4] This is the transpose of what is given in [112].

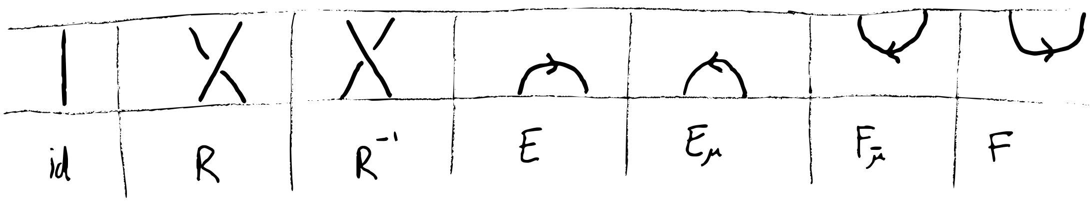

Figure 2.7: The local pieces of a tangle and the corresponding elements of the representations leading to the coloured Jones polynomial of a link.

Theorem A–15. *[185, Thm. 3.1.2] If (R, μ, a, b) is an enhanced Yang–Baxter operator, then we get a representation of the braid group, given in definition 2, by*

$$\rho_n : B_n \to \mathrm{End}(V^{\otimes n}, V^{\otimes n}) \quad s.t. \quad \rho(\sigma_i) \;=\; \underset{1}{\mathrm{Id}_V} \otimes \underset{\cdots}{\cdots} \otimes \underset{i,\, i+1}{R} \otimes \underset{\cdots}{\cdots} \otimes \underset{n}{\mathrm{Id}_V} \tag{2.14}$$

and

$$\sigma \mapsto a^{-w(\sigma)} b^{-n} \mathrm{Tr}_1(\cdots \mathrm{Tr}_k(\sigma \otimes \mu \otimes \cdots \otimes \mu) \cdots) \tag{2.15}$$

is invariant under the Markov moves given in Theorem 2.

For a knot, the assignment in Theorem 15 with enhanced Yang–Baxter operator given in Theorem 14 gives the *coloured Jones polynomial*, which we denote $\widetilde{J}_N(K; q)$ where the colour N represents the N-dimensional irreducible representation of $\mathfrak{sl}_2$. This theorem is all we need to define invariants of knots. It also defines an invariant of links, however, all components will be treated the same. Therefore, we want generalise this construction to label each component of the braid with a different representation. To do this, one needs to introduce more algebraic structures associated to tangles or ribbon graphs. For these details consult [166, 167, 112, 187, 149]. We will summarise what we need in the following theorem.

Theorem A–16. *[167, Thm. 2.5][112, Thm. 3.6] A ribbon Hopf algebra associates unique invariants of links. In particular, gluing together the local pictures in Figure 2.7 with the following operators*

- $R : V_n \otimes V_m \to V_m \otimes V_n$ *given in Theorem 14,*
- $\mu : V_n \to V_n$ *given in Theorem 14,*
- $E : V_n^* \otimes V_n \to \mathbb{C}$ *such that* $E(f \otimes x) = f(x)$*,*
- $E_\mu : V_n \otimes V_n^* \to \mathbb{C}$ *such that* $E_\mu(x \otimes f) = f(\mu(x))$*,*
- $F : \mathbb{C} \to V_n \otimes V_n^*$ *such that* $F(1) = \sum_{i=-n}^{n} e_i \otimes e^i$*,*

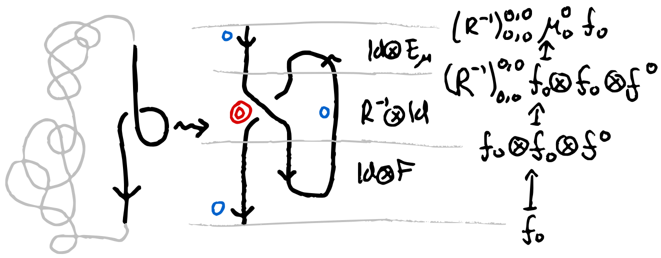

Figure 2.8: The operator associated to a local change in the framing.

- *and* $F_{\mu^{-1}} : \mathbb{C} \to V_n^* \otimes V_n$ *such that* $F(1) = \sum_{i=-n}^{n} e^i \otimes \mu^{-1}(e_i)\,,$

gives an invariant of oriented framed links, where each component is coloured by a representation V_n.

Blind computations, while they can be done, lead to very large dimensional sums. There are some tricks that can be used to simplify the computations. Firstly, for a knot, we can cut and have a start and an end which needs to be glued. When cut, this gives a map between irreducible representations and therefore, from Schur's lemma, a constant. This constant then has a simple relation to the value of the invariant. Secondly, notice that the sum of the indices the R matrix takes in is the same as the sum of the indices it spits out. Therefore, we can reduce the calculations of the contraction of these operators, by considering the possible non-zero indices at each crossing. These tricks are discussed in [137]. This method was taught to me by Stavros Garoufalidis who gave me the useful mnemonic, which I'm probably paraphrasing: "you enter with nothing; at each crossing, the strand above must pay a tax to the lower; you leave with nothing". For this to work, we need to take a different labelling set for a basis of V_n by $f_i = e_{i-n}$. Then this basis is labelled $f_0, \dots f_{2n}$. Let the various operators have indices in relation to this basis for example $R(f_k \otimes f_\ell) = \sum_{i=0}^{2n} \sum_{j=0}^{2m} R_{k,\ell}^{i,j} f_i \otimes f_j$. We can use this to see how altering the framing affects the coloured Jones polynomial.

Lemma 2. *[112, Lem. 3.27] If* L *is a framed link and we alter the framing of the* i*-th component by* ± 1 *to a framed link* $L^{(\pm)}$ *the coloured Jones polynomials for colour* $N = 2n+1$ *are related by*

$$\widetilde{J}_N(L^{(\pm)}; q) \;=\; q^{\pm(N_i^2-1)/4} \widetilde{J}_N(L; q)\,. \tag{2.16}$$

Proof. Notice that altering the framing corresponds to adding a kink of the form of Reidemeister I in Figure 1.3 into a diagram at some point of that component. We can calculate locally how this affects the operator as shown in Figure 2.8 when we shift that framing by -1. Therefore, we see that this local operator acts via multiplication by

$$(R^{-1})_{0,0}^{0,0} \mu_0^0 \;=\; q^{-(-n)^2} q^{-n} \;=\; q^{-n(n+1)} \;=\; q^{-(N^2-1)/4}\,. \tag{2.17}$$

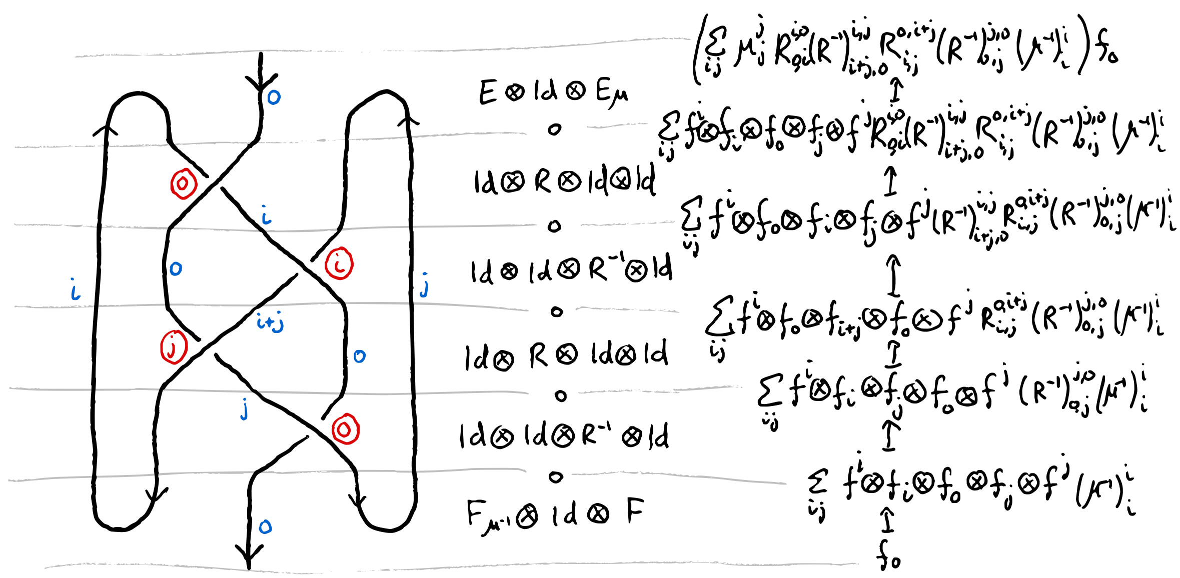

Figure 2.9: Calculation of the quantum invariant associated to 4_1 as a tangle.

Therefore, putting this into the full operator shows that the coloured Jones in simply multiplied by this. This also implies the change in framing by $+1$ as this is an inverse operation. □

Example 12 (Figure eight knot). *[137, Ex. 2.5][124, 93] From Figure 2.9, we can calculate the endomorphism associated to this tangle. The Figure 2.9 and Schur's lemma shows this tangle acts the following way*

$$f_k \mapsto f_k \sum_{i,j} \mu_j^j R_{0,i}^{i,0} (R^{-1})_{i+j,0}^{i,j} R_{i,j}^{0,i+j} (R^{-1})_{j,0}^{0,j} (\mu^{-1})_i^i \,. \tag{2.18}$$

When substituting in the expressions[5] *from Theorem 14, this endomorphism is then given by*

[5] Note of course the shift of the indices due to the change of indexing of the basis $e_{i-n} = f_i$.

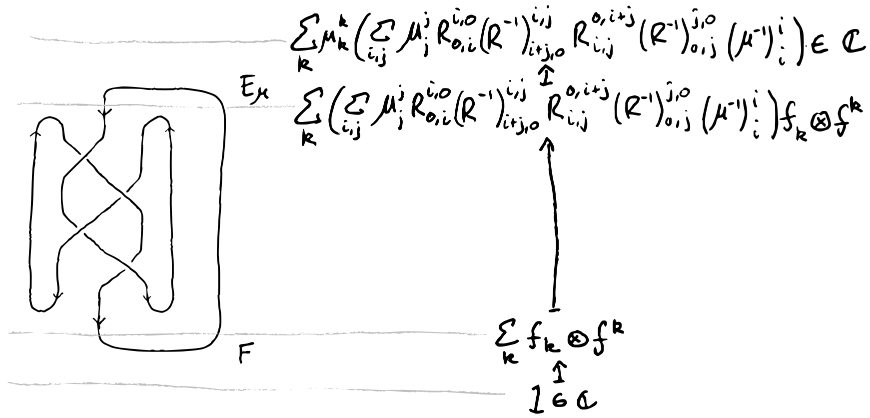

Figure 2.10: The closure of the quantum invariant associate to 4_1.

multiplication by

$$
\begin{aligned}
&\sum_{0\leq i,j\leq 2n}^{i+j\leq 2n} (-1)^j q^{-(2n+1)i+j(j+1)/2} \frac{(q;q)_{i+j}(q;q)_{2n}}{(q;q)_i (q;q)_j (q;q)_{2n-i-j}} \\
&= \sum_{k=0}^{2n} q^{-(2n+1)k} \frac{(q;q)_{2n}}{(q;q)_{2n-k}} \sum_{j=0}^{k} (-1)^j q^{(2n+1)j+j(j+1)/2} \frac{(q;q)_k}{(q;q)_j (q;q)_{k-j}} \qquad (2.19)\\
&= \sum_{k=0}^{2n} q^{-(2n+1)k} \frac{(q;q)_{2n}}{(q;q)_{2n-k}} (q^{2n+2};q)_k \;=\; \frac{1}{1-q^{2n+1}} \sum_{k=0}^{2n} q^{-(2n+1)k} \frac{(q;q)_{2n+1+k}}{(q;q)_{2n-k}}\,,
\end{aligned}
$$

where the second last equality follows from the q–binomial theorem shown previously in equation (2.11). Therefore, as $2n+1 = N$ is the dimension of the representation and including the final trace shown in Figure 2.10 we find that

$$
\widetilde{J}_N(4_1;q) \;=\; \frac{q^{-(N-1)/2}}{1-q} \sum_{k=0}^{N-1} q^{-Nk} \frac{(q;q)_{N+k}}{(q;q)_{N-1-k}}\,. \qquad (2.20)
$$

Using techniques discussed in Section 5.6 this can be shown to be equal to

$$
\widetilde{J}_N(4_1;q) \;=\; \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}} \sum_{k=0}^{N-1} q^{-kN} (q^{N-1};q^{-1})_k (q^{N+1};q)_k\,. \qquad (2.21)
$$

Notice that with this diagram the knot has the 0–framing.

Remark 5. *There are two main normalisations for the coloured Jones polynomial in the literature. The one given here is often called the TQFT normalisation. However, the coloured Jones polynomial is often also used as*

$$J_N(4_1;q) \;=\; \frac{q^{1/2}-q^{-1/2}}{q^{N/2}-q^{-N/2}}\widetilde{J}_N(4_1;q)\,. \tag{2.22}$$

This then corresponds to endomorphisms like the one that we discussed in Figure 2.9.

The form of the coloured Jones polynomial in equation (2.21) is extremely nice. We can lift this to an infinite sum as all but finitely many terms will be non-zero

$$J_N(4_1;q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q^{1-N};q)_k(q^{1+N};q)_k\,. \tag{2.23}$$

This type of expansion for knots turns out to be quite general. We have the following general theorem of Habiro.

Theorem A–17 (Cyclotomic expansion)**.** *[95, Thm. 2.1][96, Thm. 4.5] For a knot K, there exists unique $C_k(K;q)\in\mathbb{Z}[q^{\pm1}]$ such that the coloured Jones polynomial of K with the 0–framing is given by*

$$J_N(K;q) \;=\; \sum_{k=0}^{\infty}C_k(K;q)(q^{1+N};q)_k(q^{1-N};q)_k\,. \tag{2.24}$$

The uniqueness comes as this is essentially a change of basis. Indeed, from [96, Lem. 6.1] we have

$$C_k(K;q) \;=\; \sum_{N=1}^{k+1}\gamma_{k,N}(q)\widetilde{J}_N(K;q)\,. \tag{2.25}$$

where

$$\gamma_{k,N}(q) \;=\; (-1)^{N+1}\frac{q^{-k+N(N-3)/2+1}(1-q^N)(1-q^{2N})}{(q;q)_{k+N+1}(q;q)_{k-N+1}}\,. \tag{2.26}$$

Using this form, Masbaum computes the cyclotomic polynomials for all twist knots.

Example 13 (Coloured Jones polynomial of twist knots)**.** *[129, 100] The p–th twist knots K_p are depicted in Figure 2.11. For $p>0$, the cyclotomic polynomial of the p-th twist knot is given by*

$$C_k(K_p;q) \;=\; q^k\sum_{0\le s_1\le s_2\le\cdots\le s_p=k}\;\prod_{i=1}^{p-1}q^{s_i(s_i+1)}\binom{s_{i+1}}{s_i}_q\,, \tag{2.27}$$

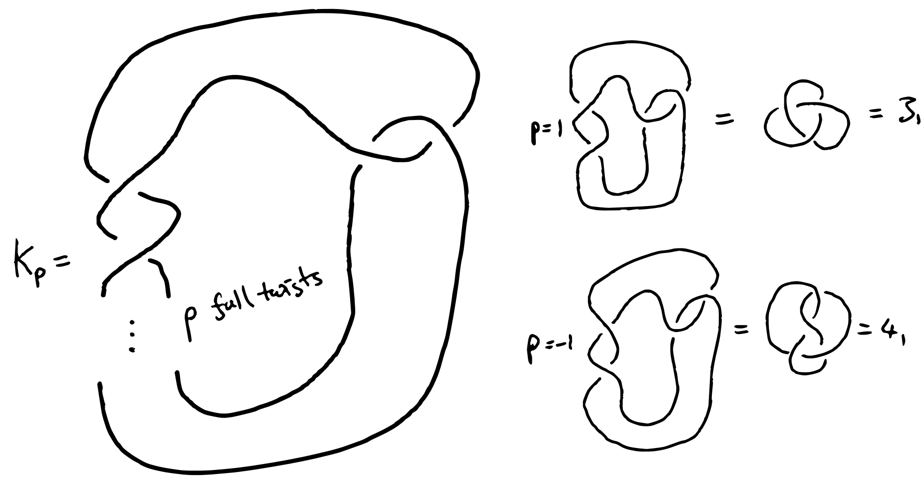

Figure 2.11: Twist knots and examples of $K_1 = 3_1$ and $K_{-1} = 4_1$.

and for $p < 0$ the cyclotomic polynomial of the p-th twist knot is given by

$$C_k(K_p; q) \;=\; (-1)^k q^{-k(k+1)/2} \sum_{0 \le s_1 \le s_2 \le \cdots \le s_{|p|} = k} \prod_{i=1}^{|p|-1} q^{-s_i(s_i+1)} \binom{s_{i+1}}{s_i}_{q^{-1}} . \tag{2.28}$$

In particular,

$$\begin{aligned} C_k(K_1; q) \;&=\; C_k(3_1; q) \;=\; q^k \,, \\ C_k(K_{-1}; q) \;&=\; C_k(4_1; q) \;=\; (-1)^k q^{-k(k+1)/2} \,. \end{aligned} \tag{2.29}$$

Notice, moreover, that in agreement with Example 11 we have

$$J_2(3_1; q) \;=\; -q^4 + q^3 + q \,, \quad \textit{and} \quad J_2(4_1; q) \;=\; q^2 - q + 1 - q^{-1} + q^{-2}. \tag{2.30}$$

There are many interesting properties of the coloured Jones polynomial. For instance, one can compute it completely from the Jones polynomial and cabellings of the original link [112, Thm. 4.15]. This allows a completely skein theoretic analysis. However, there is a much easier way to compute the coloured Jones polynomials if one can compute for a few colours with some additional information. This was realised much later for example in [91, 66]. Via an application of the theory of Zeilberger [209, 194] on q–hypergeometric sums, Garoufalidis–Lê [77] prove the following theorem.

Theorem A–18. *[77, Thm. 1] If K is a knot then there exists $r \in \mathbb{Z}_{>0}$ and $a_k^K(x; q) \in \mathbb{Z}[x^{\pm 1}, q^{\pm 1/4}]$ such that*

$$\sum_{k=0}^{r} a_k^K(q^N; q) J_{N+k}^K(q) \;=\; 0 \,. \tag{2.31}$$

This can be put into an operator form where $(\sigma J)_N = J_{N+1}$ and $(xJ)_N = q^N J_N$ and then

$$\Big(\sum_{k=0}^{r} a_k^K(x;q)\sigma^k\Big) J^K \;=\; 0\,. \tag{2.32}$$

An operator equation that is minimal is called an $\widehat{A}$*–polynomial*. The computation of such recursions is implemented in the Mathematica package [168]. Examples of computations can be found in [77, Sec. 6]. This can then be extended to all links stating that the coloured Jones polynomial is q–holonomic. See [77, Thm. 3]. This means that it satisfies a system of equations in shifts in the colours determined by some finite initial values. These kind of equations will be fundamental to this thesis and will be discussed in more detail in part III.

Example 14 ($\widehat{A}$–polynomial for 4_1)**.** *The coloured Jones polynomial for the figure eight knot given in equation (2.23) satisfies the following inhomogenous recursion*

$$\begin{aligned} J_{N+1}(q) \;&= q^{-N}\frac{(1+q^N)(1-q^{2N+1})}{1-q^{N+1}} \\ &\quad - q^{-2N}\frac{(q^N-1)^2(q^N+1)(q^{4N+1}-q^{3N+1}-q^{2N+2}-q^{2N}-q^{N+1}+q)}{(1-q^{N+1})(q^{2N}-q)}J_N(q) \\ &\quad - \frac{1-q^{2N+1}}{1-q^{2N-1}}\frac{1-q^{N-1}}{1-q^{N+1}}J_{N-1}(q)\,. \end{aligned} \tag{2.33}$$

This recursion can of course be homogenised by dividing by $q^{-N}\frac{(1+q^N)(1-q^{2N+1})}{1-q^{N+1}}$ *and then acting by* $(\sigma-1)$ *on the left.*

The reason that this operator is called the $\widehat{A}$–polynomial is that it is expected to give the A–polynomial as a specialisation that should correspond to taking a classical limit.

Conjecture 1 (AJ conjecture)**.** *[66, 91] If K is a knot then the $\widehat{A}$–polynomial specialised with $x = m^2, \sigma = \ell, q = 1$ gives the A-polynomial of the knot.*

Remark 6. *This specialisation is well defined. Noting that $\sigma x = qx\sigma$, these operators can be specialised to commuting variables when $q = 1$.*

Example 15 (AJ conjecture for 4_1)**.** *Taking the specialisation the homogenous part of the operator in equation (2.33) gives*

$$\ell - (m^{-4} - m^{-2} - 2 - m^2 + m^4) + \ell^{-1} \;=\; 0\,. \tag{2.34}$$

When the inhomogenous recursion in equation (2.33) is homogenised we get an additional factor of $(\ell-1)$, which agrees with what we previously gave in Example 2.

Remark 7. *Often the "abelian factor" of the A–polynomial $(\ell-1)$ appears as a factor on the left of the $\widehat{A}$–polynomial. This gives rise to inhomogenous recursions for the coloured Jones polynomials in many examples.*

2.4 The Witten–Reshetikhin–Turaev invariants

Following Witten's ideas, Reshetikhin–Turaev took sums over the coloured Jones polynomials evaluated at roots of unity to define invariants of closed three–manifolds. They do this by representing closed manifolds via Dehn surgery on links, which can always be done by Theorem 4. They then sum over all colours of the coloured Jones polynomial multiplied by a kernel that represents the quantum invariants of the solid tori, which are to be glued into the link complement. The invariant of the solid torus is a quantum dimension associated to the representation of $U_q(\mathfrak{sl}_2)$. They then used Kirby calculus, discussed in Theorem 5, to prove that these averages are invariant under the Kirby moves, hence proving they are invariants of three–manifolds.

Definition 8 (Witten–Reshetikhin–Turaev invariant). *[167, 112] The $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev[6] (WRT) invariant of a closed 3–manifold M, represented by a framed link L with ℓ components and the difference between the number of positive and negative eigenvalues of its linking matrix given by σ_L, evaluated at $q^{1/4} = \mathbf{e}(a/4c)$, is given by*

$$\begin{aligned}\text{Ш}(M;q) \;=\; &\left(\frac{2\mathbf{e}(-1/8)(q^{1/2}-q^{-1/2})}{\sum_{k=1}^{4c} q^{k^2/4}}\right)^{\ell} \mathbf{e}\Big(\frac{-3\sigma_L}{8}\Big)\, q^{\frac{3\sigma_L}{4}} \\ &\times \sum_{N_1,\cdots,N_\ell=1}^{c-1} \widetilde{J}_{N_1,\cdots,N_\ell}(L;q) \prod_{j=1}^{\ell} \frac{q^{N_j/2}-q^{-N_j/2}}{q^{1/2}-q^{-1/2}}\end{aligned} \tag{2.35}$$

Theorem A–19. *[167] (See also [112]) $\text{Ш}(M;q)$ is a topological invariant of M.*

We will sketch some aspects of the proof below in Theorem 1. Firstly, we will give some nice examples with simple formulae. Using the cyclotomic expansion given in Theorem 17, Beliakova–Lê [24] give a formula for the WRT invariant of a manifold obtained by rational surgery based on Laplace transform methods developed in [22].

Example 16. *[24, Thm. 7 and Cor. 5.1][100, Sec. 1] Given a knot K then*

$$(1-q)\text{Ш}(K(-1,b);q) \;=\; \sum_{k_b=0}^{\infty} C_{k_b}(K;q)(q^{k_b+1};q)_{k_b+1} \sum_{0\le k_1\le k_2\le\cdots\le k_b} \prod_{j=1}^{b-1} q^{k_j(k_j+1)} \binom{k_{j+1}}{k_j}_q . \tag{2.36}$$

For example,

$$(1-q)\text{Ш}(4_1(-1,2);q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q^{k+1};q)_{k+1} \sum_{0\le\ell\le k} q^{\ell(\ell+1)}\binom{k}{\ell}_q . \tag{2.37}$$

This expression makes it clear that these invariants are not just functions from fourth roots of roots of unity to their cyclotomic fields but elements of the Habiro ring defined in Section 5.8.

[6] I have chosen to denote by Ш pronounced "Sha".

The fact the coloured Jones polynomial satisfies a q–difference equation is of such fundamental importance to the perspectives of this thesis it is natural to ask whether there is some natural sequence that the WRT invariant of a closed manifold fits into that satisfies a difference equation. For a closed 3–manifold there are no boundary components, so it seems less natural. However, the WRT invariant is part of a family, indexed by $q^{1/4}$, of TQFTs [10, 11] and this gives it more structure than just a function from roots of unity. In particular, if $Z : \mathrm{Cob}_{n+1} \to \mathrm{Vect}_{\mathbb{C}}$ is a TQFT, Σ an n-dimensional manifold with an operator $A \in \mathrm{End}(Z(\Sigma) \otimes Z(\Sigma))$, then taking a closed $(n+1)$-dimensional oriented manifold M for every embedding $\iota : \Sigma \hookrightarrow M$ we can cut along $\iota(\Sigma)$ to get a manifold[7] $M' = \overline{M - \iota(\Sigma)}$. Then $Z(M') : \mathbb{C} \to Z(\Sigma) \otimes Z(\Sigma)$ such that for $N = \Sigma \times [0,1]$ with the orientation on the boundary so that $Z(N) : Z(\Sigma) \otimes Z(\Sigma) \to \mathbb{C}$ then

$$Z(N) \circ Z(M') = Z(M)\,. \tag{2.38}$$

We can then define

$$\langle A_\iota \rangle := Z(M, \iota, A) := Z(N) \circ A \circ Z(M')\,. \tag{2.39}$$

This means that we don't just numbers associated to closed manifolds but numbers associated to every submanifold together with an operator. This is similar to Witten's idea of Wilson loop operators used in [195].

The WRT invariant is already defined from a surgery formula. This allows us to insert operators in the sum of equation (2.35). We will do this for the TQFT associated to $q^{1/4}$ with diagonal operators that simply replace

$$\frac{q^{N_j/2} - q^{-N_j/2}}{q^{1/2} - q^{-1/2}} \rightsquigarrow \frac{q^{b_j N_j/2} - q^{-b_j N_j/2}}{q^{1/2} - q^{-1/2}} \tag{2.40}$$

in the sum of equation (2.35). Therefore, if we are in the situation of definition 8 then we define

$$\begin{aligned} Ш(M, L, b; q) \;=\; & \left(\frac{2\mathbf{e}(-1/8)(q^{1/2} - q^{-1/2})}{\sum_{k=1}^{4c} q^{k^2/4}}\right)^{\ell} \mathbf{e}\Big(\frac{-3\sigma_L}{8}\Big)\, q^{\frac{3\sigma_L}{4}} \\ & \times \sum_{N_1,\cdots,N_\ell=1}^{c-1} \widetilde{J}_{N_1,\cdots,N_\ell}(L;q) \prod_{j=1}^{\ell} \frac{q^{b_j N_j/2} - q^{-b_j N_j/2}}{q^{1/2} - q^{-1/2}}\,. \end{aligned} \tag{2.41}$$

This definition gives us a list of functions in q, which is q–Weyl finite as stated in the following corollary.

Corollary 2. *Using the fact the coloured Jones polynomial defines a q–holonomic system [77, Thm. 3], we can deduce that $Ш(M, L, b; q)$ is q–Weyl finite dimensional with respect to the operators that multiply by q^{b_i} and shift $b_i \mapsto b_i + 1$.*

[7] To do this requires the tubular neighbourhood theorem.

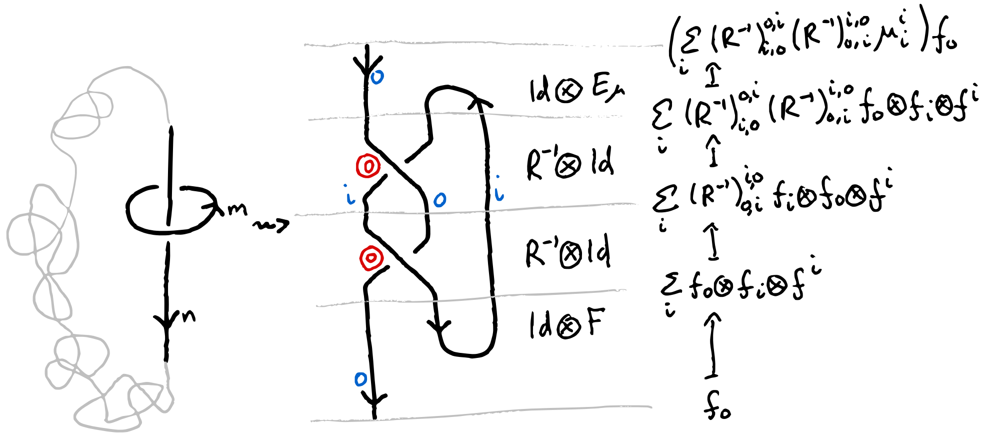

Figure 2.12: The local operator associated to a Kirby move with a different framing.

Clearly, this module depends on the presentation of the closed manifolds by a particular framed link. At the weakest level taking

$$\Upsilon_M = \mathrm{Span}_{\mathbb{Q}(q^{1/4})}\left\{Ш(M, L, b; q) : b \in \mathbb{Z}^\ell\right\} \tag{2.42}$$

gives a $\mathbb{Q}(q^{1/4})$–vector space. Notice that denominators of rational functions can lead to the function not being defined at certain roots of unity. Therefore, we view this module as the natural subspace of the $\mathbb{Q}(q^{1/4})$–vector space of functions from almost all roots of unity to $\mathbb{C}$. Here we have denoted this space without respect to L due to the following theorem. Indeed, the proof of Theorem 19 given in [112] readily adapts to give the following result and could have easily been stated there.

Theorem 1. *Υ_M is a topological invariant of M,* i.e. *as a $\mathbb{Q}(q^{1/4})$–vector space it is invariant under Kirby moves.*

Remark 8. *This result fundamentally relies on the fact that we have a family of TQFTs. For a fixed q this seems to carry almost no information. Indeed it would be either zero or one dimensional over the associated cyclotomic field.*

Proof. We will go through the proof that this is invariant under Kirby moves involving one strand. This is the base case for an induction argument used in [112], which will also apply here. Consider, an unknotted component of a link coloured by the $2m + 1$ dimensional representation that has linking number one with one component coloured by the $2n + 1$ dimensional representation and zero with all others. This is depicted in Figure 2.12. Then locally we get an operator also depicted in Figure 2.12. Using Theorem 14, we see that this

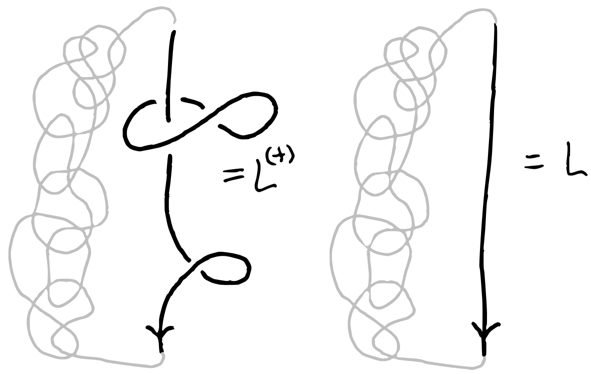

Figure 2.13: Kirby move with one strand.

this local operator is given by multiplication by

$$\sum_{i=0}^{2m}(R^{-1})_{i,0}^{0,i}(R^{-1})_{0,i}^{i,0}\mu_i^i = \sum_{i=0}^{2m} q^{n(i-m)}q^{n(i-m)}q^{(i-m)} = \sum_{i=-m}^{m} q^{Ni} = \frac{q^{NM/2}-q^{-NM/2}}{q^{N/2}-q^{-N/2}}. \tag{2.43}$$

Therefore, using Lemma 2, we see that the Kirby move in Figure 2.13 has local operator given by multiplication by

$$q^{(N^2-1)/4+(M^2-1)/4}\frac{q^{NM/2}-q^{-NM/2}}{q^{N/2}-q^{-N/2}}. \tag{2.44}$$

Therefore, the invariant associated to the framed link of the left of Figure 2.13,

$$\begin{aligned} \text{Ш}(L^{(+)},b;q) \;=\; & \left(\frac{2\mathbf{e}(-1/8)(q^{1/2}-q^{-1/2})}{\sum_{k=1}^{4c} q^{k^2/4}}\right)^{\ell+1}\mathbf{e}\Big(\frac{-3\sigma_L}{8}\Big)\, q^{3\sigma_{L+}/4} \sum_{N_1,\cdots,N_{\ell-1},N,M=1}^{c-1} \\ & \widetilde{J}_{N_1,\cdots,N_{\ell-1},N,M}(L^{(+)};q)\prod_{j=1}^{\ell-1}\frac{q^{b_jN_j/2}-q^{-b_jN_j/2}}{q^{1/2}-q^{-1/2}}\frac{q^{b_\ell N/2}-q^{-b_\ell N/2}}{q^{1/2}-q^{-1/2}}\frac{q^{b_{\ell+1}M/2}-q^{-b_{\ell+1}M/2}}{q^{1/2}-q^{-1/2}}, \end{aligned} \tag{2.45}$$

can be written in terms of L by

$$\begin{aligned}
&\text{Ш}(L^{(+)}, b; q) \\
&= \left(\frac{2\mathbf{e}(-1/8)(q^{1/2}-q^{-1/2})}{\sum_{k=1}^{4c} q^{k^2/4}}\right)^{\ell} \mathbf{e}\Big(\frac{-3\sigma_L}{8}\Big)\, q^{3\sigma_L/4} \sum_{N_1,\cdots,N_{\ell-1},N=1}^{c-1} \widetilde{J}_{N_1,\cdots,N_{\ell-1},N}(L;q) \\
&\quad \times \prod_{j=1}^{\ell-1} \frac{q^{b_j N_j/2}-q^{-b_j N_j/2}}{q^{1/2}-q^{-1/2}} \frac{q^{b_\ell N/2}-q^{-b_\ell N/2}}{q^{1/2}-q^{-1/2}} \\
&\quad \times \sum_{M=1}^{c-1} q^{(M^2-1)/4+(N^2-1)/4} \frac{q^{NM/2}-q^{-NM/2}}{q^{b_\ell N/2}-q^{-b_\ell N/2}} \frac{q^{b_{\ell+1}M/2}-q^{-b_{\ell+1}M/2}}{q^{1/2}-q^{-1/2}} \frac{-2q^{3/4}(q^{1/2}-q^{-1/2})}{\sum_{k=1}^{4c} q^{k^2/4}}\,,
\end{aligned} \tag{2.46}$$

where we note that $\sigma_{L_+} = \sigma_L + 1$. Therefore, to prove invariance under this move we notice that from a generalisation of [112, Lem. 5.1]

$$\begin{aligned}
&\sum_{M=1}^{c-1} q^{(M^2-1)/4+(N^2-1)/4} \frac{q^{NM/2}-q^{-NM/2}}{q^{N/2}-q^{-N/2}} \frac{q^{b_{\ell+1}M/2}-q^{-b_{\ell+1}M/2}}{q^{1/2}-q^{-1/2}} \\
&= \frac{-q^{-b_{\ell+1}^2/4-1/2}\sum_{k=1}^{4c} q^{k^2/4}}{2(q^{1/2}-q^{-1/2})} \frac{q^{b_{\ell+1}N/2}-q^{-b_{\ell+1}N/2}}{q^{N/2}-q^{-N/2}}\,,
\end{aligned} \tag{2.47}$$

where we use that for any j

$$\sum_{k=1}^{4c} q^{k^2/4} = 2\sum_{k=j}^{j+2c-1} q^{k^2/4}\,, \tag{2.48}$$

as $q^{(k+2c)^2/4} = q^{k^2/4+c(k+c)} = q^{k^2/4}$. Therefore, as

$$\begin{aligned}
&\frac{q^{b_\ell N/2}-q^{-b_\ell N/2}}{q^{1/2}-q^{-1/2}} \frac{q^{b_{\ell+1}N/2}-q^{-b_{\ell+1}N/2}}{q^{N/2}-q^{-N/2}} = \operatorname{sign}(b_{\ell+1}) \frac{q^{b_\ell N/2}-q^{-b_\ell N/2}}{q^{1/2}-q^{-1/2}} \sum_{k=0}^{|b_{\ell+1}|-1} q^{(|b_{\ell+1}|-1-2k)N/2} \\
&= \operatorname{sign}(b_{\ell+1}) \sum_{k=0}^{|b_{\ell+1}|-1} \frac{q^{(b_\ell+|b_{\ell+1}|-1-2k)N/2}-q^{-(b_\ell+|b_{\ell+1}|-1-2k)N/2}}{q^{1/2}-q^{-1/2}}\,,
\end{aligned} \tag{2.49}$$

we have,

$$\text{Ш}(L^{(+)}, b_1, \ldots, b_{\ell+1}; q) = \operatorname{sign}(b_{\ell+1}) q^{-(b_{\ell+1}^2-1)/4} \sum_{k=0}^{|b_{\ell+1}|-1} \text{Ш}(L, b_1, \ldots, b_\ell + |b_{\ell+1}| - 1 - 2k; q)\,. \tag{2.50}$$

Notice that of course for $b_{\ell+1} = 1$ this is simply equality, which proves the invariance of the WRT invariant and also shows that the vector spaces are equal. Therefore, we see that Υ_M

is invariant under the Kirby move in Figure 2.13. The rest of the proof can then follow along similar lines to the proof of [112, Lem. 5.6]. □

For the purpose of this thesis, the existence of such a vector space and the uncanonical module structures is enough. Indeed, often we will work with different modules, which should simply have the same span. This theorem does lead to some natural questions.

Question 1. *For a given framed link defining M, is this a q–holonomic module?*

Question 2. *What is* $\dim(\Upsilon_M)$*? Is it related to the Kauffmann bracket Skein module [160, 186]?*

Question 3. *Is there a version of the AJ conjecture for these modules and how does it depend on the presentation via a framed link?*

2.5 Stationary phase approximations in physics

Consider Witten's path integral associated to some closed 3–manifold M and some Lie group G given previously in equation (2.5) and again here

$$Z(M;\hbar)\text{ “}=\text{” }\int_{\mathcal{A}_{M\times G}/\mathcal{G}_{M\times G}}\exp\left(\frac{\mathrm{CS}(A)}{2\pi i\hbar}\right)DA\,. \tag{2.51}$$

This should be some kind of exponential integral albeit in an infinite dimensional space. Therefore, making analogies with finite dimensional integrals we could expect this to have certain asymptotic properties when $\hbar\sim 0$. Indeed, applying the method of stationary phase, discussed in Section 4.1, to such an integral we expect to find asymptotic expansions determined by the critical points of the functional CS. Given that $\mathcal{A}_{M\times G}$ is an affine space these critical points can be calculated using the following

$$\begin{aligned}
&CS(A+t\phi)+O(t^2)\\
&=\int_M \mathrm{Tr}\Big(d(A+t\phi)\wedge(A+t\phi)+\frac{2}{3}(A+t\phi)\wedge(A+t\phi)\wedge(A+t\phi)\Big)+O(t^2)\\
&=CS(A)+t\int_M \mathrm{Tr}\Big(d\phi\wedge A+dA\wedge\phi+2A\wedge A\wedge\phi\Big)+O(t^2)\\
&=CS(A)+t\int_M \mathrm{Tr}\Big(-d(A\wedge\phi)+2dA\wedge\phi+2A\wedge A\wedge\phi\Big)+O(t^2)\\
&=CS(A)+2t\int_M \mathrm{Tr}\Big((dA+A\wedge A)\wedge\phi\Big)+O(t^2)
\end{aligned} \tag{2.52}$$

where we used Stokes theorem for the last equality. Therefore, for this to vanish for all ϕ we see that the curvature $F_A=dA+A\wedge A$ must vanish. Therefore, the critical points of

CS are given by flat connections. The space of flat connection modulo gauge was given by $\mathcal{R}(M,G)$ in equation (1.79). Then using the method of stationary phase we expect[8] some $d_\rho \in \mathbb{Q}$ and $a_k^{(\rho)} \in \mathbb{C}$ such that

$$Z(M;\hbar) \sim \sum_{\rho\in\mathcal{R}(M,G)} \exp\left(\frac{\mathrm{CS}(\rho)}{2\pi i\hbar}\right)\hbar^{d_\rho}(a_0^{(\rho)} + a_{1/2}^{(\rho)}\hbar^{1/2} + a_1^{(\rho)}\hbar + \cdots), \tag{2.53}$$

as $\hbar \sim 0$. There is an important point related to this idea. As we learn very early in our mathematical careers when dealing with integrals it is natural to view real certain integrals as a special case of complex integrals. Therefore, critical points can appear outside of our initial space $\mathcal{R}(M,G)$. Indeed, $\mathcal{R}(M,\mathrm{SU}(2)) \subseteq \mathcal{R}(M,\mathrm{SL}_2(\mathbb{C}))$ and so one may expect these two theories to be intimately linked. Indeed, Witten [197] proposes that for $1/\hbar \notin \mathbb{Z}$ the contour of integration $\mathcal{R}(M,\mathrm{SU}(2))$ must be modified to some other contours in $\mathcal{R}(M,\mathrm{SL}_2(\mathbb{C}))$ to ensure convergence of the path integral.

For the rest of this section we will consider $1/\hbar \in \mathbb{Z}$. Witten's asymptotic expansion conjecture is related to the discussion in [195, Sec. 2]. This discussion was extended on in works such as [13, 14, 18] where the perturbation theory was described. From the more computational viewpoint the following conjecture was justified originally by works such as [102, 62].

Conjecture 2 (Witten's asymptotic expansion conjecture (WAEC))**.** *If M is closed 3–manifold, then there exists a finite set S of flat connections $\rho \in \mathcal{R}(M,\mathrm{SU}(2))$, $d_\rho \in \frac{1}{2}\mathbb{Z}$ and $a_k^{(\rho)} \in \mathbb{C}$ such that for $1/\hbar \in \mathbb{Z}$ and $\hbar \to 0$*

$$Ш(M,\exp(2\pi i\hbar)) \sim \sum_{\rho\in S} \exp\left(\frac{\mathrm{CS}(\rho)}{2\pi i\hbar}\right)\hbar^{d_\rho}(a_0^{(\rho)} + a_{1/2}^{(\rho)}\hbar^{1/2} + a_1^{(\rho)}\hbar^1 + \cdots) \tag{2.54}$$

This conjecture has been extensively studies by Andersen and collaborators [4, 3, 6, 5]. This is proved to leading order for many surgeries on the figure eight knot in [36]. This paper was the first to show that the WAEC held for a hyperbolic manifold. For some non-hyperbolic manifolds, this also arises with beautiful connections to partial and mock θ–functions in [118, 99].

Example 17 (WAEC for $4_1(-1,2)$)**.** *To give a flavour of this conjecture we will use the explicit example of $4_1(-1,2)$. In Example 16, we saw that the WRT invariant could be completely computed as*

$$(1-q)Ш(4_1(-1,2);q) = \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q^{k+1};q)_{k+1} \sum_{0\le\ell\le k} q^{\ell(\ell+1)}\binom{k}{\ell}_q. \tag{2.55}$$

[8] Here the form looks as though $\mathcal{R}(M,G)$ is discrete however this space has a natural measure and if it is not discrete it can be interpreted as an integral.

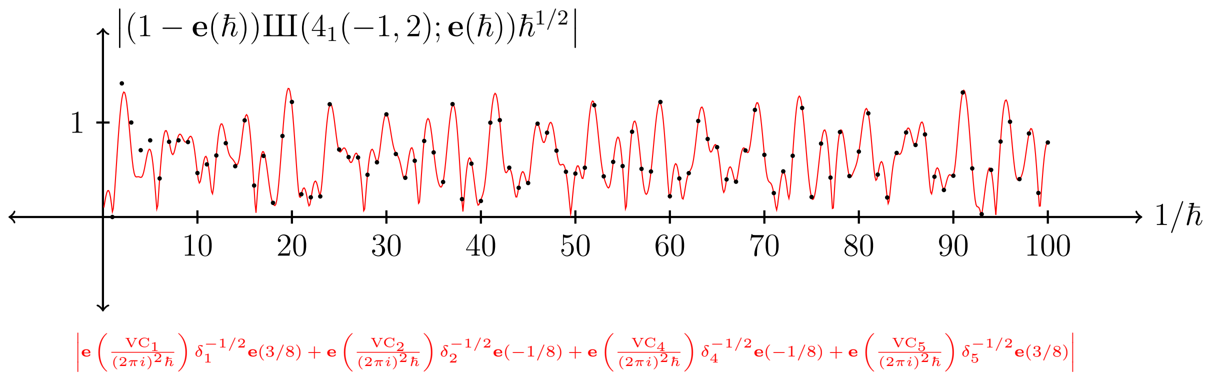

Figure 2.14: Plots of the WRT invariant of $4_1(-1,2)$ against the first order approximation in WAEC where $1/\hbar \in \mathbb{Z}$ and $\mathbf{e}(x) = \exp(2\pi i x)$.

This manifold has four non-trivial $\mathrm{SU}(2)$ *connections here labelled by* $\rho_1, \rho_2, \rho_4, \rho_5$*. They have Chern–Simons values*

$$\mathrm{VC}_1 = 20.297\cdots, \quad \mathrm{VC}_2 = -6.7857\cdots, \quad \mathrm{VC}_4 = 9.2837\cdots, \quad \mathrm{VC}_5 = 2.1292\cdots, \tag{2.56}$$

which can be computed using the methods of Section 1.15. Let δ *be numbers*[9]

$$\delta_1 = -11.578\cdots, \quad \delta_2 = -12.636\cdots, \quad \delta_4 = -7.0205\cdots, \quad \delta_5 = -5.3937\cdots. \tag{2.57}$$

In Figure 2.14, we plot the values of $|\text{Ш}(4_1(-1,2);q)|$ *for* $1/\hbar = 1, 2, \ldots, 100$ *against the expected leading order in WAEC. One can see that to agreement is very good to the naked eye. However, such graphs can be misleading, albeit fun. Indeed, there is an error of* $\hbar^{1/2}$ *coming from a contribution from the trivial flat connection. A more detailed analysis can be carried out and with some appropriate coefficients*[10] a_k^ρ *and roots of unity* $\mu_\rho^8 = 1$ *and taking* $\hbar = 1/1000$ *we find*

$$\begin{aligned}
|(1-\mathbf{e}(\hbar))\text{Ш}(4_1(-1,2);\mathbf{e}(\hbar))| &= 16.099\ldots \\
\left|(1-\mathbf{e}(\hbar))\text{Ш}(4_1(-1,2);\mathbf{e}(\hbar)) - \sum_{\rho=\rho_0,\rho_1,\rho_2,\rho_4,\rho_5} \mu_\rho \hbar^{d_\rho} \exp\left(\frac{\mathrm{VC}_\rho}{2\pi i\hbar}\right) a_0^\rho\right| &= 0.019710\ldots \\
\left|(1-\mathbf{e}(\hbar))\text{Ш}(4_1(-1,2);\mathbf{e}(\hbar)) - \sum_{\rho=\rho_0,\rho_1,\rho_2,\rho_4,\rho_5} \mu_\rho \hbar^{d_\rho} \exp\left(\frac{\mathrm{VC}_\rho}{2\pi i\hbar}\right) (a_0^\rho + a_1^\rho\hbar)\right| &= 0.00048434\ldots \\
\left|(1-\mathbf{e}(\hbar))\text{Ш}(4_1(-1,2);\mathbf{e}(\hbar)) - \sum_{\rho=\rho_0,\rho_1,\rho_2,\rho_4,\rho_5} \mu_\rho \hbar^{d_\rho} \exp\left(\frac{\mathrm{VC}_\rho}{2\pi i\hbar}\right) (a_0^\rho + a_1^\rho\hbar + a_2^\rho\hbar^2)\right| &= 6.5311\cdots \times 10^{-5}.
\end{aligned} \tag{2.58}$$

This can be continued to many orders and leads to more convincing numerical indicators. In this example, the theorem to leading order is proved in [36, Thm. 1.1].

[9] We will come back this this example in more detail in Section 8.3.
[10] See sections 8.3 and 4.9.

Given the WRT invariant, numerical computations can be carried out to find the values VC_ρ, a^ρ. These methods are discussed in Section 3.2. WAEC is actually much subtler than some of the conjectures in the following sections. For hyperbolic manifolds, considering the definition as a sum over a lattice, it would be more natural to see exponential growth. However, WAEC predicts polynomial growth as $\hbar \sim 0$. This comes from some kind of catastrophic cancellation, which seems to happen in general for all three–manifolds.

2.6 Kashaev's and Chen–Yang's volumes conjectures

Kashaev [106] introduced certain invariants of links dependingly on $N \in \mathbb{Z}_{>0}$ denoted for a link L by $\langle L \rangle_N$. He then went on to compute the associated R matrix in [107]. He then proposed a conjecture for the semi–classical behaviour of his invariant in [108], which relates the exponential growth to the volume. This appears as the invariant involves the quantum dilogarithm while the volume of a three–manifolds involves the classical dilogarithm. This conjecture went somewhat unnoticed until work of Murakhami–Murakhami [138], which showed that Kashaev's invariant was nothing but the N-th coloured Jones polynomial evaluated at the N-th root of unity,

$$\langle L \rangle_N \;=\; \widetilde{J}_{N,\dots,N}(L; \exp(2\pi i/N))\,. \tag{2.59}$$

Then a slight refinement of Kashaev's original conjecture, see for Example [139], is given as follows.

Conjecture 3 (Kashaev's volume conjecture)**.** *Let L by a hyperbolic link (with zero framing). Then there is some $d_L \in \frac{1}{2}\mathbb{Z}$ and $a_0 \in \mathbb{C}$ such that for $N \in \mathbb{Z}$ and $N \to \infty$,*

$$\widetilde{J}_N(L; \exp(2\pi i/N)) \sim \exp\left(\frac{\mathrm{VC}(L)}{2\pi i} N\right) N^{d_L}\left(a_0 + O(N^{-1})\right), \tag{2.60}$$

where $\mathrm{VC}(L)$ is the complexified volume (associated to the geometric connection) calculated in Section 1.15.

This conjecture has been extensively studied and proved in many cases. A summary of various cases it has been proved is given in [137, Sec. 3.3]. This remains one of the most important open problems in quantum topology.

Example 18. *Recall from Example 12 and in particular equation (2.23), the coloured Jones polynomial of the figure eight knot is given by*

$$\widetilde{J}_N(4_1; q) \;=\; \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} (q^{1+N}; q)_k (q^{1-N}; q)_k. \tag{2.61}$$

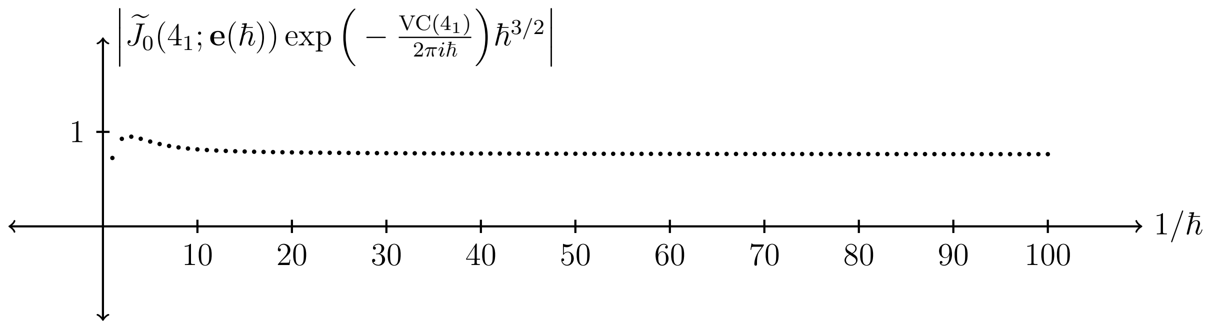

Figure 2.15: Plots of the Kashaev invariant of 4_1 divided by the expected exponential and polynomial growth from the volume conjecture.

Therefore, the Kashaev invariant for $q = \exp(2\pi i/N)$ *is then*

$$\widetilde{J}_0(4_1;q) \;=\; \langle 4_1\rangle_N \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2. \tag{2.62}$$

The complexified volume is given by

$$\mathrm{VC}(4_1) \;=\; 2.0299\cdots i\,. \tag{2.63}$$

The plot of the ratio $\widetilde{J}_0(4_1;\mathbf{e}(\hbar))\exp\left(-\frac{\mathrm{VC}(4_1)}{2\pi i\hbar}\right)\hbar^{3/2}$ *is given in Figure 2.15. Here we see it clearly seems to limit to a constant. (Again one should use more sophisticated methods than these plots. Consult Section 3.2). This example was first proved by Ekholm. See [140] for the proof.*

Remark 9. *This conjecture of course needs the appropriate normalisation of the Jones polynomial. In the TQFT normalisation the evaluation would simply vanish.*

This conjecture has a similar flavour to WAEC however here we just see one dominate term up to an error, which has exponential term smaller than the volume. More recently, Chen–Yang [38] introduced a volume conjecture for WRT invariants in contrast to WAEC. Here the trick is to evaluate at different roots of unity.

Conjecture 4 (Chen–Yang's Volume conjecture)**.** *Let* M *be a closed hyperbolic 3–manifold. Then there is some* $d_L \in \frac{1}{2}\mathbb{Z}$ *and* $a_0 \in \mathbb{C}$ *such that for* $N \in \mathbb{Z}$ *and* $N \to \infty$*,*

$$\text{Ш}\Big(M;\exp\Big(\frac{2\pi i}{N+1/2}\Big)\Big) \sim \exp\Big(\frac{\mathrm{VC}(M)}{2\pi i}\Big(N+\frac{1}{2}\Big)\Big)\Big(N+\frac{1}{2}\Big)^{3/2}\big(a_0+O(N^{-1})\big)\,. \tag{2.64}$$

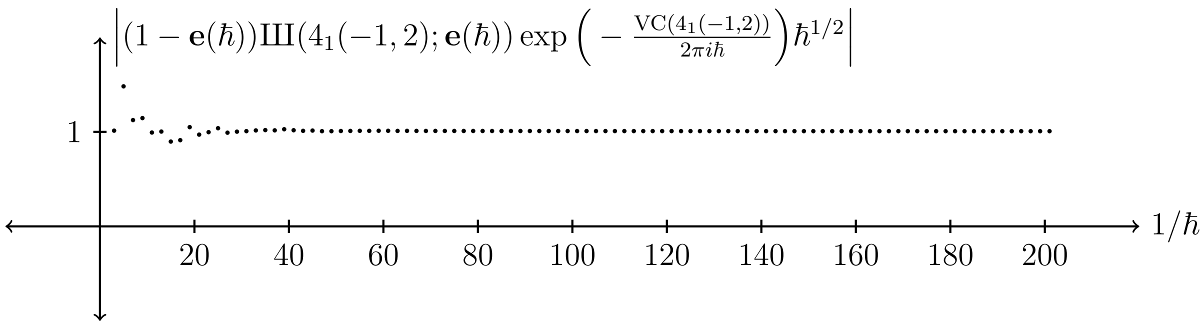

Figure 2.16: Plots of the ratio of the WRT invariant of $4_1(-1,2)$ and the leading order expected from the Chen–Yang volume conjecture where $1/\hbar \in \mathbb{Z}+1/2$.

Example 19. *We will simply continue our example* $4_1(-1,2)$*, started in Example 17, where we consider instead of roots of unity* $\exp(2\pi i\hbar)$ *with* $1/\hbar \in \mathbb{Z}$ *we have* $1/\hbar \in \mathbb{Z}+1/2$*. Here the complexified volume is given by*

$$\mathrm{VC}_7 \;=\; 4.8678\cdots + 1.3985\cdots i\,. \tag{2.65}$$

Then the ratio of the WRT invariant and that expected by the Chen–Yang volume conjecture is plotted in Figure 2.16. Again this looks like it is approaching some limit as expected. The conjecture in this example has been proved in [150]. See [199] for a discussion of the progress more generally.

2.7 Vassilev invariants and the trivial connection

Vassiliev [189] studied the spaces of maps from S^1 to S^3. This space is stratified. Indeed, considering the subspace of embeddings is open and the connected components are in bijection with knots. Vassiliev considers a sequence indexed by $\mathbb{Z}_{>0}$ of subspaces inside the zero cohomology of these spaces of knots. An element in the k-th group gives a function from the set of knots called a Vassiliev invariant of degree k. Vassiliev's invariants can be computed by reducing diagrams to the unknot but involve invariants of knotted graphs which arise when a crossing intersects to becomes a four–valent vertex of an embedded graph. These invariants were further studied by [122, 30, 17, 117, 132].

Importantly, [30] showed the replacing $q = e^{\hbar}$ in the Jones polynomial of a knot K has expansion

$$J_2(K;e^{\hbar}) \;=\; \sum_{k=0}^{\infty} a_k\hbar^k \tag{2.66}$$

where a_k are Vassiliev invariants of degree k. This was extended in [132] to give the following theorem.

Theorem A–20. *For a knot K the coloured Jones polynomial has expansion*

$$J_N(K; e^{\hbar}) = \sum_{k=0}^{\infty} \sum_{j=0}^{\lfloor k/2 \rfloor} a_{k,j}(K) N^{2j} \hbar^k \,, \tag{2.67}$$

where $a_{k,j}$ are Vassiliev invariants of degree k.

This expansion was then conjectured [132, Conj. 2] and quickly proved [171, 170, 172, 19] to satisfy the following expansion

Theorem A–21. *For a knot K there exists Laurent polynomials $a_k(x) \in \mathbb{Q}[x^{\pm 1}]$ such that $a_0(x) = 1$ and*

$$J_N(K; e^{\hbar}) = \sum_{k=0}^{\infty} \frac{a_k(\exp(N\hbar))}{\Delta_K(\exp(N\hbar))^{2k+1}} \hbar^k \,, \tag{2.68}$$

where $\Delta_K(x)$ is the Alexander polynomial discussed in Section 2.1.

This shows that the coloured Jones polynomial is strictly stronger than the Alexander polynomial. Indeed, we already saw the Jones polynomial distinguished some links and their mirrors. While Theorem 21 collects diagonal terms in the expansion of Theorem 20 we can set $N = 0$ to find a series of Vassiliev invariants, which are intimately related to the Kashaev invariant given in equation (2.59). To understand how these are related is best understood through the Habiro ring discussed in Section 5.8.

Example 20. *We have the following divergent series associated to the trivial flat connection on 4_1 given by*

$$\Phi^{(\rho_0)}(4_1; \hbar) := J_0(4_1; e^{\hbar}) = 1 - \hbar^2 + \frac{47}{12}\hbar^4 - \frac{12361}{360}\hbar^6 + \frac{10771487}{20160}\hbar^8 - \frac{23554574521}{1814400}\hbar^{10} + \cdots \,. \tag{2.69}$$

This asymptotic series that arise in this way should correspond to a contribution from a stationary phase approximation around the trivial connection [18, 171, 170]. This can also be computed for closed manifolds. Here it is more subtle and relies on results such as [136, 148, 96] in the case of integer homology spheres. For more general examples this becomes more difficult and should be the $\mathfrak{sl}_2$ LMO invariant [125]. For rational homology spheres there is a similar extended story to the integer homology spheres [23, 148].

Example 21. *We have the following divergent series associated to the trivial flat connection on* $4_1(-1,2)$ *given by*

$$\begin{aligned}\Phi^{(\rho_0)}(4_1(-1,2);-\hbar) &:= (1-e^{\hbar})\text{Ш}(4_1(-1,2);e^{\hbar}) \\ &= -\hbar-\frac{25}{2}\hbar^2-\frac{1621}{6}\hbar^3-\frac{195601}{24}\hbar^4-\frac{37907101}{120}\hbar^5-\frac{2154244133}{144}\hbar^6 \\ &\quad -\frac{4219228947781}{5040}\hbar^7-\frac{2179198982580001}{40320}\hbar^8-\frac{1434968055634260781}{362880}\hbar^9+\cdots .\end{aligned} \tag{2.70}$$

2.8 Dimofte–Garoufalidis perturbative series

We have already seen through the volume conjectures, discussed in Section 2.6, and Witten's asymptotic expansion conjecture, discussed in Section 2.5, that we expect additional asymptotic series associated to non-trivial connections to arise. These should come from the stationary phase approximation of the path integral in equation (2.5) around other critical points. Following Kashaev's definition of his invariant [106, 107] finite dimensional integrals were considered in [47, 98, 49], which ultimately led to conjectural computation/definition of these series in [45, 46] for knots using the Neumann–Zagier matrices in that were discussed in Section 1.11. These integrals were often lacking a precise contour and this was supplied and refined in many works of Andersen–Kashaev [8, 7, 9]. We will consider these precise integrals latter but for now we will apply formal Gaussian integration. We will follow closely the description given in [45, 46].

Consider some $N\times N$ Neumann–Zagier matrices, defined in Section 1.11, $\begin{pmatrix}A & B\end{pmatrix}$ *i.e.* some half symplectic matrices, $\nu\in\mathbb{Z}^N$, a solution z to the equations

$$\prod_{j=1}^{N} z_j^{A_{i,j}}(1-z_j^{-1})^{B_{i,j}} = (-1)^{\nu_i} \tag{2.71}$$

and and integral solution[11] $(f,f'')\in\mathbb{Z}^{2N}$ to

$$Af+Bf'' = \nu\,. \tag{2.72}$$

With this data we can define the *one–loop* invariant, which will be conjecturally related to the constant term of the asymptotic series arising in the volume conjecture 3. The one–loop invariant is given by

$$\delta_{A,B,\nu,z,f} = \pm\det(A\,\mathrm{diag}(z'')+B\,\mathrm{diag}(z)^{-1})z^{f''}z''^{-f}\,. \tag{2.73}$$

[11] A solution always exists from work of Neumann [144].

Example 22 (1–loop invariant of 4_1). *Recall that, after Example 6, SnapPy [43] gave Neumann–Zagier matrices*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad and \quad \nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.74}$$

which had solutions $z_1 = z_2 = \exp(2\pi i/6)$ and $(f, f'') = 0$. Therefore,

$$\delta = \pm \det \begin{pmatrix} \exp(2\pi i/6) - \exp(-2\pi i/6) & \exp(2\pi i/6) - \exp(-2\pi i/6) \\ \exp(2\pi i/6) & -\exp(-2\pi i/6) \end{pmatrix} = \pm\sqrt{-3}\,. \tag{2.75}$$

Theorem A–22. *[45, Thm. 1.3 and Thm. 1.4] Given A, B, ν, z, f as above then $\delta_{A,B,\nu,z,f}$ is invariant under all equivalences of Neumann–Zagier data given in Section 1.12.*

The higher orders in $\hbar$ of the expansion can then conjecturally be computed via Gaussian integration. Take the series

$$\psi_\hbar(x; z) = \exp\left(\sum_{n,k,n+\frac{k}{2}-1>0} \frac{\hbar^{n+\frac{k}{2}-1}(-x)^k B_n}{n!k!} \mathrm{Li}_{2-n-k}(z^{-1}) \right), \tag{2.76}$$

where B_n are the Bernoulli numbers[12] with $B_1 = 1/2$. This is related to the quantum dilogarithm, which will be another function of fundamental importance and discussed in detail in Section 8.10. Then if $\det(B) \neq 0$ take

$$\mathcal{H} = -B^{-1}A + \mathrm{diag}(z'). \tag{2.77}$$

If $\det(\mathcal{H}) \neq 0$ then taking

$$g_{A,B,\nu,z,f,\hbar}(x; z) = \exp\left(-\frac{\hbar^{1/2}}{2} x^{\mathrm{T}} B^{-1}\nu + \frac{\hbar}{8} f^{\mathrm{T}} B^{-1} A f \right) \prod_{j=1}^{N} \psi_\hbar(x_i; z_i) \in \mathbb{Q}[[x, \hbar^{1/2}]] \tag{2.78}$$

we will apply formal Gaussian integration to the following integrals to define $S_{A,B,\nu,f,k}(z)$ so that

$$\exp\left(\sum_{k=2}^{\infty} S_{A,B,\nu,f,k}(z)\hbar^{k-1} \right) = \frac{\int e^{-\frac{1}{2}x^{\mathrm{T}}\mathcal{H}x} g_{A,B,\nu,z,f,\hbar}(x; z)dx}{\int e^{-\frac{1}{2}x^{\mathrm{T}}\mathcal{H}x}dx}\,. \tag{2.79}$$

We will often drop the A, B, ν, f from the notation when it is not varying. The $S_k(z)$ can be computed using Wick's theorem and Feynman diagrams. To compute $S_k(z)$ we consider all connected graphs G such that for

$$L(G) := \#\{1 - \text{valent vertices}\} + \#\{2 - \text{valent vertices}\} + \dim(H_1(G, \mathbb{Q})) \tag{2.80}$$

[12] We will only take $B_1 = 1/2$ in this section.

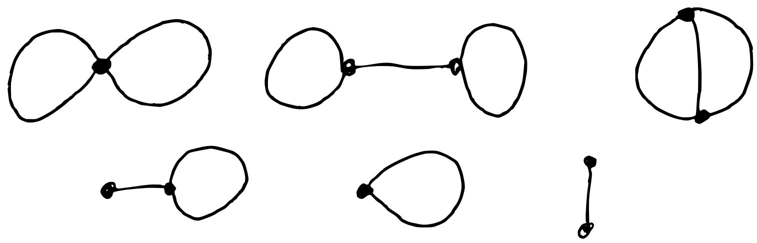

Figure 2.17: The Feynman graphs that contribute to the two loop invariant.

we have $L(G) \leq k$. Then to each edge associate the propagator,

$$\Pi \;=\; \hbar\mathcal{H}^{-1} \in \mathrm{Hom}((\mathbb{C}[[h]]^N)^{\otimes 2}, \mathbb{C}[[h]])\,, \tag{2.81}$$

and for each $\ell > 0$ and ℓ–valent vertex associate $\Gamma^{(\ell)} = \sum_{j=1}^{N} \Gamma_j^{(\ell)} e_j \otimes \cdots \otimes e_j \in (\mathbb{C}[[h]]^N)^{\otimes \ell}$ where

$$\Gamma_j^{(\ell)} \;=\; (-1)^{\ell} \sum_{p=\delta_{\ell,1}+\delta_{\ell,2}}^{\infty} \frac{B_p}{p!} \mathrm{Li}_{2-p-\ell}(z_j^{-1})\hbar^{p-1} - \frac{1}{2}(B^{-1}\nu)_j \delta_{\ell,1} \tag{2.82}$$

and as a vacuum contribution *i.e.* the graph with a single zero valent vertex take

$$\Gamma^{(0)} \;=\; \sum_{j=1}^{N} \left(\sum_{p=0}^{\infty} \frac{B_p}{p!} \mathrm{Li}_{2-p}(z_j^{-1})\hbar^{p-1} \right) + \frac{\hbar}{8} f^{\mathrm{T}} B^{-1} A f \in \mathbb{C}[[h]]\,. \tag{2.83}$$

Then we sum over all graphs weighted by the reciprocal of the order of their automorphisms where we take the various contractions of the propagators and the vertex weights. For example, if we calculate $S_2(z)$ then the graphs that contribute are shown in Figure 2.17. Therefore, $S_2(z)$ is given by

$$\begin{aligned} S_2(z) \;=\; \mathrm{coeff}\Bigg[&\Gamma^{(0)} + \sum_{i=1}^{N} \left(\frac{1}{8}\Gamma_i^{(4)}(\Pi_{i,i})^2 + \frac{1}{2}\Gamma_i^{(2)}\Pi_{i,i} \right) \\ &+ \sum_{i,j=1}^{N} \left(\frac{1}{8}\Pi_{i,i}\Gamma_i^{(3)}\Pi_{i,j}\Gamma_j^{(3)}\Pi_{j,j} + \frac{1}{12}\Gamma_i^{(3)}\Pi_{i,j}\Gamma_j^{(3)} + \frac{1}{2}\Gamma_i^{(1)}\Pi_{i,j}\Gamma_j^{(3)}\Pi_{j,j} + \frac{1}{2}\Gamma_i^{(1)}\Pi_{i,j}\Gamma_j^{(1)} \right), \hbar \Bigg]\,. \end{aligned} \tag{2.84}$$

Example 23 (2–loop invariant of 4_1)**.** *Plugging in the various formulae we find that*

$$S_2(\exp(2\pi i/6), \exp(2\pi i/6)) \;=\; \frac{11}{108}\exp(2\pi i/6) - \frac{5}{54} \in \frac{11}{24\sqrt{-3}^3} + \frac{1}{24}\mathbb{Z}\,. \tag{2.85}$$

This is computed in Code 9.

Taking the associated elements of the extended Bloch group $[z]$ corresponding to A, B, ν, f we define using (1.115)

$$S_{A,B,\nu,f,0}(z) \;=\; \mathrm{VC}([z]) \;=\; R([z])\,, \tag{2.86}$$

and then define

$$S_{A,B,\nu,f,1}(z) = -\log(\delta_{A,B,\nu,z,f})/2\,. \tag{2.87}$$

With these definitions we can take

$$\Phi_{A,B,\nu,z,f}(\hbar) \;=\; \exp\left(\sum_{k=0}^{\infty} S_{A,B,\nu,f,k}(z)\hbar^{k-1}\right). \tag{2.88}$$

Notice that there is ambiguity for the first 3 coefficients with these constructions so that

$$S_{A,B,\nu,f,0}(z) \in \mathbb{C}/(\pi^2\mathbb{Z})\,, \quad S_{A,B,\nu,f,1}(z) \in \mathbb{C}/\Big(\frac{\pi i}{2}\mathbb{Z}\Big)\,, \quad \text{and} \quad S_{A,B,\nu,f,1}(z) \in \mathbb{C}/\Big(\frac{1}{24}\mathbb{Z}\Big)\,. \tag{2.89}$$

The first arises due to the ambiguity in the Chern–Simons invariant and should be able to be lifted to $4\pi^2$ ambiguity. Then the other two come from the ability to multiple by η-functions which introduce eighth roots of unity and exponential series with $\exp(\hbar/24)$. See Section 7.2.

Conjecture 5. *[45] If K is a knot with Neumann–Zagier data A, B, ν, z, f, then the series in $\hbar$ defined in equation (2.88) should have*

$$\widetilde{J}_N(K; \exp(2\pi i/N)) \sim \Phi_{A,B,\nu,z,f}(2\pi i/N)\,. \tag{2.90}$$

Remark 10. *Although this conjecture would prove invariance at the time of writing invariance is only known for S_0 and S_1.*

A proposal for a version of this calculation in the case of closed manifolds was given by Gang–Romo–Yamazaki[13] [64]. Here it relies on a surgery kernel which came from work in [15]. While from the mathematical perspective the conjecture is more unjustified than the previous, as it relies on a conjectural surgery formula, it does seem to agree in simple examples. The construction is completely analogous to the that in the previous section.

To perform p/q–surgery on a knot K, take some extended $N \times N$ Neumann–Zagier matrices[14] for the knot K, defined in Section 1.11, $\begin{pmatrix} A & B \end{pmatrix}$ and $\begin{pmatrix} C & D \end{pmatrix}$ *i.e.* a $2N \times 2N$ symplectic matrix, $\nu, \mu \in \mathbb{Z}^N$ where for a knot $\mu = (\mu_1, 0, \dots, 0)$, a solution z to the equations

$$\prod_{j=1}^{N} z_j^{A_{i,j}}(1-z_j^{-1})^{B_{i,j}} \;=(-1)^{\nu_i} m^{2\delta_{j,N}}\,, \quad \prod_{j=1}^{N} z_j^{2C_{1,j}}(1-z_j^{-1})^{2D_{1,j}} \;=\; (-1)^{\mu_1}\ell^2\,, \tag{2.91}$$
$$\text{and} \quad m^{2p}\ell^{2q} \;=\; 1\,.$$

[13] I'm extremely grateful to the authors for the very generous sharing of their code with me. This allowed me to verify their computations matched my own very early on in my work on this subject.

[14] Note that in [64] they use a different convention to [45] and this thesis as to which row of the Neumann–Zagier matrices corresponds to the meridian. In particular, they put the equation for the boundary curve in the first row and we put it in the last row.

an integral solution $(f, f'') \in \mathbb{Z}^{2N}$ to

$$Af + Bf'' \;=\; \nu\,, \quad \text{and} \quad Cf + Df'' \;=\; \mu\,, \tag{2.92}$$

and finally $r, s \in \mathbb{Z}$ such that $rq - ps \;=\; 1$. With this data we define one–loop invariant with the auxiliary matrix

$$R_{i,j} \;=\; \frac{2q\delta_{i,N}(B^{-1})_{j,N}}{(p + 2q(DB^{-1})_{1,N})} \tag{2.93}$$

by

$$\delta_{A,B,C,D,\nu,\mu,z,f} \;=\; \frac{\pm\det((A-R)\operatorname{diag}(z'') + B\operatorname{diag}(z)^{-1})z^{f''}z''^{-f}(p + 2q(DB^{-1})_{1,N})}{(m^{1/q}\exp(\mp\pi i s/q) - m^{-1/q}\exp(\pm\pi i s/q))^2}\,. \tag{2.94}$$

Similarly to the case of knots, we then take the symetric $(N+1)\times(N+1)$–matrix $\mathcal{H}$ such that for $i, j < N+1$

$$\mathcal{H}_{i,j} \;=\; (-B^{-1}A + \operatorname{diag}(z'))_{i,j}\,, \tag{2.95}$$

for $i = N+1$ with $j \neq N+1$

$$\mathcal{H}_{i,j} \;=\; 2(B^{-1})_{j,N}\,, \tag{2.96}$$

and

$$\mathcal{H}_{N+1,N+1} \;=\; -2\frac{p}{q} - 4(DB^{-1})_{1,N}\,. \tag{2.97}$$

If $\det(\mathcal{H}) \neq 0$ then for

$$\Gamma^{(\ell)}_{N+1} \;=\; ((f^{\mathrm{T}}B^{-1})_N + q^{-1})\delta_{\ell,1} - \Big(-\frac{2}{q}\Big)^{\ell}\operatorname{Li}_{1-\ell}(\exp(\mp 2\pi i s/q)m^{-2/q}) \tag{2.98}$$

taking

$$g_{A,B,C,D,\nu,\mu,f,\hbar}(x; z) \;=\; g_{A,B,\nu,z,f,\hbar}(x; z)\exp\Bigg(-\frac{s\hbar}{2q} + \sum_{\ell=1}^{\infty}\Gamma^{(\ell)}_{N+1}\frac{x^{\ell}_{N+1}}{\ell!}\Bigg) \tag{2.99}$$

to get the higher loops we apply formal Gaussian integration

$$\exp\Bigg(\sum_{k=2}^{\infty} S_{A,B,C,D,\nu,\mu,f,k}(z)\hbar^{k-1}\Bigg) \;=\; \frac{\int e^{-\frac{1}{2}x^{\mathrm{T}}\mathcal{H}x} g_{A,B,C,D,\nu,\mu,f,\hbar}(x; z)dx}{\int e^{-\frac{1}{2}x^{\mathrm{T}}\mathcal{H}x}dx}\,. \tag{2.100}$$

Then, taking

$$\Pi \;=\; \hbar\mathcal{H}^{-1} \in \operatorname{Hom}((\mathbb{C}[[h]]^{N+1})^{\otimes 2}, \mathbb{C}[[h]])\,, \tag{2.101}$$

and for each $\ell > 0$ and ℓ–valent vertex associate $\Gamma^{(\ell)} = \sum_{j=1}^{N+1}\Gamma^{(\ell)}_j e_j \otimes \cdots \otimes e_j \in (\mathbb{C}[[h]]^{N+1})^{\otimes\ell}$ where for $j < N+1$ we have $\Gamma^{(\ell)}_j$ as given in equation (2.82) and for $j = N+1$ as in equation (2.98) and as a vacuum contribution *i.e.* the graph with a single zero valent vertex take $\Gamma^{(0)}$ as in (2.83) with an additional term $-s\hbar/4q$. With these definitions we can now define $S_k(z)$ in exactly the same way as we did in the case of knots. For example,

$$\begin{aligned} S_2(z) \;=\; \mathrm{coeff}\Big[\Gamma^{(0)} + \sum_{i=1}^{N+1}\Big(\frac{1}{8}\Gamma_i^{(4)}(\Pi_{i,i})^2 + \frac{1}{2}\Gamma_i^{(2)}\Pi_{i,i}\Big) \\ + \sum_{i,j=1}^{N+1}\Big(\frac{1}{8}\Pi_{i,i}\Gamma_i^{(3)}\Pi_{i,j}\Gamma_j^{(3)}\Pi_{j,j} + \frac{1}{12}\Gamma_i^{(3)}\Pi_{i,j}\Gamma_j^{(3)} + \frac{1}{2}\Gamma_i^{(1)}\Pi_{i,j}\Gamma_j^{(3)}\Pi_{j,j} + \frac{1}{2}\Gamma_i^{(1)}\Pi_{i,j}\Gamma_j^{(1)}\Big), \hbar\Big]\,. \end{aligned} \tag{2.102}$$

Example 24 (1–loop and 2–loop for $4_1(1,2)$)**.** *The gluing equations after surgery are given by*

$$z^2 z' w^2 w' \;=\; 1 \quad \textit{and} \quad z w^4 (1-w^{-1})^{-5} \;=\; 1\,, \tag{2.103}$$

which has solutions

$$\begin{aligned} & w^{14} + w^{13} - w^{11} - 2w^{10} + 4w^9 - 4w^8 \\ & \qquad + 3w^7 - 7w^6 + 27w^5 - 50w^4 + 49w^3 - 27w^2 + 8w - 1 \;=\; 0\,, \\ & -136w^{13} - 229w^{12} - 156w^{11} + 32w^{10} + 299w^9 - 334w^8 + 318w^7 \\ & \qquad - 191w^6 + 821w^5 - 3112w^4 + 4668w^3 - 3468w^2 + 1309w - 204 \;=\; z\,, \\ & -154 + 896w - 2142w^2 + 2595w^3 - 1561w^4 + 395w^5 - 123w^6 + 177w^7 \\ & -202w^8 + 147w^9 + 56w^{10} - 28w^{11} - 85w^{12} - 63w^{13} \;=\; m\,. \end{aligned} \tag{2.104}$$

Then we find that

$$\begin{aligned} \delta \;=\; & 3711w^{13} + 5779w^{12} + 3238w^{11} - 1863w^{10} - 8408w^9 + 10189w^8 - 9190w^7 \\ & + 6030w^6 - 22641w^5 + 87584w^4 - 136845w^3 + 105904w^2 - 41514w + 6703 \end{aligned} \tag{2.105}$$

and

$$\begin{aligned} S_2(z,w) \;=\; & -\frac{103985214498148}{52058057626129}w^{13} - \frac{184056457134922}{52058057626129}w^{12} - \frac{542317054615207}{208232230504516}w^{11} \\ & + \frac{59505081205693}{208232230504516}w^{10} + \frac{712734937701865}{156174172878387}w^9 - \frac{893320433832741}{208232230504516}w^8 \\ & + \frac{2799985419301201}{624696691513548}w^7 - \frac{1552744881772601}{624696691513548}w^6 + \frac{617002533395308}{52058057626129}w^5 \\ & - \frac{6997401920285924}{156174172878387}w^4 + \frac{40417653146722771}{624696691513548}w^3 - \frac{9551945238699689}{208232230504516}w^2 \\ & + \frac{2570480289595373}{156174172878387}w - \frac{1027130175419887}{416464461009032} \end{aligned} \tag{2.106}$$

Now letting

$$\begin{aligned}\xi \;=\; & 119 - 770w + 2004w^2 - 2599w^3 + 1647w^4 - 420w^5 + 116w^6 - 171w^7 \\ & + 192w^8 - 167w^9 - 49w^{10} + 50w^{11} + 104w^{12} + 69w^{13}\,,\end{aligned} \tag{2.107}$$

which satisfies

$$\xi^7 - \xi^6 - 2\xi^5 + 6\xi^4 - 11\xi^3 + 6\xi^2 + 3\xi - 1 \;=\; 0\,, \tag{2.108}$$

we can see that

$$\delta \;=\; 74 + 66\xi - 133\xi^2 + 74\xi^3 - 31\xi^4 - 15\xi^5 + 12\xi^6\,. \tag{2.109}$$

Then S_2 can be shown to be in

$$\frac{1497746 + 1345119\xi - 3675733\xi^2 + 2082815\xi^3 - 839488\xi^4 - 283405\xi^5 + 383432\xi^6}{24\delta^3} + \frac{1}{24}\mathbb{Z}\,. \tag{2.110}$$

This is all computed in the Code 10.

In [64], they then make the analogue to conjecture 5 in relation to the Chen–Yang volume conjecture 4. We will see later in sections 8.3 and 4.9 that this seems to hold for the previous example. While these methods provide conjectures for the first few coefficients, computing even ten coefficients with Feynman diagrams is computationally extremely costly when we have even just two tetrahedra. To explore the asymptotic behaviour of quantum invariants it then becomes of interest as to whether one can compute at least one hundred terms in these asymptotic expansions. To do this we can take a numerical approach and then explore the behaviour experimentally. This numerical analysis and associated techniques to make sense of these asymptotic series will be the content of the next part.

Part II

Asymptotics

Chapter 3

From functions to approximations and back again

Asymptotic series describe the limiting behaviour of a function or sequence. In ideal cases, one finds an infinite Taylor series that converges in some neighbourhood of the limiting point. More generally, one finds an infinite Taylor series that has zero radius of convergence. These series with zero radius of convergence can provide good approximations to a function of interest with some theoretical bound on the error. Firstly, I will describe the basic definitions of asymptotic equivalence. Then I will describe various numerical methods to compute asymptotic series. These extrapolation methods were taught to me by Don Zagier. His method is a variation on a method discovered by Richardson, which will be described first. I will then describe an array of variations. Finally, I will introduce a new variation of the method that I found, which deals with oscillatory sequences. Most of the following initial sections are also discussed in [203]. Next we consider a natural occurrence of divergent series coming from differential equations. Then we turn to the subject of making sense of the value of a divergent series at a finite point. This is done at first through the method of optimal truncation, which is also smoothed. Then we consider the more general study of approximating functions with rational functions, *i.e.* Padé approximation. Finally, we discuss Borel resummation. Combining Padé approximation with the Borel transform and Laplace integral will be a fundamental numerical method of giving a finite value to a divergent series. The key point, which is currently in most cases conjectural, is that the Borel transform of series we are interested in has endless analytic continuation.

3.1 Asymptotic series

Asymptotic series are a certain manifestation of smoothness. Roughly a function has an asymptotic series around a point if it has an infinite Taylor series, which may have zero radius of convergence. Asymptotic series appear in nature as, for example, solutions to differential equations, sending $q \to 1$ in a q–series, and from generating series related to graphs. To make precise our description of asymptotic series, we will give some basic definitions of asymptotic equivalence. The idea of asymptotic equivalence of sequences is that, up to multiplication by some number, the behaviour of two sequences at infinity is the same to leading order. See [151] for some more details.

Definition 9 ($\sim$, o, and O)**.** *[151] If $x_n, y_n \in \mathbb{C}^\times$ are sequences, then*

$$\begin{aligned} x_n &\sim y_n \quad \textit{if} \quad \lim_{n\to 0} \frac{x_n}{y_n} = 1\,, \\ x_n &= O(y_n) \quad \textit{if} \quad \left|\frac{x_n}{y_n}\right| \quad \textit{is bounded for } n > N \textit{ for some } N \in \mathbb{Z}\,, \\ x_n &= o(y_n) \quad \textit{if} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = 0\,. \end{aligned} \tag{3.1}$$

Often we will be interested in sequences such that for some $C, c, A_k \in \mathbb{C}$ and for all $K \in \mathbb{Z}$ we have

$$x_n = C^n n^c \left(\sum_{k=0}^{K} A_k n^{-k} + O(n^{-K-1}) \right) \tag{3.2}$$

In this case, we say x_n has *asymptotic expansion* [151, Ch. 1, Sec. 7.1] given by[1]

$$x_n \sim C^n n^c (A_0 + A_1 n^{-1} + \cdots)\,. \tag{3.3}$$

Example 25. *A famous example of an asymptotic expansion is given by Stirling's approximation*

$$n! \sim n^{n-\frac{1}{2}} \exp(-n)\sqrt{2\pi} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \cdots \right). \tag{3.4}$$

We will use this example to test our numerical methods.

Although it is essentially covered in the previous definitions, we can extend the previous notation to include the situation when we have multiple exponential terms. If we take $x_n, C, c, D, d, A_k, B_k \in \mathbb{C}$ and $|C| = |D|$, then

$$\begin{aligned} & x_n \sim C^n n^c (A_0 + A_1 n^{-1} + \cdots) + D^n n^d (B_0 + B_1 n^{-1} + \cdots) \\ \text{if for all } K \in \mathbb{Z}_{>0} \quad & \lim_{n\to\infty} \left(x_n - C^n n^c \sum_{k=0}^{K} A_k n^{-k} - D^n n^d \sum_{k=0}^{K} B_k n^{-k} \right) n^K = 0\,. \end{aligned} \tag{3.5}$$

This definition can be extended to any number of terms not just two.

We can treat the symbols o and O algebraically as described in the following proposition.

[1] Notice the double use of $\sim$ as noted in [151, Ch. 1, Sec. 7.1].

Proposition 1. *Suppose that* $w_n, x_n, y_n, z_n \in \mathbb{C}^\times$ *are sequences, then*

- *if* $x_n = O(y_n)$ *and* $z_n = O(y_n)$ *then* $x_n + z_n = O(y_n)$*,*
- *if* $x_n = O(y_n)$ *and* $z_n = O(w_n)$ *then* $x_n z_n = O(y_n w_n)$*,*

and the same with O *replaced by* o*. Finally,*

- *if* $x_n = o(y_n)$ *then* $x_n = O(y_n)$*.*

3.2 Richardson's and Zagier's extrapolation methods

Consider a sequence of numbers $\{x_n\}$ such that

$$x_n \sim A_0 + A_1 n^{-1} + A_2 n^{-2} + \cdots . \tag{3.6}$$

We want to answer the basic question: given x_n numerically for some set of n how can one determine the A_i numerically?

Richardson's basic idea is is to write out the asymptotics as a linear system and only partially solve this linear system. The size of the linear system is then related to the order of the error to which we can compute the coefficients A_j. Richardson, then has an inductive method that can be used to solve this linear system from smaller ones. Zagier rediscovered this approach from a slightly different point of view. His approach is closer to the second method of Richardson. He takes combinations of the values of the sequence to decrease the order of subleading terms. Then as the leading term will be much larger than the subleading, one can use the value of the new sequence as a good approximation of the limit. We will give his description discussed in [90, 203]. Then we will explain the equivalence with Richardson's method. Zagier's observation is that if x_n satisfies equation (3.6), then

$$n^k x_n \sim A_0 n^k + A_1 n^{k-1} + \cdots + A_k + A_{k+1} n^{-1} + \cdots . \tag{3.7}$$

Therefore, taking the k-th difference of this sequence will give a sequence asymptotic to

$$k! A_0 + (-1)^k k! A_{k+1} n^{-k-1} + \cdots . \tag{3.8}$$

Definition 10. *Let* $\Delta\{x_n\} = \{x_{n+1} - x_n\}$ *and* $\mathbf{n}\{x_n\} = \{n x_n\}$*. Then define the function* extrap_k*, mapping sequences to sequences, such that*

$$\mathrm{extrap}_k\{x_n\} = \frac{1}{k!}\Delta^k \mathbf{n}^k \{x_n\} . \tag{3.9}$$

The first point is that for $k \geq \ell > 0$ we have $\mathrm{extrap}_k\{n^{-\ell}\} = \{0\}$, while for $\ell > k$ we have $\mathrm{extrap}_k\{n^{-\ell}\} = O(n^{-\ell})$. Therefore, we immediately have the proposition.

Proposition 2. *If $\{x_n\}$ is a sequence such that $x_n \sim \sum_{j=0}^{\infty} A_j n^{-j}$, then*

$$\mathrm{extrap}_k\{x_n\} \;=\; A_0 + O(n^{-k-1})\,. \tag{3.10}$$

We can explore the error in more detail. Firstly, the action of extrap_k on monomials is given by

$$\begin{aligned}\mathrm{extrap}_k n^{-\ell} \;&=\; \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(n+k-j)^{k-\ell}\\ &=\; \frac{1}{k!}\sum_{j=0}^{k}\sum_{p=0}^{\infty}(-1)^j\binom{k}{j}\binom{k-\ell}{p}n^{k-\ell-p}(k-j)^p\\ &=\; \sum_{p=0}^{\infty}\binom{k-\ell}{p}n^{k-\ell-p}\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^p\\ &=\; \sum_{p=0}^{\infty}\binom{k-\ell}{p}S(p,k)n^{k-\ell-p}\end{aligned} \tag{3.11}$$

where $S(p,k)$ are the Stirling numbers of the second kind. Notice that for $k \geq \ell > 0$ and all p, or $\ell > k > 0$ and $p > k$ we have

$$\binom{k-\ell}{p}S(p,k) \;=\; 0\,. \tag{3.12}$$

Therefore,

$$\begin{aligned}\mathrm{extrap}_k\sum_{\ell=0}^{\infty}n^{-\ell}A_\ell \;&=\; A_0 + \sum_{\ell=k+1}^{\infty}\sum_{p=k}^{\infty}\binom{k-\ell}{p}S(p,k)A_\ell n^{k-\ell-p}\\ &=\; A_0 + n^{-k}\sum_{\ell=0}^{\infty}\sum_{p=0}^{\infty}\binom{-\ell-1}{p+k}S(p+k,k)A_{\ell+k+1}n^{-\ell-p-1}\\ &=\; A_0 + n^{-k-1}\sum_{r=0}^{\infty}n^{-r}\sum_{p=0}^{r}(-1)^{p+k}\binom{r+k}{p+k}S(p+k,k)A_{r-p+k+1}\end{aligned} \tag{3.13}$$

If $|A_\ell|$ is increasing then

$$\begin{aligned}\left|\sum_{p=0}^{r}(-1)^{p+k}\binom{r+k}{p+k}S(p+k,k)A_{r-p+k+1}\right| &\leq S(r+k+1,k+1)|A_{r+k+1}|\\ \leq \frac{1}{2}\binom{r+k+1}{k+1}(k+1)^r|A_{r+k+1}| &\leq 2^{r+k}(k+1)^r|A_{r+k+1}|\,.\end{aligned} \tag{3.14}$$

Therefore, if $|A_\ell|$ is increasing then

$$\left|\operatorname{extrap}_k \sum_{\ell=0}^{\infty} n^{-\ell} A_\ell\right| \leq A_0 + n^{-k-1} \sum_{r=0}^{\infty} 2^{r+k}(k+1)^r |A_r| n^{-r}\,. \tag{3.15}$$

Example 26. *To illustrate the method consider*

$$x_n \;=\; n!\, n^{\frac{1}{2}-n} \exp(n)\,, \tag{3.16}$$

which has asymptotic series deducible from equation (3.4). Using extrap_k *we can compute approximations of* $\sqrt{2\pi}$*. We have*

$$\begin{aligned}
\operatorname{extrap}_1(x)_{100} &= 2.506606727\cdots\,, & \operatorname{extrap}_1(x)_{100} - \sqrt{2\pi} &= -2.154718332\cdots \times 10^{-5}\,,\\
\operatorname{extrap}_2(x)_{100} &= 2.506628488\cdots\,, & \operatorname{extrap}_2(x)_{100} - \sqrt{2\pi} &= 2.129785028\cdots \times 10^{-7}\,,\\
\operatorname{extrap}_3(x)_{100} &= 2.506628273\cdots\,, & \operatorname{extrap}_3(x)_{100} - \sqrt{2\pi} &= -2.015005875\cdots \times 10^{-9}\,.
\end{aligned} \tag{3.17}$$

This method can be used in a variety of contexts. Firstly, notice that for x_n as in equation (3.6) we can find all A_j by inductively finding A_i for $i < j$ and then constructing the sequence

$$n^j x_n - \sum_{i=0}^{j-1} A_i n^{j-i} \;\sim\; A_j + A_{j+1} n^{-1} + \cdots\,, \tag{3.18}$$

and taking extrap_k of this sequence. Then we can consider more general sequences. For example, suppose that

$$x_n \;\sim\; n^c (A_0 + A_1 n^{-1} + \cdots)\,. \tag{3.19}$$

Then

$$\frac{x_{n+1}}{x_n} \;\sim\; 1 + c n^{-1} + \cdots\,. \tag{3.20}$$

Therefore, we can again use extrap_k to compute c then dividing by n^c we are back to the previous case. More generally we could have

$$x_n \;\sim\; (n!)^\gamma C^n n^c (A_0 + A_1 n^{-1} + \cdots)\,, \tag{3.21}$$

and then if we take quotients

$$\frac{x_{n+1}}{x_n} \;\sim\; n^\gamma C(1 + \cdots)\,, \tag{3.22}$$

which again reduces to the previous cases. These variants are discussed in [90, 203]. There is also a version discussed in [203] for $|C_i| = |C_j|$ and

$$x_n \;\sim\; \sum_{i=1}^{K} C_i^n n^{c_i} \big(A_0^{(i)} + \cdots\big)\,, \tag{3.23}$$

if we know C_i and c_i. We take

$$\mathrm{extrap}_k C_K^{-n} n^{-c_K-1} x_n\,, \tag{3.24}$$

which up to $c_K - k$ this reduces the number of terms to $K-1$ which can then continue by induction to get back to the previous case. This example in conveniently computed by considering

$$\Big(\prod_{j=1}^{K} C_j^{\mathbf{n}} \mathbf{n}^{c_j} \mathrm{extrap}_k C_j^{-\mathbf{n}} \mathbf{n}^{-c_j}\Big)\{x_n\}\,. \tag{3.25}$$

along with

$$\Big(\prod_{j=1}^{K} C_j^{\mathbf{n}} \mathbf{n}^{c_j} \mathrm{extrap}_k C_j^{-\mathbf{n}} \mathbf{n}^{-c_j}\Big)\{C_i^n n^{c_i}\} \tag{3.26}$$

Therefore,

$$\Big(\prod_{j=1}^{K} C_j^{\mathbf{n}} \mathbf{n}^{c_j} \mathrm{extrap}_k C_j^{-\mathbf{n}} \mathbf{n}^{-c_j}\Big)\{x_n\} \;=\; \sum_{i=1}^{K} A_0^{(i)} \Big(\prod_{j=1}^{K} C_j^{\mathbf{n}} \mathbf{n}^{c_j} \mathrm{extrap}_k C_j^{-\mathbf{n}} \mathbf{n}^{-c_j}\Big)\{C_i^n n^{c_i} + O(n^{-k-1})\} \tag{3.27}$$

Therefore, taking K values of n we can form a matrix and solve for $A_0^{(i)}$ up to $O(n^{-k-1})$ corrections. This matrix approach in fact works in much more general cases.

All these extrapolation methods fit into a general method called the E-algorithm. This is discussed, for example, in [152]. To describe this method suppose that

$$x_n - A_0 - A_1 g_1(n) - \cdots - A_k g_k(n) \;=\; O(f(n))\,, \tag{3.28}$$

where A_j are unknowns. Then consider the matrix equation

$$\begin{pmatrix} 1 & g_1(n) & \cdots & g_k(n) \\ 1 & g_1(n+1) & \cdots & g_k(n+1) \\ \vdots & \vdots & \cdots & \vdots \\ 1 & g_1(n+k) & \cdots & g_k(n+k) \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_k \end{pmatrix} \;=\; \begin{pmatrix} x_n + O(f(n)) \\ x_{n+1} + O(f(n+1)) \\ \vdots \\ x_{n+k} + O(f(n+k)) \end{pmatrix}\,. \tag{3.29}$$

Therefore,

$$A_0 \;=\; \frac{\det \begin{pmatrix} x_n + O(f(n)) & x_{n+1} + O(f(n+1)) & \cdots & x_{n+k} + O(f(n+k)) \\ g_1(n) & g_1(n+1) & \cdots & g_1(n+k) \\ \vdots & \vdots & \cdots & \vdots \\ g_k(n) & g_k(n+1) & \cdots & g_k(n+k) \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ g_1(n) & g_1(n+1) & \cdots & g_1(n+k) \\ \vdots & \vdots & \cdots & \vdots \\ g_k(n) & g_k(n+1) & \cdots & g_k(n+k) \end{pmatrix}} \tag{3.30}$$

This method relates to the original method using extrap_k by letting $g_j(n) = n^{-j}$ and $f(n) = n^{-k-1}$. This is illustrated in the following proposition.

Proposition 3. *Let* $\{x_n\}$ *be a sequence. Then*

$$\text{extrap}_k\{x_n\} = \left\{ \det \begin{pmatrix} x_n & x_{n+1} & \cdots & x_{n+k} \\ n^{-1} & (n+1)^{-1} & \cdots & (n+k)^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ n^{-k} & (n+1)^{-k} & \cdots & (n+k)^{-k} \end{pmatrix} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ n^{-1} & (n+1)^{-1} & \cdots & (n+k)^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ n^{-k} & (n+1)^{-k} & \cdots & (n+k)^{-k} \end{pmatrix}^{-1} \right\}. \tag{3.31}$$

Proof. Notice that $\text{extrap}_k\{x_n\}$ is given by

$$\text{extrap}_k\{x_n\} = \left\{ \sum_{m=0}^{k} (-1)^{k+m} \frac{1}{m!(k-m)!} (n+m)^k x_{n+m} \right\}. \tag{3.32}$$

Then notice that from the Vandermonde determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ n^{-1} & (n+1)^{-1} & \cdots & (n+k)^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ n^{-k} & (n+1)^{-k} & \cdots & (n+k)^{-k} \end{pmatrix} = \prod_{0 \le i < j \le k} \frac{i-j}{(n+j)(n+i)} = \prod_{j=0}^{k} (-1)^j \frac{j!}{(n+j)^k} \tag{3.33}$$

while

$$\det \begin{pmatrix} n^{-1} & (n+1)^{-1} & \cdots & (n+m-1)^{-1} & (n+m+1)^{-1} & \cdots & (n+k)^{-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ n^{-k} & (n+1)^{-k} & \cdots & (n+m-1)^{-k} & (n+m+1)^{-k} & \cdots & (n+k)^{-k} \end{pmatrix}$$
$$= \prod_{\substack{0 \le i \le k \\ i \ne m}} \frac{1}{(n+i)} \prod_{\substack{0 \le i < j \le k \\ i,j \ne m}} \frac{i-j}{(n+j)(n+i)} = \prod_{\substack{0 \le i \le k \\ i \ne m}} \frac{1}{(n+i)^k} \prod_{\substack{0 \le i < j \le k \\ i,j \ne m}} (i-j)\,. \tag{3.34}$$

Then notice that the quotient of equation (3.34) by equation (3.33) is given by

$$(n+m)^k \prod_{0 \le i < m} \frac{1}{(i-m)} \prod_{m < j \le k} \frac{1}{(m-j)} = (-1)^k (n+m)^k \frac{1}{m!(k-m)!}\,. \tag{3.35}$$

Therefore, the quotient of determinants in equation (3.31) is equal to equation (3.32) by expanding the first determinant in minors along the first row. □

This method can be used to find asymptotics of more general forms. For example, take

$$x_n \sim \sum_{i=1}^{K} (n!)^{\gamma_i} C_i^n n^{c_i} \sum_{j=0}^{\infty} A_j^{(i)} n^{-j} \,. \tag{3.36}$$

Then one can take $g_{i,j}(n) = (n!)^{\gamma_i} C_i^n n^{c_i - j}$ for $i = 1, \ldots, K$ and $j = 0, \ldots, k$. The using this method one can find solutions for $A_j^{(i)}$. One issue with all of these methods is that one needs knowledge of γ_i, C_i, c_i.

Remark 11. *When applying these numerical methods we use some finite set of values of n to evaluate our sequence x_n. Adding exponentially smaller terms will give rise to the same asymptotic series, however at a finite value of n, these exponentially small terms may in fact dominate if they are multiplied by some large pre–factor. This means that depending on the example the error of the method for some finite n is hard to quantify without further assumptions. For example, the sequence $x_n = 1 + 2^{1000-n} \sim 1 + O(2^{-n})$ will look $O(2^{-n})$ numerically until n is around* 1000*.*

3.3 A method for competing exponentials

The basic variants to extrapolation methods transform the sequence in a way which alters the asymptotics into a form that one of the previous variants can deal with. Any combination of the terms of the sequence can be used to create a new series, linear or non-linear. Don Zagier posed the question of how one could numerically determine C_i, c_i in equation (3.23) without their prior knowledge.

The following transformations give a possible answer. These transformations give the ability to change exponential growth of coefficients. This approach has many free parameters but we will give a specific version that can of course be adjusted to ones needs. The basic trick is to use the binomial theorem to change the exponential growth.

Proposition 4. *If x_n is a sequence such that for some, $C_i, A_0^{(i)}, c_i \in \mathbb{C}$ where $|C_i| = |C_j|$ for all i, j, we have*

$$x_n \sim \sum_{i=1}^{K} C_i^n n^{c_i} \big(A_0^{(i)} + \cdots\big) \,, \tag{3.37}$$

then, if for $z \in \mathbb{C}$ we have $|C_i + z| > |C_j + z|$ for $j \neq i$,

$$\sum_{m=0}^{n} \binom{n}{m} x_{n+m} z^{n-m} \sim \left(1 + \frac{C_i}{C_i + z}\right)^{c_i} n^{c_i} C_i^n (C_i + z)^n \big(A_0^{(i)} + \cdots\big) \,. \tag{3.38}$$

Proof. Using Proposition 1 we have

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}x_{n+m}z^{n-m} &\sim \sum_{i=1}^{K}\sum_{m=0}^{n}\binom{n}{m}C_i^{n+m}z^{n-m}(n+m)^{c_i}\big(A_0^{(i)}+\cdots\big)\\ &= \sum_{i=1}^{K}C_i^n n^{c_i}\big(A_0^{(i)}+\cdots\big)\sum_{m=0}^{n}\binom{n}{m}\Big(1+\frac{m}{n}\Big)^{c_i}C_i^m z^{n-m}\\ &= \sum_{i=1}^{K}\Big(1+\frac{C_i}{C_i+z}\Big)^{c_i}n^{c_i}C_i^n(C_i+z)^n\big(A_0^{(i)}+\cdots\big)\,,\end{aligned}\tag{3.39}$$

The last equality follows from the identity

$$\sum_{m=0}^{n}\binom{n}{m}\Big(1+\frac{m}{n}\Big)^{d}C^m = (C+1)^n\Big(1+\frac{C}{C+1}\Big)^{d}(1+\cdots)\,.\tag{3.40}$$

To prove this, we shift and expand using the generalised binomial theorem in a region where it converges absolutely and uniformly in n as follows:

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}\Big(1+\frac{m}{n}\Big)^{d}C^m &= \sum_{m=0}^{n}\binom{n}{m}\frac{3^d}{2^d}\Big(1-\frac{1}{3}+\frac{2}{3}\frac{m}{n}\Big)^{d}C^m\\ &= \frac{3^d}{2^d}\sum_{r=0}^{\infty}\binom{d}{r}\sum_{m=0}^{n}\binom{n}{m}\Big(-\frac{1}{3}+\frac{2}{3}\frac{m}{n}\Big)^{r}C^m\\ &= \frac{3^d}{2^d}\sum_{r,s=0}^{\infty}\binom{d}{r}\binom{r}{s}\Big(-\frac{1}{3}\Big)^{r-s}\sum_{m=0}^{n}\binom{n}{m}\Big(\frac{2}{3}\frac{m}{n}\Big)^{s}C^m\,.\end{aligned}\tag{3.41}$$

Now we can express the sum over m in terms of derivatives in C. Then expressing $(C\partial_C)^s$ in terms of $C^s\partial_C^s$ using Stirlings numbers of the second kind $S(s,k)$ we find that

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}\Big(1+\frac{m}{n}\Big)^{d}C^m &= \frac{3^d}{2^d}\sum_{r,s=0}^{\infty}\binom{d}{r}\binom{r}{s}\Big(-\frac{1}{3}\Big)^{r-s}\Big(\frac{2}{3}\frac{C}{n}\frac{\partial}{\partial C}\Big)^{s}(C+1)^n\\ &= \frac{3^d}{2^d}\sum_{r,s=0}^{\infty}\binom{d}{r}\binom{r}{s}\frac{1}{n^s}\Big(-\frac{1}{3}\Big)^{r-s}\frac{2^s}{3^s}\sum_{k=0}^{s}S(s,k)C^k\frac{\partial^k}{\partial C^k}(C+1)^n\\ &= \frac{3^d}{2^d}\sum_{r,s=0}^{\infty}\binom{d}{r}\binom{r}{s}\frac{1}{n^s}\Big(-\frac{1}{3}\Big)^{r-s}\frac{2^s}{3^s}\sum_{k=0}^{s}S(s,k)(n)_kC^k(C+1)^{n-k}\,.\end{aligned}\tag{3.42}$$

This gives the full expansion in n^{-1} and to finish the proof we notice that the n^0 coefficients are given by

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}\Big(1+\frac{m}{n}\Big)^{d}C^{m} &= \frac{3^d}{2^d}\sum_{r,s=0}^{\infty}\binom{d}{r}\binom{r}{s}\Big(-\frac{1}{3}\Big)^{r-s}\frac{2^s}{3^s}C^{s}(C+1)^{n-s}(1+\cdots)\\ &= (C+1)^{n}\frac{3^d}{2^d}\sum_{r=0}^{\infty}\binom{d}{r}\Big(-\frac{1}{3}+\frac{2}{3}\frac{C}{C+1}\Big)^{r}(1+\cdots)\\ &= (C+1)^{n}\Big(1+\frac{C}{C+1}\Big)^{d}(1+\cdots)\,.\end{aligned} \tag{3.43}$$

□

Therefore, by varying z, so that for some $z = z_i$ we have $|C_i + z_i| > |C_j + z_i|$ for $j \neq i$, we can find C_i and c_i by applying the original extrapolation method to the new sequence and solving the quadratic equation. To see that varying z will find all the exponential terms, notice that if $\arg(z_i) = \arg(C_i)$ then $|C_i+z_i| > |C_j+z_i|$ for $j \neq i$ as can be seen in Figure 3.1. Although the quadratic equation has two solutions for C_i if we slightly vary z_i we can look for the solution which remains constant. One important note that is illustrated by Figure 3.1 and Remark 11 is that, if C_i has a similar argument to another C_j then the numerical error will be much higher using the extrapolation methods. For example, in Figure 3.1, C_3 would be harder to compute than C_4 in practice.

The Airy function is given by

$$\mathrm{Ai}(x) \;=\; \frac{1}{\pi}\int_0^{\infty}\cos\Big(\frac{t^3}{3}+xt\Big)dt\,. \tag{3.44}$$

We can explore the asymptotics of this function as $x \to -\infty$. To do this take the sequence

$$x_n \;=\; \mathrm{Ai}(-n^{2/3}). \tag{3.45}$$

where we find for example

$$x_{100} \;=\; -0.26073\cdots\,,\quad x_{101} \;=\; -0.21962\cdots\,,\quad x_{102} \;=\; -0.084741\cdots\,. \tag{3.46}$$

Then applying the previous methods we find nonsense. For example, applying the previous code gives the following outputs.

```
xx=vector(250,n,airy(-n^(2/3))[1]);
asymp(xx,100,10)
/* = [2.0666 E16, -1055.8, -9.1801 E492] */
asymp(xx,100,11)
/* = [-7.3328 E17, -1169.1, 3.5239 E565] */
```

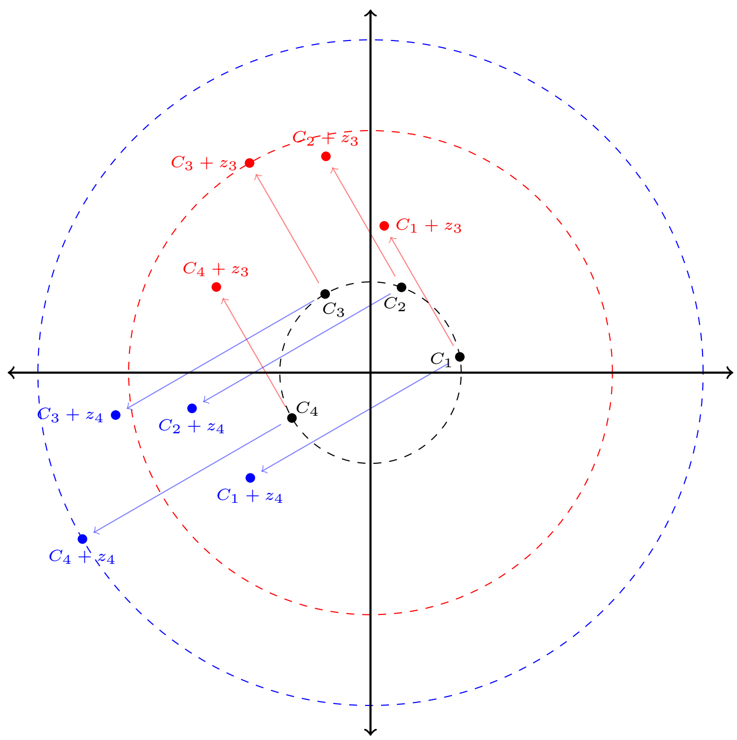

Figure 3.1: Oscillatory method's affect on exponential terms sending C_i to $C_i + z$.

Therefore, we must use the oscillatory method to probe this sequence. Testing various outcomes of the new method with $z = \mathbf{e}(k/10)$ for $k = 1, \cdots, 10$ we find the following table of outputs of potential values of $C_i^{(\ell)}$ where $\ell = 1, 2$ indexes the two solutions to the quadratic equations giving C_i. Here, to find the various limits I've used extrap$_5$.

k	$C_i^{(1)}$	$C_i^{(2)}$
0	$1.3264 \cdots \times 10^5$	$-1.3264 \cdots \times 10^5$
1	$0.90125 \cdots + 0.63470 \cdots i$	$-1.710 \cdots 3 - 1.2225 \cdots i$
2	$0.78589 \cdots + 0.61837 \cdots i$	$-1.094 \cdots 9 - 1.5694 \cdots i$
3	$0.78589 \cdots + 0.61837 \cdots i$	$-0.4768 \cdots 7 - 1.5694 \cdots i$
4	$0.78589 \cdots + 0.61837 \cdots i$	$0.023130 - 1.2062 \cdots i$
5	$0.50000 \cdots + 2.5912 \cdots i \times 10^4$	$0.50000 \cdots - 2.5912 \cdots i \times 10^4$
6	$0.78589 \cdots - 0.61837 \cdots i$	$0.023130 \cdots + 1.2062 \cdots i$
7	$0.78589 \cdots - 0.61837 \cdots i$	$-0.47687 \cdots + 1.5694 \cdots i$
8	$0.78589 \cdots - 0.61837 \cdots i$	$-1.0949 \cdots + 1.5694 \cdots i$
9	$0.90125 \cdots - 0.63470 \cdots i$	$-1.7103 \cdots + 1.2225 \cdots i$

We see that at $k = 0, 1, 5, 9$ we seem to get unstable answers, which leads us to suspect nonsense. This can be verified with more detailed checks. Then applying the method at $z = \mathbf{e}(3/10)$ we find a contribution from a series of the form

$$(0.78589 \cdots + 0.61837 \cdots i)^n n^{-0.16667} (0.19947 \cdots - 0.19947 \cdots i + \cdots) \tag{3.47}$$

while at $z = \mathbf{e}(7/10)$ we find a contribution from a series of the form

$$(0.78589 \cdots - 0.61837 \cdots i)^n n^{-0.16667} (0.19947 \cdots + 0.19947 \cdots i + \cdots) \,. \tag{3.48}$$

With some experimentation one can recognise these numbers explicitly to the find the following guess for the leading order asymptotics

$$x_n = \frac{\exp\left(\frac{2i}{3}n - \frac{\pi i}{4}\right)}{2\sqrt{\pi} n^{\frac{1}{6}}}(1+\cdots) + \frac{\exp\left(-\frac{2i}{3}n + \frac{\pi i}{4}\right)}{2\sqrt{\pi} n^{\frac{1}{6}}}(1+\cdots) = \frac{\cos\left(\frac{2}{3}n - \frac{\pi}{4}\right)}{\sqrt{\pi} n^{\frac{1}{6}}}(1+\cdots) \,. \tag{3.49}$$

This of course agrees with the known asymptotics of the Airy function. Then, subtracting off the leading order and applying the method again we find a guess for the next coefficients given by

$$\begin{aligned} x_n &\sim \frac{\exp\left(\frac{2i}{3}n - \frac{\pi i}{4}\right)}{2\sqrt{\pi}n^{\frac{1}{6}}}\Big(1 - \frac{5i}{48n} + \cdots\Big) + \frac{\exp\left(-\frac{2i}{3}n + \frac{\pi i}{4}\right)}{2\sqrt{\pi}n^{\frac{1}{6}}}\Big(1 + \frac{5i}{48n} + \cdots\Big) \\ &= \frac{\cos\left(\frac{2}{3}n - \frac{\pi}{4}\right)}{\sqrt{\pi}n^{\frac{1}{6}}}(1 + \cdots) + \frac{\sin\left(\frac{2}{3}n - \frac{\pi}{4}\right)}{\sqrt{\pi}n^{\frac{7}{6}}}\Big(\frac{5}{48} + \cdots\Big). \end{aligned} \tag{3.50}$$

These computations can be implemented in the following PARI/GP Code 15. For a more interesting application of the method see Example 44.

3.4 Linear differential equations

Homogeneous linear differential equations can be described as modules of a certain non-commutative algebras. For example, consider the algebra generated by x, D_x such that

$$[D_x, x] = x. \tag{3.51}$$

This algebra acts on smooth functions in a single variable x and on formal power series in x via

$$\begin{aligned} (D_x f)(x) &= x\frac{\partial f}{\partial x}(x), & D_x \sum_k a_k x^k &= \sum_k k a_k x^k, \\ (xf)(x) &= xf(x), & x\sum_k a_k x^k &= \sum_k a_{k-1} x^k. \end{aligned} \tag{3.52}$$

The product rule shows that this action is well defined as

$$[D_x, x]f(x) = x\frac{\partial xf}{\partial x}(x) - x^2\frac{\partial f}{\partial x}(x) = xf(x). \tag{3.53}$$

For a given problem we are interested in certain modules. These will be generated by functions or power series satisfying equations of the form

$$\sum_{i=0}^{N} \alpha_i(x) D_x^i f = 0, \quad \sum_k \sum_{i=0}^{N} \sum_{j=0}^{M_i} \alpha_{ij}(k-j)^i a_{k-j} x^k = 0, \tag{3.54}$$

for some $\alpha_i(x) = \sum_{j=0}^{M_i} \alpha_{ij} x^j$. This can naturally be extended to many variables.

Example 27 (Exponential and exponential integral)**.** *Consider functions* exp *in a single variable such that*

$$(D_x - x)\exp = 0, \quad ka_k - a_{k-1} = 0. \tag{3.55}$$

These generate a module of this non-commutative algebra. This of course has a standard generator

$$\exp(x) \ = \ \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{3.56}$$

We can consider in-homogeneous solutions and their homogenised versions

$$\begin{aligned} (D_x - x)f \ &= \ 1, \quad kb_k - b_{k-1} \ = \ \delta_k, \\ (D_x^2 - xD_x - x)f \ &= \ 0, \quad k^2 a_k - ka_{k-1} \ = \ 0. \end{aligned} \tag{3.57}$$

This also has a standard solution

$$f(x) \ = \ \mathrm{Ei}(x)\exp(x) \ = \ \exp(x)\int_x^{\infty} \exp(-t)t^{-1}dt. \tag{3.58}$$

From integration by parts, see for Example [141, Sec. 1.1], one can find the asymptotic expansion as $x \to \infty$

$$f(x) \sim \sum_{-\infty}^{-1} b_k x^k \ = \ \sum_{k=0}^{\infty} (-1)^k \frac{k!}{x^{k+1}}. \tag{3.59}$$

Notice that these b_k satisfy the same recursion for a_k except the recursion does not hold when $k = 0$ (or we could take $b_k = \infty$ for $k \geq 0$). We will see that these sequences can in fact be related via the Frobenius deformation. Moreover, we will see in Section 3.7 that this asymptotic series can be used to construct the solution $\mathrm{Ei}(x)\exp(x)$*.*

Example 28 (Apéry numbers)**.** *Consider functions in a single variable $f(x)$ such that*

$$\left(D_x^3 - x\left(2D_x + 1\right)\left(17D_x^2 + 17D_x + 5\right) + x^2\left(D_x + 1\right)^3\right) f \ = \ 0. \tag{3.60}$$

These generate a module of this non-commutative algebra. This is an interesting example coming from [89]. Equation (3.60) *for a formal power series $\sum_k a_k x^k$ is equivalent to the recursion for a_k*

$$(k+1)^3 a_{k+1} - (2k+1)(17k^2 + 17k + 5)a_k + k^3 a_{k-1} \ = \ 0. \tag{3.61}$$

We can set $a_{-1} = 0, a_0 = 1$ and this recursion, as discussed in [89], gives the Apéry numbers

$$a_k \ = \ \sum_{\ell=0}^{k} \binom{k}{\ell}^2 \binom{k+\ell}{\ell}^2. \tag{3.62}$$

Given a basis f_i of solutions to a linear differential equation one can construct the *Wronskian*

$$W(f_1, \cdots, f_N) \ = \ \begin{pmatrix} f_1 & \cdots & f_n \\ \vdots & \cdots & \vdots \\ D_x^{n-1} f_1 & \cdots & D_x^{n-1} f_n \end{pmatrix}, \tag{3.63}$$

which satisfied the first order equation

$$D_x W(f_1, \cdots, f_N) \;=\; A W(f_1, \cdots, f_N) \tag{3.64}$$

where

$$A(x;q) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -\frac{\alpha_0(x)}{\alpha_n(x)} & -\frac{\alpha_1(x)}{\alpha_n(x)} & -\frac{\alpha_2(x)}{\alpha_n(x)} & -\frac{\alpha_3(x)}{\alpha_n(x)} & \cdots & -\frac{\alpha_{n-2}(x)}{\alpha_n(x)} & -\frac{\alpha_{n-1}(x)}{\alpha_n(x)} \end{pmatrix}. \tag{3.65}$$

For f_i, g_i let $U = W(f_1, \cdots, f_N)$ and $V = W(g_1, \cdots, g_N)$ and suppose that they are two bases of solutions to the differential equation. Then we see that

$$D_x(V^{-1}U) \;=\; -V^{-1}(D_x V)V^{-1}U + V^{-1}D_x U \;=\; -V^{-1}(AV)V^{-1}U + V^{-1}AU \;=\; 0. \tag{3.66}$$

Therefore, we see that $V^{-1}U$ is a matrix of constants.

A system of equations is called *hyper-geometric* if it is linear in x. This is equivalent to recursions between a_k and a_{k-1}. For example the equation for the exponential

$$D_x - x, \quad k a_k - a_{k-1} \;=\; 0 \tag{3.67}$$

is linear in x or has a recursion in a_k and a_{k-1} and this is indeed hyper-geometric. However, equation (3.60) is quadratic in x and we see a recursion between a_{k+1}, a_k, a_{k-1}. Therefore as an equation in a single variable it is not hyper-geometric. In this example, we will see that this is in fact the specialisation of a hyper-geometric function in two variables.

Example 29 (Apéry numbers continued)**.** *As mentioned the function with a_k given by equation (3.62) is a specialisation of a hyper-geometric function in two variables. In particular, consider the function*

$$f(x,y) \;=\; \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \binom{k}{\ell}^2 \binom{k+\ell}{\ell}^2 x^k y^\ell \;=\; \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \frac{(k+\ell)!^2}{\ell!^4 (k-\ell)!^2} x^k y^\ell \;=\; \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} a_{k,\ell} x^k y^\ell. \tag{3.68}$$

We see that

$$\begin{aligned} (k-\ell)^2 a_{k,\ell} \;&=\; (k+\ell)^2 a_{k-1,\ell} \\ \ell^4 a_{k,\ell} \;&=\; (k-\ell+1)^2 (k+\ell)^2 a_{k,\ell-1} \end{aligned} \tag{3.69}$$

which implies $f(x,y)$ satisfies

$$\begin{aligned} (D_x - D_y)^2 f \;&=\; (D_x + D_y)^2 x f \\ D_y^4 f \;&=\; (D_x - D_y + 1)^2 (D_x + D_y)^2 y f. \end{aligned} \tag{3.70}$$

These are hyper-geometric as equations (3.69) only involve consecutive terms such as $a_{k,\ell}$ *and* $a_{k-1,\ell}$ *or equivalently as equations (3.70) are first order in* x *and* y*.*

When finding solutions to equations (3.69) it is important to notice that there are some special points in the k,ℓ *lattice. In particular, when* $k=\ell, \ell=0$ *and* $k=-\ell-1$ *there are vanishing of coefficients of the relations between the* $a_{k,\ell}$*. These walls break up the lattices into regions. The layout of these walls imply that finite solutions to equation (3.69) holding for all* $(k,\ell)\in\mathbb{Z}^2$ *will be forced to be zero on certain regions. These regions in this example are displayed in Figure 29. We notice that choosing* $a_{0,0}, a_{0,-1}$ *and* $a_{-1,0}$ *determines all formal solutions of the form*

$$\sum_{k,\ell\in\mathbb{Z}} a_{k,\ell} x^k y^\ell. \tag{3.71}$$

The solution for $a_{0,-1}=1$ *associated to the region* $\ell\le k$ *and* $\ell\le -k-1$ *is*

$$a_{k,\ell} \;=\; \frac{(-\ell-1)!^4}{(-k-\ell-1)!^2(k-\ell)!^2}. \tag{3.72}$$

While the solution for $a_{-1,0}=1$ *associated to the region* $\ell\le -k-1$ *and* $\ell\ge 0$ *is*

$$a_{k,\ell} \;=\; \frac{(\ell-k-1)!^2}{\ell!^4(-k-\ell-1)!^2}. \tag{3.73}$$

Given a linear differential equation we can construct it's *Newton polygon* which gives various information about the solutions to the equation. See for Example [188, Sec. 3.3]. The Newton polygon keeps track of which monomials $D_x^n x^m$ have non-zero coefficient in the operator. However, the fact that equation (3.51) is inhomogenous leads to an ambiguity in the degree of D_x. Notice that the degree in D_x can only decrease in equation (3.51). Therefore, if we have an operator as in equation (3.54) define P to be the set of $(n,m)\in\mathbb{Z}$ with $\alpha_{ij}\neq 0$ and define the Newton polygon $N(P)$ to be convex hull of the set of points with (ℓ,k) with $0\le\ell\le n$ and $k=m$ for some $(n,m)\in P$. This is then independent of the particular choice of ordering of the operators D_x and x.

Using the Newton polygon of a linear differential operator, the *Frobenius method* allows the construction of formal solutions to linear differential equations. They are of the form of coefficients of the ϵ expansion of

$$\exp(p(x^{-1}))x^{r+\epsilon}\sum_{k=0}^{\infty} a_k x^k\,. \tag{3.74}$$

where p is some polynomial.

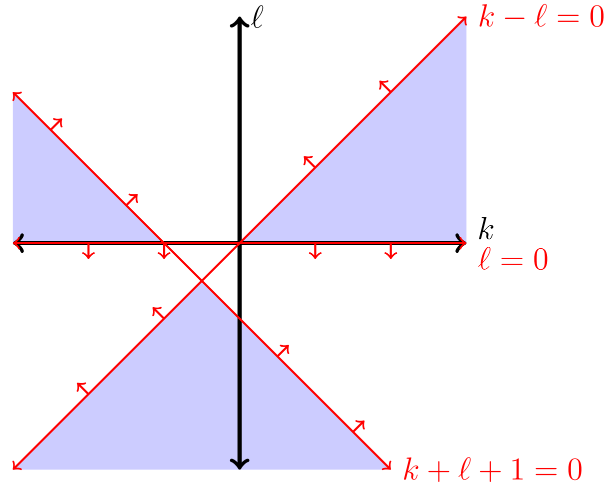

Figure 3.2: The regions determined by the recursions in $a_{k,\ell} = \frac{(k+\ell)!^2}{\ell!^4(k-\ell)!^2}$. The red lines $k - \ell = 0, \ell = 0, k + \ell + 1 = 0$ are determined by the vanishing of coefficients of $a_{k,\ell}$ in the recursions (3.69). The small arrows in red indicate that the recursions imply $a_{k,\ell}$ are zero in that region. The regions in blue indicate the regions that have no arrows pointing into them and can therefore have non-zero values of $a_{k,\ell}$.

The slopes of an edge of the Newton polygon determine the type of the exponential singularity and therefore the p of the solution. Once the p is determined we get a new differential equation for the solution with the exponential singularity removed where the left most edge of the Newton polygon is now horizontal. To see this notice that

$$\begin{aligned} D_x \exp\left(\frac{1}{kx^k}\right) \hat{f}(x) &= -\frac{1}{x^k} \exp\left(\frac{1}{kx^k}\right) \hat{f}(x) + \exp\left(\frac{1}{kx^k}\right) (D_x \hat{f})(x) \\ &= \exp\left(\frac{1}{kx^k}\right) \left(-\frac{1}{x^k} + D_x\right) \left(\hat{f}\right)(x)\,. \end{aligned} \tag{3.75}$$

Therefore any positive slope on the left of the Newton polygon of some operator can be removed by multiplying f by an appropriate exponential factor.

Using the vanishing of a_k for $k < 0$ in the Ansatz (3.74), we can find a polynomial condition on r called the *indicial polynomial*. In particular, it is the coefficient describing a_0 in terms of a_k for $k < 0$. The different roots of the indicial polynomial lead to different solutions. If the roots are not located at an integer then there will be a branch point at the origin and if it is a negative integer then there will be a pole of that order. There is some degeneracy when two roots are separated by an integer. In this case, one expands ϵ near the root and

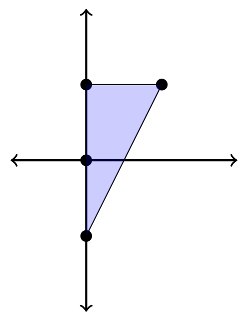

Figure 3.3: The Newton polygon associated to the operator in equation (3.78).

uses

$$x^{r+\epsilon} \;=\; x^r \left(1 + \epsilon \log(x) + \frac{\epsilon^2}{2!}\log(x)^2 + \cdots\right). \tag{3.76}$$

The effect of the Ansatz (3.74) on the action of D_x on formal series is

$$D_x x^{k+\epsilon} \;=\; (k+\epsilon)x^{k+\epsilon}. \tag{3.77}$$

Therefore, we see the Frobenius method takes the recursion for a_k and shifts $k \mapsto k+\epsilon$. We can think of this as a deformation of the recursions for a_k.

Once a solution for the left most edge is found, we apply the Theorem [188, Thm. 3.48] which states that any Newton polygon with more than one slope comes from an operator that can be factored. This factorisation then corresponds to a filtration of the space of solutions and solving for the left most edge gives a solution in the largest space in the filtration. Removing the left most factor of the factorisation we then have an operator with smaller order. We can then solve inductively by applying the same procedure.

Example 30. *Take the operator*

$$xD_x + \frac{1}{x} + 2 - 2x \tag{3.78}$$

with Newton polygon given in Figure 3.3. This has one edge of slope 2. *Therefore, we must remove the slope by multiplying by* $\exp(p(x^{-1}))$. *Suppose that*

$$f = \exp\left(\frac{1}{2x^2} + \frac{2}{x}\right)\hat{f}(x) \tag{3.79}$$

is a solution then

$$\begin{aligned} 0 &= \left(xD_x + \frac{1}{x} + 2 - 2x\right) f(x) \\ &= \left(xD_x + \frac{1}{x} + 2 - 2x\right) \exp\left(\frac{1}{2x^2} + \frac{2}{x}\right) \hat{f}(x) \\ &= \exp\left(\frac{1}{2x^2} + \frac{2}{x}\right) (xD_x - 2x)\, \hat{f}(x). \end{aligned} \tag{3.80}$$

Therefore, the equation becomes $(xD_x - 2x)\hat{f} = 0$*. Then for*

$$\hat{f}(x) = \sum_{k=0}^{\infty} a_k x^{k+r}\,, \tag{3.81}$$

we find that $(xD_x - 2x)\hat{f} = 0$ *implies that*

$$(k - 1 + r)a_{k-1} - 2a_{k-1} = 0\,, \tag{3.82}$$

and so

$$(r - 2)a_0 = 0\,. \tag{3.83}$$

Now the assumption is that $a \neq 0$ *(as of course this would amount to shifting* r *by* 1*), this gives us the indicial polynomial*

$$r - 2 = 0\,. \tag{3.84}$$

Therefore, we see that $r = 2$ *and we can set* $a_0 = 1$*. Then the recursion simply states that* $a_k = 0$ *for* $k > 0$ *and therefore we find the solution to the original differential equation*

$$f = \exp\left(\frac{1}{2x^2} + \frac{2}{x}\right) x^2, \tag{3.85}$$

which generates this module.

Example 31 (Exponential integral continued)**.** *Consider the Frobenius deformation*

$$(k + 1 + \epsilon)a_{k+1} = a_k. \tag{3.86}$$

This has a natural solution

$$a_k = \frac{1}{\Gamma(1 + k + \epsilon)}. \tag{3.87}$$

If we take

$$f(\epsilon, x) = \sum_{k=0}^{\infty} \frac{x^{k+\epsilon}}{\Gamma(1 + k + \epsilon)} = \exp(x)\epsilon\frac{\Gamma(\epsilon) - \Gamma(\epsilon, x)}{\Gamma(\epsilon + 1)} \tag{3.88}$$

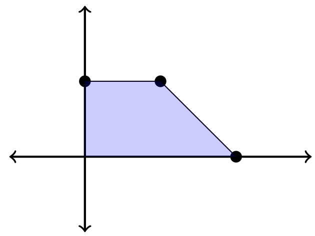

Figure 3.4: The Newton polygon associated to the operator in equation (3.94).

we find

$$(D_x - x)f \; = \; \frac{x^\epsilon}{\Gamma(\epsilon)} \; = \; \epsilon + O(\epsilon^2). \tag{3.89}$$

In fact, we find that expanding in ϵ

$$f(\epsilon, x) \; = \; \exp(x) + \exp(x)\mathrm{Ei}(x)\epsilon + O(\epsilon^2). \tag{3.90}$$

Notice that

$$\sum_{k=-\infty}^{-1} \frac{x^{k+\epsilon}}{\Gamma(1+k+\epsilon)} \; = \; \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k!}{x^{k+1}} \epsilon + O(\epsilon^2) \tag{3.91}$$

which gives the solution of the same inhomogenous equation around $x \sim \infty$*. Note we could have considered*

$$(k+\epsilon)^2 a_k - (k+\epsilon) a_{k-1} \; = \; 0. \tag{3.92}$$

The indicial polynomial in this case is ϵ^2*. Therefore, the solution to this recursion is the same as the recursion (3.86) however the indicial polynomial indicates we should expand* ϵ *to* $O(\epsilon^2)$*. Notice that the solution (3.87) has* $a_0 = \Gamma(1+\epsilon)^{-1}$*. Therefore, if we took* $a_0 = 1$ *we would find the solution*

$$\exp(x)\mathrm{Ei}(x) - \gamma \exp(x) \tag{3.93}$$

where γ *is the Euler-Mascheroni constant. This indicates the effect that choosing initial conditions has on the answer and that having the freedom to choose can lead to more natural answers in certain contexts. The second order equation that this satisfies is given by*

$$(D_x^2 - xD_x - x)f \; = \; D_x(D_x - x)f \; = \; 0\,. \tag{3.94}$$

Therefore, the Newton polygon is given in Figure 3.4. The top gives the solutions around infinity and we see the factorisation for the different slopes, the factorially divergent solution, and the exponential singularity.

Example 32 (Apéry numbers continued). *Consider equation (3.60). The indicial polynomial is given by ϵ^3. Therefore consider,*

$$(k+1+\epsilon)^3 a_{k+1}(\epsilon) - (2(k+\epsilon)+1)(17(k+\epsilon)^2+17(k+\epsilon)+5)a_k(\epsilon) + (k+\epsilon)^3 a_{k-1}(\epsilon) \; = \; 0. \tag{3.95}$$

Setting $a_0(\epsilon) = 1$ and $a_{-1}(\epsilon) = 0$, we see that when $k = -1$ only $\epsilon^3 a_0(\epsilon)$ appears in the expression for $a_{-2}(\epsilon)$. Notice that originally we had a wall so setting $a_{-1} = 0$ implied that $a_k = 0$ for $k < 0$. Here we see that $a_k = O(\epsilon^3)$ for all $k < 0$. Therefore, as indicated by the indicial polynomial, expanding in ϵ near 0 to $O(\epsilon^3)$ will give the basis of solutions to equation (3.60). This is given in [89].

Remark 12. *Notice that a translation $x \mapsto x + c$ affects the operator*

$$D_x \; = \; x\frac{\partial}{\partial x} \mapsto (x+c)\frac{\partial}{\partial(x+c)} \; = \; D_x + \frac{c}{x} D_x \, . \tag{3.96}$$

Therefore when we have positive slopes these will generally be removed by applying an affine transformation. Indeed, the positive slopes will only appear when we solve a differential equation around an irregular singularity.

3.5 Optimal truncations

Solving differential equations using the Frobenius method around an irregular singularity will lead to power series that are divergent such as

$$\sum_{k=0}^{\infty} (-1)^k \frac{k!}{x^{k+1}}. \tag{3.97}$$

These solutions can approximate certain analytic functions around the irregular singularity. Of course taking a truncated series to order N such as

$$\sum_{k=0}^{N} (-1)^k \frac{k!}{x^{k+1}}\,, \tag{3.98}$$

will lead to an $O(a_{N+1}x^{N+1})$ order approximation of an analytic solution, which for equation (3.97) is given by a function[2] such as

$$\mathrm{Ei}(x)\exp(x)\,. \tag{3.99}$$

[2]Of course any two functions with this asymptotic series could differ by exponentially small corrections.

However, when the divergent series has coefficients that behave in a certain way, we can improve this approximation by taking the *optimal truncation*. For a divergent series,

$$f(x) \;=\; \sum_{k=0}^{\infty} a_k x^k \tag{3.100}$$

the optimal truncation is defined as the piecewise analytic function

$$f^{\mathrm{op}}(x) \;=\; \sum_{k=0}^{N^{\mathrm{op}}(x)} a_k x^k \tag{3.101}$$

where

$$\left|a_{N^{\mathrm{op}}(x)} x^{N^{\mathrm{op}}(x)}\right| \;=\; \min\{a_k x^k : k \in \mathbb{Z}_{\geq 0}\}\,. \tag{3.102}$$

Note that more correctly, this should be defined for some neighbourhood of x as this smallest value needs to be a locally generic condition. Considering the example in equation (3.97), we see that for fixed x

$$(-1)^k \frac{k!}{x^{k+1}} \tag{3.103}$$

reaches it's minimum when

$$\left|(-1)^k \frac{k!}{x^{k+1}} (-1)^{k-1} \frac{x^k}{(k-1)!}\right| \;=\; \frac{k}{|x|} \sim 1\,. \tag{3.104}$$

Therefore, its optimal truncation is given roughly by

$$\sum_{k=0}^{\lfloor |x| \rfloor} (-1)^k \frac{k!}{x^{k+1}}\,. \tag{3.105}$$

Notice that this jumps as $|x|$ crosses integral radius circles around the origin by

$$\frac{|x|!}{|x|^{|x|+1}} \sim \sqrt{2\pi|x|}\exp(-|x|)(1 + O(1/|x|)) \quad \text{as} \quad |x| \to \infty\,. \tag{3.106}$$

Therefore, the associated error of the optimal truncation is exponentially small as opposed to polynomially. For this example, we can compare this analysis with the Code 16. We can see that the optimal truncation provides exactly the error with the function that we expected. Of course if we had considered

$$\mathrm{Ei}(x)\exp(x) + \exp(-7x/8) \tag{3.107}$$

then we see in Code 17. Therefore, the optimal truncation allows us to see exponentially small contributions to the asymptotics with some theoretical bound. In this example, the optimal trunctation will see exponentially small corrections of order $\exp(rx)$ when $|\exp(rx)| \leq \exp(|x|)$. Notice that the argument of rx then plays an important role.

Lets consider this more generally for examples of the form of equation (3.97). Suppose that

$$f(x) \;=\; \sum_{k=0}^{\infty} a_k x^k \tag{3.108}$$

and

$$a_k \sim \alpha \frac{\Gamma(k+\kappa)}{V^{k+\kappa}}\,. \tag{3.109}$$

Then the optimal truncation is given by

$$f^{\mathrm{op}}(x) \;=\; \sum_{k=0}^{|V-\kappa|} a_k x^k\,. \tag{3.110}$$

This has error term

$$\begin{aligned}&\alpha\frac{\Gamma(|V-\kappa|+\kappa)}{V^{|V-\kappa|+\kappa}}\\ &\quad\sim \alpha\frac{(|V-\kappa|+\kappa-1)^{|V-\kappa|+\kappa-3/2}\exp(-|V-\kappa|-\kappa+1)\sqrt{2\pi}}{V^{|V-\kappa|+\kappa}}(1+O(1/(|V-\kappa|+\kappa)))\,.\end{aligned} \tag{3.111}$$

Recently, Garoufalidis–Zagier [87] proposed a way to improve the optimal truncation for certain asymptotic series using additional information of the behaviour of the coefficients of the divergent series similar to previous works of Dingle, Berry, Howls [26, 27, 28, 50]. They call this method *smooth optimal truncation*. To describe this method we first need a function $\mathcal{E}_b(x)$ that solves the equation

$$\frac{x}{|x|}\mathcal{E}_{b+1}(x) - \mathcal{E}_b(x) \;=\; \frac{\Gamma(|x|-b)}{|x|^{|x|-b}}\,. \tag{3.112}$$

This can be solved for example by

$$\mathcal{E}_b(x) \;=\; \int_0^\infty t^{|x|-b}\exp(-t|x|)\frac{dt}{t-\frac{x}{|x|}}\,, \tag{3.113}$$

where the integral is the Cauchy principle part when $x = |x|$. In [87], this functional equation is also solved using formal asymptotic series. We will be content with this exact function.

Suppose that additionally to equation (3.109)

$$a_k \sim \sum_{\ell=0}^{\infty}\alpha_\ell\frac{\Gamma(k-\ell+\kappa)}{V^{k-\ell+\kappa}}\,. \tag{3.114}$$

Then the smooth optimal truncation is defined to be

$$f^{\text{smop}}(x;b) \;=\; \sum_{k=0}^{|V/x|-b} a_k x^k + \frac{|x|^{|V/x|-b}}{x^{|V/x|-b}} \sum_{\ell=0}^{\infty} \frac{\alpha_\ell}{V^{\kappa-\ell}} \left|\frac{x}{V}\right|^{\ell-\kappa} \mathcal{E}_{b-\kappa+\ell}(V/x)\,, \tag{3.115}$$

which is just an asymptotic expansion in general. The second sum can of course be truncated, optimally truncated or even smooth optimally truncated to give increasing levels of precision. In our previous example, we can compute the smooth optimal truncation in Code 18. Moreover, the error here is due to PARI/GP's numerical integration with working precision 1000 digits. The $\mathcal{E}_b(x)$ function is closely related to the exponential integral $\mathrm{Ei}(x)$, which in this case will give equality.

3.6 Approximating with rational functions

Padé approximates are a generalisation to Taylor series. Instead of approximating a function around a point by polynomials, they approximate by rational functions with prescribed degree of the numerator and denominator. Depending on the example, this can give a much better approximation to a function in more complicated neighbourhoods than just disks around a point. For example, if a Taylor series converges it does so with some radius of convergence. If the function we are interested in has a pole or a branch point then the radius can be at most the distance from the point we are expanding around to the singularity. Sequences of Padé approximates can give convergent functions in a larger domain. For example, any rational function will be given exactly globally by a limit of Padé approximates.

We can define the Padé approximate around a point for any analytic function. If f is an analytic function with expansion

$$f(z) \;=\; \sum_{k=0}^{\infty} a_k z^k\,, \tag{3.116}$$

then suppose that there exists polynomials $p(z), q(z)$ of degree m and n such that $q(0) = 1$ and

$$f(z) - \frac{p(z)}{q(z)} + O(z^{m+n+1}) \;=\; O(z^{m+n+1})\,, \tag{3.117}$$

or equivalently

$$f(z)q(z) - p(z) + O(z^{m+n+1}) \;=\; O(z^{m+n+1})\,. \tag{3.118}$$

If these polynomials exist the rational function is unique. To see this, notice that if there was another $p_1(z), q_1(z)$ of degree m and n such that $q_1(0) = 1$ and

$$f(z) - \frac{p_1(z)}{q_1(z)} + O(z^{m+n+1})\,, \tag{3.119}$$

then

$$p(z)q_1(z) \;=\; q(z)p_1(z) + O(z^{m+n+1})\,, \tag{3.120}$$

but $p(z)q_1(z), q(z)p_1(z)$ are both polynomials of degrees $m+n$ and therefore

$$\frac{p(z)}{q(z)} \;=\; \frac{p_1(z)}{q_1(z)}\,. \tag{3.121}$$

Definition 11. *If f is analytic and there exists $p(z), q(z)$ of degree m, n such that*

$$f(z) - \frac{p(z)}{q(z)} + O(z^{m+n+1}) \;=\; O(z^{m+n+1}) \tag{3.122}$$

then we define the $[m/n]$ Padé approximate as

$$f[m/n](z) \;=\; \frac{p(z)}{q(z)}\,. \tag{3.123}$$

Example 33 (Taylor series)**.** *The $f[n/0](x)$ Padé approximates give the Taylor series approximation.*

Example 34 (Non-existence of Padé approximates)**.** *If $f(z) = z$, then the $f[0/n](x)$ Padé approximate doesn't exist as*

$$\frac{p_0}{1 + q_1 z + \cdots} \;=\; p_0 - p_0 q_1 z + \cdots\,, \tag{3.124}$$

and therefore $p_0 = 0$ and $p_0 q_1 = 1$, which cannot be simultaneously solved.

Padé approximates can be constructed from continued fractions. Recall, the hypergeometric function for $c \notin \mathbb{Z}_{\leq 0}$

$${}_2F_1(a, b; c; z) \;=\; \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}\,, \tag{3.125}$$

where

$$(x)_k \;=\; x(x+1)\cdots(x+k-1)\,, \tag{3.126}$$

which when not a polynomial converges for $|z| < 1$. Gauss proved the following theorem.

Theorem A–23. *[105, Thm. 6.1] For $n \in \mathbb{Z}_{\geq 0}$ let*

$$\alpha_{2n+1} \;=\; -\frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)}\,, \quad \textit{and} \quad \alpha_{2n} \;=\; -\frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)}\,. \tag{3.127}$$

For $a, b, c \in \mathbb{C}$ such that $\alpha_n \notin \{0, \infty\}$, the continued fraction

$$\Phi(a,b;c;z) \;=\; 1 + \cfrac{\alpha_1 z}{1 + \cfrac{\alpha_2 z}{1 + \cfrac{\alpha_3 z}{1 + \cdots}}} \tag{3.128}$$

is convergent to a meromorphic function in the domain $\arg(z-1) \notin 2\pi\mathbb{Z}$, the convergence is uniform for any compact set not containing poles of Φ, Φ is holomorphic at $z = 0$ and $\Phi(a, b; c; 0) = 1$, and for all $|z| < 1$ we have

$$\Phi(a,b;c;z) \;=\; \frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} \tag{3.129}$$

Corollary 3. *For $a, c \in \mathbb{C}$ define*

$$\beta_{2n+1} \;=\; -\frac{(a+n)(c+n-1)}{(c+2n-1)(c+2n)}\,, \quad \textit{and} \quad \beta_{2n} \;=\; -\frac{n(c-a+n-1)}{(c+2n-2)(c+2n-1)}\,. \tag{3.130}$$

such that $\beta_n \neq 0$ for $n \in \mathbb{Z}_{\geq 0}$ then

$$f(a;c;z) \;=\; \cfrac{1}{1 + \cfrac{\beta_1 z}{1 + \cfrac{\beta_2 z}{1 + \cfrac{\beta_3 z}{1 + \cdots}}}}\,, \tag{3.131}$$

is convergent to a meromorphic function in the domain $\arg(z-1) \notin 2\pi\mathbb{Z}$, the convergence is uniform for any compact set not containing its poles, is holomorphic at $z = 0$ and $f(0) = 1$, and for all $|z| < 1$ we have

$${}_2F_1(a,1;c;z) \;=\; f(a;c;z)\,. \tag{3.132}$$

This continued fraction is given by

$$f(a;c;z) \;=\; \cfrac{1}{1 - \cfrac{az}{c - \cfrac{1(c-a)z}{c+1 - \cfrac{(a+1)cz}{c+2 - \cfrac{2(c-a+1)z}{c+3 - \cfrac{(a+2)(c+1)z}{c+4 - \cfrac{3(c-a+2)z}{c+5 - \cdots}}}}}}}\,. \tag{3.133}$$

A continued fraction of the form

$$\cfrac{z}{1+\cfrac{\alpha_1 z}{1+\cfrac{\alpha_2 z}{1+\cfrac{\alpha_3 z}{1+\cdots}}}}, \tag{3.134}$$

truncated at

$$\cfrac{z}{1+\cfrac{\alpha_1 z}{1+\cfrac{\alpha_2 z}{1+\cfrac{\alpha_3 z}{1+\cfrac{\cdots}{1+\alpha_n z}}}}}, \tag{3.135}$$

gives a rational function of degree $\lfloor (n+2)/2 \rfloor$ over $\lfloor (n+1)/2 \rfloor$ and this function is an $[\lfloor (n+2)/2 \rfloor / \lfloor (n+1)/2 \rfloor]$ Padé approximate, which can be proved by induction. We can use this to prove the convergence of certain Padé approximates of the logarithm.

Example 35 (Padé approximates of the logarithm)**.** *From Corollary 3 and equation (3.133), for* $|z| < 1$

$$z\,{}_2F_1(1,1;2;-z) \;=\; \log(1+z)\,, \tag{3.136}$$

and therefore for $z \in \mathbb{C}\backslash\mathbb{R}_{\leq -1}$ *and the principle branch*

$$\log(1+z) \;=\; \cfrac{z}{1+\cfrac{1z}{2+\cfrac{1z}{3+\cfrac{4z}{4+\cfrac{4z}{5+\cfrac{9z}{6+\cdots}}}}}}. \tag{3.137}$$

Therefore, for $f(z) = \log(1+z)$ *and* $z \in \mathbb{C}\backslash\mathbb{R}_{\leq -1}$

$$\lim_{n\to\infty} f[(n+1)/n](z) \;=\; \log(1+z) \tag{3.138}$$

with uniform convergence on any compact subset of $\mathbb{C}\backslash\mathbb{R}_{\leq -1}$*.*

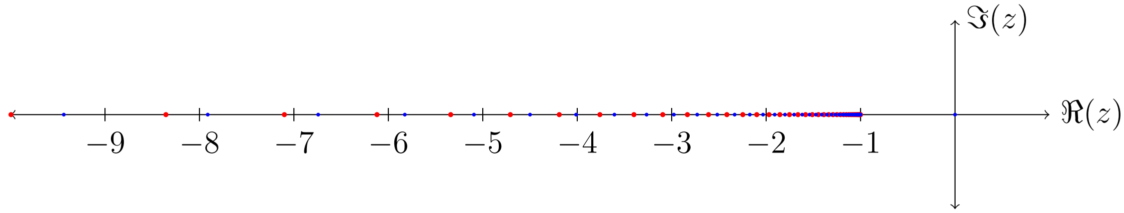

Figure 3.5: Plots of the poles and zeros of the [50/49] Padé approximate of $\log(1+z)$. The poles are in red and the zeros are in blue. One can see that, besides the zero at $z = 0$, the poles and zeros are accumulating on the branch cut concentrated mainly around the branch point $z = -1$.

To understand how the Padé approximates could possible converge to the logarithm we note that the various poles and zeros of the rational functions will accumulate around the branch cut $\mathbb{R}_{\leq -1}$. This can can be seen using the following code in PARI/GP [20], whose output is plotted in Figure 3.5.

```
polroots(numerator(bestapprPade(log(1+x+O(x^100)))))
polroots(denominator(bestapprPade(log(1+x+O(x^100)))))
```

This example shows the power of Padé approximates as opposed to Taylor series. For the logarithm, the Padé approximates converge everywhere on the cut plane as opposed to a disk of radius one. It also indicates that we can explore the behaviour of certain functions using Padé approximates to see experimentally the existence of branch cuts and singularities. Later in chapter 11, we will be interested in functions with logarithmic singularities and will approximate their analytic continuations with Padé approximates. For some discussion of the convergence of the Padé approximates in some general cases see [147, 178].

Remark 13. *When numerically exploring the possible branch points of a function, it is natural to consider various conformal maps around the point we are expanding around. This can be used to switch which critical point is closer. In examples where we expect an analytic continuation to some domain with certain singularities, these conformal maps can be used to study the singularities by making them the closest singularity to the point of expansion.*

3.7 Resummation and jumping

Factorially divergent formal power series arise in many different contexts. Importantly, they arise from stationary phase approximations of integrals as discussed in sections 2.5, 2.8 and 4.1. To make sense of these divergent functions, we will use a method that constructs analytic functions with the same asymptotics via *Borel resummation.* Suppose we have a divergent series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k . \tag{3.139}$$

The *Borel transform* of this series is defined to be

$$\mathcal{B}_1 f(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(\lambda + k + 1)} \zeta^{\lambda+k} . \tag{3.140}$$

If this function is convergent, and has an analytic continuation with certain growth conditions at infinity, then we can take it's *Laplace transform*

$$\mathcal{L}_1 \mathcal{B}_1 f(x) = \int_0^{\infty} \exp(-\zeta) \mathcal{B} f(\zeta x) d\zeta . \tag{3.141}$$

If we expand the series involved and notice that

$$\int_0^{\infty} \exp(-\zeta)(\zeta x)^k d\zeta = \Gamma(k+1) x^k , \tag{3.142}$$

we see that formally this function should have the same asymptotic series and with sufficient conditions we can see that it will satisfy the same differential equations. Indeed, having the same asymptotic series is a conseqeuce of Watson's lemma.

Theorem A–24 (Watson's lemma)**.** *[151, Thm. 3.1][141, Sec. 2.1] If $f(x)$ is a function from the positive reals with finitely many discontinuities and has an asymptotic expansion as $x \to 0$*

$$f(x) \sim \sum_{k=0}^{\infty} a_k x^k , \tag{3.143}$$

with $a_0 \neq 0$, then, if for small enough x the integral converges, we have as $x \to 0$

$$\int_0^{\infty} \exp(-\zeta)(\zeta x)^{\lambda} f(\zeta x) d\zeta \sim \sum_{k=0}^{\infty} a_k \Gamma(\lambda + k + 1) x^{\lambda+k} . \tag{3.144}$$

Therefore, we define the Borel resummation of a formal series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ to be $\mathcal{L}_1 \mathcal{B}_1 f(x)$ if it exists. By Watson's lemma this provides an analytic function with the same asymptotics as the original formal series.

Example 36 (Exponential integral)**.** *Recall the solution to the equation*

$$(D_x - x) f = 1 \tag{3.145}$$

in formal power series given by

$$f(x) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{k!}{x^{k+1}}. \tag{3.146}$$

Take the function

$$g(x) \;=\; \frac{1}{x} f\left(\frac{1}{x}\right) = \sum_{k=0}^{\infty}(-1)^k k! x^k. \tag{3.147}$$

We can take the Borel transform

$$\mathcal{B}_1 g(\zeta) \;=\; \sum_{k=0}^{\infty}(-1)^k \zeta^k \;=\; \frac{1}{1+\zeta}, \tag{3.148}$$

and then the Laplace transform

$$\mathcal{L}_1\mathcal{B}_1 g(x) \;=\; \int_0^\infty \frac{\exp(-\zeta)}{1+\zeta x} d\zeta \;=\; \int_{\frac{1}{x}}^\infty \frac{\exp\left(-\zeta + \frac{1}{x}\right)}{\zeta x} d\zeta \;=\; -\frac{1}{x}\mathrm{Ei}\left(-\frac{1}{x}\right)\exp\left(\frac{1}{x}\right), \tag{3.149}$$

and we see that $f(x) = -\mathrm{Ei}(-x)\exp(x)$ *is a solution*[3] *to the differential equation. For comparison take*

$$g(x) \;=\; f\left(\frac{1}{x}\right) = \sum_{k=0}^{\infty}(-1)^k k! x^{k+1}. \tag{3.150}$$

Then

$$\mathcal{B}_1 g(\zeta) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{\zeta^{k+1}}{k+1} \;=\; \log(1+x). \tag{3.151}$$

Similarly,

$$\mathcal{L}_1\mathcal{B}_1 g(x) \;=\; \int_0^\infty \exp(-\zeta)\log(1+\zeta x) d\zeta \;=\; -\mathrm{Ei}\left(-\frac{1}{x}\right)\exp\left(\frac{1}{x}\right), \tag{3.152}$$

When one has an asymptotic series computed numerically with a finite number of coefficients then one can attempt to numerically compute its Borel resummation. In practice, the most difficult part of this computation is giving a good approximation to the analytic continuation. This can be done using Padé approximates given in Section 3.6. For example, one could use the following PARI/GP [20] code.

```
borel(a) = serconvol(a,exp(x));
pade(a,N) = bestapprPade(a+O(x^N));
{resum(tau,a,N) = local(pb); pb=pade(borel(a),N);intnum(xi=0,[+oo,1],exp(-
   xi)*subst(pb,x,xi*tau))};
```

[3] Note that $\mathrm{Ei}(-x) = -\mathrm{Ei}(x) \pm \pi i$.

Of course when the Borel transform of a formal series is analytic but has poles or branch cuts, the Laplace transform will jump as the ray defined by input variable crosses one of these singularities. This jumping is captured by the *Stokes automorphism.* For example, if we have a basis of solutions to a linear differential equation satisfied by the Borel transform then on both sides of an isolated critical point. Therefore, on both sides there will be a matrix relating the two bases of solutions.

Example 37 (Stokes automorphism for the exponential integral)**.** *Taking*

$$g(x) \;=\; \sum_{k=0}^{\infty}(-1)^k k! x^k\,, \tag{3.153}$$

again, we see that for $x \in \mathbb{R}_{\geq 0}$

$$\begin{aligned} &\lim_{\epsilon\to\infty} \mathcal{L}_1\mathcal{B}_1 g(\exp((\pi-\epsilon)i)x) - \mathcal{L}_1\mathcal{B}_1 g(\exp((\pi+\epsilon)i)x) \\ &=\; 2\pi i\,\mathrm{Res}_{\zeta=-1/x}\frac{\exp(-\zeta)}{1+\zeta x}(z+1/x)d\zeta \;=\; \frac{2\pi i}{x}\exp\left(\frac{1}{x}\right). \end{aligned} \tag{3.154}$$

So we see that the Stokes automorphism associated to the Wronskian of $2\pi i\exp(x)$ *and* $\exp(x)\mathrm{Ei}(x)$ *is given by*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.155}$$

Chapter 4

Sums, integrals and q–hypergeometric functions

4.1 From sums to integrals to asymptotics

We will discuss three different methods that turn sums into some kind of integrals. They will start real analytic and turn more complex analytic. The basic question is to determine the behaviour of sums of the form

$$\sum_{n=0}^{M} f(n) \tag{4.1}$$

where f is a smooth function. The fact that f is a smooth function allows us to gain additional information about the behaviour of the sum. Firstly, the Bernoulli polynomials $B_n(x)$ are the unique polynomials satisfying

$$\int_y^{y+1} B_n(x)\, dx \;=\; y^n\,. \tag{4.2}$$

For $n \neq 1$ we have $B_n(0) = B_n = B_n(1)$ and note that $B_n(0) = -1/2 = -B_0(1)$ in this part we will take $B_1 = -1/2$. These have a generating function

$$\frac{\exp(xt)}{\exp(t)-1} \;=\; \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^{n-1} \;=\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{B_{n-k}}{k!(n-k)!} x^k t^{n-1}\,. \tag{4.3}$$

The first few polynomials are

$$\begin{aligned} B_0(x) \;&=\; 1\,, \quad B_1(x) \;=\; x - \frac{1}{2}\,, \quad B_2(x) \;=\; x^2 - x + \frac{1}{6}\,, \\ B_3(x) \;&=\; x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\,, \quad B_4(x) \;=\; x^4 - 2x^3 + x^2 - \frac{1}{30}\,. \end{aligned} \tag{4.4}$$

We can use integration by parts and the Bernoulli polynomials to deduce following formula.

Theorem A–25 (Euler–Maclaurin summation formula)**.** *[151, 205, 203] For f a smooth function in the positive reals with infinity and $M \in \mathbb{Z}$, we have*

$$\begin{aligned}\sum_{k=0}^{M} f(k) \;&=\; \int_0^M f(x)\,dx + \frac{f(0)}{2} + \frac{f(M)}{2} + \sum_{n=1}^{N-1} \frac{(-1)^{n+1} B_{n+1}}{(n+1)!} (f^{(n)}(M) - f^{(n)}(0)) \\ &\quad + (-1)^{N+1} \int_0^M f^{(N)}(x) \frac{B_N(x - \lfloor x \rfloor)}{N!}\,dx\,.\end{aligned} \tag{4.5}$$

In a more specific situation, we find an even better formula. This formula turns out to be extremely important for proving the basic properties of functions discussed in part IV.

Theorem A–26 (Poisson summation formula)**.** *[205] If f is a smooth function on the reals with positive and negative infinity, then for the Fourier transform*

$$\hat{f}(y) \;=\; \int_{-\infty}^{\infty} f(x) \exp(2\pi i x y)\,dx \tag{4.6}$$

we have

$$\sum_{k \in \mathbb{Z}} f(k) \;=\; \sum_{k \in \mathbb{Z}} \hat{f}(k)\,. \tag{4.7}$$

Proof. The function

$$\sum_{k=-\infty}^{\infty} f(x+k) \tag{4.8}$$

is periodic with period 1 and therefore has a Fourier series representation

$$\sum_{k=-\infty}^{\infty} f(x+k) \;=\; \sum_{\ell \in \mathbb{Z}} \exp(2\pi i \ell x) \int_0^1 \sum_{k=-\infty}^{\infty} f(\xi+k) \exp(-2\pi i \xi \ell)\,d\xi \;=\; \sum_{\ell \in \mathbb{Z}} \exp(2\pi i \ell x) \hat{f}(-\ell)\,. \tag{4.9}$$

□

The final method uses complex analysis. Assuming f is meromorphic on some domain Ω containing $0, \ldots, M$ and continuous on its boundary then from the residue theorem

$$\frac{1}{2i} \int_{\partial\Omega} \cot(\pi z) f(z)\,dz \;=\; \sum_{k=0}^{M} f(k) + \sum_{z_0 \in \Omega - \{0,\ldots,k\}} \pi \mathrm{Res}_{z=z_0} \cot(\pi z) f(z)\,dz\,. \tag{4.10}$$

This idea is quite versatile and can be adapted to ones needs. However, there are some additional tricks, which can be used. Adding some assumptions on f we can present the standard version of the Abel–Plana summation method.

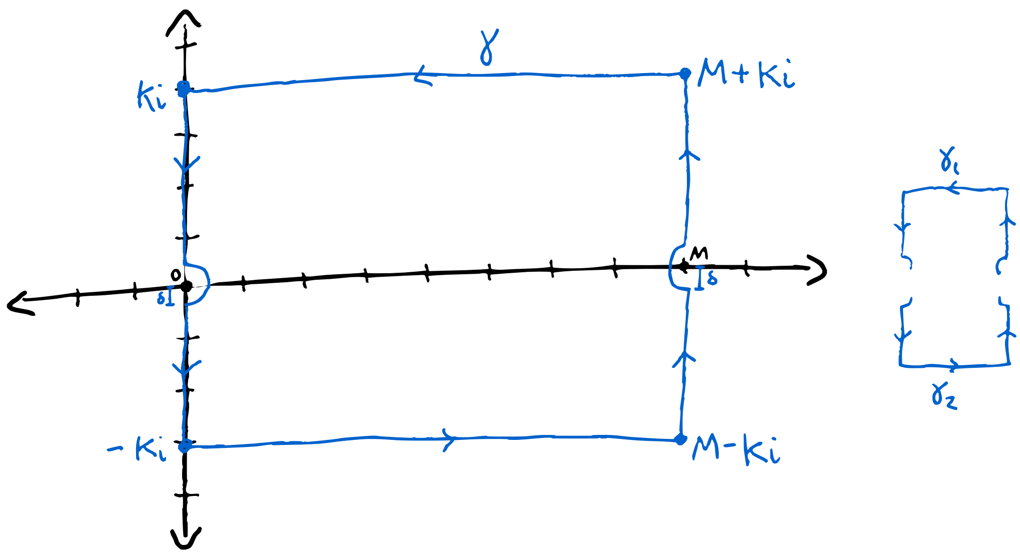

Figure 4.1: Contour for Abel–Plana summation.

Theorem A–27 (Abel–Plana summation formula)**.** *[151, Chp. 8, Sec. 3] If $f(z)$ is continuous for $0 \leq \Re(z) \leq M$ and analytic on the interior and $f(z) = o(\exp(2\pi|\Im(z)|)$ as $|\Im(z)| \to \infty$ uniformly in $0 \leq \Re(z) \leq M$, then*

$$\begin{aligned}\sum_{k=0}^{M} f(k) &= \int_0^M f(z)\,dz + \frac{1}{2}f(0) + \frac{1}{2}f(M) \\ &\quad + i\int_0^\infty \frac{f(iy) - f(M+iy) - f(-iy) + f(M-iy)}{\exp(2\pi y) - 1}\,dy\,.\end{aligned} \tag{4.11}$$

Proof. To prove this one uses the following trick. Take the contour in Figure 4.1. Then one has

$$\sum_{k=1}^{M-1} f(k) = \frac{1}{2i}\int_\gamma \cot(\pi z) f(z)\,dz\,. \tag{4.12}$$

Then adding the integral of $f(z)$ to both sides and with the assumption f is analytic[1] on

[1] For more general f, this step may be altered by the inclusion of terms coming from a residue or branch cut.

the domain we rewrite

$$\begin{aligned}\sum_{k=1}^{M-1} f(k) - \int_{\delta}^{M-\delta} f(z)\,dz &= \frac{1}{2i}\int_{\gamma}\cot(\pi z) f(z)\,dz + \frac{1}{2}\int_{\gamma_1} f(z)\,dz - \frac{1}{2}\int_{\gamma_2} f(z)\,dz \\ &= \int_{\gamma_1}\frac{f(z)}{1-\exp(-2\pi i z)}\,dz + \int_{\gamma_2}\frac{f(z)}{\exp(2\pi i z)-1}\,dz\,.\end{aligned} \tag{4.13}$$

To finish, we send $K \to \infty$ and note that the contours horizontal to the real vanish, compute the residues around the small semicircles at 0 and M, and collect the remaining vertical integrals into one integral. □

Now these methods give many ways to convert sums into integrals. This indicates that understanding the asymptotic properties of integrals through something like Watson's lemma, given in Theorem 31, will allows us to understand the asymptotics of these sums. We can apply Watson's lemma to give Laplace's method. Consider and integral of the form

$$I(z) = \int_{\gamma}\exp(-p(\xi)/z)q(\xi)d\xi \tag{4.14}$$

where γ is locally passing through 0 and $p'(0) = 0$ is simple. Then take

$$\zeta(\xi)^2 = p(\xi) - p(0)\,. \tag{4.15}$$

Supposing that

$$p(\xi) \sim p_0 + \sum_{k=2} p_k \xi^k \quad \text{and} \quad q(\xi) \sim \sum_{k=0} q_k \xi^k\,, \tag{4.16}$$

we have

$$\zeta(\xi) \sim \xi\sqrt{\sum_{k=2} p_k \xi^{k-2}} = \sqrt{p_2}\xi + \frac{p_3}{2\sqrt{p_2}}\xi^2 + \frac{4p_4p_2 - p_3^2}{8\sqrt{p_2}^3} + \cdots\,, \tag{4.17}$$

and so

$$\xi(\zeta) = \frac{1}{\sqrt{p_2}}\zeta - \frac{p_3}{2p_2^2}\zeta^2 - \frac{4p_4p_2 - 5p_3^2}{8\sqrt{p_2}^7}\zeta^3 + \cdots\,. \tag{4.18}$$

Then

$$I(z) = \exp(-p(\xi_0)/z)\int_{\gamma}\exp(-\zeta^2/z)q(\xi(\zeta))\frac{d\xi(\zeta)}{d\zeta}d\zeta\,. \tag{4.19}$$

Therefore, let

$$\begin{aligned} f(\zeta) &= q(\xi(\zeta))\frac{d\xi(\zeta)}{d\zeta} \\ &= \frac{q_0}{\sqrt{p_2}} + \frac{p_2q_1 - p_3q_0}{p_2^2}\zeta + \frac{8p_2^2q_2 - 12p_2p_3q_1 + (15p_3^2 - 12p_2p_4)q_0}{8\sqrt{p_2}^7}\zeta^2 + \cdots\,.\end{aligned} \tag{4.20}$$

To understand the asymptotics of this integral we then notice that taking $\delta \in \mathbb{R}_{>0}$ and letting $x = \pm\sqrt{\sigma}$ we have

$$\int_{-\delta}^{\delta} \exp(-x^2/z)f(x)dx \;=\; \int_{0}^{\delta^2} \exp(-\sigma/z)(f(\sqrt{\sigma}) + f(-\sqrt{\sigma}))\frac{d\sigma}{\sigma^{1/2}}\,. \tag{4.21}$$

Therefore, if $f(x) \sim \sum_{k=0}^{\infty} a_k x^k$ by Watson's lemma, given in Theorem 31, we have

$$\int_{-\delta}^{\delta} \exp(-x^2/z)f(x)dx \;\sim\; \sum_{k=0}^{\infty} \Gamma(k+1/2)a_{2k}z^{k+1/2}\,. \tag{4.22}$$

To relate the integral in equation (4.19) to the integral in equation (4.21) we deform the contour so that locally around the critical point the contour takes the path of steepest descent so that the real part of ζ is increased the most. For a given contour, choosing the best deformation to apply Watson's lemma to will depend on the context. We can give a general result here with some assumptions.

Theorem A–28 (Laplace's method)**.** *[151, Ch. 4, Sec. 7.3, Thm. 7.1] If $p(\xi), q(\xi)$ are holomorphic on some domain Ω, γ some contour with interior contained in Ω, $p'(\xi)$ has one simple zero at ξ_0 in Ω, $\theta_2 \le \arg(z) \le \theta_1$, $|z| \le M \in \mathbb{R}_{>0}$ and $\theta_2 - \theta_1 < \pi$, and $\Re(zp(\xi) - zp(\xi_0)) > 0$ on γ except at ξ_0 and bounded uniformly away from zero with respect to $\arg(z)$ at the endpoints of γ along γ, then when the integral converges uniformly and absolutely with respect to z on γ we have the following asymptotic expansion*

$$\int_{\gamma} \exp(-p(\xi)/z)q(\xi)d\xi \sim \exp(-p(\xi_0)/z)\sum_{k=0}^{\infty} \Gamma(k+1/2)a_{2k}z^{k+1/2} \tag{4.23}$$

where locally around ξ_0 we have $\zeta(\xi)^2 = p(\xi) - p(\xi_0)$ and

$$q(\xi(\zeta))\frac{d\xi(\zeta)}{d\zeta} \;=\; \sum_{k=0}^{\infty} a_k \zeta^k\,. \tag{4.24}$$

4.2 The dilogarithm function

Before applying some of the methods previously discussed, we need to introduce some important special functions. For $|z| < 1$, we define the absolutely convergent functions called the k-th polylogarithm

$$\mathrm{Li}_s(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^s}\,. \tag{4.25}$$

For $s > 1$, we see that

$$\lim_{x\to 1^-} \mathrm{Li}_s(x) \;=\; \zeta(s) \;=\; \sum_{k=1}^{\infty} \frac{1}{k^s}\,. \tag{4.26}$$

These functions satisfy the following easily seen relation

$$\mathrm{Li}_s(z) \;=\; z\frac{\partial}{\partial z}\mathrm{Li}_s(z)\,. \tag{4.27}$$

Now notice that

$$\mathrm{Li}_0(z) \;=\; \frac{z}{1-z}\,. \tag{4.28}$$

Therefore, from equation (4.27), for $k \in \mathbb{Z}_{\leq 0}$

$$\mathrm{Li}_k(z) \in \mathbb{Z}[z](1-z)^{k-1}\,. \tag{4.29}$$

For $k \in \mathbb{Z}_{\leq 0}$, these functions satisfy the relation

$$\mathrm{Li}_k(z) + (-1)^k \mathrm{Li}_k(z^{-1}) \;=\; -\delta_{k,0}\,, \tag{4.30}$$

As $\mathrm{Li}_k(0) = 0$, we can define an analytic continuation of $\mathrm{Li}_k(z)$ for $k \in \mathbb{Z}_{>0}$ for $z \in \mathbb{C} - \mathbb{R}_{\geq 1}$ via integration. For example,

$$\mathrm{Li}_1(z) \;=\; -\log(1-z)\,. \tag{4.31}$$

Then we define the dilogarithm for $z \in \mathbb{C} - \mathbb{R}_{\geq 1}$ such that for a contour contained in the same domain

$$\mathrm{Li}_2(z) \;=\; -\int_0^z \log(1-\xi)\frac{d\xi}{\xi}\,. \tag{4.32}$$

The dilogarithm has a non–commutative monodromy group. Indeed, taking any loop confined to $|z| < 1$ will give give the same valued at both ends. However, if one takes a loop in $\mathbb{C} - \mathbb{R}_{\leq 0}$ around $\xi = 1$ to the point z, as shown in Figure 4.2, then we see that as we shrink $\epsilon \to 0$ we get

$$\begin{aligned} -\int_{\gamma} \log(1-\xi)\frac{d\xi}{\xi} \;&=\; -\int_{\gamma_1+\gamma_2} \log(1-\xi)\frac{d\xi}{\xi} - \oint_{\gamma_3+\gamma_4} (\log(1-\xi) - 2\pi i)\frac{d\xi}{\xi} \\ &=\; \mathrm{Li}_2(z) + 2\pi i \int_1^z \frac{d\xi}{\xi} \;=\; \mathrm{Li}_2(z) + 2\pi i \log(z)\,. \end{aligned} \tag{4.33}$$

Now we see that although it appeared that $\mathrm{Li}_2(z)$ did not have a singularity at $z = 0$, this will acquire a singularity once we take a loop around $\xi = 1$. This also shows that the monodromy action is non–commutative. Therefore, in general we find that the various branches of the Riemann surface associated to the dilogarithm will have values in the set

$$\mathrm{Li}_2(z) + 2\pi i \log(z)\mathbb{Z} + (2\pi i)^2\mathbb{Z}\,. \tag{4.34}$$

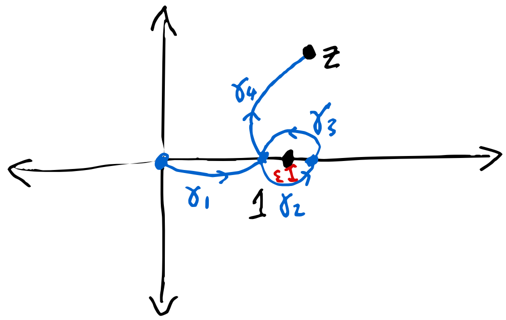

Figure 4.2: Contour giving non–commutative monodromy of Li_2.

Importantly, the dilogarithm satisfies the following functional equations

$$\begin{aligned} \mathrm{Li}_2\Big(\frac{1}{z}\Big) &= -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\log(-z)^2 \\ \mathrm{Li}_2(1-z) &= -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \log(z)\log(1-z) \\ \mathrm{Li}_2(z^2) &= 2\big(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z)\big) \end{aligned} \tag{4.35}$$

and the five term relation

$$\begin{aligned} &\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\Big(\frac{1-x}{1-xy}\Big) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\Big(\frac{1-y}{1-xy}\Big) \\ &= \frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\Big(\frac{1-x}{1-xy}\Big)\log\Big(\frac{1-y}{1-xy}\Big)\,. \end{aligned} \tag{4.36}$$

Finally, it satisfies a distribution property

$$\mathrm{Li}_2(x) = n \sum_{z^n = x} \mathrm{Li}_2(z)\,. \tag{4.37}$$

For a discussion of the many fascinating properties of these functions consult [206].

4.3 Asymptotics of the Pochhammer symbol

The asymptotics of the q–Pochhammer symbol from definition 14

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x)\,, \tag{4.38}$$

when $q \to 1$ is of fundamental importance to our examples of interest coming from quantum topology. This is due to the expression of the R matrix given in Theorem 14. To calculate this behaviour formally we can use the following lemma.

Lemma 3. *[206, Prop. 2] For $|x| < 1$ and $|q| < 1$ we have*

$$\log(x;q)_\infty \;=\; \sum_{n=1}^{\infty} \frac{x^n}{n(q^n-1)}\,. \tag{4.39}$$

The proof is completely analogous to the proof of Lemma 7. Using this expression we can formally take $q = \mathbf{e}(\tau)$, change the order of summation and expand in τ to find

$$\begin{aligned}\log(\mathbf{e}(m\tau)x;\mathbf{e}(\tau))_\infty &= \sum_{n=1}^{\infty}\frac{\mathbf{e}(mn\tau)x^n}{n(\mathbf{e}(n\tau)-1)} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{B_k(m)x^n(2\pi i n\tau)^{k-1}}{k!n}\\ &\sim \sum_{k=0}^{\infty}\frac{B_k(m)(2\pi i\tau)^{k-1}}{k!}\sum_{n=1}^{\infty}\frac{x^n}{n^{2-k}} = \sum_{k=0}^{\infty}\frac{B_k(m)(2\pi i\tau)^{k-1}}{k!}\mathrm{Li}_{2-k}(x)\,.\end{aligned} \tag{4.40}$$

For example, taking $m = 0$ and fixed x with $|x| < 1$ we have

$$\log(x;\mathbf{e}(\tau))_\infty \;\sim\; \mathrm{Li}_2(x)(2\pi i\tau)^{-1} + 2\pi i\mathbb{Z} + \frac{1}{2}\log(1-x) + \frac{x}{1-x}\frac{2\pi i\tau}{12} - \frac{x^2+x}{(1-x)^3}\frac{(2\pi i\tau)^3}{720} + \cdots\,. \tag{4.41}$$

This computation is purely formal[2] but gives the correct answer. Firstly, this can be checked numerically in the PARI/GP [20] Code 19 and then proved in the following lemma.

Lemma 4. *[88, Lem. 2.1] For $\delta \in \mathbb{R}_{>0}$, as $\hbar \to 0$ with $\delta < \arg(\tau) < \pi/2 - \delta$, and x and m such that $\delta < \arg(\mathbf{e}(m\tau)x) < -\delta$ and $\delta < \arg(x) < -\delta$ we have*

$$\log(\mathbf{e}(m\tau)x;\mathbf{e}(\tau))_\infty \sim \sum_{k=0}^{\infty}\frac{B_k(m)(2\pi i\tau)^{k-1}}{k!}\mathrm{Li}_{2-k}(x) + 2\pi i\mathbb{Z}\,. \tag{4.42}$$

Proof. We have

$$\log(\mathbf{e}(m\tau)x;\mathbf{e}(\tau))_\infty \in \sum_{n=0}^{\infty}\log(1-\mathbf{e}((n+m)\tau)x) + 2\pi i\mathbb{Z}\,. \tag{4.43}$$

Note that

$$\mathrm{Li}_1(x) \;=\; -\log(1-x)\,, \tag{4.44}$$

[2] Compare this kind of formal computation to the remarks in [151, Ch. 1, Sec.1] in relation to the exponential integral.

and

$$\frac{\partial}{\partial n}\mathrm{Li}_k(\mathbf{e}((n+m)\tau)x) \;=\; 2\pi i\tau\mathrm{Li}_{k-1}(\mathbf{e}((n+m)\tau)x)\,. \tag{4.45}$$

Applying the Euler–Marlaurin summation formula from Theorem 25, we find that

$$\begin{aligned}
&\sum_{n=0}^{M}\log(1-\mathbf{e}((n+m)\tau)x)\\
&= \int_0^M \log(1-\mathbf{e}((\xi+m)\tau)x)\,d\xi\\
&\quad+\frac{\log(1-\mathbf{e}(m\tau)x)}{2}+\frac{\log(1-\mathbf{e}((M+m)\tau)x)}{2}\\
&\quad+\sum_{n=1}^{N-1}\frac{(-1)^{n+1}B_{n+1}(2\pi i\tau)^n}{(n+1)!}(\mathrm{Li}_{1-n}(\mathbf{e}(m\tau)x)-\mathrm{Li}_{1-n}(\mathbf{e}((M+m)\tau)x))\\
&\quad+(-1)^N(2\pi i\tau)^N\int_0^M \mathrm{Li}_{1-N}(\mathbf{e}((\xi+m)\tau)x)\frac{B_N(\xi-\lfloor\xi\rfloor)}{N!}\,d\xi\,.
\end{aligned}\tag{4.46}$$

Then note that

$$\int_0^M \log(1-\mathbf{e}((\xi+m)\tau)x)\,d\xi \;=\; -\frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}((M+m)\tau)x)+\frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau)x)\,. \tag{4.47}$$

Then our assumption on $\hbar$ implies that as $M\to\infty$ we have

$$\lim_{M\to\infty}\int_0^M \log(1-\mathbf{e}((\xi+m)\tau)x)\,d\xi \;=\; \frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau)x)\,. \tag{4.48}$$

Similarly, for $k<2$

$$\lim_{M\to\infty}\mathrm{Li}_k(\mathbf{e}((M+m)\tau)x) \;=\; 0\,. \tag{4.49}$$

Finally, we note that $\mathrm{Li}_k(0)=0$ and $\mathrm{Li}_k(z)(1-z)^{1-k}$ for $k\in\mathbb{Z}_{\le 0}$ is a polynomial. Then by our assumptions on x, m and $\hbar$, there exists $L^{(N)}$ independent of $x, m, \hbar$ such that

$$|\mathrm{Li}_{1-N}(\mathbf{e}((\xi+m)\tau)x)| < L^{(N)}|\mathbf{e}(\xi\tau)|\,. \tag{4.50}$$

Therefore, there exists $P^{(N)}$ such that

$$\begin{aligned}
&\Bigg|\sum_{n=0}^{\infty}\log(1-\mathbf{e}((n+m)\tau)x)-\frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau)x)\\
&\quad-\frac{\log(1-\mathbf{e}(m\tau)x)}{2}-\sum_{n=1}^{N-1}\frac{(-1)^{n+1}B_{n+1}(2\pi i\tau)^n}{(n+1)!}\mathrm{Li}_{1-n}(\mathbf{e}(m\tau)x)\Bigg| \le |2\pi i\tau|^N P^{(N)}\,.
\end{aligned}\tag{4.51}$$

Now for $\hbar$ small (*w.r.t.* fixed m and x)

$$\mathrm{Li}_k(\mathbf{e}(m\tau)x) \;=\; \sum_{\ell=0}^{\infty} \mathrm{Li}_{k-\ell}(x)\frac{(2\pi i m\tau)^\ell}{\ell!} \tag{4.52}$$

and therefore, noting that $B_n = 0$ for odd $n > 1$, considering that

$$\begin{aligned}\sum_{n=0}^{N} \frac{B_n(2\pi i\tau)^{n-1}}{n!}\mathrm{Li}_{2-n}(\mathbf{e}(m\tau)x) \;&=\; \sum_{n=0}^{N}\sum_{\ell=0}^{\infty} \frac{B_n(2\pi i\tau)^{n+\ell-1}m^\ell}{n!\ell!}\mathrm{Li}_{2-n-\ell}(x)\\ =\; \sum_{n=0}^{N}\sum_{\ell=0}^{n} \frac{B_{n-\ell}(2\pi i\tau)^{n-1}m^\ell}{\ell!(n-\ell)!}\mathrm{Li}_{2-n}(x) + O(\tau^N) \;&=\; \sum_{n=0}^{N} \frac{B_n(m)(2\pi i\tau)^{n-1}}{n!}\mathrm{Li}_{2-n}(x) + O(\tau^N)\,,\end{aligned} \tag{4.53}$$

completes the proof. □

This analysis fails to give an answer when $x = 1$. Therefore, we need to apply a different method. Refining this can be done in a similar way described in [205]. This will be explored in the next Section 4.4. For now, we will extend this result to the case when q tends to other roots of unity $\mathbf{e}(a/c)$. The main tool to deal with this is to use the previous result with the identity

$$\Big(\mathbf{e}\Big(m\frac{a\tau+c}{c\tau+d}\Big)x;\mathbf{e}\Big(\frac{a\tau+c}{c\tau+d}\Big)\Big)_\infty \;=\; \prod_{\ell=0}^{|c|-1}\Big(\mathbf{e}\Big((m+\ell)\frac{a\tau+c}{c\tau+d}\Big)x;\mathbf{e}\Big(\frac{-|c|}{c^2\tau+cd}\Big)\Big)_\infty\,, \tag{4.54}$$

which follows from

$$\frac{a\tau+b}{c\tau+d} \;=\; \frac{ac\tau+bc}{c^2\tau+cd} \;=\; \frac{ac\tau+ad-1}{c^2\tau+cd} \;=\; \frac{a}{c} - \frac{1}{c^2\tau+cd}\,. \tag{4.55}$$

Corollary 4. *[88, Lem. 2.1] For $a/c \in \mathbb{Q}$ (with a, c coprime) and $\delta \in \mathbb{R}_{>0}$, as $\tau \to 0$ with $\pi/2+\delta < \arg(-|c|/(c^2\tau+cd)) < -\pi/2-\delta$, and x and m such that for $\ell = 0,\dots,c-1$ we have $\delta < \arg(\mathbf{e}((m+\ell)(a\tau+b)/(c\tau+d))x) < -\delta$ and $\delta < \arg(x) < -\delta$ we have*

$$\begin{aligned}&\log\Big(\mathbf{e}\Big(m\frac{a\tau+b}{c\tau+d}\Big)x;\mathbf{e}\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)_\infty\\ &\sim\; \sum_{k=0}^{\infty}\frac{(-2\pi i|c|)^{k-1}}{(c^2\tau+cd)^{k-1}k!}\sum_{\ell=0}^{|c|-1}B_k\Big(\frac{m+\ell}{|c|}\Big)\mathrm{Li}_{2-k}\Big(\mathbf{e}\Big((m+\ell)\frac{a}{c}\Big)x\Big)\end{aligned} \tag{4.56}$$

for $\gamma = [a,b;c,d] \in \mathrm{SL}_2(\mathbb{Z})$ and as $\tau \to \infty$.

This is numerically verified in Code 20. We can take for $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$

$$\tilde{q}_\gamma \;=\; \mathbf{e}\Big(\frac{a\tau + b}{c\tau + d}\Big) \;=\; \mathbf{e}\Big(\frac{a}{c} - \frac{1}{c^2\tau + cd}\Big) \;=\; \mathbf{e}\Big(\frac{a}{c}\Big)\mathbf{e}\Big(\frac{\tilde{\tau}_\gamma}{|c|}\Big)\,, \tag{4.57}$$

where

$$\tilde{\tau}_\gamma \;=\; -\frac{|c|}{c^2\tau + cd}\,. \tag{4.58}$$

Then the asymptotics take the form

$$\log\big(\tilde{q}_\gamma^m x; \tilde{q}_\gamma\big)_\infty \;\sim\; \sum_{k=0}^{\infty} \frac{(2\pi i \tilde{\tau}_\gamma)^{k-1}}{k!} \sum_{\ell=0}^{|c|-1} B_k\Big(\frac{m+\ell}{|c|}\Big) \mathrm{Li}_{2-k}\Big(\mathbf{e}\Big((m+\ell)\frac{a}{c}\Big)x\Big)\,. \tag{4.59}$$

Definition 12 (Asymptotic cyclic dilogarithm)**.** *For $m, x \in \mathbb{C}$ and $q = \mathbf{e}(a/c)$ with $a/c \in \mathbb{Q}$ define*

$$\Delta(m, x; q) \;=\; \prod_{\ell=0}^{|c|-1} \Big(1 - \mathbf{e}\Big((m+\ell)\frac{a}{c}\Big)x\Big)^{\frac{1}{2} - \frac{m+\ell}{|c|}}\,. \tag{4.60}$$

Using this function with equation (4.37), we have

$$\big(\tilde{q}_\gamma^m x; \tilde{q}_\gamma\big)_\infty \;=\; \exp\left(\frac{\mathrm{Li}_2\big(\mathbf{e}(ma|c|/c)x^{|c|}\big)}{2\pi i |c| \tilde{\tau}_\gamma}\right) \Delta\Big(m, z; \mathbf{e}\Big(\frac{a}{c}\Big)\Big)(1 + O(\tilde{\tau}_\gamma))\,. \tag{4.61}$$

In these formulae it is extremely important to use the principle branches of the various roots. Moreover, one should not combine factors in the products in ways not respecting the branching properties. Notice that that only dependence on γ as opposed to $a/c \in \mathbb{Q}$ is stored in $\tilde{\tau}_\gamma$.

4.4 Integral formula for the Pochhammer symbol

Lemma 4 does not allow for $x = 1$. This is an extremely important example. When $x = 1$ and $m = 1$ we obtain the simple formula

$$\log(\mathbf{e}(\tau); \mathbf{e}(\tau))_\infty \;\sim\; -\frac{2\pi i}{24\tau} - \frac{1}{2}\log(\tau) + \frac{2\pi i}{8} - \frac{2\pi i \tau}{24} + O(\tau^N)\,. \tag{4.62}$$

This formula can be deduced from the Euler–Maclaurin formula with some additional analysis at the end points [205]. We want to reproduce these results. In another direction, the infinite Pochhammer is only convergent when $|q| < 1$, however, there is an integral expression that captures all of the asymptotics and is convergent in a larger domain. The basic idea is to shift our analysis to the more complex world and use the Abel–Plana summation method. This results in the following theorem, which should be equivalent to the results of [29, 58, 198].

Theorem A–29. *There exists a function $\psi(\tau, m, z, M)$ such that for $m \in \mathbb{C}$, $z \in \mathbb{C} - 2\pi i\mathbb{Z}$, $M \in \mathbb{Z}$ and $\Re(\tau) > 0$ as $\tau \to 0$ we have*

$$\begin{aligned}&(\mathbf{e}(m\tau + z); e(\tau))_{M+1} \\ &\quad = (\mathbf{e}(m + z/\tau - \lceil \Re(m\tau + z)\rceil/\tau; e(-1/\tau))_{\lfloor \Re(M\tau)\rfloor} \exp\big(\psi(\tau, m, z, M)\big)\,.\end{aligned} \tag{4.63}$$

Moreover,

$$\psi(\tau, m, z, M) \;=\; \psi\Big(\tau, m + \frac{1}{\tau}, z, M\Big) \;=\; \psi(\tau, m, z + 1, M) \;=\; \psi\Big(\tau, m, z, M + \frac{1}{\tau}\Big)\,. \tag{4.64}$$

and explicitly,

$$\begin{aligned}\psi(\tau, m, z, M) \;=\;& \frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau + z)) - \frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}((M + m)\tau + z)) \\ &+ \frac{1}{2}\log(1 - \mathbf{e}(m\tau + z)) + \frac{1}{2}\log(1 - \mathbf{e}((M + m)\tau + z)) \\ &+ \frac{i}{\tau}\int_0^\infty \frac{\log(1 - \mathbf{e}((iy/\tau + m)\tau + z)) - \log(1 - \mathbf{e}((-iy/\tau + m)\tau + z))}{\mathbf{e}(-iy/\tau) - 1}dy \\ &- \frac{i}{\tau}\int_0^\infty \frac{\log(1 - \mathbf{e}((M + iy/\tau + m)\tau + z)) - \log(1 - \mathbf{e}((M - iy/\tau + m)\tau + z))}{\mathbf{e}(-iy/\tau) - 1}dy\,.\end{aligned} \tag{4.65}$$

Proof. Suppose that $\Re(\tau) > 0$. Consider the contour γ depicted in Figure 4.3. We then have

$$\sum_{n=1}^{M-1} \log(1 - \mathbf{e}((n + m)\tau + z)) \;=\; \frac{1}{2i}\int_\gamma \cot(\pi\xi)\log(1 - \mathbf{e}((\xi + m)\tau + z))d\xi\,. \tag{4.66}$$

Notice that the branch cuts of the integral are given by

$$1 - \mathbf{e}((\xi + m)\tau + z) \in \mathbb{R}_{\leq 0} \tag{4.67}$$

which is the same as

$$\xi \in -m - \frac{z}{\tau} + \frac{-i}{\tau}\mathbb{R}_{\geq 0} + \frac{1}{\tau}\mathbb{Z}\,. \tag{4.68}$$

Then applying the same trick as used the Abel–Plana summation formula of Theorem 27 we have

$$\begin{aligned}&\sum_{n=1}^{M-1} \log(1 - \mathbf{e}((n + m)\tau + z)) - \int_{\gamma_0} \log(1 - \mathbf{e}((\xi + m)\tau + z))d\xi \\ &\quad = \int_{\gamma_1} \frac{\log(1 - \mathbf{e}((\xi + m)\tau + z))}{1 - \mathbf{e}(-\xi)}d\xi + \int_{\gamma_2} \frac{\log(1 - \mathbf{e}((\xi + m)\tau + z))}{\mathbf{e}(\xi) - 1}d\xi\,.\end{aligned} \tag{4.69}$$

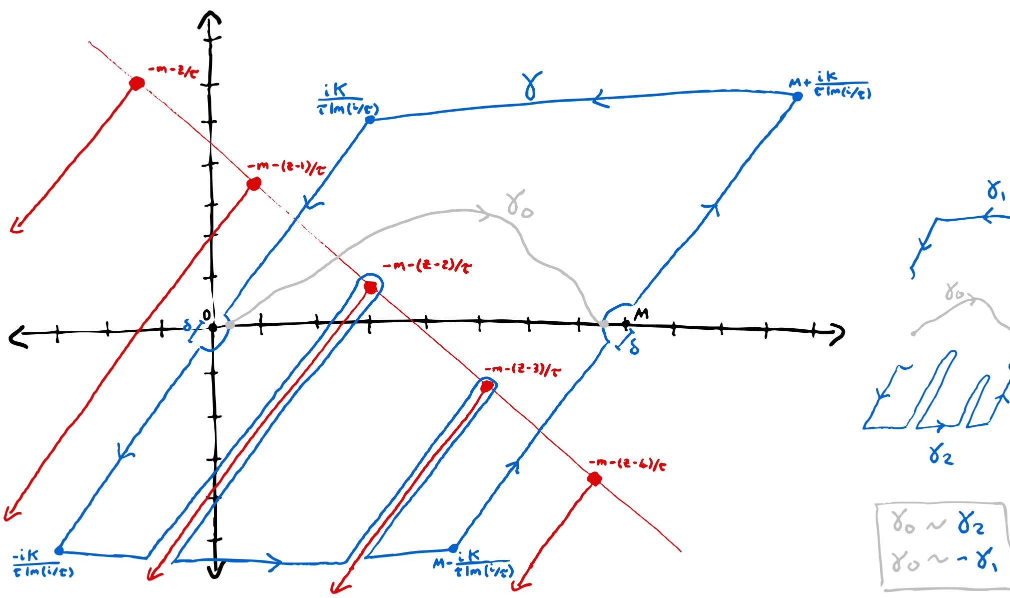

Figure 4.3: Contour used for summation methods for the Pochhammer symbol.

Firstly, notice that, using the branching of the logarithm, that the limit as $K \to \infty$ of the integrals around the branch cuts are given by

$$\begin{aligned} \int_{-\infty}^{0} \frac{2\pi i}{\mathbf{e}(iy/\tau - m - (z-k)/\tau) - 1} \frac{i}{\tau} dy &= [\log(1 - \mathbf{e}(-iy/\tau + m + (z-k)/\tau))]_{-\infty}^{0} \\ &= \log(1 - \mathbf{e}(m + z/\tau - k/\tau)) \, . \end{aligned} \quad (4.70)$$

Secondly, we notice that as we are using the principle branches of the polylogarithms that the limit of the integral along γ_0 as $\delta \to 0$ is given by

$$\int_{\gamma_0} \log(1 - \mathbf{e}((\xi + m)\tau + z)) d\xi = \frac{1}{2\pi i \tau} \mathrm{Li}_2(\mathbf{e}(m\tau + z)) - \frac{1}{2\pi i \tau} \mathrm{Li}_2(\mathbf{e}((M + m)\tau + z)) \, . \quad (4.71)$$

The integrals along the horizontal contours with imaginary part $\pm K$ both vanish as the integrand decays uniformly and exponentially as $K \to \infty$. This is also uniform in M.

Therefore,

$$\begin{aligned}
\sum_{n=0}^{M} \log(1-\mathbf{e}((n+m)\tau+z)) \;=\; & \frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau+z)) - \frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}((M+m)\tau+z)) \\
& + \frac{1}{2}\log(1-\mathbf{e}(m\tau+z)) + \frac{1}{2}\log(1-\mathbf{e}((M+m)\tau+z)) \\
& + \sum_{\substack{\Re((M+m)\tau+z)>k>\Re(m\tau+z) \\ k\in\mathbb{Z}}} \log(1-\mathbf{e}(m+z/\tau-k/\tau)) \\
& + \frac{i}{\tau}\int_0^\infty \frac{\log(1-\mathbf{e}((iy/\tau+m)\tau+z)) - \log(1-\mathbf{e}((-iy/\tau+m)\tau+z))}{\mathbf{e}(-iy/\tau)-1}dy \\
& - \frac{i}{\tau}\int_0^\infty \frac{\log(1-\mathbf{e}((M+iy/\tau+m)\tau+z)) - \log(1-\mathbf{e}((M-iy/\tau+m)\tau+z))}{\mathbf{e}(-iy/\tau)-1}dy\,.
\end{aligned} \tag{4.72}$$

□

This equality is numerically verified in the PARI/GP [20] Code 21. We can recover the previous result of Lemma 4 by analysing the asymptotics of the integrals. For this, notice that

$$\int_0^\infty \frac{y^a}{\exp(2\pi y)-1}dy \;=\; \frac{\zeta(a+1)\Gamma(a+1)}{(2\pi)^{a+1}}\,. \tag{4.73}$$

Therefore, for $\ell+1\in 2\mathbb{Z}_{>0}$ we have

$$\int_0^\infty \frac{y^\ell}{\exp(2\pi y)-1}dy \;=\; (-1)^{(\ell-1)/2}\frac{B_{\ell+1}}{2(\ell+1)}\,. \tag{4.74}$$

Therefore, we see that for $\ell\in\mathbb{Z}$

$$\int_0^\infty \frac{(iy)^\ell-(-iy)^\ell}{\exp(2\pi y)-1}dy \;=\; ((i)^\ell-(-i)^\ell)(-1)^{(\ell-1)/2}\frac{B_{\ell+1}}{2(\ell+1)} \;=\; \begin{cases} 0 & \text{if } \ell \text{ is even} \\ i\frac{B_{\ell+1}}{(\ell+1)} & \text{if } \ell \text{ is odd} \end{cases}\,. \tag{4.75}$$

Note that as the following integral exponentially converges, we can extract any finite number of polynomial terms to find

$$\begin{aligned}
&-i\int_0^\infty -\frac{\log(1-\mathbf{e}((iy+m)\tau+z))-\log(1-\mathbf{e}((-iy+m)\tau+z))}{\mathbf{e}(-iy)-1}dy\\
&= -i\sum_{\ell=0}^{N-1}\frac{\mathrm{Li}_{1-\ell}\big(\mathbf{e}(m\tau+z)\big)(2\pi i\tau)^\ell}{\ell!}\int_0^\infty\frac{(iy)^\ell-(-iy)^\ell}{\mathbf{e}(-iy)-1}dy+O(\tau^N)\\
&= \sum_{\ell=1}^{N-1}\frac{\mathrm{Li}_{1-\ell}\big(\mathbf{e}(m\tau+z)\big)(2\pi i\tau)^\ell}{\ell!}\frac{B_{\ell+1}}{\ell+1}+O(\tau^N)\\
&= \sum_{\ell=2}^{N}\frac{B_\ell(2\pi i\tau)^{\ell-1}}{\ell!}\mathrm{Li}_{2-\ell}\big(\mathbf{e}(m\tau+z)\big)+O(\tau^N)\,.
\end{aligned}\tag{4.76}$$

This together with the first few terms of equation 4.72 then we have all the terms coming from equation 4.53 and the asymptotics of Lemma 4 follow.

We can find an expression for $c<0$ (which can always be done by choosing multiplying all a,b,c,d,z by -1)

$$\begin{aligned}
&\Big(\mathbf{e}\Big(m\frac{a\tau+b}{c\tau+d}+\frac{z}{c\tau+d}\Big);e\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)_{cM+c}\\
&=\prod_{\ell=0}^{c-1}\Big(\mathbf{e}\Big((m+\ell)\frac{a\tau+b}{c\tau+d}+\frac{z}{c\tau+d}\Big);\mathbf{e}\Big(\frac{1}{c\tau+d}\Big)\Big)_{M+1}\\
&=\prod_{\ell=0}^{c-1}\Big(\mathbf{e}\Big((m+\ell)(a\tau+b)+z-\Big\lceil\Re\Big((m+\ell)\frac{a\tau+b}{c\tau+d}+\frac{z}{c\tau+d}\Big)\Big\rceil(c\tau+d)\Big);e(-c\tau)\Big)_{\lfloor\Re(M\tau)\rfloor}\\
&\qquad\times\exp\left(\psi\Big(\frac{1}{c\tau+d},z-\frac{(m-\ell)}{c},\frac{a}{c}(m+\ell),M\Big)\right)\\
&=\left(\mathbf{e}\Big(m(a\tau+b)+z-\Big\lceil\Re\Big(m\frac{a\tau+b}{c\tau+d}+\frac{z}{c\tau+d}\Big)\Big\rceil c\tau\Big);e(\tau)\right)_{\lfloor\Re(Mc\tau)\rfloor}\\
&\qquad\times\exp\left(\sum_{\ell=0}^{c-1}\psi\Big(\frac{1}{c\tau+d},z-\frac{(m-\ell)}{c},\frac{a}{c}(m+\ell),M\Big)\right)\\
&=\left(\mathbf{e}\Big(m(a\tau+b)+z-\Big\lceil\Re\Big(m\frac{a\tau+b}{c\tau+d}+\frac{z}{c\tau+d}\Big)\Big\rceil c\tau\Big);e(\tau)\right)_{\lfloor\Re(cM\tau)\rfloor}\exp\Big(\psi_\gamma\Big(\tau,m,z,cM\Big)\Big)\,.
\end{aligned}\tag{4.77}$$

where we note that $a+c\mathbb{Z}\in(\mathbb{Z}/c\mathbb{Z})^\times$.

4.5 Asymptotics of the Pochhammer symbol again

With the integral expression of Theorem 29 we can get more and prove the following asymptotics we couldn't with Euler–MacLaurin[3] when $z \in 2\pi i\mathbb{Z}$.

Lemma 5. *For fixed $m \in \mathbb{C}$ as $\tau \to 0$ we have*

$$\begin{aligned}&\frac{(\mathbf{e}(m\tau);\mathbf{e}(\tau))_\infty}{(\mathbf{e}(m-\lceil\Re(m\tau)\rceil/\tau);\mathbf{e}(-1/\tau))_\infty}\\ &\sim\ (-2\pi i m\tau)^{\frac{1}{2}-m}\frac{m^m}{\Gamma(m)}\sqrt{\frac{2\pi}{m}}\exp\left(\frac{B_0(m)(2\pi i\tau)^{-1}}{0!}\zeta(2)+\sum_{k=2}^{\infty}\frac{B_k(m)(2\pi i\tau)^{k-1}}{k!}\zeta(2-k)\right).\end{aligned} \tag{4.78}$$

Proof. Firstly, notice that

$$\log(1-\mathbf{e}((iy+m)\tau))\ =\ \log(-2\pi i(iy+m)\tau)+\sum_{\ell=1}^{\infty}\frac{B_\ell}{\ell\cdot\ell!}(iy+m)^\ell(-2\pi i\tau)^\ell\,. \tag{4.79}$$

Then we find that

$$\begin{aligned}&\int_0^\infty\frac{\log(1-\mathbf{e}((iy+m)\tau))-\log(1-\mathbf{e}((-iy+m)\tau))}{\mathbf{e}(-iy)-1}dy\\ &=\ \int_0^\infty\frac{\log(-2\pi i(iy+m)\tau)-\log(-2\pi i(-iy+m)\tau)+\pi i((iy+m)-(-iy+m))\tau}{\mathbf{e}(-iy)-1}dy\\ &\quad+\sum_{\ell=1}^{N-1}\frac{B_\ell\cdot(-2\pi i\tau)^\ell}{\ell\cdot\ell!}\int_0^\infty\frac{((iy+m)^\ell-(-iy+m)^\ell)}{\mathbf{e}(-iy)-1}+O(\tau^N)\\ &=\ im+i\frac{1}{2}\log\left(\frac{\Gamma(m)\Gamma(m+1)}{2\pi m^{2m}}\right)\\ &\qquad+i\sum_{\ell=1}^{N-1}\frac{B_\ell\cdot(-2\pi i\tau)^\ell}{\ell\cdot\ell!}\sum_{k=0}^{\lfloor(\ell-1)/2\rfloor}\binom{\ell}{2k+1}m^{\ell-2k-1}\frac{B_{2k+2}}{2k+2}+O(\tau^N)\end{aligned} \tag{4.80}$$

where we have used the identity

$$\begin{aligned}&\int_0^\infty\frac{\log(-2\pi i(iy+m)\tau)-\log(-2\pi i(-iy+m)\tau)}{\mathbf{e}(-iy)-1}dy\\ &=\ im+\frac{i}{2}\log\left(\frac{\Gamma(m)\Gamma(m+1)}{2\pi m^{2m}}\right),\end{aligned} \tag{4.81}$$

[3] This could be done directly with Euler–Maclaurin using methods discussed for example in [205] combined with the analysis in [88, 73].

which can be proved by for example differentiating in m and checking boundary conditions. Then using equation (4.35) we have

$$\frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau)) = \frac{1}{2\pi i\tau}\Big(- \mathrm{Li}_2(1-\mathbf{e}(m\tau)) + \frac{\pi^2}{6} - 2\pi i m\tau\log(1-\mathbf{e}(m\tau))\Big) \tag{4.82}$$

The noting that

$$-\mathrm{Li}_2(1-\mathbf{e}(m\tau)) = \sum_{\ell=1}^{\infty}(-1)^{\ell+1}\frac{B_{\ell-1}}{\ell!}(2\pi i m\tau)^{\ell} \tag{4.83}$$

we find that

$$\frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau)) = \sum_{\ell=1}^{\infty}\frac{B_{\ell-1}}{\ell!}m^{\ell}(-2\pi i\tau)^{\ell-1} - \frac{2\pi i}{24\tau} - m\log(1-\mathbf{e}(m\tau))\,. \tag{4.84}$$

Then finally we have

$$\log(1-\mathbf{e}(m\tau)) = \log(-2\pi i m\tau) + \sum_{\ell=1}^{\infty}\frac{B_{\ell}}{\ell\cdot\ell!}m^{\ell}(-2\pi i\tau)^{\ell} \tag{4.85}$$

and so

$$\frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau)) = \sum_{\ell=1}^{\infty}\frac{B_{\ell-1}}{\ell!}m^{\ell}(-2\pi i\tau)^{\ell-1} - \frac{2\pi i}{24\tau} - m\log(-2\pi i m\tau) - \sum_{\ell=1}^{\infty}\frac{B_{\ell}}{\ell\cdot\ell!}m^{\ell+1}(-2\pi i\tau)^{\ell}\,. \tag{4.86}$$

Therefore,

$$\begin{aligned}&\frac{1}{2\pi i\tau}\mathrm{Li}_2(\mathbf{e}(m\tau)) + \frac{1}{2}\log(1-\mathbf{e}(m\tau))\\ &= \sum_{\ell=1}^{\infty}\frac{B_{\ell-1}}{\ell!}m^{\ell}(-2\pi i\tau)^{\ell-1} - \frac{2\pi i}{24\tau} + (\frac{1}{2}-m)\log(-2\pi i m\tau) + (\frac{1}{2}-m)\sum_{\ell=1}^{\infty}\frac{B_{\ell}}{\ell\cdot\ell!}m^{\ell}(-2\pi i\tau)^{\ell}\,.\end{aligned} \tag{4.87}$$

Therefore, we find that combining the resulting series that

$$\begin{aligned}&\frac{(\mathbf{e}(m\tau);\mathbf{e}(\tau))_\infty}{(\mathbf{e}(m-\lceil\Re(m\tau)\rceil/\tau);\mathbf{e}(\tau))_\infty}\\ &\sim \exp\Bigg(- \sum_{\ell=1}^{\infty}\frac{B_{\ell}}{\ell}(-2\pi i\tau)^{\ell}\sum_{k=0}^{\ell+1}\frac{B_k}{k!(\ell+1-k)!}m^{\ell+1-k}\\ &\qquad - \frac{2\pi i}{24\tau} + \Big(\frac{1}{2}-m\Big)\log(-2\pi i m\tau) - \frac{1}{2}\log\Big(\frac{\Gamma(m)\Gamma(m+1)}{2\pi m^{2m}}\Big)\Bigg)\,.\end{aligned} \tag{4.88}$$

Taking the exponential of the last two terms and collecting the polynomials in m into the Bernoulli polynomials given in equation 4.3 gives the result. □

This is verified numerically with the PARI/GP [20] Code 22.

Corollary 5. *As $\tau \to 0$ for all $N \in \mathbb{Z}$*

$$\frac{(\mathbf{e}(\tau);\mathbf{e}(\tau))_\infty}{(\mathbf{e}(-1/\tau);\mathbf{e}(-1/\tau))_\infty} \;=\; (-i\tau)^{-\frac{1}{2}}\mathbf{e}\Big(-\frac{1}{24\tau}-\frac{\tau}{24}\Big)(1+O(\tau^N))\,. \tag{4.89}$$

Proof. We simply substitute $m=1$ into Lemma 5 and notice that for $\ell \in \mathbb{Z}_{\geq 1}$

$$B_\ell B_{\ell+1}(1) \;=\; B_\ell B_{\ell+1} \;=\; -\frac{\delta_{\ell,1}}{12}\,. \tag{4.90}$$

□

Again we can study the more general case and we find that for $j \in \{0,\dots,|c|-1\}$ that

$$\begin{aligned}
&\Big(\mathbf{e}\Big(m\frac{a\tau+b}{c\tau+d}-(j+m)\frac{a}{c}\Big);\mathbf{e}\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)_\infty \\
&= \prod_{\ell=0}^{c-1}\Big(\mathbf{e}\Big((\ell-j)\frac{a}{c}-\frac{(m+\ell)}{|c|}\frac{|c|}{c^2\tau+cd}\Big);\mathbf{e}\Big(-\frac{|c|}{c^2\tau+cd}\Big)\Big)_\infty \\
&\sim\; 2\pi\sqrt{\frac{i|c|}{c^2\tau+cd}}\Big(2\pi i\frac{m+j}{c^2\tau+cd}\Big)^{-\frac{m+j}{|c|}}\frac{\Big(\frac{m+j}{|c|}\Big)^{\frac{m+j}{|c|}}}{\Gamma\Big(\frac{m+j}{|c|}\Big)} \\
&\qquad\times\exp\Bigg(\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{-2\pi i|c|}{c^2\tau+cd}\Big)^{k-1}\sum_{0\leq\ell\leq|c|-1}^{\ell\neq j}B_k\Big(\frac{m+\ell}{|c|}\Big)\mathrm{Li}_{2-k}\Big(\mathbf{e}\Big((\ell-j)\frac{a}{c}\Big)\Big) \\
&\qquad\qquad+\frac{2\pi i(c^2\tau+cd)}{24|c|}+\sum_{k=2}^{\infty}\frac{1}{k!}\Big(\frac{-2\pi i|c|}{c^2\tau+cd}\Big)^{k-1}B_k\Big(\frac{m+j}{|c|}\Big)\zeta(2-k)\Bigg) \\
&=\; 2\pi\sqrt{-i\tilde{\tau}_\gamma}\,\Big(-2\pi i\frac{m+j}{|c|}\tilde{\tau}_\gamma\Big)^{-\frac{m+j}{|c|}}\Big(\frac{m+j}{|c|}\Big)^{\frac{m+j}{|c|}}\Gamma\Big(\frac{m+j}{|c|}\Big)^{-1} \\
&\qquad\times\exp\Bigg(\sum_{k=0}^{\infty}\frac{(2\pi i\tilde{\tau}_\gamma)^{k-1}}{k!}\sum_{0\leq\ell\leq|c|-1}^{\ell\neq j}B_k\Big(\frac{m+\ell}{|c|}\Big)\mathrm{Li}_{2-k}\Big(\mathbf{e}\Big((\ell-j)\frac{a}{c}\Big)\Big) \\
&\qquad\qquad-\frac{2\pi i}{24\tilde{\tau}_\gamma}+\sum_{k=2}^{\infty}\frac{(2\pi i\tilde{\tau}_\gamma)^{k-1}}{k!}B_k\Big(\frac{m+j}{|c|}\Big)\zeta(2-k)\Bigg)
\end{aligned} \tag{4.91}$$

Again using the cyclic dilogarithm from definition 12, we find that

$$\begin{aligned}&\Big(\mathbf{e}\Big(m\frac{a\tau+b}{c\tau+d}-(j+m)\frac{a}{c}\Big);\mathbf{e}\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)_\infty \\ &\sim\ 2\pi\sqrt{-i\tilde{\tau}_\gamma}\,\Big(-2\pi i\frac{m+j}{|c|}\tilde{\tau}_\gamma\Big)^{-\frac{m+j}{|c|}}\Big(\frac{m+j}{|c|}\Big)^{\frac{m+j}{|c|}}\Gamma\Big(\frac{m+j}{|c|}\Big)^{-1} \\ &\quad\times\mathbf{e}\Big(-\frac{1}{24|c|\tilde{\tau}_\gamma}\Big)\prod_{0\leq\ell\leq|c|-1}^{\ell\neq j}\Big(1-\mathbf{e}\Big((\ell-j)\frac{a}{c}\Big)\Big)^{\frac{1}{2}-\frac{m+\ell}{|c|}}\big(1+O(\tilde{\tau}_\gamma)\big)\end{aligned} \tag{4.92}$$

This is verified in the Code 23. Specialising $m=1, j=|c|-1$ we find

$$\begin{aligned}&\Big(\mathbf{e}\Big(\frac{a\tau+b}{c\tau+d}\Big);\mathbf{e}\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)_\infty \\ &\sim\ \sqrt{c\tau+d}\,\mathbf{e}\Big(\Big(\frac{-1}{c^2\tau+cd}\Big)\frac{-1}{24}-\frac{1}{24|c|}\frac{c^2\tau+cd}{-|c|}\Big)\sqrt{\frac{-ic}{|c|}}\prod_{\ell=1}^{|c|-1}\Big(1-\mathbf{e}\Big(\ell\frac{a}{c}\Big)\Big)^{\frac{1}{2}-\frac{\ell}{|c|}}.\end{aligned} \tag{4.93}$$

To prove this we use various properties of the dilogarithm and the Bernoulli polynomials. For the vanishing of the higher order terms we use equation (4.30) and the symmetry $B_k(x)=(-1)^kB_k(1-x)$.

4.6 General behaviour of q–hypergeometric asymptotics

Using the asymptotics of the Pochhammer symbol will allow us to determine the asymptotics of q–hypergeometric sums. The general behaviour of such sums is understood in many examples and there are many common structures observed at least numerically. We describe some of this behaviour then go through some examples coming from Nahm sums. Then we will consider some examples coming from knots, and, finally, consider a closed three–manifold.

There are multiple ways that q can tend to a root of unity. For q–hypergeometric functions defined at roots of unity such as the quantum invariants of sections 2.3 and 2.4 we can simply approach through roots of unity of increasing order. For q–series we can approach through the upper half plane in a variety of ways. Often this is done radially, which turns out to be the worst possible choice from the perspective of resurgence discussed in chapter 11. This is because this ray is an accumulation point of Stokes rays coming from a peacock pattern [69, 68]. Instead, taking the limit on some generic angle provides a better behaviour as noticed in [85]. Finally, one can also take the limit with $|\tilde{q}|$ fixed. This will be discussed in detail in section and the asymptotics is more subtle and are discussed in Section 8.2. This will be related to a weak form of quantum modularity. It was noticed in [208] that one should approach roots of unity on whatever angle using modular transformations. It is then most natural to consider asymptotics for $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$ as $\tau \to i\infty$ of the function evaluated at

$$\tilde{q}_\gamma \;=\; \mathbf{e}\Big(\frac{a\tau + b}{c\tau + d}\Big) \;=\; \mathbf{e}\Big(\frac{a}{c} - \frac{1}{c^2\tau + cd}\Big) \;=\; \mathbf{e}\Big(\frac{a}{c}\Big)\mathbf{e}\Big(\frac{\tilde{\tau}_\gamma}{|c|}\Big)\,. \tag{4.94}$$

Then the asymptotics of q–hypergeometric functions around an isolated critical point ρ takes the form

$$\begin{aligned}&\widehat{\Phi}^{\rho}_{a/c}\Big(\frac{1}{c^2\tau + cd}\Big)\\ &\;=\; \mu_\rho(a/c)(c\tau + d)^{d_\rho} \exp\Big(\frac{\mathrm{VC}_\rho(c\tau + d)}{2\pi i c}\Big)\frac{\epsilon_\rho(a/c)^{1/k}}{\sqrt{\delta_\rho}}\Big(A_{\rho,0,a/c} + A_{\rho,1,a/c}\frac{2\pi i}{c^2\tau + cd} + \cdots\Big)\end{aligned} \tag{4.95}$$

where $\mu_\rho(a/c)^{8c} = 1$, $d_\rho \in \frac{1}{2}\mathbb{Z}$, VC_ρ is a combination of values of the dilogarithm function and logarithms at points of the field $\mathbb{K}$, $\epsilon_\rho(a/c) \in \mathcal{O}^\times_{\mathbb{K}[\mathbf{e}(a/c)]}$ is a unit, $\delta_\rho \in \mathbb{K}$ and $A_{\rho,n,a/c} \in \mathbb{K}[\mathbf{e}(a/c)]$. Importantly, the series only depends on a/c and not $\widehat{\Phi}^{\rho}_{a/c}$ and just the argument will depend on d. The unit ϵ we studied in detail in [34].

We can also understand the denominators of the numbers $A_{\rho,n,a/c}$. To do this we will need a simple sequence of numbers used in [86]. Indeed, it is shown [86, Thm. 9.1] that for knots and all but finitely many primes the perterbative invariants defined in [45, 46] have universal denominators. These numbers are the denominators that appear in a half shifted Stirlings approxmation. See [177, A144618]. Take

$$D_n \;=\; 2^{2n+\sum_{\ell=0}^{\infty}\lfloor n/2^\ell\rfloor} \prod_{p>2,\ \text{prime}} p^{\sum_{\ell=0}^{\infty}\lfloor n/p^\ell(p-2)\rfloor} \tag{4.96}$$

This can be computed using Code 1. The first few values of this sequence are given by

$$D_0 \;=\; 1\,,\quad D_1 \;=\; 24\,,\quad D_2 \;=\; 1152\,,\quad D_3 \;=\; 414720\,,\quad D_4 \;=\; 39813120\,,\;\ldots \tag{4.97}$$

With the universal denominator D_n we can in general say more and it has been observed [86] that numerically when the critical points are defined over the ring of integers

$$A_{\rho,n,a/c}D_n\delta^{3n} \in \mathcal{O}_{\mathbb{K}[\mathbf{e}(a/c)]}[c^{-1}]\,. \tag{4.98}$$

Numerically this is extremely helpful as trying to recognise algebraic integers convincingly in a known number field can be done via the LLL algorithm [119]. The μ is related to the multiplier system of the Dedekind η-function 7.2.

We can view the constant terms of these series as a function from $\mathbb{Q}$ to $\mathbb{C}$ with addition properties on the image related to the denominator. Garoufalidis and Zagier [86] realised that these functions have asymptotics that behave in a similar way to the original q–hypergeometric functions. Their addition to the functions we consider will be absolutely fundamental for the later sections of the thesis on quantum modularity discussed for example in chapter 8. To get the modularity we need to use the "tweaking factor" given by

$$\lambda_{[a,b;c,d]}(r/s) \;=\; \frac{c}{s(cr+ds)} \tag{4.99}$$

which gives an additive $\mathrm{PSL}_2(\mathbb{Z})$ cocycle [86, Lem. 3.1]. We will see that this is in fact a coboundary associated to the Dedekhind η–function in Section 7.2.

There are also expectations for the behaviour of the asymptotics of the coefficients of these series. In particular, from the conjectures of [67], it is expected [86, 87] that for $a_{\rho,n} = A_{\rho,n}/\sqrt{\delta_\rho}$

$$a_{\rho,n} \sim \sum_{\rho'} \frac{M(\rho,\rho')}{2\pi i} \sum_{\ell\geq 0} \frac{\Gamma(k-\ell)}{(\mathrm{VC}_{\rho'}-\mathrm{VC}_{\rho})^{k-\ell}} a_{\rho',\ell} \tag{4.100}$$

Finally, the behaviour of the coefficients is conjectured to satisfy extremely strong properties. In particular, these series are expected to be resurgent [67, 69, 68]. The Borel transform of $\widehat{\Phi}^{\gamma}_{\rho_1}$ is expected to have analytic continuation on a star domain coming from a peacock pattern with branch cuts with branch points at

$$\mathrm{VC}_{\rho_2} - \mathrm{VC}_{\rho_1} + 4\pi^2\mathbb{Z}\,, \tag{4.101}$$

where ρ_2 ranges over the other critical points. This Borel transform is then expected, away from the singularities, to allow for Laplace transforms. Moreover, the jumping across the singularities of the Borel transform at $\mathrm{VC}_{\rho_2} - \mathrm{VC}_{\rho_1} + 4\pi^2 k$ is expected to be an integer times $\tilde{q}^k\widehat{\Phi}^{\gamma}_{\rho_2}$, which gives a finite number of asymptotic series that arise for a given hypergeometric function. Moreover, through the work in [69, 68] we expect to be able to compute these integers determining the jumping at every branch point. This leads to a compelling conjectural picture with strong computational power. This will be demonstrated in chapter 11.

Remark 14. *For non-isolated critical points defining some variety, in recent work with Garoufalidis [83] it was observed that periods can arise in the asymptotics of these q–hypergeometric functions. Therefore, somewhat speculatively, it seems like a natural generalisation of this picture would be to take periods on the various varieties given by critical points and c–fold cyclic covers when approaching other roots of unity.*

4.7 The case of rank one Nahm sums

Nahm sums are special q–hypergeometric functions. They satisfy q–difference equations, which we will see in part III. Their asymptotics were studied in detail in [88]. We won't consider their full generality here but just their one dimensional versions. Take the following sum

$$f_{A,m,n}(q) \;=\; \sum_{k=0}^{\infty}\mathbf{e}(nk)\frac{q^{\frac{A}{2}k^2+km}}{(q;q)_k}\,. \tag{4.102}$$

This can be computed efficiently with Code 30 for $A\in 2\mathbb{Z}$.

We can apply some the methods of Section 3.2 to find the leading order asymptotics as $q\to 1$ experimentally. For $A\in 2\mathbb{Z}$ we can use Code 31. To compute this leading order behaviour we can use the results of [88, 143, 142, 206]. We will assume that $A\in\mathbb{Z}$ however $A\in\mathbb{Q}$ can be handled by similar methods[4].

By the Poisson summation formula of Theorem 26, for $A>1$, we have

$$\begin{aligned} f_{A,m,n}(q) \;&=\; \frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}\mathbf{e}(nk)q^{\frac{A}{2}k^2+km}(q^{k+1};q)_\infty\\ &=\; \frac{1}{(q;q)_\infty}\sum_{\ell\in\mathbb{Z}}\int_{\mathbb{R}}\mathbf{e}\Big(\frac{A}{2}x^2\tau+xm\tau+(n+\ell)x\Big)\big(\mathbf{e}((x+1)\tau);\mathbf{e}(\tau)\big)_\infty dx\\ &=\; \frac{1}{\tau(q;q)_\infty}\sum_{\ell\in\mathbb{Z}}\int_{\tau\mathbb{R}}\mathbf{e}\Big(\frac{A}{2}\frac{x^2}{\tau}+xm+(n+\ell)\frac{x}{\tau}\Big)\big(\mathbf{e}(x+\tau);\mathbf{e}(\tau)\big)_\infty dz\,. \end{aligned} \tag{4.103}$$

The integrals can each be approximated by the saddle point method *i.e.* Laplace's method. In general, one of the integrals will dominate the others and we will find the assymptotics of $f_{A,m,n}(q)$ is determined to leading order by the asymptotics of this integral. The leading order of the integrand is given by

$$\begin{aligned} &\mathbf{e}\Big(\frac{A}{2}\frac{x^2}{\tau}+xm+(n+\ell)\frac{x}{\tau}\Big)\big(\mathbf{e}(x+\tau);\mathbf{e}(\tau)\big)_\infty\\ &\;=\;\mathbf{e}\Big(\frac{1}{(2\pi i)^2\tau}\mathrm{Li}_2(\mathbf{e}(x))-\frac{1}{(2\pi i)^2\tau}\frac{\pi^2}{6}+\frac{A}{2}\frac{x^2}{\tau}+(n+\ell)\frac{x}{\tau}+o(\tau^{-1})\Big)\,, \end{aligned} \tag{4.104}$$

and therefore, the critical points are given by solutions to

$$\begin{aligned} 0 \;&=\; \frac{\partial}{\partial x}\Big(\frac{1}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(x))-\frac{1}{(2\pi i)^2}\frac{\pi^2}{6}+\frac{A}{2}x^2+(n+\ell)x\Big)\\ &=\; -\frac{1}{2\pi i}\log(1-\mathbf{e}(x))+Az+n+\ell\,. \end{aligned} \tag{4.105}$$

The exponential of the solutions $X=\mathbf{e}(x)$ satisfies the algebraic equation

$$1-X \;=\; \mathbf{e}(n)X^A \tag{4.106}$$

[4] For $A<1$ one will need the relations given in Section 6.3.

which generically has finitely many solutions. This equation appeared in a more general form in equation (1.89) in relation to the Bloch group, which we will return to in Section 7.4. Therefore there will in general be a finite number of saddle points of each integral.

Example 38 ($A = 2, n = 0, q = 1$)**.** *This example is computed with Code 32. We have the Nahm equation*

$$1 - X \;=\; X^2 \tag{4.107}$$

which has solutions

$$X_1 \;=\; -\frac{1}{2} - \frac{\sqrt{5}}{2} \quad \text{and} \quad X_2 \;=\; -\frac{1}{2} + \frac{\sqrt{5}}{2}\,. \tag{4.108}$$

Therefore, for $k \in \mathbb{Z}$

$$\begin{aligned} x_{1,k} \;&=\; 0.50000\cdots - 0.076587\cdots i + k\,, \\ x_{2,k} \;&=\; 0.076587\cdots i + k\,. \end{aligned} \tag{4.109}$$

Then

$$\begin{aligned} -\frac{1}{2\pi i}\log(1-\mathbf{e}(x_{1,k})) + 2x_{1,k} + \ell \;&=\; 2k+\ell+1\,, \\ -\frac{1}{2\pi i}\log(1-\mathbf{e}(x_{2,k})) + 2x_{2,k} + \ell \;&=\; 2k+\ell\,. \end{aligned} \tag{4.110}$$

Therefore, the critical values are given by

$$\begin{aligned} \mathrm{VC}_{1,k} \;&=\; \mathrm{Li}_2(\mathbf{e}(x_{1,k})) - \frac{\pi^2}{6} + (2\pi i)^2 x_{1,k}^2 - (2\pi i)^2(2k+1)x_{1,k} \\ &=\; 7.2377\cdots + 4\pi^2 k(k+1) \;=\; \frac{11}{15}\pi^2 + 4\pi^2 k(k+1)\,, \\ \mathrm{VC}_{2,k} \;&=\; \mathrm{Li}_2(\mathbf{e}(x_{2,k})) - \frac{\pi^2}{6} + (2\pi i)^2 x_{2,k}^2 - (2\pi i)^2(2k+1)x_{2,k} \\ &=\; -0.65797\cdots + 4\pi^2 k^2 \;=\; -\frac{1}{15}\pi^2 + 4\pi^2 k^2\,. \end{aligned} \tag{4.111}$$

Therefore, we see that the largest contribution to the asymptotics is given by the critical point $x_{2,0}$ for any argument of τ. We then want the asymptotics of the integral

$$\frac{1}{\tau(q;q)_\infty}\int_{\tau\mathbb{R}} \mathbf{e}\Big(\frac{z^2}{\tau} + zm + \ell\frac{z}{\tau}\Big)\big(\mathbf{e}(z+\tau);\mathbf{e}(\tau)\big)_\infty dz\,. \tag{4.112}$$

When $\ell = 0$ the contour plot of

$$\Im\Big(\frac{1}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(x)) - \frac{1}{(2\pi i)^2}\frac{\pi^2}{6} + x^2\Big) \tag{4.113}$$

is given in Figure 4.4. This figure shows that we can deform the contour to the critical point. Notice that the asymptotics of the integrand at the critical point is given by

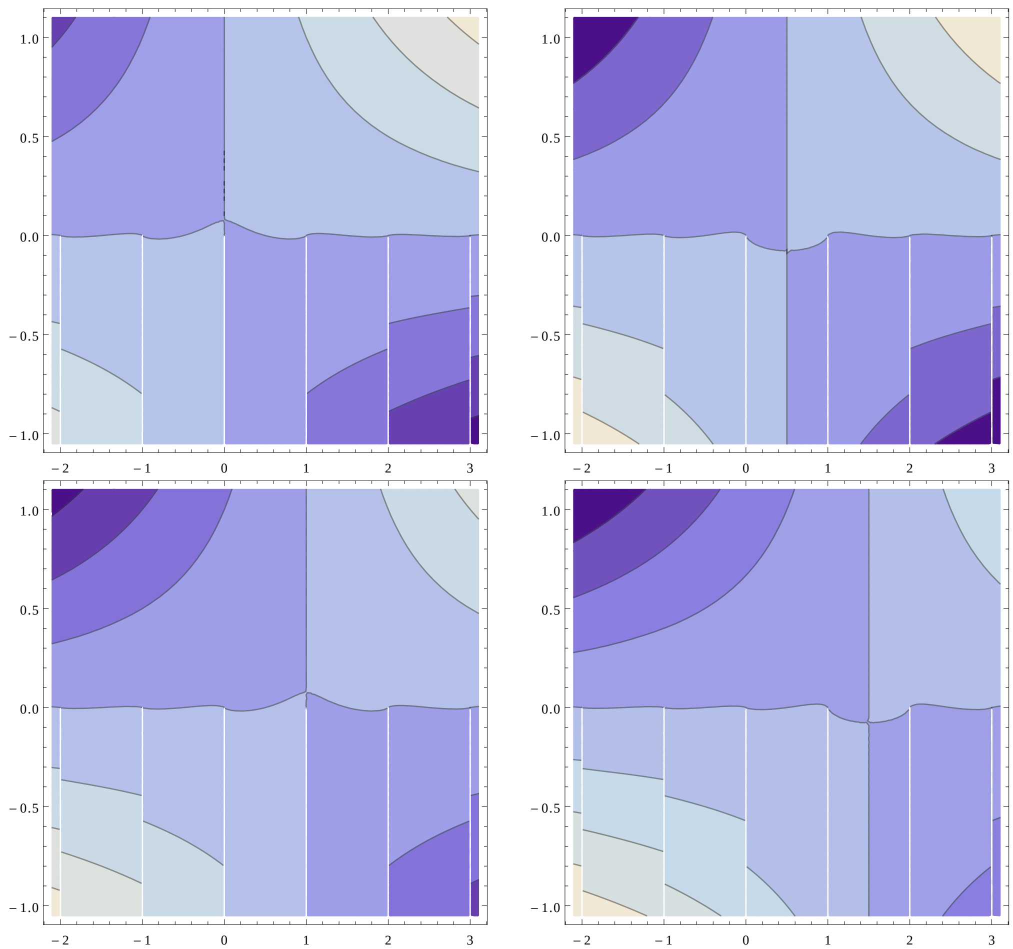

Figure 4.4: Contour plot of the imaginary part of the leading order of the logarithm of the of the integrand of the integral associated to the Nahm sum with $A = 2$ and $\ell = 0, -1, -2, -3$ respectfully. This for $\ell = 0$ has a critical point at $x_{2,0} = 0.076587 \cdots i$. This figure was made with Mathematica [101].

$$(-i\tau)^{\frac{1}{2}}\mathbf{e}\Big(x_{2,0}m+\frac{\tau}{24}+\frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{B_k(1)(2\pi i\tau)^{k-1}}{k!}\mathrm{Li}_{2-k}(\mathbf{e}(x_{2,0}))\Big)\,. \tag{4.114}$$

Therefore, we see that the method of steepest descent gives rise to the asymptotics when $\tau \to 0$ on any angle

$$f_{2,m,0}(\mathbf{e}(\tau)) \;\sim\; \mathbf{e}\Big(\frac{\mathrm{VC}_{2,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\frac{X_2^m}{\sqrt{X_2+2(1-X_2)}}(1+\cdots)\,. \tag{4.115}$$

The higher terms can similarly be computed using Laplace's method of Theorem 28. Given this is one dimensional this can be done to high order. We can also perform this computation numerically and we find that for all $N \in \mathbb{Z}$

$$f_{2,0,0}(\mathbf{e}(\tau)) \;\sim\; \mathbf{e}\Big(\frac{\mathrm{VC}_{2,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\frac{1}{\sqrt{X_2+2(1-X_2)}}\mathbf{e}\Big(\frac{\tau}{60}\Big)(1+O(\tau^N))\,. \tag{4.116}$$

Given that the asymptotics actually give rise to a convergent function we can subtract and study exponentially small corrections. This is a manifestation of modularity and we will study this example from that perspective in Section 7.4. However, the coefficients of these convergent asymptotic series are given exactly and their asymptotic behaviour is then completely determined.

Example 39 ($A = 4, n = 0, q = 1$)**.** *This example is computed with Code 33. This example has been studied in detail by Matthias Storzer in relation to work on his thesis. We have the Nahm equation*

$$1 - X \;=\; X^4 \tag{4.117}$$

which has solutions

$$\begin{aligned} X_1 &= -1.2207\cdots\,,\\ X_2 &= 0.72449\cdots\,,\\ X_3 &= 0.24813\cdots - 1.0340\cdots i\,,\\ X_4 &= 0.24813\cdots + 1.0340\cdots i\,. \end{aligned} \tag{4.118}$$

Therefore, for $k \in \mathbb{Z}$

$$\begin{aligned} x_{1,k} &= 0.50000\cdots - 0.031745\cdots i + k\,,\\ x_{2,k} &= 0.051293\cdots i + k\,,\\ x_{3,k} &= -0.21252\cdots - 0.0097740\cdots i + k\,,\\ x_{4,k} &= 0.21252\cdots - 0.0097740\cdots i + k\,. \end{aligned} \tag{4.119}$$

Then

$$\begin{aligned}
-\frac{1}{2\pi i}\log(1-\mathbf{e}(x_{1,k}))+4x_{1,k}+\ell &= 4k+\ell+2\,,\\
-\frac{1}{2\pi i}\log(1-\mathbf{e}(x_{2,k}))+4x_{2,k}+\ell &= 4k+\ell\,,\\
-\frac{1}{2\pi i}\log(1-\mathbf{e}(x_{3,k}))+4x_{3,k}+\ell &= 4k+\ell-1\,,\\
-\frac{1}{2\pi i}\log(1-\mathbf{e}(x_{4,k}))+4x_{4,k}+\ell &= 4k+\ell+1\,.
\end{aligned} \tag{4.120}$$

Therefore, the critical values are given by

$$\begin{aligned}
\mathrm{VC}_{1,k} &= \mathrm{Li}_2(\mathbf{e}(x_{1,k}))-\frac{\pi^2}{6}+2(2\pi i)^2x_{1,k}^2-(2\pi i)^2(4k+2)x_{1,k}\\
&= 8\pi^2k(k+1)+17.203\cdots\,,\\
\mathrm{VC}_{2,k} &= \mathrm{Li}_2(\mathbf{e}(x_{2,k}))-\frac{\pi^2}{6}+2(2\pi i)^2x_{2,k}^2-(2\pi i)^2(4k)x_{2,k}\\
&= 8\pi^2k^2-0.50498\cdots\,,\\
\mathrm{VC}_{3,k} &= \mathrm{Li}_2(\mathbf{e}(x_{3,k}))-\frac{\pi^2}{6}+2(2\pi i)^2x_{3,k}^2-(2\pi i)^2(4k-1)x_{3,k}\\
&= 4\pi^2k(2k-1)+3.1656\cdots-0.98137\cdots i\,,\\
\mathrm{VC}_{4,k} &= \mathrm{Li}_2(\mathbf{e}(x_{4,k}))-\frac{\pi^2}{6}+2(2\pi i)^2x_{4,k}^2-(2\pi i)^2(4k+1)x_{4,k}\\
&= 4\pi^2k(2k+1)+3.1656\cdots+0.98137\cdots i\,.
\end{aligned} \tag{4.121}$$

Therefore, we see that the largest possible contribution to the asymptotics is given by the critical points $x_{2,0}, x_{3,0}, x_{4,0}$ depending on the argument of τ. We then want the asymptotics of the integral

$$\frac{1}{\tau(q;q)_\infty}\int_{\tau\mathbb{R}}\mathbf{e}\Big(2\frac{z^2}{\tau}+zm+\ell\frac{z}{\tau}\Big)\big(\mathbf{e}(z+\tau);\mathbf{e}(\tau)\big)_\infty dz\,. \tag{4.122}$$

When $\ell=-1$ the contour plot of

$$\Im\Big(\frac{1}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(x))-\frac{1}{(2\pi i)^2}\frac{\pi^2}{6}+2x^2-x\Big) \tag{4.123}$$

is given in Figure 4.5. This figure shows that we can deform the contour to the critical point. Notice that the asymptotics of the integrand at the critical point is given by

$$(-i\tau)^{\frac{1}{2}}\mathbf{e}\Big(x_{4,0}m+\frac{\tau}{24}+\frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{B_k(1)(2\pi i\tau)^{k-1}}{k!}\mathrm{Li}_{2-k}(\mathbf{e}(x_{4,0}))\Big)\,. \tag{4.124}$$

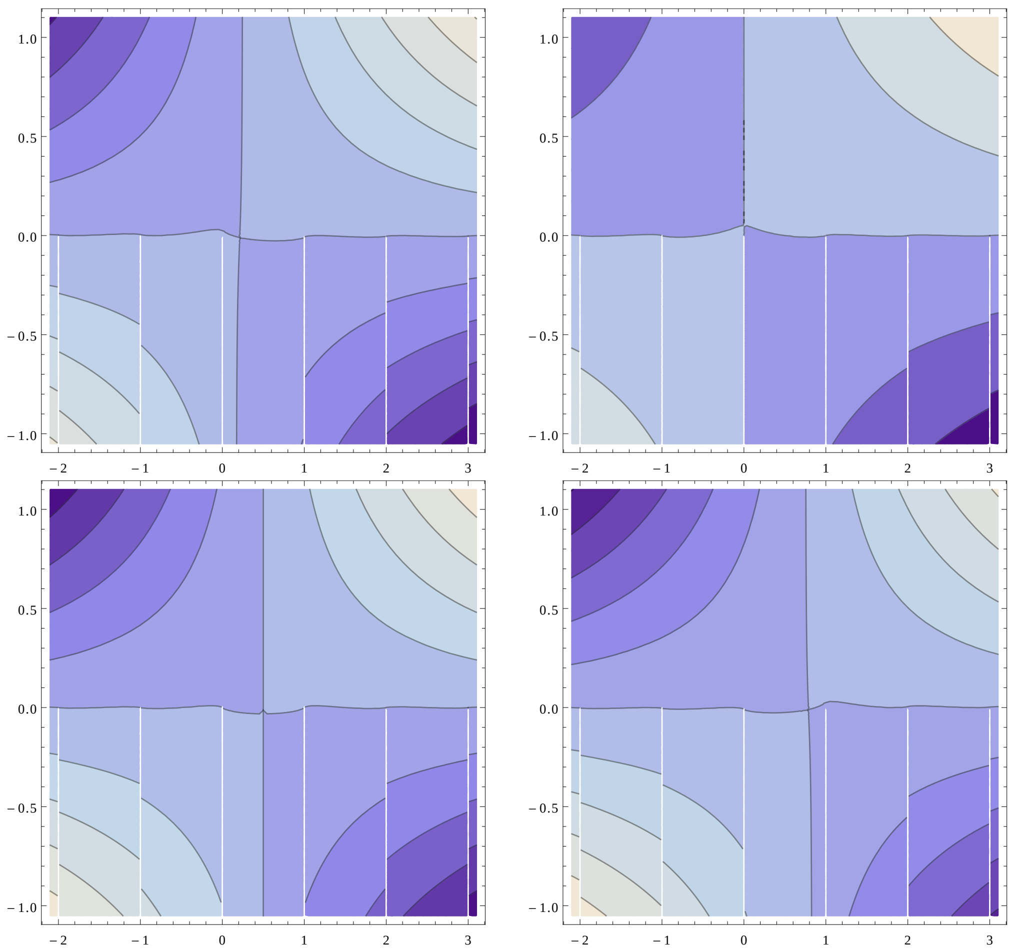

Figure 4.5: Contour plot of the imaginary part of the leading order of the logarithm of the of the integrand of the integral associated to the Nahm sum with $A = 4$ and $\ell = -1, 0, -2, -3$ respectfully where $\tau \in \mathbb{R}$. For $\ell = -1$, this has a critical point at $x_{4,0} = 0.21252\cdots - 0.0097740\cdots i$. This figure was made with Mathematica [101].

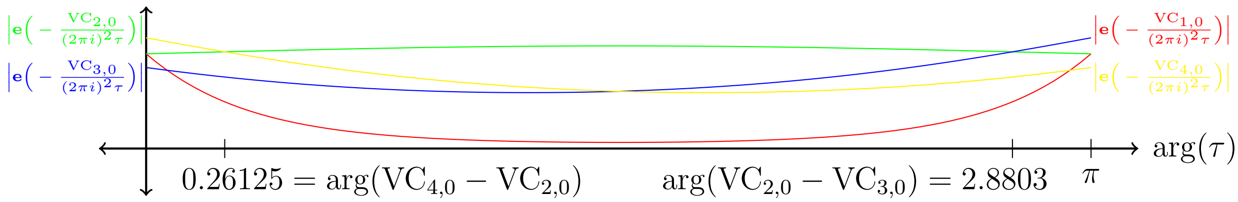

Figure 4.6: Exponential growth around each critical point with largest critical points for $A = 4$ Nahm sum.

Therefore, we see that the method of steepest descent gives rise to the asymptotics when $\tau \to 0$ is on a small angle just above the positive reals

$$f_{4,m,0}(\mathbf{e}(\tau)) \sim \mathbf{e}\Big(\frac{\mathrm{VC}_{4,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\frac{X_4^m}{\sqrt{X_4 + 4(1-X_4)}}(1+\cdots)\,. \tag{4.125}$$

The higher terms can similarly be computed using Laplace's method of Theorem 28. However, when $\tau \to 0$ nearly radially we find

$$f_{4,m,0}(\mathbf{e}(\tau)) \sim \mathbf{e}\Big(\frac{\mathrm{VC}_{2,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\frac{X_2^m}{\sqrt{X_2 + 4(1-X_2)}}(1+\cdots)\,. \tag{4.126}$$

The particular solution that dominates the asymptotics depends on the argument of τ. Indeed, whichever is larger between

$$\mathbf{e}\Big(\frac{\mathrm{VC}_{1,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\,,\quad \mathbf{e}\Big(\frac{\mathrm{VC}_{2,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\,,\quad \mathbf{e}\Big(\frac{\mathrm{VC}_{3,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\,,\quad \textit{and}\quad \mathbf{e}\Big(\frac{\mathrm{VC}_{4,0}}{(2\pi i)^2}\frac{-1}{\tau}\Big)\,. \tag{4.127}$$

We plot the absolute values as a function of $\arg(\tau)$ when $|\tau| = 1$ in Figure 4.6. Besides a global constant, the asymptotics give rise to series that are defined over the number field associated to $X^4 + X - 1 = 0$. By applying the methods of Section 3.2 and the LLL algorithm [119], we can numerically compute exact values for around one hundred of these coefficients with little effort. Letting

$$\delta = 4 - 3X\,, \tag{4.128}$$

the coefficients are all Galois conjugate and the first couple for $m = 0$ are given

$$\begin{aligned} A_0 &= 1\,,\\ A_1 &= \frac{-64 + 100\,X + 18\,X^2 - 54\,X^3}{24\,\delta^3}\,,\\ A_2 &= \frac{-104876 + 113812\,X + 29836\,X^2 + 17388\,X^3}{1152\,\delta^6}\,,\\ A_3 &= \frac{-79093616 - 1648464240\,X + 2928617760\,X^2 - 694542712\,X^3}{414720\,\delta^9}\,. \end{aligned} \tag{4.129}$$

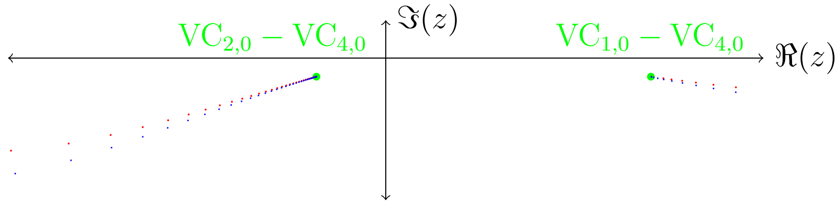

Figure 4.7: Plots of poles and zeros of the [50/50]–Padé approximate of $\Phi^{(\rho_4)}$.

Then, we can consider the asymptotics of the coeffcients $a_k = A_k/\sqrt{\delta}$ *and we find numerically that as* $k \to \infty$

$$\begin{aligned}
a_k^{(\rho_1)} &\sim \frac{1}{2\pi i}\sum_{\ell\geq 0}\frac{\Gamma(k-\ell)}{(\mathrm{VC}_{3,0}-\mathrm{VC}_{1,0})^{k-\ell}}a_\ell^{(\rho_3)} - \frac{1}{2\pi i}\sum_{\ell\geq 0}\frac{\Gamma(k-\ell)}{(\mathrm{VC}_{4,0}-\mathrm{VC}_{1,0})^{k-\ell}}a_\ell^{(\rho_4)}\\
a_k^{(\rho_2)} &\sim -\frac{1}{2\pi i}\sum_{\ell\geq 0}\frac{\Gamma(k-\ell)}{(\mathrm{VC}_{3,0}-\mathrm{VC}_{2,0})^{k-\ell}}a_\ell^{(\rho_3)} + \frac{1}{2\pi i}\sum_{\ell\geq 0}\frac{\Gamma(k-\ell)}{(\mathrm{VC}_{4,0}-\mathrm{VC}_{2,0})^{k-\ell}}a_\ell^{(\rho_4)}\\
a_k^{(\rho_3)} &\sim \frac{1}{2\pi i}\sum_{\ell\geq 0}\frac{\Gamma(k-\ell)}{(\mathrm{VC}_{2,0}-\mathrm{VC}_{3,0})^{k-\ell}}a_\ell^{(\rho_2)}\\
a_k^{(\rho_4)} &\sim -\frac{1}{2\pi i}\sum_{\ell\geq 0}\frac{\Gamma(k-\ell)}{(\mathrm{VC}_{2,0}-\mathrm{VC}_{4,0})^{k-\ell}}a_\ell^{(\rho_2)}\,.
\end{aligned} \tag{4.130}$$

This indicates that these series have factorial growth and therefore taking the Borel transform of Section 3.7 get what seem numerically to be convergent series with radius of convergence at various values $|\mathrm{VC}_{a,0} - \mathrm{VC}_{b,0}|$. *In fact, we expect branch points at all* $|\mathrm{VC}_{a,k} - \mathrm{VC}_{b,j}|$. *We can explore this numerically in a variety of ways. Firstly, we can take the Borel transform and then apply a Padé approximation from Section 3.6. As was illustrated with the logarithm in Figure 3.5, branch cuts appear as zeros and poles of the Padé approximates. For the series with embedding* ρ_4, *we get Figure 4.7. On the other hand we can use Remark 13 and take the Borel transform and then apply some conformal maps, which switch the nearest critical points. For example, the Borel transform of the series associated to* ρ_4 *has nearest critical point* $\mathrm{VC}_{2,0} - \mathrm{VC}_{4,0}$. *Therefore, we can take*

$$\psi = \frac{\xi}{\mathrm{VC}_{2,0}-\mathrm{VC}_{4,0}-\xi}\,, \quad \text{or} \quad \xi = (\mathrm{VC}_{2,0}-\mathrm{VC}_{4,0})\frac{\psi}{\psi+1}\,. \tag{4.131}$$

Expanding in ψ *we see that the critical point at* $\xi = \mathrm{VC}_{2,0} - \mathrm{VC}_{4,0}$ *is now pushed to* ∞.

Indeed, we can compute and find

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{a_k^{(\rho_4)}}{\Gamma(k)}\xi^k &= \sum_{k=1}^{\infty}\frac{a_k^{(\rho_4)}}{\Gamma(k)}\Big((\mathrm{VC}_{2,0}-\mathrm{VC}_{4,0})\frac{\psi}{\psi+1}\Big)^k \\ &= (0.016346\cdots+0.13260\cdots i)\psi+\cdots \\ &\quad +(-1.5668\cdots-3.3495\cdots i)\times 10^{10}\times\psi^{125}+O(\psi^{126})\,.\end{aligned} \tag{4.132}$$

On the other hand

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{1}{2\pi i}\sum_{\ell=0}^{k-1}\frac{\Gamma(k-\ell)}{\Gamma(k)(\mathrm{VC}_{1,0}-\mathrm{VC}_{4,0})^{k-\ell}}a_\ell^{(\rho_1)}\Big((\mathrm{VC}_{2,0}-\mathrm{VC}_{4,0})\frac{\psi}{\psi+1}\Big)^k \\ &= (-0.0050461\cdots+0.014682\cdots i)\psi+\cdots \\ &\quad +(-1.5668\cdots-3.3495\cdots i)\times 10^{10}\times\psi^{125}+O(\psi^{126})\end{aligned} \tag{4.133}$$

Therefore, we see that there is a critical point at $\mathrm{VC}_{1,0}-\mathrm{VC}_{4,0}$ *and the numerics indicate the behaviour around the branch point. The other singularities can be similarly analysed. We can do this for the singularities closest to zero and find a the matrix*

$$(M(\rho,\phi))_{\rho,\phi} = \begin{pmatrix} 0 & ? & 1 & -1 \\ ? & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}. \tag{4.134}$$

To compute the question marks we need to compute more coefficients of the asymptotic series and perform a more detailed analysis. However, with the conjectural behaviour of the Stokes phenomenon these can be compute as seen in chapter 11.

These examples and the analysis was associated to the element

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,. \tag{4.135}$$

We can extend this to the case when we take

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,, \tag{4.136}$$

Indeed, again using the Poisson summation formula, assuming $A \in 2\mathbb{Z}_{>0}$ we have

$$\begin{aligned} f_{A,m,n}(\tilde{q}_\gamma) &= \frac{1}{(\tilde{q}_\gamma;\tilde{q}_\gamma)_\infty} \sum_{k\in\mathbb{Z}} \mathbf{e}(nk)\tilde{q}_\gamma^{\frac{A}{2}k^2+km}(\tilde{q}_\gamma^{k+1};\tilde{q}_\gamma)_\infty \\ &= \frac{1}{(\tilde{q}_\gamma;\tilde{q}_\gamma)_\infty} \sum_{r=0}^{|c|-1}\sum_{k\in\mathbb{Z}} \mathbf{e}(n(k|c|+r))\tilde{q}_\gamma^{\frac{A}{2}(k|c|+r)^2+(k|c|+r)m}(\tilde{q}_\gamma^{k|c|+r+1};\tilde{q}_\gamma)_\infty \\ &= \frac{1}{(\tilde{q}_\gamma;\tilde{q}_\gamma)_\infty} \sum_{r=0}^{|c|-1} \mathbf{e}(nr)\tilde{q}_\gamma^{\frac{A}{2}r^2+rm}\sum_{k\in\mathbb{Z}} \mathbf{e}(nk|c|)\tilde{q}_\gamma^{\frac{A}{2}k^2c^2+Ak|c|r+k|c|m}(\tilde{q}_\gamma^{k|c|+r+1};\tilde{q}_\gamma)_\infty \\ &= \frac{1}{(\tilde{q}_\gamma;\tilde{q}_\gamma)_\infty} \sum_{r=0}^{|c|-1} \mathbf{e}(nr)\tilde{q}_\gamma^{\frac{A}{2}r^2+rm}\sum_{k\in\mathbb{Z}} \mathbf{e}(nk|c|)\mathbf{e}\Big(\frac{A}{2}k^2|c|\tilde{\tau}_\gamma + Akr\tilde{\tau}_\gamma + kma + km\tilde{\tau}_\gamma\Big)(\tilde{q}_\gamma^{r+1}\mathbf{e}(k\tilde{\tau}_\gamma);\tilde{q}_\gamma)_\infty \\ &= \frac{1}{(\tilde{q}_\gamma;\tilde{q}_\gamma)_\infty} \sum_{r=0}^{|c|-1} \mathbf{e}(nr)\tilde{q}_\gamma^{\frac{A}{2}r^2+rm}\sum_{\ell\in\mathbb{Z}} \int_{\mathbb{R}} \mathbf{e}\Big(\frac{A}{2}x^2|c|\tilde{\tau}_\gamma + (n|c|+ma+\ell)x + Ar\tilde{\tau}_\gamma x + m\tilde{\tau}_\gamma x\Big)(\tilde{q}_\gamma^{r+1}\mathbf{e}(x\tilde{\tau}_\gamma);\tilde{q}_\gamma)_\infty dx \\ &= \frac{1}{(\tilde{q}_\gamma;\tilde{q}_\gamma)_\infty} \sum_{r=0}^{|c|-1} \mathbf{e}(nr)\tilde{q}_\gamma^{\frac{A}{2}r^2+rm}\sum_{\ell\in\mathbb{Z}} \frac{1}{\tilde{\tau}_\gamma}\int_{\tilde{\tau}_\gamma\mathbb{R}} \mathbf{e}\Big(\frac{A}{2\tilde{\tau}_\gamma}x^2|c| + (n|c|+ma+\ell)\frac{x}{\tilde{\tau}_\gamma} + Arx + mx\Big)(\tilde{q}_\gamma^{r+1}\mathbf{e}(x);\tilde{q}_\gamma)_\infty dx \end{aligned} \tag{4.137}$$

Using the distributive property of the dilogarithm from equation (4.37), the critical points correspond to solutions of

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\frac{1}{|c|}\mathrm{Li}_2(\mathbf{e}(|c|x)) + \frac{A}{2}x^2|c| + \Big(n|c| + m|c|\frac{a}{c} + \ell\Big)x \\ &= -\frac{1}{2\pi i}\log(1-\mathbf{e}(|c|x)) + Ax|c| + n|c| + m|c|\frac{a}{c} + \ell\,. \end{aligned} \tag{4.138}$$

Again taking the exponential $X = \mathbf{e}(x)$ we find that

$$1 - X^{|c|} \;=\; \mathbf{e}(n|c| + m|c|\frac{a}{c})X^{A|c|}\,. \tag{4.139}$$

Many of the structures we explored for $\gamma = [0,-1;1,0]$ will also hold here and turn out to be the same. However, an important outcome of considering these series is that to each root of unity we can obtain such a series. The full series is of interest but most important for this thesis will the constant terms. We will compute these constant for the previous examples.

Example 40 ($A = 2, m \in \mathbb{Z}, n = 0, q = \mathbf{e}(1/c)$)**.** *This example is computed in Code 32. Consider*

$$\gamma \;=\; \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,. \tag{4.140}$$

Then we want the asymptotics of the integrals

$$\int_{\tilde{\tau}_\gamma\mathbb{R}} \mathbf{e}\Big(\frac{1}{\tilde{\tau}_\gamma}|c|x^2 + (m+\ell)\frac{x}{\tilde{\tau}_\gamma} + 2rx + mx\Big)(\tilde{q}_\gamma^{r+1}\mathbf{e}(x);\tilde{q}_\gamma)_\infty dx\,. \tag{4.141}$$

Applying the saddle point method and using

$$(\tilde{q}_\gamma^{r+1}\mathbf{e}(x);\tilde{q}_\gamma)_\infty \;\sim\; \exp\left(\frac{\mathrm{Li}_2\big(x^{|c|}\big)}{2\pi i|c|\tilde{\tau}_\gamma}\right)\prod_{s=0}^{c-1}\Big(1-\mathbf{e}\Big((r+1+s)\frac{1}{c}+x\Big)\Big)^{\frac{1}{2}-\frac{r+1+s}{|c|}}(1+O(\tilde{\tau}_\gamma))\,, \tag{4.142}$$

we find as $\tau \to \infty$ and for $X_2^{1/|c|} = \mathbf{e}(x_{2,0}/|c|)$ we have

$$\begin{aligned} f_{2,m,0}(\tilde{q}_\gamma) \;&\sim\; \mathbf{e}\Big(\frac{\mathrm{VC}_{2,0}}{(2\pi i)^2|c|\tilde{\tau}_\gamma}\Big)\frac{\prod_{\ell=1}^{|c|-1}\big(1-\mathbf{e}\big(\ell\frac{1}{c}\big)\big)^{\frac{\ell}{|c|}-\frac{1}{2}}}{\sqrt{|c|X_2/(1-X_2)+2|c|}} \\ &\quad\times\sum_{r=0}^{|c|-1}\frac{q^{r^2+rm}X_2^{(2r+m)/|c|}}{\prod_{s=0}^{|c|-1}(1-q^{r+s+1}X_2^{1/|c|})^{\frac{r+s+1}{|c|}-\frac{1}{2}}}(1+O(\tilde{\tau}_\gamma))\,. \end{aligned} \tag{4.143}$$

Then numerically we can explore and find that for all $N \in \mathbb{Z}$

$$\begin{aligned} f_{2,m,0}(\tilde{q}_\gamma) \;&\sim\; \widehat{\Phi}^{(\rho_2)}_{1/c,m}\big(\tilde{\tau}_\gamma + O(\tilde{\tau}_\gamma^{N+1})\big) \;=\; \mathbf{e}\Big(\frac{\mathrm{VC}_{2,0}}{(2\pi i)^2|c|\tilde{\tau}_\gamma}\Big)\frac{\prod_{\ell=1}^{|c|-1}\big(1-\mathbf{e}\big(\ell\frac{1}{c}\big)\big)^{\frac{\ell}{|c|}-\frac{1}{2}}}{\sqrt{|c|X_2/(1-X_2)+2|c|}} \\ &\quad\times\sum_{r=0}^{|c|-1}\frac{q^{r^2+rm}X_2^{(2r+m)/|c|}}{\prod_{s=0}^{|c|-1}(1-q^{r+s+1}X_2^{1/|c|})^{\frac{r+s+1}{|c|}-\frac{1}{2}}}\mathbf{e}\Big(\frac{\tilde{\tau}_\gamma}{60|c|}\Big)(1+O(\tilde{\tau}_\gamma^N)) \end{aligned} \tag{4.144}$$

These constants vanish for certain m when $c + 5\mathbb{Z} = 5\mathbb{Z}$ and we find exponentially small corrections again which we will return to in chapter 8. However, we can study the asymptotics the coefficients of these series as $c \to \infty$. We have for $q = \mathbf{e}(a/c)$

$$\Phi^{(\rho_j)}_{a/c,m}(0) \;=\; \frac{\prod_{\ell=1}^{|c|-1}\big(1-q^{\ell}\big)^{\frac{\ell}{|c|}-\frac{1}{2}}}{\sqrt{|c|X_j/(1-X_j)+2|c|}}\sum_{r=0}^{|c|-1}\frac{q^{r^2+rm}X_j^{(2r+m)/|c|}}{\prod_{s=0}^{|c|-1}(1-q^{r+s+1}X_j^{1/|c|})^{\frac{r+s+1}{|c|}-\frac{1}{2}}}\,. \tag{4.145}$$

Therefore, using the cocycle [86, Lem. 3.1] denoted λ_γ which has

$$\lambda_{[0,-1;1,0]}(x) \;=\; \frac{1}{\mathrm{denom}(x)\,\mathrm{numer}(x)}\,, \tag{4.146}$$

we have

$$\begin{aligned} \Phi^{(\rho_2)}_{-1/x,0}(0) \;&=\; \mathbf{e}\Big(\frac{-1}{60}x\Big)\Phi^{(\rho_2)}_{x,0}(0)\Phi^{(\rho_2)}_{1/1,0}(0)\mathbf{e}\Big(\frac{-1}{60x}\Big)\mathbf{e}\Big(\frac{-1}{60}\frac{1}{\mathrm{denom}(x)\,\mathrm{numer}(x)}\Big) \\ &\quad+\mathbf{e}\Big(\frac{11}{60}x\Big)\Phi^{(\rho_2)}_{x,1}(0)\Phi^{(\rho_1)}_{1/1,0}(0)\mathbf{e}\Big(\frac{-1}{60x}\Big)\mathbf{e}\Big(\frac{-1}{60}\frac{1}{\mathrm{denom}(x)\,\mathrm{numer}(x)}\Big)\,. \end{aligned} \tag{4.147}$$

Example 41 ($A = 4, m \in \mathbb{Z}, n = 0, c \in 1 + 2\mathbb{Z}, q = \mathbf{e}(2/c)$)**.** *Consider*

$$\gamma \;=\; \begin{pmatrix} 2 & 1 \\ c & (1+c)/2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\,. \tag{4.148}$$

Then we want the asymptotics of the integrals

$$\int_{\tilde{\tau}_\gamma\mathbb{R}} \mathbf{e}\Big(\frac{2}{\tilde{\tau}_\gamma}|c|x^2 + (2m+\ell)\frac{x}{\tilde{\tau}_\gamma} + 4rx + mx\Big)(\tilde{q}_\gamma^{r+1}\mathbf{e}(x);\tilde{q}_\gamma)_\infty dx\,. \tag{4.149}$$

Applying the saddle point method and using equation (4.142), we find as $\tau \to \infty$ just above the negative reals and for $X_4^{1/|c|} = \mathbf{e}(x_{4,0}/|c|)$ we have

$$\begin{aligned} f_{4,m,0}(\tilde{q}_\gamma) \;\sim\; \widehat{\Phi}^{(\rho_4)}_{2/c,m}\big(\tilde{\tau}_\gamma + O(\tilde{\tau}_\gamma^{N+1})\big) \;=\; \mathbf{e}\Big(\frac{\mathrm{VC}_{4,0}}{(2\pi i)^2|c|\tilde{\tau}_\gamma}\Big) \frac{\prod_{\ell=1}^{|c|-1}\big(1-q^\ell\big)^{\frac{\ell}{|c|}-\frac{1}{2}}}{\sqrt{|c|X_4/(1-X_4)+4|c|}} \\ \times \sum_{r=0}^{|c|-1} \frac{q^{2r^2+rm}X_4^{(4r+m)/|c|}}{\prod_{s=0}^{|c|-1}\big(1-q^{r+s+1}X_4^{1/|c|}\big)^{\frac{r+s+1}{|c|}-\frac{1}{2}}}\big(1+O(\tilde{\tau}_\gamma)\big)\,. \end{aligned} \tag{4.150}$$

We can also numerically calculate the asymptotics of the constant terms and find

$$\Phi^{(\rho_4)}_{1/x,0}(0) \;\sim\; \mathbf{e}\Big(\frac{V_{\rho_4}}{(2\pi i)^2\mathrm{denom}(x)\mathrm{numer}(x)}\Big)\Phi^{(\rho_4)}_{-x,1}(0)\widehat{\Phi}^{(\rho_4)}_{1/1,0}\Big(\frac{2\pi i}{x}\Big)\,. \tag{4.151}$$

4.8 The case of simple knots

Example 42 (Trefoil 3_1 [204])**.** *In Example 13, the coloured Jones polynomial of the trefoil 3_1 was given and we can therefore write down the Kashaev invariant*

$$\widetilde{J}_0(3_1;q) \;=\; \sum_{k=0}^{\infty} q^k(q;q)_k^2 \;=\; q^{-1}\sum_{k=0}^{\infty}(q^{-1};q^{-1})_k \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+3)/2}(q;q)_k\,. \tag{4.152}$$

This is an element of the Habiro ring of Section 5.8 so can be evaluated at roots of unity and formally with $q = \exp(\hbar)$ as

$$\widetilde{J}_0(3_1;\exp(\hbar)) \;=\; \Phi^{(\rho_0)}(3_1;-\hbar) \;=\; 1+\hbar^2+2\hbar^3+\frac{73}{12}\hbar^4+\frac{43}{2}\hbar^5+\frac{31861}{360}\hbar^6+\cdots\,. \tag{4.153}$$

Now applying the methods of Section 3.3 we can calculate numerically that for $n \in \mathbb{Z}$ as $n \to \infty$

$$\widetilde{J}_0(3_1;\mathbf{e}(-1/n)) \;\sim\; \mathbf{e}(1/8)\, n^{3/2}\mathbf{e}\Big(\frac{23}{24}n\Big)\mathbf{e}\Big(\frac{23}{24n}\Big) + \Phi^{(\rho_0)}\Big(3_1;\frac{-2\pi i}{n}\Big)\,. \tag{4.154}$$

One can compute the leading order behaviour as was given [204]

$$a_k^{(\rho_0)} \;\sim\; \sqrt{2\pi}\sum_{\ell=0}^{\infty}\frac{\Gamma(k-\ell+3/2)}{\ell!(\pi^2/6)}\Big(-\frac{1}{24}\Big)^\ell\,. \tag{4.155}$$

In the same paper, Zagier gave q–series that whose radial limits matched those of the series above. In particular, the strange identity of the title refers to the fact that for[5]

$$f(q) \;=\; -\frac{1}{2}\sum_{n=0}^{\infty} n\Big(\frac{12}{n}\Big)q^{(n^2-1)/24}\,, \tag{4.156}$$

[5] See, for example, [176, Def. 3.2] for the definition of the Legendre symbol.

as $q \to 1$ radially we have

$$f\big(\mathbf{e}(-1/\tau)\big) \;\sim\; \Phi^{(\rho_0)}(3_1; 2\pi i/n)\,. \tag{4.157}$$

Example 43 (Figure eight knot)**.** *This example is computed in Code 35. In Example 13, the coloured Jones polynomial of the figure eight knot 4_1 was given and we can therefore write down the Kashaev invariant as we did in equation* (2.62)

$$\widetilde{J}_0(4_1;q) \;=\; \sum_{k=0}^{\infty} q^{-k(k+1)/2}(q;q)_k^2\,. \tag{4.158}$$

Recently, [85] studied the asymptotics of q–series related to 4_1. Similarly, great detail was given in [86] on the asymptotics of functions at roots of unity. We will summarise some of these results and include the computations that lead to the series introduced in [70]. Firstly, we have

$$\mathrm{VC}^{(\rho_1)} \;=\; = \; 2.0299\cdots i \tag{4.159}$$

The asymptotic series computed to first order in [5] and to many orders in [208], and subsequentially by others for the Kashaev invariant is given by

$$\begin{aligned}\widetilde{J}_0(4_1;\mathbf{e}(-1/n)) \;&=\; \widehat{\Phi}^{(\rho_1)}(2\pi i/n)\\ \;&=\; \mathbf{e}\Big(\frac{\mathrm{VC}^{(\rho_1)}}{(2\pi i)^2}n\Big)\frac{\mathbf{e}(1/8)}{\sqrt{\sqrt{-3}}}\Big(1-\frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{n}+\frac{697}{1152\sqrt{-3}^6}\Big(\frac{2\pi i}{n}\Big)^2+\cdots\Big)\,.\end{aligned} \tag{4.160}$$

The various properties of the asymptotic series are considered in [86]. For example, the analogue of the matrix of equation 4.134 is given in [86, Eq. 39] as

$$(M(\rho,\phi))_{\rho,\phi} \;=\; \begin{pmatrix}0 & 1 & -1\\ 0 & 0 & -3\\ 0 & 3 & 0\end{pmatrix}\,. \tag{4.161}$$

They also compute the constants of the asymptotic series and give their asymptotic in Section [86, Sec. 3.1]. However, we plot the poles and zeros of a Padé approximate in Figure 4.8. Around 2010 somewhat by chance and a little help from `grep`*, Garoufalidis and Zagier made a remarkable discovery [85]. They found that this series occurred in the asymptotics of a q–series not at the time known to be related to the figure eight knot. Taking*

$$g(q) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2}}{(q;q)_k^2} \;=\; 1-q-2q^2-2q^3-2q^4+q^6+5q^7+7q^8+11q^9+\cdots\,, \tag{4.162}$$

they found that as τ vertically approaches 0

$$g(\mathbf{e}(-1/\tau)) \;\sim\; \frac{1}{\sqrt{\tau}}\big(\widehat{\Phi}^{(\rho_1)}(2\pi i/\tau)+\widehat{\Phi}^{(\rho_2)}(2\pi i/\tau)\big)\,, \tag{4.163}$$

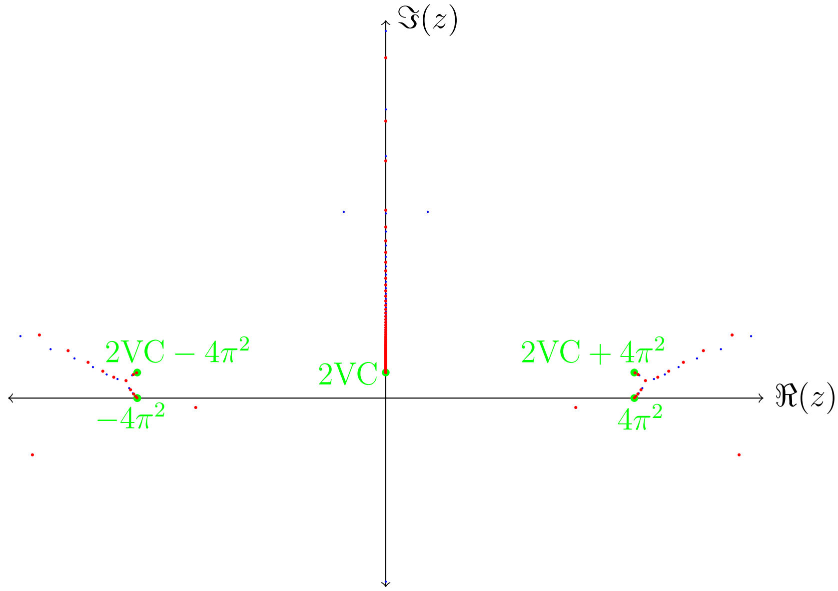

Figure 4.8: Plots of poles and zeros of the [145/145]–Padé approximate of $\Phi^{(\rho_2)}$.

where we note that $\widehat{\Phi}^{(\rho_2)}(\hbar) = -i\widehat{\Phi}^{(\rho_1)}(-\hbar)$*. This can be thought of as some kind of analogue of a strange identity for the figure eight knot. Then Garoufalidis and Kashaev factorised the state integrals of [8] in [73]. This led to the series* $g(q)$ *and an additional series*

$$\begin{aligned} G(q) &= \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_k^2}\Big(-4G_1(q) + 2\sum_{\ell=1}^{k}\frac{1+q^j}{1-q^j}\Big) \\ &= 1 - 7q - 14q^2 - 8q^3 - 2q^4 + 30q^5 + 43q^6 + 95q^7 + 109q^8 + 137q^9 + \cdots . \end{aligned} \tag{4.164}$$

where

$$G_1(q) = -\frac{1}{4} + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 4q^6 + 2q^7 + 4q^8 + 3q^9 + \cdots , \tag{4.165}$$

is the first Eisenstein series discussed and efficiently computed in 7.1. Garoufalidis and Zagier [85] then observed that

$$G(\mathbf{e}(-1/\tau)) \sim \sqrt{\tau}\big(\widehat{\Phi}^{(\rho_1)}(2\pi i/\tau) - \widehat{\Phi}^{(\rho_2)}(2\pi i/\tau)\big) . \tag{4.166}$$

They then consider various other properties of the asymptotics as we did for the $A = 4$ *Nahm sum of Example 39. While the series* $g(q), G(q)$ *have asymptotics that see* $\widehat{\Phi}^{(\rho_1)}, \widehat{\Phi}^{(\rho_2)}$*, they do not have any asymptotics that see trivial connection* i.e. *the series* $\widehat{\Phi}^{(\rho_2)}$ *of equation* (2.69)*. The important point, which was much of the initial motivation for the next part III, is that g* *and G are both special values of a rank two q–holonomic system. Their asymptotics therefore also come as part of a q–holonomic system where $q = e^{\hbar}$. However, the Kashaev invariant is part of an inhomogeneous rank two system and therefore is in an extension. Therefore, one must find a q–series that solves an associated inhomogenous rank two q–difference equation. The series that does this appears implicitly in [85] however was not noticed until this work. In particular, take*

$$\mathfrak{G}(q) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_k^2}\bigg(\frac{1}{8}\Big(-4G_1(q)+2\sum_{\ell=1}^{k}\frac{1+q^j}{1-q^j}\Big)^2-\frac{1}{24}+\sum_{\ell=1}^{k}\frac{q^\ell}{(1-q^\ell)^2}\bigg)\,. \tag{4.167}$$

Then one can observe that

$$\mathfrak{G}(q)(\mathbf{e}(-1/\tau)) \;\sim\; \widehat{\Phi}^{(\rho_0)}(-2\pi i/\tau)+\frac{\tau^{3/2}}{12}\widehat{\Phi}^{(\rho_1)}(2\pi i/\tau)+\frac{\tau^{3/2}}{12}\widehat{\Phi}^{(\rho_2)}(2\pi i/\tau)\,. \tag{4.168}$$

4.9 Half surgery on the figure eight

Example 44 $(4_1(-1,2))$**.** *This example is computed in Code 36. We will explore the asymptotics of the WRT invariant of this manifold. This was given in Example 16 so that for $q \in \mu\!\!\!/$ we have*

$$(1-q)\text{Ш}(q) \;=\; \sum_{0\le\ell\le k}(-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)}\frac{(q;q)_{2k+1}}{(q;q)_\ell(q;q)_{k-\ell}}\,. \tag{4.169}$$

We have already seen aspects of the leading asymptotics of this function in Section 2.5. In particular, the difficult case is when $q = \mathbf{e}(1/N)$ where $N \in \mathbb{Z}$ and there is polynomial behaviour. Understanding this in our case will be best done in Section 8.3. Regardless, we can use the methods of Section 3.2 to calculate the asymptotics numerically. Before this some data associated to the manifold is given. The trace field is the number field of type $[5,1]$ with discriminant -7215127 (a prime number), generated by a root of $p(x)$, where

$$p(\xi) \;=\; \xi^7-\xi^6-2\xi^5+6\xi^4-11\xi^3+6\xi^2+3\xi-1\,. \tag{4.170}$$

Then let

$$\delta \;=\; -12\xi^6+15\xi^5+31\xi^4-74\xi^3+133\xi^2-66\xi-74\,. \tag{4.171}$$

We index the solutions and therefore in particular the connections or critical points of a stationary phase approximation and the values of δ the one-loop and the critical values or the complexified volumes by

$$\begin{array}{lll}
\mathrm{VC}_1 = 20.297\dots, & \delta_1 = -11.578\dots, & \xi_1 = -2.2411\dots\\
\mathrm{VC}_2 = -6.7857\dots, & \delta_2 = -12.636\dots & \xi_2 = -0.43760\dots\\
\mathrm{VC}_3 = -0.11620\dots, & \delta_3 = -83.275\dots, & \xi_3 = 0.25599\dots\\
\mathrm{VC}_4 = 9.2837\dots, & \delta_4 = -7.0205\dots, & \xi_4 = 1.3348\dots\\
\mathrm{VC}_5 = 2.1292\dots, & \delta_5 = -5.3937\dots, & \xi_5 = 1.3483\dots\\
\mathrm{VC}_6 = 4.8678\dots - i1.3985\dots, & \delta_6 = 3.9517\dots - i0.15252\dots, & \xi_6 = 0.36981\dots - i1.4410\dots\\
\mathrm{VC}_7 = 4.8678\dots + i1.3985\dots, & \delta_7 = 3.9517\dots + i0.15252\dots, & \xi_7 = 0.36981\dots + i1.4410\dots.
\end{array} \tag{4.172}$$

Now we can apply the oscillatory asymptotic method of Section 3.3 to compute the asymptotics when $q = \mathbf{e}(-1/N)$. *Applying the method at* $N = 500$ *which needs the values at* 500 *to say* 1010 *and with parameter* $\mathbf{e}(0.1), \mathbf{e}(0.3), \mathbf{e}(0.4), \mathbf{e}(0.7)$ *we find exponential terms respectfully*

$$\begin{aligned} 0.94313\cdots + 0.33243\cdots i\,, \quad 0.093109\cdots + 0.99566\cdots i\,,\\ -0.99606\cdots - 0.088638\cdots i\,, \quad 0.47135\cdots - 0.88194\cdots i\,. \end{aligned} \tag{4.173}$$

These agree with the values

$$\exp\Big(-\frac{\mathrm{VC}_5}{2\pi i}\Big), \quad \exp\Big(-\frac{\mathrm{VC}_4}{2\pi i}\Big), \quad \exp\Big(-\frac{\mathrm{VC}_1}{2\pi i}\Big), \quad \exp\Big(-\frac{\mathrm{VC}_2}{2\pi i}\Big), \tag{4.174}$$

as predicted by Witten's asymptotic expansion conjecture as these are the SU(2) *Chern–Simons values with the exception of the missing* 1. *With more data and precision this can also be seen with the method. Similarly, we can compute the asymptotics when* $\mathbf{e}(-1/(N+1/2))$. *This has leading asymptotics*

$$(0.89275\cdots + 0.87392\cdots i)^k k^{0.50000\cdots}(-0.41564\cdots - 1.0444\cdots i)\,. \tag{4.175}$$

This can be recognised as

$$\exp\Big(-\frac{\mathrm{VC}_6}{2\pi i}(k+1/2)\Big)\sqrt{k+1/2}\frac{2\mathbf{e}(1/8)}{\sqrt{\delta_6}}\Big(1 + A_1^{(6)}\frac{-2\pi i}{k+1/2} + A_2^{(6)}\Big(\frac{-2\pi i}{k+1/2}\Big)^2 + \cdots\Big) \tag{4.176}$$

The next few coefficients can be computed as

$$\begin{pmatrix} 24\delta_6^3 A_1^{(6)} \\ 1152\delta_6^6 A_2^{(6)} \end{pmatrix} = \begin{pmatrix} 1497746 & 3014838521575 \\ 1345119 & 2732414541176 \\ -3675733 & -7414786842283 \\ 2082815 & 4197826806919 \\ -839488 & -1690529009777 \\ -283405 & -574198051621 \\ 383432 & 771765277669 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 \\ \xi_7 \\ \xi_7^2 \\ \xi_7^3 \\ \xi_7^4 \\ \xi_7^5 \\ \xi_7^6 \end{pmatrix} = \begin{pmatrix} -158.75\cdots + i57.225\ldots \\ 84862.\cdots - i924.76\ldots \end{pmatrix} \tag{4.177}$$

Before considering the asymptotics of the coefficients, we can also consider an associated q*–series. We will discuss its calculation in more detail in Section 6.9. This series was first calculated in [92] however we use the following formula proved in Proposition 10,*

$$\widehat{Z}(q) = \sum_{0\le k\le \ell}(-1)^{k+\ell}q^{\frac{1}{2}3k(k+1)+\frac{1}{2}\ell(\ell+1)-k}\frac{(q;q)_\ell}{(q;q)_{2k}(q;q)_{\ell-k}}\,. \tag{4.178}$$

The asymptotics of this functions will depend on the angle at which q *approaches* 1. *If this is done radially then we find that*

$$\widehat{Z}(q) \sim \exp\Big(-\frac{\mathrm{VC}_3}{2\pi i}\tau\Big)\sqrt{\tau}\frac{\mathbf{e}(1/8)}{\sqrt{\delta_3}}\Big(1 + A_1^{(3)}\frac{-2\pi i}{\tau} + A_2^{(3)}\Big(\frac{-2\pi i}{\tau}\Big)^2 + \cdots\Big) \tag{4.179}$$

Interestingly enough the $\mathrm{SL}_2(\mathbb{R})$ *connection has appeared. This means that we can compute numerically all the Galois conjugates of the elements* A_k*. This is of great aid in numerical computations as we can use this to guess these numbers, which now involves recognising integers once we have removed the denominators* $D_n\delta^{3n}$*. With the values at* $k = 100000, 101000$ *using these three different sets of asymptotic we can recognise around one hundred and ten coefficients. The last recognisable number with the data I have used is* A_{114}*, and* $A_{114}\delta^{342}D_{114}$ *is in the ring of integers and in this lattice the size of the integers that appear are* 10^{1041}*, which means further computations one needs a large precision.*

With these values we can compute the asymptotics of the coefficients and using optimal truncation can find the first few sub–leading terms and find numerically that for $a_k = A_k/\sqrt{\delta}$

$$a_k^{(j)} \sim \sum_{\ell\neq j} \frac{M_{j,\ell}}{2\pi i} \sum_{\ell\geq 0} \frac{\Gamma(k-\ell)}{(\mathrm{VC}_j - \mathrm{VC}_\ell)^{k-\ell}} a_\ell^{(\ell)} \tag{4.180}$$

where

$$\begin{pmatrix} ? & ? & ? & 0 & ? & -1 & 1 \\ ? & ? & ? & ? & 0 & -1 & 1 \\ 0 & ? & ? & ? & ? & -1 & 1 \\ ? & ? & ? & ? & 0 & -1 & 1 \\ ? & ? & ? & ? & ? & -1 & 1 \\ ? & ? & ? & 1 & 1 & 0 & 0 \\ ? & ? & ? & -1 & -1 & 0 & 0 \end{pmatrix} \tag{4.181}$$

Finally, for the trivial connection

$$(1-e^{\hbar})\text{Ш}(e^{\hbar}) \;=\; \sum_{k=0}^{\infty} a_k^{(0)}\hbar^k \;=\; -\hbar - \frac{25}{2}\hbar^2 - \frac{1621}{6}\hbar^3 - \frac{195601}{24}\hbar^4 + \cdots\,, \tag{4.182}$$

numerically we observe that

$$a_k^{(0)} \;\sim\; \sum_{\ell=1}^{7} \frac{M_{0,\ell}}{-i\sqrt{2\pi}} \sum_{\ell\geq 0} \frac{\Gamma(k-\ell+1/2)}{(0-\mathrm{VC}_\ell)^{k-\ell+1/2}} a_\ell^{(\ell)}\,. \tag{4.183}$$

where

$$\begin{pmatrix} ? & ? & 1 & ? & 0 & -1 & 1 \end{pmatrix} \tag{4.184}$$

The leading order of this expansion was noticed originally by Stavros Garoufalidis in a note he gave me when I started work on this subject. These matrices can be explored in more detail using conformal transformations however we will give a more detailed conjecture on their structure in chapter 11.

In another direction, we can of course consider the asymptotics as q tends to any roots of unity and we find additional series there. Again we are interested in the constants of these series. To calculate these we can use the techniques of Section 9. We use the solutions to

$$\begin{aligned} 0 &= (1-X_2)(1-X_1^{-1}X_2) - X_1\,, \\ 0 &= (1-X_1^2)^2 - X_1^2X_2(1-X_1^{-1}X_2)\,. \end{aligned} \tag{4.185}$$

These have solutions with respect to our generator of the trace field

$$\begin{aligned} X_{1,j} &= -3+11\xi_j+20\xi_j^2-15\xi_j^3+6\xi_j^4-2\xi_j^5-4\xi_j^6\,, \\ X_{2,j} &= -9+19\xi_j+76\xi_j^2-52\xi_j^3+20\xi_j^4-4\xi_j^5-13\xi_j^6\,. \end{aligned} \tag{4.186}$$

Then letting

$$\Delta = -257+806\xi+947\xi^2-749\xi^3+331\xi^4-133\xi^5-213\xi^6\,, \tag{4.187}$$

the constants of the asymptotic series can be computed as follows

$$\begin{aligned} \Phi^{(\rho_i)}_{-1/x,0}(0) &= \frac{-i}{c\varepsilon(q)}\sqrt{\frac{(1-X_1^2)^2(1-X_2)(1-X_1^{-1}X_2)}{\Delta}} \\ \times \sum_{k,\ell\in\mathbb{Z}/c\mathbb{Z}} & \frac{q^{k^2+k\ell-mk+\ell}X_1^{\frac{2k+\ell-m}{c}}X_2^{\frac{k+1}{c}}\prod_{i=0}^{c-1}(1-q^{i+1+\ell-k}X_1^{-1/c}X_2^{1/c})^{-(i+1+\ell-k)/c-1/2}}{\prod_{i=0}^{c-1}(1-q^{i+1+\ell}X_2^{1/c})^{(i+1+\ell)/c-1/2}\prod_{i=0}^{c-1}(1-q^{i+1+2k}X_1^{2/c})^{(i+1+2k)/c-1/2}}\,. \end{aligned} \tag{4.188}$$

Then we can check the asymptotics of these coefficients to find

$$\Phi^{(\rho_i)}_{-1/x,m}(0) \sim \mathbf{e}\Big(-\frac{V_j}{(2\pi i)^2\mathrm{denom}(x)\mathrm{numer}(x)}\Big)\Phi^{(\rho_j)}_{x,0}(0)\widehat{\Phi}^{(\rho_6)}_{1/1,m}\Big(-\frac{2\pi i}{x}\Big)\,. \tag{4.189}$$

Part III

q–difference equations

Chapter 5

How to solve a q–difference equation

Linear differential equations have a long history motivated from trying to understand real world physics. From the mathematical side the first examples we study are the equations satisfied by the logarithm and exponential. Besides these basic functions, Gauss's hypergeometric function ${}_2F_1$ and its second order differential equation is the next interesting example. This function was then understood from a global perspective by Riemann who used analytic continuation to define monodromy. Fuchs then continued this study to more complicated equations with what we call regular singularities. Then Frobenius put this theory in a definite form giving an algorithm to construct solutions. This can then be extended to the case of irregular singularities, where the associated Newton polygon has slopes. This leads to exponential singularities. This leads to divergent solutions, the need for resummation and Stokes phenomenon.

This theory of differential equations can be completely carried over into the world of q–difference equations. The analogue of Gauss's hypergeometric function was introduced by Heine. Basically any property of Gauss's function will have an analogue for Heine's function. For example, Barne's contour integral can be replace by Watson's. Frobenius's theory can be used to similarly construct solutions to q–difference equations. In fact, there are some aspects that are better behaved in the world of q–difference equations. However, they already behave in a more global way. Indeed, the variable of such equations naturally lives on $\mathbb{C}^\times$. Often one is interested in constructing solutions around 0 and ∞. Here irregular singularities lead to θ–functions and similarly divergent solutions. These divergent solutions can be resummed to construct meromorphic solutions to q–difference equations. Here the monodromy is stored in a matrix that takes a basis around 0 to a basis around ∞ called the monodromy matrix or Birkoff matrix.

There is also a striking difference to differential equations and that is in the description of q. In constructing some kind of solutions we need to know what q is. For examples we could have $|q| \neq 1$ which needs to some kind of q–series solutions. We could have q a root of unity or we could have $q = e^{\hbar}$ where $\hbar$ is a formal variable. This leads to many different solutions depending on the form that q takes. It also depends on the variable of the q–difference equation. If this variable is of the form q^m then this can alter the form of the algorithms that we wish to apply. The natural question that this then leads to is how all of these solutions are related. This is most beautifully described by the state integrals of part V, where functions not only satisfy a q–difference equation but an uncoupled $\tilde{q}$–difference equation.

5.1 Linear q–difference equations

The theory of q–difference equations can be developed analogously to that of linear differential equations. *Homogeneous linear q-difference equations* can be described as modules of the q–*Weyl algebra* denoted $\mathcal{W}$. This is defined as the algebra generated by[1] x, σ_x such that

$$\sigma_x x \;=\; q x \sigma_x \tag{5.1}$$

where q is central and will often just be in $\mathbb{C}^\times$. Notice that equation (5.1) is homogenous in σ_x and x as opposed to the analogous equation (3.51) for the derivative. This algebra does come with the extra structure of a natural involution $\iota : \mathcal{W} \to \mathcal{W}$ defined so that for $f(x,q) \in \mathbb{Q}(x,q)$

$$\iota(f(x,q)) \;=\; f(x,q^{-1})\,, \quad \text{and} \quad \iota(f(x,q)\sigma^j) \;=\; f(x,q^{-1})\sigma^{-j}\,. \tag{5.2}$$

Extending this to an algebra homomorphism is well defined as

$$\iota(\sigma f(x,q)) \;=\; \iota(f(qx,q)\sigma) \;=\; f(q^{-1}x,q^{-1})\sigma^{-1} \;=\; \sigma^{-1} f(x,q^{-1}) \;=\; \iota(\sigma)\iota(f(x,q))\,. \tag{5.3}$$

This algebra acts on smooth functions in variables x, q and on formal power series in x with coefficients functions in q via

$$\begin{aligned} (\sigma_x f)(x;q) \;&=\; f(qx;q), \qquad \sigma_x \sum_k a_k(q) x^k \;=\; \sum_k q^k a_k(q) x^k, \\ (xf)(x;q) \;&=\; x f(x;q), \qquad x \sum_k a_k(q) x^k \;=\; \sum_k a_{k-1}(q) x^k. \end{aligned} \tag{5.4}$$

To see this action is well defined notice that

$$(\sigma_x x f)(x;q) \;=\; (xf)(qx;q) \;=\; qxf(qx;q) \;=\; qx(\sigma_x f)(x;q) \;=\; (qx\sigma_x f)(x;q)\,. \tag{5.5}$$

For a given problem, we are interested in certain modules. These will often be generated by functions or power series satisfying equations of the form

$$\sum_{i=0}^{N} \alpha_i(x)\sigma_x^i f \;=\; 0, \quad \sum_k \sum_{i=0}^{N} \sum_{j=0}^{N_i} \alpha_{ij}(q) q^{i(k-j)} a_{k-j}(q) x^k \;=\; 0, \tag{5.6}$$

[1] To be more precise one can replace x by $f \in \mathbb{Q}(x,q)$ with the relation $\sigma f(x,q) = f(qx,q)\sigma$.

for some $a_i(x;q) = \sum_{j=0}^{M_i} a_{ij}(q)x^j$. This generator is referred to as a *cyclic vector*. This can naturally be extended to many variables.

There is a natural map from the q–Weyl algebra to two variable polynomial ring. In particular, if we formally set $q = 1$ then the relation between x and σ_x simply becomes commutation. If we have a module defined by a cyclic vector then the minimal element of the q–Weyl algebra that annihilates the cyclic vector will be set to a polynomial and the vanishing set of that polynomial will be called the *characteristic variety* and an invariant of the module.

We will see that there are many examples where we would like to specialise x to specific values. Often these will simply be $x = q^b$. These special values will be related to special points on the characteristic variety, which can be important for the links with K–theory 7.4. For interesting q–difference equations, x will not be the elliptic variable of a Jacobi form. However, at special points in x the object we get left with can have very strong modular properties for Example 57.

We see that in this case the Weyl algebra acts on functions in b, q as follows

$$(\sigma_x f)_b(q) \;=\; f_{b+1}(q)\,, \quad \text{and} \quad (xf)_b(q) \;=\; q^b f_b(q)\,. \tag{5.7}$$

This can be related to the action on power series in x, as seen in equation (5.4). This is related to the Fourier transform as discussed in [78, Sec. 5.1].

Remark 15. *One can always take the solutions with continuous variable x and construct their specialisation by taking combinations with elliptic functions so that the expansion around $x = q^b e^\epsilon$ at lowest order gives rise to solutions to the discrete equations.*

Given a basis $f^{(i)}$ of solutions to the q–difference equation (5.6), one can construct the *Wronskian*

$$W(f^{(1)}, \cdots, f^{(N)}) \;=\; \begin{pmatrix} f^{(1)} & \cdots & f^{(N)} \\ \vdots & \cdots & \vdots \\ \sigma_x^{n-1} f^{(1)} & \cdots & \sigma_x^{n-1} f^{(N)} \end{pmatrix} \tag{5.8}$$

which satisfies first order equation

$$\sigma_x W(f^{(1)}, \cdots, f^{(N)}) \;=\; AW(f^{(1)}, \cdots, f^{(N)}) \tag{5.9}$$

where

$$A(x;q) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -\frac{\alpha_0(x;q)}{\alpha_n(x;q)} & -\frac{\alpha_1(x;q)}{\alpha_n(x;q)} & -\frac{\alpha_2(x;q)}{\alpha_n(x;q)} & -\frac{\alpha_3(x;q)}{\alpha_n(x;q)} & \cdots & -\frac{\alpha_{n-2}(x;q)}{\alpha_n(x;q)} & -\frac{\alpha_{n-1}(x;q)}{\alpha_n(x;q)} \end{pmatrix} \tag{5.10}$$

is the so called *companion matrix*. Suppose we have two bases of solutions $f^{(i)}, g^{(i)}$ then letting $U = W(f^{(1)}, \cdots, f^{(N)})$, $V = W(g^{(1)}, \cdots, g^{(N)})$

$$\sigma_x(V^{-1}U) \;=\; V^{-1}A^{-1}AU \;=\; V^{-1}U. \tag{5.11}$$

Therefore, we see that $V^{-1}U$ satisfies the following equation;

$$(\sigma_x - 1)f \;=\; 0. \tag{5.12}$$

Solutions to equation (5.12) are functions f with $f(qx;q) = f(x;q)$ *i.e.* elliptic in x with respect to q. Therefore, given solutions to a q–difference equation we are free to take linear combinations with coefficients elliptic functions in x to construct more solutions. These elliptic functions are sometimes referred to as *pseudo-constants*. Notice that for $x = q^m$ we see that the elliptic functions will reduce to functions constant in m. Even in this case, in simply finding solutions, we have the freedom to multiply by functions in q.

Suppose we have a module of the q–Weyl algebra with basis $f = (f_1, \cdots f_N)^{\mathrm{T}}$. Then there is some $A \in \mathrm{GL}_N(\mathbb{Q}(x,q))$ such that $\sigma_x f \;=\; Af$. Then for $P \in \mathrm{GL}_N(\mathbb{Q}(x,q))$ we can take a new basis $g = Pf$ which satisfies $\sigma_x g \;=\; Bg$, where

$$B = (\sigma P)AP^{-1}\,. \tag{5.13}$$

Conversely, if we have two modules with bases f, g and A, B such that $\sigma_x f \;=\; Af$ and $\sigma_x g \;=\; Bf$, then the two modules are equivalent exactly when there exists a P which relates A and B as in equation (5.13). This equation is numerically testable. Indeed, assuming P has polynomial entries the equation

$$PB \;=\; (\sigma P)A \tag{5.14}$$

is linear in the coefficients of the polynomial in P.

There are two natural duals to these modules over the q–Weyl algebra. These were discussed in [82] and we will briefly mention them again here.

Definition 13 (Duals of modules of the q–Weyl algebra)**.** *Suppose that M is a module of the q–Weyl algebra. Then we define $M^\vee$ to be the module of the q–Weyl algebra with $\mathbb{Q}(x,q)$–module*

$$M^\vee \;=\; \mathrm{Hom}_{\mathbb{Q}(x,q)}(M, \mathbb{Q}(x,q))\,, \tag{5.15}$$

with action

$$\begin{aligned}(q\cdot f^\vee)(g) &= qf^\vee(g)\\ (x\cdot f^\vee)(g) &= xf^\vee(g)\\ (\sigma\cdot f^\vee)(g) &= \sigma\big(f^\vee(\sigma^{-1}g)\big)\,.\end{aligned} \tag{5.16}$$

where the inner σ^{-1} acts on M and the outer σ acts on $\mathbb{Q}(x,q)$. We also define $M^\wedge$ to be the module of the q–Weyl algebra twisted by the canonical involution ι so that it has $\mathbb{Q}$–vector space

$$M^\wedge \;=\; M\,, \tag{5.17}$$

with action

$$\begin{aligned} q\cdot f^\wedge &= q^{-1}f^\wedge \\ x\cdot f^\wedge &= xf^\wedge \\ \sigma\cdot f^\wedge &= \sigma^{-1}f^\wedge\,, \end{aligned} \tag{5.18}$$

where the action on the RHS is the action in M. In other words, we have $M^\wedge = \mathcal{W}\otimes_\iota M$.

One can check that this is well defined. The following lemma describes what happens when M is endowed with a basis.

Lemma 6. *Suppose that M has a basis $f_1,\dots,f_r$ over $\mathbb{Q}(x,q)$ then $\{f_j^\vee : j=1,\cdots,r\}$ such that $f_j^\vee(f_i)=\delta_{i,j}$ is a basis of $M^\vee$ and $\{f_i^\wedge : j=1,\dots,r\}$ such that $f_i^\wedge = 1\otimes_\iota f_i$ is a basis of $M^\wedge$. Moreover, if*

$$\sigma\begin{pmatrix} f_1\\ \vdots\\ f_r\end{pmatrix} \;=\; A(x,q)\begin{pmatrix} f_1\\ \vdots\\ f_r\end{pmatrix}\,, \tag{5.19}$$

then

$$\sigma\begin{pmatrix} f_1^\vee\\ \vdots\\ f_r^\vee\end{pmatrix} \;=\; A(x,q)^{-\mathrm{T}}\begin{pmatrix} f_1^\vee\\ \vdots\\ f_r^\vee\end{pmatrix}\,,\quad \textit{and}\quad \sigma\begin{pmatrix} f_1^\wedge\\ \vdots\\ f_r^\wedge\end{pmatrix} \;=\; A(qx,q^{-1})^{-1}\begin{pmatrix} f_1^\wedge\\ \vdots\\ f_r^\wedge\end{pmatrix}\,. \tag{5.20}$$

These dualities will be important in relation to proving quantum modularity. Next, we will consider some simple rank one modules.

5.2 The Pochhammer symbol and the θ–function

Definition 14 (Pochhammer symbol)**.** *For $x\in\mathbb{C}$ and $|q|<1$ let*

$$(x;q)_\infty \;=\; \prod_{j=0}^{\infty}(1-xq^j). \tag{5.21}$$

Then define

$$(x;q)_n \;=\; \frac{(x;q)_\infty}{(xq^n;q)_\infty}, \tag{5.22}$$

which simplifies for $n\in\mathbb{Z}$ to

$$(x;q)_n \;=\; \begin{cases} \prod_{i=0}^{n-1}(1-xq^j) & \textit{if } n\ge 0\\ \prod_{i=1}^{-n}(1-xq^{-j})^{-1} & \textit{if } n<0 \end{cases}\,. \tag{5.23}$$

The Pochhammer symbol satisfies a first order q–difference equation

$$(1-x)f(qx;q) - f(x;q) \;=\; 0\,. \tag{5.24}$$

We can calculate the Taylor series around $x = 0$ of $(x;q)_\infty$ using the q–difference equation and the initial condition at $x = 0$. This method is one of the main tools when proving identities.

Lemma 7 (q–binomial theorem)**.** *[206, Prop. 2] For $x \in \mathbb{C}$ and $|q| < 1$*

$$(x;q)_\infty \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(k-1)/2}}{(q;q)_k}x^k. \tag{5.25}$$

For $|x| < 1, |q| < 1$ we have

$$\frac{1}{(x;q)_\infty} \;=\; \sum_{k=0}^{\infty} \frac{1}{(q;q)_k}x^k. \tag{5.26}$$

Proof. Let

$$\alpha_k(q) \;=\; (-1)^k \frac{q^{k(k-1)/2}}{(q;q)_k}, \quad \text{and} \quad f(x;q) \;=\; \sum_{k=0}^{\infty} \alpha_k(q)x^k. \tag{5.27}$$

The quotient of consecutive terms in the sum and $t_k(qa;q)$ are given by

$$\frac{\alpha_{k+1}(q)}{\alpha_k(q)} \;=\; \frac{-q^k}{1-q^{k+1}}, \tag{5.28}$$

which shows that

$$f(x;q) - f(qx;q) \;=\; \sum_{k=0}^{\infty}(1-q^{k+1})\alpha_{k+1}(q)x^k \;=\; -x\sum_{k=0}^{\infty} q^k\alpha_k(x;q) \;=\; -xf(qx;q). \tag{5.29}$$

Therefore,

$$\begin{aligned} (1-x)f(qx;q) \;&=\; f(x;q), \\ f(q^n x;q)\prod_{j=0}^{n-1}(1-xq^j) \;&=\; f(x;q), \end{aligned} \tag{5.30}$$

and so

$$f(0;q)(x;q)_\infty \;=\; f(x;q). \tag{5.31}$$

Then, as $f(0;q) = 1$, Equation (5.25) follows. A similar proof works for Equation (5.26). □

These expansions allow us to give a natural definition of $(x;q)_\infty$ for $|q|>1$. Notice that for $|q|>1$ we have

$$(x;q)_\infty \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k-1)/2}}{(q;q)_k}x^k \;=\; \sum_{k=0}^{\infty}\frac{1}{(q^{-1};q^{-1})_k}(q^{-1}x)^k \;=\; \frac{1}{(q^{-1}x;q^{-1})_\infty}. \tag{5.32}$$

The next fundamental solution to a q–difference equation is given by the θ–function.

Definition 15 (Jacobi θ–function)**.** *For $x\in\mathbb{C}$ and $|q|<1$ let*

$$\theta(x;q) \;=\; \sum_{k\in\mathbb{Z}}(-1)^k q^{k(k+1)/2}x^k\,. \tag{5.33}$$

This is the Jacobi θ–function *but often we will just refer to it as the θ–function.*

The θ–function satisfies the first order equation

$$qx\theta(qx;q)+\theta(x;q) \;=\; 0\,. \tag{5.34}$$

To see this notice that

$$\begin{aligned}\theta(x;q) \;&=\; \sum_{k\in\mathbb{Z}}(-1)^k q^{k(k+1)/2+k}x^k \;=\; -q^{-1}x^{-1}\sum_{k\in\mathbb{Z}}(-1)^{k+1}q^{(k+1)(k+2)/2}x^{k+1}\\ &=\; -q^{-1}x^{-1}\theta(x;q)\,.\end{aligned} \tag{5.35}$$

Then by induction

$$\theta(q^\ell x;q) \;=\; (-1)^\ell q^{-\ell(\ell+1)/2}x^{-\ell}\theta(x;q)\,. \tag{5.36}$$

Using equation (5.32) we take

$$\theta(x;q^{-1}) \;=\; \theta(q^{-1}x;q)^{-1}\,. \tag{5.37}$$

Moreover,

$$\theta(x^{-1};q) \;=\; \theta(q^{-1}x;q)\,. \tag{5.38}$$

We can prove a product formula for the θ–function using Lemma 7.

Theorem A–30 (Jacobi triple product identity)**.** *We have the following identity*

$$\theta(x;q) \;=\; (qx;q)_\infty(x^{-1};q)_\infty(q;q)_\infty\,. \tag{5.39}$$

Proof. From Lemma 7, we see that for $1<|x|<|q^{-1}|$

$$\begin{aligned}\frac{\theta(x;q)}{(qx;q)_\infty(x^{-1};q)_\infty} \;&=\; \sum_{k\in\mathbb{Z}}\sum_{\ell,m=0}^{\infty}\frac{(-1)^k q^{k(k+1)/2+m}}{(q;q)_m(q;q)_\ell}x^{k+m-\ell}\\ &=\; \sum_{k\in\mathbb{Z}}\sum_{\ell,m=0}^{\infty}\frac{(-1)^{\ell+m+k}q^{(\ell-m+k)(\ell-m+1+k)/2+m}}{(q;q)_m(q;q)_\ell}x^k\,.\end{aligned} \tag{5.40}$$

Again using Lemma 7, for $k > 0$

$$\sum_{\ell,m=0}^{\infty} \frac{(-1)^{\ell+m+k} q^{(\ell-m+k)(\ell-m+1+k)/2+m}}{(q;q)_m (q;q)_\ell} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+k} q^{(\ell+k)(\ell+1+k)/2}}{(q;q)_\ell} (q^{1-k-\ell};q)_\infty = 0 \,, \tag{5.41}$$

for $k < 0$

$$\sum_{\ell,m=0}^{\infty} \frac{(-1)^{\ell+m+k} q^{(\ell-m+k)(\ell-m+1+k)/2+m}}{(q;q)_m (q;q)_\ell} = \sum_{m=0}^{\infty} \frac{(-1)^{m+k} q^{(k-m)(k-m+1)/2+m}}{(q;q)_m} (q^{1+k-m};q)_\infty = 0 \,, \tag{5.42}$$

and finally for $k = 0$,

$$\sum_{\ell,m=0}^{\infty} \frac{(-1)^{\ell+m} q^{(\ell-m)(\ell-m+1)/2+m}}{(q;q)_m (q;q)_\ell} = \sum_{m=0}^{\infty} \frac{(-1)^{m} q^{m(m+1)/2}}{(q;q)_m} (q^{1-m};q)_\infty = (q;q)_\infty \,. \tag{5.43}$$

Finally, analytic continuation implies the equality holds for $x \in \mathbb{C}^\times$. □

5.3 The Pochhammer symbol near roots of unity

The Pochhammer symbol satisfies a q–difference equation and therefore so will it's asymptotics. Therefore, we can phrase this for the results of the computations of Section 4.3 and Section 4.5. Firstly, the constants give rise to a function from roots of unity for generic parameters, which satisfy a q–difference equation for $q \in \mu$.

Lemma 8. *The cyclic dilogarithm of definition 12 satisfies the q–difference equation*

$$(1 - q^m x)\Delta(m+1, x; q) = \Delta(m, x; q) \,. \tag{5.44}$$

This gives a solution to the module of the infinite Pochhammer symbol at roots of unity. However, when $x = 1$, this equation become degenerate at roots of unity and we find the Pochhammer symbol has asymptotics given in Lemma 5. The asymptotics of Lemma 5 still satisfy a q–difference equation where $q = \mathbf{e}(a/c + \tilde{\tau}_\gamma/|c|)$ as a formal series in $\tilde{\tau}_\gamma$ for $a/c \in \mathbb{Q}$. We can do the same when $x \neq 1$ giving an all orders version of Lemma 8. Firstly, let

$$\Phi_{a/c}(m, x, \tilde{\tau}_\gamma) = \exp\left(\sum_{k=0}^{\infty} \frac{(2\pi i \tilde{\tau}_\gamma)^{k-1}}{k!} \sum_{\ell=0}^{|c|-1} B_k\Big(\frac{m+\ell}{|c|}\Big) \mathrm{Li}_{2-k}\Big(\mathbf{e}\Big((m+\ell)\frac{a}{c}\Big)x\Big)\right) \tag{5.45}$$

Lemma 9. *The formal series from equation 5.45 satisfies the q–difference equation*

$$(1 - \mathbf{e}(ma/c + m\tilde{\tau}_\gamma/|c|)x)\Phi_{a/c}(m+1, x, \tilde{\tau}_\gamma) = \Phi_{a/c}(m, x, \tilde{\tau}_\gamma) \,. \tag{5.46}$$

Finally, the case where $x = \mathbf{e}(-ja/c)$ for $j - m \in \{0, \dots, |c|-1\}$ we get a difference equation for the series of Lemma 5. Let

$$\begin{aligned}\Phi_{a/c}(m, j, \tilde{\tau}_\gamma) &= 2\pi\sqrt{-i\tilde{\tau}_\gamma}\Big(-2\pi i\frac{j}{|c|}\tilde{\tau}_\gamma\Big)^{-\frac{j}{|c|}}\Big(\frac{j}{|c|}\Big)^{\frac{j}{|c|}}\Gamma\Big(\frac{j}{|c|}\Big)^{-1} \\ &\quad\times\exp\Bigg(\sum_{k=0}^{\infty}\frac{(2\pi i\tilde{\tau}_\gamma)^{k-1}}{k!}\sum_{0\le\ell\le|c|-1}^{\ell\neq j}B_k\Big(\frac{m+\ell}{|c|}\Big)\mathrm{Li}_{2-k}\Big(\mathbf{e}\Big((\ell-j)\frac{a}{c}\Big)\Big) \\ &\quad-\frac{2\pi i}{24\tilde{\tau}_\gamma}+\sum_{k=2}^{\infty}\frac{(2\pi i\tilde{\tau}_\gamma)^{k-1}}{k!}B_k\Big(\frac{m+j}{|c|}\Big)\zeta(2-k)\Bigg)\end{aligned} \tag{5.47}$$

Lemma 10. *The formal series from equation 5.45 satisfies the q–difference equation*

$$(1-\mathbf{e}((m-j)a/c+m\tilde{\tau}_\gamma/|c|))\Phi_{a/c}(m+1, j, \tilde{\tau}_\gamma) = \Phi_{a/c}(m, j, \tilde{\tau}_\gamma)\,. \tag{5.48}$$

Note that in this case the power of $\tilde{\tau}_\gamma$ changes with m. This is fundamentally due to the fact the difference equation leads to multiplication by 0 at leading order and we will see that this corresponds to slopes on an associated Newton polygon discussed in Section 5.4. We can then consider the θ–function at roots of unity using the triple product. We then need the function given in equation (8.8) and again here

$$\varepsilon(q) = \sqrt{-i}\prod_{\ell=1}^{\mathrm{ord}(q)-1}\left(1-q^\ell\right)^{\frac{1}{2}-\frac{\ell}{\mathrm{ord}(q)}}, \tag{5.49}$$

Corollary 6. *We have*

$$\Delta(m+2, x; q)\Delta(-m-1, x^{-1}; q)\varepsilon(q) = -q^{-m-1}x^{-1}\Delta(m+1, x; q)\Delta(-m, x^{-1}; q)\varepsilon(q) \tag{5.50}$$

The all orders version uses

$$\begin{aligned}&\Phi_{a/c}(m+1, x, \tilde{\tau}_\gamma)\Phi_{a/c}(-m, x^{-1}, \tilde{\tau}_\gamma)\Phi_{a/c}(1, |c|-1, \tilde{\tau}_\gamma)\Phi_{a/c}(1, |c|-1, \tilde{\tau}_\gamma) \\ &= \exp\Big(-\frac{\log(-\mathbf{e}(ma|c|/c)x^{|c|})^2}{4\pi i|c|\tilde{\tau}_\gamma}\Big)\Delta(m+1, x; q)\Delta(-m, x^{-1}; q)\varepsilon(\mathbf{e}(a/c)) \\ &\quad\times\mathbf{e}\Big(-\frac{\tilde{\tau}_\gamma}{2|c|}\big((m+1)^2-(m+1)\big)-\frac{\tilde{\tau}_\gamma}{8|c|}\Big)\,.\end{aligned} \tag{5.51}$$

Corollary 7. *We have*

$$\begin{aligned}&\Phi_{a/c}(m+2, x, \tilde{\tau}_\gamma)\Phi_{a/c}(-m-1, x^{-1}, \tilde{\tau}_\gamma)\Phi_{a/c}(1, |c|-1, \tilde{\tau}_\gamma) \\ &= -\tilde{q}_\gamma^{-m-1}x^{-1}\Phi_{a/c}(m+1, x, \tilde{\tau}_\gamma)\Phi_{a/c}(-m, x^{-1}, \tilde{\tau}_\gamma)\Phi_{a/c}(1, |c|-1, \tilde{\tau}_\gamma)\,.\end{aligned} \tag{5.52}$$

5.4 The Frobenius method

Linear q–difference equations can be solved formally in a completely analogous way to differential equations. We start by computing a Newton polygon. To each edge we can construct solutions. Firstly, we flatten the edge by multiplying by a θ–function, which is the analogue of multiplying by the exponential when solving differential equations. Then we solve an associated indicial polynomial and finally a recursion for the coefficients in an expansion.

Remark 16. *We saw that the constants associated q–difference equations are elliptic functions solving equation (5.12). Therefore, for algorithmic construction of solutions we need to choose a solution to equation (5.34). We have a solution with good analytic and modular properties so all of our constructions will be relative to this convention.*

The basic Ansatz for our solutions will be functions of the form

$$\theta_\kappa(x;q)\sum_{k=0}^{\infty} a_k(q)x^k\frac{\theta(\rho^{-1}x;q)}{\theta(x;q)}\,, \tag{5.53}$$

where $a_0 \neq 0$ and (using equation (5.37)),

$$\theta_\kappa(x;q) \;=\; \begin{cases} \theta(x^\kappa;q^\kappa) & \text{if } \kappa > 0\,,\\ 1 & \text{if } \kappa = 0\,,\\ \theta(q^\kappa x^\kappa;q^{-\kappa})^{-1} & \text{if } \kappa > 0\,, \end{cases} \tag{5.54}$$

which is the analogue of equation (3.74). Take a q–difference equation

$$\sum_{i=0}^{r}\sum_{j=0}^{s_i}\alpha_{i,j}(q)x^j\sigma^i f \;=\; 0\,, \tag{5.55}$$

which we can assume has this form with the addition of some $\alpha_{i,0}(q) \neq 0$ by multiplying on the left by powers of x. The Newton polygon is defined to be the convex hull of the points (i,j) where $\alpha_{i,j}(q) \neq 0$. Suppose that f is a solution to equation (5.55) then taking $g(x;q) \;=\; \theta(x^\kappa;q^\kappa)f(x;q)$ we have

$$\begin{aligned} 0 \;&=\; \sum_{i=0}^{r}\sum_{j=0}^{s_i}\alpha_{i,j}(q)x^j\sigma^i(\theta(x^\kappa;q^\kappa)^{-1}g(x;q)) \;=\; \sum_{i=0}^{r}\sum_{j=0}^{s_i}\alpha_{i,j}(q)x^j\theta(q^{\kappa i}x^\kappa;q^\kappa)^{-1}\sigma^i g \\ &=\; \theta(x^\kappa;q^\kappa)^{-1}\sum_{i=0}^{r}\sum_{j=0}^{s_i}(-1)^i q^{\kappa i(i+1)/2}\alpha_{i,j}(q)x^{j+i\kappa}\sigma^i g\,. \end{aligned} \tag{5.56}$$

Therefore, we see that $\theta(x^\kappa;q^\kappa)f(x;q)$ satisfies a difference equation whose Newton polygon is given by the original Newton polygon sheared parallel to the vertical axis with weight

κ. Therefore, by multiplying by an appropriate θ function we can flatten any edge of the Newton polygon of a q–difference equation satisfied by a function.

This deals with the first part of our Ansatz (5.53). Therefore, assume that on the bottom of the Newton polygon of the difference equation (5.55) satisfied by f there is a flat edge and take $\kappa = 0$. Then we can substitute the Ansatz into equation (5.55) to get

$$\begin{aligned} 0 &= \frac{\theta(\rho^{-1}x;q)}{\theta(x;q)} \sum_{i=0}^{r} \sum_{j=0}^{s_i} \alpha_{i,j}(q) x^j \sum_{k=0}^{\infty} a_k(q) q^{ik} x^k \rho^i \\ &= \frac{\theta(\rho^{-1}x;q)}{\theta(x;q)} \sum_{k=0}^{\infty} \sum_{j=0}^{s_i} a_k(q) x^{j+k} \sum_{i=0}^{r} \alpha_{i,j}(q) q^{ik} \rho^i \,. \end{aligned} \tag{5.57}$$

Therefore, as we assume that $a_0 \neq 0$, taking the x^0 term divided by a_0 gives

$$\sum_{i=0}^{r} \alpha_{i,0}(q) \rho^i \;=\; 0 \,. \tag{5.58}$$

We assumed that some $\alpha_{i,0}(q) \neq 0$ and moreover that the bottom of the Newton polygon has a flat edge. This implies that there exists $i_1 \neq i_2 \in \{0, \ldots, r\}$ such that $\alpha_{i_1,0}(q) \neq 0_1$ and $\alpha_{i_2,0}(q) \neq 0$. Therefore, equation (5.58) gives a polynomial equation for ρ of order the length of the edge, which is at least one. This polynomial is called the indicial polynomial of this edge. Suppose first that ρ_0 is a root of multiplicity one such that $\rho_0 q^{\mathbb{Z}}$ is not a root. Substituting this into the q–difference equation gives the system of equations for $k \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} 0 &= \sum_{i=0}^{r} \sum_{j=0}^{\min(s_i,k)} a_{k-j}(q) \alpha_{i,j}(q) q^{i(k-j)} \rho_0^i \\ &= a_k(q) \sum_{i=0}^{r} \alpha_{i,0}(q) q^{ik} \rho_0^i + \sum_{i=0}^{r} \sum_{j=1}^{\min(s_i,k)} a_{k-j}(q) \alpha_{i,j}(q) q^{i(k-j)} \rho_0^i \,, \end{aligned} \tag{5.59}$$

which by assumption on ρ_0 gives a recursive computation of a_k in terms of lower order terms. More generally, if there are n roots of the indicial polynomial in $\rho_0 q^{\mathbb{Z}}$ counted with multiplicity, then we can get n solutions coming from these roots. Suppose that ρ_0 is a root of multiplicity m and that $\rho_0 q^{\mathbb{Z}_{<0}}$ are not roots. Then take $a_k = a_k(\epsilon; q)$ with $a_0(\epsilon; q) = O(\epsilon^{n-m})$ so that

$$\sum_{i=0}^{r} \alpha_{i,0}(\epsilon; q) \rho_0^i e^{i\epsilon} \;=\; O(\epsilon^n) \,. \tag{5.60}$$

Now substituting $\rho_0 e^{\epsilon} + O(\epsilon^n)$ into the q–difference equation we get equations

$$\begin{aligned} 0 &= \sum_{i=0}^{r} \sum_{j=0}^{\min(s_i,k)} a_{k-j}(\epsilon;q)\alpha_{i,j}(q) q^{i(k-j)} \rho_0^i e^{i\epsilon} \\ &= a_k(\epsilon;q) \sum_{i=0}^{r} \alpha_{i,0}(q) q^{ik} \rho_0^i e^{i\epsilon} + \sum_{i=0}^{r} \sum_{j=1}^{\min(s_i,k)} a_{k-j}(\epsilon;q)\alpha_{i,j}(q) q^{i(k-j)} \rho_0^i e^{i\epsilon} , \end{aligned} \tag{5.61}$$

This completely determines $a_k(\epsilon;q) + O(\epsilon^n)$ in terms of $a_0(\epsilon;q) + O(\epsilon^{2n-m})$ and moreover $a_k(\epsilon;q) = O(\epsilon^0)$. Often we will choose $a_k(\epsilon;q) = \epsilon^{n-m}$ but imposing this generally could lead to slightly less natural normalisations. These ϵ deformed solutions correspond to logarithmic solutions of differential equations coming from equation (3.76). These solutions will just be formal solutions as the a_k can give rise to divergent series. To make these into functions we need to apply some resummation procedure.

Remark 17. *If we take a discrete version where $x = q^m$ for $m \in \mathbb{Z}$ then we can apply the same techniques however we will take the Ansatz*

$$(-1)^m q^{-\kappa m(m+1)/2} \sum_{k=0}^{\infty} a_k(q) q^{km} \rho^m . \tag{5.62}$$

These methods will be illustrated in an example associated to the figure eight knot 4_1 defined in Section 6.5, which was discussed in [82].

Example 45 (A difference equation for the figure eight knot)**.** *Consider the q–difference equation,*

$$tf(t;q) + (1-3qt)f(qt;q) + (3q^2t-1)f(q^2t;q) - q^3tf(q^3t;q) \;=\; 0 . \tag{5.63}$$

This has Newton polygon depicted in Figure 5.3. We can apply the Ansatz and solve for the edge of slope minus one. this has $\kappa = -1$ and indicial polynomial

$$1 - q^{-1}\rho^{-1} \;=\; 0 . \tag{5.64}$$

Then setting $a_0(q) = -(q;q)_\infty^2$ we find the solution to the recursion

$$(1-q^{-k})a_k(q) - 2a_{k-1}(q) - q^{k-1}a_{k-2}(q) \;=\; 0 . \tag{5.65}$$

given by

$$a_k(q) \;=\; (q;q)_\infty^2 \sum_{\ell=0}^{k} (-1)^{k+1} \frac{q^{k(k+1)/2}}{(q;q)_\ell (q;q)_{k-\ell}} . \tag{5.66}$$

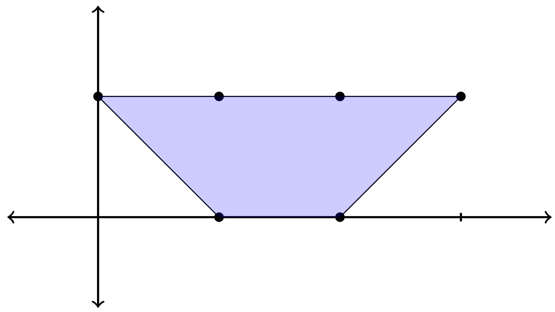

Figure 5.1: The Newton polygon of equation (5.63).

This gives a solution

$$f^{(-1)}(t;q) \;=\; \frac{(q;q)_\infty^2}{\theta(q^{-1}t;q)} \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_\ell (q;q)_{k-\ell}} t^k \,. \tag{5.67}$$

Applying the method to the edge of slope one we get a formal solution

$$f^{(1)}(t;q) \;=\; \frac{\theta(t;q)}{(q;q)_\infty^2} \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \frac{q^{\ell^2-\ell k}}{(q;q)_\ell (q;q)_{k-\ell}} t^k \,. \tag{5.68}$$

Finally, the edge of slope 0 *has solution*

$$f^{(0)}(t;q) \;=\; \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} (q;q)_k^2 t^k \,. \tag{5.69}$$

We can apply the method at infinity. We get the recursion

$$(1-3q^{k-1}\rho+3q^{2k-2}\rho^2-q^{3k-3}\rho^3)a_k(q)+q^{k-1}\rho(1-q^{k-1}\rho)a_{k-1}(q) \;=\; 0\,, \tag{5.70}$$

to find an ϵ *deformed solution*

$$g(t,\epsilon;q) \;=\; \frac{(qe^\epsilon;q)_\infty^2}{(q;q)_\infty^2} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} e^{(k+1/2)\epsilon}}{(qe^\epsilon;q)_k^2} t^{-1-k} \frac{\theta(te^{-\epsilon};q)}{\theta(t;q)} \,. \tag{5.71}$$

giving some solutions

$$
\begin{aligned}
g(t;q) &= \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_k^2} t^{-1-k} \\
G(t;q) &= \sum_{k=0}^{\infty}\Big(-2G_1(q) - \frac{1}{2} - \frac{\theta'(t^{-1};q)}{\theta(t^{-1};q)} + \sum_{j=1}^{k} \frac{1+q^j}{1-q^j}\Big)(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_k^2} t^{-1-k} \\
\mathfrak{G}(t;q) &= \sum_{k=0}^{\infty}\Big(\frac{1}{2}\Big(-2G_1(q) - \frac{1}{2} - \frac{\theta'(t^{-1};q)}{\theta(t^{-1};q)} + \sum_{j=1}^{k} \frac{1+q^j}{1-q^j}\Big)^2 \\
&\qquad + \frac{1}{2}\frac{\theta''(t^{-1};q)}{\theta(t^{-1};q)} - \frac{1}{2}\frac{\theta'(t^{-1};q)^2}{\theta(t^{-1};q)^2} - \frac{1}{24} - G_2(q)\Big)(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_k^2} t^{-1-k} .
\end{aligned}
\tag{5.72}
$$

where

$$
\theta'(t;q) = \sum_{k\in\mathbb{Z}} (-1)^k k q^{k(k+1)/2} t^k , \quad \textit{and} \quad \theta''(t;q) = \sum_{k\in\mathbb{Z}} (-1)^k k^2 q^{k(k+1)/2} t^k . \tag{5.73}
$$

5.5 Resummation of divergent solutions and monodromy

The formal solutions constructed in the previous Section 5.4 can be made into meromorphic functions via q–Borel resummation. This was studied, for example, in [51, 97, 165, 181]. We will define the *q–Borel resummation* of a series associated to a q–difference equation. With the conventions we have set, the *q–Borel transform* of a formal series of weight κ is

$$
\mathcal{B}_\kappa \sum_{k=0}^{\infty} a_k(q) x^k = \sum_{k=0}^{\infty} (-1)^k q^{\kappa k(k+1)/2} a_k(q) \xi^k \tag{5.74}
$$

When $|q| < 1$ and $\kappa > 0$ this of course improves the convergence of the series. Assuming that it is convergent, we can define the *q–Laplace transform* of a function of weight $\kappa > 0$

$$
(\mathcal{L}_\kappa f)(x,\lambda;q) = \sum_{k\in\mathbb{Z}} \frac{f(q^{\kappa k}\lambda^\kappa x;q)}{\theta(q^{\kappa k}\lambda^\kappa;q)} . \tag{5.75}
$$

Importantly, the Laplace transformation for $\kappa > 0$ introduces an additional variable. This is refered to as *q–Stokes phenomenon*. This was discussed in more detail in [82]. Notice that it is elliptic in the new variable λ

$$
(\mathcal{L}_\kappa f)(x,q\lambda;q) = (\mathcal{L}_\kappa f)(x,\lambda;q) . \tag{5.76}
$$

To define the Laplace transform when $\kappa < 0$ we take

$$
(\mathcal{L}_\kappa f)(x;q) = \oint_{|\xi|=\epsilon} f(\xi;q)\theta(x/\xi;q^{-\kappa}) \frac{d\xi}{2\pi i \xi} . \tag{5.77}
$$

Noting that

$$\sum_{\ell\in\mathbb{Z}} \frac{(q^{\kappa\ell}\lambda^{\kappa}x)^k}{\theta(q^{\kappa\ell}\lambda^{\kappa};q^{\kappa})} = \frac{x^k}{\theta(\lambda^{\kappa};q^{\kappa})}\sum_{\ell\in\mathbb{Z}}(-1)^{\ell}q^{\kappa\ell(\ell+1)/2+\kappa\ell k}\lambda^{\kappa(\ell+k)} = (-1)^k q^{-\kappa k(k+1)/2}x^k\,, \tag{5.78}$$

we have the lemma similarly for the $\kappa < 0$ by the residue theorem.

Lemma 11. *If $f(x;q)$ is a polynomial in x then*

$$\mathcal{L}_{\kappa}\mathcal{B}_{\kappa}f \;=\; \mathcal{B}_{\kappa}\mathcal{L}_{\kappa}f \;=\; f\,. \tag{5.79}$$

Note that the statement of this lemma is slightly subtle as the various vector spaces involved as domains and codomains are all different in general but agree when restricted to polynomials. The first basic analytic property of the Laplace transform is an analogue of Watson's lemma.

Theorem A–31 (q–Watson's lemma)**.** *[51, Prop. 1.9] If $f(x;q)$ is a meromorphic function for $x \in \mathbb{C}^{\times}$ and has an asymptotic expansion[2] as $x \to 0$*

$$f(x;q) \;\sim\; \sum_{k=0}^{\infty} a_k(q)x^k\,, \tag{5.80}$$

with $a_0 \neq 0$, then, if $\mathcal{L}_{\kappa}(f)$ is convergent we have

$$\mathcal{L}_{\kappa}(f)(x,\lambda;q) \;\sim\; \sum_{k=0}^{\infty}(-1)^k q^{-\kappa k(k+1)/2}a_k(q)x^k\,. \tag{5.81}$$

The next basic properties of the q–Laplace transform describe the behaviour of the additional elliptic variable. These are given in the following two lemmas, which give integral expressions for the Laplace transforms of positive weight. Moreover, the first determines the dependence on the additional elliptic variables in terms of the poles of the function on which we apply the Laplace transform. These lemmas appeared in [82].

Lemma 12. *Assuming that*

$$\lim_{\epsilon\to 0}\oint_{|\xi|=\epsilon^{\pm}} \frac{f(\xi t,q)\theta(\mu^{-1}\lambda^{-1}\xi;q)}{\theta(\xi\mu^{-1};q)\theta(\xi\lambda^{-1};q)}\frac{d\xi}{2\pi i\xi} = 0 \tag{5.82}$$

where ϵ avoids the poles of the integrand we have

$$\begin{aligned}\mathcal{L}_1(f)(t,\lambda,q) &- \mathcal{L}_1(f)(t,\mu,q)\\ &= \frac{\theta(\lambda^{-1}\mu;q)(q;q)_{\infty}^3}{\theta(\lambda^{-1};q)\theta(\mu;q)}\sum_{x\in\text{poles of }f}\operatorname{Res}_{\xi=x}\frac{f(\xi,q)\theta(\lambda^{-1}\mu^{-1}t^{-1}\xi;q)}{\theta(\xi\lambda^{-1}t^{-1};q)\theta(\xi\mu^{-1}t^{-1};q)}\,.\end{aligned} \tag{5.83}$$

[2] This involves some kind of uniform bound on the distance to the poles if they accumulate around $x = 0$. See [51] for more details.

This type of residue formula for the Laplace transform is similar to the definition of the Laplace transform for $\kappa < 0$. We can find a similar expression for a single Laplace transform using a special function called the Appell–Lerch sum studied in [211].

$$L(t, \lambda, q) \;=\; \frac{1}{\theta(\lambda; q)} \sum_{k\in\mathbb{Z}} (-1)^k \frac{q^{k(k+1)/2}\lambda^k}{1 - q^k\lambda t}. \tag{5.84}$$

Using this we have the following integral expression for the Laplace transform for $\kappa = 1$ and a similar expression can be given for $\kappa > 0$.

Lemma 13. *We have*

$$(\mathcal{L}_1 f)(t, \lambda, q) = \sum_{k\in\mathbb{Z}} \mathrm{Res}_{\xi=q^k} L(\xi^{-1}\lambda^{-1}, \lambda, q) f(\xi\lambda t, q) \frac{d\xi}{2\pi i\xi}\,. \tag{5.85}$$

Deforming the contour and the residue theorem give the following lemma.

Lemma 14. *Assuming that*

$$\lim_{\epsilon\to 0} \oint_{|\xi|=\epsilon^{\pm}} L(\xi^{-1}\lambda^{-1}, \lambda, q) f(\xi\lambda t, q) \frac{d\xi}{2\pi i\xi} = 0 \tag{5.86}$$

where ϵ avoids the poles of the integrand we have

$$(\mathcal{L}_1 f)(t, \lambda, q) = -\sum_{x\in\text{poles of } f} L(x^{-1}t, \lambda, q)\mathrm{Res}_{\xi=x} f(\xi, q) \frac{d\xi}{2\pi i\xi} \tag{5.87}$$

Notice that all the dependence on t and λ is now in the arguments of the Appell-Lerch sums. This illustrates the important role the residues of the Borel transform play in the resummation.

We can study the affect of these operations on the Newton polygon of a difference equation. Notice that

$$\begin{aligned} 0 \;=\; \mathcal{B}_\kappa 0 \;&=\; \mathcal{B}_\kappa \sum_{i=0}^{r}\sum_{j=0}^{s_i} \alpha_{i,j}(q) x^j (\sigma^i f)(x; q) \;=\; \mathcal{B}_\kappa \sum_{k=0}^{\infty}\sum_{i=0}^{r}\sum_{j=0}^{s_i} \alpha_{i,j}(q) a_k(q) q^{ik} x^{j+k} \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{r}\sum_{j=0}^{s_i} \alpha_{i,j}(q) a_k(q) (-1)^{k+j} q^{\kappa k(k+1)/2+\kappa jk+\kappa j(j+1)/2+ik} \xi^{j+k} \\ &= \sum_{i=0}^{r}\sum_{j=0}^{s_i} \alpha_{i,j}(q)(-1)^j q^{\kappa j(j+1)/2} \xi^j (\sigma^{i+\kappa j}\mathcal{B}_\kappa f)(\xi; q)\,. \end{aligned} \tag{5.88}$$

Therefore, the weight κ q–Borel transform satisfies a q–difference equations that has a Newton polygon, which is a weight κ shear parallel to the horizontal axis of the original Newton polygon. The same kind of calculations can be used to show the same shearing property of the q–Laplace transform when it is convergent.

As with the case of differential equations, the divergence of a particular solution is determined by the slopes of the Newton polygon. Suppose that we consider a solutions from a flat edge of a Newton polygon which also contains an edge with negative slope. Suppose that the largest negative slope is κ_m and i_0 is the smallest value for which $\alpha_{i_0,0} \neq 0$. Then, we have the recursion

$$\begin{aligned} a_k(\epsilon;q) &= \frac{1}{\sum_{i=i_0}^{r} \alpha_{i,0}(q) q^{ik} \rho_0^i e^{i\epsilon}} \Big(\sum_{i=0}^{i_0-1} \sum_{j=1}^{\min(s_i,k)} a_{k-j}(\epsilon;q)\alpha_{i,j}(q) q^{i(k-j)} \rho_0^i e^{i\epsilon} \\ &\qquad + \sum_{i=i_0}^{r} \sum_{j=1}^{\min(s_i,k)} a_{k-j}(\epsilon;q)\alpha_{i,j}(q) q^{i(k-j)} \rho_0^i e^{i\epsilon} \Big) \\ &= \frac{1}{\sum_{i=i_0}^{r} \alpha_{i,0}(q) q^{ik} \rho_0^i e^{i\epsilon}} \Big(\sum_{i=0}^{i_0-1} \sum_{j=\lceil (i-i_0)\kappa_m \rceil}^{\min(s_i,k)} a_{k-j}(\epsilon;q)\alpha_{i,j}(q) q^{i(k-j)} \rho_0^i e^{i\epsilon} \\ &\qquad + \sum_{i=i_0}^{r} \sum_{j=1}^{\min(s_i,k)} a_{k-j}(\epsilon;q)\alpha_{i,j}(q) q^{i(k-j)} \rho_0^i e^{i\epsilon} \Big) . \end{aligned} \tag{5.89}$$

Therefore, as $k \to \infty$ we have

$$a_k(\epsilon;q) = O\Big(q^{\frac{k^2}{2\kappa_m}}\Big) . \tag{5.90}$$

Therefore, taking Borel transforms $\mathcal{B}_{\kappa_1}, \dots, \mathcal{B}_{\kappa_n}$ with $\sum_{i=1}^{n} \kappa_i = 1/|\kappa_m|$ will lead to a convergent function. Moreover, this convergence will satisfy a q–difference equation, which can be used to explicitly construct an analytic continuation in the Borel plane. This continuation will have potential singularities whose position is determined by the zeros of $\sum_{j=0}^{s_0} \alpha_{0,j}(q)x^j$. This continuation is therefore meromorphic for $\xi \in \mathbb{C}$ *i.e.* $\xi = \infty$ could be an accumulation point of poles.

Finally, we want to apply a sequence of q–Laplace transforms to get a meromorphic function that satisfies the original q–difference equation. To apply the correct sequence of Laplace transforms we use the following lemma.

Lemma 15. *[51, Prop. 1.13] Suppose that $F(x;q) = \sum_{k=0}^{\infty} a_k(q)x^k$ is a formal solution to a difference equation with smallest negative slope κ_M and suppose that $\kappa < -1/\kappa_M$. Then, if the q–difference equation satisfied by $\mathcal{B}_\kappa F$ has a meromorphic solution f, $\mathcal{L}_\kappa f$ is convergent and satisfies the q–difference equation of F.*

Therefore, we see that at most we can only apply a weight $-1/\kappa_M$ Laplace transform to our the combinations of Borel transforms. However, after applying this we can apply the lemma to the new difference equation. Iterating this we get the following theorem.

Theorem A–32. *[51, Thm. 1.10] Suppose that $F(x;q)$ is a power series solution to a q–difference equation with negative slopes $\kappa_1, \dots, \kappa_m$. Letting*

$$\mu_1 \;=\; -\frac{1}{\kappa_1}\,, \quad \textit{and for} \quad i > 1 \quad \mu_i \;=\; \frac{1}{\kappa_{i-1}} - \frac{1}{\kappa_i}\,, \tag{5.91}$$

we have,

$$(\mathcal{L}_{\mu_m} \cdots \mathcal{L}_{\mu_1} \mathcal{B}_{\mu_1} \cdots \mathcal{B}_{\mu_m} F)(x, \lambda_1, \dots, \lambda_m; q) \tag{5.92}$$

is a meromorphic function and satisfies the q–difference equation of F.

In this way, we can construct a set of meromorphic functions solving a particular q–difference equation. Moreover, using Watson's lemma we can see that this gives a basis filtered with respect to the slopes of the Newton polygon. This shows that we have a natural construction of a basis of meromorphic functions to a q–difference equation.

Example 46 (Divergent series for the figure eight knot)**.** *Take the formal solution $f^{(1)}(t;q) = \theta(t;q)(q;q)_\infty^{-2} \hat{f}^{(1)}(t;q)$ given in equation (5.68). Then*

$$\mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) \;=\; \sum_{k,\ell=0}^{\infty} (-1)^{k+\ell} \frac{q^{k(k+1)/4 - k\ell/2 + \ell(\ell+1)/4}}{(q;q)_k (q;q)_\ell} \xi^{k+\ell} \tag{5.93}$$

is holomorphic for $|\xi| < |q^{-1/4}|$. Then, using the functional equation

$$(1 - q^{1/2}\xi^2) \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) + 2\xi \mathcal{B}_{1/2} \hat{f}^{(1)}(q^{1/2}\xi, q) - \mathcal{B}_{1/2} \hat{f}^{(1)}(q\xi, q) = 0, \tag{5.94}$$

we can analytically extend away from $\xi \in \pm q^{-1/4 + \frac{1}{2}\mathbb{Z}_{\leq 0}}$ where there are poles. Therefore, we get the meromorphic function

$$\begin{aligned} f^{(1)}(t, \lambda, q) \;&=\; \frac{\theta(t;q)}{(q;q)_\infty^2} \mathcal{L}_{1/2} \mathcal{B}_{1/2} \hat{f}^{(1)}(t, \lambda, q) \\ &=\; \frac{\theta(t;q)}{(q;q)_\infty^2} \sum_{n \in \mathbb{Z}} \frac{\mathcal{B}_{1/2} \hat{f}^{(1)}(q^{\frac{n}{2}} \lambda t, q)}{\theta(q^{\frac{n}{2}} \lambda; q^{\frac{1}{2}})} \\ &=\; \frac{\theta(t;q)}{(q;q)_\infty^2 \theta(\lambda; q^{1/2})} \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n+1)/4} \lambda^n \mathcal{B}_{1/2} \hat{f}^{(1)}(q^{\frac{n}{2}} \lambda t, q). \end{aligned} \tag{5.95}$$

where this formula uses the analytic continuation of $\mathcal{B}_{1/2} \hat{f}^{(1)}$. We also had the solution $\hat{f}^{(0)}(t,q)$ from equation (5.69). For $|\xi| < 1$, we have

$$\mathcal{B}_1 \hat{f}^{(0)}(\xi, q) \;=\; \sum_{k=0}^{\infty} (q;q)_k^2 \xi^k\,, \tag{5.96}$$

which can be analytically continued away from $\xi \in q^{\mathbb{Z}_{\leq 0}}$ using the relation

$$(1-\xi)\mathcal{B}_1\hat{f}^{(0)}(\xi,q) + 2q\xi\mathcal{B}_1\hat{f}^{(0)}(q\xi,q) - q^2\xi\mathcal{B}_1\hat{f}^{(0)}(q^2\xi,q) \;=\; 1. \tag{5.97}$$

Therefore, we define

$$f^{(0)}(t,\lambda_2,q) \;=\; \frac{1}{\theta(\lambda_2;q)}\sum_{k\in\mathbb{Z}}(-1)^k q^{k(k+1)}\lambda_2^k\mathcal{B}_1\hat{f}^{(0)}(q^k\lambda_2 t,q). \tag{5.98}$$

So up to some initial conditions we can solve q–difference equations algorithmically. We have seen the methods that apply at the two special points 0 and ∞ *i.e.* the points fixed under the action $x \mapsto qx$. This algorithm leads to two Wronskian matrices, one based at 0 and one at ∞. These satisfy the first order equation (5.9) and taking their quotient gives a matrix of elliptic functions as in equation (5.11). This matrix is called the *monodromy matrix*. This matrix stores important information such as the q–Stokes phenomenon and in general its computation can be a difficult problem. In some simple examples coming from q–hypergeometric functions, I studied this with Garoufalidis [82] so that for our working example of the third order equation associated to the figure eight knot 4_1 we find

Theorem A–33. *[82] The monodromy matrix of the difference equation (5.63) is given explicitly as*

$$M(t,\lambda_1,\lambda_2,q) \;=\; \begin{pmatrix} -1 & 0 & \wp(t,q) \\ M_{2,1}(t,\lambda_1,q) & 1 & \frac{1}{2}\frac{\wp'(t,q)-\wp'(\lambda_2,q)}{\wp(t,q)-\wp(\lambda_2,q)} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.99}$$

where

$$M_{2,1}(t,\lambda,q) \;=\; \frac{\theta(qt;q)\theta(t\lambda;q)\theta(t\lambda q^{-1/2};q)\theta(t\lambda^2 q^{-1/2};q)}{\theta(t\lambda q^{1/4};q)\theta(-t\lambda q^{1/4};q)\theta(t\lambda q^{-1/4};q)\theta(-t\lambda q^{-1/4};q)\theta(q^{-1}\lambda;q)\theta(q^{-3/2}\lambda;q)}\,, \tag{5.100}$$

and $\wp, \wp'$ are the Weierstrass elliptic $\wp$–function and its derivative.

The modularity of M gives the ability to compute this efficiently as this should lie in a space of Jacobi forms discussed in Section 7.5, which is finite dimensional once some conditions on the poles are specified.

5.6 Identities between q–series

This section involves somewhat the opposite question of solving q–difference equations, that of proving q–series identities. It is important for the previous question of computing monodromy. Proving q–series identities will be an extremely important aspect of computations in examples that come later. Besides some simple explicit computations I know essentially one catch all method for the kind of examples of which we are interested.

In practice, proving an identity between two q–series is difficult. Indeed, proving the value of two functions at a special point is potentially an extremely difficult problem. However, as with many aspects of mathematics, deformations become a powerful tool. Proving the value of two function is the same becomes a much easier problem when one can show the two functions satisfy the same differential equation and that they have the same boundary conditions. This is exactly the same with q–series identities. If one can show that two sequences of q–series of two functions in x and q satisfy the same q–difference equation and that they have the same boundary conditions then this allows for a proof of the desired identity. Although this sounds good in general there is one issue. If you start with a special value then you need to find the right deformations.

For certain identities between q–hypergeometric functions this has been thoroughly understood not only theoretically but algorithmically [209, 194]. This is implemented in [168]. The other important tool is that the constants of q–difference equations are elliptic functions. These elliptic functions are computable as we understand the algebra of elliptic functions. One of the basic and important points is that every entire holomorphic function on an elliptic curve is constant from Liouville's theorem.

5.7 Solutions when q is near one

We can solve q–difference equations when $q = e^{\hbar}$. This gives formal series in $\hbar$ with certain functions as coefficients. If we leave the variable x as the coefficient, then we will take solutions given by the WKB Ansatz

$$\widehat{\Phi}(x;\hbar) \;=\; \hbar^{\kappa \log(x)/\hbar} \exp\left(\sum_{k=-1}^{\infty} S_k(x)\hbar^k\right). \tag{5.101}$$

Here the κ will be determined by a slope of an associated Newton polygon while S_{-1} will be determined via an analogue of an indicial polynomial. Firstly, we define the Newton polygon of a q–difference equation[3] expanded in powers of $\hbar$ given by

$$\sum_{i=0}^{r}\sum_{j=0}^{\infty} \alpha_{i,j}(x)\hbar^j \sigma^i \widehat{\Phi}(x;\hbar) \;=\; 0\,, \tag{5.102}$$

[3]With all factors of $(1-q)^{-1}$ cleared.

as the convex hull of (i,j) such that $a_{i,j}(x) \neq 0$. Then choosing the correct κ we can flatten any edge of the Newton polygon. Notice that for a smooth function f we have

$$\begin{aligned} f(e^{m\hbar}x) \;&=\; \sum_{k=0}^{\infty} \frac{\hbar^k}{k!}\frac{\partial^k}{\partial \hbar^k} f(e^{m\hbar}x)\big|_{\hbar=0} \;=\; \sum_{k=0}^{\infty} \frac{\hbar^k}{k!}\Big(my\frac{\partial}{\partial y}\Big)^k f(yx)\big|_{y=1} \\ &=\; \sum_{k=0}^{\infty} \frac{(m\hbar)^k}{k!} \sum_{\ell=0}^{k} S(k,\ell) y^\ell \frac{\partial^\ell}{\partial y^\ell} f(yx)\big|_{y=1} \;=\; \sum_{k=0}^{\infty} \frac{(m\hbar)^k}{k!} \sum_{\ell=0}^{k} S(k,\ell) x^\ell f^{(\ell)}(x) \\ &=\; f(x) + xf'(x)m\hbar + (x^2 f''(x) + xf'(x))\frac{m^2\hbar^2}{2} + \cdots . \end{aligned} \tag{5.103}$$

Therefore, given a flat edge of the Newton polygon, we get the following analogue of the indicial polynomial

$$\sum_{\ell=0}^{r} \alpha_{\ell,j}(x) \exp(\ell x S'_{-1}(x)) \;=\; 0\,. \tag{5.104}$$

In particular we see that this gives an algebraic equation for $\exp(xS'_{-1}(x))$ which defines a meromorphic function on the characterisic variety of the q–difference equation.

Therefore, substituting the Ansatz into a q–difference equation and solving order by order in $\hbar$ gives a sequence of differential equations for $S_k(x)$ which uniquely determine it up to a constant. Indeed, with this Ansatz once we solve for S'_{-1} we will find equations of the form

$$S'_k(x) \;=\; f_k(x)\,, \tag{5.105}$$

where f_k only depends on the difference equation and S'_ℓ for $\ell < k$. There is an ambiguity at each power of $\hbar$ as we only get equations for the first derivative. Fixing these constants and the κ and S_{-1} then uniquely gives the solution. This method was used for the $\hat{A}$–polynomial of Theorem 18 in [47]. Assuming the AJ conjecture 1 the fact the characteristic variety appears indicates that the A–polynomial appears and gives rise to the complexified volume in a way that is also expected from the volume conjecture 3. We apply this method to the previous equation.

Example 47. *We have the q–difference equation*

$$t\widehat{\Phi}(t;\hbar) + (1-3e^{\hbar}t)\widehat{\Phi}(e^{\hbar}t;\hbar) + (3e^{2\hbar}t-1)\widehat{\Phi}(e^{2\hbar}t;\hbar) - e^{3\hbar}t\widehat{\Phi}(e^{3\hbar}t;\hbar) \;=\; 0\,. \tag{5.106}$$

This has one edge of slope zero and we get the indicial polynomial

$$\begin{aligned} 0 \;&=\; t + (1-3t)\exp(tS'_{-1}(t)) + (3t-1)\exp(2tS'_{-1}(t)) - t\exp(3tS'_{-1}(t)) \\ &=\; (1-\exp(tS'_{-1}(t)))(t + (1-2t)\exp(tS'_{-1}(t)) + t\exp(2tS'_{-1}(t)))\,. \end{aligned} \tag{5.107}$$

This gives two solutions up to constants

$$\exp(tS'_{-1}(t)) \;=\; 1 \quad \textit{and so} \quad S_{-1}(t) \;=\; 0\,, \tag{5.108}$$

and

$$\begin{aligned} \exp(tS'_{-1}(t)) &= \frac{2t-1\pm\sqrt{1-4t}}{2t} \quad \textit{and so} \\ S_{-1}(t) &= \mathrm{Li}_2\Big(\frac{1}{2}(1+\sqrt{1-4t})\Big) - \mathrm{Li}_2\Big(\frac{1}{2}(1-\sqrt{1-4t})\Big) + \frac{1}{2}\Big(\log(1+\sqrt{1-4t})^2 \\ &\quad - \log(-1+\sqrt{1-4t})^2 + 2\Big(\log(-1+\sqrt{1-4t}) - \log(1+\sqrt{1-4t}) \\ &\quad + \log\Big(\frac{1+\sqrt{1-4t}}{-1+\sqrt{1-4t}}\Big)\Big)\log(-4t) \\ &\quad - 2\Big(\log(-1+\sqrt{1-4t})\log\Big(\frac{1}{2}(1+\sqrt{1-4t})\Big) \\ &\quad + 2\Big(\log(1+\sqrt{1-4t})\log\Big(\frac{1}{2}(1-\sqrt{1-4t})\Big)\Big)\Big). \end{aligned} \tag{5.109}$$

Using this we can solve for $S'_0(t)$ *by considering the* $\hbar$ *term of*

$$t + (1-3e^{\hbar}t)\frac{\widehat{\Phi}(e^{\hbar}t;\hbar)}{\widehat{\Phi}(t;\hbar)} + (3e^{2\hbar}t-1)\frac{\widehat{\Phi}(e^{2\hbar}t;\hbar)}{\widehat{\Phi}(t;\hbar)} - e^{3\hbar}t\frac{\widehat{\Phi}(e^{3\hbar}t;\hbar)}{\widehat{\Phi}(t;\hbar)} = 0. \tag{5.110}$$

Noting that we have

$$\frac{\widehat{\Phi}(e^{m\hbar}t;\hbar)}{\widehat{\Phi}(t;\hbar)} = \exp\Big(mxS'_{-1}(x) + \Big(\frac{m^2}{2}x^2S''_{-1}(x) + \frac{m^2}{2}xS'_{-1}(x) + mxS'_0(x)\Big)\hbar + \cdots\Big) \tag{5.111}$$

we can compute the $\hbar$ *term as*

$$\frac{-16t^2(-1+6t+2t(-1+4t)S'_0(t))}{(-1+\sqrt{1-4t})^4(1+\sqrt{1-4t})\sqrt{1-4t}} = 0. \tag{5.112}$$

This gives $S'_0(t)$ *and integrating we find that*

$$S_0(t) = \frac{-1}{4}\log(1-4t) - \frac{1}{2}\log(t). \tag{5.113}$$

Following the same procedure we find the equation for $S'_1(t)$ *given by*

$$(-1+\sqrt{1-4t})(-1-5t+6t^2) + 6(1-4t)^2(-1+\sqrt{1-4t}+4t)S'_1(t) = 0, \tag{5.114}$$

which when solving and integrating gives

$$S_1(t) = \frac{-12t^2+2t-1}{24(1-4t)^{3/2}}. \tag{5.115}$$

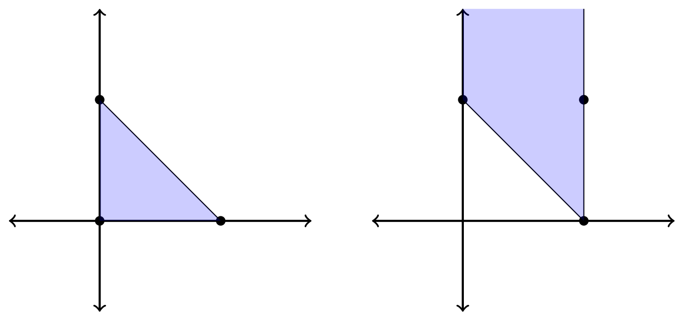

Figure 5.2: The Newton polygon of equation (5.116).

We saw that letting $q = e^\hbar$ we can solve difference equations with asymptotic series depending on x. Now a simpler method to solve these equations is to let $x = e^{m\hbar}$. Doing this, the method will again follow exactly the same method we used to solve q–difference equations in terms of q–series, however, there will be again a different Newton polygon. The Newton polygon will be the same as we used for fixed x, however, now x can contribute $\hbar$ terms. For example,

$$\sigma + (1 - x) \tag{5.116}$$

would have a flat Newton polygon shown on the left of Figure 5.2 for fixed x while for $x = e^{m\hbar}$ it would have a Newton polygon with one edge of slope -1 shown on the right of the figure. We then solve using the Ansatz

$$\widehat{\Phi}(m;\hbar) \;=\; \hbar^{\kappa_0 m}\gamma_m \exp\Big(\sum_{\ell=2}^{\infty}\alpha_{-1,\ell}m^\ell\hbar^{\ell-1}\Big)\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\alpha_{k,\ell}m^\ell\hbar^k \;=\; \hbar^{\kappa_0 m}\gamma_m\Phi(m;\hbar)\,. \tag{5.117}$$

where again κ come from the slope of the Newton polygon, γ_m solves a recursion corresponding to solving the indicial polynomial and the others can be solved recursively from these solutions. We can consider our previous example with $x = q^{m\hbar}$.

Example 48. *Consider the $q = e^\hbar$ analogue of equation (5.63)*

$$\begin{aligned} &e^{m\hbar}\widehat{\Phi}(m;\hbar) + (1 - 3e^{(m+1)\hbar})\widehat{\Phi}(m+1;\hbar) \\ &\qquad + (3e^{(m+2)\hbar} - 1)\widehat{\Phi}(m+2;\hbar) - e^{(m+3)\hbar}\widehat{\Phi}(m+3;\hbar) \;=\; 0\,. \end{aligned} \tag{5.118}$$

This has Newton polygon shown in Figure 5.3. We find the recursion for γ_m give by

$$\gamma_m - 2\gamma_{m+1} + 2\gamma_{m+2} - \gamma_{m+3} \;=\; 0\,. \tag{5.119}$$

The solutions then come in the form $\gamma_m = \rho^m$ where

$$\rho^3 - 2\rho^2 + 2\rho - 1 \;=\; (\rho^2 - \rho + 1)(\rho - 1) \;=\; 0\,. \tag{5.120}$$

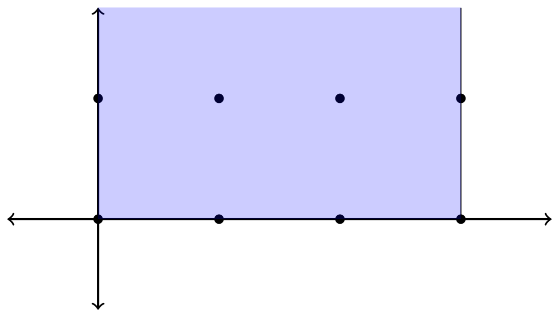

Figure 5.3: The Newton polygon of equation (5.118).

The three solutions are

$$\rho_0 \;=\; \frac{1}{2} \pm \frac{\sqrt{-3}}{2} \quad \textit{and} \quad \rho_0 \;=\; 1\,. \tag{5.121}$$

Then we have

$$\begin{aligned} &e^{m\hbar}\Phi(m;\hbar) + (1-3e^{(m+1)\hbar})\rho_0\Phi(m+1;\hbar) \\ &\qquad + (3e^{(m+2)\hbar}-1)\rho_0^2\Phi(m+2;\hbar) - e^{(m+3)\hbar}\rho_0^3\Phi(m+3;\hbar) \;=\; 0\,. \end{aligned} \tag{5.122}$$

Therefore, taking $\Phi(0;\hbar) = 1$ we can calculate the $\hbar$ term in the expansion of equation (5.122) to find for $\rho = 1/2 - \sqrt{-3}/2$ that

$$(4 - 3\sqrt{-3} - 2\sqrt{-3}m)\alpha_{-1,2} + \frac{1}{2}(-3 - 3\sqrt{-3} - 2m - 2\sqrt{-3}\alpha_{1,1}) \;=\; 0\,. \tag{5.123}$$

Then solving this triangular linear system for $\alpha_{-1,2}, \alpha_{1,1}$ we find that

$$\alpha_{-1,2} \;=\; -\frac{1}{2\sqrt{-3}}\,, \quad \textit{and} \quad \alpha_{1,1} \;=\; -\frac{5}{6}\,. \tag{5.124}$$

This gives

$$\Phi(m;\hbar) \;=\; 1 + \Big(\frac{\sqrt{-3}}{6}m^2 - \frac{5}{6}m\Big)\hbar + \cdots\,. \tag{5.125}$$

Taking the solution we found for the t dependent version in Example 47, we can substitute $t = e^{m\hbar}$ and find the same solution. However, this method is a slightly simplified complexity of course with the pay off of less information.

These algorithms give rise to efficient ways to compute solutions to q–difference equations as asymptotic series. For certain series, there is an even more efficient way to compute them. This makes use of the Habiro ring, which will be the topic of the next section.

5.8 The Habiro ring

Definition 16 (The Habiro ring)**.** *The Habiro ring is defined to be the inverse limit,*

$$\widehat{\mathbb{Z}[q]} \;=\; \lim_{\substack{\longleftarrow \\ n}} \mathbb{Z}[q]/((q;q)_n)\,. \tag{5.126}$$

This means that an element is represented by $(h_n(q) + (q;q)_n\mathbb{Z}[q])_{n\in\mathbb{Z}_{\geq 0}}$ such that for $m < n$ we have

$$h_n(q) + (q;q)_m\mathbb{Z}[q] \;=\; h_m(q) + (q;q)_m\mathbb{Z}[q]\,. \tag{5.127}$$

Therefore, all elements in $f \in \widehat{\mathbb{Z}[q]}$ can be represented, non-uniquely, via a sequence $f_k(q) \in \mathbb{Z}[q]$ by

$$f(q) \;=\; \sum_{k=0}^{\infty} f_k(q)(q;q)_k\,. \tag{5.128}$$

Example 49 (Non–uniqueness of representatives in $\widehat{\mathbb{Z}[q]}$)**.** *[94, Prop. 7.1] We have the following equality in* $\widehat{\mathbb{Z}[q]}$

$$1 \;=\; \sum_{k=0}^{\infty} q^{k+1}(q;q)_k\,. \tag{5.129}$$

This follows from the fact that

$$\sum_{k=0}^{\infty} q^{k+1}(q;q)_k \;=\; \sum_{k=0}^{\infty}(q^{k+1}-1)(q;q)_k+\sum_{k=0}^{\infty}(q;q)_k \;=\; -\sum_{k=1}^{\infty}(q;q)_k+\sum_{k=0}^{\infty}(q;q)_k \;=\; 1\,. \tag{5.130}$$

Remark 18. *As pointed out, for example in [126], the Habiro ring can be thought of as a deformation of*

$$\widehat{\mathbb{Z}} \;=\; \lim_{\substack{\longleftarrow \\ n}} \mathbb{Z}/(n!\mathbb{Z})\,. \tag{5.131}$$

To each element of the Habiro ring, we can associate a function from roots of unity, μ, to $\mathbb{C}$. For $f \in \widehat{\mathbb{Z}[q]}$ represented by $f_k(q)$ we define a function[4] $f : \mu \to \mathbb{C}$ such that for $\zeta^n = 1$ we have

$$f(\zeta) \;=\; \sum_{k=0}^{n-1} f_k(\zeta)(\zeta;\zeta)_k \in \mathbb{Z}[\zeta]\,. \tag{5.132}$$

This is well defined as the evaluation map at roots of unity sends all but finitely many of the ideals $(q;q)_n\mathbb{Z}[q]$ to zero. This function uniquely determines the element in the Habiro ring, as shown in the following theorem.

[4]This is a slight abuse of notation as we are using the same notation for both the element of the Habiro ring and the function. This is justified by Theorem 34.

Theorem A–34. *[94, Thm. 6.3] If $f \in \widehat{\mathbb{Z}[q]}$ and $f : \mu \to \mathbb{C}$ is the zero function then $f = 0$.*

We will prove this theorem shortly but first; notice that there exists $c_k \in \mathbb{Z}$ such that

$$f(q) \in \sum_{k=0}^{m-1} c_k (q-1)^k + (q-1)^m \widehat{\mathbb{Z}[q]} \,. \tag{5.133}$$

This expansion is called the *Ohtsuki expansion* [148] and its existence follows from the fact that for some $f_k(q) \in \mathbb{Z}[q]$, we have

$$f(q) \in \sum_{k=0}^{m-1} f_k(q)(q;q)_k + (q;q)_m \widehat{\mathbb{Z}[q]} \quad \text{and} \quad (q;q)_m \widehat{\mathbb{Z}[q]} \subseteq (q-1)^m \widehat{\mathbb{Z}[q]} \,. \tag{5.134}$$

Lemma 16. *Suppose that $f \in \widehat{\mathbb{Z}[q]}$. If p is a prime,*

$$f(\zeta_p) \;=\; \sum_{\ell=0}^{p-2} a_\ell \zeta_p^\ell \,, \tag{5.135}$$

and

$$f(q) \in \sum_{k=0}^{p-2} c_k (q-1)^k + (q-1)^{p-1} \widehat{\mathbb{Z}[q]} \,, \tag{5.136}$$

then,

$$c_k \equiv \sum_{\ell=k}^{p-2} \binom{\ell}{k} a_\ell \pmod{p} \,. \tag{5.137}$$

Proof. Notice that

$$(\zeta_p - 1)^{p-1} + p\mathbb{Z}[\zeta_p] \;=\; \zeta_p^{p-1} + \zeta_p^{p-2} + \cdots + 1 + p\mathbb{Z}[\zeta_p] \;=\; p\mathbb{Z}[\zeta_p] \,. \tag{5.138}$$

Therefore, reducing $\pmod{p}$

$$\begin{aligned} f(\zeta_p) + p\mathbb{Z}[\zeta_p] \;&=\; \sum_{\ell=0}^{p-2} a_\ell (\zeta_p - 1 + 1)^\ell + p\mathbb{Z}[\zeta_p] \;=\; \sum_{\ell=0}^{p-2} \sum_{k=0}^{\ell} \binom{\ell}{k} a_\ell (\zeta_p - 1)^k + p\mathbb{Z}[\zeta_p] \\ &=\; \sum_{k=0}^{p-2} (\zeta_p - 1)^k \sum_{\ell=k}^{p-2} \binom{\ell}{k} a_\ell + p\mathbb{Z}[\zeta_p] \;=\; \sum_{k=0}^{p-2} c_k (\zeta_p - 1)^k + p\mathbb{Z}[\zeta_p] \,. \end{aligned} \tag{5.139}$$

Therefore, noting that this ring is $(\mathbb{Z}/p\mathbb{Z})[\zeta_p - 1]/(\zeta_p - 1)^{p-1}$, we find that

$$c_k + p\mathbb{Z} \;=\; \sum_{\ell=k}^{p-2} \binom{\ell}{k} a_\ell + p\mathbb{Z} \,. \tag{5.140}$$

□

Proof of Theorem 34. From Lemma 16, we see that congruences classes of c_k, the coefficients in the expansion in equation (5.133), modulo all prime numbers are determined by the function $f : \mu \to \mathbb{C}$, which therefore also determines $c_k \in \mathbb{Z}$. Moreover, if $f : \mu \to \mathbb{C}$ is the zero function then $c_k = 0$ for all k. Therefore,

$$f(q) \in \bigcap_{m \in \mathbb{Z}_{\geq 0}} (q-1)^m \widehat{\mathbb{Z}[q]}\,. \tag{5.141}$$

Then we note that

$$\bigcap_{m \in \mathbb{Z}_{\geq 0}} (q-1)^m \widehat{\mathbb{Z}[q]} \;=\; 0\,, \quad \text{as} \quad \bigcap_{m \in \mathbb{Z}_{\geq 0}} (q-1)^m (\mathbb{Z}[q] + (q;q)_n \mathbb{Z}[q]) \;=\; 0\,. \tag{5.142}$$

□

Sometimes, we are given a function from roots of unity to the complex numbers. If this function takes values in the cyclotomic integers of the associated root, then it could come from an element of the Habiro ring. Lemma 16 gives additional conditions. In practice, one can construct the congruences that the hypothetical c_k should satisfy and try to lift them. If they eventually seem to have a constant lift as we increase the primes, this gives the possibility that our function is in the Habiro ring. This is illustrated in the code below for Example 50.

Example 50 (Kontsevich–Zagier function)**.** *Kontsevich considered the element of the Habiro ring*

$$F(q) \;=\; \sum_{k=0}^{\infty} (q;q)_k\,, \tag{5.143}$$

in relation to Feynman path integrals at a talk in Bonn during October 1997. This function appeared in the prior work of Kashaev as an invariant of the trefoil [106]. Kontsevich suggested this function had a relation to the Dedekind η–function. Zagier then went on to prove this relation [204], which he termed a "strange identity", and found, using [177], a relation to the dimension of spaces of Vassiliev invariants discussed in [179]. We will use this as an example to check these basic properties. Firstly, we have the expansion

$$F(q) \;=\; 1 - (q-1) + 2(q-1)^2 - 5(q-1)^3 + 15(q-1)^4 - 53(q-1)^5 + \cdots\,. \tag{5.144}$$

The first few values at roots of unity are given by

$$F(\zeta_2) \;=\; 3\,, \quad F(\zeta_3) \;=\; -\zeta_3 + 5\,, \quad F(\zeta_4) \;=\; -3\zeta_4 + 8\,, \quad F(\zeta_5) \;=\; -3\zeta_5^2 - 5\zeta_5 + 9\,. \tag{5.145}$$

Now taking the central lift of the congruence classes for c_4 coming from Lemma 16, we see that this stablises to 15 *for large enough primes. Of course large enough primes are the primes greater than* $2c_4 = 30$*. One can use the Chinese remainder theorem to lower the size of the primes needed. This is all done in Code 24.*

Remark 19. *Using the Ohtsuki expansion we can replace* $(q-1)$ *with* $\exp(\hbar)-1$. *This will produce a formal series in* $\hbar$ *with rational coefficients.*

Given a q–difference equation we can ask whether there are any solutions in the Habiro ring. This seems to be a very special property of q–difference equations however many of the examples which we study will fall into this class. We can consider our previous example.

Example 51. *The q–difference equation (5.63) we saw has solution given in equation (5.69), which is in a form similar to elements of the Habiro ring. Indeed, taking this q–difference equation and specialising to $t = q^m$ we find the equation*

$$q^m f(m;q)+(1-3q^{m+1})f(m+1;q)+(3q^{m+2}-1)f(m+2;q)-q^{m+3}f(m+3;q) \;=\; 0\,, \quad (5.146)$$

which has solution

$$f(m;q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2+mk}(q;q)_k^2 \in \widehat{\mathbb{Z}[q]}\,. \quad (5.147)$$

Therefore, using the fact that we can substitute $q = e^{\hbar}$ into elements of the Habiro ring we find a solution near $q = 1$. In this example we find the solution

$$\Phi(m;\hbar) \;=\; 1-\hbar^2-m\hbar^3+\Big(-\frac{1}{2}m^2+\frac{47}{12}\Big)\hbar^4+\Big(-\frac{1}{6}m^3+\frac{95}{12}m\Big)\hbar^5+\cdots\,. \quad (5.148)$$

Therefore, we see that solutions in the Habiro ring give rise to solutions near $q = 1$. However, they do more than this. They of course gives solutions that are functions from roots of unity to $\mathbb{C}$ but even more they give solutions which have formal series attached to each root of unity. The constants of these formal series correspond to the function from roots of unity and we have natural candidates for other solutions that are functions from roots of unity. Indeed, we saw in Section 4.6 that we can compute the constant terms of asymptotic series associated to q–hypergeomtric functions. These asymptotics should then provide solutions to the same q–difference equations. In the next section we will focus on their constant terms which correspond to functions from root of unity to $\mathbb{C}$.

5.9 “Black magic” formulae

As mentioned previously, if a function in q satisfies a q–difference equation then its asymptotics must therefore also satisfy a q–difference equation. We can then consider the leading asymptotics as it approaches roots of unity. This will provide a function from roots of unity that will also satisfy the same q–difference equation.

The form of these asymptotics leads us to explore certain Ansatz of solutions. This was explored by Garoufalidis and Zagier in unpublished work and [86] not emphasising the q–difference equations. This was implicit in their work and the q–difference equation perspective was pushed further in some unpublished joint work with Garoufalidis. The basic idea here is that we are looking for solutions to homogenous equations which are finite sums and therefore we want then to have no boundaries. This leads us to consider hypergeometric sums that are defined as sums over $\mathbb{Z}^N/c\mathbb{Z}^N$. These functions will give rise to explicit representations of the module when we specialise $q \in \mu\!\!\!/$.

We can explore this with the explicitly computed asymptotics in Example 39.

Example 52. *Consider the functions from roots of unity such that for $q = \mathbf{e}(a/c)$ we have*

$$f_j(m;q) \;=\; \sum_{r\in\mathbb{Z}/c\mathbb{Z}} \frac{q^{2r^2+rm}X_j^{(4r+m)/|c|}}{(qX_j^{1/|c|};q)_r} \tag{5.149}$$

where

$$1 - X_j \;=\; X_j^4\,. \tag{5.150}$$

First, notice that this is well defined as

$$\frac{q^{2c^2+|c|m}X_j^{4+m/|c|}}{(qX^{1/|c|};q)_c} \;=\; X_j^{m/|c|}\,. \tag{5.151}$$

Then taking the quotient of two consecutive terms in the sum gives

$$\frac{q^{4r+2+m}X^{4/|c|}}{1-q^{r+1}X^{1/|c|}}\,. \tag{5.152}$$

Therefore, we see that

$$f_j(m;q) - f_j(m+1;q) \;=\; q^{2+m}f_j(4+m;q)\,. \tag{5.153}$$

The name of this section refers to a joke of Stavros. In particular, given a q–hypergeometric sum like

$$\sum_{k=0}^{\infty} \frac{q^{2k^2+km}}{(q;q)_k}\,, \tag{5.154}$$

it is easy to guess the solution given in Example 52 as they have the same shape. This is true more generally and give remarkable formulae for the asymptotics of q–hypergeometric functions at roots of unity, which store interesting information related to the Bloch group [34].

We can push this further and take the full asymptotic series at each root of unity where $q = \mathbf{e}(a/c + \tilde{\tau}_\gamma/|c|)$ so that

$$f_j(m;q) \in \mathbb{C}[[\tilde{\tau}_\gamma]]\,. \tag{5.155}$$

Unfortunately, there is no particularly nice formula for these asymptotic series as they arise from stationary phase approximations and therefore from Gaussian integration, which in general will involve Feynman diagrams. As there is no closed formulae, after the constant and the initial condition, the solutions can be constructed using a generalisation of the methods of Example 48 to other roots. Indeed, the Ansatz will now involve powers of the roots we are expanding at. We will take a simple inhomogenous example, which determines the shape of the equation.

Example 53. *Consider an inhomogenous equation*

$$\Phi_{1/2,m-1}(\tilde{\tau}_\gamma) + ((-1)^m e^{\tilde{\tau}_\gamma m/2} - 2)\Phi_{1/2,m}(\tilde{\tau}_\gamma) + \Phi_{1/2,m+1}(\tilde{\tau}_\gamma) \;=\; 1. \tag{5.156}$$

with $\Phi_{1/2,m}(0) = \gamma_1 + \gamma_{-1}(-1)^m$*. The we find*

$$\gamma_1 - 2\gamma_1 + \gamma_{-1} + \gamma_1 + (-\gamma_{-1} + \gamma_1 - 2\gamma_{-1} - \gamma_{-1})\,(-1)^m \;=\; 1 \tag{5.157}$$

Therefore

$$\begin{aligned} \gamma_{-1} \;&=\; 1 \\ \gamma_1 - 4\gamma_{-1} \;&=\; 0 \end{aligned} \tag{5.158}$$

and therefore

$$\Phi_{1/2,m}(0) \;=\; 4 + (-1)^m\,. \tag{5.159}$$

Remark 20. *A natural question is whether these functions from roots of unity determine the associated asymptotic series as we observed for the elements of the Habiro ring. This would lead to some remarkable consequences especially if they allow for efficient computation of the asymptotic series.*

We close this section remarking on how one can prove identities between functions at roots of unity. The basic ideas discussed from proving q–series won't work the functions will no longer be elliptic with respect to some lattice as the lattice has collapsed to the reals. Therefore, we find that the basic functions we need to compute become periodic functions. These therefore come with a Fourier expansion. This means that computing this Fourier expansion will provide a means of proving identities. This can be thought of as the analogous computation on the elliptic curve with a node. This is isomorphic to $\mathbb{P}^1$ with a point identified. Indeed, if this function is bounded then it is also constant by Liouville's theorem.

5.10 "Upside down cake" and the Habiro ring

As noticed in unpublished work of Garoufalidis some years ago, we can construct some strange looking expressions for certain elements of the Habiro ring. These expressions will arise when we factorise state integrals.

Lemma 17. *We have the following identity for $q = \mathbf{e}(a/c)$ where $a/c \in \mathbb{Q}$*

$$(q;q)_k \;=\; \frac{c}{(q^{-1};q^{-1})_{c-1-k}}\,. \tag{5.160}$$

Proof. Notice that

$$\frac{(q;q)_{k+1}}{(q;q)_k} \;=\; 1-q^{k+1} \;=\; \frac{(q^{-1};q^{-1})_{c-1-k}}{(q^{-1};q^{-1})_{c-2-k}}\,, \tag{5.161}$$

and

$$(q;q)_0 \;=\; 1 \;=\; \frac{c}{(q^{-1};q^{-1})_{c-1}}\,, \tag{5.162}$$

where we note that

$$(xq^{-1};q^{-1})_{c-1} \;=\; \sum_{k=0}^{c-1} x^k\,. \tag{5.163}$$

□

For example, this can be used to find the identity

$$\sum_{k=0}^{c-1}(-1)^k q^{-k(k+1)/2}(q;q)_k^2 t^{-1-k} \;=\; c^2\sum_{k=0}^{c-1}(-1)^k\frac{q^{k(k+1)/2}}{(q;q)_k^2}t^{k-c}\,. \tag{5.164}$$

Chapter 6

q–hypergeometric equations

The main actors in this thesis are *q–hypergeometric functions*. These are defined in the simplest sense as a power series $f(x_1, \dots, x_d)$ that satisfies a system of equations for $i = 1, \dots, d$ given by

$$\big(\alpha_i(\sigma_1, \dots, \sigma_d; q) + x_i\, \beta_i(\sigma_1, \dots, \sigma_d; q)\big) f \;=\; 0\,. \tag{6.1}$$

These equations, which are linear in x_i, give first order relations between the coefficients in the expansion of f. This leads to their other definition, which states that they are a sum over a cone in a lattice where adjacent terms in the lattice are related by multiplication by a rational function in $q, q^{n_1}, \cdots, q^{n_d}$ where n_i are the coordinates of the lattice.

We will mainly be interested in *proper q–hypergeometric functions*, which is when the recursions for coefficients completely factor into linear factors. This basically means that the summands will be q to some quadratic form multiplied by a product of q–Pochhammer symbols evaluated at some linear forms.

The full description of a q–hypergeometric function is the specialisation of one of the power series above when we fix some set of x_i to constants. To do this requires some convergence, which may need some kind of resummation in general.

In this section, we will explore some simple examples of q–hypergeometric functions. The main technical points that we will explore are related to duality. This will be of fundamental importance for proving quantum modularity. Before this, we close this introduction by noting that q–hypergeometric sums are efficiently computable. Indeed, given a q–hypergeometric sum of the form

$$\sum_{k\in\mathbb{Z}^n_{\geq 0}} a_{k_1,\dots k_n}(q) \tag{6.2}$$

we note that for some rational function f we have $a_{0,\dots,0,k_n+1} = f(k;q)a_{0,\dots,0,k_n}$. Therefore,

$$\sum_{k_n\in\mathbb{Z}_{\geq 0}} a_{0,\dots,0,k_n}(q) \tag{6.3}$$

is efficiently computed from this recursion. Then as the sum is q–hypergeometric the sum $\sum_{k_n\in\mathbb{Z}_{\geq 0}} a_{0,\dots,0,k_{n-1},k_n}(q)$ satisfies a recursion in k_{n-1}. Computing this recursion and the previous sum, one can efficiently compute

$$\sum_{k_{n-1},k_n\in\mathbb{Z}_{\geq 0}} a_{0,\dots,0,k_{n-1},k_n}(q)\,. \tag{6.4}$$

Continuing this logic one can compute the whole sum in $O(N)$ time. However, the higher dimension the sum requires more initial work as one needs to compute the recursions for the sum at each stage which leads to an additional $O(1)$ which could incidentally be large.

6.1 Dualities between modules of Nahm sums

Consider the following generalised Nahm sums

$$G_{A,B,r}(t;q) \;=\; \sum_{k=0}^{\infty} a_{A,B,r,k}(t;q) \;=\; \sum_{k=0}^{\infty} \frac{q^{A(Bk+r)(Bk+r+1)/2B}t^{Bk+r}}{(q;q)_{Bk+r}}\,. \tag{6.5}$$

We have

$$\frac{a_{A,B,r,k+1}(t;q)}{a_{A,B,r,k}(t;q)} \;=\; \frac{q^{AB(k+1/2)+A(r+1/2)}t^B}{(q^{Bk+r+1};q)_B} \tag{6.6}$$

and therefore, noting that from the q–binomial theorem

$$\begin{aligned}(q^{Bk+r+1-B};q)_B \;&=\; \sum_{\ell=0}^{B}(-1)^{\ell}q^{\ell(\ell+1)/2-\ell B}\binom{B}{\ell}_q q^{(Bk+r)\ell}\\ &=\; \sum_{\ell=0}^{B}(-1)^{\ell}q^{-\ell(\ell-1)/2}\binom{B}{\ell}_{q^{-1}} q^{(Bk+r)\ell}\,,\end{aligned} \tag{6.7}$$

we have

$$\sum_{\ell=0}^{B}(-1)^{\ell}q^{-\ell(\ell-1)/2}\binom{B}{\ell}_{q^{-1}} G_{A,B,r}(q^{\ell}t;q) \;=\; q^{A(B+1)/2}t^B G_{A,B,r}(q^A t;q)\,. \tag{6.8}$$

Suppose that $A>1=B$ then we have the difference equation [206, Eq. (37)]

$$G_{A,1,0}(q^A t;q) \;=\; q^{-A/2}t^{-1}G_{A,1,0}(t;q) - q^{-A/2}t^{-1}G_{A,1,0}(qt;q)\,. \tag{6.9}$$

Therefore, the last row of the companion matrix associated to this q–difference equation is

$$\begin{pmatrix} q^{-A}t^{-1} & -q^{-A}t^{-1} & 0 & \dots & 0\end{pmatrix}\,. \tag{6.10}$$

Therefore, $A(t,q^{-1})^{\mathrm{T}}$ last column

$$\begin{pmatrix} q^A t^{-1} & -q^A t^{-1} & 0 & \dots & 0 \end{pmatrix}^{\mathrm{T}} . \tag{6.11}$$

Therefore taking

$$P(t;q) \;=\; \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ : & : & \dots & : & : \\ 1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{6.12}$$

we find that

$$A(t,q)P(t;q) \;=\; \begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ : & : & \dots & : & : \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & q^{-A}t^{-1} \end{pmatrix} \;=\; A(q^{2A}t,q^{-1})^{\mathrm{T}} P(t;q)\,. \tag{6.13}$$

This implies the following proposition.

Proposition 5. *The q–difference equation (6.15) for $A > 1 = B$ is associated to a module M with*

$$M^{\vee} \cong M^{\wedge}\,. \tag{6.14}$$

We will need a version of this later for $A < B$. So suppose that $A < B$ then we have

$$\begin{aligned} G_{A,B,r}(q^B t;q) \;=\; & q^{A/2(B+1)+B(B-1)/2}(-t)^B G_{A,B,r}(q^A t;q) \\ & + (-1)^{B-1} q^{B(B-1)/2} \sum_{k=0}^{B-1} (-1)^k q^{-k(k-1)/2} \binom{B}{k}_{q^{-1}} G_{A,B,r}(q^k t;q) \end{aligned} \tag{6.15}$$

Therefore, the last row of the companion matrix associated to this q–difference equation is

$$\begin{pmatrix} (-1)^{B-1} q^{B(B-1)/2} \\ (-1)^{B} q^{B(B-1)/2-1(1-1)/2} \binom{B}{1}_{q^{-1}} \\ (-1)^{B-1} q^{B(B-1)/2-2(2-1)/2} \binom{B}{2}_{q^{-1}} \\ \vdots \\ (-1)^{B-1-A} q^{B(B-1)/2-A(A-1)/2} \binom{B}{A}_{q^{-1}} + q^{A/2(B+1)+B(B-1)/2}(-t)^B \\ \vdots \\ q^{B(B-1)/2-(B-1)(B-2)/2} \binom{B}{B-1}_{q^{-1}} \end{pmatrix}^{\mathrm{T}} . \tag{6.16}$$

Therefore, $A(t,q)^{-\mathrm{T}}$ has first column

$$\begin{pmatrix} q^{-1(1-1)/2}\binom{B}{1}_{q^{-1}} \\ -q^{-2(2-1)/2}\binom{B}{2}_{q^{-1}} \\ \vdots \\ (-1)^{A+1}q^{-A(A-1)/2}\binom{B}{A}_{q^{-1}} + q^{A/2(B+1)}t^B \\ \vdots \\ (-1)^B q^{-(B-1)(B-2)/2}\binom{B}{B-1}_{q^{-1}} \\ (-1)^{B+1}q^{-B(B-1)/2} \end{pmatrix}. \tag{6.17}$$

and $A(q^{A+1}t, q^{-1})^{-1}$ has first row

$$\begin{pmatrix} q^{1(1-1)/2}\binom{B}{1}_q \\ -q^{2(2-1)/2}\binom{B}{2}_q \\ \vdots \\ (-1)^{A+1}q^{A(A-1)/2}\binom{B}{A}_q + q^{A(B-1)/2+B}t^B \\ \vdots \\ (-1)^B q^{(B-1)(B-2)/2}\binom{B}{B-1}_q \\ (-1)^{B+1}q^{B(B-1)/2} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} q^{(B-1)-1(1-1)/2}\binom{B}{1}_{q^{-1}} \\ -q^{2(B-1)-2(2-1)/2}\binom{B}{2}_{q^{-1}} \\ \vdots \\ (-1)^{A+1}q^{A(B-1)-A(A-1)/2}\binom{B}{A}_{q^{-1}} + q^{A(B-1)/2+B}t^B \\ \vdots \\ (-1)^B q^{(B-1)(B-1)-(B-1)(B-2)/2}\binom{B}{B-1}_{q^{-1}} \\ (-1)^{B+1}q^{B(B-1)/2} \end{pmatrix}^{\mathrm{T}}. \tag{6.18}$$

Let

$$P(t,q) = \begin{pmatrix} t^{-(B-1)} & 0 & 0 & \dots & 0 & 0 \\ 0 & P_{2,2} & P_{2,3} & \dots & P_{2,(B-1)} & P_{2,B} \\ 0 & P_{3,2} & P_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & P_{B,2} & 0 & \dots & 0 & 0 \end{pmatrix} \tag{6.19}$$

where

$$\begin{aligned} &P_{2+k,j}(t,q) \\ &\quad = (-1)^{j+k-1}q^{(j+k-1)(B-1)-(j+k)(j+k-1)/2}\binom{B}{j+k}_{q^{-1}}(q^k t)^{-B+1} + \delta_{j+k,A}q^{A(B-1)/2+1}(q^k t)\,. \end{aligned} \tag{6.20}$$

With this P we see that

$$\begin{aligned} &A(t,q)^{-\mathrm{T}}P(t,q) \\ &= \begin{pmatrix} q^{-1(1-1)/2}\binom{B}{1}_{q^{-1}}t^{-(B-1)} & P_{2,2}(t;q) & P_{2,3}(t;q) & \dots & P_{2,(B-1)}(t;q) & P_{2,B}(t;q) \\ -q^{-2(2-1)/2}\binom{B}{2}_{q^{-1}}t^{-(B-1)} & P_{3,2}(t;q) & P_{3,3}(t;q) & \dots & P_{3,(B-1)}(t;q) & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ (-1)^{A+1}q^{-A(A-1)/2}\binom{B}{A}_{q^{-1}}t^{-(B-1)} + q^{A/2(B+1)}t & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ (-1)^B q^{-(B-1)(B-2)/2}\binom{B}{B-1}_{q^{-1}}t^{-(B-1)} & P_{B,2}(t;q) & 0 & \dots & 0 & 0 \\ (-1)^{B+1}q^{-B(B-1)/2}t^{-(B-1)} & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{aligned} \tag{6.21}$$

and

$$
\begin{aligned}
&(P(qt,q)A(q^{A+1}t,q^{-1})^{-1})^{\mathrm{T}} \\
&= \begin{pmatrix}
q^{(B-1)-1(1-1)/2}\binom{B}{1}_{q^{-1}}(qt)^{-(B-1)} & P_{2,2}(qt;q) & P_{3,2}(qt;q) & \cdots & P_{(B-1),2}(qt;q) & P_{B,2}(qt;q) \\
-q^{2(B-1)-2(2-1)/2}\binom{B}{2}_{q^{-1}}(qt)^{-(B-1)} & P_{2,3}(qt;q) & P_{3,3}(qt;q) & \cdots & P_{(B-1),3}(qt;q) & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
(-1)^{A+1}q^{A(B-1)-A(A-1)/2}\binom{B}{A}_{q^{-1}}(qt)^{-(B-1)} + q^{A(B-1)/2+1}t & * & * & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
(-1)^{B}q^{(B-1)(B-1)-(B-1)(B-2)/2}\binom{B}{B-1}_{q^{-1}}(qt)^{-(B-1)} & P_{2,B}(qt;q) & 0 & \cdots & 0 & 0 \\
(-1)^{B+1}q^{B(B-1)/2}(qt)^{-(B-1)} & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\end{aligned}
\tag{6.22}
$$

Noting that

$$P_{k+1,j}(t,q) \;=\; P_{k,j+1}(qt,q) \tag{6.23}$$

and checking the equality for the first row and column we find that

$$P(qt,q)A(q^{A+1}t,q^{-1})^{-1} \;=\; A(t,q)^{-\mathrm{T}}P(t,q)\,. \tag{6.24}$$

Therefore, we have the following proposition.

Proposition 6. *The q–difference equation (6.15) for $A < B$ is associated to a module M with*

$$M^{\vee} \cong M^{\wedge}\,. \tag{6.25}$$

We want to give explicit identities between solutions to these q–difference equations associated to these dualities. These identities are strictly stronger than the duality but often the duality will assist in the proof of such identities.

6.2 Identities between functions from duality

Firstly, for $A \in 2\mathbb{Z}$ we have formulae between the functions at roots of unity $q = \mathbf{e}(a/c)$ and m

$$f_j(m;q) \;=\; \frac{\prod_{\ell=1}^{c-1}\left(1-q^{\ell}\right)^{\frac{\ell}{c}-\frac{1}{2}}\sum_{r\in\mathbb{Z}/c\mathbb{Z}}\frac{q^{Ar^2/2+rm}X_j(m)^{(Ar+m)/c}}{(qX_j(m)^{1/c};q)_r}}{\prod_{s=0}^{c-1}(1-q^{s+1}X_j(m)^{1/c})^{\frac{s+1}{c}-\frac{1}{2}}\sqrt{cX_j(m)/(1-X_j(m))+Ac}} \tag{6.26}$$

where

$$1-X_j(m) \;=\; \mathbf{e}(am)X_j(m)^{A} \tag{6.27}$$

Let

$$f(m;q) \;=\; \begin{pmatrix} f_1(m;q) & \cdots & f_A(m;q) \\ f_1(m+1;q) & \cdots & f_A(m+1;q) \\ \vdots & \cdots & \vdots \\ f_1(m+A;q) & \cdots & f_A(m+A;q) \end{pmatrix}. \tag{6.28}$$

Proposition 7 (Quadratic relations for Nahm sums)**.**

$$f(m;q)f(-m+1-A;q^{-1})^{\mathrm{T}} \;=\; \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{6.29}$$

Proof. We have

$$f(m-A-1;q^{-1})^{\mathrm{T}} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}^{-1} f(m;q) \tag{6.30}$$

is periodic as $m \mapsto m+1/c$ from Proposition 5. This defines a function on the cusped elliptic curve and similar to that case if it is bounded it must be constant. Indeed, it has a Fourier series and therefore if it has limits as $m \to \pm i\infty$ then it is constant. With this explicit formula one can compute this limit and find the identity matrix. To do this you use that for example that as $m \sim i\infty$, $a/c > 0$ and for some indexing j of solutions

$$X_1(m) \;\sim\; 1 - \mathbf{e}(am) + \cdots\,, \tag{6.31}$$

and for $j > 1$

$$X_j(m) \;\sim\; \mathbf{e}\Big(-\frac{am}{A-1}\Big)\mathbf{e}\Big(\frac{j}{A-1}+\frac{1}{2A-2}\Big)\Big(1+\frac{\mathbf{e}(\frac{am}{A-1})}{A+1}+\cdots\Big)\,. \tag{6.32}$$

Plugging in this behaviour and taking the limit can then be explicitly computed as the identity matrix using the formula for the Gauss sum. For example, one has

$$\begin{aligned} f_1(m;q) \;&=\; \frac{\prod_{\ell=1}^{c-1}(1-q^\ell)^{\frac{\ell}{c}-\frac{1}{2}} \sum_{r\in\mathbb{Z}/c\mathbb{Z}} \frac{q^{Ar^2/2+rm}(1-\mathbf{e}(am)+\cdots)^{(Ar+m)/c}}{(q(1-\mathbf{e}(am)+\cdots)^{1/c};q)_r}}{\prod_{s=0}^{c-1}(1-q^{s+1}(1-\mathbf{e}(am)+\cdots)^{1/c})^{\frac{s+1}{c}-\frac{1}{2}}\sqrt{c(1-\mathbf{e}(am)+\cdots)/(1-(1-\mathbf{e}(am)+\cdots))+Ac}} \\ &=\; \frac{\prod_{\ell=1}^{c-1}(1-q^\ell)^{\frac{\ell}{c}-\frac{1}{2}} \sum_{r\in\mathbb{Z}/c\mathbb{Z}} \frac{q^{Ar^2/2+rm}(1-\mathbf{e}(am)(Ar+m)/c+\cdots)}{(q(1-\mathbf{e}(am)/c+\cdots);q)_r}}{\prod_{s=0}^{c-1}(1-q^{s+1}(1-\mathbf{e}(am)/c+\cdots))^{\frac{s+1}{c}-\frac{1}{2}}\sqrt{c\mathbf{e}(-am)+\cdots}} \\ &\sim\; 1+O(\mathbf{e}(ma/c))\,, \end{aligned} \tag{6.33}$$

which is already enough to calculate the $(1,1)$ entry of

$$f(-m+1-A;q^{-1})^{\mathrm{T}} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & -1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & -1 & \dots & 0 & 0 \end{pmatrix} f(m;q)\,. \tag{6.34}$$

For the other entries, we use the fact that for $j > 1$ we have

$$f_j(m;q) \;=\; \mu_j(m)\mathbf{e}\Big(\log(X_j(m))\frac{m}{c} + \frac{ma}{2c(A-1)}\Big)\frac{1}{\sqrt{A-1}}\Big(1+O\Big(\mathbf{e}\Big(\frac{ma}{c(A-1)}\Big)\Big)\Big)\,, \tag{6.35}$$

where $\mu_j(m)$ is a root of unity with the properties that

$$\sum_{j=1}^{A-1}\mu_j(m) \;=\; 0\,, \quad \text{and} \quad \mu_j(m)\mu_j(-m) \;=\; 1\,. \tag{6.36}$$

This follows from the computations

$$\sum_{r\in\mathbb{Z}/c\mathbb{Z}}\frac{q^{Ar^2/2+rm}X_j(m)^{Ar/c}}{(qX_j(m)^{1/c};q)_r} \;\sim\; \sum_{r\in\mathbb{Z}/c\mathbb{Z}} q^{(A-1)r^2/2+r(m-1)}X_j(m)^{(A-1)r/c} \tag{6.37}$$

is asymptotic to a root of unity times $\sqrt{A-1}$ where we note that $q^m X_j(m)^{(A-1)r/c}$ is asymptotic to root of unity plus lower order terms while

$$\prod_{s=0}^{c-1}(1-q^{s+1}X_j(m)^{1/c})^{\frac{s+1}{c}-\frac{1}{2}} \;\sim\; X_j(m)^{\frac{1}{2c}}\prod_{s=0}^{c-1}\frac{(-q^{s+1}X_j(m)^{1/c})^{\frac{s+1}{c}-\frac{1}{2}}}{X_j(m)^{\frac{s+1}{c^2}-\frac{1}{2c}}} \tag{6.38}$$

is asymptotic to $\mathbf{e}(-\frac{ma}{2c(A-1)})$ times a root of unity. Therefore, as $f(-m+1-A,q^{-1})^{\mathrm{T}}$ has first row $1+O(\mathbf{e}(ma/c))$ we see that as the first row of the product in equation (6.34) is given by the first row of $f(m;q)$ up to exponentially small contributions. We can then choose an interval for m which makes this clearly exponentially small. Similarly, the rest are all exponentially small except when we are on the diagonal where all $A-1$ terms not in the first column combine each giving $1/(A-1)$ and exponentially small corrections. Therefore, as $m \sim i\infty$ we see that this product limits to the identity. This completes the proof. □

6.3 Nahm sums and Neumann–Zagier equivalences

Recall, that to ideal triangulations of three–manifolds we can associated the Neumann–Zagier matrices of Section 1.11. These matrices are the top half of a symplectic matrix. We then saw that under various changes to the data of the triangulations we can find equivalence between these upper half symplectic matrices, which were described in Section 1.12. There is a somewhat natural q–series we can associated to any upper half symplectic matrix (A,B) given as some kind of generalised Nahm sum

$$g_{A,B}(t;q) \;=\; \sum_{k\in\mathbb{Z}^N/\ker(B^{\mathrm{T}})}\frac{q^{\frac{1}{2}k^{\mathrm{T}}AB^{\mathrm{T}}k}t_1^{(B^{\mathrm{T}}k)_1}\cdots t_N^{(B^{\mathrm{T}}k)_N}}{(q;q)_{(B^{\mathrm{T}}k)_1}\cdots(q;q)_{(B^{\mathrm{T}}k)_N}}\,. \tag{6.39}$$

Note that as $\mathrm{Im}(B^{\mathrm{T}}) = \mathrm{Ker}(B)^{\perp}$ and the dimension of the image of B and B^{T} agree, we see that $B : \mathrm{Im}(B^{\mathrm{T}}) \to \mathrm{Im}(B)$ is an isomorphism. Therefore, there exists a matrix a such that $Ba\ell = A\ell$ for $\ell \in \mathrm{Im}(B^{\mathrm{T}})$ as $A\ell \in \mathrm{Im}(B)$ from the symplectic property $AB^{\mathrm{T}} = BA^{\mathrm{T}}$. Therefore, we have

$$g_{A,B}(t;q) \;=\; \sum_{\ell\in\mathrm{Im}(B^{\mathrm{T}})} \frac{q^{\frac{1}{2}\ell^{\mathrm{T}}a\ell}t_1^{\ell_1}\cdots t_N^{\ell_N}}{(q;q)_{\ell_1}\cdots(q;q)_{\ell_N}}\,. \tag{6.40}$$

The generators of the equivalence classes between Neumann–Zagier matrices then become relations between these q–series when they are convergent. The best way to view these relations is at the level of q–holonomic modules, however there are subtleties. In particular the $2-3$ move changes the algebra over which the module is defined and one needs to consider pull back modules.

Remark 21. *It was observed with Garoufalidis that even though the $2-3$ Pachner move induces an isomorphisms between one module and a pullback module, the ranks of these holonomic systems can change under a $2-3$ Pachner move without the pullback. This prevents a result of a similar form to Theorem 1.*

The first relation of equation (1.59) is clear for all examples. We won't consider the full generality of the other relations here and instead focus on the relations between rank one Nahm sums for equation (6.5). Firstly, notice that for the second relation of equation (1.62) with $P \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} G_{A,B,r}(t;q) \;&=\; \sum_{k=0}^{\infty} \frac{q^{A(Bk+r)(Bk+r+1)/2B}t^{Bk+r}}{(q;q)_{Bk+r}} \;=\; \sum_{s=0}^{P-1}\sum_{k=0}^{\infty} \frac{q^{A(B(Pk+s)+r)(B(Pk+s)+r+1)/2B}t^{B(Pk+s)+r}}{(q;q)_{B(Pk+s)+r}} \\ &=\; \sum_{s=0}^{P-1}\sum_{k=0}^{\infty} \frac{q^{PA(PBk+Bs+r)(PBk+Bs+r+1)/2PB}t^{PBk+Bs+r}}{(q;q)_{PBk+Bs+r}} \;=\; \sum_{s=0}^{P-1} G_{PA,PB,Bs+r}(t;q)\,. \end{aligned} \tag{6.41}$$

In a similar direction, we have

$$G_{\frac{A}{B},1,0}(\mathbf{e}(s/B)t;q) \;=\; \sum_{k=0}^{\infty}\mathbf{e}\Big(\frac{ks}{B}\Big)\frac{q^{Ak(k+1)/2B}t^k}{(q;q)_k} \;=\; \sum_{r=0}^{B-1}\mathbf{e}\Big(\frac{rs}{B}\Big)\sum_{k=0}^{\infty}\frac{q^{A(Bk+r)(Bk+r+1)/2B}t^k}{(q;q)_{Bk+r}}\,. \tag{6.42}$$

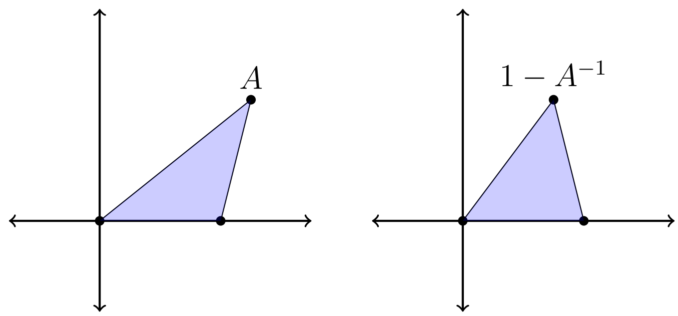

Figure 6.1: The Newton polygon of two related Nahm sums.

Then for the relation of equation 1.64 we have for $B = 1$ and $A \in \mathbb{Z}_{>1}$

$$\begin{aligned} G_{A,1,0}(t;q) &= \sum_{k=0}^{\infty} \frac{q^{Ak(k+1)/2} t^k}{(q;q)_k} = \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} q^{Ak(k+1)/2} t^k (q^{k+1};q)_\infty \\ &= \frac{1}{(q;q)_\infty} \sum_{k,\ell=0}^{\infty} (-1)^\ell \frac{q^{Ak(k+1)/2+\ell(\ell+1)/2+k\ell} t^k}{(q;q)_\ell} \\ &= \frac{1}{(q;q)_\infty} \sum_{r=0}^{A-1} \sum_{\ell=0}^{\infty} (-1)^{A\ell+r} \frac{q^{(A\ell+r)(A\ell+r+1)/2}}{(q;q)_{A\ell+r}} \theta(-q^{A\ell+r} t; q^A) \\ &= \frac{1}{(q;q)_\infty} \sum_{r=0}^{A-1} q^{r(r+A)/2A} t^{r/A} \theta(-q^r t; q^A) G_{A-1,A}(-q^{(1-A)/2A} t^{-1/A}; q)\,. \end{aligned} \tag{6.43}$$

So we see that the Nahm sums at $(A, 1)$ and $(1 - A^{-1}, 1)$ are related. This relation is extremely useful. For example, compare the two Newton polygons in Figure 6.1. Therefore, for $(1 - A^{-1}, 1)$ we can construct a basis of power series solutions whereas for $(A, 1)$ we need to apply some q–Borel resummation.

6.4 Deformations and the Rogers–Ramanujan identities

We close this section on Nahm sums with a discussion on deformation of q–hypergeometric sums. In particular, given a q–hypergeometric sum

$$\sum_{k \in \mathbb{Z}^N} a_k(q) \tag{6.44}$$

often we can naturally take the sum

$$\sum_{k \in \mathbb{Z}^N} a_{k+\epsilon}(q) \tag{6.45}$$

where for the Pochhammer symbols

$$(q;q)_{k+\epsilon} \;=\; \frac{(q^{k+\epsilon+1};q)_\infty}{(q;q)_\infty} \;=\; \frac{(q^{\epsilon+1};q)_\infty}{(q;q)_\infty (qe^{\epsilon};q)_k}\,. \tag{6.46}$$

These q–hypergeometric functions that arise from these deformations will then satisfy the same q–difference equations. Then taking the natural deformations, they seem to satisfy the same modularity properties of the original functions. These deformations are similar to the Example 31. We will consider a special example of this and prove the modularity later. Consider, the deformed Nahm sum for $(2,1)$.

$$G_m(x;q) \;=\; \frac{(qx;q)_\infty}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}\frac{q^{k^2+km}x^{2k+m}}{(qx;q)_k}\,. \tag{6.47}$$

This satisfies the q–difference equation

$$G_m(x;q) - G_{m+1}(x;q) \;=\; q^{m+1}G_{m+2}(x;q)\,, \tag{6.48}$$

which we see is independent of x as we explained. The functions for $m=0,1$ are deformations of the Roger–Ramanujan functions[164]. Take

$$F_m(x;q) \;=\; x^{2m}\frac{(qx;q)_\infty}{(q^{m+1}x;q)_\infty}\sum_k(-1)^k q^{2km}x^{5k}q^{k(5k-1)/2}(1-q^{2k+m}x^2)\frac{(q^{m+1}x;q)_{k-1}}{(qx;q)_k}\,. \tag{6.49}$$

Theorem 2. *We have the following equality*

$$G_m(x;q) \;=\; -\frac{(q;q)_\infty^2}{x^3q\theta(-x;q)\theta(q^{\frac{1}{2}}x;q)\theta(-q^{\frac{1}{2}}x;q)}F_m(x;q)\,. \tag{6.50}$$

Proof. Firstly, we can check that both sides satisfy the same recursion in m. Then noting that

$$\begin{pmatrix} G_{m+1}(x;q) & G_{m+2}(x;q)\\ F_{m+1}(x;q) & F_{m+2}(x;q)\end{pmatrix} \;=\; \begin{pmatrix} G_m(x;q) & G_{m+1}(x;q)\\ F_m(x;q) & F_{m+1}(x;q)\end{pmatrix}\begin{pmatrix} 0 & q^{-m-1}\\ 1 & -q^{-m-1}\end{pmatrix} \tag{6.51}$$

we can show that

$$\det\begin{pmatrix} q^{m^2}G_{2m}(x;q) & q^{m^2+m}G_{2m+1}(x;q)\\ q^{m^2}F_{2m}(x;q) & q^{m^2+m}F_{2m+1}(x;q)\end{pmatrix} \tag{6.52}$$

is independent of m. To show that this vanishes let

$$\begin{pmatrix} GE(x;q) & GO(x;q)\\ FE(x;q) & FO(x;q)\end{pmatrix} \;=\; \lim_{m\to-\infty}\begin{pmatrix} q^{m^2}G_{2m}(x;q) & q^{m^2+m}G_{2m+1}(x;q)\\ q^{m^2}F_{2m}(x;q) & q^{m^2+m}F_{2m+1}(x;q)\end{pmatrix} \tag{6.53}$$

Then from the formulae

$$\begin{aligned} GE(x;q) &= \frac{\theta(-q^{-1}x^2;q^2)}{(q;q)_\infty}, \qquad FE(x;q) = \frac{(q;q)_\infty}{\theta(x^{-1};q)}(\theta(-q^{-4}x^6;q^6) - x^2\theta(-q^{-2}x^6;q^6)), \\ GO(x;q) &= \frac{x\theta(-x^2;q^2)}{(q;q)_\infty}, \qquad FO(x;q) = -\frac{(q;q)_\infty}{\theta(x^{-1};q)}(x^3\theta(q^{-1}x^6;q^6) - x^5 q\theta(qx^6;q^6)). \end{aligned} \tag{6.54}$$

we see that

$$\begin{aligned} GE(qx;q) &= q^{-1}x^{-2}GE(x;q), \quad FE(qx;q) = -q^{-2}x^{-5}FE(x;q), \\ GO(qx;q) &= q^{-1}x^{-2}GO(x;q), \quad FO(qx;q) = -q^{-2}x^{-5}FO(x;q). \end{aligned} \tag{6.55}$$

Therefore we see that

$$\frac{FE(x;q)}{a^2\theta(x;q)^3 GE(x;q)} \quad \text{and} \quad \frac{FO(x;q)}{a^2\theta(x;q)^3 GO(x;q)} \tag{6.56}$$

are elliptic. With multiplicity, we see that the number of pole is equal to the number of zeros. Then notice that

- $\theta(x^3;q)GE(x;q)$ has only zeros and specifically $x \in q^{\mathbb{Z}}$ and $-q^{-1}x^2 \in q^{2\mathbb{Z}}$. Therefore $x \in q^{\mathbb{Z}}$ with multiplicity 3 or $x \in \pm i q^{\frac{1}{2}+\mathbb{Z}}$ with multiplicity 1,
- $\theta(x^3;q)GO(a;q)$ has only zeros and specifically $x \in q^{\mathbb{Z}}$ and $-x^2 \in q^{2\mathbb{Z}}$. Therefore $x \in q^{\mathbb{Z}}$ with multiplicity 3 or $x \in \pm i q^{\mathbb{Z}}$ with multiplicity 1.

Then we see that there are 5 poles and therefore as $FE(a;q)$ and $FO(a;q)$ only have a removable singularity at $x = q^{\mathbb{Z}}$ we see that we have 5 zeros. It is then easy to find all these zeros. Explicitly,

- $FE(x;q) = 0$ iff $x \in -q^{\mathbb{Z}}, \pm q^{\frac{1}{2}+\mathbb{Z}}, \pm i q^{\frac{1}{2}+\mathbb{Z}}$
- $FO(x;q) = 0$ iff $x \in -q^{\mathbb{Z}}, \pm q^{\frac{1}{2}+\mathbb{Z}}, \pm i q^{\mathbb{Z}}$.

Therefore, we see that

$$\frac{FE(x;q)GO(x;q)}{GE(x;q)FO(x;q)} \tag{6.57}$$

is holomorphic and elliptic and therefore constant in x. Noting that

$$\frac{FE(1;q)GO(1;q)}{GE(1;q)FO(1;q)} = 1, \tag{6.58}$$

which shows that

$$\det\begin{pmatrix} q^{m^2}G_{2m}(x;q) & q^{m^2+m}G_{2m+1}(x;q) \\ q^{m^2}F_{2m}(x;q) & q^{m^2+m}F_{2m+1}(x;q) \end{pmatrix} = 0, \tag{6.59}$$

we see that

$$\frac{G_m(x;q)}{F_m(x;q)} \tag{6.60}$$

is independent of m and in particular

$$\frac{G_m(x;q)}{F_m(x;q)} = \frac{GE(x;q)}{FE(x;q)} = \frac{GO(x;q)}{FO(x;q)}\,. \tag{6.61}$$

Then using elliptic functions again and the evaluation at $x = 1$ we see that

$$\frac{GE(x;q)}{FE(x;q)} = -\frac{(q;q)_\infty^2}{x^3 q\theta(-x;q)\theta(q^{\frac{1}{2}}x;q)\theta(-q^{\frac{1}{2}}x;q)} \tag{6.62}$$

Therefore we finally get to the identity

$$G_m(x;q) = -\frac{F_m(x;q)(q;q)_\infty^2}{x^3 q\theta(-x;q)\theta(q^{\frac{1}{2}}x;q)\theta(-q^{\frac{1}{2}}x;q)}\,. \tag{6.63}$$

□

6.5 Descendant Kashaev invariant and q–series

To a knot the $\widehat{A}$ polynomial from Theorem 18 associated a natural q–difference equations. There are other natural q–difference equations associated to knots more recently defined [71]. These difference equations always come from well defined invariants from roots of unity. However, it has become clear [92, 21, 68, 85] that considering solutions to these q–difference equations in q–series can give rise to functions related to the knot. They give a promising approach to analytic continuation of Chern–Simons theory expected in [197].

From the work of Habiro [94, 96] we have the cyclotomic expansion of Theorem 17. This turns the information of the coloured Jones polynomials J_N into the cyclotomic coefficients C_k. This sequence is q–holonomic and satisfies the recursion referred to as the $\widehat{C}$–polynomial. This was studied in [79]. Recall the cyclotomic expansion is of the shape

$$J_N(q) = \sum_{k=0}^{\infty} C_k(q)(q^{1+N};q)_k(q^{1-N};q)_k\,. \tag{6.64}$$

Then the $\widehat{C}$–polynomial is related to the $\widehat{A}$–polynomial in a similar way to a Fourier transform. In [71], a deformation of the coloured Jones polynomial was given as

$$J_{N,m}(q) = \sum_{k=0}^{\infty} C_k(q)(q^{1+N};q)_k(q^{1-N};q)_k q^{mk}\,. \tag{6.65}$$

This is called the *descendant coloured Jones polynomial.* This sequence is useful as we can specialise to $N=0$ to define the descendant Kashaev invariant

$$J_{0,m}(q) \;=\; \sum_{k=0}^{\infty} C_k(q)(q;q)_k^2 q^{mk}\,. \tag{6.66}$$

Before, the Kashaev invariant did not come as part of a family, and so this becomes extremely useful. Moreover, the $\widehat{C}$–polynomial give rise to a recursion for the $J_{0,m}(q)$ as a sequence of elements of the Habiro ring.

Example 54 (Descendant Kashaev invariant of 4_1)**.** *The descendant Kashaev invariant of the figure eight knot* 4_1 *is given by*

$$J_{0,m}(q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2+km}(q;q)_k^2\,. \tag{6.67}$$

This sequence satisfies the inhomogeneous recursion

$$q^{m-1}J_{0,m-1}(q)+(1-2q^m)J_{0,m}(q)+q^{m+1}J_{0,m+1}(q) \;=\; 1\,. \tag{6.68}$$

This example relates to our previous working Example 45. We see that there one can naturally compute q–series which satisfy the same homogenised recursion. This had a basis of three solutions as we saw in Example 45. This also has black magic solutions for $q=\mathbf{e}(a/c)$ given by

$$i\varepsilon(q)^2\sum_{k\in\mathbb{Z}/c\mathbb{Z}}\frac{(-1)^k q^{k(k+1)/2-(1+k)m}X_\ell^{(k-m)/c}X_\ell^{1/2c}}{\prod_{j=0}^{N-1}(1-q^{1+k+j}X_\ell^{1/c})^{2(1+j+k)/c-1}}\,, \tag{6.69}$$

where $X_\ell = 1/2+(-1)^\ell\sqrt{-3}/2$. This, together with the descendant Kashaev invariant, gives a basis of three solutions when q is a root of unity.

The important observation about the recursions associated to the descendant Kashaev invariant is that they seem to come as inhomogeneous equations. This follows from the shape of the Kashaev invariant. Previously, only the q–series associated to the homogenous part of the recursions were studied and for a long time it was wondered as to how one could construct q–series which have asymptotics related to the formal expansion of the Kashaev invariant. Given the Kashaev invariant satifies inhomogeneous q–difference equations so does its formal expansion and therefore and q–series that should have asymptotics related to this formal expansion must also satisfy an inhomogeneous recursion. For the example of 4_1, this led to the function

$$\mathfrak{G}(m;q) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2+km}}{(q;q)_k^2}\left(\frac{1}{8}\Big(2m-4G_1(q)+2\sum_{\ell=1}^{k}\frac{1+q^j}{1-q^j}\Big)^2-\frac{1}{24}+\sum_{\ell=1}^{k}\frac{q^\ell}{(1-q^\ell)^2}\right). \tag{6.70}$$

These types of functions can then similarly be constructed for other knots simply using the Frobenius method applied to the q–difference equations as exploited in [70]. The main issue here is a choice of initial condition to apply the method to. For example, for 5_2 there were different choice made in [153] and [70]. The choices made in [70] lead to the asymptotics agreeing with the Kashaev invariant, however were reached by specific example based means. Therefore, the most natural approach is to use state integrals that factorise in a way which includes the Kashaev invariant at rational numbers. This then provides a consistent set of initial conditions for the basis of solutions of the q–difference equation.

6.6 Holomorphic blocks and two variable series

The $\widehat{A}$–polynomial, the minimal q–difference equation satisfied by the coloured Jones polynomial, defines a q–holonomic module. We can construct a full basis of solutions to this equation again using the Frobenius method.

Historically, this was done for a submodule. As the A–polynomial will always contain factors of the form $(1-\ell)$, it is expected that the $\widehat{A}$–polynomial will always factor in this way so that the recursion for the coloured Jones can always be made inhomogeneous. The first examples done gave two variable series in x and q, which satisfied the homogenous part of the inhomogeneous equation. Therefore, these solutions corresponded to some submodule. This was done in examples in [21]. They start with expressions they refer to as block integrals, which are similar in nature to Watson's q–analogue of Barne's integral for Heine's q–hypergeometric sum.

Example 55 (Holomorphic block for 4_1)**.** *The two holomorphic blocks associated to 4_1 are given by the series*

$$\begin{aligned} g^{(0,x^{-1})}(x;q) &= \frac{\theta(q^{-1}x;q)\theta(x;q)(qx^2;q)_\infty}{\theta(x^2;q)(1-x)(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2}x^k}{(q;q)_k(qx^2;q)_k}\,, \\ g^{(0,x)}(x;q) &= \frac{\theta(q^{-1}x^{-1};q)\theta(x^{-1};q)(qx^{-2};q)_\infty}{\theta(x^{-2};q)(1-x^{-1})(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2}x^{-k}}{(q;q)_k(qx^{-2};q)_k}\,. \end{aligned} \tag{6.71}$$

It has long been noticed, and particularly pointed out by Gukov, that these series could not constitute a full TQFT as they don't contain information about the trivial flat connection. The reason is that they only solve the q–difference equation associated to the homogenous part of the inhomogeneous $\widehat{A}$–polynomial. Recently, in [92] an additional holomorphic block was proposed that additionally gives a solution to the inhomogeneous $\widehat{A}$–polynomial.

An important note about the inhomogeneity is that it somewhat fixes the normalisation. In particular, taking the solution from the Frobenius algorithm and insisting that it also satisfies the inhomogenous equation fixes the normalisation. Therefore, to find a natural normalisation becomes a simpler problem if the coloured Jones polynomial satisfies an inhomogenous equation. However, by manipulating the series coming from the coloured Jones polynomial itself, Gukov and Manolescu give rise to method of construction of the initial conditions. This provides a method to compute the final missing holomorphic block.

Example 56 (Two–variable series for 4_1)**.** *In [153, 70], the two variable series for 4_1 was given a q–hypergeometric formula that gave rise the numerically computed series of Gukov and Manolescu. This was given using the notation there as*

$$\begin{aligned} \Xi_{4_1}(x;q) &= (x^{1/2}-x^{-1/2})\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2}}{(x;q)_{k+1}(x^{-1};q)_{k+1}} \\ &= \sum_{k,j,\ell=0}^{\infty}(x^{k+j+\ell+1/2}-x^{k+j+\ell+3/2})\binom{k+j}{j}_q\binom{k+\ell}{\ell}_{q^{-1}} \end{aligned} \tag{6.72}$$

and so

$$\begin{aligned} F_{4_1}(x;q) &= \frac{1}{2}\sum_{k,j,\ell=0}^{\infty}(x^{k+j+\ell+1/2}-x^{k+j+\ell+3/2}-x^{-k-j-\ell-1/2}+x^{-k-j-\ell-3/2}) \\ &\qquad\times\binom{k+j}{j}_q\binom{k+\ell}{\ell}_{q^{-1}}. \end{aligned} \tag{6.73}$$

The residues of all three holomorphic blocks can be used to get the series $g, G, \mathfrak{G}$. This was explained in [69, 83].

6.7 Dualities for modules associated to simple knots

We close this section noting some dualities associated to some simple knots. These computations for 4_1 appeared in [82].

Proposition 8 (Duality of homogeneous part of descendant Kashaev invariant of 4_1)**.** *The q–difference module M associated to the q–difference equation*

$$q^{m-1}J_{0,m-1}(q)+(1-2q^m)J_{0,m}(q)+q^{m+1}J_{0,m+1}(q) = 0\,. \tag{6.74}$$

satisfies

$$M\cong M^{\wedge}\cong M^{\vee}\,. \tag{6.75}$$

Proof. The companion matrix of the q–difference equation is

$$A(t,q) = \begin{pmatrix} 0 & 1 \\ -q^{-2} & 2q^{-1}-q^{-2}t^{-1} \end{pmatrix} \tag{6.76}$$

Then take

$$P^\wedge(t,q) \;=\; \begin{pmatrix} 1 & 0 \\ 2q^{-1}-t^{-1}q^{-1} & -q^{-2} \end{pmatrix}, \quad P^\vee(t,q) \;=\; \begin{pmatrix} 0 & -q^{-1}t^{-2} \\ q^{-1}t^{-2} & 0 \end{pmatrix}. \tag{6.77}$$

□

The extra symmetry with $M \cong M^\wedge$ comes from the fact the 4_1 knot is amphichiral. This symmetry does not hold for the extension of the module to the inhomogeneous version.

Proposition 9. *The q–difference module $M^\vee$ associated to equation (6.68) is not isomorphic to $M^\wedge$.*

Proof. The companion matrices of $M^\wedge$ and $M^\vee$ are given by

$$A^\wedge(t,q) = \begin{pmatrix} 1 & 0 & 0 \\ q^{-1}t^{-1} & 2q^{-1}-t^{-1}q^{-1} & -q^{-2} \\ 0 & 1 & 0 \end{pmatrix}, \quad A^\vee(t,q) = \begin{pmatrix} 1 & t^{-1} & 0 \\ 0 & 2q-t^{-1} & 1 \\ 0 & -q^2 & 0 \end{pmatrix}. \tag{6.78}$$

If there was an isomorphism there would exist $P(t,q) \in \mathrm{GL}_3(\mathbb{Q}(t,q))$ such that

$$P(qt,q)A^\vee(t,q) = A^\wedge(t,q)P(t,q)\,. \tag{6.79}$$

It follows that

$$P_{1,1}(qt,q) = P_{1,1}(t,q), \qquad P_{1,2}(qt,q) = P_{1,3}(t,q), \tag{6.80}$$

which then implies

$$tP_{1,2}(t,q) + (1-2tq)P_{1,2}(qt,q) + q^2tP_{1,2}(q^2t,q) = P_{1,1}(qt,q). \tag{6.81}$$

Since $P_{1,1} \in \mathbb{Q}(t,q)$ satisfies (6.80), it is independent of t, i.e., $P_{1,1}(t,q) = P_{1,1}(q)$. Therefore, $P_{1,2}$ would be a $P_{1,1}(q)$ multiple of a $\mathbb{Q}(t,q)$-valued solution to Equation (6.68). The only such solution is zero, thus $P_{1,1} = P_{1,2} = 0$ which, together with (6.80) gives also $P_{1,3} = 0$, which violates the fact that P is invertible. □

6.8 The WRT module and $\widehat{Z}$ series

Until this work it was not clear how to associate a good q–difference equation to a closed 3–manifold. With this work it is still not clear, however, in Theorem 1 we saw that to a closed three–manifold presented by a link diagram, there is a canonical system of q–difference equations, which is q–Weyl finite. Moreover, we saw that the span of these functions from roots of unity is an invariant of the three–manifold. This then begs the question as to whether a basis of solutions to a q–difference equation associated to a given presentation of the closed three–manifold gives rise to invariants. This could then lead to q–series invariants similar to holomorphic blocks.

Using their two variable series associated to knots [92], Gukov and Manolescu give a conjectural surgery formula that can be used to construct q–series that give conjectural invariants of closed three–manifolds. These q–series associated to closed 3–manifolds are referred to as $\widehat{Z}$–series. These series were computed originally for non–hyperbolic manifolds [118, 99] and for hyperbolic manifolds for the first time in [92]. The definition of the surgery formula is via a Laplace transform analogous to that of [22]. The Laplace transform is defined as the map such that

$$\mathcal{L}^{(a)}_{p/r} : x^u q^v \mapsto \begin{cases} q^{-u^2r/p+v} & \text{if } ru - a \in p\mathbb{Z}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{6.82}$$

Then taking their two variable series $F_K(x;q)$, for a knot K they take, assuming it is convergent,

$$\widehat{Z}_{K(p,r),a}(q) \;=\; \mathcal{L}^{(a)}_{p/r}\big[(x^{1/2r} - x^{-1/2r})F_K(x;q)\big]\,, \tag{6.83}$$

where a indexes spin structures on the manifold. Presumably, if one could give a proof of the invariance of these series on the underlying manifold, then one could similarly prove invariance of the span of q–series constructed using these surgery formulae in a similar way to Theorem 1. It would then be natural to expect with the right definition, these q–series could give rise to a solution of the q–difference equations associated to the particular presentation of the closed three–manifold. Indeed, we will see this kind of behaviour in an example.

6.9 WRT invariant and $\widehat{Z}$ for $4_1(1,2)$

From Example 16 the WRT invariant normalised by a factor of $(1-q)$ of $4_1(-1,2)$ is given as an element of Habiro ring by

$$Ш(q) \;=\; \sum_{0\le\ell\le k} \alpha_{k,\ell}(q) \;=\; \sum_{0\le\ell\le k} (-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)} \frac{(q;q)_{2k+1}}{(q;q)_\ell (q;q)_{k-\ell}} \tag{6.84}$$

where, as usual, for $n \ge 0$ we have $(x;q)_n = (1-x)(1-qx)\dots(1-q^{n-1}x)$. This formula will be our starting point. As this is an element of the Habiro ring, we can formally evaluate at $q = e^h$ where

$$Ш(e^h) \;=\; -h - \frac{25}{2}h^2 - \frac{1621}{6}h^3 - \frac{195601}{24}h^4 - \frac{37907101}{120}h^5 + \dots \tag{6.85}$$

Using this normalisation of the WRT invariant, which matches some of the conventions used in [118], we note that

$$Ш(1) \;=\; 0\,. \tag{6.86}$$

This is of absolutely fundamental importance for relating the Witten and Chen–Yang conjectures on the asymptotics of WRT invariants.

The q–hypergeometric expression (6.84) has a natural q–holonomic module associated to it generated by

$$\mathrm{III}_{m,n}(q) \;=\; \sum_{0\le\ell\le k} \alpha_{k,\ell}(q) q^{mk+n\ell} \;=\; \sum_{0\le\ell\le k} (-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)+mk+n\ell} \frac{(q;q)_{2k+1}}{(q;q)_\ell (q;q)_{k-\ell}}\,. \tag{6.87}$$

A few values at $q = e^h$ are given by

$$\begin{aligned}
\mathrm{III}_{1,0}(e^h) &= -h - \frac{25}{2}h^2 - \frac{1693}{6}h^3 + \dots & \mathrm{III}_{0,1}(e^h) &= -h - \frac{25}{2}h^2 - \frac{1657}{6}h^3 + \dots \\
\mathrm{III}_{2,0}(e^h) &= -h - \frac{25}{2}h^2 - \frac{1765}{6}h^3 + \dots & \mathrm{III}_{0,2}(e^h) &= -h - \frac{25}{2}h^2 - \frac{1693}{6}h^3 + \dots \\
\mathrm{III}_{1,1}(e^h) &= -h - \frac{25}{2}h^2 - \frac{1729}{6}h^3 + \dots & \mathrm{III}_{2,-1}(e^h) &= -h - \frac{25}{2}h^2 - \frac{1729}{6}h^3 + \dots
\end{aligned} \tag{6.88}$$

Notice that

$$\begin{aligned}
(1-q^{k+1-\ell})\alpha_{k+1,\ell}(q) &= -q^{-k-1}(1-q^{2k+2})(1-q^{2k+3})\alpha_{k,\ell}(q) \\
(1-q^{\ell+1})\alpha_{k,\ell+1}(q) &= q^{2\ell+2}(1-q^{k-\ell})\alpha_{k,\ell}(q)
\end{aligned} \tag{6.89}$$

and so summing both sides and tracking boundary terms we see that

$$\begin{aligned}
\mathrm{III}_{m+1,n}(q) - \mathrm{III}_{m+2,n-1}(q) &= -q^m \mathrm{III}_{m,n}(q) + (q^{m+2}+q^{m+3})\mathrm{III}_{m+2,n}(q) - q^{m+5}\mathrm{III}_{m+4,n}(q)\,, \\
\mathrm{III}_{m,n}(q) - \mathrm{III}_{m,n+1}(q) &= q^{n+2}\mathrm{III}_{m,n+2}(q) - q^{n+2}\mathrm{III}_{m+1,n+1}(q)\,.
\end{aligned} \tag{6.90}$$

Using Equations (6.90) it can be shown that

$$\begin{aligned}
& q^{2m+2}\mathrm{III}_{m,n}(q) + (q^{m+1}+q^{m+2})\mathrm{III}_{m+1,n}(q) \\
& +(-q^{2m+4} - q^{2m+5} - q^{2m+6} - q^{2m+7} - q^{m+2} + 1)\mathrm{III}_{m+2,n}(q) \\
& +(q^{n+m+3} - q^{m+3} - 2q^{m+4} - q^{m+5} - 1)\mathrm{III}_{m+3,n}(q) \\
& +(q^{2m+7} + q^{2m+8} + 2q^{2m+9} + q^{2m+10} + q^{2m+11} + q^{m+4} + q^{m+5})\mathrm{III}_{m+4,n}(q) \\
& +(-q^{n+m+5} - q^{n+m+6}q^{m+6} + q^{m+7})\mathrm{III}_{m+5,n}(q) \\
& +(-q^{2m+11} - q^{2m+12} - q^{2m+13} - q^{2m+14} - q^{m+7})\mathrm{III}_{m+6,n}(q) \\
& +q^{n+m+8}\mathrm{III}_{m+7,n}(q) + q^{2m+16}\mathrm{III}_{m+8,n}(q) \;=\; 0\,.
\end{aligned} \tag{6.91}$$

This has Newton polygon depicted in Figure 6.2. Taking the classical limit of the above equation give a polynomial equation

$$\begin{aligned}
& z^8 + z^7 - 5z^6 + 8z^4 - 4z^3 + 4z^2 + 2z + 1 \\
&= (z-1)(z^7 + 2z^6 - 3z^5 - 3z^4 + 5z^3 + z^2 - 3z - 1) = 0\,.
\end{aligned} \tag{6.92}$$

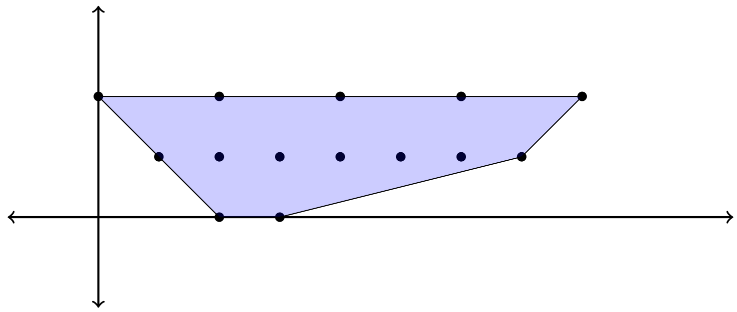

Figure 6.2: The Newton polygon of the q–difference equation satisfied by the family extending the WRT invariant of $4_1(1,2)$.

The field defined by the degree 7 polynomial is of course the trace field of M. This factorisation could suggest a refinement of this equation. One can in fact show that

$$\begin{aligned}
q^{2m+2}\text{Ш}_{m,n}(q) + (q^{2m+4}+q^{m+1}+q^{m+2})\text{Ш}_{m+1,n}(q) \\
+(-q^{2m+7}-q^{2m+5}-q^{2m+4}+q^{m+3}+1)\text{Ш}_{m+2,n}(q) \\
+(-q^{2m+9}-q^{2m+7}-q^{2m+6}+q^{m+n+3}-q^{m+5}-q^{m+4}-q^{m+3})\text{Ш}_{m+3,n}(q) \\
+(q^{2m+10}+q^{2m+9}+q^{2m+7}+q^{m+n+4}-q^{m+6})\text{Ш}_{m+4,n}(q) \\
+(q^{2m+12}+q^{2m+11}+q^{2m+9}-q^{m+n+6}+q^{m+6})\text{Ш}_{m+5,n}(q) \\
+(-q^{2m+12}-q^{m+n+7})\text{Ш}_{m+6,n}(q) - q^{2m+14}\text{Ш}_{m+7,n}(q) \;=\; (1-q)\,.
\end{aligned} \tag{6.93}$$

There is a natural q–series that pairs with $\text{Ш}(q)$. In particular, consider α from Equation (6.84) as a function

$$\alpha : \mathbb{Z}^2 \to \mathbb{Q}(q) \quad s.t. \quad \alpha_{k,\ell}(q) \;=\; \begin{cases} (-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)} \frac{(q;q)_{2k+1}}{(q;q)_\ell (q;q)_{k-\ell}} & \text{if } 0 \le \ell \le k \\ 0 & \text{otherwise} \end{cases}. \tag{6.94}$$

Equations (6.89) still hold on α with this extended domain. Moreover, Equations (6.89) completely determine the module. Therefore it is natural to ask whether there is another solution to Equations (6.89). Indeed, there is another solution given by

$$\beta : \mathbb{Z}^2 \to \mathbb{Q}(q) \quad s.t. \quad \beta_{k,\ell}(q) \;=\; \begin{cases} (-1)^{k+\ell} \frac{q^{\frac{1}{2}3k(k+1)+\frac{1}{2}\ell(\ell+1)+k+1}(q;q)_{-\ell-1}}{(q;q)_{-2k-2}(q;q)_{k-\ell}} & \text{if } \ell \le k \le -1 \\ 0 & \text{otherwise} \end{cases}. \tag{6.95}$$

We have

$$\beta_{-k-1,-\ell-1}(q) = (-1)^{k+\ell} q^{\frac{1}{2}3k(k+1)+\frac{1}{2}\ell(\ell+1)-k} \frac{(q;q)_\ell}{(q;q)_{2k}(q;q)_{\ell-k}}\,. \tag{6.96}$$

We see that the sum of these terms is convergent when $|q| < 1$, Indeed we find that

$$\begin{aligned}\sum_{k,\ell\in\mathbb{Z}} \beta_{k,\ell}(q) &= \sum_{0\le k\le \ell} (-1)^{k+\ell} q^{\frac{1}{2}3k(k+1)+\frac{1}{2}\ell(\ell+1)-k} \frac{(q;q)_\ell}{(q;q)_{2k}(q;q)_{\ell-k}} \\ &= 1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} - q^{14} - 3q^{15} + \dots .\end{aligned} \tag{6.97}$$

We will define more generally

$$\begin{aligned}Z_{m,n}(q) &= \sum_{k,\ell\in\mathbb{Z}} \beta_{k,\ell}(q) q^{mk+n\ell} \\ &= \sum_{0\le k\le \ell} (-1)^{k+\ell} q^{\frac{1}{2}3k(k+1)+\frac{1}{2}\ell(\ell+1)-(m+1)k-m-n\ell-n} \frac{(q;q)_\ell}{(q;q)_{2k}(q;q)_{\ell-k}} .\end{aligned} \tag{6.98}$$

Remarkably, the series $Z_{0,0}(q)$ has previously appeared in the literature and is, up to a factor of $-q^{1/2}$, the so called $\hat{Z}$ series associated to $4_1(-1,2)$ [92]. We can use a formula for the two variable series [153, 70] to prove the equality. Indeed, following the notation of [92, 153] we have

$$\begin{aligned}\Xi_{4_1}(x;q) &= (x^{1/2} - x^{-1/2}) \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(x;q)_{k+1}(x^{-1};q)_{k+1}} \\ &= (x^{-1/2} - x^{1/2}) \sum_{k=0}^{\infty} \frac{x^{k+1}}{(x;q)_{k+1}(x;q^{-1})_{k+1}} \\ &= (x^{-1/2} - x^{1/2}) \sum_{k,j,\ell=0}^{\infty} x^{k+j+\ell+1} \binom{k+j}{j}_q \binom{k+\ell}{\ell}_{q^{-1}} \\ &= \sum_{k,j,\ell=0}^{\infty} (x^{k+j+\ell+1/2} - x^{k+j+\ell+3/2}) \binom{k+j}{j}_q \binom{k+\ell}{\ell}_{q^{-1}}\end{aligned} \tag{6.99}$$

and so

$$\begin{aligned}F_{4_1}(x;q) &= \frac{1}{2} \sum_{k,j,\ell=0}^{\infty} (x^{k+j+\ell+1/2} - x^{k+j+\ell+3/2} - x^{-k-j-\ell-1/2} + x^{-k-j-\ell-3/2}) \\ &\qquad \times \binom{k+j}{j}_q \binom{k+\ell}{\ell}_{q^{-1}} .\end{aligned} \tag{6.100}$$

Then using the surgery formula in [92]

$$\begin{aligned}\hat{Z}(q) &= q^{3/8} \sum_{k,j,\ell=0}^{\infty} (q^{2(k+j+\ell+3/4)^2} - q^{2(k+j+\ell+7/4)^2} - q^{2(k+j+\ell+1/4)^2} + q^{2(k+j+\ell+5/4)^2}) \\ &\qquad \times \binom{k+j}{j}_q \binom{k+\ell}{\ell}_{q^{-1}} \\ &= -q^{1/2}(1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} - q^{14} - 3q^{15} + \dots) .\end{aligned} \tag{6.101}$$

Proposition 10. *Let*

$$\begin{aligned}\hat{Z}_m(q) \;&=\; q^{3/8}\sum_{k,j,\ell=0}^{\infty}\left(q^{2(k+j+\ell+3/4)^2}-q^{2(k+j+\ell+7/4)^2}-q^{2(k+j+\ell+1/4)^2}+q^{2(k+j+\ell+5/4)^2}\right)\\ &\qquad\times\binom{k+j}{j}_q\binom{k+\ell}{\ell}_{q^{-1}}q^{-mk-m}\,.\end{aligned} \tag{6.102}$$

We have the following identity

$$\hat{Z}_m(q) \;=\; -q^{1/2}Z_{m,0}(q)\,. \tag{6.103}$$

In particular,

$$\hat{Z}(q) \;=\; -q^{1/2}Z_{0,0}(q)\,. \tag{6.104}$$

Proof. One can show using holonomic function techniques discussed in Section 5.6 that $\hat{Z}_m$ satisfies the q–difference Equation (6.91) with $n=0$. Therefore, as both are power series in q^{-m} and the q–difference equations are second order in q^m, if the coefficients of q^{-m} and q^{-2m} agree this proves the result. The first of these equalities is given by

$$\begin{aligned}&-q^{-1/8}\sum_{j,\ell=0}^{\infty}(q^{2(j+\ell+3/4)^2}-q^{2(j+\ell+7/4)^2}-q^{2(-j-\ell-1/4)^2}+q^{2(-j-\ell-5/4)^2})\\ &\quad=\;\sum_{j=0}^{\infty}(-1)^jq^{j(j+1)/2}\,,\end{aligned} \tag{6.105}$$

which can be proved by direct computation and similarly for the second. □

6.10 Dualities for modules associated to $4_1(1,2)$

Then for the companion matrix

$$A(x,q) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ \frac{1}{q^{12}} & \frac{q^3x+q+1}{q^{13}x} & \frac{(-q^7-q^5-q^4)x^2+q^3x+1}{q^{14}x^2} & \frac{(-q^5-q^3-q^2)x-q-1}{q^{10}x} & \frac{(q^6+q^5+q^3)x-q^2+1}{q^{10}x} & \frac{q^3+q^2+1}{q^5} & \frac{-q^5x-1}{q^7x}\end{pmatrix} \tag{6.106}$$

we have

$$\mathbf{Z}_{m+1}(q) = A(q^m, q)\mathbf{Z}_m(q)\,. \tag{6.107}$$

This module is self dual in the sense of [82], so that $M \cong M^{\wedge\vee}$. In particular, we have a gauge transformation $P(x; q)$ whose columns respectfully are given

$$\begin{pmatrix} \frac{x^4q^{10}(2q^4+4q^3+6q^2+4q+2)+2x^3q^9+x^2q^3(2q^6+4q^5+4q^4+6q^3+4q^2+4q+2)+2)}{x^7q^{21}} \\ \frac{-2x^4q^{10}+x^2q^3(-2q^4-4q^3-4q^2-4q-2)-2}{x^6q^{15}} \\ \frac{x^2q^3(2q^2+4q+2)+2}{x^5q^{10}} \\ \frac{-2x^2q^3-2}{x^4q^6} \\ \frac{2}{x^3q^3} \\ \frac{-2}{x^2q} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{-2x^4q^{14}+x^2q^5(-2q^4-4q^3-4q^2-4q-2)-2}{x^6q^{21}} \\ \frac{x^2q^5(2q^2+4q+2)+2}{x^5q^{15}} \\ \frac{-2x^2q^5-2}{x^4q^{10}} \\ \frac{2}{x^3q^6} \\ \frac{-2}{x^2q^3} \\ 0 \\ \frac{-2}{x^2q} \end{pmatrix},$$
$$\begin{pmatrix} \frac{(x^2q^7(2q^2+4q+2)+2}{x^5q^{20}} \\ \frac{-2x^2q^7-2}{x^4q^{14}} \\ \frac{2}{x^3q^9} \\ \frac{-2}{x^2q^5} \\ 0 \\ \frac{-2}{x^2q^3} \\ \frac{2}{x^3q^3} \end{pmatrix}, \begin{pmatrix} \frac{-2x^2q^9-2}{x^4q^{18}} \\ \frac{2}{x^3q^{12}} \\ \frac{-2}{x^2q^7} \\ 0 \\ \frac{-2}{x^2q^5} \\ \frac{2}{x^3q^6} \\ \frac{(-2x^2q^3-2}{x^4q^6} \end{pmatrix}, \begin{pmatrix} \frac{2}{x^3q^{15}} \\ \frac{-2}{x^2q^9} \\ 0 \\ \frac{-2}{x^2q^7} \\ \frac{2}{x^3q^9} \\ \frac{(-2x^2q^5-2}{x^4q^{10}} \\ \frac{(x^2q^3(2q^2+4q+2)+2xq^2+2}{x^5q^{10}} \end{pmatrix}, \begin{pmatrix} \frac{-2}{x^2q^{11}} \\ 0 \\ \frac{-2}{x^2q^9} \\ \frac{2}{x^3q^{12}} \\ \frac{-2x^2q^7-2}{x^4q^{14}} \\ \frac{x^2q^5(2q^2+4q+2)+2xq^3+2}{x^5q^{15}} \\ \frac{-2x^4q^{10}+x^2q^3(-2q^4-4q^3-4q^2-4q-2)+xq^2(-2q-2)-2}{x^6q^{15}} \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ \frac{-2}{x^2q^{11}} \\ \frac{2}{x^3q^{15}} \\ \frac{-2x^2q^9-2}{x^4q^{18}} \\ \frac{x^2q^7(2q^2+4q+2)+2xq^4+2}{x^5q^{20}} \\ \frac{-2x^4q^{14}+x^2q^5(-2q^4-4q^3-4q^2-4q-2)+xq^3(-2q-2)-2}{x^6q^{21}} \\ \frac{x^4q^{10}(2q^4+4q^3+6q^2+4q+2)+x^3q^7(2q^4+2q^3+2q^2+2q+2)+x^2q^3(2q^6+4q^5+4q^4+6q^3+4q^2+4q+2)+xq^2(2q^2+2q+2)+2}{x^7q^{21}} \end{pmatrix} \tag{6.108}$$

which satisfy the following equations

$$A(x, q) \;=\; P(qx, q)A(q^7x, q^{-1})^T P(x, q)^{-1}\,. \tag{6.109}$$

This duality can be used to give rise to quadratic relations between the q–series of equation 8.154 and equation **??** that is most conveniently given in matrix form.

Lemma 18 (Quadratic relations)**.** *We have the following identity for* $\mathbf{Z}$ *from equations* (8.150) *and* (8.152)*:*

$$\mathbf{Z}_m(q)\mathbf{Z}_{-m-6}(q^{-1})^T \;=\; P(q^m, q)\,. \tag{6.110}$$

Proof. From Equation (6.109) we see that

$$\mathbf{Z}_{-m-6}(q^{-1})^T P(q^m, q)^{-1}\mathbf{Z}_m(q) \tag{6.111}$$

is independent of m. Then we find that

$$\lim_{m\to-\infty} \mathbf{Z}_{-m-6}(q^{-1})^T P(q^m, q)^{-1}\mathbf{Z}_m(q) \;=\; \mathrm{Id}_{7\times 7}\,. \tag{6.112}$$

□

An additional identity is needed. This follows from the usual method of checking the difference equations and boundary conditions at $m = \pm\infty$, which in this case vanish.

Proposition 11. *The q–series $Z^{(2)}_m(q)$ is defined for $q \neq 1$ and for $|q| > 1$, we have*

$$Z^{(2)}_m(q) \;=\; 0\,. \tag{6.113}$$

Part IV

Modularity

Chapter 7

From modular to mock

Classical modular forms originated as θ–functions. They were then implicit in the studies of Ramanujan who formulated many beautiful conjectures and illustrated many important structural properties in examples. These properties were then understood by Hecke. Their theory became most clearly described as certain functions with large and interesting symmetries related to arithmetic objects. These symmetries are strict enough that the spaces of functions that are invariant under their action are finite dimensional. They appear in almost all areas of modern mathematics and their powerful properties have led to amazing developments in algebraic geometry, mathematical physics, and number theory. This section will give some extremely elementary aspects of the theory discussing the bare minimum for what we will need. For a more detailed introduction see for Example [32, 176]. Next we move toe Jacobi forms. The theory of Jacobi forms was introduced in [56]. This combines elliptic and modular forms and captures many interesting and important examples of classical functions. The q–difference equations these satisfy are trivial and we will see that taking more interesting examples leads to quantum modular forms. As a stepping stone we consider mock modular forms

One of the most mysterious open problems over the last century was the question: what are mock θ–functions? In his famous last letter to Hardy before his untimely death, Ramanujan gave a list of some simple q–hypergeometric functions that have properties similar to those of modular forms. In particular, their asymptotics as q approaches a root of unity looks like a modular form at leading order however, at subleading order there is a different behaviour. For Example [207], consider the function

$$f(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{k^2}}{(-q;q)_k}\,, \tag{7.1}$$

which Ramanujan would refer to as an order 3 mock modular form. Then as $\tau \to i\infty$ along

$i\mathbb{R}$, one can numerically observe that for example

$$\begin{aligned} f\Big(\mathbf{e}\big(\frac{\tau}{2\tau+1}\big)\Big) \;\sim\; & \mathbf{e}\Big(-\frac{1}{60}\big(\tau+\frac{1}{2}\big)\Big)\sqrt{\tau+\frac{1}{2}}\Big(-\frac{15}{8}-\frac{5}{8}\sqrt{5}\Big)^{-\frac{1}{4}}\mathbf{e}\Big(-\frac{1}{60}\frac{1}{2(2\tau+1)}\Big) \\ & + 2 + 4\frac{-2\pi i}{2(2\tau+1)} + 36\Big(\frac{-2\pi i}{2(2\tau+1)}\Big)^2 + \frac{1640}{3}\Big(\frac{-2\pi i}{2(2\tau+1)}\Big)^3 + \cdots . \end{aligned} \tag{7.2}$$

For example, see Code 27. The leading order would be expected for a modular form, however, the subleading terms would normally involve an integer multiple of the initial series with an integer power of $\mathbf{e}(\tau)$. This is indeed a special property for a function to have however with these examples, Ramanujan gave no definition and trying to understand how he thought about them seems unfortunately impossible.

Regardless, these functions were then studied by Watson [193], who proved many of the q–series identities between them and related them to the Appell–Lerch sums. These were then studied by Zwegers in his thesis who related them to real analytic modular forms. Soon after, Bringmann and Ono used these ideas to prove various open problems in combinatorics. For an overview, see [211, 207, 31].

We close this introduction with a small remark on notation. One of the beautful developments of the theory was that special functions in a variable q should be replaced by functions in τ where $q = \mathbf{e}(\tau)$. However, we will just assume this as standard and often write

$$f(q) \;=\; f(\tau)\,. \tag{7.3}$$

Moreover, we will always take

$$q^{a/c} \;=\; \mathbf{e}(a\tau/c)\,. \tag{7.4}$$

7.1 Modular group and Eisenstein series

The basic example of a group of symmetries is the group $\mathrm{PSL}_2(\mathbb{Z})$ often referred to as the *modular group*. This group has a finite presentation

$$\Gamma_1 \;=\; \mathrm{PSL}_2(\mathbb{Z}) \;=\; \langle S, T : S^2 = 1, (ST)^3 = 1\rangle\,, \tag{7.5}$$

where the generators can be given explicitly as

$$T \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{7.6}$$

The modular group is of course the matrix group $\mathrm{SL}_2(\mathbb{Z})$ modulo its centre which is just ± 1. $\mathrm{SL}_2(\mathbb{Z})$ is also generated by the above matrices and has finite presentation where the 1s in the relations are replaced by -1. Importantly, the modular group is a subgroup of

$\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ and therefore has a natural action on $\mathfrak{h}$ via Mobius transformations. In particular, for $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\gamma \cdot \tau \; = \; \frac{a\tau + b}{c\tau + d} \,. \tag{7.7}$$

This is indeed an left action as for $\gamma' = [a', b'; c', d'] \in \mathrm{SL}_2(\mathbb{Z})$

$$\gamma' \cdot \gamma \cdot \tau \; = \; \frac{a' \frac{a\tau+b}{c\tau+d} + b'}{c' \frac{a\tau+b}{c\tau+d} + d'} \; = \; \frac{(a'a + b'c)\tau + (a'b + b'd)}{(c'a + d'c)\tau + (c'b + d'd)} \; = \; (\gamma'\gamma) \cdot \tau \,. \tag{7.8}$$

This action can be used to define an action of the space of holomorphic functions on $\mathfrak{h}$. We define the $|_k$ action as follows

$$f|_k\gamma(\tau) \; = \; (c\tau + d)^{-k} f\Big(\frac{a\tau + b}{c\tau + d}\Big) \,. \tag{7.9}$$

This is indeed a right action as

$$\begin{aligned}
(f|_k\gamma|_k\gamma')(\tau) \; &= \; (c'\tau + d')^{-k}(f|_k\gamma)\Big(\frac{a'\tau + b'}{c'\tau + d'}\Big) \\
&= \; \Big(c\frac{a'\tau + b'}{c'\tau + d'} + d\Big)^{-k}(c'\tau + d')^{-k} f\Bigg(\frac{a\frac{a'\tau+b'}{c'\tau+d'} + b}{c\frac{a'\tau+b'}{c'\tau+d'} + d}\Bigg) \\
&= \; ((ca' + dc')\tau + (cb' + dd'))^{-k} f\Big(\frac{(aa' + bc')\tau + (ab' + bd')}{(ca' + dc')\tau + (cb' + dd')}\Big) \\
&= \; (f|_k\gamma \cdot \gamma')(\tau) \,.
\end{aligned} \tag{7.10}$$

With this one can define modular forms of weight k.

Definition 17 (Modular form)**.** *A modular form of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function on the upper half plane such that it is fixed under the $|_k$ action, so in particular*

$$f|_k\gamma \; = \; f \,, \tag{7.11}$$

and such that its Fourier series

$$\sum_n c_n q^n \tag{7.12}$$

has $c_n = 0$ for $n < 0$.

As the full modular group in generated by T and S it is enough to have the equations

$$f(\tau + 1) \; = \; f(\tau) \quad \text{and} \quad f(-1/\tau) \; = \; \tau^k f(\tau) \,. \tag{7.13}$$

One can of course consider subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and this important for many interesting applications. For this thesis we will mainly restrict our attention to the full modular group. The first statement that one learns about modular forms is that to be one is a strong condition.

Theorem A–35. *[32, Cor. 1] The vector space of weight k modular forms is finite dimensional.*

Some of the simplest modular forms one encounters are the Eisenstein series. These are defined [202] for the full modular group as

$$G_n(\tau) \;=\; \sum_{\substack{k,\ell\in\mathbb{Z}\\ k>0 \text{ or } k=0,\ell>0}} \frac{1}{(m\tau+\ell)^n}\,. \tag{7.14}$$

There functions clearly satisfy the equation

$$G_n(\tau+1) \;=\; G_n(\tau) \tag{7.15}$$

and therefore have a Fouier series. This Fourier series is given as

$$G_n(q) \;=\; -\frac{B_n}{2n} + \sum_{k=1}^{\infty}\sum_{0<d|k} d^{n-1}q^k \;=\; -\frac{B_n}{2n} + \sum_{k=1}^{\infty} k^{n-1}\frac{q^k}{1-q^k}\,. \tag{7.16}$$

For example, we have

$$\begin{aligned} G_1(q) \;&=\; -\frac{1}{4} + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 4q^6 + 2q^7 + 4q^8 + 3q^9 + \cdots\,,\\ G_2(q) \;&=\; -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + 13q^9 + \cdots\,,\\ G_3(q) \;&=\; q + 5q^2 + 10q^3 + 21q^4 + 26q^5 + 50q^6 + 50q^7 + 85q^8 + 91q^9 + \cdots\,. \end{aligned} \tag{7.17}$$

For even n these functions are modular forms of weight n so that for $n \in \mathbb{Z}$

$$\begin{aligned} G_{2n}(\tau+1) \;&=\; G_{2n}(\tau)\,,\\ G_{2n}(-1/\tau) \;&=\; \tau^{2n}G_{2n}(\tau)\,. \end{aligned} \tag{7.18}$$

For the even Eisenstein series we can define

$$E_{2k}(q) \;=\; \frac{2}{\zeta(1-2k)}G_{2k}(q) \;=\; 1 + O(q)\,. \tag{7.19}$$

The odd Eisenstein series will be additive quantum modular forms discussed later. One can naturally define Eisenstein series on the lower half plane via the formal manipulation

$$\begin{aligned} G_n(q^{-1}) \;&=\; \frac{1}{2}\zeta(1-n) + \sum_{k=1}^{\infty} k^{n-1}\frac{q^{-k}}{1-q^{-k}}\\ &=\; \frac{1}{2}\zeta(1-n) - \sum_{k=1}^{\infty} k^{n-1} - \sum_{k=1}^{\infty} k^{n-1}\frac{q^k}{1-q^k} \;=\; -G_n(q)\,. \end{aligned} \tag{7.20}$$

We close this section with a little computational result. This allows the computation of the Eisenstein series of order $O(q^{k^2/2})$. This uses the following lemma.

Lemma 19. *[73, 85] We have*

$$\begin{aligned}
\frac{(qe^{\epsilon};q)_\infty}{(q;q)_\infty}\frac{1}{(qe^{\epsilon};q)_m} &= \frac{1}{(q;q)_m}\sqrt{\frac{-\epsilon}{1-\exp(\epsilon)}} \\
&\quad \times \exp\left(-\sum_{\ell=1}^{\infty}\left(G_\ell(q)-\sum_{n=1}^{m}\mathrm{Li}_{1-\ell}(q^n)\right)\frac{\epsilon^\ell}{\ell!}\right), \\
\frac{(q^{-1}e^{\epsilon};q^{-1})_\infty}{(q^{-1};q^{-1})_\infty}\frac{1}{(q^{-1}e^{\epsilon};q^{-1})_m} &= \frac{1}{(q^{-1};q^{-1})_m}\frac{-1}{\epsilon}\sqrt{\frac{-\epsilon}{1-\exp(\epsilon)}} \\
&\quad \times \exp\left(-\sum_{\ell=1}^{\infty}\left(G_\ell(q^{-1})-\sum_{n=1}^{m}\mathrm{Li}_{1-\ell}(q^{-n})\right)\frac{\epsilon^\ell}{\ell!}\right).
\end{aligned} \tag{7.21}$$

Corollary 8. *We have*

$$\frac{(qe^{\epsilon};q)_\infty}{(q;q)_\infty} = \exp\Big(-\sum_{n,k=1}^{\infty}k^{n-1}\frac{q^k}{1-q^k}\frac{\epsilon^n}{n!}\Big). \tag{7.22}$$

Expanding in the formal variable ϵ

$$\sum_{\ell=1}^{\infty}(G_\ell(q)-\zeta(1-n)/2)\frac{\epsilon^\ell}{\ell!} = -\log\Big(\frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2}}{(q;q)_k}e^{k\epsilon}\Big). \tag{7.23}$$

For example, this corollary gives the formulae

$$\begin{aligned}
G_1(q) &= -\frac{1}{4}-\frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k k\frac{q^{k(k+1)/2}}{(q;q)_k}, \\
G_2(q) &= -\frac{1}{24}-\frac{2}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\frac{k^2}{2!}\frac{q^{k(k+1)/2}}{(q;q)_k}+\frac{1}{(q;q)_\infty^2}\Big(\sum_{k=0}^{\infty}(-1)^k k\frac{q^{k(k+1)/2}}{(q;q)_k}\Big)^2,
\end{aligned} \tag{7.24}$$

where of course we can use either

$$(q;q)_\infty = \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2}}{(q;q)_k} = \sum_{k\in\mathbb{Z}}(-1)^k q^{k(3k-1)/2}, \tag{7.25}$$

for efficient computations.

Remark 22. *The main use of these formulae is evaluating odd Eisenstein series near* $q=1$*. Of course if one has an even Eisenstein series the most efficient way to do this is with the modularity.*

Remark 23. *The Eisenstein series arise when we find solutions to certain* q*–difference equations associated to roots of the indicial polynomial of edges of the Newton polygon with multiplicities. They similarly arise when factorising state integrals.*

The Eisenstein series of wieght 4 and 6 turn out to determine all other modular forms. In particular, the multiplication of two modular forms is again a modular form with weight the sum of the two previous weights. In this way, we can make the algebra of all modular forms into a graded algebra.

Theorem A–36. *The algebra of modular forms in isomorphic to the polynomial ring generated by the Eisenstein series of weight* 4 *and* 6*, that is*

$$\mathbb{C}[E_4, E_6]\,. \tag{7.26}$$

The final important property of Eisenstein series is the important transformation property of the weight 2 Eisenstein series. This function would be modular with an almost analogous proof however it lacks certain convergence properties that would be required for the proof. However, it satisfies the following transformation

$$G_2(\tilde{q}) \;=\; \tau^2 G_2(q) + \frac{i\tau}{4\pi}\,. \tag{7.27}$$

There is a space of functions that extend modular forms called *quasi modular forms* and this is gives an algebra isomoprhic to the polynomial algebra

$$\mathbb{C}[E_2, E_4, E_6]\,. \tag{7.28}$$

The functional equation for G_2 will be important when we discuss Jacobi forms.

7.2 The Dedekind η–function and θ–functions

After the Eisenstein series there is another modular form that one will encounter. Indeed, it is a major player in this thesis. This is the Dedekind η function. This is defined as

$$\eta(q) \;=\; q^{1/24}(q;q)_\infty\,. \tag{7.29}$$

This is a modular form of weight 1/2. This cannot be a usual modular form as $|_{1/2}$ is not a well defined action. Indeed, we need to twist this action. Everything is determined by the equations

$$\eta(\tau+1) \;=\; \mathbf{e}(1/24)\eta(\tau) \quad \text{and} \quad \eta(-1/\tau) \;=\; \mathbf{e}(-1/8)\sqrt{\tau}\,\eta(\tau)\,. \tag{7.30}$$

More generally, we have

$$\eta\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; \epsilon(a,b,c,d)\sqrt{c\tau+d}\,\eta(\tau) \tag{7.31}$$

where

$$\epsilon(a,b,c,d) \;=\; \begin{cases} \mathbf{e}(bd/24) & \text{if } c \;=\; 0\,, \\ \sqrt{\frac{-ic}{|c|}}\prod_{\ell=1}^{|c|-1}\Big(1-\mathbf{e}\Big(\ell\frac{a}{c}\Big)\Big)^{\frac{1}{2}-\frac{\ell}{|c|}}\mathbf{e}\Big(\frac{a+d}{24c}\Big) & \text{if } c \neq 0\,. \end{cases} \tag{7.32}$$

This $\epsilon(a,b,c,d)\sqrt{c\tau+d}$ is called an automorphy factor and ϵ a multiplier system. See definition 18. This formula for the multiplier system can be derived from the asymptotics of the Pochhammer symbol given in equation (4.93). However, it is often given in terms of the Dedekind sum which for $0 \le c$ is given by

$$\epsilon(a,b,c,d) \;=\; \begin{cases} \mathbf{e}(b/24) & \text{if } c \;=\; 0,\; d \;=\; 1\,, \\ \mathbf{e}\big(\frac{a+d}{24c} - \sum_{n=1}^{c-1} \frac{n}{c}\big(\frac{dn}{c} - \big\lfloor \frac{dn}{c} \big\rfloor - \frac{1}{2}\big) - \frac{1}{4}\big) & \text{if } c > 0\,. \end{cases} \tag{7.33}$$

This is numerically verified in Code 25. From the equations (7.30) we see that

$$\Delta(q) \;=\; \eta(q)^{24} \;=\; q\prod_{j=1}^{\infty}(1-q^j)^{24} \;=\; \sum_{k=1}^{\infty}\tau(k)q^k\,. \tag{7.34}$$

gives a modular form of weight 12. This is Ramanujan's Δ–function and the Fourier coefficients are his τ–function of which Ramanujan discovered many fascinating properties. Weight 12 is important as it is the first weight with space of functions of dimension greater than 1. Indeed, this is the first example of a cusp form, which is roughly a modular form which vanishes when $q=0$.

The most classical example of a modular form is given by the θ–functions. These were studied by Jacobi and used by Riemann to prove the functional equation of the ζ–function. The basic example is

$$\vartheta_{00}(q) \;=\; \sum_{k\in\mathbb{Z}} q^{k^2/2}\,. \tag{7.35}$$

This satisfies the functional equations

$$\vartheta_{00}(\tau+2) \;=\; \vartheta_{00}(\tau+2) \quad \text{and} \quad \vartheta_{00}(-1/\tau) \;=\; \mathbf{e}(-1/8)\sqrt{\tau}\vartheta_{00}(\tau)\,. \tag{7.36}$$

Therefore this is invariant under the group generated by T^2 and S. Hence, this subgroup is often referred to as the θ–subgroup.

To prove the modularity of each of these functions the easiest method is to apply Poisson summation. Indeed the first transform satisfied by ϑ_{00} simply follows from the Fourier expansion in $q^{1/2}$. The second follows noting that by Poisson summation given in Theorem 26 that

$$\begin{aligned} \vartheta_{00}(-1/\tau) \;&=\; \sum_{k\in\mathbb{Z}}\mathbf{e}(-k^2/2\tau) \;=\; \sum_{\ell\in\mathbb{Z}}\int_{-\infty}^{\infty}\mathbf{e}(-x^2/2\tau+\ell x)\,dx \\ &=\; \sum_{\ell\in\mathbb{Z}}\sqrt{\frac{\tau}{i}}\mathbf{e}(\ell^2/2\tau) \;=\; \mathbf{e}(-1/8)\sqrt{\tau}\vartheta_{00}(\tau)\,. \end{aligned} \tag{7.37}$$

7.3 Vector and matrix valued modular forms

The previous example of the θ function ϑ_{00} leads to a natural question: what properties does the function

$$\vartheta_{00}(\tau+1) \;=\; \sum_{k\in\mathbb{Z}}(-1)^k q^{k^2/2} \tag{7.38}$$

satisfy? For the full modular group we can in fact lift this kind of modular form on a subgroup to the full group. Let's continue with this example. We can define an additional three functions

$$\begin{aligned} \vartheta_{01}(q) \;&=\; \sum_{k\in\mathbb{Z}}(-1)^k q^{k^2/2}, \quad \vartheta_{10}(q) \;=\; q^{1/4}\sum_{k\in\mathbb{Z}} q^{k(k+1)/2}, \\ &\text{and} \quad \vartheta_{11}(q) \;=\; iq^{1/4}\sum_{k\in\mathbb{Z}}(-1)^k q^{k(k+1)/2}\,. \end{aligned} \tag{7.39}$$

These functions then satisfy the transformation

$$\begin{aligned} \vartheta_{01}(\tau+1) \;&=\; \vartheta_{00}(\tau)\,, \quad \vartheta_{01}(-1/\tau) \;=\; \mathbf{e}(-1/8)\sqrt{\tau}\vartheta_{00}(\tau)\,, \\ \vartheta_{10}(\tau+1) \;&=\; \vartheta_{10}(\tau)\,, \quad \vartheta_{10}(-1/\tau) \;=\; \mathbf{e}(-1/8)\sqrt{\tau}\vartheta_{01}(\tau)\,, \\ \vartheta_{11}(\tau+1) \;&=\; \vartheta_{11}(\tau)\,, \quad \vartheta_{11}(-1/\tau) \;=\; \mathbf{e}(-3/8)\sqrt{\tau}\vartheta_{00}(\tau)\;. \end{aligned} \tag{7.40}$$

Along with $\vartheta_{00}(\tau+1)=\vartheta_{01}(\tau)$ we see that we can form a vector–valued function

$$\vartheta(q) \;=\; \begin{pmatrix} \vartheta_{00}(\tau) \\ \vartheta_{01}(\tau) \\ \vartheta_{10}(\tau) \\ \vartheta_{11}(\tau) \end{pmatrix}\,, \tag{7.41}$$

which satisfies

$$\vartheta(\tau+1) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\vartheta(\tau)\,, \quad \vartheta(-1/\tau) \;=\; \mathbf{e}(-1/8)\sqrt{\tau}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}\vartheta(\tau)\,. \tag{7.42}$$

This kind of example can be formalised using matrix valued automorphy factors.

Definition 18. *We call $\mathbf{j}:\mathfrak{h}\times\Gamma\to \mathrm{GL}_N(\mathbb{C})$ an automorphy factor of rank N and weight k if*

$$\mathbf{j}(\tau;\gamma\gamma') \;=\; \mathbf{j}\Big(\frac{a\tau+b}{c\tau+d};\gamma\Big)\mathbf{j}(\tau;\gamma')\,, \tag{7.43}$$

and for $\gamma=[a,b;c,d]$ for some $k\in\mathbb{R}$

$$\mathbf{j}(\tau;\gamma) \;=\; (c\tau+d)^k\nu(\gamma) \tag{7.44}$$

where $|\det(\nu(\gamma))|=1$. ν is called a multiplier system.

Using this we can define modular forms with respect to automorphy factors.

Definition 19. *A holomorphic function $f : \mathfrak{h} \to \mathbb{C}^N$ is a modular form with respect to the automorphy factor $\mathbf{j}$ if for $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$*

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) \;=\; \mathbf{j}(\tau; \gamma) f(\tau)\,. \tag{7.45}$$

We can define an action on functions similarly to what we did form modular forms $|_{\mathbf{j}}$ with respect to this automorphy factor $\mathbf{j}$.

A famous and interesting example of a vector–valued modular form comes from the Rogers–Ramanujan identities [164].

Example 57. *Consider, the functions*

$$G(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}\,, \quad \textit{and} \quad H(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}\,. \tag{7.46}$$

These satisfy the following equalities proved independently and also together by Rogers–Ramanujan

$$G(q) \;=\; \frac{1}{(q;q^5)_\infty (q^4;q)_\infty}\,, \quad \textit{and} \quad H(q) \;=\; \frac{1}{(q^2;q^5)_\infty (q^3;q)_\infty}\,. \tag{7.47}$$

Then taking

$$g(q) \;=\; \begin{pmatrix} q^{-1/60} G(q) \\ q^{11/60} H(q) \end{pmatrix}\,, \tag{7.48}$$

using the product expansions, the Jacobi triple product and modularity of the θ–function from equation (7.68) one can show that

$$g(\tau + 1) \;=\; \begin{pmatrix} \mathbf{e}(-1/60) & 0 \\ 0 & \mathbf{e}(11/60) \end{pmatrix} g(\tau)\,, \quad g(-1/\tau) \;=\; \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} g(\tau)\,. \tag{7.49}$$

In this case, these two matrices associated to the generators T and S generate a representation of $\mathrm{SL}_2(\mathbb{Z})$.

7.4 q–hypergeometric functions and modularity

The previous example of the Rogers–Ramanujan functions is very special and beautiful example of a modular q–hypergeometric function. Most q–hypergeometric functions are not modular and it is an extremely interesting question of when such a function will be modular.

This seems like an impossible condition to quantify for a q–hypergeometric function. Remarkably, based on computations in conformal field theory, Nahm gave a relation between modularity and vanishing of a classes in K–theory. This can be made precise for the class of q–hypergeometric sums known as Nahm sums

$$\sum_{k\in\mathbb{Z}_{\geq 0}^N} \frac{q^{\frac{1}{2}k^{\mathrm{T}}Ak+b^{\mathrm{T}}k+c}}{(q;q)_{k_1}\cdots(q;q)_{k_N}}\,. \tag{7.50}$$

To these sums and solutions

$$1-X_i \;=\; \prod_{j=1}^{N} X_j^{A_{ij}}\,, \tag{7.51}$$

we can associate the element of the Bloch group

$$\xi_{A,i} \;=\; \sum_{j=1}^{N}[X_j^{(i)}] \in \mathcal{B}(\mathbb{C})\otimes\mathbb{Q}\,. \tag{7.52}$$

Conjecture 6 (Nahm's conjecture). *[142, 191, 206] If some $\xi_{A,i}=0$, then some combination of the sums with various b and c gives a modular form.*

For the Rogers–Ramanujan functions we have associated Bloch group element [206, Sec. 2.B]

$$[(-1\pm\sqrt{5})/2] \;=\; 0\in\mathcal{B}(\mathbb{C})\otimes\mathbb{Q} \tag{7.53}$$

and we of course have the modularity shown in Example 57. This conjecture has a natural and surprising analogue for three–manifolds. Indeed, it was noticed in [118, 99] that false θ–functions and mock modular forms appear for non–hyperbolic three–manifolds. This can be formulated in a modular setting as follows.

Conjecture 7. *If M is a non–hyperbolic 3–manifold, then the q–Weyl module associated to its $\mathfrak{sl}_2$ quantum invariants has a representation in a space of modular forms.*

7.5 Elliptic functions and the Weierstrass $\wp$–function

Elliptic functions are meromorphic functions defined on a complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ *i.e.* doubly periodic functions. From Liouville's theorem, holomorphic elliptic functions are constant. Supposing that there exists a function with a simple pole leads to a contradiction as this gives an isomorphic to the Riemann–sphere. Therefore, the first example must have two simple poles or a double pole. This leads to the Weierstrass $\wp$–function. This is defined as

$$\wp(z;\tau) \;=\; \frac{1}{z^2}+\sum_{(n,m)\in\mathbb{Z}^2-(0,0)}\Big(\frac{1}{(z-m\tau-n)^2}-\frac{1}{(m\tau+n)^2}\Big)\,. \tag{7.54}$$

This functions satisfies the functional equations

$$\wp(z + m\tau + n;\tau) \;=\; \wp(z;\tau)\,, \tag{7.55}$$

and so is elliptic. It has Laurent series expansion at $z = 0$

$$\wp(z;\tau) \;=\; \frac{1}{z^2} + 2\sum_{k=1}^{\infty} G_{2k+2}(q)\frac{(2\pi i z)^{2k}}{(2k)!}\,. \tag{7.56}$$

This expansion can be used to show that for $x = \mathbf{e}(z)$

$$\wp(z;\tau) \;=\; \frac{\theta'(x;q)^2 - \theta''(x;q)\theta(x;q)}{\theta(x;q)^2} - 2G_2(q)\,. \tag{7.57}$$

Notice that from the functional equation for the θ–function,

$$\theta(qx;q) \;=\; -q^{-1}x^{-1}\theta(x;q)\,, \tag{7.58}$$

for any set of $z_{0,j}$ and $z_{\infty,j}$ such that

$$\sum_j z_{0,j} - z_{\infty,j} \;=\; 0 \tag{7.59}$$

we find that

$$\prod_j \frac{\theta(\mathbf{e}(z + z_{0,j});q)}{\theta(\mathbf{e}(z + z_{\infty,j});q)}\,, \tag{7.60}$$

is an elliptic function. In this way if we understand the zeros and poles of an elliptic function then we can give an explict description. Note that this is not necessarily an easy problem and indeed the zeros of the Weierstrass ℘–function are interesting numbers computed in [55]. This example shows us that this natural examples has an expansion in the elliptic variable which is made up of modular forms. It is then natural to ask whether this elliptic functions satisfies any modular properties. This is indeed the case and leads to Jacobi forms. Here we see that for $\gamma = [a,b;c,d] \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\wp\Big(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\Big) \;=\; (c\tau + d)^2\wp(z;\tau)\,. \tag{7.61}$$

7.6 The Jacobi group and θ–functions

The transformation property of the Weierstrass ℘–function indicates that there are natural examples of functions that are elliptic and in some way modular. From the point of view of symmetries there is a clear way of how to combine these actions. This is through the semi–direct product. In particular, we can take the *Jacobi group*

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2\,. \tag{7.62}$$

This is the group defined to be to be the set of pairs (γ, n) for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $n \in \mathbb{Z}^2$ with multiplication

$$(\gamma, n) \cdot (\gamma', n') \;=\; (\gamma\gamma', n\gamma' + n')\,. \tag{7.63}$$

This group has a natural slash action on functions from $\mathbb{C} \times \mathfrak{h}$. This has an associated weight k and index m and we define for $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$ and $(n_1, n_2) \in \mathbb{Z}^2$

$$\begin{aligned}(f|_{k,m}\gamma)(z;\tau) \;&=\; (c\tau + d)^{-k}\mathbf{e}\Big(\frac{-mcz^2}{c\tau + d}\Big) f\Big(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\Big) \\ (f|_{k,m}n)(z;\tau) \;&=\; \mathbf{e}\big(m(n_1^2\tau + 2n_1 z)\big) f(z + n_1\tau + n_2; \tau)\end{aligned} \tag{7.64}$$

We can use this to define Jacobi forms.

Definition 20 (Jacobi form)**.** *A Jacobi form of weight k and index m is a holomorphic function from $\mathbb{C} \times \mathfrak{h}$ such that it is fixed under the $|_{k,m}$ action, so that in particular*

$$f|_{k,m}(\gamma, n) \;=\; f\,, \tag{7.65}$$

and such that its Fourier series

$$\sum_{n,r} c_{n,r} q^n x^r \tag{7.66}$$

has $c_{n,r} = 0$ for $n \geq r^2/4m$.

Much like modular forms, being a Jacobi form is a strict condition.

Theorem A–37. *[56, Thm. 1.1] The space of Jacobi forms of weight k and index m is finite dimensional.*

For example, weight k and index 0 Jacobi forms are constant in z and are therefore just modular forms of weight k. This is quite nice, however we are almost always interested in some more exotic examples with multiplier systems and vector–valued. These can analogously be defined and often we won't restrict ourselves as we will not often need these finite dimensionality results. We see that the Weierstrass $\wp$–function is then a weight 2 and index 0 meromorphic Jacobi form and so is already a slight generalisation away from holomorphicity.

The last equation (7.64), for the action of $\mathbb{Z}^2$, can be viewed as a q–difference equation. Indeed, letting $x = \mathbf{e}(z)$ and $q \;=\; \mathbf{e}(\tau)$ as usual, we find that the last equation is

$$f(q^{n_1}x; q) \;=\; q^{-mn_1^2}x^{-2m} f(x; q)\,. \tag{7.67}$$

This is similar to the q–difference equation (5.36). Indeed, the θ–function will be the most important basic example. If we consider the function $q^{\frac{1}{8}}x^{\frac{1}{2}}\theta(x;q)$ and take $\tilde{x} = \mathbf{e}(z/\tau)$, then from the Poisson summation formula of Theorem 26 we find

$$\begin{aligned}
\tilde{q}^{\frac{1}{8}}\tilde{x}^{\frac{1}{2}}\theta(\tilde{x};\tilde{q}) &= \sum_{k\in\mathbb{Z}}(-1)^k\tilde{q}^{(k+1/2)^2/2}\tilde{x}^{k+1/2} \\
&= \sum_{\ell\in\mathbb{Z}}\int_{-\infty}^{\infty}\mathbf{e}(-(\xi+1/2)^2/2\tau+(\xi+1/2)z/\tau+(\ell+1/2)\xi)d\xi \\
&= \mathbf{e}(z^2/2\tau)\sum_{\ell\in\mathbb{Z}}(-1)^{-\ell-1/2}\mathbf{e}((\ell+1/2)z)\int_{-\infty}^{\infty}\mathbf{e}(-\xi^2/2\tau+\ell\xi)d\xi \\
&= \sqrt{\tau}\,\mathbf{e}\Big(\frac{z^2}{2\tau}-\frac{3}{8}\Big)q^{\frac{1}{8}}x^{\frac{1}{2}}\theta(x;q)\,.
\end{aligned} \tag{7.68}$$

This modularity is why we choose this particular solution to the difference equation 5.34. We can use this to find modularity results for some vector–valued θ–functions. In particular, notice that for odd $\kappa \in 2\mathbb{Z}+1$

$$\begin{aligned}
\tilde{q}^{\frac{\kappa}{8}}\tilde{x}^{\frac{1}{2}}\theta(\tilde{x};\tilde{q}^{\kappa}) &= \sqrt{\frac{\tau}{\kappa}}\mathbf{e}\left(\frac{z^2}{2\kappa\tau}-\frac{3}{8}\right)q^{\frac{1}{8\kappa}}x^{\frac{1}{2\kappa}}\theta(x^{\frac{1}{\kappa}};q^{\frac{1}{\kappa}}) \\
&= \sqrt{\frac{\tau}{\kappa}}\mathbf{e}\left(\frac{z^2}{2\kappa\tau}-\frac{3}{8}\right)q^{\frac{1}{8\kappa}}x^{\frac{1}{2\kappa}}\sum_{r=\frac{\kappa-1}{2}}^{\frac{3\kappa-1}{2}}(-1)^r q^{\frac{r(r+1)}{2\kappa}}x^{\frac{r}{\kappa}}\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{\kappa k(k+1)}{2}}(q^{\frac{2r+1-\kappa}{2}}x)^k \\
&= \sqrt{\frac{\tau}{\kappa}}\mathbf{e}\left(\frac{z^2}{2\kappa\tau}-\frac{3}{8}-\frac{\kappa-1}{4}\right)\sum_{r=0}^{\kappa-1}(-1)^r q^{\frac{\kappa}{8}+\frac{r}{2}+\frac{r^2}{2\kappa}}x^{\frac{r}{\kappa}+\frac{1}{2}}\theta(q^r x;q^{\kappa})
\end{aligned} \tag{7.69}$$

Therefore we see that

$$\tilde{q}^{\frac{\kappa}{8}+\frac{s}{2}}\tilde{x}^{\frac{1}{2}}\theta(\tilde{q}^s\tilde{x};\tilde{q}^{\kappa}) = \sqrt{\frac{\tau}{\kappa}}\mathbf{e}\left(\frac{z^2}{2\kappa\tau}-\frac{1}{8}-\frac{\kappa}{4}\right)\tilde{q}^{-\frac{s^2}{2\kappa}}\tilde{x}^{-\frac{s}{\kappa}}\sum_{r=0}^{\kappa-1}(-1)^{r+s}\mathbf{e}\left(\frac{sr}{\kappa}\right)q^{\frac{\kappa}{8}+\frac{r}{2}+\frac{r^2}{2\kappa}}x^{\frac{r}{\kappa}+\frac{1}{2}}\theta(q^r x;q^{\kappa})\,. \tag{7.70}$$

Similarly, one can show that for even $\kappa \in 2\mathbb{Z}$,

$$\tilde{q}^{\frac{\kappa}{8}+\frac{s}{2}}\tilde{x}^{\frac{1}{2}}\theta(-\tilde{q}^s\tilde{x};\tilde{q}^{\kappa}) = \sqrt{\frac{\tau}{\kappa}}\mathbf{e}\left(\frac{z^2}{2\kappa\tau}-\frac{1}{8}-\frac{\kappa}{4}\right)\tilde{q}^{-\frac{s^2}{2\kappa}}\tilde{x}^{-\frac{s}{\kappa}}\sum_{r=0}^{\kappa-1}(-1)^{r+s}\mathbf{e}\left(\frac{sr}{\kappa}\right)q^{\frac{\kappa}{8}+\frac{r}{2}+\frac{r^2}{2\kappa}}x^{\frac{r}{\kappa}+\frac{1}{2}}\theta(-q^r x;q^{\kappa})\,. \tag{7.71}$$

Therefore, taking them vector–valued θ–function with the Vandermonde determinant

$$\vartheta_{\kappa}(z;\tau) = \begin{pmatrix} q^{\frac{\kappa}{8}}x^{\frac{1}{2}}\theta((-1)^{\kappa+1}x;q^{\kappa}) \\ \vdots \\ (-1)^r q^{\frac{\kappa}{8}+\frac{r}{2}+\frac{r^2}{2\kappa}}x^{\frac{r}{\kappa}+\frac{1}{2}}\theta((-1)^{\kappa+1}q^r x;q^{\kappa}) \\ \vdots \\ (-1)^{\kappa-1}q^{\frac{9\kappa^2-12\kappa+4}{8\kappa}}x^{\frac{3}{2}-\frac{1}{\kappa}}\theta((-1)^{\kappa+1}q^r x;q^{\kappa}) \end{pmatrix}, \quad V_{\kappa} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \mathbf{e}\left(\frac{1}{\kappa}\right) & \mathbf{e}\left(\frac{2}{\kappa}\right) & \dots & \mathbf{e}\left(\frac{\kappa-1}{\kappa}\right) \\ 1 & \mathbf{e}\left(\frac{2}{\kappa}\right) & \mathbf{e}\left(\frac{4}{\kappa}\right) & \dots & \mathbf{e}\left(\frac{2\kappa-2}{\kappa}\right) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \mathbf{e}\left(\frac{\kappa-1}{\kappa}\right) & \mathbf{e}\left(\frac{2\kappa-2}{\kappa}\right) & \dots & \mathbf{e}\left(\frac{(\kappa-1)^2}{\kappa}\right) \end{pmatrix}, \tag{7.72}$$

we find that

$$\vartheta_\kappa(\tilde{x};\tilde{q}) \;=\; \sqrt{\frac{\tau}{\kappa}}\mathbf{e}\left(\frac{z^2}{2\kappa\tau}-\frac{1}{8}-\frac{\kappa}{4}\right)V_\kappa\,\vartheta_\kappa(x;q)\,. \tag{7.73}$$

We have the additional relations

$$\begin{aligned}
\vartheta_\kappa(z;\tau+1) &= \operatorname{diag}\Big(\mathbf{e}\Big(\frac{\kappa}{8}\Big),\ldots,\mathbf{e}\Big(\frac{\kappa}{8}+\frac{r}{2}+\frac{r^2}{2\kappa}\Big),\ldots,\mathbf{e}\Big(\frac{9\kappa^2-12\kappa+4}{8\kappa}\Big)\Big)\vartheta_\kappa(z;\tau)\\
\vartheta_\kappa(z+1;\tau) &= \operatorname{diag}\Big(\mathbf{e}\Big(\frac{1}{2}\Big),\ldots,\mathbf{e}\Big(\frac{1}{2}+\frac{r}{\kappa}\Big),\ldots,\mathbf{e}\Big(\frac{1}{2}+\frac{\kappa-1}{\kappa}\Big)\Big)\vartheta_\kappa(z;\tau)\\
\vartheta_\kappa(z+\tau;\tau) &= -q^{-\frac{1}{2\kappa}}x^{-\frac{1}{\kappa}}\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\0&0&0&\cdots&1\\1&0&0&\cdots&0\end{pmatrix}\vartheta_\kappa(z;\tau)
\end{aligned} \tag{7.74}$$

We then see that this is a vector–valued Jacobi form with some matrix valued automorphy factor.

From Jacobi forms, we can construct modular forms. One way is by taking sums over torsion points on the elliptic curve *i.e.* sums of the form

$$\sum_{k,\ell=0}^{N-1} f((k+\tau\ell)/N;\tau)\,. \tag{7.75}$$

Also, expanding as power series in the elliptic variable near 0 and taking the coefficients we see that quasi modular forms will arise. Quasi modular forms will arise as

$$\mathbf{e}\Big(\frac{z^2}{2\tau}\Big) \;=\; \frac{\exp((2\pi iz)^2G_2(q))}{\exp((2\pi iz/\tau)^2G_2(\tilde{q}))}\,. \tag{7.76}$$

Therefore, if f is a Jacobi form of weight k and index m, we see that

$$g(z;\tau) \;=\; \exp(2m(2\pi iz)^2G_2(q))f(z;\tau) \tag{7.77}$$

satisfies

$$g\Big(\frac{z}{\tau};\frac{-1}{\tau}\Big) \;=\; \tau^k g(z;\tau)\,. \tag{7.78}$$

Therefore, the coefficients in the expansion of g in z around $z=0$ will be modular forms with weights depending on the power of z and k.

7.7 Deformations of modular q–hypergeometric functions

We can view the θ–function $\theta(x;q)$ as a deformation of the θ–function $\vartheta_{00}(q)$. Indeed, they are related by simple functions and shifting the index of summation.

$$\begin{aligned}\theta(z;\tau) &= \sum_{k\in\mathbb{Z}} \mathbf{e}(k(k+1)\tau/2+kz) = \sum_{k\in\mathbb{Z}} \mathbf{e}((k+1/2+z/\tau)^2\tau/2-(1/2+z/\tau)^2\tau/2) \\ &= \mathbf{e}(-(1/2+z/\tau)^2\tau/2) \sum_{k\in\mathbb{Z}+1/2+z/\tau} \mathbf{e}(k^2\tau/2)\,. \end{aligned} \tag{7.79}$$

It is then natural to consider the modularity of modular q–hypergeometric functions under a similar operation. From the point of view of Nahm's conjecture 6, this is natural as these deformation will not alter the original q–difference equations. While a general statement to this affect would be involved given the still somewhat mysterious nature of modular q–hypergeometric functions we can prove this in the basic example of the Roger–Ramanujan functions. Taking the deformation of the Rogers–Ramanujan functions from Example 57 as we discussed in Section 6.4 we have

$$G(x;q) = \frac{(qx;q)_\infty}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}\frac{q^{k^2}x^{2k}}{(qx;q)_k}\,, \quad \text{and} \quad H(x;q) = \frac{(qx;q)_\infty}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}\frac{q^{k^2+k}x^{2k+1}}{(qx;q)_k}\,. \tag{7.80}$$

Then let

$$g(x;q) = \begin{pmatrix} q^{-\frac{1}{60}}G(x;q) \\ q^{\frac{11}{60}}H(x;q)\end{pmatrix}\,. \tag{7.81}$$

With this we have the following result, which was seemingly known to Zwegers.

Theorem 3. *The function g is a vector–valued Jacobi form of weight 0 and index 2 associated to the representation ρ from the original Rogers–Ramanujan functions. In particular,*

$$\begin{aligned} g(z+1;\tau) &= g(z;\tau) \\ g(z+\tau;\tau) &= q^{-1}x^{-2}g(z;\tau) \\ g(z;\tau+1) &= \begin{pmatrix}\mathbf{e}\big(-\frac{1}{60}\big) & 0 \\ 0 & \mathbf{e}\big(\frac{11}{60}\big)\end{pmatrix} g(z;\tau) \\ g\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big) &= \mathbf{e}\Big(\frac{z^2}{\tau}\Big)\frac{2}{\sqrt{5}}\begin{pmatrix}\sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5}\end{pmatrix} g(z;\tau)\,. \end{aligned} \tag{7.82}$$

Proof. Firstly, notice that

$$\begin{pmatrix}0&1&0&0&-1\\0&0&1&-1&0\end{pmatrix}\begin{pmatrix}1&1&1&1&1\\1&\mathbf{e}(1/5)&\mathbf{e}(2/5)&\mathbf{e}(3/5)&\mathbf{e}(4/5)\\1&\mathbf{e}(2/5)&\mathbf{e}(4/5)&\mathbf{e}(6/5)&\mathbf{e}(8/5)\\1&\mathbf{e}(3/5)&\mathbf{e}(6/5)&\mathbf{e}(9/5)&\mathbf{e}(12/5)\\1&\mathbf{e}(4/5)&\mathbf{e}(8/5)&\mathbf{e}(12/5)&\mathbf{e}(16/5)\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{e}(1/5)-\mathbf{e}(4/5)&\mathbf{e}(2/5)-\mathbf{e}(3/5)\\\mathbf{e}(2/5)-\mathbf{e}(3/5)&-\mathbf{e}(1/5)+\mathbf{e}(4/5)\end{pmatrix}\begin{pmatrix}0&1&0&0&-1\\0&0&1&-1&0\end{pmatrix}. \tag{7.83}$$

Then, for ϑ_5 from equation (7.72), letting

$$\Theta(z;\tau) \;=\; \begin{pmatrix}0&1&0&0&-1\\0&0&1&-1&0\end{pmatrix}\vartheta_5(z;\tau)\,, \tag{7.84}$$

from the modularity of ϑ_5 and equation (7.83) we find that

$$\mathbf{e}\left(\frac{z^2}{10\tau}\right)\mathbf{e}\left(-\frac{3}{8}\right)\sqrt{5\tau}\Theta(z;\tau) \;=\; \begin{pmatrix}\mathbf{e}(1/5)-\mathbf{e}(4/5)&\mathbf{e}(2/5)-\mathbf{e}(3/5)\\\mathbf{e}(2/5)-\mathbf{e}(3/5)&-\mathbf{e}(1/5)+\mathbf{e}(4/5)\end{pmatrix}\Theta\left(\frac{z}{\tau};-\frac{1}{\tau}\right). \tag{7.85}$$

Then notice that

$$F_0(x;q) \;=\; \sum_k(-1)^k q^{k(5k-1)/2}x^{5k}(1+q^kx) \;=\; -q^2x^5\theta(q^2x^5;q^5)-q^3x^6\theta(q^3x^5;q^5) \tag{7.86}$$

and

$$F_1(x;q) \;=\; \sum_k(-1)^k q^{k(5k+3)/2}x^{5k+2}(1-q^{2k+1}x^2) \;=\; -q^4x^7\theta(q^4x^5;q^5)-qx^4\theta(qx^5;q^5). \tag{7.87}$$

Therefore, using the deformed Rogers–Ramnaujan identities of Theorem 2 and the modularity of θ and η, we find that

$$g\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big) \;=\; \mathbf{e}\Big(\frac{z^2}{\tau}\Big)\frac{2}{\sqrt{5}}\begin{pmatrix}\sin\frac{2\pi}{5}&\sin\frac{\pi}{5}\\\sin\frac{\pi}{5}&-\sin\frac{2\pi}{5}\end{pmatrix}g(z;\tau)\,. \tag{7.88}$$

The other identities for the generators of the Jacobi group can be checked directly. □

Another natural place that a kind of deformation of q–hypergeometric functions appears is through q–Borel resummation discussed in Section 5.5. This leads to additional elliptic variables when the q–series should be divergent. As a somewhat trivial example consider the sum

$$\sum_{k\in\mathbb{Z}}(-1)^k q^{-k(k+1)/2}x^k\,. \tag{7.89}$$

This is of course divergent when $|q| < 1$. We can of course use q–Borel resummation of the series

$$\sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}x^k\,, \quad \text{and} \quad \sum_{k=-\infty}^{-1}(-1)^k q^{-k(k+1)/2}x^k\,. \tag{7.90}$$

In this example, these give exactly the Appell–Lerch sums of equation (5.84) studied in [211] with the difference of a sign between the positive and negative sums. Therefore, if λ and μ represent the elliptic variables coming from the q–Laplace transforms, their sum from [82] or Lemma 12 is given by

$$\frac{(q;q)_\infty^3\theta(\lambda^{-1}\mu;q)\theta(\lambda^{-1}\mu^{-1}x^{-1};q)\theta(q^{-1}x;q)}{\theta(\lambda^{-1};q)\theta(\mu;q)\theta(\lambda^{-1}x;q)\theta(\mu^{-1}x;q)}\frac{1}{\theta(q^{-1}x;q)}\,. \tag{7.91}$$

If one multiplies by $\theta(q^{-1}x;q)$ then this is a meromorphic Jacobi form of weight 1 and index 0 in three elliptic variables. Therefore, this example of a divergent q–series that we expect to be modular, indeed turns out to be modular when q–Borel resummed.

Remark 24. *This would then give a new fully modular approach to indefinite and negative definite θ–functions. However, the combinatorial interpretation of these θ–functions at the moment seems less clear without any further investigation.*

7.8 Appell–Lerch sums

Recall the Appell–Lerch sum of equation (5.84) given by

$$L(x,\lambda,q) \;=\; \frac{1}{\theta(\lambda;q)}\sum_{k\in\mathbb{Z}}(-1)^k\frac{q^{k(k+1)/2}\lambda^k}{1-q^k\lambda x}\,. \tag{7.92}$$

This is elliptic in λ and satisfies the inhomogeneous q–difference equation in x,

$$L(qx,\lambda,q)+xL(x,\lambda,q) \;=\; \frac{1}{\theta(\lambda;q)}\sum_{k\in\mathbb{Z}}(-1)^{k-1}\frac{q^{k(k-1)/2}\lambda^{k-1}(1-q^k\lambda x)}{1-q^k\lambda x} \;=\; 1\,. \tag{7.93}$$

This q–difference equation is an inhomogenous version of the q–difference equation satisfied by

$$\frac{1}{\theta(q^{-1}x;q)}\,. \tag{7.94}$$

Therefore, the Appell–Lerch sum lives in a degree one extension of the module associated to the θ–function. To see how this related to Ramanujan's description of mock θ–functions, consider the expansion

$$L(1,\epsilon,q) \;=\; \frac{-1}{(q;q)_\infty^3}\epsilon^{-2}+\Sigma(q)+O(\epsilon^1)\,, \tag{7.95}$$

where

$$\Sigma(q) \;=\; \frac{1}{24}(2 - 90q - 462q^2 - 1540q^3 - 4554q^4 - 11592q^5 - 27830q^6 + \cdots)\,. \tag{7.96}$$

This function Σ appears in the Matthieu moonshine conjecture [54]. This function has asymptotics as $\tau \to \infty$

$$\Sigma(\tilde{q}) \sim \frac{1}{12}q^{-1/8}\sqrt{\tau}\tilde{q}^{1/8} + \frac{1}{2} + \frac{1}{8}(2\pi i/\tau) + \frac{1}{32}(2\pi i/\tau)^2 + \frac{5}{384}(2\pi i/\tau)^3 + \dots \tag{7.97}$$

which is exactly of the form Ramanujan describes where the leading asymptotics appear as they would for a modular form while the subleading appear as some new $O(1)$ series. The $O(1)$ series is given by the expansion in x of

$$\frac{1}{2}\exp(x/8)\sum_{k=0}^{\infty} A_k \frac{x^k}{8^k k!}\,, \quad \text{where} \quad \frac{1}{\cos(x)} \;=\; \sum_{k=0}^{\infty} A_k \frac{x^{2k}}{(2k)!}\,. \tag{7.98}$$

We see that this behaves exactly as Ramanujan describes. The failure of modularity of this function is given by a special function called the Mordell integral.

Theorem A–38. *[211, Prop. 1.5] We have the following equality*

$$\sqrt{\frac{i}{\tau}}\mathbf{e}\Big(\frac{z^2}{2\tau}\Big)\mathbf{e}\Big(\frac{z}{2\tau}\Big)\tilde{q}^{-\frac{1}{8}}L\Big(\frac{z}{\tau},\frac{w}{\tau};-\frac{1}{\tau}\Big) + \mathbf{e}\Big(\frac{z}{2}\Big)q^{-\frac{1}{8}}L(z,w;\tau) \;=\; \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathbf{e}(x^2\tau/2 + ixz)}{\cosh(\pi x)}dx\,. \tag{7.99}$$

Proof. The main idea of the proof is that the LHS and RHS both satisfy an inhomogenous version of the same q and $\tilde{q}$ difference equations and are holomorphic. If there existed two such functions then taking the difference and a product with $x^{1/2}\theta(x;q)$ would give a holomorphic elliptic function and therefore constant. This implies that if it is not zero that the difference has poles at $\mathbb{Z} + \tau\mathbb{Z}$ but it is holomorphic and hence must be zero. ☐

Note that by shifting the contour with the argument of τ, the Mordell integral can be used to define a holomorphic function in $\mathbb{C} - \mathbb{R}_{\leq 0}$. Letting

$$\vartheta(x;q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}x^k \tag{7.100}$$

be the negative definate partail θ–function, the Mordell integral in the lower half plane is given by

$$\sqrt{\frac{i}{\tau}}\mathbf{e}\Big(\frac{z^2}{2\tau}\Big)\mathbf{e}\Big(\frac{z}{2\tau}\Big)\tilde{q}^{-\frac{1}{8}}\vartheta\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big) + \mathbf{e}\Big(\frac{z}{2}\Big)q^{-\frac{1}{8}}\vartheta(z;\tau)\,, \tag{7.101}$$

and the proof is analogous to that in the upper half plane with the addition of checking boundary conditions as z and z/τ tend to $i\infty$. This example illustrates an interesting phenomenon. Notice that for $|q| < 1$ we have

$$(\mathcal{L}_1\mathcal{B}_1\vartheta)(x,\lambda;q) \;=\; L(x,\lambda;q)\,. \tag{7.102}$$

Therefore, we see that the mock modular behaviour of this function behaves well with respect to q–Borel resummation. This is a somewhat generalised version of the observation from the end of Section 7.7. An interesting point is that the q–Stokes phenomenon *i.e.* the variable λ, does not appear in the failure of modularity as noted by Zwegers. This again holds in more interesting examples associated to quanrtum modular forms discussed in [82].

7.9 Inhomogenous equations from modular forms

All of Ramanujan's examples of mock modular forms where q–hypergeometric of a special form. In particular, they were a sum over not a lattice but a cone. This meant that they all naturally come in families that satisfy inhomogenous q–difference equations. These inhomogenous equations require asymptotics which also satisfy the same q–difference equation. This in some sense, this is what requires the existence of the $O(1)$ asymptotic series that Ramanujan observed. This is analogous to the situation discussed in Section 6.5 and Example 43 in constructing the function $\mathfrak{G}$. Therefore, after the introduction of quantum modular forms, one can think of $\mathfrak{G}$ that appeared there as a mock quantum modular form in the vague sense of Ramanujan. However, this is somewhat redundant as we will see that mock modular forms are examples of quantum modular forms. Finally, one of the other ways that mock modular forms can be described is through indefinite θ–functions. These are again summed over a cone leading to inhomogeneous relations and the same phenomenon.

To see how the inhomogeneity fixes the $O(1)$ asymptotic series, consider again the third order mock θ–function in its natural family

$$f_m(q) \;=\; \sum_{k=0}^{\infty}\frac{q^{k^2+km}}{(-q;q)_k}\,. \tag{7.103}$$

This family satisfies the recursion

$$f_m(q)+f_{m+1}(q)-q^{m+1}f_{m+2} \;=\; 2\,. \tag{7.104}$$

We can apply the methods of Section 5.7, to find solutions that come from an asymptotic series. Indeed, with the Ansatz

$$f_m(\hbar) \;=\; \sum_{k=0}^{3}\sum_{\ell=0}^{k} a_{k,\ell}m^\ell\hbar^\ell+O(\hbar)^4 \tag{7.105}$$

we find that equation (7.104) has a unique solution

$$f_m(\hbar) \; = \; 2 + (4+2m)\hbar + (36+18m+3m^2)\hbar^2 + \Big(\frac{1640}{3} + 269m + 53m^2 + \frac{13}{3}m^4\Big)\hbar^3 + O(\hbar^4)\,. \tag{7.106}$$

Specialising to $m = 0$ we see exactly the asymptotic series we observed in the introduction to this chapter 7.

7.10 A conjecture on order 7 mock θ–functions

In joint work with Matthias Storzer, we studied various aspects of mock modular forms and noticed that using this idea of inhomogeneity when taking the expression of a mock modular form and summing over the full lattice one finds a modular form. For example,

$$f(q) \; = \; \sum_{k\in\mathbb{Z}} \frac{q^{k^2}}{(-q;q)_k}\,, \tag{7.107}$$

is a modular form. This was noticed for all of Ramaunjan's q–hypergeometric examples. Therefore, using Ramanujan's mock modular forms we see that they come with a natural partner, the sum over the rest of the lattice, so that the pair sum to a modular form. This was noticed in for example in [25, 16], which led to new examples of mock θ–functions which were a modular form plus an old one. This works for most examples, however the order seven mock modular forms lead to divergent sums and were not studied further. With access to q–Borel resummation these can be easily defined.

Consider a t deformation of Ramanujan's seventh order mock θ–functions

$$F_+(t;q) \; = \; \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n} t^{-n-1} \; = \; \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q;q)_{2n}} t^{-n-1}. \tag{7.108}$$

This satisfies the functional equations

$$tF_+(t;q) + (1+qt)F_+(qt;q) - q^2tF_+(q^2t;q) - q^4tF_0(q^3t;q) \; = \; 2-2q\,. \tag{7.109}$$

Similarly, the sum over $n < 0$ gives the function

$$f_-(t;q) \; = \; 2\sum_{n=0}^{\infty}(-1)^n \frac{q^{-n(n+1)/2}(q;q)_{2n+1}}{(q;q)_n} t^n \tag{7.110}$$

and this also satisfies Equation (7.109). F_- is divergent for $|q| < 1$ and therefore we take the q–Borel resummation. We have

$$\mathcal{B}_1 f_-(\xi;q) \; = \; 2\sum_{n=0}^{\infty} \frac{(q;q)_{2n+1}}{(q;q)_n} \xi^n\,, \tag{7.111}$$

which is convergent for $|\xi| < 1$. This satisfies the equation

$$(1-\xi)\mathcal{B}_1 f_-(\xi;q) - \mathcal{B}_1 f_-(q\xi;q) + (q^2+q^3)\xi\mathcal{B}_1 f_-(q^2\xi;q) - q^5\xi\mathcal{B}_1 f_-(q^4\xi;q) \;=\; 0\,. \tag{7.112}$$

Using this we can define an analytic extension of $\mathcal{B}_1 F_-(\xi;q)$ to all $\xi \notin q^{\mathbb{Z}_{\leq 0}}$. Using this we take the Laplace transform to get a meromorphic function

$$F_-(t,\lambda;q) \;=\; \mathcal{L}_1\mathcal{B}_1 f_-(t,\lambda;q) \;=\; \sum_{k\in\mathbb{Z}} \frac{\mathcal{B}_1 f_-(q^k\lambda t;q)}{\theta(q^k\lambda;q)}\,. \tag{7.113}$$

This function has expansion in ϵ when $\lambda = \exp(\epsilon)$ at $t=1$

$$\begin{aligned} F_-(1,\exp(\epsilon);q) \;&=\; (-2-8q-26q^2-72q^3-182q^4-422q^5-930q^6-1948q^7+\cdots)\epsilon^{-2} \\ &+\Big(\frac{1}{6}-\frac{28}{3}q-\frac{275}{6}q^2-176q^3-\frac{3389}{6}q^4-\frac{9449}{6}q^5-\frac{8125}{2}q^6-\frac{29183}{3}q^7+\cdots\Big)+O(\epsilon^2) \end{aligned} \tag{7.114}$$

and hence

$$\begin{aligned} &F_-(1,\exp(\epsilon);q) - F_+(1;q) \\ &\;=\; (-2-8q-26q^2-72q^3-182q^4-422q^5-930q^6-1948q^7+\cdots)\epsilon^{-2} \\ &\quad+\Big(-\frac{5}{6}-\frac{31}{3}q-\frac{275}{6}q^2-177q^3-\frac{3395}{6}q^4-\frac{9455}{6}q^5-\frac{8125}{2}q^6-\frac{29189}{3}q^7+\cdots\Big)+O(\epsilon^2)\,. \end{aligned} \tag{7.115}$$

We first notice that the expansion of the ϵ^{-2} coefficient matches that of the second Andrews-Gordon function

$$-2(q;q)_\infty^{-3}\sum_{k,\ell=0}^{\infty}\frac{q^{(k+\ell)^2+\ell^2+\ell}}{(q;q)_k(q;q)_\ell}\,. \tag{7.116}$$

Secondly, we observe numerically that the expansion of the ϵ^0 coefficient in (7.115) (but not in (7.114), since $F_+(1;q)$ is only mock modular) is modular of weight, and in fact numerical computations suggest the following conjecture.

Conjecture 8. *The function*

$$F_{-1}(1,\lambda;q) - F_+(1;q) \tag{7.117}$$

is a Jacobi form of index 0 *satisfying the same modular transformation as the Andrews-Gordon functions times the* η *function of different weight (as part of a vector–valued modular form).*

This cojnecture is a more interesting example of the discussion around equation (7.91). This would also imply that

$$C_{\epsilon^0}(F_{-1}(1,\exp(\epsilon);q)) \tag{7.118}$$

is a mock θ–function with the same shadow as $F_0(1;q)$.

Remark 25. *I believe this conjecture could be proved. A natural approach would be to try and find some kind of* q*–series identity with some functions related to the second Andrews–Gordon functions. This approach would involve some playing around with* q*–series, which both Matthias and I haven't done over the last year so I just report just the conjecture here.*

Chapter 8

Quantum modular forms

Quantum modular forms are a generalisation to the previous examples of modular objects we have seen. They encompass modular forms and mock modular forms as special cases. The truly new examples historically came from functions at roots of unity, whose failure of modularity has improved analyticity properties [208, Ex. 5]. These functions can roughly be thought of as functions from $\mathfrak{h} \cup \mathbb{Q} \cup \overline{\mathfrak{h}}$ to some matrix group such that some multiplicative failure of modularity defines a function that has an analytic extension to some simply connected cut plane in $\mathbb{C}$ [82, 201].

The first basic example are just modular forms. However, we need to extend these to functions at $\mathbb{Q}$ and then the lower half plane. This can be done through asymptotics and here we find that information about the cusps is stored in an interesting way. The next examples come as mock modular forms, however, we will not discuss this case in great length. Then q–hypergeometric elements of the Habiro ring were the next set of interesting examples. In particular, elements of the Habiro ring that came from knot invariants.

I will give a somewhat historical and somewhat pedagogical overview of the theory, starting with modular forms at roots of unity. Then a discussion on asymptotics and how one can prove these kind of asymptotic statements. Following this the cocycles used to define quantum modular forms will be considered, along with some of their structural properties. This will allow us to give our working definition of quantum modular forms. As usual we will then explore this in the special case of the Pochhammer symbol, which will give us the basic tool we will need when discussing state integrals.

Asymptotics of modular forms behave in very simple and beautiful ways. Here we will explore this at roots of unity and then some similar properties of q–hypergeometric examples.

We have seen that various proper q–hypergeometric functions satisfy interesting asymptotics as $\tau \to \infty$. These asymptotics possess some interesting modular properties which stem from the modular properties of the Pochhammer symbol. A famous and old problem is how to make sense of certain divergent series. Indeed, if the asymptotic series that arise are convergent then one can subtract the leading order and explore subleading behaviour. In the previous section, we numerically turned the asymptotic series into convergent functions using Borel–Padé–Laplace in the example of the WRT invariant of $4_1(1,2)$. There we saw that indeed after removing the leading order we saw subleading terms. However, the use of this resummation technique is numerical and therefore everything is conjectural. In this section, we will explore the properties that we expect these resummations to satisfy. These define a matrix valued cocycle and we will see that this gives interesting elements of some group cohomologies of $\mathrm{SL}_2(\mathbb{Z})$. This cohomological interpretation was realised in [86].

The fundamental result that will be necessary for the proofs of quantum modularity of q–hypergeometric functions is that the Pochhammer symbol is a quantum Jacobi form. We have essentially gone through the proof of this statement in Section 4.4 . We will go into the slightly more detail that we need here.

8.1 Modular forms at roots of unity

Recall the Eisenstein series $E_{2k}(q)$ of equation (7.19) for $k \in \mathbb{Z}_{>1}$. These satisfies the modular transformations

$$E_{2k}(\tau+1)E_{2k}(\tau)^{-1} \;=\; 1\,, \quad \text{and} \quad E_{2k}(-1/\tau)E_{2k}(\tau)^{-1} \;=\; \tau^{2k}\,. \tag{8.1}$$

We can define an extension of these functions to include the rationals. This is defined such that

$$\text{for} \quad x\in\mathbb{Q} \quad \text{we have} \quad E_{2k}(x) \;=\; \mathrm{denom}(x)^{2k}\,, \tag{8.2}$$

where $\mathrm{denom}(0)=1$ and $\mathrm{denom}(\infty)=0$. This extended version satisfies exactly the same modular transformations (8.1). Finally, we can extend E_{2k} to the lower half plane such that

$$\text{for} \quad \tau\in\overline{\mathfrak{h}} \quad \text{we have} \quad E_{2k}(\tau) = -E_{2k}(-\tau)\,. \tag{8.3}$$

We see that the whole extension $E_{2k}:\mathfrak{h}\cup\mathbb{Q}\cup\overline{\mathfrak{h}}\to\mathbb{C}$ satisfies the modular transformations (8.1). This is the most basic example of a (multiplicative) quantum modular form. At rationals, the following proposition states that denom^{2k} is the only rank one quantum modular form whose "failure of modularity" is given by x^{2k} up to multiplication by a constant.

Proposition 12. *The only (multiplicative) quantum modular forms $f:\mathbb{P}\mathbb{Q}\to\mathbb{C}$ that is functions with*

$$f(x+1) \;=\; f(x) \quad \textit{and} \quad f(-1/x) \;=\; x^{2k}f(x) \tag{8.4}$$

are multiples of $\mathrm{denom}(x)^{2k}$.

Proof. Notice that if

$$g(x) \;=\; \frac{f(x)}{\mathrm{denom}(x)^{2k}} \tag{8.5}$$

we have

$$g(x+1) \;=\; g(x)\,, \quad \text{and} \quad g(-1/x) \;=\; g(x)\,. \tag{8.6}$$

Then as $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}\mathbb{Q}$ we see that $g(x)$ must be constant. □

The reason that the extensions of E_{2k} is the right extension is that it comes with something similar to a strange identity. For $\tau \in \mathfrak{h}, x \in \mathbb{Z}$ with $\tau \to \infty$ with $\arg(x) > \epsilon$ and $x \to \infty$ we have

$$E_{2k}(-1/\tau) \sim \tau^{2k} + O(\tau^{2k} q)\,, \quad \text{and} \quad E_{2k}(-1/x) = x^{2k} \tag{8.7}$$

It appear that some information was lost, as we know that spaces of modular forms can have dimension greater than one. However, there is additional data at roots of unity referred to in [86] as tweaking.

To understand this, we should start with the first interesting example of a cusp form. This comes from the 24–th power of the η functions so we can start there. Define a function from rational numbers by the constant terms from equation (4.93)

$$\varepsilon\Big(\frac{r}{s}\Big) \;=\; \sqrt{-i}\prod_{\ell=1}^{s-1}\Big(1-\mathbf{e}\Big(\frac{\ell\, r}{s}\Big)\Big)^{\frac{1}{2}-\frac{\ell}{s}}, \tag{8.8}$$

where we use the convention that $\mathrm{denom}(x) > 0$ and so $s > 0$. Note that

$$\varepsilon(x)\,\varepsilon(-x) \;=\; -i\,. \tag{8.9}$$

We have the following lemma.

Lemma 20. *[208, Ex. 0] For $x \in \mathbb{Q}^\times$*

$$\mathbf{e}\Big(-\frac{1}{24x}\Big)\,\varepsilon\Big(-\frac{1}{x}\Big) \;=\; \mathbf{e}\Big(\frac{1}{24\,\mathrm{denom}(x)\,\mathrm{numer}(x)}\Big)\,\sqrt{-i}\,\mathbf{e}\Big(\frac{x}{24}\Big)\,\varepsilon(x)\,. \tag{8.10}$$

Proof. Suppose that $\gamma = [a,b;c,d] \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$.

$$\begin{aligned} \varepsilon\Big(\frac{a}{c}\Big)\,\mathbf{e}\Big(\frac{a+d}{24c}\Big) \;&=\; \epsilon(a,b,c,d)\,, \\ \varepsilon\Big(-\frac{c}{a}\Big)\,\mathbf{e}\Big(\frac{-c+b}{24a}\Big) \;&=\; \epsilon(-\mathrm{sign}(a)c, -\mathrm{sign}(a)d, |a|, \mathrm{sign}(a)\,b)\,. \end{aligned} \tag{8.11}$$

Therefore,

$$\frac{\mathbf{e}\big(-\frac{c}{24a}\big)\,\varepsilon\big(-\frac{c}{a}\big)}{\mathbf{e}\big(\frac{a}{24c}\big)\,\varepsilon\big(\frac{a}{c}\big)} \;=\; \mathbf{e}\Big(\frac{a+d}{24c}+\frac{c-b}{24a}-\frac{c}{24a}-\frac{a}{24c}\Big)\frac{\epsilon(a,b,c,d)}{\epsilon(-\mathrm{sign}(a)c, -\mathrm{sign}(a)d, |a|, \mathrm{sign}(a)\,b)}\,. \tag{8.12}$$

Then from the modularity of the η–function we have

$$\begin{aligned}&\sqrt{|a|\tau+\mathrm{sign}(a)\,b}\,\epsilon(-\mathrm{sign}(a)c,-\mathrm{sign}(a)d,|a|,\mathrm{sign}(a)\,b)\\&=\sqrt{\frac{|a|\tau+\mathrm{sign}(a)\,b}{c\tau+d}}\,\epsilon(0,-\mathrm{sign}(a),\mathrm{sign}(a),0)\sqrt{c\tau+d}\,\epsilon(a,b,c,d)\end{aligned}\tag{8.13}$$

Then we see that as

$$\begin{aligned}0<\arg(\sqrt{|a|\tau+\mathrm{sign}(a)\,b})<\frac{\pi}{2}\\-\frac{\pi}{2}<\arg\Big(\sqrt{\frac{|a|\tau+\mathrm{sign}(a)\,b}{c\tau+d}}\sqrt{c\tau+d}\Big)<\pi\end{aligned}\tag{8.14}$$

we have

$$\sqrt{|a|\tau+\mathrm{sign}(a)\,b}\;=\;\sqrt{\frac{|a|\tau+\mathrm{sign}(a)\,b}{c\tau+d}}\sqrt{c\tau+d}\tag{8.15}$$

and so

$$\frac{\mathbf{e}\big(-\frac{c}{24a}\big)\,\varepsilon\big(-\frac{c}{a}\big)}{\mathbf{e}\big(\frac{a}{24c}\big)\,\varepsilon\big(\frac{a}{c}\big)}\;=\;\mathbf{e}\Big(\frac{1}{24ac}\Big)\,\epsilon(0,-\mathrm{sign}(a),\mathrm{sign}(a),0)\;=\;\sqrt{-i}\,\mathbf{e}\Big(\frac{1}{24ac}\Big)\,.\tag{8.16}$$

□

Therefore, for $\gamma=[a,b;c,d]\in\mathrm{SL}_2(\mathbb{Z})$ we see that

$$\varepsilon\Big(\frac{a}{c}\Big)^{24}\mathbf{e}\Big(\frac{a+d}{24c}\Big)^{24}\;=\;\epsilon(a,b,c,d)^{24}\;=\;1\,.\tag{8.17}$$

Therefore, if d is any integer such that

$$a(d+c\mathbb{Z})\;=\;1+c\mathbb{Z}\,,\tag{8.18}$$

then

$$\mathbf{e}\Big(\frac{a}{c}\Big)\varepsilon\Big(\frac{a}{c}\Big)^{24}\;=\;\mathbf{e}\Big(-\frac{d}{c}\Big)\,.\tag{8.19}$$

Of course from the previous Lemma 20 we see that

$$\begin{aligned}\mathbf{e}(x+1)\varepsilon(x+1)^{24}\;&=\;\mathbf{e}(x)\varepsilon(x)^{24}\,,\\\mathbf{e}\Big(-\frac{1}{x}\Big)\varepsilon\Big(-\frac{1}{x}\Big)^{24}\;&=\;\mathbf{e}\Big(\frac{1}{\mathrm{denom}(x)\mathrm{numer}(x)}\Big)\mathbf{e}(x)\varepsilon(x)^{24}\,.\end{aligned}\tag{8.20}$$

By a completely analogous proof to Proposition 12, we find that ε is the only function satisfying these equations up to a multiplicative factor. This then indicates that modular forms at roots of unity keep track of their behaviour at cusps via the tweaking factor. For example, from this it seems somewhat natural to define

$$\Delta : \mathfrak{h} \cup \mathbb{Q} \cup \overline{\mathfrak{h}} \to \mathbb{C} \quad s.t. \quad \Delta(\tau) \;=\; \begin{cases} q\prod_{j=1}^{\infty}(1-q^j)^{24} & \text{if } \tau \in \mathfrak{h}\,, \\ q\,\varepsilon(q)^{24} & \text{if } \tau \in \mathbb{Q}\,, \\ q\prod_{j=1}^{\infty}(1-q^{-j})^{-24} & \text{if } \tau \in \overline{\mathfrak{h}}\,. \end{cases} \tag{8.21}$$

The modular transformation now becomes

$$\Delta(-1/\tau) \;=\; \mathbf{j}_{12}(\tau;S)\Delta(\tau) \tag{8.22}$$

where

$$\mathbf{j}_{12}(\tau;S) \;=\; \begin{cases} \tau^{12} & \text{if } \tau \in \mathfrak{h}\,, \\ \mathbf{e}\big(\frac{1}{\mathrm{denom}(x)\mathrm{numer}(x)}\big) & \text{if } \tau \in \mathbb{Q}\,, \\ \tau^{-12} & \text{if } \tau \in \overline{\mathfrak{h}}\,. \end{cases} \tag{8.23}$$

This automorphy factor is of a similar shape to all our examples and importantly, we need this for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 21 (Automorphy at $\mathbb{Q}$)**.** *Take the cocycle λ from [86, Lem. 3.1] defined so that for $\gamma = [a,b;c,d] \in \mathrm{SL}_2(\mathbb{Z})$ and $r/s \in \mathbb{Q}$ we have*

$$\lambda_\gamma(\tau) \;=\; \frac{c}{s(cr+ds)}\,. \tag{8.24}$$

This cocycle gives rise to automorphy factors at $\mathbb{Q}$.

It would be interesting if one could define the modular forms at the rationals in such a way that we get completely analogous results to that of classical modular forms. Importantly, some rigidity that could also encompass cusp forms. However, we won't say more on this here and close this section with one last example, which illustrates that similar ideas should also apply more generally.

The asymptotics of the $A = 2$ Nahm sums, *a.k.a* the Rogers–Ramanujan functions, were discussed in Section 4.7. Recall, the equation

$$1 - X \;=\; X^2 \tag{8.25}$$

which has solutions

$$X_1 \;=\; -\frac{1}{2} - \frac{\sqrt{5}}{2} \quad \text{and} \quad X_2 \;=\; -\frac{1}{2} + \frac{\sqrt{5}}{2}\,, \tag{8.26}$$

gave rise to two functions for $q = \mathbf{e}(a/c)$

$$f_{j,m}(q) \;=\; \frac{\prod_{\ell=1}^{|c|-1}\big(1-q^\ell\big)^{\frac{\ell}{|c|}-\frac{1}{2}}}{\sqrt{|c|X_j/(1-X_j)+2|c|}} \sum_{r=0}^{|c|-1} \frac{q^{r^2+rm}X_j^{(2r+m)/|c|}}{\prod_{s=0}^{|c|-1}(1-q^{r+s+1}X_j^{1/|c|})^{\frac{r+s+1}{|c|}-\frac{1}{2}}}\,, \tag{8.27}$$

that gave the constants of the asymptotic series. These functions satisfy the equation

$$\begin{pmatrix} \mathbf{e}\big(\frac{1}{60x}\big) f_{1,0}\big(-\frac{1}{x}\big) & \mathbf{e}\big(\frac{1}{60x}\big) f_{2,0}\big(-\frac{1}{x}\big) \\ \mathbf{e}\big(-\frac{11}{60x}\big) f_{1,1}\big(-\frac{1}{x}\big) & \mathbf{e}\big(-\frac{11}{60x}\big) f_{2,1}\big(-\frac{1}{x}\big) \end{pmatrix} \begin{pmatrix} \mathbf{e}(-\frac{11}{60}\frac{1}{\mathrm{denom}(x)\,\mathrm{numer}(x)}) & 0 \\ 0 & \mathbf{e}(\frac{1}{60}\frac{1}{\mathrm{denom}(x)\,\mathrm{numer}(x)}) \end{pmatrix}$$
$$= \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\big(\frac{2\pi}{5}\big) & \sin\big(\frac{\pi}{5}\big) \\ \sin\big(\frac{\pi}{5}\big) & -\sin\big(\frac{2\pi}{5}\big) \end{pmatrix} \begin{pmatrix} \mathbf{e}\big(-\frac{x}{60}\big) f_{1,0}(x) & \mathbf{e}\big(-\frac{x}{60}\big) f_{2,0}(x) \\ \mathbf{e}\big(\frac{11x}{60}\big) f_{1,1}(x) & \mathbf{e}\big(\frac{11x}{60}\big) f_{2,1}(x) \end{pmatrix}. \tag{8.28}$$

These equations can be proved from the modularity of the Rogers–Ramanujan functions following a similar argument we gave for the η function. This illustrates some aspects of how a theory at roots of unity should generalise. Importantly, the tweaking factor

$$\begin{pmatrix} \mathbf{e}(\frac{11}{60}\frac{1}{\mathrm{denom}(x)\,\mathrm{numer}(x)}) & 0 \\ 0 & \mathbf{e}(\frac{-1}{60}\frac{1}{\mathrm{denom}(x)\,\mathrm{numer}(x)}) \end{pmatrix}, \tag{8.29}$$

sees that behaviour at the cusp of the Rogers–Ramanujan functions as it did for the η function.

8.2 Quantum modular forms: looking back

In [208], quantum modular forms are introduced as functions $f : \mathbb{Q} \to \mathbb{C}$ such that

$$h_\gamma \;=\; f - (f|_k\gamma) \tag{8.30}$$

is "better behaved". The idea was that while f can have absolutely no continuity, h_γ could become continuous or even analytic. This definition is additive in nature whereas so far we have discussed modular forms in a sense multiplicatively. In this way, consider the function

$$f : \mathbb{Q}^\times \to \mathbb{C} \quad s.t. \quad f(x) \;=\; \log(\mathrm{denom}(x)). \tag{8.31}$$

This function has absolutely no continuity however the difference under the S matrix is given by

$$f(x) - f(-1/x) \;=\; \log(|x|). \tag{8.32}$$

To get a feeling for this change we can consider a before and after plot shown in Figure 8.1. In [208], six examples are given (from 0 to 5) and example five was by far the most interesting and surprising. This was related to a function that we have previously seen. In particular, this example was related to the Kashaev invariant of the figure eight knot 4_1. Recall the Kashaev invariant

$$\widetilde{J}_0(q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2 \;=\; \sum_{k=0}^{\infty}(q;q)_k(q^{-1};q^{-1})_k\,. \tag{8.33}$$

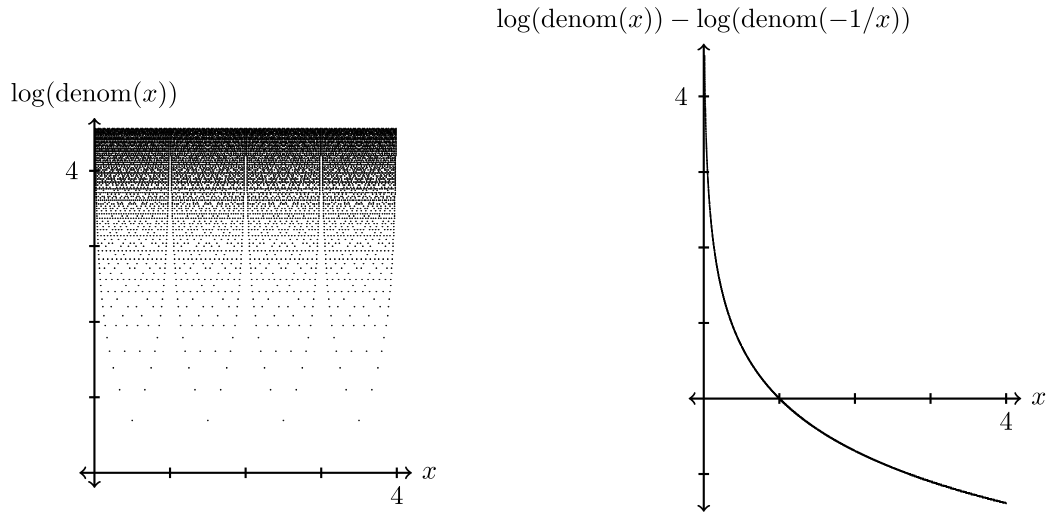

Figure 8.1: Before and after plots of quantum modularity of the log(denom) function.

We can see from the last equality this is real valued. We have seen the asymptotics of this function in Example 43 through $q = \mathbf{e}(1/n)$ for $n \in \mathbb{Z}$. However, the remarkable observation in [208] was that for some uniformity on the denominator if $n \in \mathbb{Q}$ approaches ∞ then

$$\begin{aligned} \widetilde{J}_0(4_1;\mathbf{e}(-1/n)) &= \widetilde{J}_0(4_1;\mathbf{e}(n))\widehat{\Phi}^{(\rho_1)}(2\pi i/n) \\ &= \widetilde{J}_0(4_1;\mathbf{e}(n))\mathbf{e}\Big(\frac{\mathrm{VC}^{(\rho_1)}}{(2\pi i)^2}n\Big)\frac{\mathbf{e}(1/8)}{\sqrt{\sqrt{-3}}}\Big(1 - \frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{n} + \frac{697}{1152\sqrt{-3}^6}\Big(\frac{2\pi i}{n}\Big)^2 + \cdots\Big)\,. \end{aligned} \tag{8.34}$$

Then the function

$$\log(\widetilde{J}_0(4_1;\mathbf{e}(-1/n))) - \log(\widetilde{J}_0(4_1;\mathbf{e}(n))) \tag{8.35}$$

behaves better than

$$\log(\widetilde{J}_0(4_1;\mathbf{e}(n)))\,. \tag{8.36}$$

Indeed numerically, it seems to have a smooth limit as one approaches a rational number on both sides but is discontinuous at each rational. One can consult [208] for some nice figures plotting these functions.

When Zagier originally introduced quantum modular forms, he deliberately didn't define them. His paper instead gives examples, which all have similar behaviour in slowly increasing levels of complexity. More than ten year later we have a much clearer idea about what quantum modular forms should be, and this improved analytic behaviour will be much stricter than we saw in this example. This example was refined to give rise to analytic functions on simply connected regions in $\mathbb{C}$ in [86]. There was one final computation needed, which we include in Section 9.2 based on the work of [70].

The quantum modularity observed with the previous definition is only part of a bigger story. However, when exploring examples it is often helpful to understand these leading asymptotic statements. These can often be proved for a given example by using modularity results of Pochhammer symbols, Euler–Maclaurin, and stationary phase. The methods used in [29] can be applied to many q–hypergeometric functions and the main point is to use Theorem 29. We will illustrate these methods in the example of the WRT invariant of $4_1(1,2)$ after some numerical observations. We will see that these methods work at leading order and give surprising insight into exponentially small corrections but why this works is not entirely clear to me. However, before dealing with that slightly more complicated example, we can consider Nahm sums.

Before this, one should remark that, as we have described originally, quantum modular forms were only defined for $\tau \in \mathbb{Q}$. However, $\tilde{q}$–series with certain asymptotics as $|\tau|$ tends to infinity with $\Im(\tau)$ bounded away from zero and infinity also possess similar properties. Indeed, the important observation is that $\mathbf{e}(\tau)$ is bounded away from zero. Therefore, we can see interesting $\tilde{q}$ corrections to the leading order of the asymptotics with these horizontal limits. While these statements are in a sense weaker than what we get via state integrals in part V, we include this example here to illustrate the method.

Example 58. *We can use Theorem 29 to analyse the asymptotics of this function. Firstly, suppose that $\Re(\tau) > 0$ and let $q = \mathbf{e}(\tau)$ and $\tilde{q} = \mathbf{e}(-1/\tau)$. Using ψ from the theorem given in equation (4.65), define*

$$\phi(\tau, k) \;=\; -\psi(\tau, 1, 0, k-1)\,. \tag{8.37}$$

Note that

$$\phi(\tau, k + 1/\tau) \;=\; \phi(\tau, k)\,. \tag{8.38}$$

*Then for τ small enough (*i.e. $\Re(\tau) < 1$*)*

$$\begin{aligned} f_{A,m,n}(q) \;&=\; \sum_{k=0}^{\infty} \mathbf{e}(nk) \frac{q^{\frac{A}{2}k^2+km}}{(q;q)_k} \;=\; \sum_{k=0}^{\infty} \mathbf{e}(nk) \frac{q^{\frac{A}{2}k^2+km}}{(\tilde{q};\tilde{q})_{\lfloor k\Re(\tau)\rfloor}} \exp(\phi(\tau,k)) \\ &=\; \sum_{\ell=0}^{\infty} \sum_{k\in\mathbb{Z}}^{\ell \le k\Re(\tau) < \ell+1} \mathbf{e}(nk) \frac{q^{\frac{A}{2}k^2+km}}{(\tilde{q};\tilde{q})_{\ell}} \exp(\phi(\tau,k))\,. \end{aligned} \tag{8.39}$$

Notice that for $A, k, \ell \in \mathbb{Z}$ we have $Ak\ell \in \mathbb{Z}$[1] *and so, letting $x = k - \ell/\tau$,*

$$\begin{aligned} &\mathbf{e}\Big(\frac{A}{2}k^2\tau + km\tau + nk\Big) \\ &= \mathbf{e}\Big(\frac{A}{2}(k-\ell/\tau)^2\tau + (k-\ell/\tau)m\tau + n(k-\ell/\tau) - \frac{A}{2}\ell^2/\tau + m\ell + n\ell/\tau\Big) \\ &= \mathbf{e}\Big(\frac{A}{2}x^2\tau + xm\tau + nx - \frac{A}{2}\ell^2/\tau + m\ell + n\ell/\tau\Big). \end{aligned} \tag{8.40}$$

Therefore, we find

$$f_{A,m,n}(q) = \sum_{\ell=0}^{\infty} \mathbf{e}(m\ell)\frac{\tilde{q}^{\frac{A}{2}\ell^2 - n\ell}}{(\tilde{q};\tilde{q})_\ell} \sum_{x\in\mathbb{Z}-\ell/\tau}^{0\le\Re(\tau x)<1} \mathbf{e}\Big(\frac{A}{2}x^2\tau + xm\tau + nx\Big)\exp(\phi(\tau,x)). \tag{8.41}$$

Therefore, we want to understand the asymptotics as $\tau \to 0$ of the sum

$$\mathfrak{f}(\tau,\ell) = \sum_{x\in\tau\mathbb{Z}-\ell}^{0\le\Re(x)<1} \mathbf{e}\Big(\frac{A}{2}\frac{x^2}{\tau} + xm + n\frac{x}{\tau}\Big)\exp\Big(\phi\Big(\tau,\frac{x}{\tau}\Big)\Big). \tag{8.42}$$

We can apply the Euler–Maclaurin summation formula from Theorem 25 to this sum in a form similar to that given in [203, Lec. 2]. There is one important subtly, which is that for $j \in \mathbb{Z}$ and $x \in \tau\mathbb{Z} - \ell$ we have

$$\mathbf{e}\Big(j\frac{x}{\tau}\Big) = \mathbf{e}\Big(-\frac{j\ell}{\tau}\Big), \tag{8.43}$$

therefore, we wish to apply Euler–Maclaurin to all the sums

$$\mathfrak{f}(\tau,\ell) = \mathbf{e}\Big(\frac{j\ell}{\tau}\Big)\sum_{x\in\tau\mathbb{Z}-\ell}^{0\le\Re(x)<1} \mathbf{e}\Big(\frac{A}{2}\frac{x^2}{\tau} + xm + (n+j)\frac{x}{\tau}\Big)\exp\Big(\phi\Big(\tau,\frac{x}{\tau}\Big)\Big). \tag{8.44}$$

This ambiguity can be seen to arise as the initial sum only depends on $n + \mathbb{Z}$.

Note that at the end points of the summation, as $\tau \to 0$, for certain choices of j the contribution to the Euler–Maclaurin summation formula is exponentially small. Therefore, for these j, the sum is approximated to exponentially small corrections by

$$\mathfrak{f}(\tau,\ell) \sim \frac{\mathbf{e}\big(\frac{j\ell}{\tau}\big)}{\tau}\int_{i\frac{\Im(\tau)}{\Re(\tau)}\ell}^{i\frac{\Im(\tau)}{\Re(\tau)}\ell + \frac{\tau}{\Re(\tau)}} \mathbf{e}\Big(\frac{A}{2}\frac{x^2}{\tau} + xm + (n+j)\frac{x}{\tau}\Big)\exp\Big(\phi\Big(\tau,\frac{x}{\tau}\Big)\Big)dx. \tag{8.45}$$

[1] If $A \in \mathbb{Q}$ then this condition fails. This is the only point that it is needed and we have included this assumption purely to simplify the discussion. To deal with $A \notin \mathbb{Z}$ we split the sums into congruences. This will be a finite number given by the denominator of A. These finite number of sums will again have the following decoupling between k and ℓ.

We can then apply stationary phase to these integrals. To do this we need to understand the geometry of the critical points of the logarithm of the integrand to leading order to apply a version of Laplace's method. We will choose the j so that such an analysis will work and we will find critical points near the contour. Then deforming the contour to pass through these points will give rise to the asymptotics.

We have

$$\begin{aligned} &\mathbf{e}\Big(\frac{A}{2}\frac{x^2}{\tau} + xm + (n+j)\frac{x}{\tau}\Big)\exp\Big(\phi\Big(\tau,\frac{x}{\tau}\Big)\Big) \\ &= \mathbf{e}\Big(\frac{1}{(2\pi i)^2\tau}\mathrm{Li}_2(\mathbf{e}(x)) - \frac{1}{(2\pi i)^2\tau}\frac{\pi^2}{6} + \frac{A}{2}\frac{x^2}{\tau} + (n+j)\frac{x}{\tau} + o(\tau^{-1})\Big)\,. \end{aligned} \tag{8.46}$$

and therefore, the critical points are given by solutions to

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\Big(\frac{1}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(x)) - \frac{1}{(2\pi i)^2}\frac{\pi^2}{6} + \frac{A}{2}x^2 + (n+j)x\Big) \\ &= -\frac{1}{2\pi i}\log(1-\mathbf{e}(x)) + Ax + n + j\,. \end{aligned} \tag{8.47}$$

To find solutions to this equation we take $X = \mathbf{e}(x)$ and find a polynomial equation for X,

$$1 - X \;=\; \mathbf{e}(n)X^A\,. \tag{8.48}$$

This equation is referred to as Nahm's equation. Choosing a solution to this equation and the equation

$$\mathbf{e}(x) \;=\; X \tag{8.49}$$

such that $\Re(x)$ is close to $[0,1]$ will give rise to the critical points of interest. Once x is chosen, we then choose j so that it is an honest critical point. Once this is done we deform the contour using the method of steepest descent. For the $A=4$ example, this leads to $j=1$ and as seen in Figure 8.2, we can apply stationary phase around the $x_{4,0}$ critical point to prove this form of quantum modularity, as τ approaches 0 along a horosphere

$$f_{4,m,0}(q) \;\sim\; \widehat{\Phi}^{(4)}_m(2\pi i\tau)f_{4,1,0}(\tilde{q})\,. \tag{8.50}$$

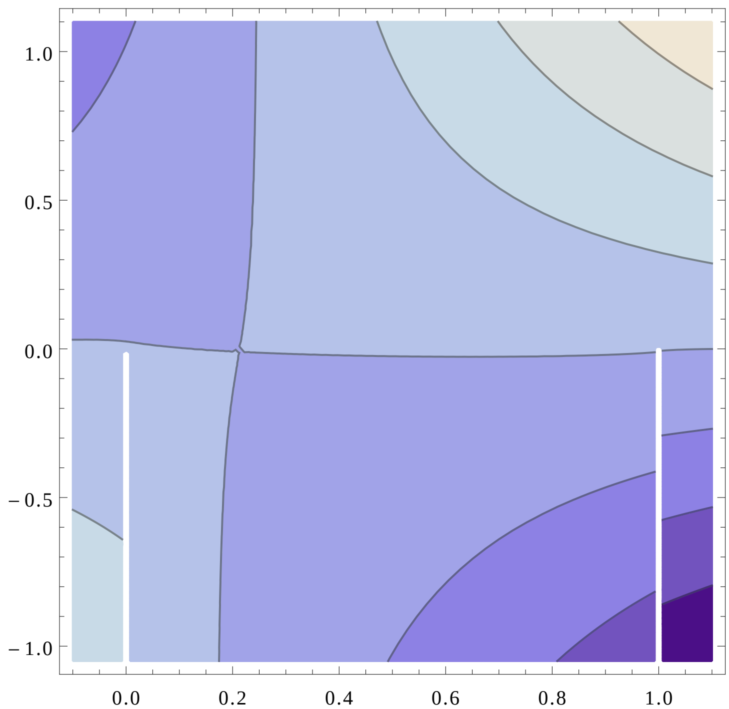

Figure 8.2: Zoomed in version of the $\ell = -1$ plot near $[0, 1]$. This shows that the integrand is exponentially smaller that the critical point at the endpoints and hence Euler–Maclaurin can be applied and stationary phase. The $\ell = 0, -2$ from the previous figure has endpoints the same size and hence Euler–Maclaurin leads to boundary terms while for $\ell = -3$ the end points are exponentially larger than the critical point.

8.3 Quantum modularity of the WRT invariant of $4_1(1,2)$

Consider the following values of the WRT invariant:

$$\begin{aligned} Ш\Big(\mathbf{e}\Big(\frac{1}{1000}\Big)\Big) &= -6.3258\cdots + i14.804\dots\,, \\ Ш\Big(\mathbf{e}\Big(\frac{1}{1000+1/2}\Big)\Big) &= 1.6433\cdots\times 10^{98} + i6.7579\cdots\times 10^{96}\,, \\ Ш\Big(\mathbf{e}\Big(\frac{1}{1000+1/3}\Big)\Big) &= 1.0551\cdots\times 10^{98} + i8.7759\cdots\times 10^{97}\,. \end{aligned} \tag{8.51}$$

The immediate observation is the order of magintude shift. The behaviour of the first two values is expected from Witten's asymptotic expansion conjecture 2 and the Chen-Yang volume conjecture 4 for WRT invariants. Indeed, notice that

$$\begin{aligned} Ш\Big(\mathbf{e}\Big(\frac{1}{1000+1/2}\Big)\Big)\mathbf{e}\Big(-\frac{\mathrm{VC}_7}{(2\pi i)^2}(1000+1/2)\Big) &= -22.044\cdots + i22.943\dots\,, \\ Ш\Big(\mathbf{e}\Big(\frac{1}{1000+1/3}\Big)\Big)\mathbf{e}\Big(-\frac{\mathrm{VC}_7}{(2\pi i)^2}(1000+1/3)\Big) &= -26.465\cdots + i7.6613\dots\,. \end{aligned} \tag{8.52}$$

More generally we have the following observation, which is in fact a theorem extending the result of Ohtsuki [150].

Theorem A–39. *For $r \in \mathbb{Q}\backslash\mathbb{Z}$ we have*

$$\lim_{k\to\infty} \frac{\log\Big(Ш\Big(\mathbf{e}\big(\frac{1}{k+r}\big)\Big)\mathbf{e}\big(-\frac{\mathrm{VC}_7}{(2\pi i)^2}(k+r)\big)\Big)}{k+r} = 0\,. \tag{8.53}$$

This theorem is a slight generalisation of the Chen–Yang volume conjecture. However, as in [208, 47, 45], we can go further and find a full asymptotic expansion. Indeed, numerically using the methods of Section 3.2, one finds that there is some $X(r) \in \mathbb{Z}[\mathbf{e}(-r)]$ such that

$$Ш\Big(\mathbf{e}\Big(\frac{1}{k+r}\Big)\Big) \sim X(r)\,\mathbf{e}(3/8)\sqrt{k+r}\,\mathbf{e}\Big(\frac{\mathrm{VC}_7}{(2\pi i)^2}(1000+r)\Big)\,\Phi^{(\sigma_7)}\Big(\frac{2\pi i}{k+r}\Big) \tag{8.54}$$

where

$$\Phi^{(\sigma_7)}(h) = a_0^{\sigma_7} + a_1^{\sigma_7}h + a_2^{\sigma_7}h^2 + \dots = \frac{1}{\sqrt{\delta_7}}\big(1 + A_1^{\sigma_7}h + A_2^{\sigma_7}h^2 + \dots\big) \tag{8.55}$$

is a factorially divergent series with $A_i \in F$. The first two values A_i are given by

$$\begin{pmatrix} 24\delta_7^3 A_1^{\sigma_7} \\ 1152\delta_7^6 A_2^{\sigma_7} \end{pmatrix} = \begin{pmatrix} 1497746 & 3014838521575 \\ 1345119 & 2732414541176 \\ -3675733 & -7414786842283 \\ 2082815 & 4197826806919 \\ -839488 & -1690529009777 \\ -283405 & -574198051621 \\ 383432 & 771765277669 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 \\ \xi_7 \\ \xi_7^2 \\ \xi_7^3 \\ \xi_7^4 \\ \xi_7^5 \\ \xi_7^6 \end{pmatrix} = \begin{pmatrix} -158.75\cdots - i57.225\dots \\ 84862.\cdots + i924.76\dots \end{pmatrix} \tag{8.56}$$

Indeed, as we saw in Section 2.6, $X(1/2) = 2$ and for $x = \frac{2\pi i}{1000+1/2}$

$$\begin{aligned} \Big|\text{Ш}(\exp(x))\Big| &= 1.6447\cdots \times 10^{98}\,, \\ \Big|\text{Ш}(\exp(x)) - 2\,\mathbf{e}(3/8)\,\mathbf{e}\Big(\frac{\mathrm{VC}_7}{2\pi i x}\Big)\sqrt{\frac{2\pi i}{\delta_7 x}}\Big| &= 1.1735\cdots \times 10^{95}\,, \\ \Big|\text{Ш}(\exp(x)) - 2\,\mathbf{e}(3/8)\,\mathbf{e}\Big(\frac{\mathrm{VC}_7}{2\pi i x}\Big)\sqrt{\frac{2\pi i}{\delta_7 x}}\big(1 + A_1^{\sigma_7}x\big)\Big| &= 1.2530\cdots \times 10^{92}\,, \\ \Big|\text{Ш}(\exp(x)) - 2\,\mathbf{e}(3/8)\,\mathbf{e}\Big(\frac{\mathrm{VC}_7}{2\pi i x}\Big)\sqrt{\frac{2\pi i}{\delta_7 x}}\big(1 + A_1^{\sigma_7}x + A_2^{\sigma_7}x^2\big)\Big| &= 3.9674\cdots \times 10^{89}\,. \end{aligned} \tag{8.57}$$

These numbers A_i can be computed from a formal Gaussian integration, as discussed in Example 24. Although these agree for this example, their topological invariance has not been established.

Lets continue, in a similar manner to [208], with a list of a few values of this not so mysterious function $X(r)$. We have the following equalities that can be computed numerically:

$$\begin{aligned} X(1/2) &= \text{Ш}(\mathbf{e}(-1/2)) = 2\,, \\ X(1/3) &= \text{Ш}(\mathbf{e}(-1/3)) = 1 - \mathbf{e}(-1/3)\,, \\ X(1/4) &= \text{Ш}(\mathbf{e}(-1/4)) = 1 - \mathbf{e}(-1/4)\,, \\ X(1/5) &= \text{Ш}(\mathbf{e}(-1/5)) = 2 - \mathbf{e}(-1/5) + 2\mathbf{e}(-2/5) + 2\mathbf{e}(-3/5)\,, \\ X(1/6) &= \text{Ш}(\mathbf{e}(-1/6)) = 1 - \mathbf{e}(-1/6)\,, \\ X(1/7) &= \text{Ш}(\mathbf{e}(-1/7)) = 2 - 2\mathbf{e}(-1/7) - \mathbf{e}(-2/7) + \mathbf{e}(-3/7)\,, \\ X(1/8) &= \text{Ш}(\mathbf{e}(-1/8)) = 3 - 3\mathbf{e}(-1/8)\,. \end{aligned} \tag{8.58}$$

This equality persists for r with larger denominator and in fact we have the following theorem.

Theorem 4. *The WRT invariant for $4_1(-1,2)$ is a quantum modular form in the sense of [208]. That is, for $r \in \mathbb{Q}\backslash\mathbb{Z}$ as $k \to \infty$,*

$$\text{Ш}(\mathbf{e}(-1/(k+r))) \sim \text{Ш}(\mathbf{e}((k+r)))\mathbf{e}(-3/8)\sqrt{\frac{k+r}{\delta_{\rho_6}}}\,\mathbf{e}\Big(-\frac{\mathrm{VC}_6}{(2\pi i)^2}(k+r)\Big)\,. \tag{8.59}$$

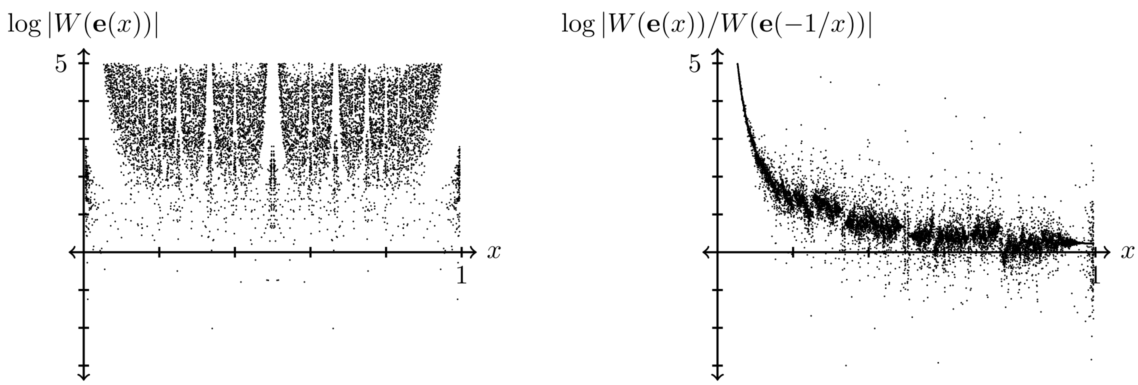

Figure 8.3: Before and after plots of quantum modularity.

We can form similar plots to those in [208] as seen in Figure 8.3. Notice that compared with the plots in [208] the second is behaving much worse than one could potentially hope. This is related to the fact that the volume of 4_1 is larger than the volume of $4_1(-1,2)$. However, if one looks close one can see that at each rational number the behaviour on either side becomes much better. We will see that the refined modularity [86] will give rise to functions which satisfy much better analytitc properties and one can compare with Figure 8.5.

Notice that this actually extends to the case that $r \in \mathbb{Z}$. We know from Witten's asymptotic expansion conjecture that the LHS grows polynomially while the RHS vanishes as $\text{Ш}(1) = 0$. Therefore, in the $r = 0$ case, the Theorem 4 states that the LHS is dominated by $\mathbf{e}(\mathrm{VC}_7/(2\pi i)^2 k)$. Already the form of Witten's asymptotic expansion conjecture and this link suggest that there could be an improvement to this result.

Proof of Theorem 4. We can rewrite the WRT invariant of $4_1(1,2)$ computed in Example 16 as

$$\begin{aligned}(1-q)\text{Ш}(q) &= \sum_{0\le\ell\le k}(-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)}\frac{(q;q)_{2k+1}}{(q;q)_\ell(q;q)_{k-\ell}} \\ &= \sum_{\ell,j=0}^{\infty}(-1)^{j+\ell}q^{-\frac{1}{2}j(j+1)-j\ell+\ell(\ell+1)/2}\frac{(q;q)_{2j+2\ell+1}}{(q;q)_\ell(q;q)_j}\,.\end{aligned} \tag{8.60}$$

Then consider the elements

$$w_{m,n,p}(q) = \sum_{\ell,j=0}^{\infty}(-1)^{j+\ell}q^{-\frac{1}{2}j(j+1)-j\ell+\ell(\ell+1)/2+mj+n\ell}\frac{(q;q)_{2j+2\ell+p}}{(q;q)_\ell(q;q)_j}\,. \tag{8.61}$$

Using the quantum modular properties of the q–Pochhammer symbol from Theorem 29, we can analyse the asymptotic behaviour of $(1-q)\text{Ш}(q)$. To do this, for ψ from Theorem 29, let

$$\phi(k,\tau) \;=\; -\psi(\tau,1,0,k-1) \tag{8.62}$$

Then, for $r=a/c\in\mathbb{Q}$, we have

$$\begin{aligned}
&(1-\mathbf{e}(-1/r))\text{Ш}(\mathbf{e}(-1/r))\\
&= \sum_{0\le\ell,0\le j}^{2\ell+2j\le a}(-1)^{j+\ell}\frac{\mathbf{e}\left(\frac{1}{2r}j(j+1)+\frac{j\ell}{r}-\frac{1}{2r}\ell(\ell+1)\right)\left(\mathbf{e}\left(-\frac{1}{r}\right);\mathbf{e}\left(-\frac{1}{r}\right)\right)_{2j+2\ell+1}}{\left(\mathbf{e}\left(-\frac{1}{r}\right);\mathbf{e}\left(-\frac{1}{r}\right)\right)_{\ell}\left(\mathbf{e}\left(-\frac{1}{r}\right);\mathbf{e}\left(-\frac{1}{r}\right)\right)_{j}}\\
&= \sum_{0\le\ell,0\le j}^{2\ell+2j\le a}(-1)^{j+\ell}\frac{\mathbf{e}\left(\frac{r}{2}\left(\frac{j}{r}\right)^2+r\frac{j}{r}\frac{\ell}{r}-\frac{r}{2}\left(\frac{\ell}{r}\right)^2+\frac{j}{2r}-\frac{\ell}{2r}\right)\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{\left\lfloor\frac{2j+2\ell+1}{r}\right\rfloor}\mathbf{e}(\phi(2j+2\ell+1,r))}{\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{\left\lfloor\frac{\ell}{r}\right\rfloor}\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{\left\lfloor\frac{j}{r}\right\rfloor}\mathbf{e}(\phi(\ell,r))\mathbf{e}(\phi(j,r))}\\
&= \sum_{0\le\ell,0\le j}^{2\ell+2j\le a}\mathbf{e}\left(\frac{r}{2}\left(\frac{j}{r}-\left\lfloor\frac{j}{r}\right\rfloor\right)^2+r\left(\frac{j}{r}-\left\lfloor\frac{j}{r}\right\rfloor\right)\left(\frac{\ell}{r}-\left\lfloor\frac{\ell}{r}\right\rfloor\right)-\frac{r}{2}\left(\frac{\ell}{r}-\left\lfloor\frac{\ell}{r}\right\rfloor\right)^2+\frac{r}{2}\left(\frac{j}{r}-\left\lfloor\frac{j}{r}\right\rfloor\right)-\frac{r}{2}\left(\frac{\ell}{r}-\left\lfloor\frac{\ell}{r}\right\rfloor\right)\right)\\
&\qquad\times(-1)^{\left\lfloor\frac{j}{r}\right\rfloor+\left\lfloor\frac{\ell}{r}\right\rfloor}\mathbf{e}\left(\frac{-r}{2}\left\lfloor\frac{j}{r}\right\rfloor^2-r\left\lfloor\frac{j}{r}\right\rfloor\left\lfloor\frac{\ell}{r}\right\rfloor+\frac{r}{2}\left\lfloor\frac{\ell}{r}\right\rfloor^2+\frac{r}{2}\left\lfloor\frac{j}{r}\right\rfloor-\frac{r}{2}\left\lfloor\frac{\ell}{r}\right\rfloor\right)\\
&\qquad\times\mathbf{e}\left(\frac{1}{2}\left(\frac{j}{r}-\left\lfloor\frac{j}{r}\right\rfloor\right)-\frac{1}{2}\left(\frac{\ell}{r}-\left\lfloor\frac{\ell}{r}\right\rfloor\right)\right)\frac{\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{\left\lfloor\frac{2j+2\ell+1}{r}\right\rfloor}\mathbf{e}(\phi(2j+2\ell+1,r))}{\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{\left\lfloor\frac{\ell}{r}\right\rfloor}\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{\left\lfloor\frac{j}{r}\right\rfloor}\mathbf{e}(\phi(\ell,r))\mathbf{e}(\phi(j,r))}\\
&= \sum_{R=0}^{3}\sum_{0\le L,0\le J}^{2L+2J+R<c}(-1)^{J+L}\mathbf{e}\left(\frac{-r}{2}J^2-rJL+\frac{r}{2}L^2+\frac{r}{2}J-\frac{r}{2}L\right)\frac{\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{2J+2L+R}}{\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{L}\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{J}}\\
&\qquad\times\sum_{(x,y)\in[0,1)^2\cap\left(\frac{1}{r}\mathbb{Z}^2-(J,L)\right)}^{R\le 2x+2y\le R+1}\frac{\mathbf{e}(\phi(2rx+2ry+1,r))}{\mathbf{e}(\phi(rx,r))\mathbf{e}(\phi(ry,r))}\mathbf{e}\left(\frac{r}{2}x^2+rxy-\frac{r}{2}y^2+\frac{r}{2}x-\frac{r}{2}y+\frac{1}{2}x-\frac{1}{2}y\right).
\end{aligned} \tag{8.63}$$

where we used the substitution

$$J \;=\; \left\lfloor\frac{j}{r}\right\rfloor,\quad L \;=\; \left\lfloor\frac{\ell}{r}\right\rfloor,\quad x \;=\; \frac{j}{r}-\left\lfloor\frac{j}{r}\right\rfloor,\quad y \;=\; \frac{\ell}{r}-\left\lfloor\frac{\ell}{r}\right\rfloor, \tag{8.64}$$

and the fact that for $n\in\mathbb{Z}$ we have

$$\phi(rx+nr,r) \;=\; \phi(rx,r). \tag{8.65}$$

Importantly,

$$\mathbf{e}(rx) \;=\; \mathbf{e}\left(r\left\lfloor\frac{j}{r}\right\rfloor\right) \;=\; \mathbf{e}(-rJ), \qquad \mathbf{e}(ry) \;=\; \mathbf{e}\left(r\left\lfloor\frac{\ell}{r}\right\rfloor\right) \;=\; \mathbf{e}(-rL). \tag{8.66}$$

Therefore we can exchange these terms between the sums over L,J and the sums over x,y. In particular, for $(a,b)\in\mathbb{Z}^2$ we have

$$\begin{aligned}
&\text{Ш}(\mathbf{e}(-1/r))\\
&= \sum_{R=0}^{3}\sum_{0\le L,0\le J}^{2L+2J+R<c}(-1)^{J+L}\mathbf{e}\left(\frac{-r}{2}J^2-rJL+\frac{r}{2}L^2+\frac{r}{2}J-\frac{r}{2}L+arJ+brL\right)\frac{\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{2J+2L+R}}{\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{L}\left(\mathbf{e}(r);\mathbf{e}(r)\right)_{J}}\\
&\qquad\times\sum_{(x,y)\in[0,1)^2\cap\left(\frac{1}{r}\mathbb{Z}^2-(J,L)\right)}^{R\le 2x+2y\le R+1}\frac{\mathbf{e}(\phi(2rx+2ry+1,r))}{\mathbf{e}(\phi(rx,r))\mathbf{e}(\phi(ry,r))}\mathbf{e}\left(\frac{r}{2}x^2+rxy-\frac{r}{2}y^2+\frac{r}{2}x-\frac{r}{2}y+\frac{1}{2}x-\frac{1}{2}y+arx+bry\right).
\end{aligned} \tag{8.67}$$

This form of the expression makes it clear the asymptotic expansions come from a stationary phase approximation applied to the sum

$$\sum_{(x,y)\in[0,1)^2\cap\left(\frac{1}{r}\mathbb{Z}^2-(J,L)\right)}^{R\le 2x+2y\le R+1}\mathbf{e}\left(\phi(2x+2y,r)-\phi(x,r)-\phi(y,r)+\frac{r}{2}x^2+rxy-\frac{r}{2}y^2+\frac{r}{2}x-\frac{r}{2}y+\frac{1}{2}x-\frac{1}{2}y+arx+bry\right). \tag{8.68}$$

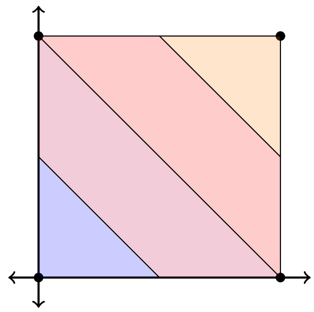

Figure 8.4: The regions R that arise from the stationary phase approximation.

As r tends to infinity, this sum is approximated by the integral

$$\int\int_R \mathbf{e}\left(\phi(2rx+2ry+1,r)-\phi(rx,r)-\phi(ry,r)+\frac{r}{2}x^2+rxy-\frac{r}{2}y^2+\frac{r}{2}x-\frac{r}{2}y+\frac{1}{2}x-\frac{1}{2}y+arx+bry\right)dxdy\,, \tag{8.69}$$

where R indexes the regions depicted in Figure 8.4. Noting that from Section 4.3 we have

$$\phi(rx,r) \;\sim\; \frac{r}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(-x))+\frac{r}{24}+O(r^0)\,, \tag{8.70}$$

we see that the critical points of the integrand are determined by the stationary points of

$$\begin{aligned} V_{a,b}(x,y) \;=\; & \frac{1}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(-2x-2y))-\frac{1}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(-x))-\frac{1}{(2\pi i)^2}\mathrm{Li}_2(\mathbf{e}(-y)) \\ & -\frac{1}{24}+\frac{1}{2}x^2+xy-\frac{1}{2}y^2+\frac{1}{2}x-\frac{1}{2}y+ax+by\,. \end{aligned} \tag{8.71}$$

These satisfy equations

$$\begin{aligned} 0 \;&=\; \frac{2}{2\pi i}\log(1-\mathbf{e}(-2x-2y))-\frac{1}{2\pi i}\log(1-\mathbf{e}(-x))+x+y+\frac{1}{2}+a \\ 0 \;&=\; \frac{2}{2\pi i}\log(1-\mathbf{e}(-2x-2y))-\frac{1}{2\pi i}\log(1-\mathbf{e}(-y))+x-y-\frac{1}{2}+b\,. \end{aligned} \tag{8.72}$$

Letting $X=\mathbf{e}(x), Y=\mathbf{e}(y)$ we find equations

$$\begin{aligned} \frac{(1-X^{-2}Y^{-2})^2}{(1-X^{-1})}XY \;&=\; -1 \\ \frac{(1-X^{-2}Y^{-2})^2}{(1-Y^{-1})}XY^{-1} \;&=\; -1\,. \end{aligned} \tag{8.73}$$

The solutions to these equations are given by

$$\begin{aligned} X \;&=\; 10-19\xi-76\xi^2+52\xi^3-20\xi^4+4\xi^5+13\xi^6 \\ Y \;&=\; -10+24\xi+87\xi^2-59\xi^3+23\xi^4-5\xi^5-15\xi^6\,. \end{aligned} \tag{8.74}$$

Now given any logarithm for (X,Y) we can choose an a,b which give rise to a solution to Equation (8.72). For example, consider the solution associated to the 3rd embedding of the trace field

$$(x_3,y_3) \;=\; (i0.0081372\dots, 1.0000\dots + i0.0090947\dots)\,. \tag{8.75}$$

We find that

$$\begin{aligned} 2+a_0 &= \frac{2}{2\pi i}\log(1-\mathbf{e}(-2x_0-2y_0)) - \frac{1}{2\pi i}\log(1-\mathbf{e}(-x_0)) + x_0 + y_0 + \frac{1}{2} + a_0 \\ -1+b_0 &= \frac{2}{2\pi i}\log(1-\mathbf{e}(-2x_0-2y_0)) - \frac{1}{2\pi i}\log(1-\mathbf{e}(-y_0)) + x_0 - y_0 - \frac{1}{2} + b_0\,. \end{aligned} \tag{8.76}$$

Choosing $(a_0,b_0)=(-2,1)$ we find that

$$V_{-2,1}(x_0,y_0) - \frac{1}{4\pi^2}\mathrm{VC}_3 \;=\; -1\,. \tag{8.77}$$

Therefore, associated to this solution to Equation (8.72) we get four elements of the Habiro ring

$$w^{(x_0,y_0,R)}(q) \;=\; q^{V_{-2,1}(x_0,y_0)-\frac{1}{4\pi^2}\mathrm{VC}_3} w_{a_0,b_0,R}(q) \;=\; q^{-1}w_{-2,1,R}(q)\,. \tag{8.78}$$

Considering ρ_6 one can show that the element of the Habiro ring is given by

$$X_{\rho_6}(q) \;=\; w_{-1,1,1}(q) \tag{8.79}$$

and with this choice one can apply Euler–Maclaurin of Theorem 25 with endpoints exponentially smaller than than the critical points, therefore allowing a stationary phase approximation. □

Following [86], we can indeed develop a conjectural improvement of this result. To do this we would like to turn the asymptotic series Φ^ρ into an analytic function. To do this we can use the sequence of Borel transformation, Padé approximation and Laplace transformation discussed in sections 3.6 and 3.7. Let $s_{2N}(\Phi)$ be the Laplace transform of the $[N/N]$ Padé approximate of the Borel transform. With 200 coefficients of Φ^ρ we can numerically compute to around order 10^{-40} at $2\pi i/100$. Indeed, the worst convergence is from Φ^{ρ_0} and the difference between the numerical values using 200 and 198 coefficents is

$$s_{200}(\Phi^{\rho_0})(2\pi i/100) - s_{198}(\Phi^{\rho_0})(2\pi i/100) \;=\; (4.8831\dots + 3.0178\dots i)\times 10^{-40}\,, \tag{8.80}$$

which give a lower bound for the numerical error. Numerically it appears that roughly that

$$\begin{aligned} \log|s_N(\Phi^{\rho_0})(2\pi i/100) - s_{500}(\Phi^{\rho_0})(2\pi i/100)| &\sim -6.3\sqrt{N} \\ \log|s_N(\Phi^{\rho_0})(2\pi i/200) - s_{500}(\Phi^{\rho_0})(2\pi i/200)| &\sim -8.8\sqrt{N} \end{aligned} \tag{8.81}$$

The values of the various numerical Borel–Padé–Laplace transforms are given by

$$\begin{aligned} s_{200}(\Phi^{\rho_0})(2\pi i/100) &= 0.022362\cdots - 0.042136\ldots i \\ \mathbf{e}(3/8)\sqrt{100}\exp\left(\mathrm{VC}_{\rho_1}100/2\pi i\right) s_{200}(\Phi^{\rho_1})(2\pi i/100) &= -0.70748\cdots - 2.8524\cdots \\ \mathbf{e}(-1/8)\sqrt{100}\exp\left(\mathrm{VC}_{\rho_2}100/2\pi i\right) s_{200}(\Phi^{\rho_2})(2\pi i/100) &= 1.0653\cdots - 2.6036\ldots i \\ \mathbf{e}(3/8)\sqrt{100}\exp\left(\mathrm{VC}_{\rho_3}100/2\pi i\right) s_{200}(\Phi^{\rho_3})(2\pi i/100) &= -0.95412\cdots + 0.53897\ldots i \\ \mathbf{e}(-1/8)\sqrt{100}\exp\left(\mathrm{VC}_{\rho_4}100/2\pi i\right) s_{200}(\Phi^{\rho_4})(2\pi i/100) &= 2.9304\cdots + 2.3781\ldots i \\ \mathbf{e}(3/8)\sqrt{100}\exp\left(\mathrm{VC}_{\rho_5}100/2\pi i\right) s_{200}(\Phi^{\rho_5})(2\pi i/100) &= -0.55386\cdots - 4.2697\ldots i \\ \mathbf{e}(3/8)\sqrt{100}\exp\left(\mathrm{VC}_{\rho_6}100/2\pi i\right) s_{200}(\Phi^{\rho_6})(2\pi i/100) &= (1.0359\cdots + 0.31226\ldots i) \times 10^{-9} \\ \mathbf{e}(3/8)\sqrt{100}\exp\left(\mathrm{VC}_{\rho_7}100/2\pi i\right) s_{200}(\Phi^{\rho_7})(2\pi i/100) &= (2.2618\cdots + 0.58777\ldots i) \times 10^{10}\,. \end{aligned} \tag{8.82}$$

The difference between the smooth optimal truncation and the Borel–Padé–Laplace resummation of the series $\Phi^{(\rho_0)}$ with two hundred coefficients at $2\pi i/100$ is

$$(-5.0344\cdots + 0.083534\ldots i) \times 10^{-5}\,. \tag{8.83}$$

So we see that the smooth optimal truncation seems to give around four digits of accuracy. This issue in this example is that the coefficients grow extremely fast. However, as we saw the Borel–Padé–Laplace resummation is stable to around forty digits.

If we sum over the SU(2) flat connections as predicted by Witten's conjecture we find that

$$\begin{aligned} &Ш(\mathbf{e}(1/100)) \;=\; 2.7567\cdots - i7.3897\ldots \\ &Ш(\mathbf{e}(1/100)) - \sum_{\rho\in\{\rho_0,\rho_1,\rho_2,\rho_4,\rho_5\}} \mu_\rho 100^{d_\rho}\exp\left(\mathrm{VC}_\rho 100/2\pi i\right) s_{200}(\Phi^\rho)(2\pi i/100) \\ &\quad= \;1.0359\ldots 10^{-9} + i3.1226\ldots 10^{-10}\,. \end{aligned} \tag{8.84}$$

Therefore, we find that additionally summing over the anti–geometric connection,

$$\begin{aligned} &Ш(\mathbf{e}(1/100)) - \sum_{\rho\in\{\rho_0,\rho_1,\rho_2,\rho_4,\rho_5,\rho_6\}} \mu_\rho 100^{d_\rho}\exp\left(\mathrm{VC}_\rho 100/2\pi i\right) s_{200}(\Phi^\rho)(2\pi i/100) \\ &\quad= \;6.1973\times 10^{-39} + i1.8705\times 10^{-38}\,. \end{aligned} \tag{8.85}$$

Notice that Equations (8.85) vanishes to the order of the error in Equation (8.80). These numerics lead to the following conjecture.

Conjecture 9. *For $k \in \mathbb{Z}_{>0}$*

$$Ш(\mathbf{e}(1/k)) \;=\; \sum_{\rho\in\{\rho_0,\rho_1,\rho_2,\rho_4,\rho_5,\rho_6\}} \mu_\rho k^{d_\rho}\exp\left(\mathrm{VC}_\rho k/2\pi i\right) s(\Phi^\rho)(2\pi i/k)\,. \tag{8.86}$$

This seems to be in contrast to Witten's asymptotic expansion conjecture where we only expect SU(2) flat connections to appear. However, as already noted in [67], the asymptotic series associated to the SU(2) flat connections contain contributions from the other $\mathrm{SL}_2(\mathbb{C})$ flat connections which leads to the series ρ_6 appearing as an exponentially small correction (which of course does not contradict the conjecture).

With some more experimentation one can naturally extend this conjecture as follows.

Conjecture 10. *For $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Q}_{\geq 0}$ there exists functions $X_\rho : \mathbb{Q} \to \mathbb{C}$ such that*

$$Ш\Big(\mathbf{e}\Big(\frac{1}{k+r}\Big)\Big) = \sum_{\rho\in\{\rho_0,\rho_1,\rho_2,\rho_4,\rho_5,\rho_6,\rho_7\}} X_\rho(r)\mu_\rho(k+r)^{d_\rho}\exp\Big(\frac{\mathrm{VC}_\rho(k+r)}{2\pi i}\Big)s(\Phi^\rho)\Big(\frac{2\pi i}{k+r}\Big), \tag{8.87}$$

and $X_\rho(r) \in \mathbb{Z}[\mathbf{e}(r)]$.

The first few values of the functions $X_\rho(r)$ are given

r	1	1/2	1/3	1/4	1/5
$X_{\rho_0}(r)$	1	1	1	1	1
$X_{\rho_1}(r)$	1	1	$-1-\mathbf{e}\left(\frac{1}{3}\right)$	1	$-2-2\mathbf{e}\left(\frac{2}{5}\right)$
$X_{\rho_2}(r)$	1	1	$-1-\mathbf{e}\left(\frac{1}{3}\right)$	$-1-2i$	$-2-\mathbf{e}\left(\frac{1}{5}\right)-\mathbf{e}\left(\frac{3}{5}\right)$
$X_{\rho_3}(r)$	0	0	$1+2\mathbf{e}\left(\frac{1}{3}\right)$	$-2i$	$-1-\mathbf{e}\left(\frac{1}{5}\right)-2\mathbf{e}\left(\frac{2}{5}\right)-\mathbf{e}\left(\frac{3}{5}\right)$
$X_{\rho_4}(r)$	1	1	$2+2\mathbf{e}\left(\frac{1}{3}\right)$	$-1+2i$	$1-\mathbf{e}\left(\frac{1}{5}\right)+\mathbf{e}\left(\frac{2}{5}\right)$
$X_{\rho_5}(r)$	1	1	$2-\mathbf{e}\left(\frac{1}{3}\right)$	1	$3+\mathbf{e}\left(\frac{1}{5}\right)+\mathbf{e}\left(\frac{2}{5}\right)+\mathbf{e}\left(\frac{3}{5}\right)$
$X_{\rho_6}(r)$	1	-1	1	i	$2-\mathbf{e}\left(\frac{3}{5}\right)$
$X_{\rho_7}(r)$	0	2	$2+\mathbf{e}\left(\frac{1}{3}\right)$	$1+i$	$3+\mathbf{e}\left(\frac{1}{5}\right)+3\mathbf{e}\left(\frac{2}{5}\right)+3\mathbf{e}\left(\frac{3}{5}\right)$

Assuming this conjecture one can numerically compute the values of X_ρ for rational numbers with denominators up to a few hundred. With this data one can recognise these functions as combinations of $Ш_{m,n}(\mathbf{e}(-1/r))$. However, these formulae can be quite complicated if not in the correct form. Indeed, it is better to study the function W in more detail to guess the correct formulae. However, considering small solutions to Equation (8.72), we can numerically find formulae for X_ρ for summarised in the following conjecture.

Conjecture 11. *The functions X_ρ in conjecture 10 are given by the following elements of the Habiro ring*

$$\begin{aligned} X_{\rho_1}(q) &= w_{0,2,0}(q)\,, \\ X_{\rho_2}(q) &= w_{-1,0,0}(q)\,, \\ X_{\rho_3}(q) &= q^{-1}w_{-1,0,1}(q) + w_{-1,2,1}(q) + q^{-1}w_{-2,1,3}(q)\,, \\ X_{\rho_4}(q) &= w_{0,1,0}(q)\,, \\ X_{\rho_5}(q) &= w_{-1,1,0}(q)\,, \\ X_{\rho_6}(q) &= w_{-1,1,1}(q)\,, \\ X_{\rho_7}(q) &= w_{-1,1,0}(q) + q^{-1}w_{-2,0,1}(q)\,. \end{aligned} \tag{8.88}$$

8.4 Non–commutative group cohomology

We will breifly recall the first group cohomology of G valued in a right G–module M. Firstly, the set of one cocycles $Z^1(G, M)$ is defined to be the set of maps $f : G \to M$ such that

$$f(gh) \;=\; f(g)h + f(h)\,. \tag{8.89}$$

The set of one coboundary $B^1(G, M)$ is defined to be the set of maps $f : G \to M$ such that for some $m \in M$ we have

$$f(g) \;=\; mg - m\,. \tag{8.90}$$

Then as always we define

$$H^1(G, M) \;=\; Z^1(G, M)/B^1(G, M)\,. \tag{8.91}$$

We will be interested in some kind of non–abelian valued first group cohomology of $\mathrm{SL}_2(\mathbb{Z})$. Firstly, we define

$$\mathbb{C}_\gamma \;=\; \begin{cases} \mathbb{C} - \mathbb{R}_{\leq -d/c} & \text{if } c > 0\,, \\ \mathbb{C} - \mathbb{R}_{\geq -d/c} & \text{if } c < 0\,, \\ \mathbb{C} - \mathbb{R} & \text{if } c = 0\,. \end{cases} \tag{8.92}$$

Consider, for $N \in \mathbb{Z}_{>0}$, the set of maps Ω from $\mathrm{SL}_2(\mathbb{Z})$ to matrix valued holomorphic functions such that $\Omega_\gamma \in GL_N(\mathcal{O}(\mathbb{C}_\gamma))$ and, where defined, we have the cocycle condition

$$\Omega_{\gamma\gamma'}(\tau) \;=\; \Omega_\gamma\Big(\frac{a'\tau + b'}{c'\tau + d'}\Big)\Omega_{\gamma'}(\tau)\,. \tag{8.93}$$

Then we say that two cocycles Ω, Ξ are equivalent if for some $f \in GL_N(\mathcal{O}(\mathbb{C}))$

$$f\Big(\frac{a\tau + b}{d\tau + c}\Big)\Omega_\gamma(\tau) \;=\; \Xi_\gamma(\tau)f(\tau)\,. \tag{8.94}$$

Unfortunately, one cannot define a group structure for cohomology theories valued in non–commutative groups. We will refer to these cocycles simple as *QM–cocycles* if rank N.

Remark 26. *While there is no group structure for a fixed N there is a natural addition between QM–cocycles defined through direct sum so that*

$$(\Omega \oplus \Xi)_\gamma \;=\; \begin{pmatrix} \Omega_\gamma & 0 \\ 0 & \Xi_\gamma \end{pmatrix}. \tag{8.95}$$

Notice that this addition is commutative, as for Ω and Ξ of rank N, M respectfully

$$\begin{pmatrix} 0 & \mathrm{Id}_M \\ \mathrm{Id}_N & 0 \end{pmatrix}\begin{pmatrix} \Omega_\gamma & 0 \\ 0 & \Xi_\gamma \end{pmatrix} \;=\; \begin{pmatrix} \Xi_\gamma & 0 \\ 0 & \Omega_\gamma \end{pmatrix}\begin{pmatrix} 0 & \mathrm{Id}_M \\ \mathrm{Id}_N & 0 \end{pmatrix}. \tag{8.96}$$

We could take some kind of Grothendieck group with respect to this operation of addition but we will not need or use this kind of structure. This operation of addition corresponds to disjoint union for the examples coming from three–manifolds.

Example 59. *Consider a representation $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_N(\mathbb{C})$ then we find that ρ is a QM–cocycle. Moreover, if ρ is equivalent to φ then as the action of $\mathrm{SL}_2(\mathbb{Z})$ has fixed points ρ and φ are conjugate representations.*

Example 60. *If we take*

$$\Omega_\gamma(\tau) \;=\; (c\tau + d)^k. \tag{8.97}$$

Then Ω is a QM–cocycle.

Note that QM–cocycles can be multiplied by QM–cocycles of rank one and remain QM–cocycles. Therefore, this last example allows us to globally change the weight of QM–cocycles so that these is no natural global weight. This is an example showing that any automorphy factor that is holomorphic for γ on $\mathbb{C}_\gamma$ then this will give rise to a QM–cocycle.

Proposition 13. *If Ω and Ξ are QM–cocycles and if for all γ with $1 \in \mathbb{C}_\gamma$*

$$\Omega_\gamma(1) \;=\; \Xi_\gamma(1)\,, \tag{8.98}$$

then $\Omega = \Xi$.

Proof. Let for $n \in \mathbb{Z}_{>1}$

$$\gamma_n \;=\; \begin{pmatrix} n+1 & 1 \\ n & 1 \end{pmatrix} \tag{8.99}$$

and note that

$$T^k S \gamma_n \;=\; \begin{pmatrix} -n + k(n+1) & -1 + k \\ n+1 & 1 \end{pmatrix} \tag{8.100}$$

From the cocycle condition

$$\Omega_{T^k S}\Big(\frac{(n+1)\tau + 1}{n\tau + 1}\Big) \;=\; \Omega_{T^k S \gamma_n}(\tau)\Omega_{\gamma_n}(\tau)^{-1}\,. \tag{8.101}$$

Then noting that $\mathbb{C}_{\gamma_n} = \mathbb{C} - \mathbb{R}_{\leq -1/n}$ and $\mathbb{C}_{T^kS\gamma_n} = \mathbb{C} - \mathbb{R}_{\leq -1/(n+1)}$ we see that we can compute

$$\Omega_{T^kS}\Big(1+\frac{1}{n+1}\Big) \;=\; \Omega_{T^kS\gamma_n}(1)\Omega_{\gamma_n}(1)^{-1} \tag{8.102}$$

Therefore, $\Omega_{T^kS} = \Xi_{T^kS}$ as their difference has a ccumulation point of zeros on the interior of their domain. Then using the cocycle condition every element of the form

$$\gamma' \;=\; T^{k_1}ST^{k_2}S\cdots T^{k_N}S \tag{8.103}$$

has $\Omega_{\gamma'} = \Xi_{\gamma'}$. Then finally, using the identity $TSTST = S$ we see that

$$\Omega_{TSTS}(\tau+1)^{-1}\Omega_S(\tau) \;=\; \Omega_T(\tau)\,. \tag{8.104}$$

and hence $\Omega_T = \Xi_T$. All other elements are now determined by the cocycle condition. □

For cocycles in the full modular group there are certain situations when we know that everything is determined by the generators of the group and the cocycle condition. For example, if the cocycle is trivial for T and satisfies the correct holomorphic properties for S then this is enough to prove the cocycle evaluated at every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ has the correct domain. This is the content of the Proposition 14.

8.5 Difference equations and rigidity

All of our examples will be related in some way to q–difference equations. This can be incorporated into the description of cocycles. This was discussed in [82]. Associated to a q–difference equation with companion matrix $A(x;q)$, we consider, for $N \in \mathbb{Z}_{>0}$, the set of maps Ω from $\mathrm{SL}_2(\mathbb{Z})$ to matrix valued meromorphic functions such that $\Omega_\gamma \in GL_N(\mathcal{M}(\mathbb{C}\times\mathbb{C}_\gamma))$ and, where defined, we have the cocycle condition

$$\Omega_{\gamma\gamma'}(z;\tau) \;=\; \Omega_\gamma\Big(\frac{z}{c'\tau+d'};\frac{a'\tau+b'}{c'\tau+d'}\Big)\Omega_{\gamma'}(z;\tau)\,, \tag{8.105}$$

along with the q and $\tilde{q}$–difference equations

$$\begin{aligned}\Omega_\gamma(z+a\tau+b;\tau) \;&=\; A\Big(\mathbf{e}\Big(\frac{z}{c\tau+d}\Big);\tilde{q}_\gamma\Big)\Omega(z;\tau)\prod_{j=0}^{a-1}A(q^j\mathbf{e}(z);q)^{-1}\\ \Omega_\gamma(z+c\tau+d;\tau) \;&=\; \Omega_\gamma(z;\tau)\prod_{j=0}^{c-1}A(q^j\mathbf{e}(z);q)^{-1}\,,,\end{aligned} \tag{8.106}$$

where $\prod$ starts on the left and if for $n>0$ we have $\prod_{j=0}^{-n-1}a_j = \prod_{j=1}^{n}a_{-j}^{-1}$. In this context, we have the following proposition, which determines the cocycle from S and T.

Proposition 14. *[82] If $\Omega_S(z;\tau)$ is meromorphic for $\tau \in \mathbb{C}_S$ and $\Omega_T(z;\tau) = \mathrm{Id}$, then $\Omega_\gamma(z;\tau)$ is meromorphic in $\mathbb{C}_\gamma$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.*

Proposition 15. *If Ω, Ξ are holomorphic and $\Omega_\gamma(0;1) = \Xi_\gamma(0;1)$ for all γ then*

$$\Omega \;=\; \Xi\,. \tag{8.107}$$

Proof. $\Omega_\gamma(0;\tau)$ and $\Xi_\gamma(0;\tau)$ define cocycles in the previous sense. Therefore, from Proposition 13 we see that

$$\Omega_\gamma(0;\tau) \;=\; \Xi_\gamma(0;\tau)\,. \tag{8.108}$$

For each $\gamma = [a,b;c,d]$ with $c \neq 0$, choose $\tau_0 \in (\mathbb{R}-\mathbb{Q}) \cap \mathbb{C}_\gamma$. Then the set

$$\mathbb{Z} + \tau_0\mathbb{Z} \tag{8.109}$$

is dense in $\mathbb{R}$. Therefore, from the q and $\tilde{q}$–difference equations we see that for $w \in \mathbb{Z}+\tau_0\mathbb{Z}$ we have

$$\Omega_\gamma(w;\tau_0) \;=\; \Xi_\gamma(w;\tau_0) \tag{8.110}$$

and therefore as they are analytic functions in z we see that for all z

$$\Omega_\gamma(z;\tau_0) \;=\; \Xi_\gamma(z;\tau_0) \tag{8.111}$$

for each $\tau_0 \in (\mathbb{R}-\mathbb{Q}) \cap \mathbb{C}_\gamma$. Therefore, for a fixed z_0 we see that for all $\tau \in (\mathbb{R}-\mathbb{Q}) \cap \mathbb{C}_\gamma$

$$\Omega_\gamma(z_0;\tau) \;=\; \Xi_\gamma(z_0;\tau) \tag{8.112}$$

and therefore for all $\tau \in \mathbb{C}_\gamma$ we have

$$\Omega_\gamma(z_0;\tau) \;=\; \Xi_\gamma(z_0;\tau)\,. \tag{8.113}$$

Finally, for T we notice again that

$$\Omega_{TSTS}(z;\tau+1)^{-1}\Omega_S(z;\tau) \;=\; \Omega_T(z;\tau)\,. \tag{8.114}$$

□

It is interesting as to how many cocycles are associated to a q–difference equation. There are some strict conditions that extension of τ to $\mathbb{C}_\gamma$ imposes.

Proposition 16. *If Ω and Ξ are QM–cocycles associated to a q–difference equation with companion matrix A then if $\gamma = [a,b;c,d] \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$ then*

$$\sum_{j=0}^{N} h_j(\tau)\big(\Omega_\gamma \Xi_\gamma^{-1}\big)^j \;=\; 0\,. \tag{8.115}$$

satisfies an order N polynomial with coefficients holomorphic functions in $\tau \in \mathbb{C}_\gamma$.

Proof. Notice that

$$\begin{aligned}\Omega_\gamma(z+a\tau+b;\tau)\Xi_\gamma(z+a\tau+b;\tau)^{-1} &= A\Big(\mathbf{e}\Big(\frac{z}{c\tau+d}\Big);\tilde{q}_\gamma\Big)\Omega_\gamma(z;\tau)\Xi_\gamma(z;\tau)^{-1}A\Big(\mathbf{e}\Big(\frac{z}{c\tau+d}\Big);\tilde{q}_\gamma\Big)^{-1},\\ \Omega_\gamma(z+c\tau+d;\tau)\Xi_\gamma(z+c\tau+d;\tau)^{-1} &= \Omega_\gamma(z;\tau)\Xi_\gamma(z;\tau)^{-1}\,.\end{aligned} \tag{8.116}$$

Therefore,

$$\det(\Omega_\gamma\Xi_\gamma^{-1}-x) = \sum_{j=0}^{N} h_j(z;\tau)x^j \tag{8.117}$$

where $h_j(z;\tau)$ are elliptic functions and holomorphic for $\tau\in\mathbb{C}_\gamma$. Therefore, from Cayley–Hamilton theorem

$$\sum_{j=0}^{N} h_j(z;\tau)\big(\Omega_\gamma\Xi_\gamma^{-1}\big)^j = 0\,, \tag{8.118}$$

and they are independent of z from the domain in τ and ellipticity. □

Corollary 9. *If Ω is a rank one QM–cocycle associated to a first order q–difference equation,*

$$\Omega_\gamma(z;\tau) = \Xi_\gamma(z;\tau)h(\tau)\,. \tag{8.119}$$

We close this section by stating that these QM–cocycles are in a sense the most mysterious and interesting objects we want to construct. For specific examples relating to quantum modular forms, these will be conjecturally be related to Borel resummation of their asymptotic series discussed in Section 11.1. Their extension properties would then be expected. However, as this is only conjectural we take a different approach. We will construct these QM–cocycles using integrals of a special function called the Faddeev quantum dilogarithm. This is the content of part V.

8.6 Quantum modular forms: looking forward

Using these analytic cocycles we can now give our definition of quantum modular forms.

Definition 22 (Quantum modular form)**.** *We say that a function $f:\mathfrak{h}\cup\mathbb{Q}\cup\overline{\mathfrak{h}}\to\mathbb{C}^N$ is a quantum modular form with automorphy factor $\mathbf{j}$ and QM–cocycle Ω if for $\gamma=[a,b;c,d]\in\mathrm{SL}_2(\mathbb{Z})$ we have*

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big) = \mathbf{j}(\tau;\gamma)\Omega_\gamma(\tau)f(\tau)\,. \tag{8.120}$$

One could immediately argue that the automorphy factor could be taken into the definition of the QM–cocycle. However, almost all examples will be associated to one QM–cocycle and different automorphy factors. For example, this could just be a shift in the weight but it could also be more exotic for a fixed function as well, for instance the automorphy factor of equation (8.23).

This can naturally be extended to functions satisfying q–difference equations.

Definition 23 (Quantum Jacobi form)**.** *We say that a function $f : \mathbb{C} \times \mathfrak{h} \cup \mathbb{Q} \cup \overline{\mathfrak{h}} \to \mathbb{C}^N$ is a quantum Jacobi form with companion matrix A, automorphy factor $\mathbf{j}$ and QM–cocycle Ω, if for $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$f\Big(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\Big) \;=\; \mathbf{j}(\tau;\gamma)\Omega_\gamma(z;\tau)f(z;\tau)\,, \tag{8.121}$$

and

$$f(z+\tau;\tau) \;=\; A(z;\tau)f(z;\tau)\,. \tag{8.122}$$

We will call f simply a quantum modular (Jacobi) form if there exists some cocycle with the properties required. Moreover, if the function is just defined on $\mathfrak{h}, \mathbb{Q}$ or $\overline{\mathfrak{h}}$ we will also refer to it as a quantum modular form.

Proposition 17. *Suppose that Ω is a rank N QM–cocycle and $f_1, \dots, f_N$ are independent quantum modular forms for Ω with automorphy factors $\mathbf{j}_1, \dots, \mathbf{j}_N$ and $g_1, \dots, g_N$ is another set with automorphy factors $\mathbf{k}_1, \dots, \mathbf{k}_N$. Then*

$$M \;=\; \begin{pmatrix} f_1 & \cdots & f_N \end{pmatrix}^{-1} \begin{pmatrix} g_1 & \cdots & g_N \end{pmatrix} \tag{8.123}$$

is a matrix valued modular form with respect to these automorphy factors so that

$$\begin{pmatrix} \mathbf{j}_1(\tau;\gamma) & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{j}_N(\tau;\gamma) \end{pmatrix} M\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; M(\tau) \begin{pmatrix} \mathbf{k}_1(\tau;\gamma) & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{k}_N(\tau;\gamma) \end{pmatrix}. \tag{8.124}$$

This shows that over a certain space of modular forms the space of quantum modular forms with a given cocycle has dimension the same as the rank. If we have a basis of quantum modular forms in a matrix U with a cocycle Ω so that

$$U\Big(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\Big) \;=\; \Omega_\gamma(z;\tau)U(z;\tau) \begin{pmatrix} \mathbf{j}_1(\tau;\gamma) & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{j}_N(\tau;\gamma) \end{pmatrix}, \tag{8.125}$$

then we see that

$$\begin{pmatrix} \mathbf{j}_1(\tau;\gamma)^{-1} & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{j}_N(\tau;\gamma)^{-1} \end{pmatrix} \;=\; U\Big(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\Big)^{-1}\Omega_\gamma(z;\tau)U(z;\tau) \tag{8.126}$$

and this makes it look like Ω_γ is equivalent to

$$\begin{pmatrix} \mathbf{j}_1(\tau;\gamma)^{-1} & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{j}_N(\tau;\gamma)^{-1} \end{pmatrix}. \tag{8.127}$$

However, U is not defined at $\tau \in \mathbb{R} - \mathbb{Q}$, which mean that it is not giving a coboundary. Finally, if we have such a U, then we can compute

$$U\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big)\begin{pmatrix} \mathbf{j}_1(\tau;\gamma)^{-1} & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{j}_N(\tau;\gamma)^{-1} \end{pmatrix} U(z;\tau)^{-1} \;=\; \Omega_\gamma(z;\tau) \tag{8.128}$$

and therefore, see that the LHS while not being defined for $\tau \in \mathbb{R}-\mathbb{R}\cap\mathbb{C}_\gamma$ it has an analytic extension.

Remark 27. *Instead of just functions from roots of unity to complex numbers, we could have taken asymptotic series of the kinds described in Section 5.8. These series can then be acted upon by* $\mathrm{SL}_2(\mathbb{Z})$ *and corresponding statements can be made. If one does this with the asymptotic series, then taking the LHS of equation (8.128) at rationals in* $\mathbb{Q}\cap\mathbb{C}_\gamma$ *gives rise to convergent series.*

With this definition we will now give a few examples of quantum modular forms deferring their proofs to the next part.

8.7 Rank one Nahm sums are quantum modular forms

For $A \in \mathbb{Z}$,

$$f_{A,m}(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{\frac{A}{2}k(k+1)+km}}{(q;q)_k}. \tag{8.129}$$

we have

$$F_{A,m}(q) \;=\; \begin{pmatrix} f_{A,m}(q) \\ f_{A,m+1}(q) \\ \vdots \\ f_{A,m+A-1}(q) \end{pmatrix}. \tag{8.130}$$

Theorem 5. *The function in the upper half plane* $F_{A,m}(q)$ *is a quantum modular form.*

This is proved in Section 10.1. From the proof one will see that this could be extended to a full matrix valued function, which will be needed in conjectural computations of the Borel resummations and Stokes constants. However, at rationals this is best done with matrices.

Recall from Section 4.7, the asymptotics of these Nahm sums at roots of unity had constant terms for the A roots of

$$1 - X_i \;=\; X_i^A\,, \tag{8.131}$$

we had functions such that for $q = \mathbf{e}(a/c)$

$$f_{A,i,m}(q) \;=\; \frac{\prod_{\ell=1}^{|c|-1}\left(1-q^\ell\right)^{\frac{\ell}{|c|}-\frac{1}{2}}}{\sqrt{|c|X_i/(1-X_i)+A|c|}}\sum_{r=0}^{|c|-1}\frac{q^{Ar(r+1)/2+rm}X_i^{(Ar+m)/|c|}X_i^{A/2|c|}}{\prod_{s=0}^{|c|-1}(1-q^{r+s+1}X_i^{1/|c|})^{\frac{r+s+1}{|c|}-\frac{1}{2}}}\,. \tag{8.132}$$

We can then define a matrix

$$F_{A,m}(q) \;=\; \begin{pmatrix} f_{A,1,m}(q) & \cdots & f_{A,A,m}(q) \\ \vdots & \cdots & \vdots \\ f_{A,1,m+A-1}(q) & \cdots & f_{A,A,m+A-1}(q) \end{pmatrix}. \tag{8.133}$$

Theorem 6. *The function $F_{A,m}(q)$ from $\mathbb{Q}$ is a quantum modular form, and in particular,*

$$F_{A,m}(\tilde{q})\begin{pmatrix} \mathbf{e}\big(\frac{\mathrm{VC}_{1,0}}{\mathrm{denom}(x)\mathrm{numer}(x)}\big) & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{e}\big(\frac{\mathrm{VC}_{A,0}}{\mathrm{denom}(x)\mathrm{numer}(x)}\big) \end{pmatrix} F_{A,m}(q)^{-1}\,, \tag{8.134}$$

extends to a holomorphic function for $\tau \in \mathbb{C}_S$.

This theorem is proved in Section 9.1.

8.8 Invariants of the figure eight knot

Consider the, functions for $|q| \neq 1$

$$\begin{aligned} g_m(q) &= \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2+km}}{(q;q)_k^2}\,, \\ G_m(q) &= \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2+km}}{(q;q)_k^2}\Big(m-2G_1(q)+\sum_{\ell=1}^{k}\frac{1+q^j}{1-q^j}\Big)\,, \\ \mathfrak{G}_m(q) &= \sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k+1)/2+km}}{(q;q)_k^2}\bigg(\frac{1}{8}\Big(2m-4G_1(q)+2\sum_{\ell=1}^{k}\frac{1+q^j}{1-q^j}\Big)^2-\frac{1}{24}+\sum_{\ell=1}^{k}\frac{q^\ell}{(1-q^\ell)^2}\bigg). \end{aligned} \tag{8.135}$$

Combining these into a matrix we can define

$$\mathfrak{g}_m(q) \;=\; \begin{pmatrix} g_m(q) & G_m(q) & \mathfrak{G}_m(q) \\ g_{m+1}(q) & G_{m+1}(q) & \mathfrak{G}_{m+1}(q) \\ g_{m+2}(q) & G_{m+2}(q) & \mathfrak{G}_{m+2}(q) \end{pmatrix}. \tag{8.136}$$

We have for $q = \mathbf{e}(a/c)$, and

$$X_j = \frac{1}{2} + (-1)^j \frac{\sqrt{-3}}{2} \tag{8.137}$$

the functions

$$J_{i,m}(q) = \sum_{k \in \mathbb{Z}/c\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2+km} X_i^{(k-m)/c} X_i^{1/2c}}{\prod_{j=0}^{N-1} (1 - q^{1+k+j} X_i^{1/c})^{2(1+j+k)/c-1}} \tag{8.138}$$

and additionally take

$$J_{0,m}(q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2-km-m} (q;q)_k^2 \,. \tag{8.139}$$

Then define

$$\mathfrak{g}_m(q) = \begin{pmatrix} J_{0,m}(q) & J_{1,m}(q) & J_{2,m}(q) \\ J_{0,m+1}(q) & J_{1,m+1}(q) & J_{2,m+1}(q) \\ J_{0,m+2}(q) & J_{1,m+2}(q) & J_{2,m+2}(q) \end{pmatrix} . \tag{8.140}$$

Finally, take the automorphy factor

$$\mathbf{j}(\tau;\gamma) = \left\{ \begin{array}{ll} \begin{pmatrix} (c\tau+d)^{-1/2} & 0 & 0 \\ 0 & (c\tau+d)^{1/2} & 0 \\ 0 & 0 & (c\tau+d)^{3/2} \end{pmatrix} & \text{if } \tau \in \mathfrak{h} \cup \overline{\mathfrak{h}} \,, \\ \begin{pmatrix} \epsilon(a,b,c,d)^3 (c\tau+d)^{3/2} & 0 & 0 \\ 0 & \mathbf{e}(\lambda_\gamma(\tau)\mathrm{VC}_1/(2\pi i)^2) & 0 \\ 0 & 0 & \mathbf{e}(\lambda_\gamma(\tau)\mathrm{VC}_2/(2\pi i)^2) \end{pmatrix} & \text{if } \tau \in \mathbb{Q} \,. \end{array} \right\} \tag{8.141}$$

Theorem 7. *[85, 86, 70]* $\mathfrak{g}_n$ *is a quantum modular form* i.e. *there exists a QM–cocycle* Ω *such that*

$$\mathfrak{g}_n(\tilde{q}_\gamma) = \Omega_\gamma(m,n;\tau)\mathfrak{g}_m(q)\mathbf{j}(\tau;\gamma) \,. \tag{8.142}$$

This result has partially appeared in the literature previously [85, 86, 73, 72, 70]. The final steps missing in the literature are outlined in Section 9.2 and an associated computation of a q–difference equation is given in Section 10.2. Importantly, this implies that

$$\mathfrak{g}_n(\tilde{q}_\gamma)\mathbf{j}(\tau;\gamma)^{-1}\mathfrak{g}_m(q)^{-1} \tag{8.143}$$

is an analytic function on $\mathbb{C}_\gamma$. See [86, Fig. 2] for the plots of these functions on the reals. We won't recreate them here.

8.9 Invariants of half surgery on the figure eight

Firstly, let

$$\begin{aligned} 0 &= (1 - X_2)(1 - X_1^{-1}X_2) - X_1 \,, \\ 0 &= (1 - X_1^2)^2 - X_1^2 X_2 (1 - X_1^{-1}X_2) \,. \end{aligned} \tag{8.144}$$

These have solutions with respect to our generator of the trace field satifying

$$p(\xi) \;=\; \xi^7 - \xi^6 - 2\xi^5 + 6\xi^4 - 11\xi^3 + 6\xi^2 + 3\xi - 1 \tag{8.145}$$

so that

$$\begin{aligned} X_{1,j} &= -3 + 11\xi_j + 20\xi_j^2 - 15\xi_j^3 + 6\xi_j^4 - 2\xi_j^5 - 4\xi_j^6 \,, \\ X_{2,j} &= -9 + 19\xi_j + 76\xi_j^2 - 52\xi_j^3 + 20\xi_j^4 - 4\xi_j^5 - 13\xi_j^6 \,. \end{aligned} \tag{8.146}$$

Then letting

$$\Delta \;=\; -257 + 806\xi + 947\xi^2 - 749\xi^3 + 331\xi^4 - 133\xi^5 - 213\xi^6 \,, \tag{8.147}$$

We define functions at roots of unity $q = \mathbf{e}(a/c)$

$$\begin{aligned} Z_m^{(j)}(q) \;&=\; \frac{-i}{c\varepsilon(q)}\sqrt{\frac{(1-X_{1,j}^2)^2(1-X_{2,j})(1-X_{1,j}^{-1}X_{2,j})}{\Delta}} \\ &\times \sum_{k,\ell\in\mathbb{Z}/c\mathbb{Z}} \frac{q^{k^2+k\ell-mk+\ell} X_1^{\frac{2k+\ell-m}{c}} X_{2,j}^{\frac{k+1}{c}} \prod_{i=0}^{c-1}(1-q^{i+1+\ell-k}X_{1,j}^{-1/c}X_{2,j}^{1/c})^{-(i+1+\ell-k)/c-1/2}}{\prod_{i=0}^{c-1}(1-q^{i+1+\ell}X_{2,j}^{1/c})^{(i+1+\ell)/c-1/2}\prod_{i=0}^{c-1}(1-q^{i+1+2k}X_{1,j}^{2/c})^{(i+1+2k)/c-1/2}} \,. \end{aligned} \tag{8.148}$$

Finally, we take

$$Z_m^{(0)}(q) \;=\; \sum_{0\le\ell\le k} (-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)+mk} \frac{(q;q)_{2k+1}}{(q;q)_\ell (q;q)_{k-\ell}} \,. \tag{8.149}$$

Then for roots of unity $q = \mathbf{e}(a/c)$ we define

$$(\mathbf{Z}_m(q))_{i,j} \;=\; Z_{m+i-1}^{(j-1)}(q) \,. \tag{8.150}$$

Also define the automorphy factor at roots of unity to be

$$\mathbf{j}(\tau;\gamma) \;=\; \mathrm{diag}(\epsilon(\gamma)^2(c\tau+d)^{3/2}, \mathbf{e}(\mathrm{VC}_1\lambda_\gamma(\tau)/(2\pi i)^2), \cdots, \mathbf{e}(\mathrm{VC}_7\lambda_\gamma(\tau)/(2\pi i)^2)) \,. \tag{8.151}$$

Next define, for $|q| \neq 1$, $k_1(q), \cdots, k_8(q) = 1, 2, 4, 5, 6, 7, 8, 9$ if $|q| < 1$ and $k_1(q), \cdots, k_8(q) = 3, 1, 4, 5, 6, 7, 8, 9$ if $|q| > 1$,

$$(\mathbf{Z}_m(q))_{i,j} \;=\; Z_{m+i-1}^{k_j(q)}(q) \tag{8.152}$$

and automorphy factor generated by

$$\mathbf{j}(\tau;T) \;=\; \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \quad \mathbf{j}(\tau;S) \;=\; \tau^{1/2} \begin{pmatrix} \tau^{2-2\delta_{|q|<1}}&0&0&0&0&0&0&0\\ 0&\tau&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \tag{8.153}$$

where

$$
\begin{aligned}
Z_m^{(1)}(q) &= (q;q)_\infty^2 \sum_{k,j=0}^{\infty} \frac{q^{k(2k+1)+jk+j-mk-m}}{(q;q)_j(q;q)_{2k}(q;q)_{k+j}} \\
Z_m^{(2)}(q) &= (q;q)_\infty^2 \sum_{k,j=0}^{\infty} \frac{q^{k(2k+1)+jk+j-mk-m}}{(q;q)_j(q;q)_{2k}(q;q)_{k+j}} \left(-k-\frac{1}{2}+2G_1(q)-\sum_{n=1}^{j}\frac{q^n}{1-q^n}-\sum_{n=1}^{k+j}\frac{q^n}{1-q^n}\right), \\
&\quad + (q;q)_\infty^2 \sum_{k=0}^{\infty}\sum_{j=-k}^{-1} \frac{q^{k(2k+1)+jk+j-mk-m}(q^{-1};q^{-1})_{-j-1}}{(q;q)_{2k}(q;q)_{k+j}}, \\
Z_m^{(3)}(q) &= (q;q)_\infty^2 \sum_{k,j=0}^{\infty} \frac{q^{k(2k+1)+jk+j-mk-m}}{(q;q)_j(q;q)_{2k}(q;q)_{k+j}} \\
&\qquad \times \left(\frac{1}{2}\left(7k+2j-2m-4G_1(q)+\sum_{n=j+1}^{k+j}\frac{q^n}{1-q^n}+4\sum_{n=1}^{2k}\frac{q^n}{1-q^n}\right)\right. \\
&\qquad\qquad \times \left(k+1/2-2G_1(q)+\sum_{n=1}^{k+j}\frac{q^n}{1-q^n}+\sum_{n=1}^{j}\frac{q^n}{1-q^n}\right) \\
&\qquad\qquad \left. + 2G_2(q)+\frac{1}{2}\sum_{n=1}^{k+j}\frac{q^n}{(1-q^n)^2}-\frac{1}{2}\sum_{n=1}^{j}\frac{q^n}{(1-q^n)^2}\right) \\
&\quad + (q;q)_\infty^2 \sum_{k=0}^{\infty}\sum_{j=-k}^{-1} \frac{q^{k(2k+1)+jk+j-mk-m}(q^{-1};q^{-1})_{-j-1}}{(q;q)_{2k}(q;q)_{k+j}} \\
&\qquad \times \left(3k+j-m-\frac{3}{4}-G_1(q)+2\sum_{n=1}^{2k}\frac{q^n}{1-q^n}+\sum_{n=1}^{-j-1}\frac{q^{-n}}{1-q^{-n}}\right), \\
Z_m^{(4)}(q) &= (q;q)_\infty (q^{3/2};q)_\infty \sum_{k,j=0}^{\infty} \frac{q^{(2k+1)(2k+2)/2+(k+1/2)j-m(k+1/2)+j-m}}{(q;q)_j(q;q)_{2k+1}(q^{\frac{3}{2}};q)_{k+j}}, \\
Z_m^{(5)}(q) &= (q;q)_\infty (-q;q)_\infty \sum_{k,j=0}^{\infty} (-1)^{j+m} \frac{q^{k(2k+1)+jk+j-mk-m}}{(q;q)_j(q;q)_{2k}(-q;q)_{k+j}}, \\
Z_m^{(6)}(q) &= (q;q)_\infty (-q^{3/2};q)_\infty \sum_{k,j=0}^{\infty} (-1)^{j+m} \frac{q^{(2k+1)(2k+2)/2+(k+1/2)j-m(k+1/2)+j-m}}{(q;q)_j(q;q)_{2k+1}(-q^{\frac{3}{2}};q)_{k+j}}, \\
Z_m^{(7)}(q) &= (q;q)_\infty (q^{1/2};q)_\infty \sum_{k,j=0}^{\infty} \frac{q^{(2k+1)(2k+2)/2+(j-k-1/2)(k+1/2)+(j-k-1/2)-m(k+1/2)-m}}{(q^{1/2};q)_{j-k}(q;q)_{2k+1}(q;q)_j}, \\
Z_m^{(8)}(q) &= (q;q)_\infty (-q;q)_\infty \sum_{k,j=0}^{\infty} (-1)^{j+m} \frac{q^{k(2k+1)+(j-k)k+(j-k)-mk-m}}{(-q;q)_{j-k}(q;q)_{2k}(q;q)_j}, \\
Z_m^{(9)}(q) &= (q;q)_\infty (-q^{1/2};q)_\infty \sum_{k,j=0}^{\infty} (-1)^{j+m} \frac{q^{(2k+1)(2k+2)/2+(j-k-1/2)(k+1/2)+(j-k-1/2)-m(k+1/2)-m}}{(-q^{1/2};q)_{j-k}(q;q)_{2k+1}(q;q)_j}.
\end{aligned}
\tag{8.154}
$$

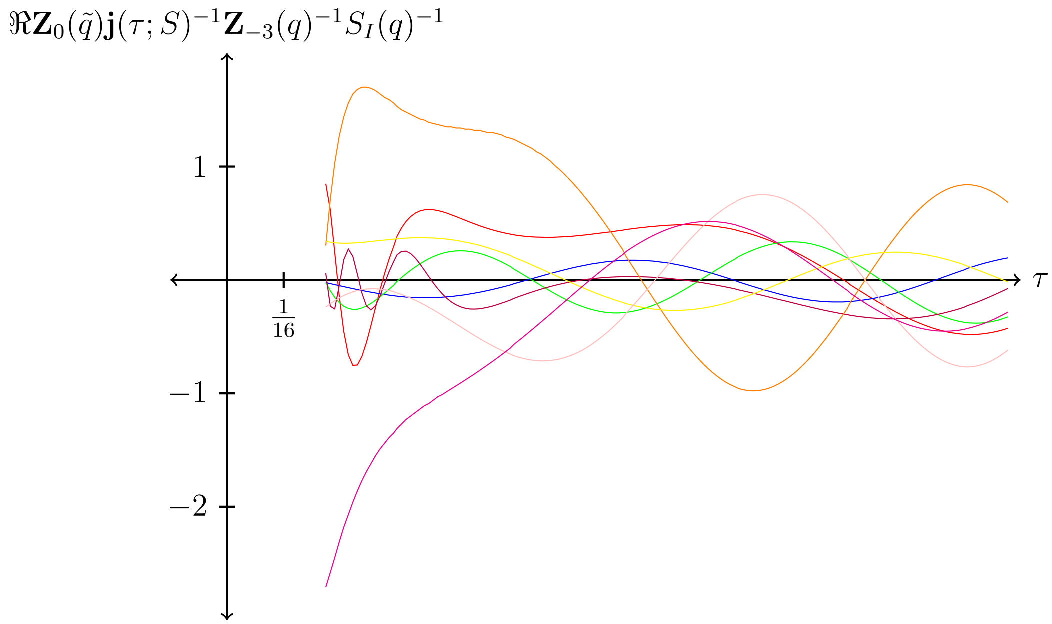

Figure 8.5: Plots of the real part of the first row of the cocycle associated to $4_1(1,2)$. Here we cut the plot off when it gets close to 0 as there is an exponential singularity there and the oscillations become large requiring many more data points to see the smooth behaviour. See the introduction to the thesis for the absolute values as opposed to real part which kills the oscillation and hence not as pretty.

Theorem 8. $\mathbf{Z}_m(q)$ *is a quantum modular form* i.e. *there exists a QM–cocycle* Ω *such that*

$$\mathbf{Z}_n(\tilde{q}_\gamma) \;=\; \Omega_\gamma(m,n;\tau)\mathbf{Z}_m(q)\mathbf{j}(\tau;\gamma)\,. \tag{8.155}$$

This is proved in sections 9.4, 10.3, and 10.4. This implies that

$$\mathbf{Z}_n(\tilde{q}_\gamma)\mathbf{j}(\tau;\gamma)^{-1}\mathbf{Z}_m(q)^{-1} \tag{8.156}$$

is an analytic function on $\mathbb{C}_\gamma$. We can make the analogous plots to those found in [86] of these functions at the positive reals as shown in Figure 8.5. Compare this figure to figure 8.3. This illustrates the improved perspective from [208] to [86].

8.10 The quantum dilogarithm

To begin with we will describe a version of the Faddeev quantum dilogarithm [58, 59]. See [8, App. A.] for a good summary. We will see that this function corresponds to a cocycle of the

Pochhammer symbol associated to the S. In what is now standard notation, this is defined as

$$\Phi_{\mathsf{b}}(x) \;=\; \exp\left(\int_{i0-\infty}^{i0+\infty} \frac{e^{-2ixw}}{4\sinh(w\mathsf{b})\sinh(w\mathsf{b}^{-1})}\frac{dw}{w}\right) \;=\; \frac{(e^{2\pi(x+c_{\mathsf{b}})\mathsf{b}};\mathbf{e}(\mathsf{b}^2))}{(e^{2\pi(x-c_{\mathsf{b}})\mathsf{b}^{-1}};\mathbf{e}(-\mathsf{b}^{-2}))}\,, \tag{8.157}$$

where $c_b = i(\mathsf{b}+\mathsf{b}^{-1})/2$ and the first expression is valid for $|\Im(x)| < |\Im(c_b)|$ while the second for $\Im(\mathsf{b}^2) > 0$. This notation and variables have certain benefits, however we are interested in modularity and this makes the natural variables

$$u \;=\; \frac{x\mathsf{b}}{i} \quad \text{and} \quad \tau \;=\; \mathsf{b}^2\,. \tag{8.158}$$

Then for certain aesthetic reasons it is also helpful to shift the variable u and take the reciprocal so we will take the function away from the reals

$$\begin{aligned} \Phi_S(u;\tau) \;&=\; \exp\left(-\int_{\sqrt{\tau}(i0+\mathbb{R})} \frac{e^{(2u-1-\tau)w/\tau}}{4\sinh(w)\sinh(w/\tau)}\frac{dw}{w}\right) \\ &=\; \frac{(\tilde{q}\mathbf{e}(u/\tau);\tilde{q})_\infty}{(\mathbf{e}(u);q)_\infty} \;=\; \Phi_{\mathsf{b}}(iu\mathsf{b}^{-1}-c_{\mathsf{b}})^{-1}\,, \end{aligned} \tag{8.159}$$

the second formula is defined for $|\Re((z-1/2-\tau/2)/\sqrt{\tau})| < |\Re((1/2+\tau/2)/\sqrt{\tau})|$ however this function then has analytic continuation to $\tau \in \mathbb{C}_S$ as we will see in Theorem 40. We also have the reverse equality

$$\Phi_{\mathsf{b}}(x) \;=\; \Phi_S(-i(x+c_{\mathsf{b}})\mathsf{b};\mathsf{b}^2)^{-1} \;=\; \Phi_S(-ix\mathsf{b}+\mathsf{b}^2/2+1/2;\mathsf{b}^2)^{-1}\,. \tag{8.160}$$

For the logic of this thesis we will take the quotient of two Pochhammers as the initial definition proving the analytic continuation later. This functional satisfies various important function equations given in the following proposition.

Proposition 18. *We have the following functional equations*

$$\Phi_S(u+1;\tau) \;=\; (1-\mathbf{e}(u/\tau))\Phi_S(u;\tau) \tag{8.161}$$
$$\Phi_S(u+\tau;\tau) \;=\; (1-\mathbf{e}(u))\Phi_S(u;\tau) \tag{8.162}$$
$$\Phi_S(u/\tau;1/\tau) \;=\; \Phi_S(u;\tau)\,. \tag{8.163}$$

An important observation is that the two q and $\tilde{q}$ difference equations are in a sense uncoupled. This leads to the corollary.

Corollary 10.

$$\Phi_S(u+m\tau+n;\tau) \;=\; (\mathbf{e}(u/\tau);\tilde{q}^{-1})_n(\mathbf{e}(u);q)_m\Phi_S(u;\tau)\,. \tag{8.164}$$

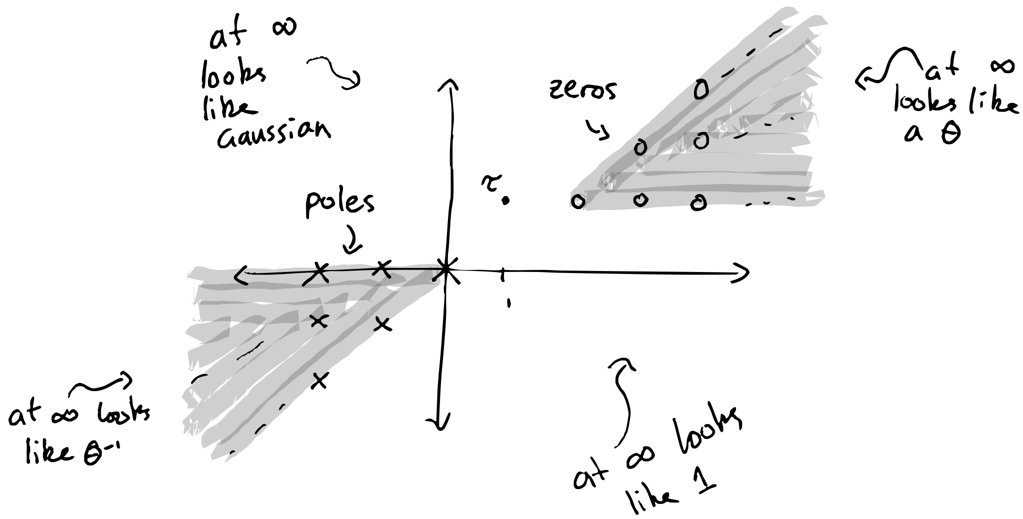

Figure 8.6: The properties of the Faddeev quantum dilogarithm. Its poles and zeros and its behaviour at infinity described in Proposition 20.

The structure of the poles and zeros of this function is easy to compute from its product formula and we find

Proposition 19. *The zeros of $\Phi_S(u;\tau)$ are first order and located at*

$$u \in \mathbb{Z}_{>0} + \tau\mathbb{Z}_{>0} \tag{8.165}$$

while the poles are also first order and located at

$$u \in \mathbb{Z}_{\leq 0} + \tau\mathbb{Z}_{\leq 0}\,. \tag{8.166}$$

The poles and zeros are depicted in Figure 8.6. Now we will give a proof of the main theorem which follows imediately from Faddeev's original definition.

Theorem A−40. *The function $\Phi_S(u;\tau)$ has an analytic continuation to a meromorphic function for $\tau \in \mathbb{C}_S$.*

Proof. From it's definition this is clear for $\tau \notin \mathbb{R}_{>0}$. Using Theorem 29 we can find an integral expression for

$$\frac{\left(\mathbf{e}\left(\frac{u-1}{\tau}\right);\mathbf{e}\left(-\frac{1}{\tau}\right)\right)_\infty}{(\mathbf{e}(u);\mathbf{e}(\tau))_\infty}\,, \tag{8.167}$$

when $\Re(\tau) > 0$. Indeed, using a version of the theorem we can write,

$$\begin{aligned}\frac{\left(\mathbf{e}\left(\frac{u-1}{\tau}\right);\mathbf{e}\left(-\frac{1}{\tau}\right)\right)_\infty}{(\mathbf{e}(u);\mathbf{e}(\tau))_\infty} &= \frac{1}{(\mathbf{e}(u);\mathbf{e}(\tau))_{-\lfloor\Re\left(\frac{u}{\tau}\right)\rfloor}\sqrt{1-\mathbf{e}(u/\tau)}} \\ &\times \exp\left(-\frac{\tau}{2\pi i}\mathrm{Li}_2\left(\mathbf{e}\left(\frac{u}{\tau}\right)\right) + i\tau\int_0^\infty \frac{\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)-\log\left(1-\mathbf{e}\left(ix+\frac{u}{\tau}\right)\right)}{1-\mathbf{e}(-i\tau x)}dx\right).\end{aligned} \tag{8.168}$$

This equality holds for all $\Re(\tau) > 0$ and $\Re\left(\frac{u}{\tau}\right) \notin \mathbb{Z}$. The RHS is an analytic function for all $\Re(\tau) > 0$ and $\Re\left(\frac{u}{\tau}\right) \notin \mathbb{Z}$ not just in the upper half plane. Moreover, we claim the function can be extended to a function for all $\Re(\tau) > 0$. Notice that we can write

$$\begin{aligned}&-\frac{\tau}{2\pi i}\mathrm{Li}_2\left(\mathbf{e}\left(\frac{u}{\tau}\right)\right)+\frac{1}{2}\log\left(1-\mathbf{e}\left(\frac{u}{\tau}\right)\right) \\ &\qquad + i\tau\int_0^\infty \frac{\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)-\log\left(1-\mathbf{e}\left(ix+\frac{u}{\tau}\right)\right)}{1-\mathbf{e}(-i\tau x)}dx \\ &= \frac{1}{2}\log\left(1-\mathbf{e}\left(\frac{u}{\tau}\right)\right) \\ &\qquad + i\tau\int_0^\infty \log\left(1-\mathbf{e}\left(ix+\frac{u}{\tau}\right)\right)+\frac{\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)-\log\left(1-\mathbf{e}\left(ix+\frac{u}{\tau}\right)\right)}{1-\mathbf{e}(-i\tau x)}dx \\ &= i\tau\int_{i0-\infty}^{i0+\infty}\frac{\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)}{1-\mathbf{e}(-i\tau x)}dx\,.\end{aligned} \tag{8.169}$$

Notice that $\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)$ has branch points at $x = -i\frac{u}{\tau} - im$. Therefore, we see that the integral jumps by the keyhole integral around the branch cut as $-im - i\frac{u}{\tau}$ crosses the real line. The keyhole integral can be computed explicitly as

$$\log\left(1-\mathbf{e}\left(m\tau+u\right)\right)\,. \tag{8.170}$$

This cancels exactly with the discontinuities of

$$\frac{1}{(\mathbf{e}(u);\mathbf{e}(\tau))_{-\lfloor\Re\left(\frac{u}{\tau}\right)\rfloor}}\,. \tag{8.171}$$

So the RHS of Equation (8.168) has two sided limits for $\Re\left(\frac{u}{\tau}\right) \in \mathbb{Z}$ which agree and therefore the RHS defines a continuous function for all $\Re(\tau) > 0$. Noting that the function is holomorphic away from the lines $\Re\left(\frac{u}{\tau}\right) \in \mathbb{Z}$ we find that this extension is holomorphic for $\Re(\tau) > 0$.

A similar computation shows that for $\tau \in \overline{\mathfrak{h}}$ with $\Re(\tau) > 0$ we can find that

$$\begin{aligned}\frac{(e(u-\tau);e(-\tau))_\infty}{\left(e\left(\frac{u}{\tau}\right);e\left(\frac{1}{\tau}\right)\right)_\infty} &= \frac{1}{(\mathbf{e}(u);\mathbf{e}(\tau))_{-\lfloor\Re\left(\frac{u}{\tau}\right)\rfloor}\sqrt{1-\mathbf{e}(u/\tau)}}\\ &\times \exp\left(-\frac{\tau}{2\pi i}\mathrm{Li}_2\left(\mathbf{e}\left(\frac{u}{\tau}\right)\right)+i\tau\int_0^\infty \frac{\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)-\log\left(1-\mathbf{e}\left(ix+\frac{u}{\tau}\right)\right)}{1-\mathbf{e}(-i\tau x)}dx\right).\end{aligned} \tag{8.172}$$

With these identities and the fact the poles in u do not accumulate as τ tends to $\mathbb{R}_{>0}$ the analytic extension and meromorphic properties are manifest. □

These identities in this form can be checked explicitly with Code 28. The final properties that we need, which will be important for the next part, are related to the behaviour as u tends to infinity.

Proposition 20. *We have as $u \to \infty$*

$$\Phi_S(u;\tau) \;\sim\; \left\{\begin{array}{cc} 1 & \textit{for } 0<\Im(u) \textit{ and } 0<\Im(u/\tau)\\ \theta(\mathbf{e}(u/\tau);\tilde{q})(\tilde{q};\tilde{q})_\infty^{-1} & \textit{for } 0<\Im(u) \textit{ and } 0>\Im(u/\tau)\\ (q;q)_\infty\theta(\mathbf{e}(u);q)^{-1} & \textit{for } 0>\Im(u) \textit{ and } 0<\Im(u/\tau)\\ -iq^{1/12}\tilde{q}^{-1/12}\mathbf{e}(u^2/2\tau+u/2+u/2\tau) & \textit{for } 0>\Im(u) \textit{ and } 0>\Im(u/\tau)\end{array}\right\} \tag{8.173}$$

Before proceeding with the proof notice that the θ functions only appear in cases where $\Im(\tau) \notin \mathbb{R}_{>0}$ so they are convergent. The various regions is depicted for $\Im(\tau) > 0$ in Figure 8.6.

Proof of Proposition 20. Notice that from Corollary 10, we see that for u in some compact every other value is determined by the asymptotic in m, n of

$$\Phi_S(u+m\tau+n) \;=\; (\mathbf{e}(u/\tau);\tilde{q}^{-1})_n(\mathbf{e}(u);q)_m\Phi_S(u;\tau)\,. \tag{8.174}$$

We see that there are essentially four cases. The first for $\Im(\tau) > 0$ is m and n tend to infinity or $0 < \arg(u) < \arg(\tau)$ and for $\Im(\tau) < 0$ is m and n tend to negative infinity or $\arg(-\tau) < \arg(u) < \pi$. This has for $\Im(\tau) > 0$

$$\begin{aligned}\Phi_S(u+m\tau+n) &\sim (-1)^n\tilde{q}^{-n(n-1)/2}\mathbf{e}(nu/\tau)(\mathbf{e}(-u/\tau);\tilde{q})_\infty(\mathbf{e}(u);q)_\infty\Phi_S(u;\tau)\\ &= \frac{\theta(\tilde{q}^{-n}\mathbf{e}(u/\tau);\tilde{q})}{(\tilde{q};\tilde{q})_\infty} = \frac{\theta(\mathbf{e}((u+m\tau+n)/\tau);\tilde{q})}{(\tilde{q};\tilde{q})_\infty}\,,\end{aligned} \tag{8.175}$$

while for $\Im(\tau) < 0$ this has (noting that $n < 0$)

$$\begin{aligned}\Phi_S(u+m\tau+n) &\sim \frac{1}{(-1)^n\tilde{q}^{n(n-1)/2}\mathbf{e}(-nu/\tau)(\tilde{q}^{-1}\mathbf{e}(-u/\tau);\tilde{q}^{-1})_\infty(q^{-1}\mathbf{e}(u);q^{-1})_\infty}\Phi_S(u;\tau)\\ &= \frac{(\tilde{q}^{-1};\tilde{q}^{-1})_\infty}{\theta(\tilde{q}^{n}\mathbf{e}(-u/\tau);\tilde{q}^{-1})} = \frac{\theta(\mathbf{e}((u+m\tau+n)/\tau);\tilde{q})}{(\tilde{q};\tilde{q})_\infty}\,.\end{aligned} \tag{8.176}$$

Secondly, for $\Im(\tau) > 0$ we can have m tend to infinity and n negative infinity or we have $\arg(\tau) < \arg(u) < \pi$ and for $\Im(\tau) < 0$ we can have n tend to infinity and m negative infinity or we have $0 < \arg(u) < \arg(-\tau)$. So for $\Im(\tau) > 0$

$$\Phi_S(u + m\tau + n) \sim \frac{(\mathbf{e}(u); q)_\infty}{(\tilde{q}\mathbf{e}(u/\tau); \tilde{q})_\infty} \Phi_S(u; \tau) = 1\,, \tag{8.177}$$

and for $\Im(\tau) < 0$

$$\Phi_S(u + m\tau + n) \sim \frac{(\mathbf{e}(u/\tau); \tilde{q}^{-1})_\infty}{(\mathbf{e}(u)q^{-1}; q^{-1})_\infty} \Phi_S(u; \tau) = 1\,, \tag{8.178}$$

while for $\tau \in \mathbb{R}_{>0}$ we can use the final expression of equation (8.169) to see that for $\Im(u) > 0$ the integral tends to 0.

The third for $\Im(\tau) > 0$ is m and n tend to negative infinity or $-\pi < \arg(u) < \arg(-\tau)$ and for $\Im(\tau) < 0$ is m and n tend to infinity or $\arg(\tau) < \arg(u) < 0$. So for $\Im(\tau) > 0$

$$\begin{aligned} \Phi_S(u + m\tau + n) &\sim \frac{1}{(\tilde{q}\mathbf{e}(u/\tau); \tilde{q})_\infty (-1)^m q^{m(m+1)/2} \mathbf{e}(mu)(q\mathbf{e}(u); q)_\infty} \Phi_S(u; \tau) \\ &= \frac{(q; q)_\infty}{\theta(q^m \mathbf{e}(u); q)} = \frac{(q; q)_\infty}{\theta(\mathbf{e}(u + m\tau + n); q)}\,, \end{aligned} \tag{8.179}$$

while for $\Im(\tau) < 0$

$$\begin{aligned} \Phi_S(u + m\tau + n) &\sim (\mathbf{e}(u/\tau); \tilde{q}^{-1})_\infty (-1)^m q^{m(m-1)/2} \mathbf{e}(mu)(\mathbf{e}(-u); q^{-1})_\infty \Phi_S(u; \tau) \\ &= \frac{\theta(q^m \mathbf{e}(-u); q^{-1})}{(q^{-1}; q^{-1})_\infty} = \frac{(q; q)_\infty}{\theta(\mathbf{e}(u + m\tau + n); q)}\,. \end{aligned} \tag{8.180}$$

Finally, for $\Im(\tau) > 0$ we can have n tend to infinity and m negative infinity or we have $\arg(-\tau) < \arg(u) < 0$ and for $\Im(\tau) < 0$ we can have m tend to infinity and n negative infinity or we have $-\pi < \arg(u) < \arg(\tau)$. So for $\Im(\tau) > 0$

$$\begin{aligned} \Phi_S(u + m\tau + n) &\sim \frac{(-1)^n \tilde{q}^{n(n-1)/2} \mathbf{e}(nu/\tau)(\mathbf{e}(-u/\tau); \tilde{q})_\infty}{(-1)^m q^{m(m-1)/2} \mathbf{e}(-mu)(q\mathbf{e}(-u); q)_\infty} \Phi_S(u; \tau) \\ &= \frac{\theta(\tilde{q}^{-n} \mathbf{e}(u/\tau); \tilde{q})}{\theta(q^{-m} \mathbf{e}(-u); q)} \frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} = \frac{\theta(\mathbf{e}((u + m\tau + n)/\tau); \tilde{q})}{\theta(q^{-1} \mathbf{e}(u + m\tau + n); q)} \\ &= -\mathbf{e}(-u - m\tau - n) \frac{\theta(\mathbf{e}((u + m\tau + n)/\tau); \tilde{q})}{\theta(\mathbf{e}(u + m\tau + n); q)} \frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} \\ &= \frac{-\mathbf{e}(-u - m\tau - n)\sqrt{\tau}\,\mathbf{e}\left(\frac{(u+n-1/2+\tau(m+1/2))^2}{2\tau} - \frac{1}{8}\right)}{\sqrt{\tau}\,\mathbf{e}(-1/8) q^{1/24} \tilde{q}^{-1/24}} \\ &= \mathbf{e}\left(\frac{(u + n - 1/2 + \tau(m - 1/2))^2}{2\tau} - \frac{\tau}{24} - \frac{1}{24\tau}\right)\,, \end{aligned} \tag{8.181}$$

and for $\Im(\tau) < 0$

$$
\begin{aligned}
\Phi_S(u+m\tau+n) &\sim \frac{(-1)^m q^{m(m-1)/2}\mathbf{e}(mu)(\mathbf{e}(-u);q^{-1})_\infty}{(-1)^n\tilde{q}^{n(n-1)/2}\mathbf{e}(-nu/\tau)(\tilde{q}^{-1}\mathbf{e}(-u/\tau);\tilde{q}^{-1})_\infty}\Phi_S(u;\tau) \\
&= \frac{\theta(q^m\mathbf{e}(u);q^{-1})(\tilde{q}^{-1};\tilde{q}^{-1})_\infty}{\theta(\tilde{q}^n\mathbf{e}(-u/\tau);\tilde{q}^{-1})(q^{-1};q^{-1})_\infty} = \frac{\theta(\mathbf{e}(u+m\tau+n);q^{-1})(\tilde{q}^{-1};\tilde{q}^{-1})_\infty}{\theta(\mathbf{e}(-(u+m\tau+n)/\tau);\tilde{q}^{-1})(q^{-1};q^{-1})_\infty} \\
&= \frac{\sqrt{-\tau}\mathbf{e}(-1/8)q^{-1/24}\tilde{q}^{1/24}}{\sqrt{-\tau}\mathbf{e}\left(\frac{-(u+n-1/2+\tau(m-1/2))^2}{2\tau}-\frac{1}{8}\right)} \\
&= \mathbf{e}\left(\frac{(u+n-1/2+\tau(m-1/2))^2}{2\tau}-\frac{\tau}{24}-\frac{1}{24\tau}\right),
\end{aligned}
\tag{8.182}
$$

while for $\tau \in \mathbb{R}_{>0}$ the using equation (8.168) and assuming for simplicity that $0 < \Re(u/\tau) < 1$ then we see that the integral

$$
i\tau\int_0^\infty \frac{\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)-\log\left(1-\mathbf{e}\left(ix+\frac{u}{\tau}\right)\right)}{1-\mathbf{e}(-i\tau x)}dx \;\sim\; \frac{2\pi i}{12\tau} \tag{8.183}
$$

while

$$
\sqrt{1-\mathbf{e}(u/\tau)} \;\sim\; -i\mathbf{e}(u/2\tau) \tag{8.184}
$$

and using equation (4.35)

$$
\mathrm{Li}_2(\mathbf{e}(u/\tau)) \;\sim\; -\frac{\pi^2}{6}+2\pi^2(u/\tau-1/2)^2 \tag{8.185}
$$

Combining these into equation (8.168) gives the result. For other u unrestricted the argument is similar and one just needs to keep track of the branching. □

We can use this to prove the agreement with the original formula of the Faddeev simply for the logical consistency of this thesis. Indeed, Faddeev's formula will be useful later in Section 8.12.

Proposition 21. *We have for* $|\Re((u-1/2-\tau/2)/\sqrt{\tau})| < |\Re((1/2+\tau/2)/\sqrt{\tau})|$

$$
\Phi_S(u;\tau) \;=\; \exp\left(-\int_{\sqrt{\tau}(i0+\mathbb{R})}\frac{e^{(2u-1-\tau)w/\tau}}{4\sinh(w)\sinh(w/\tau)}\frac{dw}{w}\right). \tag{8.186}
$$

Proof. First let and notice that

$$
\begin{aligned}
f(u;\tau) &= \exp\left(-\int_{\sqrt{\tau}(i0+\mathbb{R})}\frac{e^{(2(u+1)-1-\tau)w/\tau}}{4\sinh(w)\sinh(w/\tau)}\frac{dw}{w}\right) \\
&= \exp\left(-\int_{\sqrt{\tau}(i0+\mathbb{R})}\frac{\exp(uw/\tau)}{(1-\exp(w/\tau))(1-\exp(w))}\frac{dw}{w}\right).
\end{aligned}
\tag{8.187}
$$

For $\Im(u) > 0$ and $\Im(u/\tau) > 0$ we can push the contour to infinity and collect the residues to find that for $\tau \notin \mathbb{R}$

$$f(u;\tau) \;=\; \exp\left(\sum_{k=1}^{\infty}\frac{\mathbf{e}(ku/\tau)}{(1-\tilde{q}^{-k})k}+\sum_{\ell=1}^{\infty}\frac{\mathbf{e}(\ell u)}{(1-q^{\ell})\ell}\right). \tag{8.188}$$

Then notice that for $|\Im(\tau)| > 0$ using Lemma 3 and analytic continuation we have

$$f(u;\tau) \;=\; \exp\left(\sum_{k=1}^{\infty}\frac{\tilde{q}^{k}\mathbf{e}(ku/\tau)}{(\tilde{q}^{k}-1)k}-\sum_{\ell=1}^{\infty}\frac{\mathbf{e}(\ell u)}{(q^{\ell}-1)\ell}\right) \;=\; \Phi_S(u;\tau)\,, \tag{8.189}$$

and for $|\Im(\tau)| < 0$ using Lemma 3 and analytic continuation again we have

$$f(u;\tau) \;=\; \exp\left(-\sum_{k=1}^{\infty}\frac{\mathbf{e}(ku/\tau)}{(\tilde{q}^{-k}-1)k}+\sum_{\ell=1}^{\infty}\frac{q^{-\ell}\mathbf{e}(\ell u)}{(q^{-\ell}-1)\ell}\right) \;=\; \Phi_S(u;\tau)\,, \tag{8.190}$$

Then finally using the analytic continuation to $\tau \in \mathbb{R}_{>0}$ completes the proof. □

We will close this section noting some important equations satified by the Faddeev quantum dilogarithm. Firstly, from the modularity of the θ–function from equation 7.68 and the Jacobi triple product 30 we have the inversion relation.

Proposition 22. *[8, Eq. 47] We have the identity*

$$\Phi_S(\tau+u;\tau)\Phi_S(1-u;\tau) \;=\; \mathbf{e}\left(\frac{u^2}{2\tau}-\frac{1}{4}+\frac{u}{2}-\frac{u}{2\tau}\right)q^{\frac{1}{12}}\tilde{q}^{-\frac{1}{12}} \;=\; q^{-1/24}\tilde{q}^{1/24}\mathbf{e}\Big(\frac{(u-\frac{1}{2}+\frac{\tau}{2})^2}{2\tau}\Big) \tag{8.191}$$

The Faddeev quantum dilogarithm also satisfies some integral equations analogous to the Pochhammer symbol given in Lemma 7.

Proposition 23. *[8, Sec. 13.2] We have the following equalities so that when $\tau \in \mathfrak{h}$ then these can be written*

$$\int_{i\sqrt{\tau}\mathbb{R}-1/2-\tau/2}\frac{\mathbf{e}(wx/\tau)}{\Phi_S(x+\tau+1;\tau)}dx \;=\; \tau\frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\Phi_S(w;\tau)\,, \tag{8.192}$$

$$\int_{i\sqrt{\tau}\mathbb{R}-1/2-\tau/2}\frac{\mathbf{e}(x(x+1+\tau)/2\tau+wx/\tau)}{\Phi_S(x+\tau+1;\tau)}dx \;=\; \tau\frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\Phi_S(w+\tau+1;\tau)^{-1}\,. \tag{8.193}$$

8.11 The cocycle of the Pochhammer symbol

The cocycle of the Pochhammer symbol is a function defined for all $\gamma = [a, b; c, d] \in \mathrm{SL}_2(\mathbb{Z})$, $u \in \mathbb{C}$ and $\tau \in \mathbb{C}'_\gamma$. This cocycle is related to the modular quantum dilogarithm of [76]. We can express this function away from the reals by

$$\Phi_\gamma(u;\tau) \;=\; \frac{\left(\mathbf{e}\left(\frac{a\tau+b}{c\tau+d}\right)\mathbf{e}\left(\frac{u}{c\tau+d}\right);\mathbf{e}\left(\frac{a\tau+b}{c\tau+d}\right)\right)_\infty}{\left(\mathbf{e}\left(u\right);\mathbf{e}\left(\tau\right)\right)_\infty} \;=\; \frac{\left(\tilde{q}_\gamma\mathbf{e}\left(\frac{u}{c\tau+d}\right);\tilde{q}_\gamma\right)_\infty}{\left(\mathbf{e}\left(u\right);q\right)_\infty}, \tag{8.194}$$

where as always we use

$$(t;q^{-1})_\infty \;=\; (qt;q)_\infty^{-1}. \tag{8.195}$$

This satisfies difference equations analogous to those satisfied by Faddeev's quantum dilogarithm.

Proposition 24. *We have the following functional equations*

$$\Phi_\gamma(u+a\tau+b;\tau) \;=\; \frac{(\mathbf{e}(u);\mathbf{e}(\tau))_a}{1-\mathbf{e}\left(\frac{u}{c\tau+d}\right)}\Phi_\gamma(u;\tau) \tag{8.196}$$

$$\Phi_\gamma(u+c\tau+d;\tau) \;=\; (\mathbf{e}(u);\mathbf{e}(\tau))_c\Phi_\gamma(u;\tau) \tag{8.197}$$

$$\Phi_{[a,b;c,d]}\Big(\frac{u}{c\tau+a};-\frac{d\tau+b}{c\tau+a}\Big) \;=\; \Phi_{[d,b;c,a]}(u;\tau) \tag{8.198}$$

$$\Phi_{[a,b;c,d][a',b';c',d']}(u;\tau) \;=\; \Phi_{[a,b;c,d]}\Big(\frac{u}{c'\tau+d'};\frac{a'\tau+b'}{c'\tau+d'}\Big)\Phi_{[a',b';c',d']}(u;\tau) \tag{8.199}$$

We will show that this extends to a holomorphic function on $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $u \in \mathbb{C}$ and $\tau \in \mathbb{C}'_\gamma$. The fact that $\Phi_\gamma(u;\tau)$ extends to $\tau \in \mathbb{C}'_\gamma$ was first done in [58] in slightly less generality. This is equivalent to showing that $(x;q)_\infty$ is a quantum Jacobi form.

The proof that the Pochhammer is a quantum Jacobi form is done via a modification of the Abel-Plana summation formula. Indeed this was already used in Section 4.4. A similar argument was used in [29] to prove a modularity property of finite Pochhammers at roots of unity.

Theorem A–41. $\Phi_\gamma(u;\tau)$ *is holomorphic on* $\mathbb{C}'_\gamma$. *This is equivalent to proving that* $(\mathbf{e}(u);\mathbf{e}(\tau))_\infty$ *is a quantum Jacobi form with cocycle* $\Phi_\gamma(u;\tau)$.

This follows from Proposition 14 from [82]. We give the proof explicitly here as well.

Proof. Firstly, notice that we can write

$$\frac{a\tau+b}{c\tau+d} \;=\; \frac{ac\tau+bc}{c^2\tau+cd} \;=\; \frac{ac\tau+ad-1}{c^2\tau+cd} \;=\; \frac{a}{c}-\frac{1}{c^2\tau+cd} \tag{8.200}$$

Therefore

$$\frac{a\tau+b}{c\tau+d}(cn+r) \;=\; an - \frac{n}{c\tau+d} + \frac{a}{c}r - \frac{r}{c}\frac{1}{c\tau+d}. \tag{8.201}$$

Let $\gamma = [a,b;c,d] \in \Gamma$. Let $[ar]_c \in \{0, |c|-1\}$ with $[ar]_c = ar$ (Mod c) where we note that a is invertible modulo c. Notice that

$$\begin{aligned}
\frac{\left(\mathbf{e}\left(\frac{u}{c\tau+d}\right);\mathbf{e}\left(\frac{a\tau+b}{c\tau+d}\right)\right)_\infty}{(\mathbf{e}(u);\mathbf{e}(\tau))_\infty} &= \prod_{r=0}^{|c|-1}\frac{\left(\mathbf{e}\left(\frac{a\tau+b}{c\tau+d}r+\frac{u}{c\tau+d}\right);\mathbf{e}\left(\frac{a\tau+b}{c\tau+d}c\right)\right)_\infty}{(\mathbf{e}([ar]_c\tau+u);\mathbf{e}(c\tau))_\infty} \\
= \prod_{r=0}^{|c|-1}\frac{\left(\mathbf{e}\left(\frac{a}{c}r-\frac{r}{c}\frac{1}{c\tau+d}+\frac{u}{c\tau+d}\right);\mathbf{e}\left(-\frac{1}{c\tau+d}\right)\right)_\infty}{(\mathbf{e}([ar]_c\tau+u);\mathbf{e}(c\tau+d))_\infty} &= \prod_{r=0}^{|c|-1}\frac{\left(\mathbf{e}\left(\frac{[ar]_c}{c}-\frac{r}{c}\frac{1}{c\tau+d}+\frac{u}{c\tau+d}\right);\mathbf{e}\left(-\frac{1}{c\tau+d}\right)\right)_\infty}{(e([ar]_c\tau+u);\mathbf{e}(c\tau+d))_\infty} \\
= \prod_{r=0}^{|c|-1}\frac{\left(\mathbf{e}\left(\frac{[ar]_c c\tau+[ar]_c d-r}{c(c\tau+d)}+\frac{u}{c\tau+d}\right);\mathbf{e}\left(-\frac{1}{c\tau+d}\right)\right)_\infty}{(\mathbf{e}([ar]_c\tau+u);\mathbf{e}(c\tau+d))_\infty} & \\
= \prod_{r=0}^{|c|-1}\frac{\left(\mathbf{e}\left(\frac{[ar]_c\tau+d([ar]_c-ar)/c+b+u}{c\tau+d}\right);\mathbf{e}\left(-\frac{1}{c\tau+d}\right)\right)_\infty}{(\mathbf{e}([ar]_c\tau+d([ar]_c-ar)/c+b+u);\mathbf{e}(c\tau+d))_\infty} & \\
= \prod_{r=0}^{|c|-1}\Phi_S([ar]_c\tau+d([ar]_c-ar)/c+b+u;c\tau+d) &
\end{aligned} \tag{8.202}$$

Given that $\Phi_S(u;c\tau+d)$ is a holomorphic function for $\Re(c\tau+d) \in \mathbb{C}_S$ from Theorem 40. Notice that this is equivalent to $\tau \in \mathbb{C}'_\gamma$. □

8.12 The quantum dilogarithm at rationals

We have seen previously that the Faddeev quantum dilogarithm factors into a product of two Pochhammer symbols evaluated at q and $\tilde{q}$ or q^{-1} and $\tilde{q}^{-1}$. This factorisation gives an efficient formula when $\tau \notin \mathbb{R}$. It is natural to wonder whether a similar factorisation take place for $\tau \in \mathbb{Q}_{>0}$. Indeed, the values at $\tau \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$ are then all that is left and remain the mysterious glue that provide some of the depth captured by quantum modular forms. We will see that this is indeed true and related to the asymptotics of the Pochhammer symbol studied in Section 4.3. Firstly, we introduce an important function needed for the evlaution.

Definition 24. *We define the cyclic dilogarithm to be*

$$D_M(x;q) \;=\; \prod_{j=1}^{M-1}(1-q^j x)^{\frac{j}{M}}\,. \tag{8.203}$$

As always we take the principle branch of the roots. Notice that[2]

$$D_M(qx;q) \;=\; \prod_{j=1}^{M}(1-q^j x)^{\frac{j-1}{M}} \;=\; \frac{1-x}{(1-x^M)^{\frac{1}{M}}} D_M(x;q) \tag{8.204}$$

Using this function we have the following formula.

Theorem A–42. *[72, Thm. 1.9] For $\tau = N/M \in \mathbb{Q}$ we have*

$$\begin{aligned}\Phi_S(u;\tau) \;&=\; \exp\Big(\frac{1}{2\pi i NM}\mathrm{Li}_2(\mathbf{e}(Mu))\Big)(1-\mathbf{e}(Mu))^{\frac{u}{N}-1} D_N(\mathbf{e}(u/\tau);\tilde{q}^{-1}) D_M(\mathbf{e}(u);q)\\ &=\; \exp\Big(\frac{1}{2\pi i NM}\mathrm{Li}_2(\mathbf{e}(Mu))\Big)(1-\mathbf{e}(Mu))^{\frac{u}{N}} \prod_{j=0}^{N-1}(1-\tilde{q}^{-j}\mathbf{e}(u/\tau))^{\frac{j}{N}-\frac{1}{2}} \prod_{j=0}^{M-1}(1-q^j\mathbf{e}(u))^{\frac{j}{M}-\frac{1}{2}}\end{aligned} \tag{8.205}$$

Proof. Let $\tau = N/M \in \mathbb{Q}$, so $q^M = 1$ and $\tilde{q}^N = 1$. Then notice that for $\Im(u) > 0$ again pushing the contour to infinity and collecting the residues

$$\begin{aligned}&\exp\Big(-\int_{\sqrt{\tau}(i0+\mathbb{R})}\frac{\exp(uw/\tau)}{(1-\exp(w/\tau))(1-\exp(w))}\frac{dw}{w}\Big)\\ &= \exp\Big(-\sum_{k=1}^{\infty}\frac{\mathbf{e}(Mku)}{Nk}\Big(u-\frac{1}{2\pi i Mk}-\frac{1}{2}-\frac{\tau}{2}\Big)\\ &\qquad + \sum_{k=0}^{\infty}\sum_{n=1}^{N-1}\frac{\mathbf{e}((n+Nk)u/\tau)}{(1-\tilde{q}^{-(n+Nk)})(n+Nk)} + \sum_{\ell=0}^{\infty}\sum_{m=1}^{M-1}\frac{\mathbf{e}((m+M\ell)u)}{(1-q^{(m+M\ell)})(m+M\ell)}\Big)\\ &= \exp\Big(\frac{1}{2\pi i NM}\sum_{k=1}^{\infty}\frac{\mathbf{e}(Mu)^k}{k^2} - \frac{1}{N}\Big(u-\frac{1}{2}-\frac{\tau}{2}\Big)\sum_{k=1}^{\infty}\frac{\mathbf{e}(Mu)^k}{k}\\ &\qquad - \frac{1}{N}\sum_{k=0}^{\infty}\sum_{n=1}^{N-1}\frac{\mathbf{e}((n+Nk)u/\tau)}{n+Nk}\frac{(N-1)\tilde{q}^{-n-Nk}-N+\tilde{q}^{-n-Nk}}{(1-\tilde{q}^{-n-Nk})^2}\\ &\qquad - \frac{1}{M}\sum_{\ell=0}^{\infty}\sum_{m=1}^{M-1}\frac{\mathbf{e}((m+M\ell)u)}{m+M\ell}\frac{(M-1)q^{(m+M\ell)}-M+q^{m+M\ell}}{(1-q^{m+M\ell})^2}\Big)\end{aligned} \tag{8.206}$$

[2]We note that $\log(1-x^N) = \sum_{k=1}^{N}\log(1-\mathbf{e}(k/N)x)$ which can be proved noting that they are holomorphic and agree near $x=0$ and that they have the same branch cuts.

$$\begin{aligned}
&= \exp\Big(\frac{1}{2\pi i NM}\mathrm{Li}_2(\mathbf{e}(Mu)) + \Big(\frac{u}{N}-1\Big)\log(1-\mathbf{e}(Mu)) \\
&\qquad - \frac{1}{N}\sum_{k=1}^{\infty}\frac{\mathbf{e}(ku/\tau)}{k}\sum_{n=1}^{N-1} n\tilde{q}^{-kn} - \frac{1}{M}\sum_{\ell=1}^{\infty}\frac{\mathbf{e}(\ell u)}{\ell}\sum_{m=1}^{M-1} m q^{\ell m}\Big) \\
&= \exp\Big(\frac{1}{2\pi i NM}\mathrm{Li}_2(\mathbf{e}(Mu)) + \Big(\frac{u}{N}-1\Big)\log(1-\mathbf{e}(Mu)) \\
&\qquad + \sum_{n=1}^{N-1}\frac{n}{N}\log(1-\tilde{q}^{-n}\mathbf{e}(u/\tau)) - \sum_{m=1}^{M-1}\frac{m}{M}\log(1-q^m\mathbf{e}(u))\Big)
\end{aligned} \tag{8.207}$$

The function in the final line extends to an analytic function for $u \in \mathbb{C} - (\mathbb{Z}/M + i\mathbb{R}_{<0})$ and therefore as Φ_S extends to all $\mathbb{C}$ with poles contained in $\mathbb{Z}/M$ we see that the functions agree. □

This theorem is numerically verified in Code 29. Notice that

$$\mathrm{Li}_2(z) + \log(z)\log(1-z) \;=\; \frac{\pi^2}{6} - \mathrm{Li}_2(1-z)\,. \tag{8.208}$$

Therefore, on $\mathbf{e}(Mu) \in \mathbb{C} - \mathbb{R}_{\leq 0} - \mathbb{R}_{\geq 1}$ we have

$$\Phi_S(u;\tau) \;=\; \exp\Big(\frac{1}{2\pi i NM}\mathrm{Li}_2(1-\mathbf{e}(Mu)) - \frac{2\pi i}{24NM}\Big)\prod_{j=0}^{N-1}(1-\tilde{q}^{-j}\mathbf{e}(u/\tau))^{\frac{j}{N}-\frac{1}{2}}\prod_{j=0}^{M-1}(1-q^j\mathbf{e}(u))^{\frac{j}{M}-\frac{1}{2}} \tag{8.209}$$

Finally, we remark the following consequence on the q and $\tilde{q}$ difference equations when $\tau \in \mathbb{Q}$.

Corollary 11. *If $\tau = N/M \in \mathbb{Q}$ then we find that*

$$\Phi(u+N;\tau) \;=\; \Phi(u+M\tau;\tau) \;=\; (1-\mathbf{e}(Nu/\tau))\Phi(u;\tau) \;=\; (1-\mathbf{e}(Mu))\Phi(u;\tau)\,, \tag{8.210}$$

and

$$\begin{aligned}
\Phi_S(u+m\tau+n;\tau) \;&=\; \exp\Big(\frac{1}{2\pi i NM}\mathrm{Li}_2(\mathbf{e}(Mu))\Big)(1-\mathbf{e}(Mu))^{\frac{u}{N}} \\
&\times \prod_{j=0}^{N-1}(1-\tilde{q}^{-j-n}\mathbf{e}(u/\tau))^{\frac{j+n}{N}-\frac{1}{2}}\prod_{j=0}^{M-1}(1-q^{j+m}\mathbf{e}(u))^{\frac{j+m}{M}-\frac{1}{2}}\,.
\end{aligned} \tag{8.211}$$

Part V

State integrals and resurgence

An invitation to state integrals

State integrals originally arose as an attempt to defined mathematically $\mathrm{SL}_2(\mathbb{C})$ Chern–Simons theory [98]. However, they were then shown to factorise both as q–series [73] and as functions from roots of unity [72] without necessary regard for three–manifolds. This was then realised to give rise to proofs of a refined version of quantum modularity [85, 86]. They could seem complicated at first glance but they naturally appear as a continuous version of q–hypergeometric sums. The Pochhammer symbol is replaced by the Faddeev quantum dilogarithm, which is itself a cocycle associated to the Pochhammer. The factorisation into functions from roots of unity and q–series can then be computed with simple complex analysis. This means that for a larger class of examples there is a clear constructible method of proving a version of quantum modularity, which I hope to convey in this section. Finally, certain state integrals seem to numerically agree with Borel resummation of asymptotic series coming from q–hypergeometric functions. These numerically based conjectures, along with some basic structural conjectures, can then be used to give conjectural formulae for generating series of Stokes constants, which will be discussed in some simple examples.

Before, exploring state integrals more generally I will introduce an example that every undergraduate knowns. Indeed, the simplest state integral is just a Gaussian integral

$$\int_{\mathbb{R}} \mathbf{e}\Big(\frac{z^2}{2\tau}\Big) dz \;=\; \mathbf{e}(1/8)\sqrt{\tau}\,. \tag{8.212}$$

This integral with the contour above is convergent for $\Im(1/\tau) > 0$. However, altering the angle of the contour changes this range of convergence so that for any argument of τ we can find a convergent integral. This integral can be "factorised" when $\tau \in \mathfrak{h}$ as a product of q and $\tilde{q}$ series. Indeed, the modularity of the θ function implies that for $\vartheta_{00}(q) = \theta(-q^{-1/2}; q)$ we have

$$i\,\vartheta_{00}(\tilde{q})\vartheta_{00}(q)^{-1} \;=\; \mathbf{e}(1/8)\sqrt{\tau} \;=\; \int_{\mathbb{R}} \mathbf{e}\Big(\frac{z^2}{2\tau}\Big) dz\,. \tag{8.213}$$

In fact, the proof given in Section 7.2 of the modularity of the θ–function uses Poisson summation of Theorem 26, which exactly involves the evaluation of the Gaussian integral. This factorisation generalises but will be more complicated than just Poisson summation in general. Regardless, the fact that the Gaussian integral can easily be seen to extend to an analytic function when $\tau \in \mathbb{C} - \mathbb{R}_{\leq 0}$ gives a somewhat redundant proof that the quotient of the two θ functions extends to an analytic function. We can go further than this. We can also factorise this integral at rational numbers. In particular, for $\tau \in \mathbb{Q}$ letting

$$\vartheta_{00}(\tau) \;=\; \frac{1}{\sqrt{\mathrm{denom}(\tau)}} \sum_{k\in\mathbb{Z}/\mathrm{denom}(\tau)\mathbb{Z}} q^{k^2/2-k\,\mathrm{denom}(\tau)/2} \tag{8.214}$$

we see that

$$\sqrt{\tau}\,\mathbf{e}\Big(\frac{1}{8\,\mathrm{numer}(x)\mathrm{denom}(x)}\Big)\,\vartheta_{00}(\tilde{q})\,\vartheta_{00}(q)^{-1} \;=\; \mathbf{e}(1/8)\sqrt{\tau} \;=\; \int_{\mathbb{R}} \mathbf{e}\Big(\frac{z^2}{2\tau}\Big)dz \tag{8.215}$$

Therefore, we see that the quotient of these two Gauss sums with the tweaking also extends to an analytic function. Not only do the quotient of θ–functions and the tweaked quotient of Gauss sums extend to analytic functions, they extend to the same analytic function. This is the basic argument for proving quantum modularity and it will generalise to interesting examples.

After the Gaussian integral the next simplest example is given by the Mordell integral,

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathbf{e}(x^2\tau/2+ixz)}{\cosh(\pi x)}dx \;=\; -i\int_{i\mathbb{R}} \frac{\mathbf{e}(-x^2\tau/2+x(z+1/2))}{1-\mathbf{e}(x)}dx\,. \tag{8.216}$$

This was used by Zwegers in his description of mock modular forms [211]. This integral factorises in the upper half plane as a difference of Appell–Lerch sums given in Theorem 38. However, this also factorises in the lower half plane. Indeed, by shifting the contour with the argument of τ, the Mordell integral can be used to define a holomorphic function in $\mathbb{C}_S$. Letting

$$\vartheta(x;q) \;=\; \sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}x^k\,, \tag{8.217}$$

the Mordell integral in the lower half plane is given by

$$\sqrt{\frac{i}{\tau}}\mathbf{e}\Big(\frac{z^2}{2\tau}\Big)\mathbf{e}\Big(\frac{z}{2\tau}\Big)\tilde{q}^{-\frac{1}{8}}\vartheta\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big)+\mathbf{e}\Big(\frac{z}{2}\Big)q^{-\frac{1}{8}}\vartheta(z;\tau)\,, \tag{8.218}$$

and the proof is analogous to that in the upper half plane with the addition of checking boundary conditions as z and z/τ map to $i\infty$. Finally, this also factorises at rations described in Section 9.

As mentioned, state integrals for this thesis are an integral analogue of q–hypergeometric functions. These will roughly be integrals of the form

$$\int\cdots\int \mathbf{e}(Q(z)/2\tau+\mu(z)/\tau)\prod_j \Phi_S(\lambda_j(z);\tau)^{\pm}dz \tag{8.219}$$

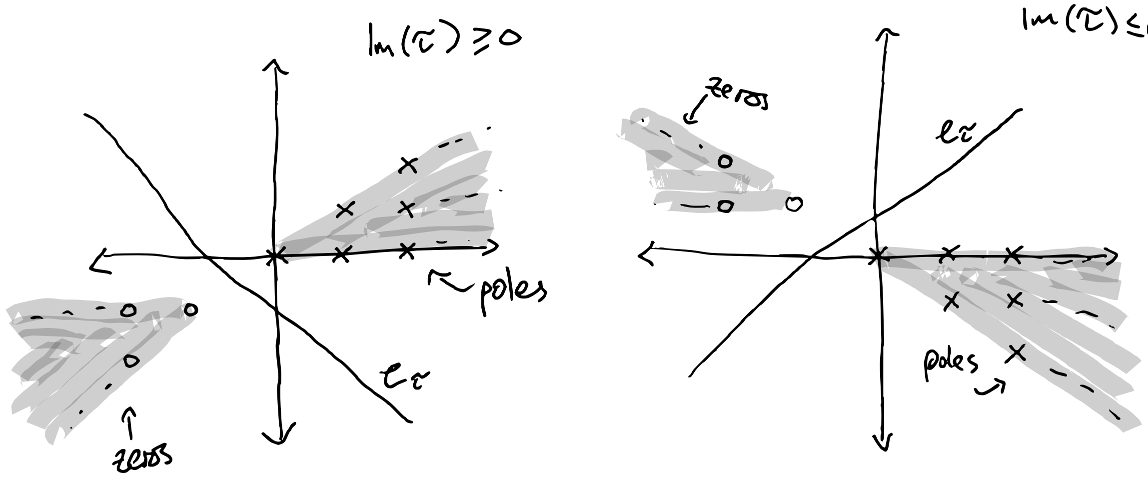

Figure 8.7: The contour $\mathcal{C}_\tau$ used for state integrals along with the usual poles and zeros of the integrand.

where $\lambda_j, \mu(z)$ are some linear forms, Q is a quadratic form and the integrals are performed over some infinite contour. Indeed, when it makes sense this integral will be related to the q–hypergeometric sum

$$\sum_{k} q^{Q(k)+\mu(k)} \prod_{j} (q;q)^{\pm}_{\lambda_j(k)} \,. \tag{8.220}$$

State integrals naturally satisfy an uncoupled system of q and $\tilde{q}$ difference equations and they behave like bilinear combinations of q–hypergeometric functions with $\tilde{q}$–hypergeometric functions. Moreover, they are defined for $\tau \in \mathbb{C}_S$. In the next sections, showing that they are bilinear combinations in the examples of interest will be the main content. Indeed, this is the main tool used to prove quantum modularity in examples. I want to stress that it is quite computational, however, it is all relatively simple complex analysis.

Finally, these state integrals with the notation of this these involve slightly more complicated contours than in previous work [8, 73]. To fix this we take the contour $\mathcal{C}_\tau = i\sqrt{\tau}\mathbb{R} - \sqrt{\tau}\epsilon$ for some small $\epsilon \in \mathbb{R}_{>0}$ depicted in Figure 8.7, which includes also the poles and zeros of $\Phi_S(z+1+\tau;\tau)^{-1}$.

Chapter 9

Factorisation of state integrals at rationals

To compute factorisations at rational numbers there is one fundamental tool. This is a simple lemma from complex analysis which has beautiful consequences. This lemma gives residue formulae for state integrals in terms of points on the characteristic variety. This then relates back to the points of stationary phase. This has similar consequences as the Duistermaat–Heckman theorem as interpreted by Atiyah and Bott so that the stationary phase gives exact results.

Lemma 21 (Fundamental lemma)**.** *[72, Lem. 2.1] If $U \subseteq \mathbb{C}$ is open, $U + a = U$ and $f : U \to \mathbb{C}$ is an analytic function such that for*

$$g(z) \;=\; \frac{f(z+a)}{f(z)} \tag{9.1}$$

we have

$$g(z+a) \;=\; g(z)\,, \tag{9.2}$$

then if γ is a contour such that $g(z) \neq 1$ on γ then

$$\int_\gamma f(z)\,dz \;=\; \Big(\int_\gamma - \int_{\gamma+a}\Big)\frac{f(z)}{1-g(z)}\,dz\,. \tag{9.3}$$

Proof.

$$\begin{aligned}\Big(\int_\gamma - \int_{\gamma+a}\Big)\frac{f(z)}{1-g(z)}\,dz \;&=\; \int_\gamma \frac{f(z)}{1-g(z)}\,dz - \int_{\gamma+a}\frac{f(z)}{1-g(z)}\,dz \\ =\; \int_\gamma \frac{f(z)}{1-g(z)}\,dz - \int_\gamma \frac{f(z+a)}{1-g(z)}\,dz \;&=\; \int_\gamma \frac{f(z)-f(z+a)}{1-g(z)}\,dz \;=\; \int_\gamma f(z)\,dz\,.\end{aligned} \tag{9.4}$$

□

We can use this to generalise to the analogous statement for integrals of many variables.

Corollary 12. *Suppose that $a \in \mathbb{C}^\times$, $U \subset \mathbb{C}$ is open, $U^n + a\mathbb{Z}^n = U^n$, $\gamma \subseteq U$ a contour, $f : U \to \mathbb{C}$ is an analytic function such that for some $a \in \mathbb{C}^\times$ and all $j = 1, \ldots n$ the functions*

$$g_j(z) \;=\; \frac{f(z_1, \ldots, z_j + a, \ldots, z_n)}{f(z_1, \ldots, z_n)} \tag{9.5}$$

are satisfy

$$g_j(z + a\mathbb{Z}) \;=\; g_j(z) \tag{9.6}$$

and $g_j(z) \neq 1$ on γ^n, then

$$\begin{aligned} &\int_\gamma \cdots \int_\gamma f(z_1, \ldots, z_n)\, dz_1 \ldots dz_n \\ &\qquad = \left(\int_\gamma - \int_{a+\gamma}\right) \cdots \left(\int_\gamma - \int_{a+\gamma}\right) \frac{f(z)\, dz_1 \ldots dz_n}{(1 - g_1(z)) \ldots (1 - g_n(z))} . \end{aligned} \tag{9.7}$$

Proof. Note that when $n = 1$ is the previous Lemma 21. So suppose that this is true to order $n - 1$. Then we find that

$$\begin{aligned} &\int_\gamma \cdots \int_\gamma f(z)\; dz_1 \ldots dz_{n-1} \\ &\qquad = \left(\int_\gamma - \int_{a+\gamma}\right) \cdots \left(\int_\gamma - \int_{a+\gamma}\right) \frac{f(z)\; dz_1 \ldots dz_{n-1}}{(1 - g_1(z)) \ldots (1 - g_{n-1}(z))} . \end{aligned} \tag{9.8}$$

Therefore, applying the $n = 1$ case we find

$$\begin{aligned} &\int_\gamma \cdots \int_\gamma \int_\gamma f(z)\; dz_1 \ldots dz_{n-1} dz_n \\ &= \left(\int_\gamma - \int_{a+\gamma}\right) \cdots \left(\int_\gamma - \int_{a+\gamma}\right) \int_\gamma \frac{f(z)\; dz_1 \ldots dz_{n-1} dz_n}{(1 - g_1(z)) \ldots (1 - g_{n-1}(z))} \\ &= \left(\int_\gamma - \int_{a+\gamma}\right) \cdots \left(\int_\gamma - \int_{a+\gamma}\right) \frac{f(z)\; dz_1 \ldots dz_{n-1} dz_n}{(1 - g_1(z)) \ldots (1 - g_{n-1}(z))} \\ &\qquad \times \frac{1}{1 - \frac{f(z_1,\ldots,z_n+a)(1-g_1(z))\ldots(1-g_{n-1}(z))}{(1-g_1(z_1,\ldots,z_{n-1},z_n+a))\ldots(1-g_{n-1}(z_1,\ldots,z_{n-1},z_n+a))f(z)}} \\ &= \left(\int_\gamma - \int_{a+\gamma}\right) \cdots \left(\int_\gamma - \int_{a+\gamma}\right) \frac{f(z)\; dz_1 \ldots dz_{n-1} dz_n}{(1 - g_1(z)) \ldots (1 - g_{n-1}(z))(1 - g_n(z))} . \end{aligned} \tag{9.9}$$

Therefore, by induction the result follows. ∎

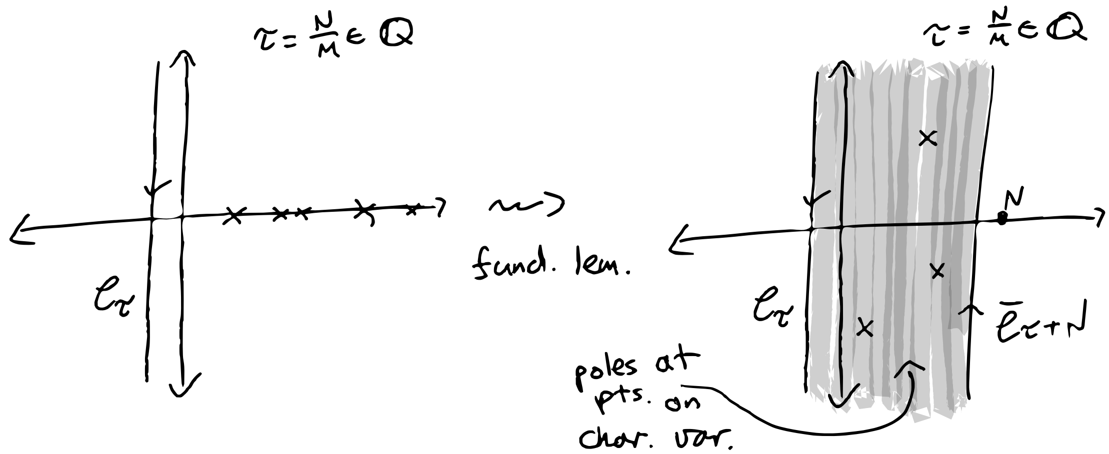

Figure 9.1: The affect of the fundamental lemma on state integrals.

The fundamental lemma has a beautiful effect on the pictures of the contours we wish to integrate as shown in Figure 9.1. As a simple warm up, we can apply this method to calculate the Mordell integral at rationals. We will calculate

$$\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(-z(z+1+\tau)/2\tau + zu/\tau))}{1-\mathbf{e}(z/\tau)} dz\,. \tag{9.10}$$

Suppose that $\tau = N/M \in \mathbb{Q}_{>0}$. Then we can use the fundamental lemma to factorise this state integral. Indeed, if

$$f(z) \;=\; \frac{\mathbf{e}(-z(z+1+\tau)/2\tau + zu/\tau)}{1-\mathbf{e}(z/\tau)} \tag{9.11}$$

then[1]

$$\frac{f(z+N)}{f(z)} \;=\; -\mathbf{e}(Mu - Mz) \;=\; \frac{f(z)}{f(z-N)}\,. \tag{9.12}$$

Therefore,

$$\begin{aligned} &\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(-z(z+1+\tau)/2\tau + zu/\tau))}{1-\mathbf{e}(z/\tau)} dz \\ &= \Big(\int_{\mathcal{C}_\tau} - \int_{\mathcal{C}_\tau+N}\Big) \frac{\mathbf{e}(-z(z+1+\tau)/2\tau + zu/\tau))}{1-\mathbf{e}(z/\tau)} \frac{1}{1+\mathbf{e}(Mu-Mz)}\, dz\,. \end{aligned} \tag{9.13}$$

[1]This follows as out of N, M and NM there is either one or three odd numbers as N and M can't both be even.

Therefore, we see that we have poles at $z \in \tau\mathbb{Z}$ and $z \in u + \frac{1}{M}(\mathbb{Z} + \frac{1}{2})$, which gives

$$\begin{aligned} &2\pi i \mathrm{Res}_{z=k\tau} \frac{\mathbf{e}(-z(z+1+\tau)/2\tau + zu/\tau))}{1-\mathbf{e}(z/\tau)} \frac{1}{1+\mathbf{e}(Mu - Mz)}\, dz \\ &= -\tau \frac{(-1)^k q^{-k(k+1)/2} \mathbf{e}(ku)}{1+\mathbf{e}(Mu)}\,. \end{aligned} \tag{9.14}$$

while

$$\begin{aligned} &2\pi i \mathrm{Res}_{z=u+\ell/M+1/2M} \frac{\mathbf{e}(-z(z+1+\tau)/2\tau + zu/\tau))}{1-\mathbf{e}(z/\tau)} \frac{1}{1+\mathbf{e}(Mu - Mz)}\, dz \\ &= \frac{-1}{M} \frac{\mathbf{e}(u(u-1-\tau)/2\tau + \ell(\ell+N+M+1)/2NM + M/4 + N/4 + NM/8}{1-\mathbf{e}(u/\tau + \ell/N + 1/2N)}\,. \end{aligned} \tag{9.15}$$

Then for the last sum we can use the Chinese remainder theorem to write the sum

$$\begin{aligned} &\sum_{\ell \in \mathbb{Z}/NM\mathbb{Z}} \frac{-1}{M} \frac{\mathbf{e}(u(u-1-\tau)/2\tau + \ell(\ell+N+M+1)/2NM + M/4 + N/4 + NM/8)}{1-\mathbf{e}(u/\tau + \ell/N + 1/2N)} \\ &= \sum_{k\in\mathbb{Z}/M\mathbb{Z}} \sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{1}{M} \frac{(-1)^k q^{k(k+1)/2+k/N} (-1)^\ell \tilde{q}^{-\ell(\ell+1)/2-\ell/M} \mathbf{e}(u(u-1-\tau)/2\tau + M/4 + N/4 + NM/8)}{1-\tilde{q}^{-\ell}\mathbf{e}(u/\tau + 1/2N)} \\ &= \sqrt{\tau}\mathbf{e}(u(u-1-\tau)/2\tau + NM/8) \sum_{k\in\mathbb{Z}/M\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2+k/2M+1/4M}}{\sqrt{M}} \sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{(-1)^\ell \tilde{q}^{-\ell(\ell+1)/2-\ell/2N-1/4N}}{\sqrt{N}(1-\tilde{q}^{-\ell}\mathbf{e}(u/\tau+1/2N))}\,. \end{aligned} \tag{9.16}$$

9.1 The case of Nahm sums

Proof of Theorem 6 for $A = 4$. Consider the following state integral

$$\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(2z^2/\tau + uz/\tau)}{\Phi_S(z;\tau)}\,. \tag{9.17}$$

Suppose that $\tau = N/M \in \mathbb{Q}_{>0}$. Then we can use the fundamental lemma to factorise this state integral. Indeed, if

$$f(z) \;=\; \frac{\mathbf{e}(2z^2/\tau + uz/\tau)}{\Phi_S(z;\tau)} \tag{9.18}$$

then

$$\frac{f(z+N)}{f(z)} \;=\; \frac{\mathbf{e}(4Mz + uM)}{(1-\mathbf{e}(Mz))} \;=\; \frac{f(z)}{f(z-N)}\,. \tag{9.19}$$

Therefore,

$$\begin{aligned} \int_{\mathcal{C}_\tau} \frac{\mathbf{e}(2z^2/\tau + uz/\tau)}{\Phi_S(z;\tau)} \;&=\; \Big(\int_{\mathcal{C}_\tau} - \int_{\mathcal{C}_\tau + N}\Big) \frac{\mathbf{e}(2z^2/\tau + uz/\tau)}{\Phi_S(z;\tau)} \frac{1}{1-\frac{\mathbf{e}(4Mz+uM)}{(1-\mathbf{e}(Mz))}}\, dz \\ &= \Big(\int_{\mathcal{C}_\tau} - \int_{\mathcal{C}_\tau + N}\Big) \frac{\mathbf{e}(2z^2/\tau + uz/\tau)}{\Phi_S(z;\tau)} \frac{1-\mathbf{e}(Mz)}{1-\mathbf{e}(Mz) - \mathbf{e}(4Mz+uM)}\, dz \end{aligned} \tag{9.20}$$

We see that the simple zeros of $\Phi_S(z;\tau)$ cancel with the zeros of $1-\mathbf{e}(Mz)$ and therefore in this strip enclosed by the contour we see that all the poles come from solutions to the equation

$$1-\mathbf{e}(Mz) \;=\; \mathbf{e}(4Mz+uM)\,, \tag{9.21}$$

which is the Nahm equation (4.106), which determined the critical points with which we applied stationary phase. Firstly, lets set $u\in\mathbb{Z}+\tau\mathbb{Z}$ so that the equation becomes

$$1-\mathbf{e}(Mz) \;=\; \mathbf{e}(4Mz)\,, \tag{9.22}$$

This has the four solutions for $\mathbf{e}(Mz)$ given by X_i from Example 39. Therefore, we see that the residues that contribute are given by the points $z=\frac{1}{M}x_{i,k}$ such that $0<\Re(z)\le N$. Therefore, let us calculate the residue at one of these points. We have

$$\begin{aligned}
&2\pi i\mathrm{Res}_{z=\frac{1}{M}x_{i,k}}\frac{\mathbf{e}(2z^2/\tau+uz/\tau)}{\Phi_S(z;\tau)}\frac{1-\mathbf{e}(Mz)}{1-\mathbf{e}(Mz)-\mathbf{e}(4Mz+uM)}\,dz\\
&=\;2\pi i\mathrm{Res}_{z=0}\frac{\mathbf{e}(2z^2/\tau+4zx_{i,k}/N+2x_{i,k}^2/MN+ux_{i,k}/N+uz/\tau)}{\Phi_S(x_{i,k}/M+z;\tau)}\\
&\qquad\qquad\times\frac{1-X_i\mathbf{e}(Mz)}{1-X_i\mathbf{e}(Mz)-X_i^4\mathbf{e}(4Mz+uM)}\,dz\\
&=\;\frac{\mathbf{e}(2x_{i,k}^2/MN+ux_{i,k}/N)}{\Phi_S(x_{i,k}/M;\tau)}\frac{1-X_i}{-MX_i-4MX_i^4\mathbf{e}(uM)}\\
&=\;\frac{\mathbf{e}(2x_{i,k}^2/MN+ux_{i,k}/N)}{\exp\left(\frac{1}{2\pi iNM}\mathrm{Li}_2(X_i)\right)(1-X_i)^{\frac{x_{i,k}}{MN}-1}D_N(\mathbf{e}(x_{i,k}/N);\tilde{q}^{-1})D_M(\mathbf{e}(x_{i,k}/M);q)}\\
&\qquad\qquad\times\frac{1-X_i}{-MX_i-4MX_i^4\mathbf{e}(uM)}
\end{aligned} \tag{9.23}$$

We will break the simplification down into pieces. First notice that from equation (4.120) we have $\frac{1}{2\pi i}\log(1-X_i)=4x_{i,k}-4k+a_i$ where

$$\begin{aligned}
\frac{1}{2\pi i}\log(1-X_1) &= 4x_{1,k}-4k-2\,,\\
\frac{1}{2\pi i}\log(1-X_2) &= 4x_{2,k}-4k\,,\\
\frac{1}{2\pi i}\log(1-X_3) &= 4x_{3,k}-4k-1\,,\\
\frac{1}{2\pi i}\log(1-X_4) &= 4x_{4,k}-4k+1\,.
\end{aligned} \tag{9.24}$$

Therefore, we see that the product

$$\begin{aligned} \frac{\mathbf{e}(2x_{i,k}^2/NM)}{\exp\Big(\frac{1}{2\pi i NM}\mathrm{Li}_2(X_i)\Big)(1-X_i)^{\frac{x_{i,k}}{NM}}} &= \mathbf{e}\Big(-\frac{\mathrm{Li}_2(X_i)}{(2\pi i)^2 NM} - (4x_{i,k}-4k+a_i)\frac{x_{i,k}}{NM} + \frac{2x_{i,k}^2}{NM}\Big) \\ &= \mathbf{e}\Big(-\frac{\mathrm{Li}_2(X_i) - \frac{\pi^2}{6} + 2(2\pi i)^2 x_{i,k}^2 - (2\pi i)^2(4k-a_i)x_{i,k}}{(2\pi i)^2 NM} - \frac{\frac{\pi^2}{6}}{(2\pi i)^2 NM}\Big) \\ &= \mathbf{e}\Big(-\frac{\mathrm{VC}_{i,k}}{(2\pi i)^2 NM} + \frac{1}{24NM}\Big) = \mathbf{e}\Big(-\frac{\mathrm{VC}_{i,0}}{(2\pi i)^2 NM} + \frac{k(2k-a_i)}{NM} + \frac{1}{24NM}\Big)\,. \end{aligned} \tag{9.25}$$

Then we have terms

$$D_N(\mathbf{e}(x_{i,k}/N);\tilde{q}^{-1}) = \prod_{j=1}^{N-1}(1-\mathbf{e}(k/N)\tilde{q}^{-j}X_i^{1/N})^{j/N}\,, \tag{9.26}$$

and

$$D_M(\mathbf{e}(x_{i,k}/M);q) = \prod_{j=1}^{N-1}(1-\mathbf{e}(k/M)q^{j}X_i^{1/M})^{j/M}\,. \tag{9.27}$$

Finally notice that

$$\frac{1-X_i}{-MX_i-4MX_i^4\mathbf{e}(uM)} = \frac{-1}{X_i/(1-X_i)+4}\frac{\sqrt{\tau}}{\sqrt{M}\sqrt{N}}\,. \tag{9.28}$$

Using the Chinese remainder theorem we can split the sum over k of residues into two decoupled sums

$$\begin{aligned} &\sum_{k\in\mathbb{Z}/MN\mathbb{Z}} \frac{\mathbf{e}((2k^2-a_ik)/NM + ux_{i,k}/N)(1-X_i)}{\prod_{j=1}^{N-1}(1-\mathbf{e}(k/N)\tilde{q}^{-j}X_i^{1/N})^{j/N}\prod_{j=1}^{N-1}(1-\mathbf{e}(k/M)q^jX_i^{1/M})^{j/M}} \\ &= \sum_{k\in\mathbb{Z}/M\mathbb{Z}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{q^{2k^2}\tilde{q}^{-2\ell^2}\mathbf{e}(-ka_i/M)\mathbf{e}(-\ell a_i/N)\mathbf{e}(ux_{i,0}/N)\mathbf{e}(ku)\mathbf{e}(\ell u/\tau)}{\prod_{j=1}^{N-1}(1-\tilde{q}^{-\ell-j}X_i^{1/N})^{j/N-1/2}\prod_{j=1}^{N-1}(1-q^{k+j}X_i^{1/M})^{j/M-1/2}} \\ &= \sum_{k\in\mathbb{Z}/M\mathbb{Z}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{q^{2k^2}\tilde{q}^{-2\ell^2}X_i^{4k/M}X_i^{4\ell/N}\mathbf{e}(ux_{i,0}/N)\mathbf{e}(ku)\mathbf{e}(\ell u/\tau)}{\prod_{j=1}^{N-1}(1-\tilde{q}^{-\ell-j}X_i^{1/N})^{j/N}(1-X_i)^{\ell/N-1/2}\prod_{j=1}^{N-1}(1-q^{k+j}X_i^{1/M})^{j/M}(1-X_i)^{k/M-1/2}} \\ &= \sum_{k\in\mathbb{Z}/M\mathbb{Z}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{q^{2k^2}\tilde{q}^{-2\ell^2}X_i^{4k/M}X_i^{4\ell/N}\mathbf{e}(ux_{i,0}/N)\mathbf{e}(ku)\mathbf{e}(\ell u/\tau)}{\prod_{j=0}^{N-1}(1-\tilde{q}^{-\ell-j}X_i^{1/N})^{(j+\ell)/N-1/2}\prod_{j=0}^{N-1}(1-q^{k+j}X_i^{1/M})^{(j+k)/M-1/2}} \\ &= \mathbf{e}(ux_{i,0}/N)\sum_{k\in\mathbb{Z}/M\mathbb{Z}} \frac{q^{2k^2}X_i^{4k/M}\mathbf{e}(ku)}{\prod_{j=0}^{N-1}(1-q^{k+j}X_i^{1/M})^{(j+k)/M-1/2}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{\tilde{q}^{-2\ell^2}X_i^{4\ell/N}\mathbf{e}(\ell u/\tau)}{\prod_{j=0}^{N-1}(1-\tilde{q}^{-\ell-j}X_i^{1/N})^{(j+\ell)/N-1/2}} \\ &= \mathbf{e}((u+4\tau+4)x_{i,0}/N)q^2\tilde{q}^{-2}\sum_{k\in\mathbb{Z}/M\mathbb{Z}} \frac{q^{2k^2}X_i^{4k/M}\mathbf{e}(k(u+4\tau+4))}{\prod_{j=0}^{N-1}(1-q^{k+j+1}X_i^{1/M})^{(j+k+1)/M-1/2}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{\tilde{q}^{-2\ell^2}X_i^{4\ell/N}\mathbf{e}(\ell(u+4\tau+4)/\tau)}{\prod_{j=0}^{N-1}(1-\tilde{q}^{-\ell-j-1}X_i^{1/N})^{(j+\ell+1)/N-1/2}} \end{aligned} \tag{9.29}$$

Then noting the identities of Proposition 7 completes the proof for $A=4$. For $A\in\mathbb{Z}$ starting with the state integral

$$\int_{\mathcal{C}_\tau}\frac{\mathbf{e}(Az(z+1+\tau)/2\tau + uz/\tau)}{\Phi_S(z+1+\tau;\tau)} \tag{9.30}$$

and applying exactly the same methods we used for the $A = 4$ example will produce the desired result. □

9.2 The case of the figure eight knot

In [70] a state integral was introduced associated to the figure eight knot. This was shown to factor in terms of the two variable series of [92]. However, the factorisation at rationals has not explicitly computed although was done at $\tau = 1$ in [82]. We will include the computation here which finalises the full proofs of quantum modularity of the various invariants of 4_1.

Consider the state integral

$$\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz\,. \tag{9.31}$$

and assume for these computations that $u \in \mathbb{Z} + \tau\mathbb{Z}$. Taking the integrand

$$f(z) \;=\; \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2}\,, \tag{9.32}$$

we have

$$\frac{f(z+N)}{f(z)} \;=\; -\frac{\mathbf{e}(Mz+uM)}{(1-\mathbf{e}(Mz))^2} \;=\; \frac{f(z)}{f(z-N)}\,. \tag{9.33}$$

The equation

$$1 \;=\; -\frac{X}{(1-X)^2} \tag{9.34}$$

has solutions

$$X_j \;=\; \frac{1}{2} + (-1)^j\frac{1}{2}\sqrt{-3}\,. \tag{9.35}$$

Therefore, from the fundamental Lemma 21 the state integral above is given by

$$\Big(\int_{\mathcal{C}_\tau} - \int_{\mathcal{C}_\tau+N}\Big)\frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2}\frac{(1-\mathbf{e}(Mz))^2}{(1-\mathbf{e}(Mz))^2+\mathbf{e}(Mz+uM)}dz\,. \tag{9.36}$$

The integrand has poles in the set $\mathbf{e}(Mz) = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$ and $z \in \tau\mathbb{Z}$. Then letting $x_{j,k} =$

$\frac{1}{2\pi i}\log(X_j)+k$ we can compute the residue at $x_{j,k}/M$ as

$$\begin{aligned}
&2\pi i\mathrm{Res}_{z=\frac{1}{M}x_{j,k}}\frac{\mathbf{e}(z(z+1+\tau)/2\tau+uz/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2}\frac{(1-\mathbf{e}(Mz))^2}{(1-\mathbf{e}(Mz))^2-\mathbf{e}(Mz+uM)}\,dz\\
&= 2\pi i\mathrm{Res}_{z=0}\frac{\mathbf{e}(z(z+1+\tau)/2\tau+zx_{j,k}/N+x_{j,k}^2/2MN+(u+1/2+\tau/2)x_{j,k}/N+uz/\tau)}{(1-\mathbf{e}(x_{j,k}/N+z/\tau))\Phi_S(x_{j,k}/M+z+1+\tau;\tau)^2}\\
&\qquad\times\frac{(1-X_i\mathbf{e}(Mz))^2}{(1-X_j\mathbf{e}(Mz))^2+X_j\mathbf{e}(Mz+uM)}\,dz\\
&= \frac{\mathbf{e}(x_{j,k}^2/2MN+(u+1/2+\tau/2)x_{j,k}/N)}{(1-\mathbf{e}(x_{j,k}/N)\Phi_S(x_{j,k}/M+1+\tau;\tau)^2}\frac{(1-X_j)^2}{M(2X_j^2+(\mathbf{e}(uM)-2)X_j)}\\
&= \frac{\mathbf{e}(x_{j,k}^2/2MN+(u+1/2+\tau/2)x_{j,k}/N)}{\exp\big(\frac{2}{2\pi iNM}\mathrm{Li}_2(X_j)\big)(1-X_j)^{2\frac{x_{j,k}}{MN}-2+\frac{2}{M}+\frac{2}{N}}D_N(\mathbf{e}(x_{j,k}/N);\tilde{q}^{-1})^2D_M(\mathbf{e}(x_{j,k}/M);q)^2}\\
&\qquad\times\frac{(1-X_j)^2}{M(1-\mathbf{e}(x_{j,k}/N))(2X_j^2+(\mathbf{e}(uM)-2)X_j)}\,.
\end{aligned}\tag{9.37}$$

Noting that

$$\frac{1}{2\pi i}\log((1-X_j)^2) \;=\; x_{j,k}-k+a_j\,,\tag{9.38}$$

where $a_j=(-1)^j/2$, we find that

$$\begin{aligned}
&\frac{\mathbf{e}(x_{j,k}^2/2NM)}{\exp\Big(\frac{2}{2\pi iNM}\mathrm{Li}_2(X_j)\Big)(1-X_j)^{2\frac{x_{j,k}}{NM}}} \;=\; \mathbf{e}\Big(-\frac{2\mathrm{Li}_2(X_j)}{(2\pi i)^2NM}-(x_{j,k}-k+a_j)\frac{x_{j,k}}{NM}+\frac{x_{j,k}^2}{2NM}\Big)\\
&\;=\; \mathbf{e}\Big(-\frac{2\mathrm{Li}_2(X_j)-\frac{\pi^2}{3}+(2\pi i)^2\frac{x_{j,k}^2}{2}-(2\pi i)^2(k-a_j)x_{j,k}}{(2\pi i)^2NM}-\frac{\frac{\pi^2}{3}}{(2\pi i)^2NM}\Big)\\
&\;=\; \mathbf{e}\Big(-\frac{\mathrm{VC}_{j,k}}{(2\pi i)^2NM}+\frac{1}{12NM}\Big) \;=\; \mathbf{e}\Big(-\frac{\mathrm{VC}_{i,0}}{(2\pi i)^2NM}+\frac{k(k/2-a_j))}{NM}+\frac{1}{12NM}\Big)\,.
\end{aligned}\tag{9.39}$$

Using the Chinese remainder theorem we can split the sum over k of residues into two decoupled sums

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}/MN\mathbb{Z}} \frac{\mathbf{e}\big((k^2/2-a_ik)/NM+(u+1/2+\tau/2)x_{i,k}/N\big)}{(1-X_i^{1/N}\mathbf{e}(k/N))\prod_{j=1}^{N-1}(1-\mathbf{e}(k/N)\tilde{q}^{-j-1}X_i^{1/N})^{2(j+1)/N-1}\prod_{j=1}^{N-1}(1-\mathbf{e}(k/M)q^{j+1}X_i^{1/M})^{2(j+1)/M-1}}\\
&= \sum_{k\in\mathbb{Z}/M\mathbb{Z}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{(-1)^kq^{k(k+1)/2}(-1)^\ell\tilde{q}^{-\ell(\ell+1)/2}\mathbf{e}(-ka_i/M)\mathbf{e}(-\ell a_i/N)\mathbf{e}((u+1/2+\tau/2)x_{i,0}/N)\mathbf{e}(ku)\mathbf{e}(\ell u/\tau)}{(1-\tilde{q}^{-\ell}X_i^{1/N})\prod_{j=1}^{N-1}(1-\tilde{q}^{-1-\ell-j}X_i^{1/N})^{2(j+1)/N-1}\prod_{j=1}^{N-1}(1-q^{1+k+j}X_i^{1/M})^{2(j+1)/M-1}}\\
&= \sum_{k\in\mathbb{Z}/M\mathbb{Z}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{(-1)^kq^{k(k+1)/2}(-1)^\ell\tilde{q}^{-\ell(\ell+1)/2}X_i^{k/M}X_i^{\ell/N}\mathbf{e}((u+1/2+\tau/2)x_{i,0}/N)\mathbf{e}(ku)\mathbf{e}(\ell u/\tau)}{(1-\tilde{q}^{-\ell}X_i^{1/N})\prod_{j=1}^{N-1}(1-\tilde{q}^{-1-\ell-j}X_i^{1/N})^{2(1+j+\ell)/N-1}\prod_{j=1}^{N-1}(1-q^{1+k+j}X_i^{1/M})^{2(1+j+k)/M-1}}\\
&= \sum_{k\in\mathbb{Z}/M\mathbb{Z}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{(-1)^kq^{k(k+1)/2}(-1)^\ell\tilde{q}^{-\ell(\ell+1)/2}X_i^{k/M}X_i^{1/2M}X_i^{\ell/N}X_i^{1/2N}\mathbf{e}(ux_{i,0}/N)\mathbf{e}(ku)\mathbf{e}(\ell u/\tau)}{(1-\tilde{q}^{-\ell}X_i^{1/N})\prod_{j=0}^{N-1}(1-\tilde{q}^{-1-\ell-j}X_i^{1/N})^{2(1+j+\ell)/N-1}\prod_{j=0}^{N-1}(1-q^{1+k+j}X_i^{1/M})^{2(1+j+k)/M-1}}\\
&= \mathbf{e}(ux_{i,0}/N)\sum_{k\in\mathbb{Z}/M\mathbb{Z}} \frac{(-1)^kq^{k(k+1)/2}X_i^{k/M}X_i^{1/2M}\mathbf{e}(ku)}{\prod_{j=0}^{N-1}(1-q^{1+k+j}X_i^{1/M})^{2(1+j+k)/M-1}}\sum_{\ell\in\mathbb{Z}/N\mathbb{Z}} \frac{(-1)^\ell\tilde{q}^{-\ell(\ell+1)/2}X_i^{\ell/N}X_i^{1/2N}\mathbf{e}(\ell u/\tau)}{(1-\tilde{q}^{-\ell}X_i^{1/N})\prod_{j=0}^{N-1}(1-\tilde{q}^{-1-\ell-j}X_i^{1/N})^{2(1+j+\ell)/N-1}}\,.
\end{aligned} \tag{9.40}$$

Note these Appell–Lerch type sums were first recognised in this context in [70] where they arise in the q–series versions. Now there is a final collection of residues at $z \in \tau\mathbb{Z}$. Here we find that

$$\begin{aligned}
&2\pi i\mathrm{Res}_{z=k\tau}\frac{\mathbf{e}(z(z+1+\tau)/2\tau+uz/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2}\frac{(1-\mathbf{e}(Mz))^2}{(1-\mathbf{e}(Mz))^2-\mathbf{e}(Mz+uM)}\,dz\\
&= 2\pi i\mathrm{Res}_{z=0}\frac{(-1)^kq^{k(k+1)/2}\mathbf{e}(ku)\mathbf{e}(z(z+1+\tau)/2\tau+kz+uz/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+(k+1)\tau;\tau)^2}\frac{(1-\mathbf{e}(Mz))^2}{(1-\mathbf{e}(Mz))^2-\mathbf{e}(Mz+uM)}\,dz\\
&= 2\pi i\mathrm{Res}_{z=0}\frac{(-1)^kq^{k(k+1)/2}\mathbf{e}(ku)\mathbf{e}(z(z+1+\tau)/2\tau+kz+uz/\tau)\exp\Big(-\frac{2}{2\pi iNM}\mathrm{Li}_2(\mathbf{e}(Mz))\Big)(1-\mathbf{e}(Mz))^{-2\frac{z}{N}}(1-\mathbf{e}(Mz))^2}{(qe(z);q)_k(1-\mathbf{e}(z/\tau))\prod_{j=0}^{N-1}(1-\tilde{q}^{-1-j}\mathbf{e}(z/\tau))^{2\frac{1+j}{N}-1}\prod_{j=0}^{M-1}(1-q^{1+j}\mathbf{e}(z))^{2\frac{1+j}{M}-1}((1-\mathbf{e}(Mz))^2-\mathbf{e}(Mz+uM))}\,dz\\
&= 2\pi i\mathrm{Res}_{z=0}\frac{(-1)^kq^{k(k+1)/2}\mathbf{e}(ku)\mathbf{e}(z(z+1+\tau)/2\tau+kz+uz/\tau)\exp\Big(-\frac{2}{2\pi iNM}\mathrm{Li}_2(\mathbf{e}(Mz))\Big)(1-\mathbf{e}(Mz))^{-2\frac{z}{N}}(1-\mathbf{e}(Mz))^2}{(qe(z);q)_k^2(1-\mathbf{e}(z/\tau))^2(1-\mathbf{e}(z))\prod_{j=1}^{N-1}(1-\tilde{q}^{-j}\mathbf{e}(z/\tau))^{2\frac{j}{N}-1}\prod_{j=1}^{M-1}(1-q^{j}\mathbf{e}(z))^{2\frac{j}{M}-1}((1-\mathbf{e}(Mz))^2-\mathbf{e}(Mz+uM))}\,dz\\
&= -\frac{(-1)^kq^{k(k+1)/2}\mathbf{e}(ku)\mathbf{e}(-\frac{1}{12NM})\varepsilon(\tilde{q}^{-1})^2\varepsilon(q)^2}{(q;q)_k^2}\frac{N^2}{\mathbf{e}(uM)}\,,
\end{aligned} \tag{9.41}$$

where we use the multiplier system of the Dedekind–η function ε from equation (8.8). Then using the identity

$$\sum_{k=0}^{M-1}(-1)^kq^{-k(k+1)/2}(q;q)_k^2\mathbf{e}(-(1+k)u) \;=\; M^2\sum_{k=0}^{M-1}(-1)^k\frac{q^{k(k+1)/2}}{(q;q)_k^2}\mathbf{e}((k-M)u)\,, \tag{9.42}$$

we see that summing over the residues at $\tau\mathbb{Z}$ gives

$$-\mathbf{e}\Big(-\frac{1}{12NM}\Big)\varepsilon(\tilde{q}^{-1})^2\varepsilon(q)^2\tau^2\sum_{k=0}^{M-1}(-1)^kq^{-k(k+1)/2}(q;q)_k^2\mathbf{e}(-(k+1)u)\,. \tag{9.43}$$

Therefore, from identities similar to those from [70, Thm. 4] this completes the proof.

9.3 A toy explanatory example

Consider, the element of the Habiro ring

$$\sum_{k=0}^{\infty} q^{-k(k+1)/2}(q;q)_{2k}\,. \tag{9.44}$$

We can use Lemma 17 to show that for $q=\mathbf{e}(N/M)$ that this element of the Habiro ring is given by

$$\sum_{k=0}^{M/2-1}(-1)^k q^{-k(k+1)/2}(q;q)_{2k+1} \;=\; -M\sum_{k=M/2}^{M-1}(-1)^k\frac{q^{k(3k+1)/2}}{(q;q)_{2k-M}}\,, \tag{9.45}$$

Then taking the state integral

$$\int_{\mathcal{C}_\tau}\frac{\mathbf{e}(z(3z+1+\tau)/2\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(2z+2+2\tau;\tau)}dz\,, \tag{9.46}$$

we see that taking the integrand

$$f(z) \;=\; \frac{\mathbf{e}(z(3z+1+\tau)/2\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(2z+2+2\tau;\tau)}\,, \tag{9.47}$$

we have

$$\frac{f(z+N)}{f(z)} \;=\; -\frac{\mathbf{e}(3Mz)}{(1-\mathbf{e}(2Mz))^2} \;=\; \frac{f(z)}{f(z-N)}\,. \tag{9.48}$$

Therefore, from the fundamental Lemma 21 the state integral above is given by

$$\Big(\int_{\mathcal{C}_\tau}-\int_{\mathcal{C}_\tau+N}\Big)\frac{\mathbf{e}(z(3z+1+\tau)/2\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(2z+2+2\tau;\tau)}\frac{(1-\mathbf{e}(2Mz))^2}{(1-\mathbf{e}(2Mz))^2+\mathbf{e}(3Mz)}dz\,. \tag{9.49}$$

Notice that this integrand has poles from the equation

$$1 \;=\; -\frac{\mathbf{e}(3Mz)}{(1-\mathbf{e}(2Mz))^2}\,, \tag{9.50}$$

and addition it has singularities at $k\in\mathbb{Z}\tau$ however for $0\le k<M/2$ these singularities a removable. Therefore, going through a similar computation as we did for the figure eight knot ending in equation 9.41, we find that this state integral factorises with a term a multiple of

$$\sum_{k=M/2}^{M-1}(-1)^k\frac{q^{k(3k+1)/2}}{(q;q)_{2k-M}}\,. \tag{9.51}$$

Importantly, the $(q;q)_{2k-M}$ appears from the periodicity of the Faddeev dilogarithm and the non–removable singularities appearing for $M/2\le k<M$.

9.4 The WRT invariant of half surgery on the figure eight

We can take the formula for the WRT invariant

$$\sum_{0\le\ell\le k}(-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)+mk}\frac{(q;q)_{2k+1}}{(q;q)_\ell(q;q)_{k-\ell}} \tag{9.52}$$

and naturally take the state integral

$$\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau}\frac{\mathbf{e}(-z_1(z_1+1+\tau)/2\tau+z_3(z_3+1+\tau)/\tau+z_1(m+m'/\tau))\Phi_S(2z_1+2+2\tau;\tau)}{\Phi_S(z_3+1+\tau;\tau)\Phi_S(z_1-z_3+1+\tau;\tau)}dz_1dz_3 \tag{9.53}$$

We can rewrite this integral by first sending $z_1\mapsto -z_1$, $z_3\mapsto -z_3$ and using Proposition 22. Then using Proposition 23 to expand the Faddeev dilogarithm in $\Phi_S(z_3+1+\tau;\tau)$, and then again to contract the integral over z_3. With these steps one can show to up to a factor of $q^m\tilde{q}^{-m}$ this integral is given by

$$\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau}\frac{\mathbf{e}(z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)}dz_1dz_2\,. \tag{9.54}$$

This integral expression has various benifits and in particular has a direction at infinity where the two dimensional contour can be pushed with vanishing contribution. However, we do not need this for the factorisation at rationals. Therefore, suppose that $\tau=N/M\in\mathbb{Q}_{>0}$. Then we can use the fundamental lemma to factorise this state integral. Indeed, if

$$\begin{aligned} f_1(z_1,z_2) &= \frac{\mathbf{e}((z_1+N)^2/\tau+(z_1+N)z_2/\tau+(z_1+N)(-m-m'/\tau)+z_2(1+1/\tau))}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2(z_1+N)+1+\tau;\tau)\Phi_S(z_2-(z_1+N)+1+\tau;\tau)}\\ &\qquad\times\frac{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)}{\mathbf{e}(z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))}\\ &= \frac{e(2Mz_1+Mz_2)(1-\mathbf{e}(-Mz_1+Mz_2))}{(1-\mathbf{e}(2Mz_1))^2}\,. \end{aligned} \tag{9.55}$$

then

$$\begin{aligned} f_2(z_1,z_2) &= \frac{\mathbf{e}(z_1^2/\tau+z_1(z_2+N)/\tau+z_1(-m-m'/\tau)+(z_2+N)(1+1/\tau))}{\Phi_S((z_2+N)+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S((z_2+N)-z_1+1+\tau;\tau)}\\ &\qquad\times\frac{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)}{\mathbf{e}(z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))}\\ &= \frac{\mathbf{e}(Mz_1)}{(1-\mathbf{e}(Mz_2))(1-\mathbf{e}(-Mz_1+Mz_2))}\,. \end{aligned} \tag{9.56}$$

Notice that

$$f_j(z_1 + kN, z_2 + \ell N) \;=\; f_j(z_1, z_2)\,, \tag{9.57}$$

and therefore,

$$\begin{aligned}
&\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z_1^2/\tau + z_1 z_2/\tau + z_1(-m - m'/\tau) + z_2(1+1/\tau))}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)} dz_1 dz_2 \\
&= \frac{\left(\int_{\mathcal{C}_\tau} - \int_{0+N+i\mathbb{R}}\right)^2 \mathbf{e}(z_1^2/\tau + z_1 z_2/\tau + z_1(-m - m'/\tau) + z_2(1+1/\tau)) dz_1 dz_2}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)\big(1 - \frac{\mathbf{e}(2Mz_1+Mz_2)(1-\mathbf{e}(-Mz_1+Mz_2))}{(1-\mathbf{e}(2Mz_1))^2}\big)\big(1 - \frac{\mathbf{e}(Mz_1)}{(1-\mathbf{e}(Mz_2))(1-\mathbf{e}(-Mz_1+Mz_2))}\big)} \\
&= \Big(\int_{\mathcal{C}_\tau} - \int_{\mathcal{C}_\tau+N}\Big)^2 \frac{\mathbf{e}(z_1^2/\tau + z_1 z_2/\tau + z_1(-m - m'/\tau) + z_2(1+1/\tau))}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)} \\
&\qquad \times \frac{(1-\mathbf{e}(2Mz_1))^2(1-\mathbf{e}(Mz_2))(1-\mathbf{e}(-Mz_1+Mz_2)) dz_1 dz_2}{\big((1-\mathbf{e}(2Mz_1))^2 - \mathbf{e}(2Mz_1+Mz_2)(1-\mathbf{e}(-Mz_1+Mz_2))\big)\big((1-\mathbf{e}(Mz_2))(1-\mathbf{e}(-Mz_1+Mz_2)) - \mathbf{e}(Mz_1)\big)}\,.
\end{aligned} \tag{9.58}$$

We see again that the zeros of the quantum dilogarithms now cancel with the zeros in the numerator. We see that the integrand has poles when

$$\begin{aligned}
0 &= (1-\mathbf{e}(Mz_2))(1-\mathbf{e}(-Mz_1+Mz_2)) - \mathbf{e}(Mz_1)\,,\\
0 &= (1-\mathbf{e}(2Mz_1))^2 - \mathbf{e}(2Mz_1+Mz_2)(1-\mathbf{e}(-Mz_1+Mz_2))\,.
\end{aligned} \tag{9.59}$$

Therefore, considering the algebraic variety

$$\begin{aligned}
0 &= (1-X_2)(1-X_1^{-1}X_2) - X_1\,,\\
0 &= (1-X_1^2)^2 - X_1^2 X_2(1-X_1^{-1}X_2)\,,
\end{aligned} \tag{9.60}$$

this can be solved exactly as notice that

$$\frac{1-X_2}{X_1} \;=\; \frac{X_1^2 X_2}{(1-X_1^2)^2} \tag{9.61}$$

and so

$$X_2 \;=\; \frac{(1-X_1^2)^2}{(1-X_1^2)^2 + X_1^3}\,, \tag{9.62}$$

and taking as in Example 44 taking

$$\xi^7 - \xi^6 - 2\xi^5 + 6\xi^4 - 11\xi^3 + 6\xi^2 + 3\xi - 1 \;=\; 0\,, \tag{9.63}$$

we have

$$\begin{aligned}
X_{1,j} &= -3 + 11\xi_j + 20\xi_j^2 - 15\xi_j^3 + 6\xi_j^4 - 2\xi_j^5 - 4\xi_j^6\,,\\
X_{2,j} &= -9 + 19\xi_j + 76\xi_j^2 - 52\xi_j^3 + 20\xi_j^4 - 4\xi_j^5 - 13\xi_j^6\,.
\end{aligned} \tag{9.64}$$

There is one additional solution when $X_1 = 1$ and $X_2 = 0$, however this is not contained at a finite point. Therefore, letting

$$x_{i,j,k} \;=\; \frac{1}{2\pi i}\log(X_{i,j}) + k \tag{9.65}$$

notice that

$$\begin{aligned}
&(1-\mathbf{e}(2M(z_1+x_1/M)))^2-\mathbf{e}(2M(z_1+x_1/M)+M(z_2+x_2/M))\\
&\qquad\times(1-\mathbf{e}(-M(z_1+x_1/M)+M(z_2+x_2/M)))\\
&=(1-X_1^2\mathbf{e}(2Mz_1))^2-X_1^2X_2\mathbf{e}(2Mz_1+Mz_2)(1-X_1^{-1}X_2\mathbf{e}(-Mz_1+Mz_2))\\
&=(4X_1^4+(-2X_2-4)X_1^2+X_2^2X_1)2\pi iMz_1+(-X_2X_1^2+2X_2^2X_1)2\pi iMz_2+\cdots
\end{aligned}\tag{9.66}$$

and

$$\begin{aligned}
&\big((1-\mathbf{e}(M(z_2+x_2/M)))(1-\mathbf{e}(-M(z_1+x_1/M)+M(z_2+x_2/M)))-\mathbf{e}(M(z_1+x_1/M))\big)\\
&=\big((1-X_2\mathbf{e}(Mz_2))(1-X_1^{-1}X_2\mathbf{e}(-Mz_1+Mz_2))-X_1\mathbf{e}(Mz_1)\big)\\
&=(-X_1+(-X_2^2+X_2)X_1^{-1})2\pi iMz_1+(-X_2+(2X_2^2-X_2)X_1^{-1})2\pi iMz_2+\cdots\,.
\end{aligned}\tag{9.67}$$

Then we can compute

$$\begin{aligned}
&\det\begin{pmatrix}4X_1^4+(-2X_2-4)X_1^2+X_2^2X_1 & -X_2X_1^2+2X_2^2X_1\\ -X_1+(-X_2^2+X_2)X_1^{-1} & -X_2+(2X_2^2-X_2)X_1^{-1}\end{pmatrix}\\
&=-257+806\xi+947\xi^2-749\xi^3+331\xi^4-133\xi^5-213\xi^6=\Delta\,.
\end{aligned}\tag{9.68}$$

Therefore, the double integral deformed near $(x_{1,j,k}/M,x_{2,j,\ell}/M)$ gives

$$\begin{aligned}
&\frac{\mathbf{e}(x_{1,j,k}^2/NM+x_{1,j,k}x_{2,j,\ell}/NM+x_{1,j,k}(-m/M-m'/N)+x_{2,j,\ell}(1/M+1/N))}{\Phi_S(x_{2,j,\ell}/M+1+\tau;\tau)\Phi_S(2x_{1,j,k}/M+1+\tau;\tau)\Phi_S(x_{2,j,\ell}/M-x_{1,j,k}/M+1+\tau;\tau)}\\
&\quad\times\frac{(1-X_{1,j}^2)^2(1-X_{2,j})(1-X_{1,j}^{-1}X_{2,j})}{M^2\Delta}\\
&=\frac{\mathbf{e}(x_{1,j,k}^2/NM+x_{1,j,k}x_{2,j,\ell}/NM+x_{1,j,k}(-m/M-m'/N)+x_{2,j,\ell}(1/M+1/N))}{(1-X_{1,j}^2)^{-2}(1-X_{2,j})^{-1}(1-X_{1,j}^{-1}X_{2,j})^{-1}M^2\Delta}\\
&\quad\times\frac{\exp\Big(-\frac{1}{2\pi iNM}\mathrm{Li}_2(X_{2,j})\Big)(1-X_{2,j})^{1-\frac{x_{2,j,\ell}}{NM}-\frac{1}{M}}}{D_N(\mathbf{e}(x_{2,j,\ell}/N);\tilde{q}^{-1})D_M(q\mathbf{e}(x_{2,j,\ell}/M);q)}\frac{\exp\Big(-\frac{1}{2\pi iNM}\mathrm{Li}_2(X_{1,j}^2)\Big)(1-X_{1,j}^2)^{1-\frac{2x_{1,j,k}}{NM}-\frac{1}{M}}}{D_N(\mathbf{e}(2x_{1,j,k}/N);\tilde{q}^{-1})D_M(q\mathbf{e}(2x_{1,j,k}/M);q)}\\
&\quad\times\frac{\exp\Big(-\frac{1}{2\pi iNM}\mathrm{Li}_2(X_{1,j}^{-1}X_{2,j})\Big)(1-X_{1,j}^{-1}X_{2,j})^{1-\frac{x_{2,j,\ell}-x_{1,j,k}}{NM}-\frac{1}{M}}}{D_N(\mathbf{e}(x_{2,j,\ell}/N-x_{1,j,k}/N);\tilde{q}^{-1})D_M(q\mathbf{e}(x_{2,j,\ell}/M-x_{1,j,k}/M);q)}\,.
\end{aligned}\tag{9.69}$$

Firstly, notice that

$$\begin{aligned}
&-\mathrm{Li}_2(X_{2,j})-\mathrm{Li}_2(X_{1,j}^2)-\mathrm{Li}_2(X_{1,j}^{-1}X_{2,j})-2\pi ix_{2,j,\ell}\log(1-X_{2,j})-2(2\pi i)x_{1,j,k}\log(1-X_{1,j}^2)\\
&-2\pi i(x_{2,j,\ell}-x_{1,j,k})\log(1-X_{1,j}^{-1}X_{2,j})+(2\pi i)^2x_{1,j,k}^2+(2\pi i)^2x_{1,j,k}x_{2,j,\ell}\\
&=-\mathrm{VC}_j-4\pi^2\Big(k^2+k\ell+a_jk+\frac{1}{24}\Big)
\end{aligned}\tag{9.70}$$

where $a_j = 2, 0, 0, 1, 1, -1, 1$ for $j = 1, \dots, 7$ or

$$2\pi i a_j \;=\; -2\log(1-X_{1,j}^2) + \log(1-X_{1,j}^{-1}X_{2,j}) + 2\log(X_1) + \log(X_2) \tag{9.71}$$

where we note that for all j

$$0 \;=\; -\log(1-X_{2,j}) - \log(1-X_{1,j}^{-1}X_{2,j}) + \log(X_1)\,. \tag{9.72}$$

Therefore, we see that

$$\begin{aligned}
&\mathbf{e}(x_{1,j,k}^2/NM + x_{1,j,k}x_{2,j,\ell}/NM)\exp\Big(-\frac{1}{2\pi i NM}\mathrm{Li}_2(X_{2,j})\Big)(1-X_{2,j})^{-\frac{x_{2,j,\ell}}{NM}}\\
&\quad\times\exp\Big(-\frac{1}{2\pi i NM}\mathrm{Li}_2(X_{1,j}^2)\Big)(1-X_{1,j}^2)^{-\frac{2x_{1,j,k}}{NM}}\\
&\quad\times\exp\Big(-\frac{1}{2\pi i NM}\mathrm{Li}_2(X_{1,j}^{-1}X_{2,j})\Big)(1-X_{1,j}^{-1}X_{2,j})^{-\frac{x_{2,j,\ell}-x_{1,j,k}}{NM}}\\
&=\;\mathbf{e}\Big(-\frac{\mathrm{VC}_j+\pi^2/6}{(2\pi i)^2 NM}\Big)\mathbf{e}\Big(\frac{k^2+k\ell}{NM}\Big)\mathbf{e}\Big(\frac{ka_j+\ell\times 0}{NM}\Big)\,.
\end{aligned} \tag{9.73}$$

With this using the residue theorem and the Chinese remainder theorem part of the final integral of equation (9.58) can then be shown to factorise as

$$\begin{aligned}
&\sum_{k,\ell\in\mathbb{Z}/M\mathbb{Z}}\frac{q^{k^2+k\ell-mk+\ell}X_1^{\frac{2k+\ell-m}{M}}X_2^{\frac{k+1}{M}}\prod_{i=0}^{M-1}(1-q^{i+1+\ell-k}X_1^{-1/M}X_2^{1/M})^{-(i+1+\ell-k)/M-1/2}}{\prod_{i=0}^{M-1}(1-q^{i+1+\ell}X_2^{1/M})^{(i+1+\ell)/M-1/2}\prod_{i=0}^{M-1}(1-q^{i+1+2k}X_1^{2/M})^{(i+1+2k)/M-1/2}}\\
&\quad\times\sqrt{\frac{(1-X_1^2)^2(1-X_2)(1-X_1^{-1}X_2)}{\Delta}}
\end{aligned} \tag{9.74}$$

Therefore, we see that the seven by seven component of the matrix of Theorem 8 gives rise to a quantum modular form after applying the identities of Lemma 18. To extend to the eight by eight, one needs to modify the original integral in a similar way to that discovered in [70]. This integral is given explicitly in (10.38). Then analogously to the situation of the figure eight knot we can factorise with use of the identity for $q = \mathbf{e}(a/c)$

$$\begin{aligned}
&\sum_{0\le\ell\le k\le c/2}(-1)^k q^{-\frac{1}{2}k(k+1)+\ell(\ell+1)+mk}\frac{(q;q)_{2k+1}}{(q;q)_\ell(q;q)_{k-\ell}}\\
&\quad=\;-\sum_{c/2\le k\le\ell\le c-1}(-1)^{k+\ell}q^{k(3k+1)/2+\ell(\ell+1)/2-mk-m}\frac{(q;q)_{\ell+k}}{(q;q)_{2k-c}(q;q)_\ell}\,,
\end{aligned} \tag{9.75}$$

where we note the $2k-c$ arises for exactly the same reason as in the toy example of Section 9.3.

Chapter 10

Factorisation of state integrals as q–series

Factorising as q–series is even simpler than the factorisation at rationals. This was studied in [21, 73]. The main difference is that we need good behaviour in some direction heading towards infinity. This relates to an untrapping procedure discovered originally in unpulished work of Garoufalidis and Kashaev [65]. The simple idea is to then use the q and $\tilde{q}$ difference equations of the quantum dilogarithm, the formula when $\tau \in \mathfrak{h}$ and to compute the residues at the poles enclosed by the contour and the direction towards infinity that has some exponential decay. These decouple leading to a bilinear combination of q and $\tilde{q}$ series. This is easily seen in a example so as a warm up lets go through a proof of Proposition 23.

Proof of Proposition 23. Consider, the state integral

$$\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(wz/\tau)}{\Phi_S(z+\tau+1;\tau)} dz \;=\; \int_{\mathcal{C}_\tau} \frac{(q\mathbf{e}(z);q)_\infty}{(\mathbf{e}(z/\tau);\tilde{q})_\infty} \mathbf{e}(wz/\tau) dz\,. \tag{10.1}$$

The contour and poles are shown in 8.7. Pushing the contour to infinity we can collect the residues which gives

$$\begin{aligned}
&- \sum_{m,n\in\mathbb{Z}_{\geq 0}} 2\pi i \mathrm{Res}_{z=m\tau+n} \Phi_S(z+\tau+1;\tau)^{-1}\mathbf{e}(wz/\tau)dz \\
&= - \sum_{m,n\in\mathbb{Z}_{\geq 0}} \mathbf{e}(mw+nw/\tau) 2\pi i \mathrm{Res}_{z=0} \Phi_S(z+(m+1)\tau+(n+1);\tau)^{-1}\mathbf{e}(wz/\tau)dz \\
&= - \sum_{m,n\in\mathbb{Z}_{\geq 0}} 2\pi i \mathrm{Res}_{z=0} \frac{(q\mathbf{e}(z);q)_\infty}{(\tilde{q}\mathbf{e}(z/\tau);\tilde{q})_\infty} \frac{\mathbf{e}(wz/\tau)}{1-\mathbf{e}(z/\tau)} \frac{\mathbf{e}(mw)}{(q\mathbf{e}(z);q)_m} \frac{\mathbf{e}(nw/\tau)}{(\tilde{q}^{-1}\mathbf{e}(z/\tau);\tilde{q}^{-1})_n} dz \qquad (10.2) \\
&= \tau \frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty} \sum_{m\in\mathbb{Z}_{\geq 0}} \frac{\mathbf{e}(mw)}{(q;q)_m} \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{(-1)^n \tilde{q}^{n(n+1)/2}\mathbf{e}(nw/\tau)}{(\tilde{q};\tilde{q})_n} \\
&= \tau \frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty} \frac{(\tilde{q}\mathbf{e}(w/\tau);\tilde{q})_\infty}{(\mathbf{e}(w);q)_\infty} \;=\; \tau \frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty} \Phi_S(w;\tau)\,.
\end{aligned}$$

A similar computation proves the other equality. □

Before continuing on to more interesting examples of this kind of computation, one should breifly discuss the methods used to improve the analyticity of state integrals. In unpublished work of Garoufalidis and Kashaev [65], they worked out a certain untrapping procedure, which in examples gave a region at infinity for which the contour could be pushed. The basic idea is to apply the identities of Proposition 23, for which we just gave the proof. The analogue of this procedure with the q–series is applying the identities of Lemma 7. For example, take the Rogers–Ramanujan function

$$H(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q;q)_k}\,. \tag{10.3}$$

Notice that

$$H(q^{-1}) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{q^{-k(k+1)/2}}{(q;q)_k} \tag{10.4}$$

is not convergent when $|q|<1$. However,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \frac{q^{k(k+1)}}{(q;q)_k} \;&=\; \frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}} q^{k(k+1)}(q^{k+1};q)_\infty \\
&=\; \frac{1}{(q;q)_\infty}\sum_{k,\ell\in\mathbb{Z}} (-1)^\ell \frac{q^{k(k+1)+k\ell+\ell(\ell+1)/2}}{(q;q)_\ell} \\
&=\; \frac{1}{(q;q)_\infty}\sum_{\ell\in\mathbb{Z}} \theta(-q^\ell;q^2)(-1)^\ell \frac{q^{\ell(\ell+1)/2}}{(q;q)_\ell} \\
&=\; \frac{1}{(q;q)_\infty}\sum_{r=0}^{1}(-1)^r \sum_{\ell\in\mathbb{Z}} \theta(-q^{2\ell+r};q^2) \frac{q^{(2\ell+r)(2\ell+r+1)/2}}{(q;q)_{2\ell+r}} \\
&=\; \frac{1}{(q;q)_\infty}\sum_{r=0}^{1}(-1)^r\theta(-q^r;q^2) \sum_{\ell\in\mathbb{Z}} q^{-\ell(\ell+1)}q^{-r\ell} \frac{q^{(2\ell+r)(2\ell+r+1)/2}}{(q;q)_{2\ell+r}} \\
&=\; \frac{1}{(q;q)_\infty}\sum_{r=0}^{1}(-1)^r\theta(-q^r;q^2)q^{r(r+1)/2} \sum_{\ell\in\mathbb{Z}} \frac{q^{\ell(\ell+r)}}{(q;q)_{2\ell+r}}
\end{aligned} \tag{10.5}$$

This is a simple example of the equation (6.43). The main point is that now there is a natural candidate for the q^{-1}–series as

$$\sum_{\ell\in\mathbb{Z}} \frac{q^{\ell(\ell+r)}}{(q;q)_{2\ell+r}} \tag{10.6}$$

is convergent when $|q| \neq 1$ and θ and η have natural extensions to $|q| \neq 1$. The most important point is that θ and η are modular functions. This means that when dealing with state integrals they will not appear to cause issue as their quotient will be some simple exact functions holomorphic for $\tau\mathbb{C}_S$. In this example, we see that the analogous computation with the state integrals gives

$$\begin{aligned}
&\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/\tau)}{\Phi_S(z+1+\tau;\tau)} dz \\
&= \frac{(\tilde{q};\tilde{q})_\infty}{\tau(q;q)_\infty} \int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(w(w+1+\tau)/2\tau + wz/\tau + z(z+1+\tau)/\tau)}{\Phi_S(w+\tau+1;\tau)} dw dz \\
&= \mathbf{e}\Big(-\frac{3}{8}\Big)\sqrt{\frac{\tau}{2}}\tilde{q}^{\frac{1}{4}} q^{-\frac{1}{4}} \frac{(\tilde{q};\tilde{q})_\infty}{\tau(q;q)_\infty} \int_{\mathcal{C}_\tau} \frac{\mathbf{e}(w^2/4\tau)}{\Phi_S(w+\tau+1;\tau)} dw \\
&= \mathbf{e}\Big(-\frac{3}{8}\Big)\sqrt{2\tau}\tilde{q}^{\frac{1}{4}} q^{-\frac{1}{4}} \frac{(\tilde{q};\tilde{q})_\infty}{\tau(q;q)_\infty} \int_{\mathcal{C}_\tau} \frac{\mathbf{e}(w^2/\tau)}{\Phi_S(2w+\tau+1;\tau)} dw
\end{aligned} \tag{10.7}$$

where we used the fact that

$$\int_{\mathcal{C}_\tau} \mathbf{e}(z(z+1+\tau)/\tau + wz/\tau) dz = \mathbf{e}\Big(\frac{1}{8}\Big)\sqrt{\frac{\tau}{2}}\mathbf{e}\Big(-\frac{w^2}{4\tau} - \frac{w(1+\tau)}{2\tau} - \frac{(1+\tau)^2}{4\tau}\Big). \tag{10.8}$$

We see that both computations are completely analogous where the θ–functions are interchanged with Gaussian integrals. These kind of computations become important when the particular shape of a q–hypergeometric function leads to state integrals whose contour cannot be pushed to infinity. However, there are some state integrals that can be evaluated without this method of pushing the contours to infinity such as the Mordell integral. There the main idea is to use the fact that the matrix of state integrals satisfies uncoupled q and $\tilde{q}$–difference equations. Then taking the a multiple of a Wronskian of the q–difference on one side and the $\tilde{q}$ version on the other, one has an elliptic function, which if holomorphic is constant in the elliptic variable. Then one can check boundary conditions to prove the equality.

10.1 The case of Nahm sums

Our first example will be the case for the $A = 4$ Nahm sum. This method works for all integral A and one could assume that it should also work for rational cases as well, however, we won't go into that detail here. Closing we will extremely briefly describe the identities that one needs for the proof for integral A.

Proof of Theorem 5 for $A = 4$. For

$$G_{A,B,r}(t;q) = \sum_{k=0}^{\infty} \frac{q^{A(Bk+r)(Bk+r+1)/2B} t^{Bk+r}}{(q;q)_{Bk+r}}, \tag{10.9}$$

let

$$
\begin{aligned}
F(t;q) &= \begin{pmatrix} G_{3/4,1,0}(t;q) & G_{3/4,1,0}(it;q) & G_{3/4,1,0}(-t;q) & G_{3/4,1,0}(-it;q) \\ G_{3/4,1,0}(qt;q) & G_{3/4,1,0}(iqt;q) & G_{3/4,1,0}(-qt;q) & G_{3/4,1,0}(-iqt;q) \\ G_{3/4,1,0}(q^2t;q) & G_{3/4,1,0}(iq^2t;q) & G_{3/4,1,0}(-q^2t;q) & G_{3/4,1,0}(-iq^2t;q) \\ G_{3/4,1,0}(q^3t;q) & G_{3/4,1,0}(iq^3t;q) & G_{3/4,1,0}(-q^3t;q) & G_{3/4,1,0}(-iq^3t;q) \end{pmatrix} \\
G(t;q) &= \begin{pmatrix} G_{3,4,0}(t;q) & G_{3,4,1}(t;q) & G_{3,4,2}(t;q) & G_{3,4,3}(t;q) \\ G_{3,4,0}(qt;q) & G_{3,4,1}(qt;q) & G_{3,4,2}(qt;q) & G_{3,4,3}(qt;q) \\ G_{3,4,0}(q^2t;q) & G_{3,4,1}(q^2t;q) & G_{3,4,2}(q^2t;q) & G_{3,4,3}(q^2t;q) \\ G_{3,4,0}(q^3t;q) & G_{3,4,1}(q^3t;q) & G_{3,4,2}(q^3t;q) & G_{3,4,3}(q^3t;q) \end{pmatrix} \qquad (10.10) \\
H(t;q) &= \begin{pmatrix} G_{1,4,0}(t;q) & G_{1,4,1}(t;q) & G_{1,4,2}(t;q) & G_{1,4,3}(t;q) \\ G_{1,4,0}(qt;q) & G_{1,4,1}(qt;q) & G_{1,4,2}(qt;q) & G_{1,4,3}(qt;q) \\ G_{1,4,0}(q^2t;q) & G_{1,4,1}(q^2t;q) & G_{1,4,2}(q^2t;q) & G_{1,4,3}(q^2t;q) \\ G_{1,4,0}(q^3t;q) & G_{1,4,1}(q^3t;q) & G_{1,4,2}(q^3t;q) & G_{1,4,3}(q^3t;q) \end{pmatrix}.
\end{aligned}
$$

We then have

$$F(qt;q) \;=\; A(t;q)F(t;q) \quad \text{and} \quad G(qt;q) \;=\; A(t;q)G(t;q) \qquad (10.11)$$

where

$$A(t;q) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q^6 & q^3+q^4+q^5+q^6 & -q-q^2-2q^3-q^4-q^5 & 1+q+q^2+q^3+q^{27/2}t^4 \end{pmatrix}. \qquad (10.12)$$

Notice that as

$$\sum_{n=0}^{\infty} \frac{q^{\frac{3}{8}n(n+1)}(i^s t)^n}{(q;q)_n} \;=\; \sum_{r=0}^{3} i^{sr} \sum_{n=0}^{\infty} \frac{q^{\frac{3}{8}(4n+r)(4n+r+1)} t^{4n+r}}{(q;q)_{4n+r}}, \qquad (10.13)$$

we have

$$F(t;q) \;=\; G(t;q) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}. \qquad (10.14)$$

Moreover, notice that as

$$
\begin{aligned}
G_{3,4,r}(t;q^{-1}) &= \sum_{k=0}^{\infty} \frac{q^{-3(4k+r)(4k+r+1)/8} t^{4k+r}}{(q^{-1};q^{-1})_{4k+r}} \\
&= \sum_{k=0}^{\infty} \frac{q^{-3(4k+r)(4k+r+1)/8} t^{4k+r}}{(-1)^{4k+r} q^{-(4k+r)(4k+r+1)/2} (q;q)_{4k+r}} \\
&= (-1)^r \sum_{k=0}^{\infty} \frac{q^{(4k+r)(4k+r+1)/8} t^{4k+r}}{(q;q)_{4k+r}} \\
&= (-1)^r G_{1,4,r}(t;q)\,,
\end{aligned}
\tag{10.15}
$$

we have

$$
G(t;q^{-1}) = H(t;q) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = H(-t;q)\,. \tag{10.16}
$$

There exists an explicit (see Proposition 5) $P(t,q)$ holomorphic for $t \in \mathbb{C}^\times$ such that

$$
A(qt;q^{-1})^{-1} = P(qt;q) A(t;q)^{-T} P(t;q)^{-1}\,. \tag{10.17}
$$

Therefore, we see that

$$
G(t;q^{-1})^{-1} P(t;q) G(t;q)^{-T} \tag{10.18}
$$

is elliptic and holomorphic for $t \in \mathbb{C}^\times$. In particular, checking boundary conditions, there is a P (with entries in $\mathbb{Q}(q)[t^{\pm}]$) such that

$$
G(t;q)^{-T} = P(t;q) G(t;q^{-1}) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,. \tag{10.19}
$$

Finally, we have

$$\begin{aligned} G_{4,1,0}(t;q) &= \sum_{k=0}^{\infty} \frac{q^{2k(k+1)}t^k}{(q;q)_k} \\ &= \frac{1}{(q;q)_\infty} \sum_{k,\ell=0}^{\infty} (-1)^\ell \frac{q^{2k(k+1)+k\ell+\ell(\ell+1)/2}}{(q;q)_\ell} t^k \\ &= \frac{1}{(q;q)_\infty} \sum_{r=0}^{3} \sum_{\ell=0}^{\infty} (-1)^{4\ell+r} \frac{q^{(4\ell+r)(4\ell+r+1)/2}}{(q;q)_{4\ell+r}} \theta(-q^{4\ell+r}t;q^4) \\ &= \frac{1}{(q;q)_\infty} \sum_{r=0}^{3} (-1)^r q^{r(r+1)/8} \theta(-q^r t;q^4) \sum_{\ell=0}^{\infty} \frac{q^{3(4\ell+r)(4\ell+r+1)/8}}{(q;q)_{4\ell+r}} (q^{3/2}t)^{-\ell} \\ &= \frac{1}{(q;q)_\infty} \sum_{r=0}^{3} (-1)^r q^{r(r+4)/8} t^{r/4} \theta(-q^r t;q^4) G_{3,4,r}(q^{-3/8}t^{-1/4},q)\,. \end{aligned} \tag{10.20}$$

Therefore,

$$\begin{pmatrix} G_{4,1,0}(t;q) \\ G_{4,1,0}(q^{-4}t;q) \\ G_{4,1,0}(q^{-8}t;q) \\ G_{4,1,0}(q^{-12}t;q) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & q^{-4}t^2 & 0 \\ 0 & 0 & 0 & q^{-12}t^3 \end{pmatrix} G(q^{-3/8}t^{-1/4};q) \begin{pmatrix} \frac{\theta(-t;q^4)}{(q;q)_\infty} \\ \frac{-q^{5/8}t^{1/4}\theta(-qt;q^4)}{(q;q)_\infty} \\ \frac{q^{12/8}t^{2/4}\theta(-q^2t;q^4)}{(q;q)_\infty} \\ \frac{-q^{21/8}t^{3/4}\theta(-q^3t;q^4)}{(q;q)_\infty} \end{pmatrix}. \tag{10.21}$$

Finally, the modularity of the vector of θ–functions is given by

$$\begin{aligned} &\mathbf{e}\left(-\frac{u^2}{8\tau}\right) \mathbf{e}\left(\frac{1}{8}\right) 2\tau^{-1/2} \tilde{q}^{1/2} \tilde{t}^{1/2} \begin{pmatrix} \theta(-\tilde{t};\tilde{q}^4) \\ -\tilde{q}^{5/8}\tilde{t}^{1/4}\theta(-\tilde{q}\tilde{t};\tilde{q}^4) \\ \tilde{q}^{12/8}\tilde{t}^{2/4}\theta(-\tilde{q}^2\tilde{t};\tilde{q}^4) \\ -\tilde{q}^{21/8}\tilde{t}^{3/4}\theta(-\tilde{q}^3\tilde{t};\tilde{q}^4) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} q^{1/2} t^{1/2} \begin{pmatrix} \theta(-t;q^4) \\ -q^{5/8}t^{1/4}\theta(-qt;q^4) \\ q^{12/8}t^{2/4}\theta(-q^2t;q^4) \\ -q^{21/8}t^{3/4}\theta(-q^3t;q^4) \end{pmatrix} \end{aligned} \tag{10.22}$$

With all this set up we can now factorise the state integral

$$\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/8\tau + uz/\tau)}{\Phi_S(z+1+\tau;\tau)} dz \tag{10.23}$$

to see that it is an elementary function holomorphic on $\tau \in \mathbb{C}_S$ times

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \mathbf{e}(mn/4)\mathbf{e}(m/8)\frac{q^{m(m+1)/8}}{(q;q)_m}t^m \mathbf{e}(-3n/8)\frac{\tilde{q}^{3n(n+1)/8}}{(\tilde{q};\tilde{q})_n}\tilde{t}^n \\
&= \sum_{r=0}^{3}\sum_{m,n=0}^{\infty} \mathbf{e}(m/2+r/8)\frac{q^{(4m+r)(4m+r+1)/8}}{(q;q)_{4m+r}}t^{4m+r}\mathbf{e}((-3+2r)n/8)\frac{\tilde{q}^{3n(n+1)/8}}{(\tilde{q};\tilde{q})_n}\tilde{t}^n \qquad (10.24)\\
&= \sum_{r=0}^{3} G_{1,4,r}\left(\mathbf{e}(1/8)t\right) G_{3/4,1,0}\left(\mathbf{e}(-3/8+r/4)\tilde{t};\tilde{q}\right) .
\end{aligned}$$

Therefore, shifting $t \mapsto q^{-3/8}\mathbf{e}(3/8)t^{-1/4}$ and $\tilde{t} \mapsto \mathbf{e}(-3/8)\tilde{q}^{-3/8}\tilde{t}^{-1/4}$, the matrix

$$\Omega^{(0)}(z;\tau) \;=\; F(i\tilde{q}^{-3/8}\tilde{t};\tilde{q})H(-q^{-3/8}t;q)^T \qquad (10.25)$$

extends for $\tau \in \mathbb{C}_S$. Then notice that

$$F(i\tilde{q}^{-3/8}\tilde{t};\tilde{q}) \;=\; F(\tilde{q}^{-3/8}\tilde{t};\tilde{q})\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} . \qquad (10.26)$$

Therefore, using Equations (10.14) (10.16) (10.19) (10.21) (10.22) (10.25),

$$
\begin{aligned}
&\begin{pmatrix} G_{4,1,0}(\tilde{t};\tilde{q}) \\ G_{4,1,0}(\tilde{q}^{-4}\tilde{t};\tilde{q}) \\ G_{4,1,0}(\tilde{q}^{-8}\tilde{t};\tilde{q}) \\ G_{4,1,0}(\tilde{q}^{-12}\tilde{t};\tilde{q}) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{t} & 0 & 0 \\ 0 & 0 & \tilde{q}^{-4}\tilde{t}^2 & 0 \\ 0 & 0 & 0 & \tilde{q}^{-12}\tilde{t}^3 \end{pmatrix} G(\tilde{q}^{-3/8}\tilde{t}^{-1/4},\tilde{q}) \begin{pmatrix} \frac{\theta(-\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \\ \frac{-\tilde{q}^{5/8}\tilde{t}^{1/4}\theta(-\tilde{q}\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \\ \frac{\tilde{q}^{12/8}\tilde{t}^{2/4}\theta(-\tilde{q}^2\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \\ \frac{-\tilde{q}^{21/8}\tilde{t}^{3/4}\theta(-\tilde{q}^3\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{t} & 0 & 0 \\ 0 & 0 & \tilde{q}^{-4}\tilde{t}^2 & 0 \\ 0 & 0 & 0 & \tilde{q}^{-12}\tilde{t}^3 \end{pmatrix} F(i\tilde{q}^{-3/8}\tilde{t}^{-1/4},\tilde{q}) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}^{-1} \begin{pmatrix} \frac{\theta(-\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \\ \frac{-\tilde{q}^{5/8}\tilde{t}^{1/4}\theta(-\tilde{q}\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \\ \frac{\tilde{q}^{12/8}\tilde{t}^{2/4}\theta(-\tilde{q}^2\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \\ \frac{-\tilde{q}^{21/8}\tilde{t}^{3/4}\theta(-\tilde{q}^3\tilde{t};\tilde{q}^4)}{(\tilde{q};\tilde{q})_\infty} \end{pmatrix} \\
&= \Omega^{(1)}(u;\tau)H(-q^{-3/8}t^{-1/4},q)^{-T} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} \frac{\theta(-t;q^4)}{(q;q)_\infty} \\ \frac{-q^{5/8}t^{1/4}\theta(-qt;q^4)}{(q;q)_\infty} \\ \frac{q^{12/8}t^{2/4}\theta(-q^2t;q^4)}{(q;q)_\infty} \\ \frac{-q^{21/8}t^{3/4}\theta(-q^3t;q^4)}{(q;q)_\infty} \end{pmatrix} \\
&= \Omega^{(1)}(u;\tau)G(q^{-3/8}t^{-1/4},q^{-1})^{-T} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\theta(-t;q^4)}{(q;q)_\infty} \\ \frac{-q^{5/8}t^{1/4}\theta(-qt;q^4)}{(q;q)_\infty} \\ \frac{q^{12/8}t^{2/4}\theta(-q^2t;q^4)}{(q;q)_\infty} \\ \frac{-q^{21/8}t^{3/4}\theta(-q^3t;q^4)}{(q;q)_\infty} \end{pmatrix} \\
&= \Omega^{(2)}(u;\tau) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & q^{-4}t^2 & 0 \\ 0 & 0 & 0 & q^{-12}t^3 \end{pmatrix} G(q^{-3/8}t^{-1/4},q) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\theta(-t;q^4)}{(q;q)_\infty} \\ \frac{-q^{5/8}t^{1/4}\theta(-qt;q^4)}{(q;q)_\infty} \\ \frac{q^{12/8}t^{2/4}\theta(-q^2t;q^4)}{(q;q)_\infty} \\ \frac{-q^{21/8}t^{3/4}\theta(-q^3t;q^4)}{(q;q)_\infty} \end{pmatrix} \\
&= \Omega^{(2)}(u;\tau) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & q^{-4}t^2 & 0 \\ 0 & 0 & 0 & q^{-12}t^3 \end{pmatrix} G(q^{-3/8}t^{-1/4},q) \begin{pmatrix} \frac{\theta(-t;q^4)}{(q;q)_\infty} \\ \frac{-q^{5/8}t^{1/4}\theta(-qt;q^4)}{(q;q)_\infty} \\ \frac{q^{12/8}t^{2/4}\theta(-q^2t;q^4)}{(q;q)_\infty} \\ \frac{-q^{21/8}t^{3/4}\theta(-q^3t;q^4)}{(q;q)_\infty} \end{pmatrix} \\
&= \Omega^{(2)}(u;\tau) \begin{pmatrix} G_{4,1,0}(t;q) \\ G_{4,1,0}(q^{-4}t;q) \\ G_{4,1,0}(q^{-8}t;q) \\ G_{4,1,0}(q^{-12}t;q) \end{pmatrix},
\end{aligned}
\tag{10.27}
$$

where $\Omega^{(i)}$ are elementary functions holomorphic for $\tau \in \mathbb{C}_S$ times $\Omega^{(0)}$. Therefore, finally, by a change of gauge we have

$$
\begin{pmatrix} G_{4,1,0}(\tilde{t};\tilde{q}) \\ G_{4,1,0}(\tilde{q}\tilde{t};\tilde{q}) \\ G_{4,1,0}(\tilde{q}^2\tilde{t};\tilde{q}) \\ G_{4,1,0}(\tilde{q}^3\tilde{t};\tilde{q}) \end{pmatrix} = \Omega(u;\tau) \begin{pmatrix} G_{4,1,0}(t;q) \\ G_{4,1,0}(qt;q) \\ G_{4,1,0}(q^2t;q) \\ G_{4,1,0}(q^3t;q) \end{pmatrix}. \tag{10.28}
$$

□

For $A \in \mathbb{Z}$ starting with the state integral

$$\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/2A\tau + uz/\tau)}{\Phi_S(z+1+\tau;\tau)}\,. \tag{10.29}$$

Applying exactly the same methods we used for the $A = 4$ example will produce the desired result. The additional steps needed are the identities of Section 6.1 and 6.3 along with the modularity of the vector–valued θ–function given in equation (7.73)

10.2 The case of the figure eight knot

For the figure eight knot, the state integral

$$\Omega(u;\tau) \;=\; \frac{(\tilde{q};\tilde{q})_\infty^2}{\tau(q;q)_\infty^2} \int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz\,, \tag{10.30}$$

was factorised in [73, 70] and the analogue of the quadratic relations were also given [70, Thm. 4]. However, we include here a computation of the q–difference equations without the factorisation. The computation is completely analogous to that of the q–series with the addition of a residue appearing as opposed to the boundary of the sum. Notice that

$$\frac{\mathbf{e}((z+\tau)(z+1+2\tau)/2\tau + zu/\tau + u)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+2\tau;\tau)^2} \frac{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2}{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)} \;=\; -\frac{\mathbf{e}(z+\tau+u)}{(1-\mathbf{e}(z+\tau))^2}\,. \tag{10.31}$$

Therefore, we see that

$$\begin{aligned}
&-\tau\frac{(q;q)_\infty^2}{(\tilde{q};\tilde{q})_\infty^2} + \int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)(1-\mathbf{e}(z))^2}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz \\
&= 2\pi i \mathrm{Res}_{z=0} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)(1-\mathbf{e}(z))^2}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz + \int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)(1-\mathbf{e}(z))^2}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz \\
&= \int_{\mathcal{C}_\tau+\tau} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)(1-\mathbf{e}(z))^2}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz \;=\; \int_{\mathcal{C}_\tau} \frac{\mathbf{e}((z+\tau)(z+1+2\tau)/2\tau + zu/\tau + u)(1-\mathbf{e}(z+\tau))^2}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+2\tau;\tau)^2} dz \\
&= -\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + zu/\tau)\mathbf{e}(z+\tau+u)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz \;=\; -qt\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(z(z+1+\tau)/2\tau + z(u+\tau)/\tau)}{(1-\mathbf{e}(z/\tau))\Phi_S(z+1+\tau;\tau)^2} dz\,.
\end{aligned} \tag{10.32}$$

Therefore, we see that

$$\Omega(u;\tau) + (q\mathbf{e}(u) - 2)\Omega(u+\tau;\tau) + \Omega(u+2\tau;\tau) \;=\; 1\,. \tag{10.33}$$

Therefore, as mentioned previously, another approach to prove the factorisation in this example is to take

$$\begin{pmatrix} \Omega(u;\tau) & \Omega(u-\tau;\tau) & \Omega(u-2\tau;\tau) \\ \Omega(u-1;\tau) & \Omega(u-1-\tau;\tau) & \Omega(u-1-2\tau;\tau) \\ \Omega(u-2;\tau) & \Omega(u-2-\tau;\tau) & \Omega(u-2-2\tau;\tau) \end{pmatrix} \tag{10.34}$$

10.3 A seven by seven matrix of q–series

Consider the state integral we introduced for the seven by seven matrix associated to $4_1(1,2)$,

$$\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau}\frac{\mathbf{e}(z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))dz_1dz_2}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)}\,. \tag{10.35}$$

This has poles at $z_1\in\frac{1}{2}\mathbb{Z}_{\geq 0}+\frac{\tau}{2}\mathbb{Z}_{\geq 0}$, $z_2-z_1\in\mathbb{Z}_{\geq 0}+\tau\mathbb{Z}_{\geq 0}$, $z_2\in\mathbb{Z}_{\geq 0}+\tau\mathbb{Z}_{\geq 0}$. We can compute by repeated applications of the residue theorem.

$$\begin{aligned}
&\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau}\frac{\mathbf{e}(z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)}dz_1dz_2\\
&=-\int_{\mathcal{C}_\tau}\sum_{j,j'=0}^{\infty}\mathrm{Res}_{z_2=j\tau+j'}\frac{\mathbf{e}(2z_1^2/\tau+z_1z_2/\tau+z_1(1-m+(1-m')/\tau)+z_2(1+1/\tau))}{\Phi_S(z_1+z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2+1+\tau;\tau)}dz_1dz_2\\
&\quad-\int_{\mathcal{C}_\tau}\sum_{j,j'=0}^{\infty}\mathrm{Res}_{z_2=j\tau+j'}\frac{\mathbf{e}(z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))}{\Phi_S(z_2+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2-z_1+1+\tau;\tau)}dz_1dz_2\\
&=\frac{\tau(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\int_{\mathbb{R}}\sum_{j,j'=0}^{\infty}\frac{q^j\tilde{q}^{-j'}\mathbf{e}(2z_1^2/\tau+z_1((j+1-m)\tau+(j'+1-m'))/\tau)}{(q;q)_j(\tilde{q}^{-1};\tilde{q}^{-1})_{j'}\Phi_S(z_1+j\tau+j'+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)}dz_1\\
&\quad+\frac{\tau(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\int_{\mathbb{R}}\sum_{j,j'=0}^{\infty}\frac{q^j\tilde{q}^{-j'}\mathbf{e}(z_1^2/\tau+z_1((j-m)\tau+(j'-m'))/\tau)}{(q;q)_j(\tilde{q}^{-1};\tilde{q}^{-1})_{j'}\Phi_S(2z_1+1+\tau;\tau)\Phi_S(j\tau+j'-z_1+1+\tau;\tau)}dz_1\\
&=-\frac{\tau(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\sum_{k,j,k',j'=0}^{\infty}\frac{\mathrm{Res}_{z_1=\frac{k}{2}\tau+\frac{k'}{2}}q^j\tilde{q}^{-j'}\mathbf{e}(2z_1^2/\tau+z_1((j+1-m)\tau+(j'+1-m'))/\tau)}{(q;q)_j(\tilde{q}^{-1};\tilde{q}^{-1})_{j'}\Phi_S(z_1+j\tau+j'+1+\tau;\tau)\Phi_S(2z_1+1+\tau;\tau)}dz_1\\
&\quad-\frac{\tau(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\sum_{k,j,k',j'=0}^{\infty}\frac{\mathrm{Res}_{z_1=\frac{k}{2}\tau+\frac{k'}{2}}q^j\tilde{q}^{-j'}\mathbf{e}(z_1^2/\tau+z_1((j-m)\tau+(j'-m'))/\tau)}{(q;q)_j(\tilde{q}^{-1};\tilde{q}^{-1})_{j'}\Phi_S(2z_1+1+\tau;\tau)\Phi_S(j\tau+j'-z_1+1+\tau;\tau)}dz_1\,.
\end{aligned} \tag{10.36}$$

These sums can now be broken up into congruences. In particular, we see that for $k,k'\in 2\mathbb{Z}$ we will get double and triple poles and otherwise we have simple poles. The values at the simple poles can easily be computed and give rise to the q–series $Z^{(j)}$ for $j>3$. Then the residues for $k,k'\in 2\mathbb{Z}$ can be computed by expanding in z_1 using Lemma 19. This gives rise to the functions $Z^{(j)}$ for $j=1,2,3$. Importantly, in the factorisation a term of the final sum proportional to

$$Z_m^{(1)}(q)Z_{m'}^{(2)}(\tilde{q}^{-1}) \tag{10.37}$$

appears and $Z_m^{(1)}(q)/2$ satisfies the inhomogeneous q–difference equation (6.93). However, as noted in Proposition 11 $Z_m^{(2)}(q^{-1})=0$ for $|q|<1$.

10.4 An eight by eight matrix of q–series

To extend this result to the eight by eight matrix, we take the state integral, which besides the factors $(1-\mathbf{e}(z_1/\tau))(1-\mathbf{e}(z_3/\tau))$ in the denominator of the integrand, appears as an intermediate step of the computation for passing from equation (9.53) to equation (9.54) and is given by

$$\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(3z_1(z_1+1+\tau)/2\tau + z_3(z_3+1+\tau)/2\tau - z_1(m+1+(m'+1)/\tau))\Phi_S(z_3+1+\tau;\tau)}{(1-\mathbf{e}(z_1/\tau))(1-\mathbf{e}(z_3/\tau))\Phi_S(z_3-z_1+\tau+1)\Phi_S(2z_1+1+\tau;\tau)} dz_1 dz_3 \,. \tag{10.38}$$

To factorise this as q–series, one expands the Faddeev in the numerator to get a triple integral

$$\tau^{-1}\frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty}\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau}\int_{\mathcal{C}_\tau} \frac{\mathbf{e}(2z_1(z_1+1+\tau)/\tau + z_1z_3/\tau + z_3(z_3+1+\tau)/2\tau - z_1(m+1+(m'+1)/\tau) + (z_1+z_3+1+\tau)z_2/\tau)}{(1-\mathbf{e}(z_1/\tau))(1-\mathbf{e}(z_3/\tau))\Phi_S(z_3+\tau+1)\Phi_S(2z_1+1+\tau;\tau)\Phi_S(z_2+1+\tau;\tau)} dz_1 dz_2 dz_3 \,. \tag{10.39}$$

Then integrating first in z_3, then z_2, and then z_1, we find exactly the q–series given in Theorem 8. The final steps are the identities similar to that of Proposition 11. We find that the final factorisation has terms with factors $Z^{(3)}$ and $Z^{(1)}$ and other factors for each of these terms vanish in opposite half planes so that $Z^{(1)}(q)$ appears when $|q|<1$ while $Z^{(3)}(q)$ appears when $|q|>1$.

Chapter 11

Conjectures on resurgence and outlook

While quantum modularity can be established in examples, there seems to be a deeper underlying structure. In particular, it seems that the QM–cocycles associated to quantum modular forms are related to the Borel resummation of asymptotic series associated to the asymptotics of the quantum modular forms. Therefore, this conjecturally states that Borel resummation in these examples factorises as q and $\tilde{q}$–series. On top of this, one finds numerically that the asymptotic series coming from q–hypergeometric functions are resurgent. Detailed conjectures on the behaviour of Stokes phenomenon then allow, when combined with the conjectural formulae for the Borel resummation, effective computation of the Stokes constants.

11.1 Borel resummation equals state integrals?

It was noticed in [69] that the Borel resummation of the asymptotic series of certain q–hypergeometric sums agrees numerically with certain combinations of associated state integrals. This conjecture can of course be extended to any of the example we have considered. As we have seen in the last sections, state integrals can be factorised at the rationals and the upper and lower half planes. This means that these conjectures can be studied using quantum modular forms as opposed to the state integrals themselves. Here we will focus on the case of the $A = 4$ Nahm sum and the WRT invariant of half surgery on the figure eight knot to test these conjectures numerically.

Conjecture 12. *[69][1] The Borel resummation of asymptotic series associated to q–hypergeometric functions is given by combinations of associated state integrals.*

[1] See the footnote for conjecture 13.

Example 61 (Borel resummation and the $A = 4$ Nahm sum)**.** *We can consider the evaluation of the Nahm sum*

$$f_m(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{2k^2+km}}{(q;q)_k}\,, \tag{11.1}$$

at $\tilde{q} = \mathbf{e}(-1/\tau)$ *where* $\tau = 1000\,\mathbf{e}(0.0001)$ *and we find that*

$$f_0(\tilde{q}) \;=\; (1.4799\cdots + 1.8058\cdots i) \times 10^{67}\,. \tag{11.2}$$

Then if we take the Borel resummation of the asymptotic series discussed in Example 39 we find that the quotient is given by

$$\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} \;=\; 1.0000\cdots - 2.7438\cdots \times 10^{-8}\,. \tag{11.3}$$

Then we see that

$$\Big(\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} - 1\Big)q^{-3} \;=\; (1.0197 - 2.4883 \times 10^{-5} \cdot i)\,, \tag{11.4}$$

and similarly,

$$\Big(\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} - 1 - q^3 - q^4 - q^5 - q^6 - q^7 - q^8 - q^9\Big)q^{-10} \;=\; (2.0397\cdots - 5.0718\cdots i \times 10^{-5})\,. \tag{11.5}$$

Indeed, continuing we can identify this q–series as

$$\begin{aligned} f_1(q) \;=\; & 1 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + 2q^{10} + 2q^{11} + 3q^{12} + 3q^{13} \\ & + 4q^{14} + 4q^{15} + 5q^{16} + 5q^{17} + 6q^{18} + 6q^{19} + 7q^{20} + 8q^{21} + 9q^{22} \\ & + 10q^{23} + 12q^{24} + 13q^{25} + 15q^{26} + 17q^{27} + 19q^{28} + 21q^{29} + \cdots\,, \end{aligned} \tag{11.6}$$

and then we find that

$$f_0(\tilde{q}) - s(\widehat{\Phi}^{(3)})(2\pi i/\tau) f_1(q) \;=\; 0.20122\cdots + 0.68776\cdots i\,. \tag{11.7}$$

Then we can continue to find that

$$\begin{aligned} & f_0(\tilde{q}) - s(\widehat{\Phi}^{(1)})(2\pi i/\tau) f_2(q) - s(\widehat{\Phi}^{(2)})(2\pi i/\tau) f_0(q) - s(\widehat{\Phi}^{(3)})(2\pi i/\tau) f_1(q) \\ & \quad - s(\widehat{\Phi}^{(3)})(2\pi i/\tau) q f_3(q) \;=\; (5.5399\cdots - 3.7010\cdots i) \times 10^{-138}\,. \end{aligned} \tag{11.8}$$

This error is exactly the order of numerical error of the Borel resummation. This leads to a conjectural identity for τ *just above the positive reals*

$$\begin{aligned} f_0(\tilde{q}) \;=\; & s(\widehat{\Phi}^{(1)})(2\pi i/\tau) f_2(q) + s(\widehat{\Phi}^{(2)})(2\pi i/\tau) f_0(q) \\ & + s(\widehat{\Phi}^{(3)})(2\pi i/\tau) f_1(q) + s(\widehat{\Phi}^{(3)})(2\pi i/\tau) q f_3(q)\,. \end{aligned} \tag{11.9}$$

Performing similar numerical checks for τ just above the negative reals, we find a similar statement

$$\begin{aligned} f_0(\tilde{q}) \;&=\; s(\widehat{\Phi}^{(1)})(2\pi i/\tau)f_2(q) + s(\widehat{\Phi}^{(2)})(2\pi i/\tau)f_0(q) \\ &\quad + s(\widehat{\Phi}^{(3)})(2\pi i/\tau)qf_3(q) + s(\widehat{\Phi}^{(3)})(2\pi i/\tau)f_1(q)\,. \end{aligned} \tag{11.10}$$

To then compute the Stokes constants we need a basis of series similar to f giving similar identities. This can be constructed using the identity from equation (10.21) and setting $t = q^m$ and replacing

$$\begin{pmatrix} \theta(-q^m;q^4) \\ -q^{5/8+m/4}\theta(-q^{m+1};q^4) \\ q^{12/8+m/2}\theta(-q^{m+2};q^4) \\ -q^{21/8+3m/4}\theta(-q^{m+3};q^4) \end{pmatrix}, \tag{11.11}$$

by coefficients in the ϵ expansion of

$$\exp\left(\frac{1}{2}\epsilon + \frac{m}{4}\epsilon + \frac{1}{4}\epsilon^2 G_2(q)\right) \begin{pmatrix} \theta(-q^m\exp(\epsilon);q^4) \\ -q^{5/8+m/4}\exp(\epsilon/4)\theta(-q^{m+1}\exp(\epsilon);q^4) \\ q^{12/8+m/2}\exp(\epsilon/2)\theta(-q^{m+2}\exp(\epsilon);q^4) \\ -q^{21/8+3m/4}\exp(3\epsilon/4)\theta(-q^{m+3}\exp(\epsilon);q^4) \end{pmatrix}, \tag{11.12}$$

which, for example, we could take the $\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^4$, which in a matrix has q expansion with columns

$$\begin{aligned} \Theta_m(q) \;=\; &\begin{pmatrix} 2+2q^4 \\ -q^{\frac{-3}{8}} - q^{\frac{5}{8}} - q^{\frac{21}{8}} \\ q^{\frac{-1}{2}} + 2q^{\frac{3}{2}} \\ -q^{\frac{-3}{8}} - q^{\frac{5}{8}} - q^{\frac{21}{8}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{4}q^{\frac{-3}{8}} - \frac{3}{4}q^{\frac{5}{8}} + \frac{5}{4}q^{\frac{21}{8}} \\ 0 \\ -\frac{1}{4}q^{\frac{-3}{8}} + \frac{3}{4}q^{\frac{5}{8}} - \frac{5}{4}q^{\frac{21}{8}} \end{pmatrix}, \begin{pmatrix} \frac{11}{48} + \frac{1}{2}q + \frac{3}{2}q^2 + 2q^3 + \frac{275}{48}q^4 + \frac{7}{2}q^5 \\ -\frac{1}{48}q^{\frac{-3}{8}} - \frac{25}{48}q^{\frac{5}{8}} - q^{\frac{13}{8}} - \frac{121}{48}q^{\frac{21}{8}} - 3q^{\frac{29}{8}} \\ -\frac{1}{96}q^{\frac{-1}{2}} + \frac{1}{4}q^{\frac{1}{2}} + \frac{83}{48}q^{\frac{3}{2}} + \frac{3}{2}q^{\frac{5}{2}} + \frac{13}{4}q^{\frac{7}{2}} \\ -\frac{1}{48}q^{\frac{-3}{8}} - \frac{25}{48}q^{\frac{5}{8}} - q^{\frac{13}{8}} - \frac{121}{48}q^{\frac{21}{8}} - 3q^{\frac{29}{8}} \end{pmatrix}, \\ &\begin{pmatrix} \frac{25}{9216} + \frac{11}{192}q + \frac{15}{64}q^2 + \frac{29}{48}q^3 + \frac{17161}{9216}q^4 + \frac{629}{192}q^5 \\ \frac{1}{9216}q^{\frac{-3}{8}} - \frac{143}{9216}q^{\frac{5}{8}} - \frac{11}{96}q^{\frac{13}{8}} - \frac{4943}{9216}q^{\frac{21}{8}} - \frac{39}{32}q^{\frac{29}{8}} \\ \frac{1}{18432}q^{\frac{-1}{2}} - \frac{1}{384}q^{\frac{1}{2}} + \frac{889}{9216}q^{\frac{3}{2}} + \frac{27}{64}q^{\frac{5}{2}} + \frac{503}{384}q^{\frac{7}{2}} \\ \frac{1}{9216}q^{\frac{-3}{8}} - \frac{143}{9216}q^{\frac{5}{8}} - \frac{11}{96}q^{\frac{13}{8}} - \frac{4943}{9216}q^{\frac{21}{8}} - \frac{39}{32}q^{\frac{29}{8}} \end{pmatrix}, +O(q^5)\,. \end{aligned} \tag{11.13}$$

Then for

$$P(q) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q^{-3} - q^{-2} - q^{-1} & q^{-3} + q^{-2} + 2q^{-1} + 1 + q & -q^2 - q^3 - q^4 - q^5 - q^{12} & q^{12} \\ q^{-5} + q^{-4} + q^{-3} & -q^{-5} - 2q^{-4} - 2q^{-3} - 2q^{-2} - q^{-1} + q^3 & 1 + 2q + 2q^2 + q^3 + q^4 + q^{10} + q^{11} & -q^{10} - q^{11} \\ -q^{-6} - q^{-1} & q^{-6} + q^{-5} + q^{-4} - q^2 & -q^{-1} - 1 - q - q^9 & q^9 \end{pmatrix} \tag{11.14}$$

we have

$$F(q) \;=\; P(q)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q^{-4} & 0 \\ 0 & 0 & 0 & q^{-12} \end{pmatrix} G(q^{-3/8};q)\Theta_0(q)\,, \tag{11.15}$$

a matrix with columns

$$\begin{aligned} &\begin{pmatrix} 1 + q^2 + q^3 + q^4 \\ 1 + q^3 + q^4 \\ 1 + q^4 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{4}q + \frac{1}{4}q^2 + \frac{1}{2}q^3 \\ \frac{1}{4} - \frac{1}{2}q - \frac{1}{2}q^2 + \frac{3}{4}q^3 + \frac{1}{2}q^4 \\ \frac{1}{4} - \frac{1}{4}q - \frac{1}{2}q^2 - \frac{1}{4}q^3 + \frac{1}{2}q^4 \\ -\frac{1}{4}q^{-1} + \frac{3}{4} - \frac{3}{4}q^2 - \frac{1}{2}q^3 - \frac{3}{4}q^4 \end{pmatrix}, \begin{pmatrix} -\frac{1}{96} + \frac{7}{32}q + \frac{67}{48}q^2 + \frac{173}{96}q^3 + \frac{299}{96}q^4 \\ \frac{1}{48} + \frac{5}{16}q + \frac{5}{8}q^2 + \frac{85}{48}q^3 + \frac{263}{96}q^4 \\ \frac{5}{24} + \frac{5}{32}q + \frac{9}{16}q^2 + \frac{19}{32}q^3 + \frac{287}{96}q^4 \\ \frac{1}{32}q^{-1} + \frac{1}{3} - \frac{1}{16}q + \frac{21}{32}q^2 + \frac{3}{4}q^3 + \frac{43}{32}q^4 \end{pmatrix}, \\ &\begin{pmatrix} \frac{1}{18432} - \frac{5}{2048}q + \frac{725}{9216}q^2 + \frac{6835}{18432}q^3 + \frac{21253}{18432}q^4 \\ -\frac{1}{9216} + \frac{13}{1024}q + \frac{103}{1536}q^2 + \frac{3107}{9216}q^3 + \frac{15745}{18432}q^4 \\ \frac{13}{4608} + \frac{275}{6144}q + \frac{153}{1024}q^2 + \frac{463}{2048}q^3 + \frac{15673}{18432}q^4 \\ -\frac{1}{6144}q^{-1} + \frac{41}{2304} + \frac{55}{1024}q + \frac{449}{2048}q^2 + \frac{221}{768}q^3 + \frac{1447}{2048}q^4 \end{pmatrix}, \end{aligned} \tag{11.16}$$

where the first column is given by $f_0(q), f_1(q), f_2(q), f_3(q)$*. Therefore, from the proof of Theorem 5 in Section 10.1 we see that the cocycle from equation (10.28)* Ω *is given by*

$$\Omega_S(0;\tau) \;=\; F(\tilde{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & \tau^4 \end{pmatrix}^{-1} F(q)^{-1}\,. \tag{11.17}$$

However, numerically we have observed that for τ *just above the positive reals*

$$s(\widehat{\Phi})(\tau) \;=\; F(\tilde{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & \tau^4 \end{pmatrix}^{-1} F(q)^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}^{-1} \;=\; \Omega_S(0;\tau) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}^{-1}. \tag{11.18}$$

We can apply a similar numerical tests for τ *just above the negative reals to find that*

$$s(\widehat{\Phi})(\tau) \;=\; F(\tilde{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & \tau^4 \end{pmatrix}^{-1} F(q)^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1} \;=\; \Omega_{-S}(0;\tau) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1}. \tag{11.19}$$

Example 62 (Half surgery on the figure eight knot and Borel resummation)**.** *Using Borel–Padé–Laplace, we can numerically find the quantum modular behaviour for* $\tau \in \mathbb{R}_{>0}$ *and* $\tau \in \mathbb{R}_{<0}$*. This can be done with q–series, functions at roots of unity, or* q^{-1} *series. There are no Stokes lines on the reals so we only need to do this once in this example. The most practical way to do this is with functions at q in roots of unity as* $\tilde{q} = O(1)$ *in this case. Then the quantum modularity can be computed without having to deal with exponentially small corrections. Numerical computations led to the functions* X_ρ *of equation (8.88). We can write this in terms of the module associated to (6.91) with* $n = 0$*. This is stored in the matrix*

$$P(q)(1-q)q^4 \;=\; \begin{pmatrix} 1 & q^3+2q^2 & -q^5+2q^4-q^3-q^2 & -q^7-q^6-2q^5-q^4 & q^8+2q^5 & q^{10}+q^9+q^7 & -q^{10}-q^8 & -q^{12} \\ 1 & q^3+2q^2 & q^4-q^3-q^2 & -q^6-2q^5-2q^4 & q^7+2q^5 & q^9+q^8+q^7 & -q^9-q^7 & -q^{11} \\ q & q^4+q^3+q^2 & -q^3 & -q^7-2q^6-q^5-q^4 & q^8+2q^6 & q^{10}+q^9+q^8 & -q^{10}-q^8 & -q^{12} \\ 0 & -q^2+q & -q^4+q^3 & q^5-q^3 & -q^5+q^4 & -q^7+q^6 & 0 & 0 \\ 1 & 2q^3+q^2 & q^5-q^3-q^2 & -q^7-q^6-3q^5 & 2q^6+q^5 & q^{10}+q^9+q^8+q^7-q^6 & -q^9-q^8 & -q^{12} \\ 1 & 2q^3+q^2 & q^5+q^4-2q^3-q^2 & -q^7-q^6-3q^5 & q^6+2q^5 & q^{10}+q^9+q^8+q^7-q^6 & -q^9-q^8 & -q^{12} \\ 0 & 0 & 0 & -q^5+q^4 & 0 & 0 & 0 & 0 \\ 1 & 2q^3+q^2 & q^5-q^3-q^2 & -q^7-q^6-3q^5 & q^6+2q^5 & q^{10}+q^9+q^8+q^7-q^6 & -q^9-q^8 & -q^{12} \end{pmatrix} \tag{11.20}$$

In particular,

$$\begin{pmatrix} X_{\rho_0}(q) \\ X_{\rho_1}(q) \\ X_{\rho_2}(q) \\ X_{\rho_3}(q) \\ X_{\rho_4}(q) \\ X_{\rho_5}(q) \\ X_{\rho_6}(q) \\ X_{\rho_7}(q) \end{pmatrix} = P(q) \begin{pmatrix} W_{-3}(q) \\ W_{-2}(q) \\ W_{-1}(q) \\ W_0(q) \\ W_1(q) \\ W_2(q) \\ W_3(q) \\ W_4(q) \end{pmatrix} \tag{11.21}$$

Then considering the eight by eight matrix of q series from Theorem 8 we find numerically the conjectural identities for τ on a small angle just above the positive reals

$$\mathbf{Z}_m(\tilde{q}) = s(\widehat{\Phi}_m)(\tau) S_I(q) \tag{11.22}$$

where

$$\begin{aligned} S_I(q) &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} P(q) \mathbf{Z}_{-3}(q) \\ &\times \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{11.23}$$

as τ on a small angle just above the negative real

$$\mathbf{Z}_m(\tilde{q}) = s(\widehat{\Phi}_m)(\tau) S_{II}(q) \tag{11.24}$$

where

$$\begin{aligned} S_{II}(q) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} P(q) \mathbf{Z}_{-3}(q) \\ &\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{11.25}$$

as τ on a small angle just below the negative reals

$$\mathbf{Z}_m(\tilde{q}) = s(\widehat{\Phi}_m)(\tau) S_{III}(q) \tag{11.26}$$

where

$$
\begin{aligned}
S_{III}(q) &= \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix} P(q)\mathbf{Z}_{-3}(q) \\
&\times \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&\tau^{-2}&0&0&0&0&0&0\\ 0&0&0&\tau^{-1}&0&0&0&0\\ 0&0&\tau^{-1}&0&0&0&0&0\\ 0&0&0&0&\tau^{-1}&0&0&0\\ 0&0&0&0&0&0&\tau^{-1}&0\\ 0&0&0&0&0&\tau^{-1}&0&0\\ 0&0&0&0&0&0&0&\tau^{-1} \end{pmatrix}
\end{aligned} \tag{11.27}
$$

as τ on a small angle just below the positive reals

$$\mathbf{Z}_m(\tilde{q}) \;=\; s(\widehat{\Phi}_m)(\tau)S_{IV}(q) \tag{11.28}$$

where

$$
\begin{aligned}
S_{IV}(q) &= \begin{pmatrix} -1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1 \end{pmatrix} P(q)\mathbf{Z}_{-3}(q) \\
&\times \begin{pmatrix} -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&\tau^{-2}&0&0&0&0&0&0\\ 0&0&0&\tau^{-1}&0&0&0&0\\ 0&0&\tau^{-1}&0&0&0&0&0\\ 0&0&0&0&\tau^{-1}&0&0&0\\ 0&0&0&0&0&0&\tau^{-1}&0\\ 0&0&0&0&0&\tau^{-1}&0&0\\ 0&0&0&0&0&0&0&\tau^{-1} \end{pmatrix}
\end{aligned} \tag{11.29}
$$

11.2 Computations near $\mathbb{R}$ give Stokes phenomenon?

Again noticed in [69], using quantum modular forms to study Borel resummation gives powerful tools to study Stoke phenomenon. In [69], there are some basic structural conjectures that relate to the Stokes phenomenon and with their assumption along with quantum modular forms one can compute conjectural formulae for all of the Stokes behaviour. To begin we will briefly give the conjectures in regards to the behaviour of the Stokes phenomenon for the asymptotics of certain q–hypergeometric functions.

Conjecture 13. *[67, 69][2] Suppose that $\widehat{\Phi}_\rho(\tau)$ is an asymptotic series around $\tau \sim \infty$ associated to a q–hypergeometric sum with discrete characteristic variety R with $\rho \in R$. Then if*

[2]The authors that gave this conjecture did not focus directly on q–hypergeometric functions so cannot be blamed if this does not hold more generally. However, this appears to be a natural extension of their conjecture as much of the structure does not seem to stem directly from knots.

V_ρ is the associated critical value so that

$$\widehat{\Phi}_\rho(2\pi i/\tau) \;=\; \mathbf{e}\Big(\frac{V_\rho}{(2\pi i)^2}\tau\Big)\Phi_\rho(2\pi i/\tau)\,, \tag{11.30}$$

the Stokes lines of the series $\widehat{\Phi}_\rho(\tau)$ are located on the rays given by

$$\frac{V_{\rho'} - V_\rho + 4\pi^2 k}{2\pi i}\,, \tag{11.31}$$

forming a Peacock pattern and the Stokes behaviour is determined by an integer $S_{\rho,\rho',k} \in \mathbb{Z}$ such that if

$$\arg(\tau) \;=\; \arg\Big(\frac{V_{\rho'} - V_\rho + (2\pi i)^2 k}{2\pi i}\Big)\,, \tag{11.32}$$

and s_+, s_- represent the Borel resummation with contours with a slight positive shift for the argument of the contour and a slight negative shift for the argument of the contour respectively, we have

$$s_+(\widehat{\Phi}_\rho)(2\pi i/\tau) - s_-(\widehat{\Phi}_\rho)(2\pi i/\tau) \;=\; S_{\rho,\rho',k}q^k\widehat{\Phi}_{\rho'}\,, \tag{11.33}$$

where of course $q = \mathbf{e}((2\pi i)^2\tau/(2\pi i)^2)$.

This conjecture allows for extremely powerful applications. Importantly, if we send τ to infinity horizontally just above the positive reals then using quantum modularity and the conjectural identities with the state integrals we can compute a canonical basis of the q–holonomic module. Then, if we do the same over the negative reals we will again find another basis however this will in general be different. We can in theiry dot he same on any ray and in each sector find a basis of the q–holonomic module. However, this implies that taking the quotient of the matrix valued quanttum modular form on adjacent sectors will give an elementray matrix wich is the identity besides from the ρ, ρ' entry where will with find a term

$$S_{\rho,\rho',k}q^k\,. \tag{11.34}$$

The beautiful result is that we can take products over all sectors in the upper or lower half planes and find that the matrix will become a matrix of q–series with each entry and power of q storing the information of the Stokes constant. Therefore, we get the following result.

Corollary 13. *Assuming that conjectures 12 and 13 are true, the generating series of Stokes constants is the quotient of matrices of quantum modular forms* i.e. *the original q–hypergeometric sums that gave the asymptotics series to begin with.*

We can now apply this to give conjectural generating series for Stokes constants for the previous two examples.

Example 63 (Stokes constants and the $A = 4$ Nahm sum)**.** *Considering equation (11.18) and equation (11.19), we see that using analytic continuation the Borel resummation just above the positive reals $s_I(\widehat{\Phi})$ and just above the negative reals $s_{II}(\widehat{\Phi})(\tau)$ are related by multiplication by the matrix*

$$
\begin{aligned}
&s_I(\widehat{\Phi})(\tau)^{-1}s_{II}(\widehat{\Phi})(\tau)\\
&= \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&q\end{pmatrix} F(q) \begin{pmatrix}1&0&0&0\\0&\tau&0&0\\0&0&\tau^2&0\\0&0&0&\tau^4\end{pmatrix} F(\tilde{q})^{-1}F(\tilde{q}) \begin{pmatrix}1&0&0&0\\0&-\tau&0&0\\0&0&\tau^2&0\\0&0&0&\tau^4\end{pmatrix}^{-1} F(q)^{-1} \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&q\\0&1&0&0\end{pmatrix}^{-1}\\
&= \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&q\end{pmatrix} F(q) \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} F(q)^{-1} \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&q\\0&1&0&0\end{pmatrix}^{-1}\\
&= \mathrm{Id} + \begin{pmatrix}-q-2q^2+ & 1+q+q^2+ & 1-q^2+ & -1-q+\\ q^2+ & -q-q^2+ & -q+ & q+q^2+\\ -q-q^2+ & 1-q^2+ & -q-2q^2+ & 2q^2+\\ q+ & -1+q+q^2+ & 2q+q^2+ & -q-2q^2\end{pmatrix} + O(q^3)
\end{aligned}
\tag{11.35}
$$

The constants of this expansion exactly agree with the matrix we computed in equation (4.134) if we only consider the terms with

$$\Im\Big(\frac{\mathrm{VC}_\phi - \mathrm{VC}_\rho}{2\pi i}\Big) > 0\,. \tag{11.36}$$

It also supplies a guess for the ? and 1 and -1.

Remark 28. *Notice that from just above the positive reals to negative reals we pass through infinitely many singularities, which gives rise toe the full q–series. Also note the extremely important point that we need the S and $-S$ cocycles to compute the Stokes phenomenon. Indeed, the Stokes constants would be quite trivial if we did not have the matrix coming from the difference between the S and $-S$ cocycles which in the previous examples was given by*

$$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{11.37}$$

Example 64 (Half surgery on the figure eight knot and Stokes constants)**.** *Therefore, assuming the conjecture on the behaviour of the Stokes phenomenon and our conjectural identities for the Borel resummation, we can compute the generating series of Stokes matrices* *as quotients of these matrices. Therefore, the Stokes constants in the upper half plane have generating functions*

$$\begin{aligned} & S_{II}(q)S_I(q)^{-1} - I \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1+q+3q^2 & -q-2q^2 & 1+q & -q^2 & q+2q^2 & -1-q+q^2 & -1+3q^2 & -1+3q^2 \\ q-q^2 & q^2 & -q-q^2 & 0 & -q^2 & q+q^2 & q & q \\ -1+2q+q^2 & -q-q^2 & 1 & -q^2 & q+q^2 & -1+q^2 & -1+q+2q^2 & -1+q+2q^2 \\ 1-2q-q^2 & q+q^2 & -1 & q^2 & -q-q^2 & 1-q^2 & 1-q-2q^2 & 1-q-2q^2 \\ -q+q^2 & -q^2 & q+q^2 & 0 & q^2 & -q-q^2 & -q & -q \\ 1-3q-q^2 & q & -1+q+2q^2 & q^2 & -q & 1-q-3q^2 & -2q-3q^2 & -2q-3q^2 \\ 1-3q-q^2 & q & -1+q+2q^2 & q^2 & -q & 1-q-3q^2 & -2q-3q^2 & -2q-3q^2 \end{pmatrix} + O(q^3) \end{aligned} \tag{11.38}$$

the Stokes constants in the lower half plane are given by

$$\begin{aligned} & S_{IV}(q^{-1})S_{III}(q^{-1})^{-1} - I \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q-2q^2 & q^2 & -q-q^2 & q+q^2 & -q^2 & -q & -q \\ -1 & 1+q & -q-q^2 & 1 & -1 & q+q^2 & 1-q-2q^2 & 1-q-2q^2 \\ -q+q^2 & -q^2 & 0 & -q^2 & q^2 & 0 & -q^2 & -q^2 \\ -q-q^2 & q+2q^2 & -q^2 & q+q^2 & -q-q^2 & q^2 & q & q \\ -2q^2 & -1-q+q^2 & q+q^2 & -1+q^2 & 1-q^2 & -q-q^2 & -1+q+3q^2 & -1+q+3q^2 \\ 2q^2 & 1-3q^2 & -q & 1-q-2q^2 & -1+q+2q^2 & q & -2q-3q^2 & -2q-3q^2 \\ q+2q^2 & 1-3q^2 & -q & 1-q-2q^2 & -1+q+2q^2 & q & -2q-3q^2 & -2q-3q^2 \end{pmatrix} + O(q^3) \end{aligned} \tag{11.39}$$

These first few coefficients agree with numerical calculations of the growth of the coefficients.

11.3 Questions for the future

It seems that one of the most important aspects that will aid in the proofs of these conjectures is that the Borel resummation of these asymptotic series satisfy $\tilde{q}$–difference equations. This appears in [75] as a result of explicit formulae for the Borel transform, however, in general for these q–hypergeometric functions is still unclear.

From the perspective of 3–manifolds, these Stokes constants are clearly interesting invariants of three–manifolds. Conjecturally [69], for knots this agrees with the 3d index [44] however this theory does not include the trivial connection as long been noticed. It would be interesting if there was some way to define q–series invariants that agreed with the Stokes constants for all asymptotic series.

In a different direction, it would be extremely interesting if there was a mathematical definition for the holomorphic blocks. These holomorphic blocks factorise state integrals of [8] but moreover they factorise the state integrals of [74] also they appear as integrals in [21]. Therefore, they seem to store not only the information of the Borel resummation of asymptotic series but the Stokes constants as well. Indeed, these holomorphic blocks should give rise to the quantum modular forms as we have taken in examples of simple three–manifolds.

Hopefully, this thesis has convinced the reader that many q–hypergeometric functions are quantum modular forms. It is of course of interest to understand the class of q–hypergeometric functions that are quantum modular forms. It seems possible that all proper q–hypergeometric functions are in fact quantum modular forms and this behaviour is extremely general. The are various approaches that may shed light on this question for future work.

Of course one of the extremely interesting questions is still in the formulation and proof of Nahm's conjecture relating q–hypergeometric sums to modular forms. A beautiful combination of themes would be if an independent construction of Stokes constants and their triviality would lead to insights into the underlying structures associated to this conjecture.

Part VI

Programs

Presets

The following functions should be loaded as they are used throughout all the following code.

Code 1 (The presets)**.**

```
E(x)=exp(2*Pi*I*x)
{DD(k)=local(t);if(k,t=2^(2*k+sum(l=0,log(k)/log(2)+1,floor(k/2^l)));
forprime(p=3,k+2,t=t*p^(sum(l=0,log(k/(p-2))/log(p)+1,floor(k/(p-2)/p^l))));t,1)};
{Gf(q,n,N)=
local(qk,t,s);
eee=exp(eps+O(eps^(n+1)));qk=1;t=1;s=1;
for(k=1,N,qk=qk*q;t=-t*qk/(1-qk)*eee;s=s+t);
-(-1)^n*bernfrac(n)/n/2-n!*polcoeff(log(s/polcoeff(s,0,eps)),n,eps)}
vecdiff(x)=vector(length(x)-1,j,x[j+1]-x[j]);
vecdiffk(x,k)=local(tv);tv=x;for(j=1,k,tv=vecdiff(tv));tv;
lim(x,n,k)=vecdiffk(vector(k+1,j,(j+n)^k*x[j+n-1]),k)[1]/k!;
{pollim(x,n,k)=local(tfo);tfo=vector(length(x));
 for(j=n,n+k+1,tfo[j]=(x[j+1]/x[j]-1)*j);lim(tfo,n,k)};
{explim(x,n,k)=local(tl);tl=vector(length(x));
 for(j=n,n+k+1,tl[j]=x[j+1]/x[j]);lim(tl,n,k)};
{asymp(x,n,k)=local(t1,t2,t3,t4,t5);t1=explim(x,n,k);t2=vector(length(x));
 for(j=n,n+k+1,t2[j]=x[j]/t1^j);t3=pollim(t2,n,k);t4=vector(length(x));
 for(j=n,n+k,t4[j]=x[j]/t1^j/j^t3);t5=lim(t4,n,k);[t1,t3,t5]}
{oscserv(a,C,d,x,t,alpha=0)=
 local(ta,llC);llC=length(C);ta=a;
 for(k=1,length(a),ta[k]=ta[k]*(x+k+alpha)^t);
 for(j=1,llC,for(k=1,length(ta),ta[k]=ta[k]*C[j]^(-(x+k+alpha))*(x+k+alpha)^(-d[j]));for(j=1,t,ta=vecdiff(ta));for(k=1,length(ta),ta[k]=ta[k]*C[j]^(x+k+alpha)*(x+k+alpha)^(d[j])));
 ta/t!}
{asympv(a,C,d,x,t,alpha=0)=
 local(lC,taa,tm);lC=length(C);
 taa=vector(lC*(t+1),k,a[x+k]);
 taa=oscserv(taa,C,d,x,t,alpha);
 tm=matrix(lC,lC);
 for(j=1,lC,tm[,j]=oscserv(vector(lC*(t+1),k,C[j]^(x+k+alpha)*(x+k+alpha)^d[j]),C,d,x,t)~);
 tm^(-1)*taa~}
{oscser(x,n,k,z)=local(t);t=vector(floor(length(x)/2));
 for(j=n,n+k+1,t[j]=sum(l=0,j,binomial(j,l)*x[j+l]*z^(j-l)));t}
{asymposc(x,n,k,z)=t1=oscser(x,n,k,z);t2=asymp(t1,n,k);
 [(-z+sqrt(z^2+4*t2[1]))/2,t2[2],
 t2[3]/(1+((-z+sqrt(z^2+4*t2[1]))/2)/((-z+sqrt(z^2+4*t2[1]))/2+z))^t2[2];
 (-z-sqrt(z^2+4*t2[1]))/2,t2[2],
 t2[3]/(1+((-z-sqrt(z^2+4*t2[1]))/2)/((-z-sqrt(z^2+4*t2[1]))/2+z))^t2[2]]}
```

Appendix A

Three dimensional topology

A.1 Representations, Neumann–Zagier matrices and volumes

Code 2 (Representation of the fundamental group of 4_1).

```
sage: from snappy import Manifold
sage: M=Manifold("4_1")
sage: p=M.ptolemy_variety(2,'all').compute_solutions()
sage: G = M.fundamental_group(simplify_presentation = False)
sage: G.peripheral_curves()
[('Ab', 'cAcBaCbAb')]
sage: p.evaluate_word('a', G)
[[], [[[Mod(2, x^2 - x + 1), Mod(1, x^2 - x + 1)], [Mod(-1, x^2 - x + 1),
   Mod(0, x^2 - x + 1)]]]]
sage: p.evaluate_word('b', G)
[[], [[[Mod(-x + 2, x^2 - x + 1), Mod(1, x^2 - x + 1)], [Mod(-1, x^2 - x +
    1), Mod(0, x^2 - x + 1)]]]]
sage: p.evaluate_word('c', G)
[[], [[[Mod(-x + 2, x^2 - x + 1), Mod(1, x^2 - x + 1)], [Mod(x, x^2 - x +
   1), Mod(x, x^2 - x + 1)]]]]
sage: p.evaluate_word('Ab', G)
[[], [[[Mod(1, x^2 - x + 1), Mod(0, x^2 - x + 1)], [Mod(-x, x^2 - x + 1),
   Mod(1, x^2 - x + 1)]]]]
sage: p.evaluate_word('cAcBaCbAb', G)
[[], [[[Mod(-1, x^2 - x + 1), Mod(0, x^2 - x + 1)], [Mod(2*x - 4, x^2 - x
   + 1), Mod(-1, x^2 - x + 1)]]]]
```

Code 3 (Neumann–Zagier matrices for 5_2).

```
In[1]: M=Manifold("5_2")
```

```
In[2]: M.gluing_equations()
Out[2]:
matrix([[ 1,  0,  1,  1,  2,  0,  1,  0,  1],
        [ 0,  1,  1,  0,  0,  2,  0,  1,  1],
        [ 1,  1,  0,  1,  0,  0,  1,  1,  0],
        [-1,  0,  0,  0,  0,  1,  0,  0,  0],
        [ 2,  0, -3,  1,  0, -2,  0,  0,  1]])
```

Code 4 (Neumann–Zagier matrices for 7_4).

```
In[1]: M=Manifold("7_4")
In[2]: M.gluing_equations(form='rect')
Out[2]:
[([1, 1, 1, 0, -1, 0], [0, -1, -1, 0, 1, 0], -1),
 ([-1, -1, 0, -1, 0, -1], [0, 0, 0, 1, -1, 1], 1),
 ([-1, 1, 0, 0, 1, 1], [1, 0, -1, -1, 0, 0], -1),
 ([0, 0, -1, 1, 0, 0], [-1, 0, 1, 0, -1, 0], -1),
 ([1, -1, -1, 1, 0, -1], [0, 1, 1, -1, 0, 0], -1),
 ([0, 0, 1, -1, 0, 1], [0, 0, 0, 1, 1, -1], 1),
 ([2, 1, 2, 0, -1, 1], [0, -1, -1, 0, 2, -1], 1),
 ([-2, 0, -2, 0, 1, -2], [0, -1, 3, 2, -1, 2], -1)]
```

Code 5 (Ptolemy variety for 7_4).

```
sage: from snappy import Manifold
sage: M=Manifold('7_4')
sage: p=M.ptolemy_variety(2,'all').compute_solutions()
sage: zs=p.cross_ratios()
sage: zs[1][0]
CrossRatios({
'z_0000_0': Mod(-x^3 - 2*x^2 - x, x^4 + 3*x^3 + 2*x^2 + 1), ...
'z_0000_1': Mod(-x^3 - 3*x^2 - 2*x + 1, x^4 + 3*x^3 + 2*x^2 + 1), ...
'z_0000_2': Mod(x + 2, x^4 + 3*x^3 + 2*x^2 + 1), ...
'z_0000_3': Mod(-x^3 - 2*x^2 - x, x^4 + 3*x^3 + 2*x^2 + 1), ...
'z_0000_4': Mod(x + 1, x^4 + 3*x^3 + 2*x^2 + 1), ...
'z_0000_5': Mod(-x^3 - x^2, x^4 + 3*x^3 + 2*x^2 + 1), ...
}, is_numerical = False, ...)
sage: zs[1][1]
CrossRatios({
'z_0000_0': Mod(-1/2*x^2 + x + 3/2, x^3 - 2*x^2 - x - 2), ...
'z_0000_1': Mod(-1/2*x^2 + 3/2*x, x^3 - 2*x^2 - x - 2), ...
'z_0000_2': Mod(-1/4*x^2 + 3/4*x + 1/2, x^3 - 2*x^2 - x - 2), ...
'z_0000_3': Mod(x, x^3 - 2*x^2 - x - 2), ...
'z_0000_4': Mod(x + 1, x^3 - 2*x^2 - x - 2), ...
'z_0000_5': Mod(-1/2*x^2 + 3/2*x, x^3 - 2*x^2 - x - 2), ...
}, is_numerical = False, ...)
```

Code 6 (Extended Bloch group elements for 5_2).

```
? z1=polroots(x^3-2*x^2+3*x-1);
? z2=vector(3,j,z1[j]^2-z1[j]+2);
? z3=z1;
? Z1(j,p,q)=[log(z1[j])+p*Pi*I,-log(1-z1[j])+q*Pi*I,-log(z1[j])+log(1-z1[j
   ])-p*Pi*I-q*Pi*I]
? Z2(j,p,q)=[log(z2[j])+p*Pi*I,-log(1-z2[j])+q*Pi*I,-log(z2[j])+log(1-z2[j
   ])-p*Pi*I-q*Pi*I]
? Z3(j,p,q)=[log(z3[j])+p*Pi*I,-log(1-z3[j])+q*Pi*I,-log(z3[j])+log(1-z3[j
   ])-p*Pi*I-q*Pi*I]
? round((Z1(3,p1,q1)[1]+Z1(3,p1,q1)[3]+Z2(3,p2,q2)[1]+2*Z2(3,p2,q2)[2]+Z3
   (3,p3,q3)[1]+Z3(3,p3,q3)[3])/Pi/I*10^10)/10^10
/* = -q1 + (p2 + (2*q2 - q3)) */
? round((Z1(3,p1,q1)[2]+Z1(3,p1,q1)[3]+2*Z2(3,p2,q2)[3]+Z3(3,p3,q3)[2]+Z3
   (3,p3,q3)[3])/Pi/I*10^10)/10^10
/* = -p1 + (-2*p2 + (-2*q2 + (-p3 - 2))) */
? round((Z1(3,p1,q1)[1]+Z1(3,p1,q1)[2]+Z2(3,p2,q2)[1]+Z3(3,p3,q3)[1]+Z3(3,
   p3,q3)[2])/Pi/I*10^10)/10^10
/* = p1 + (q1 + (p2 + (p3 + (q3 + 2)))) */
? round((-Z1(3,p1,q1)[1]+Z2(3,p2,q2)[3])/Pi/I*10^10)/10^10
/* = -p1 + (-p2 + (-q2 - 1)) */
? round((2*Z1(3,p1,q1)[1]-3*Z1(3,p1,q1)[3]+Z2(3,p2,q2)[1]-2*Z2(3,p2,q2)
   [3]+Z3(3,p3,q3)[3])/Pi/I*10^10)/10^10
/* = 5*p1 + (3*q1 + (3*p2 + (2*q2 + (-p3 + (-q3 + 4))))) */
```

Code 7 (Complexified volumes of 5_2).

```
? R(z,p,q,s=0)=log(z)*log(1-z)/2-if(s==0,intnum(x=0,1,log(1-x*z)/x),intnum
   (x=0,1,log(1-(z*(1-exp(-sign(s)*Pi*I*x))/2))/(z*(1-exp(-sign(s)*Pi*I*x)
   )/2)*((z*(sign(s)*Pi*I*exp(-sign(s)*Pi*I*x))/2))))+Pi*I/2*(p*log(1-z)+q
   *log(z))-Pi^2/6
? z1=polroots(x^3-2*x^2+3*x-1);
? z2=vector(3,j,z1[j]^2-z1[j]+2);
? z3=z1;
? R(z1[3],p3,q3)+R(z2[3],-2*p3 - 2*q3 - 2,p3 + 2*q3 + 1)+R(z3[3],p3,q3)
/* = (0.E-211 - 4.158637801 E-212*I)*p3 + ((-9.869604401 - 5.940911145 E
   -212*I)*q3 + (-6.845476025 + 2.828122088*I)) */
? R(z1[1],p3,q3)+R(z2[1],-2*p3 - 2*q3,p3 + 2*q3 + 1,-1)+R(z3[1],p3,q3)
/* = (9.8696 - 2.3764 E-212*I)*p3 + ((9.8696 - 4.7527 E-212*I)*q3 +
   (-1.1135 - 3.5645 E-212*I)) */
```

Code 8 (Complexified volumes of 4_1, 5_2 and 7_4).

```
sage: from snappy import Manifold
sage: pari.set_real_precision(20)
```

```
sage: Manifold("4_1").ptolemy_variety(2,'all').compute_solutions().complex_volume_numerical()
[[], [[2.0298832128193072500 - 3.8988952951615643 81 E-21*I, -2.0298832128193072500 - 3.8988952951615643 81 E-21*I]]]
sage: Manifold("5_2").ptolemy_variety(2,'all').compute_solutions().complex_volume_numerical()
[[], [[8.8631126875044261 53 E-20 - 0.53147951437430242645*I ,2.8281220883307831628 + 0.26573975718715121322*I, -2.8281220883307831628 + 0.26573975718715121322*I]]]
sage: Manifold("7_4").ptolemy_variety(2,'all').compute_solutions().complex_volume_numerical()
[[], [[2.0298832128193072500 + 4.5261665304785806 99 E-20*I, -2.0298832128193072500 + 4.5261665304785806 99 E-20*I ,2.0298832128193072500 + 2.2448517331957115278 E-20*I, -2.0298832128193072500 + 2.2448517331957115278 E-20*I], [9.5426447734604955 73 E-20 + 0.78719857113403445283*I, -5.1379412018734177699 + 0.42886774785709599187*I, 5.1379412018734177699 + 0.42886774785709599187*I]]]
```

A.2 Loop invariants

Code 9 (2–loop invariants of 4_1).

```
h;w;A=[1,1;1,0];B=[-1,-1;0,-1];nu=[0;0];f=[0;0];ff=[0;0];z=[Mod(w,w^2-w+1),Mod(w,w^2-w+1)];zz=vector(length(z),j,1/(1-z[j]));zzz=vector(length(z),j,1-1/z[j]);
PP=h*(-B^(-1)*A+matdiagonal(zz))^(-1);
Gam(h,ell,j,NN)=(-1)^ell*sum(p=if(ell==1,1)+if(ell==2,1),NN,if(p==1,1/2,bernfrac(p))*h^(p-1)/p!*polylog(2-p-ell,z[j]^(-1)))+if(ell==1,-1/2*(B^(-1)*nu)[j,1])
Gam0(NN)=if(NN==1,1/2,if(Mod(NN,2)==Mod(0,2),bernvec(NN/2)[NN/2+1]))/NN!*sum(j=1,length(z),polylog(2-NN,z[j]^(-1)))+if(NN==2,1/8*(f~*B^(-1)*A*f)[1,1])
S2(NN)=Gam0(2)+polcoeff(sum(i=1,length(z),1/8*Gam(h,4,i,NN)*PP[i,i]^2+1/2*Gam(h,2,i,NN)*PP[i,i]+sum(j=1,length(z),1/8*PP[i,i]*Gam(h,3,i,NN)*PP[i,j]*Gam(h,3,j,NN)*PP[j,j]+1/12*Gam(h,3,i,NN)*PP[i,j]^3*Gam(h,3,j,NN)+1/2*Gam(h,1,i,NN)*PP[i,j]*Gam(h,3,j,NN)*PP[j,j]+1/2*Gam(h,1,i,NN)*PP[i,j]*Gam(h,1,j,NN))),1,h)
S2(2)
/* = Mod(11/108*w - 5/54, w^2 - w + 1)*/
```

Code 10 (2–loop invariant of $4_1(1,2)$).

```
h;w;A=[1,1;1,0];B=[-1,-1;0,-1];C=[0,1];D=[0,-1];R=matrix(2,2,i,j,2*2*if(i==2,1,0)*(B^(-1))[j,2]/(1+2*2*(D*B^(-1))[2]));nu=[0;0];f=[0;0];ff
```

```
    =[0;0];z2=Mod(w,-1 + 8*w - 27*w^2 + 49*w^3 - 50*w^4 + 27*w^5 - 7*w^6 +
    3*w^7 - 4*w^8 + 4*w^9 - 2*w^10 - w^11 + w^13 + w^14);z1=-136*z2^13 -
    229*z2^12 - 156*z2^11 + 32*z2^10 + 299*z2^9 - 334*z2^8 + 318*z2^7 -
    191*z2^6 + 821*z2^5 - 3112*z2^4 + 4668*z2^3 - 3468*z2^2 + 1309*z2 -
    204;z=[z1,z2];zz=vector(length(z),j,1/(1-z[j]));zzz=vector(length(z),j
    ,1-1/z[j]);x=-154+896*z2-2142*z2^2+2595*z2^3-1561*z2^4+395*z2^5-123*z2
    ^6+177*z2^7-202*z2^8+147*z2^9+56*z2^10-28*z2^11-85*z2^12-63*z2^13;
matdet((A-R)*[1-1/z[1],0;0,1-1/z[2]]+B*[1/z[1],0;0,1/z[2]])*(1+2*2*(D*B
    ^(-1))[2])/(x+2+1/x)
/* Mod(3711*w^13 + 5779*w^12 + 3238*w^11 - 1863*w^10 - 8408*w^9 + 10189*w
    ^8 - 9190*w^7 + 6030*w^6 - 22641*w^5 + 87584*w^4 - 136845*w^3 + 105904*
    w^2 - 41514*w + 6703, w^14 + w^13 - w^11 - 2*w^10 + 4*w^9 - 4*w^8 + 3*w
    ^7 - 7*w^6 + 27*w^5 - 50*w^4 + 49*w^3 - 27*w^2 + 8*w - 1) */
PP=h*matrix(3,3,i,j,if(max(i,j)<3,(-B^(-1)*A+matdiagonal(zz))[i,j],if([i,j
    ]==[3,3],-(2*1/2+4*(D*B^(-1))[2]),2*(B^(-1))[if(i==3,j,i),2])))^(-1)
Gam(h,ell,j,NN)=if(j==3,if(ell==1,(f~*B^(-1))[1,2]+1/2)-(-1)^ell*polylog
    (1-ell,-1/x),(-1)^ell*sum(p=if(ell==1,1)+if(ell==2,1),NN,if(p==1,1/2,if
    (Mod(p,2)==Mod(0,2),bernvec(p/2)[p/2+1]))*h^(p-1)/p!*polylog(2-p-ell,z[
    j]^(-1)))+if(ell==1,-1/2*(B^(-1)*nu)[j,1]))
Gam0(NN)=if(NN==1,1/2,bernfrac(NN))/NN!*sum(j=1,length(z),polylog(2-NN,z[j
    ]^(-1)))+if(NN==2,1/8*(f~*B^(-1)*A*f)[1,1])
S2(NN)=Gam0(2)+polcoeff(sum(i=1,length(z)+1,1/8*Gam(h,4,i,NN)*PP[i,i
    ]^2+1/2*Gam(h,2,i,NN)*PP[i,i]+sum(j=1,length(z)+1,1/8*PP[i,i]*Gam(h,3,i
    ,NN)*PP[i,j]*Gam(h,3,j,NN)*PP[j,j]+1/12*Gam(h,3,i,NN)*PP[i,j]^3*Gam(h
    ,3,j,NN)+1/2*Gam(h,1,i,NN)*PP[i,j]*Gam(h,3,j,NN)*PP[j,j]+1/2*Gam(h,1,i,
    NN)*PP[i,j]*Gam(h,1,j,NN))),1,h)
S2(2)
/* Mod(-103985214498148/52058057626129*w^13 -
    184056457134922/52058057626129*w^12 - 542317054615207/208232230504516*w
    ^11 + 59505081205693/208232230504516*w^10 +
    712734937701865/156174172878387*w^9 - 893320433832741/208232230504516*w
    ^8 + 2799985419301201/624696691513548*w^7 -
    1552744881772601/624696691513548*w^6 + 617002533395308/52058057626129*w
    ^5 - 6997401920285924/156174172878387*w^4 +
    40417653146722771/624696691513548*w^3 -
    9551945238699689/208232230504516*w^2 +
    2570480289595373/156174172878387*w - 1027130175419887/416464461009032,
    w^14 + w^13 - w^11 - 2*w^10 + 4*w^9 - 4*w^8 + 3*w^7 - 7*w^6 + 27*w^5 -
    50*w^4 + 49*w^3 - 27*w^2 + 8*w - 1) */
xxi=Mod(119-770*w+2004*w^2-2599*w^3+1647*w^4-420*w^5+116*w^6-171*w^7+192*w
    ^8-167*w^9-49*w^10+50*w^11+104*w^12+69*w^13,w^14 + w^13 - w^11 - 2*w^10
     + 4*w^9 - 4*w^8 + 3*w^7 - 7*w^6 + 27*w^5 - 50*w^4 + 49*w^3 - 27*w^2 +
    8*w - 1)
xxi^7-xxi^6-2*xxi^5+6*xxi^4-11*xxi^3+6*xxi^2+3*xxi-1
/* Mod(0, w^14 + w^13 - w^11 - 2*w^10 + 4*w^9 - 4*w^8 + 3*w^7 - 7*w^6 +
    27*w^5 - 50*w^4 + 49*w^3 - 27*w^2 + 8*w - 1) */
matdet((A-R)*[1-1/z[1],0;0,1-1/z[2]]+B*[1/z[1],0;0,1/z[2]])*(1+2*2*(D*B
    ^(-1))[2])/(x+2+1/x)-(74+66*xxi-133*xxi^2+74*xxi^3-31*xxi^4-15*xxi
```

```
    ^5+12*xxi^6)
/* Mod(0, w^14 + w^13 - w^11 - 2*w^10 + 4*w^9 - 4*w^8 + 3*w^7 - 7*w^6 +
    27*w^5 - 50*w^4 + 49*w^3 - 27*w^2 + 8*w - 1) */
S2(2)+1/6-(1497746+1345119*xxi-3675733*xxi^2+2082815*xxi^3-839488*xxi
    ^4-283405*xxi^5+383432*xxi^6)/24/(74+66*xxi-133*xxi^2+74*xxi^3-31*xxi
    ^4-15*xxi^5+12*xxi^6)^3
/* Mod(0, w^14 + w^13 - w^11 - 2*w^10 + 4*w^9 - 4*w^8 + 3*w^7 - 7*w^6 +
    27*w^5 - 50*w^4 + 49*w^3 - 27*w^2 + 8*w - 1) */
```

Appendix B

Asymptotics

B.1 Extrapolation methods

Code 11 (Stirling's approximation with Richardson's method).

```
nextxR(x,k)=vector(floor(length(x)/2),j,x[2*j]+(x[2*j]-x[j])/(2^k-1));
limR(x,k)=local(t);t=x;for(ll=1,k-1,t=nextxR(t,ll));t
xx=vector(1000,n,n!/n^n*exp(n)/sqrt(n));
[limR(xx,1)[100],limR(xx,1)[100]-sqrt(2*Pi)]
/* = [2.508717995, 0.002089720526] */
[limR(xx,2)[100],limR(xx,2)[100]-sqrt(2*Pi)]
/* = [2.506627844, -4.301328534 E-7] */
[limR(xx,3)[100],limR(xx,3)[100]-sqrt(2*Pi)]
/* = [2.506628274, -8.413407752 E-10] */
```

Code 12 (Stirling's approximation with Zagier's method).

```
vecdiff(x)=vector(length(x)-1,j,x[j+1]-x[j]);
vecdiffk(x,k)=local(tv);tv=x;for(j=1,k,tv=vecdiff(tv));tv;
lim(x,n,k)=vecdiffk(vector(k+1,j,(j+n)^k*x[j+n-1]),k)[1]/k!;
[lim(xx,100,1),lim(xx,100,1)-sqrt(2*Pi)]
/* = [2.506606727, -2.154718332 E-5] */
[lim(xx,100,2),lim(xx,100,2)-sqrt(2*Pi)]
/* = [2.506628488, 2.129785028 E-7] */
[lim(xx,100,3),lim(xx,100,3)-sqrt(2*Pi)]
/* = [2.506628273, -2.015005875 E-9] */
```

Code 13 (Stirling's approximation with the variants).

```
xx=vector(1000,n,n^n);
{faclim(x,n,k)=local(tl);tl=vector(length(x));
```

```
    for(j=n,n+k+1,tl[j]=x[j+2]*x[j]/x[j+1]^2);lim(tl,n,k)};
faclim(xx*1.,100,1)
/* = 0.9999508222 */
faclim(xx*1.,100,3)
/* = 0.9999999981 */
faclim(xx*1.,100,10)-1
/* = 2.259629648 E-24 */
xx=vector(1000,n,n^n/n!);
{explim(x,n,k)=local(tl);tl=vector(length(x));
    for(j=n,n+k+1,tl[j]=x[j+1]/x[j]);lim(tl,n,k)};
log(explim(xx*1.,100,1))
/* = 1.000004085 */
log(explim(xx*1.,100,3))
/* = 1.000000000 */
log(explim(xx*1.,100,10))-1
/* = -1.230061846 E-25 */
xx=vector(1000,n,n^n/n!/exp(n));
{pollim(x,n,k)=local(tfo);tfo=vector(length(x));
    for(j=n,n+k+1,tfo[j]=(x[j+1]/x[j]-1)*j);lim(tfo,n,k)};
pollim(xx,100,1)
/* = -0.5000020380 */
pollim(xx,100,3)
/* = -0.5000000000 */
pollim(xx,100,10)+1/2
/* = 1.773889943 E-26 */
```

Code 14 (asymp function).

```
xx=vector(1000,n,n^n/n!);
{asymp(x,n,k)=local(t1,t2,t3,t4,t5);t1=explim(x,n,k);t2=vector(length(x));
for(j=n,n+k+1,t2[j]=x[j]/t1^j);t3=pollim(t2,n,k);t4=vector(length(x));
for(j=n,n+k,t4[j]=x[j]/t1^j/j^t3);t5=lim(t4,n,k);[t1,t3,t5]}
asymp(xx*1.,100,10)
/* = [2.718281828, -0.5000000000, 0.3989422804] */
asymp(xx*1.,100,10)-[exp(1),-1/2,1/sqrt(2*Pi)]
/* = [-3.343654765 E-25, 1.432584409 E-22, -3.772064459 E-22] */
```

Code 15 (Asymptotics of Airy function with Oscillating method).

```
xx=vector(250,n,airy(-n^(2/3))[1]);
{oscser(x,n,k,z)=local(t);t=vector(floor(length(x)/2));
for(j=n,n+k+1,t[j]=sum(l=0,j,binomial(j,l)*x[j+l]*z^(j-l)));t}
{asymposc(x,n,k,z)=t1=oscser(x,n,k,z);t2=asymp(t1,n,k);
[(-z+sqrt(z^2+4*t2[1]))/2,t2[2],
 t2[3]/(1+((-z+sqrt(z^2+4*t2[1]))/2)/((-z+sqrt(z^2+4*t2[1]))/2+z))^t2[2];
(-z-sqrt(z^2+4*t2[1]))/2,t2[2],
 t2[3]/(1+((-z-sqrt(z^2+4*t2[1]))/2)/((-z-sqrt(z^2+4*t2[1]))/2+z))^t2[2]]}
```

```
for(k=0,9,print([k,asymposc(xx,100,5,E(k/10))[,1]]))
/*
[0, [1.3264 E5 - 2.6475 E-995*I, -1.3264 E5 + 2.6475 E-995*I]~]
[1, [0.90125 + 0.63470*I, -1.7103 - 1.2225*I]~]
[2, [0.78589 + 0.61837*I, -1.0949 - 1.5694*I]~]
[3, [0.78589 + 0.61837*I, -0.47687 - 1.5694*I]~]
[4, [0.78589 + 0.61837*I, 0.023130 - 1.2062*I]~]
[5, [0.50000 + 2.5912 E4*I, 0.50000 - 2.5912 E4*I]~]
[6, [0.78589 - 0.61837*I, 0.023130 + 1.2062*I]~]
[7, [0.78589 - 0.61837*I, -0.47687 + 1.5694*I]~]
[8, [0.78589 - 0.61837*I, -1.0949 + 1.5694*I]~]
[9, [0.90125 - 0.63470*I, -1.7103 + 1.2225*I]~]
*/
asymposc(xx,100,5,E(3/10))[1,]
/* =[0.78589 + 0.61837*I, -0.16667 - 2.4163 E-10*I, 0.19947 - 0.19947*I]*/
log(asymposc(xx,100,5,E(3/10))[1,1])-2*I/3
/* = -1.7639 E-13 + 2.1943 E-13*I */
asymposc(xx,100,5,E(3/10))[1,2]+1/6
/* = 1.1732 E-10 - 2.4163 E-10*I */
asymposc(xx,100,5,E(3/10))[1,3]-E(-1/8)/2/sqrt(Pi)
/* = 1.7362 E-10 + 4.4925 E-10*I */
asymposc(xx,100,5,E(7/10))[1,]
/* =[0.78589 - 0.61837*I, -0.16667 + 2.4163 E-10*I, 0.19947 + 0.19947*I]*/
log(asymposc(xx,100,5,E(7/10))[1,1])+2*I/3
/* = -1.7639 E-13 - 2.1943 E-13*I */
asymposc(xx,100,5,E(7/10))[1,2]+1/6
/* = 1.1732 E-10 + 2.4163 E-10*I */
asymposc(xx,100,5,E(7/10))[1,3]-E(1/8)/2/sqrt(Pi)
/* = 1.7362 E-10 - 4.4925 E-10*I */
xx=vector(250,n,airy(-n^(2/3))[1]-exp((2/3*n-Pi/4)*I)/2/sqrt(Pi)/n^(1/6)
    -exp(-(2/3*n-Pi/4)*I)/2/sqrt(Pi)/n^(1/6));
asymposc(xx,100,10,E(3/10))[1,3]/(exp(-Pi/4*I)/2/sqrt(Pi))+5/48*I
/* = -4.5841 E-16 - 1.9299 E-16*I */
asymposc(xx,100,10,E(7/10))[1,3]/(exp(Pi/4*I)/2/sqrt(Pi))-5/48*I
/* = -4.5841 E-16 + 1.9299 E-16*I */
```

B.2 Optimal truncation and Pochhammer asymptotics

Code 16 (Error of optimal truncation)**.**

```
for(j=1,30,print([j,(exp(100)*eint1(100)-sum(k=0,10*j,(-1)^k*k!/100^(k+1))
   )]))
/*
[1, -3.5674 E-17]
[2, -4.1939 E-25]
[3, -6.2408 E-31]
```

```
[4, -2.3607 E-35]
[5, -1.0228 E-38]
[6, -3.1406 E-41]
[7, -4.9566 E-43]
[8, -3.1931 E-44]
[9, -7.0592 E-45]
[10, -4.6779 E-45]
[11, -8.3365 E-45]
[12, -3.6551 E-44]
[13, -3.6605 E-43]
[14, -7.8626 E-42]
[15, -3.4317 E-40]
[16, -2.9040 E-38]
[17, -4.5732 E-36]
[18, -1.2924 E-33]
[19, -6.3462 E-31]
[20, -5.2606 E-28]
[21, -7.1722 E-25]
[22, -1.5709 E-21]
[23, -5.4097 E-18]
[24, -2.8725 E-14]
[25, -2.3099 E-10]
[26, -2.7671 E-6]
[27, -0.048629]
[28, -1236.2]
[29, -4.4861 E7]
[30, -2.2959 E12]
*/
sqrt(2*Pi*100)*exp(-100)
/* 9.3248 E-43 */

for(j=1,20,print([j,(exp(1000)*eint1(1000)-sum(k=0,100*j,(-1)^k*k!/1000^(k
   +1)))]))
/*
[1, -8.5542 E-147]
[2, -1.3190 E-229]
[3, -7.0768 E-290]
[4, -1.8319 E-335]
[5, -4.0707 E-370]
[6, -4.7490 E-396]
[7, -9.9780 E-415]
[8, -3.4282 E-427]
[9, -3.1996 E-434]
[10, -2.0124 E-436]
[11, -2.7997 E-434]
[12, -3.4647 E-428]
[13, -1.7861 E-418]
[14, -2.0190 E-405]
```

```
[15, -2.8875 E-389]
[16, -3.2446 E-370]
[17, -1.8880 E-348]
[18, -3.9385 E-324]
[19, -2.1242 E-297]
[20, -2.2110 E-268]
*/
sqrt(2*Pi*1000)*exp(-1000)
/* = 4.0235 E-433 */
```

Code 17 (Error of optimal truncation with exponentially small terms)**.**

```
for(j=1,20,print([j,(exp(1000)*eint1(1000)+exp(-7*1000/8)-sum(k=0,100*j
    ,(-1)^k*k!/1000^(k+1)))]))
/*
[1, -8.5542 E-147]
[2, -1.3190 E-229]
[3, -7.0768 E-290]
[4, -1.8319 E-335]
[5, -4.0707 E-370]
[6, 9.8249 E-381]
[7, 9.8249 E-381]
[8, 9.8249 E-381]
[9, 9.8249 E-381]
[10, 9.8249 E-381]
[11, 9.8249 E-381]
[12, 9.8249 E-381]
[13, 9.8249 E-381]
[14, 9.8249 E-381]
[15, 9.8249 E-381]
[16, -3.2446 E-370]
[17, -1.8880 E-348]
[18, -3.9385 E-324]
[19, -2.1242 E-297]
[20, -2.2110 E-268]
*/
exp(-7*1000/8)
/* = 9.8249 E-381 */
```

Code 18 (Smooth optimal truncation)**.**

```
EGZ(X,b)=intnum(t=0,[oo,abs(X)],t^(abs(X)-b)*exp(-t*abs(X))/(t-X/abs(X)))

{fsmop(h)=
local(co,ot,sot);
N=floor(abs(-1/h));
b=abs(-1/h)-N;
```

```
ot=sum(k=1,N,(-1)^(k-1)*(k-1)!*h^k);
sot=((-1/h)/abs(-1/h))^(-N)*EGZ(-1/h,b);
[ot,sot,ot+sot]}

exp(1000.1)*eint1(1000.1)
/* = 0.00099890 */

fsmop(1/1000.1)
/* = [0.00099890, 1.8199 E-436, 0.00099890] */

exp(1000.1)*eint1(1000.1)-fsmop(1/1000.1)[3]
/* = 6.1620 E-656 */
```

Code 19 (Asymptotics of the Pochhammer symbol generically).

```
{qpochinfty(x,q,N)=local(s,t,qk);qk=1;t=1;s=1;for(k=1,N,qk=qk*q;t=-t*qk/q*
    x/(1-qk);s=s+t);s};
{qpochinftyasy(m,x,tau,N)=sum(k=0,N,subst(bernpol(k,yyy),yyy,m)*(2*Pi*I*
    tau)^(k-1)/k!*polylog(2-k,x))};
E(x)=exp(2*Pi*I*x);
qpochinfty(E(Pi)/2*E(I/100/Pi),E(I/100),500)
/* = 0.0053120 - 0.0085775*I */
qpochinfty(E(Pi)/2*E(I/100/Pi),E(I/100),500)/exp(qpochinftyasy(1/Pi,E(Pi)
    /2,I/100,0))
/* = 0.95359 - 0.088647*I */
qpochinfty(E(Pi)/2*E(I/100/Pi),E(I/100),500)/exp(qpochinftyasy(1/Pi,E(Pi)
    /2,I/100,10))-1
/* = -2.8664 E-16 + 1.1344 E-16*I */
```

Code 20 (Asymptotics of the Pochhammer symbol generically with gamma).

```
\p1000
default(format,"g.5")
{qpochinfty(x,q,N)=local(s,t,qk);qk=1;t=1;s=1;for(k=1,N,qk=qk*q;t=-t*qk/q*
    x/(1-qk);s=s+t);s};
{qpochinftyasygam(a,b,c,d,m,x,tau,N)=sum(k=0,N,(-2*Pi*I*abs(c))^(k-1)/(c
    ^2*tau+c*d)^(k-1)/k!*sum(ell=0,abs(c)-1,subst(bernpol(k,yyy),yyy,(m+ell
    )/abs(c))*polylog(2-k,E((m+ell)*a/c)*x)))};
E(x)=exp(2*Pi*I*x);
qpochinfty(E(E(0.2-0.11*I)*I*100/(7*I*100+1))*E(-0.55-0.2*I),E(I*100/(7*I
    *100+1)),1000)
/* = 0.26598 + 1.0302*I */
qpochinfty(E(E(0.2-0.11*I)*I*100/(7*I*100+1))*E(-0.55-0.2*I),E(I*100/(7*I
    *100+1)),1000)/exp(qpochinftyasygam(1,0,7,1,E(0.2-0.11*I),E(-0.55-0.2*I
    ),I*100,0))
/* = 0.92035 + 0.66454*I */
```

```
qpochinfty(E(E(0.2-0.11*I)*I*100/(7*I*100+1))*E(-0.55-0.2*I),E(I*100/(7*I
    *100+1)),1000)/exp(qpochinftyasygam(1,0,7,1,E(0.2-0.11*I),E(-0.55-0.2*I
    ),I*100,10))-1
/* = 1.2176 E-20 + 2.6608 E-21*I */
qpochinfty(E(E(0.2-0.11*I)*I*100/(-7*I*100+1))*E(-0.55-0.2*I),E(I*100/(-7*
    I*100+1)),10000)/exp(qpochinftyasygam(1,0,-7,1,E(0.2-0.11*I),E
    (-0.55-0.2*I),I*100,10))-1
/* = 2.0601 E-30 - 9.8181 E-30*I */
```

Code 21 (Integral formula for Pochhammer symbol)**.**

```
\p 800
default(format,"g.5")
default(parisize,"1G")
E(x)=exp(2*Pi*I*x);
qpoch(m,z,tau,M)=prod(n=0,M,1-E((n+m)*tau+z));
qpochint(m,z,tau,M)=prod(k=ceil(real(m*tau+z)),floor(real((M+m)*tau+z)),1-
    E(m+z/tau-k/tau))*exp(dilog(E(m*tau+z))/(2*Pi*I*tau)-dilog(E((M+m)*tau+
    z))/(2*Pi*I*tau)+1/2*log(1-E(m*tau+z))+1/2*log(1-E((M+m)*tau+z))-I/tau*
    intnum(y=0,[+oo, 1],(log(1-E(m*tau-I*y+z))-log(1-E(m*tau+I*y+z))-log(1-
    E((M+m)*tau-I*y+z))+log(1-E((M+m)*tau+I*y+z)))/(E(-I*y/tau)-1)));
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/1000,1500)
/* = -3.2286 E25 + 1.0241 E26*I */
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/1000,1500)-qpochint(E(1/10+I/100),
    E(0.123)/Pi,E(0.01)/1000,1500)
/* = 1.8503 E-240 + 8.6047 E-241*I */
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/100,1500)
/* = 38.503 - 20.177*I */
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/100,1500)-qpochint(E(1/10+I/100),E
    (0.123)/Pi,E(0.01)/100,1500)
/* = -3.5620 E-373 + 1.8839 E-373*I */
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/10,1500)
/* = 1.3224 - 0.0049443*I */
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/10,1500)-qpochint(E(1/10+I/100),E
    (0.123)/Pi,E(0.01)/10,1500)
/* = 2.9953 E-266 + 7.7079 E-266*I */
\p 1000
default(format,"g.5")
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/1000,1500)-qpochint(E(1/10+I/100),
    E(0.123)/Pi,E(0.01)/1000,1500)
/* = 4.9332 E-303 - 2.0625 E-303*I */
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/100,1500)-qpochint(E(1/10+I/100),E
    (0.123)/Pi,E(0.01)/100,1500)
/* = 1.1505 E-461 + 4.9936 E-462*I */
qpoch(E(1/10+I/100),E(0.123)/Pi,E(0.01)/10,1500)-qpochint(E(1/10+I/100),E
    (0.123)/Pi,E(0.01)/10,1500)
/* = -5.2424 E-266 - 2.3307 E-266*I */
```

Code 22 (Asymptotics for fixed $x = 1$)**.**

```
\p 800
default(format,"g.5")
default(parisize,"1G")
E(x)=exp(2*Pi*I*x);
{qpochinfty(m,tau,N)=local(q,x,s,t,qk);x=E(m*tau);q=E(tau);qk=1;t=1;s=1;for(k=1,N,qk=qk*q;t=-t*qk/q*x/(1-qk);s=s+t);s};
qpochasymp(m,tau,N)=2*Pi*(-I*tau)^(1/2)/(-2*Pi*I*m*tau)^(m)*m^m/gamma(m)*exp(-(2*Pi*I)/(24*tau)+sum(k=2,N,zeta(2-k)*(2*Pi*I*tau)^(k-1)*subst(bernpol(k,xxx),xxx,m)/k!))
qpochinfty(E(1/10+I/100),E(0.01)/1000,10000)/qpochasymp(E(1/10+I/100),E(0.01)/1000,10)-1
/* = -3.2289 E-32 - 2.1203 E-32*I */
qpochinfty(E(1/10+I/100),E(0.01)/1000,10000)/qpochasymp(E(1/10+I/100),E(0.01)/1000,20)-1
/* = 4.9559 E-59 + 1.2580 E-58*I */
```

Code 23 (Asymptotics for fixed x with γ)**.**

```
\p1000
default(format,"g.5")
{qpochinfty(x,q,N)=local(s,t,qk);qk=1;t=1;s=1;for(k=1,N,qk=qk*q;t=-t*qk/q*x/(1-qk);s=s+t);s};
{qpochinftyasygam(a,b,c,d,m,j,tau,N)=sum(k=0,N,(-2*Pi*I*abs(c))^(k-1)/(c^2*tau+c*d)^(k-1)/k!*sum(ell=0,abs(c)-1,if(Mod(j,abs(c))==Mod(ell,abs(c)),0,subst(bernpol(k,yyy),yyy,(m+ell)/abs(c))*polylog(2-k,E((ell-j)*a/c)))))+log(2*Pi*(-I*(-abs(c)/(c^2*tau+c*d)))^(1/2)/(2*Pi*I*((m+j))*(1/(c^2*tau+c*d)))^((m+j)/abs(c))*((m+j)/abs(c))^((m+j)/abs(c))/gamma(((m+j)/abs(c))))-(2*Pi*I)/(24*(-abs(c)/(c^2*tau+c*d)))+sum(k=2,N,zeta(2-k)*(-2*Pi*I*abs(c)/(c^2*tau+c*d))^(k-1)*subst(bernpol(k,xxx),xxx,((m+j)/abs(c)))/k!)};
E(x)=exp(2*Pi*I*x);
qpochinfty(E(E(0.2-0.11*I)*I*100/(7*I*100+1))/E(1/7*(2+E(0.2-0.11*I))),E(I*100/(7*I*100+1)),1000)
/* = -1.0733 E-12 + 1.7927 E-12*I */
qpochinfty(E(E(0.2-0.11*I)*I*100/(7*I*100+1))/E(1/7*(2+E(0.2-0.11*I))),E(I*100/(7*I*100+1)),1000)/exp(qpochinftyasygam(1,0,7,1,E(0.2-0.11*I),2,I*100,10))-1
/* = -3.0248 E-23 + 2.8187 E-23*I */
qpochinfty(E(E(0.2-0.11*I)*(3*I*100+2)/(7*I*100+5))/E(3/7*(2+E(0.2-0.11*I))),E((3*I*100+2)/(7*I*100+5)),1000)
/* = 1.5909 E-12 + 2.0595 E-12*I */
qpochinfty(E(E(0.2-0.11*I)*(3*I*100+2)/(7*I*100+5))/E(3/7*(2+E(0.2-0.11*I))),E((3*I*100+2)/(7*I*100+5)),1000)/exp(qpochinftyasygam(3,2,7,5,E(0.2-0.11*I),2,I*100,10))-1
/* = 1.0771 E-23 - 1.0059 E-23*I */
```

```
qpochinfty(E(E(0.2-0.11*I)*(5*I*100+1)/(4*I*100+1))/E(5/4*(2+E(0.2-0.11*I)
    )),E((5*I*100+1)/(4*I*100+1)),1000)/exp(qpochinftyasygam(5,1,4,1,E
    (0.2-0.11*I),2,I*100,10))-1
/* = 4.1752 E-24 - 3.7505 E-24*I */
```

Appendix C

q–difference equations and modularity

C.1 The Habiro ring, η and mock modularity

Code 24 (The Habiro ring).

```
F(q,N)=local(qk,t,s);qk=1;t=1;s=t;for(k=1,N,qk=q*qk;t=t*(1-qk);s=s+t);s
F(1+X+O(X^10),11)
/* = 1 - X + 2*X^2 - 5*X^3 + 15*X^4 - 53*X^5 + 217*X^6 - 1014*X^7 + 5335*X
   ^8 - 31240*X^9 + O(X^10) */
for(k=2,10,print([k,F(Mod(x,polcyclo(k,x)),k+1)]))
/*
[2, Mod(3, x + 1)]
[3, Mod(-x + 5, x^2 + x + 1)]
[4, Mod(-3*x + 8, x^2 + 1)]
[5, Mod(-3*x^2 - 5*x + 9, x^4 + x^3 + x^2 + x + 1)]
[6, Mod(-13*x + 17, x^2 - x + 1)]
[7, Mod(3*x^5 - 7*x^2 - 9*x + 14, x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)]
[8, Mod(-3*x^3 - 9*x^2 - 16*x + 15, x^4 + 1)]
[9, Mod(7*x^5 - 5*x^4 - x^3 - 11*x^2 - 18*x + 17, x^6 + x^3 + 1)]
[10, Mod(-23*x^2 - 15*x + 11, x^4 - x^3 + x^2 - x + 1)]
*/
local(t);forprime(p=2,50,t=lift(F(Mod(x,polcyclo(p,x)),p+1));print([p,
   centerlift(Mod(sum(ell=0,p-2,binomial(ell,4)*polcoeff(t,ell)),p))]))
/*
[2, 0]
[3, 0]
[5, 0]
[7, 1]
[11, 4]
[13, 2]
[17, -2]
[19, -4]
[23, -8]
```

```
[29, -14]
[31, 15]
[37, 15]
[41, 15]
[43, 15]
[47, 15]
*/
local(s,t,v);s=0;v=vector(3);forprime(p=prime(3),prime(5),s=s+1;t=lift(F(Mod(x,polcyclo(p,x)),p+1));v[s]=Mod(sum(ell=0,p-2,binomial(ell,4)*polcoeff(t,ell)),p));centerlift(chinese(v))
/* = 15 */
local(s,t,v);s=0;v=vector(3);forprime(p=prime(4),prime(6),s=s+1;t=lift(F(Mod(x,polcyclo(p,x)),p+1));v[s]=Mod(sum(ell=0,p-2,binomial(ell,4)*polcoeff(t,ell)),p));centerlift(chinese(v))
/* = 15 */
```

Code 25 (Multiplier system for the Dedekind η–function).

```
edeta(a,b,c,d)=eta((a*(I*1000-1000)+b)/(c*(I*1000-1000)+d),1)/eta((I*1000-1000),1)/sqrt(c*(I*1000-1000)+d)
Dq(a,b,c,d)=sqrt(-I*c/abs(c))*prod(ell=1,abs(c)-1,(1-E(ell*a/c))^(1/2-ell/abs(c)))*E((a+d)/24/c)
forvec(X=[[-10,10],[-10,10],[-10,10],[-10,10]],if(X[2]==X[3],if(X[3]==X[4],if(X[4]==-10,print([X[1],gettime()]))));if(X[1]*X[4]-X[2]*X[3]==1,if(X[3],if(round(edeta(X[1],X[2],X[3],X[4])/Dq(X[1],X[2],X[3],X[4])*10^150)/10^150==1,,print([X[1],X[2];X[3],X[4]])))))
forvec(X=[[-10,10],[-10,10],[-10,10],[-10,10]],if(X[1]*X[4]-X[2]*X[3]==1,if(X[3],print([edeta(X[1],X[2],X[3],X[4])/Dq(X[1],X[2],X[3],X[4]),[X[1],X[2];X[3],X[4]]]))))
edeta(-9, -8,-10, -9)/Dq(-9, 8,-10, -9)
edeta(7, 5,-10, -7)/Dq(7, 5,-10, -7)
forvec(X=[[-10,10],[-10,10],[-10,10],[-10,10]],if(X[1]*X[4]-X[2]*X[3]==1,if(X[3]>0,if(X[1],print([Dq(-X[3],-X[4],X[1],X[2])/Dq(X[1],X[2],X[3],X[4]),X[1]/X[3],matdet([X[1],X[2];X[3],X[4]])])))))
```

Code 26 (Asymptotics of order three mock θ–function).

```
f(q,N,m=0)=local(t,s,qk);qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=qk^2*t/q/(1+qk);s=s+t*qk^m);s
huh=vector(2000);
for(k=1000,1010,huh[k]=f(E((I*k)/(2*(I*k)+1)),k))
asymp(huh,1000,8)
\\ = [1.1104 - 4.2614 E-28*I, 0.50000 + 3.4708 E-24*I, -0.74248 + 0.038912*I]
huh0=vector(2000);
for(k=1000,1010,huh0[k]=(huh[k]+(E(-1/60*(I*k+1/2))*sqrt(I*k+1/2)*(-15/8-sqrt(5)*5/8)^(-1/4)*E(-1/(2*(2*I*k+1))/60))-2+4*(2*Pi*I/(2*(2*I*k+1)))-36*(2*Pi*I/(2*(2*I*k+1)))^2+1640/3*(2*Pi*I/(2*(2*I*k+1)))^3)/(2*Pi*I/(2*(2*I*k+1)))^4)
```

```
asymp(huh0,1000,8)
f(q+O(q^10),20,0)+f(q+O(q^10),20,1)-q^(0+1)*f(q+O(q^10),20,2)
fphi(x,m,N)=sum(k=0,N,sum(ell=0,k,eval(Str("a",k,ell))*m^ell*x^k))
fphi(x,m,5)+fphi(x,m+1,5)-exp((m+1)*x+O(x^5))*fphi(x,m+2,5)
```

Code 27 (Modularity of the η–constants)**.**

```
\p200
default(format,"g.5")
E(x)=exp(2*Pi*I*x);
Dq(x)=local(a,c);c=denominator(x);a=x*c;sqrt(-I*c/abs(c))*prod(ell=1,abs(c)-1,(1-E(ell*a/c))^(1/2-ell/abs(c)));
for(k=90,110,print([k,Dq(-1/(k+1/2))/Dq((k+1/2))/Dq(1)/E(1/24/denominator(k+1/2)/numerator(k+1/2)+1/24/(k+1/2)+(k+1/2)/24)-1]))
for(k=90,110,print([k,Dq(-1/(k+11/29))/Dq((k+11/29))/Dq(1)/E(1/24/denominator(k+11/29)/numerator(k+11/29)+1/24/(k+11/29)+(k+11/29)/24)-1]))
for(k=90,110,print([k,Dq(-1/(k-11/29))/Dq((k-11/29))/Dq(1)/E(1/24/denominator(k-11/29)/numerator(k-11/29)+1/24/(k-11/29)+(k-11/29)/24)-1]))
```

C.2 The Faddeev quantum dilogarithm

Code 28 (Integral formula for Faddeev's quantum dilogarithm)**.**

```
\p1000
default(format,"g.5")
default(parisize,"1G")
E(x)=exp(2*Pi*I*x)
qpoch(a,q,n)=if(n>-1,prod(j=0,n-1,1-a*q^j),1/prod(j=n,-1,1-a*q^(j)));
F(x,q,N)=local(t,s,qk);qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=-qk*x/q*t/(1-qk);s=s+t);s
E(x)=exp(2*Pi*I*x)
QD(z,tau,N)=if(imag(tau)>0,F(E((z-1)/tau),E(-1/tau),N)/F(E(z),E(tau),N),F(E(z-tau),E(-tau),N)/F(E(z/tau),E(1/tau),N))
f(z,tau)=local(t);t=(log(1-E(z/tau-I*(x+O(x^10))))-log(1-E(z/tau+I*(x+O(x^10)))))/(exp(2*Pi*tau*(x+O(x^10)))-1);exp(-dilog(E(z/tau))*tau/(2*Pi*I)+I*tau*intnum(xi=0, [+oo, 1],if(xi<1/10^1000,sum(i=0,7,polcoeff(t,i,x)*xi^i),(log(1-E(z/tau-I*xi))-log(1-E(z/tau+I*xi)))/(exp(2*Pi*tau*xi)-1))))/sqrt(1-E(z/tau))/qpoch(E(z),E(tau),-floor(real(z/tau)))
ze=E(Pi)*sqrt(2)
taue=-E(exp(1))*exp(1)/11
QD(ze,taue,1000)/f(ze,taue)-1
\\ = -1.1484 E-32 - 9.5559 E-33*I
ze=100+E(Pi)*sqrt(2)
```

```
taue=-E(exp(1))*exp(1)/11
QD(ze,taue,1000)/f(ze,taue)-1
\\ = -3.4468 E-55 - 9.4898 E-56*I
ze=E(Pi)*sqrt(2)
taue=-E(-exp(1))*exp(1)/11
QD(ze,taue,1000)/f(ze,taue)-1
\\ = 2.3917 E-96 - 3.2546 E-96*I
\p2000
default(format,"g.5")
ze=E(Pi)*sqrt(2)
taue=-E(exp(1))*exp(1)/11
QD(ze,taue,1000)/f(ze,taue)-1
\\ = 1.1851 E-62 - 1.3386 E-61*I
ze=100+E(Pi)*sqrt(2)
taue=-E(exp(1))*exp(1)/11
QD(ze,taue,1000)/f(ze,taue)-1
\\ = -1.0435 E-56 - 3.6018 E-56*I
ze=E(Pi)*sqrt(2)
taue=-E(-exp(1))*exp(1)/11
QD(ze,taue,1000)/f(ze,taue)-1
\\ = -9.1869 E-190 - 1.4815 E-189*I
c(b)=I/2*(b+1/b)
fK(z,tau,eps)=if(abs(imag(I*z/sqrt(tau)-c(sqrt(tau))))<abs(imag(c(sqrt(tau
    )))),exp(-intnum(w=[-oo,1],[oo,1],exp((2*z-1-tau)*(w+I*eps)/sqrt(tau))
    /4/sinh((w+I*eps)*sqrt(tau))/sinh((w+I*eps)/sqrt(tau))/(w+I*eps))),
    print("Outside domain"))
ze=0.25+0.4*I
taue=3+4*I
QD(ze,taue,1000)/f(ze,taue)-1
QD(ze,taue,2000)/fK(ze,taue,0.4)-1
fK(z,tau,eps)=if(abs(imag(I*z/sqrt(tau)-c(sqrt(tau))))<abs(imag(c(sqrt(tau
    )))),exp(-intnum(w=[-oo,1],[oo,1],exp(z*(w+I*eps)/sqrt(tau))/(1-exp((w+
    I*eps)/sqrt(tau)))/(1-exp((w+I*eps)*sqrt(tau)))/(w+I*eps))),print("
    Outside domain"))
ze=0.25+0.4*I
taue=3+4*I
QD(ze,taue,1000)/f(ze,taue)-1
QD(ze,taue,2000)/fK(ze,taue,0.4)-1
```

Code 29 (Faddeev's quantum dilogarithm at rationals)**.**

```
\p1000
default(format,"g.5")
default(parisize,"1G")
E(x)=exp(2*Pi*I*x)
qpoch(a,q,n)=if(n>-1,prod(j=0,n-1,1-a*q^j),1/prod(j=n,-1,1-a*q^(j)));
E(x)=exp(2*Pi*I*x)
```

```
f(z,tau)=local(t);t=(log(1-E(z/tau-I*(x+O(x^10))))-log(1-E(z/tau+I*(x+O(x^10)))))/(exp(2*Pi*tau*(x+O(x^10)))-1);exp(-dilog(E(z/tau))*tau/(2*Pi*I)+I*tau*intnum(xi=0, [+oo, 1],if(xi<1/10^1000,sum(i=0,7,polcoeff(t,i,x)*xi^i),(log(1-E(z/tau-I*xi))-log(1-E(z/tau+I*xi)))/(exp(2*Pi*tau*xi)-1))))/sqrt(1-E(z/tau))/qpoch(E(z),E(tau),-floor(real(z/tau)))
c(b)=I/2*(b+1/b)
fK(z,tau,eps)=if(abs(imag(I*z/sqrt(tau)-c(sqrt(tau))))<abs(imag(c(sqrt(tau)))),exp(-intnum(w=[-oo,1],[oo,1],exp(z*(w+I*eps)/sqrt(tau))/(1-exp((w+I*eps)/sqrt(tau)))/(1-exp((w+I*eps)*sqrt(tau)))/(w+I*eps))),print("Outside domain"))
ze=0.25+0.4*I
taue=13/7
f(ze,taue)
fK(ze,taue,0.4)
CD(x,q,N)=prod(k=1,N-1,(1-q^k*x)^(k/N));
frat(u,tau)=local(d,n);d=denominator(tau);n=tau*d;exp(dilog(E(d*u))/(2*Pi*I*n*d))*(1-E(d*u))^(u/n-1)*CD(E(u),E(tau),d)*CD(E(u/tau),E(1/tau),n)
frat(ze,taue)/f(ze,taue)-1
```

Appendix D

Worked examples of q–hypergeometric functions

D.1 Nahm sums

Code 30 (Computing q–expansions of Nahm sums).

```
f(q,A,m,n,N)=local(si,qk,t,s);si=Mod(x^numerator(n),polcyclo(denominator(n)));qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=si*qk^A*t/(1-qk);s=s+t*qk^(m-A/2));s
f(q+O(q^10),2,0,0,10)
/* = Mod(1 + q + q^2 + q^3 + 2*q^4 + 2*q^5 + 3*q^6 + 3*q^7 + 4*q^8 + 5*q^9 + O(q^10), x - 1) */
```

Code 31 (Asymptotics of Nahm sums).

```
prec=2000;
default(realprecision,prec)
default(format,"g.5")
E(x)=exp(2*Pi*I*x);
f(q,A,m,n,N=ceil(abs(real(sqrt(log(10^(-prec))/log(abs(q))*2/A)))))=local(si,qk,t,s);si=E(n);qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=qk^A*t/(1-qk);s=s+t*qk^(m-A/2));s
huh=vector(2000);
for(k=1000,1012,huh[k]=f(E(-1/(E(0.01)*k)),2,0))
asymp(huh,1000,10)
/* [1.0011 - 0.10501*I, 3.6671 E-7 + 8.8228 E-8*I, 0.85065 - 6.5637 E-7*I] */
huh=vector(2000);
for(k=1000,1012,huh[k]=f(E(-1/(E(0.001)*k)),4,0))
asymp(huh,1000,10)
/* [1.0200 + 0.56359*I, -1.4004 E-22 - 1.5676 E-21*I, 0.43783 - 0.17519*I] */
```

Code 32 (The leading asymptotics of the $A = 2$ Nahm sum)**.**

```
prec=200;
default(realprecision,prec)
default(format,"g.5")
f(q,A,m,n,N=ceil(abs(real(sqrt(log(10^(-prec))/log(abs(q))*2/A)))))=local(
    si,qk,t,s);si=E(n);qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=qk^A*t/(1-qk);s=s+t
    *qk^(m-A/2));s
X=polroots(x^2+x-1);
xx=log(X)/2/Pi/I;
-1/(2*Pi*I)*log(1-X[1])+2*(xx[1]+k)+j
/* = 2*k + j + 1 */
-1/(2*Pi*I)*log(1-X[2])+2*(xx[2]+k)+j
/* = 2*k + j */
dilog(X[1])-Pi^2/6+(2*Pi*I)^2*(xx[1]+k)^2-(2*Pi*I)^2*(2*k+1)*(xx[1]+k)
/* = 39.478*k^2 + 39.478*k + 7.2377 */
dilog(X[2])-Pi^2/6+(2*Pi*I)^2*(xx[2]+k)^2-(2*Pi*I)^2*(2*k)*(xx[2]+k)
/* = 39.478*k^2 - 0.65797 */
huh=vector(2000);
for(k=1000,1012,huh[k]=f(E(-1/(E(0.01)*k)),2,0));
asymp(huh,1000,10)
/* [1.0011 - 0.10501*I, 3.6671 E-7 + 8.8228 E-8*I, 0.85065 - 6.5637 E-7*I]
     */
E(1/15*Pi^2/(2*Pi*I)^2*E(0.01))
E(-1/60*E(0.01))
/* =  1.0011 - 0.10501*I */
1/sqrt(1-X[2])/sqrt(X[2]/(1-X[2])+2)
/* = 0.85065 */
huh1=vector(2000);
for(k=1000,1012,huh1[k]=(huh[k]/E(1/15*Pi^2/(2*Pi*I)^2*E(0.01)*k)*sqrt(1-X
    [2])*sqrt(X[2]/(1-X[2])+2)-1)/(2*Pi*I/(E(0.01)*k)));
lim(huh1,1000,4)
/* = -0.016667 + 8.4215 E-19*I */
huh2=vector(2000);
for(k=1000,1012,huh2[k]=(huh1[k]+1/60)/(2*Pi*I/(E(0.01)*k)));
lim(huh2,1000,4)
/* = 0.00013889 - 2.5014 E-20*I */
huh0=vector(2000);
for(k=1000,1012,huh0[k]=huh[k]/E(1/15*Pi^2/(2*Pi*I)^2*E(0.01)*k)*sqrt(1-X
    [2])*sqrt(X[2]/(1-X[2])+2)-E(-1/(E(0.01)*k)/60));
asymp(huh0,1000,3)
/* = [0.28775 + 0.87819*I, 4.4440 E-12 + 4.1924 E-10*I, 0.61803 - 2.0063 E
    -9*I] */
heh=vector(2000);
for(k=1000,1012,heh[k]=f(E(-1/(E(0.01)*k)),2,1));
asymp(huh,1000,10)
/* = [1.0011 - 0.10501*I, 3.6671 E-7 + 8.8228 E-8*I, 0.85065 - 6.5637 E-7*
    I] */
```

```
heh0=vector(2000);
for(k=1000,1012,heh0[k]=heh[k]/E(1/15*Pi^2/(2*Pi*I)^2*E(0.01)*k)/X[2]*sqrt(1-X[2])*sqrt(X[2]/(1-X[2])+2)-E(11/(E(0.01)*k)/60));
asymp(heh0,1000,3)
/* = [0.28775 + 0.87819*I, -2.9211 E-9 - 4.2488 E-9*I, -1.6180 - 5.3234 E-8*I] */
const(al,m,k,rho=2)=local(c);c=denominator(al);sum(r=0,c-1,E((r^2+r*(2*k+m))*al+(2*r+m)*xx[rho]/c)*prod(s=0,c-1,(1-E((r+s+1+k)*al+xx[rho]/c))^(1/2-(r+s+1)/c)))/sqrt(X[rho]/(1-X[rho])*c+2*c)/prod(ell=1,abs(c)-1,(1-E(ell*al))^(1/2-ell/abs(c)))
{for(m=-3,3,for(c=-10,10,if(c,
huh=vector(2000);
for(k=1000,1006,huh[k]=f(E(E(0.01)*k/(c*E(0.01)*k+1)),2,m));
print([m,c,asymp(huh,1000,3)[3]/E(1/15*Pi^2/(2*Pi*I)^2/c)/const(1/c,m,0,2)-1])))))}
{for(c=-10,10,if(c,
huh=vector(2000);
for(k=1000,1006,huh[k]=f(E(E(0.01)*k/(c*E(0.01)*k+1)),2,0));
huh0=vector(2000);
for(k=1000,1006,huh0[k]=(huh[k]/E(1/15*Pi^2/(2*Pi*I)^2*(E(0.01)*k+1/c))/const(1/c,0,0,2)/E(-1/60/(c^2*E(0.01)*k+c))-1));
print([c,asymp(huh0,1000,3)[1]])))}
hah=vector(200);
for(k=50,110,hah[k]=const(1/k,0,0,2);print([k,gettime()]))
for(k=0,10,print([k,asymposc(hah,50,3,E(k/10))]))
/*
[0, [0.88660 + 3.3481*I, -171.92 - 17.768*I, -3.9059 E303 - 1.4620 E303*I; -1.8866 - 3.3481*I, -171.92 - 17.768*I, 1.7423 E313 - 1.3758 E313*I]]
[1, [0.99452 + 0.10454*I, 0.00088587 - 0.0021848*I, 0.72060 + 0.0077394*I; -1.8035 - 0.69232*I, 0.00088587 - 0.0021848*I, 0.71976 + 0.0085990*I]]
[2, [0.99452 + 0.10453*I, 1.9822 E-6 - 6.6354 E-6*I, 0.72360 + 2.5350 E-5*I; -1.3035 - 1.0556*I, 1.9822 E-6 - 6.6354 E-6*I, 0.72360 + 2.7007 E-5*I]]
[3, [0.99452 + 0.10453*I, 6.0323 E-6 - 6.3794 E-6*I, 0.72359 + 2.6025 E-5*I; -0.68550 - 1.0556*I, 6.0323 E-6 - 6.3794 E-6*I, 0.72358 + 2.3202 E-5*I]]
[4, [0.99426 + 0.10417*I, 0.13857 + 0.020381*I, 0.34916 - 0.0026746*I; -0.18525 - 0.69195*I, 0.13857 + 0.020381*I, 0.36950 - 0.062598*I]]
[5, [0.59326 + 0.91355*I, -8.9737 E-6 - 4.8220 E-6*I, 0.27641 + 5.8938 E-6*I; 0.40674 - 0.91355*I, -8.9737 E-6 - 4.8220 E-6*I, 0.27641 + 8.2474 E-6*I]]
[6, [0.40674 - 0.91355*I, -3.2052 E-6 - 6.7678 E-6*I, 0.27640 + 1.0010 E-5*I; 0.40228 + 1.5013*I, -3.2052 E-6 - 6.7678 E-6*I, 0.27640 + 1.0232 E-5*I]]
[7, [0.40674 - 0.91355*I, 1.1832 E-5 + 4.7732 E-5*I, 0.27638 - 6.4272 E-5*I; -0.097719 + 1.8646*I, 1.1832 E-5 + 4.7732 E-5*I, 0.27637 - 7.1309 E-5*I]]
[8, [-0.022667 + 1.8571*I, -74.548 + 156.88*I, -1.8995 E184 - 1.9925 E184*
```

```
    I; -0.28635 - 0.90601*I, -74.548 + 156.88*I, -9.7656 E140 - 5.7530 E140
    *I]]
[9, [331.16 + 356.32*I, -157.35 - 0.38597*I, -1.7471 E64 + 7.9208 E62*I;
    -331.97 - 355.73*I, -157.35 - 0.38597*I, -1.6995 E64 + 6.5188 E63*I]]
[10, [0.88660 + 3.3481*I, -171.92 - 17.768*I, -3.9059 E303 - 1.4620 E303*I
    ; -1.8866 - 3.3481*I, -171.92 - 17.768*I, 1.7423 E313 - 1.3758 E313*I]]
*/
[E(1/60),E(-11/60)]
/* = [0.99452 + 0.10453*I, 0.40674 - 0.91355*I] */
asympv(hah,[E(1/60),E(-11/60)],[0,0],100,3)
/* = [0.72361 - 4.6815 E-14*I, 0.27639 + 5.5667 E-15*I]~ */
hah[100]-(const(1,0,0,2)^2*E(1/60*100)*E(2/60/100)+const(1,0,0,1)*const
    (1,1,0,2)*E(-11/60*100)*E(2/60/100))
/* = 2.1640 E-205 + 7.5473 E-206*I */
const(-1/(100+1/2),0,0,2)-(const(1,0,0,2)*const(100+1/2,0,0,2)*E
    (-1/60*(100+1/2))*E(-1/60/denominator(100+1/2)/numerator(100+1/2))*E
    (-1/60/(100+1/2))+const(1,0,0,1)*const(100+1/2,1,0,2)*E(11/60*(100+1/2)
    )*E(-1/60/(100+1/2))*E(-1/60/denominator(100+1/2)/numerator(100+1/2)))
/* = -5.7703 E-205 + 4.8749 E-205*I */
const(-1/(100+7/11),0,0,2)-(const(1,0,0,2)*const(100+7/11,0,0,2)*E
    (-1/60*(100+7/11))*E(-1/60/denominator(100+7/11)/numerator(100+7/11))*E
    (-1/60/(100+7/11))+const(1,0,0,1)*const(100+7/11,1,0,2)*E
    (11/60*(100+7/11))*E(-1/60/(100+7/11))*E(-1/60/denominator(100+7/11)/
    numerator(100+7/11)))
/* = 1.2328 E-202 - 1.1016 E-202*I */
const(-1/(100+7/11),1,0,2)-(const(1,1,0,2)*const(100+7/11,0,0,2)*E
    (-1/60*(100+7/11))*E(-1/60/denominator(100+7/11)/numerator(100+7/11))*E
    (11/60/(100+7/11))+const(1,1,0,1)*const(100+7/11,1,0,2)*E
    (11/60*(100+7/11))*E(11/60/(100+7/11))*E(-1/60/denominator(100+7/11)/
    numerator(100+7/11)))
\\ = 1.6434 E-202 - 9.6018 E-203*I
const(-1/(100+7/11),1,0,1)-(const(1,1,0,2)*const(100+7/11,0,0,1)*E
    (-1/60*(100+7/11))*E(11/60/denominator(100+7/11)/numerator(100+7/11))*E
    (11/60/(100+7/11))+const(1,1,0,1)*const(100+7/11,1,0,1)*E
    (11/60*(100+7/11))*E(11/60/(100+7/11))*E(11/60/denominator(100+7/11)/
    numerator(100+7/11)))
\\ = 2.0538 E-202 + 2.3767 E-202*I
```

Code 33 (The leading asymptotics of the $A = 4$ Nahm sum)**.**

```
prec=200;
default(realprecision,prec)
default(format,"g.5")
f(q,A,m,n,N=ceil(abs(real(sqrt(log(10^(-prec))/log(abs(q))*2/A)))))=local(
    si,qk,t,s);si=E(n);qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=qk^A*t/(1-qk);s=s+t
    *qk^(m-A/2));s
X=polroots(x^4+x-1);
xx=vector(4,j,log(X[j])/(2*Pi*I));
```

```
-1/(2*Pi*I)*log(1-X[1])+4*(xx[1]+k)+j
/* = 4*k + j + 2 */
-1/(2*Pi*I)*log(1-X[2])+4*(xx[2]+k)+j
/* = 4*k + j */
-1/(2*Pi*I)*log(1-X[3])+4*(xx[3]+k)+j
/* = 4*k + j - 1 */
-1/(2*Pi*I)*log(1-X[4])+4*(xx[4]+k)+j
/* = 4*k + j + 1 */
V=vector(4);
V[1]=dilog(X[1])-Pi^2/6+2*(2*Pi*I)^2*(xx[1]+k)^2-(2*Pi*I)^2*(4*k+2)*(xx
    [1]+k)
/* = 78.957*k^2 + 78.957*k + 17.203 */
V[2]=dilog(X[2])-Pi^2/6+2*(2*Pi*I)^2*(xx[2]+k)^2-(2*Pi*I)^2*(4*k)*(xx[2]+k
    )
/* = 78.957*k^2 - 0.50498 */
V[3]=dilog(X[3])-Pi^2/6+2*(2*Pi*I)^2*(xx[3]+k)^2-(2*Pi*I)^2*(4*k-1)*(xx
    [3]+k)
/* = 78.957*k^2 - 39.478*k + 3.1656 - 0.98137*I */
V[4]=dilog(X[4])-Pi^2/6+2*(2*Pi*I)^2*(xx[4]+k)^2-(2*Pi*I)^2*(4*k+1)*(xx
    [4]+k)
/* = 78.957*k^2 + 39.478*k + 3.1656 + 0.98137*I */
V=subst(V,k,0)
huh=vector(2000);
for(k=1000,1012,huh[k]=f(E(E(0.01)/k),4,0));
asymp(huh,1000,10)
/* [0.98674 - 0.55536*I, -9.1759 E-22 + 1.2290 E-21*I, 0.43783 + 0.17519*I
    ] */
E(V[4]/(2*Pi*I)^2/E(0.01))
/* = 0.98674 - 0.55536*I */
1/sqrt(1-X[4])/sqrt(X[4]/(1-X[4])+4)
/* = 0.43783 + 0.17519*I */
huh1=vector(2000);
for(k=1000,1012,huh1[k]=(huh[k]/E(V[4]/(2*Pi*I)^2/E(0.01)*k)*sqrt(1-X[4])*
    sqrt(X[4]/(1-X[4])+4)-1)/(2*Pi*I/(k/E(0.01))));
lim(huh1,1000,10)
/* = 0.051613 + 0.053815*I */
lindep([lim(huh1,1000,10)*((1-X[4])*(X[4]/(1-X[4])+4))^3*24,1,X[4],X[4]^2,
    X[4]^3],12)
/* = [1, -64, 100, 18, -54]~ */
hah=vector(2000);
for(k=1000,1012,hah[k]=f(E(E(0.2)/k),4,0));
asymp(hah,1000,10)
/* [1.0791 + 0.026806*I, 1.4777 E-21 + 1.0645 E-22*I, 0.73992 - 6.9681 E
    -22*I] */
E(V[2]/(2*Pi*I)^2/E(0.2))
/* = 1.0791 + 0.026806*I */
1/sqrt(1-X[2])/sqrt(X[2]/(1-X[2])+4)
/* = 0.73992 + 0.E-2003*I */
```

```
hah1=vector(2000);
for(k=1000,1012,hah1[k]=(hah[k]/E(V[2]/(2*Pi*I)^2/E(0.2)*k)*sqrt(1-X[2])*sqrt(X[2]/(1-X[2])+4)-1)/(2*Pi*I/(k/E(0.2))));
lim(hah1,1000,10)
/* = 0.018037 + 3.9407 E-26*I */
lindep([lim(hah1,1000,10)*((1-X[2])*(X[2]/(1-X[2])+4))^3*24,1,X[2],X[2]^2,X[2]^3],12)
/* = [-1, 64, -100, -18, 54]~ */
heh=vector(2000);
for(k=1000,1012,heh[k]=f(E(E(0.49)/k),4,0));
asymp(heh,1000,10)
/* = [0.98674 + 0.55536*I, -9.1759 E-22 - 1.2290 E-21*I, 0.43783 - 0.17519*I] */
E(V[3]/(2*Pi*I)^2/E(0.49))
/* = 0.98674 + 0.55536*I */
1/sqrt(1-X[3])/sqrt(X[3]/(1-X[3])+4)
/* = 0.43783 - 0.17519*I */
heh1=vector(2000);
for(k=1000,1012,heh1[k]=(heh[k]/E(V[3]/(2*Pi*I)^2/E(0.49)*k)*sqrt(1-X[3])*sqrt(X[3]/(1-X[3])+4)-1)/(2*Pi*I/(k/E(0.49))));
lim(heh1,1000,10)
/* = 0.051613 - 0.053815*I */
lindep([lim(heh1,1000,10)*((1-X[3])*(X[3]/(1-X[3])+4))^3*24,1,X[3],X[3]^2,X[3]^3],12)
/* = [1, -64, 100, 18, -54]~ */
const(al,m,k)=local(c);c=denominator(al);sum(r=0,c-1,E((2*r^2+r*(4*k+m))*al+(4*r+m)*xx[4]/c)*prod(s=0,c-1,(1-E((r+s+1+k)*al+xx[4]/c))^(1/2-(r+s+1)/c)))/sqrt(X[4]/(1-X[4])*c+4*c)/prod(ell=1,abs(c)-1,(1-E(ell*al))^(1/2-ell/abs(c)))
/*
{for(m=-3,3,for(c=-10,10,if(c,
huh=vector(2000);
for(k=1000,1006,huh[k]=f(E(E(1/2-0.01)*k/(c*E(1/2-0.01)*k+1)),4,m));
print([m,c,asymp(huh,1000,5)[3]/E(-V[4]/(2*Pi*I)^2/c)/const(1/c,m,0)-1]))))}
{for(m=-3,3,for(c=-5,5,if(c,
huh=vector(2000);
for(k=1000,1006,huh[k]=f(E((2*E(1/2-0.01)*k+1)/((2*c+1)*E(1/2-0.01)*k+(c+1))),4,m));
print([m,c,asymp(huh,1000,5)[3]/E(-V[4]/(2*Pi*I)^2*(c+1)/(2*c+1))/const(2/(2*c+1),m,0)-1]))))}
{for(m=-3,3,for(c=-3,3,if(c,
huh=vector(2000);
for(k=1000,1006,huh[k]=f(E((3*E(1/2-0.01)*k+2)/((3*c+1)*E(1/2-0.01)*k+(2*c+1))),4,m));
print([m,c,asymp(huh,1000,5)[3]/E(-V[4]/(2*Pi*I)^2*(2*c+1)/(3*c+1))/const(3/(3*c+1),m,0)-1]))))}
[7,-3;7*c+5,-3*c-2]
```

```
{for(m=0,0,for(c=-3,3,if(c,
huh=vector(2000);
for(k=1000,1006,huh[k]=f(E((7*E(1/2-0.01)*k-3)/((7*c+5)*E(1/2-0.01)*k+(-3*
    c-2))),4,m));
print([m,c,asymp(huh,1000,5)[3]/E(-V[4]/(2*Pi*I)^2*(-3*c-2)/(7*c+5))/const
    (7/(7*c+5),m,0)-1]))))}
*/
hah=vector(2000);
for(k=200,206,hah[k]=const(1/k,0,0);print([k,gettime()]))
hah0=vector(2000);
for(k=200,206,hah0[k]=(hah[k]/E(V[4]/(2*Pi*I)^2*k)/E(V[4]/(2*Pi*I)^2/k)/
    const(1,0,0)/const(1,1,0)-1)/(2*Pi*I/k))
lim(hah0,200,2)
/* = 0.051613 + 0.053814*I */
(64-100*X[4]-18*X[4]^2+54*X[4]^3)/(X[4]+4*(1-X[4]))^3/24
/* = 0.051613 + 0.053814*I */
for(k=200,206,hah[k]=const(1/(k+5/3),0,0);print([k,gettime()]))
hah0=vector(2000);
for(k=200,206,hah0[k]=(hah[k]/E(V[4]/(2*Pi*I)^2*(k+5/3))/E(V[4]/(2*Pi*I)
    ^2/denominator(k+5/3)/numerator(k+5/3))/const(1,0,0)/const(-(k+5/3)
    ,1,0)-1)/(2*Pi*I/(k+5/3)))
lim(hah0,200,2)
/* = 0.051613 + 0.053814*I */
const(al,j,m,k,thfix)=local(c);c=denominator(al);sum(r=0,c-1,E((2*r^2+r
    *(4*k+m))*al+(4*r+m)*xx[j]/c)*prod(s=0,c-1,((1-E((r+s+1+k)*al+xx[j]/c))
    *E(thfix))^(1/2-(r+s+1)/c)/E(thfix*(1/2-(r+s+1)/c)))/sqrt(X[j]/(1-X[j])
    *c+4*c)/prod(ell=1,abs(c)-1,(1-E(ell*al))^(1/2-ell/abs(c))))
constmat(al,m,k,thfix)=matrix(4,4,i,j,const(al,j,m+i-1,k,thfix))
round(constmat(1/5,1,0,-0.001)*constmat(-1/5,-1-3,0,-0.001)~*10^100)
    /10^100
round(constmat(1/5,0,0,-0.001)*constmat(-1/5,-3,0,-0.001)~*10^100)/10^100
/*
[1 1 1 1]
[1 1 1 0]
[1 1 0 0]
[1 0 0 0]
*/
constarbm(al,j,m,k,thfix,sts)=local(c,Xtemp,xxtemp);c=denominator(al);
    Xtemp=polroots(1-xvar*E(thfix)-E(al*c*m)*(xvar*E(thfix))^4);for(j=1,4,
    Xtemp[j]=Xtemp[j]*E(thfix));xxtemp=vector(4,j,(log(Xtemp[j]*E(thfix))-
    log(E(thfix)))/(2*Pi*I));sum(r=0,c-1,E((2*r^2+r*(4*k+m))*al+(4*r+m)*
    xxtemp[j]/c)*prod(s=0,c-1,((1-E((r+s+1+k)*al+xxtemp[j]/c))*E(thfix))
    ^(1/2-(r+s+1)/c)/E(thfix*(1/2-(r+s+1)/c)))/sqrt(Xtemp[j]/(1-Xtemp[j])*c
    +4*c)/prod(ell=1,abs(c)-1,(1-E(ell*al))^(1/2-ell/abs(c))))/if(sts,E(m*
    xxtemp[j]/c),1)
constarbmmat(al,m,k,thfix)=matrix(4,4,i,j,constarbm(al,j,m+i-1,k,thfix))
A(t,q)=[0,1,0,0;0,0,1,0;0,0,0,1;1/(t*q^2),-1/(t*q^2),0,0];
P=[1,1,1,1;1,1,1,0;1,1,0,0;1,0,0,0];
```

```
A(E(I/11),E(1/11))*constarbmmat(1/11,I,0,-0.001)-constarbmmat(1/11,I+1,0,-0.001)
constarbmmat(-1/5,-I*10-3,0,-0.001)~*P^(-1)*constarbmmat(1/5,I*10,0,-0.001)
for(k=0,200,print([k/100.,([constarbm(1/7,2,2+100*I+k/100,0,-0.000001,1),constarbm(1/7,3,2+100*I+k/100,0,-0.000001,1),constarbm(1/7,4,2+100*I+k/100,0,-0.000001,1)]/E((1.5+(k/100+I*100)/2)/7/3)*sqrt(3))]))
for(k=0,200,print([k/100.,([constarbm(11/13,2,2+100*I+k/100,0,-0.000001,1),constarbm(11/13,3,2+100*I+k/100,0,-0.000001,1),constarbm(11/13,4,2+100*I+k/100,0,-0.000001,1)]/E((1.5+(k/100+I*100)/2)*11/13/3)*sqrt(3))]))


{constarbmvecbits(al,j,m,k,thfix)=local(c,Xtemp,xxtemp);c=denominator(al);Xtemp=polroots(1-xvar*E(thfix)-E(al*c*m)*(xvar*E(thfix))^4);for(j=1,4,Xtemp[j]=Xtemp[j]*E(thfix));xxtemp=vector(4,j,(log(Xtemp[j]*E(thfix))-log(E(thfix)))/(2*Pi*I));
[sum(r=0,c-1,E((2*r^2+r*(4*k+m))*al+4*r*xxtemp[j]/c)/qpoch(E(al+xxtemp[j]/c),E(al),r)),
E(m*xxtemp[j]/c),
prod(s=0,c-1,((1-E((s+1+k)*al+xxtemp[j]/c))*E(thfix))^(1/2-(s+1)/c)/E(thfix*(1/2-(s+1)/c))),
1/sqrt(Xtemp[j]/(1-Xtemp[j])*c+4*c),
1/prod(ell=1,abs(c)-1,(1-E(ell*al))^(1/2-ell/abs(c)))]}

constarbmvecbits(11/13,2,2+100*I,0,-0.000001)

{constarbmvecbitsasym(al,j,m,k,thfix)=local(c,Xtemp,xxtemp);c=denominator(al);Xtemp=polroots(1-xvar*E(thfix)-E(al*c*m)*(xvar*E(thfix))^4);for(j=1,4,Xtemp[j]=Xtemp[j]*E(thfix));xxtemp=vector(4,j,(log(Xtemp[j]*E(thfix))-log(E(thfix)))/(2*Pi*I));
[sum(r=0,c-1,(-1)^r*E(3*al*r^2/2 + ((al*m*c+3*xxtemp[j])*2-al*c)/2/c*r)),
E(m*xxtemp[j]/c),
prod(s=0,c-1,((-E((s+1+k)*al+xxtemp[j]/c))*E(thfix))^(1/2-(s+1)/c)/E(thfix*(1/2-(s+1)/c))),
1/sqrt(3*c),
1/prod(ell=1,abs(c)-1,(1-E(ell*al))^(1/2-ell/abs(c)))]}

constarbmvecbitsasym(11/13,2,2+100*I,0,-0.000001)
```

```
constarbm(al,j,m,k,thfix,sts)=local(c,Xtemp,xxtemp);c=denominator(al);
    Xtemp=polroots(1-xvar*E(thfix)-E(al*c*m)*(xvar*E(thfix))^4);for(j=1,4,
    Xtemp[j]=Xtemp[j]*E(thfix));xxtemp=vector(4,j,(log(Xtemp[j]*E(thfix))-
    log(E(thfix)))/(2*Pi*I));sum(r=0,c-1,E((2*r^2+r*(4*k+m))*al+(4*r+m)*
    xxtemp[j]/c)/qpoch(E(al+xxtemp[j]/c),E(al),r))*prod(s=0,c-1,((1-E((s+1+
    k)*al+xxtemp[j]/c))*E(thfix))^(1/2-(s+1)/c)/E(thfix*(1/2-(s+1)/c)))/
    sqrt(Xtemp[j]/(1-Xtemp[j])*c+4*c)/prod(ell=1,abs(c)-1,(1-E(ell*al))
    ^(1/2-ell/abs(c)))/if(sts,E(m*xxtemp[j]/c),1)

constarbm(11/13,2,2+100*I,0,-0.000001)



\r nahm4data
delta=vector(4,j,X[j]+4*(1-X[j]));
A4=matrix(length(A4OZ),4,k,j,A4OZ[k]*[1,X[j],X[j]^2,X[j]^3]~/DD(k)/delta[j
    ]^(3*k));
a(k,j)=if(k,A4[k,j],1)/sqrt(delta[j]);
polroots(numerator(bestapprPade(serconvol(sum(k=0,120,a(k,4)*x^k)+O(x^121)
    ,exp(x+O(x^121)))))/subst(numerator(bestapprPade(serconvol(sum(k=0,120,
    a(k,4)*x^k)+O(x^121),exp(x+O(x^121))))),x,0))
polroots(denominator(bestapprPade(serconvol(sum(k=0,120,a(k,4)*x^k)+O(x
    ^121),exp(x+O(x^121)))))/subst(denominator(bestapprPade(serconvol(sum(k
    =0,120,a(k,4)*x^k)+O(x^121),exp(x+O(x^121))))),x,0))
{aa(k,j,jj,H,kk=0)=sum(l=0,H,(k-1-l)!*a(l,jj)/(V[jj]-V[j]+kk*4*Pi^2)^(k-l)
    )/2/Pi/I}
[a(100,1),a(100,2),a(100,3),a(100,4)]
/* = [-1.8298 E40 + 0.E-171*I, -8.5822 E96 + 0.E16*I, 9.8017 E96 - 6.4042
    E96*I, 9.8017 E96 + 6.4042 E96*I] */
[a(100,1)-(aa(100,1,3,0)-aa(100,1,4,0)),a(100,2)-(-aa(100,2,3,0)+aa
    (100,2,4,0)),a(100,3)-aa(100,3,2,0),a(100,4)+aa(100,4,2,0)]
/* = [-2.1917 E38 + 0.E-171*I, -1.8587 E94 + 0.E16*I, 5.3806 E93 - 5.9790
    E93*I, 5.3806 E93 + 5.9790 E93*I] */
[a(100,1)-(aa(100,1,3,10)-aa(100,1,4,10)),a(100,2)-(-aa(100,2,3,10)+aa
    (100,2,4,10)),a(100,3)-aa(100,3,2,10),a(100,4)+aa(100,4,2,10)]
/* = [1.1426 E31 + 0.E-171*I, -1.3188 E81 + 0.E16*I, -8.8103 E80 + 2.6430
    E80*I, -8.8103 E80 - 2.6430 E80*I] */
local(tx);tx=serreverse(x/(V[2]-V[4]-x)+O(x^126));confser4m2=sum(k=1,125,
    tx^k/(k-1)!*a(k,4));
local(tx);tx=serreverse(x/(V[2]-V[4]-x)+O(x^126));confser4m2a1=sum(k
    =1,125,tx^k/(k-1)!*aa(k,4,1,k-1));
polcoeff(confser4m2,125)
/* = -1.5668 E10 - 3.3495 E10*I */
polcoeff(confser4m2a1,125)
/* = -1.5668 E10 - 3.3495 E10*I */

qpoch(x,q,n)=prod(j=0,n-1,1-x*q^j);
```

```
th(x,q,N,kkk=0)=local(qk,sp,sm,bi,s);qk=1;sp=1;sm=x^(-1);bi=1;s=bi*(0^kkk*
    sp-(-1)^kkk*sm);for(k=1,N,qk=qk*q;sp=sp*x;sm=sm/x;bi=-bi*qk;s=s+bi*(k^
    kkk*sp-(-k-1)^kkk*sm));s

GG(qB,A,B,r,tt,N)=local(qk,q,t,s);q=qB^B;qk=qB^r;t=qB^(A*r*(r+1)/2)*tt^r/
    qpoch(q,q,r);s=t;for(k=1,N,for(j=1,B,qk=qk*qB;t=qk^A*tt*t/(1-qk^B));s=s
    +t);s

GGmat(q4,tt,N)=matrix(4,4,i,j,GG(q4,3,4,j-1,q4^(4*i-4)*tt,N))

TH(q8,m,N)=local(t);t=exp(eps/2+m*eps/4+eps^2*Gf(q8^8,2,N)/4+O(eps^5))*[th
    (-q8^(8*m)*exp(eps+O(eps^5)),q8^32,N),-q8^5*q8^(2*m)*exp(eps/4+O(eps^5)
    )*th(-q8^(8*m+8)*exp(eps+O(eps^5)),q8^32,N),q8^12*q8^(4*m)*exp(eps/2+O(
    eps^5))*th(-q8^(8*m+16)*exp(eps+O(eps^5)),q8^32,N),-q8^21*q8^(6*m)*exp
    (3*eps/4+O(eps^5))*th(-q8^(8*m+24)*exp(eps+O(eps^5)),q8^32,N)];matrix
    (4,4,i,j,polcoeff(t[i],j-1+if(j==4,1),eps))

TH(q8+O(q8^1000),1,600)*TH(q8+O(q8^1000),0,600)^(-1)

TH(q8+O(q8^1000),2,600)*TH(q8+O(q8^1000),1,600)^(-1)

(E((I*20)/2)*sqrt(20*I/4)*E(-1/8)*matrix(4,4,i,j,I^((i-1)*(j-1)))*TH(E(I
    *20/8),0,1000)*matdiagonal([1,(-20*I),(-20*I)^2,(-20*I)^4]))^(-1)*E
    (-1/(I*20)/2)*TH(E(-1/(I*20)/8),0,1000)

GGG(q8,m,N)=matdiagonal([1,q8^(8*m),q8^(16*m-32),q8^(24*m-96)])*GGmat(q8
    ^2,q8^(-2*m-3),N)*TH(q8,m,N)/th(1/q8^16,q8^24,N)

GGG(q8+O(q8^1000),0,100)[1,1]-f(q8^8+O(q8^1000),4,2,0,100)

PGGG2f(q)=[1,0,0,0;-q^(-3) - q^(-2) - q^(-1), q^(-3) + q^(-2) + 2*q^(-1) +
     1 + q, -q^2 - q^3 - q^4 - q^5 - q^12, q^12;q^(-5) + q^(-4) + q^(-3), -
    q^(-5) - 2*q^(-4) - 2*q^(-3) - 2*q^(-2) - q^(-1) + q^3, 1 + 2*q + 2*q^2
     + q^3 + q^4 + q^10 + q^11, -q^10 - q^11;-q^(-6) - q^(-1), q^(-6) + q
    ^(-5) + q^(-4) - q^2, -q^(-1) - 1 - q - q^9, q^9]

vectorv(4,j,f(q8^8+O(q8^1000),4,j-1,0,100))-PGGG2f(q8^8)*GGG(q8+O(q8^1000)
    ,-2,100)[,1]


\p2000
\ps200
default(format,"g.5")
X=polroots(x^4+x-1);
xx=vector(4,j,log(X[j])/(2*Pi*I));
V=vector(4);
V[1]=dilog(X[1])-Pi^2/6+2*(2*Pi*I)^2*(xx[1]+k)^2-(2*Pi*I)^2*(4*k+2)*(xx
    [1]+k)
```

```
V[2]=dilog(X[2])-Pi^2/6+2*(2*Pi*I)^2*(xx[2]+k)^2-(2*Pi*I)^2*(4*k)*(xx[2]+k
    )
V[3]=dilog(X[3])-Pi^2/6+2*(2*Pi*I)^2*(xx[3]+k)^2-(2*Pi*I)^2*(4*k-1)*(xx
    [3]+k)
V[4]=dilog(X[4])-Pi^2/6+2*(2*Pi*I)^2*(xx[4]+k)^2-(2*Pi*I)^2*(4*k+1)*(xx
    [4]+k)
V=subst(V,k,0)
delta=vector(4,j,X[j]+4*(1-X[j]));
A4=matrix(length(A40Z),4,k,j,A40Z[k]*[1,X[j],X[j]^2,X[j]^3]~/DD(k)/delta[j
    ]^(3*k));
a(k,j)=if(k,A4[k,j],1)/sqrt(delta[j]);

phi1=1+sum(k=1,128,A4[k,1]*x^k)+O(x^129);
phi2=1+sum(k=1,128,A4[k,2]*x^k)+O(x^129);
phi3=1+sum(k=1,128,A4[k,3]*x^k)+O(x^129);
phi4=1+sum(k=1,128,A4[k,4]*x^k)+O(x^129);

\\borel of a = series in x
borel(a) = serconvol(a,exp(x));

\\pade of a = series in x
pade(a,N) = local(t); t=bestapprPade(a+O(x^N))

pade1=pade(borel(phi1),128);
pade2=pade(borel(phi2),128);
pade3=pade(borel(phi3),128);
pade4=pade(borel(phi4),128);

\\borel-pade resummation at tau of a series a with N coeffs
{resum(tau,pb)= intnum(xi=0,[+oo,1],exp(-xi)*subst(pb,x,xi*tau))};

rphi1(tau)=E(-V[1]*tau/(2*Pi*I)^2)/sqrt(delta[1])*resum(2*Pi*I/tau,phi1)
rphi2(tau)=E(-V[2]*tau/(2*Pi*I)^2)/sqrt(delta[2])*resum(2*Pi*I/tau,phi2)
rphi3(tau)=E(-V[3]*tau/(2*Pi*I)^2)/sqrt(delta[3])*resum(2*Pi*I/tau,phi3)
rphi4(tau)=E(-V[4]*tau/(2*Pi*I)^2)/sqrt(delta[4])*resum(2*Pi*I/tau,phi4)

\p500
default(format,"g.5")

f(E(-1/(1000*E(0.0001))),4,0)
\\ = 1.4799 E67 + 1.8058 E67*I
rphi3(E(0.0001)*1000)
\\ = 1.4799 E67 + 1.8057 E67*I

f(E(-1/(1000*E(0.0001))),4,0)/rphi3(E(0.0001)*1000)
\\ = 1.0000 - 2.7438 E-8*I

(f(E(-1/(1000*E(0.0001))),4,0)/rphi3(E(0.0001)*1000)-1)/E(E(0.0001)*1000)
```

```
    ^3
\\ = 1.0197 - 2.4883 E-5*I

(f(E(-1/(1000*E(0.0001))),4,0)/rphi3(E(0.0001)*1000)-1-E(E(0.0001)*1000)
    ^3-E(E(0.0001)*1000)^4-E(E(0.0001)*1000)^5-E(E(0.0001)*1000)^6-E(E
    (0.0001)*1000)^7-E(E(0.0001)*1000)^8-E(E(0.0001)*1000)^9)/E(E(0.0001)
    *1000)^10
\\ = 2.0397 - 5.0718 E-5*I

f(E(-1/(1000*E(0.0001))),4,0)-rphi3(E(0.0001)*1000)*f(E(1000*E(0.0001))
    ,4,1,0,100)
\\ = 0.20122 + 0.68776*I

f(E(-1/(1000*E(0.0001))),4,0)-rphi3(E(0.0001)*1000)*f(E(1000*E(0.0001))
    ,4,1,0,100)-rphi2(E(0.0001)*1000)*f(E(1000*E(0.0001)),4,0,0,100)-rphi1(
    E(0.0001)*1000)*f(E(1000*E(0.0001)),4,2,0,100)-rphi4(E(0.0001)*1000)*E
    (1000*E(0.0001))*f(E(1000*E(0.0001)),4,3,0,100)
\\ = 5.5399 E-138 - 3.7010 E-138*I

f(E(-1/(1000*E(0.00001))),4,0)-rphi3(E(0.00001)*1000)*f(E(1000*E(0.00001))
    ,4,1,0,100)-rphi2(E(0.00001)*1000)*f(E(1000*E(0.00001)),4,0,0,100)-
    rphi1(E(0.00001)*1000)*f(E(1000*E(0.00001)),4,2,0,100)-rphi4(E(0.00001)
    *1000)*E(1000*E(0.00001))*f(E(1000*E(0.00001)),4,3,0,100)

GGG(E(-1/(1000*E(0.00001))/8),-2,100000)[1,]-[rphi1(E(0.00001)*1000),rphi2
    (E(0.00001)*1000),rphi3(E(0.00001)*1000),rphi4(E(0.00001)*1000)
    ]*[0,0,1,0;1,0,0,0;0,1,0,0;0,0,0,E(1000*E(0.00001))]*PGGG2f(E(1000*E
    (0.00001)))*GGG(E(1000*E(0.00001)/8),-2,100)*matdiagonal([1,1000*E
    (0.00001),(1000*E(0.00001))^2,(1000*E(0.00001))^4])

f(E(-1/(1000*E(1/2-0.00001))),4,0)-rphi4(E(1/2-0.00001)*1000)*f(E(1000*E
    (1/2-0.00001)),4,1,0,100)-rphi2(E(1/2-0.00001)*1000)*f(E(1000*E
    (1/2-0.00001)),4,0,0,100)-rphi1(E(1/2-0.00001)*1000)*f(E(1000*E
    (1/2-0.00001)),4,2,0,100)-rphi3(E(1/2-0.00001)*1000)*E(1000*E
    (1/2-0.00001))*f(E(1000*E(1/2-0.00001)),4,3,0,100)

GGG(E(-1/(1000*E(1/2-0.00001))/8),-2,100000)[1,]-[rphi1(E(1/2-0.00001)
    *1000),rphi2(E(1/2-0.00001)*1000),rphi3(E(1/2-0.00001)*1000),rphi4(E
    (1/2-0.00001)*1000)]*[0,0,1,0;1,0,0,0;0,0,0,E(1000*E(1/2-0.00001))
    ;0,1,0,0]*PGGG2f(E(1000*E(1/2-0.00001)))*GGG(E(1000*E(1/2-0.00001)/8)
    ,-2,100)*[1,0,0,0;0,-1,0,0;0,0,1,0;0,0,0,1]*matdiagonal([1,1000*E
    (1/2-0.00001),(1000*E(1/2-0.00001))^2,(1000*E(1/2-0.00001))^4])


[0,0,1,0;1,0,0,0;0,1,0,0;0,0,0,q8^8]*PGGG2f(q8^8)*GGG(q8+O(q8^200),-2,100)
    *([0,0,1,0;1,0,0,0;0,0,0,q8^8;0,1,0,0]*PGGG2f(q8^8)*GGG(q8+O(q8^200)
    ,-2,100)*[1,0,0,0;0,-1,0,0;0,0,1,0;0,0,0,1])^(-1)

```

```
/*
[1 - q8^8 - 2*q8^16 - 2*q8^24 - q8^32 + q8^40 + 3*q8^48 + 7*q8^56 + 10*q8
    ^64 + 14*q8^72 + 16*q8^80 + 17*q8^88 + 15*q8^96 + 11*q8^104 + 2*q8^112
    - 12*q8^120 - 30*q8^128 + O(q8^135) 1 + q8^8 + q8^16 - q8^32 - 3*q8^40
    - 6*q8^48 - 8*q8^56 - 10*q8^64 - 10*q8^72 - 9*q8^80 - 5*q8^88 + q8^96 +
     11*q8^104 + 23*q8^112 + 38*q8^120 + 55*q8^128 + O(q8^135) 1 - q8^16 -
    3*q8^24 - 3*q8^32 - 4*q8^40 - 4*q8^48 - 2*q8^56 + q8^64 + 6*q8^72 + 12*
    q8^80 + 20*q8^88 + 27*q8^96 + 36*q8^104 + 42*q8^112 + 46*q8^120 + 45*q8
    ^128 + O(q8^135) -1 - q8^8 + 2*q8^24 + 4*q8^32 + 5*q8^40 + 7*q8^48 + 6*
    q8^56 + 5*q8^64 - 5*q8^80 - 14*q8^88 - 23*q8^96 - 36*q8^104 - 48*q8^112
     - 60*q8^120 - 69*q8^128 + O(q8^135)]

[q8^16 + 2*q8^24 + 2*q8^32 + 2*q8^40 - 2*q8^56 - 5*q8^64 - 9*q8^72 - 13*q8
    ^80 - 17*q8^88 - 20*q8^96 - 21*q8^104 - 19*q8^112 - 15*q8^120 - 5*q8
    ^128 + O(q8^135) 1 - q8^8 - q8^16 - q8^24 - q8^32 + q8^40 + 2*q8^48 +
    5*q8^56 + 8*q8^64 + 10*q8^72 + 12*q8^80 + 13*q8^88 + 10*q8^96 + 7*q8
    ^104 - 11*q8^120 - 24*q8^128 + O(q8^135) -q8^8 + q8^24 + 2*q8^32 + 4*q8
    ^40 + 4*q8^48 + 5*q8^56 + 4*q8^64 + 2*q8^72 - 2*q8^80 - 7*q8^88 - 16*q8
    ^96 - 23*q8^104 - 33*q8^112 - 42*q8^120 - 51*q8^128 + O(q8^135) q8^8 +
    q8^16 - q8^32 - 4*q8^40 - 5*q8^48 - 7*q8^56 - 8*q8^64 - 7*q8^72 - 5*q8
    ^80 - q8^88 + 7*q8^96 + 16*q8^104 + 27*q8^112 + 42*q8^120 + 55*q8^128 +
     O(q8^135)]

[-q8^8 - q8^16 + q8^24 + 2*q8^32 + 4*q8^40 + 5*q8^48 + 6*q8^56 + 5*q8^64 +
     3*q8^72 - 7*q8^88 - 14*q8^96 - 24*q8^104 - 34*q8^112 - 47*q8^120 - 55*
    q8^128 + O(q8^135) 1 - q8^16 - q8^24 - 3*q8^32 - 3*q8^40 - 4*q8^48 - 2*
    q8^56 + 4*q8^72 + 9*q8^80 + 16*q8^88 + 22*q8^96 + 29*q8^104 + 35*q8^112
     + 38*q8^120 + 39*q8^128 + O(q8^135) 1 - q8^8 - 2*q8^16 - q8^24 - q8^32
     + q8^40 + 2*q8^48 + 7*q8^56 + 8*q8^64 + 12*q8^72 + 14*q8^80 + 16*q8^88
     + 12*q8^96 + 10*q8^104 + q8^112 - 11*q8^120 - 27*q8^128 + O(q8^135) 2*
    q8^16 + 2*q8^24 + 3*q8^32 + q8^40 - 4*q8^56 - 8*q8^64 - 12*q8^72 - 17*
    q8^80 - 21*q8^88 - 24*q8^96 - 23*q8^104 - 22*q8^112 - 11*q8^120 - q8
    ^128 + O(q8^135)]

[q8^8 - 2*q8^24 - 3*q8^32 - 5*q8^40 - 4*q8^48 - 4*q8^56 - q8^64 + 2*q8^72
    + 9*q8^80 + 16*q8^88 + 25*q8^96 + 33*q8^104 + 42*q8^112 + 49*q8^120 +
    52*q8^128 + O(q8^135) -1 + q8^8 + q8^16 + 2*q8^24 + 3*q8^32 + 2*q8^40 +
     2*q8^48 - q8^56 - 5*q8^64 - 10*q8^72 - 15*q8^80 - 22*q8^88 - 25*q8^96
    - 30*q8^104 - 30*q8^112 - 27*q8^120 - 19*q8^128 + O(q8^135) 2*q8^8 + q8
    ^16 + q8^24 - q8^32 - 3*q8^40 - 5*q8^48 - 8*q8^56 - 10*q8^64 - 12*q8^72
     - 11*q8^80 - 9*q8^88 - q8^96 + 5*q8^104 + 20*q8^112 + 34*q8^120 + 55*
    q8^128 + O(q8^135) 1 - q8^8 - 2*q8^16 - 2*q8^24 - 2*q8^32 + 2*q8^40 +
    3*q8^48 + 8*q8^56 + 11*q8^64 + 16*q8^72 + 17*q8^80 + 20*q8^88 + 15*q8
    ^96 + 12*q8^104 + q8^112 - 14*q8^120 - 34*q8^128 + O(q8^135)]
*/
```

The file **nahm4data** begins as follows.

```
{A40Z=vector(129)};
```

```
{A40Z[1]=[-64, 100, 18, -54]};
{A40Z[2]=[-104876, 113812, 29836, 17388]};
{A40Z[3]=[-79093616, -1648464240, 2928617760, -694542712]};
...
```

D.2 Knots

Code 34 (Asymptotics of the Kashaev invariant of the trefoil)**.**

```
prec=2000;
default(realprecision,prec)
default(format,"g.5")
E(x)=exp(2*Pi*I*x);
J(x)=local(dx,qk,qq,t,s);qq=E(x);dx=denominator(x);qk=1;t=1;s=t;for(k=1,dx-1,qk=qk*qq;t=t*(1-qk)^2;s=s+qk*t);s
Jq(q,N)=local(qk,t,s);qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=t*(1-qk)^2;s=s+qk*t);s
huh=vector(2000);
for(k=100,220,huh[k]=J(-1/k))
asymposc(huh,100,5,E(0.42))
/* =
[1.0000 - 3.2503 E-9*I 5.0322 E-6 + 1.5038 E-6*I 0.99996 - 5.6844 E-6*I]

[ -0.12369 - 0.48175*I 5.0322 E-6 + 1.5038 E-6*I 0.99997 - 1.1274 E-5*I]
*/
asymposc(huh,100,5,E(0.57))
/* =
[  0.96593 - 0.25882*I 1.5000 - 4.3098 E-11*I 0.70711 + 0.70711*I]

[-0.061099 + 0.68460*I 1.5000 - 4.3098 E-11*I -1.5175 + 0.88145*I]
*/
huh0=vector(2000);
for(k=100,220,huh0[k]=huh[k]-k^(3/2)*E(23/24*k)*E(1/8)*E(23/24/k)-subst(bestapprPade(Jq(exp(x+O(x^10)),10),0),x,-2*Pi*I/k))
asymp(huh0,200,18)
/* = [1.0000 - 2.9525 E-12*I, -11.000 + 1.4843 E-8*I, 4.2882 E7 + 3.5833 E14*I] */
phi0=Jq(exp(x+O(x^101)),100);
a0(k)=polcoeff(phi0,k);
{aa0(k,H,kk=0)=sqrt(2*Pi)*sum(l=0,H,gamma(k-l+3/2)*(-1/24)^(l)/l!/(4*Pi^2/24+kk*4*Pi^2)^(k-l+3/2))}
a0(100)
/* = 2.6597 E137 */
a0(100)-aa0(100,200)
/* = -0.00011742 */
a0(100)-aa0(100,200)+aa0(100,200,1)*5
```

```
/* = -2.4509 E-34 */
a0(100)-aa0(100,200)+aa0(100,200,1)*5+aa0(100,500,2)*7
/* = 1.7936 E-74 */


/*********************************************/
\\q-series for 3_1
/*********************************************/

{g0(q,m,N)=
local(qk,s,t);
qk=1;
t=1;
s=t;
for(k=1,N,qk=q*qk;t=-qk*t/(1-qk);s=s+t*qk^m);
s}

{g1(q,m,N)=
local(qk,s,t,G1);
G1=Gf(q,1,N);
qk=1;
t=1;
ha=0;
s=t*(m+1/4-G1+ha);
for(k=1,N,qk=q*qk;t=-qk*t/(1-qk);ha=ha+qk/(1-qk);s=s+qk^m*t*(k+m+1/4-G1+ha
   ));
s}

{g(q,eps,N)=
local(qk,s,t);
qk=1;
t=1;
s=t*exp(eps/2);
for(k=1,N,qk=q*qk;t=-qk*t/(1-qk*exp(eps));s=s+t*exp((k+1/2)*eps));
s*qpoch(q*exp(eps),q,N)/qpoch(q,q,N)}

g(q+O(q^10),x+O(x^3),30)-(g0(q+O(q^10),0,30)+x*g1(q+O(q^10),0,30)+O(x^2))

g0(q+O(q^50),0,50)
\\ = 1 - q - q^2 + q^5 + q^7 - q^12 - q^15 + q^22 + q^26 - q^35 - q^40 + O
   (q^50)

g1(q+O(q^50),0,50)*2
\\ = 1 - 5*q - 7*q^2 + 11*q^5 + 13*q^7 - 17*q^12 - 19*q^15 + 23*q^22 + 25*
   q^26 - 29*q^35 - 31*q^40 + O(q^50)

{gL(q,m,N)=
local(qk,s,t);
```

```
qk=1;
t=1;
s=0;
for(k=1,N,qk=q*qk;t=-qk*t/(1-qk);s=s+qk^m*t/(1-qk));
1/4-Gf(1/q,1,N)-m+s}

/**********************************************/
\\Habiro element
/**********************************************/

{F(q,N)=local(qk,s,t);qk=1;t=1;s=t;for(k=1,N,qk=q*qk;t=t*(1-qk);s=s+t);s}
\\ = 1 - x + 3/2*x^2 - 19/6*x^3 + 69/8*x^4 - 3451/120*x^5 + 27221/240*x^6
    - 2602699/5040*x^7 + 35825749/13440*x^8 - 5581680571/362880*x^9 + O(x
    ^10)

/**********************************************/
\\matrix of q-series
/**********************************************/


G(q,m,N)=[1,0;g1(q,m,N),g0(q,m,N)]

A(q,m)=[1,0;1/(1-q^(m+1)),1/(1-q^(m+1))]

G(q+O(q^30),1,30)-A(q+O(q^30),0)*G(q+O(q^30),0,30)

GL(q,m,N)=[1,0;-gL(q,-m,N),g0(q,-m-1,N)]

G(q+O(q^30),0,30)^(-1)*A(q+O(q^30),0)^(-1)-G(q+O(q^30),1,30)^(-1)
GL(1/q+O(q^30),0,30)*A(q+O(q^30),0)^(-1)-GL(1/q+O(q^30),1,30)
```

Code 35 (Asymptotics of the Kashaev invariant of the figure eight knot).

```
prec=2000;
default(realprecision,prec)
default(format,"g.5")
E(x)=exp(2*Pi*I*x);
J(x)=local(dx,qk,qq,t,s);qq=E(x);dx=denominator(x);qk=1;t=1;s=t;for(k=1,dx
    -1,qk=qk*qq;t=-t*(1-qk)^2/qk;s=s+t);s
Jq(q,N,m=0)=local(qk,t,s);qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=-t*(1-qk)^2/qk;
    s=s+t*qk^m);s
huh=vector(2000);
for(k=200,220,huh[k]=J(-1/k))
asymp(huh,200,10)
/* = [1.3814 - 1.9035 E-1982*I, 1.5000 - 2.6077 E-1978*I, 0.75984 + 2.5320
     E-1978*I] */
V=I*imag(dilog(E(1/6)))*2
/* = 2.0299*I */
```

```
E(V/(2*Pi*I)^2)
/* = 1.3814 */
E(1/8)/sqrt(sqrt(-3))
huh1=vector(2000);
for(k=200,220,huh1[k]=(huh[k]/E(V/(2*Pi*I)^2*k)/(E(1/8)/sqrt(sqrt(-3)))-1)
    /(2*Pi*I/k))
lim(huh1,200,10)*sqrt(-3)^3*DD(1)
/* = -11.000 - 2.2834 E-1979*I */
g(q,N=ceil(abs(real(sqrt(log(10^(-prec))/log(abs(q))*2)))))=local(qk,t,s);
    qk=1;t=1;s=t;for(k=1,N,qk=qk*q;t=-t*qk/(1-qk)^2;s=s+t);s
hoh=vector(2000);
for(k=100,220,hoh[k]=g(E(-1/(I*k)))*sqrt(I*k))
asymposc(hoh,100,5,E(0.3))
/* =
[0.94827 + 0.31748*I -1.6319 E-8 - 1.7181 E-8*I 0.75984 + 7.9416 E-8*I]
[-0.63925 - 1.2685*I -1.6319 E-8 - 1.7181 E-8*I 0.75984 + 9.3682 E-8*I] */
asymposc(hoh,100,5,E(-0.3))
/* =
[0.94827 - 0.31748*I -1.6319 E-8 + 1.7181 E-8*I 7.9416 E-8 + 0.75984*I]
[-0.63925 + 1.2685*I -1.6319 E-8 + 1.7181 E-8*I 9.3682 E-8 + 0.75984*I] */
asympv(hoh,[E(V/(2*Pi*I)^2*I),E(-V/(2*Pi*I)^2*I)],[0,0],200,5)
G(q,N=ceil(abs(real(sqrt(log(10^(-prec))/log(abs(q))*2)))))=local(qk,t,s,
    ts,G1);G1=Gf(q,1,N);qk=1;t=1;ts=0;s=t*(-4*G1+2*ts);for(k=1,N,qk=qk*q;ts
    =ts+(1+qk)/(1-qk);t=-t*qk/(1-qk)^2;s=s+t*(-4*G1+2*ts));s
hah=vector(2000);
for(k=100,220,hah[k]=G(E(-1/(I*k)))/sqrt(I*k))
asymposc(hah,100,5,E(0.3))
/* =
[0.94827 + 0.31748*I -1.6770 E-8 - 1.7203 E-8*I 0.75984 + 7.9405 E-8*I]
[-0.63925 - 1.2685*I -1.6770 E-8 - 1.7203 E-8*I 0.75984 + 9.3944 E-8*I] */
asymposc(hah,100,5,E(-0.3))
/* =
[0.94827 - 0.31748*I -1.6770 E-8 + 1.7203 E-8*I -7.9405 E-8 - 0.75984*I]
[-0.63925 + 1.2685*I -1.6770 E-8 + 1.7203 E-8*I -9.3944 E-8 - 0.75984*I]
    */
asympv(hah,[E(V/(2*Pi*I)^2*I),E(-V/(2*Pi*I)^2*I)],[0,0],200,5)
Ggoth(q,N=ceil(abs(real(sqrt(log(10^(-prec))/log(abs(q))*2)))))=local(qk,t
    ,s,ts,tss,G1);G1=Gf(q,1,N);qk=1;t=1;ts=0;tss=0;s=t*((-4*G1+2*ts)
    ^2/8-1/24+tss);for(k=1,N,qk=qk*q;ts=ts+(1+qk)/(1-qk);tss=tss+qk/(1-qk)
    ^2;t=-t*qk/(1-qk)^2;s=s+t*((-4*G1+2*ts)^2/8-1/24+tss));s
heh=vector(2000);
for(k=200,220,heh[k]=Ggoth(E(-1/(I*k))))
asympv(heh,[1,E(V/(2*Pi*I)^2*I),E(-V/(2*Pi*I)^2*I)],[0,3/2,3/2],200,5)
/* = [1.0000 + 7.9151 E-1986*I, -0.044774 + 0.044774*I, -0.044774 -
    0.044774*I]~ */
\r 41data
heh1=vector(2000);
for(k=200,220,heh1[k]=(heh[k]-E(V/(2*Pi*I)^2*I*k)*(E(1/8)/sqrt(sqrt(-3)))*
```

```
sqrt(I*k)^3/12*(1+sum(ell=1,10,(-1)^ell*AK41OZ[ell]/DD(ell)/sqrt(-3)^(3*ell)*(2*Pi*I/(I*k))^ell))-E(-V/(2*Pi*I)^2*I*k)*(E(1/8)/sqrt(-sqrt(-3)))*sqrt(I*k)^3/12*(1+sum(ell=1,10,AK41OZ[ell]/DD(ell)/sqrt(-3)^(3*ell)*(2*Pi*I/(I*k))^ell))-1)/(2*Pi*I/(I*k))^2)
asympv(heh1,[1,E(V/(2*Pi*I)^2*I),E(-V/(2*Pi*I)^2*I)],[0,1/2,1/2],200,5)
/* = [-1.0000 - 9.1164 E-1981*I, -2.1472 E-11 + 2.7900 E-12*I, -2.1472 E-11 - 2.7900 E-12*I]~ */
heh2=vector(2000);
for(k=200,220,heh2[k]=(heh1[k]+1)/(2*Pi*I/(I*k))^2)
asympv(heh2,[1,E(V/(2*Pi*I)^2*I),E(-V/(2*Pi*I)^2*I)],[0,1/2,1/2],200,5)*12
/* = [47.000 - 1.6569 E-1975*I, 5.0548 E-9 + 7.5165 E-10*I, 5.0548 E-9 - 7.5165 E-10*I]~ */
a(k,j)=1/sqrt(-(-1)^j*sqrt(-3))*if(k,AK41OZ[k]/sqrt(-3)^(3*k)/DD(k)*(-1)^(k+j+1),1)
polroots(numerator(bestapprPade(serconvol(sum(k=0,290,a(k,1)*x^k)+O(x^291),exp(x+O(x^291)))))/subst(numerator(bestapprPade(serconvol(sum(k=0,290,a(k,1)*x^k)+O(x^291),exp(x+O(x^291))))),x,0))
polroots(denominator(bestapprPade(serconvol(sum(k=0,290,a(k,1)*x^k)+O(x^291),exp(x+O(x^291)))))/subst(denominator(bestapprPade(serconvol(sum(k=0,290,a(k,1)*x^k)+O(x^291),exp(x+O(x^291))))),x,0))
```

The file **41data** begins as follows.

```
{AK41OZ=vector(296)};
{AK41OZ[1]=11};
{AK41OZ[2]=697};
{AK41OZ[3]=724351};
...
```

D.3 Half surgery on the figure eight knot

Code 36 (Asymptotics of the WRT invariant and $\widehat{Z}$ series)**.**

```
/**********************************************/
\\setup
/**********************************************/
\p 2000
\ps 200
default(parisize,120000000)
default(format,"g.5")

\\qpochhammer
qpoch(a,q,n)=local(p,qn);p=1;qn=1;if(n>-1,for(k=1,n,p=p*(1-qn*a);qn=qn*q),for(k=1,-n,qn=qn/q;p=p/(1-qn*a)));p;

qpochinfty(a,q,N,ss=1,WMW=ss)=local(s,t,qn);t=1;s=t;qn=1;for(k=1,N,qn=qn*q^ss;t=-qn/q^(1/2+ss/2)*t/(1-qn)*a*q^(ss/2-WMW/2);s=s+t);s^ss;
```

```

/***********************************************/
\\fields
/***********************************************/
\\ trace field, type [5,1] with discriminant -7215127 (a prime)
P(x)=x^7-x^6-2*x^5+6*x^4-11*x^3+6*x^2+3*x-1;
xi=polroots(P(T));
\\
\\[-2.2411, -0.43760, 0.25599, 1.3348, 1.3483, 0.36981 - 1.4410*I, 0.36981 + 1.4410*I] where xi[3] is SL2R xi[6] and xi[7] are SL2C and others are SU2

\\ finds elements of the n-th embedding of the trace field
find(x,n,e)=lindep([x,1,xi[n],xi[n]^2,xi[n]^3,xi[n]^4,xi[n]^5,xi[n]^6],e)

\\ shape field
PP(z)=z^14+2*z^13+z^12-4*z^10-8*z^9-10*z^8-13*z^7-10*z^6-8*z^5-4*z^4+z^2+2*z+1;
xixi=polroots(PP(T));
findsh(x,n,e)=lindep([x,1,xixi[n],xixi[n]^2,xixi[n]^3,xixi[n]^4,xixi[n]^5,xixi[n]^6,xixi[n]^7,xixi[n]^8,xixi[n]^9,xixi[n]^10,xixi[n]^11,xixi[n]^12,xixi[n]^13],e)
\\
\\[0.61642, 1.6223, -0.69314 - 0.019415*I, -0.69314 + 0.019415*I, -1.4416 - 0.040379*I, -1.4416 + 0.040379*I, 0.47669 - 0.87907*I, 0.47669 + 0.87907*I, -0.42361 - 0.90585*I, -0.42361 + 0.90585*I, -0.19405 - 0.98099*I, -0.19405 + 0.98099*I, 0.15633 - 0.98770*I, 0.15633 + 0.98770*I]

\\ trace field from shape field
s2t(x)=-15-8*x-3*x^2+4*x^3+55*x^4+40*x^5+91*x^6+66*x^7+56*x^8+43*x^9+x^10-14*x^12-11*x^13;
\\
\\vector(14,j,s2t(xixi[j])) = [0.25599, 0.25599, 0.36981 + 1.4410*I, 0.36981 - 1.4410*I, 0.36981 - 1.4410*I, 0.36981 + 1.4410*I, -2.2411, -2.2411, 1.3483, 1.3483, 1.3348, 1.3348, -0.43760, -0.43760]
\\therefore see that 1P->7,8PP|2P->13,14PP|3P->1,2PP|4P->11,12PP|5P->9,10PP|6P->4,5PP|7P->3,6PP|

\\ shape field is quadratic extension
sot(x,j)=x^2+x*(-1-4*xi[j]-4*xi[j]^2+3*xi[j]^3-xi[j]^4+xi[j]^5+xi[j]^6)+1

\\ shapes from the trace field
t2s=vector(7,j,(-(-1-4*xi[j]-4*xi[j]^2+3*xi[j]^3-xi[j]^4+xi[j]^5+xi[j]^6)+sqrt((-1-4*xi[j]-4*xi[j]^2+3*xi[j]^3-xi[j]^4+xi[j]^5+xi[j]^6)^2-4))/2)

/***********************************************/
\\complex volumes
```

```
/***********************************************/
R(z,p,q)=dilog(z)+(log(z)+2*Pi*I*p)*(log(1-z)-2*Pi*I*q)/2;

\\ shapes
z1vec=vector(7,j,2*t2s[j]^13+3*t2s[j]^12+t2s[j]^11-9*t2s[j]^9-12*t2s[j]^8-15*t2s[j]^7-20*t2s[j]^6-10*t2s[j]^5-13*t2s[j]^4-t2s[j]^3+t2s[j]^2+2*t2s[j]+4);
z2vec=vector(7,j,-2*t2s[j]^13-3*t2s[j]^12-t2s[j]^11+9*t2s[j]^9+12*t2s[j]^8+15*t2s[j]^7+20*t2s[j]^6+10*t2s[j]^5+12*t2s[j]^4+t2s[j]^3-2*t2s[j]-3);

\\ vols
vol=[-R(z1vec[1],0,0)+R(z2vec[1],-2,-3),-R(z1vec[2],0,1)+R(z2vec[2],0,0),-R(z1vec[3],-1,-1)+R(z2vec[3],0,1),-R(z1vec[4],0,0)+R(z2vec[4],1,1),-R(z1vec[5],0,-1)+R(z2vec[5],-1,-2),-R(z1vec[6],0,0)+R(z2vec[6],-1,-2),-R(z1vec[7],0,0)+R(z2vec[7],1,2)];
\\[99.254, -6.7857, -0.11620, 9.2837, 41.608, 44.346 - 1.3985*I, 44.346 + 1.3985*I]

\\ the shifted volumes
V = vol-4*Pi^2*[2,0,-1,0,1,1,1]
\\[20.296, -6.7856, 39.362, 9.2837, 2.1292, 4.8678 - 1.3985*I, 4.8678 + 1.3985*I]

\\1-loop
delta=vector(7,j,[-74,-66,133,-74,31,15,-12]*vectorv(7,l,xi[j]^(l-1)));

/***********************************************/
\\formula based WRT
/***********************************************/
\\WRT invariant for 4_1(-1,2) see Hecke..., Lovejoy, Hikami
Wex(q,m,n,N)=sum(k=0,N,(-1)^k*q^(-k*(k+1)/2+m*k)*qpoch(q,q,2*k+1)*sum(ell=0,k,q^(ell*(ell+1)+n*ell)/qpoch(q,q,ell)/qpoch(q,q,k-ell)));

\\fast version
{W(q,m,n,N)=
local(q2,t,qbc,qbs,qp,qk,q2k,tqt,mo,temp);
q2=q^2;t=1-q;qbc=1;qbs=0;qp=1-q;qk=1;q2k=1;tqt=1;mo=1;
for(k=1,N,temp=qbc;mo=(-1)*mo;qk=q*qk;q2k=(q2)*q2k;tqt=qk*tqt;qbc=(1+q-qk+q2k*q^n)*qbc-(q-qk)*qbs;qbs=temp;qp=qp*(1+qk)*(1-q*q2k);t=t+mo*qk^m*qp*qbc/tqt);
t};

\\check they agree
W(exp(h+O(h^20)),7,-11,60)-Wex(exp(h+O(h^20)),7,-11,60)

/***********************************************/
\\numerical asymptotics
```

```
/***********************************************/
huh=vector(2000);
for(k=500,1020,huh[k]=W(E(-1/k),0,0,k+10);print([k,gettime()]))

asymposc(huh,500,2,E(0.1))
\\ = [0.94313 + 0.33243*I 0.50001 + 1.0554 E-5*I 0.30443 - 0.30448*I]

asymposc(huh,500,2,E(0.3))
\\ = [0.093109 + 0.99566*I 0.49999 + 3.3768 E-6*I -0.26688 + 0.26690*I]

asymposc(huh,500,2,E(0.4))
\\ = [-0.99606 - 0.088638*I 0.50000 + 9.4365 E-6*I 0.20780 - 0.20783*I]

asymposc(huh,500,2,E(0.7))
\\ = [0.47135 - 0.88194*I 0.50000 + 1.2317 E-5*I -0.19891 + 0.19894*I]

exp(-vol/2/Pi/I)
\\ = [-0.99606 - 0.088638*I, 0.47135 - 0.88194*I, 0.99983 - 0.018493*I,
    0.093109 + 0.99566*I, 0.94313 + 0.33243*I, 0.89275 + 0.87392*I, 0.57201
     + 0.55994*I]

hah=vector(2000);
for(k=1000,1020,hah[k]=W(E(-1/(k+1/2)),0,0,2*k+10))

asymp(hah,1000,18)
\\ = [0.89275 + 0.87392*I, 0.50000 - 7.1669 E-33*I, -0.41564 - 1.0444*I]

/***********************************************/
\\reading in data
/***********************************************/

A=vector(200,k,vector(7,l,0));
\r 4112data
\r data_41_12surgery

HH=114 \\ current number of a(k,j)

am=matrix(HH,7,k,j,(delta[j])^(-1/2)*A[k]*vectorv(7,l,xi[j]^(l-1))/delta[j
    ]^(3*k)/DD(k));
a(k,j)=if(k,am[k,j],(delta[j])^(-1/2));

hah1=vector(2000);
for(k=1000,1020,hah1[k]=(hah[k]/exp(-vol[6]/2/Pi/I*(k+1/2))/sqrt(k+1/2)*
    sqrt(delta[6])/E(1/8)/2-1)/(2*Pi*I/(k+1/2)))

lim(hah1,1000,18)
\\ = 0.11069 - 0.025945*I
```

```
-A[1]*vectorv(7,l,xi[6]^(l-1))/delta[6]^(3*1)/DD(1)
\\ = 0.11069 - 0.025945*I

/*********************************************/
\\zhat
/*********************************************/

\\Zhat invariant for 4_1(-1,2) see Two-Var..., Gukov, Manolescu for numerics
Zhatex(q,m,n,N)=q^(-m-n)*sum(ell=0,N,(-1)^ell*q^(ell*(ell+1)/2-n*ell)*qpoch(q,q,ell)*sum(k=0,ell,(-1)^k*q^(3*k*(k+1)/2-(m+1)*k)/qpoch(q,q,ell-k)/qpoch(q,q,2*k)));

\\quick computation
{Zhat(q,m,n,N)=
local(q2,t,qbc,qbs1,qbs2,qbs3,qp,qell,tqt,temp);
t=q^(-n-m);qbc=q^(-m);qbs1=0;qbs2=0;qbs3=0;qp=1;qell=1;tqt=1;mo=1;
for(ell=1,N,temp=qbc;qell=qell*q;tqt=tqt*qell;qbc=-((-qell^(-4)*q^(-4)-qell^(-4)*q^(-3)-qell^(-4)*q^(-2)-qell^(-4)*q^(-1)+q^(-5)*qell^(-2)+2*q^(-4)*qell^(-2)+q^(-3)*qell^(-2)+q^(-5-m)*qell^(-1))/(qell^(-4)*q^(-4)-q^(-5)*qell^(-2)-q^(-4)*qell^(-2)-q^(-6-m)*qell^(-1)+q^(-5)+q^(-m-6)*qell^(-1))*qbc+(qell^(-4)*q^(-3)+qell^(-4)*q^(-2)+2*qell^(-4)*q^(-1)+qell^(-4)-q^(-4)*qell^(-2)-q^(-3)*qell^(-2)-q^(-5-m)*qell^(-1)+qell^(-4)*q^(+1))/(qell^(-4)*q^(-4)-q^(-5)*qell^(-2)-q^(-4)*qell^(-2)-q^(-6-m)*qell^(-1)+q^(-5)+q^(-m-6)*qell^(-1))*qbs1-(qell^(-4)*q^(-1)+qell^(-4)+qell^(-4)*q^(+1)+qell^(-4)*q^(+2))/(qell^(-4)*q^(-4)-q^(-5)*qell^(-2)-q^(-4)*qell^(-2)-q^(-6-m)*qell^(-1)+q^(-5)+q^(-m-6)*qell^(-1))*qbs2+qell^(-4)*q^(+2)/(qell^(-4)*q^(-4)-q^(-5)*qell^(-2)-q^(-4)*qell^(-2)-q^(-6-m)*qell^(-1)+q^(-5)+q^(-m-6)*qell^(-1))*qbs3);qbs3=qbs2;qbs2=qbs1;qbs1=temp;qp=qp*(1-qell);t=t+(-1)^ell/qell^n/q^(n)*qp*qbc*tqt);
t};

\\checks
Zhat(q+O(q^50),0,0,50)-Zhatex(q+O(q^50),0,0,50)
Zhat(q+O(q^50),7,-11,50)-Zhatex(q+O(q^50),7,-11,50)
Zhat(q+O(q^50),2,3,50)-Zhatex(q+O(q^50),2,3,50)

heh=vector(2000);
for(k=1000,1020,heh[k]=Zhat(E(-1/(I*k)),0,0,2*k);print([k,gettime()]))

asymp(heh,1000,18)
\\ = [1.0187, 0.50000, 0.10958]

exp(-vol[3]/2/Pi/I*I)
\\ = 1.0187

heh1=vector(2000);
```

```
for(k=1000,1020,heh1[k]=(heh[k]/exp(-vol[3]/2/Pi/I*I*k)/sqrt(I*k)*sqrt(delta[3])/E(1/8)-1)/(2*Pi*I/(I*k)))

lim(heh1,1000,15)
\\ = 0.11778 - 3.2040 E-1959*I

-A[1]*vectorv(7,l,xi[3]^(l-1))/delta[3]^(3*1)/DD(1)
\\ = 0.11778 + 0.E-2004*I

/***********************************************/
\\asymptotics series
/***********************************************/
Phi0=W(exp(x),0,0,201);
Phi1=sum(k=0,HH,a(k,1)*x^k)+O(x^(HH+1));
Phi2=sum(k=0,HH,a(k,2)*x^k)+O(x^(HH+1));
Phi3=sum(k=0,HH,a(k,3)*x^k)+O(x^(HH+1));
Phi4=sum(k=0,HH,a(k,4)*x^k)+O(x^(HH+1));
Phi5=sum(k=0,HH,a(k,5)*x^k)+O(x^(HH+1));
Phi6=sum(k=0,HH,a(k,6)*x^k)+O(x^(HH+1));
Phi7=sum(k=0,HH,a(k,7)*x^k)+O(x^(HH+1));

{B=[abs(V[1]-V[6]),abs(V[2]-V[6]),abs(V[3]-V[6]),abs(V[4]-V[6]),
abs(V[5]-V[6]),abs(V[6]-V[5]),abs(V[7]-V[5])]} \\ asympt growth rate of a(k,j)/k!

{BB=[(V[1]-V[6]),(V[2]-V[6]),(V[3]-V[6]),(V[4]-V[6]),
(V[5]-V[6]),(V[6]-V[5]),(V[7]-V[5])]}

a0(k)=polcoeff(Phi0,k);

/***********************************************/
\\asymptotics of the coefficients
/***********************************************/

\\optimal truncation asymp of coeffs a(k,j)
ako(k,j,jj)=floor(k*B[j]/(abs(V[j]-V[jj])+B[j]))

\\optimal truncated asymp of coeffs a(k,j)
{aa(k,j,jj,H=min(HH,ako(k,j,jj)))=
sum(l=0,H,(k-1-l)!*a(l,jj)/(V[j]-V[jj])^(k-l))/2/Pi/I}

{aaa(k,j,jj,H=HH)=
sum(l=0,H,(k-1-l)!*a(l,jj)/(-V[j]+V[jj])^(k-l))/2/Pi/I}

{err(k,j,jj,tt=0)=H=min(HH,ako(k,j,jj));
(k-1-H)!*a(H,jj)/(-V[j]+V[jj]+tt*Pi^2)^(k-H)/2/Pi/I}

a(80,1) \\ = 0.E-1801 - 5.2055 E20*I
```

```
a(80,1)-aa(80,1,6,20)-aa(80,1,7,20) \\ = 0.E-1809 - 3.8570 E14*I

a(80,2) \\ = 0.E-1813 + 3.9225 E30*I
a(80,2)-aa(80,2,6,20)-aa(80,2,7,20) \\ = 0.E-1820 - 1.1333 E21*I

a(110,3) \\ = 0.E-1997 + 4.0035 E5*I
a(110,3)-aa(110,3,6,13)-aa(110,3,7,13) \\ = 0.E-1978 + 5.8422*I

a(80,4) \\ = 0.E-1750 - 6.3552 E62*I
a(80,4)-aa(80,4,6,20)-aa(80,4,7,20) \\ = 0.E-1767 - 1.2429 E45*I

a(80,5) \\ = 0.E-1722 - 1.3417 E77*I
a(80,5)-aa(80,5,6,20)-aa(80,5,7,20) \\ = 0.E-1735 + 8.3409 E55*I

a(80,6) \\ = 5.6878 E76 + 4.3293 E75*I
a(80,6)+aa(80,6,5) \\ = 2.4379 E62 + 1.6582 E62*I
a(80,6)+aa(80,6,5)+aa(80,6,4) \\ = -4.9759 E51 + 1.0774 E51*I
err(80,6,5) \\ = -9.9868 E51 - 3.6041 E50*I
aa(80,6,7) \\ = -2.0196 E78 - 1.3147 E80*I

a(80,7) \\ = 5.6878 E76 - 4.3293 E75*I
a(80,7)+aa(80,7,5) \\ = 2.4379 E62 - 1.6582 E62*I
a(80,7)+aa(80,7,5)+aa(80,7,4) \\ = -4.9759 E51 - 1.0774 E51*I

V0=vol-4*Pi^2*[3,0,0,0,1,1,1];

\\optimal truncation asymp of coeffs a0per(k)
a0ko(k,j)=floor(k*abs(V0[3])/(abs(V0[j])+abs(V0[3])))

\\optimal truncated asymp of coeffs a0(k)
{a0a(k,j,H=min(HH,a0ko(k,j)))=1/(-V0[j])^(k)*sum(l=0,H,gamma(k-l+1/2)*(-
    sqrt(-1/2/Pi))*a(l,j)*(-V0[j])^(l-1/2))}

a0(200)*1. \\ = -6.4904 E559
a0(200)-a0a(200,3,110) \\ = 1.1057 E232 + 2.9568 E-1437*I
a0(200)-a0a(200,3,110)+a0a(200,6)-a0a(200,7) \\ = 3.5233 E224 + 2.9568 E
    -1437*I

/***********************************************/
\\one loop invariants at roots of unity
/***********************************************/
\p200
default(format,"g.5")

X1=vector(7,j,[-3, 11, 20, -15, 6, -2, -4]*vectorv(7,k,xi[j]^(k-1)))
X2=vector(7,j,[-9, 19, 76, -52, 20, -4, -13]*vectorv(7,k,xi[j]^(k-1)))
x1(j,k)=log(X1[j])/2/Pi/I+k
x2(j,k)=log(X2[j])/2/Pi/I+k
```

```
Delta=vector(7,j,[-257, 806, 947, -749, 331, -133, -213]*vectorv(7,k,xi[j]^(k-1)))
Dq(x)=local(a,c);c=denominator(x);a=x*c;sqrt(-I*c/abs(c))*prod(ell=1,abs(c)-1,(1-E(ell*a/c))^(1/2-ell/abs(c)));

{const(al,j,m,n,thfix=0)=local(c,q);c=denominator(al);q=E(al);
sum(k=0,c-1,sum(ell=0,c-1,q^(k^2+k*ell-m*k-n*ell-m)*X1[j]^((2*k+ell-m)/c)*X2[j]^((k-n)/c)/prod(i=0,c-1,((1-q^(i+1+ell-k)*X1[j]^(-1/c)*X2[j]^(1/c))*E(thfix))^((i+1+ell-k)/c-1/2)/E(thfix*((i+1+ell-k)/c-1/2)))/prod(i=0,c-1,((1-q^(i+1+ell)*X2[j]^(1/c))*E(thfix))^((i+1+ell)/c-1/2)/E(thfix*((i+1+ell)/c-1/2)))/prod(i=0,c-1,((1-q^(i+1+2*k)*X1[j]^(2/c))*E(thfix))^((i+1+2*k)/c-1/2)/E(thfix*((i+1+2*k)/c-1/2)))))*sqrt((1-X1[j]^2)^2*(1-X2[j])*(1-X2[j]/X1[j])/(Delta[j]))/I/c/Dq(al)}

{constterm(k,al,j,m,n,thfix=0)=local(c,q);c=denominator(al);q=E(al);
sum(ell=0,c-1,q^(k^2+k*ell-m*k-n*ell-m)*X1[j]^((2*k+ell-m)/c)*X2[j]^((k-n)/c)/prod(i=0,c-1,((1-q^(i+1+ell-k)*X1[j]^(-1/c)*X2[j]^(1/c))*E(thfix))^((i+1+ell-k)/c-1/2)/E(thfix*((i+1+ell-k)/c-1/2)))/prod(i=0,c-1,((1-q^(i+1+ell)*X2[j]^(1/c))*E(thfix))^((i+1+ell)/c-1/2)/E(thfix*((i+1+ell)/c-1/2)))/prod(i=0,c-1,((1-q^(i+1+2*k)*X1[j]^(2/c))*E(thfix))^((i+1+2*k)/c-1/2)/E(thfix*((i+1+2*k)/c-1/2))))*sqrt((1-X1[j]^2)^2*(1-X2[j])*(1-X2[j]/X1[j])/(Delta[j]))/I/c/Dq(al)}

{consttermf(k,al,j,m,n,thfix=0)=local(c,q,tl,sl);c=denominator(al);q=E(al);
tl=q^(k^2-m*k-m)*X1[j]^((2*k-m)/c)*X2[j]^((k-n)/c)/prod(i=0,c-1,((1-q^(i+1-k)*X1[j]^(-1/c)*X2[j]^(1/c))*E(thfix))^((i+1-k)/c-1/2)/E(thfix*((i+1-k)/c-1/2)))/prod(i=0,c-1,((1-q^(i+1)*X2[j]^(1/c))*E(thfix))^((i+1)/c-1/2)/E(thfix*((i+1)/c-1/2)))/prod(i=0,c-1,((1-q^(i+1+2*k)*X1[j]^(2/c))*E(thfix))^((i+1+2*k)/c-1/2)/E(thfix*((i+1+2*k)/c-1/2)))*sqrt((1-X1[j]^2)^2*(1-X2[j])*(1-X2[j]/X1[j])/(Delta[j]))/I/c/Dq(al);
sl=tl;
for(ell=1,c-1,tl=q^(k-n)*X1[j]^(1/c)*tl/(1-q^(ell-k)*X1[j]^(-1/c)*X2[j]^(1/c))/(1-q^(ell)*X2[j]^(1/c));sl=sl+tl);
sl}

{constf(al,j,m,thfix=0)=local(c,q,t0,t1,s,temp0);c=denominator(al);q=E(al);
t0=consttermf(0,al,j,0,-1,thfix);
t1=consttermf(1,al,j,0,-1,thfix);
s=t0/q^m+t1/q^(2*m);
qk=1;
for(k=2,c-1,qk=qk/q;temp0=t1;t1=(-(qk/X1[j]^(1/c))^2*t0-(q*(qk/X1[j]^(1/c)) + (qk/X1[j]^(1/c))^3 - (qk/X1[j]^(1/c))^2 + (qk/X1[j]^(1/c)))*(1+(qk/X1[j]^(1/c)))*(1-q*(qk/X1[j]^(1/c))^2)*temp0)/(q^2*(1+(qk/X1[j]^(1/c)))*(1-q*(qk/X1[j]^(1/c))^2)*(1-(qk/X1[j]^(1/c))^2/q^2)*(1-(qk/X1[j]^(1/c))^2/q));t0=temp0;s=s+t1*(qk/q^2)^m);
s/X1[j]^(m/c)/if(c==1,2,1)}
```

```

gettime()
const(1/11,6,3,-1,-0.001)
gettime()
constf(1/11,6,3,-0.001)
gettime()

heh=vector(2000);
for(k=200,210,heh[k]=constf(1/k,6,0,-0.001);print([k,gettime()]))

heh0=vector(2000);
for(k=200,205,heh0[k]=(heh[k]/E(V[7]/(2*Pi*I)^2*k)/constf(11,6,0,-0.001)/
    constf(11,7,0,-0.001)/E(V[6]/(2*Pi*I)^2/k)/I-1)/(2*Pi*I/k))

lim(heh0,200,4)
a(1,7)*sqrt(delta[7])

heeh=vector(2000);
for(k=200,205,heeh[k]=constf(1/k,2,0,-0.001);print([k,gettime()]))

heeh0=vector(2000);
for(k=200,210,heeh0[k]=(heeh[k]/E(V[7]/(2*Pi*I)^2*k)/constf(11,2,0,-0.001)
    /constf(11,7,0,-0.001)/E(V[2]/(2*Pi*I)^2/k)/I-1)/(2*Pi*I/k))

lim(heeh0,200,4)
a(1,7)*sqrt(delta[7])

heeeh=vector(2000);
for(k=200,205,heeeh[k]=constf(1/(k+1/2),2,0,-0.001);print([k,gettime()]))

heeeh0=vector(2000);
for(k=200,210,heeeh0[k]=(heeeh[k]/E(V[7]/(2*Pi*I)^2*(k+1/2))/constf(k
    +1/2,2,0,-0.001)/constf(1,7,0,-0.001)/E(V[2]/(2*Pi*I)^2/(2*k+1)/2)/I-1)
    /(2*Pi*I/k))

lim(heeeh0,200,4)
a(1,7)*sqrt(delta[7])

heeeeh=vector(2000);
for(k=200,204,heeeeh[k]=constf(1/(k+1/5),2,0,-0.001);print([k,gettime()]))

heeeeh0=vector(2000);
for(k=200,204,heeeeh0[k]=(heeeeh[k]/E(V[7]/(2*Pi*I)^2*(k+1/5))/constf(-(k
    +1/5),2,0,-0.001)/constf(1,7,0,-0.001)/E(V[2]/(2*Pi*I)^2/(5*k+1)/5)/I
    -1)/(2*Pi*I/k))

lim(heeeeh0,200,3)
a(1,7)*sqrt(delta[7])
```

```

heeeeeh=vector(2000);
for(k=200,204,heeeeeh[k]=constf(1/(k+1/5),2,1,-0.001);print([k,gettime()])
    )

heeeeeh0=vector(2000);
for(k=200,204,heeeeeh0[k]=(heeeeeh[k]/E(V[7]/(2*Pi*I)^2*(k+1/5))/constf(-(
    k+1/5),2,0,-0.001)/constf(1,7,1,-0.001)/E(V[2]/(2*Pi*I)^2/(5*k+1)/5)/I)
    )

lim(heeeeeh0,200,3)

constmat(al,m,n,thfix=0)=matrix(7,7,i,j,const(al,j,m+i-1,-1,thfix))

constmatf7x7(al,m,thfix=0)=matrix(7,7,i,j,constf(al,j,m+i-1,thfix))

gettime();constmatf(1/3,0,-0.001);gettime()

/**************************************************/
\\quadratic relations
/**************************************************/

{P(x,q)=[
(x^4*q^(10)*(2*q^4 + 4*q^3 + 6*q^2 + 4*q + 2) + 2*x^3*q^(9) + x^2*q^(3)
    *(2*q^6 + 4*q^5 + 4*q^4 + 6*q^3 + 4*q^2 + 4*q + 2)+2)/(x^7*q^(21)),
(-2*x^4*q^(10) +x^2*q^(3)*(- 2*q^4 - 4*q^3 - 4*q^2 - 4*q - 2) - 2)/(x^6*q
    ^(15)),
(x^2*q^(3)*(2*q^2 + 4*q + 2) + 2)/(x^5*q^(10)),
(-2*x^2*q^(3) - 2)/(x^4*q^(6)),
2/(x^3*q^(3)),
-2/(x^2*q^(1)),
0;
(-2*x^4*q^(14) + x^2*q^(5)*(- 2*q^4 - 4*q^3 - 4*q^2 - 4*q - 2) - 2)/(x^6*q
    ^(21)),
(x^2*q^(5)*(2*q^2 + 4*q + 2) + 2)/(x^5*q^(15)),
(-2*x^2*q^(5) - 2)/(x^4*q^(10)),
2/(x^3*q^(6)),
-2/(x^2*q^(3)),
0,
-2/(x^2*q^(1));
(x^2*q^(7)*(2*q^2 + 4*q + 2) + 2)/(x^5*q^(20)),
(-2*x^2*q^(7) - 2)/(x^4*q^(14)),
2/(x^3*q^(9)),
-2/(x^2*q^(5)),
0,
-2/(x^2*q^(3)),
2/(x^3*q^(3));
(-2*x^2*q^(9) - 2)/(x^4*q^(18)),
```

```
2/(x^3*q^(12)),
-2/(x^2*q^(7)),
0,
-2/(x^2*q^(5)),
2/(x^3*q^(6)),
(-2*x^2*q^(3) - 2)/(x^4*q^(6));
2/(x^3*q^(15)),
-2/(x^2*q^(9)),
0,
-2/(x^2*q^(7)),
2/(x^3*q^(9)),
(-2*x^2*q^(5) - 2)/(x^4*q^(10)),
(x^2*q^(3)*(2*q^2 + 4*q + 2) + 2*x*q^(2) + 2)/(x^5*q^(10));
-2/(x^2*q^(11)),
0,
-2/(x^2*q^(9)),
2/(x^3*q^(12)),
(-2*x^2*q^(7) - 2)/(x^4*q^(14)),
(x^2*q^(5)*(2*q^2 + 4*q + 2) + 2*x*q^(3) + 2)/(x^5*q^(15)),
(-2*x^4*q^(10) +x^2*q^(3)*(- 2*q^4 - 4*q^3 - 4*q^2 - 4*q - 2) +x*q^(2)*(-
    2*q - 2) - 2)/(x^6*q^(15));
0,
-2/(x^2*q^(11)),
2/(x^3*q^(15)),
(-2*x^2*q^(9) - 2)/(x^4*q^(18)),
(x^2*q^(7)*(2*q^2 + 4*q + 2) + 2*x*q^(4) + 2)/(x^5*q^(20)),
(-2*x^4*q^(14) + x^2*q^(5)*(- 2*q^4 - 4*q^3 - 4*q^2 - 4*q - 2) +x*q^(3)*(-
     2*q - 2) - 2)/(x^6*q^(21)),
(x^4*q^(10)*(2*q^4 + 4*q^3 + 6*q^2 + 4*q + 2) + x^3*q^(7)*(2*q^4 + 2*q^3 +
     2*q^2 + 2*q + 2) + x^2*q^(3)*(2*q^6 + 4*q^5 + 4*q^4 + 6*q^3 + 4*q^2 +
    4*q + 2) + x*q^(2)*(2*q^2 + 2*q + 2) + 2)/(x^7*q^(21))]}

for(k=1,10,print(round(constmatf7x7(-1/k,-6,-0.001)~*P(1,E(1/k))^(-1)*
    constmatf7x7(1/k,0,-0.001)*2*I*10^100)/10^100))

/*
for(k=1,10,print(round(constmat(-1/k,-6,-1,-0.001)~*P(1,E(1/k))^(-1)*
    constmat(1/k,0,-1,-0.001)*2*I*10^100)/10^100))
*/

/***********************************************/
\\vols
/***********************************************/

VVV(j,k)=-dilog(X2[j])-dilog(X1[j]^2)-dilog(X1[j]^(-1)*X2[j])-2*Pi*I*x2(j,
    k)*log(1-X2[j])-2*(2*Pi*I)*x1(j,k)*log(1-X1[j]^2)-2*Pi*I*(x2(j,k)-x1(j,
    k))*log(1-X1[j]^(-1)*X2[j])+(2*Pi*I)^2*x1(j,k)^2+(2*Pi*I)^2*x1(j,k)*x2(
    j,k)
```

```
VV=vector(7,j,-VVV(j,0)-Pi^2/6)

/*********************************************/
\\cocycle
/*********************************************/

constmatf(al,m,thfix=0)=matrix(8,8,i,j,if(j==1,W(E(al),m+i-1,0,denominator(al)/2+10),constf(al,j-1,m+i-1,thfix)))

{SI8x8(q)=[
-1/(q^5 - q^4),((-q - 2)/(q^3 - q^2)),((q^3 - 2*q^2 + q + 1)/(q^3 - q^2)),((q^3 + q^2 + 2*q + 1)/(q - 1)),((-q^4 - 2*q)/(q - 1)),((-q^6 - q^5 - q^3)/(q - 1)),((q^6 + q^4)/(q - 1)),q^8/(q - 1);
-1/(q^5 - q^4),((-q - 2)/(q^3 - q^2)),((-q^2 + q + 1)/(q^3 - q^2)),((q^2 + 2*q + 2)/(q - 1)),((-q^3 - 2*q)/(q - 1)),((-q^5 - q^4 - q^3)/(q - 1)),((q^5 + q^3)/(q - 1)),q^7/(q - 1);
-1/(q^4 - q^3),((-q^2 - q - 1)/(q^3 - q^2)),1/(q^2 - q),((q^3 + 2*q^2 + q + 1)/(q - 1)),((-q^4 - 2*q^2)/(q - 1)),((-q^6 - q^5 - q^4)/(q - 1)),((q^6 + q^4)/(q - 1)),q^8/(q - 1);
0,1/q^3,1/q,((-q - 1)/q),1,q^2,0,0;
-1/(q^5 - q^4),((-2*q - 1)/(q^3 - q^2)),((-q^3 + q + 1)/(q^3 - q^2)),((q^3 + q^2 + 3*q)/(q - 1)),((-2*q^2 - q)/(q - 1)),((-q^6 - q^5 - q^4 - q^3 + q^2)/(q - 1)),((q^5 + q^4)/(q - 1)),q^8/(q - 1);
-1/(q^5 - q^4),((-2*q - 1)/(q^3 - q^2)),((-q^3 - q^2 + 2*q + 1)/(q^3 - q^2)),((q^3 + q^2 + 3*q)/(q - 1)),((-q^2 - 2*q)/(q - 1)),((-q^6 - q^5 - q^4 - q^3 + q^2)/(q - 1)),((q^5 + q^4)/(q - 1)),q^8/(q - 1);
0,0,0,1,0,0,0,0;
-1/(q^5 - q^4),((-2*q - 1)/(q^3 - q^2)),((-q^3 + q + 1)/(q^3 - q^2)),((q^3 + q^2 + 3*q)/(q - 1)),((-q^2 - 2*q)/(q - 1)),((-q^6 - q^5 - q^4 - q^3 + q^2)/(q - 1)),((q^5 + q^4)/(q - 1)),q^8/(q - 1)]}

cocycle(al,m,n,thfix)=[matrix(7,7,i,j,const(-1/al,j,m+i-1,-1,thfix)),matrix(7,7,i,j,const(al,j,n+i-1,-1,thfix))]

/*cocyclepicc=vector(500);
for(k=2,100,cocyclepicc[k]=[constmatf(k/101,0,-0.0001),constmatf(-101/k,-3,-0.0001)];print([k,gettime()]))*/

/*********************************************/
\\pade and borel
/*********************************************/

\\borel of a = series in x
borel(a) = serconvol(a,exp(x));

\\pade of a = series in x
pade(a,N) = bestapprPade(a+O(x^N));
```

```
\\borel-pade resummation at tau of a series a with N coeffs
{resum(tau,a,N)= local(pb); pb=pade(borel(a),N);
intnum(xi=0,[+oo,1],exp(-xi)*subst(pb,x,xi*tau))};

/***********************************************/
\\ the resummed phi series
/***********************************************/

{rphi0(tau,N=200)=resum(2*Pi*I/tau,Phi0*1.,N);}
{rphi1(tau,N=HH)=E(3/8)*sqrt(tau)*E(V[1]*tau/(2*Pi*I)^2)*resum(2*Pi*I/tau,
    Phi1,N);}
{rphi2(tau,N=HH)=E(-1/8)*sqrt(tau)*E(V[2]*tau/(2*Pi*I)^2)*resum(2*Pi*I/tau
    ,Phi2,N);}
{rphi3(tau,N=HH)=E(3/8)*sqrt(tau)*E(V[3]*tau/(2*Pi*I)^2)*resum(2*Pi*I/tau,
    Phi3,N);}
{rphi4(tau,N=HH)=E(-1/8)*sqrt(tau)*E(V[4]*tau/(2*Pi*I)^2)*resum(2*Pi*I/tau
    ,Phi4,N);}
{rphi5(tau,N=HH)=E(3/8)*sqrt(tau)*E(V[5]*tau/(2*Pi*I)^2)*resum(2*Pi*I/tau,
    Phi5,N);}
{rphi6(tau,N=HH)=E(3/8)*sqrt(tau)*E(V[6]*tau/(2*Pi*I)^2)*resum(2*Pi*I/tau,
    Phi6,N);}
{rphi7(tau,N=HH)=E(3/8)*sqrt(tau)*E(V[7]*tau/(2*Pi*I)^2)*resum(2*Pi*I/tau,
    Phi7,N);}

phivec8(tau)=[rphi0(tau),rphi1(tau),rphi2(tau),rphi3(tau),rphi4(tau),rphi5
    (tau),rphi6(tau),rphi7(tau)]

/***********************************************/
\\ q series for 4_1(1,2)
/***********************************************/

{Zhatf1(q,m,N,ss=1)=
local(t0,t1,s,temp0,qk,qj);
t0=(-1+q)*(ss+1);t1=1;qj=1;tj=1;
for(j=1,N,qj=qj*q;tj=q*tj/(1-qj)^2;t1=t1+tj);
t1=qpochinfty(q,q,N,ss,1)^2*t1;
s=t1/q^m;qk=q;
for(k=1,N,qk=qk/q;temp0=t1;t1=-(qk^2*t0+(q*qk + qk^3 - qk^2 + qk)*(1+qk)
    *(1-q*qk^2)*temp0)/(q^2*(1+qk)*(1-q*qk^2)*(1-qk^2/q^2)*(1-qk^2/q));t0=
    temp0;s=s+(qk/q^2)^m*t1);
s}

{Zhatf2(q,m,N,ss=1)=
local(t0,t1,tj,tjn,tjc,s,temp0,qk,qj);
t0=0;qj=1;tj=1;tjc=-1/2+2*ss*Gf(q^ss,1,N);tjn=0;t1=tj*(tjc-2*tjn);
for(j=1,N,qj=qj*q;tj=q*tj/(1-qj)^2;tjn=tjn+qj/(1-qj);t1=t1+tj*(tjc-2*tjn))
    ;
```

```
t1=qpochinfty(q,q,N,ss,1)^2*t1;
s=t1/q^m;
qk=q;
for(k=1,N,qk=qk/q;temp0=t1;t1=-(qk^2*t0+(q*qk + qk^3 - qk^2 + qk)*(1+qk)
    *(1-q*qk^2)*temp0)/(q^2*(1+qk)*(1-q*qk^2)*(1-qk^2/q^2)*(1-qk^2/q));t0=
    temp0;s=s+(qk/q^2)^m*t1);
s}

{Zhatf3(q,m,N,ss=1)=
local(t0,t1,tj,tjn,tjc1,tjc2,tG1,tG2,s,temp0,qk,qj);
t0=-2*(1-q);qj=1;tj=1;tG1=ss*Gf(q^ss,1,N);tG2=ss*Gf(q^ss,2,N);tjc1=(1/2-2*
    tG1);tjc2=(-2*m-4*tG1);tjn=0;
t1=tj*((tjc1+2*tjn)*tjc2/2+2*tG2);
for(j=1,N,qj=qj*q;tj=q*tj/(1-qj)^2;tjn=tjn+qj/(1-qj);tjc2=tjc2+2;t1=t1+tj
    *((tjc1+2*tjn)*tjc2/2+2*tG2));
t1=qpochinfty(q,q,N,ss,1)^2*t1;s=t1/q^m;qk=q;
for(k=1,N,qk=qk/q;temp0=t1;t1=-(qk^2*t0+(q*qk + qk^3 - qk^2 + qk)*(1+qk)
    *(1-q*qk^2)*temp0)/(q^2*(1+qk)*(1-q*qk^2)*(1-qk^2/q^2)*(1-qk^2/q));t0=
    temp0;s=s+(qk/q^2)^m*t1);
s}

{Zhatf4(q2,m,N,ss=1)=
local(t0,t1,t0j,t1j,s,temp0,qk,qkj,qpc);
qpc=qpochinfty(q2^2,q2^2,N,ss,1)*qpochinfty(q2^3,q2^2,N,ss);
t0j=q2^(2)/qpoch(q2^2,q2^2,1);t1j=q2^(12)/qpoch(q2^3,q2^2,1)/qpoch(q2^2,q2
    ^2,3);t0=t0j;t1=t1j;qj=1;
for(j=1,N,qj=qj*q2^2;t0j=t0j*q2^3/(1-qj)/(1-qj*q2);t1j=t1j*q2^5/(1-qj)/(1-
    qj*q2^3);t0=t0+t0j;t1=t1+t1j);
t0=qpc*t0;t1=qpc*t1;s=t0/q2^(3*m)+t1/q2^(5*m);qk=1;
for(k=1,N,qk=qk/q2^2;temp0=t1;t1=-((qk/q2)^2*t0+(q2^2*(qk/q2) + (qk/q2)^3
    - (qk/q2)^2 + (qk/q2))*(1+(qk/q2))*(1-q2^2*(qk/q2)^2)*temp0)/(q2^4*(1+(
    qk/q2))*(1-q2^2*(qk/q2)^2)*(1-(qk/q2)^2/q2^4)*(1-(qk/q2)^2/q2^2));t0=
    temp0;s=s+(qk/q2^5)^m*t1);
s}

{Zhatf5(q,m,N,ss=1)=
local(t0,t1,t0j,t1j,s,temp0,qk,qj,qpc);
qpc=qpochinfty(q,q,N,ss,1)*qpochinfty(-q,q,N,ss);
t0j=(-1)^m;t0=t0j;t1j=(-1)^m*q^(3)/qpoch(-q,q,1)/qpoch(q,q,2);t1=t1j;qj=1;
for(j=1,N,qj=qj*q;t0j=-t0j*q/(1-qj^2);t0=t0+t0j;t1j=-t1j*q^2/(1-qj)/(1+qj*
    q);t1=t1+t1j);
t0=qpc*t0;t1=qpc*t1;s=t0/q^m+t1/q^(2*m);qk=1;
for(k=1,N,qk=qk/q;temp0=t1;t1=-((-qk)^2*t0+(q*(-qk) + (-qk)^3 - (-qk)^2 +
    (-qk))*(1+(-qk))*(1-q*(-qk)^2)*temp0)/(q^2*(1+(-qk))*(1-q*(-qk)^2)
    *(1-(-qk)^2/q^2)*(1-(-qk)^2/q));t0=temp0;s=s+(qk/q^2)^m*t1);
s}

{Zhatf6(q2,m,N,ss=1)=
```

```
local(t0,t1,t0j,t1j,s,temp0,qk,qj,qpc);
qpc=qpochinfty(q2^2,q2^2,N,ss,1)*qpochinfty(-q2^3,q2^2,N,ss);
t0j=(-1)^(m)*q2^2/qpoch(q2^2,q2^2,1);t0=t0j;
t1j=(-1)^(m)*q2^12/qpoch(-q2^3,q2^2,1)/qpoch(q2^2,q2^2,3);
t1=t1j;qj=1;
for(j=1,N,qj=qj*q2^2;t0j=-t0j*q2^3/(1-qj)/(1+qj*q2);t0=t0+t0j;t1j=-t1j*q2
    ^5/(1-qj)/(1+qj*q2^3);t1=t1+t1j);
t0=qpc*t0;t1=qpc*t1;s=t0/q2^(3*m)+t1/q2^(5*m);qk=1;
for(k=1,N,qk=qk/q2^2;temp0=t1;t1=-((-qk/q2)^2*t0+(q2^2*(-qk/q2) + (-qk/q2)
    ^3 - (-qk/q2)^2 + (-qk/q2))*(1+(-qk/q2))*(1-q2^2*(-qk/q2)^2)*temp0)/(q2
    ^4*(1+(-qk/q2))*(1-q2^2*(-qk/q2)^2)*(1-(-qk/q2)^2/q2^4)*(1-(-qk/q2)^2/
    q2^2));t0=temp0;s=s+(qk/q2^5)^m*t1);
s}

{Zhatf7(q4,m,N,ss=1)=
local(t0,t1,t0j,t1j,s,temp0,qk,qj,q22,qpc);
q22=q4^2;qpc=qpochinfty(q22^2,q22^2,N,ss,1)*qpochinfty(q22,q22^2,N,ss);
t0j=q4/qpoch(q22^2,q22^2,1);t0=t0j;t1j=q4^9/qpoch(q22,q22^2,-1)/qpoch(q22
    ^2,q22^2,3);t1=t1j;qj=1;
for(j=1,N,qj=qj*q22^2;t0j=t0j*q22^3/(1-qj)/(1-qj/q22);t0=t0+t0j;t1j=t1j*
    q22^5/(1-qj)/(1-qj/q22^3);t1=t1+t1j);
t0=t0*qpc;t1=t1*qpc;s=t0/q22^(3*m)+t1/q22^(5*m);qk=1;
for(k=1,N,qk=qk/q22^2;temp0=t1;t1=-((qk/q22)^2*t0+(q22^2*(qk/q22) + (qk/
    q22)^3 - (qk/q22)^2 + (qk/q22))*(1+(qk/q22))*(1-q22^2*(qk/q22)^2)*temp0
    )/(q22^4*(1+(qk/q22))*(1-q22^2*(qk/q22)^2)*(1-(qk/q22)^2/q22^4)*(1-(qk/
    q22)^2/q22^2));t0=temp0;s=s+(qk/q22^5)^m*t1);
s}

{Zhatf8(q,m,N,ss=1)=
local(t0,t1,t0j,t1j,s,temp0,qk,qj,qpc);
qpc=qpochinfty(q,q,N,ss,1)*qpochinfty(-q,q,N,ss);
t0j=(-1)^(m);t0=t0j;t1j=(-1)^(m)*q/qpoch(-q,q,-1)/qpoch(q,q,2);t1=t1j;qj
    =1;
for(j=1,N,qj=qj*q;t0j=-t0j*q/(1-qj^2);t0=t0+t0j;t1j=-t1j*q^2/(1-qj)/(1+qj/
    q);t1=t1+t1j);
t0=t0*qpc;t1=t1*qpc;s=t0/q^m+t1/q^(2*m);qk=1;
for(k=1,N,qk=qk/q;temp0=t1;t1=-((-qk)^2*t0+(q*(-qk) + (-qk)^3 - (-qk)^2 +
    (-qk))*(1+(-qk))*(1-q*(-qk)^2)*temp0)/(q^2*(1+(-qk))*(1-q*(-qk)^2)
    *(1-(-qk)^2/q^2)*(1-(-qk)^2/q));t0=temp0;s=s+(qk/q^2)^m*t1);
s}

{Zhatf9(q4,m,N,ss=1)=
local(t0,t1,t0j,t1j,s,temp0,qk,q22,qpc);
q22=q4^2;qpc=qpochinfty(q22^2,q22^2,N,ss,1)*qpochinfty(-q22,q22^2,N,ss);
t0j=(-1)^(m)*q4/qpoch(q22^2,q22^2,1);t0=t0j;t1j=(-1)^(m)*q4^9/qpoch(-q22,
    q22^2,-1)/qpoch(q22^2,q22^2,3);t1=t1j;qj=1;
for(j=1,N,qj=qj*q22^2;t0j=-t0j*q22^3/(1-qj)/(1+qj/q22);t0=t0+t0j;t1j=-t1j*
    q22^5/(1-qj)/(1+qj/q22^3);t1=t1+t1j);
```

```
t0=t0*qpc;t1=t1*qpc;s=t0/q22^(3*m)+t1/q22^(5*m);qk=1;
for(k=1,N,qk=qk/q22^2;temp0=t1;t1=-((-qk/q22)^2*t0+(q22^2*(-qk/q22) + (-qk
    /q22)^3 - (-qk/q22)^2 + (-qk/q22))*(1+(-qk/q22))*(1-q22^2*(-qk/q22)^2)*
    temp0)/(q22^4*(1+(-qk/q22))*(1-q22^2*(-qk/q22)^2)*(1-(-qk/q22)^2/q22^4)
    *(1-(-qk/q22)^2/q22^2));t0=temp0;s=s+(qk/q22^5)^m*t1);
s}


{ZHAT7x7(q4,m,N,s=1)=
if(s==1,
[
Zhatf2(q4^4,m,N,s),Zhatf4(q4^2,m,N,s),Zhatf5(q4^4,m,N,s),Zhatf6(q4^2,m,N,s
    ),Zhatf7(q4,m,N,s),Zhatf8(q4^4,m,N,s),Zhatf9(q4,m,N,s);
Zhatf2(q4^4,m+1,N,s),Zhatf4(q4^2,m+1,N,s),Zhatf5(q4^4,m+1,N,s),Zhatf6(q4
    ^2,m+1,N,s),Zhatf7(q4,m+1,N,s),Zhatf8(q4^4,m+1,N,s),Zhatf9(q4,m+1,N,s);
Zhatf2(q4^4,m+2,N,s),Zhatf4(q4^2,m+2,N,s),Zhatf5(q4^4,m+2,N,s),Zhatf6(q4
    ^2,m+2,N,s),Zhatf7(q4,m+2,N,s),Zhatf8(q4^4,m+2,N,s),Zhatf9(q4,m+2,N,s);
Zhatf2(q4^4,m+3,N,s),Zhatf4(q4^2,m+3,N,s),Zhatf5(q4^4,m+3,N,s),Zhatf6(q4
    ^2,m+3,N,s),Zhatf7(q4,m+3,N,s),Zhatf8(q4^4,m+3,N,s),Zhatf9(q4,m+3,N,s);
Zhatf2(q4^4,m+4,N,s),Zhatf4(q4^2,m+4,N,s),Zhatf5(q4^4,m+4,N,s),Zhatf6(q4
    ^2,m+4,N,s),Zhatf7(q4,m+4,N,s),Zhatf8(q4^4,m+4,N,s),Zhatf9(q4,m+4,N,s);
Zhatf2(q4^4,m+5,N,s),Zhatf4(q4^2,m+5,N,s),Zhatf5(q4^4,m+5,N,s),Zhatf6(q4
    ^2,m+5,N,s),Zhatf7(q4,m+5,N,s),Zhatf8(q4^4,m+5,N,s),Zhatf9(q4,m+5,N,s);
Zhatf2(q4^4,m+6,N,s),Zhatf4(q4^2,m+6,N,s),Zhatf5(q4^4,m+6,N,s),Zhatf6(q4
    ^2,m+6,N,s),Zhatf7(q4,m+6,N,s),Zhatf8(q4^4,m+6,N,s),Zhatf9(q4,m+6,N,s)
],
[
Zhatf1(q4^4,m,N,s),Zhatf4(q4^2,m,N,s),Zhatf5(q4^4,m,N,s),Zhatf6(q4^2,m,N,s
    ),Zhatf7(q4,m,N,s),Zhatf8(q4^4,m,N,s),Zhatf9(q4,m,N,s);
Zhatf1(q4^4,m+1,N,s),Zhatf4(q4^2,m+1,N,s),Zhatf5(q4^4,m+1,N,s),Zhatf6(q4
    ^2,m+1,N,s),Zhatf7(q4,m+1,N,s),Zhatf8(q4^4,m+1,N,s),Zhatf9(q4,m+1,N,s);
Zhatf1(q4^4,m+2,N,s),Zhatf4(q4^2,m+2,N,s),Zhatf5(q4^4,m+2,N,s),Zhatf6(q4
    ^2,m+2,N,s),Zhatf7(q4,m+2,N,s),Zhatf8(q4^4,m+2,N,s),Zhatf9(q4,m+2,N,s);
Zhatf1(q4^4,m+3,N,s),Zhatf4(q4^2,m+3,N,s),Zhatf5(q4^4,m+3,N,s),Zhatf6(q4
    ^2,m+3,N,s),Zhatf7(q4,m+3,N,s),Zhatf8(q4^4,m+3,N,s),Zhatf9(q4,m+3,N,s);
Zhatf1(q4^4,m+4,N,s),Zhatf4(q4^2,m+4,N,s),Zhatf5(q4^4,m+4,N,s),Zhatf6(q4
    ^2,m+4,N,s),Zhatf7(q4,m+4,N,s),Zhatf8(q4^4,m+4,N,s),Zhatf9(q4,m+4,N,s);
Zhatf1(q4^4,m+5,N,s),Zhatf4(q4^2,m+5,N,s),Zhatf5(q4^4,m+5,N,s),Zhatf6(q4
    ^2,m+5,N,s),Zhatf7(q4,m+5,N,s),Zhatf8(q4^4,m+5,N,s),Zhatf9(q4,m+5,N,s);
Zhatf1(q4^4,m+6,N,s),Zhatf4(q4^2,m+6,N,s),Zhatf5(q4^4,m+6,N,s),Zhatf6(q4
    ^2,m+6,N,s),Zhatf7(q4,m+6,N,s),Zhatf8(q4^4,m+6,N,s),Zhatf9(q4,m+6,N,s)
])};

{ZHAT8x8(q4,m,N,s=1)=
if(s==1,
[
Zhatf1(q4^4,m,N,s),Zhatf2(q4^4,m,N,s),Zhatf4(q4^2,m,N,s),Zhatf5(q4^4,m,N,s
    ),Zhatf6(q4^2,m,N,s),Zhatf7(q4,m,N,s),Zhatf8(q4^4,m,N,s),Zhatf9(q4,m,N,
```

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    s);
Zhatf1(q4^4,m+1,N,s),Zhatf2(q4^4,m+1,N,s),Zhatf4(q4^2,m+1,N,s),Zhatf5(q4
    ^4,m+1,N,s),Zhatf6(q4^2,m+1,N,s),Zhatf7(q4,m+1,N,s),Zhatf8(q4^4,m+1,N,s
    ),Zhatf9(q4,m+1,N,s);
Zhatf1(q4^4,m+2,N,s),Zhatf2(q4^4,m+2,N,s),Zhatf4(q4^2,m+2,N,s),Zhatf5(q4
    ^4,m+2,N,s),Zhatf6(q4^2,m+2,N,s),Zhatf7(q4,m+2,N,s),Zhatf8(q4^4,m+2,N,s
    ),Zhatf9(q4,m+2,N,s);
Zhatf1(q4^4,m+3,N,s),Zhatf2(q4^4,m+3,N,s),Zhatf4(q4^2,m+3,N,s),Zhatf5(q4
    ^4,m+3,N,s),Zhatf6(q4^2,m+3,N,s),Zhatf7(q4,m+3,N,s),Zhatf8(q4^4,m+3,N,s
    ),Zhatf9(q4,m+3,N,s);
Zhatf1(q4^4,m+4,N,s),Zhatf2(q4^4,m+4,N,s),Zhatf4(q4^2,m+4,N,s),Zhatf5(q4
    ^4,m+4,N,s),Zhatf6(q4^2,m+4,N,s),Zhatf7(q4,m+4,N,s),Zhatf8(q4^4,m+4,N,s
    ),Zhatf9(q4,m+4,N,s);
Zhatf1(q4^4,m+5,N,s),Zhatf2(q4^4,m+5,N,s),Zhatf4(q4^2,m+5,N,s),Zhatf5(q4
    ^4,m+5,N,s),Zhatf6(q4^2,m+5,N,s),Zhatf7(q4,m+5,N,s),Zhatf8(q4^4,m+5,N,s
    ),Zhatf9(q4,m+5,N,s);
Zhatf1(q4^4,m+6,N,s),Zhatf2(q4^4,m+6,N,s),Zhatf4(q4^2,m+6,N,s),Zhatf5(q4
    ^4,m+6,N,s),Zhatf6(q4^2,m+6,N,s),Zhatf7(q4,m+6,N,s),Zhatf8(q4^4,m+6,N,s
    ),Zhatf9(q4,m+6,N,s);
Zhatf1(q4^4,m+7,N,s),Zhatf2(q4^4,m+7,N,s),Zhatf4(q4^2,m+7,N,s),Zhatf5(q4
    ^4,m+7,N,s),Zhatf6(q4^2,m+7,N,s),Zhatf7(q4,m+7,N,s),Zhatf8(q4^4,m+7,N,s
    ),Zhatf9(q4,m+7,N,s)
],
[
Zhatf3(q4^4,m,N,s),Zhatf1(q4^4,m,N,s),Zhatf4(q4^2,m,N,s),Zhatf5(q4^4,m,N,s
    ),Zhatf6(q4^2,m,N,s),Zhatf7(q4,m,N,s),Zhatf8(q4^4,m,N,s),Zhatf9(q4,m,N,
    s);
Zhatf3(q4^4,m+1,N,s),Zhatf1(q4^4,m+1,N,s),Zhatf4(q4^2,m+1,N,s),Zhatf5(q4
    ^4,m+1,N,s),Zhatf6(q4^2,m+1,N,s),Zhatf7(q4,m+1,N,s),Zhatf8(q4^4,m+1,N,s
    ),Zhatf9(q4,m+1,N,s);
Zhatf3(q4^4,m+2,N,s),Zhatf1(q4^4,m+2,N,s),Zhatf4(q4^2,m+2,N,s),Zhatf5(q4
    ^4,m+2,N,s),Zhatf6(q4^2,m+2,N,s),Zhatf7(q4,m+2,N,s),Zhatf8(q4^4,m+2,N,s
    ),Zhatf9(q4,m+2,N,s);
Zhatf3(q4^4,m+3,N,s),Zhatf1(q4^4,m+3,N,s),Zhatf4(q4^2,m+3,N,s),Zhatf5(q4
    ^4,m+3,N,s),Zhatf6(q4^2,m+3,N,s),Zhatf7(q4,m+3,N,s),Zhatf8(q4^4,m+3,N,s
    ),Zhatf9(q4,m+3,N,s);
Zhatf3(q4^4,m+4,N,s),Zhatf1(q4^4,m+4,N,s),Zhatf4(q4^2,m+4,N,s),Zhatf5(q4
    ^4,m+4,N,s),Zhatf6(q4^2,m+4,N,s),Zhatf7(q4,m+4,N,s),Zhatf8(q4^4,m+4,N,s
    ),Zhatf9(q4,m+4,N,s);
Zhatf3(q4^4,m+5,N,s),Zhatf1(q4^4,m+5,N,s),Zhatf4(q4^2,m+5,N,s),Zhatf5(q4
    ^4,m+5,N,s),Zhatf6(q4^2,m+5,N,s),Zhatf7(q4,m+5,N,s),Zhatf8(q4^4,m+5,N,s
    ),Zhatf9(q4,m+5,N,s);
Zhatf3(q4^4,m+6,N,s),Zhatf1(q4^4,m+6,N,s),Zhatf4(q4^2,m+6,N,s),Zhatf5(q4
    ^4,m+6,N,s),Zhatf6(q4^2,m+6,N,s),Zhatf7(q4,m+6,N,s),Zhatf8(q4^4,m+6,N,s
    ),Zhatf9(q4,m+6,N,s);
Zhatf3(q4^4,m+7,N,s),Zhatf1(q4^4,m+7,N,s),Zhatf4(q4^2,m+7,N,s),Zhatf5(q4
    ^4,m+7,N,s),Zhatf6(q4^2,m+7,N,s),Zhatf7(q4,m+7,N,s),Zhatf8(q4^4,m+7,N,s
    ),Zhatf9(q4,m+7,N,s)])};
```

```

{ZHAT1x8(q4,m,N,s=1)=
if(s==1,
[Zhatf1(q4^4,m,N,s),Zhatf2(q4^4,m,N,s),Zhatf4(q4^2,m,N,s),Zhatf5(q4^4,m,N,
    s),Zhatf6(q4^2,m,N,s),Zhatf7(q4,m,N,s),Zhatf8(q4^4,m,N,s),Zhatf9(q4,m,N
    ,s)],
[Zhatf3(q4^4,m,N,s),Zhatf1(q4^4,m,N,s),Zhatf4(q4^2,m,N,s),Zhatf5(q4^4,m,N,
    s),Zhatf6(q4^2,m,N,s),Zhatf7(q4,m,N,s),Zhatf8(q4^4,m,N,s),Zhatf9(q4,m,N
    ,s)])};

\\check of quad rels
ZHAT7x7(q4+O(q4^500),-3,500)*ZHAT7x7(1/q4+O(q4^500),-3,500,-1)~-P(1/q4^12,
    q4^4)


/***********************************************/
\\ computing stokes matrices
/***********************************************/

\\canonical basis I
ZHAT1x7(E(-1/(E(0.01)*100)/4),0,100*500)-phivec7(-E(0.01)*100)
    *[-1,0,0,0,0,0,0; 0,-1,0,0,0,0,0; 0,0,-1,0,0,0,0; 0,0,0,-1,0,0,0;
    0,0,0,0,-1,0,0; 0,0,0,0,0,1,0; 0,0,0,0,0,0,1]*SI(E(E(0.01)*100))*
    ZHAT8x7(E(E(0.01)*100/4),-3,100)*matdiagonal([-1,1,1,-1,-1,-1,1])*[E
    (0.01)*100,0,0,0,0,0,0; 0,0,1,0,0,0,0; 0,1,0,0,0,0,0; 0,0,0,1,0,0,0;
    0,0,0,0,0,1,0; 0,0,0,0,1,0,0; 0,0,0,0,0,0,1]
\\ = [-1.5264 E-49 - 1.5231 E-49*I, -1.6190 E-51 - 1.4242 E-51*I, -1.3758
    E-68 + 7.8474 E-69*I, 1.3758 E-68 - 7.8474 E-69*I, 1.6190 E-51 + 1.4242
     E-51*I, 1.0288 E-55 + 4.4904 E-56*I, -1.0288 E-55 - 4.4904 E-56*I]

\\canonical basis II
ZHAT1x7(E(-1/(E(1/2-0.01)*100)/4),0,100*500)-phivec7(-E(1/2-0.01)*100)
    *[1,0,0,0,0,0,0; 0,1,0,0,0,0,0; 0,0,1,0,0,0,0; 0,0,0,1,0,0,0;
    0,0,0,0,1,0,0; 0,0,0,0,0,0,1; 0,0,0,0,0,1,0]*SI(E(E(1/2-0.01)*100))*
    ZHAT8x7(E(E(1/2-0.01)*100/4),-3,100)*matdiagonal([-1,-1,-1,1,1,1,-1])*[
    E(1/2-0.01)*100,0,0,0,0,0,0; 0,0,1,0,0,0,0; 0,1,0,0,0,0,0;
    0,0,0,1,0,0,0; 0,0,0,0,0,1,0; 0,0,0,0,1,0,0; 0,0,0,0,0,0,1]
\\ = [-1.5264 E-49 + 1.5231 E-49*I, -1.6190 E-51 + 1.4242 E-51*I, -1.3758
    E-68 - 7.8474 E-69*I, 1.3758 E-68 + 7.8474 E-69*I, 1.6190 E-51 - 1.4242
     E-51*I, 1.0288 E-55 - 4.4904 E-56*I, -1.0288 E-55 + 4.4904 E-56*I]

\\I
SMI8x8(q4,N)=[-1,0,0,0,0,0,0,0; 0,-1,0,0,0,0,0,0; 0,0,-1,0,0,0,0,0;
    0,0,0,-1,0,0,0,0; 0,0,0,0,-1,0,0,0; 0,0,0,0,0,-1,0,0; 0,0,0,0,0,0,1,0;
    0,0,0,0,0,0,0,1]*SI8x8(q4^4)*ZHAT8x8(q4,-3,100)*matdiagonal
    ([-1,-1,1,1,-1,-1,-1,1])

\\II
```

```
SMII8x8(q4,N)=[1,0,0,0,0,0,0,0; 0,1,0,0,0,0,0,0; 0,0,1,0,0,0,0,0;
    0,0,0,1,0,0,0,0; 0,0,0,0,1,0,0,0; 0,0,0,0,0,1,0,0; 0,0,0,0,0,0,0,1;
    0,0,0,0,0,0,1,0]*SI8x8(q4^4)*ZHAT8x8(q4,-3,100)*matdiagonal
    ([1,-1,-1,-1,1,1,1,-1])

SMI8x8(q4+O(q4^100),100)*SMII8x8(q4+O(q4^100),100)^(-1)-matrix(8,8,i,j,if(
    i==j,1,0)+O(q4^50))
/*
[O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10)]

[-1 + q4^4 + 3*q4^8 + O(q4^10) -q4^4 - 2*q4^8 + O(q4^10) 1 + q4^4 + O(q4
    ^10) -q4^8 + O(q4^10) q4^4 + 2*q4^8 + O(q4^10) -1 - q4^4 + q4^8 + O(q4
    ^10) 1 - 3*q4^8 + O(q4^10) 1 - 3*q4^8 + O(q4^10)]

[q4^4 - q4^8 + O(q4^10) q4^8 + O(q4^10) -q4^4 - q4^8 + O(q4^10) O(q4^10) -
    q4^8 + O(q4^10) q4^4 + q4^8 + O(q4^10) -q4^4 + O(q4^10) -q4^4 + O(q4
    ^10)]

[-1 + 2*q4^4 + q4^8 + O(q4^10) -q4^4 - q4^8 + O(q4^10) 1 + O(q4^10) -q4^8
    + O(q4^10) q4^4 + q4^8 + O(q4^10) -1 + q4^8 + O(q4^10) 1 - q4^4 - 2*q4
    ^8 + O(q4^10) 1 - q4^4 - 2*q4^8 + O(q4^10)]

[1 - 2*q4^4 - q4^8 + O(q4^10) q4^4 + q4^8 + O(q4^10) -1 + O(q4^10) q4^8 +
    O(q4^10) -q4^4 - q4^8 + O(q4^10) 1 - q4^8 + O(q4^10) -1 + q4^4 + 2*q4^8
     + O(q4^10) -1 + q4^4 + 2*q4^8 + O(q4^10)]

[-q4^4 + q4^8 + O(q4^10) -q4^8 + O(q4^10) q4^4 + q4^8 + O(q4^10) O(q4^10)
    q4^8 + O(q4^10) -q4^4 - q4^8 + O(q4^10) q4^4 + O(q4^10) q4^4 + O(q4^10)
    ]

[-1 + 3*q4^4 + q4^8 + O(q4^10) -q4^4 + O(q4^10) 1 - q4^4 - 2*q4^8 + O(q4
    ^10) -q4^8 + O(q4^10) q4^4 + O(q4^10) -1 + q4^4 + 3*q4^8 + O(q4^10) -2*
    q4^4 - 3*q4^8 + O(q4^10) -2*q4^4 - 3*q4^8 + O(q4^10)]

[-1 + 3*q4^4 + q4^8 + O(q4^10) -q4^4 + O(q4^10) 1 - q4^4 - 2*q4^8 + O(q4
    ^10) -q4^8 + O(q4^10) q4^4 + O(q4^10) -1 + q4^4 + 3*q4^8 + O(q4^10) -2*
    q4^4 - 3*q4^8 + O(q4^10) -2*q4^4 - 3*q4^8 + O(q4^10)]
*/

\\ IV
ZHAT1x8(E(-1/(E(-0.001)*100)/4),0,100*3000,-1)-phivec8(-E(-0.001)*100)
    *[-1,0,0,0,0,0,0,0; 0,1,0,0,0,0,0,0; 0,0,1,0,0,0,0,0; 0,0,0,1,0,0,0,0;
    0,0,0,0,1,0,0,0; 0,0,0,0,0,1,0,0; 0,0,0,0,0,0,-1,0; 0,0,0,0,0,0,0,-1]*
    SI8x8(E(E(-0.001)*100))*ZHAT8x8(E(E(-0.001)*100/4),-3,100,-1)*
    matdiagonal([-1,-1,1,1,-1,-1,-1,1])*[1,0,0,0,0,0,0,0; 0,1/(E(-0.001)
    *100)^2,0,0,0,0,0,0; 0,0,0,1/(E(-0.001)*100),0,0,0,0; 0,0,1/(E(-0.001)
    *100),0,0,0,0,0; 0,0,0,0,1/(E(-0.001)*100),0,0,0; 0,0,0,0,0,0,1/(E
    (-0.001)*100),0; 0,0,0,0,0,1/(E(-0.001)*100),0,0; 0,0,0,0,0,0,0,1/(E
```

```
(-0.001)*100)]
\\ = [2.2937 E-38 - 3.5670 E-38*I, -2.3174 E-53 - 7.7430 E-53*I, 1.1822 E-51 + 3.8614 E-51*I, 4.3954 E-52 + 1.4803 E-51*I, 2.5418 E-52 + 8.4512 E-52*I, -1.1838 E-51 - 3.8671 E-51*I, 2.9963 E-52 + 1.0185 E-51*I, -1.9835 E-52 - 6.6765 E-52*I]

\\III
ZHAT1x8(E(-1/(E(1/2+0.001)*100)/4),0,100*3000,-1)-phivec8(-E(1/2+0.001)*100)*[1,0,0,0,0,0,0,0; 0,1,0,0,0,0,0,0; 0,0,1,0,0,0,0,0; 0,0,0,1,0,0,0,0; 0,0,0,0,1,0,0,0; 0,0,0,0,0,1,0,0; 0,0,0,0,0,0,0,1; 0,0,0,0,0,0,1,0]*SI8x8(E(E(1/2+0.001)*100))*ZHAT8x8(E(E(1/2+0.001)*100/4),-3,100,-1)*matdiagonal([1,1,1,1,-1,-1,-1,1])*[1,0,0,0,0,0,0,0; 0,1/(E(1/2+0.001)*100)^2,0,0,0,0,0,0; 0,0,0,1/(E(1/2+0.001)*100),0,0,0,0; 0,0,1/(E(1/2+0.001)*100),0,0,0,0,0; 0,0,0,0,1/(E(1/2+0.001)*100),0,0,0; 0,0,0,0,0,0,1/(E(1/2+0.001)*100),0; 0,0,0,0,0,1/(E(1/2+0.001)*100),0,0; 0,0,0,0,0,0,0,1/(E(1/2+0.001)*100)]
\\ = [2.2937 E-38 + 3.5670 E-38*I, -2.3174 E-53 + 7.7430 E-53*I, 1.1822 E-51 - 3.8614 E-51*I, 4.3954 E-52 - 1.4803 E-51*I, 2.5418 E-52 - 8.4512 E-52*I, -1.1838 E-51 + 3.8671 E-51*I, 2.9963 E-52 - 1.0185 E-51*I, -1.9835 E-52 + 6.6765 E-52*I]

\\IV
SMIV8x8(q4,N)=[-1,0,0,0,0,0,0,0; 0,1,0,0,0,0,0,0; 0,0,1,0,0,0,0,0; 0,0,0,1,0,0,0,0; 0,0,0,0,1,0,0,0; 0,0,0,0,0,1,0,0; 0,0,0,0,0,0,-1,0; 0,0,0,0,0,0,0,-1]*SI8x8(1/q4^4)*ZHAT8x8(1/q4,-3,N,-1)*matdiagonal([-1,-1,1,1,-1,-1,-1,1])

\\III
SMIII8x8(q4,N)=[1,0,0,0,0,0,0,0; 0,1,0,0,0,0,0,0; 0,0,1,0,0,0,0,0; 0,0,0,1,0,0,0,0; 0,0,0,0,1,0,0,0; 0,0,0,0,0,1,0,0; 0,0,0,0,0,0,0,1; 0,0,0,0,0,0,1,0]*SI8x8(1/q4^4)*ZHAT8x8(1/q4,-3,N,-1)*matdiagonal([1,1,1,1,-1,-1,-1,1])

SMIV8x8(q4+O(q4^100),100)*SMIII8x8(q4+O(q4^100),100)^(-1)-matrix(8,8,i,j,if(i==j,1,0)+O(q4^10))
/*
[O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10) O(q4^10)]

[O(q4^10) -q4^4 - 2*q4^8 + O(q4^10) q4^8 + O(q4^10) -q4^4 - q4^8 + O(q4^10) q4^4 + q4^8 + O(q4^10) -q4^8 + O(q4^10) -q4^4 + O(q4^10) -q4^4 + O(q4^10)]

[-1 + O(q4^10) 1 + q4^4 + O(q4^10) -q4^4 - q4^8 + O(q4^10) 1 + O(q4^10) -1 + O(q4^10) q4^4 + q4^8 + O(q4^10) 1 - q4^4 - 2*q4^8 + O(q4^10) 1 - q4^4 - 2*q4^8 + O(q4^10)]

[-q4^4 + q4^8 + O(q4^10) -q4^8 + O(q4^10) O(q4^10) -q4^8 + O(q4^10) q4^8 + O(q4^10) O(q4^10) -q4^8 + O(q4^10) -q4^8 + O(q4^10)]
```

```

[-q4^4 - q4^8 + O(q4^10) q4^4 + 2*q4^8 + O(q4^10) -q4^8 + O(q4^10) q4^4 + q4^8 + O(q4^10) -q4^4 - q4^8 + O(q4^10) q4^8 + O(q4^10) q4^4 + O(q4^10) q4^4 + O(q4^10)]

[-2*q4^8 + O(q4^10) -1 - q4^4 + q4^8 + O(q4^10) q4^4 + q4^8 + O(q4^10) -1 + q4^8 + O(q4^10) 1 - q4^8 + O(q4^10) -q4^4 - q4^8 + O(q4^10) -1 + q4^4 + 3*q4^8 + O(q4^10) -1 + q4^4 + 3*q4^8 + O(q4^10)]

[2*q4^8 + O(q4^10) 1 - 3*q4^8 + O(q4^10) -q4^4 + O(q4^10) 1 - q4^4 - 2*q4^8 + O(q4^10) -1 + q4^4 + 2*q4^8 + O(q4^10) q4^4 + O(q4^10) -2*q4^4 - 3*q4^8 + O(q4^10) -2*q4^4 - 3*q4^8 + O(q4^10)]

[q4^4 + 2*q4^8 + O(q4^10) 1 - 3*q4^8 + O(q4^10) -q4^4 + O(q4^10) 1 - q4^4 - 2*q4^8 + O(q4^10) -1 + q4^4 + 2*q4^8 + O(q4^10) q4^4 + O(q4^10) -2*q4^4 - 3*q4^8 + O(q4^10) -2*q4^4 - 3*q4^8 + O(q4^10)]
*/
```

The file 4112data begins as follows.

```
{A[1]=[1497746, 1345119, -3675733, 2082815, -839488, -283405, 383432]};
{A[2]=[3014838521575, 2732414541176, -7414786842283, 4197826806919, -1690529009777, -574198051621, 771765277669]};
...
```

Bibliography

[1] James Alexander. A lemma on systems of knotted curves. *Proceedings of the National Academy of Sciences of the United States of America*, 9(3):93–95, 1923.

[2] James Alexander. Topological invariants of knots and links. *Transactions of the American Mathematical Society*, 30:275–306, 1928.

[3] Jørgen Andersen. The Witten–Reshetikhin–Turaev invariants of finite order mapping tori II. *Quantum topology*, 3(3/4):377–421, 2011.

[4] Jørgen Andersen and *et al*. Problems on invariants of knots and 3–manifolds. Preprint 2002, arXiv:0406190.

[5] Jørgen Andersen and Søren Hansen. Asymptotics of the quantum invariants for surgeries on the figure 8 knot. *Journal of Knot Theory and Its Ramifications*, 15:479–548, 2005.

[6] Jørgen Andersen, Benjamin Himpel, Søren Jørgensen, Johan Martens, and Brendan McLellan. The Witten–Reshetikhin–Turaev invariant for links in finite order mapping tori I. *Advances in Mathematics*, 304:131–178, 2017.

[7] Jørgen Andersen and Rinat Kashaev. A new formulation of the Teichmüller TQFT. Preprint 2013, arXiv:1305.4291.

[8] Jørgen Andersen and Rinat Kashaev. A TQFT from quantum Teichmüller theory. *Communications in Mathematical Physics*, 330:887–934, 2014.

[9] Jørgen Andersen and Rinat Kashaev. The Teichmüller TQFT. *International Congress of Mathematicians*, pages 2527–2552, 2018.

[10] Michael Atiyah. Topological quantum field theories. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 68:175–186, 1988.

[11] Michael Atiyah. *The Geometry and Physics of Knots.* Lezioni Lincee. Cambridge University Press, 1990.

[12] Michael Atiyah and Raoul Bott. The Yang–Mills equations over Riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308:523–615, 1983.

[13] Scott Axelrod and Isadore Singer. Chern–Simons perturbation theory. In *International Conference on Differential Geometric Methods in Theoretical Physics*, pages 3–45, 1991.

[14] Scott Axelrod and Isadore Singer. Chern–Simons perturbation theory II. *Journal of Differential Geometry*, 39:173–213, 1994.

[15] Jinbeom Bae, Dongmin Gang, and Jaehoon Lee. 3d $\mathcal{N} = 2$ minimal SCFTs from wrapped M5–branes. *Journal of High Energy Physics*, 2017:1–23, 2017.

[16] Jitendra Bajpai, Susie Kimport, Liang Jie, Ding Ma, and James Ricci. Bilateral series and Ramanujan's radial limits. *Proceedings of the American Mathematical Society*, 143(2):479–492, 2015.

[17] Dror Bar-Natan. On the Vassiliev knot invariants. *Topology*, 34:423–472, 1995.

[18] Dror Bar-Natan. Perturbative Chern–Simons theory. *Journal of Knot Theory and Its Ramifications*, 04:503–547, 1995.

[19] Dror Bar-Natan and Stavros Garoufalidis. On the Melvin–Morton–Rozansky conjecture. *Inventiones Mathematicae*, 125:103–133, 1996.

[20] Christian Batut, Karim Belabas, Dominique Benardi, Henri Cohen, and Michel Olivier. *User's Guide to PARI–GP*. By anonymous ftp from `ftp://megrez.math.u-bordeaux.fr/pub/pari`.

[21] Christopher Beem, Tudor Dimofte, and Sara Pasquetti. Holomorphic blocks in three dimensions. *Journal of High Energy Physics*, 2014:1–120, 2012.

[22] Anna Beliakova, Christian Blanchet, and Thang Lê. Laplace transform and universal sl_2 invariants. Preprint 2005, arXiv:0509394.

[23] Anna Beliakova, Irmgard Bühler, and Thang Lê. A unified quantum SO(3) invariant for rational homology 3–spheres. *Inventiones Mathematicae*, 185:121–174, 2008.

[24] Anna Beliakova and Thang Lê. Integrality of quantum 3–manifold invariants and a rational surgery formula. *Compositio Mathematica*, 143:1593–1612, 2007.

[25] Bruce Berndt and Song Heng Chan. Sixth order mock theta functions. *Advances in Mathematics*, 216:771–786, 2007.

[26] Michael Berry. Uniform asymptotic smoothing of Stokes's discontinuities. *Proceedings of the Royal Society of London. Series A*, 422(1862):7–21, 1989.

[27] Michael Berry and Christopher Howls. Hyperasymptotics. *Proceedings of the Royal Society of London. Series A*, 430(1880):653–668, 1990.

[28] Michael Berry and Christopher Howls. Hyperasymptotics for integrals with saddles. *Proceedings of the Royal Society of London. Series A*, 434(1892):657–675, 1991.

[29] Sandro Bettin and Sary Drappeau. Modularity and value distribution of quantum invariants of hyperbolic knots. *Mathematische Annalen*, 382:1631–1679, 2019.

[30] Joan Birman and Xiao-Song Lin. Knot polynomials and Vassiliev's invariants. *Inventiones Mathematicae*, 111:225–270, 1993.

[31] Kathrin Bringmann, Amanda Folsom, Ken Ono, and Larry Rolen. *Harmonic Maass forms and mock modular forms: theory and applications*, volume 64. American Mathematical Society, 2017.

[32] Jan Bruinier, Gerard van der Geer, Günter Harder, Kristian Ranestad, and Don Zagier. *The 1–2–3 of Modular Forms*. 2008.

[33] Gerhard Burde and Heiner Zieschang. *Knots*. De Gruyter studies in mathematics. Walter de Gruyter, 2003.

[34] Frank Calegari, Stavros Garoufalidis, and Don Zagier. Bloch groups, algebraic K–theory, units, and Nahm's Conjecture. Preprint 2017, arXiv:1712.04887, to appear in Annales scientifiques de l'École normale supérieure.

[35] Burtis Casler. An imbedding theorem for connected 3–manifolds with boundary. *Proceedings of the American Mathematical Society*, 16(4):559–566, 1965.

[36] Laurent Charles and Julien Marché. Knot state asymptotics II: Witten conjecture and irreducible representations. *Publications mathématiques de l'IHÉS*, 121:323–361, 2011.

[37] Jeff Cheeger and James Simons. Differential characters and geometric invariants. *Geometry and Topology*, pages 50–80, 1985.

[38] Qingtao Chen and Tian Yang. Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants. *Quantum Topology*, 9(3):419–460, 2018.

[39] Miranda Cheng, Sungbong Chun, Francesca Ferrari, Sergei Gukov, and Sarah Harrison. 3d modularity. *Journal of High Energy Physics*, 2019(10):1–95, 2019.

[40] Shiing-Shen Chern and James Simons. Characteristic forms and geometric invariants. *Annals of Mathematics*, 99(1):48–69, 1974.

[41] John Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational problems in abstract algebra*, pages 329–358, 1970.

[42] Daryl Cooper, Marc Culler, Henry Gillet, Darren Long, and Peter Shalen. Plane curves associated to character varieties of 3–manifolds. *Inventiones Mathematicae*, 118:47–84, 1994.

[43] Marc Culler, Nathan Dunfield, Matthias Goerner, and Jeffrey Weeks. SnapPy, a computer program for studying the geometry and topology of 3–manifolds. Available at http://snappy.computop.org (15/10/2022).

[44] Tudor Dimofte, Davide Gaiotto, and Sergei Gukov. 3–manifolds and 3d indices. *Advances in Theoretical and Mathematical Physics*, 17:975–1076, 2011.

[45] Tudor Dimofte and Stavros Garoufalidis. The quantum content of the gluing equations. *Geometry & Topology*, 17(3):1253–1315, 2013.

[46] Tudor Dimofte and Stavros Garoufalidis. Quantum modularity and complex Chern–Simons theory. *Communications in Number Theory and Physics*, 12:1–52, 2018.

[47] Tudor Dimofte, Sergei Gukov, Jonatan Lenells, and Don Zagier. Exact results for perturbative Chern–Simons theory with complex gauge group. *Communications in Number Theory and Physics*, 3:363–443, 2009.

[48] Tudor Dimofte and Roland van der Veen. A spectral perspective on Neumann–Zagier. Preprint 2014, arXiv:1403.5215.

[49] Tudor Dan Dimofte. Quantum Riemann surfaces in Chern–Simons theory. *Advances in Theoretical and Mathematical Physics*, 17:479–599, 2011.

[50] Robert Dingle. *Asymptotic expansions: their derivation and interpretation*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London–New York, 1973.

[51] Thomas Dreyfus. Building meromorphic solutions of q–difference equations using a Borel–Laplace summation. *International Mathematics Research Notices*, 2015:6562–6587, 2014.

[52] Johan Dupont. The dilogarithm as a characteristic class for flat bundles. *Journal of Pure and Applied Algebra*, 44:137–164, 1987.

[53] Johan Dupont and Chih-Han Sah. Scissors congruences, II. *Journal of Pure and Applied Algebra*, 25(2):159–195, 1982.

[54] Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the K3 surface and the Mathieu group M_{24}. *Experimental Mathematics*, 20:91–96, 2010.

[55] Martin Eichler and Don Zagier. On the zeros of the Weierstrass $\wp$–function. *Mathematische Annalen*, 258:399–407, 1982.

[56] Martin Eichler and Don Zagier. *The Theory of Jacobi Forms*. Progress in Mathematics (55). Birkhäuser–Verlag, Basel–Boston, 1985.

[57] Jürgen Elstrodt, Fritz Grunewald, and Jens Mennicke. *Groups Acting on Hyperbolic Space: Harmonic Analysis and Number Theory*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1997.

[58] Ludvig Faddeev. Discrete Heisenberg–Weyl group and modular group. *Letters in Mathematical Physics*, 34:249–254, 1995.

[59] Ludwig Faddeev and Rinat Kashaev. Quantum dilogarithm. *Modern Physics Letters A*, 9(05):427–434, 1994.

[60] Roger Fenn and Colin Rourke. On Kirby's calculus of links. *Topology*, 18(1):1–15, 1979.

[61] Daniel Freed. Classical Chern–Simons theory, 1. *Advances in Mathematics*, 113:237–303, 1995.

[62] Daniel Freed and Robert Gompf. Computer calculation of Witten's 3–manifold invariant. *Communications in Mathematical Physics*, 141:79–117, 1991.

[63] Peter Freyd, David Yetter, Jim Hoste, William Lickorish, Kenneth Millett, and Adrian Ocneanu. A new polynomial invariant of knots and links. *Bulletin of the American Mathematical Society*, 12:239–246, 1985.

[64] Dongmin Gang, Mauricio Romo, and Masahito Yamazaki. All–order volume conjecture for closed 3–manifolds from complex Chern–Simons theory. *Communications in Mathematical Physics*, 359:915–936, 2017.

[65] Stavros Garoufalidis. State integrals, the quantum dilog, and knots. Lecture 18th of September 2018, Max Planck Institute for Mathematics.

[66] Stavros Garoufalidis. On the characteristic and deformation varieties of a knot. *Geometry & Topology Monographs*, 7: Proceedings of the Casson Fest:291–309, 2004.

[67] Stavros Garoufalidis. Chern–Simons theory, analytic continuation and arithmetic. *Acta Mathematica Vietnamica*, 33:335–362, 2008.

[68] Stavros Garoufalidis, Jie Gu, and Marcos Mariño. Peacock patterns and resurgence in complex Chern–Simons theory. Preprint 2020, arXiv:2012.00062.

[69] Stavros Garoufalidis, Jie Gu, and Marcos Mariño. The resurgent structure of quantum knot invariants. *Communications in Mathematical Physics*, 386:469–493, 2020.

[70] Stavros Garoufalidis, Jie Gu, Marcos Mariño, and Campbell Wheeler. Resurgence of Chern–Simons theory at the trivial flat connection. Preprint 2021, arXiv:2111.04763.

[71] Stavros Garoufalidis and Rinat Kashaev. The descendant colored Jones polynomials. Preprint 2021, arXiv:2108.07553.

[72] Stavros Garoufalidis and Rinat Kashaev. Evaluation of state integrals at rational points. *Communications between number theory and physics*, 9:549–582, 2015.

[73] Stavros Garoufalidis and Rinat Kashaev. From state integrals to q–series. *Mathematical Research Letters*, 24:781–801, 2017.

[74] Stavros Garoufalidis and Rinat Kashaev. A meromorphic extension of the 3d index. *Research in the Mathematical Sciences*, 6, 2018.

[75] Stavros Garoufalidis and Rinat Kashaev. Resurgence of Faddeev's quantum dilogarithm. *Topology and Geometry*, 2020.

[76] Stavros Garoufalidis, Rinat Kashaev, and Don Zagier. A modular extension of Faddeev's quantum dilogarithm. In preparation.

[77] Stavros Garoufalidis and Thang Lê. The colored Jones function is q–holonomic. *Geometry & Topology*, 9:1253–1293, 2005.

[78] Stavros Garoufalidis and Thang Lê. A survey of q–holonomic functions. *L'Enseignement Mathematique*, 62:501–525, 2016.

[79] Stavros Garoufalidis and Xinyu Sun. The C–polynomial of a knot. *Algebraic & Geometric Topology*, 6:1623–1653, 2005.

[80] Stavros Garoufalidis, Dylan Thurston, and Christian Zickert. The complex volume of $\mathrm{SL}(n, \mathbb{C})$–representations of 3–manifolds. *Duke Mathematical Journal*, 164:2099–2160, 2011.

[81] Stavros Garoufalidis and Roland van der Veen. Asymptotics of quantum spin networks at a fixed root of unity. *Mathematische Annalen*, 352:987–1012, 2010.

[82] Stavros Garoufalidis and Campbell Wheeler. Modular q–holonomic modules. Preprint 2022, arXiv:2203.17029.

[83] Stavros Garoufalidis and Campbell Wheeler. Periods, the meromorphic 3d–index and the Turaev–Viro invariant. Preprint 2022, arXiv:2209.02843.

[84] Stavros Garoufalidis and Don Zagier. Hyperbolic 3–manifolds, the Bloch group, and the work of Walter Neumann. Preprint 2022.

[85] Stavros Garoufalidis and Don Zagier. Knots and their related q–series. Preprint 2023.

[86] Stavros Garoufalidis and Don Zagier. Knots, perturbative series and quantum modularity. Preprint 2021, arXiv:2111.06645.

[87] Stavros Garoufalidis and Don Zagier. Smooth truncation of factorially divergent series. In preparation.

[88] Stavros Garoufalidis and Don Zagier. Asymptotics of Nahm sums at roots of unity. *The Ramanujan Journal*, 55:219–238, 2020.

[89] Vasily Golyshev and Don Zagier. Proof of the gamma conjecture for Fano 3–folds of Picard rank 1. *Izvestiya: Mathematics*, 80:24–49, 2016.

[90] Daniel B. Grünberg and Pieter Moree. Sequences of Enumerative Geometry: Congruences and Asymptotics, with an appendix by Don Zagier. *Experimental Mathematics*, 17(4):409–426, 2008.

[91] Sergei Gukov. Three–dimensional quantum gravity, Chern–Simons theory, and the A–polynomial. *Communications in Mathematical Physics*, 255:577–627, 2003.

[92] Sergei Gukov and Ciprian Manolescu. A two–variable series for knot complements. *Quantum Topology*, 12(1):1–109, 2019.

[93] Kazuo Habiro. On the colored Jones polynomials of some simple links. *Surikaisekikenkyusho Kokyuroku*, 1172:34–43, 2000.

[94] Kazuo Habiro. Cyclotomic completions of polynomial rings. *Publications of The Research Institute for Mathematical Sciences*, 40:1127–1146, 2002.

[95] Kazuo Habiro. On the quantum $\mathfrak{sl}_2$ invariants of knots and integral homology spheres. *Geometry & Topology Monographs*, 4, 12 2002.

[96] Kazuo Habiro. A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres. *Inventiones Mathematicae*, 171:1–81, 2007.

[97] Charlotte Hardouin, Jacques Sauloy, and Michael Singer. *Galois Theories of Linear Difference Equations: An Introduction*, volume 211. 2016.

[98] Kazuhiro Hikami. Hyperbolic structure arising from a knot invariant. *International Journal of Modern Physics A*, 16:3309–3333, 2001.

[99] Kazuhiro Hikami. Mock (false) theta functions as quantum invariants. *Regular & Chaotic Dynamics*, 10:509–530, 2005.

[100] Kazuhiro Hikami and Jeremy Lovejoy. Hecke–type formulas for families of unified Witten–Reshetikhin–Turaev invariants. *Communications in Number Theory and Physics*, 11:249–272, 2017.

[101] Wolfram Research, Inc. Mathematica, Version 13.1. Champaign, IL, 2022.

[102] Lisa Jeffrey. Chern–Simons–Witten invariants of lens spaces and torus bundles, and the semiclassical approximation. *Communications in Mathematical Physics*, 147:563–604, 1992.

[103] Vaughan Jones. A polynomial invariant for knots via von Neumann algebras. *Bulletin of the American Mathematical Society*, 12:103–111, 1985.

[104] Vaughan Jones. Hecke algebra representations of braid groups and link polynomials. *Annals of Mathematics*, 126(2):335–388, 1987.

[105] William Jones and Wolfgang Thron. *Continued Fractions: Analytic Theory and Applications*. 1984.

[106] Rinat Kashaev. Quantum dilogarithm as a 6j–symbol. *Modern Physics Letters A*, 09:3757–3768, 1994.

[107] Rinat Kashaev. A link invariant from quantum dilogarithm. *Modern Physics Letters A*, 10:1409–1418, 1995.

[108] Rinat Kashaev. The hyperbolic volume of knots from the quantum dilogarithm. *Letters in Mathematical Physics*, 39:269–275, 1996.

[109] Christian Kassel and Vladimir Turaev. *Braid Groups*. Graduate Texts in Mathematics. Springer New York, 2008.

[110] Rinat Kedem, Timothy Klassen, Barry McCoy, and Ezer Melzer. Fermionic sum representations for conformal field theory characters. *Physics Letters B*, 307:68–76, 1993.

[111] Robion Kirby. A calculus for framed links in S^3. *Inventiones Mathematicae*, 45:35–56, 1978.

[112] Robion Kirby and Paul Melvin. The 3–manifold invariants of Witten and Reshetikhin–Turaev for $\mathfrak{sl}_2(\mathbb{C})$. *Inventiones Mathematicae*, 105:473–545, 1991.

[113] Anatol Kirillov and Nicolai Reshetikhin. Representations of the algebra $U_q(\mathfrak{sl}_2)$, q–orthogonal polynomials and invariants of links. *New developments in the theory of knots*, 11:202, 1990.

[114] Paul Kirk and Eric Klassen. Chern–Simons invariants of 3–manifolds and representation spaces of knot groups. *Mathematische Annalen*, 287:343–367, 03 1990.

[115] Paul Kirk and Eric Klassen. Chern–Simons invariants of 3–manifolds decomposed along tori and the circle bundle over the representation space of T^2. *Communications in Mathematical Physics*, 153:521–557, 1993.

[116] Joachim Kock. *Frobenius algebras and 2–d topological quantum field theories*. Number 59. Cambridge University Press, 2004.

[117] Maxim Kontsevich. Vassiliev's knot invariants. *Advances in Soviet Math*, 16(2):137–150, 1993.

[118] Ruth Lawrence and Don Zagier. Modular forms and quantum invariants of 3–manifolds. *Asian Journal of Mathematics*, 3:93–107, 1999.

[119] Arjen Lenstra, Hendrik Lenstra, and László Lovász. Factoring polynomials with rational coefficients. *Mathematische Annalen*, 261:515–534, 1982.

[120] William Lickorish. A representation of orientable combinatorial 3–manifolds. *Annals of Mathematics*, 76(3):531–540, 1962.

[121] William Lickorish. *An Introduction to Knot Theory*. Graduate Texts in Mathematics. Springer New York, 1997.

[122] Xiao-Song Lin. Vertex models, quantum groups and Vassiliev's knot invariants. *Columbia University preprint*, 1991.

[123] Charles Livingston. *Knot theory*, volume 24. Cambridge University Press, 1993.

[124] Thang Lê. Quantum invariants of 3–manifolds: Integrality, splitting, and perturbative expansion. *Topology and its Applications*, 127:125–152, 2000.

[125] Thang Lê, Jun Murakami, and Tomotada Ohtsuki. On a universal perturbative invariant of 3–manifolds. *Topology*, 37(3):539–574, 1998.

[126] Yuri Manin. Cyclotomy and analytic geometry over $\mathbb{F}_1$. *Quanta of maths*, 385:408, 2010.

[127] Julien Marché. Geometric interpretation of simplicial formulas for the Chern–Simons invariant. *Algebraic & Geometric Topology*, 12(2):805–827, 2012.

[128] Andrei Markoff. Über die freie Äquivalenz der geschlossenen Zöpfe. *Matematicheskii Sbornik*, 1(43)(1):73–78, 1936.

[129] Gregor Masbaum. Skein–theoretical derivation of some formulas of Habiro. *Algebraic & Geometric Topology*, 3:537–556, 2003.

[130] Sergei Matveev. Transformations of special spines and the zeeman conjecture. *Mathematics of the USSR–Izvestiya*, 31(2):423, 1988.

[131] Sergei Matveev. *Algorithmic Topology and Classification of 3–Manifolds*. Algorithms and Computation in Mathematics. Springer Berlin Heidelberg, 2013.

[132] Paul Melvin and Hugh Morton. The coloured Jones function. *Communications in Mathematical Physics*, 169:501–520, 1995.

[133] Gregory Moore and Nathan Seiberg. Polynomial equations for rational conformal field theories. *Physics Letters B*, 212(4):451–460, 1988.

[134] Siegfried Moran. *The Mathematical Theory of Knots and Braids An Introduction*, volume 82 of *North–Holland Mathematics Studies*. North–Holland, 1983.

[135] George Mostow. *Strong rigidity of locally symmetric spaces*. Number 78. Princeton University Press, 1973.

[136] Hitoshi Murakami. Quantum SO(3)–invariants dominate the SU(2)–invariant of Casson and Walker. *Mathematical Proceedings of the Cambridge Philosophical Society*, 117:237–249, 1995.

[137] Hitoshi Murakami. An introduction to the volume conjecture. *Interactions between hyperbolic geometry, quantum topology and number theory*, 541:1–40, 2011.

[138] Hitoshi Murakami and Jun Murakami. The colored Jones polynomials and the simplicial volume of a knot. *Acta Mathematica*, 186:85–104, 1999.

[139] Hitoshi Murakami, Jun Murakami, Miyuki Okamoto, Toshie Takata, and Yoshiyuki Yokota. Kashaev's conjecture and the Chern-simons invariants of knots and links. *Experimental Mathematics*, 11:427–435, 2002.

[140] Hitoshi Murakami and Yoshiyuki Yokota. *Volume conjecture for knots*, volume 30. Springer, 2018.

[141] James Murray. *Asymptotic analysis*, volume 48. Springer Science & Business Media, 2012.

[142] Werner Nahm. Conformal field theory and torsion elements of the Bloch group. In *Les Houches School of Physics: Frontiers in Number Theory, Physics and Geometry*, pages 67–132, 2007.

[143] Werner Nahm, Andreas Recknagel, and Michael Terhoeven. Dilogarithm identities in conformal field theory. *Modern Physics Letters A*, 8(19):1835–1847, 1993.

[144] Walter Neumann. Combinatorics of triangulations and the Chern–Simons invariant for hyperbolic 3–manifolds. *Topology 90, Proceedings of the Research Semester in Low Dimensional Topology at Ohio State*, page 243–271, 1990.

[145] Walter Neumann. Extended Bloch group and the Cheeger–Chern–Simons class. *Geometry & Topology*, 8:413–474, 2004.

[146] Walter Neumann and Don Zagier. Volumes of hyperbolic three–manifolds. *Topology*, 24(3):307–332, 1985.

[147] John Nuttall. The convergence of Padé approximantes to functions with branch points. In Edward Saff and Richard Varga, editors, *Pade and Rational Approximation*, pages 101–109. Academic Press, 1977.

[148] Tomotada Ohtsuki. A polynomial invariant of rational homology 3–spheres. *Inventiones Mathematicae*, 123:241–257, 1996.

[149] Tomotada Ohtsuki. *Quantum Invariants: A Study of Knots, 3–manifolds, and Their Sets*. K & E series on knots and everything. World Scientific, 2002.

[150] Tomotada Ohtsuki. On the asymptotic expansion of the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ for closed hyperbolic 3–manifolds obtained by integral surgery along the figure–eight knot. *Algebraic & Geometric Topology*, 2018.

[151] Frank Olver. *Asymptotics and special functions*. AK Peters/CRC Press, 1997.

[152] Naoki Osada. The E–algorithm and the Ford–Sidi algorithm. *Journal of Computational and Applied Mathematics*, 122(1):223–230, 2000. Numerical Analysis in the 20th Century Vol. II: Interpolation and Extrapolation.

[153] Sunghyuk Park. Inverted state sums, inverted Habiro series, and indefinite theta functions. Preprint 2021, arXiv:2106.03942.

[154] Sunghyuk Park. Large color R–matrix for knot complements and strange identities. *Journal of Knot Theory and Its Ramifications*, 2020.

[155] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. Preprint 2002, arXiv:0211159.

[156] Grisha Perelman. Finite extinction time for the solutions to the Ricci flow on certain three–manifolds. Preprint 2003, arXiv:0307245.

[157] Grisha Perelman. Ricci flow with surgery on three–manifolds. Preprint 2003, arXiv:0303109.

[158] Riccardo Piergallini. Standard moves for standard polyhedra and spines. *Supplemento ai Rendiconti del Circolo Matemàtico di Palermo. Serie II.*, 18, 01 1988.

[159] Gopal Prasad. Strong rigidity of Q–rank 1 lattices. *Inventiones Mathematicae*, 21:255–286, 1973.

[160] Józef Przytycki. Skein modules of 3–manifolds. *Bulletin of the Polish Academy of Sciences*, 39, 12 2006.

[161] Józef Przytycki and Paweł Traczyk. Invariants of links of Conway type, 1988. Preprint 1988, arXiv:1610.06679.

[162] Trivandrum Ramadas, Isadore Singer, and Jonathan Weitsman. Some comments on Chern–Simons gauge theory. *Communications in Mathematical Physics*, 126:409–420, 1989.

[163] Srinivasa Ramanujan. Ramanujan's last letter to Hardy. 1920.

[164] Srinivasa Ramanujan. Collected papers of Srinivasa Ramanujan. *Notes and Queries*, CLVI(feb11):107–108, 02 1928.

[165] Jean-Pierre Ramis, Jacques Sauloy, and Changgui Zhang. *Local analytic classification of q–difference equations.* Société mathématique de France, 2013.

[166] Nicolai Reshetikhin and Vladimir Turaev. Ribbon graphs and their invaraints derived from quantum groups. *Communications in Mathematical Physics*, 127:1–26, 1990.

[167] Nicolai Reshetikhin and Vladimir Turaev. Invariants of 3–manifolds via link polynomials and quantum groups. *Inventiones Mathematicae*, 103:547–597, 1991.

[168] Axel Riese. qMultiSum–a package for proving q–hypergeometric multiple summation identities. *Journal of Symbolic Computation*, 35:349–376, 2003.

[169] Dale Rolfsen. *Knots and links*, volume 346. American Mathematical Society, 2003.

[170] Lev Rozansky. A contribution of the trivial connection to the Jones polynomial and Witten's invariant of 3d manifolds, II. *Communications in Mathematical Physics*, 175:297–318, 1994.

[171] Lev Rozansky. A contribution of the trivial connection to the Jones polynomial and Witten's invariant of 3d manifolds, I. *Communications in Mathematical Physics*, 175:275–296, 1996.

[172] Lev Rozansky. The universal R–matrix, Burau representation, and the Melvin–Morton expansion of the colored Jones polynomial. *Advances in Mathematics*, 134:1–31, 1996.

[173] Nikolai Saveliev. *Lectures on the Topology of 3–manifolds: An Introduction to the Casson Invariant*. De Gruyter textbook. Walter de Gruyter, 1999.

[174] Nancy Scherich. The Alexander polynomial. Masters thesis 2013, https://nancyscherich.com/research/.

[175] Graeme Segal. The definition of conformal field theory. In Konrad Bleuler and Marcus Werner, editors, *Differential Geometrical Methods in Theoretical Physics*, pages 165–171. Springer Netherlands, Dordrecht, 1988.

[176] Jean-Pierre Serre. A course in arithmetic. In *Graduate texts in mathematics*, 1973.

[177] Neil Sloane and The OEIS Foundation Inc. The on–line encyclopedia of integer sequences, 2020.

[178] Herbert Stahl. The convergence of Padé approximants to functions with branch points. *Journal of Approximation Theory*, 91(2):139–204, 1997.

[179] Alexander Stoimenow. Enumeration of chord diagrams and an upper bound for Vassiliev invariants. *Journal of Knot Theory and Its Ramifications*, 07:93–114, 1998.

[180] Andrei Suslin. K_3 of a field and the Bloch group. In *Proceedings of the Steklov Institute of Mathematics*, 1991. null ; Conference date: 01-01-1991.

[181] Hidetoshi Tahara. q–analogues of Laplace and Borel transforms by means of q–exponentials. *Annales de l'Institut Fourier*, 67:1865–1903, 2017.

[182] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.0)*, 2022. `https://www.sagemath.org`.

[183] William Thurston. *The Geometry and Topology of Three–manifolds*. Princeton University, 1979.

[184] William Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bulletin (New Series) of the American Mathematical Society*, 6(3):357–381, 1982.

[185] Vladimir Turaev. The Yang–Baxter equation and invariants of links. *Inventiones Mathematicae*, 92:527–553, 1988.

[186] Vladimir Turaev. Conway and Kauffman modules of a solid torus. *Journal of Soviet Mathematics*, 52:2799–2805, 1990.

[187] Vladimir Turaev. *Quantum Invariants of Knots and 3–Manifolds*. De Gruyter Studies in Mathematics. De Gruyter, 2016.

[188] Marius van der Put and Michael Singer. *Galois theory of linear differential equations*, volume 328. Springer Science & Business Media, 2012.

[189] Victor Vassiliev. Cohomology of knot spaces. volume 1, pages 23–70. Providence, RI. American Mathematical Society, 12 1990.

[190] Erik Verlinde. Fusion rules and modular transformations in 2d conformal field theory. *Nuclear Physics B*, 300:360–376, 1988.

[191] Masha Vlasenko and Sander Zwegers. Nahm's conjecture: asymptotic computations and counterexamples. *Communications in Number Theory and Physics*, 5(3):617–642, 2011.

[192] Andrew Wallace. Modifications and cobounding manifolds, IV. *Journal of Mathematics and Mechanics*, 12(3):445–484, 1963.

[193] George Watson. The final problem: An account of the mock theta functions. *Journal of The London Mathematical Society–second Series*, 11:55–80, 1936.

[194] Herbert Wilf and Doron Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Inventiones Mathematicae*, 108:575–633, 1992.

[195] Edward Witten. Quantum field theory and the Jones polynomial. *Communications in Mathematical Physics*, 121:351–399, 1989.

[196] Edward Witten. Quantization of Chern–Simons gauge theory with complex gauge group. *Communications in Mathematical Physics*, 137:29–66, 1991.

[197] Edward Witten. Analytic continuation of Chern–Simons theory. *AMS/IP Studies in Advanced Mathematics*, 50:347, 2011.

[198] Stanisław Woronowicz. Quantum exponential function. *Reviews in Mathematical Physics*, 12:873–920, 2000.

[199] Tian Yang. Recent progresses on the volume conjectures for Reshetikhin–Turaev and Turaev–Viro invariants. *Acta Mathematica Vietnamica*, 46:389–398, 2021.

[200] Tomoyoshi Yoshida. The η–invariant of hyperbolic 3–manifolds. *Inventiones Mathematicae*, 81:473–514, 1985.

[201] Don Zagier. Holomorphic quantum modular forms. In preparation.

[202] Don Zagier. Holomorphic quantum modular forms. Lecture 28th of Febuary 2020, Hausdorff Center for Mathematics, youtube.

[203] Don Zagier. Standard and less standard asymptotic methods. Lectures series 2022, International Centre for Theoretical Physics, youtube.

[204] Don Zagier. Vassiliev invariants and a strange identity related to the Dedekind eta–function. *Topology*, 40:945–960, 2001.

[205] Don Zagier. Appendix: The Mellin transformation and other useful analytic techniques. In *Quantum Field Theory I: Basics in Mathematics and Physics*, 2006.

[206] Don Zagier. The dilogarithm function. In Pierre Cartier, Pierre Moussa, Bernard Julia, and Pierre Vanhove, editors, *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization*, pages 3–65. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

[207] Don Zagier. Ramanujan's mock theta functions and their applications [d'après Zwegers and Bringmann–Ono]. *Séminaire Bourbaki*, 60ème année:143–164, 2007–2008.

[208] Don Zagier. Quantum modular forms. *Quanta of maths*, 11:659–675, 2010.

[209] Doron Zeilberger. A holonomic systems approach to special functions identities. *Journal of Computational and Applied Mathematics*, 32:321–368, 1990.

[210] Christian Zickert. The extended Bloch group and algebraic K–theory. *Crelle's Journal*, 2015:21–54, 2009.

[211] Sander Zwegers. Mock theta functions. Thesis 2008, arXiv:0807.4834.