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Introduction

Group decisions are ubiquitous in democratic societies. They range from general elections, and referenda to voting inside various organizations such as political institutions, and corporate boards. Modern social choice theory, “the study of collective decision processes and procedures” (List (2022)), is shaped by the celebrated Arrow (1963) as well as Gibbard (1973), and Satterthwaite (1975) impossibility results: These seminal findings suggest that there is no universally good voting rule as soon as there are more than two alternatives (e.g., candidates, policies). Consequently, appropriate procedures to reach collective decisions are context-dependent, and there is scope to design good voting rules. The present dissertation consists of four self-contained essays that study the design of voting mechanisms in settings with multiple alternatives. The essays identify reasonable voting rules for different contexts. Therefore, they represent normative contributions. Methodologically, the analyses are theoretical, and mathematical. They build upon tools from microeconomic theory, and, in particular, game theory, and mechanism design.

The “one person, one vote” criterion is a principle that is central to democracy, but it is arguably with justification violated in certain situations, where the voters are asymmetric. Chapters 1 and 2 share the following overarching research question: How to assign voting weights to heterogeneous agents?

Chapter 1, *Sequential Voting and the Weights of Nations*, focuses on institutions of representative democracy such as the Council of the European Union, where representatives vote on behalf of groups of citizens of heterogeneous size. The chapter develops a model of representative democracy, and studies the design of welfare-maximizing voting mechanisms for the collective decision-making process involving representatives. There are multiple policy alternatives, and the citizens’ preferences are assumed to be consistent with a unidimensional political spectrum, that is, they satisfy the condition of single-peakedness. For this setting, the chapter attempts to answer the following questions: How to optimally organize the voting process of representatives who vote on behalf of groups of citizens that differ in population size? How should the voting weights or the weights of nations be assigned to the representatives as a function of the group size? Which majority quotas should be used depending on the nature of the proposal? What is the interaction between voting weights and majority quotas in an optimal mechanism? The welfare-maximizing voting rule among a large class of sequential voting procedures can be implemented via a sequence of binary weighted majority voting decisions. The utilitarian weights of nations feature an overweighting of smaller groups that lies between the benchmark, where weights are proportional to group sizes, and a power law with an exponent of $1 - \frac{\ln 2}{\ln 3} \approx 0.37$. The main insight of the chapter is as follows: Under the optimal mechanism, the vote on more extreme alternatives does not only require majority quotas that are further away from simple majority, but it also involves voting weights that feature less overweighting of smaller groups. Finally, the chapter discusses the implications of these theoretical results for voting in the Council of the European Union. Until 2014, in the Council of the European Union, most collective decisions were reached according to a weighted majority rule exhibiting voting weights that are approximately proportional to the square root of the population sizes of the member states, and a majority quota of about 74% (see European Union (2001), and European Union (2012)). The model developed in the chapter can rationalize such a majority quota. However, the quantitative analysis in the chapter suggests that the use of it should be accompanied by voting weights that feature less overweighting of small member states compared to the weights that were applied in the Council of the European Union.

Chapter 2, *Public Goods Provision and Weighted Majority Voting*, considers the problem of optimally providing public goods, that is, non-excludable, and non-rivalrous goods. It analyzes the utilitarian voting mechanism for the provision of a costly public good. The setting features multiple public good levels, and the voters are asymmetric in the following two ways: On the one hand, the asymmetry of the voters arises from heterogeneous distributions of the benefits of the public good. This type of asymmetry covers, for example, situations, where the public good is an infrastructure project, implying that people living near the location of the infrastructure project are concerned differently about the project in comparison with people living in the same state, but far away from the location of the project (see Fleurbaey (2008)). How should this heterogeneity be incorporated in the voting mechanism? Should collective decisions about such public goods be reached according to local or statewide referenda? How to assign voting weights to agents that are asymmetric because of heterogeneous benefit distributions? On the other hand, voters are asymmetric because of an unequal sharing of the costs of the public good. Member countries of international organizations such as the International Monetary Fund or the World Bank contribute to these organizations differently in financial terms, and, therefore, collective decisions about the provision of public goods inside these institutions are taken by voters who are asymmetric because of an unequal sharing of the costs of the public good (see International Monetary Fund (2021), and World Bank (2021)). How should this heterogeneity be reflected in the voting rules? Is it optimal to assign voting weights to the member states of these international organizations that are increasing in the financial contribution or the cost share? The voting mechanism that maximizes the ex-ante utilitarian welfare among all strategy-proof, and surjective social choice functions involves a sequence of weighted majority decisions going gradually from low to high public good levels. The utilitarian majority requirements decrease monotonically in the public good level under consideration. If the benefits of the public good are drawn from heterogeneous distributions, the optimal mechanism assigns higher voting weights to voters whose benefit distributions are more variable, and the utilitarian voting weights are more equal for more extreme public good levels. If the costs of the public good are shared unequally, the optimal voting weights are generally not increasing in the voters' cost shares.

Chapter 3, *Committee Search Design*, is joint work with Christina Luxen. Search processes in many organizations such as hiring procedures share the following two characteristics: The decisions are taken collectively by a committee via voting, and the committee evaluates multiple items simultaneously. The present chapter investigates sequential search by committee, where, in each period of time, K items can be sampled, and at least M out of N committee members have to approve an item in order to stop search. The focus of the chapter is on the design of the sample size per period, K. Designing the sample size per period amounts to determining certain aspects of the voting procedure. First, it can be viewed as delayed voting: Assume that one item per period arrives. Then, simultaneously reviewing K items corresponds to taking voting decisions only every K periods instead of every single period. Thus, choosing the sample size K can be interpreted as selecting voting times.[1] Second, determining the sample size can be seen as designing the number of alternatives that are put to a vote in each period of time, while holding the number of committee members and the required degree of approval to stop search fixed. For example, in the hiring context, one or multiple candidates could be evaluated simultaneously. Hiring on a rolling basis corresponds to the case in which voting is not delayed, but, in every period of time, the committee takes a collective decision. Therefore, the chapter attempts to provide answers under which circumstances a hiring process should be conducted on a rolling basis. The main insight of the chapter is as follows: The welfare assessment of the sample sizes per period differs if the search committee operates under unanimity versus qualified majority voting. Under unanimity voting, the welfare ranking of the sample sizes depends on how the search costs vary with the sample size. In contrast, under qualified majority voting distinct from unanimity, independently of the shape of the cost function, reviewing more items per period of time improves welfare as long as the magnitude of the search costs is sufficiently small.

Chapter 4, *Optimal Voting Mechanisms on Generalized Single-Peaked Domains*, examines welfare-maximizing voting mechanisms in settings with multiple alternatives, and voters who have generalized single-peaked preferences derived from median spaces as introduced in Nehring and Puppe (2007). These preference structures are

1. We thank Olivier Compte for suggesting this interpretation.

considerably larger than the class of single-peaked preferences on a line considered in chapters 1 and 2 of this dissertation. However, in contrast to these chapters, the voters are not asymmetric here. The utilitarian voting rule among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity is composed of binary votes on subsets of alternatives involving flexible majority requirements. The chapter discusses an application of this general optimality result to the design of voting mechanisms for the provision of two costly public goods subject to the constraint that the provided level of the former good is weakly higher than the provided level of the latter good. For example, if the public goods represent expansions of the rail, and the road network respectively, this constraint might reflect the fight against climate change. The more general classes of single-peaked preferences considered in this chapter make it possible to analyze this application featuring two instead of one public good as in chapter 2 as well as a constraint on the set of feasible public good levels.

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Chapter 1

Sequential Voting and the Weights of Nations

1.1 Introduction

A defining feature of modern democracies is that citizens appoint representatives who then engage in a collective decision-making process on behalf of groups of citizens of heterogeneous size. Prominent examples are the United States Congress as well as the Council of the European Union.

In the United States, the seats in the House of Representatives are distributed among the states such that they are approximately proportional to the population sizes of the states; in contrast, in the Senate, each state has two seats independently of the population size; depending on the proposal that is up for a vote, different majority quotas apply: Standard bills or motions pass whenever a simple majority votes in favour of them, whereas constitutional amendments require the support of a two-thirds majority (see United States (2022a), and United States (2022b)).

In the Council of the European Union, currently, for most policy areas, the following voting rules are used: If the vote is on a proposal from the European Commission or the High Representative of the Union for Foreign Affairs and Security Policy, it is collectively approved whenever it is supported by at least 55% of the member states, representing at least 65% of the population of the European Union; otherwise, the support of at least 72% instead of 55% of the member states is required for a proposal to pass (see European Union (2007)). In contrast, until 2014, collective decisions in the Council of the European Union were reached according to a weighted majority voting rule, where the voting weights were roughly proportional to the square root of the population sizes and the majority quota was approximately 74% (see European Union (2001), and European Union (2012)).

The variety of majority quotas and seat or voting weight distributions observed in the described institutions of the United States and the European Union raises the question of how to optimally organize the voting process of representatives who vote on behalf of groups of citizens that differ in population size. How should the voting weights or the weights of nations be assigned to the representatives as a function of the group size? This issue is called the apportionment problem. Which majority quotas should be used depending on the nature of the proposal? What is the interaction between voting weights and majority quotas in an optimal mechanism?

To answer these questions, I develop a model of representative democracy and study the design of welfare-maximizing voting mechanisms for the collective decision-making process involving representatives. There are multiple policy alternatives and the citizens' preferences are single-peaked. To the best of my knowledge, I provide the first analytical results for the apportionment problem in a setting featuring more than two alternatives, single-peaked preferences and a utilitarian objective criterion. The core economic problem that the utilitarian designer faces is one of Bayesian inference: The designer has to make inferences about the citizens' preferences based on the vote choices of the representatives. Given these inferences, how does the optimal voting mechanism look like? The optimal voting rule can be implemented via a sequence of binary weighted majority voting decisions. The main insight of this chapter is as follows: Under the optimal mechanism, the vote on more extreme alternatives does not only require majority quotas that are further away from simple majority, but it also involves relatively higher voting weights for large groups.

In more detail, to keep the analysis tractable, I make the following key assumptions. Regarding preferences, I assume in the benchmark model that the citizens' preferences are distributed independently and identically across all citizens. In particular, the citizens' preferences are independent within groups. The analysis of the benchmark model concentrates on the case of finite population sizes in the sense that I do not invoke any limit arguments. The reason for this focus is as follows: It turns out that the model featuring independent preferences as well as finite population sizes is similar to a model, following Barberà and Jackson (2006), in which population sizes are large and preferences are correlated within groups. The latter model seems to be the empirically relevant model.[1] In an extension contained in section 1.8, I show that my findings for the former model extend to the latter model.

Moreover, I suppose that each group has exactly one representative whose most preferred alternative coincides with the median of all most preferred alternatives or the Condorcet winner in the group. In other words, within groups, preferences are aggregated according to simple majority voting. Maaser and Napel (2007), Maaser and Napel (2012), Maaser and Napel (2014), and Kurz, Maaser, and Napel (2017) also assume that representatives are median citizens. The rationale behind this assumption is twofold. On the one hand, from a theoretical viewpoint, employing the median rule to select the representative's peak alternative is in a certain sense op-

1. In fact, Kurz, Maaser, and Napel (2017) argue that correlated preferences within groups seem to be the main cause why there is no redistricting in practice such that representatives vote on behalf of equally-sized groups of citizens.

timal.[2] On the other hand, from an applied perspective, I aim to match the within-group preference aggregation as close as possible to reality, and simple majority voting is the most frequent voting rule observed in practice. In this way, I make recommendations how to improve collective decision-making at the representatives' level.

Furthermore, the representatives play an ex-post perfect equilibrium that is induced by a sequential voting procedure. I assume that the set of voting mechanisms over which the utilitarian designer optimizes coincides with the class of successive voting procedures analysed in Gershkov, Moldovanu, and Shi (2017), and Kleiner and Moldovanu (2017). The reasons for this assumption are twofold. First, from a theoretical perspective, there is a one-to-one relationship between, on the one hand, direct mechanisms that are dominant-strategy incentive-compatible and surjective on the full single-peaked domain, and, on the other hand, the equilibria of the successive voting procedures that I consider.[3] Second, from a pragmatic standpoint, the sequential voting procedures are a common feature of many real-world institutions (see Rasch (2000)). These dynamic voting rules are described in detail in section 1.4.

In order to outline the main results, for concreteness, consider the example of three policy alternatives.[4] The three alternatives are 1, 2, and 3, and the citizens' preferences are single-peaked with respect to the ordering $1 < 2 < 3$. The welfare-maximizing mechanism takes the form of a weighted successive voting procedure illustrated in Figure 1.1. Assign to each representative or group of citizens two alternative-dependent voting weights: For any group $j \in \{1, ..., c\}$ with $c \geq 2$, the weight $w_j(1)$ is related to alternative 1, and the other weight $w_j(2)$ is linked to alternative 2. Further, introduce two alternative-dependent majority quotas $q(1)$ and $q(2)$. Then, a weighted successive voting procedure works as follows: First, put alternative 1 on the agenda, and perform a binary vote whether to implement alternative 1 or not. The weights and quotas related to alternative 1, i.e., $[w_1(1), ..., w_c(1); q(1)]$, determine when alternative 1 is collectively accepted: Alternative 1 is implemented if the sum of weights related to alternative 1 associated with representatives voting "Yes" at alternative 1 exceeds the majority quota $q(1)$ linked to alternative 1.

2. Under the restriction on the preference distribution that I impose in the characterization of the optimal mechanism for the representatives' voting process, the median mechanism is optimal in the following sense: It maximizes the welfare of a group's own citizens among all direct mechanisms that are dominant-strategy incentive-compatible, and anonymous on the full single-peaked domain, and that satisfy the additional constraint that the outcome always coincides with some citizen's peak alternative. Gershkov, Moldovanu, and Shi (2017) contains a related finding. The result follows by combining the finding that is mentioned directly before Footnote 3 as well as Theorem 1.1 while setting all population sizes equal to 1.

3. Similar results for the anonymous case are contained in Gershkov, Moldovanu, and Shi (2017), and Kleiner and Moldovanu (2017). The stated finding follows from the characterization of strategy-proof social choice functions in Nehring and Puppe (2007).

4. All the presented results hold for any finite number of alternatives.

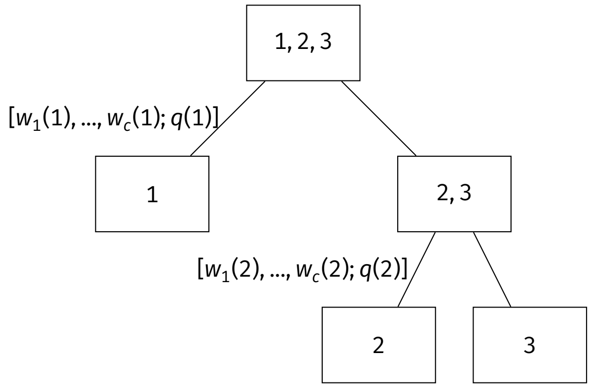

Figure 1.1. Weighted Successive Voting Procedure

Otherwise, alternative 2 is put on the agenda, and there is a binary vote on this alternative. Again, representatives can either vote "Yes" or "No", and the weights and quotas $[w_1(2), ..., w_c(2); q(2)]$ linked to alternative 2 govern when alternative 2 is collectively approved in a way that is analogous to the vote on alternative 1. Otherwise, alternative 3 is implemented. Now, define a representative's strategy in the game induced by such weighted successive voting procedures as sincere if this representative plays action "Yes" if and only if the alternative on the agenda lies weakly to the right of the representative's most preferred alternative.

The main results are as follows: First, I show that the welfare-maximizing mechanism can be implemented via the sincere equilibrium induced by a weighted successive voting procedure, and I derive closed-form expressions for the utilitarian voting weights and majority quotas (Theorem 1.1). The optimal weights and quotas are sensitive to the alternative that is on the agenda through the cdf of the citizens' most preferred alternatives denoted by G. Concretely, in the three-alternatives case, this means that the welfare-maximizing weights $w_j(1)$ and quota $q(1)$ depend on the value of $G(1)$, i.e., the probability that 1 is any citizen's most preferred alternative. Similarly, the optimal $w_j(2)$ and $q(2)$ are sensitive to the value of $G(2)$, i.e., the probability that any citizen does not have 3 as his or her most preferred alternative. Second, I derive analytically several properties of the optimal weights and quotas. To present these features, say that an alternative k' is more moderate or less extreme than another alternative k'' if $|G(k') - \frac{1}{2}| < |G(k'') - \frac{1}{2}|$.

For the majority quotas, I find that the vote on more extreme alternatives requires majority quotas that are further away from simple majority. In particular, the welfare-maximizing quotas do not coincide with simple majority.

The utilitarian weights of nations have two key properties. First, for any alternative, the corresponding optimal weights feature a degree of overweighting of smaller groups that lies between the benchmark, where weights are linear in group sizes, and a power law with exponent $1 - \frac{\ln 2}{\ln 3} \approx 0.37$ (Proposition 1.2). In particular, this finding implies that the optimal weights exhibit degressive proportionality, that is, the weights themselves are increasing, but the weights per citizen are decreasing in the group size. Second, I investigate how the magnitude of overweighting smaller groups varies across alternatives. I find that the overweighting of smaller groups is larger for more moderate compared to more extreme alternatives (Theorem 1.2).

Taking together these properties, the above mentioned insight that the vote on more extreme alternatives involves majority quotas further away from simple majority and, at the same time, relatively higher voting weights for large groups emerges.

The features of the optimal weights and quotas differ from what the previous literature has found: Barberà and Jackson (2006) find voting weights that are proportional to the square root of the group sizes as well as a simple majority quota. Maaser and Napel (2007), Maaser and Napel (2012), Maaser and Napel (2014), and Kurz, Maaser, and Napel (2017) assume simple majority quotas and also find square root weights.[5]

Finally, I discuss the implications of my theoretical findings for the design of voting mechanisms in real-world institutions while focusing on the European Union. Consider the weighted majority voting rule that was in place in the Council of the European Union until 2014 exhibiting approximately square root weights, and a majority quota of 74%. While the stated previous literature can rationalize the square root weights, it cannot explain the use of a qualified majority quota of 74%. In contrast, my model can rationalize such a majority quota. However, my quantitative analysis suggests that the use of it should be accompanied by higher voting weights for large member states or, in other words, less overweighting of small member states compared to the weights that were applied in the Council of the European Union.

The remainder of this chapter is organized as follows. The subsequent section 1.2 discusses the related literature. The model is introduced in section 1.3. Next, in section 1.4, I present the class of successive voting procedures. Then, in section 1.5, I characterize the welfare-maximizing mechanism for the collective decision-making process of the representatives. Based on this characterization, in sections 1.6 and 1.7, I study in detail the features of the utilitarian quotas and weights of nations. The following section 1.8 treats the extension to preferences that are correlated within groups. Subsequently, in section 1.9, I analyse the design of voting mechanisms for institutions of the European Union. The final section 1.10 concludes. The proofs are contained in Appendix 1.A.

1.2 Literature

This chapter builds upon and contributes to the following two strands of the literature: The literature on the apportionment problem as well as the literature on sequential voting and mechanism design.

5. The reasons for these differences are discussed in the literature review in section 1.2.

The literature on the apportionment problem starts with Penrose (1946) who argues in a two-alternatives setting, where every citizen prefers one over the other alternative with an independent probability of $\frac{1}{2}$, that the voting power of each group of citizens should be proportional to the square root of the group size. The argument relies on the observation that, in this case, the probability that a citizen is pivotal in a within-group simple majority election is asymptotically proportional to the inverse of the square root of the group size.[6]

More recently, Barberà and Jackson (2006) derive the utilitarian voting mechanism for the case of two alternatives, supposing that the key assumptions of the benchmark model described in the introduction hold.[7] In addition, assume that the citizens' preference intensities are symmetric across the two alternatives.[8] Barberà and Jackson (2006) exclusively study the case in which every citizen prefers one over the other alternative with probability $\frac{1}{2}$. Then, they find a simple majority quota, and, for large population sizes, square root weights.

I contribute to the literature on the apportionment problem with two alternatives in the following ways: First, I generalize Barberà and Jackson (2006) by characterizing optimal mechanisms for any probability with which any citizen prefers one over the other alternative. Second, my analysis reveals that the assumption in Barberà and Jackson (2006) that each citizen prefers one over the other alternative with probability $\frac{1}{2}$ is crucial for their findings. Allowing for probabilities with which any citizen prefers one over the other alternative that are different from $\frac{1}{2}$, I obtain that the optimal majority quotas do not coincide with simple majority and that the degree of overweighting of smaller groups is lower compared to what Barberà and Jackson (2006) find.

The apportionment problem with an interval of alternatives, single-peaked preferences, and representatives being selected according to the median mechanism has already received some attention in the literature. While employing different objective criteria, Maaser and Napel (2007), Maaser and Napel (2012), and Maaser and Napel (2014) provide numerical results how weights should be assigned to representatives or groups of citizens. Maaser and Napel (2007) aim at equalizing pivot probabilities across citizens from different groups, Maaser and Napel (2012) minimize the direct democracy deficit, and Maaser and Napel (2014) rely on the utilitarian principle. In addition, Kurz, Maaser, and Napel (2017) confirm the numerical

6. Chamberlain and Rothschild (1981) show that this pivot probability is asymptotically proportional to the inverse of the group size if there is uncertainty about the probability with which any citizen prefers one over the other alternative. Gelman, Katz, and Bafumi (2004) is a related empirical analysis.

7. Other contributions on the apportionment problem with two alternatives that also rely on the utilitarian principle include Beisbart and Bovens (2007), Fleurbaey (2008), and Macé and Treibich (2021). Also, Koriyama, Laslier, Macé, and Treibich (2013) study the apportionment problem in a setting featuring repeated binary decisions.

8. I impose a generalization of this restriction when characterizing optimal mechanisms.

findings in Maaser and Napel (2007) by providing analytical results when, again, the objective is to equalize pivot probabilities.[9] Apart from the aspect that these contributions assume that the set of alternatives constitutes an interval, whereas in my model there is a finite number of alternatives, the set of feasible mechanisms for the collective decision-making process of the representatives that these papers allow for is a strict subset of the set of sequential voting procedures I allow for. In the stated contributions, the mechanisms are not described in terms of successive voting procedures, but the mechanisms they consider can nevertheless be implemented by successive voting procedures. Then, starting from the set of successive voting procedures, these authors impose three additional constraints. First, they restrict attention to weighted successive voting procedures as defined in the introduction. Second, they assume that the weights and majority quotas are not sensitive to the alternative that is on the agenda. Third, they suppose that the majority requirements amount to simple majority. Taking this class of mechanisms, under the assumption that the preferences are distributed independently and identically across all citizens, Maaser and Napel (2007), Maaser and Napel (2012), Maaser and Napel (2014), and Kurz, Maaser, and Napel (2017) find that the resulting alternative-independent weights should be proportional to the square root of the population sizes.

I contribute to literature on the apportionment problem with more than two alternatives, and single-peaked preferences as follows: To the best of my knowledge, I provide the first analytical results for the apportionment problem in a setting featuring more than two alternatives, single-peaked preferences, and a utilitarian objective criterion. Moreover, in contrast to Maaser and Napel (2007), Maaser and Napel (2012), Maaser and Napel (2014), and Kurz, Maaser, and Napel (2017), I find that the optimal weights in my model do, generally, not give rise to a square root rule, and the welfare-maximizing quotas do not amount to simple majority.

The literature on sequential voting and mechanism design studies equilibrium voting behaviour under sequential voting procedures, and the utilitarian efficiency of voting rules. Early contributions include Farquharson (1969), and Rae (1969) respectively.

More recently, Kleiner and Moldovanu (2017) analyse sequential voting procedures when agents have single-peaked preferences while restricting attention to anonymous voting rules. They identify conditions on the voting procedures under which the induced dynamic games admit an ex-post perfect equilibrium in which the agents vote sincerely. The successive voting procedures that I consider are part of the class of voting rules identified in Kleiner and Moldovanu (2017), but I generalize them to allow for non-anonymous voting. While allowing for non-anonymous voting, Nehring and Puppe (2007) characterize dominant-strategy incentive-compatible, and surjective social choice functions on the full single-peaked domain in terms

9. Also, for this objective criterion, Kurz, Maaser, and Napel (2018) show how population sizes relate to the Shapley value of a specific cooperative game.

of voting by issues mechanisms. There is a one-to-one relationship between these static voting rules and the dynamic successive voting procedures. Therefore, the successive voting procedures represent dynamic versions of the mechanisms identified in Nehring and Puppe (2007).

Moreover, Gershkov, Moldovanu, and Shi (2017) characterize utilitarian voting mechanisms under direct democracy in a setting with more than two alternatives, single-peaked preferences, and agents that are ex-ante identical while imposing anonymity as an additional constraint. They show that, in this case, the optimal voting rule among all anonymous, unanimous, and dominant-strategy incentive-compatible mechanisms takes the form of an anonymous successive voting procedure with majority thresholds that are decreasing along the sequence of ballots. My characterization of welfare-maximizing mechanisms for the voting process of representatives extends the findings from Gershkov, Moldovanu, and Shi (2017) from the case of direct democracy, and ex-ante identical agents to the case of indirect democracy, and ex-ante heterogeneous agents. The proof for my characterization result builds upon the proof of Gershkov, Moldovanu, and Shi (2017)'s characterization of utilitarian mechanisms, but the fact that the collective decision-making is indirect and that voting might be non-anonymous requires different arguments as well as distinct assumptions.

Finally, on a technical note, I employ two main analytical tools in my analysis: A characterization of the binomial distribution in terms of truncated expectations due to Ahmed (1991) as well as a recurrence relation for the symmetric incomplete beta function due to Saunders (1992).

1.3 Model

There is a finite set of alternatives $M := \{1, ..., m\}$ with $m \geq 2$, and a finite set of countries $C := \{1, ..., c\}$ with $c \geq 2$. In each country $j \in C$, there is an odd number of citizens $n_j \geq 1$. I assume that not all countries have a population size of 1.[10]

Let me recall the well-known definition of a single-peaked preference relation. A strict preference ordering $\succ$ over M is single-peaked with respect to the ordering $1 < 2 < ... < m-1 < m$ if there exists some alternative $p \in M$ such that, for all distinct alternatives $k, k' \in M$ with $k \neq k'$,

$$[k' < k \leq p \vee p \leq k < k'] \Rightarrow k \succ k'.$$

The preference domain restriction of single-peakedness relies on the assumption that there is a unidimensional political spectrum that might range from left-wing to right-wing policies. Then, single-peakedness amounts to the following constraint: If an alternative k' is further away to the left or the right from the most preferred or peak

10. For simplicity, from now on, the framing is in terms of countries.

policy p than alternative k, policy k must be preferred to alternative k'. Let $\boldsymbol{P}_{SP}$ denote the set of all single-peaked preference relations. Citizens' preferences are assumed to be single-peaked with respect to the stated ordering $1 < 2 < ... < m-1 < m$, and they are supposed to be cardinal as utilitarianism is the objective criterion.

More specifically, citizens' have types that are governed by the random variable T. The distribution of T has full support on some non-empty set $S \neq \emptyset$. All subsequent expectations are taken with respect to this distribution. Types are distributed independently and identically across all citizens. If alternative $k \in M$ is implemented, the utility that a citizen with type realization $t \in S$ derives from this alternative is denoted by $u^k(t)$.

I collect three restrictions on the utility function and the type distribution. First, utilities are assumed to be bounded, meaning, there exists some bound $B \in \mathbb{R}$ such that, for almost all type realizations $t \in S$ and for every alternative $k \in M$, $|u^k(t)| < B$. Second, I exclude indifferences, that is, for almost all type realizations $t \in S$ and for every pair of distinct alternatives $k, k' \in M$ with $k \neq k'$, $u^k(t) \neq u^{k'}(t)$. Third, utilities have to be, of course, consistent with single-peakedness. Formally, this condition requires that, for almost all type realizations $t \in S$, there exists a single-peaked preference relation $\succ \in \boldsymbol{P}_{SP}$ such that, for every pair of distinct alternatives $k, k' \in M$ with $k \neq k'$,

$$k \succ k' \Leftrightarrow u^k(t) > u^{k'}(t).$$

Further, I impose a weak richness assumption on the preference domain requiring that every alternative is the most preferred alternative for some type. Formally, I assume that, for every alternative $k \in M$, there exists a set of types $Z \subset S$ arising with positive probability such that, for every element in this set $t \in Z$, it holds, for all alternatives $k' \in M$ with $k \neq k'$,

$$u^k(t) > u^{k'}(t).$$

Finally, let G describe the cdf of the citizens' most preferred or peak alternatives, meaning, for all $k \in M$, define

$$G(k) := Pr(\underset{k' \in M}{\operatorname{argmax}} u^{k'}(t) \leq k).$$

Note that the domain richness assumption implies that $G(k)$ is strictly increasing in k. In order to characterize the optimal mechanism, I impose further constraints on the preference distribution that I present along with the characterization result.

Now, let me describe the collective decision-making process. Again, every country has exactly one representative. I assume that the representative of country $j \in C$ is a median citizen in this country, that is, his or her most preferred alternative coincides with the median of all most preferred alternatives in country j. In other words, I impose that the representatives' most preferred alternatives are consistent with the

Condorcet winners in the countries. The representatives play an equilibrium that is induced by a sequential voting procedure. Each representative's equilibrium strategy conditions only on his or her peak alternative, but it does not depend on further preference information. Therefore, it is sufficient to specify how the most preferred alternatives of the representatives are determined. I introduce the set of feasible sequential voting procedures, and the equilibrium in the following section. Finally, the mechanism designer maximizes the citizens' ex-ante utilitarian welfare over these sequential voting procedures, taking into account that representatives are median citizens.

1.4 Sequential Voting

In this section, I present the sequential voting procedures over which the utilitarian designer optimizes. I focus on the class of successive voting procedures that has been studied previously by Gershkov, Moldovanu, and Shi (2017), and Kleiner and Moldovanu (2017) who restrict attention to anonymous procedures. Again, static versions of these voting procedures are contained in Nehring and Puppe (2007) who allow for non-anonymous voting.

For simplicity, identify the representative of country $j \in C$ directly with the label of this country. To describe the class of successive voting procedures, I rely on simple games studied in Taylor and Zwicker (1999). Specifically, following Nehring and Puppe (2007), define a family of winning coalitions $\mathcal{W}$ to be a non-empty collection of non-empty subsets of the set of countries or representatives C that is closed under taking supersets, meaning,

$$[W \in \mathcal{W} \wedge W \subseteq W'] \Rightarrow W' \in \mathcal{W}.$$

Every successive voting procedure is characterized by $m-1$ families of winning coalitions, and each of these families of winning coalition is associated with a unique alternative from the set $M \setminus \{m\}$. More formally, for each $1 \leq k < m$, there is a family of winning coalitions $\mathcal{W}_k$. Moreover, assume that these families of winning coalitions are ordered by set inclusion, that is, suppose that, for all $1 \leq k < m-1$,

$$\mathcal{W}_k \subseteq \mathcal{W}_{k+1}.$$

This set inclusion restriction has the interpretation that it is more difficult to collectively approve alternatives that are put earlier on the agenda.

The basic idea behind the successive voting procedure is to perform a sequence of binary votes and to go gradually from left-wing to right-wing alternatives. At each stage of the dynamic procedure, representatives vote simultaneously, and they can either approve or reject the alternative that is currently on the agenda. In other words, the available actions are "Yes" and "No". The successive voting procedure with families of winning coalitions $\{\mathcal{W}_k\}_{k=1}^{m-1}$ works as follows:

(1) To begin with, the leftmost alternative 1 is put on the agenda, and representatives can either approve or reject alternative 1. If the set of representatives who play “Yes” coincides with some element of the family of winning coalitions $\mathscr{W}_1$, alternative 1 is implemented. Otherwise, alternative 2 is considered.

(2) Representatives either vote in favor or against alternative 2. If the set of representatives voting in favor of alternative 2 coincides with some element of the family of winning coalitions $\mathscr{W}_2$, alternative 2 is selected. Otherwise, continue the voting process.

(3) Consider alternative 3 and possibly more alternatives to the right of 3, and treat them in the same way as alternatives 1 and 2. Eventually, either some alternative $1 \leq k < m$ is selected or if the set of agents approving alternative $m-1$ does not coincide with some element of $\mathscr{W}_{m-1}$, implement alternative m.

For the case of three alternatives, i.e., $m = 3$, Figure 1.2 illustrates the successive voting procedure with families of winning coalitions $\mathscr{W}_1$ and $\mathscr{W}_2$ satisfying $\mathscr{W}_1 \subseteq \mathscr{W}_2$ by means of a tree. The three alternatives might be interpreted as follows: Alternative 3 constitutes the status quo policy. In the context of the United States Congress, alternatives 1 and 2 might be a constitutional amendment and a standard bill respectively. With regard to the Council of the European Union, alternative 2 might represent a policy proposal from the European Commission or the High Representative of the Union for Foreign Affairs and Security Policy, whereas alternative 1 relates to the same political matter, but it is proposed by someone else.

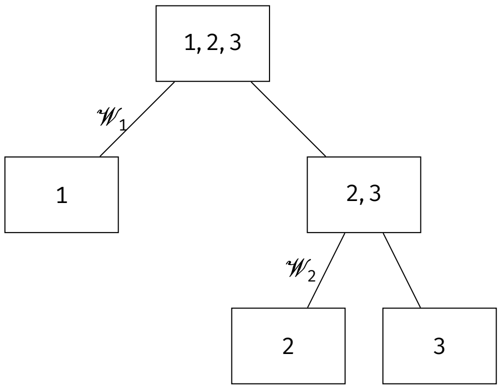

Figure 1.2. Successive Voting Procedure

Having described the class of successive voting procedures, I study equilibrium behavior. Following Gershkov, Moldovanu, and Shi (2017), call a strategy sincere if some representative plays action “Yes” if and only if the alternative on the agenda lies weakly to the right of this representative’s most preferred alternative. The solution concept is ex-post perfect equilibrium. Following, again, Gershkov, Moldovanu, and Shi (2017), this equilibrium concept can be defined in words as follows: For every profile of type realizations and at each stage of the dynamic procedure, the continuation strategies constitute a Nash equilibrium of the subgame where the profile of type realizations is common knowledge.[11] Similar to the anonymous case studied in Gershkov, Moldovanu, and Shi (2017), and Kleiner and Moldovanu (2017), it turns out that sincere voting constitutes an ex-post perfect equilibrium.

Proposition 1.1. Sincere voting constitutes an ex-post perfect equilibrium in the game induced by any successive voting procedure.

From now on, I suppose that the representatives play the sincere voting equilibrium, and I assume that the utilitarian designer optimizes over the class of successive voting procedures that I just presented.
Before turning to welfare maximization, I introduce a subset of the class of successive voting procedures that I call weighted successive voting procedures. I emphasize that I optimize over all successive voting procedures, but it turns out that the welfare-maximizing mechanism can be implemented via a weighted successive voting procedure. A successive voting procedure with collections of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{m-1}$ constitutes a weighted successive voting procedure if there exist weights $w_j(k) \in \mathbb{R}_{>0}$ and quotas $q(k) \in \mathbb{R}_{>0}$ with $1 \leq k < m$ and $j \in C$ such that, for all $1 \leq k < m$ and every set of representatives $D \subseteq C$, it holds

$$D \in \mathscr{W}_k \Leftrightarrow \sum_{j \in D} w_j(k) \geq q(k).$$

In words, for each alternative $1 \leq k < m$, there is a majority quota $q(k)$ as well as a vector of voting weights $[w_1(k), ..., w_c(k)]$. In particular, weights and quotas might be sensitive to the alternative that is on the agenda. Then, when it comes to the binary vote on alternative $1 \leq k < m$, this alternative is approved if and only if the sum of weights $w_j(k)$ with $j \in C$ associated with representatives who vote "Yes" at alternative k exceeds the majority requirement $q(k)$.

1.5 Welfare Maximization

In this section, I optimize over the successive voting procedures, and characterize the welfare-maximizing mechanism for the collective decision-making process involving the representatives, taking into account that each representative is a median citizen in the respective country. The optimization problem amounts to finding the families of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{m-1}$ with $\mathscr{W}_k \subseteq \mathscr{W}_{k+1}$ for all $1 \leq k < m-1$ that maximize the citizens' utilitarian welfare. Since utilities are bounded by assumption, a bounded function is maximized over a finite set of elements. Thus, the existence of a solution is guaranteed.
Recall that the utilitarian designer has to make inferences about the citizens' preferences based on the vote choices of the representatives. I introduce two objects that

11. For a formal definition, I refer to Kleiner and Moldovanu (2017).

reflect this inference, and that determine the basic trade-off the designer is facing. On the one hand, for any alternative $1 \leq k < m$, define

$$\mu^N_{n_j}(k) \ := \ \mathbb{E}[u^{k+1}(T) - u^k(T) | \text{j votes “No” at } k].$$

Intuitively, $\mu^N_{n_j}(k)$ describes how much an arbitrary citizen from country j cares about continuing the voting process of the representatives one more step and implementing alternative $k+1$ versus stopping the voting process right now and implementing the current alternative k, conditional on the event that the representative of country j votes “No” at alternative k. Now, observe that j votes “No” at k if and only if there are at least $\frac{n_j+1}{2}$ citizens from country j having peaks weakly to the right of $k+1$. The object $\mu^N_{n_j}(k)$ can be expressed as

$$\begin{aligned}\mu^N_{n_j}(k) \ =& \mathbb{E}[u^{k+1}(T) - u^k(T) | \text{j votes “No” at } k] \\ =& w^N_{n_j}(k)\mathbb{E}[u^{k+1} - u^k | u^{k+1} \ > \ u^k] + [1 - w^N_{n_j}(k)][-1]\mathbb{E}[u^k - u^{k+1} | u^k \ > \ u^{k+1}].\end{aligned}$$

The term $w^N_{n_j}(k)$ appearing in the expression is defined as

$$w^N_{n_j}(k) \ := \ \sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{r(n_j, k, s)}{1 - R(n_j, k, \frac{n_j-1}{2})} \frac{s}{n_j},$$

where $r(n_j, k, \cdot)$ and $R(n_j, k, \cdot)$ denote the pmf and the cdf of the binomial distribution with parameters n_j and $1 - G(k)$.[12] In words, the term $w^N_{n_j}(k)$ describes the expected share of citizens from country j who would want their representative to vote “No” at alternative k, i.e., who have peaks weakly to the right of $k+1$, conditional on the representative of country j voting “No” at alternative k. The derivation of the expression for $\mu^N_{n_j}(k)$ is contained in Appendix 1.B.

On the other hand, for any $1 \leq k < m$, define

$$\mu^Y_{n_j}(k) \ := \ \mathbb{E}[u^k(T) - u^{k+1}(T) | \text{j votes “Yes” at } k].$$

In intuitive terms, $\mu^Y_{n_j}(k)$ captures how much an arbitrary citizen from country j cares about stopping the voting process of the representatives right now and implementing alternative k versus continuing the voting process for one more step and implementing alternative $k+1$, conditional on the event that the representative of country j votes “Yes” at k. Note that j votes “Yes” at k if and only if there are at least $\frac{n_j+1}{2}$ citizens from country j having peaks weakly to the left of k. Now, define the expected share

$$w^Y_{n_j}(k) \ := \ \sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{l(n_j, k, s)}{1 - L(n_j, k, \frac{n_j-1}{2})} \frac{s}{n_j},$$

12. Recall that G denotes the cdf of the citizens’ peaks.

where $l(n_j,k,\cdot)$ and $L(n_j,k,\cdot)$ denote the pmf and the cdf of the binomial distribution with parameters n_j and $G(k)$. The term $w^Y_{n_j}(k)$ represents the expected share of citizens from country j who would want their representative to vote “Yes” at alternative k, i.e., who have peaks weakly to the left of alternative k, conditional on the representative of country j voting “Yes” at this alternative. Then, the object $\mu^Y_{n_j}(k)$ can be written as

$$\begin{aligned}&\mu^Y_{n_j}(k)\\&= w^Y_{n_j}(k)\mathbb{E}[u^k - u^{k+1}|u^k > u^{k+1}] + [1 - w^Y_{n_j}(k)][-1]\mathbb{E}[u^{k+1} - u^k|u^{k+1} > u^k].\end{aligned}$$ [13]

It turns out that the optimal mechanism is determined through a comparison of the two objects $\mu^N_{n_j}(k)$ and $\mu^Y_{n_j}(k)$. These objects might be interpreted as the citizens' inferred preference intensities that are induced by indirect democracy.
Theorem 1.1 characterizes the welfare-maximizing mechanism. It reveals that the optimal mechanism can be implemented via a weighted successive voting procedure. Without loss of generality, I normalize the weights such that, for all $1 \le k < m$, it holds $\sum_{j\in C} w_j(k) = c$. The proof of Theorem 1.1 makes use of several lemmata that are contained in the appendix.

Theorem 1.1. *Suppose that, for all $1 \le k' < m$,*

$$\mathbb{E}[u^{k'} - u^{k'+1}|u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'}|u^{k'+1} > u^{k'}].$$

The weighted successive voting procedure with weights

$$w_j(k) = \frac{n_j \cdot [\mu^Y_{n_j}(k) + \mu^N_{n_j}(k)]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [\mu^Y_{n_j}(k) + \mu^N_{n_j}(k)]} = \frac{n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]}$$

and quotas

$$q(k) = \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j \cdot \mu^Y_{n_j}(k)}{\frac{1}{c}\sum_{j\in C} n_j \cdot \mu^N_{n_j}(k)}} = \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j \cdot [2w^Y_{n_j}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [2w^N_{n_j}(k) - 1]}}$$

with $j \in C$ and $1 \le k < m$ implements the optimal mechanism among all successive voting procedures.

Let me discuss the assumption on the preference distribution. To begin with, the restriction on the preference distribution, meaning, for all $1 \le k' < m$,

$$\mathbb{E}[u^{k'} - u^{k'+1}|u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'}|u^{k'+1} > u^{k'}]$$

13. The derivation of this expression is omitted because it is analogous to the derivation of the term for $\mu^N_{n_j}(k)$.

imposed in Theorem 1.1 represents a symmetry condition on the citizens' preference intensities within all pairs of alternatives that are neighbors according to the ordering $1 < 2 < ... < m-1 < m$ from which single-peakedness is derived. In that sense, it extends the assumption in Barberà and Jackson (2006) from the two-alternatives case to scenarios with more than two alternatives. In general, the assumption constitutes a joint restriction on the utility function and the type distribution. Let me outline a simple example of a preference distribution that satisfies the discussed assumption. Suppose that each citizen's type is governed by $T := P \times \Theta_L \times \Theta_R$, where P captures the citizen's peak, and Θ_L as well as Θ_R determine how much the utility decreases when deviating from the peak to the left and to the right respectively. The support of the distribution of T is given by $S := \{p_1, ..., p_m\} \times [0, \bar{\theta}]^2$ with $p_1, ..., p_m \in \mathbb{R}$, $p_1 < p_2 < ... < p_{m-1} < p_m$ and $\bar{\theta} > 0$. Also, assume that the random variable P is independent of the random variable $\Theta_L \times \Theta_R$, and impose that $\mathbb{E}[\Theta_L] = \mathbb{E}[\Theta_R]$. Now, if a citizen's type realization is $t \in S$ and alternative $1 \leq k \leq m$ is implemented, this citizen derives the following utility:

$$u^k(t) \ := \ u^k(p, \theta_L, \theta_R) \ = \begin{cases} -\theta_R(p_k - p), & p_k \ \geq \ p \\ -\theta_L(p - p_k), & p_k \ < \ p \end{cases} .$$

In words, the policy alternatives are spatially located in a one-dimensional policy spectrum, the utility a citizen derives from an alternative is shaped by the absolute difference between the locations of this citizen's peak and the implemented alternative, and the slope with which this citizen's utility decreases when moving away from the peak is specific to the direction of the deviation from the peak. Finally, observe that this preference distribution satisfies the assumption of Theorem 1.1.

Further, in general, as emphasized by the notation, the optimal voting weights and majority quotas are sensitive to the alternative on the agenda through the cdf of the citizens' peaks G. More specifically, the optimal weights and quotas associated with alternative $k \in M$ depend on $G(k)$, which is, again, the probability that an arbitrary citizen's most preferred alternative lies weakly to the left of k. However, the utilitarian weights and quotas do not depend on the precise value of the preference intensities $\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}]$ with $1 \leq k' < m$.

To interpret the optimal weights and quotas, consider some alternative $1 \leq k < m$, and let $X^k_{n_j} \sim Binomial(n_j, 1 - G(k))$ as well as $Y^k_{n_j} \sim Binomial(n_j, G(k))$. The random variable $X^k_{n_j}$ describes the number of citizens in country j whose most preferred alternatives lie weakly to the right of $k+1$. Similarly, the random variable $Y^k_{n_j}$ captures the number of citizens in country j with peak alternatives weakly to the left of k. Rewriting the expression appearing in Theorem 1.1 as done in the proof of Theorem 1.2 reveals that the optimal weights $w_j(k)$ are proportional to

$$\mathbb{E}[X^k_{n_j} | X^k_{n_j} \ \geq \ \frac{n_j + 1}{2}] - \mathbb{E}[X^k_{n_j}] + \mathbb{E}[Y^k_{n_j} | Y^k_{n_j} \ \geq \ \frac{n_j + 1}{2}] - \mathbb{E}[Y^k_{n_j}].$$

This expression shows how the inference problem that the designer faces is resolved. The object $\mathbb{E}[X^k_{n_j}|X^k_{n_j} \geq \frac{n_j+1}{2}]$ captures the expected number of citizens from country j who would want their representative to vote "No" at k, i.e., who have peaks weakly to the right of $k+1$, conditional on j actually voting "No" at k. The object $\mathbb{E}[Y^k_{n_j}|Y^k_{n_j} \geq \frac{n_j+1}{2}]$ represents the expected number of citizens of country j who want the representative of country j to vote "Yes", i.e, who have peaks weakly to the left of k, conditional on j voting "Yes". Consequently, intuitively, optimal weights reflect how much the designer learns about the citizens' preferences from the representatives' voting decisions relative to the prior expectations $\mathbb{E}[X^k_{n_j}]$ and $\mathbb{E}[Y^k_{n_j}]$.
Moreover, using the introduced notation, the optimal quota $q(k)$ can be expressed as

$$q(k) = \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C}\mathbb{E}[Y^k_{n_j}|Y^k_{n_j} \geq \frac{n_j+1}{2}] - \frac{1}{2}\cdot n_j}{\frac{1}{c}\sum_{j\in C}\mathbb{E}[X^k_{n_j}|X^k_{n_j} \geq \frac{n_j+1}{2}] - \frac{1}{2}\cdot n_j}}.$$

Before analysing the features of the optimal weights and quotas for the case of finite population sizes comprehensively, to develop some intuition, it is instructive to briefly consider the scenario of large population sizes here. Suppose that the population sizes n_j with $j \in C$ are large. I perform a case distinction depending on the value of $G(k)$. First, I consider the case in which $G(k) < \frac{1}{2}$. Here, if n_j is large, by the law of large numbers, the optimal weight $w_j(k)$ is approximately proportional to the term

$$\begin{aligned}
&\mathbb{E}[X^k_{n_j}|X^k_{n_j} \geq \frac{n_j+1}{2}] - \mathbb{E}[X^k_{n_j}] + \mathbb{E}[Y^k_{n_j}|Y^k_{n_j} \geq \frac{n_j+1}{2}] - \mathbb{E}[Y^k_{n_j}] \\
&\approx \mathbb{E}[X^k_{n_j}] - \mathbb{E}[X^k_{n_j}] + \frac{n_j+1}{2} - G(k)\cdot n_j \\
&= \frac{1}{2} + [\frac{1}{2} - G(k)]\cdot n_j,
\end{aligned}$$

which means that the weights are effectively linear in population sizes. Intuitively, the designer expects the median peak in country j to lie to the right of alternative k or, in other words, the designer expects that there are more than $\frac{n_j+1}{2}$ citizens in country j with peaks strictly to the right of alternative k. Therefore, he or she essentially learns nothing about the citizens' preferences from a representative voting "No" at k. Further, the designer expects that there are $G(k)\cdot n_j$ citizens in country j with peaks weakly to the left of alternative k. Consequently, a "Yes" vote of a representative at k reveals that there are $\frac{n_j+1}{2} - G(k)\cdot n_j$ more citizens in country j with peaks weakly to the left of k than expected. Moreover, again, by the law of large numbers, if $G(k) < \frac{1}{2}$ and population sizes are large, the optimal quota $q(k)$ approximately equals

$$
\begin{aligned}
q(k) &= \frac{c}{1+\frac{\frac{1}{c}\sum_{j\in C}\mathbb{E}[Y^k_{n_j}|Y^k_{n_j}\geq\frac{n_j+1}{2}]-\frac{1}{2}n_j}{\frac{1}{c}\sum_{j\in C}\mathbb{E}[X^k_{n_j}|X^k_{n_j}\geq\frac{n_j+1}{2}]-\frac{1}{2}n_j}}\\
&\approx \frac{c}{1+\frac{\frac{1}{c}\sum_{j\in C}\frac{n_j+1}{2}-\frac{1}{2}\cdot n_j}{\frac{1}{c}\sum_{j\in C}\mathbb{E}[X^k_{n_j}]-\frac{1}{2}\cdot n_j}}\\
&= \frac{c}{1+\frac{\frac{1}{c}\sum_{j\in C}\frac{1}{2}}{\frac{1}{c}\sum_{j\in C}[\frac{1}{2}-G(k)]\cdot n_j}}\\
&\approx c,
\end{aligned}
$$

implying that almost unanimous consent is required in order to implement alternative k.

Second, if $G(k) > \frac{1}{2}$, and population sizes are large, an analogous reasoning implies that the optimal weights $w_j(k)$ are again effectively linear in population sizes, and that the optimal quota approximately satisfies $q(k) \approx 0$. The latter aspect means that almost unanimous consent is needed in order to reject alternative k.

Third, if $G(k) = \frac{1}{2}$, the expression for the optimal quota implies that $q(k) = \frac{1}{2}c$ for all population sizes. In other words, the optimal quota coincides with the simple majority threshold. Furthermore, while employing the characterization result from Ahmed (1991), the proof of Theorem 1.2 reveals that the weight assigned to country j is proportional to

$$\frac{G(k)\frac{n_j+1}{2}Pr(X^k_{n_j} = \frac{n_j+1}{2})}{Pr(X^k_{n_j} \leq \frac{n_j-1}{2})Pr(X^k_{n_j} \geq \frac{n_j+1}{2})}.$$

Now, because of $G(k) = \frac{1}{2}$, this expression simplifies to

$$n_j \cdot Pr(X^k_{n_j-1} = \frac{n_j-1}{2}).$$

Observe that the term $Pr(X^k_{n_j-1} = \frac{n_j-1}{2})$ represents the probability that a citizen is pivotal in a simple majority election within country j involving two alternatives, where each of them is preferred by any citizen in country j with an independent probability of $\frac{1}{2}$. For large population sizes, by the De Moivre-Laplace theorem, this probability is approximately proportional to the inverse of the square root of the population size. Therefore, the optimal weights are effectively proportional to the square root of the population sizes. This is Penrose (1946)'s early insight, and his square root law is recovered in the special case in which $G(k) = \frac{1}{2}$. The case of $G(k) = \frac{1}{2}$ corresponds also to the scenario studied in Barberà and Jackson (2006).

Overall, this discussion of the scenario in which population sizes are large can be summarized as follows: Say that an alternative $1 \leq k < m$ is maximally moderate if $G(k) = \frac{1}{2}$. Then, unless the alternative is maximally moderate, the corresponding optimal weights are effectively linear in population sizes, and the optimal quotas are approximately equal to unanimity or anti-unanimity. In contrast, if the alternative is maximally moderate, the optimal weights are effectively proportional to the square root of the population sizes, and the optimal quota equals simple majority.

In the following sections 1.6 and 1.7, I study in detail the properties of the optimal quotas and weights while concentrating on finite population sizes. Again, the justification for this focus is that the model incorporating correlated preferences essentially corresponds to the model featuring independent preferences as well as finite population sizes. If population sizes are finite, it turns out that the optimal weights and quotas have similar features as in the case of large population sizes, but, informally speaking, the properties are in some ways less extreme or, in other words, the features for the case of large population sizes are in a sense smoothed out.

1.6 Majority Quotas

In this section, I discuss the main properties of the utilitarian majority quotas. To begin with, the optimal quotas are strictly decreasing in the alternative k, that is, they are strictly decreasing along the sequence of ballots. More precisely, they strictly decrease in the value of $G(k)$. Depending on the value of $G(k)$, the welfare-maximizing quotas represent supermajority, simple majority or submajority requirements:

$$\begin{aligned} G(k) &< \frac{1}{2} \Rightarrow q(k) > c \cdot \frac{1}{2}, \\ G(k) &= \frac{1}{2} \Rightarrow q(k) = c \cdot \frac{1}{2}, \text{ and} \\ G(k) &> \frac{1}{2} \Rightarrow q(k) < c \cdot \frac{1}{2}. \end{aligned}$$

These features follow from the fact that, for all $j \in C$, $w^Y_{n_j}(k)$ is increasing in k whereas $w^N_{n_j}(k)$ is decreasing in k[14] together with the aspect that $G(k) = \frac{1}{2}$ if and only if $w^Y_{n_j}(k) = w^N_{n_j}(k)$ with $j \in C$.

Except for the knife-edge case in which $G(k) = \frac{1}{2}$,[15] the welfare-maximizing majority requirements do not coincide with simple majority. This stands in contrast to the assumption of simple majority quotas in Maaser and Napel (2007), Maaser and Napel (2012), Maaser and Napel (2014), and Kurz, Maaser, and Napel (2017), but also the simple majority finding of Barberà and Jackson (2006) for the case of two alternatives.

Again, define an alternative k' to be more moderate or less extreme than another

14. In the proofs of Lemma 1.2, and Lemma 1.4 appearing in Appendix 1.A, I argue that these two aspects hold.

15. Again, Barberà and Jackson (2006) analyse only this case.

alternative k'' if $|G(k') - \frac{1}{2}| < |G(k'') - \frac{1}{2}|$. The vote on more extreme alternatives requires majority quotas that are further away from simple majority. In other words, a larger value of $|G(k) - \frac{1}{2}|$ implies a majority quota that is more distant from simple majority. This property follows from the discussion above together with the following observation: Take two distinct alternatives k' and k'' such that $G(k') = 1 - G(k'')$. Then, it holds that $q(k') = c - q(k'')$.

In order to better understand the shape of the welfare-maximizing quotas, let me begin by emphasizing that the property of strictly decreasing quotas does not arise because population sizes are heterogeneous: It is also present if all countries have the same population size.[16] In contrast, the feature is driven by the aspect that the collective decision-making is indirect. To see this, compare the optimal mechanism under indirect versus direct democracy. Direct democracy corresponds to the scenario ruled out in the model, where all population sizes are equal to one, i.e., $n_j = 1$ for all $j \in C$. Assume that c is odd such that, in the direct democracy case, the total number of citizens is odd. Under the symmetry restriction on the preference intensities from Theorem 1.1, if democracy is direct, the optimal mechanism assigns an alternative-independent weight of 1 to all citizens.[17] More importantly, the optimal mechanism under direct democracy features an alternative-independent and, thus, constant, quota amounting to simple majority.[18] Therefore, the finding that the vote on more extreme alternatives requires majority quotas that are further away from simple majority is not driven by asymmetries in the citizens' preference intensities since I precisely assume in Theorem 1.1 that these intensities are not asymmetric, but it arises because of indirect democracy. In particular, this feature arises here for a different reason than in Gershkov, Moldovanu, and Shi (2017) who obtain a similar finding. In Gershkov, Moldovanu, and Shi (2017), the result is driven by the fact that they assume that the citizens' preference intensities are asymmetric.

For an intuition behind the result that alternatives that are put to a vote earlier require higher majority quotas under indirect democracy, compare two arbitrary alternatives $k', k'' \in M$ with $k' < k''$. Then, conditional on the representative of country $j \in C$ voting "Yes", the expected number of citizens of country j who would want the representative to vote "Yes", i.e., who have peaks weakly to the left of the respective alternative, is smaller at alternative k' compared to k''. In addition, conditional on j voting "No", the expected number of citizens from country j who would want the representative to vote "No", i.e., who have peaks strictly to the right of the respective alternative, is larger at alternative k' than at k''. It follows that $\mu^Y_{n_j}(k') < \mu^Y_{n_j}(k'')$ as well as $\mu^N_{n_j}(k') > \mu^N_{n_j}(k'')$. Therefore, the asymmetries in these induced preference

16. In this case, the utilitarian mechanism assigns an alternative-independent weight of 1 to all countries.

17. Since citizens are ex-ante identical, there is no reason to discriminate among citizens.

18. Theorem 1.1 also applies to the direct democracy case, and the stated finding follows from it when setting all population sizes equal to one.

intensities imply that the designer imposes a higher majority requirement at k' compared to k''.

1.7 Weights of Nations

In this section, I study in detail how the utilitarian weights of nations vary with the population sizes, and how the relationship between population sizes and weights interacts with the alternative on the agenda.
To begin with, I analyse whether and how much smaller countries are overweighted relative to the linear benchmark. Proposition 1.2 reveals that, whatever the alternative, the associated optimal weights feature a degree of overweighting of smaller countries that lies between the linear benchmark and a power law with exponent $1 - \frac{\ln 2}{\ln 3} \approx 0.37$.

Proposition 1.2. Suppose that, for all $1 \le k' < m$,

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

For all $0 \le \alpha < 1 - \frac{\ln 2}{\ln 3}$ as well as any $1 \le k < m$ and $j', j'' \in C$ such that $n_{j'} < n_{j''}$, the optimal weights satisfy

$$\frac{w_{j'}(k)}{n_{j'}^{\alpha}} < \frac{w_{j''}(k)}{n_{j''}^{\alpha}}, \text{ and } \frac{w_{j'}(k)}{n_{j'}} > \frac{w_{j''}(k)}{n_{j''}}.$$

The first part of Proposition 1.2 can be interpreted as an upper bound on the magnitude of overweighting of smaller countries in the following sense: Whatever the alternative, even when dividing the associated optimal weights by the population size to the power of some $0 \le \alpha < 1 - \frac{\ln 2}{\ln 3} \approx 0.37$, the resulting ratio is nevertheless strictly increasing in the population size. In particular, when setting $\alpha = 0$, this finding shows that, for all alternatives, the corresponding optimal weights are strictly increasing in the population size. Moreover, the proof of Proposition 1.2 reveals that the discussed upper bound on the degree of overweighting is tight in the following sense: Suppose that $G(k) = \frac{1}{2}$, $n_{j'} = 1$, $n_{j''} = 3$, and $\alpha = 1 - \frac{\ln 2}{\ln 3}$. Then, it holds that $\frac{w_{j'}(k)}{n_{j'}^{\alpha}} = \frac{w_{j''}(k)}{n_{j''}^{\alpha}}$.
The second part of Proposition 1.2 means that, whatever the alternative, the associated optimal weights feature an overweighting of smaller countries in the sense that the optimal weights per citizen are strictly decreasing in the population size. This finding also represents a lower bound on the degree of overweighting of smaller countries. Taking both aspects together, I conclude that, for all alternatives, the corresponding optimal weights feature a magnitude of overweighting of smaller countries that falls between the linear benchmark and a power law with exponent $1 - \frac{\ln 2}{\ln 3} \approx 0.37$.

For the special case of $\alpha = 0$, Proposition 1.2 implies that the optimal weights related to any alternative exhibit degressive proportionality, that is, for any alternative, the associated weights are strictly increasing, but the weights per citizen are strictly decreasing in the population size. Let me discuss this implication of Proposition 1.2 more in detail. It might be argued that the property that the weights are increasing in the population size represents a basic requirement any reasonable model should predict. However, there is a trade-off, implying that this result is not trivial. The proof of Theorem 1.2 reveals that the optimal weight of some country with population size n related to alternative $1 \leq k < m$ is proportional to

$$\mathbb{E}[X_n^k | X_n^k \geq \frac{n+1}{2}] - \mathbb{E}[X_n^k | X_n^k \leq \frac{n-1}{2}],$$

where, again, $X_n^k \sim Binomial(n, 1-G(k))$. Now, both objects $\mathbb{E}[X_n^k|X_n^k \geq \frac{n+1}{2}]$ and $\mathbb{E}[X_n^k|X_n^k \leq \frac{n-1}{2}]$ are increasing in n for two reasons. First, note that the distribution of X_n^k varies with n. Increasing n yields a stochastic increase according to the likelihood ratio order which, in turn, implies a stochastic increase according to the hazard rate as well as reversed hazard rate order (see e.g. Shaked and Shanthikumar (2007)). Therefore, both truncated expectations increase as n rises even for fixed truncation points. Now, truncation points also become larger as n increases which reinforces the rise of these two objects. Consequently, since both involved objects are increasing in n, there is a trade-off, and, a priori, it is ambiguous how the difference of these objects evolves as a function of n. Proposition 1.2 is established by applying Theorem 1.2 as well as invoking additional claims that are shown as part of the proof of Theorem 1.2. In the proof of the latter theorem, I employ the following two analytical tools: I make use of a relationship between the lower truncated expectation and the hazard rate for binomial random variables due to Ahmed (1991). In addition, I apply a recurrence relation for the symmetric incomplete beta function due to Saunders (1992).

Intuitively, larger population sizes have two effects pushing in opposite directions. On the one hand, higher population sizes imply a greater importance in the utilitarian welfare function. On the other hand, the informativeness of the representatives' vote choices declines in relative terms in the sense that the share of citizens of a country, who would want to vote in the same way as their representative does, decreases as the population size increases. Formally, for every alternative $1 \leq k < m$ and for any two countries $j', j'' \in C$ with $n_{j'} < n_{j''}$, it holds $w^Y_{n'_j}(k) > w^Y_{n''_j}(k)$ as well as $w^N_{n'_j}(k) > w^N_{n''_j}(k)$. In fact, this finding is a technical result contained in Albrecht, Anderson, and Vroman (2010) who study committee search. Therefore, the second part of Proposition 1.2 can also be derived from Albrecht, Anderson, and Vroman (2010).[19]

19. The reason that allows me to invoke this finding from Albrecht, Anderson, and Vroman (2010) is the observation that the inferred preference intensities $\mu^Y_{n_j}(k)$ and $\mu^N_{n_j}(k)$ have a similar structure

To illustrate the degressive proportionality result implied by Proposition 1.2 more concretely, consider an extreme situation in which there are a several small countries, say 9 countries $\{1, ..., 9\}$ with population size 1, and a single larger country 10 with population size 9. In this case, for all $1 \leq k < m$, up to a constant, the optimal weights of the small countries are given by $w_j(k) = 1$ with $j \in \{1, ..., 9\}$ because

$$\mathbb{E}[X_1^k | X_1^k \geq 1] - \mathbb{E}[X_1^k] + \mathbb{E}[Y_1^k | Y_1^k \geq 1] - \mathbb{E}[Y_1^k] = 1.$$

Here, the only citizen in the respective country is also the representative. Therefore, conditional on a “Yes” or “No” vote of the representative, all citizens in such a country want to vote in the same way as the respective representative does. In contrast, for all $1 \leq k < m$, up to the same constant, the optimal weights of the large country 10 satisfy $1 < w_{10}(k) < 9$ because

$$\begin{aligned}
1 &= 5 + 5 - 9 \\
&< \mathbb{E}[X_9^k | X_9^k \geq 5] + \mathbb{E}[Y_9^k | Y_9^k \geq 5] - 9 \\
&= \mathbb{E}[X_9^k | X_9^k \geq 5] - \mathbb{E}[X_9^k] + \mathbb{E}[Y_9^k | Y_9^k \geq 5] - \mathbb{E}[Y_9^k] \\
&= \mathbb{E}[X_9^k | X_9^k \geq 5] + \mathbb{E}[Y_9^k | Y_9^k \geq 5] - 9 \\
&< 9 + 9 - 9 = 9.
\end{aligned}$$

In words, the designer learns that the sum of the expected numbers of citizens in the large country, who would want to vote in the same way as the representative does, conditional on a “Yes” or “No” vote of the representative, exceeds the sum of the priors by more than 1. Therefore, the optimal mechanism assigns a higher weight to the large country compared to the small countries. Further, conditional on a “Yes” or “No” vote of the representative, with positive probability, there are citizens in the large country, who would want to vote in the opposite way as the representative does. To put this differently, the expected number of citizens in the large country, who would want to vote in the same way as the representative does, is strictly smaller than the population size, that is,

$$\mathbb{E}[X_9^k | X_9^k \geq 5] < 9 \text{ and } \mathbb{E}[Y_9^k | Y_9^k \geq 5] < 9.$$

Consequently, the weights per citizen of the small countries that coincide with the weights of these countries are larger than the respective weights per citizen of the large country. In other words, the small countries are overweighted, and the large country is underweighted relative to the linear benchmark.

Having established bounds on the degree of overweighting of smaller countries that are valid for all alternatives, I study next how the degree of overweighting smaller

as the expected values conditional on stopping in the committee search model of Albrecht, Anderson, and Vroman (2010).

countries interacts with the alternative that is on the agenda or, in other words how it varies along the sequence of ballots. Theorem 1.2 addresses this point. Recall that G denotes the cdf of the citizens' peaks.

Theorem 1.2. *Suppose that, for all $1 \leq k' < m$,*

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

Consider any $j', j'' \in C$ such that $n_{j'} < n_{j''}$ as well as any $1 \leq k', k'' < m$ with $k' \neq k''$ such that $|G(k') - \frac{1}{2}| < |G(k'') - \frac{1}{2}|$. The ratio of optimal weights satisfies

$$\frac{w_{j'}(k')}{w_{j''}(k')} > \frac{w_{j'}(k'')}{w_{j''}(k'')}.$$

In words, Theorem 1.2 can be expressed as follows. Take two countries $j', j'' \in C$ such that $n_{j'} < n_{j''}$ as well as two alternatives $1 \leq k', k'' < m$ with $k' \neq k''$ such that $|G(k') - \frac{1}{2}| < |G(k'') - \frac{1}{2}|$. Again, I interpret the latter property as alternative k' being more moderate or less extreme than alternative k''. Then, the degree of overweighting smaller countries is larger at alternative k' compared to k''. To put it differently, the more moderate the alternative the higher the degree of overweighting smaller countries. In particular, the degree of overweighting smaller countries relative to the linear benchmark might vary non-monotonically along the sequence of ballots.
Consider some population size n. The proof of Theorem 1.2 relies on studying the behavior of the ratio of optimal weights for countries with population sizes n and $n + 2$

$$r(G) = r(G(k)) := \frac{\mathbb{E}[X_n^k | X_n^k \geq \frac{n+1}{2}] - \mathbb{E}[X_n^k | X_n^k \leq \frac{n-1}{2}]}{\mathbb{E}[X_{n+2}^k | X_{n+2}^k \geq \frac{n+3}{2}] - \mathbb{E}[X_{n+2}^k | X_{n+2}^k \leq \frac{n+1}{2}]}$$

as a function of G.[20] Since the second parameter of the binomial random variables X_n^k and X_{n+2}^k is $1 - G$, the distributions of X_n^k and X_{n+2}^k vary with G. In particular, an increase in G yields a stochastic decrease according to likelihood ratio order, which, in turn, implies a stochastic decrease according to the hazard rate as well as reversed hazard rate order (see e.g. Shaked and Shanthikumar (2007)). Thus, all four truncated expectations involved in the ratio r are decreasing in G. Hence, there is a non-trivial trade-off, and, in the first place, the behavior of the ratio r as a function of G is not clear. Again, I make use of the analytical tools from Ahmed (1991) and Saunders (1992) in order to resolve this trade-off.
Let me provide some intuition for the result in Theorem 1.2 by comparing the polar

20. Recall that population sizes are assumed to be odd.

cases of an arbitrarily extreme alternative, i.e. $G(k) \approx 0$ or $G(k) \approx 1$ and a maximally moderate alternative, i.e., $G(k) = \frac{1}{2}$. Recall that the optimal weights $w_j(k)$ are proportional to

$$\mathbb{E}[X^k_{n_j} | X^k_{n_j} \geq \frac{n_j + 1}{2}] - \mathbb{E}[X^k_{n_j}] + \mathbb{E}[Y^k_{n_j} | Y^k_{n_j} \geq \frac{n_j + 1}{2}] - \mathbb{E}[Y^k_{n_j}].$$

If $G(k) \approx 0$, the designer knows that virtually all citizens want their respective representative to vote "No" at alternative k. Therefore, the designer essentially learns nothing about the citizens' preferences from a representative voting "No" at k, meaning,

$$\mathbb{E}[X^k_{n_j} | X^k_{n_j} \geq \frac{n_j + 1}{2}] - \mathbb{E}[X^k_{n_j}] \approx 0,$$

but only a "Yes" vote at k of a representative is informative for the designer. However, if $G(k) \approx 0$, it holds that

$$\mathbb{E}[Y^k_{n_j} | Y^k_{n_j} \geq \frac{n_j + 1}{2}] - \mathbb{E}[Y^k_{n_j}] \approx \frac{n_j + 1}{2}.$$

Consequently, if $G(k) \approx 0$, the optimal weights are proportional to a term that is affine in the population size. Note that the reasoning in section 1.5 for the case of $G(k) < \frac{1}{2}$ and large population sizes is similar to the argument presented here. However, since population sizes are finite here, the conclusion from section 1.5 is valid here only for $G(k) \approx 0$ instead of $G(k) < \frac{1}{2}$. The shape of the optimal weights and the intuition for the case in which $G(k) \approx 1$ is analogous.
In contrast, if an alternative is maximally moderate, i.e., $G(k) = \frac{1}{2}$, the shape of the optimal weights is different. In this case, as discussed in section 1.5, the optimal weights are proportional to

$$n \cdot Pr(X^k_{n-1} = \frac{n-1}{2}).$$

Again, the term $Pr(X^k_{n-1} = \frac{n-1}{2})$ represents the probability that a citizen is pivotal in a within-country simple majority election involving two alternatives that are each preferred by any citizen with an independent probability of $\frac{1}{2}$. Therefore, the weights initially proposed by Penrose (1946) are recovered here.
From the derived expressions, it can be inferred that the degree of overweighting of smaller countries is larger if $G(k) = \frac{1}{2}$ compared to $G(k) \approx 0$ or $G(k) \approx 1$. Consider the ratio

$$\frac{n \cdot Pr(X^k_{n-1} = \frac{n-1}{2})}{\frac{n+1}{2}} = 2 \cdot Pr(X^k_n = \frac{n-1}{2}),$$

where $X^k_n \sim Binomial(n, \frac{1}{2})$. In this ratio, the optimal weight if $G(k) = \frac{1}{2}$ is divided by the optimal weight if $G(k) \approx 0$ or $G(k) \approx 1$. It can be verified that this ratio is decreasing in the population size n, implying that the magnitude of overweighting smaller countries is larger for $G(k) = \frac{1}{2}$ in comparison with the cases of $G(k) \approx 0$ or $G(k) \approx 1$. Now, of course, Theorem 1.2 is much more general than this comparison of the two polar cases of a maximally moderate, and an arbitrarily extreme alternative.

Let me discuss how these findings about the optimal weights relate to previous results in the literature. On the one hand, Barberà and Jackson (2006)'s result amounts to the following corollary implied by Theorem 1.1 as well as the large population approximation discussed in section 1.5.[21]

Corollary 1.1. Barberà and Jackson (2006)
Suppose that $m = 2$ as well as that $G(1) = 1 - G(1) = \frac{1}{2}$, and assume that

$$\mathbb{E}[u^1 - u^2 | u^1 > u^2] = \mathbb{E}[u^2 - u^1 | u^2 > u^1].$$

The optimal weights satisfy, for all $j', j'' \in C$ with $n_{j'} + 2 = n_{j''}$,

$$\frac{w_{n_{j'}}(1)}{w_{n_{j''}}(1)} = \frac{n_{j'} \cdot Pr(X^k_{n_{j'}-1} = \frac{n_{j'}-1}{2})}{n_{j''} \cdot Pr(X^k_{n_{j''}-1} = \frac{n_{j''}-1}{2})} = \frac{n_{j'} + 1}{n_{j'} + 2} \overset{n_{j'} \text{ and } n_{j''} \text{ large}}{\approx} \frac{\sqrt{n_{j'}}}{\sqrt{n_{j''}}},$$

and the optimal quota coincides with simple majority, i.e., $q(1) = c \cdot \frac{1}{2}$.

Apart from the knife-edge case of $G(k) = \frac{1}{2}$ studied in Barberà and Jackson (2006), Proposition 1.2 and Theorem 1.2 imply that there is an overweighting of smaller countries relative to the linear benchmark, but the magnitude of overweighting is smaller compared to what Barberà and Jackson (2006) find for the case of $G(k) = \frac{1}{2}$. Therefore, allowing for probabilities with which any citizen prefers one alternative over the other distinct from $\frac{1}{2}$ leads to different results as far as the optimal weights, but also the welfare-maximizing quotas are concerned as discussed in section 1.6.

On the other hand, for the case of more than two alternatives, and single-peaked preferences, Maaser and Napel (2007), Maaser and Napel (2012), Maaser and Napel (2014), and Kurz, Maaser, and Napel (2017) find that the weights should be proportional to the square root of the population sizes. The optimal weights that I obtain, generally, do not give rise to a square root rule as, for instance, the discussion of optimal weights associated with arbitrarily extreme alternatives, i.e., $G(k) \approx 0$ or $G(k) \approx 1$, reveals.

Finally, I specialize to the case of three alternatives, i.e., $m = 3$, and I impose a symmetry assumption. Specifically, I assume that the probability that any citizen's most preferred alternative amounts to 1 equals the corresponding probability for alternative 3. I obtain the following corollary.

21. However, I emphasize that the simple closed form expressions $r(\frac{1}{2}) = \frac{n+1}{n+2}$ as well as $r(\frac{1}{2}) = \frac{n \cdot Pr(X^k_{n-1} = \frac{n-1}{2})}{[n+2] \cdot Pr(X^k_{n+1} = \frac{n+1}{2})}$ do not appear in Barberà and Jackson (2006).

Corollary 1.2. Suppose that $m = 3$ as well as that $G(1) = 1 - G(2)$, and assume that, for all $k' \in \{1, 2\}$,

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

The optimal mechanism features constant weights, i.e., $w_j(1) = w_j(2)$ with $j \in C$ as well as quotas that do not coincide with simple majority, and that are decreasing along the sequence of ballots, i.e., $q(1) > c \cdot \frac{1}{2} > q(2)$ while $q(1) = c - q(2)$.

The additional symmetry assumption on the extreme alternatives 1 and 3 allows me to rationalize weights that are constant along the sequence of ballots.[22]

1.8 Correlated Preferences

The benchmark model assumes that the citizens' types or preferences are distributed independently and identically across all citizens. In this part, I relax the assumption that preferences are distributed independently within countries. However, preferences remain to be independent across countries. In order to introduce within-country correlation among the citizens' preferences, I rely on the block model from Barberà and Jackson (2006). Let me recall their block model. Suppose that each country $j \in C$ is composed of an odd number of equally-sized blocks $b_{n_j} \geq 1$, and that not all numbers of blocks are equal to 1. Let $s_{n_j} := \frac{n_j}{b_{n_j}}$ describe the block size in country $j \in C$. Moreover, I impose that, for all countries $j', j'' \in C$ having the same population size, i.e., $n_{j'} = n_{j''}$, it holds that $b_{n'_j} = b_{n''_j}$ and, hence, $s_{n'_j} = s_{n''_j}$. Now, citizens' preferences are assumed to be perfectly correlated within blocks, but independent across blocks. All other aspects of the benchmark model are maintained. In particular, in every country, the peak alternative of the median block determines the representative's most preferred alternative. Then, of course, the benchmark model corresponds here to the special case, where, for each country $j \in C$, it holds $b_{n_j} = n_j$ and, thus, $s_{n_j} = 1$.
Theorem 1.3 shows how the characterization of the welfare-maximizing voting mechanisms for the collective decision-making process of the representatives extends to preferences that are correlated within countries.

Theorem 1.3. *Consider the block model. Further, suppose that, for all $1 \leq k' < m$,*

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

22. Note that the rationalization of constant weights is not possible if there are more than three alternatives.

The weighted successive voting procedure with weights

$$w_j(k) = \frac{n_j \cdot [\mu^Y_{b_{n_j}}(k) + \mu^N_{b_{n_j}}(k)]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [\mu^Y_{b_{n_j}}(k) + \mu^N_{b_{n_j}}(k)]} = \frac{n_j \cdot [w^Y_{b_{n_j}}(k) + w^N_{b_{n_j}}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{b_{n_j}}(k) + w^N_{b_{n_j}}(k) - 1]}$$

and quotas

$$q(k) = \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j \cdot \mu^Y_{b_{n_j}}(k)}{\frac{1}{c}\sum_{j\in C} n_j \cdot \mu^N_{b_{n_j}}(k)}} = \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j \cdot [2w^Y_{b_{n_j}}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [2w^N_{b_{n_j}}(k) - 1]}}$$

with $j \in C$ and $1 \leq k < m$ implements the optimal mechanism among all successive voting procedures.

Note that the symmetry assumption on the preference intensities within pairs of neighboring alternatives is the same as in Theorem 1.1. The presence of blocks alters the inference of the utilitarian designer about the citizens' preferences based on the representatives' vote choices. More specifically, for all countries $j \in C$, the preference intensities $\mu^Y_{n_j}(k)$, and $\mu^N_{n_j}(k)$ that are induced by indirect democracy, and that determine the trade-off the designer is facing become $\mu^Y_{b_{n_j}}(k)$, and $\mu^N_{b_{n_j}}(k)$ respectively. This reflects the fact that, in each country $j \in C$, b_{n_j} instead of n_j independent draws of types are conducted. The proof of Theorem 1.1 applies to Theorem 1.3 when replacing the objects $\mu^Y_{n_j}(k)$, $\mu^N_{n_j}(k)$, $w^Y_{n_j}(k)$, and $w^N_{n_j}(k)$ with $\mu^Y_{b_{n_j}}(k)$, $\mu^N_{b_{n_j}}(k)$, $w^Y_{b_{n_j}}(k)$, and $w^N_{b_{n_j}}(k)$ respectively. Therefore, a separate proof is omitted here.

Since there is at least one country that has more than one block, the welfare-maximizing majority requirements for more extreme alternatives continue to be more distant from simple majority. Thus, the main insight regarding the optimal quotas discussed in section 1.6 extends to the block model.[23]

Similar to Barberà and Jackson (2006), subsequently, I distinguish two forms of within-country correlation of preferences or, in other words, two variants of the block model. To begin with, I consider the case where, for all countries $j', j'' \in C$ with $n_{j'} < n_{j''}$, it holds that

$$s_{n'_j} \leq s_{n''_j} \text{ and } b_{n'_j} < b_{n''_j}.$$

In words, this case corresponds to a scenario, where the size of the blocks weakly increases, and the number of blocks strictly increases in the population size.[24] It

23. I emphasize the assumption that not all countries have one block because the discussed finding would break if $b_{n_j} = 1$ for all $j \in C$, meaning, preferences are fully correlated within countries. In this case, the optimal quotas would be constant along the sequence of ballots, and they would amount to simple majority.

24. This variation of the block model generalizes the fixed-size-block model studied in Barberà and Jackson (2006). The latter variant corresponds to the special case, where, for all countries $j', j'' \in C$ with $n_{j'} < n_{j''}$, it holds that $s_{n'_j} = s_{n''_j}$ and $b_{n'_j} < b_{n''_j}$.

turns out that the findings concerning the utilitarian weights of nations presented in section 1.7 for the case of independent preferences within countries carry over to this form of correlated preferences.

First, Proposition 1.3 generalizes Proposition 1.2: Whatever the alternative, the associated optimal weights continue to feature a degree of overweighting of smaller countries that falls between the linear benchmark and a power law with exponent $1-\frac{\ln 2}{\ln 3}\approx 0.37$.

Proposition 1.3. Consider the block model, and assume that $s_{n_j'} \leq s_{n_j''}$ and $b_{n_j'} < b_{n_j''}$ for all $j',j'' \in C$ with $n_{j'} < n_{j''}$. Further, suppose that, for all $1 \leq k' < m$,

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

For all $0 \leq \alpha < 1-\frac{\ln 2}{\ln 3}$ as well as any $1 \leq k < m$ and $j',j'' \in C$ such that $n_{j'} < n_{j''}$, the optimal weights satisfy

$$\frac{w_{j'}(k)}{n_{j'}^{\alpha}} < \frac{w_{j''}(k)}{n_{j''}^{\alpha}}, \text{ and } \frac{w_{j'}(k)}{n_{j'}} > \frac{w_{j''}(k)}{n_{j''}}.$$

Second, Theorem 1.4 shows that Theorem 1.2 extends to the discussed form of correlated preferences: Again, the magnitude of overweighting of smaller countries is larger for more moderate compared to more extreme alternatives.

Theorem 1.4. *Consider the block model, and assume that $s_{n_j'} \leq s_{n_j''}$ and $b_{n_j'} < b_{n_j''}$ for all $j',j'' \in C$ with $n_{j'} < n_{j''}$. Further, suppose that, for all $1 \leq k' < m$,*

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

Consider any $j',j'' \in C$ such that $n_{j'} < n_{j''}$ as well as any $1 \leq k',k'' < m$ with $k' \neq k''$ such that $|G(k')-\frac{1}{2}| < |G(k'')-\frac{1}{2}|$. The ratio of optimal weights satisfies

$$\frac{w_{j'}(k')}{w_{j''}(k')} > \frac{w_{j'}(k'')}{w_{j''}(k'')}.$$

Recall that the proof of Theorem 1.2 relies on studying the behavior of the ratio of optimal weights for countries with population sizes n and $n+2$ denoted by $r(G)$ as a function of G. Here, consider instead two countries with numbers of blocks b and $b+2$, and let s and s' be the corresponding block sizes respectively. Since the numbers of blocks are assumed to be odd, and because population sizes are strictly increasing in the number of blocks, it is sufficient to determine the behavior of the ratio of optimal weights for the two countries with numbers of blocks b and $b+2$ as a function of G in order to establish Theorem 1.4. However, this ratio is given by $\frac{s}{s'}r(G)$ when replacing in the ratio $r(G)$ the variable n with the variable b. Since $\frac{s}{s'} > 0$ is a positive constant, it does not affect the behavior of the ratio $\frac{s}{s'}r(G)$ as a

function of G. Therefore, the proof of Theorem 1.2 can be used to establish Theorem 1.4. Hence, I omit a separate proof.
The second variant of the block model that I consider is characterized by the following feature: For all countries $j', j'' \in C$ with $n_{j'} < n_{j''}$, it holds $b_{n'_j} = b_{n''_j}$. This means that the number of blocks is constant across countries, and it is independent of the population sizes.[25] In this case, the qualitative features of the weights of nations change compared to the case, where preferences are independent within countries. It follows from Theorem 1.3 that, in this case, the welfare-maximizing weights are constant along the sequence of ballots, and they are linear in the population sizes.

1.9 European Union

The purpose of this section is twofold: On the one hand, I explicitly compute welfare-maximizing weights and quotas for the collective decision-making process in institutions of the European Union such as the Council of the European Union. On the other hand, I compare the optimal mechanism with the voting rules that are actually used in the Council of the European Union.
In order to calculate optimal weights and quotas, I employ the block model discussed in section 1.8. Moreover, as in Barberà and Jackson (2006), I assume that the block size is approximately one million in all member states of the European Union. To be precise, the number of blocks for each member state of the European Union is given by the odd natural number that minimizes the distance between the induced block size and one million. Also, the population data is taken from Eurostat (2021), and I use the numbers from January 1, 2020.[26]
Figure 1.3 shows the optimal weights for different values of G.[27] For instance, suppose that $G = 0.4$, and consider the largest as well as the smallest member states Germany and Malta: The corresponding voting weight of Germany is about 20 times higher than the weight of Malta, whereas Germany's population size is about 160 times higher than Malta's population size. Observe that the weights satisfy the theoretically established properties apart from a few minor exceptions. The reason for these irregularities is that the theoretical analysis assumes that the block size is weakly increasing in the population size, but this property does not hold here because of the discussed integer problem regarding the number of blocks.
Figure 1.4 depicts the optimal quotas for different values of G.[28] For example, if $G = 0.4$, the corresponding welfare-maximizing quota is approximately 69% of the

25. This variation is also discussed in Barberà and Jackson (2006), and they call it fixed-number-of-blocks model.
26. Throughout, the United Kingdom is excluded from the analysis.
27. The sum of weights assigned is normalized to 27, representing the number of member states of the European Union.
28. Recall that the sum of weights assigned is normalized to 27.

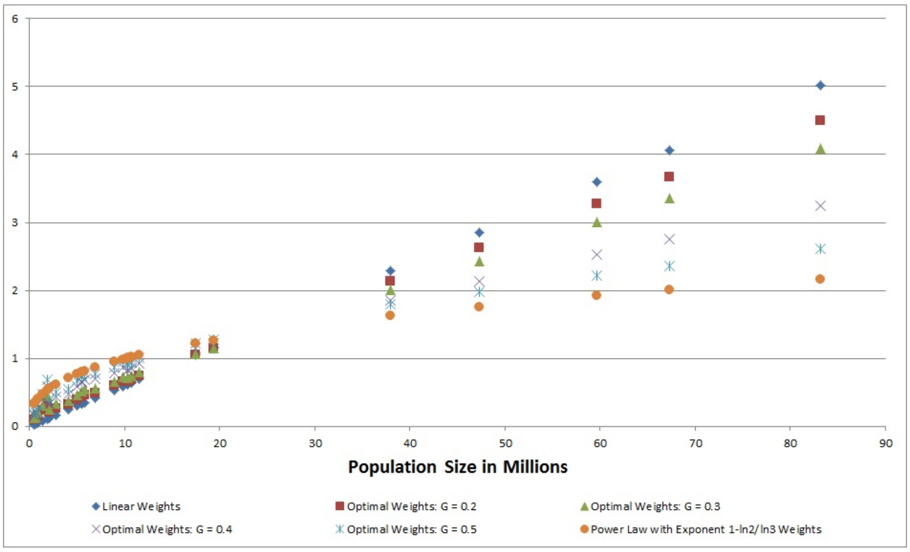

Figure 1.3. Optimal Voting Weights for the European Union

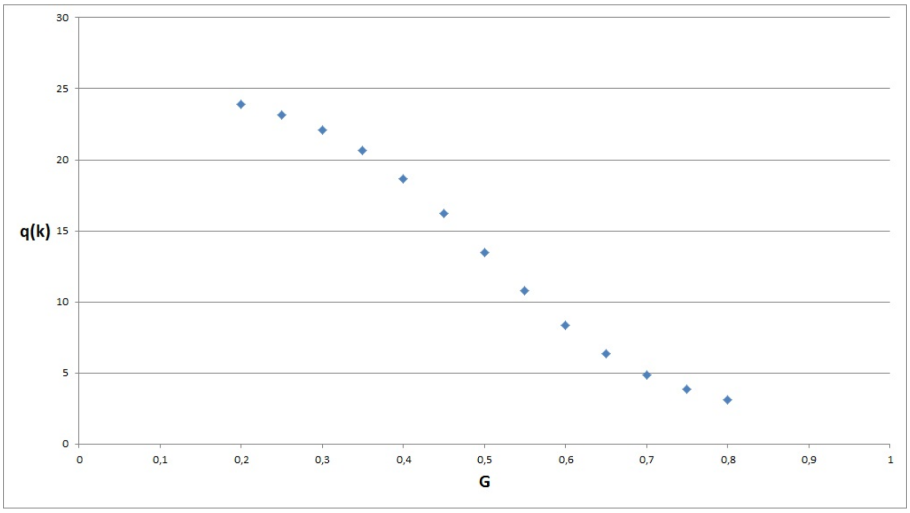

Figure 1.4. Optimal Majority Quotas for the European Union

total weights assigned. Note that, as shown theoretically, the optimal quotas are strictly decreasing in G, and that, except for the knife-edge case of $G = \frac{1}{2}$, they do not coincide with simple majority. In addition, the numerical investigation suggests that the quotas are concave in G if $G < 0.5$, and convex in G if $G > 0.5$.
In the following, I compare the optimal mechanism with the voting rules that are actually used in the Council of the European Union for most policy areas.[29] Until 2014, according to the Treaty of Nice, the Council of the European Union employed

29. There are some exceptions including, for example, the foreign and security policy, and the issue of tax harmonisation, where the voting rule is unanimity (see European Union (2015)).

a weighted majority rule (see European Union (2001)). Then, the Treaty of Lisbon stipulated a change of the voting rule to a double majority system: If the proposal is made by the European Commission or the High Representative of the Union for Foreign Affairs and Security Policy, it passes whenever it is approved by at least 55% of the member states, representing at least 65% of the population of the European Union; otherwise, the support of at least 72% instead of 55% of the member states is required (see European Union (2007)).[30] Since the voting rule that was in place until 2014 involves as the welfare-maximizing mechanism a weighted majority rule, the subsequent comparison focuses on this voting rule. Concretely, the Treaty of Accession of Croatia specifies the voting weights for every member state that were most recently in place, and it stipulates that the majority threshold amounts to approximately 74% of the total weights assigned (see European Union (2012)).[31]

Figure 1.5 presents the voting weights that were employed in the Council of the European Union as well as the benchmarks of linear weights and weights following a power law with exponent $1 - \frac{\ln 2}{\ln 3} \approx 0.37$. Fitting the population data and the weights used in the Council of the European Union to a power law yields an exponent of about 0.48. Thus, these weights approximately follow a square root rule. Now, I consider again the block model while imposing that the block size is approximately one million in all member states of the European Union. Figure 1.5 displays the corresponding optimal voting weights for a value of $G = 0.37$, inducing an optimal quota of about 74%, that is, the majority quota that was actually used in the Council of the European Union.[32]

Several observations can be made: To begin with, the voting weights in the Council of the European Union are qualitatively consistent with the theoretically predicted bounds on the magnitude of overweighting of smaller countries, that is, the degree of overweighting in the Council of the European Union lies between the linear benchmark and a power law with exponent $1 - \frac{\ln 2}{\ln 3} \approx 0.37$. However, the voting weights used in the Council of the European Union are inconsistent with the employed majority quota.

If the majority quota has to be for some exogenous reason the quota that was employed in the Council of the European Union, that is, approximately 74%, the voting weights in the Council of the European Union feature too much overweighting of smaller member states relative to the weights that would be optimal in this case.

30. There is a qualification: The blocking minority must include at least four member states, representing more than 35% of population of the European Union.

31. There are two qualifications: The approval of a proposal also requires the support of a majority of the member states if the proposal was made by the European Commission, and the support of at least two thirds of the member states otherwise. Further, the member states that support a proposal have to represent at least 62% of the population of the European Union. I abstract from these two additional aspects.

32. The fact that the exponent of the power law benchmark and the inferred value of G are both roughly equal to 0.37 is just a coincidence.

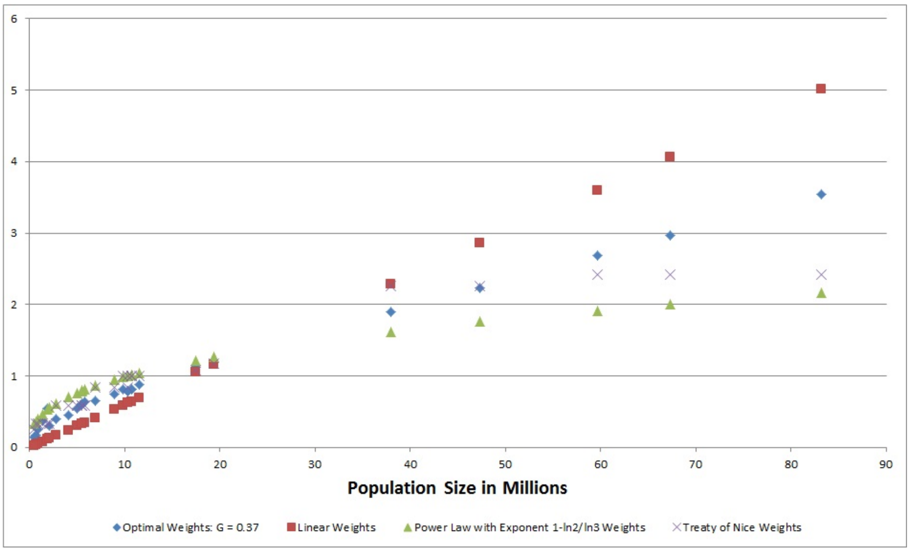

Figure 1.5. Voting in the Council of the European Union

Specifically, fitting the population data and the optimal voting weights, inducing an optimal quota of about 74%, to a power law yields an exponent of roughly 0.57, which is higher than the exponent of 0.48 that, again, results from fitting the population data and the weights used in the Council of the European Union to a power law.

If the voting weights have to take for some exogenous reason the form that they had in the Council of the European Union, the employed majority quota of 74% is too high compared to the quota that would be optimal in this case. More precisely, when fitting the population data and optimal voting weights to a power law, a value of $G = 0.47$ implies that the exponent is roughly 0.48 which is, again, approximately the exponent that results from fitting the population data and the weights that were used in the Council of the European Union to a power law. However, the optimal quota corresponding to a value of $G = 0.47$ is roughly 56%, and, thus, it is lower than the quota of about 74% that was employed in the Council of the European Union.

1.10 Conclusion

In this chapter, I study the design of welfare-maximizing voting mechanisms for institutions of representative democracy when there are multiple alternatives, and the citizens' preferences are single-peaked. The welfare-maximizing mechanism takes the form of a weighted successive voting procedure with alternative-dependent voting weights and majority quotas. The main insight of this chapter is that the vote on more extreme alternatives involves majority quotas further away from simple majority, and, in addition, relatively higher voting weights for large groups.

I derive several features of the optimal majority quotas, and the weights of nations. For the majority quotas, I find that indirect democracy implies that the vote on more extreme alternatives requires majority quotas that are further away from simple majority. The utilitarian weights of nations have the following two key properties: First, whatever the alternative, the associated optimal weights exhibit a magnitude of overweighting of smaller countries that falls between the linear benchmark and a power law with exponent $1 - \frac{\ln 2}{\ln 3} \approx 0.37$, implying that the weights feature degressive proportionality. Second, the degree of this overweighting of smaller countries is larger for more moderate compared to more extreme alternatives.

Finally, as illustrated in the previous section on the European Union, I believe that these insights provide some guidance for the design of voting mechanisms for real-world institutions of representative democracy.

Appendix 1.A Proofs

Proof of Proposition 1.1.
Suppose that all agents except for agent j play the sincere strategy and assume that agent j's peak is $1 \leq p \leq m$. First, consider voting stages such that some alternative $l < p$ is on the agenda. Let Z be the set of agents distinct from j who play "Yes" at this stage. Agent j's action matters if and only if he or she is pivotal, that is, if and only if $Z \notin \mathscr{W}_l$, but $Z \cup \{j\} \in \mathscr{W}_l$. Since the agents distinct from j play the sincere strategy, the number of "Yes" weakly increases along the sequence of ballots. Moreover, by assumption, $\mathscr{W}_l \subseteq W_k$ for all $l < k \leq m$. Therefore, if agent j votes "Yes", alternative l is selected and if he or she votes "No", some alternative $l' \in M$ with $l < l' \leq p$ is chosen. Hence, voting "No", i.e., playing the action prescribed by the sincere strategy, is optimal because preferences are single-peaked implying that agent j prefers alternative l' over alternative l. Second, consider voting stages such that some alternative $p \leq l$ is on the agenda. Again, let Z be the set of agents distinct from j who play "Yes" at this stage. Similar to the first case, agent j's action matters if and only if he or she is pivotal, that is, if and only if $Z \notin \mathscr{W}_l$, but $Z \cup \{j\} \in \mathscr{W}_l$. Hence, if agent j votes "Yes", alternative l is selected while voting "No" results in the implementation of some alternative $l' \in M$ with $p \leq l < l'$. Thus, voting "Yes", i.e., choosing the action prescribed by the sincere strategy, is optimal because preferences are single-peaked implying that agent j prefers alternative l over alternative l'. Taking both cases together, if all agents distinct from j adopt the sincere strategy, playing the sincere strategy is a best response for agent j. Consequently, by symmetry, sincere voting constitutes an ex-post perfect equilibrium. □

In the following, I provide several lemmata, having the purpose to derive a characterization of the utilitarian mechanism.

Consider any optimal mechanism, meaning, any families of winning coalitions $\{\mathscr{W}_k^*\}_{k=1}^{m-1}$ that are welfare-maximizing. For any alternative $1 \leq k < m$, define

$$\mathscr{W}_{k,min}^* := \{W \in \mathscr{W}_k^* : \forall W' \text{ with } W' \subset W : W' \notin \mathscr{W}_k^*\}$$

In words, a coalition of representatives is contained in the set $\mathscr{W}_{k,min}^*$ if and only if it is winning at k, but any proper subset of this coalition is not winning at k. Thus, $\mathscr{W}_{k,min}^*$ can be called the set of minimally winning coalitions at alternative k. I obtain the following lemma.

Lemma 1.1. Suppose that, for all $1 \leq k' < m$,

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

Consider any $1 \leq k < m$. For any $W \subseteq C$ such that either $W \in \mathscr{W}_{1,min}^*$ (if $k = 1$) or $W \in \mathscr{W}_{k,min}^*$ and $W \notin \mathscr{W}_{k-1}^*$ (if $k > 1$),

$$\sum_{j \in W} n_j \cdot \mu_{n_j}^Y(k) \geq \sum_{j \in C \setminus W} n_j \cdot \mu_{n_j}^N(k)$$
$$\Leftrightarrow \sum_{j \in W} n_j \cdot [2w_{n_j}^Y(k) - 1] \geq \sum_{j \in C \setminus W} n_j \cdot [2w_{n_j}^N(k) - 1].$$

The basic idea behind Lemma 1.1 is as follows: Start with an optimal mechanism, and study the implications of dropping minimally winning coalitions from the optimal families of winning coalitions. However, for $k > 1$, removing coalitions $W \in \mathscr{W}_{k,min}^*$ from the family of winning coalitions $\mathscr{W}_k^*$ is not feasible if $W \in \mathscr{W}_{k-1}^*$ because families of winning coalitions have to meet the set inclusion restriction $\mathscr{W}_{k'} \subseteq \mathscr{W}_{k'+1}$ for all $1 \leq k' < m-1$. The subsequent lemma addresses this point. I show that the inequalities derived in Lemma 1.1 remain to be valid even if the considered alteration of the optimal mechanism is not feasible.

Lemma 1.2. Suppose that, for all $1 \leq k' < m$,

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

Consider any $1 < k < m$. For any $W \subseteq C$ such that $W \in \mathscr{W}_{k,min}^*$, but $W \in \mathscr{W}_{k-1}^*$, it, nevertheless, holds that

$$\sum_{j \in W} n_j \cdot \mu_{n_j}^L(k) \geq \sum_{j \in C \setminus W} n_j \cdot \mu_{n_j}^R(k)$$
$$\Leftrightarrow \sum_{j \in W} n_j \cdot [2w_{n_j}^Y(k) - 1] \geq \sum_{j \in C \setminus W} n_j \cdot [2w_{n_j}^N(k) - 1].$$

Next, for any alternative $1 \leq k < m$, define

$$\mathscr{W}_{\neg k,max}^* := \{W \notin \mathscr{W}_k^* : \forall W' \text{ with } W \subset W' : W' \in \mathscr{W}_k^*\}.$$

In words, the set $\mathscr{W}_{\neg k,max}^*$ contains all coalitions that are maximally loosing at alternative k, meaning, a coalition is part of $\mathscr{W}_{\neg k,max}^*$ if and only if this coalition is not winning at alternative k, but any proper superset is winning at k. The following lemma holds.

Lemma 1.3. Suppose that, for all $1 \leq k' < m$,

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

Consider any $1 \leq k < m$. For any $W' \subseteq C$ such that either $W' \in \mathscr{W}_{\neg(m-1),max}^*$ (if $k = m-1$) or $W' \in \mathscr{W}_{\neg k,max}^*$ and $W' \in \mathscr{W}_{k+1}^*$ (if $k < m-1$),

$$\sum_{j \in W'} n_j \cdot \mu_{n_j}^Y(k) \leq \sum_{j \in C \setminus W'} n_j \cdot \mu_{n_j}^N(k)$$
$$\Leftrightarrow \sum_{j \in W'} n_j \cdot [2w_{n_j}^Y(k) - 1] \leq \sum_{j \in C \setminus W'} n_j \cdot [2w_{n_j}^N(k) - 1].$$

Lemma 1.3 relies on the idea of studying the implications of adding maximally loosing coalitions to the optimal families of winning coalitions. However, for $k < m-1$, adding coalitions $W' \in \mathscr{W}^*_{\neg k,max}$ to $\mathscr{W}^*_k$ is not feasible if $W' \notin \mathscr{W}^*_{k+1}$ because of the set inclusion restriction on the families of winning coalitions, i.e., $\mathscr{W}_{k'} \subseteq \mathscr{W}_{k'+1}$ for all $1 \leq k' < m-1$. Nevertheless, the subsequent lemma reveals that the inequalities derived in Lemma 1.3 are also valid if the considered modification of the optimal mechanism is not feasible.

Lemma 1.4. Suppose that, for all $1 \leq k' < m$,

$$\mathbb{E}[u^{k'} - u^{k'+1} | u^{k'} > u^{k'+1}] = \mathbb{E}[u^{k'+1} - u^{k'} | u^{k'+1} > u^{k'}].$$

Consider any $1 \leq k < m-1$. For any $W' \subseteq C$ such that $W' \in \mathscr{W}^*_{\neg k,max}$, but $W' \notin \mathscr{W}^*_{k+1}$ it, nevertheless, holds that

$$\begin{aligned} \sum_{j \in W'} n_j \cdot \mu^Y_{n_j}(k) &\leq \sum_{j \in C \setminus W'} n_j \cdot \mu^N_{n_j}(k) \\ \Leftrightarrow \sum_{j \in W'} n_j \cdot [2w^Y_{n_j}(k) - 1] &\leq \sum_{j \in C \setminus W'} n_j \cdot [2w^N_{n_j}(k) - 1]. \end{aligned}$$

Consequently, to summarize, the inequalities derived in Lemma 1.1 and Lemma 1.2 as well as the inequalities obtained from Lemma 1.3 and Lemma 1.4 represent necessary conditions for optimality. I make use of these lemmata in the proof of Theorem 1.1.

Proof of Lemma 1.1.
First of all, if $W = C$, it must be that $k = 1$. In this case, the desired inequality reduces to

$$\sum_{j \in C} n_j \cdot \mu^Y_{n_j}(1) \geq 0$$

which is true since, for all $j \in C$, it holds $w^Y_{n_j}(1) \geq \frac{n_j+1}{2n_j} = \frac{1}{2} + \frac{1}{2n_j} > \frac{1}{2}$. Therefore, subsequently, suppose that $W \neq C$.
Take some $W \subset C$ such that either $W \in \mathscr{W}^*_{1,min}$ (if $k = 1$) or $W \in \mathscr{W}^*_{k,min}$ and $W \notin \mathscr{W}^*_{k-1}$ (if $k > 1$) and modify the optimal families of winning coalitions such that $W \notin \mathscr{W}^*_k$. Since by assumption either $W \in \mathscr{W}^*_{1,min}$ (if $k = 1$) or $W \in \mathscr{W}^*_{k,min}$ and $W \notin \mathscr{W}^*_{k-1}$ (if $k > 1$) and, in addition, $W \neq C$, this modification of the optimal families of winning coalitions is feasible. Moreover, since $W \in \mathscr{W}^*_{k,min}$, this alteration matters only if all $j \in W$ vote "Yes" at k and all $j \in C \setminus W$ vote "No" at k. In this case, under the optimal families of winning coalitions, if $k = 1$, alternative 1 is selected and, if $k > 1$, alternative k is chosen because $W \notin \mathscr{W}^*_{k-1}$ implies that $W \notin \mathscr{W}^*_{k'}$ for all $1 \leq k' < k$. Further, since $W \in \mathscr{W}^*_k \subseteq \mathscr{W}^*_{k'}$ for all $k < k' < m$, I have that $W \in \mathscr{W}^*_{k'}$ for all $k < k' < m$ and, thus, under the modification, alternative $k+1$ is chosen.

Since the modification of the optimal families of winning coalitions should weakly decrease welfare, I obtain the following condition that is necessary for optimality whenever the considered alteration is feasible:

$$\begin{aligned}
&\mathbb{E}[\sum_{j\in C} n_j \cdot u^k(T)|(\forall j \in W : \text{“Yes” vote at } k)\\
&\quad\wedge (\forall j \in C \setminus W : \text{“No” vote at } k)]\\
\geq &\mathbb{E}[\sum_{j\in C} n_j \cdot u^{k+1}(T)|(\forall j \in W : \text{“Yes” vote at } k)\\
&\quad\wedge (\forall j \in C \setminus W : \text{“No” vote at } k)]\\
\Leftrightarrow&\\
&\sum_{j\in W} n_j \cdot \mathbb{E}[u^k(T)|\text{j votes “Yes” at } k]\\
&\quad+ \sum_{j\in C\setminus W} n_j \cdot \mathbb{E}[u^k(T)\text{j votes “No” at } k]\\
\geq &\sum_{j\in W} n_j \cdot \mathbb{E}[u^{k+1}(T)|\text{j votes “Yes” at } k]\\
&\quad+ \sum_{j\in C\setminus W} n_j \cdot \mathbb{E}[u^{k+1}(T)|\text{j votes “No” at } k]\\
\Leftrightarrow&\\
&\sum_{j\in W} n_j \cdot \mathbb{E}[u^k(T) - u^{k+1}(T)|\text{j votes “Yes” at } k]\\
\geq &\sum_{j\in C\setminus W} n_j \cdot \mathbb{E}[u^{k+1}(T) - u^k(T)|\text{j votes “No” at } k]\\
\Leftrightarrow&\\
&\sum_{j\in W} n_j \cdot \mu^Y_{n_j}(k) \geq \sum_{j\in C\setminus W} n_j \cdot \mu^N_{n_j}(k)\\
\Leftrightarrow&\\
&\sum_{j\in W} n_j \cdot [2w^Y_{n_j}(k) - 1] \geq \sum_{j\in C\setminus W} n_j \cdot [2w^N_{n_j}(k) - 1].
\end{aligned}$$

I derived the desired inequality completing the proof. □

Proof of Lemma 1.2.
Consider any $1 < k < m$ and suppose that $W \in \mathscr{W}^*_{k,min}$ and $W \in \mathscr{W}^*_{k-1}$. To start, I claim that $W \in \mathscr{W}^*_{k-1,min}$. Suppose not, i.e., $W \notin \mathscr{W}^*_{k-1,min}$. Because of $W \in \mathscr{W}^*_{k-1}$, there must be some $W' \subset W$ such that $W' \in \mathscr{W}^*_{k-1}$. Further, since $W' \in \mathscr{W}^*_{k-1} \subseteq \mathscr{W}^*_k$, I have that $W' \in \mathscr{W}^*_k$. However, this contradicts $W \in \mathscr{W}^*_{k,min}$.
Employing the property $W \in \mathscr{W}^*_{k-1,min}$, in the following, I perform a case distinction:
1) $W \notin \mathscr{W}^*_{k-2}$

Here, I have that $W \in \mathscr{W}^*_{k-1,min}$ as well as $W \notin \mathscr{W}^*_{k-2}$ and, therefore, Lemma 1.1 applies to $k-1$.
2) $W \in \mathscr{W}^*_{k-2}$
In this case, apply the argument from the previous claim to $k-2$ yielding $W \in \mathscr{W}^*_{k-2,min}$. When iterating the same case distinction for at most finitely many times, two cases are possible:
(i) There exists some $1 < k'' < k$ such that $W \in \mathscr{W}^*_{k'',min}$ and $W \notin \mathscr{W}^*_{k''-1}$. In this case, Lemma 1.1 applies to k''.
(ii) There exists no such k'' implying that I must have $W \in \mathscr{W}^*_{1,min}$. Here, Lemma 1.1 applies to 1.
Consequently, taking the cases (i) and (ii) together and applying Lemma 1.1, there must be some $1 \leq k' < k$ (either $k' = 1$ or $k' = k''$) such that

$$\begin{aligned} &\sum_{j \in W} n_j \cdot \mu^Y_{n_j}(k') \geq \sum_{j \in C \setminus W} n_j \cdot \mu^N_{n_j}(k') \\ \Leftrightarrow &\sum_{j \in W} n_j \cdot [2w^Y_{n_j}(k) - 1] \geq \sum_{j \in C \setminus W} n_j \cdot [2w^N_{n_j}(k) - 1]. \end{aligned}$$

Since $k' < k$ and because of the richness assumption on the preference domain, it holds that $G(k') < G(k)$. Therefore, for any $j \in C$, the binomial distributions with parameters n_j and $G(k')$ is stochastically dominated by the binomial distribution with parameters n_j and $G(k)$ according to the likelihood ratio order. This implies that the latter distribution also dominates the former distribution in the hazard rate order. For a reference for these two claims, see e.g. Shaked and Shanthikumar (2007). However, this means that $w^Y_{n_j}(k') < w^Y_{n_j}(k)$. Moreover, for any $j \in C$, the binomial distributions with parameters n_j and $1 - G(k')$ stochastically dominates the binomial distribution with parameters n_j and $1 - G(k)$ according to the likelihood ratio order. Thus, the latter distribution is also dominated by the former distribution in the hazard rate order. Again, for a reference for these two claims, see e.g. Shaked and Shanthikumar (2007). Hence, I have that $w^N_{n_j}(k') > w^N_{n_j}(k)$. Therefore, I obtain that

$$\begin{aligned} &\sum_{j \in W} n_j \cdot [2w^Y_{n_j}(k) - 1] > \sum_{j \in W} n_j \cdot [2w^Y_{n_j}(k') - 1] \\ \geq &\sum_{j \in C \setminus W} n_j \cdot [2w^N_{n_j}(k') - 1] > \sum_{j \in C \setminus W} n_j \cdot [2w^N_{n_j}(k) - 1]. \end{aligned}$$

Consequently, I derived the desired inequality

$$\sum_{j \in W} n_j \cdot [2w^Y_{n_j}(k) - 1] \geq \sum_{j \in C \setminus W} n_j \cdot [2w^N_{n_j}(k) - 1].$$

□

Proof of Lemma 1.3.
To begin with, if $W' = \emptyset$, it must hold that $k = m - 1$ and the desired inequality reduces to

$$0 \leq \sum_{j \in C} n_j \cdot \mu^N_{n_j}(m-1)$$

which is satisfied because $w^N_{n_j}(m-1) \geq \frac{n_j+1}{2n_j} = \frac{1}{2} + \frac{1}{2n_j} > \frac{1}{2}$. Hence, in the following, suppose that $W' \neq \emptyset$.
Consider some $W' \subset C$ such that either $W' \in \mathscr{W}^*_{\neg(m-1),max}$ (if $k = m-1$) or $W' \in \mathscr{W}^*_{\neg k,max}$ and $W' \in \mathscr{W}^*_{k+1}$ (if $k < m-1$) and modify the optimal families of winning coalitions such that $W' \in \mathscr{W}^*_k$. Since either $W' \in \mathscr{W}^*_{\neg(m-1),max}$ (if $k = m-1$) or $W' \in \mathscr{W}^*_{\neg k,max}$ and $W' \in \mathscr{W}^*_{k+1}$ (if $k < m-1$), this modification is feasible. Furthermore, because $W' \in \mathscr{W}^*_{\neg k,max}$, this alteration matters only if all $j \in W'$ vote "Yes" at k and all $j \in C \setminus W'$ vote "No" at k. In this case, under the optimal families of winning coalitions, if $k = m-1$, alternative m is selected since $W' \notin \mathscr{W}^*_{m-1}$ and, if $k < m-1$, alternative $k+1$ is chosen because $W' \in \mathscr{W}^*_{k+1}$. Moreover, since $W' \notin \mathscr{W}^*_k$, I have $W' \notin \mathscr{W}^*_{k'}$ for all $1 \leq k' < k$ and, hence, under the modified families of winning coalitions, alternative k is selected.
The modification of the optimal families of winning coalitions weakly decreases welfare and, therefore, I obtain the following condition that is necessary for optimality whenever the considered alteration is feasible:

$$\begin{aligned}
&\mathbb{E}[\sum_{j \in C} n_j \cdot u^k(T) | (\forall j \in W' : \text{“Yes” vote at } k) \\
&\quad \wedge (\forall j \in C \setminus W' : \text{“No” vote at } k)] \\
\leq &\mathbb{E}[\sum_{j \in C} n_j \cdot u^{k+1}(T) | (\forall j \in W' : \text{“Yes” vote at } k) \\
&\quad \wedge (\forall j \in C \setminus W' : \text{“No” vote at } k)].
\end{aligned}$$

Rearranging this inequality while employing the same steps as in the proof of Lemma 1.1, I arrive at the expression

$$\sum_{j \in W'} n_j \cdot \mu^Y_{n_j}(k) \leq \sum_{j \in C \setminus W'} n_j \cdot \mu^N_{n_j}(k),$$

which is equivalent to

$$\sum_{j \in W'} n_j \cdot [2w^Y_{n_j}(k) - 1] \leq \sum_{j \in C \setminus W'} n_j \cdot [2w^N_{n_j}(k) - 1].$$

This constitutes the desired inequality completing the proof. □

Proof of Lemma 1.4.
Take any $1 \leq k < m-1$ and assume that $W' \in \mathscr{W}^*_{\neg k,max}$ and $W' \notin \mathscr{W}^*_{k+1}$. First of all, I argue that $W' \in \mathscr{W}^*_{\neg k+1,max}$. Towards a contradiction, suppose that $W' \notin \mathscr{W}^*_{\neg k+1,max}$. Since $W' \notin \mathscr{W}^*_{k+1}$, there must be some $W' \subset W''$ such that $W'' \notin \mathscr{W}^*_{k+1}$. Moreover, because of $\mathscr{W}^*_k \subseteq \mathscr{W}^*_{k+1}$, it follows that $W'' \notin \mathscr{W}^*_k$. However, this contradicts $W' \in \mathscr{W}^*_{\neg k,max}$ because $W' \subset W''$.
Making use of the property that $W' \in \mathscr{W}^*_{\neg k+1,max}$, subsequently, I perform a case distinction:
1) $W' \in \mathscr{W}^*_{k+2}$
In this case, I have that $W' \in \mathscr{W}^*_{\neg k+1,max}$ as well as $W' \in \mathscr{W}^*_{k+2}$ and, thus, Lemma 1.3 applies to $k+1$.
2) $W' \notin \mathscr{W}^*_{k+2}$
Here, I apply the reasoning from the previous claim to $k+2$ which implies that $W' \in \mathscr{W}^*_{\neg k+2,max}$. When repeating the same case distinction for at most finitely many times, two case can occur:
(i) There exists some $k < k'' < m-1$ such that $W' \in \mathscr{W}^*_{\neg k'',max}$ and $W' \in \mathscr{W}^*_{k''+1}$. Then, Lemma 1.3 applies to k''.
(ii) There exists no such k'' which means that it must be true that $W' \in \mathscr{W}^*_{\neg(m-1),max}$.
In this case, Lemma 1.3 applies to $m-1$.
Taking the two cases (i) and (ii) together and employing Lemma 1.3, there must be some $k < k' < m-1$ (either $k' = m-1$ or $k' = k''$) such that

$$\begin{aligned} \sum_{j \in W'} n_j \cdot \mu^Y_{n_j}(k') &\leq \sum_{j \in C \setminus W'} n_j \cdot \mu^N_{n_j}(k') \\ \Leftrightarrow \sum_{j \in W'} n_j \cdot [2w^Y_{n_j}(k) - 1] &\leq \sum_{j \in C \setminus W'} n_j \cdot [2w^N_{n_j}(k) - 1]. \end{aligned}$$

Since $k < k'$ and because of the richness assumption on the preference domain, it holds that $G(k) < G(k')$. Therefore, for any $j \in C$, the binomial distribution with parameters n_j and $G(k')$ stochastically dominates the binomial distribution with parameters n_j and $G(k)$ according to the likelihood ratio order. This implies that the latter distribution is also dominated by the former distribution in the hazard rate order. Again, for a reference for these two claims, see e.g. Shaked and Shanthikumar (2007). However, this means that $w^Y_j(k) < w^Y_j(k')$. Moreover, for any $j \in C$, the binomial distribution with parameters n_j and $1 - G(k')$ is stochastically dominated by the binomial distribution with parameters n_j and $1 - G(k)$ according to the likelihood ratio order. Thus, the latter distribution also dominates the former distribution in the hazard rate order. Again, for a reference for these two claims, see e.g. Shaked and Shanthikumar (2007). Hence, it holds that $w^N_j(k) > w^N_j(k')$. Therefore, I have that

$$\begin{aligned}\sum_{j\in W'} n_j\cdot[2w^Y_{n_j}(k)-1] &< \sum_{j\in W'} n_j\cdot[2w^Y_{n_j}(k')-1]\\ \le \sum_{j\in C\setminus W'} n_j\cdot[2w^N_{n_j}(k')-1] &< \sum_{j\in C\setminus W'} n_j\cdot[2w^N_{n_j}(k)-1].\end{aligned}$$

Thus, I derived the desired inequality

$$\begin{aligned}&\sum_{j\in W'} n_j\cdot\mu^Y_{n_j}(k) \le \sum_{j\in C\setminus W'} n_j\cdot[2w^Y_{n_j}(k)-1]\\ \Leftrightarrow &\sum_{j\in W'} n_j\cdot\mu^Y_{n_j}(k) \le \sum_{j\in C\setminus W'} n_j\cdot[2w^N_{n_j}(k)-1].\end{aligned}$$

□

Proof of Theorem 1.1.
First of all, there exist optimal families of winning coalitions $\{W^*_k\}_{k=1}^{m-1}$ because a bounded function is maximized over a finite set of elements. Further, all optimal collections of winning coalitions must have the following properties. Taking together Lemma 1.1 and Lemma 1.2, for any $1\le k<m$ and for all $W\in W^*_{k,min}$, it holds that

$$\sum_{j\in W} n_j\cdot\mu^Y_{n_j}(k) \ge \sum_{j\in C\setminus W} n_j\cdot\mu^N_{n_j}(k).$$

Note that this inequality is equivalent to

$$\sum_{j\in W}\frac{n_j\cdot[w^Y_{n_j}(k)+w^N_{n_j}(k)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[w^Y_{n_j}(k)+w^N_{n_j}(k)-1]} \ge \frac{c}{1+\frac{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^Y_{n_j}(k)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^N_{n_j}(k)-1]}}.$$

Lemma 1.3 and Lemma 1.4 imply together that, for any $1\le k<m$ and $W'\in W^*_{\neg k,max}$, the inequality

$$\sum_{j\in W'} n_j\cdot\mu^Y_{n_j}(k) \le \sum_{j\in C\setminus W'} n_j\cdot\mu^N_{n_j}(k)$$

is satisfied. Rearranging this inequality yields

$$\sum_{j\in W'}\frac{n_j\cdot[w^Y_{n_j}(k)+w^N_{n_j}(k)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[w^Y_{n_j}(k)+w^N_{n_j}(k)-1]} \le \frac{c}{1+\frac{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^Y_{n_j}(k)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^N_{n_j}(k)-1]}}.$$

In other words, these inequalities are necessary for optimality.
In the following, I argue that these inequalities pin down a generically unique solution, i.e., families of coalitions that are feasible and optimal. In other words, I establish that these inequalities are also sufficient for optimality. To begin with, I claim that all families of coalitions $\{\mathcal{S}_k\}_{k=1}^{m-1}$ satisfying these inequalities constitute families of winning coalitions respecting the constraint $\mathcal{S}_k\subseteq\mathcal{S}_{k+1}$ for all $1\le k<m-1$,

meaning, they are feasible. Suppose not, i.e., assume that there exists families of coalitions $\{\mathscr{S}_k\}_{k=1}^{m-1}$ satisfying the previous inequalities, but they are not feasible. First, $\{S_k\}_{k=1}^{m-1}$ might not be feasible because there exists some $1 \leq k < m$ such that $\emptyset \in \mathscr{S}_k$. This means that the inequality

$$0 \ \geq \ \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]$$

must be met. However, this cannot be the case since, for all $j \in C$, $w_{n_j}^N(k) > \frac{1}{2}$. Second, there might be some $1 \leq k < m$ such that $\mathscr{S}_k = \emptyset$. Hence, in particular, it holds that $C \notin \mathscr{S}_k$. Therefore, the inequality

$$\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1] \ \leq \ 0$$

must be satisfied. However, there is a contradiction because $w_{n_j}^Y(k) > \frac{1}{2}$ for all $j \in C$. Third, $\{\mathscr{S}_k\}_{k=1}^{m-1}$ might not be closed under taking supersets, i.e., there exist $1 \leq k < m$ and $S \subseteq S'$ such that $S \in \mathscr{S}_k$, but $S' \notin \mathscr{S}_k$. If $S = S'$, there is a contradiction. Thus, focus on the case in which $S \subset S'$. Since $S \in \mathscr{S}_k$, there exists $S'' \subseteq S$ such that $S'' \in \mathscr{S}_{k,min}$. Thus, S'' must meet the inequality

$$\sum_{j \in S''} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \ \geq \ \frac{c}{1 + \frac{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Further, since $S'' \subseteq S \subset S'$ and $w_{n_j}^Y(k) + w_{n_j}^N(k) - 1 > 0$ for all $j \in C$, it holds that

$$\sum_{j \in S'} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \ > \ \frac{c}{1 + \frac{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Moreover, because $S' \notin \mathscr{S}_k$, there exists $S' \subseteq S'''$ such that $S''' \in \mathscr{S}_{\neg k,max}$. Thus, S''' meets the inequality

$$\sum_{j \in S'''} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \ \leq \ \frac{c}{1 + \frac{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Then, since $S' \subseteq S'''$ and, again, $w_{n_j}^Y(k) + w_{n_j}^N(k) - 1 > 0$ for all $j \in C$, it holds that

$$\sum_{j \in S'} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \ \leq \ \frac{c}{1 + \frac{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c} \sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

This is the desired contradiction. Fourth, the set inclusion restriction $\mathscr{S}_k \subseteq \mathscr{S}_{k+1}$ for all $1 \leq k < m-1$ might be violated, meaning, there exists some $1 \leq k < m-1$ and S such that $S \in \mathscr{S}_k$, but $S \notin \mathscr{S}_{k+1}$. Since $S \in \mathscr{S}_k$, there exists $S' \subseteq S$ such that $S' \in \mathscr{S}_{k,min}$. Thus, S' must meet the inequality

$$\sum_{j\in S'} \frac{n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]} \geq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^Y_{n_j}(k)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^N_{n_j}(k)-1]}}.$$

Further, since $S' \subseteq S$ and $w^Y_{n_j}(k) + w^N_{n_j}(k) - 1 > 0$ for all $j \in C$, it holds that

$$\sum_{j\in S} \frac{n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]} \geq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^Y_{n_j}(k)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^N_{n_j}(k)-1]}}$$

or, equivalently,

$$\sum_{j\in S} n_j \cdot [2\cdot w^Y_{n_j}(k) - 1] \geq \sum_{j\in C\setminus S} n_j \cdot [2\cdot w^N_{n_j}(k) - 1].$$

As argued in the proofs of the Lemma 1.2 and Lemma 1.4, it holds that $w^Y_{n_j}(k) < w^Y_{n_j}(k+1)$ as well as $w^N_{n_j}(k+1) < w^N_{n_j}(k)$. Hence, I obtain that

$$\begin{aligned} &\sum_{j\in S} n_j \cdot [2\cdot w^Y_{n_j}(k+1) - 1] > \sum_{j\in S} n_j \cdot [2\cdot w^Y_{n_j}(k) - 1] \\ &\geq \sum_{j\in C\setminus S} n_j \cdot [2\cdot w^N_{n_j}(k) - 1] > \sum_{j\in C\setminus S} n_j \cdot [2\cdot w^N_{n_j}(k+1) - 1], \end{aligned}$$

yielding

$$\sum_{j\in S} \frac{n_j \cdot [w^Y_{n_j}(k+1) + w^N_{n_j}(k+1) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k+1) + w^N_{n_j}(k+1) - 1]} > \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^Y_{n_j}(k+1)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^N_{n_j}(k+1)-1]}}.$$

Further, since $S \notin \mathscr{S}_{k+1}$, there exists $S \subseteq S''$ such that $S'' \in \mathscr{S}_{\neg k+1,max}$. Thus, S'' meets the inequality

$$\sum_{j\in S''} \frac{n_j \cdot [w^Y_{n_j}(k+1) + w^N_{n_j}(k+1) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k+1) + w^N_{n_j}(k+1) - 1]} \leq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^Y_{n_j}(k+1)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^N_{n_j}(k+1)-1]}}.$$

Then, since $S \subseteq S''$ and $w^Y_{n_j}(k+1) + w^N_{n_j}(k+1) - 1 > 0$ for all $j \in C$, it follows that

$$\sum_{j\in S} \frac{n_j \cdot [w^Y_{n_j}(k+1) + w^N_{n_j}(k+1) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k+1) + w^N_{n_j}(k+1) - 1]} \leq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^Y_{n_j}(k+1)-1]}{\frac{1}{c}\sum_{j\in C} n_j\cdot[2\cdot w^N_{n_j}(k+1)-1]}}.$$

However, this contradicts the previous inequality. Consequently, I conclude that all families of coalitions satisfying the discussed set of inequalities are feasible. Finally, I establish that, among the feasible collections of coalitions, there are generically unique families of winning coalitions $\{W_k^*\}_{k=1}^{m-1}$ that meet the discussed inequalities. Hence, $\{W_k^*\}_{k=1}^{m-1}$ must be optimal because, again, there exists a solution and the necessary conditions for optimality determine essentially unique families of winning coalitions.

Towards a contradiction, suppose that there are two distinct families of winning coalitions $\{W_k^*\}_{k=1}^{m-1}$ and $\{V_k^*\}_{k=1}^{m-1}$ that satisfy both the discussed inequalities. This means that there must be some $1 \leq k < m$ and some S such that $S \in W_k^*$, but $S \notin V_k^*$. Since $S \in W_k^*$, there exists $S' \subseteq S$ such that $S' \in W_{k,min}^*$. Hence, S' must meet the inequality

$$\sum_{j \in S'} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \geq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Further, since $w_{n_j}^Y(k) + w_{n_j}^N(k) - 1 > 0$ for all $j \in C$ and because of $S' \subseteq S$, I have that

$$\sum_{j \in S} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \geq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Next, because $S \notin V_k^*$, there exists $S \subseteq S''$ such that $S'' \in V_{\neg k,max}^*$. Thus, S'' satisfies the inequality

$$\sum_{j \in S''} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \leq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Moreover, since $S \subseteq S''$ and $w_{n_j}^Y(k) + w_{n_j}^N(k) - 1 > 0$ for all $j \in C$, it also holds that

$$\sum_{j \in S} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} \leq \frac{c}{1 + \frac{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Hence, there is a contradiction, unless

$$\sum_{j \in S} \frac{n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [w_{n_j}^Y(k) + w_{n_j}^N(k) - 1]} = \frac{c}{1 + \frac{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^Y(k) - 1]}{\frac{1}{c}\sum_{j \in C} n_j \cdot [2 \cdot w_{n_j}^N(k) - 1]}}.$$

Rearranging this equality yields

$$\sum_{j\in S} n_j \cdot [2 \cdot w^Y_{n_j}(k) - 1] \;=\; \sum_{j\in C\setminus S} n_j \cdot [2 \cdot w^N_{n_j}(k) - 1].$$

However, this equality fails generically because any perturbation of $G(k)$ would imply that this equality cannot hold. Therefore, I conclude that the discussed inequalities are not only necessary, but also sufficient for optimality.
Now, for all $j \in C$ and $1 \leq k < m$, set

$$w_j(k) \;:=\; \frac{n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]}$$

as well as

$$q(k) \;:=\; \frac{c}{1 + \frac{\frac{1}{c}\sum_{j\in C} n_j \cdot [2w^Y_{n_j}(k) - 1]}{\frac{1}{c}\sum_{j\in C} n_j \cdot [2w^N_{n_j}(k) - 1]}},$$

and consider the weighted successive voting procedure associated with these weights and quotas. By construction, the families of winning coalitions induced by this weighted successive voting procedure meet all inequalities that are necessary and sufficient for optimality. Hence, this weighted successive voting procedure implements the optimal mechanism. □

Proof of Proposition 1.2.
Consider any $0 \leq \alpha < 1 - \frac{\ln 2}{\ln 3}$ as well as any $1 \leq k < m$. I argue that an increase in the population size by 2, say from n to $n+2$, implies an increase in the ratio, where the corresponding optimal weights are divided by the population size to the power of α. Since population sizes are assumed to be odd, this implies that $\frac{w_{j'}(k)}{n^\alpha_{j'}} < \frac{w_{j''}(k)}{n^\alpha_{j''}}$ for any two countries $j', j'' \in C$ with $n_{j'} < n_{j''}$. Towards a contradiction, assume that an increase in the population size from n to $n+2$ does not yield an increase in the discussed ratio, that is,

$$\frac{w_{n+2}(k)}{[n+2]^\alpha} \;\leq\; \frac{w_n(k)}{n^\alpha},$$

where $w_n(k)$ and $w_{n+2}(k)$ are the optimal weights linked to alternative k for countries with population sizes n and $n+2$ respectively. Using the notation from the main text, this inequality is equivalent to

$$\frac{n^\alpha}{[n+2]^\alpha} \;\leq\; \frac{w_n(k)}{w_{n+2}(k)} \;=\; r(G(k)).$$

The proof of Theorem 1.2 reveals that

$$r(G(k)) \;\leq\; r(\frac{1}{2}) \;=\; \frac{n+1}{n+2}.$$

Hence, I obtain that

$$\frac{n^\alpha}{[n+2]^\alpha} \leq \frac{n+1}{n+2},$$

which is equivalent to

$$\alpha \geq \frac{\ln(\frac{n+1}{n+2})}{\ln(\frac{n}{n+2})} =: f(n).$$

Taking the derivative of the function f yields

$$f'(n) = \frac{n\ln(\frac{n}{n+2}) - 2[n+1]\ln(\frac{n+1}{n+2})}{n[n+1][n+2][\ln(\frac{n}{n+2})]^2}.$$

I claim that $f'(n) \geq 0$ for all $n \geq 1$. Towards a contradiction, assume that there exists $n \geq 1$ such that $f'(n) < 0$. This means that

$$h(n) := n\ln(\frac{n}{n+2}) - 2[n+1]\ln(\frac{n+1}{n+2}) < 0.$$

When taking the derivative of the function h, I obtain that

$$h'(n) = \ln(\frac{n}{n+2}) - \ln((\frac{n+1}{n+2})^2).$$

I have that $h'(n) \geq 0$ for all $n \geq 1$ because $\frac{n}{n+2} < (\frac{n+1}{n+2})^2$ for all $n \geq 1$. This means that the function h is weakly decreasing and, thus, the previous inequality implies that

$$\lim_{l\to\infty} h(l) \leq h(n) = n\ln(\frac{n}{n+2}) - 2[n+1]\ln(\frac{n+1}{n+2}) < 0.$$

Applying L'Hôpital's rule yields $\lim_{l\to\infty} h(l) = 0$, which is the desired contradiction. Therefore, I obtain that $f'(n) \geq 0$ for all $n \geq 1$. Further, the inequality $\alpha \geq f(n)$ above implies that

$$\alpha \geq \frac{\ln(\frac{n+1}{n+2})}{\ln(\frac{n}{n+2})} = f(n) \geq f(1) = \frac{\ln(\frac{2}{3})}{\ln(\frac{1}{3})} = 1 - \frac{\ln 2}{\ln 3}.$$

However, this contradicts the assumption that $\alpha < 1 - \frac{\ln 2}{\ln 3}$. Consequently, I infer that the ratio involving the optimal weights and the population size to the power of α with $0 \leq \alpha < 1 - \frac{\ln 2}{\ln 3}$ is increasing in the population size, establishing the first part of the proposition.

In order to show the second part of the proposition, I argue that an increase in the population size by 2, say from n to $n+2$, implies a decline in the corresponding optimal weights per citizen. Since population sizes are assumed to be odd, this yields the desired conclusion that $\frac{w_{j'}(k)}{n_{j'}} > \frac{w_{j''}(k)}{n_{j''}}$ for any two countries $j', j'' \in C$ with $n_{j'} <$

$n_{j''}$. To the contrary, suppose that an increase in the population size from n to $n+2$ does not yield a decline in the associated weights per citizen, meaning,

$$\frac{w_{n+2}(k)}{n+2} \geq \frac{w_n(k)}{n},$$

where $w_n(k)$ and $w_{n+2}(k)$ describe, again, the optimal weights related to alternative k for countries with population sizes n and $n+2$ respectively. Rearranging this inequality while employing the notation from the main text, I obtain that

$$\frac{n}{n+2} \geq \frac{w_n(k)}{w_{n+2}(k)} = r(G(k)).$$

Now, the proof of Theorem 1.2 shows that

$$r(G(k)) \geq \lim_{G(k)\to 0} r(G(k)) = \lim_{G(k)\to 1} r(G(k)) = \frac{n+1}{n+3}.$$

Therefore, it follows that

$$\frac{n}{n+2} \geq \frac{n+1}{n+3},$$

which is equivalent to $0 \geq 2$. This is the desired contradiction.

□

Proof of Theorem 1.2.
To begin with, I derive an alternative expression for the optimal weights. Consider any $1 \leq k < m$ and any country $j \in C$ and focus on the weight $w_j(k)$. Since the constant $\frac{1}{c}\sum_{j\in C} n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1]$ cancels out, I ignore it and, hence, with abuse of notation, the expression for $w_j(k)$ reduces to

$$w_j(k) = n_j \cdot [w^Y_{n_j}(k) + w^N_{n_j}(k) - 1] = n_j \cdot w^N_{n_j}(k) - n_j \cdot [1 - w^Y_{n_j}(k)].$$

Recall that

$$w^N_{n_j}(k) = \sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{r(n_j,k,s)}{1-R(n_j,k,\frac{n_j-1}{2})} \frac{s}{n_j}$$

and, therefore, I obtain that

$$n_j \cdot w^N_{n_j}(k) = \sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{r(n_j,k,t)}{1-R(n_j,k,\frac{n_j-1}{2})} \cdot s = \mathbb{E}[X^k_{n_j} | X^k_{n_j} \geq \frac{n_j+1}{2}]$$

where the random variable $X^k_{n_j}$ is distributed according to the cdf $R(n_j, k, \cdot)$, that is, $X^k_{n_j}$ follows a binomial distribution with parameters n_j and $1-G(k)$. Additionally, note that

$$w^Y_{n_j}(k) \;=\; \sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{l(n_j,k,t)}{1-L(n_j,k,\frac{n_j-1}{2})}\frac{s}{n_j}$$

and, hence, I get that

$$n_j(1-w^Y_{n_j}(k)) \;=\; n_j - \sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{l(n_j,k,s)}{1-L(n_j,k,\frac{n_j-1}{2})}\cdot s \;=\; \mathbb{E}[n_j - Y^k_{n_j} | Y^k_{n_j} \;\geq\; \frac{n_j+1}{2}]$$

where the random variable $Y^k_{n_j}$ is distributed according to the cdf $L(n_j, k, \cdot)$, that is, $Y^k_{n_j}$ follows a binomial distribution with parameters n_j and $G(k)$. In order to simplify the notation, subsequently, I write G instead of $G(k)$. Since $L(n_j, k, \cdot)$ and $R(n_j, k, \cdot)$ are binomial distributions, the random variable $n_j - Y^k_{n_j}$ has the same distribution as the random variable $X^k_{n_j}$ and, thus, it follows that

$$\mathbb{E}[n_j - Y^k_{n_j} | Y^k_{n_j} \;\geq\; \frac{n_j+1}{2}] \;=\; \mathbb{E}[n_j - Y^k_{n_j} | \frac{n_j-1}{2} \;\geq\; n_j - Y^k_{n_j}] \;=\; \mathbb{E}[X^k_{n_j} | X^k_{n_j} \;\leq\; \frac{n_j-1}{2}].$$

By the law of total expectation, I have that

$$\begin{aligned} n_j(1-G) \;&=\; \mathbb{E}[X^k_{n_j}] \\ =&Pr(X^k_{n_j} \;\geq\; \frac{n_j+1}{2})\mathbb{E}[X^k_{n_j}|X^k_{n_j} \;\geq\; \frac{n_j+1}{2}] + Pr(X^k_{n_j} \;\leq\; \frac{n_j-1}{2})\mathbb{E}[X^k_{n_j}|X^k_{n_j} \;\leq\; \frac{n_j-1}{2}], \end{aligned}$$

which is equivalent to

$$\mathbb{E}[X^k_{n_j}|X^k_{n_j} \;\leq\; \frac{n_j-1}{2}] \;=\; \frac{n_j(1-G) - Pr(X^k_{n_j} \;\geq\; \frac{n_j+1}{2})\mathbb{E}[X^k_{n_j}|X^k_{n_j} \;\geq\; \frac{n_j+1}{2}]}{Pr(X^k_{n_j} \;\leq\; \frac{n_j-1}{2})}.$$

Hence, $w_j(k)$ can be expressed as

$$\begin{aligned} w_j(k) \;=&\mathbb{E}[X^k_{n_j}|X^k_{n_j} \;\geq\; \frac{n_j+1}{2}] - \mathbb{E}[X^k_{n_j}|X^k_{n_j} \;\leq\; \frac{n_j-1}{2}] \\ =&\frac{\mathbb{E}[X^k_{n_j}|X^k_{n_j} \;\geq\; \frac{n_j+1}{2}] - n_j(1-G)}{Pr(X^k_{n_j} \;\leq\; \frac{n_j-1}{2})}. \end{aligned}$$

Ahmed (1991) characterizes the binomial distribution in terms of truncated expectations, and, in particular, it is shown in Ahmed (1991) that the subsequent equation holds for random variables following a binomial distribution:

$$\mathbb{E}[X^k_{n_j} | X^k_{n_j} \geq \frac{n_j+1}{2}] = n_j(1-G) + G\frac{n_j+1}{2}\frac{Pr(X^k_{n_j} = \frac{n_j+1}{2})}{Pr(X^k_{n_j} \geq \frac{n_j+1}{2})}.$$

Therefore, the weight $w_j(k)$ satisfies

$$w_j(k) = \frac{G\frac{n_j+1}{2}Pr(X^k_{n_j} = \frac{n_j+1}{2})}{Pr(X^k_{n_j} \leq \frac{n_j-1}{2})Pr(X^k_{n_j} \geq \frac{n_j+1}{2})}.$$

Now, observe that, for any $j \in C$ and $1 \leq k < m$, the optimal weight $w_j(k)$ depends on k only through G. In the following, write $w_{n_j}(G)$ instead of $w_j(k)$ as well as $w^N_{n_j}(G)$ and $w^Y_{n_j}(G)$ instead of $w^N_{n_j}(k)$ and $w^Y_{n_j}(k)$ respectively. Also, recall that population sizes are odd. I consider an arbitrary population size n and study the behaviour of the ratio of optimal weights involving two countries with population sizes n and $n+2$. In other words, using the notation from the main text, I analyse the behaviour of

$$r(G) = \frac{w_n(G)}{w_{n+2}(G)}$$

as a function of G. I establish that the function $r(G)$ satisfies the subsequent four properties:

$$\begin{aligned}
&\text{(i) } r'(G) > 0 \; \forall G \in (0, \frac{1}{2}),\\
&\text{(ii) } r'(\frac{1}{2}) = 0,\\
&\text{(iii) } r'(G) < 0 \; \forall G \in (\frac{1}{2}, 1),\\
&\text{(iv) } r(G) = r(1-G) \; \forall G \in (0,1).
\end{aligned}$$

The theorem follows directly from these four properties.
Employing the alternative expression for the optimal weights derived above, I have that

$$w_n(G) = \frac{G\frac{n+1}{2}Pr(X^k_n = \frac{n+1}{2})}{Pr(X^k_n \leq \frac{n-1}{2})Pr(X^k_n \geq \frac{n+1}{2})}$$

as well as

$$w_{n+2}(G) = \frac{G\frac{n+3}{2}Pr(X^k_{n+2} = \frac{n+3}{2})}{Pr(X^k_{n+2} \leq \frac{n+1}{2})Pr(X^k_{n+2} \geq \frac{n+3}{2})}.$$

Therefore, I obtain that

$$r(G) = \frac{w_n(G)}{w_{n+2}(G)} = \frac{[G\frac{n+1}{2}Pr(X_n^k = \frac{n+1}{2})][Pr(X_{n+2}^k \le \frac{n+1}{2})Pr(X_{n+2}^k \ge \frac{n+3}{2})]}{[Pr(X_n^k \le \frac{n-1}{2})Pr(X_n^k \ge \frac{n+1}{2})][G\frac{n+3}{2}Pr(X_{n+2}^k = \frac{n+3}{2})]}$$
$$= \frac{\frac{n+1}{2}Pr(X_n^k = \frac{n+1}{2})}{\frac{n+3}{2}Pr(X_{n+2}^k = \frac{n+3}{2})} \frac{Pr(X_{n+2}^k \le \frac{n+1}{2})}{Pr(X_n^k \le \frac{n-1}{2})} \frac{Pr(X_{n+2}^k \ge \frac{n+3}{2})}{Pr(X_n^k \ge \frac{n+1}{2})}.$$

Furthermore, the ratio $\frac{Pr(X_n^k = \frac{n+1}{2})}{Pr(X_{n+2}^k = \frac{n+3}{2})}$ satisfies

$$\frac{Pr(X_n^k = \frac{n+1}{2})}{Pr(X_{n+2}^k = \frac{n+3}{2})} = \frac{\frac{n!}{(\frac{n+1}{2})!(\frac{n-1}{2})!}(1-G)^{\frac{n+1}{2}}G^{\frac{n-1}{2}}}{\frac{(n+2)!}{(\frac{n+1}{2}+1)!(\frac{n+1}{2})!}(1-G)^{\frac{n+1}{2}+1}G^{\frac{n+1}{2}}} = \frac{1}{(1-G)G\frac{(n+2)(n+1)}{(\frac{n+1}{2}+1)\frac{n+1}{2}}}.$$

Hence, the first factor of the expression for $r(G)$ simplifies to

$$\frac{\frac{n+1}{2}}{\frac{n+3}{2}} \frac{Pr(X_n^k = \frac{n+1}{2})}{Pr(X_{n+2}^k = \frac{n+3}{2})} = \frac{\frac{n+1}{2}}{\frac{n+3}{2}} \frac{1}{(1-G)G\frac{(n+2)(n+1)}{(\frac{n+1}{2}+1)\frac{n+1}{2}}} = \frac{1}{4(1-G)G}\frac{n+1}{n+2}.$$

Moreover, I employ the relationship between the binomial distribution and the beta distribution in order to rewrite the other two factors of the expression for $r(G)$. For a reference treating this relationship, see e.g. Abramowitz and Stegun (1965). First, consider the ratio $\frac{Pr(X_{n+2}^k \le \frac{n+1}{2})}{Pr(X_n^k \le \frac{n-1}{2})}$ which can be expressed as

$$\frac{Pr(X_{n+2}^k \le \frac{n+1}{2})}{Pr(X_n^k \le \frac{n-1}{2})} = \frac{\frac{(n+2)!}{(\frac{n+1}{2})!(\frac{n+1}{2})!}\int_0^G [t(1-t)]^{\frac{n+1}{2}}dt}{\frac{n!}{(\frac{n-1}{2})!(\frac{n-1}{2})!}\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt} = 4\frac{n+2}{n+1}\frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}}dt}{\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt}.$$

Second, take the ratio $\frac{Pr(X_{n+2}^k \ge \frac{n+3}{2})}{Pr(X_n^k \ge \frac{n+1}{2})}$ which can be written as

$$\frac{Pr(X_{n+2}^k \ge \frac{n+3}{2})}{Pr(X_n^k \ge \frac{n+1}{2})} = 4\frac{n+2}{n+1}\frac{\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}}dt}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}}dt}.$$

Consequently, $r(G)$ can be expressed as

$$r(G) = 4\frac{n+2}{n+1}\frac{1}{(1-G)G}\frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}}dt}{\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt}\frac{\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}}dt}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}}dt}$$
$$= 4\frac{n+2}{n+1}\frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}}dt}{G\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt}\frac{\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}}dt}{[1-G]\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}}dt}.$$

Hence, it is immediate that property (iv), i.e., $r(G) = r(1-G)$ for all $G \in (0,1)$, holds.
Define the term

$$a(G) := \frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt}{G \int_0^G [t(1-t)]^{\frac{n-1}{2}} dt}.$$

Clearly, it holds that

$$r(G) = 4\frac{n+2}{n+1} a(G)a(1-G)$$

as well as

$$r'(G) = 4\frac{n+2}{n+1}[a'(G)a(1-G) - a(G)a'(1-G)].$$

Thus, it is immediate that property (ii), i.e., $r'(\frac{1}{2}) = 0$, holds. In the following, I show that the equation $r'(G) = 0$ has no other solution. Compute the derivative

$$\begin{aligned} a'(G) = & \frac{G[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt][G(1-G)]^{\frac{n+1}{2}}}{G^2[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]^2} \\ & - \frac{[\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt]\{G[G(1-G)]^{\frac{n-1}{2}} + [\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]\}}{G^2[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]^2}. \end{aligned}$$

Further, I have that $a(1-G) \neq 0$ and $a'(1-G) \neq 0$; To see the latter claim, towards a contradiction, assume that $a'(1-G) = 0$ which is equivalent to

$$0 = [G(1-G)]^{\frac{n+1}{2}} - \frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt][G(1-G)]^{\frac{n-1}{2}}}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]} - \frac{1}{1-G}.$$

Rearranging yields

$$\frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt][G(1-G)]^{\frac{n-1}{2}}}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]} = [G(1-G)]^{\frac{n+1}{2}} - \frac{1}{1-G}.$$

However, the left hand side of this equation is positive, whereas the right hand side is negative. This is the desired contradiction.
Consequently, $r'(G) = 0$ is equivalent to

$$\frac{a'(G)}{a'(1-G)} = \frac{a(G)}{a(1-G)}.$$

To the contrary, suppose that there exists some $G \in (0,1)$ with $G \neq \frac{1}{2}$ such that

$$\frac{a'(G)}{a'(1-G)} = \frac{a(G)}{a(1-G)}.$$

Plugging in the relevant expressions and simplifying yields

$$\frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt}{\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt} = \frac{[G(1-G)]^{\frac{n+1}{2}} - \frac{[\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt][G(1-G)]^{\frac{n-1}{2}}}{[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]} - \frac{1}{G}}{[G(1-G)]^{\frac{n+1}{2}} - \frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt][G(1-G)]^{\frac{n-1}{2}}}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]} - \frac{1}{1-G}}$$

Now, by property (iv), without loss of generality, suppose that $G > \frac{1}{2}$. In this case, the left hand side of the equation is strictly larger than 1. Subsequently, I establish that the right hand side is weakly smaller than 1, implying the desired contradiction. First of all, it holds that

$$[G(1-G)]^{\frac{n+1}{2}} - \frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt][G(1-G)]^{\frac{n-1}{2}}}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]} - \frac{1}{1-G} < 0.$$

To the contrary, suppose that this claim is not true. Then, the reversed inequality can be written as

$$[G(1-G)]^{\frac{n+1}{2}} - \frac{1}{1-G} \geq \frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt][G(1-G)]^{\frac{n-1}{2}}}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]}.$$

However, the left hand side of this inequality is negative, whereas the right hand side is positive. This is the desired contradiction.
Now, towards a contradiction, assume that the right hand side of the equality above is strictly larger than 1. Then, I obtain that

$$\begin{aligned} &\frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt]}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]}[G(1-G)]^{\frac{n-1}{2}} + \frac{1}{1-G} \\ &< \frac{[\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt]}{[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]}[G(1-G)]^{\frac{n-1}{2}} + \frac{1}{G} \end{aligned}$$

or, equivalently,

$$\begin{aligned} &2[G - \frac{1}{2}] \\ &< \{\frac{[\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt]}{[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]} - \frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt]}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]}\}[G(1-G)]^{\frac{n+1}{2}}. \end{aligned}$$

Saunders (1992) provides a recurrence relation for the symmetric incomplete beta function that reads

$$\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt \;=\; [G(1-G)]^{\frac{n+1}{2}} \frac{G-\frac{1}{2}}{n+2} + \frac{n+1}{4n+8}[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]$$

as well as

$$\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt \;=\; [G(1-G)]^{\frac{n+1}{2}} \frac{\frac{1}{2}-G}{n+2} + \frac{n+1}{4n+8}[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt].$$

Thus, the recurrence relation for the symmetric incomplete beta function implies

$$\frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt} \;=\; [G(1-G)]^{\frac{n+1}{2}} \frac{G-\frac{1}{2}}{n+2} \frac{1}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt} + \frac{1}{4}\frac{n+1}{n+2}$$

as well as

$$\frac{\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt} \;=\; [G(1-G)]^{\frac{n+1}{2}} \frac{\frac{1}{2}-G}{n+2} \frac{1}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt} + \frac{1}{4}\frac{n+1}{n+2}.$$

Hence, I have that

$$\begin{aligned}
&\frac{[\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt]}{[\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt]} - \frac{[\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt]}{[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt]} \\
=&[G(1-G)]^{\frac{n+1}{2}} \frac{G-\frac{1}{2}}{n+2} \frac{1}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt} \\
&+ [G(1-G)]^{\frac{n+1}{2}} \frac{G-\frac{1}{2}}{n+2} \frac{1}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt} \\
=&[G(1-G)]^{\frac{n+1}{2}} \frac{G-\frac{1}{2}}{n+2}[\frac{1}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt} + \frac{1}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt}].
\end{aligned}$$

Therefore, the inequality above reduces to

$$\begin{aligned}
&2[G-\frac{1}{2}] \\
&<\; [G(1-G)]^{n+1} \frac{G-\frac{1}{2}}{n+2}[\frac{1}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt} + \frac{1}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt}].
\end{aligned}$$

Since $G > \frac{1}{2}$, this inequality is equivalent to

$$2 \;<\; \frac{[G(1-G)]^{n+1}}{n+2}[\frac{1}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt} + \frac{1}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt}].$$

Rearranging yields

$$2\Big[\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt\Big]\Big[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}}dt\Big]$$
$$<\frac{[G(1-G)]^{n+1}}{n+2}\Big\{\Big[\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt\Big]+\Big[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}}dt\Big]\Big\}.$$

Note that the incomplete beta function satisfies

$$\Big[\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt\Big]+\Big[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}}dt\Big]$$
$$=\int_0^1 [t(1-t)]^{\frac{n-1}{2}}dt \;=\; 2\Big[\int_0^{\frac{1}{2}} [t(1-t)]^{\frac{n-1}{2}}dt\Big].$$

For a reference for this claim, see e.g. Abramowitz and Stegun (1965). Therefore, the inequality above simplifies to

$$b(G)\;:=\Big[\int_0^G [t(1-t)]^{\frac{n-1}{2}}dt\Big]\Big[\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}}dt\Big]$$
$$-[G(1-G)]^{n+1}\frac{[\int_0^{\frac{1}{2}}[t(1-t)]^{\frac{n-1}{2}}dt]}{n+2}\;<\;0.$$

Now, observe that

$$b'(G)\;=-[G(1-G)]^{\frac{n-1}{2}}\Big[\int_{1-G}^G [t(1-t)]^{\frac{n-1}{2}}dt\Big]$$
$$+\Big[\int_0^{\frac{1}{2}} [t(1-t)]^{\frac{n-1}{2}}dt\Big]\frac{n+1}{n+2}[G(1-G)]^n[2G-1].$$

I claim that $b'(G)\le 0$ for all $G\in(\frac{1}{2},1]$. Suppose not, i.e., there exists $G\in(\frac{1}{2},1]$ such that

$$-[G(1-G)]^{\frac{n-1}{2}}\Big[\int_{1-G}^G [t(1-t)]^{\frac{n-1}{2}}dt\Big]$$
$$+\Big[\int_0^{\frac{1}{2}} [t(1-t)]^{\frac{n-1}{2}}dt\Big]\frac{n+1}{n+2}[G(1-G)]^n[2G-1]\;>\;0.$$

This inequality is equivalent to

$$\Big[\int_0^{\frac{1}{2}} [t(1-t)]^{\frac{n-1}{2}}dt\Big]\frac{n+1}{n+2}[G(1-G)]^n[2G-1]$$
$$>[G(1-G)]^{\frac{n-1}{2}}\Big[\int_{1-G}^G [t(1-t)]^{\frac{n-1}{2}}dt\Big].$$

Since the minimizers of the function $[t(1-t)]^{\frac{n-1}{2}}$ on the interval $[1-G, G]$ amount to G as well as $1-G$, I have that

$$
\begin{aligned}
&[\int_0^{\frac{1}{2}} [t(1-t)]^{\frac{n-1}{2}} dt] \frac{n+1}{n+2} [G(1-G)]^n [2G-1] \\
>&[G(1-G)]^{\frac{n-1}{2}} [\int_{1-G}^{G} [t(1-t)]^{\frac{n-1}{2}} dt] \\
\geq&[G(1-G)]^{\frac{n-1}{2}} [\int_{1-G}^{G} [G(1-G)]^{\frac{n-1}{2}} dt] \; = \; [G(1-G)]^{n-1}[2G-1].
\end{aligned}
$$

Hence, because $G > \frac{1}{2}$, this inequality reduces to

$$
[\int_0^{\frac{1}{2}} [t(1-t)]^{\frac{n-1}{2}} dt] \frac{n+1}{n+2} [G(1-G)] \; > \; 1.
$$

However, the left hand side of this inequality is smaller than 1. Thus, there is a contradiction and I conclude that $b'(G) \leq 0$ for all $G \in (\frac{1}{2}, 1]$.
Consequently, I obtain that

$$
\begin{aligned}
0 \; >&[\int_0^{G} [t(1-t)]^{\frac{n-1}{2}} dt][\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt] \\
&- [G(1-G)]^{n+1} \frac{[\int_0^{\frac{1}{2}} [t(1-t)]^{\frac{n-1}{2}} dt]}{n+2} \\
=&b(G) \; \geq \; b(1) \; = \; 0
\end{aligned}
$$

This constitutes the desired contradiction and I infer that the right hand side of the inequality discussed above is weakly smaller than 1. Hence, it follows that the equation $r'(G) = 0$ has no solution distinct from $G = \frac{1}{2}$.
Next, observe that

$$
\lim_{G \to 0} r(G) \; = \; \lim_{G \to 0} \frac{n[w_n^Y(G) + w_n^N(G) - 1]}{[n+2][w_{n+2}^Y(G) + w_{n+2}^N(G) - 1]} \; = \; \frac{n+1}{n+3}
$$

as well as

$$
\lim_{G \to 1} r(G) \; = \; \lim_{G \to 1} \frac{n[w_n^Y(G) + w_n^N(G) - 1]}{[n+2][w_{n+2}^Y(G) + w_{n+2}^N(G) - 1]} \; = \; \frac{n+1}{n+3}.
$$

Now, consider again the function r, that is,

$$
\begin{aligned}
r(G) \; =& \; 4 \frac{n+2}{n+1} \frac{1}{(1-G)G} \frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt \int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt \int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt} \\
=& \; 4 \frac{1}{(1-G)G} \frac{n+1}{n+2} \frac{n+2}{n+1} \frac{\int_0^G [t(1-t)]^{\frac{n+1}{2}} dt}{\int_0^G [t(1-t)]^{\frac{n-1}{2}} dt} \frac{n+2}{n+1} \frac{\int_0^{1-G} [t(1-t)]^{\frac{n+1}{2}} dt}{\int_0^{1-G} [t(1-t)]^{\frac{n-1}{2}} dt}.
\end{aligned}
$$

Employing, again, the recurrence relation for the symmetric incomplete beta function due to Saunders (1992) yields

$$\frac{n+2}{n+1}\frac{\int_0^{1-G}[t(1-t)]^{\frac{n+1}{2}}dt}{\int_0^{1-G}[t(1-t)]^{\frac{n-1}{2}}dt}$$
$$=\frac{[\frac{1}{2}-G][G(1-G)]^{\frac{n+1}{2}}}{[n+1][\int_0^{1-G}[t(1-t)]^{\frac{n-1}{2}}dt]}+\frac{1}{4}$$

as well as

$$\frac{n+2}{n+1}\frac{\int_0^{G}[t(1-t)]^{\frac{n+1}{2}}dt}{\int_0^{G}[t(1-t)]^{\frac{n-1}{2}}dt}$$
$$=\frac{[G-\frac{1}{2}][G(1-G)]^{\frac{n+1}{2}}}{[n+1][\int_0^{G}[t(1-t)]^{\frac{n-1}{2}}dt]}+\frac{1}{4}.$$

Therefore, I obtain that

$$\begin{aligned} r(G) =& 4\frac{1}{(1-G)G}\frac{n+1}{n+2}\Big[\frac{[G-\frac{1}{2}][G(1-G)]^{\frac{n+1}{2}}}{[n+1][\int_0^{G}[t(1-t)]^{\frac{n-1}{2}}dt]}+\frac{1}{4}\Big] \\ &\cdot\Big[\frac{[\frac{1}{2}-G][G(1-G)]^{\frac{n+1}{2}}}{[n+1][\int_0^{1-G}[t(1-t)]^{\frac{n-1}{2}}dt]}+\frac{1}{4}\Big] \end{aligned}$$

Hence, it is immediate that $r(\frac{1}{2})=\frac{n+1}{n+2}$.
Now, note that

$$r(\frac{1}{2}) = \frac{n+1}{n+2} > \frac{n+1}{n+3} = \lim_{G\to 0} r(G) = \lim_{G\to 1} r(G).$$

Therefore, this inequality together with the claim that $r'(G)=0$ has no solution distinct from $G=\frac{1}{2}$ shown above, implies property (i), i.e., $r'(G)>0\ \forall G\in(0,\frac{1}{2})$ as well as property (iii), i.e., $r'(G)<0\ \forall G\in(\frac{1}{2},1)$, of the function r. Consequently, I established all four desired properties of the function r, completing the proof of the theorem. □

Proof of Proposition 1.3.
Take any $0\le\alpha<1-\frac{\ln 2}{\ln 3}$ as well as any $1\le k<m$. I show that an increase in the number of blocks from b to $b+2$ yields a rise in the ratio, where the corresponding optimal weights are divided by the population size to the power of α. Since the numbers of blocks are assumed to be odd, and because population sizes are strictly increasing in the number of blocks, I directly obtain the desired conclusion that $\frac{w_{j'}(k)}{n_{j'}^{\alpha}}<\frac{w_{j''}(k)}{n_{j''}^{\alpha}}$ for any two countries $j',j''\in C$ with $n_{j'}<n_{j''}$. To the contrary, suppose

that a rise in the number of blocks from b to $b+2$ does not imply an increase in the discussed ratio, that is,

$$\frac{w_{b+2}(k)}{[n']^{\alpha}} \leq \frac{w_b(k)}{n^{\alpha}},$$

where $w_b(k)$ and $w_{b+2}(k)$ are the optimal weights linked to alternative k for two countries with numbers of blocks b and $b+2$ respectively, and the variables n and n' denote the population sizes of two such countries respectively. Let s and s' be the corresponding block sizes, and note that $s \leq s'$ by assumption. This inequality is equivalent to

$$\frac{[b \cdot s]^{\alpha}}{[(b+2) \cdot s']^{\alpha}} \leq \frac{w_b(k)}{w_{b+2}(k)}.$$

Applying the proof of Theorem 1.2 implies that

$$\frac{w_b(k)}{w_{b+2}(k)} \leq \frac{s}{s'}\frac{b+1}{b+2}.$$

Consequently, I obtain that

$$\frac{b^{\alpha}}{[b+2]^{\alpha}} \leq [\frac{s}{s'}]^{1-\alpha}\frac{b+1}{b+2} \leq \frac{b+1}{b+2},$$

where the latter inequality follows from $\frac{s}{s'} \leq 1$ and $1-\alpha \geq 0$. Now, observe that this inequality is the same as the inequality

$$\frac{n^{\alpha}}{[n+2]^{\alpha}} \leq \frac{n+1}{n+2},$$

which appears in the proof of Proposition 1.2. Therefore, the arguments presented in the latter proof imply the desired contradiction.

In order to establish the second part of the proposition, I show that a rise in the number of blocks from b to $b+2$ yields a decline in the corresponding optimal weights per citizen. Since the numbers of blocks are assumed to be odd, and because population sizes are strictly increasing in the number of blocks, I obtain the desired conclusion that $\frac{w_{j'}(k)}{n_{j'}} > \frac{w_{j''}(k)}{n_{j''}}$ for any two countries $j', j'' \in C$ with $n_{j'} < n_{j''}$. Towards a contradiction, assume that a rise in the number of blocks from b to $b+2$ does not imply a decline in the associated weights per citizen, meaning,

$$\frac{w_{b+2}(k)}{n'} \geq \frac{w_b(k)}{n},$$

where the notation is the same as in the first part of this proof. Rearranging this inequality, I have that

$$\frac{n}{n'} \geq \frac{w_b(k)}{w_{b+2}(k)}.$$

Now, again, applying the proof of Theorem 1.2 reveals that

$$\frac{w_b(k)}{w_{b+2}(k)} \geq \frac{s}{s'}\frac{b+1}{b+3}$$

Therefore, it follows that

$$\frac{b+1}{b+3} \leq \frac{n}{s}\frac{s'}{n'} = \frac{b}{b+2},$$

which is equivalent to $0 \geq 2$. This is the desired contradiction. □

Appendix 1.B Derivation

I provide the derivation of the expression for the object $\mu_{n_j}^N(k)$ appearing in the main text. The stated term is obtained as follows:

$$\begin{aligned}
&\mu_{n_j}^N(k) \\
=&\mathbb{E}[u^{k+1}(T) - u^k(T)|\text{j votes “No” at } k] \\
=&\sum_{s=\frac{n_j+1}{2}}^{n_j} Pr(\text{“\#peaks} \geq k+1\text{”} = s|\text{“\#peaks} \geq k+1\text{”} \geq \frac{n_j+1}{2}) \\
&\cdot \mathbb{E}[u^{k+1}(T) - u^k(T)|\text{“\#peaks} \geq k+1\text{”} = s] \\
=&\sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{r(n_j,k,s)}{1-R(n_j,k,\frac{n_j-1}{2})} \\
&\{Pr(\text{peak} \geq k+1|\text{“\#peaks} \geq k+1\text{”} = s) \\
&\cdot \mathbb{E}[u^{k+1}(T) - u^k(T)|\text{peak} \geq k+1] \\
&+ Pr(\text{peak} \leq k|\text{“\#peaks} \geq k+1\text{”} = s)\mathbb{E}[u^{k+1}(T) - u^k(T)|\text{peak} \leq k]\} \\
=&\sum_{s=\frac{n_j+1}{2}}^{n_j} \frac{r(n_j,k,s)}{1-R(n_j,k,\frac{n_j-1}{2})} \\
&\{\frac{s}{n_j}\mathbb{E}[u^{k+1}(T) - u^k(T)|\text{peak} \geq k+1] \\
&+ \frac{n_j-s}{n_j}[-1]\mathbb{E}[u^k(T) - u^{k+1}(T)|\text{peak} \leq k]\} \\
=&w_{n_j}^N(k)\mathbb{E}[u^{k+1} - u^k|u^{k+1} > u^k] + [1 - w_{n_j}^N(k)][-1]\mathbb{E}[u^k - u^{k+1}|u^k > u^{k+1}].
\end{aligned}$$

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Chapter 2

Public Goods Provision and Weighted Majority Voting

2.1 Introduction

Frequently, collective decisions about the provision of costly public goods are taken according to weighted majority rules. Two examples are the geographic decentralization of collective decisions as discussed in Fleurbaey (2008) as well as voting in international organizations such as the International Monetary Fund or the World Bank (see International Monetary Fund (2021) and World Bank (2021)).

First, consider the geographic decentralization of collective decisions. Suppose that the public good under consideration is an infrastructure project such as the extension of an airport or the construction of a new train station. Typically, these projects are predominantly financed by the national or the state government. However, as Fleurbaey (2008) explains, people, who live near the location where the infrastructure project might be implemented, are concerned differently about the project compared to people who live far away. Fleurbaey (2008) argues that these systematic differences in the benefits of the public good are the reason why sometimes, instead of nationwide or statewide referenda, collective decisions about these projects are taken according to local referenda. Local referenda are nothing else than an extreme form of a weighted majority rule. In the described situations, there is approximately an equal sharing of the costs of the public good across the voters, but the distributions of the benefits of the public good are heterogeneous across voters. This raises the question of how to assign voting weights to voters that are asymmetric because of non-identical benefit distributions.

Second, consider voting in international organizations such as the International Monetary Fund or the World Bank. In these institutions, collective decisions about the provision of costly public goods, say financial stability in the case of the International Monetary Fund, and poverty reduction in the case of the World Bank, are taken according to weighted majority rules, where the voting weights of the member countries are affine in the capital or payment shares of the member countries (again, see International Monetary Fund (2021) and World Bank (2021)). Therefore, here, it seems to be plausible to assume that the benefits of the public good are roughly distributed identically across member states, but the agents voting on behalf of the member countries are asymmetric because of an unequal sharing of the costs of the public good. This raises the question whether voting weights that are affine in cost shares are optimal, and even more basic whether it is optimal that the voting weights are increasing in the cost shares.

In this chapter, I study the design of welfare-maximizing voting mechanisms for the provision of a costly public good. The setting features multiple public good levels, and the voters are ex-ante asymmetric. In line with the two discussed applications, the asymmetry arises from heterogeneous distributions of the benefits of the public good or from an unequal sharing of the costs of the public good. The heterogeneity of the voters justifies the violation of the important democratic principle of anonymity or "one person, one vote".[1] Therefore, the question of how the asymmetry of the voters should be optimally reflected in the voting weights arises. Gershkov, Moldovanu, and Shi (2017) study the same mechanism design problem, but they assume voters to be ex-ante identical, and restrict attention to anonymous mechanisms.

The main economic problem that is at work here is a Bayesian inference problem: The utilitarian designer has to make inferences about the voters' preference intensities based on their vote choices. These inferred preference intensities depend on the vote choice under consideration as well as on the identity of the voter as they are asymmetric. The designer has to trade-off the different inferred preference intensities in order to determine the welfare-maximizing mechanism.

The main results are as follows: First, I characterize the welfare-maximizing mechanism among all strategy-proof, and surjective social choice functions. It is composed of a sequence of binary weighted majority decisions going gradually from low to high public good levels, and I derive closed-form expressions for the optimal voting weights and majority quotas (Theorem 2.1).[2] In general, these weights and quotas depend on the public good level under consideration.

Second, I study the properties of the welfare-maximizing voting weights. If the benefits of the public good are drawn from heterogeneous distributions, the optimal mechanism assigns higher voting weights to voters whose benefit distributions are more variable (Proposition 2.3), and the utilitarian voting weights are more equal for more extreme public good levels (Proposition 2.4).[3] If the costs of the public good are shared unequally, the optimal voting weights are generally not increasing

1. Note that, in practice, the anonymity criterion is violated in the two applications presented above.

2. The characterization is shown for benefit distributions that admit log-concave densities.

3. The latter result is shown for a large class of benefit distributions derived in Deimen and Szalay (2019) that includes the uniform and the symmetric triangular distribution.

in the voters' cost shares (Proposition 2.5).
The remainder of this chapter is organized as follows: The following section 2.2 discusses the related literature. In section 2.3, I introduce the model. Then, in section 2.4, I present a description of the class of incentive-compatible mechanisms that is convenient for the ensuing optimization task. In section 2.5, I characterize the welfare-maximizing mechanism. In the subsequent section 2.6, I analyse how the asymmetry of the voters is reflected in the welfare-maximizing voting weights. The final section 2.7 concludes. Appendix 2.A contains the proofs.

2.2 Literature

This chapter contributes to the vast literature on the design of mechanisms for the provision of public goods. Green and Laffont (1979) provide an early overview of this literature. For a recent textbook treatment, I refer to Börgers (2015).
More specifically, the present work relates to the literature on the analysis of the utilitarian efficiency of voting mechanisms going back to Rae (1969). Several papers including Barberà and Jackson (2006), Fleurbaey (2008), Azrieli and Kim (2014), and Azrieli (2018) evaluate voting mechanisms according to the utilitarian principle in settings with two alternatives and asymmetric voters. Shao and Zhou (2016) argue in a two-alternatives setting that, under some conditions, voting mechanisms are welfare-maximizing even when monetary transfers are allowed.[4] Similar to the present work, in Shao and Zhou (2016), voters are also asymmetric because of non-identical benefit distributions or unequal cost sharing. The utilitarian voting rule they identify coincides with the optimal voting mechanism derived in Theorem 2.1 for the special case in which there are only two public good levels, but they do not investigate the properties of the optimal mechanism.
I contribute to the literature studying utilitarian voting rules in settings with two alternatives and asymmetric voters as follows: While focusing on public goods provision, I provide comparative statics results how the optimal voting weights vary with the benefit distributions, and the cost shares.
Kleiner and Moldovanu (2017) investigate sequential voting procedures for settings with more than two alternatives and single-peaked preferences. They derive conditions on the dynamic voting procedures under which the induced games admit an ex-post perfect equilibrium in which agents vote sincerely. The successive voting procedures that I consider in the present chapter satisfy these conditions except that the voting rules might be non-anonymous. Also, the successive voting procedures constitute dynamic representations of the static voting by issues mechanisms studied in Nehring and Puppe (2007), who allow for non-anonymous voting.

4. While also considering the case of two alternatives, Drexl and Kleiner (2018) obtain a conceptually similar finding, but they focus on symmetric voters, and anonymous mechanisms.

Gershkov, Moldovanu, and Shi (2017) derive the utilitarian voting rules among all anonymous, unanimous, and strategy-proof mechanisms for settings with more than two alternatives, single-peaked preferences, and ex-ante identical voters. They show that the optimal mechanisms take the form of anonymous successive voting procedures involving majority quotas that are decreasing along the sequence of ballots.[5]
In independent work, Jennings, Laraki, Puppe, and Varloot (2022) consider a model featuring an interval of alternatives, single-peaked preferences, and symmetric voters. For this setting, they characterize non-anonymous strategy-proof social choice functions that maximize utilitarian welfare functions involving agent-specific weights. Since these welfare weights translate one-to-one into alternative-independent voting weights, their maximization effectively amounts to finding the optimal majority quotas, making it similar to Gershkov, Moldovanu, and Shi (2017).
I contribute to the literature on utilitarian voting rules in settings with more than two alternatives, and single-peaked preferences as follows: While concentrating on public goods provision, I characterize the optimal voting mechanisms among all strategy-proof, and surjective social choice functions for settings in which the agents are asymmetric, and the mechanisms are non-anonymous. The presence of heterogeneous voters, and the relaxation of anonymity imply that the voting weights arise as a new design tool in addition to the majority quotas. Moreover, again, I analyze how the resulting alternative-dependent voting weights are shaped by the voters' benefit distributions and cost shares.
In chapter 1 of this dissertation, I study the apportionment problem, meaning, the question of how to assign voting weights to representatives of differently-sized groups of citizens. In contrast, in the present chapter, voters are asymmetric because of preference heterogeneity. Both chapters share that the setting features multiple alternatives, and single-peaked preferences, and that the objective criterion is utilitarian. In terms of results, Proposition 2.1 appears in both chapters. The proof of Theorem 2.1 is, to a large extent, identical to the proof of the characterization of welfare-maximizing mechanisms in chapter 1. However, the two results are not logically related because the assumptions are different. There is no overlap as far as all other findings in the two chapters are concerned.
Finally, regarding dominant-strategy incentive-compatibility, the analysis in the present chapter takes as a starting point a characterization of strategy-proof social choice functions due to Achuthankutty and Roy (2018).[6] In terms of analytical tools, I employ a technical result about stochastic orders from Belzunce, Martínez-Riquelme, and Ruiz (2013). This technical finding involves a stochastic variability ordering, which appeared previously in Szalay (2012) in the context of strategic information transmission. Also, again, in some parts of the analysis, I focus on a

5. Gersbach (2017) surveys related contributions. Again, these contributions focus on ex-ante identical agents as well as anonymous voting rules.

6. They generalize previous results from Moulin (1980), and Saporiti (2009).

class of benefit distributions derived in Deimen and Szalay (2019), who also study strategic information transmission, and compare its performance to delegation.

2.3 Model

The model extends the linear utility model in Gershkov, Moldovanu, and Shi (2017) to allow for voters that are ex-ante asymmetric, and mechanisms that are not anonymous.

There is a finite set of public good levels $\mathcal{K} := \{1, \ldots, K\}$ with $K \geq 2$, and a finite set of voters $N := \{1, \ldots, n\}$ with $n \geq 2$. The voters' preferences feature independent private values. Each voter $i \in N$ has a type governed by the random variable X_i that is distributed according to the cdf F_i. The distribution F_i admits a density f_i, and it has full support on some bounded interval $[\underline{x}_i, \overline{x}_i]$ with $0 \leq \underline{x}_i < \overline{x}_i < \infty$. The types are distributed independently across voters. Each voter $i \in N$ is privately informed about his or her type realization.

The voters' utility functions are affine in types, that is, voter $i \in N$ having type realization $x_i \in [\underline{x}_i, \overline{x}_i]$ derives the following utility from public good level $k \in \mathcal{K}$:

$$u_i^k(x_i) := G^k \cdot x_i - c_i^k.$$

The involved parameters c_i^k and G^k with $k \in \mathcal{K}$ are common knowledge, and they satisfy the restrictions

$$\begin{aligned} c_i^1 < c_i^2 < \ldots < c_i^{K-1} < c_i^K, \text{ and} \\ G^K > G^{K-1} > \ldots > G^2 > G^1 \geq 0. \end{aligned}$$

The function G^k maps public good level indices into utilities and, therefore, the constraint on G^k imposes that higher public good level indices are associated with higher benefits of the public good. The function c_i^k represents the costs of the public good that voter $i \in N$ has to bear and, hence, the assumption on c_i^k means that the costs are increasing in the public good level.

For all voters $i \in N$ and any two public good levels $m, l \in \mathcal{K}$ with $m < l$, define the cutoff

$$x_i^{m,l} := \frac{c_i^l - c_i^m}{G^l - G^m}.$$

This cutoff describes the type realization at which voter i is indifferent between public good levels m and l. Suppose that, for all voters $i \in N$, there are no two distinct cutoffs $x_i^{m,l}$ and $x_i^{m',l'}$ with $m, m', l, l' \in \mathcal{K}$ as well as $m < l$ and $m' < l'$ that coincide. In addition, assume that the cutoffs involving adjacent public good levels are ordered, meaning, for all voters $i \in N$ and any public good level $1 \leq k < K - 1$, it holds that

$$x_i^{k,k+1} < x_i^{k+1,k+2}.$$

Also, the smallest of these cutoffs is larger than the lower bound of the support of the type distribution and the largest of these cutoffs is smaller than the upper bound of the support of the type distribution, that is, $\underline{x}_i < x_i^{1,2}$, and $x_i^{K-1,K} < \overline{x}_i$.
The constraints on the cutoffs involving adjacent public good levels imply that the ordinal preferences induced by the utility representation are single-peaked with respect to the order of the public good levels $1 < ... < K$.[7] The restrictions constitute a mild assumption on the underlying functions G^k and c_i^k. Together with sufficiently large supports, the following conditions are sufficient for the discussed constraints: The function G^k is concave, meaning, $G^{k+1} - G^k$ is weakly decreasing in k, and, for all voters i, the function c_i^k is convex, that is, $c_i^{k+1} - c_i^k$ is weakly increasing in k, and at least one of these two aspects holds strictly.[8]
Finally, let me describe the designer's optimization problem. Monetary transfers are not feasible, the solution concept is dominant-strategy equilibrium, and mechanisms are deterministic. The set of feasible mechanisms is equal to the set of all possibly indirect, surjective, and deterministic mechanisms $\Gamma = (M_1, ..., M_n, h)$, inducing a game that possesses an equilibrium in dominant strategies, where M_i is the message set of voter $i \in N$ and $h : \times_{i \in N} M_i \to \mathcal{K}$ is the outcome function.[9]
The designer maximizes the voters' utilitarian welfare over the described set of mechanisms. Within the class of deterministic mechanisms considered here, it is without loss to restrict attention to direct mechanisms: Deterministic direct mechanisms do not allow the implementation of stochastic outcomes, whereas it is possible to implement such outcomes via mixed strategies of deterministic indirect mechanisms. However, since the solution concept is dominant-strategy equilibrium, all pure message profiles in the support of the mixed strategies are also equilibria. Hence, when choosing the pure action profile yielding the highest welfare among those that are in the support of the mixed strategies, the resulting welfare is not lower compared to the welfare of the equilibrium in mixed strategies. This point has been made by Jarman and Meisner (2017). In other words, they provide a revelation principle in terms of payoffs implying the following result: For any possibly indirect, and surjective mechanism Γ that admits a dominant-strategy equilibrium, there exists a direct mechanism $\Gamma' = ([\underline{x}_1, \overline{x}_1], ..., [\underline{x}_n, \overline{x}_n], h')$ that is dominant-strategy incentive-compatible, and surjective, and the utilitarian welfare under Γ' is weakly higher than under Γ. Therefore, from now on, I restrict attention to direct mechanisms that are dominant-strategy incentive-compatible, and surjective.

7. Gershkov, Moldovanu, and Shi (2017) contains an argument for the stated claim.
8. For example, together with sufficiently large supports, the sufficient conditions are satisfied if G^k is linear in the public good level, i.e., $G^k = k$, the costs take a quadratic form and each voter i bears a positive share s_i of the costs, that is, $c_i^k = s_i \frac{1}{2} k^2$ with $s_i > 0$ and $\sum_{i \in N} s_i = 1$.
9. Surjectivity requires that every public good level is in the image of the outcome function. It represents a mild condition ensuring that no public good level is ex-ante excluded from the collective decision-making process.

2.4 Incentive Compatibility

In this section, I establish a one-to-one relationship between, on the one hand, dominant-strategy incentive-compatible, and surjective direct mechanisms, and, on the other hand, sincere equilibria of successive voting procedures. The successive voting procedure is a sequential voting rule that is frequently used in real-world institutions (see Rasch (2000)). This dynamic implementation generalizes previous results from Gershkov, Moldovanu, and Shi (2017), and Kleiner and Moldovanu (2017), who restrict attention to anonymous mechanisms. Nehring and Puppe (2007) contains static variants of these voting procedures, while allowing for non-anonymous voting. The description of these mechanisms in terms of sequential voting procedures is convenient for the optimization over these voting rules.

The following presentation of the class of successive voting procedures as well as Proposition 2.1 below are also contained in chapter 1 of this dissertation. The material is included here for completeness because the subsequent analysis relies on it.

The definition of successive voting procedures makes use of simple games as discussed in Taylor and Zwicker (1999). More precisely, following Nehring and Puppe (2007), define a family of winning coalitions $\mathscr{W}$ to be a non-empty collection of non-empty subsets of the set of voters N that is closed under taking supersets, meaning, $[W \in \mathscr{W} \wedge W \subseteq W'] \Rightarrow W' \in \mathscr{W}$.

Every successive voting procedure is characterized by $K-1$ families of winning coalitions. For each public good level $1 \leq k < K$, there is a family of winning coalitions $\mathscr{W}_k$. Suppose that these families of winning coalitions are ordered by set inclusion, meaning, for all $1 \leq k < K-1$, it holds that $\mathscr{W}_k \subseteq \mathscr{W}_{k+1}$. This set inclusion restriction means that it is more difficult to collectively accept lower public good levels.

Now, a successive voting procedure is composed of a sequence of binary votes while going gradually from lower to higher public good levels. At each stage of the dynamic procedure, agents vote simultaneously, and they can either approve (action "Yes") or reject (action "No") the public good level that is currently on the agenda. Concretely, a successive voting procedure with families of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{K-1}$ can be described as follows:

(1) To begin with, the lowest public good level 1 is put to a vote, and voters can either approve or reject this public good level. If the set of voters who play "Yes" coincides with some element of the family of winning coalitions $\mathscr{W}_1$, public good level 1 is implemented. Otherwise, public good level 2 is considered.

(2) Agents either vote in favour or against public good level 2. If the set of agents voting in favour of this public good level coincides with some element of the family of winning coalitions $\mathscr{W}_2$, public good level 2 is collectively approved. Otherwise, continue the voting process.

(3) Consider public good level 3, and possibly higher public good levels, and treat them in the same way as public good levels 1 and 2. Eventually, either some

alternative $1 \leq k < K$ is collectively selected or if the set of agents approving public good level $K-1$ does not coincide with some element of $\mathcal{W}_{K-1}$, implement alternative K.

Following Gershkov, Moldovanu, and Shi (2017), an ex-post perfect equilibrium can be defined in words as follows: For every profile of type realizations and at each stage of the dynamic procedure, the continuation strategies constitute a Nash equilibrium of the subgame, where the profile of type realizations is common knowledge.[10] Also, a strategy is called sincere if a voter plays action "Yes" if and only if the public good level on the agenda lies weakly to the right of this voter's most preferred public good level. Proposition 2.1 states that any successive voting procedure admits an ex-post perfect equilibrium in sincere strategies. The same result for the anonymous case is contained in Gershkov, Moldovanu, and Shi (2017), and Kleiner and Moldovanu (2017).

Proposition 2.1. Sincere voting constitutes an ex-post perfect equilibrium in the game induced by any successive voting procedure.

The proof of Proposition 2.1 contained in chapter 1 of this dissertation applies to the present setting: It only requires that the ordinal preferences induced by the utility representation are single-peaked, but the precise utility representation is irrelevant. Therefore, I omit the proof here and refer to chapter 1.
In order to establish the connection between, on the one hand, the sincere equilibria of the successive voting procedures, and, on the other hand, dominant-strategy incentive-compatible, and surjective direct mechanisms, I introduce several well-known definitions. A direct mechanism or social choice function $h : \times_{i \in N}[\underline{x}_i, \overline{x}_i] \to \mathcal{K}$ maps type profiles into public good levels. A social choice function h is dominant-strategy incentive-compatible or strategy-proof if the following condition is met: For all voters $i \in N$ and for all type realizations $x_i, x_i' \in [\underline{x}_i, \overline{x}_i]$ and $x_{-i} \in \times_{j \in N: j \neq i}[\underline{x}_j, \overline{x}_j]$, it holds that

$$u^{h(x_i, x_{-i})}(x_i) \geq u^{h(x_i', x_{-i})}(x_i).^{11}$$

A social choice function h is surjective if, for all public good levels $k \in \mathcal{K}$, there exists a set of type profiles $(Z_1, \ldots, Z_n) \subset \times_{i \in N}[\underline{x}_i, \overline{x}_i]$ arising with positive probability such that, for all $(x_1, \ldots, x_n) \in (Z_1, \ldots, Z_n)$, it holds that $h(x_1, \ldots, x_n) = k$.
It turns out that there is a one-to-one relationship between the sincere equilibria of the dynamic successive voting procedures, and the dominant-strategy equilibria of strategy-proof and surjective direct mechanisms. This result is captured in Proposition 2.2. Gershkov, Moldovanu, and Shi (2017), and Kleiner and Moldovanu (2017)

10. For a formal definition, I refer to Kleiner and Moldovanu (2017).
11. Note that strategy-proofness implies that two distinct type realizations of a voter inducing the same ordinal preference relation are not treated differently.

obtained the same finding while assuming anonymity.
I give a direct proof for Proposition 2.2 based on a characterization of strategy-proof social choice functions due to Achuthankutty and Roy (2018) while explicitly linking the parameters in their characterization and the families of winning coalitions of the successive voting procedures. In independent work, Jennings et al. (2022) establish a similar connection. Achuthankutty and Roy (2018)'s result is contained in Appendix 2.A. However, it should be noted that Proposition 2.2 can, essentially, be also derived by combining existing results from Moulin (1980), Nehring and Puppe (2007), and Achuthankutty and Roy (2018).

Proposition 2.2. (i) Given any successive voting procedure with collections of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{K-1}$, there is a strategy-proof and surjective social choice function such that the outcomes coincide for any realization of type profiles.
(ii) Conversely, given any strategy-proof and surjective social choice function, there is a successive voting procedure with collections of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{K-1}$ such that the outcomes coincide for any realization of type profiles.

Therefore, when focusing on implementation in dominant strategies while insisting on surjectivity, there is no loss in considering successive voting procedures. In other words, when varying the families of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{K-1}$ while maintaining the set inclusion restriction $\mathscr{W}_k \subseteq \mathscr{W}_{k+1}$ for all $1 \le k < K-1$, all strategy-proof and surjective social choice functions can be replicated by the sincere equilibria of the successive voting procedures. Consequently, in the following, I restrict attention to successive voting procedures when it comes to the set of feasible mechanisms.

2.5 Optimal Mechanism

In this section, I provide a characterization of the mechanism that maximizes utilitarian welfare among all dominant-strategy incentive-compatible, and surjective social choice functions.
First of all, Proposition 2.2 implies that the optimization problem reduces to finding the optimal collections of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{K-1}$. Also, because a bounded function is maximized over a finite set of elements, the existence of a solution is guaranteed.
It turns out that the optimal mechanism can be implemented by a weighted successive voting procedure. A successive voting procedure with collections of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{K-1}$ constitutes a weighted successive voting procedure if there exist weights $w_i(k) \in \mathbb{R}_{>0}$ and quotas $q(k) \in \mathbb{R}_{>0}$ with $1 \le k < K$ and $i \in N$ such that, for all $1 \le k < K$ and every set of voters $C \subseteq N$, it holds

$$C \in \mathscr{W}_k \Leftrightarrow \sum_{i \in C} w_i(k) \ge q(k).$$

In words, for each public good level $1 \le k < K$, there is a majority requirement $q(k)$ as well as a vector of voting weights $[w_1(k), ..., w_n(k)]$. In particular, weights and quotas might be sensitive to the public good level that is on the agenda. Then, when it comes to the binary vote on public good level $1 \le k < K$, this public good level is collectively approved if and only if the sum of weights $w_i(k)$ with $i \in N$ associated with agents voting "Yes" at public good level k exceeds the majority requirement $q(k)$. Based on this definition, Theorem 2.1 characterizes the welfare-maximizing mechanism.

Theorem 2.1. *Suppose that the density f_i is log-concave for all $i \in N$. The weighted successive voting procedure with weights*

$$w_i(k) = \frac{\mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}] + \mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\{\mathbb{E}[X_i | X_i \ge x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \le x_i^{k,k+1}]\}}$$

and quotas

$$q(k) = \frac{n}{1 + \frac{\frac{1}{n}\sum_{i \in N}\mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}]}}$$

with $i \in N$ and $1 \le k < K$ implements the optimal mechanism among all strategy-proof and surjective social choice functions.

Theorem 2.1 reveals that the optimal mechanism among all dominant-strategy incentive-compatible, and surjective social choice functions can be implemented via a weighted successive voting procedure, and it provides closed-form expressions for the welfare-maximizing voting weights and majority quotas. In general, the optimal weights and quotas are sensitive to the public good level that is on the agenda, and they reveal how the designer's Bayesian inference problem is resolved. The optimal weights and quotas related to public good level $1 \le k < K$ are determined by comparing the welfare of the two adjacent public good levels k and $k+1$, but they do not depend on the utility the voters derive from other public good levels. Essentially, the regularity condition that the densities f_i with $i \in N$ are log-concave ensures that the comparison of these two public good levels only is valid.
The optimal weights $w_i(k)$ with $i \in N$ and $1 \le k < K$ are proportional to the sum

$$\mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}] + \mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}].$$ [12]

The expression $\mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}]$ reflects the designer's inference about voter i's preference intensity given a "No" vote at public good level $k \in \mathcal{K}$. The term $\mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}]$ captures the inference from a "Yes" vote of voter i at k.

12. Recall that $x_i^{k,k+1}$ denotes the cutoff type at which voter i is different between public good levels k and $k+1$.

The welfare-maximizing quotas $q(k)$ with $1 \leq k < K$ are decreasing in the public good level. To see this, note that the log-concavity of the densities f_i with $i \in N$ implies that the random variables X_i have the decreasing mean residual life as well as the increasing mean inactivity time property (see e.g. Bagnoli and Bergstrom (2005)). Gershkov, Moldovanu, and Shi (2017) also obtain in the symmetric case that the majority quotas are decreasing along the sequence of ballots. Therefore, this feature of the optimal mechanism extends to the asymmetric case.

Before studying, in the next section, the features of the welfare-maximizing voting weights in detail, in order to provide some intuition, let me briefly discuss the special case of uniformly distributed types. Specifically, suppose that $X_i \sim \mathscr{U}[\underline{x}_i, \overline{x}_i]$.[13] In this case, the expressions for the optimal weights $w_i(k)$ simplify, and the weights are proportional to $\overline{x}_i - \underline{x}_i$. This means that the optimal weights are proportional to the length of the support interval. In particular, a larger support interval implies a higher voting weight. The weights depend only on the type distribution via the length of the support interval, and they are independent of the cost sharing structure. Also, the weights are not sensitive to the public good level on the agenda. Of course, these properties are special to the uniform distribution. However, the aspect that a larger support interval implies weights that are uniformly higher for all public good levels generalizes to all distributions admitting a log-concave density in the following way: A larger support interval means that the distribution is more variable, and it turns out that more variable distributions imply uniformly higher weights. The following section makes this claim precise.

2.6 Weights of Asymmetric Voters

In this section, I study properties of the welfare-maximizing voting weights. First, I analyse how the optimal weights vary with the type distribution capturing the benefits of the public good. Second, I investigate how the cost sharing structure shapes the optimal weights.

2.6.1 Heterogeneous Benefit Distributions

The first part of this section focuses on the case of heterogeneous benefit or type distributions, that is, for this part, suppose that the costs of the public good are shared equally. Formally, for all $k \in \mathscr{K}$, there exists $c^k \in \mathbb{R}$ such that $c^k = c_i^k$ for all $i \in N$. This implies that the cutoffs $x_i^{k,k+1}$ that enter the expressions of the optimal weights do not depend on i, and, hence, for simplicity, I write $x^{k,k+1}$ instead of $x_i^{k,k+1}$. It turns out that a more variable benefit distribution implies voting weights that are

13. Clearly, the densities of the uniform distributions are log-concave, and, therefore, Theorem 2.1 applies.

uniformly higher for all public good levels. For any two voters $i, j \in N$, define the following stochastic order:

$$
\begin{aligned}
X_i \geq_v X_j \; :\Leftrightarrow \quad & \\
& \mathbb{E}[X_i] = \mathbb{E}[X_j], \text{ and} \\
& \exists y \in (\min\{\underline{x}_i, \underline{x}_j\}, \max\{\bar{x}_i, \bar{x}_j\}) \; : \; \frac{f_j(x)}{f_i(x)} \\
& \text{non-decreasing for } \min\{\underline{x}_i, \underline{x}_j\} < x < y \text{ and} \\
& \text{non-increasing for } y < x < \max\{\bar{x}_i, \bar{x}_j\}.
\end{aligned}
$$

In words, $X_i \geq_v X_j$ if and only if the expected values of both distributions coincide and the likelihood ratio $\frac{f_j(x)}{f_i(x)}$ is single-peaked over the union of the supports. This stochastic order is not new to the economics literature: In the context of strategic information transmission, Szalay (2012) considers this order while focusing on symmetric distributions. He calls the resulting order mean reverting monotone likelihood ratio property, and, for the case of symmetric distributions, he establishes, for example, the following relation: If $X_i \geq_v X_j$, F_i is a mean preserving spread of F_j. Therefore, intuitively, $X_i \geq_v X_j$ means that F_i is in a strong sense more variable than F_j. For instance, X_i and X_j are ranked in this way if X_i is drawn from the uniform distribution and X_j follows a symmetric triangular distribution both supported on the unit interval $[0, 1]$.
Proposition 2.3 relates the stochastic order $\geq_v$ to the optimal voting weights. The proof of it relies on a technical result about stochastic orders due to Belzunce, Martínez-Riquelme, and Ruiz (2013).

Proposition 2.3. Suppose that $X_i \geq_v X_j$ for some $i, j \in N$. For all $1 \leq k < K$, the optimal voting weights satisfy

$$w_i(k) \geq w_j(k).$$

In words, if a benefit distribution F_i is more variable than a type distribution F_j in the sense of $X_i \geq_v X_j$, the voting weights related to all public good levels are higher for voter i compared to voter j. To put it differently, a stochastic ranking of the type distributions in the sense of $\geq_v$ constitutes a sufficient condition for a uniform ranking of the optimal voting weights. Intuitively, if $X_i \geq_v X_j$, the designer infers from both a “Yes” and a “No” vote at any public good level higher preference intensities for voter i than for voter j and, therefore, the optimal mechanism assigns higher voting weights to voter i compared to voter j.
Next, I address how much the voting weights are larger for voter i than for voter j as a function of the public good level on the agenda given $X_i \geq_v X_j$. Concretely, while assuming that $X_i \geq_v X_j$, I analyse how the ratio

$$\frac{w_j(k)}{w_i(k)}$$

varies with the public good level $1 \leq k < K$, which enters this ratio via the cutoff $x^{k,k+1}$.

As a first step, I specialize to the case, where the corresponding benefit distributions F_i and F_j share a common support in addition to having the same expected value. Let the common support of the two distributions be $[\underline{x}, \overline{x}]$. Under these assumptions, it can be inferred from the expressions of the optimal weights that the ratio discussed above satisfies

$$\lim_{x^{k,k+1} \to \underline{x}} \frac{w_j(k)}{w_i(k)} = \lim_{x^{k,k+1} \to \overline{x}} \frac{w_j(k)}{w_i(k)} = 1.$$

This means that the optimal mechanism stipulates approximately anonymous voting on public good levels $1 \leq k < K$ that are arbitrarily extreme in the sense that the corresponding cutoff $x^{k,k+1}$ converges to the boundaries of the support.

Subsequently, for tractability reasons, I focus on a class of distributions derived in Deimen and Szalay (2019). These authors characterize all symmetric distributions supported on a bounded interval[14] that satisfy the following property: The expectation truncated from below is affine in the truncation point for all truncations points that are larger than the expected value. Specifically, take a voter $i \in N$, let the expected value of his or her type distribution be $\mu := \mathbb{E}[X_i] > 0$, and denote by $[\mu - d, \mu + d] := [\underline{x}_i, \overline{x}_i]$ with $d > 0$ and $\mu - d \geq 0$ the support of F_i. Deimen and Szalay (2019) show that F_i meets the described condition on the truncated expectations if and only if its cdf satisfies

$$F_i(x) = F^{\alpha_i}(x) := \begin{cases} \frac{1}{2}[1 - \frac{\mu - x}{d}]^{\frac{\alpha_i}{1-\alpha_i}}, & x \leq \mu \\ 1 - \frac{1}{2}[1 - \frac{x - \mu}{d}]^{\frac{\alpha_i}{1-\alpha_i}}, & x > \mu \end{cases}$$

for some $0 < \alpha_i < 1$. Let f^{α_i} be the density corresponding to the cdf F^{α_i}. Given this parametrization, for all $\mu \leq t \leq \mu + d$, the stated truncated expectations are given by

$$\mathbb{E}[X_i | X_i \geq t] = \alpha_i \cdot t + [1 - \alpha_i][\mu + d] = \mu + d - \alpha_i \cdot [\mu + d - t].$$

The characterization of welfare-maximizing mechanisms from Theorem 2.1 requires that the density f_i is log-concave. It can be verified that the density f^{α_i} is log-concave if and only if $\alpha_i \geq \frac{1}{2}$. Therefore, in the following, I suppose that, for all voters $i \in N$, the associated benefit distribution coincides with a distribution F^{α_i} derived in Deimen and Szalay (2019) for some parameter $\frac{1}{2} \leq \alpha_i < 1$. In particular, I assume that all type distributions have the common mean μ, and the common support is given by $[\mu - d, \mu + d]$. Observe that the class of distributions from Deimen and Szalay (2019) encompasses, for example, the uniform distribution corresponding to

14. They also allow for the case in which the support is an unbounded interval. This case is ruled out by assumption here.

the parameter $\alpha_i = \frac{1}{2}$. Also, setting $\alpha_i = \frac{2}{3}$ yields the symmetric triangular distribution.

It can be verified that the class of distributions considered here can be ranked according to the stochastic order $\geq_v$: Take any parameters $\frac{1}{2} \leq \alpha_i, \alpha_j < 1$ such that $\alpha_j > \alpha_i$. Then, it holds that $F^{\alpha_i} \geq_v F^{\alpha_j}$, meaning, a smaller parameter induces a more variable distribution in the sense of the presented ordering $\geq_v$. Therefore, Proposition 2.3 yields $\frac{w_j(k)}{w_i(k)} \leq 1$ for all public good levels $1 \leq k < K$.

Go back to the example, where X_i is drawn from the uniform distribution and X_j follows a symmetric triangular distribution with $i, j \in N$, and both distributions are supported on the unit interval $[0, 1]$. For this case, Figure 2.1 pictures the shape of the ratio $\frac{w_j(k)}{w_i(k)}$ as a function of the cutoff $x^{k,k+1}$.[15] This means that the ratio $\frac{w_j(k)}{w_i(k)}$ is

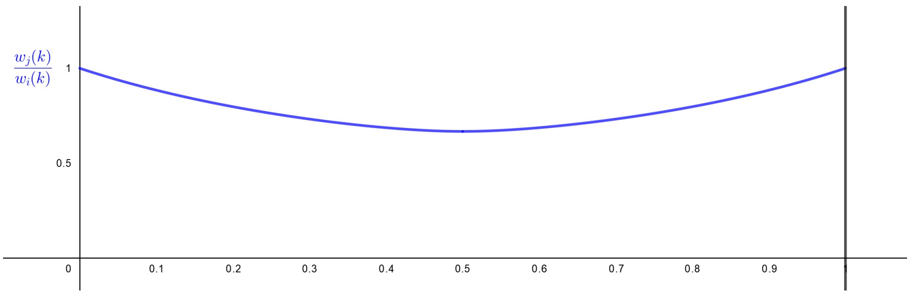

Figure 2.1. Ratio $\frac{w_j(k)}{w_i(k)}$ as a function of $x^{k,k+1}$ if X_i uniform and X_j symmetric triangular on $[0, 1]$

U-shaped when interpreted as a function of the cutoff $x^{k,k+1}$. Proposition 2.4 generalizes this observation to any two distributions from Deimen and Szalay (2019) $F_i = F^{\alpha_i}$ and $F_j = F^{\alpha_j}$ such that $\frac{1}{2} \leq \alpha_i < \alpha_j < 1$.

Proposition 2.4. Suppose that $F_i = F^{\alpha_i}$ and $F_j = F^{\alpha_j}$ with $\frac{1}{2} \leq \alpha_i < \alpha_j < 1$ for some $i, j \in N$, and consider any public good levels $1 \leq k', k'' < K$ with $k' \neq k''$ such that

$$|x^{k',k'+1} - \mu| < |x^{k'',k''+1} - \mu|.$$

The ratio of optimal weights satisfies

$$\frac{w_j(k')}{w_i(k')} < \frac{w_j(k'')}{w_i(k'')}.$$

Proposition 2.4 reveals that the welfare-maximizing voting weights are more equal for more extreme public good levels k in the sense that the associated cutoff $x^{k,k+1}$ is more far away from the common mean of the benefit distributions μ.

15. Straightforward calculations yield this plot.

2.6.2 Unequal Cost Sharing

The second part of this section concentrates on unequal cost sharing, meaning, in this part, I assume that the types are distributed identically across voters. Formally, there is a distribution F such that $F = F_i$ for all voters $i \in N$. As in the previous part, for tractability reasons, I focus on the class of distributions derived in Deimen and Szalay (2019), that is, the common distribution satisfies $F = F^{\alpha}$ for some parameter $\frac{1}{2} < \alpha < 1$. I exclude the case of $\alpha = \frac{1}{2}$ yielding the uniform distribution because, as discussed in section 2.5, the optimal weights do not depend on the cost sharing structure if types are distributed uniformly.

For concreteness, suppose that voter $i \in N$ bears a strictly positive share $s_i > 0$ of the total costs regardless of the provided level of the public good. To put it differently, assume that, for all public good levels $k \in \mathcal{K}$, there exists some $c^k \in \mathbb{R}$ such that

$$c_i^k \;=\; s_i \cdot c^k,$$

where $s_i > 0$ for all $i \in N$ as well as $\sum_{i \in N} s_i = 1$. For all public good levels $1 \leq k < K$, define

$$y^{k,k+1} \;:=\; \frac{1}{s_i} x_i^{k,k+1} \;=\; \frac{c^{k+1} - c^k}{G^{k+1} - G^k},$$

and note that $y^{k,k+1}$ is independent of i.

The following result describes how the optimal weights are shaped by the cost sharing structure.

Proposition 2.5. Suppose that $F = F^{\alpha}$ with $\frac{1}{2} < \alpha < 1$. Fix some $1 \leq k < K$, and consider any $i, j \in N$ such that

$$|s_j \cdot y^{k,k+1} - \mu| \;<\; |s_i \cdot y^{k,k+1} - \mu|.$$

The optimal weights satisfy

$$w_j(k) \;<\; w_i(k).$$

Proposition 2.5 is shown by invoking Proposition 2.4. It reveals that the welfare-maximizing voting weights are, generally, not increasing in the cost share a voter has to bear. Instead, for low public good levels, i.e., small cutoffs $y^{k,k+1}$, the optimal weights might even be decreasing in the cost share. The precise shape of the voting weights depends on all parameters of the model. Intuitively, for small public good levels, voters bearing small cost shares are protected from an underprovision of the public good, and, for large public good levels, voters paying large cost shares are secured against an overprovision of the public good.

Finally, let me mention that Proposition 2.5 can be extended to distributions that go beyond those from Deimen and Szalay (2019) by using sufficient conditions for the convexity and concavity of different truncated expectations due to Gardner (2020).

2.7 Conclusion

In this chapter, I characterize the voting mechanisms for the provision of a costly public good that maximize the utilitarian welfare among all strategy-proof and surjective social choice functions. The setting involves multiple public good levels, and the voters are ex-ante asymmetric because of heterogeneous benefit distributions or unequal cost sharing. The optimal mechanism consists of a sequence of binary weighted majority decisions going gradually from small to large public good levels.
Let me outline the implications of my theoretical results for the two applications discussed in the introduction. First, consider the geographic decentralization of collective decisions as discussed in Fleurbaey (2008), and suppose that the costly public good under consideration is an infrastructure project. Here, it seems plausible to assume that the benefit distributions of people who live closer to the location, where the project might be implemented, is more variable. People who live in the city, where a new train station is supposed to be constructed or the airport is supposed to be extended, might benefit much more from a new train station or a larger airport because of the increased number of nearby mobility opportunities. However, at the same time, they might also be harmed much more because of the intervention in the urban environment or because of more noise. Under this assumption, my theoretical findings suggest to assign weakly higher voting weights to people who live in the city, where the infrastructure project is possibly done, compared to people from the same state or country who live far away. For extreme public good levels, i.e., either high or low public good levels, anonymous voting or voting that respects the democratic principle of "one person, one vote" seems to be approximately optimal. In contrast, the local population should have some discretion when it comes to intermediate public good levels: In these cases, the results suggest to give strictly higher weights to people who live near the location of the public good, but they do not suggest to go all the way to local referenda.
Second, consider voting in international organizations like the International Monetary Fund or the World Bank. Again, here, the costs of the public good are shared unequally across the member states. In general, the welfare-maximizing voting weights are not increasing in the cost shares. This theoretical finding stands in contrast to the voting rules that are used in practice in the International Monetary Fund, and the World Bank (see International Monetary Fund (2021), and World Bank (2021)). The optimal weights depend on all the parameters of the model, but the general tendencies implied by my theoretical results are as follows: If the public good level is low, the optimal weights tend to be decreasing in the cost shares. Instead, for high public good levels, the welfare-maximizing weights tend to be increasing in the cost shares.

Appendix 2.A Proofs

The proof of Proposition 2.2 relies on a characterization of strategy-proof social choice functions due to Achuthankutty and Roy (2018). In order to present this characterization, I collect some definitions and observations contained in Gershkov, Moldovanu, and Shi (2017) and Achuthankutty and Roy (2018).

To begin with, the ordinal preferences induced by the utility representation satisfy by construction single-crossing, that is, for any voter $i \in N$ and any two distinct public good levels $m, l \in \mathcal{K}$ with $m < l$, there is a unique cutoff $x_i^{m,l}$ such that $u_i^m(x_i^{m,l}) = u_i^l(x_i^{m,l})$, $u_i^m(x_i) > u_i^l(x_i)$ for any $x_i \in [\underline{x}_i, x_i^{m,l})$, and $u_i^m(x_i) < u_i^l(x_i)$ for any $x_i \in (x_i^{m,l}, \bar{x}_i]$.

Now, for all voters $i \in N$, define

$$\tau_i(x_i) := \begin{cases} 1, & \underline{x}_i \leq x_i < x_i^{1,2} \\ k, & x_i \in [x_i^{k-1,k}, x_i^{k,k+1}) \text{ with } 1 < k < K \\ K, & x_i^{K-1,K} \leq x_i \leq \bar{x}_i \end{cases}.$$

Observe that $\tau_i(x_i)$ describes voter i's most-preferred public good level if his or her type realization is $x_i \in [\underline{x}_i, \bar{x}_i]$.

Due to the restrictions on the cutoffs involving adjacent public good levels stated in section 2.3, for any voter, every public good level is the most preferred one for some non-degenerate interval of types. Hence, the induced preference domain is said to be regular. Because, for any voter, no two cutoffs coincide, the set of ordinal preferences generated by the utility representation forms a maximal single-crossing domain, that is, the inclusion of any additional preference relation would violate single-crossing. Overall, the utility specification induces a set of ordinal preferences that constitutes a maximal and regular single-crossing domain.

Achuthankutty and Roy (2018) provide a characterization of strategy-proof social choice functions for maximal and regular single-crossing domains. Their result is stated as Theorem 2.2.[16]

Theorem 2.2. *Achuthankutty and Roy (2018)*
A social choice function h is strategy-proof and surjective if and only if it is a min-max rule, that is, there is a family of parameters $(\alpha_S)_{S \subseteq N}$ with

$$\begin{aligned} &\text{(i) } \alpha_S \in \mathcal{K} \text{ for all } S \subseteq N, \\ &\text{(ii) } \alpha_\emptyset = K, \\ &\text{(iii) } \alpha_N = 1 \text{ and} \\ &\text{(iv) } \alpha_T \leq \alpha_S \text{ for all } S, T \subseteq N \text{ with } S \subseteq T \end{aligned}$$

16. In fact, Achuthankutty and Roy (2018)'s finding applies not only to maximal and regular single-crossing domains, but it covers more general classes of domains. Also, these authors impose the condition of unanimity instead of surjectivity. However, it can be verified that, in the presence of strategy-proofness, both properties are equivalent on maximal and regular single-crossing domains.

such that

$$h(x_1,\dots,x_n) \;=\; \min_{S\subseteq N}\{\max_{i\in S}\{\tau_i(x_i),\alpha_S\}\}.$$

Proof of Proposition 2.2.

(i)

Take any successive voting procedure with collections of winning coalitions $\{\mathscr{W}_k\}_{k=1}^{K-1}$. For any $S \subseteq N$, set

$$\alpha_S \;:=\; \begin{cases} 1, & S \in \mathscr{W}_1 \\ k, & S \notin \mathscr{W}_{k-1} \wedge S \in \mathscr{W}_k \text{ with } 1 < k < K \\ K, & S \notin \mathscr{W}_{K-1} \end{cases}.$$

First of all, observe that α_S is well-defined. If $S \in \mathscr{W}_1$, by assumption, $S \in \mathscr{W}_l$ for all $1 \le l < K$ and, hence, the second and third case do not apply. If $S \notin \mathscr{W}_1$, either there is some minimal $1 < k < K$ such that $S \in \mathscr{W}_k$ or $S \notin \mathscr{W}_l$ for all $1 \le l < K$. In the former scenario, the second case applies, but the first and third case do not apply because $S \notin \mathscr{W}_1$ as well as $S \in \mathscr{W}_{K-1}$. In the latter scenario, only the third case applies. Hence, I conclude that α_S is well-defined.

Next, I argue that the four conditions that are required for $(\alpha_S)_{S\subseteq N}$ to be a family of parameters of some min-max rule are satisfied. The restriction $\alpha_S \in \mathscr{K}$ for all $S \subseteq N$, i.e., condition (i), is met by construction. Condition (ii) requires $\alpha_\emptyset = K$. Since families of winning coalitions are by definition collections of non-empty subsets of N, I have that $\emptyset \notin \mathscr{W}_l$ for all $1 \le l < K$. Hence, in particular, $\emptyset \notin \mathscr{W}_{K-1}$ and the third case applies. Thus, as desired, I obtain $\alpha_\emptyset = K$. Because families of winning coalitions are by definition non-empty collections of subsets of N and closed under taking supersets, I infer that $N \in \mathscr{W}_l$ for all $1 \le l < K$. Therefore, in particular, $N \in \mathscr{W}_1$ and the first case applies. Consequently, I obtain $\alpha_N = 1$ as required by condition (iii). Finally, concerning condition (iv), take any $S, T \subseteq N$ such that $S \subseteq T$. If $S \in \mathscr{W}_1$, by assumption, $T \in \mathscr{W}_1$ and, thus, I have $1 = \alpha_T \le \alpha_S = 1$. If $S \notin \mathscr{W}_1$, either there is some minimal $1 < k < K$ such that $S \in \mathscr{W}_k$ or $S \notin \mathscr{W}_l$ for all $1 \le l < K$. In the former case, I obtain $\alpha_S = k$. Moreover, since $S \subseteq T$, $T \in \mathscr{W}_k$ as well as $T \in \mathscr{W}_l$ for all $k \le l < K$. This implies that $\alpha_T > k$ is impossible and, thus, I infer that $\alpha_T \le k = \alpha_S$. In the latter case, in particular, it holds that $S \notin \mathscr{W}_{K-1}$ and, thus, I have $\alpha_S = K$. Hence, the restriction $\alpha_T \le \alpha_S = K$ is met because, by construction, $\alpha_T \le K$. Overall, I conclude that $(\alpha_S)_{S\subseteq N}$ constitutes a family of parameters of some min-max rule.

It remains to verify that the outcomes of both procedures coincide. Let h denote the constructed min-max rule. Take any profile of type realizations $x = (x_1,\dots,x_n) \in \times_{i\in N}[\underline{x}_i, \overline{x}_i]$.

First, suppose that the outcome of the successive voting procedure is $k \in \mathscr{K}$. This means that there must be some set of agents $Z \subseteq N$ such that $Z \in \mathscr{W}_k$ and $\tau_i(x_i) \le k$

for all $i \in Z$. Moreover, since $Z \in \mathscr{W}_k$, $Z \in \mathscr{W}_l$ for all $k \leq l < K$ and, hence, $\alpha_Z \leq k$. Therefore, I obtain

$$h(x) \;=\; \min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \;\leq\; \max_{i \in Z}\{\tau_i(x_i), \alpha_Z\} \;=\; \max\{k, k\} \;=\; k.$$

Further, I claim that $h(x) \geq k$ or, equivalently, for all $S \subseteq N$,

$$\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \;\geq\; k.$$

Subsequently, I perform a case distinction. If $\tau_i(x_i) \geq k$ for some $i \in S$, I obtain

$$\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \;\geq\; \max\{k, \alpha_S\} \;\geq\; k.$$

If $\tau_i(x_i) < k$ or, equivalently, $\tau_i(x_i) \leq k-1$ for all $i \in S$, it must be that $S \notin \mathscr{W}_{k-1}$ because, otherwise, the outcome of the successive voting procedure cannot be k. However, since $S \notin \mathscr{W}_{k-1}$, it holds that $S \notin \mathscr{W}_l$ for all $1 \leq l \leq k-1$ and, thus, I infer that $\alpha_S > k-1$ or, equivalently, $\alpha_S \geq k$. Therefore, I obtain

$$\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \;\geq\; \max_{i \in S}\{\tau_i(x_i), k\} \;\geq\; k.$$

Taking both aspects together, I have $h(x) \leq k$ as well as $h(x) \geq k$ and, consequently, I conclude $h(x) = k$.
Second, assume that $h(x) = k \in \mathscr{K}$ which is equivalent to the two inequalities

$$\min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \;\leq\; k \text{ and } \min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \;\geq\; k.$$

The first inequality implies that there must be some $Z \subseteq N$ such that

$$\max_{i \in Z}\{\tau_i(x_i), \alpha_Z\} \;\leq\; k.$$

This inequality is equivalent to the two inequalities

$$\max_{i \in Z} \tau_i(x_i) \;\leq\; k \text{ and } \alpha_Z \;\leq\; k.$$

Now, $\alpha_Z \leq k$ yields $Z \in \mathscr{W}_k$. However, $\max_{i \in Z} \tau_i(x_i) \leq k$ and $Z \in \mathscr{W}_k$ imply together that the outcome of the successive voting procedure must lie weakly to the left of k. Next, I argue that this outcome must also lie weakly to the right of k. The inequality

$$\min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \;\geq\; k$$

is equivalent to $\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \geq k$ for all $S \subseteq N$. Towards a contradiction, suppose that the outcome of the successive voting procedure lies weakly to the left of $k-1$. Define the set of agents $R := \{i \in N : \tau_i(x_i) \leq k-1\}$. This means that $R \in \mathscr{W}_{k-1}$ which implies that $\alpha_R \leq k-1$. Therefore,

$$\max_{i \in R}\{\tau_i(x_i), \alpha_R\} \;\leq\; \max\{k-1, k-1\} \;=\; k-1$$

which contradicts the inequality $\max_{i \in R}\{\tau_i(x_i), \alpha_R\} \geq k$. Thus, I infer that the outcome of the successive procedure must lie weakly to the right of k. Taking both aspects together, I conclude that this outcome must coincide with k.
(ii)
Take any min-max rule h with family of parameters $(\alpha_S)_{S \subseteq N}$. For any $1 \leq k < K$, set

$$\mathscr{W}_k := \{S \subseteq N : \alpha_S \leq k\}.$$

First of all, I argue that $\mathscr{W}_k$ is a family of winning coalitions. By construction, all elements in the set $\mathscr{W}_k$ are subsets of N. In addition, all these subsets are non-empty because $\emptyset \in \mathscr{W}_k$ would require that $\alpha_\emptyset \leq k$ contradicting condition (ii) satisfied by $(\alpha_S)_{S \subseteq N}$, that is, $\alpha_\emptyset = K$. Further, by condition (iii) imposed on the family of parameters $(\alpha_S)_{S \subseteq N}$, $\alpha_N = 1$ and, hence, $N \in \mathscr{W}_k$ implying that $\mathscr{W}_k \neq \emptyset$. Finally, I claim that $\mathscr{W}_k$ is closed under taking supersets completing the argument that $\mathscr{W}_k$ constitutes a family of winning coalitions. Towards a contradiction, suppose that there are $S, T \subseteq N$ with $S \subseteq T$ such that $S \in \mathscr{W}_k$, but $T \notin \mathscr{W}_k$. Since $S \in \mathscr{W}_k$, $\alpha_S \leq k$. By condition (iv) a family of parameters $(\alpha_S)_{S \subseteq N}$ has to satisfy, it holds that $\alpha_T \leq \alpha_S$ and, thus, $\alpha_T \leq k$. However, this means that $T \in \mathscr{W}_k$ which constitutes the desired contradiction.
Next, I show that $\mathscr{W}_k \subseteq W_{k+1}$ for all $1 \leq k < K$. Suppose not, meaning, there exist some $1 \leq k < K$ and $S \subseteq N$ such that $S \in \mathscr{W}_k$, but $S \notin \mathscr{W}_{k+1}$. By definition, $S \in \mathscr{W}_k$ implies that $\alpha_S \leq k$ and, at the same time, $S \notin \mathscr{W}_{k+1}$ yields $\alpha_S > k$. This is the desired contradiction. Consequently, I conclude that the constructed collections of winning coalitions are valid in the sense that they gives rise to some successive voting procedure.
It remains to verify that the outcomes of both procedures coincide. Take any profile of type realizations $x = (x_1, \ldots, x_n) \in \times_{i \in N}[\underline{x}_i, \overline{x}_i]$.
First, suppose that the outcome of the successive voting procedure is $k \in \mathscr{K}$. This means that there must be some set of agents $Z \subseteq N$ such that $Z \in \mathscr{W}_k$ and $\tau_i(x_i) \leq k$ for all $i \in Z$. Moreover, by definition, since $Z \in \mathscr{W}_k$, $\alpha_Z \leq k$. Therefore, I obtain

$$h(x) = \min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \leq \max_{i \in Z}\{\tau_i(x_i), \alpha_Z\} = \max\{k, k\} = k.$$

Further, I claim that $h(x) \geq k$ or, equivalently, for all $S \subseteq N$,

$$\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \geq k.$$

Subsequently, I perform a case distinction. If $\tau_i(x_i) \geq k$ for some $i \in S$, I obtain

$$\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \geq \max\{k, \alpha_S\} \geq k.$$

If $\tau_i(x_i) < k$ or, equivalently, $\tau_i(x_i) \leq k-1$ for all $i \in S$, it must be that $S \notin \mathscr{W}_{k-1}$ because, otherwise, the outcome of the successive voting procedure cannot be k.

However, by definition, since $S \notin \mathscr{W}_{k-1}$, it holds that $\alpha_S > k-1$ or, equivalently, $\alpha_S \geq k$. Therefore, I obtain

$$\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \;\geq\; \max_{i \in S}\{\tau_i(x_i), k\} \;\geq\; k.$$

Taking both aspects together, I have $h(x) \leq k$ as well as $h(x) \geq k$ and, consequently, I conclude $h(x) = k$.
Second, assume that $h(x) = k \in \mathscr{K}$ which is equivalent to the two inequalities

$$\min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \;\leq\; k \text{ and } \min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \;\geq\; k.$$

The first inequality implies that there must be some $Z \subseteq N$ such that

$$\max_{i \in Z}\{\tau_i(x_i), \alpha_Z\} \;\leq\; k.$$

This inequality is equivalent to the two inequalities

$$\max_{i \in Z} \tau_i(x_i) \;\leq\; k \text{ and } \alpha_Z \;\leq\; k.$$

Now, by definition, $\alpha_Z \leq k$ yields $Z \in \mathscr{W}_k$. However, $\max_{i \in Z} \tau_i(x_i) \leq k$ and $Z \in \mathscr{W}_k$ imply together that the outcome of the successive voting procedure must lie weakly to the left of k. Next, I argue that this outcome must also lie weakly to the right of k. The inequality

$$\min_{S \subseteq N}\{\max_{i \in S}\{\tau_i(x_i), \alpha_S\}\} \;\geq\; k$$

is equivalent to $\max_{i \in S}\{\tau_i(x_i), \alpha_S\} \geq k$ for all $S \subseteq N$. Towards a contradiction, suppose that the outcome of the successive voting procedure lies weakly to the left of $k-1$. Define the set of agents $R := \{i \in N : \tau_i(x_i) \leq k-1\}$. This means that $R \in \mathscr{W}_{k-1}$ which, by definition, implies that $\alpha_R \leq k-1$. Therefore,

$$\max_{i \in R}\{\tau_i(x_i), \alpha_R\} \;\leq\; \max\{k-1, k-1\} \;=\; k-1$$

which contradicts the inequality $\max_{i \in R}\{\tau_i(x_i), \alpha_R\} \geq k$. Thus, I infer that the outcome of the successive procedure must lie weakly to the right of k. Taking both aspects together, I conclude that this outcome must coincide with k. □

Towards characterizing the utilitarian mechanism, subsequently, I derive several lemmata. Take any optimal family of winning coalitions related to alternative $1 \leq k < K$ and let this set be $\mathscr{W}_k^*$. For any $1 \leq k < K$, define

$$\mathscr{W}_{k,min}^* \;:=\; \{W \;\in\; \mathscr{W}_k^* \;:\; \forall W' \text{ with } W' \;\subset\; W \;:\; W' \;\notin\; \mathscr{W}_k^*\}.$$

Lemma 2.1. Fix any $1 \leq k < K$. For any $W \subseteq N$ such that either $W \in \mathscr{W}^*_{1,min}$ (if $k = 1$) or $W \in \mathscr{W}^*_{k,min}$ and $W \notin \mathscr{W}^*_{k-1}$ (if $k > 1$),

$$\sum_{i \in W} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \geq \sum_{i \in N \setminus W} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}].$$

Lemma 2.2. Fix any $1 < k < K$ and assume that the density f_i is log-concave for all $i \in N$. For any $W \subseteq N$ such that $W \in \mathscr{W}^*_{k,min}$, but $W \in \mathscr{W}^*_{k-1}$, it, nevertheless, holds that

$$\sum_{i \in W} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \geq \sum_{i \in N \setminus W} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}].$$

Further, for all $1 \leq k < K$, introduce the set

$$\mathscr{W}^*_{\neg k,max} := \{W \notin \mathscr{W}^*_k : \forall W' \text{ with } W \subset W' : W' \in \mathscr{W}^*_k\}.$$

Lemma 2.3. Fix any $1 \leq k < K$. For any $W' \subseteq N$ such that either $W' \in \mathscr{W}^*_{\neg(K-1),max}$ (if $k = K - 1$) or $W' \in \mathscr{W}^*_{\neg k,max}$ and $W' \in \mathscr{W}^*_{k+1}$ (if $k < K - 1$),

$$\sum_{i \in W'} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \leq \sum_{i \in N \setminus W'} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}].$$

Lemma 2.4. Fix any $1 \leq k < K - 1$ and assume that the density f_i is log-concave for all $i \in N$. For any $W' \subseteq N$ such that $W' \in \mathscr{W}^*_{\neg k,max}$, but $W' \notin \mathscr{W}^*_{k+1}$ it, nevertheless, holds that

$$\sum_{i \in W'} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \leq \sum_{i \in N \setminus W'} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}].$$

Proof of Lemma 2.1.
To start, if $W = N$, it must be that $k = 1$. Hence, here, the desired inequality simplifies to

$$\sum_{i \in N} \mathbb{E}[x_i^{1,2} - X_i | X_i \leq x_i^{1,2}]. \geq 0$$

This inequality is true since $\mathbb{E}[x_i^{1,2} - X_i | X_i \leq x_i^{1,2}] \geq 0$ for all $i \in N$. Thus, in the following, assume that $W \neq N$.
Consider any $W \subset N$ such that either $W \in \mathscr{W}^*_{1,min}$ (if $k = 1$) or $W \in \mathscr{W}^*_{k,min}$ and $W \notin \mathscr{W}^*_{k-1}$ (if $k > 1$) and modify the optimal collections of winning coalitions such that $W \notin \mathscr{W}^*_k$. Because by assumption it holds that either $W \in \mathscr{W}^*_{1,min}$ (if $k = 1$) or $W \in \mathscr{W}^*_{k,min}$ and $W \notin \mathscr{W}^*_{k-1}$ (if $k > 1$) and, in addition, $W \neq N$, this modification of the optimal collections of winning coalitions is feasible. Further, since $W \in \mathscr{W}^*_{k,min}$, this

alteration matters only if $\tau(x_i) \leq k$ for all $i \in W$ and $\tau(x_i) \geq k+1$ for all $i \in N \setminus W$. In this case, under the optimal collections of winning coalitions, if $k=1$, alternative 1 is selected and, if $k>1$, alternative k is chosen because $W \notin \mathscr{W}^*_{k-1}$ implies that $W \notin \mathscr{W}^*_{k'}$ for all $1 \leq k' < k$. Moreover, because $W \in \mathscr{W}^*_k \subseteq \mathscr{W}^*_{k'}$ for all $k < k' < K$, I have that $W \in \mathscr{W}^*_{k'}$ for all $k < k' < K$ and, thus, under the modification, alternative $k+1$ is chosen.

Because the alteration of the optimal collections of winning coalitions should weakly decrease welfare, I get the subsequent condition that is necessary for optimality whenever the considered alteration is feasible:

$$
\begin{aligned}
&\mathbb{E}[\sum_{i\in N} u_i^k(X_i)|(\forall i \in W : X_i \leq x_i^{k,k+1}) \wedge (\forall i \in N\setminus W : X_i \geq x_i^{k,k+1})]\\
\geq&\mathbb{E}[\sum_{i\in N} u_i^{k+1}(X_i)|(\forall i \in W : X_i \leq x_i^{k,k+1}) \wedge (\forall i \in N\setminus W : X_i \geq x_i^{k,k+1})]\\
\Leftrightarrow&\\
&\mathbb{E}[\sum_{i\in W} u_i^k(X_i)|(\forall i \in W : X_i \leq x_i^{k,k+1}) \wedge (\forall i \in N\setminus W : X_i \geq x_i^{k,k+1})]\\
+&\mathbb{E}[\sum_{i\in N\setminus W} u_i^k(X_i)|(\forall i \in W : X_i \leq x_i^{k,k+1}) \wedge (\forall i \in N\setminus W : X_i \geq x_i^{k,k+1})]\\
\geq&\mathbb{E}[\sum_{i\in W} u_i^{k+1}(X_i)|(\forall i \in W : X_i \leq x_i^{k,k+1}) \wedge (\forall i \in N\setminus W : X_i \geq x_i^{k,k+1})]\\
+&\mathbb{E}[\sum_{i\in N\setminus W} u_i^{k+1}(X_i)|(\forall i \in W : X_i \leq x_i^{k,k+1}) \wedge (\forall i \in N\setminus W : X_i \geq x_i^{k,k+1})]\\
\Leftrightarrow&\\
&\sum_{i\in W}\mathbb{E}[u_i^k(X_i)|X_i \leq x_i^{k,k+1}]\\
&+\sum_{i\in N\setminus W}\mathbb{E}[u_i^k(X_i)|X_i \geq x_i^{k,k+1}]\\
\geq&\sum_{i\in W}\mathbb{E}[u_i^{k+1}(X_i)|X_i \leq x_i^{k,k+1}]\\
&+\sum_{i\in N\setminus W}\mathbb{E}[u_i^{k+1}(X_i)|X_i \geq x_i^{k,k+1}]\\
\Leftrightarrow&\\
&\sum_{i\in W}\mathbb{E}[u_i^k(X_i)-u_i^{k+1}(X_i)|X_i \leq x_i^{k,k+1}]\\
\geq&\sum_{i\in N\setminus W}\mathbb{E}[u_i^{k+1}(X_i)-u_i^k(X_i)|X_i \geq x_i^{k,k+1}]\\
\Leftrightarrow&
\end{aligned}
$$

$$\Leftrightarrow$$
$$\sum_{i \in W} \mathbb{E}[(G^k \cdot X_i - c_i^k) - (G^{k+1} \cdot X_i - c_i^{k+1}) | X_i \leq x_i^{k,k+1}]$$
$$\geq \sum_{i \in N \setminus W} \mathbb{E}[(G^{k+1} \cdot X_i - c_i^{k+1}) - (G^k \cdot X_i - c_i^k) | X_i \geq x_i^{k,k+1}]$$
$$\Leftrightarrow$$
$$\sum_{i \in W} \{(c_i^{k+1} - c_i^k) - (G^{k+1} - G^k) \cdot \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}]\}$$
$$\geq \sum_{i \in N \setminus W} \{-(c_i^{k+1} - c_i^k) + (G^{k+1} - G^k) \cdot \mathbb{E}[X_i | X_i \geq x_i^{k,k+1}]\}$$
$$\Leftrightarrow$$
$$\sum_{i \in W} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \geq \sum_{i \in N \setminus W} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}]$$

This constitutes the desired inequality, completing the proof. □

Proof of Lemma 2.2.
Fix any $1 < k < K$ and assume that $W \in \mathcal{W}_{k,min}^*$ and $W \in \mathcal{W}_{k-1}^*$. To begin with, I claim that $W \in \mathcal{W}_{k-1,min}^*$. Suppose not, that is, $W \notin \mathcal{W}_{k-1,min}^*$. Since $W \in \mathcal{W}_{k-1}^*$, there must be some $W' \subset W$ such that $W' \in \mathcal{W}_{k-1}^*$. Moreover, because $W' \in \mathcal{W}_{k-1}^* \subseteq \mathcal{W}_k^*$, I obtain that $W' \in \mathcal{W}_k^*$. This contradicts $W \in \mathcal{W}_{k,min}^*$.
Making use of the feature $W \in \mathcal{W}_{k-1,min}^*$, subsequently, I distinguish different cases:
1) $W \notin \mathcal{W}_{k-2}^*$
In this case, it holds that $W \in \mathcal{W}_{k-1,min}^*$ and $W \notin \mathcal{W}_{k-2}^*$. Consequently, Lemma 2.1 applies to $k-1$.
2) $W \in \mathcal{W}_{k-2}^*$
Take the argument from the previous claim and apply it to $k-2$, implying that $W \in \mathcal{W}_{k-2,min}^*$. When iterating the same case distinction for at most finitely many times, two scenarios are possible:
(i) There exists some $1 < k'' < k$ such that $W \in \mathcal{W}_{k'',min}^*$ and $W \notin \mathcal{W}_{k''-1}^*$. In this case, Lemma 2.1 applies to k''.
(ii) There is no such k'' which implies that $W \in \mathcal{W}_{1,min}^*$. However, then, Lemma 2.1 applies to 1.
Consequently, when merging the two cases (i) and (ii) and applying Lemma 2.1, there must be some $1 \leq k' < k$ (either $k' = 1$ or $k' = k''$) such that

$$\sum_{i \in W} \mathbb{E}[x_i^{k',k'+1} - X_i | X_i \leq x_i^{k',k'+1}] \geq \sum_{i \in N \setminus W} \mathbb{E}[X_i - x_i^{k',k'+1} | X_i \geq x_i^{k',k'+1}].$$

Since $k' < k$, it holds that $x_i^{k',k'+1} < x_i^{k,k+1}$ for all $i \in N$. Moreover, for any $i \in N$, the log-concavity of the density f_i implies that the associated random variable X_i has the decreasing mean residual life as well as the increasing mean inactivity time property (see e.g. Bagnoli and Bergstrom (2005)). Therefore, I obtain that

$$\begin{aligned}
&\sum_{i\in W}\mathbb{E}[x_i^{k,k+1}-X_i|X_i \leq x_i^{k,k+1}]\\
\geq &\sum_{i\in W}\mathbb{E}[x_i^{k',k'+1}-X_i|X_i \leq x_i^{k',k'+1}]\\
\geq &\sum_{i\in N\setminus W}\mathbb{E}[X_i-x_i^{k',k'+1}|X_i \geq x_i^{k',k'+1}]\\
\geq &\sum_{i\in N\setminus W}\mathbb{E}[X_i-x_i^{k,k+1}|X_i \geq x_i^{k,k+1}].
\end{aligned}$$

Consequently, I derived the desired inequality

$$\sum_{i\in W}\mathbb{E}[x_i^{k,k+1}-X_i|X_i \leq x_i^{k,k+1}] \geq \sum_{i\in N\setminus W}\mathbb{E}[X_i-x_i^{k,k+1}|X_i \geq x_i^{k,k+1}].$$

□

Proof of Lemma 2.3.
First of all, if $W'=\emptyset$, it must hold that $k=K-1$ and the desired inequality reduces to

$$0 \leq \sum_{i\in N}\mathbb{E}[X_i-x_i^{K-1,K}|X_i \geq x_i^{K-1,K}]$$

which is true because by assumption $\mathbb{E}[X_i-x_i^{K-1,K}|X_i \geq x_i^{K-1,K}]\geq 0$ for all $i\in N$. Consequently, in the following, suppose that $W'\neq\emptyset$.
Take some $W'\subset N$ such that either $W'\in\mathscr{W}^*_{\neg(K-1),max}$ (if $k=K-1$) or $W'\in\mathscr{W}^*_{\neg k,max}$ and $W'\in\mathscr{W}^*_{k+1}$ (if $k<K-1$) and modify the optimal collections of winning coalitions such that $W'\in\mathscr{W}^*_k$. Since either $W'\in\mathscr{W}^*_{\neg(K-1),max}$ (if $k=K-1$) or $W'\in\mathscr{W}^*_{\neg k,max}$ and $W'\in\mathscr{W}^*_{k+1}$ (if $k<K-1$), this modification is feasible. Moreover, since $W'\in\mathscr{W}^*_{\neg k,max}$, this alteration matters only if $\tau(x_i)\leq k$ for all $i\in W'$ and $\tau(x_i)\geq k+1$ for all $i\in N\setminus W'$. In this case, under the optimal collections of winning coalitions, if $k=K-1$, alternative K is selected since $W'\notin\mathscr{W}^*_{K-1}$ and, if $k<K-1$, alternative $k+1$ is chosen because $W'\in\mathscr{W}^*_{k+1}$. Furthermore, because $W'\notin\mathscr{W}^*_k$, it holds that $W'\notin\mathscr{W}^*_{k'}$ for all $1\leq k'<k$ and, therefore, under the modified collections of winning coalitions, alternative k is selected.
The modification of the optimal collections of winning coalitions weakly decreases welfare and, hence, the following condition is necessary for optimality whenever the considered alteration is feasible:

$$\begin{aligned}
&\mathbb{E}[\sum_{i\in N}u_i^{k+1}(X_i)|(\forall i\in W' : X_i \leq x_i^{k,k+1})\wedge(\forall i\in N\setminus W : X_i \geq x_i^{k,k+1})]\\
\geq&\mathbb{E}[\sum_{i\in N}u_i^{k}(X_i)|(\forall i\in W' : X_i \leq x_i^{k,k+1})\wedge(\forall i\in N\setminus W : X_i \geq x_i^{k,k+1})].
\end{aligned}$$

Rearranging this inequality while making use of the same steps as in the proof of Lemma 2.1, I arrive at the expression

$$\sum_{i\in W'} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \leq \sum_{i\in N\setminus W'} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}].$$

This represents the desired inequality and the proof is completed. □

Proof of Lemma 2.4.
Fix any $1 \leq k < K-1$ and suppose that $W' \in \mathscr{W}^*_{\neg k,max}$ and $W' \notin \mathscr{W}^*_{k+1}$. To start, I argue that $W' \in \mathscr{W}^*_{\neg k+1,max}$. To the contrary, assume that $W' \notin \mathscr{W}^*_{\neg k+1,max}$. Because $W' \notin \mathscr{W}^*_{k+1}$, there must be some $W' \subset W''$ such that $W'' \notin \mathscr{W}^*_{k+1}$. Further, it follows from $\mathscr{W}^*_k \subseteq \mathscr{W}^*_{k+1}$ that $W'' \notin \mathscr{W}^*_k$. However, this contradicts $W' \in \mathscr{W}^*_{\neg k,max}$ because $W' \subset W''$.
Employing the feature that $W' \in \mathscr{W}^*_{\neg k+1,max}$, I perform a case distinction:
1) $W' \in \mathscr{W}^*_{k+2}$
Here, it holds that $W' \in \mathscr{W}^*_{\neg k+1,max}$ as well as $W' \in \mathscr{W}^*_{k+2}$. Therefore, Lemma 2.3 applies to $k+1$.
2) $W' \notin \mathscr{W}^*_{k+2}$
I apply the reasoning from the previous claim to $k+2$ yielding that $W' \in \mathscr{W}^*_{\neg k+2,max}$.
When repeating the same case distinction for at most finitely many times, the following two scenarios can happen:
(i) There exists some $k < k'' < K-1$ such that $W' \in \mathscr{W}^*_{\neg k'',max}$ and $W' \in \mathscr{W}^*_{k''+1}$. However, this means that Lemma 2.3 applies to k''.
(ii) There exists no such k'' implying that $W' \in \mathscr{W}^*_{\neg(K-1),max}$. In this case, Lemma 2.3 applies to $K-1$.
Merging case (i) and (ii) and making use of Lemma 2.3, there must be some $k < k' < K-1$ (either $k' = K-1$ or $k' = k''$) such that

$$\sum_{i\in W'} \mathbb{E}[x_i^{k',k'+1} - X_i | X_i \leq x_i^{k',k'+1}] \leq \sum_{i\in N\setminus W'} \mathbb{E}[X_i - x_i^{k',k'+1} | X_i \geq x_i^{k',k'+1}].$$

Now, again, $k < k'$ yields $x_i^{k,k+1} < x_i^{k',k'+1}$ for all $i \in N$. Moreover, again, for all $i \in N$, since the density f_i is log-concave, the associated random variable X_i has the decreasing mean residual life as well as the increasing mean inactivity time property (see e.g. Bagnoli and Bergstrom (2005)). Consequently, I obtain that

$$\begin{aligned}
&\sum_{i\in W'} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \\
\leq &\sum_{i\in W'} \mathbb{E}[x_i^{k',k'+1} - X_i | X_i \leq x_i^{k',k'+1}] \\
\leq &\sum_{i\in N\setminus W'} \mathbb{E}[X_i - x_i^{k',k'+1} | X_i \geq x_i^{k',k'+1}] \\
\leq &\sum_{i\in N\setminus W'} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}].
\end{aligned}$$

Therefore, I derived the desired inequality

$$\sum_{i \in W'} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}] \le \sum_{i \in N \setminus W'} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}].$$

$\square$

Proof of Theorem 2.1.
To start, the existence of optimal families of winning coalitions $\{W_k^*\}_{k=1}^{K-1}$ is guaranteed since a bounded function is maximized over a finite set of elements. Moreover, all optimal families of winning coalitions have the subsequent features. Merging Lemma 2.1 and Lemma 2.2, for any $1 \le k < K$ and for all $W \in W_{k,min}^*$, it holds that

$$\sum_{i \in W} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}] \ge \sum_{i \in N \setminus W} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}].$$

Rearranging this inequality yields

$$\sum_{i \in W} \left\{ \frac{\mathbb{E}[X_i | X_i \ge x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \le x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \{\mathbb{E}[X_i | X_i \ge x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \le x_i^{k,k+1}]\}} \right\} \ge \frac{n}{1 + \frac{\frac{1}{n}\sum_{i \in N} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}]}}$$

Lemma 2.3 and Lemma 2.4 imply together that, for any $1 \le k < K$ and $W' \in W_{\neg k,max}^*$, the inequality

$$\sum_{i \in W'} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}] \le \sum_{i \in N \setminus W'} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}].$$

is satisfied. This inequality is equivalent to

$$\sum_{i \in W'} \left\{ \frac{\mathbb{E}[X_i | X_i \ge x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \le x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \{\mathbb{E}[X_i | X_i \ge x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \le x_i^{k,k+1}]\}} \right\} \le \frac{n}{1 + \frac{\frac{1}{n}\sum_{i \in N} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \le x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \ge x_i^{k,k+1}]}}$$

This means that these inequalities are necessary for optimality.
Subsequently, I establish that these inequalities determine a generically unique solution, that is, families of winning coalitions that are feasible and optimal. To put it differently, I argue that the inequalities above are also sufficient for optimality. To start, I claim that all families of coalitions $\{\mathcal{S}_k\}_{k=1}^{K-1}$ satisfying these inequalities constitute families of winning coalitions respecting the constraint $\mathcal{S}_k \subseteq \mathcal{S}_{k+1}$ for all $1 \le k < K-1$, meaning, they are feasible. Towards a contradiction, suppose that there exists families of coalitions $\{\mathcal{S}_k\}_{k=1}^{K-1}$ satisfying the previous inequalities, but

they are not feasible. First, $\{S_k\}_{k=1}^{K-1}$ might not be feasible because there exists some $1 \leq k < K$ such that $\emptyset \in \mathscr{S}_k$. This means that the inequality

$$0 \geq \sum_{i \in N} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}].$$

must be met. However, this inequaly cannot be true because, for all $i \in N$, it holds that $\mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}] > 0$. Second, there might be some $1 \leq k < K$ such that $\mathscr{S}_k = \emptyset$. Hence, in particular, it holds that $N \notin \mathscr{S}_k$. Therefore, the inequality

$$\sum_{i \in N} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] \leq 0.$$

must be satisfied. However, there is a contradiction because $\mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}] > 0$ for all $i \in N$. Third, $\{\mathscr{S}_k\}_{k=1}^{K-1}$ might not be closed under taking supersets, i.e., there exist $1 \leq k < K$ and $S \subseteq S'$ such that $S \in \mathscr{S}_k$, but $S' \notin \mathscr{S}_k$. If $S = S'$, there is a contradiction. Thus, focus on the case in which $S \subset S'$. Because $S \in \mathscr{S}_k$, there exists $S'' \subseteq S$ such that $S'' \in \mathscr{S}_{k,min}$. Therefore, S'' must meet the inequality

$$\sum_{i \in S''} \left\{ \frac{\mathbb{E}[X_i | X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \{\mathbb{E}[X_i | X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}]\}} \right\} \geq \frac{n}{1 + \frac{\frac{1}{n} \sum_{i \in N} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}]}}.$$

Because $S'' \subseteq S \subset S'$ and $\mathbb{E}[X_i | X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}] > 0$ for all $i \in N$, it holds that

$$\sum_{i \in S'} \left\{ \frac{\mathbb{E}[X_i | X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \{\mathbb{E}[X_i | X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}]\}} \right\} > \frac{n}{1 + \frac{\frac{1}{n} \sum_{i \in N} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}]}}.$$

Further, since $S' \notin \mathscr{S}_k$, there exists $S' \subseteq S'''$ such that $S''' \in \mathscr{S}_{\neg k,max}$. Thus, S''' meets the inequality

$$\sum_{i \in S'''} \left\{ \frac{\mathbb{E}[X_i | X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \{\mathbb{E}[X_i | X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i | X_i \leq x_i^{k,k+1}]\}} \right\} \leq \frac{n}{1 + \frac{\frac{1}{n} \sum_{i \in N} \mathbb{E}[x_i^{k,k+1} - X_i | X_i \leq x_i^{k,k+1}]}{\frac{1}{n} \sum_{i \in N} \mathbb{E}[X_i - x_i^{k,k+1} | X_i \geq x_i^{k,k+1}]}}.$$

Then, since $S' \subseteq S'''$ and, again, $\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}] > 0$ for all $i \in N$, it holds that

$$\sum_{i\in S'} \{ \frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}} \} \leq \frac{n}{1 + \frac{\frac{1}{n}\sum_{i\in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}.$$

This is the desired contradiction. Fourth, the set inclusion restriction $\mathscr{S}_k \subseteq \mathscr{S}_{k+1}$ for all $1 \leq k < K-1$ might be violated, that is, there exists some $1 \leq k < K-1$ and S such that $S \in \mathscr{S}_k$, but $S \notin \mathscr{S}_{k+1}$. Since $S \in \mathscr{S}_k$, there exists $S' \subseteq S$ such that $S' \in \mathscr{S}_{k,min}$. Thus, S' must meet the inequality

$$\sum_{i\in S'} \{ \frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}} \} \geq \frac{n}{1 + \frac{\frac{1}{n}\sum_{i\in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}.$$

Because of $S' \subseteq S$ and $\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}] > 0$ for all $i \in N$, it holds that

$$\sum_{i\in S} \{ \frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}} \} \geq \frac{n}{1 + \frac{\frac{1}{n}\sum_{i\in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}$$

which is equivalent to

$$\sum_{i\in S}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}] \geq \sum_{i\in N\setminus S}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}].$$

As argued in the proofs of the Lemma 2.2 and Lemma 2.4, for all $i \in N$, it holds that

$$\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}] \leq \mathbb{E}[x_i^{k+1,k+2} - X_i|X_i \leq x_i^{k+1,k+2}]$$

as well as

$$\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}] \geq \mathbb{E}[X_i - x_i^{k+1,k+2}|X_i \geq x_i^{k+1,k+2}].$$

Consequently, I have that

$$\begin{aligned}&\sum_{i\in S}\mathbb{E}[x_i^{k+1,k+2}-X_i|X_i\ \le\ x_i^{k+1,k+2}]\\ \ge&\sum_{i\in S}\mathbb{E}[x_i^{k,k+1}-X_i|X_i\ \le\ x_i^{k,k+1}]\\ \ge&\sum_{i\in N\setminus S}\mathbb{E}[X_i-x_i^{k,k+1}|X_i\ \ge\ x_i^{k,k+1}]\\ \ge&\sum_{i\in N\setminus S}\mathbb{E}[X_i-x_i^{k+1,k+2}|X_i\ \ge\ x_i^{k+1,k+2}],\end{aligned}$$

implying that

$$\sum_{i\in S}\mathbb{E}[x_i^{k+1,k+2}-X_i|X_i\ \le\ x_i^{k+1,k+2}]\ \ge\ \sum_{i\in N\setminus S}\mathbb{E}[X_i-x_i^{k+1,k+2}|X_i\ \ge\ x_i^{k+1,k+2}].$$

Moreover, since $S\notin\mathscr{S}_{k+1}$, there exists $S\subseteq S''$ such that $S''\in\mathscr{S}_{\neg k+1,max}$. Thus, S'' satisfies the inequality

$$\begin{aligned}&\sum_{i\in S''}\{\frac{\mathbb{E}[X_i|X_i\ \ge\ x_i^{k+1,k+2}]-\mathbb{E}[X_i|X_i\ \le\ x_i^{k+1,k+2}]}{\frac{1}{n}\sum_{i\in N}\{\mathbb{E}[X_i|X_i\ \ge\ x_i^{k+1,k+2}]-\mathbb{E}[X_i|X_i\ \le\ x_i^{k+1,k+2}]\}}\}\\ \le&\frac{n}{1+\frac{\frac{1}{n}\sum_{i\in N}\mathbb{E}[x_i^{k+1,k+2}-X_i|X_i\le x_i^{k+1,k+2}]}{\frac{1}{n}\sum_{i\in N}\mathbb{E}[X_i-x_i^{k+1,k+2}|X_i\ge x_i^{k+1,k+2}]}}.\end{aligned}$$

Then, since $S\subseteq S''$ and $\mathbb{E}[X_i|X_i\ge x_i^{k+1,k+2}]-\mathbb{E}[X_i|X_i\le x_i^{k+1,k+2}]>0$ for all $i\in N$, it follows that

$$\begin{aligned}&\sum_{i\in S}\{\frac{\mathbb{E}[X_i|X_i\ \ge\ x_i^{k+1,k+2}]-\mathbb{E}[X_i|X_i\ \le\ x_i^{k+1,k+2}]}{\frac{1}{n}\sum_{i\in N}\{\mathbb{E}[X_i|X_i\ \ge\ x_i^{k+1,k+2}]-\mathbb{E}[X_i|X_i\ \le\ x_i^{k+1,k+2}]\}}\}\\ \le&\frac{n}{1+\frac{\frac{1}{n}\sum_{i\in N}\mathbb{E}[x_i^{k+1,k+2}-X_i|X_i\le x_i^{k+1,k+2}]}{\frac{1}{n}\sum_{i\in N}\mathbb{E}[X_i-x_i^{k+1,k+2}|X_i\ge x_i^{k+1,k+2}]}}\end{aligned}$$

or, equivalently,

$$\sum_{i\in S}\mathbb{E}[x_i^{k+1,k+2}-X_i|X_i\ \le\ x_i^{k+1,k+2}]\ \le\ \sum_{i\in N\setminus S}\mathbb{E}[X_i-x_i^{k+1,k+2}|X_i\ \ge\ x_i^{k+1,k+2}].$$

Combining this inequality with the reversed inequality above, I get the equality

$$\sum_{i\in S}\mathbb{E}[x_i^{k+1,k+2}-X_i|X_i\ \le\ x_i^{k+1,k+2}]\ =\ \sum_{i\in N\setminus S}\mathbb{E}[X_i-x_i^{k+1,k+2}|X_i\ \ge\ x_i^{k+1,k+2}].$$

However, this equality fails generically because any perturbation of the distributions F_i or the cutoffs $x_i^{k+1,k+2}$ with $i\in N$ would imply that this equality cannot hold. Consequently, I conclude that all families of coalitions satisfying the discussed set of inequalities are feasible.

In the following, I argue that, among the feasible collections of coalitions, there are generically unique families of winning coalitions $\{W_k^*\}_{k=1}^{K-1}$ that satisfy the discussed inequalities. Hence, $\{W_k^*\}_{k=1}^{K-1}$ must be optimal because, again, there exists a solution and the necessary conditions for optimality determine generically unique families of winning coalitions. To the contrary, assume that there are two distinct families of winning coalitions $\{W_k^*\}_{k=1}^{K-1}$ and $\{V_k^*\}_{k=1}^{K-1}$ that meet both the discussed inequalities. Without loss of generality, there must be some $1 \leq k < K$ and some S such that $S \in W_k^*$, but $S \notin V_k^*$.

Because $S \in W_k^*$, there exists $S' \subseteq S$ such that $S' \in W_{k,min}^*$. Thus, S' must satisfy the inequality

$$\sum_{i \in S'} \{\frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}}\}$$
$$\geq \frac{n}{1 + \frac{\frac{1}{n}\sum_{i \in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}.$$

Since $\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}] > 0$ for all $i \in N$ and because of $S' \subseteq S$, I have that

$$\sum_{i \in S} \{\frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}}\}$$
$$\geq \frac{n}{1 + \frac{\frac{1}{n}\sum_{i \in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}.$$

Next, because $S \notin V_k^*$, there exists $S \subseteq S''$ such that $S'' \in V_{\neg k,max}^*$. Hence, S'' meets the inequality

$$\sum_{i \in S''} \{\frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}}\}$$
$$\leq \frac{n}{1 + \frac{\frac{1}{n}\sum_{i \in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}.$$

Moreover, since $S \subseteq S''$ and $\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}] > 0$ for all $i \in N$, it also holds that

$$\sum_{i \in S} \{\frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}}\}$$
$$\leq \frac{n}{1 + \frac{\frac{1}{n}\sum_{i \in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i \in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}.$$

Consequently, there is a contradiction, unless

$$\sum_{i\in S}\{\frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}}\}$$
$$= \frac{n}{1 + \frac{\frac{1}{n}\sum_{i\in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}.$$

Rearranging this equality yields

$$\sum_{i\in S}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}] = \sum_{i\in N\setminus S}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}].$$

However, this equality fails generically because any perturbation of the distributions F_i or the cutoffs $x_i^{k,k+1}$ with $i \in N$ would imply that this equality is violated. Therfore, I conclude that the discussed inequalities are not only necessary, but also sufficient for optimality.

Now, for all $i \in N$ and $1 \leq k < K$, set

$$w_i(k) := \frac{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\{\mathbb{E}[X_i|X_i \geq x_i^{k,k+1}] - \mathbb{E}[X_i|X_i \leq x_i^{k,k+1}]\}}$$

as well as

$$q(k) := \frac{n}{1 + \frac{\frac{1}{n}\sum_{i\in N}\mathbb{E}[x_i^{k,k+1} - X_i|X_i \leq x_i^{k,k+1}]}{\frac{1}{n}\sum_{i\in N}\mathbb{E}[X_i - x_i^{k,k+1}|X_i \geq x_i^{k,k+1}]}}$$

and consider the weighted successive voting procedure associated with these weights and quotas. By construction, the families of winning coalitions induced by this weighted successive voting procedure satisfy all inequalities that are necessary and sufficient for optimality. Therefore, this weighted successive voting procedure implements the optimal mechanism. □

The proof of Proposition 2.3 is based on a technical result due to Belzunce, Martínez-Riquelme, and Ruiz (2013). To present this technical finding, introduce the notions of the mean residual life order $\geq_{MRL}$ and the mean inactivity time order $\geq_{MIT}$. The definitions are contained in Belzunce, Martínez-Riquelme, and Ruiz (2013). Two random variables X_i, X_j satisfy $X_i \geq_{MRL} X_j$ if, for all $\min\{\underline{x}_i, \underline{x}_j\} \leq t \leq \min\{\overline{x}_i, \overline{x}_j\}$, it holds that

$$\mathbb{E}[X_i - t|X_i \geq t] \geq \mathbb{E}[X_j - t|X_j \geq t].$$

Two random variables X_i, X_j meet the condition of $X_i \geq_{MIT} X_j$ if, for all $\max\{\underline{x}_i, \underline{x}_j\} \leq t \leq \max\{\overline{x}_i, \overline{x}_j\}$, it holds that

$$\mathbb{E}[t - X_i|X_i \leq t] \leq \mathbb{E}[t - X_j|X_j \leq t]$$

Then, Belzunce, Martínez-Riquelme, and Ruiz (2013) obtain the following result.

Lemma 2.5. Belzunce, Martínez-Riquelme, and Ruiz (2013)
For any $i, j \in N$, the following implications hold:

$$X_i \geq_v X_j \Rightarrow X_i \geq_{MRL} X_j, \text{ and}$$
$$X_i \geq_v X_j \Rightarrow X_i \leq_{MIT} X_j.$$

Proof of Proposition 2.3.
Theorem 2.1 yields that the optimal weights $w_i(k)$ with $i \in N$ related to public good level $1 \leq k < K$ are proportional to the sum

$$\mathbb{E}[X_i - x^{k,k+1} | X_i \geq x^{k,k+1}] + \mathbb{E}[x^{k,k+1} - X_i | X_i \leq x^{k,k+1}].$$

Take two voters $i, j \in N$ such that $X_i \geq_v X_j$. On the one hand, by Lemma 2.5, $X_i \geq_v X_j$ implies $X_i \geq_{MRL} X_j$, and, hence, it follows that

$$\mathbb{E}[X_i - x^{k,k+1} | X_i \geq x^{k,k+1}] \geq \mathbb{E}[X_j - x^{k,k+1} | X_j \geq x^{k,k+1}].$$

On the other hand, again by Lemma 2.5, $X_i \geq_v X_j$ yields $X_i \leq_{MIT} X_j$, and, thus, it holds that

$$\mathbb{E}[x^{k,k+1} - X_i | X_i \leq x^{k,k+1}] \geq \mathbb{E}[x^{k,k+1} - X_j | X_j \leq x^{k,k+1}].$$

Taking both aspects together, I obtain $w_i(k) \geq w_j(k)$ which is the desired conclusion. □

Proof of Proposition 2.4.
Take any two voters $i, j \in N$, and suppose that $F_i = F^{\alpha_i}$ and $F_j = F^{\alpha_j}$ with $\frac{1}{2} \leq \alpha_i < \alpha_j < 1$. By Theorem 2.1, the ratio

$$\frac{w_j(k)}{w_i(k)}$$

depends on the public good level $1 \leq k < K$ only through the cutoff $x^{k,k+1}$. Therefore, I study the behaviour of this ratio as a function of the cutoff. For simplicity, let $r(t)$ denote this ratio if the cutoff is t. Subsequently, I show that r is increasing in t for all $\mu < t \leq \mu + d$. The aspect that r is decreasing in t for all $\mu - d \leq t < \mu$ follows from $r(t') = r(t'')$ if and only if $|t' - \mu| = |t'' - \mu|$ implied by the symmetry of the involved distributions F^{α_i} and F^{α_j}. Hence, I omit the proof of this aspect. The claim of the proposition, then, follows from these two features together with the aspect that $r(t') = r(t'')$ if and only if $|t' - \mu| = |t'' - \mu|$.
Consider any $\mu \leq t \leq \mu + d$. The law of total expectation yields

$$-\mathbb{E}[X_j | X_j \leq t] = \frac{(1 - F^{\alpha_j})\mathbb{E}[X_j | X_j \geq t] - \mu}{F^{\alpha_j}}, \text{ and}$$
$$-\mathbb{E}[X_i | X_i \leq t] = \frac{(1 - F^{\alpha_i})\mathbb{E}[X_i | X_i \geq t] - \mu}{F^{\alpha_i}}.$$

Hence, I have that

$$\begin{aligned}\mathbb{E}[X_j|X_j \geq t] - \mathbb{E}[X_j|X_j \leq t] &= \frac{\mathbb{E}[X_j|X_j \geq t] - \mu}{F^{\alpha_j}}\\ \mathbb{E}[X_i|X_i \geq t] - \mathbb{E}[X_i|X_i \leq t] &= \frac{\mathbb{E}[X_i|X_i \geq t] - \mu}{F^{\alpha_i}}\end{aligned}$$

Consequently, while invoking Theorem 2.1 as well as the formula for the involved truncated expectations from Deimen and Szalay (2019), the stated ratio $r(t)$ satisfies

$$\begin{aligned}r(t) &= \frac{\mathbb{E}[X_j|X_j \geq t] - \mu}{\mathbb{E}[X_i|X_i \geq t] - \mu}\frac{F^{\alpha_i}}{F^{\alpha_j}}\\ &= \frac{d - \alpha_j(\mu + d - t)}{d - \alpha_i(\mu + d - t)}\frac{1 - \frac{1}{2}[1 - \frac{t-\mu}{d}]^{\frac{\alpha_i}{1-\alpha_i}}}{1 - \frac{1}{2}[1 - \frac{t-\mu}{d}]^{\frac{\alpha_j}{1-\alpha_j}}}\\ &= \frac{d - \alpha_j(\mu + d - t)}{d - \alpha_i(\mu + d - t)}\frac{2 - [1 - \frac{t-\mu}{d}]^{\frac{\alpha_i}{1-\alpha_i}}}{2 - [1 - \frac{t-\mu}{d}]^{\frac{\alpha_j}{1-\alpha_j}}}.\end{aligned}$$

Define $k := \frac{\alpha_i}{1-\alpha_i}$ and $m := \frac{\alpha_j}{1-\alpha_j}$, and observe that $k, m \in [1, \infty)$ as well as $k < m$. Then, up to a constant that is independent of t, it holds that

$$r(t) = \frac{d + m(t - \mu)}{d + k(t - \mu)}\frac{2 - [1 - \frac{t-\mu}{d}]^k}{2 - [1 - \frac{t-\mu}{d}]^m}.$$

Define $y := \frac{t-\mu}{d}$, and note that r is increasing in t if and only if it is increasing in y. Also, it holds that $y \in [0, 1)$. Again, up to a constant that is independent of y, the ratio $r(y)$ satisfies

$$r(y) = \frac{1 + my}{1 + ky}\frac{2 - [1 - y]^k}{2 - [1 - y]^m}.$$

Compute the derivative of the ratio r:

$$\begin{aligned}r'(y) =&\frac{[(1 + ky)(2 - [1 - y]^m)][(1 + my)(k)(1 - y)^{k-1} + (2 - [1 - y]^k)m]}{(1 + ky)^2(2 - [1 - y]^m)^2}\\ &-\frac{[(1 + my)(2 - [1 - y]^k)][(1 + ky)(m)(1 - y)^{m-1} + (2 - [1 - y]^m)k]}{(1 + ky)^2(2 - [1 - y]^m)^2}.\end{aligned}$$

Clearly, it holds that $r'(0) = 0$ which is equivalent to $r'(t) = 0$ if $t = \mu$.
Now, towards a contradiction, suppose that there exists some $y \in (0, 1)$ such that $r'(y) = 0$. This implies that

$$\begin{aligned}&[(1 + ky)(2 - [1 - y]^m)][(1 + my)(k)(1 - y)^{k-1} + (2 - [1 - y]^k)m]\\ =&[(1 + my)(2 - [1 - y]^k)][(1 + ky)(m)(1 - y)^{m-1} + (2 - [1 - y]^m)k].\end{aligned}$$

Rearranging yields

$$g(m) := \frac{m}{1+my} - \frac{m(1-y)^{m-1}}{2-[1-y]^m} = \frac{k}{1+ky} - \frac{k(1-y)^{k-1}}{2-[1-y]^k}$$

Subsequently, I establish that $g'(m) > 0$, which yields the desired contradiction. To the contrary, assume that there exists some $m \in [1, \infty)$ such that $g'(m) \leq 0$. Computing the derivative yields

$$g'(m) = -\frac{(1-y)^{m-1}}{2-(1-y)^m} + \frac{1}{(1+my)^2} + \frac{2m(1-y)^{m-1}[-\ln(1-y)]}{(2-(1-y)^m)^2}.$$

Since $-\ln(1-y) > y$ as well as $1+my > 2-(1-y)^m$, it follows that

$$\begin{aligned} 0 \geq g'(m) &> -\frac{(1-y)^{m-1}}{2-(1-y)^m} + \frac{1}{(1+my)^2} + \frac{2m(1-y)^{m-1}y}{(1+my)^2} \\ &= \frac{[1+2m(1-y)^{m-1}y][2-(1-y)^m] - [(1-y)^{m-1}][(1+my)^2]}{[2-(1-y)^m][(1+my)^2]} \end{aligned}$$

While simplifying and rearranging this inequality, I obtain that

$$\begin{aligned} 0 >&[1+2m(1-y)^{m-1}y][2-(1-y)^m] - [(1-y)^{m-1}][(1+my)^2] \\ =&2-(1-y)^{m-1} - (1-y)^m + 2m(1-y)^{m-1}y \\ &- 2m(1-y)^{m+m-1}y - m^2(1-y)^{m-1}y^2 \\ \geq&2-(1-y)^{m-1} - (1-y)^m - m^2(1-y)^{m-1}y^2 =: h(m). \end{aligned}$$

The latter inequality follows from $(1-y)^{m-1} \geq (1-y)^{m+m-1}$.Now, observe that $h(1) = y - y^2 > 0$. Hence, showing $h'(m) > 0$ is sufficient to show that the inequality cannot hold. Towards a contradiction, suppose that there exists some $m \in [0, \infty)$ such that $h'(m) \leq 0$. Compute the derivative of the function h:

$$\begin{aligned} h'(m) =& -2m(1-y)^{m-1}y^2 + (1-y)^{m-1}[-\ln(1-y)] + (1-y)^m[-\ln(1-y)] \\ &+ m^2(1-y)^{m-1}y^2[-\ln(1-y)]. \end{aligned}$$

While using $-\ln(1-y) > y$, I have that

$$0 \geq h'(m) > -2m(1-y)^{m-1}y^2 + (1-y)^{m-1}y + (1-y)^m y + m^2(1-y)^{m-1}y^3,$$

which is equivalent to

$$0 > -2my + 1 + (1-y) + m^2y^2 = m^2y^2 - 2my + 2 - y.$$

When interpreted as a function of m, the right-hand side of this inequality is a parabola that is opening to the top. Therefore, the minimizer over $\mathbb{R}$ solves $m^* := \frac{1}{y}$.

Since $y \in (0,1)$, it holds that $m^* \in [1,\infty)$, and, thus, m^* must also be the minimizer of the discussed function over the interval $[1,\infty)$. Hence, it follows that

$$0 > (m^*)^2y^2 - 2m^*y + 2 - y = (\frac{1}{y})^2y^2 - 2\frac{1}{y}y + 2 - y = 1 - y > 0,$$

which is the desired contradiction.
Consequently, overall, I conclude that there exists no $y \in (0,1)$ such that $r'(y) = 0$, which is equivalent to $r'(t) \neq 0$ for all $\mu < t \leq \mu + d$. Also, recall that $r'(t) = 0$ if $t = \mu$. Now, observe that

$$r(\mu) = \frac{1-\alpha_j}{1-\alpha_i} < 1$$

as well as

$$\lim_{t \to \mu + d} r(t) = 1.$$

From these two observations and the established property $r'(t) \neq 0$ for all $\mu < t \leq \mu + d$, it follows that $r'(t) > 0$ for all $\mu < t \leq \mu + d$. This is the desired claim. □

Proof of Proposition 2.5.
Take two voters $i', j' \in N$ and suppose that $F_{i'} = F^{\frac{1}{2}}$ and $F_{j'} = F^{\alpha}$ with $\frac{1}{2} < \alpha < 1$. In particular, this means that $X_{i'} \sim \mathscr{U}[\underline{x}_{i'}, \overline{x}_{i'}]$. As discussed in section 2.5, in this case, the optimal weights $w_{i'}(k)$ do not depend on the public good level $1 \leq k < K$, but they are are proportional to $\overline{x}_{i'} - \underline{x}_{i'} = 2d$. Therefore, the ratio of optimal weights satisfies

$$\frac{w_{j'}(k)}{w_{i'}(k)} = \frac{w_{j'}(k)}{2d},$$

and, hence, it is sensitive to the public good level k only through $w_{j'}(k)$. By Theorem 2.1, the optimal weight $w_{j'}(k)$ depends on k only through the cutoff $x^{k,k+1}$. Therefore, Proposition 2.4 yields

$$|x^{k',k'+1} - \mu| < |x^{k'',k''+1} - \mu| \Rightarrow w_{j'}(k') < w_{j'}(k'').$$

This means that cutoffs that are further away from the expected value μ imply optimal voting weights that are larger. However, this is precisely the claim that needs to be established. To see this, consider some public good level $1 \leq k < K$, and take two voters $i, j \in N$. Replace $x^{k',k'+1}$ by $s_j \cdot y^{k,k+1}$ and $x^{k'',k''+1}$ by $s_i \cdot y^{k,k+1}$. Then, it follows that

$$w_j(k) = w_{j'}(k') < w_{j'}(k'') = w_i(k).$$

□

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Chapter 3

Committee Search Design★

Joint with Christina Luxen

3.1 Introduction

The decision-making processes in various organizations share the following two characteristics: The decisions are taken collectively by a committee via voting, and the committee evaluates multiple items simultaneously instead of one item at a time. Examples include hiring decisions for high-skill positions and long-term contracts such as academic hiring, project selection in firms, elections of board members or politicians by assemblies as well as legislative decision-making.

In contrast to these examples, the literature on committee or collective search mainly focused on the search process, where items are reviewed "one at a time".[1] The present chapter studies sequential search by committee, where, in each period of time, K items instead of one item can be sampled, and at least M out of N committee members have to approve an item in order to stop search. The focus of the chapter is on the design of the sample size per period, K. We compare different sample sizes in terms of acceptance standards and welfare for the search committee. Moreover, we derive the welfare-maximizing sample size per period for small magnitudes of search costs.

If there are multiple items in each period of time, committee members can directly compare these items. This has two implications: On the one hand, the expected value of an item conditional on accepting increases with the sample size. On the other hand, the probability of approving a particular item decreases with the sample size, and, thus, the expected search costs are altered. Generally, there is a trade-off between these two objects that determine the committee's welfare. The resolution

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1. We only know of one exception: Cao and Zhu (2022). We discuss the relationship to this chapter in section 3.2.

of this trade-off depends on the voting rule and the specification of search costs associated with the simultaneous evaluation of multiple items.

We find that under unanimity voting, the welfare ranking of the discussed search procedures depends on how the search costs vary with the sample size. In contrast, under qualified majority voting distinct from unanimity, this sensitivity to the shape of the cost function partly disappears. In this case, we show that, independently of the shape of the cost function, reviewing more items per period of time improves welfare as long as the magnitude of the search costs is sufficiently small.

Sequential search conducted by a single decision-maker is a special case of committee search with unanimity voting. Our results imply that the classic finding for the single decision-maker case where evaluating multiple items at a time instead of only one does not improve welfare if there are no related economies of scale (see e.g. Manning and Morgan (1985)) does not extend to committee search with qualified majority voting other than unanimity voting. In other words, unless the search committee operates under unanimity voting, the economic trade-offs determining a good search rule for committees are significantly different from those in the single decision-maker case. Thus, treating the committee as a single agent would lead to systematically wrong predictions.

The sample size per period is a relevant design parameter that can be interpreted in different ways. First, it can be viewed as delayed voting: Suppose that one item per period arrives. Then, simultaneously evaluating K items corresponds to taking voting decisions only every K periods instead of every single period. In other words, choosing the sample size K can be interpreted as selecting voting times.[2] Second, choosing the sample size can be seen as designing the number of alternatives that are put to a vote in each period of time, while holding the number of committee members and the required degree of approval to stop search fixed.

The chapter speaks to several relevant questions or issues. Consider the examples mentioned above. In the hiring context, one or multiple candidates could be evaluated simultaneously. Reviewing candidates "one at a time" or, in other words, hiring on a rolling basis corresponds to the case in which the sample size per period equals one. Therefore, the chapter provides some answers to the question under which circumstances a hiring process should be conducted on a rolling basis. When it comes to the election of board members or politicians by assemblies, there are often several candidates running in the election and the following issue arises: Should the number of candidates that are still allowed to participate in the election be reduced after a few election rounds if no candidate receives the required support in these rounds? In legislative decision-making, when there are multiple bills relating to the same policy issue, the following question occurs: Should there be a separate vote on each bill or should the vote on the bills be bundled? The chapter might provide

2. We thank Olivier Compte for suggesting this interpretation.

some guidance for these issues.

In our model, a committee consisting of $N \geq 1$ members searches for one item. The committee reviews in each time period a fixed number of items $K \geq 1$ simultaneously. The time horizon is infinite, and rejected items cannot be recalled. The committee members' preferences feature independent private values. For every member, the value of an item is a random variable, which is distributed independently and identically across time, members, and items. Each committee member observes his or her own value realization for every item and has distributional knowledge about the other members' values. Since the collective decisions in the above mentioned examples are taken rather infrequently, there might be a lot of uncertainty about the members' preferences, making the incomplete information assumption plausible.

We consider a class of voting rules where each member may either vote for one of the available items or may opt to continue search. An item is then approved if and only if at least M out of N members vote in favor of it, where the qualified majority threshold M ranges from simple majority to unanimity. This class of voting rules is frequently used in practice, making it a natural choice when adopting an approach that is positive with regard to the voting rule, but normative with respect to the search technology.[3] For example, when abstracting from abstention, the default voting rule for collective decisions by the general assembly of registered associations in Germany prescribes that any resolution requires the support of a simple majority, independently of the number of alternatives (cf. Bundesrepublik Deutschland (2019)).

If an item is accepted, search stops; otherwise, search continues, and each committee member bears an additive search cost $c \cdot h(K) > 0$. We restrict the committee members' voting strategies to symmetric and neutral[4] stationary Markov strategies. Then, a member votes in favor of an item if and only if the item's value is the highest among all observed K values and it exceeds some cutoff representing the member's acceptance standard. Acceptance standards coincide with welfare because values are private.[5]

To begin with, we prove the existence and uniqueness of a symmetric and neutral stationary Markov equilibrium for all $K \geq 1$ and for all qualified majority voting rules including unanimity voting. The uniqueness of equilibrium is shown for value distributions that admit a log-concave density. In the subsequent comparison of search procedures, we maintain this distributional assumption.

Consider two search procedures with $K' \geq 1$ and $K \geq 1$ items per period, and suppose that $K' > K$. First, we study the case of unanimity voting in detail. We find that if the cost function h satisfies $\frac{h(K')}{K'} \geq \frac{h(K)}{K}$, i.e., the search costs per item are weakly

3. As soon as there is more than one item per period, other voting rules are also conceivable. We discuss this point in section 3.7.

4. A strategy is *neutral* if it does not condition on the identity of the item.

5. To be precise, this holds if and only if the equilibrium cutoff is interior.

higher if there are K' versus K items at a time, evaluating K items per period yields higher acceptance standards and welfare than reviewing K' items at a time.[6] Intuitively, given some acceptance standard, the expected value of an item conditional on stopping is higher if there are K' than if there are K items per period. However, at the same time, expected search costs are also higher because the probability of hiring a particular item is lower and the function h satisfies $\frac{h(K')}{K'} \geq \frac{h(K)}{K}$. We show that the increase in the expected value conditional on stopping is limited and that the overall trade-off is resolved in favor of the search procedure with K items per period. This result implies in particular that sequential search with one item per period is welfare-maximizing if the function h meets the condition $\frac{h(K')}{K'} \geq h(1)$ for all $K' > 1$. In contrast, if $\frac{h(K')}{K'} < \frac{h(K)}{K}$, reviewing K' items at a time yields higher welfare than evaluating K items in every period if the magnitude of search costs quantified by the parameter c is sufficiently small. Here, if c is small, acceptance standards are close to the upper bound of the support of the value distribution. Hence, while the probability of hiring a particular item is higher under the search technology featuring K items per period, it is low for both search procedure. Therefore, if c is sufficiently small, expected search costs are actually lower if there are K' items at a time because h is assumed to satisfy $\frac{h(K')}{K'} < \frac{h(K)}{K}$. In addition, as before, the expected value conditional on stopping for a sample size K' is not lower than the respective value for a sample size K. Moreover, if the search costs per item are minimal for some sample size per period $K \geq 2$ and for exogenous reasons at most $\bar{K} < \infty$ items can be reviewed simultaneously, the two discussed results imply together that sequential search with a sample size that coincides with the smallest minimizer of the search costs per item is welfare-maximizing as long as the magnitude of the search costs c is sufficiently small. Extensions to interdependent values and correlated values contained in section 3.7 show the robustness of these findings.

Second, we investigate qualified majority voting rules that do not require full unanimity. Again, consider two search procedures with $K' \geq 1$ and $K \geq 1$ items per period, and assume that $K' > K$. We find that reviewing K' items at a time yields a higher welfare than evaluating K items per period for all cost functions h as long as c is sufficiently small. Thus, the sensitivity to the shape of the cost function h that we find for the unanimity rule partly disappears. To prove this result, we first establish that the ranking of the expected values conditional on stopping from the unanimity voting case carries over to qualified majority, meaning, the respective expected value is higher if there are K' compared to K items per period. Then, we show that if c is sufficiently small, this increase in the expected value conditional on stopping outweighs the potential rise in expected costs.[7] Furthermore, this result has the following implication for the welfare-maximizing sample size per period. Suppose

6. This result does not require the density of the value distribution to be log-concave.

7. Depending on the shape of the cost function h, expected search costs might also decrease. Of course, this only reinforces our reasoning.

that for exogenous reasons at most $\bar{K} < \infty$ items can be evaluated simultaneously in each period of time. Then, whatever the shape of the cost function h, sequential search with $\bar{K}$ items at a time is welfare-maximizing as long as the magnitude of the search costs c is sufficiently small.

Consequently, the comparison of sequential search featuring different sample sizes per period differs considerably if the search committee operates under qualified majority voting instead of unanimity voting. This is the main insight of this chapter. Moreover, as alluded to above, our results imply in particular that the conclusions for search conducted by a single decision-maker which is a special form of committee search with unanimity voting do not carry over to committee search with qualified majority voting.

The chapter is organized as follows: Section 3.2 reviews the related literature, section 3.3 introduces the model, and section 3.4 proves the existence and uniqueness of the equilibrium. Section 3.5 treats the unanimity voting case, and section 3.6 contains the results for qualified majority voting rules. The next section 3.7 contains the extensions to interdependent and correlated values, and we discuss other voting rules. The final section 3.8 concludes. Appendix 3.B contains the proofs, and Appendix 3.C derives expressions for the probability of approving a particular item and the expected value conditional on stopping.

3.2 Related Literature

The present chapter contributes to the growing literature on committee search where a committee conducts search dynamically over time.[8] Albrecht, Anderson, and Vroman (2010), Compte and Jehiel (2010), and Moldovanu and Shi (2013) study different aspects of committee search while focusing exclusively on sequential search with one item per period. For example, Albrecht, Anderson, and Vroman (2010) study the implications of different voting rules or committee sizes while holding the search technology, i.e., one item per period, fixed. In contrast, this chapter focuses on the effect of different search procedures on acceptance standards and welfare given the voting rule. Therefore, we contribute to the literature on committee search by introducing sequential search by committee featuring multiple items per period and by comparing search protocols with different sample sizes per period in terms of acceptance standards and welfare.

We know of only one other contribution that is concerned with the comparison of different search technologies in the committee search environment.[9] In independent work, Cao and Zhu (2022) compare sequential search with one item per period to a

8. The static case of committee decision-making has also been analyzed in depth, cf. the survey by Li and Suen (2009).

9. In the literature on auctions, the comparison between different selling technologies has been studied before. Wang (1993) compares auctions to posted-price selling in terms of revenue and prices

fixed-sample-size search technology that can be described as follows: First, the committee determines collectively the total sample size. Then, the items are drawn until the predetermined sample size is reached. Finally, the committee selects collectively one out of these items. There are two main differences between Cao and Zhu (2022) and our chapter: First, our model is much more general. Second, more importantly, fixed-sample-size search is conceptually different from the search technologies we study, and, therefore, their results are also different.

In the literature on search conducted by a single decision-maker, not only sequential search with one item at a time due to McCall (1970), but also other search technologies have been treated.[10] In Morgan (1983) as well as Manning and Morgan (1985), search is conducted by a single decision-maker, and they consider general classes of search procedures, where, in each period, the single agent decides how many items to draw in the following period if search continues and whether to stop search in the current period. Thus, sequential search with a fixed number of items per period conducted by a single decision-maker is part of the search protocols studied in Morgan (1983) as well as Manning and Morgan (1985).

Morgan (1983) derives properties of the optimal sample size in each time period depending on the searcher's recall, time horizon, and outside option, but he does not analytically identify conditions on the primitives of the model under which sequential search with one item per period is optimal. However, he mentions numerical simulations indicating that sequential search with one item at a time might not be optimal if there is no recall and there are intraperiodic economies of scale in the simultaneous evaluation of multiple items. Our analytical result for committee search with unanimity voting and cost functions h satisfying $\frac{h(K')}{K'} < \frac{h(K)}{K}$ with $K' > K \geq 1$ specialized to the single-agent case addresses this point.

Manning and Morgan (1985) show analytically that sequential search with one item per period conducted by a single agent is optimal if the time horizon is infinite, there is full recall, and the single searcher bears additive search costs that are increasing and convex in the number of items per period. This result resembles our finding for committee search with unanimity voting and cost functions h satisfying $\frac{h(K')}{K'} \geq h(1)$ for all $K' > 1$ when specializing it to the single-searcher case. Note that Manning and Morgan (1985) assume full recall, whereas we assume that rejected alternatives cannot be recalled. Yet, as long as the sample size per period does not depend on calendar time as it is the case in our model, in the single-agent case, the no recall assumption is without loss.[11] Therefore, our finding for committee search with

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and finds that the ranking of the two technologies depends on the seller's auctioning costs and on the steepness of the marginal revenue curve.

10. See for instance Stigler (1961), Rothschild (1974), and Burdett and Judd (1983).

11. For sequential search with one item at a time and a single decision-maker, this point has been made previously by Albrecht, Anderson, and Vroman (2010).

unanimity voting and functions h satisfying $\frac{h(K')}{K'} \geq h(1)$ for all $K' > 1$ specialized to the single-agent case can be derived from Manning and Morgan (1985)'s result.[12]

3.3 Model

From now on, for simplicity, the framing is in terms of the hiring problem. A committee consisting of members $\mathcal{N} := \{1, ..., N\}$ with $N \geq 1$, who are indexed by i, seeks to hire one candidate. In each discrete period of time t, a set of candidates $\mathcal{K} := \{1, ..., K\}$ with $1 \leq K < \infty$ arrives. K is denoted as sample size per period.

Preferences feature independent private values. For each committee member $i \in \mathcal{N}$, the value of hiring candidate $k \in \mathcal{K}$ is governed by the random variable X_i^k, where X_i^k is distributed independently and identically across time periods, candidates, and members according to the cumulative distribution function F with density f. We assume that the distribution of X_i^k has full support on the bounded interval $[0, \bar{x}]$ with $\bar{x} > 0$. Let μ denote the mean of the random variable X_i^k. For all candidates $k \in \mathcal{K}$, committee member $i \in \mathcal{N}$ observes the realization of X_i^k perfectly and has only distributional knowledge about the value X_j^k that any committee member j other than i assigns to candidate k.

The timing is as follows: In every time period, member i observes a realization of the vector of random variables $(X_i^1, \ldots, X_i^K)$, that is, K values. Then, members simultaneously cast a vote, voting either for one candidate k (action k) or for the option to continue search (action 0). Candidate k is hired and search is stopped if and only if the number of votes in favor of k is larger than or equal to the (qualified) majority threshold $M \in \{1, \ldots, N\}$, with $M > \frac{N}{2}$.[13] This class of voting rules encompasses, for instance, unanimity voting corresponding to the case where $M = N$ or simple majority voting with an odd number of members, that is, $M = \frac{N+1}{2}$. If search is continued, each committee member incurs a per period cost of $c \cdot h(K) > 0$, where $h(K)$ is the value of some function $h : \mathbb{N}_+ \to \mathbb{R}_{>0}$ evaluated at K, and $c > 0$ represents a scaling parameter. Finally, we assume that the search horizon is infinite, and that rejected candidates cannot be recalled.

3.4 Equilibrium Analysis

Committee member i's strategy is a sequence of functions $\sigma_i = \{\sigma_i(H_t)\}_t$, mapping from any history H_t until period t to $\Delta(\{0\} \cup \mathcal{K})$, i.e., all probability distributions over the set of actions $\{0\} \cup \mathcal{K}$ that are available in each period. As is common in the literature on committee search, we restrict strategies to be (1) Markovian, meaning,

12. However, note that our assumption on the shape of the cost function is more general.

13. The assumption $M > \frac{N}{2}$ ensures that no two distinct candidates meet the (qualified) majority requirement at the same time.

the action that member i's strategy prescribes in period t does not depend on the entire history up to period t, but only on the evaluation of the most recent K candidates, and we focus on (2) stationary and (3) symmetric equilibria, that is, the equilibrium strategies are neither sensitive to calendar time nor to the identity of the committee member. In addition, we assume strategies to be (4) neutral, that is, they have to be invariant with respect to permutations of the candidates' labels.[14] Essentially, neutrality rules out stationary and symmetric equilibria in Markov strategies in which voters coordinate on ignoring one or more candidates. Apart from conditions (1) - (4), we also impose that search terminates in finite time, excluding dominated equilibria in which all members always vote to continue search, independently of the value realizations. Subsequently, we simply write equilibrium when referring to a stationary and symmetric Markov equilibrium in neutral strategies.
Strategies that satisfy these refinements are characterized by cutoffs $z \in [0, \bar{x})$. More specifically, in any time period, upon observing the value realizations $(x_i^1, ..., x_i^K) \in [0, \bar{x}]^K$, member $i \in \mathcal{N}$ votes in favor of candidate $k \in \mathcal{K}$ if and only if

$$x_i^k \geq \max_{l \neq k} x_i^l \text{ and } x_i^k \geq z.$$

We call these strategies *maximum-strategies with cutoff*. In words, every member chooses the best among the K available candidates and approves this candidate whenever the associated value exceeds the cutoff, or acceptance standard, z. Intuitively, since candidates are identical ex ante and because members treat candidates in a neutral way, all candidates have the same chance to be elected from the perspective of an individual member. Consequently, no member has an incentive to vote in favor of any candidate but the best.[15]
Interior equilibrium cutoffs $z \in (0, \bar{x})$ solve $z = v$, where v is the continuation value implied by this strategy profile.[16] The continuation value which coincides with the ex ante utilitarian welfare per committee member is given by

$$v = -\frac{c \cdot h(K)}{K \cdot \Pr(\text{candidate } k \text{ hired})} + \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}].$$

The continuation value amounts to the difference between the expected value conditional on stopping $\mathbb{E}[X_i^k|\text{candidate } k \text{ hired}]$ and the expected search costs $\frac{c \cdot h(K)}{K \cdot \Pr(\text{candidate } k \text{ hired})}$. Let $Q^K(z, N, M)$ be the cumulative distribution function of the

14. Any stationary Markov strategy can be described by a mapping $s : [0, \bar{x}]^K \to \Delta(\{0\} \cup \mathcal{K})$. A strategy s satisfies neutrality if, for all $(x^1, \ldots, x^K) \in [0, \bar{x}]^K$, it holds that $s(x^{\rho(1)}, \ldots, x^{\rho(K)}) = (s^0(x^1, \ldots, x^K), s^{\rho(1)}(x^1, \ldots, x^K), \ldots, s^{\rho(K)}(x^1, \ldots, x^K))$ for any permutation ρ of the set $\mathcal{K}$.

15. Note that mixed strategies do not arise in equilibrium.

16. Boundary solutions, i.e., equilibria involving some maximum strategy with cutoff $z = 0$, may arise if the search costs $c \cdot h(K)$ are large. Subsequently, we take care of this issue.

Binomial distribution with parameters N and $\Pr(X_i^k \geq z \text{ and } X_i^k \geq \max_{l \neq k} X_i^l)$ evaluated at $M-1$. Also, for any $b \in \mathbb{N}_0$ with $b \leq N$, $q^K(z, N, b)$ denotes the corresponding probability mass function evaluated at b. Further, we argue in Appendix 3.C.2 that

$$\Pr(X_i^k \geq z \text{ and } X_i^k \geq \max_{l \neq k} X_i^l) = \frac{1}{K}[1 - F(z)^K].$$

Then, the equilibrium equation can be written as

$$z = -\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z, N, M)]} + \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}]. \tag{3.1}$$

Intuitively, acceptance standards z arising in equilibrium are calibrated in a way such that a member is indifferent between stopping and continuing search whenever the value of some candidate coincides with the cutoff z. A derivation of the equilibrium strategies and the equation characterizing the equilibrium cutoffs can be found in Appendix 3.A.

3.4.1 Equilibrium Existence

We claim that there exists an equilibrium. The reasoning in the previous part implies that there exists an equilibrium if and only if there either exists $0 \leq z < \bar{x}$ that solves equation (3.1), or there is a boundary equilibrium, in which the maximum-strategy with cutoff $z = 0$ forms an equilibrium.

Proposition 3.1. There exists an equilibrium.

We prove the existence of an equilibrium while making use of the intermediate value theorem. Similar existence arguments appear in Albrecht, Anderson, and Vroman (2010), Compte and Jehiel (2010), and Moldovanu and Shi (2013).

3.4.2 Equilibrium Uniqueness

We turn to the problem of equilibrium uniqueness. Apart from being of interest in itself, the uniqueness of equilibrium is important for a transparent comparison between search procedures featuring different sample sizes per period. It turns out that the equilibrium is unique if we impose the assumption that the density f is log-concave.

Proposition 3.2. If the density f is log-concave, the equilibrium is unique.

Many well-known distributions including, for instance, the uniform distribution or the truncated normal distribution meet this requirement.[17]
Conceptually, the proof strategy follows Albrecht, Anderson, and Vroman (2010),

17. For a comprehensive list of distributions that admit a log-concave density, we refer to Bagnoli and Bergstrom (2005).

but, as discussed below, the presence of more than one candidate per period requires a substantial amount of supplementary steps that are not needed if $K = 1$. The arguments from the previous parts imply that there is a unique equilibrium if and only if either equation (3.1) admits exactly one solution and there is no supplementary boundary equilibrium or there is a boundary equilibrium and the equilibrium equation has no solution. Rearrange equation (3.1):

$$\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z,N,M)]} = \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}] - z.$$

The essential part of the proof is to establish that the left-hand side of this equation is increasing in z, whereas the right-hand side is decreasing in z. Then, the uniqueness result follows from the opposite monotonicities of the discussed functions.
First, it is straightforward to derive that the left-hand side is increasing in z. Intuitively, if the acceptance standard z increases, the probability of voting in favor of some candidate k decreases, and, hence, the probability of hiring this candidate k and the overall probability of stopping decrease as well. Thus, the expected search costs increase. Consequently, it remains to show that $\mathbb{E}[X_i^k | \text{candidate } k \text{ hired}] - z$ is decreasing in z. This claim is stated as Lemma 3.1.[18] Define $S^K(z,N,M) := \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}]$ to emphasize that the expected value conditional on hiring depends on K and M.

Lemma 3.1. Consider any $K \geq 1$. If the density f is log-concave, the function

$$S^K(z,N,M) - z$$

is decreasing in z.

Subsequently, we discuss the proof of Lemma 3.1. Introduce the following two objects:

$$\begin{aligned} \mu_a^K(z) &:= \mathbb{E}[X_i^k | X_i^k \geq z \text{ and } X_i^k \geq \max_{l \neq k} X_i^l], \text{ and} \\ \mu_r^K(z) &:= \mathbb{E}[X_i^k | X_i^k < z \text{ or } X_i^k < \max_{l \neq k} X_i^l]. \end{aligned}$$

These conditional expectations capture the expected value of an arbitrary candidate $k \in \mathcal{K}$ for an arbitrary member $i \in \mathcal{N}$ conditional on approving or rejecting this candidate, respectively. We argue in Appendix 3.C.1 that

$$\mathbb{E}[X_i^k | \text{candidate } k \text{ hired}] = w^K(z)\mu_a^K(z) + [1 - w^K(z)]\mu_r^K(z), \tag{3.2}$$

18. For sequential search with one candidate at a time, i.e., $K = 1$, this property has been shown in Albrecht, Anderson, and Vroman (2010).

with $w^K(z)$ being defined as

$$w^K(z) := \sum_{l=M}^{N} \frac{q^K(z,N,l)}{1-Q^K(z,N,M)} \frac{l}{N}.$$ [19]

Intuitively, conditional on stopping, the accepted candidate k might be supported or rejected by an arbitrary member. Therefore, the expected value of k conditional on stopping amounts to an average of the expected values conditional on supporting as well as rejecting candidate k. The weight $w^K(z)$ represents the expected share of members supporting k conditional on k meeting the majority requirement. Note that under unanimity voting, hired candidates must be accepted by every member. Thus, in this case, the expected value conditional on hiring simplifies to $\mathbb{E}[X_i^k|\text{candidate } k \text{ hired}] = \mu_a^K(z)$.
After some intermediate steps that are similar to those in the proof of Albrecht, Anderson, and Vroman (2010) we obtain that, for $z \in (0, \bar{x})$,

$$\frac{d\mathbb{E}[X_i^k|\text{candidate } k \text{ hired}]}{dz} < w^K(z)\frac{d\mu_a^K(z)}{dz} + [1-w^K(z)]\frac{d\mu_r^K(z)}{dz}.$$

Hence, the key proof step is to show that $\frac{d\mu_a^K(z)}{dz} \leq 1$ and $\frac{d\mu_r^K(z)}{dz} \leq 1$. Notice that if $K = 1$, these conditional expected values are truncated means:

$$\mu_a^1(z) = \mathbb{E}[X_i^k|X_i^k \geq z], \text{ and } \mu_r^1(z) = \mathbb{E}[X_i^k|X_i^k < z].$$

It is well-known that log-concavity of f implies the desired Lipschitz conditions on the truncated means, i.e., $\frac{d\mu_a^1(z)}{dz} \leq 1$ and $\frac{d\mu_r^1(z)}{dz} \leq 1$ (see e.g. Bagnoli and Bergstrom (2005)). However, for $K > 1$, the discussed implications are not standard because the involved expected values conditional on rejecting or supporting a candidate do no longer constitute truncated means. To obtain that $\frac{d\mu_a^K(z)}{dz} \leq 1$, we establish that the conditional density $\Pr(X_i^k = x|X_i^k \geq \max_{l \neq k} X_i^l)$ is log-concave by employing the fact that log-concavity is preserved under integration, which has been shown in Prékopa (1973). Then, like in the case of $K = 1$, log-concavity implies the desired Lipschitz condition on $\mu_a^K(z)$. Next, we show that $\frac{d\mu_r^K(z)}{dz} \leq 1$ by directly invoking the log-concavity of f as well as its implications. Again, the preservation of log-concavity under integration due to Prékopa (1973) is important. Taking both aspects together, Lemma 3.1 follows, and we obtain that the right-hand side of the equation above is decreasing in z. When comparing the welfare induced by different sample sizes per period, we repeatedly make use of Lemma 3.1. We believe that the technical property established in Lemma 3.1 might be useful beyond its application in this chapter.

19. This kind of representation of the expected value conditional on stopping is due to Albrecht, Anderson, and Vroman (2010).

3.5 Unanimity Voting

Having established equilibrium existence and uniqueness, in this section we assume that the committee employs unanimity voting, i.e., we set $M = N$. We compare search procedures induced by different sample sizes in terms of acceptance standards and welfare and show how the superiority of different search technologies depends on the search cost structure. Moreover, for sufficiently small magnitudes of search costs, we derive the welfare-maximizing sample size, depending on the shape of the search cost function. In particular, we identify conditions on the search cost function under which sequential search with one candidate per period is optimal and suboptimal respectively.
Consider search with $K' \geq 1$ and $K \geq 1$ candidates per period, and assume that $K' > K$.
To begin with, as a first cost regime, we study cost functions h that satisfy $\frac{h(K')}{K'} \geq \frac{h(K)}{K}$. This restriction on the function h means that the search costs per candidate when there are K' candidates per period are at least as high as under the search technology featuring K candidates per period. For instance, this condition is met if $h(K') = (K')^\alpha$ and $h(K) = (K)^\alpha$ for some $\alpha \geq 1$.
Denote the ex ante utilitarian welfare per committee member in the game with K' and K candidates per period by $v_{K'}$ and v_K respectively. Proposition 3.3 establishes that the welfare under search with K' candidates per period is strictly lower than the welfare when there are K candidates per period.

Proposition 3.3. Suppose that the voting rule is unanimity, i.e., $M = N$, and consider any $K', K \geq 1$ with $K' > K$.
If the function h satisfies

$$\frac{h(K')}{K'} \geq \frac{h(K)}{K},$$

the committee's ex ante utilitarian welfare is higher under sequential search with K candidates per period relative to sequential search with K' candidates per period, i.e., $v_K > v_{K'}$.

In words, under unanimity voting, weakly higher search costs per candidate when the sample size is larger imply that the welfare of the search procedure featuring a larger sample size is strictly lower. This conclusion holds for all magnitudes of search costs as quantified by the parameter c. Moreover, the result does not require the density of the value distribution f to be log-concave[20] and it applies to all equilibria of the discussed search procedures in case a search technology admits more than one equilibrium.
The basic trade-off when moving from K to K' candidates per period is that, on the

20. We thank an anonymous referee for pointing this out.

one hand, the expected value conditional on stopping rises, but on the other hand, expected search costs rise, too. The former effect arises because unanimity voting means that, conditional on stopping, all members vote in favor of the hired candidate, and, when there K' instead of K candidates per period, members only approve some candidate if the associated value is the maximum out of the K' instead of the K values they observe. The latter effect is due to two aspects: First, the probability of hiring an arbitrary candidate k is smaller if K' versus K candidates are reviewed simultaneously, and, second, the search costs per candidate are weakly higher if there are K' compared to K candidates per period. Thus, a priori, the ranking of the two search procedures in terms of welfare is ambiguous. The key proof step is to show that the increase in the expected value conditional on stopping is limited when moving from sequential search with K to K' candidates per period. This aspect is captured in Lemma 3.2.

Lemma 3.2. Consider any $K', K \geq 1$ with $K' > K$. For all $z_K, z_{K'} \in [0, \bar{x})$ such that $z_K \leq z_{K'}$, it holds

$$\frac{\mu_a^{K'}(z_{K'}) - z_{K'}}{\mu_a^K(z_K) - z_K} < \frac{\frac{1}{K}[1 - F(z_K)^K]}{\frac{1}{K'}[1 - F(z_{K'})^{K'}]}.$$

Take any possibly non-equilibrium cutoffs $z_K, z_{K'} \in [0, \bar{x})$ such that $z_K \leq z_{K'}$, and consider the ratio of the expected values conditional on stopping net of a cutoff when there are K' candidates and the cutoff is $z_{K'}$ versus having K candidates and the cutoff being z_K. Lemma 3.2 reveals that an upper bound of this ratio is given by the ratio of the probability that an individual member votes in favor of a candidate k if there K candidates and the cutoff is z_K to this probability if there are K' candidates and the cutoff is $z_{K'}$. We believe that this technical property might be useful beyond its application in this chapter.

Now, let us sketch the proof of Proposition 3.3 for interior cutoffs. In this case, acceptance standards coincide with welfare.[21] Consider the ratio of the expected value conditional on stopping net of the cutoff when there are K' candidates compared to the net value when there are K candidates, that is,

$$\frac{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}] - z_{K'}}{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K] - z_K},$$

where $z_{K'}$ and z_K denote equilibrium cutoffs when there are K' and K candidates, respectively. Towards a contradiction, assume that $z_K \leq z_{K'}$. By the equilibrium equation, i.e., equation (3.1), the considered ratio is equal to the ratio of the expected search costs when there are K' versus K candidates. Then, the assumption

21. We emphasize that the result also holds if some equilibria constitute boundary solutions.

$\frac{h(K')}{K'} \geq \frac{h(K)}{K}$ on the search cost function yields a lower bound on this ratio of expected search costs. Moreover, while invoking $z_K \leq z_{K'}$ and applying Lemma 3.2, we obtain an upper bound on the discussed ratio of expected values conditional on stopping. It turns out that the derived lower bound is larger than the upper bound, which constitutes the desired contradiction.

Recall that $K' > K$. Let us turn now to the second cost regime and focus on cost functions h that satisfy $\frac{h(K')}{K'} < \frac{h(K)}{K}$. This assumption is reasonable if there are fixed costs associated with the hiring process or if there are cost savings when multiple candidates can be considered. For example, it is satisfied if $h(K') = (K')^\beta$ and $h(K) = (K)^\beta$ for some $\beta < 1$. Proposition 3.4 reveals that under this assumption on the search costs, the conclusion of the previous part of this section is partly reversed: If the magnitude of the search costs as quantified by the parameter c is sufficiently small, evaluating K' candidates at a time improves welfare relative to reviewing K candidates at a time.

Proposition 3.4. Suppose that the voting rule is unanimity, i.e., $M = N$, assume that the density f is log-concave, and consider any $K', K \geq 1$ with $K' > K$.

If the function h satisfies

$$\frac{h(K')}{K'} < \frac{h(K)}{K}$$

there exists $\bar{c}_{K',K} > 0$ such that for all $c < \bar{c}_{K',K}$, the committee's ex ante utilitarian welfare is higher under sequential search with K' candidates per period relative to sequential search with K candidates per period, i.e., $v_{K'} > v_K$.

To verbalize Proposition 3.4, under unanimity voting, strictly lower search costs per candidate if the sample size is larger imply that the welfare of the search technology with a larger sample size is strictly higher as long as the magnitude of search costs is sufficiently low.

Intuitively, again, the expected value conditional on stopping is not lower when there are K' relative to K candidates at a time. However, in contrast to the previous cost regime, here, for sufficiently small magnitudes of search costs c, the expected search costs are actually lower if there K' compared to K candidates per period, yielding a higher welfare for the committee if K' instead of K candidates are evaluated simultaneously in every period of time.

Let us sketch the proof of Proposition 3.4 in more detail. Assume, by contradiction, that for all $\bar{c}_{K',K} > 0$, there exists $c < \bar{c}_{K',K}$ such that $v_K \geq v_{K'}$. Without loss of generality, suppose that both cutoffs are interior. Then, they coincide with welfare and, thus, we have that $z_K \geq z_{K'}$. First, we show that given $z_K \geq z_{K'}$, the expected value conditional on stopping is increasing when moving from K to K' candidates per period. This is a consequence of the log-concavity of f and, more precisely, the Lipschitz condition $\frac{d\mu_a^{K'}(z)}{dz} \leq 1$ we derived in Lemma 3.1. The equilibrium condition (3.1) then implies that the expected search costs have to be higher if K' compared to K candidates are evaluated simultaneously. However, if c becomes small, under both search procedures, the equilibrium acceptance standards are close to the upper bound of the support of the value distribution, $\bar{x}$. This conclusion crucially relies on the fact that the voting rule is unanimity and fails in the case of qualified majority rules distinct from unanimity. Then, even though the probability of hiring an arbitrary candidate k is higher for K than for K', this probability is small for K and for K'. In fact, if c is small enough, the difference is low enough such that, given $\frac{h(K')}{K'} < \frac{h(K)}{K}$, the expected search costs are overall actually smaller for K' than for K candidates at a time. This is the desired contradiction.

Finally, Propositions 3.3 and 3.4 allow us to characterize the welfare-maximizing sample size if the magnitude of search costs is sufficiently small. First, if $\frac{h(K')}{K'} \geq h(1)$ for all $K' > 1$, meaning, the search costs per candidate are minimal if one candidate is evaluated at a time, it is immediate from Proposition 3.3 that single-option sequential search is optimal for all magnitudes of costs as measured by the parameter c. This finding is stated as Corollary 3.1.

Corollary 3.1. Suppose that the voting rule is unanimity, i.e., $M = N$. If the function h satisfies $\frac{h(K')}{K'} \geq h(1)$ for all $K' > 1$, the committee's ex ante utilitarian welfare is maximized for sequential search with one candidate per period, i.e., $v_1 > v_{K'}$ for all $K' > 1$.

In contrast, if the search costs per candidate are minimal for some sample size larger than one and for exogenous reasons at most $\bar{K} < \infty$ candidates can be reviewed simultaneously, Propositions 3.3 and 3.4 together imply that sequential search with a sample size that coincides with the smallest minimizer of the search costs per candidate is welfare-maximizing as long as the magnitude of the search costs c is sufficiently small. Corollary 3.2 captures this result.

Corollary 3.2. Suppose that the voting rule is unanimity, i.e., $M = N$, assume that the density f is log-concave, and impose that $h(1) > \min_{1 \leq K \leq \bar{K}} \frac{h(K)}{K}$ for some $1 < \bar{K} < \infty$. Consider the smallest $1 < K' \leq \bar{K}$ such that

$$\frac{h(K')}{K'} = \min_{1 \leq K \leq \bar{K}} \frac{h(K)}{K}.$$

There exists $\bar{c} > 0$ such that for all $c < \bar{c}$, the committee's ex ante utilitarian welfare is higher under sequential search with K' candidates per period relative to any other form of sequential search featuring at most $\bar{K}$ candidates at a time, i.e., $v_{K'} > v_K$ for all $1 \leq K \leq \bar{K}$ such that $K \neq K'$.

Overall, we conclude that the ranking of different sample sizes in terms of welfare as well as the welfare-maximizing number of candidates per period is mainly determined by the shape of the search cost function.

3.6 Qualified Majority Voting

Having studied the case of unanimity voting, in this section, we turn to qualified majority voting, considering a majority requirement M such that $M < N$. We compare the unique equilibria of different forms of sequential search in terms of acceptance standards and welfare, and, again, we derive the welfare-maximizing number of candidates per period for small magnitudes of search costs.
As before, consider search with $K' \geq 1$ and $K \geq 1$ candidates per period, and suppose that $K' > K$. Again, let $v_{K'}$ and v_K be the ex ante utilitarian welfare per committee member if there are K' and K candidates per period respectively. As already stated, the welfare induced by a search procedure is determined by two ingredients: The expected value conditional on hiring and the expected search costs. To start, we compare in Lemma 3.3 the expected values conditional on stopping when there are K' versus K candidates per period. Recall that $S^{K'}(z,N,M)$ and $S^K(z,N,M)$ denote the expected value conditional on hiring if there are K' and K candidates at a time respectively.

Lemma 3.3. Consider any $K', K \geq 1$ with $K' > K$. For all $z \in [0, \bar{x})$, it holds

$$S^K(z,N,M) \; < \; S^{K'}(z,N,M).$$

Lemma 3.3 reveals that, when fixing a cutoff value z, the expected value conditional on stopping when the sample size is K' is higher than the corresponding object when the sample size is K. If the voting rule is unanimity, this conclusion is immediate because, in this case, $K' > K$ directly yields

$$\begin{aligned} S^K(z,N,N) \; &= \mathbb{E}[X_i^k | X_i^k \; \geq \; z \text{ and } X_i^k \; \geq \max_{l \in \{1,\ldots,K\}: l \neq k} X_i^l] \\ &< \mathbb{E}[X_i^k | X_i^k \; \geq \; z \text{ and } X_i^k \; \geq \max_{l \in \{1,\ldots,K'\}: l \neq k} X_i^l] \; = \; S^{K'}(z,N,N). \end{aligned}$$

Yet, if the voting rule is qualified majority, the conclusion is not obvious because there are two forces pulling in opposite directions. Consider the average representations of the expected values conditional on hiring for both discussed search technologies as introduced in equation (3.2):

$$\begin{aligned} S^{K'}(z,N,N) \; &= \; w^{K'}(z)\mu_a^{K'}(z) + [1 - w^{K'}(z)]\mu_r^{K'}(z) \text{ and} \\ S^K(z,N,N) \; &= \; w^K(z)\mu_a^K(z) + [1 - w^K(z)]\mu_r^K(z). \end{aligned}$$

Note that for $M < N$, in contrast to unanimity, it does neither hold that $w^{K'}(z) = 1$ nor $w^K(z) = 1$, but these objects depend non-trivially on the number of candidates per period. Fix a potentially non-equilibrium cutoff value z. Observe that $\mu_a^{K'}(z) > \mu_a^K(z)$ as well as $\mu_r^{K'}(z) > \mu_r^K(z)$, that is, both the expected value conditional on approving as well as conditional on rejecting an arbitrary candidate k are higher if there are K' versus K candidates at a time. Similar to the case of unanimity voting, $\mu_a^{K'}(z) > \mu_a^K(z)$ holds since a member approves a candidate only if the candidate's value is the highest among the K' versus K values that this member observes. Further, the intuition behind $\mu_r^{K'}(z) > \mu_r^K(z)$ is as follows: If the value of some candidate is above the cutoff z, but some member does not vote in favor of this candidate, this means that this candidate's value is not the maximum out of the K' versus K values this member observes, implying that the considered expected value is lower in the latter case. However, at the same time, we have that $w^{K'}(z) < w^K(z)$: Conditional on stopping, the expected share of members who approve some candidate k decreases when increasing the sample size from K to K'. This holds because the probability that a single member approves a candidate k decreases when moving from K to K', since the candidate's value has to be the maximum out of K' instead of K values in addition to being above the cutoff z. Finally, since $\mu_a^{K'}(z) > \mu_r^{K'}(z)$ as well as $\mu_a^K(z) > \mu_r^K(z)$, the overall effect on the expected value conditional on stopping is a priori ambiguous. We prove Lemma 3.3 by employing a technical result from Albrecht, Anderson, and Vroman (2010) related to the expected share of members who approve some candidate k conditional on stopping. In Proposition 3.5, we claim that sequential search with more candidates at a time increases welfare independently of the shape of the cost function as long as the magnitude of the search costs is sufficiently small.

Proposition 3.5. Suppose that the voting rule is qualified majority distinct from unanimity, i.e., $M < N$, assume that the density f is log-concave, and consider any $K', K \geq 1$ with $K' > K$.
There exists $\bar{c}_{K',K} > 0$ such that for all $c < \bar{c}_{K',K}$, the committee's ex ante utilitarian welfare is higher under sequential search with K' candidates per period relative to sequential search with K candidates per period, i.e., $v_{K'} > v_K$.

Intuitively, the increase in the expected value conditional on hiring when increasing the sample size from K to K' as revealed by Lemma 3.3 outweighs the potential rise of expected search costs[22] if the magnitude of the search costs c is sufficiently small. We emphasize once again that this result does not depend on the form of the cost function. For any function h, there are cost levels c such that evaluating K candidates at a time is dominated by reviewing K' candidates in each period of time.[23] Let us discuss the proof of Proposition 3.5. To the contrary, suppose that for all $\bar{c}_{K',K} > 0$, there exists $c < \bar{c}_{K',K}$ such that $v_K \geq v_{K'}$. Again, without loss of generality, focus on interior cutoffs. Thus, we have that $z_K \geq z_{K'}$ where, again, z_K and $z_{K'}$ denote the equilibrium cutoffs if there are K and K' candidates per period respectively. Recall

22. We write potential rise of expected search costs because depending on the shape of the function h the expected search costs might also be lower if there K' versus K candidates in each period of time. Of course, if that is the case, this only reinforces our reasoning.

23. However, as indicated in Proposition 3.5, the threshold $\bar{c}_{K',K}$ depends on the precise values of K' and K as well as on the shape of the function h.

Lemma 3.1: The log-concavity of f is sufficient for $\frac{dS^K(z,N,M)}{dz} \leq 1$. When employing this Lipschitz condition, we obtain that the difference $S^{K'}(z_K,N,M) - S^K(z_K,N,M)$ is bounded above by the difference in expected search costs between the search procedures involving K' and K candidates at a time. Now, in contrast to unanimity voting, if $M < N$, the equilibrium cutoffs arising under both discussed search technologies do not converge to the upper bound of the support of the value distribution as the magnitude of the search costs c becomes small, but they remain bounded away from $\bar{x}$. For the case of sequential search with one candidate per period, this observation has been made previously in Albrecht, Anderson, and Vroman (2010) as well as Compte and Jehiel (2010). The intuition for this result is as follows: Under qualified majority voting, conditional on stopping, a candidate might be hired even though some member did not vote in favor of this candidate. Taking that scenario, which does not arise under unanimity voting, into account, members do not become arbitrarily picky if search costs become small. Consequently, if c goes to 0, the difference in expected search costs discussed above vanishes. However, due to Lemma 3.3, the difference $S^{K'}(z_K,N,M) - S^K(z_K,N,M)$ remains strictly positive.[24] This is the desired contradiction.

Moreover, Proposition 3.5 allows us to characterize the welfare-maximizing sample size per period for small magnitudes of search costs. Suppose that for exogenous reasons at most $\bar{K} < \infty$ candidates can be reviewed simultaneously in each period of time. Then, Proposition 3.5 implies the following: Whatever the shape of the cost function h, sequential search with $\bar{K}$ candidates per period is welfare-maximizing as long as the magnitude of the search costs c is sufficiently small. Corollary 3.3 records this implication.

Corollary 3.3. Suppose that the voting rule is qualified majority distinct from unanimity, i.e., $M < N$, assume that the density f is log-concave, and consider any $1 < \bar{K} < \infty$.

There exists $\bar{c} > 0$ such that for all $c < \bar{c}$, the committee's ex ante utilitarian welfare is higher under sequential search with $\bar{K}$ candidates per period relative to any other form of sequential search featuring at most $\bar{K}$ candidates at a time, i.e., $v_{\bar{K}} > v_K$ for all $1 \leq K < \bar{K}$.

Our analysis reveals that the ranking of different sample sizes as well as the welfare-maximizing number of candidates per period for the single-searcher case do not generally extend to the committee search case. Again, note that the single decision-maker case is equivalent to the case of a committee with size $N = 1$ operating under the unanimity voting rule. Thus, our results from section 3.5 apply to the single-agent case. To emphasize the drastically different findings, again, suppose

24. This step fails if the voting rule is unanimity because, in this case, if c goes to 0, z_K converges to $\bar{x}$ and, thus, the difference $S^{K'}(z_K,N,N) - S^K(z_K,N,N)$ would vanish as well.

that for exogenous reasons at most $\bar{K} < \infty$ candidates can be evaluated simultaneously in every period. If the function h satisfies $\frac{h(K')}{K'} \geq h(1)$ for all $K' > 1$ and the magnitude of search costs c is small, sequential search with one candidate per period is welfare-maximizing under unanimity voting, whereas search featuring $\bar{K}$ candidates at a time is optimal under qualified majority voting. What drives these considerably different conclusions? Again, consider sequential search with $K' \geq 1$ and $K \geq 1$ candidates per period, and assume that $K' > K$. If the voting rule is unanimity, there is a race between the difference in the expected values conditional on stopping and the difference in the expected search costs between the search technologies involving K' and K candidates at a time: If c becomes small, the difference in expected search costs between K' and K vanishes, and, in addition, the difference in the expected values conditional on hiring also goes to 0. In contrast, under qualified majority voting, if c becomes small, as in the unanimity voting case, the difference in the expected search costs goes to 0. However, in contrast to the unanimity voting case, the difference in the expected values conditional on stopping does not vanish because equilibrium cutoffs do not converge to the upper bound of the support of the value distribution, but they stay bounded away from $\bar{x}$. This discrepancy explains why the ranking of the two types of search procedures is different when the voting rule is qualified majority instead of unanimity. Therefore, when comparing the single-searcher case with the committee search case, the choice of the voting rule crucially matters.

3.7 Extensions and Discussion

In the main model, the committee members' preferences feature independent private values. For the case of unanimity voting, we provide extensions to interdependent as well as correlated values. Moreover, we briefly discuss other voting rules.

3.7.1 Interdependent and Correlated Values

For the unanimity voting rule, we explore the robustness of our results via two extensions: Allowing for interdependent values instead of private values, and allowing for correlated values instead of independent values.[25]

First, regarding interdependent values, we follow the approach in Moldovanu and Shi (2013), assuming that the value a member derives from hiring some candidate is a weighted average of his or her own observed signal and the signals of all other members. We establish that the ranking of the acceptance standards implied by different search technologies carries over from the analysis under private values. As far as welfare is concerned, note that under the assumption of interdependent values, acceptance standards and welfare no longer coincide even if the equilibrium

25. The arguments for these extensions are available on request from the authors.

cutoff is interior (cf. Moldovanu and Shi (2013)). We find that Proposition 3.4 extends from the private-values case to interdependent values. Overall, this suggests that our results concerning unanimity voting are not driven by the private-values assumption on preferences.

Second, to relax the assumption that candidates' values are distributed independently across committee members, we introduce an unknown state of the world s_k for each candidate $k \in \mathscr{K}$, which we assume to be independently and identically distributed across time and candidates. Conditional on the state realization s_k, the values associated with candidate k are then independently and identically distributed across committee members. The state-dependent value distributions are assumed to be stochastically ranked according to the likelihood-ratio ordering. While relaxing the independence of values across members, we maintain the assumption that committee members' preferences feature private values. Thus, acceptance standards and welfare again coincide whenever the equilibrium is interior. We find that all results for the unanimity voting rule carry over from the private-values case to correlated values. Therefore, we conclude that, while the assumption of independently distributed values is admittedly strong, it does not drive our results for the unanimity voting rule.

3.7.2 Other Voting Rules

Let us discuss the class of simple voting rules on which we focus. Recall that each member may either vote for one of the available candidates or may opt to continue search, and a candidate is hired if and only if the number of votes he or she receives exceeds some threshold. Again, as argued in the introduction, considering these voting rules is a natural choice when adopting an approach that is positive with regard to the voting rule, but normative with respect to the search technology. That being said, since members have to decide about more than two alternatives as soon as the sample size per period is larger than one, other voting rules are also conceivable. Inspired by approval voting,[26] one might allow the members to approve any number of candidates instead of only one candidate, and assume that, subject to some tie-breaking rule, a candidate is hired if and only if he or she is approved by more members than any other candidate and the number of supporters of this candidate exceeds some threshold. However, the analysis of the equilibrium voting behavior under these approval-based voting rules is much more complicated. Suppose that there are two candidates per period, i.e., $K = 2$, and assume that the mentioned threshold coincides with unanimity, meaning, it equals the committee size N. Even in this simple case, for instance, the strategy "approve all candidates above some cutoff" does not constitute an equilibrium: Whether approving the second-best can-

26. For an overview about several aspects related to approval voting, we refer to Laslier and Sanver (2010).

didate is beneficial for a member does not only depend on the aspect whether the value of this candidate is above or below the cutoff, but it also matters how much the value of this candidate falls below the value of the best candidate and how much it exceeds the cutoff or continuation value. If the values of the two candidates are both above the cutoff and they are very close to each other, members might want to approve both candidates. In contrast, if the two values are above the cutoff, the value of the best candidate is close to the upper bound of the support of the value distribution, but the value of the second-best candidate is only slightly above the continuation value, members might want to approve only their best candidate. This discussion reveals that already the analysis of the equilibrium voting behavior under these alternative voting rules is rather involved. Consequently, studying the ranking of the search procedures in terms of welfare, which is the focus of this chapter, under these alternative voting rules does not seem to be tractable.

3.8 Conclusion

In this chapter, we contrast the well-known sequential search procedure, in which candidates are evaluated “one at a time”, and other forms of sequential search, where, in each period, committees simultaneously evaluate a set of candidates of fixed size. We study the equilibrium behavior under these search procedures and show equilibrium existence as well as equilibrium uniqueness within a reasonably restricted class of equilibria. Based on the equilibrium analysis, we compare sequential search featuring different sample sizes in terms of acceptance standards and welfare. We identify circumstances under which the “one at a time” policy commonly studied in the committee search literature is not optimal. Generally, the superiority of one or the other search technology depends on two important ingredients of the search problem: The voting rule and the specification of the search costs associated with the simultaneous evaluation of multiple candidates.

If the committee operates under the unanimity rule, the comparison of different search protocols is sensitive to the shape of the cost function. This dependence on the form of the cost function partly vanishes when committees employ a qualified majority rule different from unanimity. In this case, evaluating more candidates at a time improves welfare for any type of cost function as long as the magnitude of the search costs is sufficiently small. Consequently, the assessment of the studied search procedures as well as the underlying trade-offs considerably change when moving from the unanimity rule to qualified majority rules. This is the main insight of this chapter. Again, note that search conducted by a single agent is a special case of committee search with unanimity voting. Consequently, our analysis reveals that the results for the single decision-maker case (see e.g. Manning and Morgan (1985)) do not carry over to the committee setting, but the presence of a committee alters the search design problem and implies different rankings of search procedures.

Appendix 3.A Equilibrium Characterization

To begin with, we claim that the best response of any member $i \in \mathcal{N}$ against an arbitrary neutral stationary Markov strategy that is symmetric across all other members amounts to a maximum-strategy with cutoff, that is, member i votes in favor of candidate $k \in \mathcal{K}$ if and only if

$$x_i^k \geq \max_{l \neq k} x_i^l \text{ and } x_i^k \geq z$$

with $z \in [0, \bar{x})$ being some cutoff.

Assume that all members except for member $i \in \mathcal{N}$ in some period t behave according to a common Markovian strategy that is stationary and neutral. First of all, let v be the continuation value member i obtains when search continues. Note that v does not depend on past or current actions or value realizations since the continuation strategy adopted by all members in periods following t is Markovian. Also, it is neither sensitive to the identity i of the member nor to calendar time because continuation strategies are symmetric across members and stationary. Now, suppose that member i observes the value realizations $(x_i^1, \dots, x_i^K)$ in period t. Member i is pivotal for candidate k if and only if exactly $M-1$ out of the other $N-1$ members choose action k in the given period, that is, approve candidate k.

Let $p_k(a, b) > 0$ with $a \in \mathbb{N}$, $b \in \mathbb{N}_0$ and $b \leq a$ denote the probability that exactly b out of a members choose action k in the given period. Similarly, $P_k(a, b) > 0$ with $a, b \in \mathbb{N}$ and $b \leq a$ describes the probability that at most $b-1$ out of a members select action k. Then, the probability that member i is pivotal in favor of candidate k is given by $p_k(N-1, M-1)$.

The expected utility that member i obtains when approving candidate k can be expressed as follows:

$$\begin{aligned}
&[(1 - P_k(N-1, M)) + p_k(N-1, M-1)][x_i^k] + \sum_{l \in \{1,\dots,K\}: l \neq k} [1 - P_l(N-1, m)][x_i^l] \\
&\quad + [P_k(N-1, M) - p_k(N-1, M-1) - \sum_{l \in \{1,\dots,K\}: l \neq k} (1 - P_l(N-1, M))][v].
\end{aligned}$$

The expected payoff of member i when voting in favor of continuing search, i.e., selecting action 0, amounts to

$$\sum_{l \in \{1,\dots,K\}} [1 - P_l(N-1, M)][x_i^l] + [1 - \sum_{l \in \{1,\dots,K\}} (1 - P_l(N-1, M))][v].$$

Since the stationary Markov strategy that is commonly adopted by members distinct from i is neutral, it holds that $P_d(a, b) = P_e(a, b)$ as well as $p_d(a, b) = p_e(a, b)$ for all $d, e \in \mathcal{K}$. For simplicity, write $P(a, b)$ and $p(a, b)$ to denote these probabilities.

Consequently, the expected utility of choosing action k can be reformulated in the following way:

$$p(N-1,M-1)[x_i^k]+[1-P(N-1,M)][\sum_{l\in\{1,\dots,K\}} x_i^l] \\ +[1-K(1-P(N-1,M))-p(N-1,M-1)][v].$$

Similarly, the expected payoff of action 0 simplifies to the expression

$$[1-P(N-1,M)][\sum_{l\in\{1,\dots,K\}} x_i^l]+[1-K(1-P(N-1,M))][v].$$

Thus, voting in favor of candidate k is optimal for member i if and only if, for all $m\in\mathscr{K}$ with $m\neq k$,

$$\begin{aligned} & p(N-1,M-1)[x_i^k]+[1-P(N-1,M)][\sum_{l\in\{1,\dots,K\}} x_i^l] \\ &\qquad +[1-K(1-P(N-1,M))-p(N-1,M-1)][v] \\ \geq\ & p(N-1,M-1)[x_i^m]+[1-P(N-1,M)][\sum_{l\in\{1,\dots,K\}} x_i^l] \\ &\qquad +[1-K(1-P(N-1,M))-p(N-1,M-1)][v], \end{aligned}$$

and, at the same time,

$$\begin{aligned} & p(N-1,M-1)[x_i^k]+[1-P(N-1,M)][\sum_{l\in\{1,\dots,K\}} x_i^l] \\ & +[1-K(1-P(N-1,M))-p(N-1,M-1)][v] \\ \geq & [1-P(N-1,M)][\sum_{l\in\{1,\dots,K\}} x_i^l]+[1-K(1-P(N-1,M))][v]. \end{aligned}$$

The former condition is equivalent to requiring that $x_i^k\geq\max_{l\neq j}x_i^l$. The latter condition reduces to $x_i^k\geq v$. This means that there exists a cutoff value $z_i(t)\in[0,\bar{x})$ such that this condition is met if and only if $x_i^j\geq z_i(t)$. Moreover, the cutoff value solves $z_i(t)=v$ whenever it is interior. Hence, given an arbitrary neutral stationary Markov strategy commonly adopted by all members except for member i in period t, it is optimal for member i to employ a maximum-strategy with cutoff $z_i(t)$ in this period.
In the following, we make use of this claim, and we establish the sufficiency and the necessity part separately.
With regard to necessity, it is immediate from the previous claim that any symmetric stationary Markov equilibrium in neutral strategies must involve a maximum-strategy with cutoff $z\in[0,\bar{x})$ solving $z=v$ whenever being interior, and that this strategy is commonly adopted by all members since, otherwise, at least one member has a profitable deviation. In particular, the cutoffs are neither sensitive to the

members' identities nor to calendar time because, by assumption, equilibria are symmetric and stationary. Moreover, the consistency of continuation values and equilibrium strategies implies that v must satisfy

$$\begin{aligned} v &= -c \cdot h(K) + [1 - K(1 - P(N,M))]v \\ &\quad + K \cdot [1 - P(N,M)]\mathbb{E}[X_i^k | \text{candidate } k \text{ hired}]. \end{aligned}$$

Rearranging this equation yields

$$v = -\frac{c \cdot h(K)}{K \cdot [1 - P(N,M)]} + \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}].$$

Therefore, equilibrium cutoffs solve the equation

$$z = -\frac{c \cdot h(K)}{K \cdot [1 - P(N,M)]} + \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}]$$

whenever they are interior. Finally, recall that $P(N,M)$ denotes the probability that at most $M-1$ out of N members approve some candidate k. Thus, when using the notation introduced in the main text, we have that $P(N,M) = Q^K(z,N,M)$. This concludes the proof of the necessity part.

Next, we turn to sufficiency. First of all, observe that strategy profiles in which all members adopt the same maximum-strategy with cutoff $z \in [0,\bar{x})$ are symmetric, neutral, and stationary Markov. Furthermore, as argued in the necessity part of this proof, these strategy profiles give rise to continuation values satisfying

$$v = -\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z,N,M)]} + \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}].$$

Consequently, it remains to verify that these strategy profiles constitute equilibria. To this end, consider any strategy with cutoff $z \in [0,\bar{x})$ solving

$$z = v = -\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z,N,M)]} + \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}]$$

whenever the cutoff z is interior. First, by construction, the consistency of continuation values and strategies is fulfilled. Second, if all members apart from member $i \in \mathcal{N}$ in period t adopt the discussed strategy, the claim above implies that it is optimal for member i to follow the same strategy in period t, that is, the maximum-strategy with cutoff $z_i(t)$ solving $z_i(t) = v = z$ whenever it is interior. Now, the one-shot deviation principle implies that no member has a profitable deviation. Thus, the maximum-strategy with cutoff z solving

$$z = v = -\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z,N,M)]} + \mathbb{E}[X_i^k | \text{candidate } k \text{ hired}]$$

whenever being interior constitutes an equilibrium. This completes the sufficiency part.

Appendix 3.B Proofs

Proof of Proposition 3.1.
Recall that $S^K(z,N,M) = \mathbb{E}[X_i^k|\text{candidate } k \text{ accepted}]$. Rewriting equation (3.1) which characterizes equilibrium cutoff values yields

$$\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z,N,M)]} = S^K(z,N,M) - z.$$

Suppose that $z = 0$. In this case, the left-hand side amounts to $\frac{c \cdot h(K)}{K \cdot [1-Q^K(0,N,M)]} = \frac{c\frac{h(K)}{K}}{1-Q^K(0,N,M)}$ and the right-hand side reduces to $S^K(0,N,M)$. In contrast, if $z \to \bar{x}$, the left-hand side goes to ∞ whereas the right-hand side amounts to $S^K(\bar{x},N,M) - \bar{x} \leq 0$.
Depending on the magnitude of the search costs c, we perform a case distinction:
1) $\frac{c\frac{h(K)}{K}}{1-Q^K(0,N,M)} < S^K(0,N,M)$
In this case, we observe that the left-hand side is strictly smaller than the right-hand side of the equilibrium equation when evaluating both sides at $z = 0$. In contrast, if z is sufficiently close to $\bar{x}$, the left-hand side is strictly larger than the right-hand side. Moreover, note that both sides of the equation involve functions that are continuous in z. Hence, the intermediate value theorem yields the existence of a cutoff z that solves equation (3.1).
2) $\frac{c\frac{h(K)}{K}}{1-Q^Kk(0,N,M)} = S^K(0,N,M)$
Here, the cutoff $z = 0$ solves the equilibrium equation which means that the maximum-strategy with cutoff $z = 0$ constitutes an equilibrium.
3) $\frac{c\frac{h(K)}{K}}{1-Q^K(0,N,M)} > S^K(0,N,M)$
In this case, suppose that all members apart from member $i \in \mathcal{N}$ in period t adopt the maximum-strategy with cutoff $z = 0$. In this case, the arguments in Appendix 3.A still apply, and, thus, it is optimal for member i to follow some maximum-strategy with cutoff. However, since $v = -\frac{c\frac{h(K)}{K}}{1-Q^K(0,N,M)} + S^K(0,N,M) < 0$ by assumption, the optimal cutoff for member i in the given period is $z = 0$. The reason is that member i wants to stop search as quickly as possible, and the probability of voting in favor of some candidate k is maximized at $z = 0$. Alluding to the one-deviation-principle, this shows that there exists a boundary equilibrium such that the maximum-strategy with cutoff amounting to $z = 0$ forms an equilibrium. □

Proof of Lemma 3.1.
We establish that $S_z^K(z,N,M) \leq 1$ which implies that the function $S^K(z,N,M) - z$ is non-increasing in z. Subsequently, again, we make use of the notation

$$\mu_a^K(z) = \mathbb{E}[X_i^k|X_i^k \geq z \text{ and } X_i^k \geq \max_{l \neq k} X_i^l] \text{ and}$$
$$\mu_r^K(z) = \mathbb{E}[X_i^k|X_i^k < z \text{ or } X_i^k < \max_{l \neq k} X_i^l].$$

Then, as shown in Appendix 3.C.1, $S^K(z,N,M)$ can be expressed as

$$S^K(z,N,M) = w^K(z)\mu_a^K(z) + (1-w^K(z))\mu_r^K(z),$$

where $w^K(z)$ is given by

$$w^K(z) = \sum_{l=M}^{N} \frac{q^K(z,N,l)}{1-Q^K(z,N,M)} \frac{l}{N}.$$

Further, to simplify the notation, define

$$1-R^K(z) := \Pr(X_i^k \geq \max_{l\neq k} X_i^l, X_i^k \geq z).$$

First, we obtain that $\frac{dw^K(z)}{dz} \leq 0$.[27] Observe that $w^K(z)$ constitutes the average of terms of form $\frac{l}{N}$ with weights

$$w_l^K(z) := \frac{q^K(z,N,l)}{1-Q^K(z,N,M)}.$$

We claim that, for all $l < l'$, $\frac{w_l^K(z)}{w_{l'}^K(z)}$ is non-decreasing in z. This means that increasing z yields a stochastic decrease according to the likelihood-ratio ordering which, as is well-known, implies a stochastic decrease in terms of first-order stochastic dominance. Hence, exploiting the average structure of $w^K(z)$, when increasing z, the average $w^K(z)$ decreases. In other words, we have $\frac{dw^K(z)}{dz} \leq 0$. In order to see that $\frac{w_l^K(z)}{w_{l'}^K(z)}$ is increasing in z, note that

$$\frac{w_l^K(z)}{w_{l'}^K(z)} = \frac{\binom{N}{l}}{\binom{N}{l'}} R^K(z)^{l'-l}(1-R^K(z))^{l-l'},$$

and, therefore, straightforward differentiation yields

$$\frac{d\frac{w_l^K(z)}{w_{l'}^K(z)}}{dz} = \frac{\binom{N}{l}}{\binom{N}{l'}} \frac{dR^K(z)}{dz}(l'-l)R^K(z)^{l'-l-1}(1-R^K(z))^{l-l'-1}.$$

The derivation in Appendix 3.C.2 reveals that

$$1-R^K(z) = \frac{1}{K}[1-F(z)^K].$$

Thus, $\frac{dR^K(z)}{dz} = F(z)^{K-1}f(z) \geq 0$ and we obtain that $\frac{d\frac{w_l^K(z)}{w_{l'}^K(z)}}{dz} \geq 0$ which is the desired claim. Therefore, we conclude that $\frac{dw^K(z)}{dz} \leq 0$.

27. The argument yielding $\frac{dw^K(z)}{dz} \leq 0$ is analogous to step 2 in the proof of Lemma 1 in Albrecht, Anderson, and Vroman (2010).

Second, we show that $\mu_a^K(z) - z$ is non-increasing or, in other words, $\frac{d\mu_a^K(z)}{dz} \leq 1$. Consider the density

$$\begin{aligned} g^K(x) \; := \; & \Pr(X_i^k = x | X^k \geq \max_{l \neq k} X_i^l) \\ = \; & \frac{\Pr(X_i^k = x, X_i^k \geq \max_{l \neq k} X_i^l)}{\Pr(X_i^k \geq \max_{l \neq k} X_i^l)} \\ = \; & \frac{\Pr(X_i^k = x, x \geq \max_{l \neq k} X_i^l)}{\Pr(X_i^k \geq \max_{l \neq k} X_i^l)} \\ = \; & \frac{\Pr(X_i^k = x) \Pr(x \geq \max_{l \neq k} X_i^l)}{\Pr(X_i^k \geq \max_{l \neq k} X_i^l)} \\ = \; & K f(x) [F(x)]^{K-1}. \end{aligned}$$

We know from Prékopa (1973) that the log-concavity of the density f implies that the cdf F is also log-concave. Moreover, since the product of log-concave functions must be again log-concave, we obtain that the density g^K is log-concave as well. Therefore, as is well-known, the log-concavity of g^K implies that the random variable $X_i^k | X_i^k \geq \max_{l \neq k} X_i^l$ has the decreasing mean residual life property which means that $\mu_a^K(z) - z$ is non-increasing.[28] Thus, we conclude that $\frac{d\mu_a^K(z)}{dz} \leq 1$.

Third, we establish that $\frac{d\mu_r^K(z)}{dz} \leq 1$. By the law of total expectation, we obtain

$$\mu \; = \; \mathbb{E}[X_i^k] \; = \; \mu_a^K(z)[1 - R(z)] + \mu_r^K(z) R(z).$$

Again, in Appendix 3.C.2, we derive that

$$1 - R^K(z) \; = \; \frac{1}{K}[1 - F(z)^K].$$

Thus,

$$\mu \; = \; \mu_a^K(z)[\frac{1}{K}(1 - F(z)^K)] + \mu_r^K(z)[1 - \frac{1}{K}(1 - F(z)^K)].$$

Let G^K be the cdf of the random variable $X_i^k | X_i^k \geq \max_{l \neq k} X_i^l$. Hence, rearranging yields

$$\begin{aligned} \mu_r^K(z) \; = \; & \frac{\mu - \mu_a^K(z)[\frac{1}{K}(1 - F(z)^K)]}{1 - \frac{1}{K}(1 - F(z)^K)} \\ = \; & \frac{\int_0^{\bar{x}} s f(s) ds - [\frac{1}{K}(1 - F(z)^K)] \int_z^{\bar{x}} s \frac{g^K(s)}{1 - G^K(z)} ds}{1 - \frac{1}{K}(1 - F(z)^K)} \\ = \; & \frac{\int_0^{\bar{x}} s f(s) ds - \int_z^{\bar{x}} s f(s) F(s)^{K-1} ds}{1 - \frac{1}{K}(1 - F(z)^K)}. \end{aligned}$$

28. Bagnoli and Bergstrom (2005) discuss the relationship between log-concave densities and concepts from reliability theory.

Taking the derivative of $\mu_r^K(z)$ with respect to z yields

$$\begin{aligned}
&\frac{d\mu_r^K(z)}{dz}\\
&= \frac{(zf(z)F(z)^{K-1}))\cdot(1-\frac{1}{K}(1-F(z)^K))}{[1-\frac{1}{K}(1-F(z)^K)]^2}\\
&- \frac{(\int_0^{\bar{x}} sf(s)ds - \int_z^{\bar{x}} sf(s)F(s)^{K-1}ds)\cdot f(z)F(z)^{K-1}}{[1-\frac{1}{K}(1-F(z)^K)]^2}\\
&= \frac{f(z)F(z)^{K-1}}{[1-\frac{1}{K}(1-F(z)^K)]^2}\\
&\cdot[z(1-\frac{1}{K}) + z\frac{1}{K}F(z)^K - sF(s)\Big|_0^{\bar{x}} + \int_0^{\bar{x}} F(s)ds + s\frac{1}{K}F(s)^K\Big|_z^{\bar{x}} - \int_z^{\bar{x}} \frac{1}{K}F(s)^K ds]\\
&= \frac{f(z)F(z)^{K-1}}{[1-\frac{1}{K}(1-F(z)^K)]^2}\\
&\cdot[z(1-\frac{1}{K}) + z\frac{1}{K}F(z)^K - \bar{x}(1-\frac{1}{K}) - z\frac{1}{K}F(z)^K + \int_0^{\bar{x}} F(s)ds - \int_z^{\bar{x}} \frac{1}{K}F(s)^K ds]\\
&= \frac{f(z)F(z)^{K-1}[(z-\bar{x})(1-\frac{1}{K}) + \int_0^{\bar{x}} F(s)ds - \int_z^{\bar{x}} \frac{1}{K}F(s)^K ds]}{[1-\frac{1}{K}(1-F(z)^K)]^2}\\
&= \frac{f(z)F(z)^{K-1}[\int_0^{\bar{x}} F(s)ds - \int_z^{\bar{x}} [1-\frac{1}{K}(1-F(s)^K)]ds]}{[1-\frac{1}{K}(1-F(z)^K)]^2}\\
&= \frac{f(z)F(z)^{K-1}[\int_0^{z} F(s)ds + \int_z^{\bar{x}} F(s)ds - \int_z^{\bar{x}} [1-\frac{1}{K}(1-F(s)^K)]ds]}{[1-\frac{1}{K}(1-F(z)^K)]^2}\\
&= \frac{f(z)F(z)^{K-1}[\int_0^{z} F(s)ds + \int_z^{\bar{x}} F(s) - [1-\frac{1}{K}(1-F(s)^K)]ds]}{[1-\frac{1}{K}(1-F(z)^K)]^2}.
\end{aligned}$$

Since we have $\frac{d\mu_r^K(z)}{dz}\Big|_{z=0} = 0 \leq 1$, for the remainder of the proof of $\frac{d\mu_r^K(z)}{dz} \leq 1$, suppose that $z \neq 0$.

Again, due to Prékopa (1973), log-concavity is preserved under integration. Hence, since the density f is log-concave, the cdf $F(z) = \int_0^z f(s)ds$ is also log-concave and, consequently, the left-hand integral $\int_0^z F(s)ds$ must be log-concave as well. By definition of log-concavity, this means that $\int_0^z F(s)ds \leq \frac{F(z)^2}{f(z)}$.[29]

Moreover, note that, for all $s \in [0, \bar{x}]$,

$$\begin{aligned}
\frac{1}{K}(1-F(s)^K) &= 1-R^K(s) = \Pr(X^k \geq \max_{l\neq k} X^l \text{ and } X^k \geq s)\\
&\leq \Pr(X^k \geq s) = 1-F(s).
\end{aligned}$$

29. Again, for a discussion of these kinds of implications, we refer to Bagnoli and Bergstrom (2005).

Thus, we obtain, for all $s \in [0, \bar{x}]$, that

$$F(s) - [1 - \frac{1}{K}(1 - F(s)^K)] \;\leq\; 0,$$

and, in particular, it holds that

$$\int_z^{\bar{x}} F(s) - [1 - \frac{1}{K}(1 - F(s)^K)]ds \;\leq\; 0.$$

Also, observe that $F(z) - [1 - \frac{1}{K}(1 - F(z)^K)] \leq 0$ is equivalent to

$$\frac{1}{1 - \frac{1}{K}(1 - F(z)^K)} \;\leq\; \frac{1}{F(z)}.$$

Employing the derived inequalities yields

$$\begin{aligned}
\frac{d\mu_r^K(z)}{dz} &= \frac{f(z)F(z)^{K-1}[\int_0^z F(s)ds + \int_z^{\bar{x}} F(s) - [1 - \frac{1}{K}(1 - F(s)^K)]ds]}{[1 - \frac{1}{K}(1 - F(z)^K)]^2} \\
&\leq \frac{f(z)F(z)^{K-1}\int_0^z F(s)ds}{[1 - \frac{1}{K}(1 - F(z)^K)]^2} \\
&\leq \frac{f(z)F(z)^{K-1}\frac{F(z)^2}{f(z)}}{[1 - \frac{1}{K}(1 - F(z)^K)]^2} \\
&= \frac{F(z)^{K+1}}{[1 - \frac{1}{K}(1 - F(z)^K)]^2} \\
&\leq \frac{F(z)^{K+1}}{F(z)^2} \\
&= F(z)^{K-1} \\
&\leq 1.
\end{aligned}$$

Therefore, we conclude that $\frac{d\mu_r^K(z)}{dz} \leq 1$.
Further, note that $\mu_a^K(z) > \mu_r^K(z)$ or, equivalently, $\mu_a^K(z) - \mu_r^K(z) > 0$. Taking together the three ingredients $\frac{dw^K(z)}{dz} \leq 0$, $\frac{d\mu_a^K(z)}{dz} \leq 1$ and $\frac{d\mu_r^K(z)}{dz} \leq 1$, we have

$$
\begin{aligned}
S_z^K(z,N,M) &= \frac{d[w^K(z)\mu_a^K(z) + (1-w^K(z))\mu_r^K(z)]}{dz} \\
&= \frac{d[w^K(z)[\mu_a^K(z) - \mu_r^K(z)] + \mu_r^K(z)]}{dz} \\
&= \frac{dw^K(z)}{dz}[\mu_a^K(z) - \mu_r^K(z)] + w^K(z)[\frac{d\mu_a^K(z)}{dz} - \frac{d\mu_r^K(z)}{dz}] + \frac{d\mu_r^K(z)}{dz} \\
&= \frac{dw^K(z)}{dz}(\mu_a^K(z) - \mu_r^K(z)) + w^K(z)\frac{d\mu_a^K(z)}{dz} + [1-w^K(z)]\frac{d\mu_r^K(z)}{dz} \\
&\leq w^K(z)\frac{d\mu_a^K(z)}{dz} + [1-w^K(z)]\frac{d\mu_r^K(z)}{dz} \\
&\leq w^K(z) + [1-w^K(z)] \\
&= 1.
\end{aligned}
$$

In conclusion, as desired, we infer that $S_z^K(z,N,M) \leq 1$ which, implies that the function $S^K(z,N,M) - z$ is non-increasing in z. Additionally, the argument reveals that $S_z^K(z,N,M) < 1$ whenever $z \neq 0$ and, thus, $S^K(z,N,M) - z$ is strictly decreasing in z. □

Proof of Proposition 3.2.

To begin with, by Proposition 3.1, there exists an equilibrium. Moreover, we know from Lemma 3.1 that the function $S^K(z,N,M) - z$ is decreasing in z. Next, we show that the function

$$\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z,N,M)]}$$

is increasing in z.

Again, to simplify the notation, define

$$1 - R^K(z) := \Pr(X_i^k \geq \max_{l \neq k} X_i^l, X_i^k \geq z).$$

Taking the derivative of the discussed function with respect to z yields

$$\frac{d}{dz}[\frac{c \cdot h(K)}{K \cdot [1 - Q^K(z,N,M)]}] = \frac{c \cdot h(K) \cdot Q_z^K(z,N,M)}{K \cdot [1 - Q^K(z,N,M)]^2}.$$

Further, using the relationship between the Binomial and the Beta distribution,[30] we have

$$
\begin{aligned}
Q^K(z,N,M) &= \sum_{l=0}^{M-1} \binom{N}{l} (1 - R^K(z))^l \cdot R^K(z)^{N-l} \\
&= \frac{N!}{(N-M)! \cdot (M-1)!} \int_0^{R^K(z)} s^{N-M}(1-s)^{M-1} ds.
\end{aligned}
$$

30. cf. Casella and Berger (2002)

Taking the derivative of $Q^K(z,N,M)$ with respect to z yields

$$Q_z^K(z,N,M) \;=\; \frac{N!}{(N-M)!\cdot(M-1)!}\frac{dR^K(z)}{dz}R^K(z)^{N-M}(1-R^K(z))^{M-1}.$$

Again, the derivation in Appendix 3.C.2 reveals that

$$1-R^K(z) \;=\; \frac{1}{K}[1-F(z)^K].$$

Thus, we have that $\frac{dR^K(z)}{dz}=F(z)^{K-1}f(z)\geq 0$. Hence, we obtain that $Q_z^K(z,N,M)\geq 0$, yielding the desired inference that

$$\frac{d}{dz}\Big[\frac{c\cdot h(K)}{K\cdot[1-Q^K(z,N,M)]}\Big] \;=\; \frac{c\cdot h(K)\cdot Q_z^K(z,N,M)}{K\cdot[1-Q^K(z,N,M)]^2} \;\geq\; 0.$$

Additionally, the argument shows that this derivative is strictly larger than 0 whenever $z\neq 0$ and, hence, $\frac{c\cdot h(K)}{K\cdot[1-Q^K(z,N,M)]}$ is strictly increasing in z.
Consider the equation characterizing equilibrium cutoff values

$$S^K(z,N,M)-z \;=\; \frac{c\cdot h(K)}{K\cdot[1-Q^K(z,N,M)]} \;=\; \frac{c\frac{h(K)}{K}}{1-Q^K(z,N,M)}.$$

Depending on the magnitude of the search costs, we perform a case distinction:
1) $\frac{c\frac{h(K)}{K}}{1-Q^K(0,N,M)}<S^K(0,N,M)$
In this case, all cutoffs associated with equilibrium strategies are interior, satisfying $z\neq 0$. In particular, these cutoffs must solve the equilibrium equation. However, due to Lemma 3.1, the left-hand side of the discussed equation is strictly decreasing and the right-hand side is strictly increasing. Therefore, both sides of the equation have at most one intersection which establishes uniqueness of equilibrium.
2) $\frac{c\frac{h(K)}{K}}{1-Q^K(0,N,M)}\geq S^K(0,N,M)$
Here, the cutoff $z=0$ is part of an equilibrium. Either $z=0$ solves the equilibrium equation or there is a boundary equilibrium involving the cutoff $z=0$. To the contrary, suppose that there is another equilibrium with some cutoff $z'>0$. This cutoff must solve the equilibrium equation because it is interior. However, employing the monotonicity properties of the functions involved in the equilibrium equation that are partly derived in Lemma 3.1, we have

$$\frac{c\frac{h(K)}{K}}{1-Q^K(z',N,M)} \;>\; \frac{c\frac{h(K)}{K}}{1-Q^K(0,N,M)} \;\geq\; S^K(0,N,M) \;>\; S^K(z',N,M)-z'.$$

Hence, the cutoff $z'>0$ cannot be part of an equilibrium which constitutes the desired contradiction. □

Proof of Lemma 3.2.
Consider any $K', K \geq 1$ with $K' > K$. Suppose, by contradiction, that there exist some $z_K, z_{K'} \in [0, \bar{x})$ with $z_K \leq z_{K'}$ such that

$$\begin{aligned}\frac{\mu_a^{K'}(z_{K'}) - z_{K'}}{\mu_a^K(z_K) - z_K} &= \frac{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}] - z_{K'}}{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K] - z_K} \\ &\geq \frac{\frac{1}{K}[1 - F(z_K)^K]}{\frac{1}{K'}[1 - F(z_{K'})^{K'}]} = \frac{K'}{K}\frac{[1 - F(z_K)^K]}{[1 - F(z_{K'})^{K'}]}.\end{aligned}$$

Rewriting the left-hand side of the inequality yields

$$\begin{aligned}\frac{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}] - z_{K'}}{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K] - z_K} &= \frac{\frac{\int_{z_{K'}}^{\bar{x}} f(s)F(s)^{K'-1} sds}{\frac{1}{K'}[1-F(z_{K'})^{K'}]} - z_{K'}}{\frac{\int_{z_K}^{\bar{x}} f(s)F(s)^{K-1} sds}{\frac{1}{K}[1-F(z_K)^K]} - z_K} \\ = \frac{K'}{K}\frac{[1 - F(z_K)^K]}{[1 - F(z_{K'})^{K'}]}\frac{\int_{z_{K'}}^{\bar{x}} f(s)F(s)^{K'-1} sds - z_{K'}[\frac{1}{K'}(1 - F(z_{K'})^{K'})]}{\int_{z_K}^{\bar{x}} f(s)F(s)^{K-1} sds - z_K[\frac{1}{K}(1 - F(z_K)^K)]},\end{aligned}$$

where the first step uses the fact that

$$\begin{aligned}\Pr(X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}) &= \frac{1}{K'}[1 - F(z_{K'})^{K'}] \text{ and} \\ \Pr(X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K) &= \frac{1}{K}[1 - F(z_K)^K],\end{aligned}$$

which is derived in Appendix 3.C.2.
Thus, we obtain that

$$\frac{\int_{z_{K'}}^{\bar{x}} f(s)F(s)^{K'-1} sds - z_{K'}[\frac{1}{K'}(1 - F(z_{K'})^{K'})]}{\int_{z_K}^{\bar{x}} f(s)F(s)^{K-1} sds - z_K[\frac{1}{K}(1 - F(z_K)^K)]} \geq 1. \tag{3.B.1}$$

Observe that $\int_{z_K}^{\bar{x}} f(s)F(s)^{K-1} sds - z_K[\frac{1}{K}(1 - F(z_K)^K)] > 0$ because this inequality is equivalent to $\mu_a^K(z_K) > z_K$. Therefore, inequality (3.B.1) is equivalent to

$$\begin{aligned}&\int_{z_{K'}}^{\bar{x}} f(s)F(s)^{K'-1} sds - z_{K'}[\frac{1}{K'}(1 - F(z_{K'})^{K'})] \\ \geq &\int_{z_K}^{\bar{x}} f(s)F(s)^{K-1} sds - z_K[\frac{1}{K}(1 - F(z_K)^K)] =: g(z_K).\end{aligned}$$

The right-hand side of this inequality is non-increasing in z_K. To see this, compute the derivative

$$\begin{aligned}g'(z_K) &= -f(z_K)F(z_K)^{K-1} z_K - \frac{1}{K}[1 - F(z_K)^K] + z_K F(z_k)^{K-1} f(z_k) \\ &= -\frac{1}{K}[1 - F(z_K)^K] \leq 0.\end{aligned}$$

Therefore, because of $z_K \leq z_{K'}$, it follows that

$$\begin{aligned}&\int_{z_{K'}}^{\bar{x}} f(s)F(s)^{K'-1} s ds - z_{K'}[\frac{1}{K'}(1-F(z_{K'})^{K'})]\\ \geq &\int_{z_{K'}}^{\bar{x}} f(s)F(s)^{K-1} s ds - z_{K'}[\frac{1}{K}(1-F(z_{K'})^{K})].\end{aligned}$$

Rearranging yields

$$\int_{z_{K'}}^{\bar{x}} f(s)s[F(s)^{K'-1} - F(s)^{K-1}]ds \;\geq\; z_{K'}[\frac{1}{K'}(1-F(z_{K'})^{K'}) - \frac{1}{K}(1-F(z_{K'})^{K})].$$

Since $z_{K'} < \bar{x}$ and $K' > K$,

$$\begin{aligned}\int_{z_{K'}}^{\bar{x}} f(s)s[F(s)^{K'-1} - F(s)^{K-1}]ds \;&<\; z_{K'}\int_{z_{K'}}^{\bar{x}} f(s)[F(s)^{K'-1} - F(s)^{K-1}]ds\\ &=\; z_{K'}[\frac{1}{K'}(1-F(z_{K'})^{K'}) - \frac{1}{K}(1-F(z_{K'})^{K})].\end{aligned}$$

Hence, we have that

$$\begin{aligned}&z_{K'}[\frac{1}{K'}(1-F(z_{K'})^{K'}) - \frac{1}{K}(1-F(z_{K'})^{K})]\\ > &\int_{z_{K'}}^{\bar{x}} f(s)s[F(s)^{K'-1} - F(s)^{K-1}]ds\\ \geq & z_{K'}[\frac{1}{K'}(1-F(z_{K'})^{K'}) - \frac{1}{K}(1-F(z_{K'})^{K})],\end{aligned}$$

which is the desired contradiction. □

Proof of Proposition 3.3.
We begin by deriving conditions for when boundary solutions of either of the search procedures arise.
First of all, note that the proof of Proposition 3.1 reveals that under sequential search with K candidates per period, there is a boundary equilibrium if and only if

$$c \;\geq\; \frac{S^K(0,N,N)[1-Q^K(0,N,N)]}{\frac{h(K)}{K}} \;=\; \frac{\mu_a^K(0)[\frac{1}{K}]^N}{\frac{h(K)}{K}} \;=:\; c^K.$$

Similarly, if there are K' candidates per period, a corner solution arises if and only if

$$c \;\geq\; \frac{S^{K'}(0,N,N)[1-Q^{K'}(0,N,N)]}{\frac{h(K')}{K'}} \;=\; \frac{\mu_a^{K'}(0)[\frac{1}{K'}]^N}{\frac{h(K')}{K'}} \;=:\; c^{K'}.$$

We claim that $c^{K'} < c^K$.
Suppose not, i.e., assume that $c^{K'} \geq c^K$. By definition, this means that

$$\frac{\mu_a^{K'}(0)[\frac{1}{K'}]^N}{\frac{h(K')}{K'}} \geq \frac{\mu_a^K(0)[\frac{1}{K}]^N}{\frac{h(K)}{K}}.$$

Applying Lemma 3.2 while setting $z_K = z_{K'} = 0$ yields

$$\frac{\mu_a^{K'}(0)}{\mu_a^K(0)} < \frac{K'}{K}.$$

Combining the two inequalities, we obtain

$$\frac{\mu_a^{K'}(0)[\frac{1}{K'}]^N}{\frac{h(K')}{K'}} \geq \frac{\mu_a^K(0)[\frac{1}{K}]^N}{\frac{h(K)}{K}} > \frac{\mu_a^{K'}(0)[\frac{1}{K}]^N}{\frac{h(K)}{K}} \frac{K}{K'}.$$

Hence, since, by assumption, $\frac{h(K')}{K'} \geq \frac{h(K)}{K}$, we have that

$$\frac{\mu_a^{K'}(0)[\frac{1}{K'}]^N}{\frac{h(K')}{K'}} > \frac{\mu_a^{K'}(0)[\frac{1}{K}]^N}{\frac{h(K)}{K}} \frac{K}{K'} \geq \frac{\mu_a^{K'}(0)[\frac{1}{K}]^N}{\frac{h(K')}{K'}} \frac{K}{K'}.$$

Simplifying yields

$$[\frac{1}{K'}]^{N-1} > [\frac{1}{K}]^{N-1}.$$

If $N = 1$, the inequality reduces to $1 > 1$ and, in the case where $N \geq 2$, we must have that $K > K'$. Thus, in both cases, we derived the desired contradiction.
We are now ready to perform a case distinction depending on the magnitude of the scaling parameter c:
1) $c \geq c^K > c^{K'}$
In this case, both search procedures give rise to a unique boundary equilibrium with equilibrium cutoffs $z_K = z_{K'} = 0$. In order to see that there are no additional interior equilibria, consider the search procedure with K' candidates per period. The argument for the search protocol with K candidates at a time is analogous. Towards a contradiction, suppose that there exists an equilibrium with cutoff $z'_{K'} \in (0, \overline{x})$. Since this cutoff is interior, it solves the equilibrium equation

$$c\frac{h(K')}{K'} = [\frac{1}{K'}(1 - F(z'_{K'})^{K'})]^N[\mu_a^{K'}(z'_{K'}) - z'_{K'}].$$

Making use of $z'_{K'} > 0$ and rewriting yield the inequality

$$c\frac{h(K')}{K'}$$
$$< [\frac{1}{K'}(1 - F(z'_{K'})^{K'})]^N \mu_a^{K'}(z'_{K'}) = [\frac{1}{K'}(1 - F(z'_{K'})^{K'})]^{N-1} \int_{z'_{K'}}^{\overline{x}} f(s)F(s)^{K'-1} sds.$$

Now, observe that the right-hand side of this inequality is decreasing in the cutoff. Therefore, it follows that

$$c\frac{h(K')}{K'} \;<\; [\frac{1}{K'}]^N \mu_a^{K'}(0),$$

which is equivalent to $c < c^{K'}$. This is the desired contradiction.
The respective welfare levels induced by the unique boundary equilibria of the two search procedures amount to

$$\begin{aligned} v_K &= \mu_a^K(0) - \frac{c \cdot h(K)}{K[\frac{1}{K}]^N} = \mu_a^K(0) - (K)^N c\frac{h(K)}{K} \text{ and} \\ v_{K'} &= \mu_a^{K'}(0) - \frac{c \cdot h(K')}{K'[\frac{1}{K'}]^N} = \mu_a^{K'}(0) - (K')^N c\frac{h(K')}{K'}. \end{aligned}$$

Towards a contradiction, suppose $v_{K'} \geq v_K$. Applying Lemma 3.2 while setting $z_K = z_{K'} = 0$ and using $\frac{h(K')}{K'} \geq \frac{h(K)}{K}$, we obtain that

$$\begin{aligned} \mu_a^K(0) - (K)^N c\frac{h(K)}{K} &= v_K \leq v_{K'} = \mu_a^{K'}(0) - (K')^N c\frac{h(K')}{K'} \\ &< \mu_a^K(0)\frac{K'}{K} - (K')^N c\frac{h(K)}{K}. \end{aligned}$$

Thus, we conclude that

$$\mu_a^K(0)[\frac{K'}{K} - 1] > c\frac{h(K)}{K}[(K')^N - (K)^N].$$

Since $K' > K$ and $c \geq c^K = \frac{\mu_a^K(0)[\frac{1}{K}]^N}{\frac{h(K)}{K}}$, we have that

$$\mu_a^K(0)[\frac{K'}{K} - 1] > c\frac{h(K)}{K}[(K')^N - (K)^N] \geq \frac{\mu_a^K(0)[\frac{1}{K}]^N}{\frac{h(K)}{K}}\frac{h(K)}{K}[(K')^N - (K)^N].$$

Simplifying yields

$$\frac{K'}{K} > (\frac{K'}{K})^N.$$

In the case of $N = 1$, there is a contradiction. If $N \geq 2$, we must have that $K' < K$ which constitutes a contradiction as well.
2) $c^K > c \geq c^{K'}$
Here, if there are K candidates per period, the corresponding search procedure admits only interior equilibria described by cutoff values $z_K > 0$. In contrast, if there are K' candidates per period, as argued above, there is a unique boundary equilibrium

with cutoff $z_{K'} = 0$. Therefore, the resulting welfare levels of both search procedures are given by

$$\begin{aligned} v_K &= z_K \text{ and} \\ v_{K'} &= \mu_a^{K'}(0) - \frac{c \cdot h(K')}{K'[\frac{1}{K'}]^N} = \mu_a^{K'}(0) - (K')^N c \frac{h(K')}{K'}. \end{aligned}$$

By definition of $c^{K'}$ and because of $c \geq c^{K'}$, we directly obtain that $v_{K'} \leq 0$. In contrast, it holds that $v_K = z_K > 0$, directly implying $v_{K'} < v_K$.
3) $c^K > c^{K'} > c$
In this case, both search technologies give rise to interior equilibria. Denote the equilibrium cutoff values in the game with K' candidates per period by $z_{K'}$ and the cutoff values in the search game with K candidate per period by z_K. Given private value preferences, cutoff values, or acceptance standards, coincide with welfare, i.e., $v_{K'} = z_{K'}$ and $v_K = z_K$.
Assume, by contradiction, that there are equilibria such that $v_K = z_K \leq z_{K'} = v_{K'}$. The equilibrium cutoff values satisfy the following equations:

$$\begin{aligned} S^K(z_K, N, N) - z_K &= \frac{c \cdot h(K)}{K \cdot [1 - Q^K(z_K, N, N)]} \text{ and} \\ S^{K'}(z_{K'}, N, N) - z_{K'} &= \frac{c \cdot h(K')}{K' \cdot [1 - Q^{K'}(z_{K'}, N, N)]}. \end{aligned}$$

In the following, we derive bounds on the ratio

$$\frac{S^{K'}(z_{K'}, N, N) - z_{K'}}{S^K(z_K, N, N) - z_K} = \frac{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}] - z_{K'}}{\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K] - z_K}.$$

First, since $z_K \leq z_{K'}$, Lemma 3.2 yields

$$\frac{S^{K'}(z_{K'}, N, N) - z_{K'}}{S^K(z_K, N, N) - z_K} < \frac{K'}{K} \frac{[1 - F(z_K)^K]}{[1 - F(z_{K'})^{K'}]}.$$

Second, by the equilibrium conditions, we have that

$$\begin{aligned} \frac{S^{K'}(z_{K'}, N, N) - z_{K'}}{S^K(z_K, N, N) - z_K} &= \frac{\frac{c \frac{h(K')}{K'}}{1 - Q^{K'}(z_{K'}, N, N)}}{\frac{c \frac{h(K)}{K}}{1 - Q^K(z_K, N, N)}} \\ &= \frac{h(K')}{K'} \frac{K}{h(K)} \frac{[\Pr(X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K)]^N}{[\Pr(X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'})]^N}. \end{aligned}$$

Since $\frac{h(K')}{K'} \geq \frac{h(K)}{K}$, we obtain

$$\begin{aligned}
&\frac{h(K')}{K'} \frac{K}{h(K)} \frac{[\Pr(X_i^k \geq \max_{l \in \{1,\dots,K\}:l \neq k} X_i^l, X_i^k \geq z_K)]^N}{[\Pr(X_i^k \geq \max_{l \in \{1,\dots,K'\}:l \neq k} X_i^l, X_i^k \geq z_{K'})]^N} \\
\geq &\frac{[\Pr(X_i^k \geq \max_{l \in \{1,\dots,K\}:l \neq k} X_i^l, X_i^k \geq z_K)]^N}{[\Pr(X_i^k \geq \max_{l \in \{1,\dots,K'\}:l \neq k} X_i^l, X_i^k \geq z_{K'})]^N} \\
= &[\frac{K'(1-F(z_K)^K)}{K(1-F(z_{K'})^{K'})}]^N,
\end{aligned}$$

where the last step uses expressions for the involved probabilities that are derived in Appendix 3.C.2. Therefore, we get that

$$\frac{S^{K'}(z_{K'}, N, N) - z_{K'}}{S^K(z_K, N, N) - z_K} \geq [\frac{K'(1-F(z_K)^K)}{K(1-F(z_{K'})^{K'})}]^N.$$

Putting both bounds on $\frac{S^{K'}(z_{K'},N,N)-z_{K'}}{S^K(z_K,N,N)-z_K}$ together, we conclude that

$$\frac{K'(1-F(z_K)^K)}{K(1-F(z_{K'})^{K'})} > [\frac{K'(1-F(z_K)^K)}{K(1-F(z_{K'})^{K'})}]^N.$$

If $N = 1$, there is a contradiction. If $N \geq 2$, this inequality is equivalent to

$$1 > \frac{\frac{1}{K}[1-F(z_K)^K]}{\frac{1}{K'}[1-F(z_{K'})^{K'}]}.$$

Because of $z_K \leq z_{K'}$, it follows that

$$1 > \frac{\frac{1}{K}[1-F(z_K)^K]}{\frac{1}{K'}[1-F(z_K)^{K'}]}.$$

Now, observe that the term on the right-hand side of this inequality is the ratio of the probabilities of voting in favor of a candidate k when there K compared to K' candidates per period for a fixed cutoff z_K. Since this probability is smaller for K than for K' candidates per period, this ratio must be strictly larger than 1. This is the desired contradiction. Consequently, it must be true that $v_K = z_K > z_{K'} = v_{K'}$. □

Proof of Proposition 3.4.
To begin with, denote the unique equilibrium cutoff values in the games with K' and K candidates per period by $z_{K'}$ and z_K respectively. To the contrary, suppose that for all $\bar{c}_{K',K} > 0$ there exists $c < \bar{c}_{K',K}$ such that $v_K \geq v_{K'}$. Without loss of generality, restrict attention to sufficiently small values of c such that the equilibria under both procedures are interior. Then, cutoff values coincide with welfare, i.e., $v_K = z_K$ and

$v_{K'} = z_{K'}$.
The respective equilibrium thresholds satisfy the following equations:

$$\begin{aligned} S^K(z_K, N, N) - z_K &= \frac{c \cdot h(K)}{K \cdot [1 - Q^K(z_K, N, N)]} \text{ and} \\ S^{K'}(z_{K'}, N, N) - z_{K'} &= \frac{c \cdot h(K')}{K' \cdot [1 - Q^{K'}(z_{K'}, N, N)]}. \end{aligned}$$

Lemma 3.1 implies that

$$S^{K'}(z_K, N, N) - z_{K'} = \mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}] - z_{K'}$$

is non-increasing in $z_{K'}$.
Therefore, the assumption $z_K \geq z_{K'}$ yields the inequality

$$\begin{aligned} &\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_K] - z_K \\ \leq &\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}] - z_{K'}. \end{aligned}$$

Moreover, since $K' > K$, it holds that

$$\begin{aligned} &\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K] \\ \leq &\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_K]. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} &\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K] - z_K \\ \leq &\mathbb{E}[X_i^k | X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'}] - z_{K'}. \end{aligned}$$

This inequality is the same as

$$S^K(z_K, N, N) - z_K \leq S^{K'}(z_{K'}, N, N) - z_{K'}.$$

Exploiting the equilibrium equations, we get that

$$\frac{c \frac{h(K)}{K}}{1 - Q^K(z_K, N, N)} \leq \frac{c \frac{h(K')}{K'}}{1 - Q^{K'}(z_{K'}, N, N)}.$$

Rewriting this inequality yields

$$\begin{aligned} &[\Pr(X_i^k \geq \max_{l \in \{1,\dots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'})]^N \\ \leq &\frac{h(K')}{K'} \frac{K}{h(K)} [\Pr(X_i^k \geq \max_{l \in \{1,\dots,K\}: l \neq k} X_i^l, X_i^k \geq z_K)]^N. \end{aligned}$$

Furthermore, because of $z_K \geq z_{K'}$, we have that

$$\begin{aligned} &\Pr(X_i^k \geq \max_{l \in \{1,\ldots,K\}: l \neq k} X_i^l, X_i^k \geq z_K) \\ \leq &\Pr(X_i^k \geq \max_{l \in \{1,\ldots,K\}: l \neq k} X_i^l, X_i^k \geq z_{K'}). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} &[\Pr(X_i^k \geq \max_{l \in \{1,\ldots,K'\}: l \neq k} X_i^l, X_i^k \geq z_{K'})]^N \\ \leq &\frac{h(K')}{K'} \frac{K}{h(K)} [\Pr(X_i^k \geq \max_{l \in \{1,\ldots,K\}: l \neq k} X_i^l, X_i^k \geq z_{K'})]^N. \end{aligned}$$

Rearranging this inequality while employing expressions for the involved probabilities derived in Appendix 3.C.2 implies that

$$[\frac{\frac{1}{K'}(1 - F(z_{K'})^{K'})}{\frac{1}{K}(1 - F(z_{K'})^K)}]^N \leq \frac{h(K')}{K'} \frac{K}{h(K)}.$$

Since, by assumption, $\frac{h(K')}{K'} < \frac{h(K)}{K}$, we have that the right-hand side of this inequality is strictly smaller than 1. We claim that, no matter the fixed value of $0 < \frac{h(K')}{K'} \frac{K}{h(K)} < 1$, as long as the cost parameter c is sufficiently small, the left-hand side of the inequality is below 1, but arbitrarily close to it. The first part of this statement is true because the discussed term is the ratio of the probabilities of accepting a candidate k when there K' compared to K candidates per period for a fixed cutoff $z_{K'}$. To see the second part, note that as $c \to 0$, $z_{K'} \to \bar{x}$, implying that $F(z_{K'}) \to 1$. Then, l'Hôpital's rule yields

$$\lim_{c \to 0} [\frac{\frac{1}{K'}(1 - F(z_{K'})^{K'})}{\frac{1}{K}(1 - F(z_{K'})^K)}]^N = [\lim_{c \to 0} \frac{-F(z_{K'})^{K'-1} f(z_{K'})}{-F(z_{K'})^{K-1} f(z_{K'})}]^N = [\lim_{c \to 0} F(z_{K'})^{K'-K}]^N = 1.$$

Thus, as $c \to 0$, the left-hand side of the inequality converges to 1. Consequently, eventually, for small c, the left-hand side of the inequality exceeds the right-hand side because $\frac{h(K')}{K'} \frac{K}{h(K)} < 1$. This is the desired contradiction. □

Proof of Lemma 3.3.
To begin with, take any $K', K \geq 1$ with $K' > K$, and fix any value $z \in [0, \bar{x})$. In order to improve readability, we often drop the dependence of the involved functions on z. The subsequent argument does not apply to the case in which $K = 1$ and $z = 0$. We tackle this case separately at the end of this proof.
First, we derive an expression for $S^K(z, N, M)$ in terms of $w^K(z)$, $\mu_a^K(z)$, $F(z)$ and μ. By the law of total expectation, we have

$$\mu_r^K = \frac{\mu - \frac{1}{K}(1 - F^K)\mu_a^K}{1 - \frac{1}{K}(1 - F^K)}$$

and, consequently, we obtain

$$\mu_a^K - \mu_r^K = \mu_a^K - \frac{\mu - \frac{1}{K}(1-F^K)\mu_a^K}{1-\frac{1}{K}(1-F^K)} = \frac{\mu_a^K - \mu}{1-\frac{1}{K}(1-F^K)}.$$

Therefore, $S^K(z,N,M)$ can be written as

$$\begin{aligned} S^K(z,N,M) &= w^K\mu_a^K + [1-w^K]\mu_r^K \\ &= \mu_r^K + w^K[\mu_a^K - \mu_r^K] \\ &= \frac{\mu - \frac{1}{K}(1-F^K)\mu_a^K}{1-\frac{1}{K}(1-F^K)} + w^K\frac{\mu_a^K - \mu}{1-\frac{1}{K}(1-F^K)} \\ &= \mu[\frac{1-w^K}{1-\frac{1}{K}(1-F^K)}] + \mu_a^K[\frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)}] \\ &= \mu + [\frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)}][\mu_a^K - \mu]. \end{aligned}$$

Further, the law of total expectation yields

$$S^K(z,N,M) = \frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)}[\mu_a^K - \mu] + \frac{1}{K'}(1-F^{K'})\mu_a^{K'} + [1-\frac{1}{K'}(1-F^{K'})]\mu_r^{K'}.$$

Second, we develop an expression for $\mu_a^{K'} - \mu_r^{K'}$ as well as a lower bound on this term. The law of total expectation implies

$$\mu_r^{K'} = \frac{\mu - \frac{1}{K'}(1-F^{K'})\mu_a^{K'}}{1-\frac{1}{K'}(1-F^{K'})}.$$

Thus, we obtain

$$\begin{aligned} \mu_a^{K'} - \mu_r^{K'} &= \mu_a^{K'} - \frac{\mu - \frac{1}{K'}(1-F^{K'})\mu_a^{K'}}{1-\frac{1}{K'}(1-F^{K'})} \\ &= \frac{\mu_a^{K'} - \mu}{1-\frac{1}{K'}(1-F^{K'})} \\ &\geq \frac{\mu_a^{K} - \mu}{1-\frac{1}{K'}(1-F^{K'})}, \end{aligned}$$

where the inequality follows from the assumption $K' > K$ which implies $\mu_a^{K'} \geq \mu_a^K$. Now, suppose to the contrary that $S^K(z,N,M) \geq S^{K'}(z,N,M)$. This means that

$$\begin{aligned} & S^K(z,N,M) \\ =& \frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)}[\mu_a^K - \mu] + \frac{1}{K'}(1-F^{K'})\mu_a^{K'} + [1-\frac{1}{K'}(1-F^{K'})]\mu_r^{K'} \\ \geq& \mu_a^{K'}w^{K'} + \mu_r^{K'}[1-w^{K'}] = S^{K'}(z,N,M). \end{aligned}$$

Rearranging this inequality yields

$$\begin{aligned}&\frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)}[\mu_a^K - \mu] + \mu_r^{K'}[1 - \frac{1}{K'}(1-F^{K'}) - 1 + w^{K'}] \\ \geq &\mu_a^{K'}[w^{K'} - \frac{1}{K'}(1-F^{K'})],\end{aligned}$$

which is equivalent to

$$\frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)}[\mu_a^K - \mu] \geq [\mu_a^{K'} - \mu_r^{K'}][w^{K'} - \frac{1}{K'}(1-F^{K'})].$$

Employing the lower bound on $\mu_a^{K'} - \mu_r^{K'}$, we have

$$\frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)}[\mu_a^K - \mu] \geq \frac{\mu_a^K - \mu}{1-\frac{1}{K'}(1-F^{K'})}[w^{K'} - \frac{1}{K'}(1-F^{K'})]$$

because $w^{K'} - \frac{1}{K'}(1-F^{K'}) > 0$. To see the latter point, observe that

$$\begin{aligned} w^{K'} &= \sum_{l=M}^{N} \frac{q^{K'}(z,N,l)}{1-Q^{K'}(z,N,M)} \frac{l}{N} \\ &\geq \frac{M}{N} \sum_{l=M}^{N} \frac{q^{K'}(z,N,l)}{1-Q^{K'}(z,N,M)} = \frac{M}{N} > \frac{1}{2}. \end{aligned}$$

Moreover, since $K' > K \geq 1$, we have

$$\frac{1}{K'}(1-F^{K'}) \leq \frac{1}{2}(1-F^{K'}) \leq \frac{1}{2}.$$

Hence, it holds that $w^{K'} - \frac{1}{K'}(1-F^{K'}) > 0$.
Next, we note that $[\mu_a^K - \mu] > 0$ because F has full support and, by assumption, $z > 0$. Thus, we arrive at the following expression:

$$\frac{w^K - \frac{1}{K}(1-F^K)}{1-\frac{1}{K}(1-F^K)} \geq \frac{w^{K'} - \frac{1}{K'}(1-F^{K'})}{1-\frac{1}{K'}(1-F^{K'})}.$$

Rewriting this inequality yields

$$1 - w^K \leq \frac{1-\frac{1}{K}(1-F^K)}{1-\frac{1}{K'}(1-F^{K'})}[1-w^{K'}]. \tag{3.B.2}$$

Now, Albrecht, Anderson, and Vroman (2010) provide an alternative expression for the weight w^1 as a function of the probability that some member votes in favor of the available candidate. They rely on the Gaussian hypergeometric function as well as the Euler integral.[31] We apply those expressions to the weights w^K and $w^{K'}$. In order to simplify the notation, let A^K and $A^{K'}$ be the probability of approving some candidate k if there are K or K' candidates respectively. In other words, define

$$\begin{aligned}A^K(z) &:= \frac{1}{K}(1-F^K), \text{ as well as}\\ A^{K'}(z) &:= \frac{1}{K'}(1-F^{K'}).\end{aligned}$$

Making use of this notation, the expressions in Albrecht, Anderson, and Vroman (2010) read as follows:[32]

$$\begin{aligned}w^K &= A^K + \frac{M}{N}(1-A^K)\{\int_0^1 [1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1} \text{ and}\\ w^{K'} &= A^{K'} + \frac{M}{N}(1-A^{K'})\{\int_0^1 [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}1-w^K &= 1-A^K - \frac{M}{N}(1-A^K)\{\int_0^1 [1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}\\ &= [1-A^K]\cdot[1-\frac{M}{N}\{\int_0^1 [1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}]\\ &= [1-\frac{1}{K}(1-F^K)]\cdot[1-\frac{M}{N}\{\int_0^1 [1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}],\end{aligned}$$

as well as

$$\begin{aligned}1-w^{K'} &= 1-A^{K'} - \frac{M}{N}(1-A^{K'})\{\int_0^1 [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}\\ &= [1-A^{K'}]\cdot[1-\frac{M}{N}\{\int_0^1 [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}]\\ &= [1-\frac{1}{K'}(1-F^{K'})]\cdot[1-\frac{M}{N}\{\int_0^1 [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}].\end{aligned}$$

Then, inequality (3.B.2) becomes

$$\begin{aligned}&[1-\frac{1}{K}(1-F^K)]\cdot[1-\frac{M}{N}\{\int_0^1 [1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}]\\ \leq&\frac{1-\frac{1}{K}(1-F^K)}{1-\frac{1}{K'}(1-F^{K'})}\cdot[1-\frac{1}{K'}(1-F^{K'})]\\ &\cdot[1-\frac{M}{N}\{\int_0^1 [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M}dy\}^{-1}].\end{aligned}$$

31. See for example Abramowitz and Stegun (1965).
32. The derivation can be found on pages 1403 f. in Albrecht, Anderson, and Vroman (2010).

Simplifying and rearranging this inequality yields

$$\int_0^1 [1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M}dy \;\leq\; \int_0^1 [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M}dy.$$

In the following, we claim that, for all $y \in [0,1)$,

$$[1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M} \;>\; [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M},$$

which implies that the former inequality cannot be true.
To begin with, note that $A^K = A^K(z) > A^{K'}(z) = A^{K'}$ since $z \neq \bar{x}$ and $K' > K$. Now, take any $y \in [0,1)$ and observe that

$$\begin{aligned}
& A_K \;>\; A_{K'} \\
\Leftrightarrow \quad & \frac{A^K}{1-A^K} \;>\; \frac{A^{K'}}{1-A^{K'}} \\
\Leftrightarrow \quad & 1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}}) \;>\; 1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}}) \\
\Leftrightarrow \quad & [1+\frac{A^K}{1-A^K}(1-y^{\frac{1}{M}})]^{N-M} \;>\; [1+\frac{A^{K'}}{1-A^{K'}}(1-y^{\frac{1}{M}})]^{N-M}.
\end{aligned}$$

This establishes the claim, yielding the desired contradiction. Therefore, overall, we conclude that $S^K(z,N,M) < S^{K'}(z,N,M)$ for all $z \in (0,\bar{x})$.
Finally, it remains to tackle the case in which $K=1$ and $z=0$. Here, observe that $S^1(0,N,M) = \mu$. Towards a contradiction, suppose that $\mu = S^1(0,N,M) \geq S^{K'}(0,N,M)$. By the law of total expectation, we obtain

$$\begin{aligned}
\mu \;&= [\frac{1}{K'}(1-[F(0)]^{K'})]\mu_a^{K'} + [1-\frac{1}{K'}(1-[F(0)]^{K'})]\mu_r^{K'} \\
&\geq S^{K'}(0,N,M) \;=\; \mu_a^{K'}w^{K'} + \mu_r^{K'}[1-w^{K'}].
\end{aligned}$$

Rearranging this inequality yields

$$0 \;\geq\; [\mu_a^{K'} - \mu_r^{K'}][w^{K'} - \frac{1}{K'}].$$

However, we have that

$$0 \;\geq\; [\mu_a^{K'} - \mu_r^{K'}][w^{K'} - \frac{1}{K'}] \;>\; 0$$

because $\mu_a^{K'} - \mu_r^{K'} > 0$ as well as $w^{K'} - \frac{1}{K'}(1-[F(0)]^{K'}) > 0$. The latter point is implied by $K' > 1$ and it has been established in the first part of this proof. Hence, we arrive at the desired contradiction. $\square$

Proof of Proposition 3.5.
Consider any $K', K \geq 1$ with $K' > K$. To the contrary, suppose that for all $\bar{c}_{K',K} > 0$ there exists $c < \bar{c}_{K',K}$ such that $v_K \geq v_{K'}$. Without loss of generality, restrict attention to sufficiently small values of c such that the unique equilibria under both procedures are interior. Let z_K and $z_{K'}$ denote the equilibrium cutoffs corresponding to sequential search with K and K' candidates per period respectively. These cutoffs solve the respective equilibrium equations

$$\begin{aligned} S^K(z_K, N, M) - z_K &= \frac{c\frac{h(K)}{K}}{1 - Q^K(z_K, N, M)} \\ S^{K'}(z_{K'}, N, M) - z_{K'} &= \frac{c\frac{h(K')}{K'}}{1 - Q^{K'}(z_{K'}, N, M)}, \end{aligned}$$

and they coincide with welfare, meaning, $z_K = v_K$ as well as $z_{K'} = v_{K'}$. Thus, by assumption, $z_K \geq z_{K'}$. Lemma 3.1 implies that the function $S^K(z, N, M) - z$ is decreasing in z. Making use of this property and employing the equilibrium equations as well as $z_K \geq z_{K'}$, we obtain

$$\begin{aligned} \frac{c\frac{h(K)}{K}}{1 - Q^K(z_K, N, M)} &= S^K(z_K, N, M) - z_K \\ &\leq S^K(z_{K'}, N, M) - z_{K'} \\ &= S^K(z_{K'}, N, M) + \frac{c\frac{h(K')}{K'}}{1 - Q^{K'}(z_{K'}, N, M)} - S^{K'}(z_{K'}, N, M). \end{aligned}$$

Rearranging this inequality yields

$$S^{K'}(z_{K'}, N, M) - S^K(z_{K'}, N, M) \leq \frac{c\frac{h(K')}{K'}}{1 - Q^{K'}(z_{K'}, N, M)} - \frac{c\frac{h(K)}{K}}{1 - Q^K(z_K, N, M)}. \quad (3.B.3)$$

Now, we claim that there exists $B < \bar{x}$ such that for all $c > 0$, it holds $z_K < B$ and $z_{K'} < B$.
First, towards a contradiction, suppose that for all $B^K < \bar{x}$ there exist $c > 0$ such that $z_K \geq B^K$. By the equilibrium equation and the monotonicity properties of the involved functions established in the proofs of Lemma 3.1 and Proposition 3.2, we have that z_K is weakly decreasing in c. Thus, the previous assumption requires that $z_K \to \bar{x}$ as $c \to 0$. Consider the following rearranged version of the equilibrium equation:

$$z_K = S^K(z_K, N, M) - \frac{c\frac{h(K)}{K}}{1 - Q^K(z_K, N, M)}.$$

If we take the limit on both sides of the equation as $c \to 0$, we obtain

$$\begin{aligned} \bar{x} &= \lim_{c\to 0}[z_K] = \lim_{c\to 0}[S^K(z_K, N, M) - \frac{c\frac{h(K)}{K}}{1 - Q^K(z_K, N, M)}] \\ &\leq \lim_{c\to 0}[S^K(z_K, N, M)] < \bar{x}, \end{aligned}$$

which constitutes the desired contradiction. Recalling the average representation of $S^K(z_K, N, M)$, the final inequality holds because $\lim_{c\to 0} \mu_r^K(z_K) = \mu < \bar{x}$ as well as $\lim_{c\to 0} w^K(z_K) = \frac{M}{N} < 1$ which is implied by $M < N$. Therefore, there exists $B^K < \bar{x}$ such that for all $c > 0$, it holds that $z_K < B^K$.

Second, applying the same argument in an analogous way to sequential search with K' candidates per period, we infer that there exists $B^{K'} < \bar{x}$ such that for all $c > 0$, it holds that $z_{K'} < B^{K'}$.

Consequently, setting $B := \max\{B^K, B^{K'}\}$, we conclude that $z_K < B$ and $z_{K'} < B$ for all $c > 0$.

Making use of this feature, we obtain the following upper bound on the right-hand side of inequality (3.B.3):

$$\begin{aligned}
&\frac{c\frac{h(K')}{K'}}{1 - Q^{K'}(z_{K'}, N, M)} - \frac{c\frac{h(K)}{K}}{1 - Q^K(z_K, N, M)} \\
<&\frac{c\frac{h(K')}{K'}}{1 - Q^{K'}(B, N, M)} - \frac{c\frac{h(K)}{K}}{1 - Q^K(0, N, M)} \\
=&c[\frac{\frac{h(K')}{K'}}{1 - Q^{K'}(B, N, M)} - \frac{\frac{h(K)}{K}}{1 - Q^K(0, N, M)}].
\end{aligned}$$

Note that this upper bound does not depend on the equilibrium cutoffs of the two considered procedures z_K and $z_{K'}$.

Let us perform a case distinction:

1) $\frac{\frac{h(K')}{K'}}{1-Q^{K'}(B,N,M)} - \frac{\frac{h(K)}{K}}{1-Q^K(0,N,M)} \leq 0$

In this case, inequality (3.B.3) and the upper bound on the right-hand side of this inequality yield

$$S^{K'}(z_{K'}, N, M) - S^K(z_{K'}, N, M) \;\leq\; 0,$$

which contradicts Lemma 3.3 because of $K' > K$. Let $\bar{c}_{K',K}$ be the cost value such that for all $c < \bar{c}_{K',K}$, the unique equilibrium under both search procedures is interior. That is, set

$$\bar{c}_{K',K} \;:=\; \min\{\frac{S^{K'}(0, N, M)[1 - Q^{K'}(0, N, M)]}{\frac{h(K')}{K'}}, \frac{S^K(0, N, M)[1 - Q^K(0, N, M)]}{\frac{h(K)}{K}}\} \;>\; 0,$$

recalling the proofs of Propositions 3.1 and 3.2. Then, the established contradiction implies that, for all these levels of c, we have $v_K < v_{K'}$.

2) $\frac{\frac{h(K')}{K'}}{1-Q^{K'}(B,N,M)} - \frac{\frac{h(K)}{K}}{1-Q^K(0,N,M)} > 0$

To begin with, define

$$r \;:=\; \min_{s\in[0,B]} [S^{K'}(s, N, M) - S^K(s, N, M)].$$

Observe that r is well-defined because the involved minimum exists due to the extreme value theorem. Further, Lemma 3.3 implies that $r > 0$. Also, note that r does not depend on z_K, $z_{K'}$ and c. Moreover, we have that the left-hand side of inequality (3.B.3) is bounded below by r, meaning,

$$S^{K'}(z_{K'}, N, M) - S^K(z_{K'}, N, M) \;\geq\; r.$$

Taking the upper bound on the right-hand side of inequality inequality (3.B.3) together with this lower bound on the left hand-side of the discussed inequality, we arrive at the following inequality:

$$r \;<\; c\Big[\frac{\frac{h(K')}{K'}}{1 - Q^{K'}(B, N, M)} - \frac{\frac{h(K)}{K}}{1 - Q^K(0, N, M)}\Big].$$

Now, set

$$\bar{c}_{K',K} \;:=\; \frac{r}{\frac{\frac{h(K')}{K'}}{1 - Q^{K'}(B,N,M)} - \frac{\frac{h(K)}{K}}{1 - Q^K(0,N,M)}}.$$

Note that $\bar{c}_{K',K} > 0$ since $\frac{\frac{h(K')}{K'}}{1-Q^{K'}(B,N,M)} - \frac{\frac{h(K)}{K}}{1-Q^K(0,N,M)} > 0$ by assumption and, again, $r > 0$ because of Lemma 3.3. Then, for all $c < \bar{c}_{K',K}$, we have that

$$\begin{aligned} r \;&<\; c\Big[\frac{\frac{h(K')}{K'}}{1 - Q^{K'}(B, N, M)} - \frac{\frac{h(K)}{K}}{1 - Q^K(0, N, M)}\Big] \\ &< \frac{r}{\frac{\frac{h(K')}{K'}}{1 - Q^{K'}(B,N,M)} - \frac{\frac{h(K)}{K}}{1 - Q^K(0,N,M)}} \cdot \Big[\frac{\frac{h(K')}{K'}}{1 - Q^{K'}(B, N, M)} - \frac{\frac{h(K)}{K}}{1 - Q^K(0, N, M)}\Big] \\ &= r. \end{aligned}$$

This constitutes the desired contradiction. □

Appendix 3.C Derivations

3.C.1 Expected Value Conditional on Stopping

First, we derive the expression for the value quality of some candidate $k \in \mathscr{K}$ for some member $i \in \mathscr{N}$ conditional on stopping:

$$
\begin{aligned}
S^K(z,N,M) &= \mathbb{E}[X_i^k|\text{candidate } k \text{ hired}] \\
&= \sum_{l=M}^{N} \Pr(\#k\ supporters\ =\ l|k\ hired)\mathbb{E}[X_i^k|k\ hired \text{ and } \#k\ supporters\ =\ l] \\
&= \sum_{l=M}^{N} \frac{q^K(z,N,l)}{1-Q^K(z,N,M)}\mathbb{E}[X_i^k|\#k\ supporters\ =\ l] \\
&= \sum_{l=M}^{N} \frac{q^K(z,N,l)}{1-Q^K(z,N,M)}\cdot \\
&\{\Pr(voter\ i\ supports\ k|\#k\ supporters\ =\ l)\mathbb{E}[X_i^k|voter\ i\ supports\ k] \\
&+\Pr(voter\ i\ rejects\ k|\#k\ supporters\ =\ l)\mathbb{E}[X_i^k|voter\ i\ rejects\ k]\} \\
&= \sum_{l=M}^{N} \frac{q^K(z,N,l)}{1-Q^K(z,N,M)}[\frac{l}{N}\mu_a^K(z) + \frac{N-l}{N}\mu_r^K(z)] \\
&= w^K(z)\mu_a^K(z) + [1-w^K(z)]\mu_r^K(z),
\end{aligned}
$$

where $w^K(z)$ is defined as

$$
w^K(z) := \sum_{l=M}^{N} \frac{q^K(z,N,l)}{1-Q^K(z,N,M)}\frac{l}{N}.
$$

3.C.2 Probability of Acceptance

Second, we derive the expression for the probability that some member $i \in \mathcal{N}$ votes in favor of some candidate $k \in \mathcal{K}$ as a function of K, F and the employed cutoff z:

$$
\begin{aligned}
&\Pr(X_i^k \geq \max_{l \neq k} X_i^l, X_i^k \geq z) \\
&= \int_0^{\bar{x}} \Pr(X_i^k \geq s, X_i^k \geq z) \Pr(\max_{l \neq k} X_i^l = s) ds \\
&= \int_0^{\bar{x}} \Pr(X_i^k \geq \max\{s, z\}) \Pr(\max_{l \neq k} X_i^l = s) ds \\
&= \int_0^{z} \Pr(X_i^k \geq z) \Pr(\max_{l \neq k} X_i^l = s) ds + \int_z^{\bar{x}} \Pr(X_i^k \geq s) \Pr(\max_{l \neq k} X_i^l = s) ds \\
&= [1 - F(z)] \int_0^{z} \frac{dF(s)^{K-1}}{ds} ds + \int_z^{\bar{x}} [1 - F(s)](K-1)F(s)^{K-2} f(s) ds \\
&= [1 - F(z)]F(z)^{K-1} + \int_z^{\bar{x}} (K-1)F(s)^{K-2} f(s) ds \\
&- \int_z^{\bar{x}} (K-1)F(s)^{K-1} f(s) ds \\
&= [1 - F(z)]F(z)^{K-1} + \int_z^{\bar{x}} \frac{dF(s)^{K-1}}{ds} ds - \int_z^{\bar{x}} \frac{d[\frac{K-1}{K} F(s)^K]}{ds} ds \\
&= [1 - F(z)]F(z)^{K-1} + [1 - F(z)^{K-1}] - \frac{K-1}{K} + \frac{K-1}{K} F(z)^K \\
&= F(z)^{K-1} - F(z)^K + 1 - F(z)^{K-1} - 1 + \frac{1}{K} + F(z)^K - \frac{1}{K} F(z)^K \\
&= \frac{1}{K}[1 - F(z)^K].
\end{aligned}
$$

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Chapter 4

Optimal Voting Mechanisms on Generalized Single-Peaked Domains*

4.1 Introduction

In this chapter, I characterize the optimal utilitarian voting mechanisms, meaning, the voting rules that maximize the ex-ante utilitarian welfare, among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity. The setting features more than two alternatives, and the voters have generalized single-peaked preferences derived from median spaces as introduced in Nehring and Puppe (2007b), henceforth NP (2007b). This class of preferences is much larger than the well-known class of preferences that are single-peaked on a line.

NP (2007b) provide a characterization of strategy-proof social choice functions for generalized single-peaked domains giving rise to median spaces. This characterization constitutes the starting point of my analysis of welfare-maximizing voting rules. Gershkov, Moldovanu, and Shi (2017) study the stated mechanism design problem for preferences which are single-peaked on a line.[1] For these preferences, they derive the utilitarian mechanism, and they show that, in this case, the optimal voting rule takes the form of a successive procedure with weakly decreasing thresholds that depend on the intensities of preferences. Therefore, the present chapter extends the work of Gershkov, Moldovanu, and Shi (2017) to a considerably larger class of preferences. This extension is important because it covers a much wider range of economically relevant preferences. For instance, the following collective decision-making problems are covered: Collective choice when preferences are single-peaked

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1. To be more precise, they assume that preferences are single-crossing and single-peaked on a line.

with respect to trees as introduced in Demange (1982),[2] and voting on hypercubes, that is, voting on multiple binary decisions as studied in Barberà, Sonnenschein, and Zhou (1991).[3]

The characterization of optimal mechanisms for preference domains that are generalized single-peaked with respect to a median space essentially involves the following three assumptions: First, voters have private types that are distributed independently and identically across the voters. Second, the utility function that is common to all voters satisfies an additive separability condition, constituting a natural constraint in settings, where alternatives might be multidimensional. This condition is vacuously met in the special case of single-peaked preferences on trees. Third, I impose a constraint on the preference intensities that represents a joint restriction on type distribution and the utility function. This condition is vacuously satisfied in the special case of hypercubes.

The utilitarian mechanism takes the form of voting by properties, that is, the social choice is determined through a collection of binary votes on subsets of alternatives involving flexible majority requirements that reflect the characteristics of these subsets of alternatives. The characterization of optimal mechanisms for preference domains that are generalized single-peaked with respect to a median space constitutes the main contribution of this chapter. To illustrate this finding, before introducing the general model, I discuss an application to the design of voting mechanisms for the provision of two costly public goods α and β subject to the constraint that the provided level of α is weakly higher than the provided level of β. For example, if α and β represent expansions of the rail and the road network respectively, this constraint might reflect the fight against climate change. Therefore, to get a more concrete idea how optimal mechanisms look like, I directly refer to section 4.3. Also, when developing the general result, I repeatedly revisit this application in order to illustrate the concepts and assumptions I employ in the general analysis in a less abstract setting.

The structure of this chapter is as follows: In the following section 4.2, I discuss the related literature, and, in section 4.3, I present the public goods application. Next, in section 4.4, I introduce the general model, and, in section 4.5, I review the characterization of strategy-proof social choice functions from NP (2007b). Then, in section 4.6, I present my general optimality finding. The following section 4.7 discusses the two special cases of trees and hypercubes. The final section 4.8 concludes. The proofs are contained in Appendix 4.A.

2. Kleiner and Moldovanu (2020) argue that in some real-world voting problems from the German as well as the British parliament preferences were single-peaked on a tree.

3. As part of section 4.2 on the related literature, I discuss more comprehensively which kind of collective decision-making problems are captured by my analysis.

4.2 Literature

The present chapter relates to work on social choice and mechanism design and it contributes, specifically, to the literature on the evaluation of the utilitarian efficiency of voting rules. This literature starts with Rae (1969), who focuses on *binary decisions*. More recent contributions that also consider binary decisions include Nehring (2004), Schmitz and Tröger (2012), and Drexl and Kleiner (2018).

NP (2007b) provide a characterization of strategy-proof social choice functions for all rich generalized single-peaked domains.[4] Nehring and Puppe (2005), and Nehring and Puppe (2007a) also treat strategy-proof social choice on generalized single-peaked domains, having each a different focus. Among many other preference structures, the unrestricted domain as well as domains that give rise to median spaces constitute generalized single-peaked domains. The latter preference domain admits a large class of well-behaved strategy-proof social choice functions, circumventing the *Gibbard-Satterthwaite-Theorem* (Gibbard (1973), Satterthwaite (1975)). Again, I take NP (2007b)'s characterization of strategy-proof social choice functions for these preference structures as the starting point for my analysis of utilitarian mechanisms. In this way, I show how results from strategy-proof social choice can be leveraged to solve mechanism design problems, representing a contribution of the present chapter.

The preference domains from the literature discussed below are among many other preference structures instances of generalized single-peaked domains derived from median spaces (see NP (2007b)).[5] My optimality analysis covers all these preference domains, and, therefore, I unify and generalize previous results in the mechanism design literature. The main contribution of this chapter constitutes the optimization over strategy-proof mechanisms on generalized single-peaked domains giving rise to median spaces while relying on the utilitarian principle. The chapter shares the research question with my master thesis Rachidi (2019), and it mainly extends Rachidi (2019) in the following way: The characterization of utilitarian mechanisms in the present chapter is more general than in the master thesis for all median spaces except for the special case of single-peaked preferences on trees. In particular, the optimality result in the master thesis does not cover the public goods application presented in this chapter.[6] As far as the formal results are concerned, Lemma 4.1, Lemma 4.3, and Corollary 4.1 are directly taken from the master thesis. Nevertheless, for completeness, I include the proofs of the two stated lemmata in Appendix 4.A. All other findings presented here are more general than in the master thesis.

One strand of the literature investigates *hypercubes* or coupled binary decisions,

4. NP (2007b) generalize previous work by Barberà, Massó, and Neme (1997).
5. I refer to NP (2007b) for an overview and a classification of median spaces.
6. In the master thesis, I discuss a conceptually similar public goods application, but the utility specification is different.

meaning, voters face a collection of binary decisions. In terms of strategy-proof social choice, Barberà, Sonnenschein, and Zhou (1991) provide a characterization of strategy-proof and surjective mechanisms when preferences are separable or additively separable across the binary issues. When it comes to mechanism design, Jackson and Sonnenschein (2007) offer a mechanism that is based on the idea of budgeting. For sufficiently many decisions, their mechanism is approximately Bayesian incentive-compatible as well as nearly ex-ante Pareto efficient. Other voting rules in the context of Bayesian mechanism design where voters report cardinal utility information include qualitative voting studied in Hortala-Vallve (2012) as well as storable votes due to Casella (2005). In contrast to Jackson and Sonnenschein (2007), Hortala-Vallve (2010) considers finitely many decision problems as well as strategy-proof mechanisms. Allowing for random mechanisms, he finds that ex-ante Pareto efficiency cannot be attained and, moreover, in the presence of strategy-proofness, there is no unanimous mechanism which is sensitive to preference intensities.

Another branch of the literature considers preferences which are *single-peaked on a line*. Moulin (1980) characterizes in his seminal contribution peaks-only and strategy-proof social choice functions for the full domain of preferences which are single-peaked on a line. His elegant characterization involves min-max rules or generalized median mechanisms when restricting attention to anonymous social choice functions. Again, Gershkov, Moldovanu, and Shi (2017) characterize the utilitarian mechanism when preferences are single-crossing and single-peaked on a line. In contrast to their work, I allow for a considerably larger class of economically relevant preferences, going beyond single-peaked preferences on a line. As far as the proof of my optimality result is concerned, I build upon their proof, but the much larger class of preferences requires additional arguments as well as different assumptions. Further, Gersbach (2017) also emphasizes the importance of flexible majority rules. Moreover, Kleiner and Moldovanu (2017) analyze dynamic, binary, and sequential voting procedures. They identify conditions on the voting procedures under which the induced dynamic games possess an ex-post perfect equilibrium in which voters behave sincerely. Moreover, they illustrate their theoretical findings by means of several empirical case studies involving collective decisions from different parliaments. Building on preferences which are single-peaked on a line, *products of lines*, the coupling of unidimensional decisions or, as Barberà, Gul, and Stacchetti (1993) put it, multidimensional single-peaked preferences have also received attention in the literature. Removing the peaks-only assumption in Moulin (1980), Border and Jordan (1983) as well as Barberà, Gul, and Stacchetti (1993) provide characterizations of strategy-proof social choice functions for the stated class of voting problems. Despite considering each somewhat different preferences, the main conclusion following from these contributions is that any strategy-proof social choice function is peaks-only and it can be decomposed into unidimensional functions such that each dimension is treated in a separate way. In other words, any strategy-proof social choice function is composed of a collection of the mechanisms that Moulin (1980) identified for the unidimensional case. Finally, regarding mechanism design, Gershkov, Moldovanu, and Shi (2019) consider a spatial voting environment, but they keep the voting procedure fixed in the sense that, essentially, the collective choice in each coordinate of the multidimensional setting is determined via simple majority voting. They show that the redefinition of the involved issues or, in other words, the rotation of the initial coordinate axes leads, generally, to improvements in terms of welfare.

While extending single-peaked preferences on a line in a somewhat different direction compared to products of lines, but maintaining the general idea of single-peakedness, Demange (1982) investigates preferences which are *single-peaked on trees*. She establishes that these domains ensure the existence of a Condorcet winner. However, when it comes to aggregation theory instead of voting, the majority relation need not be transitive. Moreover, Kleiner and Moldovanu (2020) study dynamic, binary, and sequential voting procedures in the context of single-peaked preferences on trees. They derive conditions on the voting procedures such that voting sincerely constitutes an ex-post perfect equilibrium and the Condorcet winner is implemented in this equilibrium. Also, again, they apply their theoretical findings to real-world voting problems from the German and the British parliament.

4.3 Public Goods Provision

The main purpose of this section is to illustrate the general optimality result presented in Theorem 4.2 below by means of an application to the design of voting mechanisms for the provision of two public goods subject to a constraint, but this application is also of interest in itself. Again, when developing the general optimality result subsequently, I repeatedly go back to this application in order to illustrate the concepts and assumptions I employ in the general analysis in a less abstract setting. There is a finite set of voters $N := \{1, ..., n\}$ with $n \geq 2$. Suppose that there are two public goods α and β, and that, for each public good, there are three possible levels $\{1, 2, 3\}$. Further, assume that there is an exogenously given constraint imposing that the provided level of α has to be weakly higher than the provided level of β.[7] Again, for instance, if α and β represent expansions of the rail and the road network respectively, this constraint might reflect the fight against climate change. Therefore, the set of alternatives A amounts to

$$A := \{(k_\alpha, k_\beta) \in \{1, 2, 3\} \times \{1, 2, 3\} : k_\alpha \geq k_\beta\}.$$

7. Similar applications appear in Barberà, Massó, and Neme (1997), Nehring and Puppe (2005), Nehring and Puppe (2007a), Block (2010), and Block de Priego (2014). However, these authors are not concerned about welfare maximization, but they focus on characterizing strategy-proof social choice functions.

The subsequent specification of types and utilities suitably extends the linear utility model contained in Gershkov, Moldovanu, and Shi (2017) from one to two public goods.

The voters' types are governed by the two-dimensional random variable $T := X \times Y$. The support of the type distribution S is given by the right triangle

$$S := \{(x,y) \in \mathbb{R}^2 : l \leq x \leq u, l \leq y \leq u, y \leq x\}$$

for some $0 \leq l < u < \infty$. In particular, note that the set S is convex. Denote by G and g the cdf and density of the bivariate distribution of T and let G_X and g_X as well as G_Y and g_Y be the marginal cdfs and densities corresponding to the random variables X and Y respectively. Types are distributed independently and identically across voters, and each voter is privately informed about his or her type realization. Now, a voter having type realization $(x,y) \in S$ receives utility

$$u^{(k_\alpha,k_\beta)}(x,y) := O^{k_\alpha} \cdot x - c^{k_\alpha} + O^{k_\beta} \cdot y - c^{k_\beta}$$

from alternative $(k_\alpha, k_\beta) \in A$. The involved parameters satisfy $c^1 < c^2 < c^3$ and $0 \leq O^1 < O^2 < O^3$, and they are common knowledge. In words, utilities are additively separable across the two public goods, the realizations of X and Y capture the valuation of public good α and β respectively, the valuation for α is always weakly higher than the value for β, the function O^k with $k \in \{1,2,3\}$ translates public good level indices into utilities, and the function c^k with $k \in \{1,2,3\}$ represents the cost function. Take any public good $\gamma \in \{\alpha, \beta\}$ and consider two public good levels $k, m \in \{1,2,3\}$ with $k > m$: The cutoff

$$z^{m,k} := \frac{c^k - c^m}{O^k - O^m}$$

describes the valuation corresponding to the public good γ at which a voter is indifferent between providing level k and m of the good γ for any fixed level of the other public good. Note that these cutoffs are homogenous across the two public goods because the functions O^k and c^k are assumed to be homogenous across the two goods. Suppose that the cutoffs involving neighboring public good levels are ordered, that is, suppose that

$$z^{0,1} := l < z^{1,2} < z^{2,3} < u =: z^{3,4}.$$

This is a mild assumption on the involved parameters: For example, it is satisfied if the function O^k is linear in k, the cost function c^k is strictly convex in k, and the support of the type distribution S is sufficiently large, meaning, l and u are sufficiently small and large respectively. It implies that any alternative is the most preferred or peak alternative for some types. In particular, the most preferred alternative of a voter constitutes $(p_\alpha, p_\beta) \in A$ if and only if the type realization $(x,y) \in S$ satisfies $x \in [z^{p_\alpha-1,p_\alpha}, z^{p_\alpha,p_\alpha+1}]$ and $y \in [z^{p_\beta-1,p_\beta}, z^{p_\beta,p_\beta+1}]$.

Finally, following Nehring and Puppe (2007a) and NP (2007b), observe that any type realization induces an ordinal preference relation that is generalized single-peaked with respect to a median space.[8] Specifically, the requirement of single-peakedness amounts here to the following condition: There exists an alternative $(p_\alpha, p_\beta) \in A$, which is the most preferred alternative, such that for all alternatives $(k_\alpha, k_\beta), (m_\alpha, m_\beta) \in A$ with $(k_\alpha, k_\beta) \neq (m_\alpha, m_\beta)$ it holds that whenever (k_α, k_β) lies on a shortest path in the graph shown in Figure 4.1 connecting (p_α, p_β) and (m_α, m_β), the voter must prefer (k_α, k_β) over (m_α, m_β). For instance, suppose that a

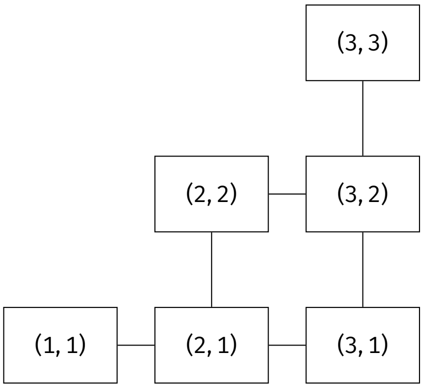

Figure 4.1. Public Goods Provision

voter's most preferred alternative is $(2,2)$. Then, single-peakedness requires, among other things, that this voter must prefer $(2,1)$ and $(3,2)$ over $(3,1)$, but it does not impose whether $(2,1)$ is preferred to $(3,2)$ or the other way around.

In the following, I present the direct mechanism that maximizes the utilitarian welfare among all strategy-proof, anonymous, and surjective mechanisms for the outlined setting.[9] In order to apply the general optimality result in Theorem 4.2 below, I impose three regularity assumptions on the type distribution. Specifically, assume that both marginal densities g_X and g_Y are log-concave and that $G_X \geq_{lr} G_Y$, where $\geq_{lr}$ denotes the likelihood ratio order. For instance, it can be verified that the three assumptions are met if the joint distribution G is the uniform distribution.[10]

The structure of the optimal mechanism can be described by means of four majority

8. Note that this would not be true if the support of the type distribution were given by the square $\{(x,y) \in \mathbb{R}^2 : l \leq x \leq u, l \leq y \leq u\}$.

9. There is an issue concerning the set of ordinal preferences generated by the utility representation introduced above: This set of ordinal preferences does not satisfy NP (2007b)'s richness condition on the preference domain. Therefore, the strategy-proof social choice functions they identify are strategy-proof for the outlined setting, but there might be other strategy-proof direct mechanisms in addition to those identified in their paper. However, NP (2007b)'s proof goes nevertheless through in the present setting, that is, there are no such other strategy-proof social choice functions. The argument for this claim is available on request from the author.

10. Note that if the joint density g is log-concave, the marginal densities g_X and g_Y must be log-concave as well (see Prékopa (1973)).

quotas $q_\alpha(1), q_\alpha(2), q_\beta(1)$, and $q_\beta(2)$, that is, four natural numbers weakly between 1 and n. Consider any public good $\gamma \in \{\alpha, \beta\}$ and public good level $k \in \{1,2\}$: If there are at least $q_\gamma(k)$ voters with most preferred alternatives sharing the feature that the public good level of γ is weakly smaller than k, the social choice features the same property, that is, the provided level of γ is at most k. Otherwise, the provided level of γ is strictly larger than k. In other words, for every public good $\gamma \in \{\alpha, \beta\}$ and each level $k \in \{1,2\}$, there is a binary vote between the following two subsets of alternatives: Alternatives sharing the feature that the public good level of γ is weakly smaller versus strictly larger than k. It follows from Nehring and Puppe (2007a) and NP (2007b) that, in the present setting, any strategy-proof, anonymous, and surjective social choice function takes this form subject to some constraints on the majority quotas. Now, under the regularity assumptions on the type distribution stated above, the described collection of binary votes on subsets of alternatives involving the majority quotas

$$q^*_\alpha(k) := \lceil \frac{n}{1 + \frac{\mathbb{E}[z^{k,k+1} - X | X \leq z^{k,k+1}]}{\mathbb{E}[X - z^{k,k+1} | X \geq z^{k,k+1}]}} \rceil$$

and

$$q^*_\beta(k) := \lceil \frac{n}{1 + \frac{\mathbb{E}[z^{k,k+1} - Y | Y \leq z^{k,k+1}]}{\mathbb{E}[Y - z^{k,k+1} | Y \geq z^{k,k+1}]}} \rceil,$$

where $k \in \{1,2\}$, implements the welfare-maximizing mechanism among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity. In particular, this collection of binary votes represents a proper social choice function in the sense that it yields a unique alternative for all profiles of type realizations. The two main features of the optimal majority quotas are as follows: First, for both public goods, the associated quotas are decreasing in the public good level that determines the respective partition of the set of alternatives into two subsets, i.e., $q^*_\alpha(1) \geq q^*_\alpha(2)$ and $q^*_\beta(1) \geq q^*_\beta(2)$. Second, the majority quotas corresponding to public good α are higher than the quotas linked to public good β, i.e., $q^*_\alpha(k) \geq q^*_\beta(k)$ for all $k \in \{1,2\}$.[11] The designer faces a Bayesian inference problem, that is, he or she has to make inferences about the voters' preference intensities based on their vote choices in the described collection of binary votes. The optimal majority quotas that are shaped by ratios of preference intensities show how this inference problem is resolved. For concreteness, consider for example the welfare-maximizing quota $q^*_\alpha(2)$: Rearranging the equation determining this quota while ignoring the integer problem yields

$$\frac{q^*_\alpha(2)}{n} \mathbb{E}[X | X \leq z^{2,3}] + \frac{n - q^*_\alpha(2)}{n} \mathbb{E}[X | X \geq z^{2,3}] = z^{2,3}.$$

11. Recall that the exogenously given constraint imposes that the provided level of α has to be weakly higher than the provided level of β.

Say that the designer is pivotal if there are exactly $q^*_\alpha(2)$ out of the n voters having most preferred alternatives that share the feature that the public good level of α is weakly smaller than 2, meaning, there are exactly $q^*_\alpha(2)$ voters with most preferred alternatives from the set $\{(1,1),(2,1),(2,2)\}$. Then, the quota $q^*_\alpha(2)$ is calibrated such that, conditional on being pivotal, the designer infers that the type component X governing the valuation for public good α equals the cutoff $z^{2,3}$, that is, the value at which a voter is indifferent between providing level 2 and 3 of α for any fixed level of β.[12] Furthermore, when rewriting the equation above once more, I obtain that, for all $k_\beta \in \{1,2\}$, it holds

$$\frac{q^*_\alpha(2)}{n}\mathbb{E}[u^{(2,k_\beta)}(X,Y)|X \leq z^{2,3}] + \frac{n-q^*_\alpha(2)}{n}\mathbb{E}[u^{(2,k_\beta)}(X,Y)|X \geq z^{2,3}]$$
$$= \frac{q^*_\alpha(2)}{n}\mathbb{E}[u^{(3,k_\beta)}(X,Y)|X \leq z^{2,3}] + \frac{n-q^*_\alpha(2)}{n}\mathbb{E}[u^{(3,k_\beta)}(X,Y)|X \geq z^{2,3}].$$

In words, this equation means that, conditional on being pivotal, the designer is indifferent between any two alternatives such that the provided level of α is 2 versus 3, that is, it differs by exactly one, but the provided level of β is the same in both alternatives. In other words, the designer is indifferent between alternatives $(2,1)$ and $(3,1)$ as well as between $(2,2)$ and $(3,2)$. This characteristic of optimal quotas is not special to this public goods application, but it turns out that a generalization of it holds for all median spaces.

Having presented the public goods application, in the following section, I introduce the general model.

4.4 Model

There is a finite set of voters $N := \{1,\ldots,n\}$ with $n \geq 2$ and a finite set of alternatives A with $|A| \geq 2$. Following NP (2007b), the set of alternatives is endowed with a property space structure. Elements of A are distinguished by *properties* which are described by $\mathscr{H} \subseteq \mathscr{P}(A)$, where $\mathscr{H} \neq \emptyset$, and $\mathscr{P}(A)$ denotes the power set of A. Each $H \in \mathscr{H}$ captures the property shared by all elements in $H \subseteq A$, but violated by all alternatives in $H^c := A \setminus H$. In other words, properties are subsets of the set of alternatives A. The set of properties $\mathscr{H}$ satisfies the regularity conditions

$$H \in \mathscr{H} \Rightarrow H \neq \emptyset \text{ (non-triviality)},$$
$$H \in \mathscr{H} \Rightarrow H^c \in \mathscr{H} \text{ (closedness under negation), and}$$
$$\forall k,m \in A, k \neq m : \exists H \in \mathscr{H} : k \in H \wedge m \notin H \text{ (separation).}$$

Given some alternative $k \in A$, let $\mathscr{H}_k$ be the set of all properties shared by alternative k, meaning, define $\mathscr{H}_k := \{H \in \mathscr{H} : k \in H\}$. Due to separation, it holds that

12. This feature holds because utilities are affine in types.

$\cap_{H \in \mathscr{H}_k} H = \{k\}$. Further, each pair (H, H^c) involving some property and its complement forms an *issue*, and the tuple $(A, \mathscr{H})$ is called *property space*. The property space $(A, \mathscr{H})$ induces a ternary relation on A, denoted by $B_{\mathscr{H}}$, in the following way: For all $(a, b, c) \in A \times A \times A$,

$$(a, b, c) \in B_{\mathscr{H}} :\Longleftrightarrow [\forall H \in \mathscr{H} : \{a, c\} \subseteq H \Rightarrow b \in H].$$

The relation $B_{\mathscr{H}}$ is called *betweenness relation*. This means that some alternative b is between the alternatives a and c if and only if all properties that are jointly shared by a and c are also shared by b.

Moreover, I suppose that any property space constitutes a *median space* as introduced in NP (2007b).[13] This requires that the betweenness relation $B_{\mathscr{H}}$ satisfies the following constraint: For any $a, b, c \in A$, there exists some alternative $m = m(a, b, c) \in A$, called the median, such that

$$\{(a, m, b), (a, m, c), (b, m, c)\} \subseteq B_{\mathscr{H}}.$$

Take any set that is composed of three alternatives. The restriction of being a median space demands that there must be some alternative having the feature that it is between all pairs of alternatives that can be formed from the given set of three alternatives.

Based on these concepts, I introduce preferences. Following NP (2007b), an ordinal preference relation $\succ$ is said to be *generalized single-peaked* with respect to the underlying betweenness relation $B_{\mathscr{H}}$ if it satisfies the following condition: There exists some alternative $p \in A$ such that, for all $k, m \in A$ with $k \neq m$, it holds

$$(p, k, m) \in B_{\mathscr{H}} \Rightarrow k \succ m.$$

Intuitively, a generalized single-peaked preference relation is characterized by two main ingredients. On the one hand, the alternative p describes the peak of that preference relation. On the other hand, the constraint formalizing the generalized notion of single-peakedness requires that any alternative k distinct from m which is between the peak p and alternative m according to the betweenness relation $B_{\mathscr{H}}$ must be preferred to m. Let $\mathscr{P}_{\mathscr{H}}$ denote the set of all preference relations that are generalized single-peaked with respect to $B_{\mathscr{H}}$.

Public Goods Provision. Go back to the public goods application. Denote by $A_{Public\,Goods}$ the set of alternatives for this application.[14] While following Nehring and Puppe (2007a), consider, for all $k \in \{1, 2\}$, the properties

$$H^{\alpha}_{\leq k} := \{(m_\alpha, m_\beta) \in A_{Public\,Goods} : m_\alpha \leq k\}$$
$$H^{\alpha}_{\geq k+1} := \{(m_\alpha, m_\beta) \in A_{Public\,Goods} : m_\alpha \geq k+1\}$$

13. Again, this assumption ensures that there is a rich class of non-degenerate incentive-compatible social choice functions.

14. Recall that the set of alternatives is given by $\{(k_\alpha, k_\beta) \in \{1, 2, 3\} \times \{1, 2, 3\} : k_\alpha \geq k_\beta\}$.

as well as

$$H^{\beta}_{\leq k} := \{(m_\alpha, m_\beta) \in A_{Public\ Goods} : m_\beta \leq k\}$$
$$H^{\beta}_{\geq k+1} := \{(m_\alpha, m_\beta) \in A_{Public\ Goods} : m_\beta \geq k+1\}.$$

Let $\mathscr{H}_{Public\ Goods}$ be the collection of these properties. The betweenness relation $B_{\mathscr{H}_{Public\ Goods}}$ induced by the property space $(A_{Public\ Goods}, \mathscr{H}_{Public\ Goods})$ satisfies the following condition: Alternative b is between alternatives a and c, meaning, $(a, b, c) \in B_{\mathscr{H}_{Public\ Goods}}$ if and only if b lies on a shortest path connecting a and c in the graph shown in Figure 4.1. Therefore, for this public goods application, the general definition of a generalized single-peaked preference relation introduced here exactly reduces to the definition based on the graphic notion of betweenness given in section 4.3. Moreover, it can be inferred from Figure 4.1 that the property space $(A_{Public\ Goods}, \mathscr{H}_{Public\ Goods})$ constitutes a median space.

Since I rely on the utilitarian principle as far as the objective criterion of the designer is concerned, I have to introduce a utility representation of ordinal preferences. Voters have types that are governed by the random variable T. Each voter is privately informed about his or her type realization. The distribution of T has full support on some non-empty set $S \neq \emptyset$. All subsequent expectations are taken with respect to this distribution. I suppose that types are distributed independently and identically across voters.

Assumption 4.1. The types T are distributed independently and identically across voters.

Now, $u^k(t)$ denotes the utility that a voter with type realization $t \in S$ receives if alternative $k \in A$ is implemented. I impose several constraints on the utility function and the type distribution. First, utilities are bounded, meaning, there exists some bound $B \in \mathbb{R}$ such that, for all type realizations $t \in S$ and for every alternative $k \in A$, $|u^k(t)| < B$. Second, I exclude indifferences, that is, for almost all type realizations $t \in S$ and for every pair of distinct alternatives $k, m \in A$ with $k \neq m$, it holds $u^k(t) \neq u^m(t)$. Third, of course, utilities must be consistent with generalized single-peakedness, that is, for almost all type realizations $t \in S$, there exists a generalized single-peaked preference relation $\succ \in \mathscr{P}_{\mathscr{H}}$ such that, for every pair of distinct alternatives $k, m \in A$ with $k \neq m$, it holds $k \succ m \Leftrightarrow u^k(t) > u^m(t)$. Fourth, I assume that the richness condition on the preference domain from NP (2007b) is satisfied. This means that the following two restrictions are met: First, for all $k, m \in A$ such that $\{k, m\} = \{l \in A : (k, l, m) \in B_{\mathscr{H}}\}$, there exists a set of type realizations $Z \subset S$ arising with positive probability such that, for every element in this set $t \in Z$, it holds $u^k(t) > u^m(t) > u^l(t)$ for all $l \in A \setminus \{k, m\}$. Second, for all $p, k, m \in A$ such that $(p, k, m) \notin B_{\mathscr{H}}$, there exists a set of type realizations $Z \subset S$ arising with positive probability such that, for every element in this set $t \in Z$, it holds $u^m(t) > u^k(t)$

and $u^p(t) > u^l(t)$ for all $l \in A \setminus \{p\}$. The important point here is that there are no strategy-proof, anonymous, and surjective social choice functions apart from those identified in NP (2007b)'s characterization. In this sense, the utility representation of the public goods application is covered as well.[15]
Finally, the designer maximizes the voters' ex-ante utilitarian welfare over all social choice functions that are strategy-proof, anonymous, and surjective. The timing is as follows:

(1) The designer announces and commits to some strategy-proof, anonymous, and surjective direct mechanism.
(2) The voters observe their type realizations and report them to the designer.
(3) The designer implements an alternative according to the announced mechanism.

4.5 Incentive Compatibility

In this section, for completeness, I review the characterization of strategy-proof, anonymous, and surjective social choice functions for generalized single-peaked domains giving rise to median spaces due to NP (2007b).
First of all, I assume that the set of feasible mechanisms coincides with the set all possibly indirect deterministic mechanisms $\Gamma = (M, ..., M, f)$ inducing a game that admits a dominant-strategy equilibrium, where M is the voters' message set and $f : M^n \to A$ is the outcome function. Also, I suppose that the mechanisms Γ are anonymous[16] and surjective.[17] Now, invoking a revelation principle in terms of payoffs due to Jarman and Meisner (2017) implies the following: For each such anonymous and surjective mechanism Γ, there exists a direct mechanism $\Gamma' = (S, ..., S, f')$ that is dominant-strategy incentive-compatible, anonymous, and surjective, and the utilitarian welfare under Γ' is weakly higher than under Γ. In this sense, within the class of deterministic mechanisms, it is without loss to restrict attention to direct mechanisms.
From now on, I restrict attention to deterministic direct mechanisms that are anonymous, surjective, and dominant-strategy incentive-compatible or, in other words, strategy-proof. A direct mechanism or, equivalently, a *social choice function* f is a mapping assigning to each type profile an alternative from the set A. In formal terms, this mapping amounts to $f : S^n \to A$. In the following, I recall some well-known properties of social choice functions.

15. Recall the discussion in footnote 9.
16. A mechanism Γ is anonymous if, for all $(m_1, ..., m_n) \in M^n$, $f(m_1, ..., m_n) = f(m_{\sigma(1)}, ..., m_{\sigma(n)})$, where σ is an arbitrary permutation of the set of voters N.
17. A mechanism Γ is surjective if, for all $k \in A$, there exists $(m_1, ..., m_n) \in M^n$ such that $f(m_1, ..., m_n) = k$.

Definition 4.1. A social choice function f is strategy-proof if, for all $i \in N$ and for all $t_i, t_i' \in S$ and $t_{-i} \in S^{n-1}$, it holds that

$$u^{f(t_i,t_{-i})}(t_i) \geq u^{f(t_i',t_{-i})}(t_i).$$

In words, *strategy-proofness* requires that all voters have a weakly dominant strategy to truthfully reveal their types. Further, observe that strategy-proofness implies the following: Consider any voter $i \in N$ and take two type realizations $t_i, t_i' \in S$ inducing the same ordinal preference relation. Then, a strategy-proof direct mechanism f must treat both types in the same way, that is, for any type realizations of the other voters $t_{-i} \in S^{n-1}$, it must hold that $f(t_i, t_{-i}) = f(t_i', t_{-i})$.

Definition 4.2. A social choice function f is anonymous if, for all $(t_1, ..., t_n) \in S^n$, it holds that $f(t_1, ..., t_n) = f(t_{\sigma(1)}, ..., t_{\sigma(n)})$, where σ is an arbitrary permutation of the set of voters N.

Intuitively, *anonymity* imposes that mechanisms treat all voters equally. To put it differently, anonymity ensures that mechanisms respect the democratic principle of "one person, one vote".

Definition 4.3. A social choice function f is surjective if, for all $k \in A$, there exists $(t_1, ..., t_n) \in S^n$ such that $f(t_1, ..., t_n) = k$.

The requirement that social choice functions are *surjective* represents a mild condition ensuring that no alternative is a priori excluded from the set of outcomes.
For the preference domains considered in this chapter, NP (2007b) show that strategy-proof and surjective social choice functions must be *peaks-only*, meaning, the outcome of any strategy-proof and surjective social choice function depends only on the voters' most preferred alternatives. Therefore, in the following, with abuse of notation, social choice functions are simply mappings $f : A^n \to A$ assigning to every profile of most preferred or peak alternatives $(p_1, ..., p_n) \in A^n$ some winning alternative from the set A.
In order to be able to state NP (2007b)'s characterization result, I need the following supplementary definitions from their paper. To begin with, introduce the notion of a *family of quotas* relative to some property space $(A, \mathscr{H})$.

Definition 4.4. NP (2007b)
Given some property space $(A, \mathscr{H})$, a family of quotas $\{q_H : H \in \mathscr{H}\}$ is a function that assigns an integer-valued quota $1 \leq q_H \leq n$ to each property $H \in \mathscr{H}$ such that, for all $H \in \mathscr{H}$, the associated quotas satisfy $q_H + q_{H^c} = n + 1$.

Take any property $H \in \mathscr{H}$. The associated absolute quota, threshold or majority requirement q_H describes the minimal number of votes that are needed in order to ensure that some alternative sharing property H is winning. Furthermore, the condition $q_H + q_{H^c} = n+1$ reflects that whenever the quota q_H linked to property H is reached, the quota associated with the complementary property H^c cannot be attained, and vice versa. Thus, exactly one of these two quotas is always achieved. On the basis of the definition of families of quotas, consider the following class of functions which is termed anonymous *voting by properties*. These functions are central for the ensuing characterization result.

Definition 4.5. NP (2007b)
Given some property space $(A, \mathscr{H})$ and associated family of quotas $\{q_H : H \in \mathscr{H}\}$, voting by properties is the function $f_{\{q_H : H \in \mathscr{H}\}} : A^n \to \mathscr{P}(A)$ such that, for all profiles of peak alternatives $p = (p_1, ..., p_n) \in A^n$, it holds that

$$k \in f_{\{q_H : H \in \mathscr{H}\}}(p) \;:\Leftrightarrow\; [\forall H \in \mathscr{H}_k \;:\; |\{i \in N \;:\; p_i \in H\}| \geq q_H].$$

Intuitively, under voting by properties, the social choice is determined through a collection of binary votes on subsets of alternatives involving qualified majority requirements. In more detail, it works as follows: Take some family of quotas $\{q_H : H \in \mathscr{H}\}$. For any issue (H, H^c), it is collectively decided according to the quotas q_H and q_{H^c} whether the winning alternative is supposed to share property H or its complement H^c. These binary decisions yield a collection of properties that the winning alternative is supposed to share. However, it has to be ensured that this set of, loosely speaking, winning properties is consistent in the sense that the intersection of these properties is not empty, but it contains exactly one alternative which, then, constitutes the winning alternative. Thus, in general, the considered mapping needs not represent a proper social choice function. However, as the following result reveals, under some conditions on the family of quotas, the stated mapping forms a social choice function.
I state NP (2007b)'s characterization of strategy-proof, anonymous, and surjective social choice functions for generalized single-peaked domains derived from median spaces.

Theorem 4.1. *NP (2007b)*
A social choice function f is strategy-proof, anonymous, and surjective if and only if it is voting by properties $f_{\{q_H : H \in \mathscr{H}\}} : A^n \to A$ with a family of quotas $\{q_H : H \in \mathscr{H}\}$ such that, for all properties $H, K \in \mathscr{H}$, it holds

$$H \subseteq K \Rightarrow q_H \geq q_K.$$

Theorem 4.1 implies that, when searching for the optimal mechanism among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity, it is sufficient to optimize over the set of quotas $\{q_H : H \in \mathscr{H}\}$ related to voting by properties while respecting the collection of inequalities stated in Theorem 4.1. I tackle this problem in the subsequent section.

Public Goods Provision. Before that, go again back to the public goods application. Recall that I described in section 4.3 strategy-proof, anonymous, and surjective social choice functions in terms of a collection of binary votes that are determined by the four majority quotas $q_\alpha(1)$, $q_\alpha(2)$, $q_\beta(1)$, and $q_\beta(2)$ subject to some constraints on these majority quotas that I did not specify there explicitly. Now, any such collection of binary votes coincides with a voting by properties mechanism, and the stated constraints are the restrictions from Theorem 4.1. To see this, for all $k \in \{1, 2\}$, set

$$q_{H^\alpha_{\leq k}} := q_\alpha(k), \text{ and } q_{H^\alpha_{\geq k+1}} := n + 1 - q_\alpha(k)$$

as well as

$$q_{H^\beta_{\leq k}} := q_\beta(k), \text{ and } q_{H^\beta_{\geq k+1}} := n + 1 - q_\beta(k).$$

4.6 Welfare Maximization

In this section, I characterize the mechanism that maximizes the voters' ex-ante utilitarian welfare among all social choice functions satisfying strategy-proofness, anonymity, and surjectivity, constituting the main result of this chapter.

By Theorem 4.1, it is sufficient to find the optimal quotas related to voting by properties. Also, the existence of a solution is ensured since a bounded function is maximized over a finite set of elements. The structure of the proof of the main theorem below is as follows: First, consider some optimal mechanism and derive necessary conditions for optimality by means of studying the implications of alterations of this optimal mechanism. Second, argue that these necessary conditions are also sufficient for optimality and conclude that they determine a unique optimal mechanism.

When deriving the discussed necessary conditions for optimality, it turns out that I have to compare the welfare induced by the following two sets of alternatives: For every property $H \in \mathscr{H}$, define the sets of alternatives

$$A_H := H \cap [\cap_{\{M \in \mathscr{H} : M \subset H\}} M^c],$$

and

$$A_{H^c} := H^c \cap [\cap_{\{M \in \mathscr{H} : M \subset H^c\}} M^c].$$

Alternatives contained in the set A_H share property H, but these alternatives violate all properties that are subsets of H. Likewise, alternatives from the set A_{H^c} satisfy property H^c, but properties that are subsets of H^c are violated. In Lemma 4.1, I establish that the sets A_H and A_{H^c} have a particular tuple structure.

Lemma 4.1. Rachidi (2019)
Consider any property $H \in \mathscr{H}$. The sets A_H and A_{H^c} satisfy $A_H \neq \emptyset$ and $A_{H^c} \neq \emptyset$. Moreover, all elements in both sets can be uniquely matched into tuples having the form (k, m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H, meaning, $\{H\} = \{K \in \mathscr{H} : k \notin K \wedge m \in K\}$.

The proof of Lemma 4.1 employs a characterization of median spaces in terms of the involved properties instead of relying on the induced betweenness relation due to NP (2007b). Let Z_H denote the set of tuples implied by Lemma 4.1. It is clear that $|A_H| = |A_{H^c}|$, but, in general, it does not hold that $|A_H| = |A_{H^c}| = 1$.

Public Goods Provision. Revisit again the public goods application. For instance, consider the property $H^{\beta}_{\leq 1}$. In this case, I have that $\{M \in \mathscr{H}_{Public\ Goods} : M \subset H^{\beta}_{\leq 1}\} = \{H^{\alpha}_{\leq 1}\}$. Hence, the set $A_{H^{\beta}_{\leq 1}}$ amounts to

$$A_{H^{\beta}_{\leq 1}} = \{(2, 1), (3, 1)\}.$$

Similarly, the set $A_{H^{\beta}_{\geq 2}}$ satisfies

$$A_{H^{\beta}_{\geq 2}} = \{(2, 2), (3, 2)\}.$$

In particular, it holds that $|A_{H^{\beta}_{\leq 1}}| = |A_{H^{\beta}_{\geq 2}}| \neq 1$. Moreover, the set of tuples $Z_{H^{\beta}_{\leq 1}}$ is given by

$$Z_{H^{\beta}_{\leq 1}} = \{((2, 2), (2, 1)), ((3, 2), (3, 1))\}.$$

This is precisely the tuple structure established in Lemma 4.1: Alternatives $(2, 2)$ and $(2, 1)$ as well as $(3, 2)$ and $(3, 1)$ are each separated only by property $H^{\beta}_{\leq 1}$. Further, this tuple structure is unique for the following reason: When matching $(2, 2)$ and $(3, 1)$ as well as $(3, 2)$ and $(2, 1)$, the matched alternatives are each separated by more properties than just $H^{\beta}_{\leq 1}$, meaning, $\{H^{\beta}_{\leq 1}\} \subset \{K \in \mathscr{H} : (2, 2) \notin K \wedge (3, 1) \in K\}$ and $\{H^{\beta}_{\leq 1}\} \subset \{K \in \mathscr{H} : (3, 2) \notin K \wedge (2, 1) \in K\}$, violating the condition that the matched alternatives are each separated only by property $H^{\beta}_{\leq 1}$. More generally, for the public goods application, it can be verified that two alternatives form a tuple (k, m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H if and only if, for one public good, the provided level is the same in k and m, and, for the other good, the levels differ by exactly one.

The tuple structure established in Lemma 4.1 implies that the comparison of the welfare generated by the sets A_H and A_{H^c} reduces to contrasting a collection of pairs of alternatives such that the elements within each pair are separated by one property only. Furthermore, in order to characterize the optimal quotas in a separable way, I have to make sure that the welfare gains and losses involved in the welfare comparison within the discussed tuples do not depend on the tuple under consideration, but that they are the same across all tuples. The purpose of Assumption 4.2 on the utility function is to ensure exactly that.

Assumption 4.2.
Consider any property $H \in \mathscr{H}$. For any two tuples of alternatives (k,m) and (k',m') with $k,k' \in A_{H^c}$ and $m,m' \in A_H$ such that k and m as well as k' and m' are each separated only by property H, that is, $\{H\} = \{K \in \mathscr{H} : k \notin K \wedge m \in K\} = \{K \in \mathscr{H} : k' \notin K \wedge m' \in K\}$, and, for all type realizations $t \in S$, the utility function satisfies

$$u^k(t) - u^m(t) \;=\; u^{k'}(t) - u^{m'}(t).$$

Assumption 4.2 represents essentially an additive separability restriction on the utility function, making it a natural assumption in contexts, where alternatives might be multidimensional.[18] Moreover, this assumption is vacuously met in the special case of trees that I discuss in section 4.7.

Public Goods Provision. Go back to the public goods application. Clearly, Assumption 4.2 is satisfied in the public goods application because the utility function is additively separable across the two public goods. For example, consider again the set of tuples $Z_{H^{\beta}_{\leq 1}} = \{((2,2),(2,1)),((3,2),(3,1))\}$. In this case, for all type realizations $(x,y) \in S$, it holds that

$$\begin{aligned} u^{(2,2)}(x,y) - u^{(2,1)}(x,y) \;&=\; u^{(3,2)}(x,y) - u^{(3,1)}(x,y) \\ =& O^2 \cdot y - c^2 - O^1 \cdot y + c^1. \end{aligned}$$

Finally, Assumption 4.3 takes care of the fact that the alterations of some optimal mechanism that build the basis for the derivation of necessary conditions for optimality need not be feasible due to the constraints on the family of quotas appearing in Theorem 4.1. Essentially, it implies that the discussed necessary conditions remain valid even if the considered alterations are not feasible.

Assumption 4.3.
Consider two arbitrary properties $H,K \in \mathscr{H}$ satisfying $H \subseteq K$. For any two tuples of alternatives (k,m) and (j,l) with $k \in A_{H^c}$ and $m \in A_H$ as well as $j \in A_{K^c}$ and $l \in A_K$ such that k and m are separated only by property H as well as j and l are separated only by property K, that is, $\{H\} = \{L \in \mathscr{H} : k \notin L \wedge m \in L\}$ and $\{K\} = \{L \in \mathscr{H} : j \notin L \wedge l \in L\}$, the following inequality holds:

$$\begin{aligned} \delta_H \;:=& \frac{\mathbb{E}[u^k - u^m | u^k \;>\; u^m]}{\mathbb{E}[u^k - u^m | u^k \;>\; u^m] + \mathbb{E}[u^m - u^k | u^m \;>\; u^k]} \\ \geq& \frac{\mathbb{E}[u^j - u^l | u^j \;>\; u^l]}{\mathbb{E}[u^j - u^l | u^j \;>\; u^l] + \mathbb{E}[u^l - u^j | u^l \;>\; u^j]} \;=:\; \delta_K. \end{aligned}$$

18. NP (2007b) present a similar condition for a different purpose: They argue that such a restriction is economically natural and show that their characterization of strategy-proof social choice functions applies when restricting the preferences in such a way.

Note that in the presence of Assumption 4.2, for any two tuples of alternatives (k,m) and (k',m') with $k,k' \in A_{H^c}$ and $m,m' \in A_H$ such that k and m as well as k' and m' are each separated only by property H, it holds that

$$\begin{aligned}\delta_H &= \frac{\mathbb{E}[u^k - u^m | u^k > u^m]}{\mathbb{E}[u^k - u^m | u^k > u^m] + \mathbb{E}[u^m - u^k | u^m > u^k]} \\ &= \frac{\mathbb{E}[u^{k'} - u^{m'} | u^{k'} > u^{m'}]}{\mathbb{E}[u^{k'} - u^{m'} | u^{k'} > u^{m'}] + \mathbb{E}[u^{m'} - u^{k'} | u^{m'} > u^{k'}]}.\end{aligned}$$

Therefore, the notation δ_H is justified because δ_H does not depend on the considered tuple of alternatives that are separated only by property H.
Moreover, observe that the following requirement constitutes a sufficient condition for Assumption 4.3: For all $H,K \in \mathscr{H}$ such that $H \subseteq K$ or, equivalently, $K^c \subseteq H^c$, it holds that

$$\begin{aligned}\mathbb{E}[u^m - u^k | u^m > u^k] &\leq \mathbb{E}[u^l - u^j | u^l > u^j], \text{ and} \\ \mathbb{E}[u^k - u^m | u^k > u^m] &\geq \mathbb{E}[u^j - u^l | u^j > u^l],\end{aligned}$$

where $k \in A_{H^c}$ and $m \in A_H$ as well as $j \in A_{K^c}$ and $l \in A_K$ such that k and m are separated only by property H as well as j and l are separated only by property K. Further, this assumption is vacuously satisfied in the special case of hypercubes that I discuss in section 4.7.

Public Goods Provision. Consider again the public goods application. In this case, for all $k \in \{1,2\}$, it can be verified that the considered ratios simplify to the following expressions:

$$\delta_{H^\alpha_{\leq k}} = \frac{1}{1 + \frac{\mathbb{E}[z^{k,k+1} - X | X \leq z^{k,k+1}]}{\mathbb{E}[X - z^{k,k+1} | X \geq z^{k,k+1}]}}, \text{ and } \delta_{H^\alpha_{\geq k+1}} = 1 - \delta_{H^\alpha_{\leq k}}$$

as well as

$$\delta_{H^\beta_{\leq k}} = \frac{1}{1 + \frac{\mathbb{E}[z^{k,k+1} - Y | Y \leq z^{k,k+1}]}{\mathbb{E}[Y - z^{k,k+1} | Y \geq z^{k,k+1}]}}, \text{ and } \delta_{H^\beta_{\geq k+1}} = 1 - \delta_{H^\beta_{\leq k}}.$$

Now, the interrelations between properties in the sense that one property is a subset of another property that are relevant for Assumption 4.3 are as follows: First, for any public good $\gamma \in \{\alpha, \beta\}$, $H^\gamma_{\leq 1} \subset H^\gamma_{\leq 2}$. Hence, Assumption 4.3 requires that $\delta_{H^\gamma_{\leq 1}} \geq \delta_{H^\gamma_{\leq 2}}$. However, this feature is implied by the regularity condition imposed in section 4.3 that the marginal densities g_X and g_Y are log-concave.[19] Second, for any public good level $k \in \{1,2\}$, $H^\alpha_{\leq k} \subset H^\beta_{\leq k}$. Thus, Assumption 4.3 demands that

19. To see this, recall that a random variable satisfies the decreasing mean residual life property as well as the increasing mean inactivity time property if its density is log-concave (see Bagnoli and Bergstrom (2005)).

$\delta_{H^{\alpha}_{\leq k}} \geq \delta_{H^{\beta}_{\leq k}}$. In section 4.3, I assumed that $G_X \geq_{lr} G_Y$.[20] This condition is sufficient for $\delta_{H^{\alpha}_{\leq k}} \geq \delta_{H^{\beta}_{\leq k}}$.[21] The conditions associated with all other interrelations of properties are automatically satisfied if the constraints related to the discussed relevant interrelations are met. Overall, the regularity conditions on the type distribution from section 4.3 ensure that Assumption 4.3 is satisfied in the public goods application. This suggests that, at least in the public goods application, Assumption 4.3 constitutes a rather mild constraint.

Having presented the required assumptions as well as some preliminary steps for the analysis, I state the main result of this chapter, that is, I provide a characterization of the welfare-maximizing mechanism among all strategy-proof, anonymous, and surjective social choice functions.

Theorem 4.2.
Suppose that Assumptions 4.1, 4.2 and 4.3 hold.
The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q^*_H = \lceil n\delta_H \rceil \text{ for all } H \in \mathscr{H}.$$

While taking into account that mechanisms have to be dominant-strategy incentive-compatible, Theorem 4.2 characterizes the optimal utilitarian mechanism for generalized single-peaked domains derived from median spaces. In particular, Theorem 4.2 provides closed-form expressions for the welfare-maximizing quotas related to voting by properties.
The intuition behind the optimal quotas $q^*_H = \lceil n\delta_H \rceil$ is as follows: Take any tuple of alternatives (k, m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H. Now, first of all, observe that the quota q^*_H is shaped by the ratio of preference intensities

$$\frac{\mathbb{E}[u^m - u^k | u^m > u^k]}{\mathbb{E}[u^k - u^m | u^k > u^m]},$$

reflecting the utilitarian objective of the designer. Also, regarding comparative statics, the quota q^*_H decreases in the discussed ratio of preference intensities. For the purpose of a more detailed understanding, ignore that the quotas must be integer-valued. Plugging in the term for δ_H and rearranging yields

$$\begin{aligned}
&\frac{q^*_H}{n}\mathbb{E}[u^k | u^m > u^k] + \frac{n - q^*_H}{n}\mathbb{E}[u^k | u^k > u^m] \\
=&\frac{q^*_H}{n}\mathbb{E}[u^m | u^m > u^k] + \frac{n - q^*_H}{n}\mathbb{E}[u^m | u^k > u^m].
\end{aligned}$$

20. Again, the order $\geq_{lr}$ denotes the likelihood ratio order.

21. The reason is that, if $G_X \geq_{lr} G_Y$, the same ordering holds in terms of the hazard as well as reversed hazard rate ordering, implying the claim (see Shaked and Shanthikumar (2007)).

This expression shows how the designer's Bayesian inference problem is resolved: The optimal quota q_H^* is calibrated such that the designer is indifferent between implementing alternatives k and m conditional on being pivotal, that is, conditional on the event that exactly q_H^* out of the n voters prefer alternative m over alternative k. The latter event coincides with the event that there are exactly q_H^* voters whose most preferred alternatives share property H. This point follows from generalized single-peakedness (see NP (2007b)). Consequently, the optimal quota q_H^* is set such that the designer is indifferent between any pair of alternatives separated only by property H conditional on being there exactly q_H^* voters with peaks from the set H.

Public Goods Provision. Revisit the public goods application, and recall the following: First, again, in the public goods application, two alternatives form a tuple (k,m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H if and only if, for one public good, the provided level is the same in k and m, and, for the other good, the levels in k and m differ by exactly one. Second, as discussed in section 4.3, the optimal quotas in the public goods application are calibrated in the following way: Conditional on being pivotal, the designer is indifferent between any two alternatives such that the provided level of one good is the same in both alternatives, but the levels of the other good differ by exactly one. This discussion shows how the indifference property of the optimal quotas in the public goods application described in section 4.3 generalizes to all median spaces.

Subsequently, I outline the proof of Theorem 4.2. Conceptually, the proof strategy follows Gershkov, Moldovanu, and Shi (2017), but, again, the much larger class of preferences requires supplementary arguments and distinct assumptions. In particular, Lemma 4.1 as well as Assumption 4.2 are completely absent in their paper. The reason is as follows: If preferences are single-peaked on a line, essentially, the welfare induced by two single alternatives is compared. This feature continues to hold for preferences that are single-peaked on trees, but it fails for all other median spaces, where the welfare generated by two sets of alternatives needs to be contrasted. In section 4.7, in the context of trees, I discuss how Theorem 4.2 extends the main result in Gershkov, Moldovanu, and Shi (2017).

To begin with, by Theorem 4.1, it is sufficient to optimize over the set of quotas related to voting by properties.[22] Furthermore, again, due to Theorem 4.1, for all $H, K \in \mathscr{H}$, the optimal quotas must satisfy

$$H \subseteq K \Rightarrow q_H \geq q_K.$$

Consider some property $H \in \mathscr{H}$ as well as the associated quota q_H^* which is supposed to be part of an optimal mechanism. To simplify the exposition, I divide the proof of Theorem 4.2 into two lemmata.

22. Again, since a bounded function is maximized over a finite set of elements, the existence of a solution is ensured.

Lemma 4.2.
Suppose that Assumptions 4.1 and 4.2 hold. Consider any property $H \in \mathscr{H}$.
(i) If $H' \subset H \Rightarrow q^*_{H'} > q^*_H$ for all $H' \in \mathscr{H}$ such that $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$, the inequality

$$q^*_H \geq n \cdot \delta_H$$

constitutes a necessary condition for optimality.
(ii) If $H \subset H' \Rightarrow q^*_H > q^*_{H'}$ for all $H' \in \mathscr{H}$ such that $\nexists H'' \in \mathscr{H} : H \subset H'' \subset H'$, any optimal mechanism meets the inequality

$$q^*_H \leq n \cdot \delta_H + 1.$$

Suppose that increasing q^*_H by 1 is feasible, meaning, this alteration does not violate the inequalities from Theorem 4.1. This change matters only if there are q^*_H voters having some peak from the set H and $n - q^*_H$ voters with peaks from the set H^c. In this case, since $q^*_L \leq q^*_H$ for all $H \subset L$, the properties $\{L : H \subset L\}$ or, equivalently, $\{L^c : L \subset H^c\}$ are accepted whenever there are such properties. Additionally, since increasing q^*_H by 1 is feasible, I must have that $q^*_M > q^*_H$ for all $M \subset H$. Thus, the properties $\{M : M \subset H\}$ are rejected or, equivalently, the properties $\{M^c : M \subset H\}$ are winning whenever there are such properties. Putting these features together and using the introduced notation, if the quota is q^*_H, some element of the set $A_H \neq \emptyset$ is the winning alternative. However, if the quota amounts to $q^*_H + 1$, some element of the set $A_{H^c} \neq \emptyset$ is selected. Since q^*_H is part of an optimal mechanism, the modification of this quota should weakly decrease welfare. In other words, the expected welfare induced by alternatives from the set A_H must be weakly higher compared to the welfare generated by alternatives from the set A_{H^c}. This observation translates into a condition which is necessary for optimality whenever the considered change in the optimal quota q^*_H is feasible. Exploiting the tuple structure derived in Lemma 4.1, the comparison of the expected welfare induced by the two sets of alternatives reduces to contrasting a collection of tuples of alternatives such that the elements within each tuple are separated only by property H. Now, imposing Assumption 4.2 implies, as discussed above, that these within-tuple welfare comparisons are not sensitive to the tuple under consideration. This simplifies the involved expressions and leads to the inequality appearing in part (i) of Lemma 4.2.
Studying the effect of a decrease of q^*_H by 1 yields via an analogous argument the inequality appearing in part (ii) of Lemma 4.2. This inequality is necessary for optimality as long as the considered decrease in the optimal quota q^*_H is feasible.
The second step of the proof of Theorem 4.2 is summarized in Lemma 4.3.

Lemma 4.3. Rachidi (2019)
Suppose that Assumptions 4.1, 4.2 and 4.3 hold. Consider any properties $H', H \in \mathscr{H}$

such that $H' \subset H$ and $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$.
If $q^*_{H'} = q^*_H$, any optimal mechanism nevertheless satisfies

$$q^*_H \geq n \cdot \delta_H$$

as well as

$$q^*_{H'} \leq n \cdot \delta_{H'} + 1.$$

The two alterations of the quota q^*_H that is part of an optimal mechanism considered above might not be feasible. Lemma 4.3 addresses this issue. Making use of Assumption 4.3, I show that the two inequalities derived in Lemma 4.2 still hold even if these alterations are not feasible.
Finally, it turns out that these inequalities are not only necessary, but also sufficient for optimality, and they determine the generically unique optimal mechanism featuring the quotas appearing in Theorem 4.2.

4.7 Applications

In this section, I apply the general characterization of welfare-maximizing mechanisms developed in Theorem 4.2 to the special cases of trees and hypercubes. In these settings Assumption 4.2 and Assumption 4.3 are vacuously met respectively. NP (2007b) identify trees and hypercubes as distinguished instances of median spaces.[23] The purpose of this section is to present these two instances of median spaces as in both settings one of the three assumptions in Theorem 4.2 is vacuously met.

4.7.1 Trees

To begin with, I consider the special case of single-peaked preferences on trees as introduced in Demange (1982). Take any tree (A, E), that is, take any undirected graph that is connected and acyclic. The set of alternatives A coincides with the set of nodes and the set E captures the set of edges corresponding to the tree. In particular, the set E satisfies $E \subseteq \{V \in \mathscr{P}(A) : |V| = 2\}$. Following Nehring and Puppe (2007a), for any edge $V = \{k, m\} \in E$, define the two properties

$$H_{V,k} := \{a \in A : \text{“}a \text{ lies in direction of } k\text{”}\} \text{ and}$$
$$H_{V,m} := \{a \in A : \text{“}a \text{ lies in direction of } m\text{”}\}.$$[24]

23. For any median space, NP (2007b) characterize the requirement of generalized single-peakedness in terms of a separability and a convexity condition, and they argue that, on trees and hypercubes, generalized single-peakedness reduces to convexity and separability respectively.

Note that any property coincides with a set of nodes corresponding to a connected component of the underlying tree. Also, the properties of the form $(H_{V,k}, H_{V,m})$ constitute an issue. Let $\mathscr{H}_{Tree}$ denote the collection of all these properties. Further, observe that a preference relation is single-peaked with respect to the underlying tree as defined in Demange (1982) if and only if it is generalized single-peaked with respect to the betweenness relation $B_{\mathscr{H}_{Tree}}$. The former definition reads as follows: There exists an alternative $p \in A$, which is the most preferred or peak alternative, such that for all alternatives $k, m \in A$ with $k \neq m$ it holds that whenever k lies on the shortest path in the underlying tree connecting p and m, the voter must prefer k over m.

To illustrate this class of property spaces more concretely, take the simplest tree that is not a line: Suppose that there are four alternatives $\{1, 2, 3, 4\}$ and take the tree that is shown in Figure 4.2. In this case, the collection of properties $\mathscr{H}_{Tree}$ amounts

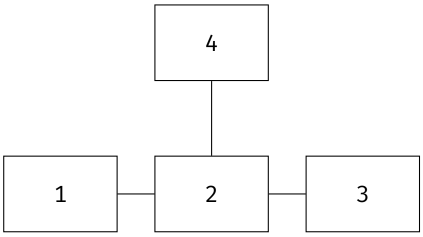

Figure 4.2. Single-Peaked Preferences on a Tree

to

$$\begin{aligned} H_{\{2,4\},2} &= \{1,2,3\}, H_{\{2,4\},4} = \{4\}, \\ H_{\{1,2\},1} &= \{1\}, H_{\{1,2\},2} = \{2,3,4\}, \text{ and} \\ H_{\{2,3\},2} &= \{1,2,4\}, H_{\{2,3\},3} = \{3\}. \end{aligned}$$

For instance, if some voter's most preferred alternative is 1, generalized single-peakedness requires here that alternative 2 is preferred over 3 and 4, but it does not impose whether 3 is preferred to 4 or the other way around.

Moreover, if preferences are single-peaked with respect to a tree (A, E), a voting by properties mechanism can be intuitively described as "voting by edges": Take any edge of the tree (A, E), cut this edge yielding two subsets of alternatives or, more precisely, two connected components of the tree. Then, perform a binary vote determining which of the two connected components is winning. This binary vote yields one winning connected component, that is, the social choice must be contained in

24. More formally, following Nehring and Puppe (2007a), the property $H_{V,k}$ is composed of all alternatives $a \in A$ such that k lies on the shortest path from a to m in the tree (A, E). Similarly, $H_{V,m}$ comprises all alternatives $a \in A$ such that m lies on the shortest path from a to k in the tree (A, E).

the set of nodes associated with the connected component that is winning. These binary voting decisions are conducted for all edges yielding a collection of connected components that are winning. Eventually, the final outcome is given by the intersection of the sets of nodes linked to the connected components that are winning.[25]
Now, if preferences are single-peaked with respect to a tree (A, E), the following feature follows from NP (2007b): Take any edge $V = \{k, m\} \in E$ with $k, m \in A$. Any two alternatives form a tuple (j, l) with $j \in A_{H_{V,k}}$ and $l \in A_{H^c_{V,k}} = A_{H_{V,m}}$ such that j and l are separated only by property $H_{V,m}$ if and only if $j = k$ and $l = m$. This implies that, for any property $H \in \mathscr{H}_{Tree}$, the sets A_H and A_{H^c} considered in section 4.6 are singletons, i.e., $|A_H| = |A_{H^c}| = 1$. Consequently, Assumption 4.2 is vacuously met on trees, and I have the subsequent corollary of Theorem 4.2.

Corollary 4.1. Rachidi (2019)
Consider the median space $(A, \mathscr{H}_{Tree})$, and suppose that Assumptions 4.1 and 4.3 are satisfied.
The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q^*_H = \lceil n\delta_H \rceil \text{ for all } H \in \mathscr{H}_{Tree}.$$

The general indifference property of the welfare-maximizing quotas discussed in section 4.6 reduces here to the following feature: Take any edge $V = \{k, m\} \in E$ with $k, m \in A$. Then, the corresponding optimal quotas $q^*_{H_{V,k}}$ and $q^*_{H_{V,m}}$ are calibrated such that, conditional on being pivotal, the designer is indifferent between the two graph neighbors k and m in the tree (A, E).
Now, let me discuss how Corollary 4.1 extends the main result in Gershkov, Moldovanu, and Shi (2017). For concreteness, without loss of generality, suppose that the set of alternatives amounts to $A := \{1, \ldots, l\}$ with $l \geq 2$. Following NP (2007b), assume that, for all $1 \leq k < l$, the properties are given by

$$H_{\leq k} := \{m \in \{1, \ldots, l\} : m \leq k\} \text{ as well as}$$
$$H_{\geq k+1} := \{m \in \{1, \ldots, l\} : m \geq k+1\}.$$

Let $\mathscr{H}_{Line}$ denote the set of all these properties. Note that this collection of properties exactly coincides with $\mathscr{H}_{Tree}$ if the underlying tree (A, E) constitutes the line shown in Figure 4.3. In particular, a preference relation is generalized single-peaked with respect to the betweenness relation $B_{\mathscr{H}_{Line}}$ if and only if it is in the classical sense single-peaked on a line and, more precisely, it is single-peaked on a line with respect to the natural ordering $1 < 2 < \ldots < l - 1 < l$.
Specializing Corollary 4.1 to the case of single-peaked preferences on a line, I immediately obtain the following corollary.

25. Of course, the quotas involved in the described binary votes on subsets of the set of alternatives have to satisfy the restrictions from Theorem 4.1.

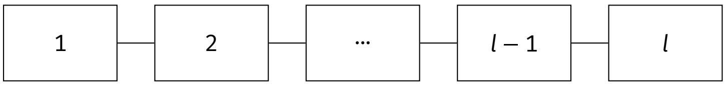

Figure 4.3. Single-Peaked Preferences on a Line

Corollary 4.2. Gershkov, Moldovanu, and Shi (2017)
Consider the median space $(\{1,\ldots,l\},\mathscr{H}_{Line})$, and suppose that Assumptions 4.1 and 4.3 are satisfied.
The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q_H^* = \lceil n\delta_H \rceil \text{ for all } H \in \mathscr{H}_{Line}.$$

Observe that Corollary 4.2 coincides with the main result in Gershkov, Moldovanu, and Shi (2017): Assumption 4.1 is assumption A in their paper, and Assumption 4.3 reduces exactly to assumption B in their work.[26]

4.7.2 Hypercubes

Having treated collective choice when preferences are single-peaked on trees, I continue with the discussion of voting on hypercubes, that is, voting on multiple binary decisions as studied in Barberà, Sonnenschein, and Zhou (1991). To start, assume that the set of alternatives A is given by $A := \{0,1\}^l$, where $l \geq 1$ is a natural number. This means that there are l binary decisions, each coordinate of an alternative corresponds to a binary decision, and, without loss of generality, each binary decision amounts either to 0 or 1. Following NP (2007b), suppose that, for all $1 \leq k \leq l$, the properties are given by

$$\begin{aligned} H_{0,k} &:= \{(m_1,\ldots,m_l) \in \{0,1\}^l : m_k = 0\}, \text{ and} \\ H_{1,k} &:= \{(m_1,\ldots,m_l) \in \{0,1\}^l : m_k = 1\}. \end{aligned}$$

Let $\mathscr{H}_{Hypercube}$ denote the collection of these properties. Moreover, it follows from NP (2007b) that the requirement of generalized single-peakedness reduces here to the restriction of separable preferences imposed in Barberà, Sonnenschein, and

26. Unless there are at most three alternatives, i.e., unless $l \leq 3$, there is the following caveat: Because Gershkov, Moldovanu, and Shi (2017) assume that preferences are single-crossing and single-peaked, the set of ordinal preferences induced by their utility representation does not satisfy NP (2007b)'s richness condition on the preference domain. Hence, the strategy-proof social choice functions NP (2007b) identify are strategy-proof in Gershkov, Moldovanu, and Shi (2017)'s setting, but there might be more strategy-proof direct mechanisms. However, when combining results from Moulin (1980), NP (2007b), and Saporiti (2009), it can be inferred that also in Gershkov, Moldovanu, and Shi (2017)'s model there are no other strategy-proof social choice functions apart from those identified in NP (2007b)'s characterization.

Zhou (1991). The latter requirement reads as follows: For any $1 \leq k \leq l$ and all sequences $m \in \{0,1\}^{k-1}$ and $m' \in \{0,1\}^{l-k}$, a voter prefers alternative $(0^{k-1}, 1, 0^{l-k})$ over $(0^{k-1}, 0, 0^{l-k})$ if and only if he or she prefers alternative $(m, 1, m')$ over $(m, 0, m')$.

Furthermore, it can be verified that, on hypercubes, there are no properties $H, K \in \mathscr{H}$ that are interrelated in the sense that $H \subsetneq K$. Therefore, the restrictions from Theorem 4.1 as well as Assumption 4.3 are vacuously met. Consequently, I obtain the following corollary of Theorem 4.2.[27]

Corollary 4.3.
Consider the median space $(\{0,1\}^l, \mathscr{H}_{Hypercube})$, and suppose that Assumptions 4.1 and 4.2 are satisfied.

The optimal mechanism among all strategy-proof, anonymous, and surjective social choice functions takes the form of voting by properties with quotas

$$q_H^* = \lceil n\delta_H \rceil \text{ for all } H \in \mathscr{H}_{Hypercube}.$$

Here, any voting by properties social choice function amounts to performing qualified majority voting separately for each binary decision. Also, it follows from NP (2007b) that Assumption 4.2 is satisfied if and only if the voters' utilities are additively separable across the binary decisions, making it a natural assumption on hypercubes. Moreover, the general indifference property of the welfare-maximizing quotas from section 4.6 simplifies here to the following feature: Take any $1 \leq k \leq l$. Then, the associated optimal quotas $q^*_{H_{0,k}}$ and $q^*_{H_{1,k}}$ are set such that, conditional on being pivotal, the designer is indifferent between any two alternatives that differ only with respect to the outcome in the k-th binary decision, that is, any two alternatives $(m, 1, m')$ and $(m, 0, m')$ with $m \in \{0,1\}^{k-1}$ and $m' \in \{0,1\}^{l-k}$.

4.8 Conclusion

In this chapter, I present a welfare analysis of voting rules. Specifically, I derive the optimal utilitarian mechanism among all strategy-proof, anonymous, and surjective social choice functions for generalized single-peaked domains giving rise to median spaces. The optimal mechanism takes the form of voting by properties, meaning, the social choice is determined through a collection of binary votes on subsets of alternatives involving flexible majority requirements that incorporate the characteristics of these subsets of alternatives. Consequently, my results emphasize, for a broad range of economically relevant preference domains, the importance of flexible and qualified majority requirements for utilitarian welfare in voting.

27. Corollary 4.3 can also be obtained by combining, on the one hand, the results from NP (2007b) or Barberà, Sonnenschein, and Zhou (1991) and, on the other hand, the optimality findings for the two-alternatives case from, for example, Nehring (2004) or Drexl and Kleiner (2018).

Appendix 4.A Proofs

The proof of Lemma 4.1 employs a result from NP (2007b) that is stated as Lemma 4.4 below. In order to present this result, I need to introduce the notion of *critical families of properties* from their paper. These sets are collections of properties having the following characteristic.

Definition 4.6. NP (2007b)
A set of properties $\mathscr{F} \subseteq \mathscr{H}$ is a critical family of properties if

$$\cap_{\bar{F}\in\mathscr{F}} \bar{F} = \emptyset \text{ and}$$
$$\forall F \in \mathscr{F} : \cap_{\bar{F}\in\mathscr{F}:\bar{F}\neq F}\bar{F} \neq \emptyset.$$

In words, a collection of properties constitutes a critical family of properties if the intersection of all involved properties is empty, but these properties have a non-empty intersection whenever an arbitrary single property of the collection is removed. Also, note that any critical family of properties involves at least two elements. Based on this definition, NP (2007b) obtain the following result about the size of critical families of properties in median spaces.

Lemma 4.4. NP (2007b)
If $(A, \mathscr{H})$ constitutes a median space, all critical families of properties have length two.

Lemma 4.4 says that median spaces share the characteristic that there are no critical families of properties involving more than two properties.

Proof of Lemma 4.1.
Take any property $H \in \mathscr{H}$ and consider the related sets A_H and A_{H^c} as defined in the main text.
Concerning the non-emptiness of the sets A_H and A_{H^c}, consider the set A_H. The argument for the set A_{H^c} is analogous. Towards a contradiction, suppose that $A_H = \emptyset$. If $\nexists M \in \mathscr{H} : M \subset H$, I have that $H \subseteq A_H$. Since $H \neq \emptyset$, it follows that $A_H \neq \emptyset$. If $\exists M \in \mathscr{H} : M \subset H$, $A_H = \emptyset$ implies $H \cap (\cap_{\{M\in\mathscr{M}:M\subset H\}} M^c) = \emptyset$. In other words, the collection of properties $\{H\} \cup \{M^c : M \subset H\}$ is not consistent. However, this means that there must be some subset of the set of these properties which constitutes a critical family of properties. If this critical family involves at least three elements, the desired contradiction is derived since, due to Lemma 4.4, all critical families have length two in a median space. In case this critical family involves only two properties, there are two possibilities. On the one hand, if H is part of this critical family, the other element must be some single property M^c such that $M \subset H$. However, the collection of these two properties cannot be inconsistent and, hence, not critical since the intersection of H and M^c must be non-empty. On the other hand, if H is not

part of the critical family, this family must be composed of two properties from the set $\{M^c : M \subset H\}$, but both of them are by definition supersets of H^c which means that they are consistent and, thus, not critical since $H^c \neq \emptyset$. Therefore, in the two possible cases, I derived the desired contradiction.
Regarding the unique tuple structure, take any $k \in A_{H^c}$, and consider the following intersection of properties:

$$\mathscr{K}_k := (\cap_{\{K \in \mathscr{H}_k : K \neq H^c\}} K) \cap H.$$

Now, because of separation, the set $\mathscr{K}_k$ is either empty or it contains exactly one alternative, but it does not contain more than one alternative. I claim that $\mathscr{K}_k$ cannot be empty. Towards a contradiction, assume that $\mathscr{K}_k$ is empty. This means that the collection of properties $(\mathscr{H}_k \setminus \{H^c\}) \cup \{H\}$ is not consistent. To begin with, if $\mathscr{H}_k \setminus \{H^c\} = \emptyset$, the set of properties $(\mathscr{H}_k \setminus \{H^c\}) \cup \{H\}$ must be consistent since $H \neq \emptyset$. Thus, subsequently, assume that $\mathscr{H}_k \setminus \{H^c\}$ contains at least one property. $(\mathscr{H}_k \setminus \{H^c\}) \cup \{H\}$ being inconsistent implies that there must be some subset of this collection of properties which constitutes a critical family of properties. Since all property spaces are median spaces, due to Lemma 4.4, this critical family must involve exactly two properties. Again, there are two possibilities. On the one hand, if H is part of this critical family, the other element must be some single property $K \in \mathscr{H}_k \setminus \{H^c\}$ satisfying $K \subset H^c$. In particular, it must hold that $k \in K$. However, by definition of A_{H^c}, because of $K \subset H^c$, I have $k \in K^c$. This contradicts $k \in K$. On the other hand, if H is not part of the critical family, this family must be composed of two properties from the set $\mathscr{H}_k \setminus \{H^c\}$, but, by construction, the alternative k shares both of them which means that they are consistent and, thus, not critical. Hence, in both possible cases, I obtain a contradiction. Therefore, I infer that $\mathscr{K}_k$ is not empty, but it contains exactly one alternative. Denote this alternative by m. Now, by construction, k and m are separated only by property H. Further, I obtain that $m \in A_H$ for the following reason: If $\nexists M \in \mathscr{H} : M \subset H$, by definition of A_H, I have that $\mathscr{K}_k \cap A_H = \mathscr{K}_k \cap H = \mathscr{K}_k$. If $\exists M \in \mathscr{H} : M \subset H$, again by definition of A_H, it holds that

$$\mathscr{K}_k \cap A_H = \mathscr{K}_k \cap H \cap (\cap_{\{M \in \mathscr{H} : M \subset H\}} M^c) = \mathscr{K}_k \cap (\cap_{\{M \in \mathscr{H} : H^c \subset M^c\}} M^c) = \mathscr{K}_k.$$

Consequently, I conclude that there exists some $m \in A_H$ such that k and m are separated only by property H. Moreover, there cannot be another alternative $m' \in A_H$ with $m \neq m'$ such that k and m' are also separated only by property H since this would contradict separation. The argument for the other direction, meaning, starting with some $m \in A_H$ and showing that there is some unique $k \in A_{H^c}$ such that both alternatives are separated only by property H works in the same way. This establishes the claimed unique tuple structure. □

Proof of Lemma 4.2.
Take any property $H \in \mathscr{H}$.

Assume that $H' \subset H \Rightarrow q^*_{H'} > q^*_H$ for all $H' \in \mathscr{H}$ such that $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$. Consider the quota q^*_H being part of an optimal mechanism and suppose that it is increased by 1, i.e. the quota linked to property H moves to $q^*_H + 1$. In particular, as long as $q^*_H \neq n$, the modified quota $q^*_H + 1$ is still feasible because $q^*_{H'} \geq q^*_H + 1 > q^*_H$ for all $H' \in \mathscr{H}$ such that $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$.

This alteration matters only if there are q^*_H voters having some peak from the set H and $n - q^*_H$ voters with peaks from the set H^c. For simplicity, call this event "piv_H". In this case, since $q^*_L \leq q^*_H$ for all $H \subset L$, the properties $\{L : H \subset L\}$ or, equivalently, $\{L^c : L \subset H^c\}$ are accepted whenever there are such properties. Additionally, $q^*_{H'} > q^*_H$ for all $H' \in \mathscr{H}$ such that $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$ implies that $q^*_M > q^*_H$ for all $M \subset H$. Thus, the properties $\{M : M \subset H\}$ are rejected or, equivalently, the properties $\{M^c : M \subset H\}$ are winning whenever there are such properties. Putting these features together and using the notation introduced in the main text, if the quota is q^*_H, some element of the set $A_H \neq \emptyset$ is the winning alternative. However, if the quota amounts to $q^*_H + 1$, some element of the set $A_{H^c} \neq \emptyset$ is selected.

Therefore, for both quotas, employing Assumption 4.1, the expected welfare conditional on the event where the alteration of q^*_H matters, i.e., the expected welfare conditional on the event "piv_H", can be expressed in the following way: If the quota is q^*_H, the resulting welfare amounts to

$$\sum_{l \in A_H} Pr(l\, wins | piv_H) \cdot \{n \cdot \mathbb{E}[u^l(T) | piv_H \wedge l\, wins]\}.$$

In contrast, if the quota is $q^*_H + 1$, the induced welfare satisfies

$$\sum_{j \in A_{H^c}} Pr(j\, wins | piv_H) \cdot \{n \cdot \mathbb{E}[u^j(T) | piv_H \wedge j\, wins]\}.$$

Because q^*_H is part of an optimal mechanism, it must be that the former expression is weakly higher than the latter term. This necessary condition for optimality translates into the inequality

$$\sum_{l \in A_H} Pr(l\, wins | piv_H) \mathbb{E}[u^l(T) | piv_H \wedge l\, wins] \geq$$
$$\sum_{j \in A_{H^c}} Pr(j\, wins | piv_H) \mathbb{E}[u^j(T) | piv_H \wedge j\, wins].$$

Now, consider the tuple structure derived in Lemma 4.1 and, with abuse of notation, suppose that (j, l) constitutes a tuple of alternatives such that j and l are separated only by property H. This means that the events "$l\, wins \wedge piv_H$" and "$j\, wins \wedge piv_H$" must coincide, meaning, they refer to the same set of type realizations. This is true for the following reason: The event "$j\, wins \wedge piv_H$" means that the properties $\mathscr{H}_j \setminus H^c$ are winning and the number of voters having peaks from the set H amounts to q^*_H. The event "$l\, wins \wedge piv_H$" means that the properties $\mathscr{H}_l \setminus H$ are winning and the number of voters having peaks from the set H is q_H^*. However, since j and l are separated only by property H, it holds that $\mathscr{H}_j \setminus H^c = \mathscr{H}_l \setminus H$. Therefore, the events "$j\,wins \wedge piv_H$" and "$l\,wins \wedge piv_H$" coincide. Call this event "$j/l\,win \wedge piv_H$". In particular, I have that

$$Pr(j/l\,win|piv_H) \;=\; Pr(l\,wins|piv_H) \;=\; Pr(j\,wins|piv_H).$$

Therefore, the inequality above can be rewritten as follows:

$$\sum_{(j,l)\in Z_H} Pr(j/l\,win|piv_H)\{\mathbb{E}[u^l(T) - u^j(T)|j/l\,win \wedge piv_H]\} \;\geq\; 0.$$

Now, take any pair of alternatives (k,m) with $k \in A_{H^c}$ and $m \in A_H$ such that k and m are separated only by property H. Due to Assumption 4.2, it holds that

$$u^k(t) - u^m(t) \;=\; u^l(t) - u^j(t)$$

for all tuples of alternatives $(j,l) \in Z_H$ and for all type realizations $t \in S$. Thus, the previous inequality can be written as

$$\sum_{(j,l)\in Z_H} Pr(j/l\,win|piv_H)\{\mathbb{E}[u^m(T) - u^k(T)|j/l\,win \wedge piv_H]\} \;\geq\; 0.$$

Next, by the law of total expectation, I obtain that

$$\mathbb{E}[u^m(T) - u^k(T)|piv_H] \;\geq\; 0.$$

Moreover, applying again the law of total expectations, this inequality can be written in the following way:

$$\begin{aligned} &Pr(\text{``}peak \in H\text{''}|piv_H)\mathbb{E}[u^m(T) - u^k(T)|\text{``}peak \in H\text{''} \wedge piv_H] \\ &+Pr(\text{``}peak \in H^c\text{''}|piv_H)\mathbb{E}[u^m(T) - u^k(T)|\text{``}peak \in H^c\text{''} \wedge piv_H] \;\geq\; 0, \end{aligned}$$

where "$peak \in H$" and "$peak \in H^c$" refer to the events that an arbitrary voter's most-preferred alternative or peak shares property H and H^c respectively. While using the definition of the event "piv_H", Assumption 4.1 implies that the probabilities involved in the inequality satisfy

$$\begin{aligned} Pr(\text{``}peak \in H\text{''}|piv_H) \;&=\; \frac{q_H^*}{n} \text{ and} \\ Pr(\text{``}peak \in H^c\text{''}|piv_H) \;&=\; \frac{n - q_H^*}{n}. \end{aligned}$$

Also, Assumption 4.1 yields

$$\begin{aligned} \mathbb{E}[u^m(T) - u^k(T)|\text{``}peak \in H\text{''} \wedge piv_H] \;&=\; \mathbb{E}[u^m(T) - u^k(T)|\text{``}peak \in H\text{''}] \text{ and} \\ \mathbb{E}[u^m(T) - u^k(T)|\text{``}peak \in H^c\text{''} \wedge piv_H] \;&=\; \mathbb{E}[u^m(T) - u^k(T)|\text{``}peak \in H^c\text{''}]. \end{aligned}$$

Further, it follows from generalized single-peakedness that the events "$peak \in H$" and "$peak \in H^c$" are equivalent to the events "$u^m(T) > u^k(T)$" and "$u^k(T) > u^m(T)$" respectively (see Fact 2.1 in NP (2007b)). Taking these three features together, the inequality above simplifies to

$$\frac{q_H^*}{n}\mathbb{E}[u^m(T) - u^k(T)|u^m(T) > u^k(T)]$$
$$+\frac{n-q_H^*}{n}\mathbb{E}[u^m(T) - u^k(T)|u^k(T) > u^m(T)] \geq 0.$$

Hence, rearranging yields

$$q_H^* \geq n \cdot \delta_H$$

while I use the notation introduced in the main text. In addition, if $q_H^* = n$, the derived inequality still holds since $\delta_H \in (0, 1)$. This establishes the first claim of the lemma.

Turning to the second point of the lemma, suppose that $H \subset H' \Rightarrow q_H^* > q_{H'}^*$ for all $H' \in \mathscr{H}$ such that $\nexists H'' \in \mathscr{H} : H \subset H'' \subset H'$.

Consider again the quota q_H^* related to an optimal mechanism and suppose that it is decreased by 1, i.e. the quota q_H^* moves to $q_H^* - 1$. In particular, the altered quota is still feasible as long as $q_H^* \neq 1$. This change matters only if there are $q_H^* - 1$ voters with peak alternatives from the set H and $n - q_H^* + 1$ voters having peaks that share property H^c.

Following the steps employed in the reasoning above in an analogous way, it can be verified that the inequality

$$q_H^* \leq n \cdot \delta_H + 1$$

constitutes a necessary condition for optimality. Additionally, observe that the derived inequality also holds if $q_H^* = 1$ since $n \cdot \delta_H > 0$. □

Proof of Lemma 4.3.

Assume that there are properties $H', H \in \mathscr{H}$ with $H' \subset H$ as well as $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$ and the quotas related to an optimal mechanism satisfy $q_{H'}^* = q_H^*$.

Define

$$\begin{aligned}\mathscr{Q} :=\{K \in \mathscr{H} : [(K \subseteq H' \vee H \subseteq K) \wedge q_K^* = n] \textit{ and} \\ \nexists K' \in \mathscr{H} : [K \subset K' \wedge (K' \subseteq H' \vee H \subseteq K') \wedge q_{K'}^* = n]\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{R} :=\{K \in \mathscr{H} : [(K \subseteq H' \vee H \subseteq K) \wedge q_K^* = 1] \textit{ and} \\ \nexists K' \in \mathscr{H} : [K' \subset K \wedge (K' \subseteq H' \vee H \subseteq K') \wedge q_{K'}^* = 1]\}.\end{aligned}$$

In the following, I perform a case distinction:
1) Suppose that $\mathscr{Q} \neq \emptyset$ and $\mathscr{R} \neq \emptyset$.
1a) $\exists \bar{Q} \in \mathscr{Q} : H \subseteq \bar{Q}$
By definition of $\mathscr{Q}$, it holds that $q^*_{\bar{Q}} = n$ and, because $H \subseteq \bar{Q}$, it follows that $q^*_H = n$. Thus, the inequality $q^*_H \geq n \cdot \delta_H$ is met.
Moreover, I obtain $H \subseteq Q$ for all $Q \in \mathscr{Q}$ since otherwise, $Q \subseteq H'$ or, equivalently, $Q \subset H$ which would imply $Q \notin \mathscr{Q}$ because $q^*_H = n$.
Take some $Q' \in \mathscr{Q}$ and consider the related set

$$\mathscr{S} := \{K \in \mathscr{H} : Q' \subset K \, and \, \nexists K' \in \mathscr{H} : Q' \subset K' \subset K\}$$

of properties.
If $\mathscr{S} = \emptyset$, this means that there are no properties $Q'' \in \mathscr{H}$ such that $Q' \subset Q''$. Consequently, decreasing the quota $q^*_{Q'} = n$ by 1 is feasible and, thus, Lemma 4.2 implies that the inequality $q^*_{Q'} \leq n \cdot \delta_{Q'} + 1$ holds.
If $\mathscr{S} \neq \emptyset$, it must be that $q^*_S < q^*_{Q'}$ for all $S \in \mathscr{S}$. Suppose not, meaning, there exists some $S \in \mathscr{S}$ such that $q^*_S \geq q^*_{Q'}$. Since, by construction $Q' \subset S$, I obtain $q^*_S = q^*_{Q'} = n$. But, then, it holds that $H \subseteq Q' \subset S$ and $q^*_S = n$ and, thus, it follows that $Q' \notin \mathscr{Q}$ which is the desired contradiction.
Now, the feature $q^*_S < q^*_{Q'}$ for all $S \in \mathscr{S}$ implies that decreasing the quota $q^*_{Q'}$ by 1 is feasible and, therefore, by Lemma 4.2, the inequality $q^*_{Q'} \leq n \cdot \delta_{Q'} + 1$ is met.
Hence, in both possible cases, the inequality $q^*_{Q'} \leq n \cdot \delta_{Q'} + 1$ is true. Furthermore,

$$q^*_{H'} = q^*_H = n = q^*_{Q'} \leq n \cdot \delta_{Q'} + 1 \leq n \cdot \delta_{H'} + 1$$

by Assumption 4.3 since $H' \subset H \subseteq Q'$. Thus, the inequality $q^*_{H'} \leq n \cdot \delta_{H'} + 1$ holds.

1b) $\exists \bar{R} \in \mathscr{R} : \bar{R} \subseteq H'$
By definition of $\mathscr{R}$, it holds $q^*_{\bar{R}} = 1$ and, since $\bar{R} \subseteq H'$, I obtain $q^*_{H'} = 1$. Therefore, the second inequality $q^*_{H'} \leq n \cdot \delta_{H'} + 1$ is true.
Furthermore, I obtain $R \subseteq H'$ for all $R \in \mathscr{R}$ because $H' \subset H \subseteq R$ would imply $R \notin \mathscr{R}$ since $q^*_{H'} = 1$.
Take some $R' \in \mathscr{R}$ and consider the related set

$$\mathscr{J} := \{K \in \mathscr{H} : K \subset R' \, and \, \nexists K' \in \mathscr{H} : K \subset K' \subset R'\}$$

of properties.
If $\mathscr{J} = \emptyset$, this means that there are no properties $R'' \in \mathscr{H}$ satisfying $R'' \subset R'$. Consequently, increasing the quota $q^*_{R'} = 1$ by 1 must be feasible yielding the inequality $q^*_{R'} \geq n \cdot \delta_{R'}$ because of Lemma 4.2.
If $\mathscr{J} \neq \emptyset$, it must be that $q^*_J > q^*_{R'}$ for all $J \in \mathscr{J}$. To see this point, suppose that the contrary is true, meaning, there exists some $J \in \mathscr{J}$ such that $q^*_J \leq q^*_{R'}$. Thus, because of $J \subset R'$, I obtain $q^*_J = q^*_{R'} = 1$. However, since $J \subset R' \subseteq H'$ and $q^*_J = 1$, the property

R' cannot be part of the set $\mathscr{R}$ which contradicts $R' \in \mathscr{R}$.
Employing the feature $q_J^* > q_{R'}^*$ for all $J \in \mathscr{J}$, I observe that increasing the quota $q_{R'}^*$ by 1 is feasible and, therefore, by Lemma 4.2, the inequality $q_{R'}^* \geq n \cdot \delta_{R'}$ holds.
Hence, in both possible scenarios, I obtain that the inequality $q_{R'}^* \geq n \cdot \delta_{R'}$ is satisfied.
Consequently, since $R' \subseteq H' \subset H$, Assumption 4.3 implies

$$1 = q_H^* = q_{H'}^* = q_{R'}^* \geq n \cdot \delta_{R'} \geq n \cdot \delta_H.$$

Therefore, the inequality $q_H^* \geq n \cdot \delta_H$ is also true.

1c) $\forall \bar{Q} \in \mathscr{Q} : \bar{Q} \subseteq H'$ and $\forall \bar{R} \in \mathscr{R} : H \subseteq \bar{R}$
Define

$$\begin{aligned} \mathscr{O} :=\{K \in \mathscr{H} : [K \subseteq H' \wedge q_K^* > q_H^*] \; and \\ \nexists K' \in \mathscr{H} : [K \subset K' \subseteq H' \wedge q_{K'}^* > q_H^*]\} \end{aligned}$$

and

$$\begin{aligned} \mathscr{P} :=\{K \in \mathscr{H} : [H \subseteq K \wedge q_K^* < q_H^*] \; and \\ \nexists K' \in \mathscr{H} : [H \subseteq K' \subset K \wedge q_{K'}^* < q_H^*]\}. \end{aligned}$$

In particular, $\mathscr{O} \neq \emptyset$ as well as $\mathscr{P} \neq \emptyset$ since $\mathscr{Q} \neq \emptyset$ and $\mathscr{R} \neq \emptyset$.
Take some $O \in \mathscr{O}$. By construction, I have $q_L^* = q_H^*$ for all $L \in \mathscr{H}$ such that $O \subset L \subseteq H'$.
Also, since $\mathscr{Q} \neq \emptyset$ and $\bar{Q} \subseteq H'$ for all $\bar{Q} \in \mathscr{Q}$, it must be that $q_H^* \neq n$.
Moreover, there exists some $L' \in \mathscr{H}$ such that $\nexists L'' \in \mathscr{H} : O \subset L'' \subset L' \subseteq H'$. Consider the set

$$\mathscr{I} := \{K \in \mathscr{H} : K \subset L' \; and \; \nexists K' \in \mathscr{H} : K \subset K' \subset L'\}$$

of properties. In particular, I have $\mathscr{I} \neq \emptyset$ because, by construction, $O \in \mathscr{I}$.
If $q_I^* > q_{L'}^*$ for all $I \in \mathscr{I}$, increasing $q_{L'}$ by 1 is feasible and, therefore, by Lemma 4.2, the inequality $q_{L'}^* \geq n \cdot \delta_{L'}$ holds.
If there exists $I' \in \mathscr{I}$ such that $q_{I'}^* \leq q_{L'}^*$, it follows that $q_{I'}^* = q_{L'}^* = q_H^*$ because $I' \subset L' \subseteq H'$. Now, employ the reasoning that I used to tackle L' and apply it to I'. Again, there are two possibilities: Either increasing $q_{I'}^*$ is feasible or there must be some property $I'' \in \mathscr{H}$ such that $I'' \subset I' \subseteq H'$ satisfying $q_{I''}^* = q_{L'}^* = q_H^*$. If necessary, since there are finitely many properties, repeat this argument for a finite number of times.
This yields that there exist either some property $I''' \in \mathscr{H}$ with $I''' \subseteq H'$ such that increasing $q_{I'''}^*$ satisfying $q_{I'''}^* = q_{L'}^* = q_H^*$ is feasible or, otherwise, there must be some property $I'''' \in \mathscr{H}$ with $I'''' \subseteq H'$, $q_{I''''}^* = q_{L'}^* = q_H^*$ and $\nexists I''''' \in \mathscr{H} : I''''' \subset I''''$. However, concerning the latter case, increasing $q_{I''''}^*$ by 1 is feasible.
Therefore, in any scenario, there must be some $\tilde{I} \in \mathscr{H}$ with $\tilde{I} \subseteq L' \subseteq H' \subset H$ such that increasing $q_{\tilde{I}}$ by 1 is feasible and $q_{\tilde{I}}$ satisfies $q_{\tilde{I}}^* = q_{L'}^* = q_H^*$. Employing Lemma 4.2,

this means that the inequality $q^*_{\tilde{I}} \geq n \cdot \delta_{\tilde{I}}$ is met. But, then, since $\tilde{I} \subset H$, Assumption 4.3 implies

$$q^*_H \;=\; q^*_{\tilde{I}} \;\geq\; n \cdot \delta_{\tilde{I}} \;\geq\; n \cdot \delta_H$$

and, thus, the inequality $q^*_H \geq n \cdot \delta_H$ is met.
Consider some arbitrary $P \in \mathscr{P}$. By construction, I have $q^*_M = q^*_{H'}$ for all $M \in \mathscr{H}$ such that $H \subseteq M \subset P$. Further, since $\mathscr{R} \neq \emptyset$ and $H \subseteq \bar{R}$ for all $\bar{R} \in \mathscr{R}$, it must be that $q^*_{H'} \neq 1$. Additionally, there exists some $M' \in \mathscr{H}$ such that $\nexists M'' \in \mathscr{H} : H \subseteq M' \subset M'' \subset P$. Focus on the set

$$\mathscr{C} \;:=\; \{K \;\in\; \mathscr{H} \;:\; M' \;\subset\; K \,and\, \nexists K' \;\in\; \mathscr{H} \;:\; M' \;\subset\; K' \;\subset\; K\}$$

of properties. In particular, I have $\mathscr{C} \neq \emptyset$ because, by construction, $P \in \mathscr{C}$.
If $q^*_C < q^*_{M'}$ for all $C \in \mathscr{C}$, decreasing $q_{M'}$ by 1 is feasible and, therefore, due to Lemma 4.2, the inequality $q^*_{M'} \leq n \cdot \delta_{M'} + 1$ holds.
If there exists $C' \in \mathscr{C}$ such that $q^*_{C'} \geq q^*_{M'}$, it follows that $q^*_{C'} = q^*_{M'} = q^*_{H'}$ because $H \subseteq M' \subset C'$. Now, employ the reasoning that I used to tackle M' and apply it to C'. Again, there are two possibilities: Either decreasing $q^*_{C'}$ is feasible or there must be some property $C'' \in \mathscr{H}$ such that $H \subseteq C' \subset C''$ satisfying $q^*_{C''} = q^*_{M'} = q^*_{H'}$. If necessary, since there are finitely many properties, repeat this argument for a finite number of times. This yields that there exist either some property $C''' \in \mathscr{H}$ with $H \subseteq C'''$ such that increasing $q^*_{C'''}$ satisfying $q^*_{C'''} = q^*_{M'} = q^*_{H'}$ is feasible or, otherwise, there must be some property $C'''' \in \mathscr{H}$ with $H \subseteq C''''$, $q^*_{C''''} = q^*_{M'} = q^*_{H'}$ and $\nexists C''''' \in \mathscr{H} : C'''' \subset C'''''$. However, concerning the latter case, decreasing $q^*_{C''''}$ by 1 is feasible.
Therefore, in any scenario, there must be some $\tilde{C} \in \mathscr{H}$ with $H' \subset H \subseteq M' \subseteq \tilde{C}$ such that decreasing $q_{\tilde{C}}$ by 1 is feasible and $q_{\tilde{C}}$ satisfies $q^*_{\tilde{C}} = q^*_{M'} = q^*_{H'}$. Invoking Lemma 4.2, this means that the inequality $q^*_{\tilde{C}} \leq n \cdot \delta_{\tilde{C}} + 1$ is met. But, then, since $H' \subset \tilde{C}$, Assumption 4.3 implies

$$q^*_{H'} \;=\; q^*_{\tilde{C}} \;\leq\; n \cdot \delta_{\tilde{C}} + 1 \;\leq\; n \cdot \delta_{H'} + 1$$

and, thus, the inequality $q^*_{H'} \leq n \cdot \delta_{H'} + 1$ is met.
In conclusion, as desired, despite $q^*_{H'} = q^*_H$, both relevant inequalities are met at q^*_H.
2) If $\mathscr{Q} = \emptyset$ and $\mathscr{R} = \emptyset$, the argument from case 1c applies.
3) Suppose that $\mathscr{Q} \neq \emptyset$, but $\mathscr{R} = \emptyset$.
If $\exists \bar{Q} \in \mathscr{Q} : H \subseteq \bar{Q}$, the reasoning in case 1a yields the desired conclusion; in case $Q \subset H$ for all $Q \in \mathscr{Q}$, take the argument from case 1c.
4) Suppose that $\mathscr{R} \neq \emptyset$, but $\mathscr{Q} = \emptyset$.
In case $H \subset R$ for all $R \in \mathscr{R}$, replicate the steps in case 1c; if $\exists \bar{R} \in \mathscr{Q} : \bar{R} \subseteq H$, the argument from case 1b applies.
Taking all four cases together, this shows that the two relevant inequalities

$$\begin{aligned} q^*_H \;&\geq\; n \cdot \delta_H \text{ and} \\ q^*_{H'} \;&\leq\; n \cdot \delta_{H'} + 1 \end{aligned}$$

determining q_H^* as well as $q_{H'}^*$ hold despite $q_{H'}^* = q_H^*$. Therefore, overall, the claim in the lemma follows. □

Proof of Theorem 4.2.
It is sufficient to find the quotas related to voting by issues that are part of an optimal mechanism. The existence of a solution is ensured since a bounded function is optimized over a finite set of elements.

Recall, by Theorem 4.1, the optimal quotas must satisfy

$$H \subseteq K \Rightarrow q_H^* \geq q_K^*$$

for all $K', K \in \mathscr{H}$.
Consider some arbitrary property $H \in \mathscr{H}$ and the associated quota q_H^* being part of an optimal mechanism. Subsequently, I perform case distinctions.
1a) If $\forall H' \in \mathscr{H}$ with $H' \subset H$ and $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$, it holds that $q_{H'}^* > q_H^*$, part (i) of Lemma 4.2 yields that the inequality $q_H^* \geq n \cdot \delta_H$ is met.
1b) If there is some $H' \in \mathscr{H}$ with $H' \subset H$ and $\nexists H'' \in \mathscr{H} : H' \subset H'' \subset H$ such that $q_{H'}^* = q_H^*$, Lemma 4.3 implies that the inequality $q_H^* \geq n \cdot \delta_H$ holds.
Therefore, no matter the shape of the optimal mechanism, the inequality $q_H^* \geq n \cdot \delta_H$ constitutes a necessary condition for optimality.
2a) If $\forall H' \in \mathscr{H}$ with $H \subset H'$ and $\nexists H'' \in \mathscr{H} : H \subset H'' \subset H'$, it holds $q_H^* > q_{H'}^*$, part (ii) of Lemma 4.2 yields that the inequality $q_H^* \leq n \cdot \delta_H + 1$ is true.
2b) If there is some $H' \in \mathscr{H}$ with $H \subset H'$ and $\nexists H'' \in \mathscr{H} : H \subset H'' \subset H'$ such that $q_H^* = q_{H'}^*$, Lemma 4.3 implies that the inequality $q_H^* \leq n \cdot \delta_H + 1$ is satisfied.
Thus, no matter the shape of the optimal mechanism, the inequality $q_H^* \leq n \cdot \delta_H + 1$ is necessary for optimality.
Taking both inequalities together, since the quotas are integer-valued, the quotas q_H^* satisfying these inequalities are, generically, unique and they amount to $q_H^* = \lceil n\delta_H \rceil$ with $H \in \mathscr{H}$. Consequently, the equalities $q_H^* = \lceil n\delta_H \rceil$ with $H \in \mathscr{H}$ are necessary for optimality. Finally, it remains to be verified that these equalities are also sufficient for optimality. The quotas determined by these equalities are feasible in the sense that they constitute a family of quotas and that they meet the inequalities from Theorem 4.1. First, note that, for all $H \in \mathscr{H}$, since $0 < \delta_H < 1$, I have that $1 \leq q_H^* = \lceil n\delta_H \rceil \leq n$. Second, observe that, for any $H \in \mathscr{H}$, it holds that $q_H^* + q_{H^c}^* = n + 1$. Thus, the discussed quotas constitute a family of quotas. Also, it is immediate from Assumption 4.3 that these quotas satisfy the inequalities from Theorem 4.1. Moreover, the quotas determined by the equalities $q_H^* = \lceil n\delta_H \rceil$ with $H \in \mathscr{H}$ must be optimal because, again, there exists a solution and this solution has to meet these equalities. Consequently, the discussed equalities are also sufficient for optimality, and the theorem follows. □

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