Modularity of special motives of rank four associated with Calabi-Yau threefolds

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Summary

In this thesis, we study modularity properties of pure motives and mixed period matrices of rank four. The pure motives that we consider are associated with Calabi-Yau threefolds, while the mixed period matrices are associated with limit mixed Hodge structures of hypergeometric families of Calabi-Yau threefolds. By modularity properties, we mean two possible things. First, whether associated Galois representations are given by Galois representations of modular forms. And, second, whether associated period matrices are given by period matrices of modular forms. We do not only consider elliptic modular forms, but also Hilbert modular forms and Bianchi modular forms.

After a short introduction about modularity properties of algebraic varieties, we review how one can associate motives with algebraic varieties in the second chapter. Here, we consider motives as purely linear algebraic structures which contain information about Galois representations and periods of the underlying variety. In the third chapter, we introduce elliptic modular forms and review that some of these have associated motives of rank two. As generalizations, we also discuss Hilbert modular forms and Bianchi modular forms. In the fourth chapter, we introduce Calabi-Yau threefolds and review how families of these can be studied using differential equations.

We present new results in the last two chapters. This starts in the fifth chapter, where we consider four Calabi-Yau threefolds whose associated pure motives of rank four are (conjecturally) given by sums or products of motives associated with modular forms. While the results on the level of the periods are numerical, we can prove the modularity of the Galois representations in two cases. For instance, we give the first example of a Calabi-Yau threefold whose associated Galois representations are proven to be associated with Bianchi modular forms of weight 4 and weight 2. In the last chapter, we consider mixed period matrices associated with limit mixed Hodge structures of hypergeometric families of Calabi-Yau threefolds. It is known that there are fourteen such families and we study these using the method of “fibering out”, which has recently been introduced by Vasily Golyshev. For twelve examples, we prove that the period matrices can be expressed completely in terms of integrals of modular forms. It has been expected that this is possible for a certain submatrix, but the result for the whole period matrix is surprising and leads to an interesting new class of periods of meromorphic modular forms. While our computations allow to give many examples of these modular forms and to prove their special properties, we do not have a more general understanding which is independent of the relation with families of Calabi-Yau threefolds. We conclude this thesis by giving experimental identities of a new type which relate mixed periods to central values of derivatives of L-functions.

List of publications

Parts of this thesis are based on and complete the results of the following preprint:

- K. Bönisch, A. Klemm, E. Scheidegger and D. Zagier. *D-brane masses at special fibres of hypergeometric families of Calabi-Yau threefolds, modular forms, and periods*, arXiv:2203.09426, 2022.

Further publications by the author of this thesis, which are about related topics but whose results have not been included, are:

- K. Bönisch, C. Duhr, F. Fischbach, A. Klemm and C. Nega. *Feynman integrals in dimensional regularization and extensions of Calabi-Yau motives*, J. High Energy Phys. **156** (2022).
- K. Bönisch, M. Elmi, A.-K. Kashani-Poor and A. Klemm. *Time reversal and CP invariance in Calabi-Yau compactifications*, J. High Energy Phys. **19** (2022).
- K. Bönisch, F. Fischbach, A. Klemm, C. Nega and R. Safari. *Analytic structure of all loop banana integrals*, J. High Energy Phys. **66** (2021).

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Contents

1 Introduction

In this thesis, we study modularity properties of pure motives and mixed period matrices of rank four. This involves the analysis of Galois representations and periods. While this might sound rather abstract, our results can be seen as very concrete formulas for the number of solutions of polynomial equations over finite fields and integrals of algebraic functions. To illustrate the nature of such formulas, we consider a simple example related to the polynomial

$$P(x, y) \;=\; x^3 - x^2 - y^2 - y\,.$$

We can now ask how many solutions the equation $P(x, y) = 0$ has. If we demand that x and y are integers, then there are only the four solutions (x, y) given by the tuples $(0, 0)$, $(0, -1)$, $(1, 0)$ and $(1, -1)$. It turns out that there are no additional solutions if we allow for rational values of x and y. Also, there are clearly infinitely many solutions if we allow x and y to be real or imaginary numbers. To get more interesting numbers of solutions, we can consider prime numbers p and define N_p as the number of solutions of $P(x, y) = 0$ with x and y taking values in the finite field with p elements. In other words, N_p is the number of tuples (x, y) with integers $x \in \{0, 1, ..., p-1\}$, $y \in \{0, 1, ..., p-1\}$ that satisfy $P(x, y) \equiv 0 \bmod p$. Counting the number of solutions for small primes, we obtain the following table:

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
N_p	4	4	4	9	10	9	19	19	24	29	24	34	49	49	39

For the primes above, it is straightforward to check that $N_p = p - a_p$, where the numbers a_p are defined by the power series

$$\sum_{n=1}^{\infty} a_n\, q^n \;=\; q \prod_{n=1}^{\infty} (1 - q^n)^2\, (1 - q^{11\,n})^2 \;=\; q - 2\,q^2 - q^3 + 2\,q^4 + q^5 + 2\,q^6 - 2\,q^7 + \cdots\,.$$

In fact, this is true for all primes and this is implied by a famous modularity theorem. The term "modularity" refers to the fact that the power series above defines a function with very special properties, a so-called modular form. More precisely, we can define a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ on the complex upper half-plane $\mathfrak{h} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\ \tau \;>\; 0\}$ by setting

$$f(\tau) \;=\; \sum_{n=1}^{\infty} a_n\, q^n \quad \text{with} \quad q \;=\; e^{2\pi i\,\tau}\,.$$

This function has special transformation properties with respect to elements of a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, which is the group of all 2×2 matrices with integer entries and determinant 1. In the present example, these transformation properties are

$$f\left(\frac{a\,\tau + b}{c\,\tau + d}\right) \;=\; (c\,\tau + d)^2\, f(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \equiv 0 \bmod 11$. The relations given above are one manifestation of the modularity of the polynomial P, i.e. that the number of solutions over finite fields can be given in terms of coefficients of a modular form. We now explain the second manifestation, which is concerned with complex numbers, so-called periods, which can be defined as certain integrals associated with the polynomial P. To motivate the form of these integrals, we see the vanishing set $P(x, y) = 0$ in $\mathbb{C}^2$ as a complex one-dimensional manifold. Rewriting the equation $P(x, y) = 0$ as $(y + \frac{1}{2})^2 = x^3 - x^2 + \frac{1}{4}$, we see that (away from the points where $x^3 - x^2 + \frac{1}{4} = 0$) this manifold is just a double cover of the complex plane $\mathbb{C}$ with coordinate x. One example of a period is now given by the integral

$$\int_{x_0}^{\infty} \frac{\mathrm{d}x}{y + \frac{1}{2}} \;=\; \int_{x_0}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^3 - x^2 + \frac{1}{4}}} \;=\; 6.346046521397767108443 97\cdots\,,$$

where x_0 is the unique real solution of $x^3 - x^2 + \frac{1}{4} = 0$. This number can be given as an integral of the modular form f. More precisely, since $f(\tau)$ decreases exponentially for $\mathrm{Im}\,\tau \to \infty$ and satisfies the additional transformation property $f(-1/11\,\tau) = -11\,\tau^2\,f(\tau)$, one can integrate $f(i\,u)$ from $u = 0$ to $u = \infty$ and it turns out that

$$\int_{x_0}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^3 - x^2 + \frac{1}{4}}} = 50\,\pi \int_0^{\infty} f(i\,u)\,\mathrm{d}u\,.$$

We now embed the findings of the previous paragraph in a more general context. The vanishing sets of systems of polynomial equations define algebraic varieties. These have associated motives, which are purely linear algebraic structures which contain Galois representations (that contain information about the number of solutions over finite fields) and periods. One can also associate motives with certain modular forms. In particular, there are Galois representations and periods associated with these modular forms. One can now study, whether an algebraic variety and a modular form define the same motive, or at least whether they define the same Galois representations or periods. This is what we sketched in the previous paragraph for the algebraic variety defined by the polynomial $x^3 - x^2 - y^2 - y$ which is an example of an elliptic curve. For elliptic curves, there are very general modularity theorems. However, less is known for more general algebraic varieties. In our thesis, we study motives associated with Calabi-Yau threefolds, which can be seen as generalizations of elliptic curves (which are one-dimensional) to three dimensions.

We begin in the next chapter by reviewing how one can associate motives with algebraic varieties. For this, we start by discussing the Hodge theory of compact Kähler manifolds and then specialize to smooth projective varieties defined over number fields to introduce associated periods, Galois representations and, finally, motives. In the third chapter, we introduce elliptic modular forms and review that some of these, the so-called newforms, have associated motives of rank two. As generalizations, we also discuss Hilbert modular forms and Bianchi modular forms. For these, there is again a notion of newforms, which have associated Galois representations and which are further expected to have associated motives. In the fourth chapter, we introduce Calabi-Yau threefolds. In particular, we explain how periods and Galois representations of members of families of Calabi-Yau threefolds can be studied by using differential equations. In the fifth chapter, we consider four Calabi-Yau threefolds with associated pure motives of rank four which are (conjecturally) given by sums or products of motives of rank two associated with elliptic modular forms, Hilbert modular forms and Bianchi modular forms. In two examples, we present numerical evidence which suggests that the complete period matrix can be expressed in terms of periods and quasiperiods of elliptic modular forms. On the level of the Galois representations, our results can be summarized in two conjectures and two theorems. For instance, we give the first example of a Calabi-Yau threefold whose associated Galois representations are given by that of Bianchi modular forms of weight 4 and weight 2. In the last chapter, we consider mixed period matrices associated with limit mixed Hodge structures of hypergeometric families of Calabi-Yau threefolds. It is known that there are fourteen such families and each gives a limit mixed Hodge structure associated with the so-called conifold fiber. Using the method of "fibering out", which has recently been introduced by Vasily Golyshev, we prove that for twelve cases the period matrix of the limit mixed Hodge structure can be expressed in terms of integrals of elliptic modular forms of weight 4. Previously, there was already numerical evidence which suggests that this is true for a certain submatrix, but the result for the complete period matrix is surprising. In particular, the computations gives rise to an interesting new class of periods of meromorphic modular forms with non-vanishing residues. We can give many examples of such forms and also prove their special properties, but we do not have a more general understanding which does not rely on the geometric origin. We conclude the last chapter with numerical evidence for identities of a new type between certain combinations of the mixed periods and central values of derivatives of L-functions.

We have used Magma [13] and Pari [69] for several computations. The scripts are available in the repository [17]. All numerical results have been checked to at least 100 digits and computing with higher accuracy can be done without any problems.

2 From Hodge structures to motives

The cohomology groups of compact Kähler manifolds together with their Hodge decompositions are examples of Hodge structures. These are linear algebraic structures which, in the geometric context, capture information about the complex structure. In the arithmetic algebraic context, i.e. if one works with an algebraic variety defined over some number field, one can consider other cohomology groups which contain finer information. These cohomology groups come with extra structures and certain compatibilities. For example, the comparison between Betti cohomology groups and algebraic de Rham cohomology groups gives complex numbers called periods. Also, ℓ-adic cohomology groups come with Galois representations and these contain information about the number of points of the variety over finite fields. To combine these structures, it was proposed by Grothendieck that all reasonable cohomology groups should be unified to one object, a so-called motive, which captures the cohomological structure of the variety. In particular, it should contain information about the periods and the Galois representations.

In the first section of this chapter, we review the Hodge theory of compact Kähler manifolds, more general Hodge structures and their variations. We also discuss the Hodge theory of algebraic varieties and the definition of periods. In the second section, we review the arithmetic structure of algebraic varieties defined over number fields. In particular, we review the Weil conjectures and their relation to ℓ-adic cohomology groups. In the last section, we use the previously defined cohomology groups of algebraic varieties to collect them in one structure: the associated motive. The exposition in this chapter is based on that of [12], but with several additions.

2.1 Hodge theory and Hodge structures

In the first part of this section, we review the Hodge theory of compact Kähler manifolds and discuss more general pure Hodge structures. In the second part, we specialize to algebraic varieties. These have associated algebraic de Rham cohomology groups which have Hodge filtrations. In particular, we define interesting complex numbers, so-called periods, for varieties defined over number fields. In the third part, we consider variations of Hodge structures, which in the geometric context arise from families of compact Kähler manifolds. In the last part, we conclude this section by generalizing pure Hodge structures to mixed Hodge structures.

Hodge theory and pure Hodge structures

Let X be a compact Kähler manifold of dimension n. We have for each integer $0 \leq k \leq 2\,n$ the kth homology group $H_k(X,\mathbb{Q})$, whose elements are represented by closed k-dimensional chains modulo boundaries of $(k+1)$-dimensional chains[1]. The dimension of this space is the kth Betti number $b_k(X)$. Considering the cochain complex, we also get the associated cohomology groups $H^k(X,\mathbb{Q})$. By de Rham's theorem, we can represent elements of $H^k(X,\mathbb{Q})$ by elements of the de Rham cohomology group $H^k_{\mathrm{dR}}(X)$ whose elements are represented by closed k-forms modulo exact k-forms. More concretely, by Stokes's theorem, the integration of differential forms over chains gives a well-defined pairing

$$\int : \quad H_k(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^k_{\mathrm{dR}}(X) \to \mathbb{C}$$

and, by de Rham's theorem, this pairing is non-degenerate. This induces a canonical isomorphism

$$H^k_{\mathrm{dR}}(X) \;\cong\; H^k(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}\,.$$

[1]Note that one could also work with the $\mathbb{Z}$-module $H_k(X,\mathbb{Z})$, which can have torsion.

The complex structure of X further allows us, by a theorem of Hodge, to decompose $H^k_{\mathrm{dR}}(X)$ into subspaces whose elements can be represented by forms of Hodge type[2] (p, q) with $p + q = k$. This gives the Hodge decomposition

$$H^k_{\mathrm{dR}}(X) \;=\; \bigoplus_{p+q=k} H^{p,q}(X)$$

on which complex conjugation gives an isomorphism between $H^{p,q}(X)$ and $H^{q,p}(X)$.

We can replace the Hodge decomposition by the decreasing Hodge filtration

$$H^k_{\mathrm{dR}}(X) \;=\; F^0 H^k_{\mathrm{dR}}(X) \;\supseteq\; F^1 H^k_{\mathrm{dR}}(X) \;\supseteq \cdots \supseteq F^k H^k_{\mathrm{dR}}(X) \;\supseteq\; F^{k+1} H^k_{\mathrm{dR}}(X) \;=\; 0$$

with

$$F^i H^k_{\mathrm{dR}}(X) \;=\; \bigoplus_{p \geq i} H^{p,k-p}(X)\,.$$

This is not necessary at this stage, since the Hodge decomposition gives a canonical splitting and can be recovered from the filtration by $H^{p,q}(X) = F^p H^k_{\mathrm{dR}}(X) \cap \overline{F^q H^k_{\mathrm{dR}}(X)}$ for $p+q = k$. However, later it will be more convenient to talk about Hodge filtrations instead of Hodge decompositions. There are several reasons for this. For example, if X is algebraic then the Hodge decomposition is not defined algebraically, but one can define algebraic de Rham cohomology groups with Hodge filtrations. Also, when we later in this section consider families of compact Kähler manifolds, then the Hodge filtration will lead to holomorphic subbundles while the Hodge decomposition will not.

The combination of the rational structure of $H^k(X, \mathbb{Q})$ and the Hodge filtration of $H^k_{\mathrm{dR}}(X)$ is abstractly captured by a so-called pure $\mathbb{Q}$-Hodge structure of weight k. This is, by definition, a finite-dimensional $\mathbb{Q}$-vector space V together with a filtration of the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$

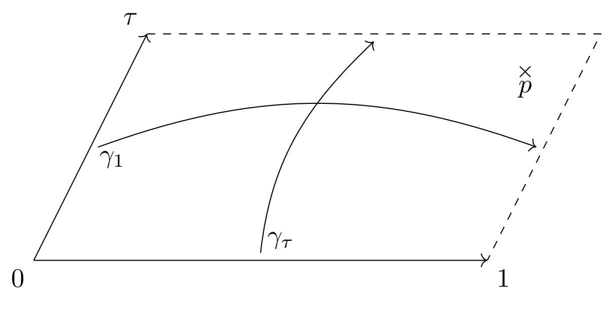
$$\ldots \;\supseteq\; F^{-1} V_{\mathbb{C}} \;\supseteq\; F^0 V_{\mathbb{C}} \;\supseteq\; F^1 V_{\mathbb{C}} \;\supseteq\; \ldots$$

so that for all p, one has $F^p V_{\mathbb{C}} \cap \overline{F^{k+1-p} V_{\mathbb{C}}} = 0$ and $F^p V_{\mathbb{C}} \oplus \overline{F^{k+1-p} V_{\mathbb{C}}} = V_{\mathbb{C}}$. Analogously, one defines pure $\mathbb{R}$-Hodge structures and pure $\mathbb{Z}$-Hodge structures. Note that an equivalent definition can be given in terms of a Hodge decomposition $V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}_{\mathbb{C}}$, where one requires that $\overline{V^{p,q}_{\mathbb{C}}} = V^{q,p}_{\mathbb{C}}$. Here, the complex conjugation acts trivially on V and as usual complex conjugation on the complex numbers. The equivalence of the two definitions is obtained from the identification $V^{p,q}_{\mathbb{C}} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$ for $p + q = k$ and the identification $F^i V_{\mathbb{C}} = \oplus_{p \geq i} V^{p,k-p}_{\mathbb{C}}$.

To give some geometric examples for pure Hodge structures, we consider the complex torus $T^2 = \mathbb{C}/\langle 1, \tau \rangle_{\mathbb{Z}}$ with some point τ in the complex upper half-plane. The associated singular cohomology groups are given by

$$H^0(T^2, \mathbb{Z}) \;=\; \langle \widehat{p} \rangle_{\mathbb{Z}}\,, \quad H^1(T^2, \mathbb{Z}) \;=\; \langle \widehat{\gamma}_1, \widehat{\gamma}_\tau \rangle_{\mathbb{Z}}\,, \quad H^2(T^2, \mathbb{Z}) \;=\; \langle \widehat{T^2} \rangle_{\mathbb{Z}}$$

in terms of the duals of the classes of the chains depicted below:



[2] A differential form is said to have Hodge type (p, q) if it can locally be written as a linear combination of products of p holomorphic differentials and q antiholomorphic differentials with smooth functions as coefficients.

This gives the following pure $\mathbb{Z}$-Hodge structures:

- In weight 0, we have $V = \langle \widehat{p} \rangle_{\mathbb{Z}}$, $F^0 V_{\mathbb{C}} = \langle \widehat{p} \rangle_{\mathbb{C}}$ and $F^1 V_{\mathbb{C}} = 0$.
- In weight 1, we have $V = \langle \widehat{\gamma}_1, \widehat{\gamma}_\tau \rangle_{\mathbb{Z}}$, $F^0 V_{\mathbb{C}} = \langle \widehat{\gamma}_1, \widehat{\gamma}_\tau \rangle_{\mathbb{C}}$, $F^1 V_{\mathbb{C}} = \langle \widehat{\gamma}_1 + \tau \widehat{\gamma}_\tau \rangle_{\mathbb{C}}$ and $F^2 V_{\mathbb{C}} = 0$. In terms of the coordinate z of $\mathbb{C}$, one can identify $\widehat{\gamma}_1 + \tau \widehat{\gamma}_\tau$ with $\mathrm{d}z$ and $\widehat{\gamma}_1 + \overline{\tau} \widehat{\gamma}_\tau$ with $\mathrm{d}\overline{z}$.
- In weight 2, we have $V = \langle \widehat{T^2} \rangle_{\mathbb{Z}}$, $F^1 V_{\mathbb{C}} = \langle \widehat{T^2} \rangle_{\mathbb{C}}$ and $F^2 V_{\mathbb{C}} = 0$.

Algebraic de Rham cohomology groups and periods

Now assume that the Kähler manifold is algebraic, in which case we write $X(\mathbb{C})$ instead of X, where X is a smooth projective variety defined over some subfield K of $\mathbb{C}$. The cohomology of $X(\mathbb{C})$ has a Hodge decomposition, but its definition is not purely algebraic since it involves complex conjugation, which is not an algebraic operation. For instance, for an elliptic curve E with affine equation $y^2 = x^3 + a\,x + b$, the space $H^{1,0}(E(\mathbb{C}))$ is generated by $\frac{\mathrm{d}x}{y}$, which is algebraic, but the space $H^{0,1}(E(\mathbb{C}))$ is generated by $\frac{\mathrm{d}\overline{x}}{\overline{y}}$, which is not. However, one can also represent $H^1_{\mathrm{dR}}(E(\mathbb{C}))$ by the holomorphic form $\frac{\mathrm{d}x}{y}$ and the meromorphic form $x\,\frac{\mathrm{d}x}{y}$ with vanishing residues. The same works for any curve, i.e. one can represent the first cohomology by algebraic holomorphic differentials (differentials of the first kind) and algebraic meromorphic differentials with vanishing residues (differentials of the second kind). There is still no algebraically defined Hodge decomposition, but the notion of differentials of the first and second kind naturally give a filtration and this agrees with the usual Hodge filtration. The generalization to higher dimensional varieties, where it is less obvious how to proceed, has been given by Grothendieck [39]. He defined the algebraic de Rham cohomology groups $H^k_{\mathrm{dR}}(X)$ as the hypercohomology groups of the algebraic de Rham complex

$$0 \,\to\, \Omega^0_X \,\to \Omega^1_X \,\to\, \ldots \,\to\, \Omega^n_X \,\to\, 0\,.$$

The cohomology groups are now K-vector spaces and Grothendieck proves that there is a canonical isomorphism

$$H^k_{\mathrm{dR}}(X) \otimes_K \mathbb{C} \cong H^k_{\mathrm{dR}}(X(\mathbb{C}))$$

and a Hodge filtration

$$H^k_{\mathrm{dR}}(X) \,=\, F^0 H^k_{\mathrm{dR}}(X) \,\supseteq\, F^1 H^k_{\mathrm{dR}}(X) \,\supseteq\, \ldots \supseteq F^k H^k_{\mathrm{dR}}(X) \,\supseteq\, F^{k+1} H^k_{\mathrm{dR}}(X) \,=\, 0$$

which is compatible with the Hodge filtration of $H^k_{\mathrm{dR}}(X(\mathbb{C}))$.

More generally, if X is defined over any field K, the algebraic de Rham cohomology groups $H^k_{\mathrm{dR}}(X)$ are K-vector spaces and they still have Hodge filtrations. This is particularly useful in the case where K is a number field, to which we restrict now. For every embedding $\sigma : K \hookrightarrow \mathbb{C}$, we then obtain a compact Kähler manifold $X_\sigma(\mathbb{C})$ and canonical isomorphisms

$$H^k_{\mathrm{dR}}(X) \otimes_\sigma \mathbb{C} \,\cong\, H^k_{\mathrm{dR}}(X_\sigma(\mathbb{C})) \,\cong H^k(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}\,.$$

By choosing bases for $H^k(X_\sigma(\mathbb{C}), \mathbb{Q})$ and $H^k_{\mathrm{dR}}(X)$, this gives rise to a complex $b_k(X) \times b_k(X)$ matrix, which we call a period matrix and whose entries we call periods. These periods are periods in the sense of Kontsevich and Zagier [51]. This means that they are integrals of algebraic functions over domains described by algebraic equations or inequalities, where both the integrand and the domain are defined over $\overline{\mathbb{Q}}$. If one considers families of varieties over some parameter space, one obtains period functions. These are integrals of elements of the algebraic de Rham cohomology (varying algebraically with the parameter) over locally constant cycles. More generally, this leads to the notion of variations of Hodge structures, to which we turn after the following example.

To give one example of a period matrix, we consider the elliptic curve E described by the affine equation $y^2 = x^3 - x$. The algebraic de Rham cohomology can be represented by the two differentials $\frac{\mathrm{d}x}{y}$ and $x\,\frac{\mathrm{d}x}{y}$. The first singular homology of $E(\mathbb{C})$ is generated by double covers of the intervals $-1 \le x \le 0$ and $0 \le x \le 1$. An associated period matrix is then given by

$$\begin{pmatrix} \int_{-1}^0 \frac{\mathrm{d}x}{\sqrt{x^3-x}} & \int_{-1}^0 x\,\frac{\mathrm{d}x}{\sqrt{x^3-x}} \\ \int_0^1 \frac{\mathrm{d}x}{i\,\sqrt{x-x^3}} & \int_0^1 x\,\frac{\mathrm{d}x}{i\,\sqrt{x-x^3}} \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(1/4)^2}{2\,\sqrt{2\pi}} & -\frac{\sqrt{2\pi}^3}{\Gamma(1/4)^2} \\ \frac{-i\,\Gamma(1/4)^2}{2\,\sqrt{2\pi}} & -\frac{i\,\sqrt{2\pi}^3}{\Gamma(1/4)^2} \end{pmatrix} = \begin{pmatrix} 2.62205\cdots & 1.19814\cdots \\ -2.62205\cdots i & -1.19814\cdots i \end{pmatrix}.$$

Variations of Hodge structures

It is often useful to consider variations of complex structures. To give an analytic description of these, let $f : X \to B$ be a family of compact Kähler manifolds of dimension n, i.e. B is a complex manifold, X is a connected Kähler manifold of dimension dim $B+n$ and f is a proper holomorphic submersion. Then each fiber X_b with the induced Kähler structure is a compact Kähler manifold of dimension n. By Ehresmann's lemma, $f : X \to B$ is a locally trivial fibration and this allows us to define for any $0 \leq k \leq 2\,n$ a local system $\mathcal{H}^k$ with stalk $\mathcal{H}^k_b = H^k(X_b, \mathbb{Q})$ over b. The sections are given by constant classes of k-dimensional cochains. More precisely, any point $b \in B$ has an open neighborhood U together with a trivialization

$$\begin{aligned} f^{-1}(U) \;&\to\; U \times X_b \\ x \;&\mapsto\; (f(x), \phi(x)) \end{aligned}$$

and the sections over U are given by the functions

$$\begin{aligned} s : U \;&\to\; \bigcup_{b' \in U} H^k(X_{b'}, \mathbb{Q}) \\ b' \;&\mapsto\; (\phi|_{X_{b'}})^*\, h \end{aligned}$$

for any $h \in H^k(X_b, \mathbb{Q})$. From the local system, we obtain a monodromy representation

$$\rho : \pi_1(B, b) \;\to\; \mathrm{Aut}(H^k(X_b, \mathbb{Q}))\,.$$

Tensoring with the sheaf of holomorphic functions, we get a holomorphic bundle $\mathcal{H}^k_{\mathcal{O}_B} = \mathcal{H}^k \otimes_{\mathbb{Q}} \mathcal{O}_B$ whose fiber over b is given by the complexification of $H^k(X_b, \mathbb{Q})$. The Hodge filtrations on these fibers give a decreasing filtration into subbundles

$$\mathcal{H}^k_{\mathcal{O}_B} \;=\; \mathcal{F}^0\mathcal{H}^k_{\mathcal{O}_B} \;\supseteq\; \mathcal{F}^1\mathcal{H}^k_{\mathcal{O}_B} \;\supseteq \cdots \supseteq \mathcal{F}^k\mathcal{H}^k_{\mathcal{O}_B} \;\supseteq\; \mathcal{F}^{k+1}\mathcal{H}^k_{\mathcal{O}_B} \;=\; 0$$

and local computations show that these subbundles are holomorphic. On $\mathcal{H}^k_{\mathcal{O}_B}$, we further have a flat connection ∇ (called the Gauss-Manin connection) which acts trivially on $\mathcal{H}^k$ and as usual derivatives on $\mathcal{O}_B$. Local computations show that the connection ∇ maps sections of $\mathcal{F}^p\mathcal{H}^k_{\mathcal{O}_B}$ to sections of $\mathcal{F}^{p-1}\mathcal{H}^k_{\mathcal{O}_B} \otimes T^*\mathcal{O}_B$. This property is called Griffiths transversality.

The combination of the local system $\mathcal{H}^k$ with the Hodge filtration on $\mathcal{H}^k_{\mathcal{O}_B}$ is abstractly captured by a so-called variation of pure $\mathbb{Q}$-Hodge structures of weight k. This is, by definition, a local system $\mathcal{V}$ of finite-dimensional $\mathbb{Q}$-vector spaces over a complex manifold B together with a filtration of $\mathcal{V}_{\mathcal{O}_B} = \mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_B$ into holomorphic subbundles

$$\ldots \;\supseteq\; \mathcal{F}^{-1}\mathcal{V}_{\mathcal{O}_B} \;\supseteq\; \mathcal{F}^0\mathcal{V}_{\mathcal{O}_B} \;\supseteq\; \mathcal{F}^1\mathcal{V}_{\mathcal{O}_B} \;\supseteq\; \ldots$$

so that all fibers together with the induced filtration are pure $\mathbb{Q}$-Hodge structures of weight k and so that the natural flat connection ∇ on $\mathcal{V}_{\mathcal{O}_B}$ maps sections of $\mathcal{F}^p\mathcal{V}_{\mathcal{O}_B}$ to sections of $\mathcal{F}^{p-1}\mathcal{V}_{\mathcal{O}_B} \otimes T^*\mathcal{O}_B$. Analogously, one defines variations of pure $\mathbb{R}$-Hodge structures and pure $\mathbb{Z}$-Hodge structures.

For us, variations of pure Hodge structures are particularly useful because, to a large extent, they are described by differential equations. To see this, let Ω be a non-trivial holomorphic section of $\mathcal{V}_{\mathcal{O}_B}$. Acting with sufficiently many derivatives, one finds that Ω satisfies a differential equation (which in an algebraic geometric context is called a Picard-Fuchs equation). Assuming that the monodromy representation associated with $\mathcal{V}$ is irreducible, it follows that this representation is given by the monodromy representation associated with the differential equation.

To give a geometric example of a variation of pure Hodge structures, consider the family of complex tori with the fiber over a point τ in the complex upper half-plane $\mathfrak{H}$ given by $\mathbb{C}/\langle\, 1, \tau\,\rangle_{\mathbb{Z}}$. The associated local system $\mathcal{H}^1$ is trivialized by $\widehat{\gamma}_1$ and $\widehat{\gamma}_\tau$. The holomorphic subbundle $\mathcal{F}^1\mathcal{H}^1_{\mathcal{O}_{\mathfrak{H}}}$ is trivialized by $\Omega_\tau = \widehat{\gamma}_1 + \tau\,\widehat{\gamma}_\tau$ and this section satisfies $\nabla^2_\tau \Omega = 0$. Note also that the subbundle $\overline{\mathcal{F}^1\mathcal{H}^1_{\mathcal{O}_{\mathfrak{H}}}}$ of Hodge type $(0,1)$ is not holomorphic.

Mixed Hodge structures

Pure Hodge structures can be generalized to mixed Hodge structures. A mixed $\mathbb{Q}$-Hodge structure is, by definition, a finite-dimensional $\mathbb{Q}$-vector space V with an increasing weight filtration

$$\ldots \subseteq W^{-1}V \subseteq W^0V \subseteq W^1V \subseteq \ldots$$

and a decreasing Hodge filtration of the complexification $V_\mathbb{C} = V \otimes_\mathbb{Q} \mathbb{C}$

$$\ldots \supseteq F^{-1}V_\mathbb{C} \supseteq F^0V_\mathbb{C} \supseteq F^1V_\mathbb{C} \supseteq \ldots$$

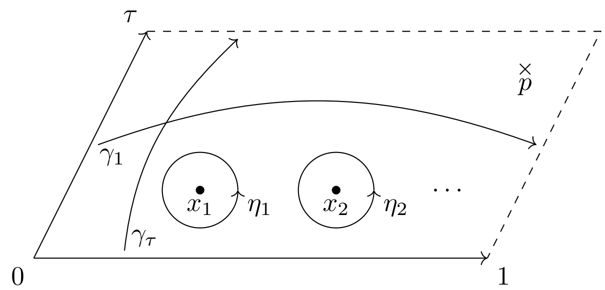
so that every graded piece $W^kV/W^{k-1}V$ with the filtration induced by F defines a pure $\mathbb{Q}$-Hodge structure of weight k. Hence, one can view a mixed Hodge structure as iterated extensions of pure Hodge structures.

Mixed Hodge structures arose from Deligne's discovery [27, 28] that Hodge theory can be generalized to include varieties X defined over $\mathbb{C}$ that are not necessarily smooth or projective, but that in this case the cohomology groups $H^k(X(\mathbb{C}), \mathbb{Q})$ have a canonical mixed rather than pure Hodge structure. These are important occurrences of mixed Hodge structures in algebraic geometry, but there are others, for instance limit mixed Hodge structures associated with variations of pure Hodge structures, to which we come back in Section 4.2.

To give one explicit geometric example of a mixed Hodge structure, we consider a complex torus $T^2 = \mathbb{C}/\langle 1, \tau \rangle_\mathbb{Z}$ minus a finite set of points $\Sigma = \{x_1, x_2, ..., x_n\}$. The associated singular cohomology groups are given by

$$H^0(T^2 \setminus \Sigma, \mathbb{Z}) = \langle \widehat{p} \rangle_\mathbb{Z}\,, \quad H^1(T^2 \setminus \Sigma, \mathbb{Z}) = \langle \widehat{\gamma}_1, \widehat{\gamma}_\tau, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1} \rangle_\mathbb{Z}\,, \quad H^2(T^2 \setminus \Sigma, \mathbb{Z}) = 0$$

in terms of the duals of the classes of the chains depicted below:



We want to define a mixed $\mathbb{Z}$-Hodge structure associated with $H^1(T^2 \setminus \Sigma, \mathbb{Z})$. To do so, we first note that there is an exact sequence

$$0 \to H^1(T^2, \mathbb{Z}) \to H^1(T^2 \setminus \Sigma, \mathbb{Z}) \to H^0(\Sigma, \mathbb{Z}) \to H^2(T^2, \mathbb{Z}) \to 0\,,$$

where the first (non-trivial) map is the pullback of the inclusion $i : T^2 \setminus \Sigma \to T^2$, the second map is the residue map r, which maps $a_1\, \widehat{\eta}_1 + \cdots + a_{n-1}\, \widehat{\eta}_{n-1}$ to $a_1\, \widehat{x}_1 + \cdots + a_{n-1}\, \widehat{x}_{n-1} - (a_1 + \cdots a_{n-1})\, \widehat{x}_n$ and the third map is the pushforward of the inclusion $\Sigma \to T^2$, which maps $a_1\, \widehat{x}_1 + \cdots + a_n\, \widehat{x}_n$ to $(a_1 + \cdots + a_n)\, \widehat{T^2}$. We have a non-canonical isomorphism $H^1(T^2 \setminus \Sigma, \mathbb{Z}) \cong \mathrm{Im}\ i^* \oplus \mathrm{Im}\ r$ and on $H^1(T^2, \mathbb{Z})$ and $H^0(\Sigma, \mathbb{Z})$ we have pure Hodge structures of weight 1 and weight 0. However, to be consistent with the filtration $\mathrm{Im}\ i^* \subseteq H^1(T^2 \setminus \Sigma, \mathbb{Z})$, we shift the weight of $\mathrm{Im}\ r$ from 0 to 2 (and the Hodge type from $(0, 0)$ to $(1, 1)$). Then, a mixed Hodge structure can be defined by

$$\begin{aligned}
W^0H^1(T^2 \setminus \Sigma, \mathbb{Z}) &= 0 \\
W^1H^1(T^2 \setminus \Sigma, \mathbb{Z}) &= \mathrm{Im}\ i^* = \langle \widehat{\gamma}_1, \widehat{\gamma}_\tau \rangle_\mathbb{Z} \\
W^2H^1(T^2 \setminus \Sigma, \mathbb{Z}) &= H^1(T^2 \setminus \Sigma, \mathbb{Z}) = \langle \widehat{\gamma}_1, \widehat{\gamma}_\tau, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1} \rangle_\mathbb{Z}
\end{aligned}$$

and

$$\begin{aligned}
F^0H^1(T^2 \setminus \Sigma, \mathbb{Z})_\mathbb{C} &= H^1(T^2 \setminus \Sigma, \mathbb{Z})_\mathbb{C} = \langle \widehat{\gamma}_1, \widehat{\gamma}_\tau, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1} \rangle_\mathbb{C} \\
F^1H^1(T^2 \setminus \Sigma, \mathbb{Z})_\mathbb{C} &= \langle \widehat{\gamma}_1 + \tau\, \widehat{\gamma}_\tau, \widehat{\eta}_1, \widehat{\eta}_2, \ldots, \widehat{\eta}_{n-1} \rangle_\mathbb{C} \\
F^2H^1(T^2 \setminus \Sigma, \mathbb{Z})_\mathbb{C} &= 0\,.
\end{aligned}$$

2.2 Varieties over number fields and their zeta functions

We now specialize to algebraic varieties which are defined over number fields and discuss their arithmetic properties. More precisely, we consider the number of points of such varieties over finite fields. Generating functions of these numbers are called zeta functions and, according to the Weil conjectures, these functions have remarkable properties. For the proof of the Weil conjectures, one uses cohomology groups on which Galois groups act.

Let X be a smooth projective variety of dimension n defined over some number field K. Since X is given as a subspace of some projective space by equations with coefficients in K, we can reduce these defining equations (after multiplication by an element of $\mathcal{O}_K$ to clear the denominators) modulo any prime $\mathfrak{p}$, leading to a variety $X_\mathfrak{p}$ defined over the finite field $\mathbb{F}_q = \mathcal{O}_K/\mathfrak{p}$ of order q. We restrict to the case that this variety is smooth, which happens for all but finitely many $\mathfrak{p}$, called the primes of good reduction. For any $i \geq 1$, we consider the number $|X_\mathfrak{p}(\mathbb{F}_{q^i})|$ of solutions of the defining equations with the variables taking their values in the field $\mathbb{F}_{q^i}$. The local zeta function of $X_\mathfrak{p}$ is a generating function of these numbers

$$Z(X_\mathfrak{p}, T) \,=\, \exp\left(\sum_{i=1}^{\infty} |X_\mathfrak{p}(\mathbb{F}_{q^i})| \frac{T^i}{i}\right) .$$

A deep theorem says that $Z(X_\mathfrak{p}, T)$ is not just a power series but a rational function in T with integral coefficients. Moreover, Weil conjectured that this rational function has the form

$$Z(X_\mathfrak{p}, T) \,=\, \prod_{k=0}^{2\,n} P_k(X_\mathfrak{p}, T)^{(-1)^{k+1}} ,$$

where $P_k(X_\mathfrak{p}, T)$ is a polynomial of degree $b_k(X)$ with integral coefficients and with all roots of absolute value $q^{-k/2}$ ("local Riemann hypothesis") and satisfies the functional equation

$$P_{2\,n-k}(X_\mathfrak{p}, 1/q^n T) \,=\, \pm P_k(X_\mathfrak{p}, T)/(q^{n/2}T)^{b_k(X)} .$$

He further conjectured that it should be possible to prove this by finding an appropriate cohomology theory for the variety $X_\mathfrak{p}$ defined over $\mathbb{F}_q$. This was later realized through the work of Grothendieck, Artin and others by introducing, in general for any smooth projective variety Y defined over any field L with seperable closure L^{sep}, the ℓ-adic cohomology groups $H^k_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell)$ for primes $\ell \neq \operatorname{char} L$. Here, $\overline{Y}$ stands for the variety Y regarded as a variety over L^{sep}. There is a right action of the Galois group $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ on $\overline{Y}$ (corresponding to the inverse of the left action on $Y(L^{\mathrm{sep}})$) and this action induces a continuous left action on $H^k_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell)$. In the case $Y = X_\mathfrak{p}$ and $L = \mathbb{F}_q$, the Galois group is topologically generated by the arithmetic Frobenius automorphism $\mathrm{Frob}_\mathfrak{p} : x \mapsto x^q$ and we denote the inverse, the so-called geometric Frobenius, by $F_\mathfrak{p}$. The fixed points of the ith power of $\mathrm{Frob}_\mathfrak{p}$ on $X_\mathfrak{p}(\overline{\mathbb{F}_q})$ are precisely the points defined over $\mathbb{F}_{q^i}$. This can be used to relate $|X_\mathfrak{p}(\mathbb{F}_{q^i})|$ to the traces of the Frobenius automorphism, since, as proven by Grothendieck, the Lefschetz trace formula can be applied also to the ℓ-adic cohomology groups and one obtains

$$|X_\mathfrak{p}(\mathbb{F}_{q^i})| \,=\, \sum_{k=0}^{2\,n} (-1)^k \,\mathrm{Tr}\big((F_\mathfrak{p}^*)^i \,|\, H^k_{\text{ét}}(\overline{X_\mathfrak{p}}, \mathbb{Q}_\ell)\big) .$$

A direct consequence is that the local zeta function has the form

$$Z(X_\mathfrak{p}, T) \,=\, \prod_{k=0}^{2\,n} \det\big(1 - T\,F_\mathfrak{p}^* \,|\, H^k_{\text{ét}}(\overline{X_\mathfrak{p}}, \mathbb{Q}_\ell)\big)^{(-1)^{k+1}} .$$

In particular, the product on the right is independent of the chosen prime ℓ. Due to the local Riemann hypothesis, which was proven by Deligne, the same holds for each factor, giving the desired polynomial $P_k(X_\mathfrak{p}, T) \in \mathbb{Z}[T]$.

The considerations above apply to any smooth projective variety defined over $\mathbb{F}_q$ and not only to the reduction $X_\mathfrak{p}$ of a variety X defined over a number field K. However, having a global variety X

defined over K allows us to define the ℓ-adic cohomology group $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ for all primes ℓ. An important fact is that for every embedding $\overline{\sigma} : \overline{K} \hookrightarrow \mathbb{C}$ there is a canonical isomorphism of vector spaces

$$H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; H^k(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \,.$$

Here, σ is the restriction of $\overline{\sigma}$ to K. In particular, this implies that $P_k(X_\mathfrak{p}, T)$ is a polynomial of degree $b_k(X)$. Another important fact is that for all primes $\mathfrak{p}$ of good reduction which are coprime to ℓ and any embedding $\overline{K} \hookrightarrow \overline{K_\mathfrak{p}}$ (where $\overline{K_\mathfrak{p}}$ denotes the algebraic closure of the $\mathfrak{p}$-adic completion of K) there is a canonical isomorphism of representations of $\mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})$

$$H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; H^k_{\text{ét}}(\overline{X_\mathfrak{p}}, \mathbb{Q}_\ell) \,.$$

The Frobenius automorphism $F_\mathfrak{p} \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ thus corresponds to a well-defined conjugacy class in the action of $\mathrm{Gal}(\overline{K}/K)$ on $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$, which we also denote by $F^*_\mathfrak{p}$, and we have

$$P_k(X_\mathfrak{p}, T) \;=\; \det\bigl(1 - T\, F^*_\mathfrak{p} \,|\, H^k_{\text{ét}}(\overline{X_\mathfrak{p}}, \mathbb{Q}_\ell)\bigr) \;=\; \det\bigl(1 - T\, F^*_\mathfrak{p} \,|\, H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)\bigr) \,.$$

The fact that all local zeta functions come from the same variety X allows us to define the Hasse-Weil zeta function

$$\zeta(X, s) \;=\; \prod_{\mathfrak{p}} Z(X_\mathfrak{p}, q^{-s}) \qquad (\mathrm{Re}\; s \gg 0)$$

which may also be written as an alternating product of the L-functions

$$L_k(X, s) \;=\; \prod_{\mathfrak{p}} P_k(X_\mathfrak{p}, q^{-s})^{-1} \qquad (\mathrm{Re}\; s \gg 0) \,.$$

Note that both equations have to be supplemented with definitions of the local factors for primes of bad reduction, but we will not discuss these. One of the most important conjectures in modern arithmetic algebraic geometry is that each L_k has remarkable analytic properties. For example, it is expected that L_k can be analytically continued to a meromorphic function on the complex plane which has a functional equation with respect to the symmetry $s \mapsto k + 1 - s$. For a few varieties, these properties can be proven but in almost all cases they are conjectural. For a more detailed treatment, we refer to [45].

Apart from issues with local L-factors for ramified primes, the L-function $L_k(X, \cdot)$ is determined by any of the representations $\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell))$. This is useful, since we can use powerful theorems about Galois representations. In particular, by a theorem of Faltings [31], the semisimplification of ρ_ℓ is uniquely determined from the characteristic polynomials of finitely many Frobenius elements. For the case of two-dimensional representations with even trace, this has been refined by Serre [63] and Livné [52]. As a result, the so-called Faltings-Serre-Livné method (see theorem 4.3 in [52]) gives a simple method to decide whether two such representations have isomorphic semisimplifications.

To give an example for the structures reviewed above, we consider again the elliptic curve E with affine equation $y^2 = x^3 - x$. This has good reduction for all primes $p \neq 2$ and the local zeta function is

$$Z(E_p, T) \;=\; \frac{1 - a_p\, T + p\, T^2}{(1 - T)\,(1 - p\, T)} \,,$$

where $a_p = 0$ for $p \equiv 3 \bmod 4$ and $a_p = (-1)^{\frac{\alpha+\beta-1}{2}}\, 2\,\beta$ for $p = \alpha^2 + \beta^2$ with even α and odd $\beta > 0$. The numerator of the local zeta function equals the polynomial $\det\bigl(1 - T\, F^*_p \,|\, H^k_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell)\bigr)$ for any prime $\ell \neq p$. In this example (and for elliptic curves in general), there is a simple description of $H^k_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell)$ and its Galois representation, because it is isomorphic to the dual of the Tate module (the inverse limit of the ℓ^n-torsion subgroups $E[\ell^n]$) tensored by $\mathbb{Q}_\ell$. These subgroups are isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})^2$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ naturally acts on them since E is defined over $\mathbb{Q}$.

2.3 Motives

The idea of motives was proposed by Grothendieck to capture the cohomological structure of varieties. We want to briefly explain this idea without going much into detail. For more details, we refer to [3] and [45].

We start by explaining pure Chow motives. Let X be again a smooth projective variety of dimension n defined over some number field K. In the two previous sections, we recalled that for every integer $0 \leq k \leq 2\,n$, we can associate multiple cohomology groups with X:

- For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, we obtain the complex manifold $X_\sigma(\mathbb{C})$ which gives rise to the Betti cohomology group $H^k(X_\sigma(\mathbb{C}), \mathbb{Q})$. The complexification of this has a Hodge filtration.
- Using the algebraic structure, we obtain the algebraic de Rham cohomology group $H^k_{\mathrm{dR}}(X)$ and its Hodge filtration.
- For any prime ℓ, we obtain the ℓ-adic cohomology group $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$. On this, the Galois group $\mathrm{Gal}(\overline{K}/K)$ acts continuously.

We also saw that these have several compatibilities. For example, there are canonical isomorphisms between $H^k(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ and $H^k_{\mathrm{dR}}(X) \otimes_\sigma \mathbb{C}$ and between $H^k(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$.

A motive should capture all of these structures and informally one can think of a motive as an algebraically defined part of the cohomology (so that it makes sense in any reasonable cohomology theory). To make this more precise, let H be any Weil cohomology theory and let X and Y be smooth projective varieties defined over a number field K. Any morphism $f : Y \to X$ induces a linear map $f^* : H(X) \to H(Y)$. However, we can be more general by allowing correspondences of degree zero, i.e. elements of the Chow group

$$\mathrm{Corr}(X, Y) \,=\, CH^{\dim X}(X \times Y)\,.$$

Any element of $\mathrm{Corr}(X, Y)$ gives a class in $H^{2\dim X}(X \times Y)$ and using the Künneth isomorphism and Poincaré duality this also gives a linear map $H(X) \to H(Y)$. The correspondence associated with a morphism $f : Y \to X$ is just the transpose of the graph of f. The composition of correspondences $\gamma_1 \in \mathrm{Corr}(X, Y)$ and $\gamma_2 \in \mathrm{Corr}(Y, Z)$ is defined by

$$\gamma_2 \circ \gamma_1 \,=\, \mathrm{pr}_{X\times Z,*}(\mathrm{pr}^*_{X\times Y}\gamma_1 \cdot \mathrm{pr}^*_{Y\times Z}\gamma_2)$$

in terms of the projections from $X \times Y \times Z$ to products of two factors. One could now define a motive as a pair (X, p), where X is a smooth projective variety defined over a number field K and p is an idempotent element of $\mathrm{Corr}(X, X)$. One also says that p cuts out the motive (X, p). For any Weil cohomology theory H, one then defines the cohomology of (X, p) as the image of the linear map $H(X) \to H(X)$ induced by p. As morphisms from (X, p) to another motive (Y, q), one can consider elements $\gamma \in \mathrm{Corr}(X, Y)$. The associated action on Weil cohomology theories is then obtained from the composition $q \circ \gamma \circ p$. The motives defined in this way are called pure Chow motives. Three simple examples are given as follows:

- The diagonal $\Delta \subset X \times X$ defines the motive (X, Δ), which corresponds to the complete cohomology of X.
- Any point $x \in X(K)$ defines the motive $(X, x \times X)$, which corresponds to the cohomology of weight 0 of X.
- Any point $x \in X(K)$ defines a motive $(X, X \times x)$, which corresponds to the cohomology of weight $2\dim X$ of X.

While the definition of motives as pure Chow motives is very geometric, it can be difficult to work with in practice. For example, even the existence of a weight decomposition of (X, Δ) is only conjectural in general. Thus, we continue to work with the more practical approach followed in section 9.2 of [45]. Here, a pure motive of weight k (and with a fixed number field K) is defined as a collection

$$V \,=\, (V_\sigma, V_{\mathrm{dR}}, V_\ell; I_{\infty,\sigma}, I_{\ell,\overline{\sigma}})\,,$$

where $\sigma, \overline{\sigma}$ run through all embeddings $K, \overline{K} \hookrightarrow \mathbb{C}$ and ℓ runs through all rational primes. Each V_σ is a pure Hodge structure of weight k, V_{dR} is a K-vector space with a decreasing Hodge filtration, each V_ℓ is a $\mathbb{Q}_\ell$-vector space with a continuous action ρ_ℓ by $\mathrm{Gal}(\overline{K}/K)$, $I_{\infty,\sigma}$ is an isomorphism between $V_\sigma \otimes_{\mathbb{Q}} \mathbb{C}$ and $V_{\mathrm{dR}} \otimes_\sigma \mathbb{C}$ that is compatible with the filtrations and $I_{\ell,\overline{\sigma}}$ is an isomorphism between $V_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and V_ℓ. We further require that the Galois representations ρ_ℓ are compatible with each other and have the correct weight. By this, we mean that there is a finite set S of primes so that each ρ_ℓ is unramified outside of the union of S and the prime factors of ℓ and that for unramified primes $\mathfrak{p}$ the characteristic polynomials $\det(1 - T\,\rho_\ell(F_{\mathfrak{p}}))$ are in $\mathbb{Z}[T]$, do not depend on ℓ and have complex roots of absolute value $q^{-k/2}$. We abbreviate the characteristic polynomials by $\det\bigl(1 - T\,F_{\mathfrak{p}}^*|V\bigr)$. Note that the motives defined in this way are purely linear algebraic structures and do not refer to any algebraic variety. However, it is expected that these motives always have a geometric realization, i.e. that they can be realized as a subspace of the cohomology of an algebraic variety. Even stronger, it is expected that they are cut out by correspondences.

An important motive, the so-called Tate motive, is obtained from the cohomology of weight 2 of $\mathbb{P}^1$. The associated period matrix is just $2\pi i$ and the associated Galois representations are just the duals of ℓ-adic cyclotomic characters. Given any pure motive V of weight k, we denote the tensor product with the Tate motive by $V(-1)$. This is also called the (-1)th Tate twist of V. We use the same notation on the level of Galois representations.

Given a smooth projective variety X defined over a number field K, we denote the associated pure motive of weight k by $H^k(X)$. If there is a splitting $H^k(X_\sigma(\mathbb{C}), \mathbb{Q}) = V_{\sigma,1} \oplus V_{\sigma,2}$ with sub-Hodge structures $V_{\sigma,1}$ and $V_{\sigma,2}$, one can ask whether these are always parts of some pure motives $V_1, V_2 \subseteq H^k(X)$. This is suggested by the Hodge conjecture. More precisely, we obtain projectors $\sigma_1, \sigma_2 : H^k(X_\sigma(\mathbb{C}), \mathbb{Q}) \to H^k(X_\sigma(\mathbb{C}), \mathbb{Q})$ and by Poincaré duality and the Künneth isomorphism these give elements in $H^{2\,n}(X_\sigma(\mathbb{C}) \times X_\sigma(\mathbb{C}), \mathbb{Q}) \cap H^{n,n}(X_\sigma(\mathbb{C}) \times X_\sigma(\mathbb{C}))$. The Hodge conjecture predicts that these are the image of some algebraic cycles, which would then give motives $V_1, V_2 \subseteq H^k(X)$ (for a sufficiently large extension of K). In the same way, if there is a splitting $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) = V_{\ell,1} \oplus V_{\ell,2}$ with subrepresentation $V_{\ell,1}$ and $V_{\ell,2}$, one can ask whether these are always parts of pure motives $V_1, V_2 \subseteq H^k(X)$. This is suggested by the Tate conjecture. More precisely, we obtain projectors $\sigma_1, \sigma_2 : H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \to H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ and by Poincaré duality and the Künneth isomorphism these give elements in $H^{2\,n}_{\text{ét}}(\overline{X} \times \overline{X}, \mathbb{Q}_\ell)(n)$. The Tate conjecture predicts that these come from algebraic cycles, which would then give motives $V_1, V_2 \subseteq H^k(X)$. In summary, we obtain the following diagram:

$$\begin{tikzcd}
& \begin{array}{c} H^k(X) \text{ splits into submotives} \\ \text{cut out by correspondences} \end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c} H^k(X_\sigma(\mathbb{C}), \mathbb{Q}) \text{ splits into} \\ \text{sub-Hodge structures} \end{array} \arrow[ur, bend left, "\begin{array}{c}\text{Hodge}\\ \text{conjecture}\end{array}"] & & \begin{array}{c} H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \text{ splits into} \\ \text{subrepresentations} \end{array} \arrow[ul, bend right, "\begin{array}{c}\text{Tate}\\ \text{conjecture}\end{array}"']
\end{tikzcd}$$

There are two important generalizations of our definition of motives. First, one can consider finite extensions of the coefficient fields $\mathbb{Q}$, K and $\mathbb{Q}_\ell$ of the various vector spaces. This can be useful, for example, in the case where one has a pure motive which splits to a sum of pure motives after some finite extension of the coefficient fields. Second, one can generalize to mixed motives. These are collections $V = (V_\sigma, V_{\mathrm{dR}}, V_\ell; I_{\infty,\sigma}, I_{\ell,\overline{\sigma}})$ analogous to the case of pure motives, but now each V_σ is only a mixed Hodge structure and V_{dR} and V_ℓ also carry an increasing weight filtration. The action of the Galois group and the isomorphisms $I_{\infty,\sigma}$ and $I_{\ell,\overline{\sigma}}$ should be compatible with these weight filtrations. As in the case of pure and mixed Hodge structures, one can view a mixed motive as iterated extensions of pure motives.

3 Modular forms and associated motives

Modular forms are functions on the complex upper half-plane with certain analyticity properties and certain transformation properties with respect to an action by subgroups of $\mathrm{SL}_2(\mathbb{R})$. They enjoy remarkable properties and they occur in several areas of mathematics. For us, the most important fact is that one can associate motives with a certain class of modular forms, the so-called newforms.

In the first section, we review the general theory of modular forms and their associated period polynomials. This leads to the definition of Hecke operators, newforms and their associated periods. In the second section, we define merormorphic modular forms, which leads to the definitions of meromorphic partners and quasiperiods of newforms. In the third section, we review that associated with any newform there is a pure motive of rank two. The corresponding traces of Frobenius elements are given by Hecke eigenvalues and the associated period matrices are given by the periods and quasiperiods of the newform. The expositions in these first three sections are based on that of [12], but with several refinements. In the last two sections, we discuss two generalizations of modular forms: Hilbert modular forms and Bianchi modular forms.

3.1 Holomorphic modular forms and periods

In this section, we define the periods associated with modular forms[1] for discrete and cofinite subgroups Γ of $\mathrm{SL}_2(\mathbb{R})$. For us, the relevant examples are the level N subgroups $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$. We start the section by reviewing a few basic facts about these groups and the properties of modular forms. In particular, we define Hecke operators and describe how one can associate period polynomials with modular forms. We conclude by reviewing a method for the computation of period polynomials.

Holomorphic modular forms

We review some elementary facts about holomorphic modular forms. For further details, see e.g. [72] or [23]. For an introduction emphasizing computational aspects, we refer to [66].

The group $\mathrm{SL}_2(\mathbb{R})$ of real 2×2 matrices of determinant 1 acts as usual on the complex upper half-plane $\mathfrak{H} = \{\tau \in \mathbb{C} |\, \mathrm{Im}\,\tau > 0\}$ by $\tau \mapsto g\tau = \frac{a\tau+b}{c\tau+d}$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and this action also extends to $\mathfrak{H} \cup \mathbb{P}^1(\mathbb{R})$. Elements in $\mathrm{SL}_2(\mathbb{R})$ which have exactly one fixed point in $\mathbb{P}^1(\mathbb{R})$ are called parabolic elements and every parabolic element is conjugate to $\pm T$, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Now let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ that is cofinite, i.e. $\Gamma \backslash \mathfrak{H}$ has finite hyperbolic area. The fixed points in $\mathbb{P}^1(\mathbb{R})$ with respect to parabolic elements of Γ are called the cusps of Γ and we denote the union of $\mathfrak{H}$ and the set of cusps of Γ by $\overline{\mathfrak{H}}$. The action of Γ can be restricted to $\overline{\mathfrak{H}}$ and two cusps are said to be equivalent if they are in the same Γ orbit. There are only finitely many equivalence classes of cusps.

For any function $f : \mathfrak{H} \to \mathbb{C}$, any integer k and any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, one writes

$$(f|_k g)(\tau) \;=\; (c\tau + d)^{-k} f(g\tau)$$

and calls $|_k$ the weight k slash operator. For any $k \in \mathbb{Z}$, we define the vector space $M_k(\Gamma)$ of (holomorphic) modular forms by

$$M_k(\Gamma) \;=\; \{f : \mathfrak{H} \to \mathbb{C} \mid f|_k\gamma = f \;\forall\, \gamma \in \Gamma, f \text{ holomorphic on } \overline{\mathfrak{H}}\}\,,$$

where f is said to be holomorphic (vanish) at a cusp fixed by $\pm gTg^{-1} \in \Gamma$ if $(f|_k g)(x + iy)$ is bounded (vanishes) for $y \to \infty$. A modular form $f \in M_k(\Gamma)$ is a cusp form if it vanishes at all

[1] To distinguish these from various generalizations, they are also called elliptic modular forms or classical modular forms.

cusps. We denote the subspace of cusp forms by $S_k(\Gamma)$. The spaces $M_k(\Gamma)$ and hence $S_k(\Gamma)$ are finite-dimensional and there are standard formulas for the dimensions of $M_k(\Gamma)$ and $S_k(\Gamma)$.

Modular forms have Fourier expansions around each cusp, i.e. for a cusp fixed by $\pm gTg^{-1} \in \Gamma$ one finds that $(f|_k g)(\tau+1) = (\pm 1)^k (f|_k g)(\tau)$ and hence there is an expansion

$$(f|_k g)(\tau) \;=\; \sum_m a_{g,m}\, q^m \qquad \text{with} \qquad q \;=\; e^{2\pi i \tau}\,,$$

where, depending on $(\pm 1)^k$, the sum runs over positive integers or positive half integers. If f is a cusp form, we further have $a_{g,0} = 0$. If $T \in \Gamma$, we abbreviate $a_{1,m}$ by a_m and we then have

$$f(\tau) \;=\; \sum_{m=0}^{\infty} a_m\, q^m\,.$$

Holomorphic modular forms for $\Gamma_1(N)$

We are particularly interested in the subgroups

$$\Gamma_0(N) \;=\; \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N\}$$

of level $N \in \mathbb{N}$ and their normal subgroups

$$\Gamma_1(N) \;=\; \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid a, d \equiv 1 \bmod N,\; c \equiv 0 \bmod N\}$$

which are the kernel of the homomorphism $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$. The weight k slash operator gives an action of $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ on the space $M_k(\Gamma_1(N))$ and its subspace $S_k(\Gamma_1(N))$. This gives the decompositions

$$M_k(\Gamma_1(N)) \;=\; \bigoplus_\chi M_k(\Gamma_0(N),\chi) \qquad \text{and} \qquad S_k(\Gamma_1(N)) \;=\; \bigoplus_\chi S_k(\Gamma_0(N),\chi)\,,$$

where χ runs over all characters of $(\mathbb{Z}/N\mathbb{Z})^\times$. To work with these spaces, we generalize the weight k slash operator by

$$(f|_k^\chi g)(\tau) \;=\; \chi(d)^{-1}\,(c\tau+d)^{-k} f(g\tau)\,.$$

Then, one can write

$$M_k(\Gamma_0(N),\chi) \;=\; \{f : \mathfrak{H} \to \mathbb{C} \mid f|_k^\chi \gamma = f \;\forall\, \gamma \in \Gamma_0(N),\; f \text{ holomorphic on } \overline{\mathfrak{H}}\}\,.$$

There is a canonical splitting

$$M_k(\Gamma_0(N),\chi) \;=\; E_k(\Gamma_0(N),\chi) \oplus S_k(\Gamma_0(N),\chi)\,,$$

where $E_k(\Gamma_0(N),\chi)$ has a basis given by so-called Eisenstein series. To describe this basis, let k be a positive integer and let χ_1 and χ_2 be two Dirichlet characters modulo N_1 and N_2 which satisfy $\chi_1(-1)\,\chi_2(-1) = (-1)^k$. We define

$$E_{k,\chi_1,\chi_2}(\tau) \;=\; a_0 + \sum_{m=1}^{\infty} \left(\sum_{n|m} \chi_1(n)\,\chi_2(m/n)\,n^{k-1} \right) q^m\,,$$

where the second sum runs over all positive divisors of m. Here, a_0 equals 0 if $N_1 > 1$ and is given in terms of generalized Bernoulli numbers by $a_0 = -\frac{B_{k,\chi_1}}{2k}$ otherwise. If $k=2$ and $\chi_1 = \chi_2 = 1$, the differences $E_{k,\chi_1,\chi_2}(\tau) - m\,E_{k,\chi_1,\chi_2}(m\tau)$ give elements in $E_2(\Gamma_0(N))$ for any multiple N of m. Otherwise, $E_{k,\chi_1,\chi_2}(m\tau)$ gives an element of $E_2(\Gamma_0(N),\chi_1\chi_2)$ for any multiple N of $m\,N_1\,N_2$. The elements described in this way give a basis of $E_k(\Gamma_0(N),\chi)$.

Hecke operators and Atkin-Lehner involutions

For each $n \in \mathbb{N}$ with $(n, N) = 1$, we define the Hecke operator T_n, acting on $M_k(\Gamma_0(N), \chi)$, as follows. Let

$$\mathcal{M}_{n,N} \;=\; \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) \mid \det(g) = n, c \equiv 0 \bmod N\}\;,$$

where $\mathrm{M}_2(\mathbb{Z})$ denotes the set of integral 2×2 matrices. Note that this set is stabilized under left and right multiplication by any $\gamma \in \Gamma_0(N)$. For $f \in M_k(\Gamma_0(N), \chi)$, we then define

$$f|_k^\chi T_n \;=\; n^{k-1} \sum_{M \in \Gamma_0(N) \backslash \mathcal{M}_{n,N}} f|_k^\chi M\;,$$

where the weight k slash operator on the right is defined as before even though the matrices M do not have determinant 1. The sum is over any set of representatives for the left action of $\Gamma_0(N)$ on $\mathcal{M}_{n,N}$, a convenient choice being

$$\mathcal{M}_n^{[\infty]} \;=\; \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) \mid ad = n,\; 0 \le b < d\}\;.$$

Note that the cardinality of this set equals $\sigma_1(n)$, the sum of divisors of n. In particular, the sum in the definition of the Hecke operators is finite and does not depend on the choice of representatives since f is modular. It is easy to see that $f|_k^\chi T_n$ is again modular since the set $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ is invariant under right multiplication by any $\gamma \in \Gamma_0(N)$. We further see that T_n maps cusp forms to cusp forms. Since $T \in \Gamma_1(N)$, we have a Fourier expansion and if one chooses the representatives $\mathcal{M}_n^{[\infty]}$, one gets a formula for the action of T_n on the Fourier expansion of f. For cusp forms this gives

$$(f|_k^\chi T_n)(\tau) \;=\; \sum_{m=1}^{\infty} \sum_{\substack{r|(m,n) \\ r>0}} \chi(r)\, r^{k-1}\, a_{mn/r^2}\, q^m\;.$$

Since the T_n for different n commute with each other and with their adjoints with respect to a certain scalar product on $S_k(\Gamma_0(N), \chi)$, one can choose a common basis of eigenforms f of $S_k(\Gamma_0(N), \chi)$ such that

$$f|_k^\chi T_n \;=\; \lambda_n f$$

for all n that are coprime to N. From the action of the Hecke operators on the Fourier expansion, one then gets $a_n = \lambda_n a_1$ for $(n, N) = 1$. In particular, for $N = 1$ any eigenform is (up to a multiplicative constant) uniquely determined by its Hecke eigenvalues. For $N > 1$, this is not true in general and a form $f \in S_k(\Gamma_0(N), \chi)$ is a so-called newform if it is uniquely determined by its Hecke eigenvalues and the normalization $a_1 = 1$. All elements of $S_k(\Gamma_0(N), \chi)$ can be written as linear combinations $\sum_i a_i\, f_i(m_i\tau)$ for integers m_i and newforms f_i of levels dividing N.

We remark that for any $f \in M_k(\Gamma_0(N), \chi)$ with Fourier expansion $f(\tau) = \sum_{n=0}^\infty a_n\, q^n$ and any Dirichlet character ψ modulo N_ψ, one can define the twist $f_\psi \in M_k(\Gamma_0(N\, N_\psi^2), \chi\, \psi^2)$ by the Fourier expansion $f_\psi(\tau) = \sum_{n=0}^\infty \psi(n)\, a_n\, q^n$. Further, if f is a newform, then there is a unique newform $f \otimes \psi$ (which is in general not equal to f_ψ) so that for all n which do not divide $N\, N_\psi^2$ the eigenvalue of $f \otimes \psi$ under T_n is $\psi(n)\, a_n$.

In the case of the trivial character, there is a further set of operators on $M_k(\Gamma_0(N))$ that are relevant for us. For any exact divisor Q of N, i.e. $Q|N$ and $(Q, N/Q) = 1$, any element in the set

$$\mathcal{W}_Q \;=\; \frac{1}{\sqrt{Q}} \begin{pmatrix} Q\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & Q\mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{R})$$

normalizes $\Gamma_0(N)$ and the product of any two elements of $\mathcal{W}_Q$ is in $\Gamma_0(N)$. Hence, any $W_Q \in \mathcal{W}_Q$ induces an involution on $\Gamma_0(N)\backslash\mathfrak{H}$ via the action of W_Q on $\mathfrak{H}$. These involutions do not depend on the choice of $W_Q \in \mathcal{W}_Q$ and are called the Atkin-Lehner involutions. They generate a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\ell$, where ℓ is the number of prime factors of N. The subgroup of $\mathrm{SL}_2(\mathbb{R})$ obtained by adjoining all Atkin-Lehner involutions to $\Gamma_0(N)$ is denoted by $\Gamma_0^*(N)$, i.e.

$$\Gamma_0^*(N) \;=\; \bigcup_{\substack{Q|N\\(Q,N/Q)=1}} W_Q\Gamma_0(N)\,.$$

It normalizes $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{R})$ and permutes the cusps of $\Gamma_0(N)$. Each Atkin-Lehner involution on $\Gamma_0(N)\backslash\mathfrak{H}$ induces an involution (also called Atkin-Lehner involution) on the space $M_k(\Gamma_0(N))$ by $f \mapsto f|_k W_Q$, which is again independent of the choice of W_Q. These involutions commute with each other and define a decomposition into eigenspaces $M_k(\Gamma_0(N)) = \bigoplus_\epsilon M_k^\epsilon(\Gamma_0(N))$, where the sum ranges over the characters of $(\mathbb{Z}/2\mathbb{Z})^\ell$. Since the Atkin-Lehner involutions also commute with the Hecke operators, every newform automatically belongs to one of these eigenspaces.

Eichler integrals and period polynomials

We consider the normalized derivative $D = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}$, where the factor $\frac{1}{2\pi i}$ is introduced so that D sends periodic functions with rational Fourier coefficients to periodic functions with rational Fourier coefficients. The operator D does not preserve modularity. Instead, we have the following elementary but not obvious proposition.

Proposition 1 (Bol's identity [11])**.** *For all meromorphic functions $f : \mathfrak{H} \to \mathbb{P}^1(\mathbb{C})$, integers $k \geq 2$ and $g \in \mathrm{SL}_2(\mathbb{R})$, we have*

$$D^{k-1}(f|_{2-k}\, g) \;=\; (D^{k-1}f)|_k\, g\,.$$

If f is modular of weight k on some group Γ, then any holomorphic function $\widetilde{f} : \mathfrak{H} \to \mathbb{C}$ with the property that $D^{k-1}\widetilde{f} = f$ is called an Eichler integral of f. The Eichler integral exists, but is well–defined only up to a degree $k-2$ polynomial $p \in V_{k-2}(\mathbb{C})$, where $V_{k-2}(K) = \langle 1, \ldots, \tau^{k-2}\rangle_K$. For instance, we can take $\widetilde{f}$ to be $\widetilde{f}_{\tau_0}$, where

$$\widetilde{f}_{\tau_0}(\tau) \;=\; \frac{(2\pi i)^{k-1}}{(k-2)!}\int_{\tau_0}^{\tau}(\tau - z)^{k-2}\, f(z)\,\mathrm{d}z$$

for any $\tau_0 \in \mathfrak{h}$, or even $\tau_0 \in \overline{\mathfrak{h}}$ if f is a cusp form. In particular, if $T \in \Gamma$, then we have

$$\widetilde{f}_\infty(\tau) \;=\; \sum_{m=1}^{\infty}\frac{a_m}{m^{k-1}}\, q^m \quad \text{if} \quad f(\tau) \;=\; \sum_{m=1}^{\infty} a_m\, q^m \;\in S_k(\Gamma)\,.$$

For later purposes, we observe that $\widetilde{f}_\infty$ is related to $\widetilde{f}_{\tau_0}$ for any $\tau_0 \in \mathfrak{h}$ by

$$\widetilde{f}_\infty(\tau) - \widetilde{f}_{\tau_0}(\tau) \;=\; \frac{(2\pi i)^{k-1}}{(k-1)!}\int_{\tau_0}^{\tau_0-1} B_{k-1}(\tau - z)\, f(z)\,\mathrm{d}z\,,$$

where B_n is the nth Bernoulli polynomial. Indeed, using that $B_n(x+1) = B_n(x) + nx^{n-1}$ and that f is periodic, we find that this equation does not depend on τ_0 and since it is true for $\tau_0 = \infty$ it is true for all τ_0.

For a fixed choice of Eichler integral $\tilde{f}$, it follows from Bol's identity that for all $\gamma \in \Gamma$

$$r_f(\gamma) \;:=\; \widetilde{f}|_{2-k}(\gamma - 1)(\tau) \;\in\; V_{k-2}(\mathbb{C})\,,$$

i.e. $r_f(\gamma)$ is a polynomial of degree $k-2$, which is called a period polynomial of f for $\gamma \in \Gamma$. Here, we extended the action of the slash operator to the group algebra $\mathbb{C}[\mathrm{SL}_2(\mathbb{R})]$ in the obvious way (by setting $f|_k \sum g_i = \sum f|_k g_i$, where we write $\sum g_i$ instead of the more correct $\sum[g_i]$). The period polynomials measure the failure of modularity of the Eichler integral. An immediate consequence of the definition is that the period polynomials satisfy the cocycle condition

$$r_f(\gamma\gamma') \;=\; r_f(\gamma)|_{2-k}\gamma' + r_f(\gamma')\,,$$

where we define an action of $\mathrm{SL}_2(\mathbb{R})$ on $V_{k-2}(\mathbb{C})$ by extending the slash operator to polynomials in the obvious way.

Since the Eichler integral $\widetilde{f}$ is unique only up to the addition of polynomials $p \in V_{k-2}(\mathbb{C})$, it follows that r_f is unique only up to the addition of maps of the form $\gamma \mapsto p|_{2-k}(\gamma - 1)$ with polynomials $p \in V_{k-2}(\mathbb{C})$. The dependence on p is described in terms of group cohomology. Let K be any field so that Γ is contained in $\mathrm{SL}_2(K)$. We define the group of cocycles

$$Z^1(\Gamma, V_{k-2}(K)) \;=\; \{r : \Gamma \to V_{k-2}(K) \mid r(\gamma\gamma') = r(\gamma)|_{2-k}\gamma' + r(\gamma') \;\forall\, \gamma, \gamma' \in \Gamma\}$$

and the group of coboundaries

$$B^1(\Gamma, V_{k-2}(K)) \;=\; \{\Gamma \ni \gamma \mapsto p|_{2-k}(\gamma - 1) \mid p \in V_{k-2}(K)\}\ .$$

Then, the (first) group cohomology is defined as the quotient

$$H^1(\Gamma, V_{k-2}(K)) \;=\; \frac{Z^1(\Gamma, V_{k-2}(K))}{B^1(\Gamma, V_{k-2}(K))}\ .$$

It follows from the definition of r_f that the freedom in the choice of the Eichler integral $\widetilde{f}$ results in a coboundary. Therefore, we can associate to f a unique cohomology class $[r_f] \in H^1(\Gamma, V_{k-2}(\mathbb{C}))$. Furthermore, we define the group of parabolic cocycles

$$Z^1_{\mathrm{par}}(\Gamma, V_{k-2}(K)) \;=\; \{r \in Z^1(\Gamma, V_{k-2}(K)) \mid r(\gamma) \in V_{k-2}(K)|_{2-k}(\gamma - 1) \;\forall \text{ parabolic } \gamma \in \Gamma\}\ .$$

Trivially, one has $B^1 \subseteq Z^1_{\mathrm{par}} \subseteq Z^1$. Hence, one can define the parabolic cohomology group

$$H^1_{\mathrm{par}}(\Gamma, V_{k-2}(K)) \;=\; \frac{Z^1_{\mathrm{par}}(\Gamma, V_{k-2}(K))}{B^1(\Gamma, V_{k-2}(K))} \;\subseteq\; H^1(\Gamma, V_{k-2}(K))\,,$$

where the codimension of the embedding is in general less than or equal to the number of equivalence classes of cusps times the dimension of $V_{k-2}(K)$. We have the following proposition.

Proposition 2. *For any $f \in S_k(\Gamma)$ one has $r_f \in Z^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))$.*

Proof. First note that the statement does not depend on the choice of Eichler integral $\widetilde{f}$ of f. Hence, we only have to show that for any parabolic $\gamma \in \Gamma$ there is a choice of Eichler integral so that $r_f(\gamma) = 0$. We can write $\gamma = \pm gTg^{-1} \in \Gamma$ for some $g \in \mathrm{SL}_2(\mathbb{R})$. Then we have a Fourier expansion

$$(f|_k g)(\tau) \;=\; \sum_m a_{g,m}\, q^m\,,$$

where $a_{g,0}$ vanishes since f is a cusp form. The function

$$\left(\sum_m \frac{a_{g,m}}{m^{k-1}} q^m\right)\bigg|_{2-k} g^{-1}$$

is annihilated by $\gamma - 1$ and, using Bol's identity, we find that it is an Eichler integral of f. □

The importance of the parabolic cohomology group stems from a theorem due to Eichler. To state this, we define the space of $\overline{S_k(\Gamma)}$ of antiholomorphic cusp forms as the space of all functions $\overline{f}$ for $f \in S_k(\Gamma)$, where we define $\overline{f}(\tau) = \overline{f(\tau)}$.

Theorem 1 (Eichler-Shimura isomorphism)**.** *The map $f \mapsto [r_f]$ and its conjugate $\overline{f} \mapsto [r_{\overline{f}}] := [\overline{r_f}]$ (obtained by complex conjugating the coefficients) induce an isomorphism*

$$H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C})) \;\cong\; S_k(\Gamma) \oplus \overline{S_k(\Gamma)}\ .$$

Proof. For even k, a first result of this type was given by Eichler in [30], who in particular showed that the dimensions of both sides agree. For the complete proof for even and odd k, we refer to Shimura [65]. □

We now assume that Γ is normalized by $\varepsilon = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$. We then get an involution $r \mapsto r|_{2-k}\varepsilon$ on $Z^1(\Gamma, V_{k-2}(K))$, where we define the action of any normalizer $W \in \mathrm{GL}_2(K)$ of Γ on elements in $Z^1(\Gamma, V_{k-2}(K))$ by

$$(r|_{2-k}W)(\gamma) \;=\; r(W\gamma W^{-1})|_{2-k}W\,.$$

Here, we generalize that the slash operator acts on polynomials as defined previously even when the determinant of W is negative. The eigenvalues of the involution are ± 1 and we get an induced decomposition

$$H^1_{\mathrm{par}}(\Gamma, V_{k-2}(K)) \;=\; H^1_{\mathrm{par}}(\Gamma, V_{k-2}(K))^+ \oplus H^1_{\mathrm{par}}(\Gamma, V_{k-2}(K))^-\,.$$

It is straightforward to verify that, with respect to the Eichler-Shimura isomorphism, the action of the involution ε on $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))$ corresponds to the involution on $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ induced by $f \mapsto (-1)^{k-1}f^*$, where $f^*(\tau) = f(-\overline{\tau})$. In particular, the restriction of period polynomials to the eigenspaces $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(K))^{\pm}$ gives isomorphisms

$$S_k(\Gamma) \cong H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))^{\pm}\,.$$

We now fix $\Gamma = \Gamma_0(N)$ and a character χ of $(\mathbb{Z}/N\mathbb{Z})^\times$. The Eichler-Shimura isomorphism then gives an isomorphism

$$H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(\mathbb{C})) \;\cong\; S_k(\Gamma_0(N),\chi) \oplus \overline{S_k(\Gamma_0(N),\overline{\chi})}\,,$$

where $V^{\chi}_{k-2}(\mathbb{C})$ corresponds to the representation $V_{k-2}(\mathbb{C})$ twisted by χ. Since any $S_k(\Gamma_0(N),\chi)$ admits an action by Hecke operators, the isomorphism induces an action of the Hecke operators on $H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(\mathbb{C}))$. This can be described as follows. For a map $r : \Gamma_0(N) \to V^{\chi}_{k-2}(K)$ and for $n \in \mathbb{N}$ with $(n, N) = 1$, we define a map $r|^{\chi}_{2-k}T_n : \Gamma_0(N) \to V^{\chi}_{k-2}(K)$ by

$$(r|^{\chi}_{2-k}T_n)(\gamma) \;=\; \sum_{i=1}^{\sigma_1(n)} r(\gamma_i)|^{\chi}_{2-k}M_{\pi_\gamma(i)}\,,$$

where M_i, $i = 1,\ldots,\sigma_1(n)$ are chosen representatives of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ and the $\gamma_i \in \Gamma_0(N)$ are determined by the identity

$$M_i\gamma \;=\; \gamma_i M_{\pi_\gamma(i)}\,.$$

Here, $\pi_\gamma(i)$ denotes a permutation of the indices $i = 1,\ldots,\sigma_1(n)$. Using the cocycle property, it is straightforward to show that this map can be restricted to Z^1, Z^1_{par} and B^1. Further, the map depends on the choice of representatives of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$, but we have the following propositions.

Proposition 3. *For any $r \in Z^1(\Gamma_0(N), V^{\chi}_{k-2}(K))$ the cohomology class $[r|^{\chi}_{2-k}T_n]$ does not depend on the chosen representatives of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$.*

Proof. Let $r|^{\chi}_{2-k}T'_n$ be defined with respect to a second choice M'_i, $i = 1,...,\sigma_1(n)$ of representatives of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$. We order these so that $M'_i = \gamma'_i M_i$ for uniquely determined $\gamma'_i \in \Gamma_0(N)$. By using the cocycle property, one finds that for all $\gamma \in \Gamma_0(N)$

$$(r|^{\chi}_{2-k}T'_n - r|^{\chi}_{2-k}T_n)(\gamma) \;=\; \Big(\sum_{i=1}^{\sigma_1(n)} r(\gamma'_i)|^{\chi}_{2-k}M_i\Big)|^{\chi}_{2-k}(\gamma - 1)$$

and thus $[r|^{\chi}_{2-k}T'_n] = [r|^{\chi}_{2-k}T_n]$. □

Proposition 4. *For $f \in S_k(\Gamma_0(N),\chi)$ we have*

$$r_{f|^{\chi}_{k}T_n} \;=\; r_f|^{\chi}_{2-k}T_n\,,$$

where the same set of representatives of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ has been chosen on both sides and the Eichler integral on the left side has been chosen as $\widetilde{f|^{\chi}_{k}T_n} = n^{k-1}\widetilde{f}|^{\chi}_{2-k}T_n$.

Proof. Using Bol's identity, we find that

$$D^{k-1}(n^{k-1}\widetilde{f}|^{\chi}_{2-k}T_n) \;=\; (D^{k-1}\widetilde{f})|^{\chi}_{k}T_n \;=\; f|^{\chi}_{k}T_n$$

and thus our choice of the Eichler integral is indeed valid. We then get

$$\begin{aligned} r_{f|^{\chi}_{k}T_n}(\gamma) \;&=\; \widetilde{f|^{\chi}_{k}T_n}\Big|^{\chi}_{2-k}(\gamma-1) \;=\; n^{k-1}\,\widetilde{f}|^{\chi}_{2-k}T_n\Big|^{\chi}_{2-k}(\gamma-1) \\ &=\; \sum_{i=1}^{\sigma_1(n)} \widetilde{f}|^{\chi}_{2-k}(M_i\gamma - M_i) \;=\; \sum_{i=1}^{\sigma_1(n)} \widetilde{f}|^{\chi}_{2-k}(\gamma_i M_{\sigma_\gamma(i)} - M_i) \\ &=\; \sum_{i=1}^{\sigma_1(n)} r_f(\gamma_i)|^{\chi}_{2-k}M_{\sigma_\gamma(i)}\,. \end{aligned}$$

□

We conclude that the action of Hecke operators defined on cocycles induces a well-defined action on $H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K))$ which does not depend on the chosen representatives of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ and is compatible with the Eichler-Shimura isomorphism for $K=\mathbb{C}$. Completely analogously, we can define the action of Atkin-Lehner operators W_Q on $Z^1(\Gamma_0(N), V_{k-2}(K))$ (for suitable K) by $r \mapsto r|_{2-k}W_Q$. This gives a well-defined action on $H^1_{\mathrm{par}}(\Gamma_0(N), V_{k-2}(K))$ which does not depend on the chosen element of $\mathcal{W}_Q$ and is compatible with the Eichler-Shimura isomorphism for $K=\mathbb{C}$.

We conclude this introduction to period polynomials with an important proposition about the period polynomials associated with newforms.

Proposition 5. *Let $f \in S_k(\Gamma_0(N),\chi)$ be a newform and let K_f be the number field generated by the Hecke eigenvalues of f. Then the Eichler integral can be chosen such that*

$$r_f \;\in\; \omega_f^+ Z^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_f))^+ \oplus \omega_f^- Z^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_f))^-$$

for some $\omega_f^{\pm} \in \mathbb{C}$. If K_f is totally real, one has $\omega_f^+ \in \mathbb{R}$ and $\omega_f^- \in i\mathbb{R}$.

Proof. Let $K\chi$ be the number field generated by the image of χ. First note that

$$H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(\mathbb{C})) \cong H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_\chi)) \otimes_{K_\chi} \mathbb{C}$$

and that the Hecke operators and the involution ε act on $H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K\chi))$. Since f is uniquely determined by its Hecke eigenvalues in K_f (which contains $K\chi$, see e.g. corollary 3.6 in [16]), we can define two one-dimensional eigenspaces $V^{\pm} \subseteq H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_f))^{\pm}$ with the same eigenvalues as $f \pm (-1)^{k-1}f^*$. Then, the first statement directly follows. If K_f is totally real, we have $f^* = \overline{f}$ and then the second statement also follows. □

We call the numbers $\omega_f^{\pm}$, which are unique only up to multiplication by K_f, the periods of f. For $N=1$ the proposition was first proved by Manin [54].

Computation of $H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_\chi))$

We conclude this section by explaining how one can compute a basis of $H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_\chi))$ and the action of various operators on this space. As for the computation of the first group cohomology of any finitely presented group, this reduces to linear algebra once we have good control over the group $\Gamma_0(N)$.

We start by discussing the case $N=1$ corresponding to the full modular group $\mathrm{SL}_2(\mathbb{Z})$. A presentation for $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\mathrm{SL}_2(\mathbb{Z}) \;=\; \langle S, T \mid S^4 \;=\; 1\,,\; (S\,T)^3 \;=\; S^2\rangle$$

with

$$S \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Any cocycle $r \in Z^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}(\mathbb{Q}))$ is determined by its values on S and T. Conversely, given elements $v_S, v_T \in V_{k-2}(\mathbb{Q})$, we can define a cocycle r on the free group generated by symbols S and T by setting $r(S) = v_S$, $r(T) = v_T$ and $r(g_1\, g_2) = r(g_1)|_{2-k}\, g_2 + r(g_2)$. This gives a well defined cocycle on $\mathrm{SL}_2(\mathbb{Z})$ if the relations $r(S^4) = 0$ and $r((S\,T)^3) = r(S^2)$ are satisfied. Since the action of S^2 on $V_{k-2}(\mathbb{Q})$ is trivial, we can identify $Z^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}(\mathbb{Q}))$ with the kernel of the map

$$\begin{aligned} V_{k-2}(\mathbb{Q}) \oplus V_{k-2}(\mathbb{Q}) \;&\rightarrow\; V_{k-2}(\mathbb{Q}) \oplus V_{k-2}(\mathbb{Q}) \\ (v_S, v_T) \;&\mapsto\; (v_S|_{2-k}\, S + v_S, (v_S|_{2-k}\, T + v_T)|_{2-k}((S\,T)^2 + S\,T + 1))\,. \end{aligned}$$

Similarly, the space $B^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}(\mathbb{Q}))$ can be identified with the image of the map

$$\begin{aligned} V_{k-2}(\mathbb{Q}) \;&\rightarrow\; V_{k-2}(\mathbb{Q}) \oplus V_{k-2}(\mathbb{Q}) \\ v \;&\mapsto\; (v|_{2-k}(S-1), v|_{2-k}(T-1))\,. \end{aligned}$$

To compute $H^1(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}(\mathbb{Q}))$ it only remains to take the quotient of these two subspaces of $V_{k-2}(\mathbb{Q}) \oplus V_{k-2}(\mathbb{Q})$. Then, since all cusps of $\mathrm{SL}_2(\mathbb{Z})$ are equivalent to ∞, the parabolic subspace $H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}(\mathbb{Q}))$ can be obtained by restricting to elements whose restriction to $H^1(\langle T\rangle, V_{k-2}(\mathbb{Q}))$ vanishes. Concretely, this just means that one restricts to classes which can be represented by cocycles r which satisfy $r(T) \in V_{k-2}(\mathbb{Q})|_{2-k}(T-1)$. To evaluate the cocycles (corresponding to elements of $V_{k-2}(\mathbb{Q}) \oplus V_{k-2}(\mathbb{Q})$) on arbitrary elements of $\mathrm{SL}_2(\mathbb{Z})$, and to compute the action of the involution ε and the Hecke operators, it just remains to be able to write any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ as a word in S and T. This can be done inductively. If $|a| \geq |c|$, one multiplies from the left by suitable powers of T to achieve that $|a| < |c|$. Then, one multiplies from the left by S and repeats these steps until $c = 0$. Up to a sign, the resulting matrix is a power of T and the sign can be changed by multiplying with S^2.

For the computation of $H^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_\chi))$, one option is to use Shapiro's lemma to relate this space to the group cohomology of $\mathrm{SL}_2(\mathbb{Z})$ and the induced representation. Another option is to directly generalize the algorithm given above. This is straightforward, once:

- we have a finite presentation of the abstract group $\Gamma_0(N)$ and matrices corresponding to the abstract generators
- we can write any matrix in $\Gamma_0(N)$ as a word in the generators
- we have generators of the stabilizer group of representatives of all equivalence classes of cusps

We now sketch how this can be achieved. As a first step, we compute a set R of coset representatives for $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$ by using the bijection

$$\begin{aligned} \Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z}) \;&\rightarrow\; \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \\ \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \;&\mapsto\; (c:d) \end{aligned}$$

and the Chinese remainder theorem for the computation of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. After choosing lifts from the quotient $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z})$, this gives a possible choice for R. Now any equivalence class of cusps contains $r\cdot\infty$ for some $r \in R$ (but these are in general not all inequivalent). Writing $r = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the stabilizer group of that cusp is then generated by -1 and $r\,T^h\,r^{-1}$, where $h = N/(N, c^2)$ is the so-called width of the cusp. To obtain generators of $\Gamma_0(N)$, we follow a procedure outlined for example in [66]. For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and any $r \in R$, there is a unique $\alpha_r \in R$ such that $r\,\gamma\,\alpha_r^{-1}$ lies in $\Gamma_0(N)$. This allows us to define the map $f_\gamma : R \rightarrow \Gamma_0(N)$ which maps r to $r\,\gamma\,\alpha_r^{-1}$. A set of generators of $\Gamma_0(N)$ is then given by $f_S(R) \cup f_T(R)$. This fact can be proven constructively in the sense that one gives a method to write any element of $\Gamma_0(N)$, given as a word in S and T, as a word in the elements of $f_S(R) \cup f_T(R)$. Following the steps of the previous paragraph, we can also write the elements of $f_S(R) \cup f_T(R)$ as words in S and T. Now we are left with a well studied problem in computational group theory. Namely, we have a finitely presented group G and a finite index subgroup $H \subseteq G$ specified by generators given by words in the generators of G and we want a finite presentation of H with generators given as words in the generators of G and a method to express elements of H (given as words in the generators of G) as words in the generators of the finite presentation of H. This can be done using the Reidemeister-Schreier rewriting process. For our computations, we use the implementation in Magma [13].

3.2 Meromorphic modular forms and quasiperiods

In the previous subsection, we have seen that there is the Eichler-Shimura isomorphism

$$H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C})) \;\cong\; S_k(\Gamma) \oplus \overline{S_k(\Gamma)}\,.$$

For the case $k = 2$, this corresponds to the usual Hodge decomposition $H^1 = H^{1,0} \oplus H^{0,1}$ for complex curves and the complex conjugation makes this decomposition non-algebraic. In this case, the algebraic version of H^1 can be realized by holomorphic differentials (differentials of the first kind) and meromorphic differentials with vanishing residues (differentials of the second kind). The integration of these forms gives a well-defined pairing with the homology. Taking the quotient by derivatives of meromorphic forms one obtains a space that is isomorphic to H^1 and defined algebraically. Instead of the Hodge decomposition, we then have a filtration into classes that can be represented by differentials of the first and second kind, respectively. In this section, we discuss the algebraic analogue of the Eichler-Shimura isomorphism by considering meromorphic modular forms. This will allow us to define quasiperiods as the periods of certain meromorphic modular forms. The theory of meromorphic cusp forms and their associated period polynomials was first introduced by Eichler [30] and later independently rediscovered by Brown [15] and Zagier in the context of [37]. We conclude this section by explaining concrete computations of spaces of meromorphic modular forms.

Meromorphic cusp forms and their period polynomials

We want to extend the period map $r : S_k(\Gamma) \to H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))$ to the space of meromorphic modular forms

$$M_k^{\mathrm{mero}}(\Gamma) \;=\; \{F : \overline{\mathfrak{H}} \to \mathbb{P}^1(\mathbb{C}) \mid F \text{ meromorphic and } F|_k\gamma = F \;\forall \gamma \in \Gamma\}\,.$$

However, to have an Eichler integral, we need to restrict to forms that are $(k-1)$th derivatives. By simple connectivity, it is enough to require that they are locally $(k-1)$th derivatives and we thus define

$$S_k^{\mathrm{mero}}(\Gamma) \;=\; \{F \in M_k^{\mathrm{mero}}(\Gamma) \mid F \text{ is locally a } (k-1)\text{th derivative}\}\,.$$

Concretely, this means that for each $\tau_0 \in \mathfrak{H}$, the coefficients of $(\tau - \tau_0)^n$ in the Laurent expansion around τ_0 vanish for $n = -1, ..., -(k-1)$ and that for each cusp the constant coefficient in the Fourier expansion around that cusp vanishes. For any $F \in S_k^{\mathrm{mero}}(\Gamma)$, one can then choose an Eichler integral $\widetilde{F}$, i.e. a meromorphic function such that $D^{k-1}\widetilde{F} = F$, and compute the period polynomials $r_F(\gamma) = \widetilde{F}|_{2-k}(\gamma - 1)(\tau)$ for $\gamma \in \Gamma$. These are polynomials by Bol's identity and as in the case of holomorphic cusp forms one finds that they define parabolic cocycles and induce a well-defined class $[r_F] \in H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))$ which does not depend on the choice of Eichler integral. Bol's identity also implies that $D^{k-1}M_{2-k}^{\mathrm{mero}}(\Gamma) \subseteq S_k^{\mathrm{mero}}(\Gamma)$ and of course the classes in $H^1(\Gamma, V_{k-2}(\mathbb{C}))$ associated with elements in $D^{k-1}M_{2-k}^{\mathrm{mero}}(\Gamma)$ are trivial. This motivates introducing the quotient

$$\mathbb{S}_k(\Gamma) \;=\; S_k^{\mathrm{mero}}(\Gamma)/D^{k-1}M_{2-k}^{\mathrm{mero}}(\Gamma)\,.$$

Note that the Riemann-Roch theorem implies that one can choose the representatives to have poles only in an arbitrary non-zero subset of $\overline{\mathfrak{H}}$ closed under the action of Γ, for instance the set of all cusps (if there are cusps) or the set of cusps equivalent to ∞ (if ∞ is a cusp). For suitable Γ, we therefore have canonical isomorphisms

$$\mathbb{S}_k^{[\infty]}(\Gamma) \;\cong\; \mathbb{S}_k^{!}(\Gamma) \;\cong\; \mathbb{S}_k(\Gamma)\,,$$

where the first two spaces are defined similar to $\mathbb{S}_k(\Gamma)$ but restricting to forms with possible poles only at $[\infty]$ or only at the cusps, respectively.

In the following, we explain that the period map gives an isomorphism between the spaces $\mathbb{S}_k(\Gamma)$ and $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))$. We start by defining a useful pairing.

Proposition 6. *There is a pairing* $\{\ ,\ \} : S_k^{\mathrm{mero}}(\Gamma) \times S_k^{\mathrm{mero}}(\Gamma) \to \mathbb{C}$ *defined by*

$$\{F, G\} \;=\; (2\pi i)^k \sum_{\tau \in \Gamma\backslash\overline{\mathfrak{H}}} \mathrm{Res}_\tau(\widetilde{F} G \, \mathrm{d}\tau)\,.$$

This pairing is $(-1)^{k+1}$*–symmetric and descends to* $\mathbb{S}_k(\Gamma) \times \mathbb{S}_k(\Gamma)$.

Proof. First note that the definition of $\{F, G\}$ makes sense because the sum is finite (only finitely many orbits have poles) and the individual residues do not depend on the choice of τ in the Γ-orbit (the difference of the residues at τ and $\gamma\tau$ is $r_F(\gamma)G\,\mathrm{d}\tau$ which cannot have any residues since G is a $(k-1)$th derivative and $r_F(\gamma)$ is a polynomial of degree at most $k-2$). Similarly, the pairing does not depend on the choice of Eichler integral since $\widetilde{F}$ is unique up to a polynomial p of degree $k-2$ and $p\,G\,\mathrm{d}\tau$ again has no residues. The $(-1)^{k+1}$–symmetry follows since $\widetilde{F}G - (-1)^{k+1}F\widetilde{G}$ is a derivative. Using this symmetry, it just remains to prove that $\{F, G\}$ vanishes when F is in $D^{k-1}M_{2-k}^{\mathrm{mero}}(\Gamma)$. This is clear since one can choose $\widetilde{F}$ to be in $M_{2-k}^{\mathrm{mero}}(\Gamma)$ and then $\widetilde{F}G\,\mathrm{d}\tau$ is a well-defined meromorphic differential on the compact curve $\Gamma\backslash\overline{\mathfrak{H}}$ and hence the sum of its residues vanishes. $\square$

Theorem 2. *The map* $S_k(\Gamma) \to \mathbb{S}_k(\Gamma)$ *induced by inclusion and the map* $F \mapsto \{F,\ \}$ *give a short exact sequence*

$$0 \longrightarrow S_k(\Gamma) \longrightarrow \mathbb{S}_k(\Gamma) \xrightarrow{\{\ ,\ \}} S_k(\Gamma)^\vee \longrightarrow 0\,,$$

where $S_k(\Gamma)^\vee$ *denotes the dual space of* $S_k(\Gamma)$.

Proof. The first (non-trivial) map is injective since the period polynomial of a holomorphic cusp form determines the form uniquely. The composite of the first two maps is trivial since holomorphic functions have no poles. Eichler [30] shows that the kernel of the second map is exactly the image of the first map and that the second map is surjective. $\square$

This theorem implies that $\mathbb{S}_k(\Gamma)$ is (non-canonically) isomorphic to $S_k(\Gamma) \oplus S_k(\Gamma)^\vee$. Hence, the domain and the codomain of the period map $r : \mathbb{S}_k(\Gamma) \to H^1_{par}(\Gamma, V_{k-2}(\mathbb{C}))$ have the same dimension and since the map is injective it gives an isomorphism

$$\mathbb{S}_k(\Gamma) \;\cong\; H^1_{par}(\Gamma, V_{k-2}(\mathbb{C}))\,.$$

From now on we restrict to $\Gamma = \Gamma_1(N)$. As in the case of holomorphic modular forms, we have a decomposition

$$M_k^{\mathrm{mero}}(\Gamma_1(N)) \;=\; \bigoplus_\chi M_k^{\mathrm{mero}}(\Gamma_0(N), \chi)\,,$$

where χ runs over all characters of $(\mathbb{Z}/N\mathbb{Z})^\times$. Analogously, we obtain decompositions of the two spaces $S_k^{\mathrm{mero}}(\Gamma_1(N))$ and $\mathbb{S}_k(\Gamma_1(N))$. In particular, the period map gives an isomorphism

$$\mathbb{S}_k(\Gamma_0(N), \chi) \;\cong\; H^1_{par}(\Gamma_0(N), V^\chi_{k-2}(\mathbb{C}))\,.$$

We define the action of Hecke operators on $M_k^{\mathrm{mero}}(\Gamma_0(N), \chi)$ in the same way as we did for holomorphic modular forms. By Bol's identity, it follows that the action descends to $\mathbb{S}_k(\Gamma_0(N), \chi)$ and it is easy to see that this action is compatible with the isomorphism between $\mathbb{S}_k(\Gamma_0(N), \chi)$ and $H^1_{par}(\Gamma_0(N), V^\chi_{k-2}(\mathbb{C}))$. Using the Eichler-Shimura isomorphism, we can conclude from this that the eigenvalues of a Hecke operator T_n on $\mathbb{S}_k(\Gamma_0(N), \chi)$ are just given by two copies of the eigenvalues of T_n on $S_k(\Gamma_0(N), \chi)$. Associated with any newform $f \in S_k(\Gamma_0(N), \chi)$, we thus have a two-dimensional subspace of $\mathbb{S}_k(\Gamma_0(N), \chi)$ with the same Hecke eigenvalues. Now let $F \in S_k^{\mathrm{mero}}(\Gamma_0(N), \chi)$ be such that $[f]$ and $[F]$ generate this subspace. We can choose F to have poles only at cusps equivalent to ∞ and Fourier coefficients in K_f and we then call F or (any representative of) the class $[F]$ a meromorphic partner of f. Previously, we showed that

$$[r_f] \;=\; \omega_f^+[r^+] + \omega_f^-[r^-]$$

for $r^{\pm} \in Z^1_{\mathrm{par}}(\Gamma_0(N), V^{\chi}_{k-2}(K_f))^{\pm}$ and used this to define the periods $\omega_f^{\pm}$, which are unique up to multiplication by K_f. Completely analogously, we have

$$[r_F] \,=\, \eta_f^+[r^+] + \eta_f^-[r^-]$$

for the same $r^{\pm}$, which defines the quasiperiods $\eta_f^{\pm}$. Note that these only depend on the class of the meromorphic partner F. We thus obtain a period matrix

$$\begin{pmatrix} \omega_f^+ & \eta_f^+ \\ \omega_f^- & \eta_f^- \end{pmatrix}$$

associated with any newform $f \in S_k(\Gamma_0(N), \chi)$. This is unique up to rescalings of the rows and columns by K_f and shifts of the second column by the first column. The determinant of this matrix is an algebraic multiple[2] of $(2\pi i)^{k-1}$. This generalizes the Legendre relations for the periods of elliptic curves. In general, the determinant being an algebraic multiple of $(2\pi i)^{k-1}$ follows from the motivic considerations in the next section, but for each given case one can prove this by a concrete computation. Below we exemplify this for the case of level 1.

Proposition 7. *Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a newform with meromorphic partner F. Then the associated periods and quasiperiods satisfy*

$$\det \begin{pmatrix} \omega_f^+ & \eta_f^+ \\ \omega_f^- & \eta_f^- \end{pmatrix} \,\in\, (2\pi i)^{k-1} K_f\,.$$

Proof. First note that clearly $\{f, F\} \in (2\pi i)^{k-1} K_f$. The idea now is to relate the pairing $\{\cdot,\cdot\}$ to a pairing on $H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}(\mathbb{C}))$. This goes along the lines of similar calculations in [40] and [50]. Recall that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T. A standard (non-strict) fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\overline{\mathfrak{H}}$ is given by $\mathcal{F} = \{\tau \in \overline{\mathfrak{H}} \mid |\mathrm{Re}\ \tau| \le \frac{1}{2}\} \smallsetminus \{\tau \in \overline{\mathfrak{H}} \mid |\tau| < \frac{1}{2}\}$. In the following, we assume that F has no poles on the boundary of $\mathcal{F}$. We then have

$$\begin{aligned}\{f, F\} &= (2\pi i)^{k-1} \int_{\partial\mathcal{F}} \widetilde{f} F \,\mathrm{d}\tau \\ &= (2\pi i)^{k-1} \int_{\frac{i\sqrt{3}-1}{2}}^{\infty} (\widetilde{f}|_{2-k}(T-1)) F \,\mathrm{d}\tau \\ &\quad + (2\pi i)^{k-1} \int_{i}^{\frac{i\sqrt{3}-1}{2}} (\widetilde{f}|_{2-k}(S-1)) F \,\mathrm{d}\tau\,.\end{aligned}$$

For $\tau_0 = \frac{i\sqrt{3}+1}{2}$ we have $T^{-1}\tau_0 = S^{-1}\tau_0$ and with the choice $\widetilde{f} = \widetilde{f}_{\tau_0}$ this gives $r_{f,\tau_0}(S) = r_{f,\tau_0}(T)$ and thus

$$\{f, F\} \,= (2\pi i)^{k-1} \int_i^{\infty} r_{f,\tau_0}(T) F \,\mathrm{d}\tau\,.$$

From $S^2 = -1$ we further get $r_{f,\tau_0}(S)|_{2-k}S = -r_{f,\tau_0}(S)$ and so

$$\begin{aligned}\{f, F\} \,&=\, \frac{1}{2}(2\pi i)^{k-1} \int_0^{\infty} r_{f,\tau_0}(T) F \,\mathrm{d}\tau \\ &=\, -\frac{(k-2)!}{2} \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i}^{-1} r_{f,\tau_0}(T)_i r_{F,\infty}(S)_{k-2-i} \\ &=:\, -\frac{(k-2)!}{2} \langle r_{f,\tau_0}(T), r_{F,\infty}(S)\rangle\,.\end{aligned}$$

Here, p_i denotes the coefficient of τ^i for $p \in V_{k-2}(\mathbb{C})$ and it is straightforward to show that the pairing $\langle\cdot,\cdot\rangle : V_{k-2}(\mathbb{C}) \times V_{k-2}(\mathbb{C}) \to \mathbb{C}$ that we implicitly defined above is $\mathrm{SL}_2(\mathbb{Z})$-invariant. We

[2] For all computations that we performed, the multiplicative factor was in fact in the compositum of K_f and the field fixed under the kernel of χ. Here, we identify χ with a representation of the Galois group of the Nth cyclotomic field.

now want to replace r_{f,τ_0} by $r_{f,\infty}$. Using the T-invariance of $\widetilde{f}_\infty$, the $\mathrm{SL}_2(\mathbb{Z})$-invariance of the pairing $\langle\cdot,\cdot\rangle$ and the cocycle relations associated with the identities $S^2 = (ST)^3 = -1$ gives

$$\begin{aligned}
\langle r_{f,\tau_0}(T), r_{F,\infty}(S)\rangle &= \langle(\widetilde{f}_{\tau_0} - \widetilde{f}_\infty)|_{2-k}(T-1), r_{F,\infty}(S)\rangle \\
&= \langle \widetilde{f}_{\tau_0} - \widetilde{f}_\infty, r_{F,\infty}(S)|_{2-k}(T^{-1}-1)\rangle \\
&= -\langle \widetilde{f}_{\tau_0} - \widetilde{f}_\infty, r_{F,\infty}(S)|_{2-k}(ST^{-1}+1)\rangle \\
&= -\langle \widetilde{f}_{\tau_0} - \widetilde{f}_\infty, r_{F,\infty}(S)|_{2-k}((TS)^2+1)\rangle \\
&= -\frac{1}{3}\langle \widetilde{f}_{\tau_0} - \widetilde{f}_\infty, r_{F,\infty}(S)|_{2-k}((TS)^2+1-2TS)\rangle \\
&= -\frac{1}{3}\langle \widetilde{f}_{\tau_0} - \widetilde{f}_\infty, r_{F,\infty}(S)|_{2-k}(TST-T)(S-T^{-1})\rangle \\
&= \frac{1}{3}\langle(\widetilde{f}_{\tau_0} - \widetilde{f}_\infty)|_{2-k}(T-S), r_{F,\infty}(S)|_{2-k}(TST-T)\rangle \\
&= \frac{1}{3}\langle r_{f,\infty}(S), r_{F,\infty}(S)|_{2-k}(ST^{-1}S-T)\rangle \\
&= \frac{1}{3}\langle r_{f,\infty}(S)|_{2-k}(T-T^{-1}), r_{F,\infty}(S)\rangle\,.
\end{aligned}$$

We note that any coboundary which vanishes on T comes from a constant polynomial and hence this expression is invariant under shifting the parabolic cocycles by such coboundaries. In particular, we can define a pairing $\langle\cdot,\cdot\rangle$ on $H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), V_{k-2}(K))$ by

$$\langle [r_1],[r_2]\rangle \;=\; -\frac{(k-2)!}{6}\langle r_1(S)|_{2-k}(T-T^{-1}), r_2(S)\rangle\,,$$

where r_1, r_2 must be chosen such that $r_1(T) = r_2(T) = 0$. We see that this pairing is ε-invariant and we conclude that

$$\{f,F\} \;=\; (\omega_f^+\eta_f^- - \omega_f^-\eta_f^+)\underbrace{\langle r^+, r^-\rangle}_{\in K_f} \;\in\; (2\pi i)^{k-1}K_f\,.$$

$\square$

Computation of $\mathbb{S}_k(\Gamma_0(N),\chi)$

We conclude this section by explaining how one can compute representatives for $\mathbb{S}_k(\Gamma_0(N),\chi)$ and the action of the Hecke operators on these. We will work with representatives F which only have poles at cusps equivalent to ∞. To do so, we first compute a modular form $h \in M_l(\Gamma_0(N))$ with the maximal vanishing order $\frac{l}{12}\cdot[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)] = \frac{l}{12}\cdot N\prod_{p|N}(1+\frac{1}{p})$ at ∞, which exists for l large enough. Here, the vanishing order at ∞ is the leading exponent of q in the Fourier expansion. As explained in [59], the form h necessarily can be realized as an eta quotient

$$h(\tau) \;=\; \prod_{m|N}\eta(m\tau)^{r_m}$$

with the Dedekind eta function η and exponents $r_m \in \mathbb{Z}$. For n large enough, we can then represent any element of $\mathbb{S}_k(\Gamma_0(N),\chi)$ by elements

$$F \;=\; \frac{G}{h^n}$$

with $G \in S_{k+n\,l}(\Gamma_0(N),\chi)$. The only restriction for an element G/h^n to lie in $S_k^{\mathrm{mero}}(\Gamma_0(N),\chi)$ is that there is no constant term in the Fourier expansion. Further, two elements G_1/h^n and G_2/h^n only define the same class in $\mathbb{S}_k(\Gamma_0(N),\chi)$ if they differ by the $(k-1)$th derivative of b/h^n with some $b \in M^{\mathrm{mero}}_{2-k+n\,l}(\Gamma_0(N),\chi)$. The spaces that are relevant for the construction are finite-dimensional and, working with Fourier expansions of bases of these spaces (which we compute using PARI [69]), finding representatives for $\mathbb{S}_k(\Gamma_0(N),\chi)$ reduces to linear algebra. Computing the action of Hecke operators on these spaces is also straightforward, but we remark that acting with T_m multiplies the pole order at infinity by m and hence one has to work with numerators of higher weight.

3.3 Associated motives and L-functions

In this section, we review that associated with any newform $f \in S_k(\Gamma_0(N), \chi)$, there is a pure motive of weight $k-1$ and rank two. We discuss possible geometric realizations of such motives and properties of the associated L-functions.

The simplest situation arises for modular forms of weight 2 and some level N. In this case, one can consider the modular curve $X_1(N)$ which is obtained by giving the quotient $\Gamma_1(N)\backslash\mathfrak{H}$ the structure of a Riemann surface. Then there is an isomorphism

$$\begin{aligned} \mathbb{S}_2(\Gamma_1(N)) &\to H^1_{\mathrm{dR}}(X_1(N)(\mathbb{C})) \\ [F] &\mapsto [2\pi i\, F\, \mathrm{d}\tau]\,. \end{aligned}$$

In fact, $X_1(N)$ can be given the structure of a smooth projective variety defined over $\mathbb{Q}$ and if one restricts to classes that can be represented by forms in $S_2^{[\infty]}(\Gamma_1(N))$ with rational Fourier coefficients, this gives an isomorphism with the algebraic de Rham cohomology $H^1_{\mathrm{dR}}(X_1(N))$. Hence, as a natural motive, one can consider the motive $V = H^1(X_1(N))$. For any divisor N' of N, there are $[\Gamma_1(N') : \Gamma_1(N)]$ correspondences on $X_1(N) \times X_1(N)$ and acting on V one obtains a splitting $V = V^{\mathrm{new}} \oplus V^{\mathrm{old}}$ corresponding to the splitting into old forms and new forms. We further have the action of $\Gamma_0(N)/\Gamma_1(N)$ and correspondences associated with the Hecke operators, which further split V^{new} so that attached to any newform $f \in S_2(\Gamma_0(N), \chi)$ we obtain a two-dimensional motive V_f (with coefficients in some extension). From the work of Eichler and Shimura, it follows that for all primes $p \nmid N$

$$\det\bigl(1 - T\, F_p^* | V_f\bigr) \;=\; 1 - a_p\, T + \chi(p)\, p\, T^2\,,$$

where a_p is the eigenvalue of f under T_p. We conclude that attached to f there is a motive V_f so that the periods of V_f are the periods and quasiperiods of f and the traces of the Frobenius elements are just the eigenvalues of the Hecke operators.

For newforms $f \in S_k(\Gamma_0(N), \chi)$ of weight $k > 2$, Deligne [26] showed that the Hecke eigenvalues coincide with the traces of Frobenius elements acting on the $(k-1)$th cohomology group of an appropriate Kuga-Sato variety, defined as a suitable compactification of a fiber bundle over $\Gamma_1(N)\backslash\mathfrak{H}$ whose fiber over a point τ is the $(k-2)$th Cartesian product of an elliptic curve E_τ with a point of order N. This results in the following theorem.

Theorem 3. *Let $f \in S_k(\Gamma_0(N), \chi)$ be a newform of weight $k \geq 2$ and let K_f be the number field generated by the Hecke eigenvalues a_n. For any rational prime ℓ and any prime λ over ℓ, there is a continuous representation*

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \;\to\; \mathrm{GL}_2(K_{f,\lambda})$$

satisfying

$$\det(1 - T\, \rho_{f,\lambda}(F_p)) \;=\; 1 - a_p\, T + \chi(p)\, p^{k-1}\, T^2\,,$$

where $K_{f,\lambda}$ denotes the λ-adic completion of K_f. This representation is unramified at all primes which do not divide ℓN.

Scholl [62] further used the construction by Deligne to associate not only Galois representations but a complete motive V_f with f. This motive is again two-dimensional and the periods of V_f are given by the periods and quasiperiods of f.

We remark that newforms of weight 1 are also motivic. Geometrically, the associated motives are not very interesting since the relevant varieties are zero-dimensional. In particular, the Galois representations $\rho_{f,\ell}$ associated with a newform $f \in S_1(\Gamma_0(N), \chi)$ factor through the Galois group of some finite extension of $\mathbb{Q}$. As an example, consider the unique newform $f \in S_1(\Gamma_0(23), (\frac{-23}{\cdot}))$. In terms of the Dedekind eta function, this can be given by $f(\tau) = \eta(\tau)\eta(23\tau)$. The newform f is associated with the variety defined by $x^3 - x - 1$ and this manifests in the number of roots of this polynomial over the finite field $\mathbb{F}_p$ for primes $p \neq 23$ being $a_p + 1$, where a_p is the eigenvalue of the Hecke operator T_p. This example was given by van der Blij in [70].

While the constructions above give an explicit geometric realization of the motive V_f associated with a newform f, these motives can also arise in other varieties. A famous example for this is the modularity theorem for elliptic curves defined over $\mathbb{Q}$. This has been proved in a somewhat less general form by Wiles. The proof of the more general theorem has been completed in [14] and the theorem reads as follows.

Theorem 4. *Let E be an elliptic curve defined over $\mathbb{Q}$. Then the Galois representations*

$$\rho_{E,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \;\to\; \mathrm{Aut}(H^1_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell))$$

are isomorphic to the representations $\rho_{f,\ell}$ associated with a newform $f \in S_2(\Gamma_0(N))$. The level N equals the conductor of E.

Geometrically, the statement of the modularity theorem is that any elliptic curve E defined over $\mathbb{Q}$ has a modular parametrization $X_0(N) \to E$, where $X_0(N)$ is the modular curve corresponding to $\Gamma_0(N)$. In particular, the motives $H^1(E)$ and V_f are isomorphic. Another general modularity theorem has been given for Calabi-Yau threefolds (to be defined in the next chapter) that are rigid and defined over $\mathbb{Q}$. Here, rigidity means that the Hodge number $h^{2,1}$ (which equals the dimension of the complex structure deformation space) vanishes. The Galois representations associated with the third cohomology are then two-dimensional and the following theorem was proven in [38].

Theorem 5. *Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. Up to semisimplification, the Galois representations*

$$\rho_{\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \;\to\; \mathrm{Aut}(H^3_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell))$$

are isomorphic to the representations $\rho_{f,\ell}$ associated with a newform $f \in S_4(\Gamma_0(N))$.

We remark that for rigid Calabi-Yau threefolds, the nature of the associated level N is not completely understood. There are numerous examples of Galois representations of newforms which arise in other varieties. We will study two cases with non-rigid Calabi-Yau threefolds in Section 5.1 and Section 5.2.

Associated with a motive V_f of a newform $f \in S_k(\Gamma_0(N), \chi)$, we have the L-function defined by

$$L(f,s) \;=\; \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for Re $s > \frac{k+1}{2}$. As the Hecke operators satisfy a certain multiplicativity and f is an eigenfunction under all Hecke operators, the L-function has the Euler product

$$L(f,s) \;=\; \prod_p \frac{1}{1 - a_p\, p^{-s} + \chi(p)\, p^{k-1-2\,s}}\,,$$

where we replaced $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ by the associated Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$. The fact that f is a cusp form implies that the associated L-functions can be analytically continued to a holomorphic function on the whole complex plane. More precisely, we can write

$$L(f,s) \;=\; \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(ix)\, x^s\, \frac{\mathrm{d}x}{x}$$

and the right hand side defines a holomorphic function on the complex plane since $f(\tau)$ is sufficiently small for $\tau \to 0$ and $\tau \to \infty$. The action by the Atkin-Lehner involution W_N (which for a general character χ relate f with the form $\overline{f^*} \in S_k(\Gamma_0(N), \overline{\chi})$ defined by $\overline{f^*}(\tau) = \overline{f(-\overline{\tau})}$) further gives a functional equation between $L(f,s)$ and $L(\overline{f^*}, k-s)$. From the integral expression above, it also follows that the values $\frac{1}{(2\pi i)^s}\, L(f,s)$ for $s = 1, 2, ..., k-1$ are periods of f. This is easy to see in the case $N = 1$, because then 0 and ∞ are equivalent cusps. For general N, one can use the action of Hecke operators to show that also any integral between non-equivalent cusps evaluates to a linear combination of periods.

3.4 Hilbert modular forms

In the previous section, we saw that associated with elliptic newforms there are continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In this section, we consider Hilbert modular forms. These can be seen as a generalization of elliptic modular forms. Hilbert modular forms can be defined for any totally real number field K, but for simplicity we will restrict to real quadratic fields with narrow class number one. For us, the relevance of Hilbert modular forms comes from the expectation that one can associate motives with a certain class of them, the so-called newforms. In this case, the associated Galois representations are representations of $\mathrm{Gal}(\overline{K}/K)$.

In the first part of this section, we review the definition of Hilbert modular forms for real quadratic fields with narrow class number one. We also comment on definitions of Hecke operators and newforms. In the second part, we review motivic aspects of Hilbert newforms.

General definitions

We review some elementary facts about Hilbert modular forms for real quadratic fields. For simplicity, we restrict to fields with narrow class number one. For further details, see e.g. [34].

Let K be a real quadratic field with narrow class number one and embeddings $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{C}$. Using these embeddings, we can embed $\mathrm{SL}_2(K)$ into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. The latter acts by Möbius transformations on the product $\mathfrak{H}^2 = \mathfrak{H} \times \mathfrak{H}$ of two complex upper half-planes. Thus, we obtain an action of $\mathrm{SL}_2(K)$ on $\mathfrak{H}^2$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \left(\frac{\sigma_1(a)\,\tau_1 + \sigma_1(b)}{\sigma_1(c)\,\tau_1 + \sigma_1(d)}, \frac{\sigma_2(a)\,\tau_2 + \sigma_2(b)}{\sigma_2(c)\,\tau_2 + \sigma_2(d)} \right),$$

where we write $\tau = (\tau_1, \tau_2)$. This action extends to $\mathfrak{H}^2 \cup \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ and the elements in the image of the embedding $\mathbb{P}^1(K) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ are called cusps. In our situation, all cusps are in the same $\mathrm{SL}_2(\mathcal{O}_K)$-orbit, while for general real fields the number of orbits is given by the class number. In the following, we will be particularly interested in the level $\mathfrak{N}$ subgroups

$$\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_K) \mid c \equiv 0 \bmod \mathfrak{N} \},$$

where $\mathfrak{N}$ is an ideal of $\mathcal{O}_K$.

For any function $f : \mathfrak{H}^2 \to \mathbb{C}$, any tuple $k = (k_1, k_2) \in \mathbb{Z}^2$ and any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(K)$ one writes

$$(f|_k g)(\tau) = (\sigma_1(c)\tau_1 + \sigma_1(d))^{-k_1} (\sigma_2(c)\tau_2 + \sigma_2(d))^{-k_2} f(g\tau)$$

and calls $|_k$ the weight k slash operator. The vector space $M_k(\Gamma_0(\mathfrak{N}))$ of Hilbert modular forms of level $\mathfrak{N}$ and weight k is then defined by

$$M_k(\Gamma_0(\mathfrak{N})) = \{ f : \mathfrak{H}^2 \to \mathbb{C} \mid f|_k\gamma = f \; \forall\, \gamma \in \Gamma_0(\mathfrak{N}), f \text{ holomorphic on } \mathfrak{H}^2 \}.$$

The elements of $M_k(\Gamma_0(\mathfrak{N}))$ have Fourier expansions around all cusps. To make this precise, first note that for any cusp we can choose a $g \in \mathrm{SL}_2(K)$ and a $\mathbb{Z}$-module $M \subseteq K$ of rank two such that

$$g \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} g^{-1}$$

is in $\Gamma_0(\mathfrak{N})$ and fixes the cusp for any $\mu \in M$. For any $f \in M_k(\Gamma_0(\mathfrak{N}))$, it then follows that $(f|_k g)(\tau + \mu) = (f|_k g)(\tau)$, where we write $\tau + \mu = (\tau_1 + \sigma_1(\mu), \tau_2 + \sigma_2(\mu))$. This shows that there is a Fourier expansion

$$(f|_k g)(\tau) = \sum_{m \in M^\vee} a_{g,m}\, q^m,$$

where $q^m = \exp(2\pi i\, (\sigma_1(m)\,\tau_1 + \sigma_2(m)\,\tau_2))$ and

$$M^\vee = \{ m \in K \mid \sigma_1(m\,\mu) + \sigma_2(m\,\mu) \in \mathbb{Z} \text{ for all } \mu \in M \}.$$

The spaces $M_k(\Gamma_0(\mathfrak{N}))$ are finite-dimensional, even though, in contrast to the case of elliptic modular forms, the definition of $M_k(\Gamma_0(\mathfrak{N}))$ does not involve any holomorphicity at the cusps. It turns out that for Hilbert modular forms this is not required since, by the so-called Götzky-Koecher principle, the Fourier coefficients $a_{g,m}$ automatically vanish if $\sigma_1(m) < 0$ or $\sigma_2(m) < 0$. A Hilbert modular form is a cusp form if it vanishes at all cusps and we denote the subspace of cusp forms by $S_k(\Gamma_0(\mathfrak{N})) \subseteq M_k(\Gamma_0(\mathfrak{N}))$. The conjugate $\overline{f} \in M_{(k_2,k_1)}(\Gamma_0(\mathfrak{N}))$ of a Hilbert modular form $f \in M_{(k_1,k_2)}(\Gamma_0(\mathfrak{N}))$ is defined by $\overline{f}(\tau_1, \tau_2) = f(\tau_2, \tau_1)$.

Because we restricted to fields of narrow class number one, one can define Hecke operators $T_\mathfrak{n}$ in the same way as we did for elliptic modular forms. These are now labeled by ideals $\mathfrak{n}$ of $\mathcal{O}_K$ which are coprime to $\mathfrak{N}$. They can again be diagonalized simultaneously and there is a notion of newforms, which are normalized Hecke eigenforms $f \in S_k(\Gamma_0(\mathfrak{N}))$ which are uniquely determined by their Hecke eigenvalues. Given a newform $f \in S_k(\Gamma_0(\mathfrak{N}))$, we denote the eigenvalues under the Hecke operator $T_\mathfrak{n}$ by $a_\mathfrak{n}$. For the conjugate $\overline{f}$ of a newform f, the eigenvalue of $T_\mathfrak{n}$ is the eigenvalue of f under $T_{\overline{\mathfrak{n}}}$.

Motivic aspects

In the case of elliptic modular forms, there is a pure motive of rank two associated with each newform. The same is expected for Hilbert modular forms and there are several theorems in this direction. For example, the work of several people (see for example [67]) resulted in the following.

Theorem 6. *Let $f \in S_k(\Gamma_0(\mathfrak{N}))$ be a Hilbert newform and let K_f be the number field generated by the Hecke eigenvalues $a_\mathfrak{n}$. For any rational prime ℓ and any prime λ over ℓ, there is a continuous representation*

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_2(K_{f,\lambda})$$

satisfying

$$\det(1 - T\,\rho_{f,\lambda}(F_\mathfrak{p})) \;=\; 1 - a_\mathfrak{p}\,T + \mathcal{N}(\mathfrak{p})^{\max(k_1,k_2)-1}\,T^2\,,$$

where $K_{f,\lambda}$ denotes the λ-adic completion of K_f and $\mathcal{N}$ denotes the norm. This representation is unramified at primes $\mathfrak{p} \not| \,\ell\,\mathfrak{N}$.

As for elliptic modular forms, there is a definition of an L-function associated with any Hilbert newform $f \in S_k(\Gamma_0(\mathfrak{N}))$. This is analytic on the whole complex plane and satisfies a functional equation. Also, the local L-factors for primes $\mathfrak{p}$ which do not divide $\mathfrak{N}$ agree with the ones that one obtains from the representations $\rho_{f,\lambda}$, i.e. the local L-factor at a prime $\mathfrak{p} \not| \,\mathfrak{N}$ is given by

$$\frac{1}{1 - a_\mathfrak{p}\,\mathcal{N}(\mathfrak{p})^{-s} + \mathcal{N}(\mathfrak{p})^{\max(k_1,k_2)-1-2\,s}}\,.$$

Motives associated with Hilbert newforms have been studied in [8], but we are not aware of a concrete description of the associated period matrices (as we have it for elliptic newforms). For the case of weight $k = (2,2)$, some results in this direction have been given in [57].

In the simplest case, Galois representations of Hilbert newforms are associated with elliptic curves. This is explained by the following modularity theorem from [32].

Theorem 7. *Let E be an elliptic curve defined over a real quadratic field K. Then the Galois representations*

$$\rho_{E,\ell} : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{Aut}(H^1_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell))$$

are isomorphic to the representations $\rho_{f,\ell}$ associated with a Hilbert newform $f \in S_{(2,2)}(\Gamma_0(\mathfrak{N}))$ of some level $\mathfrak{N}$.

There are also examples for Galois representations of Hilbert newforms which arise in more complicated varieties. For example, the appearance of Galois representations of a newform of weight $(4,2)$ and its conjugate of weight $(2,4)$ in a motive of a Calabi-Yau threefold have been studied in [25]. In Section 5.3, we discuss another case where this also seems to be true.

3.5 Bianchi modular forms

In this section, we consider another generalization of elliptic modular forms, namely Bianchi modular forms. These are associated with imaginary quadratic fields K and for simplicity we will restrict to the case that the class number is one. We are interested in Bianchi modular forms because one expects that one can again associate motives with a certain class of them, the so-called newforms. The associated Galois representations are then representations of $\mathrm{Gal}(\overline{K}/K)$.

In the first part of this section, we sketch the definition of Bianchi modular forms. There is again a notion of Hecke operators and newforms, but we will not give a direct definition of these. Instead, we will introduce an action of Hecke operators on group cohomologies which, by a generalized Eichler-Shimura isomorphism, are isomorphic to spaces of Bianchi modular forms. In the second part, we explain how one can compute these cohomology groups. In the last part, we review motivic aspects of Bianchi newforms.

General definitions

We give a short sketch of the definition of Bianchi modular forms. For more details, we refer to the review [24] by Şengün, who we would also like to thank for several clarifications.

The group $\mathrm{GL}_2(\mathbb{C})$ acts on the hyperbolic three-dimensional space

$$\mathbb{H} \;=\; \{(x,y)\in\mathbb{C}\times\mathbb{R}\mid y>0\}$$

by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}(x,y) \;=\; \left(\frac{(a\,x+b)\overline{(c\,x+d)}+a\,\overline{c}\,y^2}{|c\,x+d|^2+|c|^2\,y^2},\frac{|a\,d-b\,c|\,y}{|c\,x+d|^2+|c|^2\,y^2}\right).$$

Now let K be an imaginary quadratic field, which, for simplicity of this exposition, we will assume to have class number one. Thinking of K as a subfield of $\mathbb{C}$, we obtain an action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\mathbb{H}$. In the following, we will be particularly interested in the level $\mathfrak{N}$ subgroups

$$\Gamma_0(\mathfrak{N}) \;=\; \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathcal{O}_K)\mid c\equiv 0 \bmod \mathfrak{N}\}\;,$$

where $\mathfrak{N}$ is an ideal of $\mathcal{O}_K$. Bianchi modular forms of weight k and level $\mathfrak{N}$ are real analytic functions $f:\mathbb{H}\to\mathbb{C}^{k+1}$ that satisfy certain transformation properties under $\Gamma_0(\mathfrak{N})$, are annihilated by certain second order differential operators and satisfy certain growth conditions. A precise definition of these conditions can be found in [33]. The space $M_k(\Gamma_0(\mathfrak{N}))$ of Bianchi modular forms of weight k and level $\mathfrak{N}$ is finite-dimensional and the elements have an expansion similar to the Fourier expansion of elliptic modular forms and Hilbert modular forms, the so-called Fourier-Bessel expansion. There is also a subspace $S_k(\Gamma_0(\mathfrak{N}))$ of cusp forms, a definition of Hecke operators and a notion of newforms. We will not discuss these on the level of functions, but instead we will now introduce a group cohomology that is isomorphic to the space of cusp forms. This will later allow us to do concrete computations.

For $k\geq 2$ there is a generalized Eichler-Shimura isomorphism

$$S_k(\Gamma_0(\mathfrak{N}))\;\cong\; H^1_{\mathrm{par}}(\Gamma_0(\mathfrak{N}),V_{k-2,k-2}(\mathbb{C}))$$

studied by Harder [42, 43]. Here, $V_{k-2,k-2}(\mathbb{C})$ denotes the tensor product $V_{k-2}(\mathbb{C})\otimes\overline{V_{k-2}(\mathbb{C})}$, where the space $V_{k-2}(\mathbb{C})=\langle 1,\tau,...,\tau^{k-2}\rangle_{\mathbb{C}}$ is equipped with the right-action of $\mathrm{GL}_2(\mathbb{C})$ given by

$$p(\tau)|_{2-k}\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \;=\; (c\,\tau+d)^{k-2}p\left(\frac{a\,\tau+b}{c\,\tau+d}\right)$$

and $\overline{V_{k-2}(\mathbb{C})}$ is the complex conjugate representation. As in the case of elliptic modular forms, the parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma,V_{k-2,k-2}(\mathbb{C}))$ is the kernel of the restriction

$$H^1(\Gamma,V_{k-2,k-2}(\mathbb{C}))\;\to\bigoplus_{[c]\in\Gamma\backslash\mathbb{P}^1(K)} H^1(\Gamma_c,V_{k-2,k-2}(\mathbb{C}))\,,$$

where the sum runs over representatives of equivalence classes of cusps, i.e. over representatives c of elements of the quotient $\Gamma\backslash\mathbb{P}^1(K)$, and $\Gamma_c \subset \Gamma_0(\mathfrak{N})$ denotes the stabilizer group of c.

The action of Hecke operators on the parabolic cohomology groups is defined analogously as for elliptic modular form, i.e. we first define for any ideal $\mathfrak{n} \subseteq \mathcal{O}_K$ coprime to $\mathfrak{N}$

$$\mathcal{M}_{\mathfrak{n},\mathfrak{N}} = \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O}_K) \mid (\det(g)) = \mathfrak{n}, c \equiv 0 \bmod \mathfrak{N}\} ,$$

where $\mathrm{M}_2(\mathcal{O}_K)$ denotes the set of 2×2 matrices with entries in $\mathcal{O}_K$. Note that this set is stabilized under left and right multiplication by any $\gamma \in \Gamma_0(\mathfrak{N})$. After choosing representatives M_i for the quotient $\Gamma_0(\mathfrak{N})\backslash\mathcal{M}_{\mathfrak{n},\mathfrak{N}}$, we then define an action on $Z^1(\Gamma_0(\mathfrak{N}), V_{k-2,k-2}(\mathbb{C}))$ by

$$(r|_{2-k}T_{\mathfrak{n}})(\gamma) = \sum_i r(\gamma_i)|_{2-k}M_{\pi_\gamma(i)} ,$$

where the $\gamma_i \in \Gamma_0(\mathfrak{N})$ are determined by the identity

$$M_i\gamma = \gamma_i M_{\pi_\gamma(i)}$$

with a unique permutation π_γ of the indices. Using the cocycle property, it is straightforward to show that this map can be restricted to Z^1_{par}, Z^1 and B^1. Further, it is straightforward to show that we get a well-defined action of $T_{\mathfrak{n}}$ on $H^1_{\mathrm{par}}(\Gamma_0(\mathfrak{N}), V_{k-2,k-2}(\mathbb{C}))$ that does not depend on the chosen representatives of $\Gamma_0(\mathfrak{N})\backslash\mathcal{M}_{\mathfrak{n},\mathfrak{N}}$. The Hecke operators commute with each other and can be diagonalized simultaneously. An element $f \in S_k(\Gamma_0(\mathfrak{N}))$ is called a newform if the associated class in $H^1_{\mathrm{par}}(\Gamma_0(\mathfrak{N}), V_{k-2,k-2}(\mathbb{C}))$ is an eigenvector of all Hecke operators and uniquely determined (up to normalization) by its eigenvalues. For a given newform f, we denote the associated eigenvalue of $T_{\mathfrak{n}}$ by $a_{\mathfrak{n}}$.

Computation of $H^1_{\mathrm{par}}(\Gamma_0(\mathfrak{N}), V_{k-2,k-2}(K))$

A basis of the space $H^1_{\mathrm{par}}(\Gamma_0(\mathfrak{N}), V_{k-2,k-2}(K))$, as well as the action of Hecke operators, can be computed exactly as for elliptic modular forms, once:

- we have a finite presentation of the abstract group $\mathrm{GL}_2(\mathcal{O}_K)$ and matrices corresponding to the abstract generators
- we can write any matrix in $\mathrm{GL}_2(\mathcal{O}_K)$ as a word in the generators
- we have generators of the stabilizer group of representatives of all $\mathrm{GL}_2(\mathcal{O}_K)$-equivalence classes of cusps
- we have representatives for $\Gamma_0(\mathfrak{N})\backslash\mathrm{GL}_2(\mathcal{O}_K)$

We show how this can be achieved for the case $K = \mathbb{Q}(\sqrt{-1})$ (which is the only case that we will need later). In this case, there is the presentation

$$\mathrm{PSL}_2(\mathcal{O}_K) = \left\langle S, T_1, T_2 \;\middle|\; \begin{smallmatrix} S^2=T_2^{-1}T_1^{-1}T_2T_1=(ST_1)^3=(T_2^{-1}ST_2^2S)^2 \\ =(T_2^{-1}ST_2S)^3=(ST_2^{-1}T_1ST_2T_1)^2=1 \end{smallmatrix} \right\rangle$$

with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \quad T_2 = \begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 1 \end{pmatrix} .$$

In terms of the transpose of the generators, this is given in [55]. From this, we can easily obtain a presentation for $\mathrm{GL}_2(\mathcal{O}_K)$. To write any matrix in $\mathrm{GL}_2(\mathcal{O}_K)$ as a word in the generators, we then follow the same procedure as for $\mathrm{SL}_2(\mathbb{Z})$ (replacing powers of T by powers of T_1 and T_2 and using that $\left(\begin{smallmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{smallmatrix}\right) = T_2\,S\,T_2^{-1}\,S\,T_2\,S$). There is just one orbit of cusps and this can be represented by ∞. It is again straightforward to write down generators of the stabilizer group of ∞. It only remains to be able to find representatives for the quotient $\Gamma_0(\mathfrak{N})\backslash\mathrm{GL}_2(\mathcal{O}_K)$ and this can be done using the isomorphism

$$\begin{aligned} \Gamma_0(\mathfrak{N})\backslash\mathrm{GL}_2(\mathcal{O}_K) &\to \mathbb{P}^1(\mathcal{O}_K/\mathfrak{N}) \\ \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] &\mapsto (c:d) . \end{aligned}$$

Motivic aspects

As for elliptic modular forms and Hilbert modular forms, it is expected that there are pure motives of rank two associated with Bianchi newforms. There are several theorems in this direction. For instance, the work of several people (e.g. [5]) resulted in the following.

Theorem 8. *Let $f \in S_k(\Gamma_0(\mathfrak{N}))$ be a Bianchi newform of some even weight $k \geq 2$ and let K_f be the number field generated by the Hecke eigenvalues $a_{\mathfrak{n}}$. There is a finite extension E_f of K_f of degree at most four so that for any rational prime ℓ and any prime λ over ℓ, there is a continuous representation*

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{K}/K) \;\rightarrow\; \mathrm{GL}_2(E_{f,\lambda})$$

satisfying

$$\det(1 - T\,\rho_{f,\lambda}(F_{\mathfrak{p}})) \;=\; 1 - a_{\mathfrak{p}}\,T + \mathcal{N}(\mathfrak{p})^{k-1}\,T^2\,,$$

where $E_{f,\lambda}$ denotes the λ-adic completion of E_f and $\mathcal{N}$ denotes the norm. This representation is unramified at all primes which do not divide $\ell\,\mathcal{N}(\mathfrak{N})$ and which are unramified in K.

We remark that possible choices for the field E_f are explained in corollary 1 in [68]. For example, let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be distinct primes of K_f which do not divide $\ell\,\mathfrak{N}$ and which are unramified in K_f. If for both primes $a_{\overline{\mathfrak{p}_i}}$ does not vanish and $1 - a_{\mathfrak{p}_i}\,T + \mathcal{N}(\mathfrak{p}_i)^{k-1}\,T^2$ has two distinct complex roots $\alpha_{\mathfrak{p}_i}$ and $\beta_{\mathfrak{p}_i}$, then one can choose $E_f = K_f(\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2})$.

As in the case of elliptic modular forms and Hilbert modular forms, there are L-functions associated with Bianchi newforms $f \in S_k(\Gamma_0(\mathfrak{N}))$. These are analytic on the whole complex plane and satisfy a functional equation. Also, the local L-factors for primes $\mathfrak{p}$ which do not divide $\mathfrak{N}$ agree with the ones that one obtains from the representations $\rho_{f,\lambda}$, i.e. the local L-factor at a prime $\mathfrak{p} \nmid \mathfrak{N}$ is given by

$$\frac{1}{1 - a_{\mathfrak{p}}\,\mathcal{N}(\mathfrak{p})^{-s} + \mathcal{N}(\mathfrak{p})^{k-1-2\,s}}\,.$$

In the simplest case, Galois representations of Bianchi newforms are associated with elliptic curves. For example, there is the following modularity theorem which is proven in more generality in [22].

Theorem 9. *Let E be an elliptic curve defined over $K = \mathbb{Q}(\sqrt{-1})$ which does not have complex multiplication. Then the Galois representations*

$$\rho_{E,\ell} : \mathrm{Gal}(\overline{K}/K) \;\rightarrow\; \mathrm{Aut}(H^1_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell))$$

are isomorphic (after embedding into $\mathrm{GL}_2(E_{f,\lambda})$) to the representations $\rho_{f,\lambda}$ associated with a Bianchi newform $f \in S_2(\Gamma_0(\mathfrak{N}))$.

There are also examples for Galois representations of Bianchi newforms which arise in more complicated varieties. For example, we will study one instance of this with Bianchi newforms of weight 4 and weight 2 in Section 5.4.

4 Calabi-Yau threefolds and Calabi-Yau operators

Calabi-Yau varieties can be seen as generalizations of elliptic curves to higher dimensions. They are well studied because of their relevance in mathematical physics. In this work, we use families of three-dimensional Calabi-Yau varieties for all geometric realizations of rank four motives. This is convenient for two reasons. First, the cohomology of Calabi-Yau threefolds is simpler than that of generic varieties. Second, to a great extend, motives associated with families of Calabi-Yau threefolds can be described by differential operators and there is a large list of such operators.

In the first section of this chapter, we introduce Calabi-Yau varieties and review some of their properties. In particular, we discuss some general properties of motives associated with Calabi-Yau threefolds. Such motives are often easier to study by considering families of Calabi-Yau threefolds, which we discuss in the second chapter. The associated variations of Hodge structures give rise to Picard-Fuchs operators and limit mixed Hodge structures. The Picard-Fuchs operators associated with families of Calabi-Yau threefolds are expected to have very special properties and differential operators with these properties, so-called Calabi-Yau operators, have been studied extensively. We review their definition and a conjectural method which only uses Calabi-Yau operators for the computation of characteristic polynomials of the action of Frobenius elements on the cohomology of associated varieties. In the last section, we discuss Calabi-Yau operators which are built out of simpler Picard-Fuchs operators. These are called Hadamard products and we also review a geometric construction of the corresponding families of Calabi-Yau threefolds.

4.1 Calabi-Yau varieties and their associated motives

Calabi-Yau varieties are important objects in mathematics and mathematical physics. In the physical context, this originates from their application in the description of spacetime in string theories. We call an n-dimensional smooth projective variety defined over $\mathbb{C}$ a Calabi-Yau variety[1] if $\Omega_X^n \cong \mathcal{O}_X$ and $H^0(X, \Omega_X^p) = 0$ for all $0 < p < n$. Calabi-Yau varieties are named after a conjecture by Calabi and its proof by Yau. In our context, the conjecture states that the complex manifold $X(\mathbb{C})$ associated with a Calabi-Yau variety X has a unique Ricci-flat metric in each Kähler class. In particular, the holonomy group is contained in $\mathrm{SU}(n)$. Calabi-Yau varieties exist in every dimension. For example, using the adjunction formula and some information about Hodge numbers, one can show that a smooth hypersurface of degree $n+2$ in $\mathbb{P}^{n+1}$ is a Calabi-Yau n-fold.

In Figure 4.1, we give the structure of the Hodge diamonds for Calabi-Yau varieties of dimension one, two and three. In dimension one, Calabi-Yau varieties are just elliptic curves. In dimension two, Calabi-Yau varieties are K3 surfaces. Only starting in dimension three, there are Calabi-Yau varieties whose associated complex manifolds are not diffeomorphic. In fact, it is not known whether there are only finitely many Calabi-Yau threefolds whose associated complex manifolds are not diffeomorphic.

$$
\begin{array}{ccc}
 & 1 & \\
1 & & 1 \\
 & 1 &
\end{array}
\qquad
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 4.1: Structure of the Hodge diamonds of Calabi-Yau varieties of dimensions $n = 1, 2, 3$.

[1] In the literature, other definitions of Calabi-Yau varieties can be found. For example, some authors do not require any vanishing of Hodge numbers.

If X is a Calabi-Yau variety of dimension n defined over some number field K (so that for every embedding $\sigma : K \hookrightarrow \mathbb{C}$ the base change X_σ is a Calabi-Yau variety defined over $\mathbb{C}$), then we get associated motives $H^k(X)$ for every $0 \le k \le 2n$. These are simpler than the ones of generic varieties and in the following we discuss some of their properties for $n = 3$. From the structure of the Hodge diamond, it is clear that the only interesting motives are the ones of weights 2, 3 and 4. For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, the short exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_{X_\sigma(\mathbb{C})} \xrightarrow{\exp} \mathcal{O}^\times_{X_\sigma(\mathbb{C})} \to 0$$

gives a long exact sequence of sheaf cohomology groups. If we now use that both $H^0(X, \Omega^1_X)$ and $H^0(X, \Omega^2_X)$ vanish, we obtain a canonical isomorphism between $H^2(X_\sigma(\mathbb{C}), \mathbb{Z})$ and the Picard group $H^1(X_\sigma(\mathbb{C}), \mathcal{O}^\times_{X_\sigma(\mathbb{C})})$. It follows that the motive $H^2(X)$ is generated by the image of divisors under the cycle class map. As a consequence, $H^4(X)$ is also generated by the image of algebraic cycles under the cycle class map and $H^3(X)$ remains as the only motive that can have a more interesting structure. The intersection pairing $H^3(X) \times H^3(X) \to H^6(X)$ gives $H^3(X)$ a symplectic structure. On the level of the periods, this implies that the period matrices W_σ (for suitably chosen bases of $H^3(X_\sigma(\mathbb{C}), \mathbb{Q})$ and $H^3_{\mathrm{dR}}(X)$) satisfy

$$W_\sigma^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} W_\sigma \;=\; (2\pi i)^3 \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

On the level of the characteristic polynomials of Frobenius elements for primes $\mathfrak{p}$ with norm q, the symplectic pairing gives the identity

$$\det\left(1 - \frac{1}{q^3\,T}\,F_\mathfrak{p}^* \mid H^3(X)\right) \;=\; \frac{1}{(q^3)^{b_3(X)/2}\,T^{b_3(X)}}\,\det\bigl(1 - T\,F_\mathfrak{p}^* \mid H^3(X)\bigr)$$

with the third Betti number $b_3(X)$. For example, in the case $b_3(X) = 2$, this gives

$$\det\bigl(1 - T\,F_\mathfrak{p}^* \mid H^3(X)\bigr) \;=\; 1 - \mathrm{Tr}\bigl(F_\mathfrak{p}^* \mid H^3(X)\bigr)\,T + q^3\,T^2\,.$$

In the case $b_3(X) = 4$, one finds that the characteristic polynomials of Frobenius elements have the form

$$\det\bigl(1 - T\,F_\mathfrak{p}^* \mid H^3(X)\bigr) \;=\; 1 + A_\mathfrak{p}\,T + B_\mathfrak{p}\,q\,T^2 + A_\mathfrak{p}\,q^3\,T^3 + q^6\,T^4$$

with integers $A_\mathfrak{p}$ and $B_\mathfrak{p}$ defined by

$$A_\mathfrak{p} \;=\; -\,\mathrm{Tr}\bigl(F_\mathfrak{p}^* \mid H^3(X)\bigr) \qquad \text{and} \qquad B_\mathfrak{p} = \frac{\mathrm{Tr}\bigl(F_\mathfrak{p}^* \mid H^3(X)\bigr)^2 - \mathrm{Tr}\bigl((F_\mathfrak{p}^2)^* \mid H^3(X)\bigr)}{2\,q}\,.$$

4.2 Families of Calabi-Yau threefolds and Calabi-Yau operators

In many contexts, both in algebraic geometry and mathematical physics, it is useful to consider families of Calabi-Yau threefolds. For simplicity, we will restrict to projective families $X \to B$ defined over $\mathbb{Q}$, where $B = \mathbb{P}^1 \smallsetminus \Delta$ with a finite set of points Δ. Concretely, this means that X is defined by some homogeneous ideal in $\mathbb{Q}[s,t][x_0, ..., x_n]$ which defines a Calabi-Yau threefold, given as a subset of $\mathbb{P}^n$, for any $(s : t) \in B$. Usually, we choose the coordinates $(s : t)$ of $\mathbb{P}^1$ so that Δ contains the point $(0 : 1)$ and then we work with the affine coordinate $z = t/s$ and homogeneous ideals in $\mathbb{Q}(z)[x_0, ..., x_n]$. As our running example, we will consider the famous Dwork family $M \to \mathbb{P}^1 \smallsetminus \{0, 1/5^5, \infty\}$ defined by the vanishing of the polynomial

$$P \;=\; z\,x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - x_0\,x_1\,x_2\,x_3\,x_4\,.$$

In the literature, the Dwork family is usually understood to be the family of hypersurfaces

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\,\psi\,x_0\,x_1\,x_2\,x_3\,x_4 \;=\; 0\,,$$

which is also smooth for $\psi = 0$. The relation with our family is obtained by replacing z by $1/(5\,\psi)^5$ and x_0 by $5\,\psi\,x_0$. For us, it will be more convenient to work with the variable z.

Associated variations of Hodge structures

For every family X as above, we obtain the variation of Hodge structures $\mathcal{H}^3$ associated with the middle cohomology. The line bundle $\mathcal{F}^3\mathcal{H}^3_{\mathcal{O}_{B(\mathbb{C})}}$ can be trivialized by a holomorphic section Ω, since any holomorphic vector bundle over a non-compact Riemann surface is trivial. Taking derivatives of Ω, we generate a subspace of sections of $\mathcal{H}^3_{\mathcal{O}_{B(\mathbb{C})}}$, which is also called the horizontal subspace. This subspace necessarily has a finite dimension h and hence Ω satisfies a differential equation of order h. From now on, we will assume that Ω is algebraic, by which we mean that it corresponds to an element of Ω^3_X, where we regard X as defined over $\mathbb{Q}(z)$. Then the associated differential equation is called a Picard-Fuchs equation and it can be written as

$$\sum_{i=0}^{h} p_i(z)\,(z\,\nabla_z)^i\,\Omega \;=\; 0$$

with polynomials p_i. We say that

$$\mathcal{L} \;=\; \sum_{i=0}^{h} p_i(z)\,\Theta^i \qquad \text{with} \qquad \Theta \;=\; z\,\frac{\mathrm{d}}{\mathrm{d}z}$$

is the Picard-Fuchs operator associated with Ω.

As an example, consider the Dwork family M defined by $P = 0$. Taking residues, we obtain a map $H^4_{\mathrm{dR}}(\mathbb{P}^4 \smallsetminus M) \to H^3_{\mathrm{dR}}(M)$, where the varieties are regarded as varieties over $\mathbb{Q}(z)$. This turns out to be an isomorphism. The space $H^4_{\mathrm{dR}}(\mathbb{P}^4 \smallsetminus M)$ is generated by classes of differential forms

$$\omega(f) \;=\; \frac{f}{P^{n+1}}\,\sum_{i=0}^{4}(-1)^i\,x_i\,\mathrm{d}x_0\wedge\cdots\wedge\widehat{\mathrm{d}x_i}\wedge\cdots\wedge\mathrm{d}x_4$$

with homogeneous $f \in \mathbb{Q}(z)[x_0,...,x_4]$ of some degree $5n$. Here, $\widehat{\mathrm{d}x_i}$ stands for the omission of $\mathrm{d}x_i$. The space of these differentials has a filtration coming from the degree of the numerator f and the differences

$$\omega(f\,\partial_{x_i}\,P) - \frac{1}{n}\,\omega(\partial_{x_i}\,f)$$

(where f has degree $5n-4$) are exact. The upshot is that there are isomorphisms

$$F^p H^3_{\mathrm{dR}}(M) \;\cong\; \mathbb{Q}(z)[x_0,...,x_4]_{\text{degrees } 0,5,\ldots,5\,(3-p)}/\sim\,,$$

where the equivalence is generated by the relations $f\,\partial_{x_i}\,P \sim \frac{1}{n}\,\partial_{x_i}\,f$ for homogeneous polynomials f of degree $5n-4$. Using this isomorphism, the computation of all $F^pH^3_{\mathrm{dR}}(M)$ reduces to simple algebra and one finds 204 representatives which generate $H^3_{\mathrm{dR}}(M)$. In particular, we can choose the trivialization Ω of $\mathcal{H}^3_{\mathcal{O}_{B(\mathbb{C})}}$ that corresponds to $\omega(1)$. The action of the derivative ∇_z on a differential $\omega(f)$ corresponds to the action of the usual derivative ∂_z on the quotient f/P^{n+1} and one finds that the Picard-Fuchs operator associated with Ω is given by the hypergeometric operator

$$\mathcal{L} \;=\; \Theta^4 - 5^5\,z\,(\Theta+1/5)\,(\Theta+2/5)\,(\Theta+3/5)\,(\Theta+4/5)\,.$$

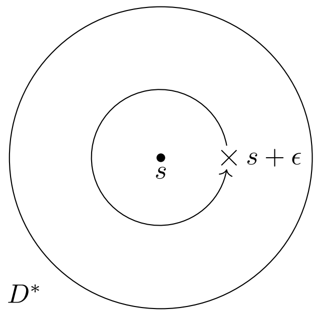
We continue with a general family X. Over any contractible open subset $U \subset B(\mathbb{C})$ we can choose a basis $\widehat{\gamma}_1,...,\widehat{\gamma}_{b_3}$ of $\mathcal{H}^3(U)$ so that

$$\Omega|_U \;=\; \sum_{i=1}^{b_3}\Pi_i\,\widehat{\gamma}_i$$

for holomorphic functions $\Pi_i \in \mathcal{O}_{B(\mathbb{C})}(U)$. These are called period functions and they are annihilated by the Picard-Fuchs operator $\mathcal{L}$. Due to the fact that the differential operator $\mathcal{L}$ comes from geometry, it has several special properties. For example, it is a Fuchsian operator, which means that all singularities are regular singularities. Other properties follow from the intersection pairing $\mathcal{H}^3 \times \mathcal{H}^3 \to \mathcal{H}^6 \cong \mathbb{Z}$. For example, the action of the monodromy on the vector of period functions Π is given by a representation of $\mathrm{SP}_{b_3(X)}(\mathbb{Q})$. For the remainder of this section, we restrict to the case that $\mathcal{H}^3$ has rank four and is irreducible[2]. Then the associated Hodge structure is 1 1 1 1 and the Picard-Fuchs operator $\mathcal{L}$ necessarily has rank four.

Associated limit mixed Hodge structures

For any point $z \in B(\mathbb{C})$, the variation of Hodge structures $\mathcal{H}^3$ gives a pure $\mathbb{Q}$-Hodge structure of weight 3 and rank four. In the following, we explain that associated with any point $s \in \Delta$ one obtains a mixed Hodge structure of rank four, a so-called limit mixed Hodge structure. For this, we follow the review [41]. Let D be a small disk centered at s and let $D^* = D \setminus \{s\}$. We can fix a basis $\widehat{\gamma} = (\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\gamma}_4)$ of $\mathcal{H}^3_{s+\epsilon}$ at some $s+\epsilon$ and consider the local monodromy if we encircle s counterclockwise:



The basis transforms as

$$\widehat{\gamma} \mapsto \widehat{\gamma}\, M_s^{-1}$$

for some matrix M_s with rational entries. The matrix M_s is necessarily quasi unipotent, which means that there are positive integers a and b such that $(M_s^a - 1)^b = 0$. For simplicity, we will assume that M_s is unipotent, i.e. $a = 1$ (more generally, this can be accomplished by going to an a-fold cover). The basis $\widehat{\gamma}$ can be continued to a basis of sections over some neighborhood of the point $s+\epsilon$ and in terms of

$$N_s = \log(M_s) = \sum_{k=1}^{b-1} (-1)^{k+1} \frac{(M_s - 1)^k}{k}$$

one finds that

$$\widehat{\gamma}(z) \exp\left(N_s \frac{\log(z-s)}{2\pi i} \right)$$

defines sections of $\mathcal{H}^3(D^*) \otimes_{\mathbb{Q}} \mathcal{O}_{D^*}$. We use these sections to define a new local system $\tilde{\mathcal{H}}^3$ over D. The holomorphic bundle $\tilde{\mathcal{H}}^3 \otimes_{\mathbb{Q}} \mathcal{O}_D$ has a filtration into holomorphic subbundles induced by the filtration of $\mathcal{H}^3_{\mathcal{O}_{B(\mathbb{C})}}$. Using the multiplication by N_s, one can further define a filtration

$$W^0 \tilde{\mathcal{H}}^3_s \subseteq \ldots \subseteq W^6 \tilde{\mathcal{H}}^3_s = \tilde{\mathcal{H}}^3_s$$

which is uniquely determined by the following requirements:

- N_s maps $W^k \tilde{\mathcal{H}}^3_s$ to $W^{k-2} \tilde{\mathcal{H}}^3_s$.
- N_s^k is an isomorphism between graded pieces of weight $3+k$ and $3-k$.

[2]More generally, one can consider the case that $\mathcal{H}^3$ contains an irreducible local system of rank four. This is what happens for the example of the Dwork family.

This turns $\tilde{\mathcal{H}}^3_s$ into a mixed $\mathbb{Q}$-Hodge structure, the so-called limit mixed Hodge structure. Since the section Ω and its derivatives lift (after possible rescalings by powers of $z-s$) to sections of the bundle $\tilde{\mathcal{H}}^3 \otimes_{\mathbb{Q}} \mathcal{O}_D$, one can also associate a period matrix with the limit mixed Hodge structure. The entries corresponding to the graded pieces are called pure periods and the other entries are called mixed periods. We remark that these periods depend on the choice of the coordinate z (more precisely, they depend on the tangent vector ∂_z at s).

As an example, we consider the limit mixed Hodge structure of a so-called MUM point. Here, MUM stands for maximal unipotent monodromy, which corresponds to the local monodromy matrix being unipotent and having a Jordan block of maximal size. Up to conjugation, we then have

$$M_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad N_s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

It follows that the graded pieces with weights 0, 2, 4 and 6 are respectively generated by the four vectors $(1,0,0,0)^T$, $(0,1,0,0)^T$, $(0,0,1,0)^T$ and $(0,0,0,1)^T$. On the level of the Hodge diamond, going from the pure Hodge structure of weight 3 associated with a regular point z to the limit mixed Hodge structure associated with the MUM point s thus has the following effect:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \quad \to \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The period matrix of the limit mixed Hodge structure depends on the variation that one considers. As an example, we consider the Dwork family M with the Picard-Fuchs operator $\mathcal{L}$. The variation $\mathcal{H}^3$ has rank 204, but it contains a subvariation of rank four corresponding to Ω and its first three derivatives. All local exponents (roots of the indicial equation) of $\mathcal{L}$ at $z=0$ vanish and hence $z=0$ is a MUM point. Around this point, a basis of solutions of the Picard-Fuchs equation is given by

$$\varpi(z) = \begin{pmatrix} f_1(z) \\ \log(z)\, f_1(z) + f_2(z) \\ \frac{1}{2}\log(z)^2 f_1(z) + \log(z)\, f_2(z) + f_3(z) \\ \frac{1}{6}\log(z)^3 f_1(z) + \frac{1}{2}\log(z)^2 f_2(z) + \log(z)\, f_3(z) + f_4(z) \end{pmatrix}$$

with convergent power series normalized by $f_1(0)=1$ and $f_2(0)=f_3(0)=f_4(0)=0$. It turns out that these solutions combine to a basis of period functions (corresponding to the rational structure of $\mathcal{H}^3$) by

$$\Pi = \begin{pmatrix} (2\pi i)^3 & 0 & 0 & 0 \\ 0 & (2\pi i)^2 & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -40\,\zeta(3) & 0 & 0 & 1 \end{pmatrix} \varpi\,.$$

The period matrix of the limit mixed Hodge structure is just the matrix above. One can see that the pure periods are powers of $2\pi i$ and that the only additional mixed period is $\zeta(3)$. Note that this depends on the choice of the coordinate z. If we would rescale the coordinate z by some rational number, the period matrix of the associated limit mixed Hodge would also contain logarithms of rational numbers.

Calabi-Yau operators

One attempt to understand families of Calabi-Yau threefolds is to first understand the associated Picard-Fuchs operators. This has been followed in [2], where a class of differential operators, the so-called Calabi-Yau operators, is introduced. As reviewed in [71], the idea is to give properties of differential operators which are so constraining that the operators that have these properties always come from families of Calabi-Yau threefolds. For example, from the existence of the intersection pairing $\mathcal{H}^3 \times \mathcal{H}^3 \to \mathcal{H}^6 \cong \mathbb{Z}$ it follows that $\mathcal{L}$ is self-dual. This means that there is a rational function α such that

$$\mathcal{L}^\vee = \alpha(z)^{-1} \, \mathcal{L} \, \alpha(z) \,,$$

where the dual $\mathcal{L}^\vee$ of

$$\mathcal{L} = \sum_{i=0}^{4} a_i(z) \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^i$$

is defined as

$$\mathcal{L}^\vee = \sum_{i=0}^{4} \left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^i a_i(z) \,.$$

From this consideration, one sees that Calabi-Yau operators should be Fuchsian operators of rank four that are self-dual. However, these two properties are true also for Picard-Fuchs operators which do not come from families of Calabi-Yau threefolds and to have a smaller class of operators one wants to impose more conditions. These come from the field of mirror symmetry, which is concerned with certain pairs of families of Calabi-Yau threefolds and gives precise conjectures for Picard-Fuchs operators of families of Calabi-Yau threefolds. To give an explicit example, we consider again the Dwork family M with the associated Picard-Fuchs operator $\mathcal{L}$. This example was studied in the famous paper [20] by Candelas, de la Ossa, Green and Parkes and this can be seen as the beginning of mirror symmetry. The so-called family of mirror quintics is obtained as a resolution of the quotient of M by the symmetries

$$(x_0 : \ldots : x_4) \mapsto (\alpha_0 \, x_0 : \ldots : \alpha_4 \, x_4)$$

for fifth roots of unity $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfying $\alpha_0 \, \alpha_1 \, \alpha_2 \, \alpha_3 \, \alpha_4 = 1$. The fibers of this family have the Hodge numbers $h^{2,1} = 1$ and $h^{1,1} = 101$, i.e. the two Hodge numbers of the Dwork family get exchanged. The form Ω is invariant under the symmetries and hence Ω and its first three derivatives generate the middle cohomology of the fibers of the family of mirror quintics. In particular, the Picard-Fuchs operator $\mathcal{L}$ describes the complete variation of Hodge structures of rank four. One of the predictions of mirror symmetry is that certain enumerative invariants of generic quintic hypersurfaces in $\mathbb{P}^4$ can be computed from the solutions ϖ of $\mathcal{L}$. To make this more precise, we review some computations from [20]. One observes that the holomorphic solution

$$\varpi_1(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} \, z^n = 1 + 120\, z + 113400\, z^2 + 168168000\, z^3 + 305540235000\, z^4 + \cdots$$

has integral coefficients. Also, the so-called q-coordinate

$$q = \exp\!\left(\frac{\varpi_2(z)}{\varpi_1(z)}\right) = z + 770\, z^2 + 1014275\, z^3 + 1703916750\, z^4 + 3286569025625\, z^5 + \cdots$$

is believed to have integral coefficients. As a consequence of the self-duality, one can find a unique power series normalized by $K(q) = 5 + O(q)$ such that

$$\left(q \frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \frac{1}{K(q)} \left(q \frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \frac{\varpi(z)}{\varpi_1(z)} = 0.$$

The so-called instanton numbers n_d are now defined by the expansion

$$K(q) \;=\; 5 + \sum_{d=1}^{\infty} n_d\, d^3\, \frac{q^d}{1-q^d}$$

and one obtains

$$n_1 \;=\; 2875\;, \quad n_2 \;=\; 609250\;, \quad n_3 \;=\; 317206375\;, \quad n_4 \;=\; 242467530000\;, \quad \ldots .$$

For $d \leq 9$, the number n_d equals the number of rational curves of degree d on a generic quintic threefold. This is not true in general (in particular, it is wrong for $d = 10$), but Givental [35] proved a precise relation between the instanton numbers defined above and certain Gromov-Witten invariants.

Computations similar to the ones above have been carried out for more general Picard-Fuchs operators of families of Calabi-Yau threefolds. Motivated by these computations, one says that a rank four differential operator $\mathcal{L} \in \mathbb{Q}[z, \Theta]$ is a Calabi-Yau operator if the following properties are fulfilled:

- The operator is Fuchsian.
- The operator is self-dual.
- The operator has a MUM point at $z = 0$.
- The holomorphic solution ϖ_1 has integral coefficients, the q-coordinate has integral coefficients and the instanton numbers n_d are integers. Here, the instanton numbers are defined as above but, in general, one has a different normalization $K(q) = c + O(q)$ for some integer c.

Note that these properties depend on the choice of the coordinate z. A list of Calabi-Yau operators has been given in [1]. This list is also called the AESZ list, named after the authors Almkvist, van Enckevort, van Straten and Zudilin. Strictly speaking, most of the operators in this list are not proven to be Calabi-Yau operators, since the integrality properties have been checked only numerically. We will nevertheless refer to operators from this list as Calabi-Yau operators. We remark that it is known that not all Picard-Fuchs operators of families of Calabi-Yau threefolds have a MUM point, hence the conditions above are strictly speaking to strong. However, as the structure of the associated limit mixed Hodge structure is so simple, the existence of a MUM point is very convenient from the practical point of view. For example, it is expected that the period matrix of the limit mixed Hodge structure of $z = 0$ always has the same form as that for the Dwork family, except that in general the number -40 that multiplies $\zeta(3)$ is some other rational number. In practice, it is easy to determine this number numerically by requiring rational monodromy matrices.

The deformation method

In the previous parts, we saw that the limit mixed Hodge structure at MUM points is particularly simple. Conjecturally, the associated period matrix is determined by one rational number. If this number is known, one can numerically compute period matrices at all points by analytically continuing. We now review conjectures which suggest that from similar computations one can obtain characteristic polynomials of Frobenius elements on all regular fibers. We follow [21] to sketch the ideas that lead to these conjectures. For a more general review of related ideas, we refer to [47].

Let $\mathcal{L}$ be a Calabi-Yau operator coming from some variation $\mathcal{H}^3$. There are similar variations which correspond to so-called p-adic cohomology groups. To give an heuristic description of these, we fix some prime p and define (purely symbolically) the vector space

$$V \;=\; \langle \Omega, \nabla_\Theta \Omega, \nabla_\Theta^2 \Omega, \ldots \rangle_{\mathbb{Q}_p((z))} / \mathcal{L}\Omega\,,$$

where $\mathbb{Q}_p((z))$ denotes the ring of formal Laurent series with coefficients in the p-adic numbers $\mathbb{Q}_p$. On $\mathbb{Q}_p((z))$, we define the usual logarithmic derivative $\Theta : f(z) \mapsto z\, f'(z)$ and the Frobenius action $F_p^* : f(z) \mapsto f(z^p)$. On V, we define in the natural way the action of ∇_Θ. Additionally, we want to equip V with a symmetric pairing $\Sigma : V \times V \to \mathbb{Q}_p((z))$ and a F_p^*-linear Frobenius (denoted by the same symbol) $F_p^* : V \to V$ satisfying the following compatibilities:

- $\Theta\Sigma(v,w) = \Sigma(\nabla_\Theta v, w) + \Sigma(v, \nabla_\Theta w)$
- $\Sigma(\Omega, \nabla_\Theta^k \Omega) = 0$ for $k < 3$
- $\Sigma(F_p^* v, F_p^* w) = p^3\, F_p^*(\Sigma(v,w))$
- $\nabla_\Theta\, F_p^* = p\, F_p^*\, \nabla_\Theta$

The first two properties are identical to the ones for the symplectic pairing from the variation of Hodge structures and they allow us to compute Σ up to a multiplicative constant. The last property can be seen as a differential equation for F_p^*, which allows us to compute F_p^* up to some undetermined constants (some of which can be fixed from the third property). In [21], it is argued that, for a suitable choice of the constants determining F_p^*, one expects the following properties to hold:

- The entries of the matrix corresponding to F_p^* are rational functions of z up to any finite p-adic order.
- The entries of the matrix corresponding to F_p^* do not have poles at $z = 0$.
- For any smooth fiber X_{z_0} over $z_0 \in \mathbb{F}_p$ and any prime $\ell \neq p$, the characteristic polynomial

$$\det\bigl(1 - T\, F_p^* \mid H^3_{\text{ét}}(\overline{X_{z_0}}, \mathbb{Q}_\ell)\bigr)$$

  is given by the characteristic polynomial of F_p^* evaluated at the Teichmüller lift $\tilde{z}_0$ of z_0. Here, the Teichmüller lift of z_0 is the unique $\tilde{z}_0 \in \mathbb{Z}_p$ satisfying $\tilde{z}_0^p = \tilde{z}_0$ and $\tilde{z}_0 \equiv z_0 \mod p$.

The complex roots of the characteristic polynomials of the action of F_p^* on $H^3_{\text{ét}}(\overline{X_{z_0}}, \mathbb{Q}_\ell)$ have absolute value $p^{-3/2}$ and, hence, if the properties above hold, it is sufficient to work up to finite p-adic order. Simple local computations show that, in terms of the fundamental matrix $W(z)$ defined by $W(z) = (\varpi(z), \Theta\,\varpi(z), \Theta^2\,\varpi(z), \Theta^3\,\varpi(z))$, one can write

$$F_p^* \;=\; W(z)^{-1}\, \varepsilon_p \begin{pmatrix} p^3 & 0 & 0 & 0 \\ p^2\,\alpha_p & p^2 & 0 & 0 \\ p\,\beta_p & p\,\alpha_p & p & 0 \\ \gamma_p & \beta_p & \alpha_p & 1 \end{pmatrix} W(z^p)$$

with $\varepsilon_p = \pm 1$ and some α_p, β_p and γ_p satisfying $\alpha_p^2 = 2\,\beta_p$. Note that in this expression all logarithms cancel, so that one gets a matrix in $\mathbb{Q}_p[[z]]$. Based on numerical computations, it is conjectured in [21] that one can choose $\alpha_p = \beta_p = 0$ and that γ_p is a rational multiple of $p^3\,\zeta_p(3)$. Here, the p-adic zeta function is defined by

$$\zeta_p(3) \;=\; -\tfrac{1}{2}(\Gamma_p'''(0) - \Gamma_p'(0)^3)$$

in terms of the p-adic gamma function. It is further conjectured that the rational constant is the same constant that multiplies $\zeta(3)$ in the period matrix of the limit mixed Hodge structure at the MUM point $z = 0$. In particular, it does not depend on p. This gives a second way to numerically compute this constant, i.e. by requiring that F_p^* is a rational function of z up to any finite p-adic order.

We conclude this section with three remarks. First, the constant ε_p is not determined from the Calabi-Yau operator alone, but depends on the underlying family. For our practical computations, this will not be relevant and we always set $\varepsilon_p = 1$. Second, one runs into problems if the Calabi-Yau operator has apparent singularities. These are singularities of the differential operator which do not correspond to singularities of the underlying family. The local monodromy at these points is trivial and a typical example for the local exponents is 0, 1, 3, 4. Then F_p^* might have poles at these apparent singularities, which does not allow a direct evaluation. In practice, we observe that this can be fixed by modifying the fundamental matrix W, e.g. by replacing $W(z)$ by $W(\varpi(z), \Theta\,\varpi(z), \Theta^3\,\varpi(z), \Theta^4\,\varpi(z))$. This is suggested from the fact that Ω, $\nabla_z\Omega$ and $\nabla_z^2\Omega$ are linearly dependent at apparent singularities with local exponents 0, 1, 3, 4. As a last remark, we note that the computations can be made more efficient by computing the inverse of F_p^*, since inverting $W(z^p)$ up to a finite order in z is more efficient than inverting $W(z)$.

4.3 Hadamard products and their geometric realization

Many Calabi-Yau operators can be obtained as so-called Hadamard products of Picard-Fuchs operators of rank two. This is explained in [2] and we review this construction in the first part of this section. In the second part, we review the construction of (families of) Calabi-Yau threefolds from fiber products of elliptic surfaces. This allows to give a geometric construction corresponding to Hadamard products.

Hadamard products of differential operators

Let

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n\, z^n \qquad \text{and} \qquad g(z) \;=\; \sum_{n=0}^{\infty} b_n\, z^n$$

be two power series which are solutions of linear differential equations of finite order and with polynomial coefficients. Then the Hadamard product defined by

$$(f * g)(z) \;=\; \sum_{n=0}^{\infty} a_n\, b_n\, z^n$$

again satisfies a linear differential equation of finite order and with polynomial coefficients. In this way, one can obtain higher order differential equations from ones of lower order. Many of the Calabi-Yau operators in [1] arise in this way, where f and g are period functions of families of elliptic curves. However, note that not all Hadamard products of differential operators of rank two give differential operators of rank four.

For $|z|$ sufficiently small, one can express the Hadamard product as

$$(f * g)(z) \;=\; \frac{1}{2\pi i} \oint f(t)\, g(z/t)\, \frac{\mathrm{d}t}{t}\,,$$

where the contour is chosen so that $|t|$ and $|z/t|$ are sufficiently small. This gives a hint towards geometric realizations of Hadamard products. For this, assume that f and g are period functions of some families of varieties, so that we can locally write

$$f(z) \;=\; \int_{\gamma_1(z)} \omega_z \qquad \text{and} \qquad g(z) \;=\; \int_{\gamma_2(z)} \eta_z\,.$$

Then we can write the Hadamard product as

$$(f * g)(z) \;=\; \frac{1}{2\pi i} \oint \int_{\gamma_1(z)} \int_{\gamma_2(z/t)} \omega_t \wedge \eta_{z/t} \wedge \frac{\mathrm{d}t}{t}$$

and this suggests that geometrically the Hadamard product corresponds to a family of twisted fiber products of the two families. In the next part of this section, we study such fiber products for the case that f and g are period functions of elliptic surfaces.

Fiber products of rational elliptic surfaces with section

Let $S \to \mathbb{P}^1$ be a rational elliptic surface. By this, we mean that S is a smooth projective variety which is birational to $\mathbb{P}^2$, that all fibers are connected and that all but finitely many fibers are elliptic curves. We further assume that the surface is relatively minimal, i.e. that the fibers do not contain smooth rational curves of self intersection -1. The fibers that are not elliptic curves are called singular fibers and these have been classified (also for more general elliptic surfaces) by Kodaira in [48, 49]. As an example, consider the family of cubics which consists of all points $((s:t),(X:Y:Z)) \in \mathbb{P}^1 \times \mathbb{P}^2$ satisfying

$$s\, X\, Y\, Z - t\,(X^3 + Y^3 + Z^3) \;=\; 0\,.$$

This has singular fibers at $(1:0)$ and at the three points $(s:t)$ where $s^3 = (3\,t)^3$. The singular fibers are all unions of three rational curves with three distinct intersection points, i.e. they are given by an equation of the form $X\,Y\,Z = 0$. In the notation of Kodaira, such fibers are called I_3 fibers. More generally, a union of n rational curves with n distinct intersection points is called an I_n fiber.

Given two rational elliptic surfaces $S_1 \to \mathbb{P}^1$ and $S_2 \to \mathbb{P}^1$, one can consider the threefold given by their fiber product $S_1 \times_{\mathbb{P}^1} S_2$. This is only singular at the points where singular fibers of S_1 and S_2 meet. For the case where both elliptic surfaces are relatively minimal and have sections, such fiber products have been studied in [61]. We review some of the results, but for simplicity we restrict to the case that all singular fibers of S_1 and S_2 are of the type I_n. Then their fiber product $S_1 \times_{\mathbb{P}^1} S_2$ has $n \cdot m$ double points in fibers of the form $I_n \times I_m$. In local affine coordinates, each of these can be described by the equation

$$x_1\,x_2 \;=\; x_3\,x_4\,.$$

Such singularities can be resolved by a so-called small resolution. For this, one introduces homogeneous coordinates $(y_1 : y_2)$ of $\mathbb{P}^1$ and replaces the singular variety in $\mathbb{A}^4$ by the variety

$$\begin{aligned} x_1\,x_2 &= x_3\,x_4 \\ x_1\,y_1 &= x_3\,y_2 \\ x_2\,y_2 &= x_4\,y_1 \end{aligned}$$

in $\mathbb{A}^4 \times \mathbb{P}^1$. Note that there is also a second resolution, which can be obtained by exchanging x_1 and x_2. Outside of the double point, the resolution and the singular variety are isomorphic, but the double point gets replaced by a $\mathbb{P}^1$. In [61], it is shown that for any choice of small resolutions one obtains a smooth variety X which is again projective and a Calabi-Yau threefold. As for any Calabi-Yau threefold, $H^2(X)$ and $H^4(X)$ are generated by algebraic cycles. The fourth homology does not change under small resolutions and the structure of the fibration allows to write down an explicit basis of divisors of $S_1 \times_{\mathbb{P}^1} S_2$. This is given by:

- one fiber of the fibration
- all but one components of each fiber
- $S_1 \times \{0\}$
- $\{0\} \times S_2$
- if there is an isogeny between S_1 and S_2, then the corresponding graph

This gives a basis of the fourth homology of X and in particular the Hodge number $h^{1,1}(X)$. On the other hand, the Euler number $\chi(\mathrm{X})$ is twice the number of nodes that one resolves and then one can compute the remaining Hodge number by

$$h^{2,1}(X) \;=\; h^{1,1}(X) - \frac{\chi(X)}{2}\,.$$

Further, for all fibers of X that have the form $I_n \times E$ with an elliptic curves E, one obtains n inclusions of the form

$$\mathbb{P}^1 \times E \;\to\; X\,.$$

It is shown in section 5 of [44] that the pushforward with respect to $n-1$ of these gives a $2(n-1)$-dimensional image in the middle homology of X.

5 Modularity of some pure motives of rank four

In Chapter 3, we reviewed that there is a pure motive of rank two and weight $k-1$ associated with any elliptic newform in $S_k(\Gamma_0(N), \chi)$. The Hodge structure of these motives is

$$1 \quad \underbrace{0 \cdots 0}_{(k-2)\text{-times}} \quad 1\,.$$

By taking sums and tensor products, we can construct modular motives of higher rank. We are particularly interested in modular motives with the Hodge structure 1 1 1 1. These can be constructed as sums of a motive of a newform of weight 4 and the (-1)th Tate twist of a motive of a newform of weight 2. On the level of the Hodge structure, this reads

$$1\;1\;1\;1 = 1\;0\;0\;1 \oplus 0\;1\;1\;0\,.$$

Another possibility is to take the product of a motive of a newform of weight 3 and a motive of a newform of weight 2. On the level of the Hodge structure, this reads

$$1\;1\;1\;1 = 1\;0\;1 \otimes 1\;1\,.$$

In this chapter, we study occurrences of such motives (and generalizations with Hilbert modular forms and Bianchi modular forms) in the middle cohomology of Calabi-Yau threefolds. To do so, we consider families X of Calabi-Yau threefolds and then search for fibers X_z whose associated motives $H^3(X_z)$ we conjecture to be modular. If the additive splitting above happens over $\mathbb{Q}$, the intersection $H^3(X_z(\mathbb{C}), \mathbb{Q}) \cap (H^{2,1}(X_z(\mathbb{C})) \oplus H^{1,2}(X_z(\mathbb{C})))$ is two-dimensional and in the physical context one calls z a rank two attractor point. The physical relevance of these is discussed in [56] and an efficient method to find such points has been given in [19]. This method starts with a Calabi-Yau operator $\mathcal{L}$ associated with X and proceeds as follows:

- Using the deformation method from Section 4.2, one computes the polynomials which are expected to be characteristic polynomials of Frobenius elements F_p acting on the cohomology of X_z for all regular $z \in \mathbb{F}_p$ and many primes p.
- One looks for persistent factorizations, i.e. one tries to reconstruct points z defined over some number field so that the characteristic polynomials for the reductions of z to $\mathbb{F}_p$ always factor over $\mathbb{Z}$.

Note that for this method one only needs the Calabi-Yau operator $\mathcal{L}$ and not an associated family of Calabi-Yau threefolds. However, to make more precise and more rigorous statements, one should eventually work with an associated family.

Our approach is the same in each section. First, we start with a Calabi-Yau operator and use the method from above to identify points where associated Galois representations seem to split. To make this more rigorous, we continue by giving a geometric realization of the corresponding motives and then study the modularity of the associated Galois representations and period matrices.

In the first section, we consider a motive which seems to be the sum of motives of elliptic newforms of weight 4 and weight 2. In the second section, we consider a motive which seems to be the product of motives of elliptic newforms of weight 3 and weight 2. In this case, we can prove the modularity of the associated Galois representations. In the third section, we consider a motive which seems to be the sum of motives associated with a Hilbert newform of weight $(4, 2)$ and its conjugate of weight $(2, 4)$. In the last section, we consider a motive which seems to be the sum of motives associated with Bianchi newforms of weight 4 and weight 2. In this case, we can again prove the modularity of the associated Galois representations.

The numerical results of Section 5.1 are part of the collaboration [12] with Albrecht Klemm, Emanuel Scheidegger and Don Zagier. The motives in Section 5.2 and Section 5.4 correspond to fixed points of symmetries of Hadamard products and the motivation to study these originates from observations made by Mohamed Elmi.

5.1 A sum of motives of elliptic modular forms

In the first part of this section, we consider a hypergeometric Calabi-Yau operator $\mathcal{L}$. Experimental computations suggest that the semisimplifications of Galois representations

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$$

associated with the point $z_* = -1/2^3 3^6$ split to sums of two-dimensional Galois representations. To make this more rigorous, we construct in the second part a family X of Calabi-Yau threefolds associated with $\mathcal{L}$. The middle cohomology of each fiber X_z contains a four-dimensional part V_z whose variation with z is described by $\mathcal{L}$. In the third part, we compute characteristic polynomials of the representations ρ_ℓ associated with V_{z_*}. The results suggest that the representations ρ_ℓ are up to semisimplification the sum of representations associated with a newform f of weight 4 and a newform g of weight 2. In fact, we compute characteristic polynomials of the action of Frobenius elements on the motives $H^k(X_{z_*})$ for all weights $0 \le k \le 6$ and give a precise conjecture for the Hasse-Weil zeta function of X_{z_*}. In the last part, we present numerical computations which suggest that the period matrix of V_{z_*} is given by a sum of the period matrix of f and the period matrix of g multiplied by $2\pi i$. In particular, the period matrix is in block diagonal form, which gives further evidence for the splitting of the motive V_{z_*}.

Experimental computations

We start with an experimental study of the Calabi-Yau operator

$$\mathcal{L} = \Theta^4 - 3^6 z (\Theta + 1/3)^2 (\Theta + 2/3)^2 \qquad \text{with} \qquad \Theta = z\frac{\mathrm{d}}{\mathrm{d}z}.$$

The operator is hypergeometric and it is the fourth operator from the list [1]. Using the script `DeformationMethod.gp` from [17], we compute the associated characteristic polynomials for all primes $11 \le p \le 997$ and all parameters $z \in \mathbb{F}_p$ of good reduction. To find Galois representations which split, we look at factorizations of these polynomials over $\mathbb{Z}$. The number of factorizations that occur for each prime p is depicted in Figure 5.1. We observe that the polynomials always factor at least once. Looking at the cases of exactly one factorization we see that these occur for the reductions of $z_* = -1/2^3 3^6$ to $\mathbb{F}_p$. All computed polynomials of this fiber factor and we thus expect that the semisimplifications of associated Galois representations split.

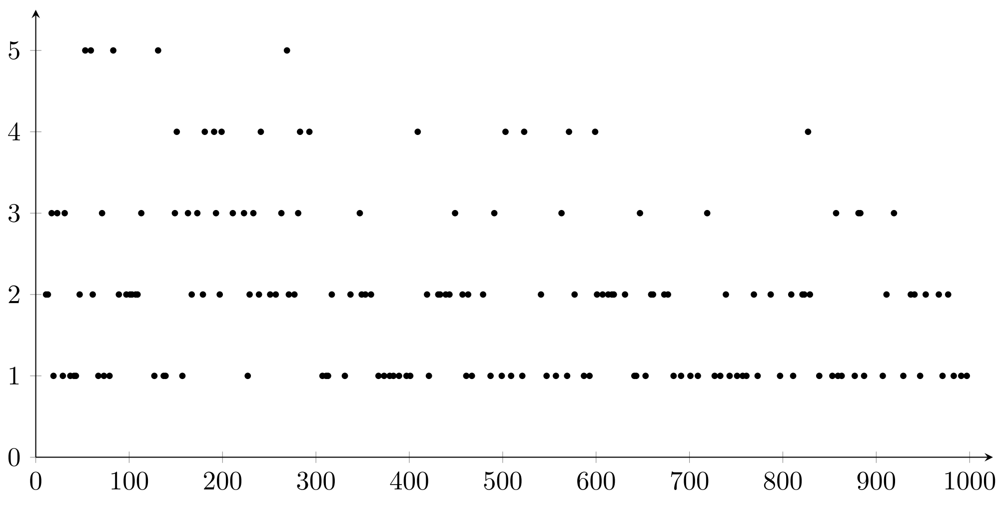


Figure 5.1: Number of factorizations over $\mathbb{Z}$ of characteristic polynomials for small primes p.

Geometric construction

To construct a family $X \to \mathbb{P}^1 \smallsetminus \{0, 1/3^6, \infty\}$ of Calabi-Yau threefolds associated with $\mathcal{L}$, we first note that for $|z| < 1/3^6$ the holomorphic solution has the form

$$F(z) \;=\; \sum_{n=0}^{\infty} \left(\frac{(3\,n)!}{n!^3}\right)^2 z^n\,.$$

Comparing this with

$$f(t) \;=\; \sum_{n=0}^{\infty} \frac{(3\,n)!}{n!^3}\, t^{3\,n} \;=\; \frac{1}{(2\pi i)^3} \oint \mathrm{d}X \oint \mathrm{d}Y \oint \mathrm{d}Z\, \frac{1}{X\,Y\,Z - t\,(X^3 + Y^3 + Z^3)}$$

for $|t| < 1/3$, we see that (after replacing z by z^3) F is the Hadamard product of f with itself, i.e.

$$F(z^3) \;=\; \frac{1}{2\pi i} \oint f(t)\, f(z/t)\, \frac{\mathrm{d}t}{t}\,.$$

This suggests that we can construct X_{z^3} from a fiber product of the two elliptic surfaces. In order not to work with a triple cover of $\mathbb{P}^1 \smallsetminus \{0, 1/3^6, \infty\}$, we consider the elliptic surface S_z that consists of all points $((s:t), (X:Y:Z)) \in \mathbb{P}^1 \times \mathbb{P}^2$ that satisfy

$$s\,X\,Y\,Z - t\,(X^3 + Y^3 + z\,Z^3) = 0\,,$$

together with the projection on the first factor. This has singular fibers at $(1:0)$ and at the three points $(3\,\xi_z : 1)$ where $\xi_z^3 = z$. Each of these has the Kodaira classification I_3. Choosing for example the point $(1:-1:0)$, we give S_z the structure of a rational elliptic surface with section. The differential form

$$\omega_z \;=\; \mathrm{Res}\; s\, \frac{X\,\mathrm{d}Y \wedge \mathrm{d}Z + Y\,\mathrm{d}Z \wedge \mathrm{d}X + Z\,\mathrm{d}X \wedge \mathrm{d}Y}{s\,X\,Y\,Z - t\,(X^3 + Y^3 + z\,Z^3)}$$

defines a holomorphic 1-form on all regular fibers and for $|t/s| < 1/3$ one of its periods is

$$2\pi i \sum_{n=0}^{\infty} \frac{(3\,n)!}{n!^3} (z\,(t/s)^3)^n\,.$$

We now define the family of Calabi-Yau threefolds $X \to \mathbb{P}^1 \smallsetminus \{0, 1/3^6, \infty\}$ with the fibers

$$X_z \;=\; S_1 \times_{\mathbb{P}^1} \epsilon^* S_z$$

with $\epsilon : (s:t) \mapsto (t:s)$. We can study its cohomology as reviewed in Section 4.3. The second homology of X_z is generated by one fiber, $S_1 \times_{\mathbb{P}^1} \{0\}$, $\{0\} \times_{\mathbb{P}^1} \epsilon^* S_z$ and all but one components of all fibers. This gives the Hodge number

$$h^{1,1}(X_z) \;=\; 1 + 2 + 8 \cdot (3 \cdot 1 - 1) \;=\; 19\,.$$

On the other hand, the Euler number $\chi(X_z)$ equals 0 and thus

$$h^{2,1}(X_z) \;=\; h^{1,1}(X_z) - \frac{\chi(X_z)}{2} \;=\; 19\,.$$

Further, X_z has eight fibers of the form $I_3 \times E$ with different elliptic curves E. Choosing two of the components of each I_3 fiber, we get sixteen inclusions of the form

$$\mathbb{P}^1 \times E \;\to\; X_z$$

and the pushforward with respect to each of these gives a two-dimensional image in the middle homology. In total this gives a 32-dimensional subspace of the middle homology of X_z. We obtain a surjection

$$i^* : H^3(X_z) \;\to\; \bigoplus_E H^2(\mathbb{P}^1) \otimes H^1(E)$$

and we use this to define the motive $U_z = \ker i^*$. This has the Hodge structure 1 3 3 1. To obtain a submotive with Hodge structure 1 1 1 1, we fix a non-trivial third root of unity ξ and define an automorphism $\phi : X_z \to X_z$ by

$$\phi : ((s:t),(X_1:Y_1:Z_1),(X_2:Y_2:Z_2)) \mapsto ((s:\xi\, t),(\xi\, X_1:Y_1:Z_1),(\xi^{-1}\, X_2:Y_2:Z_2)) .$$

This generates an action of the cyclic group C_3 on X_z. The action of ϕ on $H^0(X_z)$ and $H^6(X_z)$ is clearly trivial. We can also explicitly work out the action on $H^2(X_z)$ and $H^4(X_z)$, which both contain an eleven-dimensional trivial representation. Similarly, we can compute the action on the quotient $H^3(X_z)/U_z$, which contains an eight-dimensional trivial representation. From the Lefschetz trace formula it follows that the number of fixed points of ϕ is given by

$$\Lambda(\phi) \;=\; 1 + (11 - 8/2) - (8 - 24/2 + \mathrm{Tr}(\phi^* \,|\, U_z)) + (11 - 8/2) + 1 .$$

On the other hand, $\Lambda(\phi)$ is twice the number of fixed points on $(S_1)_{(0:1)} \times (S_z)_{(1:0)}$, which is just a product of an elliptic curve (with complex multiplication by $\mathbb{Q}(\sqrt{-3})$) and a union of three rational curves. The Lefschetz numbers of both factors are 3 and thus $\Lambda(\phi) = 2 \cdot 3 \cdot 3 = 18$. Hence, we have $\mathrm{Tr}(\phi^* \,|\, U_z) = 2 = 4 - 4/2$ and it follows that U_z contains a four-dimensional trivial representation, which we denote by V_z. More explicitly, we define the motive V_z as the kernel of

$$\frac{2 - \phi^* - (\phi^{-1})^*}{3} : U_z \to U_z .$$

The de Rham realization of V_z contains the holomorphic form

$$\Omega_z \;=\; \omega_1 \wedge \epsilon^* \omega_z \wedge \frac{s\,\mathrm{d}t - t\,\mathrm{d}s}{s\,t} .$$

An explicit computation shows that $\mathcal{L}$ is the Picard-Fuchs operator of Ω. In particular, V_z has the Hodge structure 1 1 1 1 and its de Rham realization is generated by Ω and its first three derivatives (evaluated at z).

Modularity of the Galois representations

To study the Galois representations of the fiber X_{z_*} over $z_* = -1/2^3 3^6$, we compute Lefschetz numbers of F_q and $\phi^{\pm 1} \circ F_q$ for $q = p$ and $q = p^2$ with small primes p. We do this by counting fixed points over $\overline{\mathbb{F}_p}$. For this, we use that the fixed points of F_q are defined over $\mathbb{F}_q$ while the ones of $\phi^{\pm 1} \circ F_q$ are defined over $\mathbb{F}_{q^3}$ or $\mathbb{F}_{q^2}$ depending on whether -3 is a square in $\mathbb{F}_q$ or not (since then $(\phi^{\pm 1} \circ F_q)^3 = F_q^3$ or $(\phi^{\pm} \circ F_q)^2 = F_q^2$). The primes of bad reduction are 2 and 3 and using the script `PointCounting.gp` from [17] we obtain the following numbers:

p	$\Lambda(F_p)$	$\Lambda(\phi^{\pm 1} \circ F_p)$	$\Lambda(F_p^2)$	$\Lambda(\phi^{\pm 1} \circ F_p^2)$
5	318	318	32292	19917
7	1512	693	176148	133371
11	2346	2346	2098764	1868985
13	4752	3582	5384448	5031576
17	7128	7128	25921296	24709230
19	16200	9360	49504608	47971080
23	15996	15996	153762192	149943870

The action of the Galois group on the cohomologies of even weight can be worked out explicitly by looking at the action on the corresponding algebraic cycles. The action on the ℓ-adic realizations of the quotient $H^3(X_{z_*})/U_{z_*}$ can be worked out as well by looking at the action on the corresponding products $\mathbb{P}^1 \times E$. It follows that

$$\begin{aligned}&\frac{1}{3}(\Lambda(F_q) + \Lambda(\phi \circ F_q) + \Lambda(\phi^{-1} \circ F_q))\\ &= 1 + (9 + 2\,(\tfrac{-3}{q}))\, q - \mathrm{Tr}\big(F_q^* | V_{z_*}\big) - 4\, q\; \mathrm{Tr}\big(F_q^* | H^1(E)\big) + (9 + 2\,(\tfrac{-3}{q}))\, q^2 + q^3 ,\end{aligned}$$

where the Legendre symbol $(\frac{-3}{q})$ is ± 1 depending on whether -3 is a square in $\mathbb{F}_q$ and E is the elliptic curve $X^3+Y^3+Z^3$, which has complex multiplication by $\mathbb{Q}(\sqrt{-3})$. Counting points of E over $\mathbb{F}_q$, we obtain $\mathrm{Tr}\big(F_q^*|H^1(E)\big)$. We can then compute the traces of F_p^* and $F_{p^2}^*$ on V_{z_*} and combining this with Poincaré duality, we get the characteristic polynomials listed below:

p	$\det\big(1-T\,F_p^*\vert V_{z_*}\big)$
5	$(1-3\,T+125\,T^2)\,(1-15\,T+125\,T^2)$
7	$(1-29\,T+343\,T^2)\,(1+7\,T+343\,T^2)$
11	$(1+57\,T+1331\,T^2)\,(1+33\,T+1331\,T^2)$
13	$(1-20\,T+2197\,T^2)\,(1+52\,T+2197\,T^2)$
17	$(1+72\,T+4913\,T^2)\,(1+4913\,T^2)$
19	$(1+106\,T+6859\,T^2)\,(1-38\,T+6859\,T^2)$
23	$(1-174\,T+12167\,T^2)\,(1+138\,T+12167\,T^2)$

We remark that these polynomials agree with the polynomials obtained from the deformation method. In terms of the unique newform $f\in S_4(\Gamma_0(54))$ with Hecke eigenvalue $a_5=3$ and the unique newform $g\in S_2(\Gamma_0(54))$ with Hecke eigenvalue $a_5=3$, whose Hecke eigenvalues can be computed using the script `HeckeEigenvalues.gp` from [17], we see that for the primes considered above

$$\det\big(1-T\,F_p^*|V_{z_*}\big) \;=\; \det(1-T\,\rho_{f,\ell}(F_p))\,\det(1-p\,T\,\rho_{g,\ell}(F_p))\,.$$

This suggests that up to semisimplification the Galois representations on the ℓ-adic realizations of V_{z_*} are isomorphic to $\rho_{f,\ell}\oplus\rho_{g,\ell}(-1)$. Using

$$\Lambda(F_q) \;=\; 1+(13+6\,(\tfrac{-3}{q}))\,q-\mathrm{Tr}\big(F_q^*|U_{z_*}\big)-16\,q\;\mathrm{Tr}\big(F_q^*|H^1(E)\big)+(13+6\,(\tfrac{-3}{q}))\,q^2+q^3\,,$$

we can also compute the characteristic polynomials of Frobenius elements on U_{z_*}/V_{z_*}. These are given in the following table:

p	$\det\big(1-T\,F_p^*\vert U_{z_*}/V_{z_*}\big)$
5	$(1-15\,T+125\,T^2)\,(1+15\,T+125\,T^2)$
7	$(1+7\,T+343\,T^2)\,(1+7\,T+343\,T^2)$
11	$(1+33\,T+1331\,T^2)\,(1-33\,T+1331\,T^2)$
13	$(1+52\,T+2197\,T^2)\,(1+52\,T+2197\,T^2)$
17	$(1+4913\,T^2)\,(1+4913\,T^2)$
19	$(1-38\,T+6859\,T^2)\,(1-38\,T+6859\,T^2)$
23	$(1+138\,T+12167\,T^2)\,(1-138\,T+12167\,T^2)$

For the primes considered above, we observe that

$$\det\big(1-T\,F_p^*|U_{z_*}/V_{z_*}\big) \;=\; \det(1-p\,T\,\rho_{g,\ell}(F_p))\,\det(1-p\,T\,\rho_{g_1,\ell}(F_p))\,,$$

where g_1 is the unique newform in $S_2(\Gamma_0(54))$ with Hecke eigenvalue $a_5=-3$ (i.e. the twist of g by $(\frac{-3}{\cdot})$). Since the elliptic curve E given by $X^3+Y^3+Z^3=0$ is associated with the unique newform $g_2\in S_2(\Gamma_0(27))$, we further know that

$$\det\big(1-T\,F_p^*|H^3(X_{z_*})/U_{z_*}\big) \;=\; \det(1-p\,T\,\rho_{g_2,\ell}(F_p))^{16}\,.$$

We can thus collect our observations in the conjecture that, for a suitable choice of local L-factors for primes of bad reduction, the Hasse-Weil zeta function is given by

$$\zeta(X_{z_*},s) \;=\; \frac{L(f,s)\,L(g,s-1)^2\,L(g_1,s-1)\,L(g_2,s-1)^{16}}{\zeta(s)\,\zeta(s-1)^{13}\,L((\frac{-3}{\cdot}),s-1)^6\,\zeta(s-2)^{13}\,L((\frac{-3}{\cdot}),s-2)^6\,\zeta(s-3)}\,.$$

Modularity of the periods

We end this section with a numerical study of the period matrix of V_{z_*}. The computations were done using the script `PeriodIdentities.gp` from [17].

To compute a period matrix of V_{z_*}, we first note that for $|z| < 1/3^6$ one of the periods of Ω_z is given by

$$\Pi_1(z) \;=\; (2\pi i)^3 \sum_{n=0}^{\infty} \left(\frac{(3\,n)!}{n!^3}\right)^2 z^n\,.$$

To get a basis of period functions of Ω, we can look at the monodromy of this period with respect to z. This is particularly well understood for hypergeometric period functions (see e.g. [36]). The upshot for our situation is that we can describe a basis of period functions of Ω as follows. First, we define a local basis of solutions of $\mathcal{L}$ by[1]

$$\varpi(z) \;=\; \begin{pmatrix} f_1(z) \\ \log(-z)\,f_1(z) + f_2(z) \\ \frac{1}{2}\,\log(-z)^2 f_1(z) + \log(-z)\,f_2(z) + f_3(z) \\ \frac{1}{6}\,\log(-z)^3 f_1(z) + \frac{1}{2}\,\log(-z)^2 f_2(z) + \log(-z)\,f_3(z) + f_4(z) \end{pmatrix}$$

with convergent power series normalized by $f_1(0) = 1$ and $f_2(0) = f_3(0) = f_4(0) = 0$. A basis of period functions of Ω is then given by (an analytic continuation of)

$$\Pi \;=\; \begin{pmatrix} (2\pi i)^3 & 0 & 0 & 0 \\ 0 & (2\pi i)^2 & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -16\,\zeta(3) & 0 & 0 & 1 \end{pmatrix} \varpi\,.$$

Taking derivatives, we obtain the matrix of period functions $T = (\Pi, \Theta\Pi, \Theta^2\Pi, \Theta^3\Pi)$. After evaluating at z_* and rounding to ten digits of precision, this gives the period matrix

$$T(z_*) \;=\; \begin{pmatrix} -246.5748210\,i & 1.422681792\,i & 1.322994856\,i & 1.139691091\,i \\ 341.4537969 & -40.08652727 & -0.3421868473 & -0.07458997994 \\ 236.3008701\,i & -54.26089900\,i & 6.584390256\,i & 0.3184178716\,i \\ -128.0451738 & 37.46236623 & -8.758887010 & 0.9736207171 \end{pmatrix}.$$

We now compare the period matrix $T(z_*)$ with the period matrices of the newforms f and g. Choosing the Eichler integral $\widetilde{f}_\infty$, we define the periods $\omega_f^\pm$ by

$$\begin{aligned} r_f\left(\left(\begin{smallmatrix} 23 & -3 \\ 54 & -7 \end{smallmatrix}\right)\right) \;&=\; \frac{(2\pi i)^3}{2} \int_{7/54}^{\infty} (\tau - z)^2\, f(z)\,\mathrm{d}z \\ &=\; \omega_f^+\,(-5 + 75\,\tau - 288\,\tau^2) + \omega_f^-\,(13 - 207\,\tau + 756\,\tau^2)\,. \end{aligned}$$

Choosing the Eichler integral $\widetilde{g}_\infty$, we define the periods $\omega_g^\pm$ by

$$r_g\left(\left(\begin{smallmatrix} 43 & -4 \\ 54 & -5 \end{smallmatrix}\right)\right) \;=\; 2\pi i \int_{5/54}^{\infty} g(z)\,\mathrm{d}z \;=\; \omega_g^+ - \omega_g^-\,.$$

Rounded to ten digits of precision, we have

$$\begin{aligned} \omega_f^+ &= 6.323218461\,, & \omega_f^- &= 0.7610333982\,i\,, \\ \omega_g^+ &= 1.052362238\,, & \omega_g^- &= 0.8924581010\,i\,. \end{aligned}$$

[1] We choose $-z$ as the argument of the logarithms since we eventually want to evaluate the solutions at the point z_* on the negative real axis and this makes the structure of the solutions simpler.

To compute the quasiperiods of f and g, we need meromorphic partners F and G. We define these as quotients of holomorphic modular forms by the form $h \in M_4(\Gamma_0(54))$ with the maximal vanishing order 36 at ∞. This can be expressed as the eta quotient

$$h(\tau) = \frac{\eta(9\tau)^4\eta(54\tau)^{24}}{\eta(18\tau)^8\eta(27\tau)^{12}}\,.$$

We choose the meromorphic partners F and G specified by their numerators $h\,F \in S_8(\Gamma_0(54))$ and $h\,G \in S_6(\Gamma_0(54))$ with the Fourier expansions

$$\begin{aligned}
h(\tau)\,F(\tau) = {} & \frac{158171}{9}q^{13} + \frac{10648}{9}q^{14} + \frac{32000}{3}q^{16} + \frac{54872}{9}q^{17} + \frac{142477}{9}q^{19} - \frac{4096}{9}q^{20} \\
& - \frac{566828}{9}q^{22} - \frac{42592}{9}q^{23} - \frac{297485}{9}q^{25} - \frac{219488}{9}q^{26} - \frac{539188}{9}q^{28} + \frac{13640}{9}q^{29} \\
& + \frac{1320661}{9}q^{31} + \frac{106352}{9}q^{32} + \frac{613940}{9}q^{34} + \frac{548480}{9}q^{35} + \frac{1394015}{9}q^{37} + \frac{51392}{3}q^{38} \\
& - \frac{293248}{3}q^{40} + \frac{148336}{9}q^{41} + \frac{2404846}{9}q^{43} + \frac{91744}{9}q^{44} - 121536\,q^{46} - \frac{5962616}{9}q^{47} \\
& + \frac{301006}{3}q^{49} + \frac{4376200}{9}q^{50} - \frac{8529572}{9}q^{52} - \frac{3064016}{3}q^{53} - \frac{8464213}{9}q^{55} \\
& + \frac{24447008}{9}q^{56} - \frac{12319156}{9}q^{58} - \frac{678016}{3}q^{59} + O(q^{60}) \\
h(\tau)\,G(\tau) = {} & 10\,q^{31} + 8\,q^{32} + 4\,q^{35} + 32\,q^{37} - 38\,q^{38} - 6\,q^{40} + 66\,q^{41} + O(q^{42})\,.
\end{aligned}$$

Choosing the Eichler integrals $\widetilde{F}_\infty$ and $\widetilde{G}_\infty$, we define the quasiperiods $\eta_f^\pm$ and $\eta_g^\pm$ by

$$\begin{aligned}
r_F\left(\left(\begin{smallmatrix} 23 & -3 \\ 54 & -7 \end{smallmatrix}\right)\right) &= \eta_f^+\,(-5 + 75\,\tau - 288\,\tau^2) + \eta_f^-\,(13 - 207\,\tau + 756\,\tau^2) \\
r_G\left(\left(\begin{smallmatrix} 43 & -4 \\ 54 & -5 \end{smallmatrix}\right)\right) &= \eta_g^+ - \eta_g^-\,.
\end{aligned}$$

Rounded to ten digits of precision, we have

$$\begin{aligned}
\eta_f^+ &= 64915.70758\,, & \eta_f^- &= 7773.726564\,i\,, \\
\eta_g^+ &= 32.63160583\,, & \eta_g^- &= 33.64385855\,i
\end{aligned}$$

and the Legendre relations read

$$\det\begin{pmatrix} \omega_f^+ & \eta_f^+ \\ \omega_f^- & \eta_f^- \end{pmatrix} = (2\pi i)^3 \qquad \text{and} \qquad \det\begin{pmatrix} \omega_g^+ & \eta_g^+ \\ \omega_g^- & \eta_g^- \end{pmatrix} = 2\pi i\,.$$

From the modularity of the Galois representations, we expect that $T(z_*)$ can be expressed in terms of the periods and quasiperiods of f and g. Numerically, we indeed observe that

$$T(z_*) = A \begin{pmatrix} \begin{pmatrix} \omega_f^+ & \eta_f^+ \\ \omega_f^- & \eta_f^- \end{pmatrix} & 0 \\ 0 & 2\pi i \begin{pmatrix} \omega_g^+ & \eta_g^+ \\ \omega_g^- & \eta_g^- \end{pmatrix} \end{pmatrix} B$$

with

$$A = \begin{pmatrix} 0 & -48 & -48 & 0 \\ 8 & 0 & 0 & -24 \\ 0 & 46 & -18 & 0 \\ -3 & 0 & 0 & -23 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \frac{27}{4} & -\frac{9}{8} & \frac{1}{8} & \frac{21}{8} \\ 0 & 0 & 0 & -\frac{1}{3888} \\ 0 & \frac{1}{8} & \frac{1}{8} & -\frac{7}{162} \\ 0 & 0 & -\frac{1}{216} & \frac{5}{3888} \end{pmatrix}.$$

5.2 A product of motives of elliptic modular forms

In the first part of this section, we consider a hypergeometric Calabi-Yau operator $\mathcal{L}$ with a symmetry $z \mapsto 1/2^{16}z$. Experimental computations suggest that Galois representations

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$$

associated with the regular fixed point $z_* = -1/2^8$ of the symmetry split (after restriction to the absolute Galois group of $K = \mathbb{Q}(\sqrt{-1})$, embedding into $\mathrm{GL}_4(K_\lambda)$ for primes λ over ℓ and semisimplifying) to sums of Galois representations $\rho_{\pm,\lambda} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(K_\lambda)$. To make this more rigorous, we construct in the second part a family X of Calabi-Yau threefolds associated with $\mathcal{L}$. The middle cohomology of each fiber X_z contains a four-dimensional part V_z whose variation with z is described by $\mathcal{L}$. We define isomorphisms $\phi_z : X_z \to X_{1/2^{16}z}$ corresponding to the symmetry of $\mathcal{L}$. The automorphism on the fixed fiber X_{z_*} can be used to split V_{z_*} into two parts which give the representations $\rho_{\pm,\lambda}$. In the third part, we prove that the representations $\rho_{\pm,\lambda}$ are up to semisimplification the tensor product of representations associated with a newform f of weight 3 and a Größencharakter ψ. As a consequence, the four-dimensional representations ρ_ℓ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are up to semisimplification tensor products of representations associated with the newform f and a newform g of weight 2 with complex multiplication by K. In the last part, we present numerical computations which suggest that the period matrix of V_{z_*} is given by the tensor product of period matrices of f and g.

Experimental computations

We start with an experimental study of the Calabi-Yau operator

$$\mathcal{L} = \Theta^4 - 2^8\, z\,(\Theta + 1/2)^4 \qquad \text{with} \qquad \Theta = z\frac{\mathrm{d}}{\mathrm{d}z}\,.$$

The operator is hypergeometric and it is the third operator from the list [1]. Using the script `DeformationMethod.gp` from [17], we compute the associated characteristic polynomials for all primes $11 \le p \le 997$ and all parameters $z \in \mathbb{F}_p$ of good reduction. To find Galois representations which split, we first look at factorizations of these polynomials over $\mathbb{Z}$. The number of factorizations that occur for each prime p is depicted in Figure 5.2. We observe that almost always the polynomials factor an even number of times. The reason for this is the symmetry $z \mapsto 1/2^{16}z$ of the differential operator $\mathcal{L}$ which, if we ignore the regular fixed point $z_* = -1/2^8$, causes the factorizations to come in pairs. With the available data we are not able to observe persistent factorizations over $\mathbb{Z}$. We continue by looking for factorizations over $\mathbb{Z}[\sqrt{-1}]$.

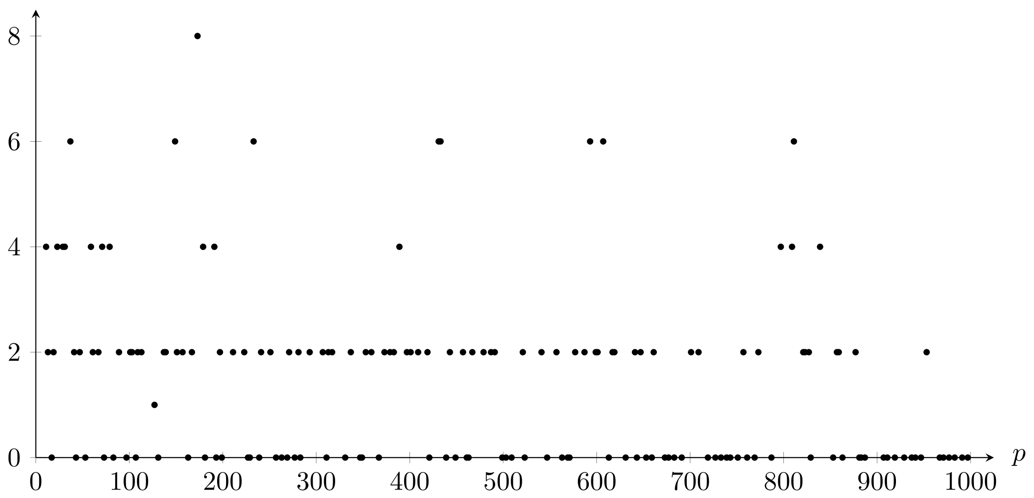


Figure 5.2: Number of factorizations over $\mathbb{Z}$ of characteristic polynomials for small primes p.

The number of factorizations over $\mathbb{Z}[\sqrt{-1}]$ is depicted in Figure 5.3. Now there are more primes for which at least one factorization occurs and for about half of the primes there is an odd number of factorizations. More precisely, there is an odd number of factorizations if $p \equiv 1 \bmod 4$ and, by the symmetry, this implies that the Frobenius polynomials for the regular fixed point z_* factor if $p \equiv 1 \bmod 4$. These are exactly the primes which split in the ring of integers of $K = \mathbb{Q}(\sqrt{-1})$ and we thus expect that Galois representations $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$ associated with z_* split after restriction to $\mathrm{Gal}(\overline{K}/K)$, embedding into $\mathrm{GL}_4(K_\lambda)$ for primes λ over ℓ and semisimplifying.

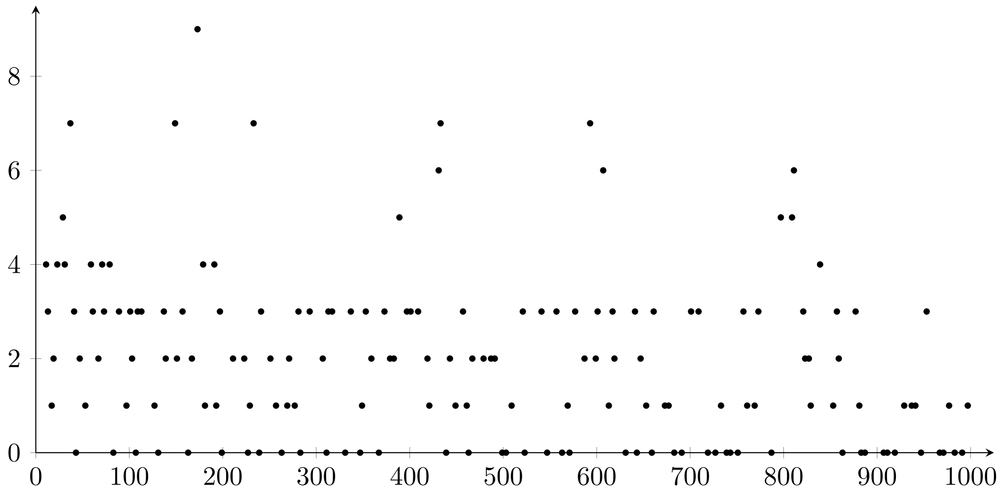


Figure 5.3: Number of factorizations over $\mathbb{Z}[\sqrt{-1}]$ of characteristic polynomials for small primes p.

Geometric construction

To construct a family $X \to \mathbb{P}^1 \smallsetminus \{0, 1/2^8, \infty\}$ of Calabi-Yau threefolds associated with $\mathcal{L}$, we first note that for $|z| < 1/2^8$ the holomorphic solution has the form

$$F(z) \;=\; \sum_{n=0}^{\infty} \binom{2\,n}{n}^4 z^n\,.$$

Comparing this with

$$f(t) \;=\; \sum_{n=0}^{\infty} \binom{2\,n}{n}^2 t^n \;=\; \frac{1}{\pi} \int_{-\infty}^{-1} \frac{\mathrm{d}x}{\sqrt{-x\,(x+1)\,(x+16\,t)}}$$

for $|t| < 1/2^4$, we see that F is the Hadamard product of f with itself, i.e.

$$F(z) \;=\; \frac{1}{2\pi i} \oint f(t)\, f(z/t)\, \frac{\mathrm{d}t}{t}\,.$$

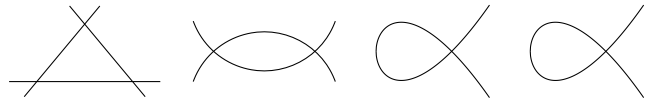
This suggests that we can construct X_z from a fiber product of elliptic surfaces described by the affine equations $y^2 = x\,(x+1)\,(x+16\,t)$ and $y^2 = x\,(x+1)\,(x+16\,z/t)$. We follow this approach, but it turns out to be beneficial to replace the variable t by t^2. To obtain for any $z \in \mathbb{P}^1 \smallsetminus \{0, \infty\}$ an elliptic surface S_z associated with the affine equation $y^2 = x\,(x+1)\,(x+16\,z\,t^2)$, we consider the fibration given by the hypersurface consisting of all $((s:t),(X:Y:Z)) \in \mathbb{P}^1 \times \mathbb{P}^2$ satisfying

$$s\,X\,(X^2 + 2\,Z\,(8\,z\,Z + Y)) + t\,Z\,(16\,z\,X^2 - Y^2) \;=\; 0\,,$$

together with the projection on the first factor. A birational relation with the affine equation is obtained by the identification

$$(x, y) \;=\; \left(\frac{t}{s}\frac{X}{Z}, \frac{t}{s}\frac{X}{Z} - \left(\frac{t}{s}\right)^2 \frac{Y}{Z} \right).$$

The singular fibers of this fibration are as follows:

base point	$(0:1)$	$(1:0)$	$(-4\sqrt{z}:1)$	$(4\sqrt{z}:1)$
sketch of fiber				

Each singular fiber contains one surface singularity and blowing these up once we obtain the elliptic surface S_z. The singular fibers of S_z have the following Kodaira classification:

base point	$(0:1)$	$(1:0)$	$(-4\sqrt{z}:1)$	$(4\sqrt{z}:1)$
Kodaira classification	I_4	I_4	I_2	I_2

Choosing for example the point $(0:1:0)$, we give S_z the structure of a rational elliptic surface with section.

We can now construct a family of Calabi-Yau threefolds $X \to \mathbb{P}^1 \smallsetminus \{0, 1/2^8, \infty\}$ and study its cohomology as reviewed in Section 4.3. For any $z \in \mathbb{P}^1 \smallsetminus \{0, 1/2^8, \infty\}$, we consider the fiber product

$$S_1 \times_{\mathbb{P}^1} \epsilon^* S_z$$

with $\epsilon : (s:t) \mapsto (t:s)$. After a small projective resolution this gives a Calabi-Yau threefold X_z. The second homology of $S_1 \times_{\mathbb{P}^1} \epsilon^* S_z$ is generated by one fiber, $S_1 \times_{\mathbb{P}^1} \{0\}$, $\{0\} \times_{\mathbb{P}^1} \epsilon^* S_z$ and all but one components of all fibers. This gives the Hodge number

$$h^{1,1}(X_z) = 1 + 2 + 2 \cdot (4 \cdot 4 - 1) + 4 \cdot (2 \cdot 1 - 1) = 37 .$$

On the other hand, the Euler number $\chi(X_z)$ equals 64 (twice the number of nodes that we resolved) and thus

$$h^{2,1}(X_z) = h^{1,1}(X_z) - \frac{\chi(X_z)}{2} = 5 .$$

Further, X_z has four fibers of the form $I_2 \times E$ with different elliptic curves E. Choosing one of the components of each I_2 fiber, we get four inclusions of the form

$$\mathbb{P}^1 \times E \to X_z$$

and the pushforward with respect to each of these gives a two-dimensional image in the middle homology. In total, this gives an eight-dimensional subspace of the middle homology of X_z. We obtain a surjection

$$i^* : H^3(X_z) \to \bigoplus_E H^2(\mathbb{P}^1) \otimes H^1(E)$$

and we use this to define the motive $V_z = \ker i^*$. This has the Hodge structure 1 1 1 1. The de Rham realization of V_z contains the holomorphic form Ω_z which, expressed in terms of the affine coordinates t, (x_1, y_1) and (x_2, y_2), has the form

$$\frac{\mathrm{d}x_1}{y_1} \wedge \frac{\mathrm{d}x_2}{y_2} \wedge \frac{\mathrm{d}t}{t} .$$

An explicit computation shows that $\mathcal{L}$ is the Picard-Fuchs operator of Ω. In particular, Ω and its first three derivatives (evaluated at z) generate the de Rham realization of V_z.

To finish the geometric construction, we give a geometric realization of the symmetry $z \mapsto 1/2^{16}z$ of $\mathcal{L}$ and study its regular fixed point. To do so, we use isogenies $S_z \to S_z$ of degree one which map the fiber over $(s:t)$ to the fiber over $(-16\,z\,t : s)$. After choosing a square root of z, these can be given in affine coordinates t, (x, y) by

$$(x, y) \mapsto \left(\frac{x}{(4\sqrt{z}\,t)^2}, \frac{y}{(4\sqrt{z}\,t)^3} \right) .$$

Acting with these on the factors of the fiber product, identifying $(S_z)_{(-16\,z\,s:t)}$ with $(S_{1/2^{16}z})_{(s:-16\,t)}$ and lifting to the resolution gives isomorphisms $\phi_z : X_z \to X_{1/2^{16}z}$. Restricting to the fixed fiber X_{z_*} over $z_* = -1/2^8$, we get an automorphism $\phi_{z_*} : X_{z_*} \to X_{z_*}$ defined over $K = \mathbb{Q}(\sqrt{-1})$, where we choose 1 as the square root of 1 and $\sqrt{-1}/16$ as the square root of z_*. Acting on Ω and its derivatives, we see that ϕ_{z_*} splits V_{z_*} into two parts with eigenvalues $\pm\sqrt{-1}$ and Hodge structures 0 1 0 1 and 1 0 1 0. For the ℓ-adic realizations of V_{z_*}, this implies that the associated Galois representations $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$ split to sums of two-dimensional representations

$$\rho_{\pm,\lambda} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(K_\lambda)$$

for any prime λ lying over ℓ. Since X_{z_*} has good reduction for all rational primes $p \neq 2$, these representations are unramified at all primes $\mathfrak{p} \neq 1 + \sqrt{-1}$ which do not divide ℓ.

Modularity of the Galois representations

We follow similar steps from [25] to prove the modularity of the representations ρ_ℓ. We start by giving two propositions which allow us to prove modularity by comparing finitely many characteristic polynomials of Frobenius elements.

Proposition 8. *Let* $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F}_2)$ *be a continuous representation unramified outside* $\{1 + \sqrt{-1}\}$. *Then* $\mathrm{Tr}(\rho) = 0$.

Proof. The kernel of ρ is an open normal subgroup of $\mathrm{Gal}(\overline{K}/K)$ and this gives a finite Galois extension L/K defined by $L = \overline{K}^{\ker\rho}$. This extension is unramified outside $\{1 + \sqrt{-1}\}$ and its Galois group is isomorphic to the image of ρ and hence (since $\mathrm{GL}_2(\mathbb{F}_2)$ is isomorphic to S_3) to a subgroup of S_3. Now suppose that $\mathrm{Tr}(\rho) \neq 0$. The elements in $\mathrm{GL}_2(\mathbb{F}_2)$ with trace 1 are exactly the elements of order three and thus $\mathrm{Gal}(L/K)$ is isomorphic to S_3 or C_3. It follows that L/K is the Galois closure of a degree three extension M/K. The corresponding degree six extension $M/\mathbb{Q}$ is unramified outside $\{2\}$. Using the database [46], we find that such extensions do not exist. □

Proposition 9. *Let* $\rho_1, \rho_2 : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{Z}_2[\sqrt{-1}])$ *be continuous representations unramified outside* $S = \{1+\sqrt{-1}\}$. *Suppose that the characteristic polynomials of* $\rho_1(F_\mathfrak{p})$ *and* $\rho_2(F_\mathfrak{p})$ *are equal for all* $\mathfrak{p} \in T = \{2+\sqrt{-1}, 3, 2+3\sqrt{-1}\}$. *Then* ρ_1 *and* ρ_2 *have isomorphic semisimplifications.*

Proof. Using Proposition 8, we find that the reductions of the representations to $\mathrm{GL}_2(\mathbb{F}_2)$ have vanishing traces. Hence, the claim follows from theorem 4.3 in [52] once we have shown that the image of $\{F_\mathfrak{p}\}_{\mathfrak{p}\in T}$ in the $\mathbb{F}_2$-vector space $\mathrm{Gal}(K_S/K)$ is non-cubic. Here, K_S is the compositum of all quadratic extensions of K unramified outside S and a subset U of a vector space V is called non-cubic if every homogeneous function of degree three on V vanishes if it vanishes on U. The extension K_S/K is generated by $\sqrt[4]{-1}$ and $\sqrt{1+\sqrt{-1}}$ and thus $\mathrm{Gal}(K_S/K)$ is isomorphic to C_2^2. It only remains to show that the image of $\{F_\mathfrak{p}\}_{\mathfrak{p}\in T}$ in $\mathrm{Gal}(K_S/K)$ contains all three non-trivial elements. This can be seen from the following table, which gives the action of the elements $\{F_\mathfrak{p}\}_{\mathfrak{p}\in T}$ on the chosen generators of K_S/K:

$\mathfrak{p}$	$\sqrt[4]{-1}$	$\sqrt{1+\sqrt{-1}}$
$2+\sqrt{-1}$	-1	-1
3	1	-1
$2+3\sqrt{-1}$	-1	1

□

To use the previous propositions, we need to compute characteristic polynomials of $\rho_{\pm,\lambda}(F_\mathfrak{p})$ and find modular Galois representations with the same characteristic polynomials. For the first part, we compute Lefschetz numbers of $F_\mathfrak{p}$, $F_{\mathfrak{p}^2}$ and $\phi_{z_*} \circ F_\mathfrak{p}$ by counting fixed points over $\overline{\mathbb{F}_{\mathcal{N}(\mathfrak{p})}}$ and use the Lefschetz trace formula to obtain characteristic polynomials. For the counting, we use that the fixed points of $F_\mathfrak{p}^k$ are defined over $\mathbb{F}_{\mathcal{N}(\mathfrak{p})^k}$ and the ones of $\phi_{z_*} \circ F_\mathfrak{p}$ are defined over $\mathbb{F}_{\mathcal{N}(\mathfrak{p})^4}$

since $(\phi_{z_*} \circ F_{\mathfrak{p}})^4 = F_{\mathfrak{p}}^4$. Using the script `PointCounting.gp` from [17] we obtain the following numbers:

$\mathfrak{p}$	$\Lambda(F_{\mathfrak{p}})$	$\Lambda(F_{\mathfrak{p}}^2)$	$\Lambda(\phi_{z_*} \circ F_{\mathfrak{p}})$
$2+\sqrt{-1}$	1280	40000	208
3	4288	774208	1000
$2+3\sqrt{-1}$	8704	5884480	2688

The action of the Galois group on the cohomologies of even weight can be worked out explicitly by looking at the action on the corresponding algebraic cycles. The action on the ℓ-adic realizations of the quotient $H^3(X_{z_*})/V_{z_*}$ can be worked out as well by looking at the action on the corresponding products $\mathbb{P}^1 \times E$. It follows that

$$\Lambda(F_{\mathfrak{p}}^k) \;=\; 1 + 37\mathcal{N}(\mathfrak{p})^k - \mathrm{Tr}\big((F_{\mathfrak{p}}^k)^* | V_{z_*}\big) - 4\mathcal{N}(\mathfrak{p})^k\ \mathrm{Tr}\big((F_{\mathfrak{p}}^k)^* | H^1(E)\big) + 37\mathcal{N}(\mathfrak{p})^{2k} + \mathcal{N}(\mathfrak{p})^{3k}\,,$$

where E is the elliptic curve with affine equation $y^2 = x^3 - x$, which has complex multiplication by K. Counting points of E over $\mathbb{F}_{\mathcal{N}(\mathfrak{p})^k}$, we obtain $\mathrm{Tr}\big((F_{\mathfrak{p}}^k)^* | H^1(E)\big)$. We can then compute the traces of $F_{\mathfrak{p}}^*$ and $(F_{\mathfrak{p}}^2)^*$ on V_{z_*} and combining this with Poincaré duality, we get the characteristic polynomials listed below:

$\mathfrak{p}$	$\det\big(1 - T\,F_{\mathfrak{p}}^* | V_{z_*}\big)$
$2+\sqrt{-1}$	$1 + 4\,T - 130\,T^2 + 500\,T^3 + 15625\,T^4$
3	$1 + 12\,T + 1494\,T^2 + 8748\,T^3 + 531441\,T^4$
$2+3\sqrt{-1}$	$1 + 84\,T + 4238\,T^2 + 184548\,T^3 + 4826809\,T^4$

We remark that these agree with the polynomials obtained from the deformation method. Over $\mathcal{O}_K$, each of the polynomials factors to the product of the characteristic polynomials of $\rho_{+,\lambda}(F_{\mathfrak{p}})$ and $\rho_{-,\lambda}(F_{\mathfrak{p}})$. To find out which factor corresponds to which representation, we relate the number of fixed points of $\phi_{z_*} \circ F_{\mathfrak{p}}$ to the difference of the traces of $\rho_{+,\lambda}(F_{\mathfrak{p}})$ and $\rho_{-,\lambda}(F_{\mathfrak{p}})$. The action of ϕ_{z_*} on the cohomology groups of even weight can be worked out explicitly by looking at the action on the corresponding algebraic cycles and we find that $\mathrm{Tr}\big(\phi_{z_*}^* | H^0(X_{z_*})\big) = \mathrm{Tr}\big(\phi_{z_*}^* | H^6(X_{z_*})\big) = 1$ and $\mathrm{Tr}\big(\phi_{z_*}^* | H^2(X_{z_*})\big) = \mathrm{Tr}\big(\phi_{z_*}^* | H^4(X_{z_*})\big) = 3$. We also see that $\mathrm{Tr}\big(F_{\mathfrak{p}}^* \phi_{z_*}^* | H^3(X_{z_*})/V_{z_*}\big) = 0$ and the Lefschetz trace formula then gives

$$\Lambda(\phi_{z_*} \circ F_{\mathfrak{p}}) \;=\; 1 + 3\mathcal{N}(\mathfrak{p}) - \mathrm{Tr}\big(F_{\mathfrak{p}}^*\,\phi_{z_*}^* | V_{z_*}\big) + 3\mathcal{N}(\mathfrak{p})^2 + \mathcal{N}(\mathfrak{p})^3\,.$$

Combining this with

$$\mathrm{Tr}\big(F_{\mathfrak{p}}^*\,\phi_{z_*}^* | V_{z_*}\big) \;=\; \sqrt{-1}\,(\mathrm{Tr}(\rho_{+,\lambda}(F_{\mathfrak{p}})) - \mathrm{Tr}(\rho_{-,\lambda}(F_{\mathfrak{p}})))\,,$$

we end up with the following characteristic polynomials:

$\mathfrak{p}$	$\det(1 - T\,\rho_{\pm,\lambda}(F_{\mathfrak{p}}))$
$2+\sqrt{-1}$	$1 + (2 \pm 4\sqrt{-1})\,T - (75 \mp 100\sqrt{-1})\,T^2$
3	$1 + 6\,T + 729\,T^2$
$2+3\sqrt{-1}$	$1 + (42 \pm 28\sqrt{-1})\,T + (845 \pm 2028\sqrt{-1})\,T^2$

We proceed by defining Galois representations of a modular form and a Größencharakter whose products give the same characteristic polynomials as in the table above. The space of elliptic modular forms $S_3(\Gamma_0(32), (\frac{-4}{\cdot}))$ contains a newform f that is unique up to conjugation. This has the Fourier-expansion

$$f(\tau) \;=\; q + 4\sqrt{-1}\,q^3 + 2\,q^5 - 8\sqrt{-1}\,q^7 - 7\,q^9 - 4\sqrt{-1}\,q^{11} - 14\,q^{13} + \cdots$$

and for every prime λ of K we have an associated Galois representation

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \;\to\; \mathrm{GL}_2(K_\lambda)\,.$$

We define a Größencharakter ψ modulo $(1+\sqrt{-1})^4$ associated with the elliptic curve E with affine equation $y^2 = x^3 - x$. The value on ideals coprime to $1+\sqrt{-1}$ is given by $\psi((a)) = \chi(a)\,a$ with the character $\chi : (\mathcal{O}_K/(1+\sqrt{-1})^4)^\times \to \mathcal{O}_K$ determined by $\chi(\sqrt{-1}) = -\sqrt{-1}$ and $\chi(1+2\sqrt{-1}) = -1$. In terms of this Größencharakter, we have $\mathrm{Tr}\big((F_{\mathfrak{p}}^k)^*|H^1(E)\big) = \psi(\mathfrak{p})^k + \overline{\psi(\mathfrak{p})^k}$. Associated with the Größencharakter ψ, we have a Galois representation

$$\rho_{\psi,\lambda} : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_1(K_\lambda)$$

which satisfies

$$\rho_{\psi,\lambda}(F_{\mathfrak{p}}) \;=\; \psi(\mathfrak{p})\,.$$

The Hecke eigenvalues of f and the values of χ can be computed using the script `HeckeEigenvalues.gp` from [17] and for the primes in the table above we observe that

$$\begin{aligned}
\det(1 - T\,\rho_{+,\lambda}(F_{\mathfrak{p}})) \;&=\; \det\Big(1 - T\,\overline{\rho_{\psi,\lambda}(F_{\mathfrak{p}})}\,\rho_{f,\lambda}(F_{\mathcal{N}(\mathfrak{p})})\Big)\\
\det(1 - T\,\rho_{-,\lambda}(F_{\mathfrak{p}})) \;&=\; \det\big(1 - T\,\rho_{\psi,\lambda}(F_{\mathfrak{p}})\,\rho_{f,\lambda}(F_{\mathcal{N}(\mathfrak{p})})\big)\,.
\end{aligned}$$

As we show now, this suffices to conclude the equality for all primes $\mathfrak{p}$.

Theorem 10. *The representations*

$$\rho_{+,\lambda}\,,\;\; \overline{\rho_{\psi,\lambda}} \otimes \rho_{f,\lambda} : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_2(K_\lambda)$$

have isomorphic semisimplifications. The same holds for the representations

$$\rho_{-,\lambda}\,,\;\; \rho_{\psi,\lambda} \otimes \rho_{f,\lambda} : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_2(K_\lambda)\,.$$

Proof. By the compatibility of the representations, we can restrict to the case $\lambda = 1+\sqrt{-1}$. We can further conjugate the representations so that the image is in $\mathrm{GL}_2(\mathcal{O}_{K,\lambda})$ (see e.g. section 1 in [64]). From the computed characteristic polynomials and Proposition 9, it then follows that the representations have isomorphic semisimplifications. □

This result can be lifted to the representations ρ_ℓ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. To do so, we denote the unique newform in $S_2(\Gamma_0(32))$ by g. The associated representations $\rho_{g,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$ are then isomorphic to $\rho_{\psi,\lambda} \oplus \overline{\rho_{\psi,\lambda}}$ after restricting to $\mathrm{Gal}(\overline{K}/K)$ and embedding into $\mathrm{GL}_2(K_\lambda)$.

Corollary 1. *The representation $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$ (embedded into $\mathrm{GL}_4(K_\lambda)$) and the representation $\rho_{f,\lambda} \otimes \rho_{g,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(K_\lambda)$ have isomorphic semisimplifications.*

Proof. For rational primes p which are inert in $\mathcal{O}_K$, F_p^* exchanges the two two-dimensional parts of V_{z_*}. It follows that

$$\det\big(1 - T\,F_p^*|V_{z_*}\big) \;=\; 1 - \frac{\mathrm{Tr}\big((F_p^2)^*|V_{z_*}\big)}{2}\,T^2 + p^6\,T^4\,.$$

Combining this with Theorem 10, we find that the characteristic polynomials of $\rho_\ell(F_p)$ and of $\rho_{f,\lambda}(F_p) \otimes \rho_{g,\ell}(F_p)$ are equal for all rational primes $p \neq 2$. Hence, the representations have isomorphic semisimplifications. □

In summary, we can conclude that, for suitable choices of local L-factors for $p = 2$, the Hasse-Weil zeta function of X_{z_*} is given by

$$\zeta(X_{z_*}, s) \;=\; \frac{L(f \otimes g, s)\,L(g, s-1)^4}{\zeta(s)\,\zeta(s-1)^{36}\,L((\frac{-4}{\cdot}), s-1)\,\zeta(s-2)^{36}\,L((\frac{-4}{\cdot}), s-2)\,\zeta(s-3)}\,.$$

We remark that we got the idea that the representations ρ_ℓ have the structure of tensor products from Don Zagier.

Modularity of the periods

We end this section with a numerical study of the period matrix of V_{z_*}. The computations were done using the script `PeriodIdentities.gp` from [17].

For $|z| < 1/2^8$, one of the periods of Ω_z is given by

$$\oint \frac{\mathrm{d}t}{t} \int_{-\infty}^{-1} \frac{\mathrm{d}x_1}{\sqrt{x_1\,(x_1+1)\,(x_1+16\,t^2)}} \int_{-\infty}^{-1} \frac{\mathrm{d}x_2}{\sqrt{x_2\,(x_2+1)\,(x_2+16\,z/t^2)}}$$
$$= \frac{1}{4}\,(2\pi i)^3 \sum_{n=0}^{\infty} \binom{2\,n}{n}^4 z^n\,.$$

To get a basis of period functions of Ω_z, we can look at the monodromy of this period with respect to z. This can again be done with the analysis described in [36]. We can conclude that we can describe a basis of period functions of Ω as follows. First, we define a local basis of solutions of $\mathcal{L}$ by[2]

$$\varpi(z) \;=\; \begin{pmatrix} f_1(z) \\ \log(-z)\,f_1(z) + f_2(z) \\ \frac{1}{2}\,\log(-z)^2 f_1(z) + \log(-z)\,f_2(z) + f_3(z) \\ \frac{1}{6}\,\log(-z)^3 f_1(z) + \frac{1}{2}\,\log(-z)^2 f_2(z) + \log(-z)\,f_3(z) + f_4(z) \end{pmatrix}$$

with convergent power series normalized by $f_1(0) = 1$ and $f_2(0) = f_3(0) = f_4(0) = 0$. A basis of period functions of Ω is then given by (an analytic continuation of)

$$\Pi \;=\; \begin{pmatrix} (2\pi i)^3 & 0 & 0 & 0 \\ 0 & (2\pi i)^2 & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -8\,\zeta(3) & 0 & 0 & 1 \end{pmatrix} \varpi\,.$$

Taking derivatives, we obtain the matrix of period functions $T = (\Pi, \Theta\Pi, \Theta^2\Pi, \Theta^3\Pi)$. After evaluating at z_* and rounding to ten digits of precision, this gives the period matrix

$$T(z_*) \;=\; \begin{pmatrix} -235.9349240\,i & 9.718877474\,i & 6.246763306\,i & 1.994528703\,i \\ 215.6263142 & -40.43037144 & 0.8617089734 & 1.672830554 \\ 97.98253527\,i & -33.28643960\,i & 7.429239846\,i & 0.2851530083\,i \\ -38.47629526 & 15.21567188 & -5.413165819 & 1.285052944 \end{pmatrix}.$$

We now compare the period matrix $T(z_*)$ with the period matrices of the newforms f and g. Choosing the Eichler integral $\widetilde{f}_\infty$, we define the periods $\omega_f^\pm$ by

$$r_f\left(\left(\begin{smallmatrix} 129 & -25 \\ 160 & -31 \end{smallmatrix}\right)\right) \;=\; (2\pi i)^2 \int_{31/160}^{\infty} (\tau - z)\,f(z)\,\mathrm{d}z$$
$$= \omega_f^+\,((-28+20\,i)\,\tau + (5-4\,i)) + \omega_f^-\,((-88-48\,i)\,\tau + (17+9\,i))\,.$$

We remark that, in order to get a real and a purely imaginary period, we have chosen the splitting into $\omega_f^\pm$ that corresponds to the action of the involution ε on the period polynomials of $f + \overline{f}$ instead of f. Choosing the Eichler integral $\widetilde{g}_\infty$, we define the periods $\omega_g^\pm$ by

$$r_g\left(\left(\begin{smallmatrix} 27 & -11 \\ 32 & -13 \end{smallmatrix}\right)\right) \;=\; 2\pi i \int_{13/32}^{\infty} g(z)\,\mathrm{d}z \;=\; \omega_g^+ - \omega_g^-\,.$$

Rounded to ten digits of precision, we have

$$\omega_f^+ = 5.139721139\,, \qquad \omega_f^- = 3.975811091\,i\,,$$
$$\omega_g^+ = 1.311028777\,, \qquad \omega_g^- = 1.311028777\,i\,.$$

[2] As in the previous section, we choose $-z$ as the argument of the logarithms since we eventually want to evaluate the solutions at the point z_* on the negative real axis and this makes the structure of the solutions simpler.

Due to the complex multiplication of g, we have $\omega_g^- = i\,\omega_g^+$ and we can further express these in terms of values of the gamma function by

$$\omega_g^+ \;=\; \frac{1}{4\sqrt{2\pi}}\,\Gamma\left(\tfrac{1}{4}\right)^2 .$$

To compute the quasiperiods of f and g, we need meromorphic partners F and G. We define these as quotients of holomorphic modular forms by the form $h \in M_4(\Gamma_0(32))$ with the maximal vanishing order 16 at ∞. This can be expressed as the eta quotient

$$h(\tau) \;=\; \frac{\eta(32\tau)^{16}}{\eta(16\tau)^8} .$$

We choose the meromorphic partners F and G specified by their numerators $h\,F \in S_7(\Gamma_0(32), (\frac{-4}{\cdot}))$ and $h\,G \in S_6(\Gamma_0(32))$ with the Fourier expansions

$$\begin{aligned} h(\tau)\,F(\tau) \;&=\; \frac{25\sqrt{-1}}{2}\,q^{11} - 9\,q^{13} + 3\sqrt{-1}\,q^{15} - \frac{\sqrt{-1}}{2}\,q^{19} - q^{21} + O(q^{22}) \\ h(\tau)\,G(\tau) \;&=\; 2\,q^{15} + O(q^{18}) . \end{aligned}$$

Choosing the Eichler integrals $\widetilde{F}_\infty$ and $\widetilde{G}_\infty$, we define the quasiperiods $\eta_f^\pm$ and $\eta_g^\pm$ by

$$\begin{aligned} r_F\left(\left(\begin{smallmatrix} 129 & -25 \\ 160 & -31 \end{smallmatrix}\right)\right) \;&=\; \eta_f^+\,((-28+20\,i)\,\tau + (5-4\,i)) + \eta_f^-\,((-88-48\,i)\,\tau + (17+9\,i)) \\ r_G\left(\left(\begin{smallmatrix} 27 & -11 \\ 32 & -13 \end{smallmatrix}\right)\right) \;&=\; \eta_g^+ - \eta_g^- . \end{aligned}$$

Rounded to ten digits of precision, we have

$$\begin{aligned} \eta_f^+ &= 7.651002004\,, &\qquad \eta_f^- &= -1.762640177\,i\,, \\ \eta_g^+ &= -2.396280469\,, &\qquad \eta_g^- &= 2.396280469\,i \end{aligned}$$

and the Legendre relations read

$$\det\begin{pmatrix} \omega_f^+ & \eta_f^+ \\ \omega_f^- & \eta_f^- \end{pmatrix} \;=\; (2\pi i)^2\,i \qquad\text{and}\qquad \det\begin{pmatrix} \omega_g^+ & \eta_g^+ \\ \omega_g^- & \eta_g^- \end{pmatrix} \;=\; 2\pi i\,.$$

As a consequence of the complex multiplication of g, we have $\eta_g^- = -i\,\eta_g^+$ and thus

$$\eta_g^+ \;=\; -2\,\frac{(2\pi)^{3/2}}{\Gamma\left(\tfrac{1}{4}\right)^2} .$$

Due to the splitting of V_{z_*}, we know that the period matrix $T(z_*)$ can be brought to a block diagonal form after extending by i. From the modularity of the Galois representations, we further expect that it can be expressed in terms of products of the periods and quasiperiods of f and g. Numerically, we indeed observe that

$$T(z_*) \;=\; A\begin{pmatrix} \omega_g^+\begin{pmatrix} \omega_f^+ & \eta_f^+ \\ \omega_f^- & \eta_f^- \end{pmatrix} & 0 \\ 0 & \eta_g^+\begin{pmatrix} \omega_f^+ & \eta_f^+ \\ \omega_f^- & \eta_f^- \end{pmatrix} \end{pmatrix} B$$

with

$$A \;=\; \begin{pmatrix} 48\,i & -96 & -48\,i & 0 \\ 24 & 0 & -24 & -48\,i \\ 14\,i & -4 & 10\,i & -24 \\ 5 & 12\,i & 7 & 2\,i \end{pmatrix} \qquad\text{and}\qquad B \;=\; \begin{pmatrix} 4/3 & -1/3 & 1/16 & -1/192 \\ 0 & 0 & -1/96 & 1/128 \\ 0 & -1/12 & 1/24 & -5/768 \\ 0 & 0 & 0 & -1/1536 \end{pmatrix} .$$

5.3 A sum of motives of Hilbert modular forms

In the first part of this section, we consider again the Calabi-Yau operator $\mathcal{L}$ from the previous section. Experimental computations suggest that representations of the absolute Galois group of $K = \mathbb{Q}(\sqrt{2})$

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$$

associated with the point $z_* = \frac{17+12\sqrt{2}}{2^8}$ split (after embedding into $\mathrm{GL}_4(K_\lambda)$ for primes λ over ℓ and semisimplifying) to sums of two-dimensional Galois representations. To make this more rigorous, we use in the second part the geometric realization of a family X of Calabi-Yau threefolds associated with $\mathcal{L}$ from Section 5.2. We compute characteristic polynomials of the representations ρ_ℓ associated with the four-dimensional part V_{z_*} of the middle cohomology. The results suggest that the representations ρ_ℓ are up to semisimplification the sum of representations associated with a Hilbert newform f of weight $(4, 2)$ and its conjugate $\overline{f}$ of weight $(2, 4)$. In the third part, we present numerical computations which suggest that the period matrix can be brought to a block diagonal form. We are not aware of a general definition of periods of Hilbert newforms and hence we do not have periods which we can compare with the periods of V_{z_*}.

Experimental computations

We consider again the hypergeometric Calabi-Yau operator

$$\mathcal{L} = \Theta^4 - 2^8 z (\Theta + 1/2)^4 \qquad \text{with} \qquad \Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$$

from Section 5.2. Using the script `DeformationMethod.gp` from [17], we compute the associated characteristic polynomials for all primes $11 \leq p \leq 997$ and all parameters $z \in \mathbb{F}_p$ of good reduction. Now we look for factorizations of these polynomials over $\mathbb{Z}[\sqrt{2}]$. The number of factorizations that occur for each prime p is depicted in Figure 5.3. We observe that for all primes $p \equiv \pm 1 \bmod 8$ there are at least two factorizations. Restricting to primes $p \equiv \pm 1 \bmod 8$ with exactly two factorizations, we see that these come from the two reductions of $z_* = \frac{17+12\sqrt{2}}{2^8}$ to $\mathbb{F}_p$. Since all computed polynomials of this fiber factor, we expect that (after embedding into $\mathrm{GL}_4(K_\lambda)$ for primes λ over ℓ and semisimplifying) the associated Galois representations $\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$ split. We remark that the fibers over z_* and its Galois conjugate are related by the symmetry $z \mapsto 1/2^{16}z$. In particular, the associated characteristic polynomials agree.

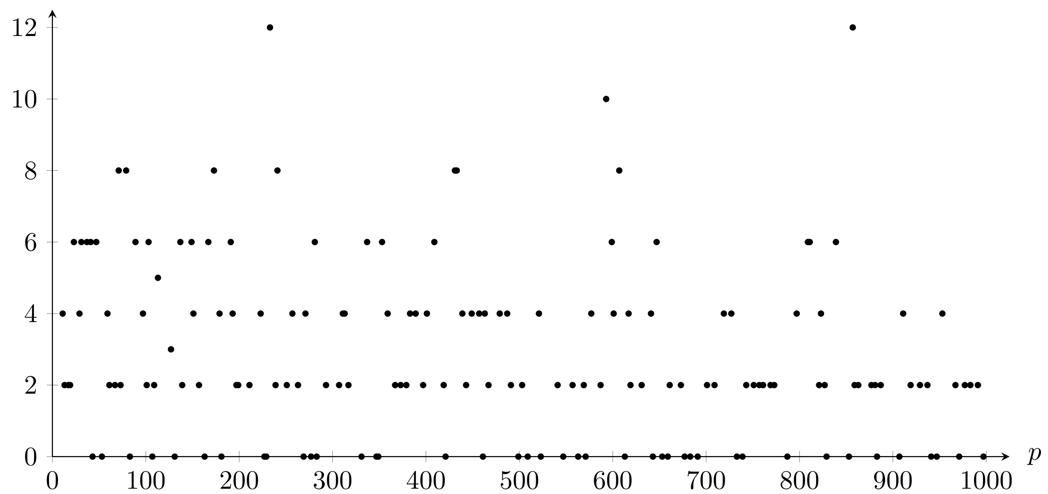


Figure 5.4: Number of factorizations over $\mathbb{Z}[\sqrt{2}]$ of characteristic polynomials for small primes p.

Modularity of the Galois representations

In Section 5.2, we constructed a family of Calabi-Yau threefolds X associated with $\mathcal{L}$. Now, we are interested in the fiber over $z_* = \frac{17+12\sqrt{2}}{2^8}$, which is defined over $K = \mathbb{Q}(\sqrt{2})$. We remark that this is isomorphic to its Galois conjugate, since the isomorphism $\phi_{z_*} : X_{z_*} \to X_{1/2^{16} z_*}$ is defined over K and relates z_* with its Galois conjugate. To study the Galois representations associated with X_{z_*}, we compute Lefschetz numbers of $F_\mathfrak{p}$ and $F_\mathfrak{p}^2$ for small primes p. Due to the isomorphism between X_{z_*} and its Galois conjugate, these only depend on the norm of $\mathfrak{p}$. The only prime of bad reduction is $\sqrt{2}$ and using the script `PointCounting.gp` from [17] we obtain the following numbers:

$\mathcal{N}(\mathfrak{p})$	$\Lambda(F_\mathfrak{p})$	$\Lambda(F_{\mathfrak{p}^2})$
7	2432	211264
3^2	4288	774208
17	16192	27284544
23	32384	158532928
5^2	40256	258714176
31	66560	921999424
41	130624	4854788672
47	187136	10961041472
71	547712	129044412736
73	589760	152388652096
79	728832	244532846656
89	997824	499306354752
97	1259584	836243840064

The action of the Galois group on the cohomologies of even weight can be worked out explicitly by looking at the action on the corresponding algebraic cycles. The action on the ℓ-adic realizations of the quotient $H^3(X_{z_*})/V_{z_*}$ can be worked out by looking at the action on the corresponding products $\mathbb{P}^1 \times E$, too. It follows that

$$\Lambda(F_\mathfrak{p}^k) \;=\; 1 + 37\mathcal{N}(\mathfrak{p})^k - \mathrm{Tr}\big((F_\mathfrak{p}^k)^*|V_{z_*}\big) - 4\mathcal{N}(\mathfrak{p})^k\ \mathrm{Tr}\big((F_\mathfrak{p}^k)^*|H^1(E)\big) + 37\mathcal{N}(\mathfrak{p})^{2k} + \mathcal{N}(\mathfrak{p})^{3k}\,,$$

where E is the elliptic curve with affine equation $y^2 = x^3 - x$. Counting points of E over $\mathbb{F}_{\mathcal{N}(\mathfrak{p})^k}$, we obtain $\mathrm{Tr}\big((F_\mathfrak{p}^k)^*|H^1(E)\big)$. We can then compute the traces of $F_\mathfrak{p}^*$ and $(F_\mathfrak{p}^2)^*$ on V_{z_*} and, combining this with Poincaré duality, we get the characteristic polynomials listed below:

$\mathcal{N}(\mathfrak{p})$	$\det\big(1 - T\,F_\mathfrak{p}^*|V_{z_*}\big)$
7	$(1 + (-16\sqrt{2} + 8)\,T + 343\,T^2)\,(1 + (16\sqrt{2} + 8)\,T + 343\,T^2)$
3^2	$(1 + 6\,T + 729\,T^2)\,(1 + 6\,T + 729\,T^2)$
17	$(1 + (-32\sqrt{2} + 46)\,T + 4913\,T^2)\,(1 + (32\sqrt{2} + 46)\,T + 4913\,T^2)$
23	$(1 + (-16\sqrt{2} - 104)\,T + 12167\,T^2)\,(1 + (16\sqrt{2} - 104)\,T + 12167\,T^2)$
5^2	$(1 - 10\,T + 15625\,T^2)\,(1 - 10\,T + 15625\,T^2)$
31	$(1 + (-128\sqrt{2} + 32)\,T + 29791\,T^2)\,(1 + (128\sqrt{2} + 32)\,T + 29791\,T^2)$
41	$(1 + (-64\sqrt{2} - 186)\,T + 68921\,T^2)\,(1 + (64\sqrt{2} - 186)\,T + 68921\,T^2)$
47	$(1 + (-96\sqrt{2} - 80)\,T + 103823\,T^2)\,(1 + (96\sqrt{2} - 80)\,T + 103823\,T^2)$
71	$(1 + (-304\sqrt{2} + 328)\,T + 357911\,T^2)\,(1 + (304\sqrt{2} + 328)\,T + 357911\,T^2)$
73	$(1 + (-64\sqrt{2} - 442)\,T + 389017\,T^2)\,(1 + (64\sqrt{2} - 442)\,T + 389017\,T^2)$
79	$(1 + (-32\sqrt{2} + 976)\,T + 493039\,T^2)\,(1 + (32\sqrt{2} + 976)\,T + 493039\,T^2)$
89	$(1 + (-704\sqrt{2} + 22)\,T + 704969\,T^2)\,(1 + (704\sqrt{2} + 22)\,T + 704969\,T^2)$
97	$(1 + (-352\sqrt{2} + 1086)\,T + 912673\,T^2)\,(1 + (352\sqrt{2} + 1086)\,T + 912673\,T^2)$

We remark that these polynomials agree with the ones obtained from the deformation method. In terms of the unique Hilbert newform $f \in S_{(4,2)}(\Gamma_0(2\sqrt{2}))$ and the conjugate $\overline{f} \in S_{(2,4)}(\Gamma_0(2\sqrt{2}))$, whose Hecke eigenvalues can be computed using the script `HeckeEigenvalues.mgm` from [17], we see that

$$\det\bigl(1 - T\, F_{\mathfrak{p}}^{*}|V_{z_*}\bigr) \;=\; \det(1 - T\,\rho_{f,\lambda}(F_{\mathfrak{p}}))\,\det\Bigl(1 - T\,\rho_{\overline{f},\lambda}(F_{\mathfrak{p}})\Bigr)\,.$$

This suggests that (after embedding into $\mathrm{GL}_2(K_\lambda)$ and semisimplifying) the Galois representations on the ℓ-adic realizations of V_{z_*} are isomorphic to $\rho_{f,\lambda} \oplus \rho_{\overline{f},\lambda}$. For a suitable choice of local L-factors for primes of bad reduction, the Hasse-Weil zeta function of X_{z_*} would then be given by

$$\zeta(X_{z_*}, s) \;=\; \frac{L(f,s)\,L(\overline{f},s)\,L_K(g,s-1)^4}{\zeta_K(s)\,\zeta_K(s-1)^{37}\,\zeta_K(s-2)^{37}\,\zeta_K(s-3)}\,.$$

Modularity of the period matrix

We end this section with a numerical study of the period matrix of V_{z_*}. The computations were done using the script `PeriodIdentities.gp` from [17].

To study the period matrix of V_{z_*}, we first define a local basis of solutions of $\mathcal{L}$ by

$$\varpi(z) \;=\; \begin{pmatrix} f_1(z) \\ \log(z)\, f_1(z) + f_2(z) \\ \frac{1}{2}\,\log(z)^2 f_1(z) + \log(z)\, f_2(z) + f_3(z) \\ \frac{1}{6}\,\log(z)^3 f_1(z) + \frac{1}{2}\,\log(z)^2 f_2(z) + \log(z)\, f_3(z) + f_4(z) \end{pmatrix}$$

with convergent power series normalized by $f_1(0) = 1$ and $f_2(0) = f_3(0) = f_4(0) = 0$. From the discussion in Section 5.2, it follows that a basis of period functions of Ω (which is different to the one defined in Section 5.2) is given by (an analytic continuation of)

$$\Pi \;=\; \begin{pmatrix} (2\pi i)^3 & 0 & 0 & 0 \\ 0 & (2\pi i)^2 & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -8\,\zeta(3) & 0 & 0 & 1 \end{pmatrix} \varpi\,.$$

Taking derivatives, we obtain the matrix of period functions $T = (\Pi, \Theta\Pi, \Theta^2\Pi, \Theta^3\Pi)$. Evaluating at $\sigma_\pm(z_*)$, where $\sigma_\pm : K \hookrightarrow \mathbb{C}$ corresponds to the embedding that maps $\sqrt{2}$ to $\pm\sqrt{2}$, we obtain two period matrices. Since X_{z_*} is isomorphic to its Galois conjugate, we just study the period matrix $T(\sigma_-(z_*))$. After rounding to ten digits of precision, we have

$$T(\sigma_-(z_*)) \;=\; A \begin{pmatrix} \omega_1^+ & \eta_1^+ & 0 & 0 \\ \omega_1^- & \eta_1^- & 0 & 0 \\ 0 & 0 & \omega_2^+ & \eta_2^+ \\ 0 & 0 & \omega_2^- & \eta_2^- \end{pmatrix} B$$

with

$$A = \begin{pmatrix} 0 & -6 & 0 & -6 \\ 6 & 0 & 6 & 0 \\ 0 & 2+3\sqrt{2} & 0 & 2-3\sqrt{2} \\ 2-3\sqrt{2} & 0 & 2+3\sqrt{2} & 0 \end{pmatrix},\; B = \begin{pmatrix} 32 & 2\sqrt{2}-8 & 2-\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 32 & 0 & 7\sqrt{2}-10 \\ 0 & 0 & -16 & 12-7\sqrt{2} \end{pmatrix}$$

and

$$\begin{pmatrix} \omega_1^+ & \eta_1^+ \\ \omega_1^- & \eta_1^- \end{pmatrix} = \begin{pmatrix} 3.734032182 & 81.31701305 \\ 2.588655160\, i & 55.86969204\, i \end{pmatrix}$$
$$\begin{pmatrix} \omega_2^+ & \eta_2^+ \\ \omega_2^- & \eta_2^- \end{pmatrix} = \begin{pmatrix} 0.3907313043 & 0.3272298509 \\ 0.8464023516\, i & 0.5081025185\, i \end{pmatrix}.$$

As expected, the period matrix can be brought to a block diagonal form. We further expect that there is a suitable definition of periods of Hilbert newforms which reproduces the numbers given above, but we have not made any analysis in this direction. As a first step, one could consider values of the L-functions of f and $\overline{f}$ as well as twists of these L-functions.

5.4 A sum of motives of Bianchi modular forms

In the first part of this section, we consider a Calabi-Yau operator $\mathcal{L}$ with a symmetry $z \mapsto -1/2^8 z$. Experimental computations suggest that semisimplifications of representations of the absolute Galois group of $K = \mathbb{Q}(\sqrt{-1})$

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_4(\mathbb{Q}_\ell)$$

associated with the fixed point $z_* = \sqrt{-1}/2^4$ of the symmetry split to sums of two Galois representations $\rho_{\pm,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$. To make this more rigorous, we construct in the second part a family X of Calabi-Yau threefolds associated with $\mathcal{L}$. The middle cohomology of each fiber X_z contains a four-dimensional part V_z whose variation with z is described by $\mathcal{L}$. We construct morphisms $\phi_z : X_z \to X_{-1/2^8 z}$ corresponding to the symmetry of $\mathcal{L}$. The endomorphism on the fixed fiber X_{z_*} can be used to split V_{z_*} into two parts which give the representations $\rho_{\pm,\ell}$. In the third part, we prove that the representations $\rho_{\pm,\ell}$ are up to semisimplification given by representations associated with a Bianchi newform f of weight 4 and a a Bianchi newform g of weight 2. In the last part, we present numerical computations which suggest that the period matrix can be brought to a block diagonal form. We are not aware of a general definition of periods of Bianchi newforms, but we find that half of the periods of V_{z_*} agree with periods of an elliptic curve associated with g.

Experimental computations

We consider the Calabi-Yau operator $\mathcal{L}$ which annihilates the power series $\sum_{n=0}^{\infty} a_n\, z^n$ with

$$a_n \;=\; \sum_{k=0}^{n} \binom{n}{k}^3 \cdot \sum_{k=0}^{n} \binom{n}{k} \binom{2\,k}{k} \binom{2\,n-2\,k}{n-k}.$$

This operator is the 105th operator from the list [1]. Using the script `DeformationMethod.gp` from [17], we compute the associated characteristic polynomials for all primes $19 \le p \le 997$ and all parameters $z \in \mathbb{F}_p$ of good reduction. To find Galois representations which split, we look at factorizations of these polynomials over $\mathbb{Z}$. The number of factorizations that occur for each prime p is depicted in Figure 5.5. Due to the symmetry $z \mapsto -1/2^8 z$ of the differential operator $\mathcal{L}$, the polynomials always factor an even number of times. We observe that for all primes $p \equiv 1 \bmod 4$ there are always at least two factorizations. Restricting to these primes and the cases with exactly two factorizations, we see that these come from the two reductions of the fixed point $z_* = \sqrt{-1}/2^4$ to $\mathbb{F}_p$. All computed polynomials of this fiber factor and, thus, we expect that semisimplifications of the associated Galois representations split.

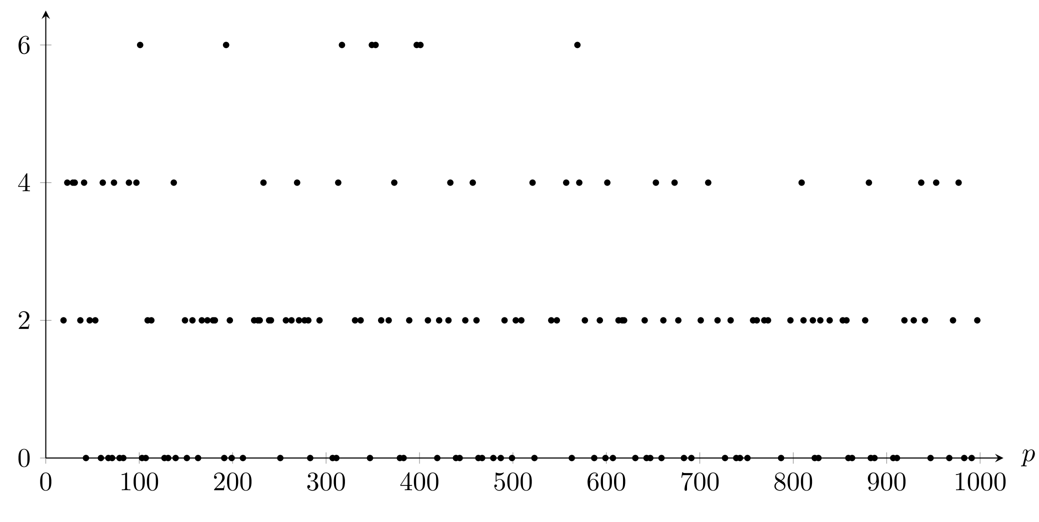


Figure 5.5: Number of factorizations over $\mathbb{Z}$ of characteristic polynomials for small primes p.

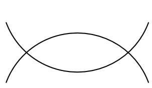
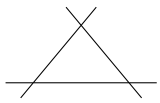
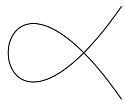
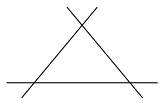
Geometric construction

The multiplicative structure of the coefficients of the power series that is annihilated by $\mathcal{L}$ suggests that $\mathcal{L}$ is a Hadamard product and that we can construct an associated family of Calabi-Yau threefolds from fiber products of elliptic surfaces. This turns out to be true and in the following we start by defining the relevant surfaces. These are famous elliptic surfaces that have been studied in [4].

To define the first elliptic surface, we consider the fibration obtained from the hypersurface given by all $((s:t),(X:Y:Z)) \in \mathbb{P}^1 \times \mathbb{P}^2$ which satisfy

$$s\,X\,Y\,Z - t\,(X+Y)\,(Y+Z)\,(Z+X) \;=\; 0\,,$$

together with the projection on the first factor. The singular fibers of this fibration are as follows:

base point	$(1:-1)$	$(1:0)$	$(8:1)$	$(0:1)$
sketch of fiber				

The only surface singularities are the three singularities in the fiber over $(1:0)$. Blowing up each of these once, we obtain a rational elliptic surface S_1. The Kodaira classification of the singular fibers is as follows:

base point	$(1:-1)$	$(1:0)$	$(8:1)$	$(0:1)$
Kodaira classification	I_2	I_6	I_1	I_3

Choosing the point $(1:-1:0)$, we give S_1 the structure of a rational elliptic surface with section. The differential form

$$\omega_1 \;=\; \operatorname{Res}\, s\,\frac{X\,\mathrm{d}Y \wedge \mathrm{d}Z + Y\,\mathrm{d}Z \wedge \mathrm{d}X + Z\,\mathrm{d}X \wedge \mathrm{d}Y}{s\,X\,Y\,Z - t\,(X+Y)\,(Y+Z)\,(Z+X)}$$

defines a holomorphic 1-form on all regular fibers and for $|t/s| < 1/8$ one of its periods evaluates to

$$2\pi i \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^3 \left(\frac{t}{s}\right)^n .$$

In our later analysis, we use the morphism $\phi_1 : S_1 \to S_1$ defined by

$$\left(\begin{pmatrix} -8\,t \\ s \end{pmatrix}, \begin{pmatrix} \psi_1(X+Y,Z) + (X-Y)\,\psi_2(X+Y,Z) \\ \psi_1(X+Y,Z) - (X-Y)\,\psi_2(X+Y,Z) \\ \psi_3(X+Y,Z) \end{pmatrix} \right)$$

with

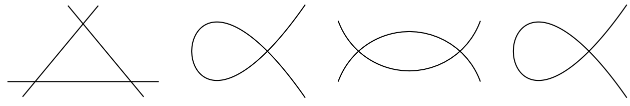
$$\begin{aligned} \psi_1(X,Z) &= X\,(s\,X\,Z - t\,(X+2\,Z)^2) \\ \psi_2(X,Z) &= -(s\,Z^2 - t\,X\,(X+2\,Z)) \\ \psi_3(X,Z) &= -X\,(s\,Z\,(X-Z) - 2\,t\,X^2)\,. \end{aligned}$$

On regular fibers this gives an isogeny of degree two.

For the second elliptic surface, we consider the fibration obtained from the hypersurface consisting of all $((s:t),(X:Y:Z)) \in \mathbb{P}^1 \times \mathbb{P}^2$ which satisfy

$$(s-4\,t)\,X\,Y\,Z - t\,(X+Y)\,(X\,Y+Z^2) \;=\; 0\,,$$

together with the projection on the first factor. The singular fibers of this fibration are as follows:

base point	$(1:0)$	$(8:1)$	$(4:1)$	$(0:1)$
sketch of fiber				

The only surface singularities are the three singularities in the fiber over $(1:0)$. Blowing up each of these once, we obtain a rational elliptic surface S_2. The Kodaira classification of the singular fibers is as follows:

base point	$(1:0)$	$(8:1)$	$(4:1)$	$(0:1)$
Kodaira classification	I_8	I_1	I_2	I_1

Choosing the point $(1:-1:0)$, we give S_2 the structure of a rational elliptic surface with section. The differential form

$$\omega_2 \;=\; \mathrm{Res}\; s\,\frac{X\,\mathrm{d}Y\wedge\mathrm{d}Z + Y\,\mathrm{d}Z\wedge\mathrm{d}X + Z\,\mathrm{d}X\wedge\mathrm{d}Y}{(s-4\,t)\,X\,Y\,Z - t\,(X+Y)\,(X\,Y+z^2)}$$

defines a holomorphic 1-form on all regular fibers and for $|t/s| < 1/8$ one of its periods evaluates to

$$2\pi i\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\binom{2\,n-2\,k}{n-k}\binom{2\,k}{k}\left(\frac{t}{s}\right)^{n}.$$

In our later analysis, we use the morphism $\phi_2 : S_2 \to S_2$ defined by

$$\left(\begin{pmatrix}32\,t\\ s\end{pmatrix},\begin{pmatrix}\psi_1(X+Y,Z)+(X-Y)\,\psi_2(X+Y,Z)\\ \psi_1(X+Y,Z)-(X-Y)\,\psi_2(X+Y,Z)\\ \psi_3(X+Y,Z)\end{pmatrix}\right)$$

with

$$\begin{aligned}
\psi_1(X,Z) \;=&\; -\,X\,Z\,(s-8\,t)\,(s\,Z-2\,t\,(X+2\,Z))^2\,(s\,X\,Z-t\,(X+2Z)^2)\\
&(s\,Z\,(X+Z)-t\,(X+2\,Z)^2)^2\,(s\,(X-Z)\,Z-t\,(X^2+4\,X\,Z-4\,Z^2))\\
\psi_2(X,Z) \;=&\,2\sqrt{-1}\,(s\,Z-2\,t\,(X+2\,Z))\,(s\,Z\,(X+Z)-t\,(X+2\,Z)^2)\\
&(t^5\,X^8-4\,(s-4\,t)\,t^4\,X^7\,Z+2\,t^3\,(s-4\,t)\,(3\,s-10\,t)\,X^6\,Z^2\\
&-4\,(s-4\,t)^2\,(s-t)\,t^2\,X^5\,Z^3+t\,(s-4\,t)\,(s^3+s^2\,t-58\,s\,t^2+168\,t^3)\,X^4\,Z^4\\
&-2\,(s-4\,t)^2\,t\,(3\,s^2-26\,s\,t+72\,t^2)\,X^3\,Z^5+(s-4\,t)^3\,(s^2-10\,s\,t+44\,t^2)\,X^2\,Z^6\\
&-4\,(s-4\,t)^4\,t\,X\,Z^7+(s-4\,t)^4\,t\,Z^8)\\
\psi_3(X,Z) \;=&\,s\,X\,Z\,(s\,Z\,(-X+Z)+t\,(X^2+4\,X\,Z-4\,Z^2))\\
&(t^2\,(X+2\,Z)^4+s^2\,Z^2\,(X^2+Z^2)-2\,s\,t\,Z\,(X^3+4\,X^2\,Z+4\,X\,Z^2+4\,Z^3))\\
&(2\,t^3X^4-4\,t^2(s-4\,t)\,X^3\,Z+3\,(s-4\,t)^2\,t\,X^2\,Z^2\\
&-(s-4\,t)^3\,X\,Z^3+2\,(s-4\,t)^2\,t\,Z^4)\,.
\end{aligned}$$

On regular fibers this gives an isogeny of degree eight.

We can now construct a family of Calabi-Yau threefolds $X \to \mathbb{P}^1 \smallsetminus \{-\frac{1}{4},-\frac{1}{8},0,\frac{1}{64},\frac{1}{32}\}$ and study its cohomology as reviewed in Section 4.3. For any $z \in \mathbb{P}^1 \smallsetminus \{-\frac{1}{4},-\frac{1}{8},0,\frac{1}{64},\frac{1}{32}\}$, we consider the fiber product

$$S_1 \times_{\mathbb{P}^1} \epsilon_z^*\,S_2$$

with $\epsilon_z : (s : t) \mapsto (t : z\, s)$. After a small projective resolution this gives a Calabi-Yau threefold X_z. The second cohomology of $S_1 \times_{\mathbb{P}^1} \epsilon_z^* S_2$ is generated by one fiber, $S_1 \times_{\mathbb{P}^1} \{0\}$, $\{0\} \times_{\mathbb{P}^1} \epsilon_z^* S_z$ and all but one components of all fibers. This gives the Hodge number

$$h^{1,1}(X_z) \;=\; 1 + 2 + (8 \cdot 3 - 1) + (6 \cdot 1 - 1) + 2 \cdot (2 \cdot 1 - 1) \;=\; 33\,.$$

On the other hand, the Euler number $\chi(X_z)$ equals 60 (twice the number of nodes that we resolved) and thus

$$h^{2,1}(X_z) \;=\; h^{1,1}(X_z) - \frac{\chi(X_z)}{2} \;=\; 3\,.$$

Further, X_z has two fibers of the form $I_2 \times E$ with different elliptic curves E. Choosing one of the components of each I_2 fiber, we get two inclusions of the form

$$\mathbb{P}^1 \times E \;\to\; X_z$$

and the pushforward with respect to each of these gives a two-dimensional image in the middle homology. In total, this gives a four-dimensional subspace of the middle homology of X_z. We obtain a surjection

$$i^* : H^3(X_z) \;\to\; \bigoplus_E H^2(\mathbb{P}^1) \otimes H^1(E)$$

and we use this to define the motive $V_z = \ker i^*$. This has the Hodge structure 1 1 1 1. The de Rham realization of V_z contains the holomorphic form

$$\Omega_z \;=\; \omega_1 \wedge \epsilon_z^* \omega_2 \wedge \frac{s\, \mathrm{d}t - t\, \mathrm{d}s}{s\, t}\,.$$

An explicit computation shows that $\mathcal{L}$ is the Picard-Fuchs operator of Ω. In particular, Ω and its first three derivatives (evaluated at z) generate the de Rham realization of V_z. For $|z| < 1/64$, one of the period functions of Ω is given by the power series $(2\pi i)^3 \sum_{n=0}^{\infty} a_n\, z^n$ with the coefficients

$$a_n \;=\; \sum_{k=0}^{n} \binom{n}{k}^3 \cdot \sum_{k=0}^{n} \binom{n}{k} \binom{2\,k}{k} \binom{2\,n - 2\,k}{n - k}\,.$$

The morphisms ϕ_1 and ϕ_2 induce morphisms $\phi_z : X_z \;\to\; X_{-1/2^8 z}$. If we restrict to the fiber over $z_* = \sqrt{-1}/16$, we get a Calabi-Yau threefold X_{z_*} defined over $K = \mathbb{Q}(\sqrt{-1})$ together with an endomorphism $\phi_{z_*} : X_{z_*} \to X_{z_*}$. Acting on Ω and its derivatives, we see that ϕ_{z_*} splits V_{z_*} into parts with eigenvalues ± 4 and Hodge structures 0 1 1 0 and 1 0 0 1. For the ℓ-adic realizations of V_{z_*}, this implies that the associated Galois representations $\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_4(\mathbb{Q}_\ell)$ split to sums of two-dimensional representations $\rho_{\pm,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$. These are unramified for all primes of good reduction, i.e. for all $\mathfrak{p} \notin \{1 + \sqrt{-1}, 2 + \sqrt{-1}, 4 + \sqrt{-1}\}$.

Modularity of the Galois representations

We proceed as in Section 5.2 to prove the modularity of the representations $\rho_{\pm,\ell}$. We start by giving two propositions which allow us to prove modularity by comparing finitely many characteristic polynomials of Frobenius elements.

Proposition 10. *Let $\rho : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_2(\mathbb{F}_2)$ be a continuous representation unramified outside $S = \{1 + \sqrt{-1}, 2 - \sqrt{-1}, 2 + \sqrt{-1}, 4 - \sqrt{-1}, 4 + \sqrt{-1}\}$. Then $\mathrm{Tr}(\rho) \;=\; 0$ if and only if*

$$\mathrm{Tr}(\rho(F_\mathfrak{p})) \;=\; 0 \quad \textit{for all} \quad \mathfrak{p} \;\in\; \{3 - 2\sqrt{-1}, 5 + 2\sqrt{-1}, 6 + \sqrt{-1}, 5 + 4\sqrt{-1}\}\,.$$

Proof. The kernel of ρ is an open normal subgroup of $\mathrm{Gal}(\overline{K}/K)$ and this gives a finite Galois extension L/K defined by $L \;=\; \overline{K}^{\ker \rho}$. This extension is unramified outside S and its Galois group is isomorphic to the image of ρ and hence, since $\mathrm{GL}_2(\mathbb{F}_2)$ is isomorphic to S_3, to a subgroup of S_3. Now, suppose that $\mathrm{Tr}(\rho) \neq 0$. The elements in $\mathrm{GL}_2(\mathbb{F}_2)$ with trace 1 are exactly the elements of order three and thus $\mathrm{Gal}(L/K)$ is isomorphic to S_3 or C_3. It follows that L/K is the Galois closure of a degree three extension M/K. The corresponding degree six extension $M/\mathbb{Q}$ is unramified outside $\{2, 5, 17\}$. The database [46] lists 1525 such extensions represented as quotients of $\mathbb{Q}[x]$ by monic polynomials in $\mathbb{Z}[x]$. Using that M must contain K and that M must be unramified outside S, we end up with 44 candidates for the extension L/K. We represent these as splitting fields of monic polynomials $f \in \mathcal{O}_K[x]$. Below we list the polynomials f together with a chosen prime $\mathfrak{p} \notin S$ of $\mathcal{O}_K$ such that the reduction of f to $\mathcal{O}_K/\mathfrak{p}$ does not factor:

f	$\mathfrak{p}$
$x^3 + (-1 - \sqrt{-1})\,x^2 - \sqrt{-1}\,x + (1 - \sqrt{-1})$	$5 + 4\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + \sqrt{-1}\,x + (1 + \sqrt{-1})$	$5 + 4\sqrt{-1}$
$x^3 - 5\sqrt{-1}\,x + (-5 + 5\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + 5\sqrt{-1}\,x + (-5 - 5\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 - 5\sqrt{-1}\,x + (-10 + 10\sqrt{-1})$	$5 + 4\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + 5\sqrt{-1}\,x + (-10 - 10\sqrt{-1})$	$5 + 4\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (1 + \sqrt{-1})\,x + (1 - \sqrt{-1})$	$6 + \sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (1 - \sqrt{-1})\,x + (1 + \sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + 5\sqrt{-1}\,x + (1 - \sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 - 5\sqrt{-1}\,x + (1 + \sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 - \sqrt{-1}\,x^2 + 11\,x - 5\sqrt{-1}$	$5 + 2\sqrt{-1}$
$x^3 + \sqrt{-1}\,x^2 + 11\,x + 5\sqrt{-1}$	$5 + 2\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (2 + 4\sqrt{-1})\,x + (-3 - \sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (2 - 4\sqrt{-1})\,x + (-3 + \sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 - \sqrt{-1}\,x^2 + (-1 + 3\sqrt{-1})\,x + (1 - \sqrt{-1})$	$5 + 4\sqrt{-1}$
$x^3 + \sqrt{-1}\,x^2 + (-1 - 3\sqrt{-1})\,x + (1 + \sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + 17\,x - 34\sqrt{-1}$	$3 - 2\sqrt{-1}$
$x^3 + 17\,x + 34\sqrt{-1}$	$3 - 2\sqrt{-1}$
$x^3 - x^2 + (-2 - 2\sqrt{-1})\,x + (-4 + 2\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 - x^2 + (-2 + 2\sqrt{-1})\,x + (-4 - 2\sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + 3\,x + (1 - \sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + 3\,x + (1 + \sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (1 - 2\sqrt{-1})\,x + (-4 - 2\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (1 + 2\sqrt{-1})\,x + (-4 + 2\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (2 + 3\sqrt{-1})\,x + (-1 - 5\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (2 - 3\sqrt{-1})\,x + (-1 + 5\sqrt{-1})$	$6 + \sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (-17 - 5\sqrt{-1})\,x + (24 + 10\sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (-17 + 5\sqrt{-1})\,x + (24 - 10\sqrt{-1})$	$6 + \sqrt{-1}$
$x^3 - \sqrt{-1}\,x^2 + (-6 + 17\sqrt{-1})\,x + (-17 - 56\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + \sqrt{-1}\,x^2 + (-6 - 17\sqrt{-1})\,x + (-17 + 56\sqrt{-1})$	$6 + \sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (7 + \sqrt{-1})\,x + (2 - 4\sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (7 - \sqrt{-1})\,x + (2 + 4\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (11 + 2\sqrt{-1})\,x + (-9 - 11\sqrt{-1})$	$6 + \sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (11 - 2\sqrt{-1})\,x + (-9 + 11\sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (1 - 9\sqrt{-1})\,x + (2 - 4\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (1 + 9\sqrt{-1})\,x + (2 + 4\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 - \sqrt{-1}\,x^2 + (-12 + 10\sqrt{-1})\,x - 22\sqrt{-1}$	$3 - 2\sqrt{-1}$
$x^3 + \sqrt{-1}\,x^2 + (-12 - 10\sqrt{-1})\,x + 22\sqrt{-1}$	$6 + \sqrt{-1}$
$x^3 + (-1 - \sqrt{-1})\,x^2 + (-5 + 9\sqrt{-1})\,x + (26 - 16\sqrt{-1})$	$3 - 2\sqrt{-1}$
$x^3 + (-1 + \sqrt{-1})\,x^2 + (-5 - 9\sqrt{-1})\,x + (26 + 16\sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 - x^2 + (-8 - 5\sqrt{-1})\,x + (-10 - 9\sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 - x^2 + (-8 + 5\sqrt{-1})\,x + (-10 + 9\sqrt{-1})$	$5 + 2\sqrt{-1}$
$x^3 - x^2 + (1 - 3\sqrt{-1})\,x + (11 + 7\sqrt{-1})$	$5 + 4\sqrt{-1}$
$x^3 - x^2 + (1 + 3\sqrt{-1})\,x + (11 - 7\sqrt{-1})$	$6 + \sqrt{-1}$

For any of these pairs, it follows that any element in the conjugacy class $F_{\mathfrak{p}}$ in the Galois group of the candidate for L/K has order three and hence $\mathrm{Tr}(\rho(F_{\mathfrak{p}})) = 1$ for one of the six primes $\mathfrak{p}$. $\square$

Proposition 11. *Let $\rho_1, \rho_2 : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{Z}_2(\sqrt{-1}))$ be continuous representations unramified outside $S = \{1+\sqrt{-1}, 2-\sqrt{-1}, 2+\sqrt{-1}, 4-\sqrt{-1}, 4+\sqrt{-1}\}$. Suppose that the traces of both representations vanish modulo $1+\sqrt{-1}$ and that the characteristic polynomials of $\rho_1(F_{\mathfrak{p}})$ and $\rho_2(F_{\mathfrak{p}})$ are equal for all*

$$\begin{aligned}\mathfrak{p} \in T = \{&3-2\sqrt{-1}, 3+2\sqrt{-1}, 5-2\sqrt{-1}, 5+2\sqrt{-1}, 6-\sqrt{-1}, 6+\sqrt{-1}, 5-4\sqrt{-1},\\ &5+4\sqrt{-1}, 7, 7-2\sqrt{-1}, 7+2\sqrt{-1}, 6-5\sqrt{-1}, 6+5\sqrt{-1}, 8-3\sqrt{-1}, 8+3\sqrt{-1},\\ &8-5\sqrt{-1}, 9-4\sqrt{-1}, 9+4\sqrt{-1}, 10-\sqrt{-1}, 10+\sqrt{-1}, 10-3\sqrt{-1}, 10+3\sqrt{-1},\\ &8-7\sqrt{-1}, 8+7\sqrt{-1}, 11-4\sqrt{-1}, 11+4\sqrt{-1}, 10-7\sqrt{-1}, 10+7\sqrt{-1}, 11-6\sqrt{-1},\\ &11+6\sqrt{-1}, 10-9\sqrt{-1}, 10+9\sqrt{-1}, 14-\sqrt{-1}, 14+\sqrt{-1}, 15-4\sqrt{-1}, 15+4\sqrt{-1},\\ &14-9\sqrt{-1}, 14+9\sqrt{-1}, 16-5\sqrt{-1}, 14-11\sqrt{-1}, 17-8\sqrt{-1}\}\,.\end{aligned}$$

Then ρ_1 and ρ_2 have isomorphic semisimplifications.

Proof. The claim follows from theorem 4.3 in [52] once we have shown that the image of $\{F_{\mathfrak{p}}\}_{\mathfrak{p}\in T}$ in the $\mathbb{F}_2$-vector space $\mathrm{Gal}(K_S/K)$ is non-cubic. Here, K_S is the compositum of all quadratic extensions of K unramified outside S and a subset U of a vector space V is called non-cubic if every homogeneous function of degree three on V vanishes if it vanishes on U. The extension K_S/K is generated by $\{\sqrt{2}, \sqrt{1+\sqrt{-1}}, \sqrt{2-\sqrt{-1}}, \sqrt{2+\sqrt{-1}}, \sqrt{4-\sqrt{-1}}, \sqrt{4+\sqrt{-1}}\}$ and hence $\mathrm{Gal}(K_S/K)$ is isomorphic to C_2^6. From the following table, which gives the action of the elements $\{F_{\mathfrak{p}}\}_{\mathfrak{p}\in T}$ on the chosen generators of K_S/K, it is straightforward to show that the image of $\{F_{\mathfrak{p}}\}_{\mathfrak{p}\in T}$ in $\mathrm{Gal}(K_S/K)$ is non-cubic:

$\mathfrak{p}$	$\sqrt{2}$	$\sqrt{1+\sqrt{-1}}$	$\sqrt{2-\sqrt{-1}}$	$\sqrt{2+\sqrt{-1}}$	$\sqrt{4-\sqrt{-1}}$	$\sqrt{4+\sqrt{-1}}$
$3-2\sqrt{-1}$	-1	1	-1	1	1	1
$3+2\sqrt{-1}$	-1	-1	1	-1	1	1
$5-2\sqrt{-1}$	-1	-1	-1	-1	1	-1
$5+2\sqrt{-1}$	-1	1	-1	-1	-1	1
$6-\sqrt{-1}$	-1	1	1	-1	-1	1
$6+\sqrt{-1}$	-1	-1	-1	1	1	-1
$5-4\sqrt{-1}$	1	1	-1	-1	-1	1
$5+4\sqrt{-1}$	1	1	-1	-1	1	-1
7	1	1	-1	-1	-1	-1
$7-2\sqrt{-1}$	-1	-1	1	-1	-1	-1
$7+2\sqrt{-1}$	-1	1	-1	1	-1	-1
$6-5\sqrt{-1}$	-1	-1	1	1	1	-1
$6+5\sqrt{-1}$	-1	1	1	1	-1	1
$8-3\sqrt{-1}$	1	-1	1	-1	1	-1
$8+3\sqrt{-1}$	1	-1	-1	1	-1	1
$8-5\sqrt{-1}$	1	-1	1	1	-1	-1
$9-4\sqrt{-1}$	1	-1	1	-1	-1	1
$9+4\sqrt{-1}$	1	-1	-1	1	1	-1
$10-\sqrt{-1}$	-1	-1	-1	-1	1	1
$10+\sqrt{-1}$	-1	1	-1	-1	1	1
$10-3\sqrt{-1}$	-1	-1	1	1	-1	1
$10+3\sqrt{-1}$	-1	1	1	1	1	-1
$8-7\sqrt{-1}$	1	1	-1	1	-1	1
$8+7\sqrt{-1}$	1	1	1	-1	1	-1
$11-4\sqrt{-1}$	1	1	-1	1	-1	-1
$11+4\sqrt{-1}$	1	1	1	-1	-1	-1
$10-7\sqrt{-1}$	-1	1	1	1	-1	-1

$10+7\sqrt{-1}$	-1	-1	1	1	-1	-1
$11-6\sqrt{-1}$	-1	-1	-1	1	-1	-1
$11+6\sqrt{-1}$	-1	1	1	-1	-1	-1
$10-9\sqrt{-1}$	-1	-1	-1	-1	-1	1
$10+9\sqrt{-1}$	-1	1	-1	-1	1	-1
$14-\sqrt{-1}$	-1	1	-1	1	1	-1
$14+\sqrt{-1}$	-1	-1	1	-1	-1	1
$15-4\sqrt{-1}$	1	-1	-1	-1	1	-1
$15+4\sqrt{-1}$	1	-1	-1	-1	-1	1
$14-9\sqrt{-1}$	-1	1	1	-1	1	-1
$14+9\sqrt{-1}$	-1	-1	-1	1	-1	1
$16-5\sqrt{-1}$	1	-1	-1	-1	1	1
$14-11\sqrt{-1}$	-1	1	-1	1	-1	1
$17-8\sqrt{-1}$	1	1	1	-1	1	1

□

To use the previous propositions, we need to compute characteristic polynomials of $\rho_{\pm,\ell}(F_{\mathfrak{p}})$ and find modular Galois representations with the same characteristic polynomials. For the first part, we compute Lefschetz numbers of $F_{\mathfrak{p}}$, $F_{\mathfrak{p}^2}$ and $\phi_{z_*} \circ F_{\mathfrak{p}}$ by counting fixed points over $\overline{\mathbb{F}_{\mathcal{N}(\mathfrak{p})}}$ and use the Lefschetz trace formula to obtain characteristic polynomials. For the first two, we use that the fixed points of $F_{\mathfrak{p}}^k$ are defined over $\mathbb{F}_{\mathcal{N}(\mathfrak{p})^k}$. Computing the fixed points of $\phi_{z_*} \circ F_{\mathfrak{p}}$ is computationally expensive since we do not know a bound for the cardinality of the field over which these are defined. But it will be sufficient for us to know the number of fixed points of $\phi_{z_*} \circ F_{\mathfrak{p}}$ modulo some suitable rational prime ℓ. Since the fixed points can be computed from the action on the fixed fibers (which are just products of elliptic curves), we can compute $\phi_{z_*} \circ F_{\mathfrak{p}}$ modulo ℓ by computing the action on the torsion group $E[\ell]$ for different elliptic curves E. Using the script `PointCounting.gp` from [17] we obtain the following numbers:

$\mathfrak{p}$	$\Lambda(F_{\mathfrak{p}})$	$\Lambda(F_{\mathfrak{p}}^2)$	$\Lambda(\phi_{z_*} \circ F_{\mathfrak{p}})$
$3-2\sqrt{-1}$	8202	5783112	$4+O(5)$
$3+2\sqrt{-1}$	8184	5788332	$3+O(5)$
$5-2\sqrt{-1}$	52976	618260652	$6+O(7)$
$5+2\sqrt{-1}$	53552	618263820	$3+O(7)$
$6-\sqrt{-1}$	97668	2627742348	$4+O(5)$
$6+\sqrt{-1}$	96864	2627907084	$4+O(5)$
$5-4\sqrt{-1}$	125852	4843603356	$1+O(5)$
$5+4\sqrt{-1}$	126322	4843739304	—
7	198534	14032058040	$2+O(11)$
$7-2\sqrt{-1}$	243988	22425517740	$O(7)$
$7+2\sqrt{-1}$	242148	22425678060	$1+O(5)$
$6-5\sqrt{-1}$	353400	51977892108	$3+O(5)$
$6+5\sqrt{-1}$	352794	51978146376	$2+O(5)$
$8-3\sqrt{-1}$	566634	152271900072	$3+O(5)$
$8+3\sqrt{-1}$	566208	152272776636	$4+O(5)$
$8-5\sqrt{-1}$	970992	499054541244	$4+O(5)$
$9-4\sqrt{-1}$	1225992	835897027644	$2+O(5)$
$9+4\sqrt{-1}$	1227180	835894419324	$3+O(5)$
$10-\sqrt{-1}$	1371784	1064961203532	$O(5)$
$10+\sqrt{-1}$	1368600	1064960493324	$3+O(5)$
$10-3\sqrt{-1}$	1690482	1681759340040	$4+O(5)$
$10+3\sqrt{-1}$	1693800	1681764264972	$3+O(5)$
$8-7\sqrt{-1}$	1867716	2087339071452	$2+O(5)$
$8+7\sqrt{-1}$	1872798	2087338171704	$3+O(5)$
$11-4\sqrt{-1}$	3196340	6623481837372	$1+O(5)$

$11+4\sqrt{-1}$	3195270	6623484870840	$3+O(5)$
$10-7\sqrt{-1}$	4040292	10958810189676	$3+O(5)$
$10+7\sqrt{-1}$	4051404	10958801726220	$2+O(5)$
$11-6\sqrt{-1}$	4686132	14996141175180	$1+O(5)$
$11+6\sqrt{-1}$	4686564	14996148363084	$3+O(5)$
$10-9\sqrt{-1}$	7014300	35197283281452	$1+O(7)$
$10+9\sqrt{-1}$	7018788	35197261712748	$2+O(5)$
$14-\sqrt{-1}$	8935636	58501484347500	$3+O(5)$
$14+\sqrt{-1}$	8939396	58501454790156	$2+O(5)$
$15-4\sqrt{-1}$	15916398	196041994383864	$2+O(5)$
$15+4\sqrt{-1}$	15917508	196041949742556	$1+O(5)$
$14-9\sqrt{-1}$	23792328	451923974944044	$1+O(5)$
$14+9\sqrt{-1}$	23797434	451923989726280	$6+O(7)$
$16-5\sqrt{-1}$	24815584	492515028547068	$1+O(5)$
$14-11\sqrt{-1}$	35181582	1015075324856760	$4+O(5)$
$17-8\sqrt{-1}$	48104544	1935366582948252	$2+O(5)$

The action of the Galois group on the cohomologies of even weight can be worked out explicitly by looking at the action on the corresponding algebraic cycles. The action on the ℓ-adic realizations of the quotient $H^3(X_{z_*})/V_{z_*}$ can be worked out by looking at the action on the corresponding products $\mathbb{P}^1 \times E$, too. It follows that

$$\begin{aligned}\Lambda(F_{\mathfrak{p}}^k) &= 1 + 33\,\mathcal{N}(\mathfrak{p})^k - \mathrm{Tr}\big((F_{\mathfrak{p}}^k)^*|V_{z_*}\big) - \mathcal{N}(\mathfrak{p})^k\,\big(\mathrm{Tr}\big((F_{\mathfrak{p}}^k)^*|H^1(E_1)\big) + \mathrm{Tr}\big((F_{\mathfrak{p}}^k)^*|H^1(E_2)\big)\big)\\ &\quad + 33\,\mathcal{N}(\mathfrak{p})^{2\,k} + \mathcal{N}(\mathfrak{p})^{3\,k}\,,\end{aligned}$$

where E_1 and E_2 are the elliptic curves $(S_1)_{(4:\sqrt{-1})}$ and $(S_2)_{(16:-\sqrt{-1})}$. Counting points of E_i over $\mathbb{F}_{\mathcal{N}(\mathfrak{p})^k}$, we obtain $\mathrm{Tr}\big((F_{\mathfrak{p}}^k)^*|H^1(E_i)\big)$. We can then compute the traces of $(F_{\mathfrak{p}})^*$ and $(F_{\mathfrak{p}}^2)^*$ on V_{z_*} and, combining this with Poincaré duality, we get the characteristic polynomials listed below:

$\mathfrak{p}$	$\det\big(1 - T\,F_{\mathfrak{p}}^*\vert V_{z_*}\big)$
$3-2\sqrt{-1}$	$1+24\,T+4394\,T^2+52728\,T^3+4826809\,T^4$
$3+2\sqrt{-1}$	$1-20\,T+3198\,T^2-43940\,T^3+4826809\,T^4$
$5-2\sqrt{-1}$	$1-356\,T+66062\,T^2-8682484\,T^3+594823321\,T^4$
$5+2\sqrt{-1}$	$1+220\,T+28478\,T^2+5365580\,T^3+594823321\,T^4$
$6-\sqrt{-1}$	$1-124\,T+105006\,T^2-6280972\,T^3+2565726409\,T^4$
$6+\sqrt{-1}$	$1+108\,T+75998\,T^2+5470524\,T^3+2565726409\,T^4$
$5-4\sqrt{-1}$	$1+268\,T+109142\,T^2+18470828\,T^3+4750104241\,T^4$
$5+4\sqrt{-1}$	$1+656\,T+245426\,T^2+45212176\,T^3+4750104241\,T^4$
7	$1-456\,T+157682\,T^2-53647944\,T^3+13841287201\,T^4$
$7-2\sqrt{-1}$	$1+452\,T+334430\,T^2+67292404\,T^3+22164361129\,T^4$
$7+2\sqrt{-1}$	$1-540\,T+368350\,T^2-80393580\,T^3+22164361129\,T^4$
$6-5\sqrt{-1}$	$1+1612\,T+1101294\,T^2+365893372\,T^3+51520374361\,T^4$
$6+5\sqrt{-1}$	$1+152\,T+141642\,T^2+34501112\,T^3+51520374361\,T^4$
$8-3\sqrt{-1}$	$1-1088\,T+684594\,T^2-423250496\,T^3+151334226289\,T^4$
$8+3\sqrt{-1}$	$1+676\,T+855414\,T^2+262975492\,T^3+151334226289\,T^4$
$8-5\sqrt{-1}$	$1-444\,T+1361878\,T^2-313006236\,T^3+496981290961\,T^4$
$9-4\sqrt{-1}$	$1-380\,T+1713990\,T^2-346815740\,T^3+832972004929\,T^4$
$9+4\sqrt{-1}$	$1-356\,T-502266\,T^2-324911588\,T^3+832972004929\,T^4$
$10-\sqrt{-1}$	$1-100\,T+1999598\,T^2-103030100\,T^3+1061520150601\,T^4$
$10+\sqrt{-1}$	$1-1668\,T+2704174\,T^2-1718542068\,T^3+1061520150601\,T^4$
$10-3\sqrt{-1}$	$1+872\,T+1069290\,T^2+1129265288\,T^3+1677100110841\,T^4$
$10+3\sqrt{-1}$	$1+1356\,T+3049166\,T^2+1756059324\,T^3+1677100110841\,T^4$
$8-7\sqrt{-1}$	$1+1068\,T+3150214\,T^2+1541013996\,T^3+2081951752609\,T^4$
$8+7\sqrt{-1}$	$1+2760\,T+4789618\,T^2+3982395720\,T^3+2081951752609\,T^4$

$11-4\sqrt{-1}$	$1+4924\,T+10899446\,T^2+12661342172\,T^3+6611856250609\,T^4$
$11+4\sqrt{-1}$	$1+840\,T+1059010\,T^2+2159936520\,T^3+6611856250609\,T^4$
$10-7\sqrt{-1}$	$1-3420\,T+8874142\,T^2-11313185580\,T^3+10942526586601\,T^4$
$10+7\sqrt{-1}$	$1+4116\,T+10461886\,T^2+13615518084\,T^3+10942526586601\,T^4$
$11-6\sqrt{-1}$	$1+2036\,T+8471406\,T^2+7879102148\,T^3+14976071831449\,T^4$
$11+6\sqrt{-1}$	$1-44\,T+7627374\,T^2-170275292\,T^3+14976071831449\,T^4$
$10-9\sqrt{-1}$	$1-1804\,T+12381486\,T^2-10697252764\,T^3+35161828327081\,T^4$
$10+9\sqrt{-1}$	$1+4132\,T+13224222\,T^2+24501689812\,T^3+35161828327081\,T^4$
$14-\sqrt{-1}$	$1+2276\,T+16032254\,T^2+17400868948\,T^3+58451728309129\,T^4$
$14+\sqrt{-1}$	$1+2884\,T+13999214\,T^2+22049255732\,T^3+58451728309129\,T^4$
$15-4\sqrt{-1}$	$1+1480\,T+25639026\,T^2+20716331080\,T^3+195930594145441\,T^4$
$15+4\sqrt{-1}$	$1+5964\,T+34528070\,T^2+83481215244\,T^3+195930594145441\,T^4$
$14-9\sqrt{-1}$	$1+10492\,T+68156958\,T^2+222996265036\,T^3+451729667968489\,T^4$
$14+9\sqrt{-1}$	$1-6008\,T+24283482\,T^2-127693629464\,T^3+451729667968489\,T^4$
$16-5\sqrt{-1}$	$1+1316\,T+44799830\,T^2+29199461956\,T^3+492309163417681\,T^4$
$14-11\sqrt{-1}$	$1+1872\,T+56360698\,T^2+59632584336\,T^3+1014741853230169\,T^4$
$17-8\sqrt{-1}$	$1+2292\,T-48446426\,T^2+100818151284\,T^3+1934854145598529\,T^4$

We remark that these agree with the polynomials obtained from the deformation method. Each of the characteristic polynomials factors to the product of the characteristic polynomials of $\rho_{+,\ell}(F_\mathfrak{p})$ and $\rho_{-,\ell}(F_\mathfrak{p})$. To find out which factor corresponds to which representation, we relate the number of fixed points of $\phi_{z_*} \circ F_\mathfrak{p}$ to the difference of the traces of $\rho_{+,\ell}(F_\mathfrak{p})$ and $\rho_{-,\ell}(F_\mathfrak{p})$. The action of ϕ_{z_*} on the fourth cohomology can be worked out explicitly and this gives $\mathrm{Tr}\big(\phi_{z_*}^*|H^4(X_{z_*})\big)=26$. The degree of ϕ_{z_*} is $2\cdot 8=16$ and thus $\mathrm{Tr}\big(\phi_{z_*}^*|H^6(X_{z_*})\big)=16$. On the other hand, we see that $\mathrm{Tr}\big(\phi_{z_*}^*|H^3(X_{z_*})\big)=0$ and since $\Lambda(\phi_{z_*})=54$, we also get $\mathrm{Tr}\big(\phi_{z_*}^*|H^2(X_{z_*})\big)=11$. We also see that $\mathrm{Tr}\big(F_\mathfrak{p}^*\phi_{z_*}^*|H^3(X_{z_*})/V_{z_*}\big)=0$ and the Lefschetz trace formula then gives

$$\Lambda(\phi_{z_*}\circ F_\mathfrak{p}) \;=\; 1+11\,\mathcal{N}(\mathfrak{p})-\mathrm{Tr}\big(F_\mathfrak{p}^*\,\phi_{z_*}^*|V_{z_*}\big)+26\,\mathcal{N}(\mathfrak{p})^2+16\,\mathcal{N}(\mathfrak{p})^3\,.$$

From $\mathrm{Tr}\big(F_\mathfrak{p}^*\,\phi_{z_*}^*|V_{z_*}\big)=4\,(\mathrm{Tr}(\rho_{+,\lambda}(F_\mathfrak{p}))-\mathrm{Tr}(\rho_{-,\lambda}(F_\mathfrak{p})))$, we obtain the polynomials below:

$\mathfrak{p}$	$\det(1-T\,\rho_{-,\ell}(F_\mathfrak{p}))$	$\det(1-T\,\rho_{+,\ell}(F_\mathfrak{p}))$
$3-2\sqrt{-1}$	$1+24\,T+2197\,T^2$	$1+2197\,T^2$
$3+2\sqrt{-1}$	$1-46\,T+2197\,T^2$	$1+26\,T+2197\,T^2$
$5-2\sqrt{-1}$	$1-298\,T+24389\,T^2$	$1-58\,T+24389\,T^2$
$5+2\sqrt{-1}$	$1-70\,T+24389\,T^2$	$1+290\,T+24389\,T^2$
$6-\sqrt{-1}$	$1-50\,T+50653\,T^2$	$1-74\,T+50653\,T^2$
$6+\sqrt{-1}$	$1-114\,T+50653\,T^2$	$1+222\,T+50653\,T^2$
$5-4\sqrt{-1}$	$1+350\,T+68921\,T^2$	$1-82\,T+68921\,T^2$
$5+4\sqrt{-1}$	$1+328\,T+68921\,T^2$	$1+328\,T+68921\,T^2$
7	$1+132\,T+117649\,T^2$	$1-588\,T+117649\,T^2$
$7-2\sqrt{-1}$	$1+346\,T+148877\,T^2$	$1+106\,T+148877\,T^2$
$7+2\sqrt{-1}$	$1-222\,T+148877\,T^2$	$1-318\,T+148877\,T^2$
$6-5\sqrt{-1}$	$1+758\,T+226981\,T^2$	$1+854\,T+226981\,T^2$
$6+5\sqrt{-1}$	$1+640\,T+226981\,T^2$	$1-488\,T+226981\,T^2$
$8-3\sqrt{-1}$	$1+80\,T+389017\,T^2$	$1-1168\,T+389017\,T^2$
$8+3\sqrt{-1}$	$1+530\,T+389017\,T^2$	$1+146\,T+389017\,T^2$
$8-5\sqrt{-1}$	$1+90\,T+704969\,T^2$	$1-534\,T+704969\,T^2$
$9-4\sqrt{-1}$	$1-574\,T+912673\,T^2$	$1+194\,T+912673\,T^2$
$9+4\sqrt{-1}$	$1-1714\,T+912673\,T^2$	$1+1358\,T+912673\,T^2$
$10-\sqrt{-1}$	$1-302\,T+1030301\,T^2$	$1+202\,T+1030301\,T^2$
$10+\sqrt{-1}$	$1-1062\,T+1030301\,T^2$	$1-606\,T+1030301\,T^2$
$10-3\sqrt{-1}$	$1-872\,T+1295029\,T^2$	$1+1744\,T+1295029\,T^2$
$10+3\sqrt{-1}$	$1+702\,T+1295029\,T^2$	$1+654\,T+1295029\,T^2$
$8-7\sqrt{-1}$	$1+390\,T+1442897\,T^2$	$1+678\,T+1442897\,T^2$

$8+7\sqrt{-1}$	$1+1404\,T+1442897\,T^2$	$1+1356\,T+1442897\,T^2$
$11-4\sqrt{-1}$	$1+1910\,T+2571353\,T^2$	$1+3014\,T+2571353\,T^2$
$11+4\sqrt{-1}$	$1+2484\,T+2571353\,T^2$	$1-1644\,T+2571353\,T^2$
$10-7\sqrt{-1}$	$1-2526\,T+3307949\,T^2$	$1-894\,T+3307949\,T^2$
$10+7\sqrt{-1}$	$1+1434\,T+3307949\,T^2$	$1+2682\,T+3307949\,T^2$
$11-6\sqrt{-1}$	$1+466\,T+3869893\,T^2$	$1+1570\,T+3869893\,T^2$
$11+6\sqrt{-1}$	$1-358\,T+3869893\,T^2$	$1+314\,T+3869893\,T^2$
$10-9\sqrt{-1}$	$1-1442\,T+5929741\,T^2$	$1-362\,T+5929741\,T^2$
$10+9\sqrt{-1}$	$1+3770\,T+5929741\,T^2$	$1+362\,T+5929741\,T^2$
$14-\sqrt{-1}$	$1+1882\,T+7645373\,T^2$	$1+394\,T+7645373\,T^2$
$14+\sqrt{-1}$	$1+3278\,T+7645373\,T^2$	$1-394\,T+7645373\,T^2$
$15-4\sqrt{-1}$	$1+2444\,T+13997521\,T^2$	$1-964\,T+13997521\,T^2$
$15+4\sqrt{-1}$	$1+4518\,T+13997521\,T^2$	$1+1446\,T+13997521\,T^2$
$14-9\sqrt{-1}$	$1+6614\,T+21253933\,T^2$	$1+3878\,T+21253933\,T^2$
$14+9\sqrt{-1}$	$1-8224\,T+21253933\,T^2$	$1+2216\,T+21253933\,T^2$
$16-5\sqrt{-1}$	$1+754\,T+22188041\,T^2$	$1+562\,T+22188041\,T^2$
$14-11\sqrt{-1}$	$1-1932\,T+31855013\,T^2$	$1+3804\,T+31855013\,T^2$
$17-8\sqrt{-1}$	$1+12882\,T+43986977\,T^2$	$1-10590\,T+43986977\,T^2$

We now want to find a Bianchi newform f_- of weight 4 and a Bianchi newform f_+ of weight 2 such that the traces of Frobenius elements with respect to the representations $\rho_{\pm,\ell}$ agree with the ones for $\rho_{f_-,\lambda}$ and $\rho_{f_+,\lambda}(-1)$ for all primes that we considered above. Computing Hecke eigenvalues using the script `HeckeEigenvalues.mgm` from [17], we find that this is true for the unique newform $f_- \in S_4(\Gamma_0(26-2\sqrt{-1}))$ with Hecke eigenvalue $a_{2-\sqrt{-1}} = 4$ and the unique newform $f_+ \in S_2(\Gamma_0(26-2\sqrt{-1}))$ with Hecke eigenvalue $a_{2-\sqrt{-1}} = -4$. As we show now, this suffices to conclude the equality for all primes $\mathfrak{p}$.

Theorem 11. *The representations*

$$\rho_{-,\ell}\,,\rho_{f_-,\lambda} : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_2(E_{f_-,\lambda})$$

have isomorphic semisimplifications. The same holds for the representations

$$\rho_{+,\ell}\,,\;\rho_{f_+,\lambda}(-1) : \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{GL}_2(E_{f_+,\lambda})\,.$$

Proof. We first determine suitable choices for the number fields E_{f_-} and E_{f_+}. Since f_- has Hecke eigenvalues $a_{3-2\sqrt{-1}} = -24$, $a_{5+2\sqrt{-1}} = 70$, $a_{3+2\sqrt{-1}} = 46 \neq 0$ and $a_{5-2\sqrt{-1}} = 298 \neq 0$, we can choose E_{f_-} to be the compositum of the splitting fields of $1+24\,T+2197\,T^2$ and $1-70\,T+24389\,T^2$, i.e. $E_{f_-} = \mathbb{Q}(\sqrt{-417},\sqrt{-5791})$. In this field, we have the prime factorization $2 = \lambda_1^2\,\lambda_2^2$ and the completion of E_{f_-} at both of the primes λ_1 and λ_2 is isomorphic to $\mathbb{Q}_2(\sqrt{-1})$. Analogously, to obtain a possible choice for the number field E_{f_+}, we use the Hecke eigenvalues $a_{5-2\sqrt{-1}} = 2 \neq 0$ and $a_{5+2\sqrt{-1}} = -10 \neq 0$. We can thus choose E_{f_+} to be the compositum of the splitting fields of the polynomials $1-2\,T+29\,T^2$ and $1+10\,T+29\,T^2$, i.e. $E_{f_+} = \mathbb{Q}(\sqrt{-7},\sqrt{-1})$. In this field, we have the prime factorization $2 = \lambda_1^2\,\lambda_2^2$ and the completion of E_{f_+} at both of the primes λ_1 and λ_2 is again isomorphic to $\mathbb{Q}_2(\sqrt{-1})$. We can further conjugate the representations so that the image is in $\mathrm{GL}_2(\mathbb{Z}_2[\sqrt{-1}])$ (see e.g. section 1 in [64]) and from the computed characteristic polynomials, Proposition 10 and Proposition 11, it then follows that the representations have isomorphic semisimplifications. □

Analogously, it follows that the elliptic curves E_1 and E_2 are associated with the unique Bianchi newform $g_1 \in S_2(\Gamma_0(12-14\sqrt{-1}))$ with Hecke eigenvalue $a_{2-\sqrt{-1}} = 0$ and the unique Bianchi newform $g_2 \in S_2(\Gamma_0(26-2\sqrt{-1}))$ with Hecke eigenvalue $a_{2-\sqrt{-1}} = -2$. For a suitable choice of local L-factors for primes of bad reduction, one finds that the Hasse-Weil zeta function is given by

$$\zeta(X_{z_*},s) \;=\; \frac{L(f_-,s)\,L(f_+,s-1)\,L(g_1,s-1)\,L(g_2,s-1)}{\zeta_K(s)\,\zeta_K(s-1)^{33}\,\zeta_K(s-2)^{33}\,\zeta_K(s-3)}\,.$$

Modularity of the periods

We end this section with a numerical study of the period matrix of V_{z_*}. The computations were done using the script `PeriodIdentities.gp` from [17].

To study the period matrices of V_{z_*}, we first note that for $|z| < 1/2^6$ one of the periods of Ω_z is given by

$$\Pi_1(z) \;=\; (2\pi i)^3 \sum_{n=0}^{\infty} a_n\, z^n\,.$$

To get a basis of period functions of Ω, we can look at the monodromy of this period with respect to z. To state numerical results about the monodromy, we define a local basis of solutions of $\mathcal{L}$ by

$$\varpi(z) \;=\; \begin{pmatrix} f_1(z) \\ \log(z)\, f_1(z) + f_2(z) \\ \frac{1}{2}\,\log(z)^2 f_1(z) + \log(z)\, f_2(z) + f_3(z) \\ \frac{1}{6}\,\log(z)^3 f_1(z) + \frac{1}{2}\,\log(z)^2 f_2(z) + \log(z)\, f_3(z) + f_4(z) \end{pmatrix}$$

with convergent power series normalized by $f_1(0) = 1$ and $f_2(0) = f_3(0) = f_4(0) = 0$. Numerically, we then find that, when we encircle the point $1/2^6$ counterclockwise, Π_1 transforms by

$$(2\pi i)^3\, \varpi_1 \;\mapsto\; \left(\frac{7}{2}\,(2\pi i)^3 - 12\,\zeta(3)\right)\, \varpi_1 - 9\,(2\pi i)^2\, \varpi_2 + 24\,(2\pi i)\, \varpi_3 - 48\, \varpi_4\,.$$

This suggests that a basis of period functions of Ω is given by (an analytic continuation of)

$$\Pi \;=\; \begin{pmatrix} (2\pi i)^3 & 0 & 0 & 0 \\ 0 & (2\pi i)^2 & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ \frac{1}{4}\,\zeta(3) & 0 & 0 & 1 \end{pmatrix} \varpi\,.$$

Taking derivatives, we obtain the matrix of period functions $T = (\Pi, \Theta\Pi, \Theta^3\Pi, \Theta^4\Pi)$. Evaluating at $\sigma_\pm(z_*)$, where $\sigma_\pm : K \hookrightarrow \mathbb{C}$ corresponds to the embedding that maps $\sqrt{-1}$ to $\pm i$, we obtain two period matrices. After rounding to ten digits of precision, we have

$$T(\sigma_+(z_*)) \;=\; A \begin{pmatrix} \omega_-^1 & \eta_-^1 & 0 & 0 \\ \omega_-^2 & \eta_-^2 & 0 & 0 \\ 0 & 0 & \omega_+^1 & \eta_+^1 \\ 0 & 0 & \omega_+^2 & \eta_+^2 \end{pmatrix} B$$

with

$$A \;=\; \begin{pmatrix} 48 & 0 & 48 & 0 \\ 0 & 48 & 0 & 48 \\ -9 & 16 & -1 & 16 \\ -2 & 1 & -2 & 9 \end{pmatrix}, \qquad B \;=\; \begin{pmatrix} 14 + 12\sqrt{-1} & -7 - 6\sqrt{-1} & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

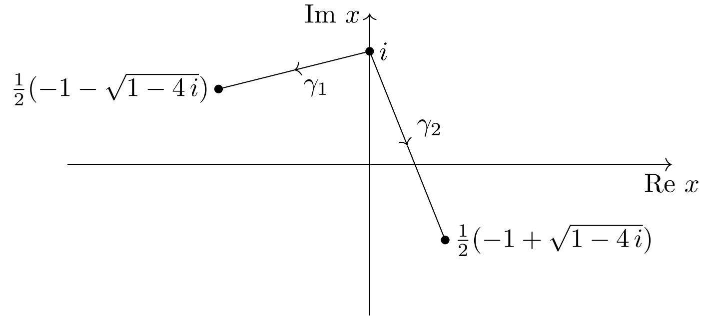
and

$$\begin{aligned} \begin{pmatrix} \omega_-^1 & \eta_-^1 \\ \omega_-^2 & \eta_-^2 \end{pmatrix} &= \begin{pmatrix} -0.1011243985 - 0.2021879387\, i & -0.1183200150 + 0.1021205156\, i \\ 0.07172909004 - 0.1221585214\, i & -0.08873323456 + 0.03077960964\, i \end{pmatrix} \\ \begin{pmatrix} \omega_+^1 & \eta_+^1 \\ \omega_+^2 & \eta_+^2 \end{pmatrix} &= \begin{pmatrix} 0.5957500779 - 1.069965099\, i & -0.1724046435 - 0.5900557656\, i \\ 0.4701143224 - 0.2328055773\, i & 0.1169688027 - 0.1569580219\, i \end{pmatrix}. \end{aligned}$$

The period matrix of the conjugate point is given by

$$T(\sigma_-(z_*)) \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overline{T(\sigma_+(z_*))}\,.$$

As expected, the period matrix can be brought to a block diagonal form. Since we are not aware of a fully developed theory of periods associated with Bianchi modular forms, we can just give some partial results for a possible modularity of the periods. For this, we consider an elliptic curve E associated with f_+ and define the period matrix of f_+ by the period matrix of E. We choose the elliptic curve E with affine equation $y^2 = x^3 + (1 - \sqrt{-1})\, x^2 + 1$. This is easily seen to be associated with f_+ by counting the number of points over finite fields. We have a period matrix associated with E_{σ_+} and we expect that (after mutliplication by $2\pi i$) this gives the four periods ω_+^1, ω_+^2, η_+^1 and η_+^2. The first singular homology of $E_{\sigma_+}(\mathbb{C})$ is generated by double covers of the lines γ_1 and γ_2 depicted below:



Numerically, we indeed find that

$$\begin{pmatrix} \omega_+^1 & \eta_+^1 \\ \omega_+^2 & \eta_+^2 \end{pmatrix} = 2\pi i \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \int_{\gamma_1} \frac{\mathrm{d}x}{y} & \int_{\gamma_1} x\, \frac{\mathrm{d}x}{y} \\ \int_{\gamma_2} \frac{\mathrm{d}x}{y} & \int_{\gamma_2} x\, \frac{\mathrm{d}x}{y} \end{pmatrix} \begin{pmatrix} \frac{1}{48} & \frac{4111}{346800} - \frac{499}{173400}\, i \\ 0 & -\frac{13}{14450} + \frac{42}{7225}\, i \end{pmatrix},$$

where in the integrand we choose the root $y = \sqrt{x^3 + (1 - \sqrt{-1})\, x^2 + 1}$ with non-negative real part.

6 Modularity of some mixed period matrices of rank four

In this chapter, we find new classes of elliptic modular forms and their periods. More concretely, we find these from the study of fourteen specific limit mixed Hodge structures of rank four. The limit mixed Hodge structures are associated with families of Calabi-Yau threefolds, one of which is the family of mirror quintics, which led to the discovery of mirror symmetry and which was studied in the famous paper [20] by Candelas, de la Ossa, Green and Parkes. In this introduction, we give an overview of our results for this specific case. The corresponding Picard-Fuchs operator is given by

$$\Theta^4 - 5^5\, z\, (\Theta + 1/5)\, (\Theta + 2/5)\, (\Theta + 3/5)\, (\Theta + 4/5) \qquad \text{with} \qquad \Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$$

and, for $0 < z < 1/5^5$, a basis of solutions is given by

$$\varpi(z) \;=\; \begin{pmatrix} f_1(z) \\ \log(z)\, f_1(z) + f_2(z) \\ \frac{1}{2}\, \log(z)^2\, f_1(z) + \log(z)\, f_2(z) + f_3(z) \\ \frac{1}{6}\, \log(z)^3\, f_1(z) + \frac{1}{2}\, \log(z)^2\, f_2(z) + \log(z)\, f_3(z) + f_4(z) \end{pmatrix}$$

with convergent power series normalized by $f_1(0) = 1$ and $f_2(0) = f_3(0) = f_4(0) = 0$. These solutions combine to a basis of period functions by

$$\Pi \;=\; \begin{pmatrix} (2\pi i)^3 & 0 & 0 & 0 \\ 0 & (2\pi i)^2 & 0 & 0 \\ 50\, \frac{(2\pi i)^3}{24} & \frac{1}{2}\, (2\pi i)^2 & -5\, (2\pi i) & 0 \\ -200\, \zeta(3) & 50\, \frac{(2\pi i)^2}{24} & 0 & 5 \end{pmatrix} \varpi\,.$$

In terms of the variable $\delta = 1 - 5^5 z$, the period matrix of the limit mixed Hodge structure of the so-called conifold point $\delta = 0$ can now be obtained from the expansion

$$\Pi(z) \;=\; \begin{pmatrix} -2\pi i\,\sqrt{5} & b & d & c \\ 0 & w^+ & e^+ & a^+ \\ 0 & \frac{1}{2}\, w^+ + w^- & \frac{1}{2}\, e^+ + e^- & \frac{1}{2}\, a^+ + a^- \\ 0 & 0 & 0 & (2\pi i)^2\,\sqrt{5} \end{pmatrix} \begin{pmatrix} \log(\delta)\, \nu(\delta) + O(\delta^3) \\ 1 + O(\delta^3) \\ \delta^2 + O(\delta^3) \\ \nu(\delta) \end{pmatrix}$$

with the so-called vanishing period function $\nu(\delta) = \delta + O(\delta^2)$, real constants w^+, e^+, a^+ and purely imaginary constants w^-, e^-, a^-, b, d, c. These constants satisfy the quadratic relations

$$\begin{aligned} \det \begin{pmatrix} w^+ & e^+ \\ w^- & e^- \end{pmatrix} &= -(2\pi i)^3\, \frac{5}{2} \\ \det \begin{pmatrix} w^+ & a^+ \\ w^- & a^- \end{pmatrix} &= -(2\pi i)^2\, \sqrt{5}\, b \\ \det \begin{pmatrix} e^+ & a^+ \\ e^- & a^- \end{pmatrix} &= -(2\pi i)^3\, \frac{9}{4} - (2\pi i)^2\, \sqrt{5}\, d\,. \end{aligned}$$

In [60], it was shown that the Galois representations of a resolution of the conifold fiber are associated with the unique newform $f \in S_4(\Gamma_0^*(25))$ with Hecke eigenvalue $a_2 = 1$. Because of this, one expects that the periods of the limit mixed Hodge structure are also related with f. Evidence for this has been given in [12], where numerical results suggest that the pure periods $w^\pm$ and $e^\pm$ are periods and quasiperiods of f. In this chapter, we prove this and give similar identifications for the remaining periods. To state our results for the example of the family of mirror quintics, we define several modular forms. First, we define $f_{50}(\tau) = 5\,f(\tau) - 20\,f(2\,\tau)$ and the modular function $t_{50} = \frac{(1-h)\,(3+h)^2}{5\,(1-h-h^2)}$ in terms of the normalized Hauptmodul $h(\tau) = q^{-1} + O(q)$ of $\Gamma_0^*(50)$. We further define the two meromorphic modular forms

$$F_{50} \;=\; \frac{t_{50}\,(7 + 13\,t_{50} + 5\,t_{50}^2 - 25\,t_{50}^3)}{2\,(1 - 5\,t_{50})^4}\,f_{50} \qquad \text{and} \qquad g_{50} \;=\; -\frac{5\,t_{50}\,(1 - t_{50})}{(1 - 5\,t_{50})^2}\,f_{50}\,.$$

The form F_{50} is a meromorphic partner of f_{50}. In particular, the residues of $\tau^n\,F_{50}(\tau)$ vanish for $n = 0, 1, 2$. The residues of $\tau^n\,g_{50}(\tau)$ for $n = 0, 1, 2$ do not vanish, but they lie in $\frac{2}{(2\pi i)^4}\Lambda$ with the lattice $\Lambda = \frac{\sqrt{5}\,(2\pi i)^2}{500}\,\mathbb{Z}$. This allows to define associated period polynomials with coefficients in $\mathbb{C}/\Lambda$. Our three main results are the following:

- The classes

$$\begin{aligned} r_{f_{50}}, r_{F_{50}} \;&\in\; H^1_{\mathrm{par}}(\Gamma_0(50), \langle 1, \tau, \tau^2\rangle_{\mathbb{C}}) \\ r_{g_{50}} \;&\in\; H^1_{\mathrm{par}}(\Gamma_0(50), \langle 1, \tau, \tau^2\rangle_{\mathbb{C}/\Lambda}) \end{aligned}$$

  decompose as

$$(r_{f_{50}}, r_{F_{50}}, r_{g_{50}}) \;=\; (r^+, r^-)\begin{pmatrix} w^+ & e^+ & a^+ \\ w^- & e^- & a^- \end{pmatrix}$$

  for some $r^\pm \in H^1_{\mathrm{par}}(\Gamma_0(50), \langle 1, \tau, \tau^2\rangle_{\mathbb{Q}})^\pm$.

- In terms of the CM points $\tau_\pm = \pm\frac{2}{5} + i\,\frac{\sqrt{2}}{10}$, we have

$$\begin{aligned} b \;&=\; (2\pi i)^3 \int_{\tau_-}^{\tau_+} f_{50}(\tau)\,\mathrm{d}\tau \\ d \;&=\; \lim_{\epsilon \downarrow 0}\left((2\pi i)^3 \int_{\tau_-}^{\tau_+} F_{50}(\tau + i\tfrac{\sqrt{2}}{5}\,\epsilon)\,\mathrm{d}\tau + 2\pi i\,\sqrt{5}\,\left(-\tfrac{1}{5\,t_{50}'(\tau_-)\,t_{50}'(\tau_+)}\,(\tfrac{1}{\epsilon^3} + 1) - \tfrac{257}{480}\right)\right) \\ c \;&=\; \lim_{\epsilon \downarrow 0}\left((2\pi i)^3 \int_{\tau_-}^{\tau_+} g_{50}(\tau + i\tfrac{\sqrt{2}}{5}\,\epsilon)\,\mathrm{d}\tau + 2\pi i\,\sqrt{5}\,\left(\tfrac{1}{\epsilon} + \log\bigl(-5\,t_{50}'(\tau_-)\,t_{50}'(\tau_+)\,\epsilon^2\bigr)\right)\right)\,, \end{aligned}$$

  where

$$t_{50}'(\tau_\pm) \;=\; \mp\frac{\Gamma(\frac{1}{8})^2\Gamma(\frac{3}{8})^2}{2\,\sqrt{10}\,\pi^2}\,.$$

  Here, the integral in the expression for c is along the straight line between τ_- and τ_+.

- Numerical computations suggest that

$$1 + \frac{1}{2\pi i\,\sqrt{5}\,w^-}\det\begin{pmatrix} w^- & a^- \\ b & c \end{pmatrix} \;=\; -\frac{5}{3}\log 5 - \frac{125}{6}\frac{2\pi i\,\sqrt{5}\,L'(f \otimes \chi, 2)}{w^-}$$

  in terms of the quadratic character χ associated with $\mathbb{Q}(\sqrt{5})$.

The structure of this chapter is as follows. In the first section, we introduce the fourteen hypergeometric variations of Hodge structures and review the modularity of two-dimensional Galois representations associated with resolutions of the conifold fibers. In the second section, we review the limit mixed Hodge structure of the conifold fiber. In the third section, we use the method of "fibering out" to prove the modularity of pure periods of the limit mixed Hodge structure for twelve cases. In the third section, we make further use of this method to express, for the same twelve cases, all mixed periods in terms of integrals of modular forms. Some of these modular forms have residues and these give a new class of modular forms which is yet to be understood in more generality. In the last section, we give new numerical identities between the mixed periods and central values of derivatives of L-functions.

The results of Section 6.3 and Section 6.4 are part of work in progress with Vasily Golyshev and Albrecht Klemm [18]. The ideas that led us to the numerical computations in Section 6.5 originate from discussions with Spencer Bloch, Vasily Golyshev and Matt Kerr.

6.1 Hypergeometric families and their conifold fibers

There are fourteen hypergeometric variations associated with one-parameter families X of Calabi-Yau threefolds. The corresponding Picard-Fuchs operators are of the form

$$\Theta^4 - \tfrac{1}{\mu}\, z\,(\Theta+a_1)\,(\Theta+a_2)\,(\Theta+a_3)\,(\Theta+a_4) \qquad \text{with} \qquad \Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$$

for the hypergeometric indices $\{a_1, a_2, a_3, a_4\}$ and μ given in Table 6.1. Using that the hypergeometric indices come in pairs which add up to one, one finds that the Riemann symbol associated with the Picard-Fuchs operator is

$$\mathcal{P}\left\{\begin{array}{ccc} 0 & \mu & \infty \\ \hline 0 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 2 & a_4 \end{array}\right\}.$$

In Section 5.1, Section 5.2 and Section 5.3, we considered regular points of two of the hypergeometric operators, but now we are particularly interested in the so-called conifold fiber X_μ of each model. In section 5 of [53], geometric realizations of the fourteen families are summarized and it is shown that the singular fiber X_μ can be resolved to a rigid Calabi-Yau threefold $\widehat{X}_\mu$ defined over $\mathbb{Q}$. Here, rigidity means that the Hodge number $h^{2,1}(\widehat{X}_\mu)$ (which equals the dimension of the complex structure moduli space) vanishes. It is known that the associated two-dimensional Galois representations $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$ are modular. For the case with the hypergeometric indices $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ (corresponding to the family of mirror quintics), this was shown in [60]. The result for all fourteen cases follows from the modularity theorem for rigid Calabi-Yau threefolds defined over $\mathbb{Q}$ [38]. In particular, in each case there is an elliptic newform $f \in S_4(\Gamma_0(N))$ of some level N such that

$$\det(1 - T\,\rho_\ell(F_p)) \;=\; 1 - a_p\,T + p^3\,T^2$$

in terms of the Hecke eigenvalues a_p of f. The levels can be found in Table 6.1 and a simple way to compute the Hecke eigenvalues for primes p not dividing $1/\mu$ is by using the supercongruences

$$a_p \;\equiv\; \sum_{k=0}^{p-1} \frac{(a_1)_k\,(a_2)_k\,(a_3)_k\,(a_4)_k}{k!^4} \mod p^3$$

with the Pochhammer symbol $(a)_k = a\,(a+1)\cdots(a+k-1)$ together with the bound $|a_p| \le 2\,p^{3/2}$. These congruences were discovered in [58] and later proven in [53].

N	a_1, a_2, a_3, a_4	$1/\mu$	N	a_1, a_2, a_3, a_4	$1/\mu$
8	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	2^8	72	$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	$2^8 3^3$
9	$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	$2^6 3^3$	108	$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	$2^4 3^6$
16	$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	2^{10}	128	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	2^{16}
25	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	5^5	144	$\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}$	$2^{10} 3^3$
27	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	3^6	200	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$2^8 5^5$
32	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	2^{12}	216	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$2^8 3^6$
36	$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	$2^4 3^3$	864	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$2^{12} 3^6$

Table 6.1: Hypergeometric indices and the level N of the newform associated with the conifold fiber.

6.2 The limit mixed Hodge structure of the conifold fibers

We begin by briefly reviewing the limit mixed Hodge structure of the conifold fiber as discussed in detail in [12]. Consider any of the fourteen sets of hypergeometric indices. For $0 < z < \mu$, we define a basis of solutions of the associated Picard-Fuchs operator by

$$\varpi(z) \;=\; \begin{pmatrix} f_1(z) \\ \log(z)\, f_1(z) + f_2(z) \\ \frac{1}{2}\,\log(z)^2\, f_1(z) + \log(z)\, f_2(z) + f_3(z) \\ \frac{1}{6}\,\log(z)^3\, f_1(z) + \frac{1}{2}\,\log(z)^2\, f_2(z) + \log(z)\, f_3(z) + f_4(z) \end{pmatrix}$$

with convergent power series normalized by $f_1(0) = 1$ and $f_2(0) = f_3(0) = f_4(0) = 0$. One actual period function of the underlying family of Calabi-Yau threefolds is given by $(2\pi i)^3\, \varpi_1$. To obtain a basis of period functions, one can consider the monodromy of this period function with respect to the path encircling the conifold point counterclockwise. This has the form

$$(2\pi i)^3\, \varpi_1 \;\mapsto\; (2\pi i)^3\, \varpi_1 - \Big(\chi\, \zeta(3)\, \varpi_1 + c_2 \cdot D\, \frac{(2\pi i)^2}{24}\, \varpi_2 + \kappa\, \varpi_4\Big)$$

with integers χ, $c_2 \cdot D$, κ. In the context of mirror symmetry, these integers arise as topological invariants and the values are listed in table 1 in [12]. It follows that a basis of period functions is given by

$$\Pi \;=\; \begin{pmatrix} (2\pi i)^3 & 0 & 0 & 0 \\ 0 & (2\pi i)^2 & 0 & 0 \\ c_2 \cdot D\, \frac{(2\pi i)^3}{24} & \sigma\, (2\pi i)^2 & -\kappa\, 2\pi i & 0 \\ \chi\, \zeta(3) & c_2 \cdot D\, \frac{(2\pi i)^2}{24} & 0 & \kappa \end{pmatrix} \varpi\,,$$

where σ is 0 or 1/2 depending on whether κ is even or odd. In this basis, the monodromy matrices (acting by $\Pi \mapsto M \Pi$) are integral and symplectic with respect to the intersection matrix

$$\Sigma \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

For loops counterclockwise around 0 and μ, the monodromy matrices are given by

$$M_0 \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \sigma - \frac{\kappa}{2} & -\kappa & 1 & 0 \\ \frac{c_2 D + 2\,\kappa}{12} & \sigma + \frac{\kappa}{2} & -1 & 1 \end{pmatrix} \quad \text{and} \quad M_\mu \;=\; \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the fundamental matrix $W = (\Pi, \Theta\Pi, \Theta^2\Pi, \Theta^3\Pi)$, the intersection pairing gives the quadratic relation

$$W(z)^T\, \Sigma\, W(z) \;=\; \kappa\, (2\pi i)^3 \begin{pmatrix} 0 & 0 & 0 & \frac{1}{1-z/\mu} \\ 0 & 0 & -\frac{1}{1-z/\mu} & -\frac{z/\mu}{(1-z/\mu)^2} \\ 0 & \frac{1}{1-z/\mu} & 0 & \frac{(1-\frac{4}{3}\,\alpha)\, z/\mu}{(1-z/\mu)^2} \\ -\frac{1}{1-z/\mu} & \frac{z/\mu}{(1-z/\mu)^2} & -\frac{(1-\frac{4}{3}\,\alpha)\, z/\mu}{(1-z/\mu)^2} & 0 \end{pmatrix}$$

with $\alpha = \frac{3}{8} \sum_i a_i^2$.

We can now describe the limit mixed Hodge structure for $\delta = 1 - z/\mu \to 0$. This can be computed as explained in Section 4.2. On the level of the Hodge diamond, going from the pure

Hodge structure of weight 3 associated with a regular point to the limit mixed Hodge structure associated with the conifold point $\delta = 0$ has the following effect:

$$
\begin{array}{ccccccccccccccc}
 & & & 0 & & & & & & & & 0 & & & \\
 & & 0 & & 0 & & & & & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & & & 0 & & 1 & & 0 & \\
1 & & 1 & & 1 & & 1 & \to & 1 & & 0 & & 0 & & 1 \\
 & 0 & & 0 & & 0 & & & & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & & & & & 0 & & 0 & & \\
 & & & 0 & & & & & & & & 0 & & &
\end{array}
$$

Here, the parts of the weight filtration with weight 2, 3 and 4 consist of the image of $M_\mu - 1$, the kernel of $M_\mu - 1$ and the whole space. We define an associated period matrix T_μ by

$$
\Pi(z) \;=\; T_\mu \begin{pmatrix} \log(\delta)\,\nu(\delta) + O(\delta^3) \\ 1 + O(\delta^3) \\ \delta^2 + O(\delta^3) \\ \nu(\delta) \end{pmatrix}
$$

with the so-called vanishing period function $\nu(\delta) = \delta + O(\delta^2)$. Using the monodromy matrices, the quadratic relations from the intersection pairing and that $f_1(z) > 0$ for $0 < z < \mu$, one finds that

$$
T_\mu \;=\; \begin{pmatrix} -2\pi i\,\sqrt{\kappa} & b & d & c \\ 0 & w^+ & e^+ & a^+ \\ 0 & \sigma\, w^+ + w^- & \sigma\, e^+ + e^- & \sigma\, a^+ + a^- \\ 0 & 0 & 0 & (2\pi i)^2\,\sqrt{\kappa} \end{pmatrix}
$$

with real constants w^+, e^+, a^+ and purely imaginary constants w^-, e^-, a^-, b, d, c satisfying

$$
\begin{aligned}
\det \begin{pmatrix} w^+ & e^+ \\ w^- & e^- \end{pmatrix} &\;=\; -(2\pi i)^3\,\frac{\kappa}{2} \\
\det \begin{pmatrix} w^+ & a^+ \\ w^- & a^- \end{pmatrix} &\;=\; -(2\pi i)^2\,\sqrt{\kappa}\, b \\
\det \begin{pmatrix} e^+ & a^+ \\ e^- & a^- \end{pmatrix} &\;=\; -(2\pi i)^3\,\kappa\,\alpha - (2\pi i)^2\,\sqrt{\kappa}\, d\,.
\end{aligned}
$$

We remark that the pure period matrices of the graded pieces are given by

$$
2\pi i\,\sqrt{\kappa}\,, \qquad \begin{pmatrix} w^+ & e^+ \\ w^- & e^- \end{pmatrix}, \qquad (2\pi i)^2\,\sqrt{\kappa}\,.
$$

In particular, the pure period matrix of weight 3 corresponds to the period matrix of $H^3(\widehat{X}_\mu)$. It is expected that the limit mixed Hodge structure of the conifold fiber is motivic, i.e. that it is part of a mixed motive, but we are not aware of a proof of this.

6.3 Modularity of the pure periods

The periods $w^\pm$ and $e^\pm$ are periods of the rigid Calabi-Yau threefold $\widehat{X}_\mu$. Due to the modularity of the corresponding Galois representations, it is expected that these periods are given by periods and quasiperiods of the associated newform f. Numerical results in this direction were given in [73], where for each case the values of $L(f,s)$ for $s = 1,2,3$ are given in terms of w^+ and w^-. In [12], stronger numerical results have been given, which express $w^\pm$ and $e^\pm$ as periods and quasiperiods of f. For the case with hypergeometric indices $\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$, the numerical results further have been proven by constructing a correspondence with a Kuga-Sato threefold. In this section, we use another method, the so-called fibering out, to prove the modularity of $w^\pm$ and $e^\pm$ for twelve cases.

Theorem 12. *Consider any of the twelve hypergeometric variations with hypergeometric indices not equal to $\{\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}\}$ or $\{\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}\}$. Then the numbers $w^\pm$ and $e^\pm$ occurring in the period matrix T_μ are periods and quasiperiods of the associated newform f.*

Proof. The proof uses the method of fibering out from [37] and is analogous for all twelve cases. We give it in detail for the case with hypergeometric indices $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ (i.e. for the family of mirror quintics) and later comment on the other cases. The first important ingredient is an expression of the period functions of X in terms of integrals of period functions of a rank three variation of Hodge structures. To obtain this, we use the elementary identity

$$\begin{aligned}
{}_4F_3\left(\begin{matrix}\frac{1}{5}\ \frac{2}{5}\ \frac{3}{5}\ \frac{4}{5}\\ 1\ 1\ 1\end{matrix};5^5 z\right) &= \sum_{n=0}^{\infty} \frac{(5\,n)!}{n!^5}\, z^n \\
&= \frac{1}{2\pi i} \oint\limits_{|t|=1/5} \left(\sum_{n=0}^{\infty}\sum_{k=-n}^{\infty} \frac{(5\,n+k)!}{n!^4\,(n+k)!}\, t^k\, z^n\right) \frac{\mathrm{d}t}{t} \\
&= \frac{1}{2\pi i} \oint\limits_{|t|=1/5} \frac{1}{1-t} \left(\sum_{n=0}^{\infty} \frac{(4\,n)!}{n!^4} \left(\frac{z}{t\,(1-t)^4}\right)^n\right) \frac{\mathrm{d}t}{t} \\
&= \frac{1}{2\pi i} \oint\limits_{|t|=1/5} \frac{1}{1-t}\ {}_3F_2\left(\begin{matrix}\frac{1}{2}\ \frac{1}{4}\ \frac{3}{4}\\ 1\ 1\end{matrix};2^8 \frac{z}{t\,(1-t)^4}\right)\frac{\mathrm{d}t}{t}
\end{aligned}$$

which holds for $|z| \le 1/5^5$. The rank three hypergeometric function in the integral is associated with a family of K3 surfaces and we say that we have fibered out a period function of the family of mirror quintics. Note that the contour of integration can be deformed (without crossing singularities of the integrand) and we have just chosen the circle $|t| = 1/5$ so that the identity is valid for all $|z| \le 1/5^5$. To derive similar identities for a basis of period functions, we consider the monodromy with respect to z. To do so, we first write our identity in a more conceptual form. For this, consider the operator $\Theta^3 - 2^8\, t\,(\Theta + 1/2)\,(\Theta + 1/4)\,(\Theta + 3/4)$ with $\Theta = t\frac{\mathrm{d}}{\mathrm{d}t}$. This annihilates the rank three hypergeometric function and for $|t| < 1/2^8$, a basis of solutions is given by

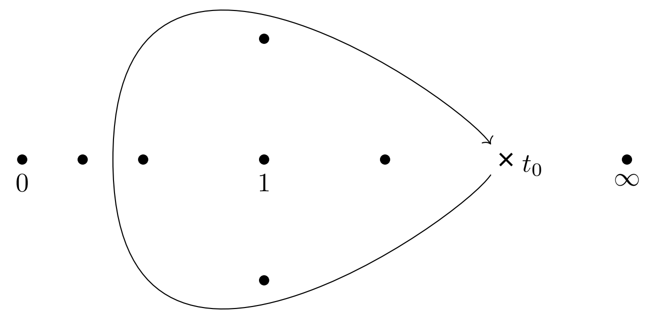
$$\varrho(t) = \begin{pmatrix}(2\pi i)^2 & 0 & 0\\ 0 & 2\pi i & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} f_1(t)\\ f_1(t)\,\log(t) + f_2(t)\\ \frac{1}{2} f_1(t)\log(t)^2 + f_2(t)\,\log(t) + f_3(t)\end{pmatrix}$$

with power series normalized by $f_1(0) = 1$ and $f_2(0) = f_3(0) = 0$. Now fix some $t_0 \gg 0$ and define

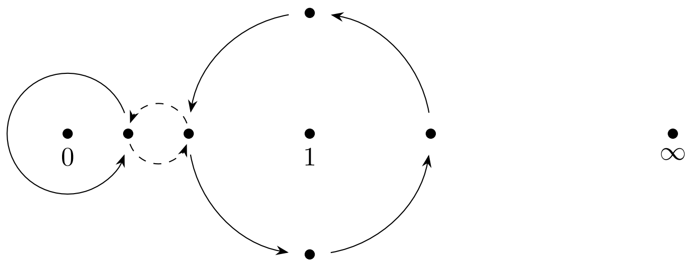
$$\begin{aligned}
I_z :\ \pi_1(\mathcal{M}_z, t_0) \times \mathbb{Q}^3 &\to \mathbb{C}\\
(\gamma, v) &\mapsto \int_\gamma \frac{1}{t\,(1-t)}\, v\cdot\phi_z^*\varrho(t)\,\mathrm{d}t\,,
\end{aligned}$$

where $\phi_z(t) = \frac{z}{t\,(1-t)^4}$ and $\mathcal{M}_z = \mathbb{P}^1 \smallsetminus \phi_z^{-1}(\{0, 1/2^8, \infty\})$. Here, the integrand is understood to be analytically continued along γ. Our identity can then be rewritten as $\Pi_1(z) = I_z(\gamma_1, (1,0,0))$ with γ_1 as depicted below for $0 < z < 1/5^5$:

γ_1 :

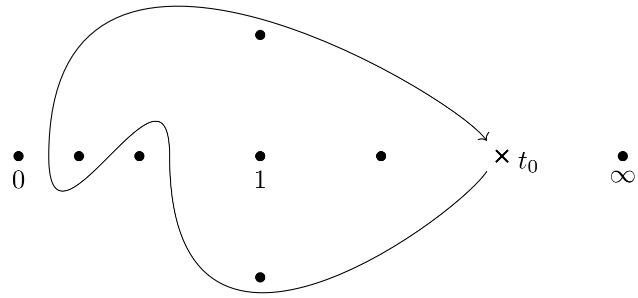


Applying any monodromy M, we obtain an identity $(M\,\Pi)_1(z) = I_z(M\gamma_1, (1,0,0))$ for some (not necessarily unique) $M\gamma_1 \in \pi_1(\mathcal{M}_z, t_0)$. To determine possible choices of $M\gamma_1$ for different monodromies M, we look at the variation of $\mathcal{M}_z$ with z. For a loop which starts at some $0 < z < 1/5^5$ and encircles 0 (respectively $1/5^5$) counterclockwise, the action on the holes of $\mathcal{M}_z$ is depicted by the solid (respectively dashed) arrows below:

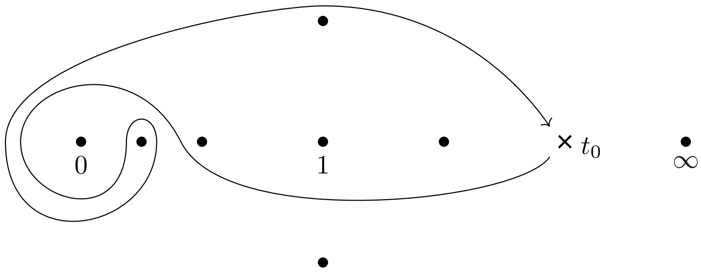


Using this action, we can look at the corresponding deformation of γ_1 to obtain the following loops:

$M_{1/5^5}\gamma_1$:



$M_{1/5^5}M_0\gamma_1$:



More generally, for any $\gamma \in \pi_1(\mathcal{M}_z, t_0)$ and any $v \in \mathbb{Q}^3$ that is invariant under the monodromy associated with γ, $I_z(\gamma, v)$ is locally a holomorphic function of z which does not depend on the choice of t_0 and solves the Picard-Fuchs equation associated with the family of mirror quintics. Integrating by parts, one then finds that the first two derivatives with respect to z are given by

$$I'_z(\gamma, v) \;=\; \frac{1}{z}\int_\gamma \frac{5}{(1-5\,t)^2}\, v\cdot \phi_z^*\varrho(t)\,\mathrm{d}t \qquad \text{and} \qquad I''_z(\gamma, v) \;=\; \frac{1}{z^2}\int_\gamma \frac{30\,t\,(3-5\,t)}{(1-5\,t)^4}\, v\cdot \phi_z^*\varrho(t)\,\mathrm{d}t\,.$$

Even more generally, it will turn out to be useful to consider sums of pairs (γ_i, v_i) that are invariant under the monodromy in the sense that $\sum_i v_i\,(\rho(\gamma_i) - 1) = 0$ in terms of the monodromy representation ρ. In this case, $\sum_i I_z(\gamma_i, v_i)$ is again locally a holomorphic function of z which does not depend on the choice of t_0 and solves the Picard-Fuchs equation associated with the family of mirror quintics.

To evaluate our integral identities in the limit $z \to 1/5^5$, we use the second important ingredient for the proof, which is the well-known modularity of ϱ. More precisely, in terms of the normalized Hauptmodul $h(\tau) = q^{-1} + O(q)$ of $\Gamma_0^*(2)$ and $t_2 = \frac{1}{h+104}$, we have

$$t_2^*\varrho(\tau) \;=\; (2\pi i)^2 \begin{pmatrix} 1 \\ \tau \\ \frac{1}{2}\tau^2 \end{pmatrix} E(\tau)$$

with the unique Eisenstein series $E \in M_2(\Gamma_0(2))$ normalized by $E(\tau) = 1 + O(q)$. It is not clear that this helps since we need to evaluate the pullback $\phi_z^*\varrho$ and not ϱ. However, in the limit $z \to 1/5^5$, we can use that there are modular solutions to $\phi_{1/5^5}(t(\tau)) = t_2(5\,\tau)$. We fix the solution that is given in terms of the normalized Hauptmodul $h(\tau) = q^{-1} + O(q)$ of $\Gamma_0^*(50)$ by $t_{50} = \frac{(1-h)\,(3+h)^2}{5\,(1-h-h^2)}$. In terms of the Dedekind eta function, one can express the Hauptmodul h as

$$h(\tau) \;=\; \frac{\eta(\tau)\,\eta(50\,\tau)}{\eta(2\,\tau)\,\eta(25\,\tau)} + \frac{\eta(2\,\tau)\,\eta(25\,\tau)}{\eta(\tau)\,\eta(50\,\tau)} - 1\,.$$

We obtain the pullback

$$t_{50}^*\left(\frac{1}{t\,(1-t)}\;\phi_{1/5^5}^*\,\varrho(t)\,\mathrm{d}t\right) \;=\; (2\pi i)^3\begin{pmatrix}1\\ 5\,\tau\\ \frac{25}{2}\,\tau^2\end{pmatrix} f_{50}\,\mathrm{d}\tau\,,$$

where $f_{50}(\tau) = 5\,f(\tau) - 20\,f(2\,\tau)$ in terms of the newform f. We now use this pullback to express

$$\begin{aligned}
&I_z(M_{1/5^5}M_0\gamma_1,(1,0,0)) + 4\,I_z(\gamma_1,(1,0,0))\\
&=(M_{1/5^5}M_0\,\Pi)_1(z) + 4\,\Pi_1(z)\\
&=(-\tfrac{5}{2}\,w^+ + w^-, -\tfrac{5}{2}\,e^+ + e^-, -\tfrac{5}{2}\,a^+ + a^- - (2\pi i)^2\sqrt{5})\begin{pmatrix}1+O(\delta^3)\\ \delta^2+O(\delta)\\ \delta+\frac{7}{10}\,\delta^2+O(\delta^3)\end{pmatrix}
\end{aligned}$$

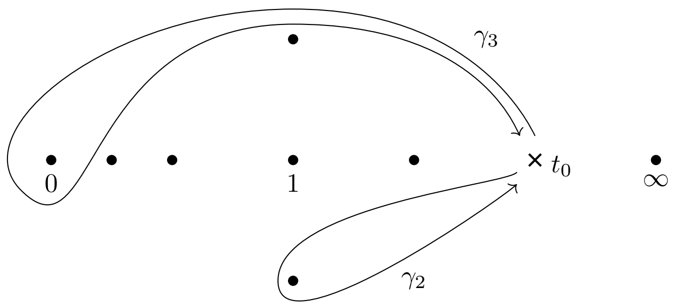
in terms of integrals over the upper half-plane. One has to be careful with the limit $z \to 1/5^5$ because in this limit the two holes of $\mathcal{M}_z$ that are between 0 and 1 collide at $1/5$ and pinch the loops there. To circumvent this problem, we decompose the loops into loops that do not get pinched and loops that get pinched but where the integrand is holomorphic at the two points that collide. The latter can then be pushed outside of the region that gets pinched. To do this, we need the monodromy matrices which act on ϱ. For loops which encircle 0 and $1/2^8$ counterclockwise, these are given by

$$\begin{pmatrix}1&0&0\\1&1&0\\\frac{1}{2}&1&1\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}0&0&-4\\0&1&0\\-\frac{1}{4}&0&0\end{pmatrix}.$$

Neglecting an integral around ∞ that vanishes since $\phi_z^*\varrho_1$ is holomorphic at ∞, one then obtains

$$\begin{aligned}
&I_z(M_{1/5^5}M_0\gamma_1,(1,0,0)) + 4\,I_z(\gamma_1,(1,0,0))\\
&=I_z(\gamma_2,(1,0,0)) + I_z(\gamma_3,(-18,-12,-4)) + I_z(\gamma_1,(1,4,-4))
\end{aligned}$$

with the paths γ_2 and γ_3 sketched below:



The integrand of $I_z(\gamma_1,(1,4,-4))$ is holomorphic at the two points which collide for $z \to 1/5^5$. Hence, we can push γ_1 outside of the region that gets pinched. Now one can safely take the limit $z \to 1/5^5$ and pullback the contours by t_{50}. Taking the limit $t_0 \to \infty$, we can read off the corresponding paths on the upper half-plane from the monodromy matrices and we obtain

$$\begin{aligned}
-\frac{5}{2}\,w^+ + w^- \;=\; (2\pi i)^3\Big(&\int_\infty^{1/5} f_{50}(\tau)\,\mathrm{d}\tau + \int_\infty^{-3/5}(-18-60\,\tau-50\,\tau^2)\,f_{50}(\tau)\,\mathrm{d}\tau\\
&+\int_\infty^{2/5}(1+20\,\tau-50\,\tau^2)\,f_{50}(\tau)\,\mathrm{d}\tau\Big)\,.
\end{aligned}$$

This proves that $w^{\pm}$ are periods of f. To obtain an equation for $e^{\pm}$, we just need to differentiate with $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\delta^2} - \frac{7}{10}\frac{\mathrm{d}}{\mathrm{d}\delta}$ before taking the limit $z \to 1/5^5$. One then finds that in the equation above, replacing on the left hand side $w^{\pm}$ by $e^{\pm}$ corresponds to replacing on the right hand side f_{50} by

$$F_{50} \;=\; \frac{t_{50}\,(7 + 13\,t_{50} + 5\,t_{50}^2 - 25\,t_{50}^3)}{2\,(1-5\,t_{50})^4}\,f_{50}\,.$$

This modular form is a meromorphic partner of f_{50} and hence $e^{\pm}$ are quasiperiods of f. This finishes the proof of the theorem for the mirror quintic.

For the other cases, we use that identities between our rank four hypergeometric functions and integrals involving rank three hypergeometric functions exist for all fourteen cases. The identitites are given in table 12 in [29] and, in Table 6.2, we reproduce a part of this table for our applications. Each row in the table contains hypergeometric indices $\{a_1, a_2, a_3, a_4\}$, hypergeometric indices $\{b_1, b_2, b_3\}$ and parameters k, l, β and corresponds to the identity

$${}_4F_3\!\left(\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ & 1 & 1 & 1 \end{matrix};z\right) \;=\; \frac{1}{2\pi i}\oint_{|t|=\frac{k}{k+l}} \frac{1}{(1-t)^{\beta}}\;{}_3F_2\!\left(\begin{matrix} b_1 & b_2 & b_3 \\ & 1 & 1 \end{matrix};\frac{k^k\,l^l}{(k+l)^{k+l}}\frac{z}{t^k\,(1-t)^l}\right)\frac{\mathrm{d}t}{t}$$

for $|z| \leq 1$.

a_1,a_2,a_3,a_4	b_1,b_2,b_3	k,l,β	$\widetilde{N}$
$\frac12,\frac12,\frac12,\frac12$	$\frac12,\frac12,\frac12$	$1,1,1$	16
$\frac14,\frac13,\frac23,\frac34$	$\frac13,\frac12,\frac23$	$2,2,1$	48
	$\frac13,\frac12,\frac23$	$1,1,\frac12$	48
	$\frac14,\frac12,\frac34$	$2,1,1$	18
$\frac14,\frac12,\frac12,\frac34$	$\frac12,\frac12,\frac12$	$2,2,1$	64
	$\frac12,\frac12,\frac12$	$1,1,\frac12$	64
	$\frac13,\frac12,\frac23$	$3,1,1$	48
	$\frac14,\frac12,\frac34$	$1,1,1$	8
$\frac15,\frac25,\frac35,\frac45$	$\frac13,\frac12,\frac23$	$3,2,1$	75
	$\frac14,\frac12,\frac34$	$4,1,1$	50
$\frac13,\frac13,\frac23,\frac23$	$\frac13,\frac12,\frac23$	$2,1,1$	27
$\frac14,\frac14,\frac34,\frac34$	$\frac14,\frac12,\frac34$	$2,2,1$	32
	$\frac14,\frac12,\frac34$	$1,1,\frac12$	32
$\frac13,\frac12,\frac12,\frac23$	$\frac12,\frac12,\frac12$	$2,1,1$	—
	$\frac13,\frac12,\frac23$	$1,1,1$	12
$\frac16,\frac12,\frac12,\frac56$	$\frac12,\frac12,\frac12$	$2,1,\frac12$	—
	$\frac13,\frac12,\frac23$	$3,3,1$	—
	$\frac14,\frac12,\frac34$	$1,2,\frac12$	72
	$\frac16,\frac12,\frac56$	$1,1,1$	—
$\frac16,\frac13,\frac23,\frac56$	$\frac13,\frac12,\frac23$	$2,1,\frac12$	108
	$\frac14,\frac12,\frac34$	$4,2,1$	—
	$\frac16,\frac12,\frac56$	$2,1,1$	—
$\frac18,\frac38,\frac58,\frac78$	$\frac13,\frac12,\frac23$	$3,1,\frac12$	—
	$\frac14,\frac12,\frac34$	$4,4,1$	128
	$\frac14,\frac12,\frac34$	$2,2,\frac12$	128
	$\frac16,\frac12,\frac56$	$1,3,\frac12$	—
$\frac16,\frac14,\frac34,\frac56$	$\frac14,\frac12,\frac34$	$2,1,\frac12$	72
	$\frac16,\frac12,\frac56$	$2,2,1$	—
	$\frac16,\frac12,\frac56$	$1,1,\frac12$	—
$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$\frac14,\frac12,\frac34$	$4,1,\frac12$	200
	$\frac16,\frac12,\frac56$	$2,3,\frac12$	—
$\frac16,\frac16,\frac56,\frac56$	$\frac16,\frac12,\frac56$	$2,1,\frac12$	—
$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$\frac14,\frac12,\frac34$	$4,2,\frac12$	—

Table 6.2: Data for integral identities involving hypergeometric functions of rank four and rank three. The last column gives the level $\widetilde{N}$ in the cases where the identity directly allows to prove the modularity of pure periods of the conifold fiber.

Using these identities, one proceeds as for the example with hypergeometric indices $\{\frac15,\frac25,\frac35,\frac45\}$. However, it is not guaranteed that the pullback of the integrands to the upper half-plane gives modular forms. If this is the case, then the last column in Table 6.2 gives the level $\widetilde{N}$ and it is easy to see that the periods that one obtains are periods and quasiperiods of f. For example, the modular form $f_{\widetilde{N}} \in S_4(\Gamma_0(\widetilde{N}))$ has either rational Fourier coefficients or Fourier coefficients in some quadratic field and either is in the same twist class as f or, in the case of $N = 9$, both f and $f_{\widetilde{N}}$ have complex multiplication by $\mathbb{Q}(\sqrt{-3})$. The forms $f_{\widetilde{N}}$ can be found in the script `ModularityFiberingOut.gp` from [17]. $\square$

We remark that the identities in Table 6.2 which do not give cusp forms $f_{\widetilde{N}}$ still give interesting identities. In general, one obtains instead of a cusp form a modular form multiplied by an algebraic function of a modular function. To give one simple example, we consider the identity

$$ {}_4F_3\left(\begin{matrix} \frac{1}{2} \; \frac{1}{2} \; \frac{1}{6} \; \frac{5}{6} \\ 1 \; 1 \; 1 \end{matrix}; z\right) = \frac{1}{2\pi i} \oint_{|t|=1} \frac{1}{\sqrt{1-t}} \, {}_3F_2\left(\begin{matrix} \frac{1}{2} \; \frac{1}{6} \; \frac{5}{6} \\ 1 \; 1 \end{matrix}; \frac{z}{t}\right) \frac{\mathrm{d}t}{t}, $$

which is actually not featured in Table 6.2. For $z = 1$, we can substitute $t_1(\tau) = j(\tau)/1728$ with the j-invariant $j(\tau) = q^{-1} + 744 + 196884\, q + \cdots$. The integrand then becomes

$$ \frac{E_4(\tau)}{\sqrt{-j(\tau)/1728}} \, \mathrm{d}\tau $$

with the Eisenstein series $E_4(\tau) = 1 + 240\, q + \cdots$ of level 1. As a consequence, integrals of this object (which is not even a well-defined function on the upper half-plane) must evaluate to the periods $w^{\pm}$ of the newform f associated with the conifold fiber (i.e. the unique newform in $S_4(\Gamma_0(72))$ with Hecke eigenvalue $a_5 = -14$). This can also be checked numerically and, for example, the script `PeriodIdentityLevel1.gp` from [17] gives the numerical identity

$$ \frac{(2\pi i)^3}{2} \int_0^{\infty} (\tau - z)^2 \, \frac{E_4(z)}{\sqrt{-j(z)/1728}} \, \mathrm{d}z = -w_+ \, \tau + \frac{w_-}{8} \, (\tau^2 - 1) \, . $$

6.4 Modularity of the mixed periods

In this section, we extend the result from the previous section to the mixed periods. We start with the following theorem.

Theorem 13. *Consider any of the twelve hypergeometric variations with hypergeometric indices not equal to* $\{\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}\}$ *or* $\{\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}\}$*. Then there are modular forms*

$$ \begin{aligned} f_{\widetilde{N}} &\in S_4(\Gamma_0(\widetilde{N})) \\ F_{\widetilde{N}} &\in S_4^{\mathrm{mero}}(\Gamma_0(\widetilde{N})) \\ g_{\widetilde{N}} &\in M_4^{\mathrm{mero}}(\Gamma_0(\widetilde{N})) \end{aligned} $$

and a one-dimensional $\mathbb{Z}$*-lattice* $\Lambda \subset \sqrt{\kappa}\,(2\pi i)^2\,\mathbb{Q}$ *such that the residues of* $\tau^n g_{\widetilde{N}}(\tau)$ *for* $n = 0, 1, 2$ *are in* $\frac{2}{(2\pi i)^4}\Lambda$ *and the associated classes*

$$ \begin{aligned} r_{f_{\widetilde{N}}}, r_{F_{\widetilde{N}}} &\in H^1_{\mathrm{par}}(\Gamma_0(\widetilde{N}), \langle 1, \tau, \tau^2 \rangle_{\mathbb{C}}) \\ r_{g_{\widetilde{N}}} &\in H^1_{\mathrm{par}}(\Gamma_0(\widetilde{N}), \langle 1, \tau, \tau^2 \rangle_{\mathbb{C}/\Lambda}) \end{aligned} $$

decompose as

$$ (r_{f_{\widetilde{N}}}, r_{F_{\widetilde{N}}}, r_{g_{\widetilde{N}}}) = (r^+, r^-) \begin{pmatrix} w^+ & e^+ & a^+ \\ w^- & e^- & a^- \end{pmatrix} $$

for some $r^{\pm} \in H^1_{\mathrm{par}}(\Gamma_0(\widetilde{N}), \langle 1, \tau, \tau^2 \rangle_{\mathbb{Q}})^{\pm}$.

Proof. The forms $f_{\widetilde{N}}$ and $F_{\widetilde{N}}$ are defined in the proof of Theorem 12. In the same way, $g_{\widetilde{N}}$ is obtained from the pullback of the derivative of the integrand of I_z at $z = \mu$, which, for the example of the family of mirror quintics, gives

$$ g_{50} = -\frac{5\, t_{50}\, (1 - t_{50})}{(1 - 5\, t_{50})^2} \, f_{50} \, . $$

The residues of $\tau^n g_{\widetilde{N}}(z)$ for $n = 0, 1, 2$ can only be non-vanishing at points in the preimage of $\frac{k}{k+l}$. These residues contribute to the vanishing period function Π_4 and thus they must lie in $\frac{2}{(2\pi i)^4}\Lambda$ for some one-dimensional $\mathbb{Z}$-lattice $\Lambda \subset \sqrt{\kappa}\,(2\pi i)^2\,\mathbb{Q}$. In particular, we can choose an Eichler integral of $g_{\widetilde{N}}$ which is only well-defined modulo $\langle 1, \tau, \tau^2\rangle_\Lambda$. The classes $r_{f_{\widetilde{N}}}$, $r_{F_{\widetilde{N}}}$ and $r_{g_{\widetilde{N}}}$ are uniquely determined by the integrals

$$\frac{(2\pi i)^3}{2}\int_{\gamma^{-1}\tau_0}^{\tau_0} p(\tau)\,(f_{\widetilde{N}}(\tau), F_{\widetilde{N}}(\tau), g_{\widetilde{N}}(\tau))\,\mathrm{d}\tau$$

with $\gamma \in \Gamma_0(\widetilde{N})$ and polynomials $p(\tau) \in \langle 1, \tau, \tau^2\rangle_{\mathbb{Q}}$ that are invariant under $|_{-2}\gamma$. Here, the integrals of $g_{\widetilde{N}}$ again have to be understood modulo $\langle 1, \tau, \tau^2\rangle_\Lambda$. That these integrals determine the cohomology classes follows from the fact that the map

$$H^1_{\mathrm{par}}(\Gamma_0(\widetilde{N}), \langle 1, \tau, \tau^2\rangle_{\mathbb{Q}}) \;\to \bigoplus_{\gamma\in\Gamma_0(\widetilde{N})} H^1_{\mathrm{par}}(\langle\gamma\rangle, \langle 1, \tau, \tau^2\rangle_{\mathbb{Q}})$$

is injective (and that π is transcendental), which we checked for each level $\widetilde{N}$ by using the script `CohomologyRestrictionInjectivity.mgm` from [17]. What is convenient about these integrals is that the integrand is invariant under the action of γ and thus the integral does not depend on the choice of τ_0. To prove the statement of the theorem, we want to express these integrals in terms of the periods of the limit mixed Hodge structure. For this, we push the integrals forward by $t_{\widetilde{N}}$ to express them in terms of I_z. It follows that the three classes are determined by the expansion of integrals $I_z(\gamma, v)$ for $z \to \mu$, where γ is not pinched for $z \to \mu$ and v is invariant under the monodromy with respect to γ. In this case, we can write

$$I_z(\gamma, v) \;=\; c_2\cdot\Pi_2(z) + c_3\cdot\Pi_3(z) + c_4\cdot\Pi_4(z)$$

and after choosing Λ sufficiently large it only remains to show that always $c_2, c_3, c_4 \in \mathbb{Q}$. To prove this, one acts with sufficiently many monodromy matrices on $\Pi_1(z) = I_z(\gamma_1, (1,0,0))$. We explain this in detail for the family of mirror quintics. After acting with words of length ≤ 5 in M_0 and $M_{1/5^5}$, one obtains a sixteen-dimensional space spanned by monodromy invariant linear combinations of pairs (γ, v). Restricting to linear combinations that do not get pinched for $z \to 1/5^5$, one obtains a thirteen-dimensional space. We can supplement this by pairs where the first component is a loop around the two points that collide for $z \to 1/5^5$. This loop has trivial monodromy and hence the integrals correspond to rational multiples of the vanishing period function. One ends up with a fifteen-dimensional space, i.e. with the complete space of all monodromy invariant linear combinations of pairs (γ, v) that do not get pinched for $z \to 1/5^5$. For the other cases, one proceeds in the same way, except that sometimes one has to add pairs (γ, v) where γ is a loop around a single point and the integrand corresponding to v is holomorphic at that point. The associated integral clearly vanishes. In each case, one then obtains the complete space of all monodromy invariant linear combinations that do not get pinched for $z \to \mu$ (which has dimension $3\cdot(k+l)$). The computations were done using the script `MonodromyActionFundamentalGroup.gp` and the choices of cycles from `FundamentalGroupBases.pdf`, both of which can be found in [17]. □

There are several remarks we would like to make about this theorem. First, the nature of $f_{\widetilde{N}}$ and $F_{\widetilde{N}}$ and their associated periods is well understood. In particular, we can obtain these periods from the newform f of level N and a meromorphic partner F. The forms f and F (modulo third derivatives) belong to the finite-dimensional space $\mathbb{S}_4(\Gamma_0(N))$ and the subspace that they span is determined from the Galois representations associated with $\widehat{X}_\mu$. However, we do not have a general understanding of $g_{\widetilde{N}}$. The theorem shows that $g_{\widetilde{N}}$ contains the information about the Hecke eigenvalues, but we were not able to write down a finite-dimensional space to which $g_{\widetilde{N}}$ belongs to and which has an action of Hecke operators. An attempt which fails is to consider the extension of $\mathbb{S}_4(\Gamma_0(\widetilde{N}))$ by $g_{\widetilde{N}}$. This is certainly finite-dimensional, but it is not preserved by the usual Hecke operators. Another attempt would be to first understand the nature of

$$g_{\widetilde{N}} - c_1\,f_{\widetilde{N}} - c_2\,F_{\widetilde{N}}$$

with

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} w^+ & e^+ \\ w^- & e^- \end{pmatrix}^{-1} \begin{pmatrix} a^+ \\ a^- \end{pmatrix} = -\frac{2}{(2\pi i)^3\,\kappa}\begin{pmatrix} (2\pi i)^3\,\kappa\,\alpha + (2\pi i)^2\sqrt{\kappa}\,d \\ -(2\pi i)^2\sqrt{\kappa}\,b \end{pmatrix}.$$

The periods of this form are rational multiples of $(2\pi i)^2\sqrt{\kappa}$, but the Fourier coefficients are now in general more complicated. Only for the case of level $N = 8$, numerical computations suggest that the coefficients are still rational. In terms of the unique newform f of $S_4(\Gamma_0(8))$ and the Hauptmodul t of $\Gamma_0(8)$ given by $t(\tau) = \eta(\tau)^8\,\eta(4\tau)4/\eta(2\tau)^{12}$, one then finds that the periods of

$$\frac{1-6\,t^2+t^4}{(1+t^2)^2}\,f$$

are rational multiples of $(2\pi i)^2$. Algebraically, this gives for example the identity

$$\frac{1}{2}\int_0^1 \frac{1-6\,t^2+t^4}{(1+t^2)^2}\left(\int_1^\infty \frac{1}{\sqrt{x\,(x-1)\,(x-(1-t^2))}}\,\mathrm{d}x\right)^2 \mathrm{d}t \;=\; -\frac{(2\pi i)^2}{8}\,.$$

In the general case, one can replace $g_{\widetilde{N}} - c_1\,f_{\widetilde{N}} - c_2\,F_{\widetilde{N}}$ by a form with simpler Fourier coefficients by applying suitable linear combinations of Hecke operators (so that only the action on $g_{\widetilde{N}}$ remains). All of this suggests that it would be beneficial to have an understanding of modular forms with simple periods (in our case rational multiples of $(2\pi i)^2\sqrt{\kappa}$). In the case of weight 2, this would be simple to understand, since one can consider derivatives of logarithms of modular functions. In the case of weight 4, we have not made progress in this direction.

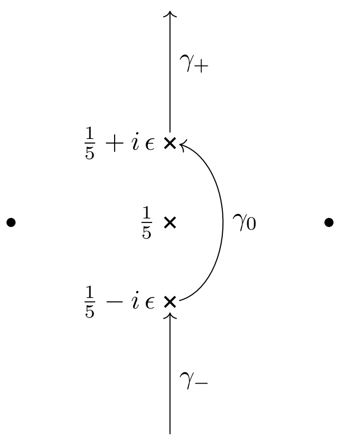
We continue with a discussion of the remaining mixed periods b, d, c.

Theorem 14. *Consider any of the twelve hypergeometric variations with hypergeometric indices not equal to* $\{\frac16,\frac16,\frac56,\frac56\}$ *or* $\{\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}\}$. *Then the constants* b, d, c *can be expressed as (suitably regularized) integrals of* $f_{\widetilde{N}}, F_{\widetilde{N}}, g_{\widetilde{N}}$ *between CM points* $\tau_\pm$. *For example, for the family of mirror quintics, one can choose* $\tau_\pm = \pm\frac25 + i\,\frac{\sqrt2}{10}$ *and then*

$$\begin{aligned}
b \;&=\; (2\pi i)^3\int_{\tau_-}^{\tau_+} f_{50}(\tau)\,\mathrm{d}\tau\\
d \;&=\; \lim_{\epsilon\downarrow 0}\left((2\pi i)^3\int_{\tau_-}^{\tau_+} F_{50}(\tau + i\tfrac{\sqrt2}{5}\,\epsilon)\,\mathrm{d}\tau + 2\pi i\,\sqrt5\,\left(-\tfrac{1}{5\,t'_{50}(\tau_-)\,t'_{50}(\tau_+)}\,(\tfrac{1}{\epsilon^3}+1) - \tfrac{257}{480}\right)\right)\\
c \;&=\; \lim_{\epsilon\downarrow 0}\left((2\pi i)^3\int_{\tau_-}^{\tau_+} g_{50}(\tau + i\tfrac{\sqrt2}{5}\,\epsilon)\,\mathrm{d}\tau + 2\pi i\,\sqrt5\,\left(\tfrac{1}{\epsilon} + \log\!\left(-5\,t'_{50}(\tau_-)\,t'_{50}(\tau_+)\,\epsilon^2\right)\right)\right)\,,
\end{aligned}$$

where $t'_{50}(\tau_\pm) \;=\; \mp\Gamma(\frac18)^2\,\Gamma(\frac38)^2/2\sqrt{10}\,\pi^2$. *Here, the integral in the expression for* c *is along the straight line between* τ_- *and* τ_+.

Proof. The proof is again analogous for all twelve cases and we only give it in detail for the family of mirror quintics. Consider the setup of the proof of Theorem 12. In the limit $\delta \to 0$, the loop γ_1 gets pinched at $1/5$ and we have to be careful with divergencies. We take the limit $t_0 \to \infty$ and decompose γ_1 into the three parts γ_-, γ_0, γ_+ depicted below:



For fixed ϵ, we can take the limit $\delta \to 0$ for the integrals over $\gamma_\pm$ and pull these back to the upper half-plane. The asymptotics of the integral over γ_0 in the limit where we first take $\delta \to 0$ and then $\epsilon \to 0$ can be obtained from the expansion

$$\varrho(t) = \begin{pmatrix} -i\,\Gamma(1/8)^2\,\Gamma(3/8)^2 & 8i\pi^2\sqrt{2} & -4i\,\Gamma(5/8)^2\,\Gamma(7/8)^2 \\ \frac{1}{2}\sqrt{2}\,\Gamma(1/8)^2\,\Gamma(3/8)^2 & 0 & -2\sqrt{2}\,\Gamma(5/8)^2\,\Gamma(7/8)^2 \\ \frac{i}{4}\,\Gamma(1/8)^2\,\Gamma(3/8)^2 & 2i\pi^2\sqrt{2} & i\,\Gamma(5/8)^2\,\Gamma(7/8)^2 \end{pmatrix} \begin{pmatrix} 1+\frac{3}{16}x+O(x^2) \\ \sqrt{x}\,(1+O(x)) \\ x+O(x^2) \end{pmatrix}$$

in terms of $x = 1 - 2^8\,t$. From this expansion and the monodromy matrices, we also find that for $\epsilon \to 0$ we can pull back the sum of the paths $\gamma_\pm$ to the path from $\tau_- = -\frac{2}{5}+i\,\frac{\sqrt{2}}{10}$ to $\tau_+ = \frac{2}{5}+i\,\frac{\sqrt{2}}{10}$. The evaluation of b is now straightforward since γ_0 does not contribute in the limit $\epsilon, \delta \to 0$ and thus

$$b = \lim_{z\to 1/5^5} \Pi_1(z) = (2\pi i)^3 \int_{\tau_-}^{\tau_+} f_{50}(\tau)\,\mathrm{d}\tau\,.$$

To obtain d and c, we need to subtract divergent contributions and include the contributions of γ_0. After reparametrizing ϵ, this gives

$$\begin{aligned} d &= \lim_{z\to 1/5^5} \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\delta^2}\Pi_1(z) - \frac{7}{10}\frac{\mathrm{d}}{\mathrm{d}\delta}\Pi_1(z) + 2\pi i\sqrt{5}\left(\frac{1}{2\,\delta}+\frac{7}{20}\right)\right) \\ &= \lim_{\epsilon\downarrow 0}\left((2\pi i)^3\int_{\tau_-}^{\tau_+} F_{50}(\tau+i\tfrac{\sqrt{2}}{5}\,\epsilon)\,\mathrm{d}\tau + 2\pi i\sqrt{5}\,\left(-\tfrac{1}{5\,t'_{50}(\tau_-)\,t'_{50}(\tau_+)}\,(\tfrac{1}{\epsilon^3}+1) - \tfrac{257}{480}\right)\right) \\ c &= \lim_{z\to 1/5^5}\left(\Pi_1(z) + 2\pi i\sqrt{5}\,\log(\delta)\right) \\ &= \lim_{\epsilon\downarrow 0}\left((2\pi i)^3\int_{\tau_-}^{\tau_+} g_{50}(\tau+i\tfrac{\sqrt{2}}{5}\,\epsilon)\,\mathrm{d}\tau + 2\pi i\sqrt{5}\,\left(\tfrac{1}{\epsilon}+\log\left(-5\,t'_{50}(\tau_-)\,t'_{50}(\tau_+)\,\epsilon^2\right)\right)\right) \end{aligned}$$

and from the expansion of ϱ we obtain

$$t'_{50}(\tau_\pm) = \mp\frac{\Gamma(\frac{1}{8})^2\Gamma(\frac{3}{8})^2}{2\sqrt{10}\,\pi^2}\,.$$

Note that due to the residues of g_{50}, variations of the contour of integration can shift the integral in the expression for c by integer multiples of $(2\pi i)^2\sqrt{5}$. The equality above holds for the straight line between τ_- and τ_+.

□

6.5 Relations with central values of derivatives of L-functions

We conclude this chapter by giving new numerical identities for the mixed periods. We start with an heuristic explanation of the steps that led us to these identities. After division by $(2\pi i)^2\sqrt{\kappa}$, the period matrix T_μ has the structure of period matrices of biextensions studied in [10]. There, a real number (called the height) associated with such period matrices is defined. It is shown that for period matrices of suitable nodal degenerations of curves this height coincides (up to rational multiples of logarithms of rational numbers) with a value of the Beilinson-Bloch height defined in [7] and [9]. According to a conjecture by Beilinson [6], the latter should be a rational multiple of the leading coefficient of the Taylor expansion of an associated L-function at its central point divided by a specified period. In our situtation, the height associated with T_μ is given by

$$1 + \frac{1}{2\pi i\sqrt{\kappa}\,w^-}\,\det\begin{pmatrix} b & c \\ w^- & a^- \end{pmatrix}$$

and we look for relations with

$$\frac{2\pi i\sqrt{\kappa}\cdot\text{leading coefficient of Taylor expansion of } L(f\otimes\chi,s) \text{ at } s=2}{w^-}\,.$$

Here, χ is the Dirichlet character associated with $\mathbb{Q}(\sqrt{\kappa})$ (which is trivial if κ is a square). We twist f by this character since this corresponds to a rescaling of the associated periods by $\sqrt{\kappa}$. Using the script `HeightIdentities.gp}` from [17] we obtain the numerical relations given in Table 6.3. Only for the case with hypergeometric indices $\{\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}\}$, where the analytic rank (the order of vanishing of $L(f \otimes \chi, s)$ at $s = 2$) is equal to 2, we were not able to identify the leading coefficient of the associated L-function.

a_1, a_2, a_3, a_4	$\mathrm{ord}_{s=2} L(f \otimes \chi, s)$	height
$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	0	$-3 \log 2$
$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	1	$-6 \log 2 - 3 \log 3 - 144 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	1	$-6 \log 2 - 128 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	1	$-\frac{5}{3} \log 5 - \frac{125}{6} \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	1	$-3 \log 3 - \frac{243}{2} \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	1	$-5 \log 2 - 64 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	1	$-4 \log 2 - 3 \log 3 - 288 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	1	$-3 \log 2 - 3 \log 3 - 216 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	1	$-4 \log 2 - 3 \log 3 - 108 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	1	$-7 \log 2 - 128 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}$	1	$-6 \log 2 - 3 \log 3 - 144 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	1	$-3 \log 2 - \frac{5}{3} \log 5 - \frac{250}{3} \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	1	$-3 \log 2 - 3 \log 3 - 54 \frac{2\pi i \sqrt{\kappa} L'(f\otimes\chi,2)}{w^-}$
$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	2	?

Table 6.3: Numerical identities for the height.

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