A PEL-type Igusa Stack and the p-adic Geometry of Shimura Varieties

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ABSTRACT. Let (G, X) be a PEL-Shimura datum of type AC in Kottwitz's classification. Assume $G_{\mathbb{Q}_p}$ is unramified. We show in this thesis that the good reduction locus of the infinite p-level Shimura variety attached to this datum, considered as a diamond, can be described as the fiber product of a certain v-stack (which we call "Igusa stack") with a Schubert cell of the corresponding B_{dR}^+-affine Grassmannian, over the stack of $G_{\mathbb{Q}_p}$-torsors on the Fargues-Fontaine curve. We also construct a minimal compactification of the Igusa stack and show that this fiber product structure extends to the minimal compactification of the Shimura variety. When the Schubert cell of the affine Grassmannian is replaced by a bounded substack of $\mathcal{G}$-shtukas, where $\mathcal{G}$ is a reductive model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$, we show that this fiber product recovers the integral model of the Shimura variety. This result on integral models, if specialized to a Newton polygon stratum, recovers the fiber product formula of Mantovan. Similar fiber product structures are conjectured by Scholze to exist on general Shimura varieties.

CONTENTS

Notations and conventions

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{R}$: the integers, the rational, complex and real numbers
- $\mathbb{A}_f$: the ring of finite adeles of $\mathbb{Q}$
- p: a fixed rational prime
- $\mathbb{F}_q$: the finite field of cardinality q, which is a power of p
- $\mathbb{Q}_p$, $\mathbb{Z}_p$: the p-adic numbers and the p-adic integers
- $\breve{\mathbb{Q}}_p$, $\breve{\mathbb{Z}}_p$: the maximal unramified extension of $\mathbb{Q}_p$ and the its ring of integers
- A non-archimedean field is a nondiscrete topological field K whose topology is induced by a nonarchimedean norm $|\cdot| : K \to \mathbb{R}_{\geq 0}$. We denote by $\mathcal{O}_K$ its ring of integers, i.e. where the norm is no more than one.
- For a complete non-archimedean field K, we write $\mathrm{Spa}K$ for the adic space $\mathrm{Spa}(K, \mathcal{O}_K)$ and $\mathrm{Spa}\mathcal{O}_K$ for $\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$.
- We use covariant Dieudonné theory and follow the convention of [CS17] to divide the Frobenius in the usual convention by p. So the covariant Dieudonné module of $\mathbb{Q}_p/\mathbb{Z}_p$ is $(\mathbb{Z}_p, F = 1)$.
- Our definition of Breuil-Kisin-Fargues module follows [PR21, 2.2.4], which differs from [SW20, 11.4.3].
- For a geometric object X (e.g. scheme, formal scheme, diamond etc.), we use $|X|$ to mean its underlying topological space.
- Underlined objects denote sheaves, e.g. $\underline{\mathrm{Hom}}$, $\underline{\mathrm{Aut}}$. For a topological space X, $\underline{X}$ means we view it as a sheaf on some site that sends a test object S to continuous maps from $|S|$ to X.

Acknowledgements

1. Introduction: Scholze's fiber product conjecture

The motivating question of this thesis is to understand the geometry of Shimura varieties as p-adic analytic objects and the relation to that of their local counterparts. Instances of such relations can be dated back to the p-adic uniformization of Rapoport and Zink [RZ96]: it relates an open part (the basic Newton stratum) of PEL type Shimura varieties as p-adic rigid analytic spaces to simpler rigid spaces (Rapoport-Zink spaces). The formula for this uniformization formally resembles the complex uniformization expressing the Shimura varieties as adelic double quotients.

As for a general Newton stratum labelled by an element b in the corresponding Kottwitz set, Mantovan [Man05] (c.f. [CS17] for this reformulation and notation) discovered that up to quotienting by the action of a certain group $\tilde{G}_b$, it is a product of a corresponding Rapoport-Zink space $\mathcal{M}^b_\infty$ and a so-called Igusa variety $(\mathrm{Ig}^b_{\mathcal{O}_K})^{\mathrm{ad}}_\eta$. On the basic stratum, this takes the form of a p-adic "uniformization", since in that case the Igusa variety is merely a profinite set.

We show in this thesis that for some PEL type Shimura varieties, via constructing a p-adic analytic stack, which we call "Igusa stack", it is possible to interpolate between the strata and obtain a similar "product structure" on the whole Shimura variety. In order to do this, we need to work relatively over a stack that interpolates the classifying stacks for the groups $\tilde{G}_b$. This base turns out to be provided by the classifying stack of G-bundles on the Fargues-Fontaine curve, which appeared in the work of Fargues-Scholze [FS21]. Correspondingly the role of a p-adic symmetric space is played by a minuscule Schubert cell of the B^+_{dR}-affine Grassmannian of Scholze-Weinstein [SW20], which interpolates the quotients $[\mathcal{M}^b_\infty/\tilde{G}_b]$. Since a general formalism of stacks fibered over adic spaces is not available and might not behave well at all, we work in the category of small v-stacks on perfectoid spaces in characteristic p in the framework of Scholze [Sch18].

In very rough terms, the fiber product structure we seek for is a separation of the geometric information of a p-adic Shimura variety into a p-part and a prime-to-p part, where the minuscule Schubert cell models the local geometry of the Shimura variety at p, while the Igusa stack records the global prime-to-p information. Although in this work we only deal with certain PEL-type Shimura varieties, a similar fiber product structure is conjectured by Scholze to exist on general Shimura varieties. Let us give a precise formulation of this conjecture, before stating our results towards it.

1.1. **The fiber product conjecture.** Let $(G/\mathbb{Q}, X)$ be a Shimura datum, which determines a $G(\mathbb{C})$-conjugacy class of minuscule cocharacters $[\mu^{-1}]$ with field of definition E_0. Fix a rational prime p and let E be the completion of E_0 at a prime above p with residue field $\mathbb{F}_q$. Take a compact open subgroup $K = K_p K^p \subset G(\mathbb{A}_f)$. Consider the category Perf of perfectoid spaces in characteristic p and equip it with the v-topology. Let $\mathcal{S}_{K_p K^p}$ denote the diamond over $\mathrm{Spd} E$ attached to the corresponding Shimura variety at level $K_p K^p$ and $\mathcal{S}_{K^p} := \varprojlim_{K_p} \mathcal{S}_{K_p K^p}$. Let Gr_G be the B^+_{dR}-affine Grassmannian attached to $G_{\mathbb{Q}_p}$, considered as a diamond over $\mathrm{Spd} E$. Fix an isomorphism $\mathbb{C} \cong \bar{\mathbb{Q}}_p$ over E_0, where $\bar{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$ containing E. Fixing a maximal torus inside a Borel subgroup of $G_{\bar{\mathbb{Q}}_p}$, we choose a dominant cocharacter μ representing the $G(\bar{\mathbb{Q}}_p)$-conjugacy class $[\mu]$. Denote by $\mathrm{Gr}_{G,\mu}$ the Schubert cell labelled by μ. Let $\mathrm{Bun}_G := \mathrm{Bun}_{G_{\mathbb{Q}_p}}$ be the

small v-stack on Perf of $G_{\mathbb{Q}_p}$-bundles on the Fargues-Fontaine curve. The affine Grassmannian maps to Bun_G via the Beauville-Laszlo map $BL : \text{Gr}_G \to \text{Bun}_G$. The Shimura variety $\mathcal{S}_{K^p}$ maps to the affine Grassmannian via the Hodge-Tate period map $\pi_{HT} : \mathcal{S}_{K^p} \to \text{Gr}_G$, with image lying in $\text{Gr}_{G,\mu}$.

Conjecture 1.1. *(Scholze)[1] There exists a construction of a system of small v-stacks $\{\text{Igs}_{K^p}\}_{K^p}$ on* Perf*, together with maps $\mathcal{S}_{K^p}/\text{Spd}E \to \text{Igs}_{K^p}$ and $\text{Igs}_{K^p} \xrightarrow{\bar{\pi}_{HT}} \text{Bun}_G$ such that*

(1) (Cartesian diagram) For each K^p, the diagram

$$
\begin{array}{ccc}
\mathcal{S}_{K^p} & \xrightarrow{\pi_{HT}} & \text{Gr}_{G,\mu} \\
\downarrow & & \downarrow{\scriptstyle BL} \\
\text{Igs}_{K^p} & \xrightarrow{\bar{\pi}_{HT}} & \text{Bun}_G
\end{array}
$$

is cartesian.

(2) (Hecke action) There exists a $G(\mathbb{A}_f)$-action on $\{\text{Igs}_{K^p}\}_{K^p}$ (where $G(\mathbb{Q}_p)$ acts trivially) descending that on $\{S_{K^p}\}_{K^p}$. In particular for any compact open subgroup K_p of $G(\mathbb{Q}_p)$, we have a similar Cartesian diagram at level K_p, with the top row replaced by

$$
\mathcal{S}_{K_p K^p} \xrightarrow{\pi_{HT,K_p}} [\text{Gr}_{G,\mu}/K_p].
$$

(3) (Minimal compactification) There exist compactifications $\text{Igs}_{K^p} \hookrightarrow \text{Igs}^*_{K^p}$ over Bun_G, extending the above cartesian diagram to the minimal compactification $\mathcal{S}^*_{K^p}$'s of the Shimura varieties.

(4) (Integral model) For $\mathcal{G}$ being a smooth parahoric model of G over $\mathbb{Z}_p$, the cartesian diagram at level $K_p = \mathcal{G}(\mathbb{Z}_p)$ has a canonical integral model

$$
\begin{array}{ccc}
S^{\Diamond}_K & \xrightarrow{\pi_{crys}} & \text{Sht}_{\mathcal{G},\mu} \\
\downarrow & & \downarrow \\
\text{Igs}_{K^p} & \xrightarrow{\bar{\pi}_{HT}} & \text{Bun}_G
\end{array}
$$

where $S^{\Diamond}_K$ is the v-sheaf[2] associated with the (conjectural) schematic canonical integral model of the Shimura variety at level K over $\mathcal{O}_E$, uniquely characterized by Conjecture 4.2.2 of [PR21], $\text{Sht}_{\mathcal{G},\mu}/\text{Spd}\mathcal{O}_E$ is the moduli stack of (p-adic) $\mathcal{G}$-shtukas with one leg bounded by μ, and the map π_{crys} is given by the universal $\mathcal{G}$-shtuka on $S^{\Diamond}_K$.

(5) (Functoriality) The construction is functorial in Shimura data.

Here Bun_G, the affine Grassmannian Gr_G and the Beauville-Laszlo map are explained in detail in [FS21, III], [SW20, Lecture 19]. The construction of the Hodge-Tate period map is originally due to Scholze [Sch15] and rewritten in [CS17] for Hodge type Shimura varieties. In this generality, it is recorded in a preliminary draft of Hansen [Han16], c.f.[PR21, 2], whose existence relies on the fact that the

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[1]The conjecture is made during the current project and this is the first written formulation of it.
[2]There are two ways of attaching a v-sheaf to an $\mathcal{O}_E$-scheme, see Definition 2.26 and here we are using the one that views a test perfectoid space as a locally ringed space with its structure sheaf (instead of the integral structure sheaf) as sheaf of rings.

tautological $G(\mathbb{Q}_p)$-local system on the Shimura variety is de Rham, established by the work of Liu and Zhu [LZ17].

To comment on the motivation and some features of conjecture 1.1, we mention that it arises in the context of the geometrization of the local Langlands conjectures due to Fargues [Far16] and Fargues-Scholze [FS21]. Conjecturally, to any local Langlands parameter, there is a certain corresponding perverse sheaf on Bun_G, thus realizing the local Langlands correspondence as a geometric Langlands correspondence on the Fargues-Fontaine curve. It is expected that the complex $R\pi_{HT,!}\overline{\mathbb{Q}}_l$, obtained by pushing forward a constant local system from the Shimura variety along the Hodge-Tate map, descends to Bun_G and relates to the conjectural perverse sheaves in some form of compatibility to the global Langlands correspondence [CS17, 1.18], [Far16, 7]. The current conjecture 1.1 is a geometric and hence more robust version of the weaker conjecture that $R\pi_{HT,!}\overline{\mathbb{Q}}_l$ descends.

Part (4) of the conjecture can also be formulated by saying that the Igusa stack constructed from part (1) using the generic fiber, when pulled back to the moduli stack of p-adic $\mathcal{G}$-shtukas bounded by μ is representable by a flat normal $\mathcal{O}_E$-scheme with certain properties and similarly for its minimal compactification. This seems to provide a new way of constructing canonical integral models of Shimura varieties, even though our current approach to this part of the conjecture in the PEL case uses the existence of integral models as an input. It also supports the idea that shtukas in the sense of Scholze-Weinstein are the correct incarnation of motives in p-adic situations.

Also, having the construction of Igusa stacks at hand, we can take their fiber products with various objects over Bun_G, not necessarily the affine Grassmannian. This provides new semi-global companions of Shimura varieties. As pointed out to me by Tamiozzo, the conjecture could be potentially applied to a local version of the plectic conjectures by taking fiber product of the Igusa stack with a moduli stack of shtukas with several legs. This idea will be pursued in a later work.

Remark 1.2. Our formulation of the conjecture does not uniquely characterize the system of v-stacks $\{\mathrm{Igs}_{K^p}\}_{K^p}$. For example it does not predict their images in Bun_G under $\overline{\pi}_{HT}$. A more idealized version of the conjecture would require each Igs_{K^p} to surject onto Bun_G. Yet given that currently we can only approach the construction of Igs_{K^p} via Shimura varieties, in this thesis we will be content with having a v-stack that is covered by the Shimura variety, whose image in Bun_G is therefore bounded by μ. At first sight this would lead to the Igusa stacks being dependent on $[\mu]$ and hence being defined over the residue field of E. However it is expected that they only depend on the Kottwitz set $B(G,\mu)$, not the conjugacy class $[\mu]$ itself, and hence Igs_{K^p} should already be defined over $\mathrm{Spd}\mathbb{F}_p$.

1.2. **Main results and organization of the thesis.** The aim of this thesis is to prove conjecture 1.1 for PEL Shimura varieties of type AC in the classification of Kottwitz, and the main result is the following. (We leave functoriality and the compactification part of point (4) in the conjecture to an upcoming project which extends these results to Hodge type Shimura varieties, where the argument is cleaner.)

Theorem 1.3 (Theorem 8.13, Theorem 9.38, Proposition 10.1, Theorem 11.23)**.** *If (G, X) is a PEL Shimura datum of type AC, assuming $G_{\mathbb{Q}_p}$ is unramified and $\mathcal{G}$*

is reductive (see assumption 5.5), then part (1)(2)(4) of conjecture 1.1 is true on the good reduction locus $\mathcal{S}^{\circ}_{K^p}$ of the Shimura variety. If we further assume that the minimal compactification of the Shimura variety has boundary codimension at least two, then part (3) of the conjecture is true.[3]

Our proof relies heavily on the fact that the Shimura variety in concern is a moduli space of abelian varieties with additional structures. In short, in this case the Igusa stack can be constructed as a moduli stack of abelian varieties up to isogenies in characteristic p. Upon relating points of the Schubert cell $\mathrm{Gr}_{G,\mu}$ (respectively Bun_G) to p-divisible groups with additional structure via Dieudonné theory, the desired cartesian property of the diagrams in part $(1),(4)$ of the conjecture follows from Serre-Tate theory of lifting abelian varieties.

In section 2 to section 7 we review small v-stacks, p-divisible groups, the Hodge-Tate period map, the stack Bun_G and the Beauville-Laszlo map. This collects results for later use and in parallel presents our specific global PEL setup.

In section 8 we give a construction of the Igusa stack and show part (1) of the conjecture on the good reduction locus. More precisely, we define:

Definition 1.4 (Definition 8.1). Let $\mathrm{Igs}^{\circ}_{K^p}$ be the v-stackification of $\mathrm{Igs}^{\mathrm{pre}}$, which takes an affinoid perfectoid space $T = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$ to the groupoid $\mathrm{Igs}^{\mathrm{pre}}(T)$ whose objects are $\mathrm{Spec}(R^+/\varpi)$-points of S_K, where ϖ is a pseudo-uniformizer of R^+ and S_K is the schematic Shimura variety at level K over $\mathcal{O}_E$. Isomorphisms between two objects are quasi-isogenies between abelian schemes compatible with extra structures.

The map $\mathcal{S}^{\circ}_{K^p}/\mathrm{Spd}E \to \mathrm{Igs}^{\circ}_{K^p}$ is constructed by taking the reduction of abelian schemes over R^+ to R^+/ϖ, and we denote it by red. The map $\bar{\pi}^{\circ}_{HT} : \mathrm{Igs}^{\circ}_{K^p} \to \mathrm{Bun}_G$ is constructed by taking the G-bundle on the Fargues-Fontaine curve attached to the rational Dieudonné module of the objects in $\mathrm{Igs}^{\circ}_{K^p}$. With these we show:

Proposition 1.5 (Theorem 8.13). *For PEL Shimura varieties of type AC, with the above definitions, part (1) of the conjecture is true on the good reduction locus $\mathcal{S}^{\circ}_{K^p} \subset \mathcal{S}_{K^p}$.*

The proof uses the moduli interpretation as alluded to earlier, except that the relation of $\mathrm{Gr}_{G,\mu}$ and Bun_G to p-divisible groups is only clean on rank one geometric points. Hence some effort is paid to extend the result from rank one points to a basis of the v-topology called "product of points" by I. Gleason in his thesis.

This geometric relation leads to the following sheaf theoretic corollary.

Corollary 1.6 (Proposition 8.20). *For any ring of coefficients Λ such that $n\Lambda = 0$ for some n prime to p, we have natural base change equivalence*

$$BL^* R\bar{\pi}^{\circ}_{HT,*} \cong R\pi^{\circ}_{HT,*}\mathrm{red}^*$$

of functors $D_{\text{ét}}(\mathrm{Igs}^{\circ}_{K^p}, \Lambda) \to D_{\text{ét}}(\mathrm{Gr}_{G,\mu}, \Lambda)$. In particular, the complex $R\pi^{\circ}_{HT,*}\Lambda$ on $\mathrm{Gr}_{G,\mu}$ descends to the complex $R\bar{\pi}^{\circ}_{HT,*}\Lambda$ on Bun_G. The same statement for pushforward with compact support is true.

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[3]We do not obtain the optimal base field. In our construction the Igusa stacks live over the residue field of E. See Proposition 9.25 for a classification of the (simple) Shimura varieties that are excluded by the codimension condition.

Section 9 deals with the minimal compactification and along the way investigates the geometry of Igusa varieties. This section is more technical, but arguably novel.

The idea (due to Scholze) of constructing a minimal compactification $\mathrm{Igs}^*_{K^p}$ is based on the fact that the fibers of the Hodge-Tate period map are affinoid. So upon imposing the condition that the boundary of the minimal compactification of the Shimura variety has codimension at least two, we can mimic a construction of a relative spectrum "$\underline{\mathrm{Spa}}_{\mathrm{Bun}_G}(\overline{\pi}^\circ_{\mathrm{HT},*}(\mathcal{O}, \mathcal{O}^+))$" over Bun_G. That this relative spectrum, when taken fiber product with $\mathrm{Gr}_{G,\mu}$, recovers the minimal compactification of the Shimura variety would be a consequence of the algebraic Hartogs' extension lemma. To carry this out, we define the affinization of a small v-stack X to be the v-sheaf

$$X_0 : S \mapsto \mathrm{Hom}((\mathcal{O}_X, \mathcal{O}_X^+)(X), (\mathcal{O}_S, \mathcal{O}_S^+)(S)).$$

Proposition 1.7 (Definition/Proposition 9.36). *The functor on strictly totally disconnected perfectoid spaces over $\mathrm{Bun}_G^\sharp$*

$$\mathrm{Igs}^* : T \mapsto \mathrm{Hom}_T(T, \overline{(T \times_{\mathrm{Bun}_G} \mathrm{Igs}^\circ_{K^p})_0}^{/T}),$$

where $^{/T}$ denotes the canonical compactification towards T, is a sheaf for the v-topology, and hence extends to a v-stack $\mathrm{Igs}^*_{K^p}$ with a 0-truncated map to Bun_G.

It contains $\mathrm{Igs}^\circ_{K^p}$ as an open substack and the fiber product $\mathrm{Igs}^*_{K^p} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}$ is isomorphic to the minimal compactification $\mathcal{S}^*_{K^p}$ of $\mathcal{S}_{K^p}$. Its structure morphism to Bun_G is pulled back to the Hodge-Tate period map on $\mathcal{S}^*_{K^p}$ under this identification.

The main effort here is to show the pullback of Igs^*_T along a map of strictly totally disconnected spaces $T' \to T$ is indeed isomorphic to $\mathrm{Igs}^*_{T'}$. Write Igs_T for $T \times_{\mathrm{Bun}_G} \mathrm{Igs}^\circ_{K^p}$. Using perfectoid machinery and almost mathematics, this eventually boils down to a comparison between the global sections of the sheaf $\mathcal{O}^+/\varpi$ for some pseudo-uniformizer ϖ on Igs_T with $\mathcal{O}^+(\mathrm{Igs}_T)/\varpi$. We first made a reduction to the case $T = \mathrm{Spa}(C, C^+)$ is a geometric point. Then using the comparison between the fibers of the Hodge-Tate period map with Igusa varieties due to [CS19] and [San23], we are reduced to show the natural map

$$\mathcal{O}^+(\mathrm{Ig}^b_C)/\varpi \to (\mathcal{O}^+/\varpi)(\mathrm{Ig}^b_C)$$

is an almost isomorphism. Here Ig^b_C is a perfectoid Igusa variety corresponding to some element b in the Kottwitz set. This is constructed as the adic generic fiber of a formal deformation to $\mathrm{Spf}\mathcal{O}_C$ of a perfect scheme Ig^b over the residue field of C. Using the short exact sequence for multiplication by ϖ on the integral structure sheaf, what we need to show becomes the almost vanishing of the ϖ-torsion in $H^1(\mathrm{Ig}^b_C, \mathcal{O}^+)$. This is almost isomorphic to the Witt vector cohomology of the perfect scheme Ig^b. We found surprisingly (although easy to prove) that in the generality of any perfect scheme, we have torsion-vanishing phenomenon in its first Witt vector cohomology:

Proposition 1.8 (Proposition 9.28). *Let X be a perfect scheme in characteristic p. Denote by $W(\cdot)$ the p-typical Witt vectors. Then the Witt vector cohomology $H^1(X, W\mathcal{O}_X)$ on the Zariski site of X is p-torsionfree.*

This fulfills our purpose. The rest, namely to check that the fiber product recovers the minimal compactification of the Shimura variety, is easy and is again
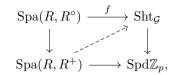
reduced to test objects being geometric points. Here we need to compare the global sections of the structure sheaves on Ig_C^b and its partial minimal compactification $\mathrm{Ig}_C^{b,*}$, which reduces to comparing those of their special fibers Ig^b, $\mathrm{Ig}^{b,*}$. We thus make the assumption that the codimension of the boundary of $\mathrm{Ig}^{b,*}$ is at least two, so that using affiness and normality of $\mathrm{Ig}^{b,*}$, we can apply algebraic Hartogs lemma to show that they agree. As a side result, we classify the situations that we exclude. The assumption on codimension turns out to be rather mild.

Proposition 1.9 (Proposition 9.25). *If the boundary of the partial minimal compactification of an Igusa variety on a (simple) Shimura variety of PEL-type AC has codimension one, then the Igusa variety must lie over the ordinary locus and the Shimura variety is either the modular curve, or a unitary Shimura curve attached to an imaginary quadratic extension of $\mathbb{Q}$ as in example 5.12.*

The short section 10 deals with the Hecke action. This is direct, given the Hecke action on Shimura varieties.

In the final section 11, we introduce an integral model of the cartesian diagram for $\mathcal{G}$ being a reductive model of $G_{\mathbb{Q}_p}$ (though only for $\mathrm{Igs}_{K^p}^\circ$ and not its compactification). Here we first define the moduli stack $\mathrm{Sht}_{\mathcal{G}}$ of $\mathcal{G}$-shtukas and study its geometry. The main result is

Theorem 1.10 (Proposition 11.13,11.16, Corollary 11.14, Theorem 11.19). *The structure map $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Spd}\mathbb{Z}_p$ is quasi-separated, with proper diagonal, and for any perfectoid Tate algebra R with an open bounded integrally closed subring R^+ and any commutative diagram with solid arrows*

$$
\begin{array}{ccc}
\mathrm{Spa}(R, R^\circ) & \xrightarrow{\ f\ } & \mathrm{Sht}_{\mathcal{G}} \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Spa}(R, R^+) & \longrightarrow & \mathrm{Spd}\mathbb{Z}_p,
\end{array}
$$

there is a unique (up to isomorphism) dotted arrow making the whole diagram commute up to a natural transform given by an automorphism of f. For any dominant cocharacter λ of $G_{\overline{\mathbb{Q}}_p}$, the bounded substack $\mathrm{Sht}_{\mathcal{G},\lambda}$ is quasi-compact.

Moreover, the fiber $\mathrm{Sht}_{\mathcal{G},\mathbb{Q}_p}$ over $\mathrm{Spd}\mathbb{Q}_p$ identifies with the quotient $[\mathrm{Gr}_G/\underline{K}_p]$ of the affine Grassmannian.

The proof relies on a recent result of Gleason-Ivanov [GI23] on extending shtukas to Breuil-Kisin-Fargues modules over products of rank one geometric points, as well as a result of Anschütz about triviality of torsors on the spectrum of the ring $W(R^+)[1/p]$, where R^+ is the integral subring of such a test object.

Later we introduce the crystalline period map on the formal integral model of the Shimura variety. The existence of the map is a consequence of the existence of a universal $\mathcal{G}$-shtuka on it. Pappas-Rapoport [PR21] showed this for Hodge-type Shimura varieties and we rephrased their construction in our situation. The cartesian property of the desired diagram is easy to prove in this case, since using qcqsness of the map $\overline{\pi}_{HT}^\circ$ established as a corollary of proposition 1.5 and the quasi-separatedness of $\mathrm{Sht}_{\mathcal{G}}$, the map from the Shimura variety to the fiber product is qcqs. Therefore it suffices to check on geometric points, which was done in section 8.

Finally we discussed the Newton stratification on our cartesian diagram. This recovers the fiber product formula of Mantovan, and in the specific case of a basic stratum, the p-adic uniformization of Rapoport and Zink.

1.3. Example of the modular curve.

Let us discuss the example of the modular curve in detail to illustrate the content of the conjecture. This also clarifies our conventions on Dieudonné theory. The general case shares great similarity.

Consider the Shimura datum $G = \mathrm{GL}_2$, $X = \mathfrak{h}^+ \coprod \mathfrak{h}^-$ the union of the complex upper and lower half plane, identified with the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of the map

$$h : \mathbb{C} \to \mathrm{GL}_2(\mathbb{R}) : a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We also fix the diagonal torus T and standard (upper triangular) Borel B of $\mathrm{GL}_{2,\overline{\mathbb{Q}}}$. The root datum is

$$(\mathbb{Z}^2, \{\pm\alpha\}, \mathbb{Z}^2, \{\pm\alpha^\vee\}),$$

where the character lattice is trivialized by a basis e_1, e_2 with dual basis $e_1^\vee$, $e_2^\vee$ and $\alpha = e_1 - e_2$, $\alpha^\vee = e_1^\vee - e_2^\vee$. Then the minuscule cocharacter μ^{-1} can be chosen to be $(1,0)$ and a dominant cocharacter representing its inverse is $\mu = (0,-1)$.

Fix the level subgroup $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ at p and a prime-to-p principal level $K^p = K(N)$, $p \nmid N \geq 3$. We let $K := K_p K^p \subset \mathrm{GL}_2(\mathbb{A}_f)$. The Shimura variety we obtain is the modular curve at level K. It is defined over $\mathbb{Q}$ and parametrizes isomorphism classes of pairs consisting of an elliptic curve and a trivialization of its N-torsion points. We consider its base change to $\mathbb{Q}_p$ and take the diamond $\mathcal{S}_K$ attached to its p-adic analytification. By trivializing the Tate module of the universal elliptic curve, we obtain $\mathcal{S}_{K^p}$, the modular curve with infinite level at p. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. In this case the Schubert cell $\mathrm{Gr}_{\mathrm{GL}_2,\mu}$ for $\mathrm{GL}_2/\mathbb{Q}_p$ is the diamond over $\mathbb{Q}_p$ attached to the flag variety for the opposite of the standard Borel, which is a projective line $\mathbb{P}^1$.

Here the Hodge-Tate period map measures the relative position of the Hodge-Tate filtration on the Tate-module of the universal elliptic curve $\mathcal{E}$, which is of the form

$$Lie\mathcal{E} \hookrightarrow T_p\mathcal{E} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{S}_{K^p}} \cong \mathcal{O}_{\mathcal{S}_{K^p}}^{\oplus 2}.$$

Here we use that Tate module is tautologically trivialized on $\mathcal{S}_{K^p}$. Hence this defines a map

$$\mathcal{S}_{K^p} \to \mathbb{P}^{1,\diamond}.$$

The stack $\mathrm{Bun}_G = \mathrm{Bun}_2$ classifies rank two vector bundles on the Fargues-Fontaine curve. To define the Beauville-Laszlo map

$$BL : \mathbb{P}^{1,\diamond} \to \mathrm{Bun}_2,$$

consider a test object $S \in \mathrm{Perf}$ with an untilt $S^\sharp$ over $\mathrm{Spa}\mathbb{Q}_p$, a map $x : S \to \mathbb{P}^{1,\diamond}/\mathrm{Spd}\mathbb{Q}_p$ gives a injection $\mathcal{L} \hookrightarrow \mathcal{O}_{S^\sharp}^{\oplus 2}$ for some line bundle $\mathcal{L}$. The untilt $S^\sharp$ defines a closed Cartier divisor on the relative Fargues-Fontaine curve over S and we denote the closed immersion by $i : S^\sharp \hookrightarrow X_S$. Then we define the image $BL(x)$ to be the limit $\mathcal{K}$ of the diagram

$$i_*\mathcal{L} \hookrightarrow i_*\mathcal{O}_{S^\sharp}^{\oplus 2} \leftarrow \mathcal{O}_{X_S}(1)^{\oplus 2}.$$

We construct the stack $\mathrm{Igs}^\circ_{K^p}$ on Perf by sheafifying the presheaf of groupoids of isogeny classes of elliptic curves with N-level structures:

$$\mathrm{Spa}(R, R^+) \mapsto \{E/(R^+/\varpi), E[N] \cong (\mathbb{Z}/N)^2\}/ \sim .$$

This maps to Bun_2 by taking the (rational) crystalline Dieudonné module of E, which is a rank two projective $B^+_{\mathrm{cris}}(R^+/\varpi)$-module with Frobenius and hence a rank two vector bundle on the Fargues-Fontaine curve X_S. We have a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{S}^\circ_{K^p} & \xrightarrow{\pi^\circ_{HT}} & \mathbb{P}^{1,\diamond} \\
\downarrow & & \downarrow{\scriptstyle BL} \\
\mathrm{Igs}^\circ_{K^p} & \xrightarrow{\bar{\pi}^\circ_{HT}} & \mathrm{Bun}_2.
\end{array}
$$

Let us explain the Newton stratification on the diagram. Let $k = \overline{\mathbb{F}}_p$. The Kottwitz set $B(\mathrm{GL}_2)$ for $\mathrm{GL}_2/\mathbb{Q}_p$ is in bijection to the dominant cocharacters and can described by a pair of half integers (slopes) with nonincreasing order. The subset $B(G, \mu)$ of μ-admissible elements consists of two points $[b_0], [b_1]$ with $[b_0] \leq [b_1]$ under the partial order, whose images under the Newton map are respectively $(-\frac{1}{2}, -\frac{1}{2})$ and $(0, -1)$.

Let $\mathbb{X}/k$ be a formal p-divisible group of height two and dimension one. Such a p-divisible group is unique up to isomorphisms. Let D_p be the nonsplit quaternion algebra over $\mathbb{Q}_p$ and D be the endomorphism ring of a supersingular elliptic curve over k, tensored with $\mathbb{Q}$. This is a division algebra of dimension four over $\mathbb{Q}$ whose p-adic completion is D_p. We consider the special fiber $S_{K,k}$ of the integral model of the modular curve over $\mathbb{Z}_p$. For any $b \in B(\mathrm{GL}_2, \mu)$, Ig^b is the corresponding Igusa variety, which is a perfect k-scheme. We denote by $\overline{\mathrm{Ig}^{b,\diamond}}$ the canonical compactification of the v-sheaf attached to Ig^b towards $\mathrm{Spd}k$. Also let $\mathcal{BC}(\mathcal{O}(1))$ denote the Banach-Colmez space as in [FS21, II] that sends a perfectoid space S to the global sections of $\mathcal{O}(1)$ on the relative Fargues-Fontaine curve X_S.

We list on the next page descriptions of the Newton strata on each object appearing in the cartesian diagram, as well as their corresponding invariants. In the table, we use $S^b_{K,k}$ to denote the usual Newton strata on the special fiber of the schematic Shimura variety over k; and $\mathcal{S}^{\circ;b}_{K^p}$ denotes the Newton strata on the good reduction locus of the diamond Shimura variety with infinite level at p, which is defined by pulling back the Newton stratification on Bun_G. We caution the reader that latter is not the same as the stratification on $\mathcal{S}^\circ_{K^p}$, defined by pulling back that on $S^b_{K,k}$ using the specialization map. They agree on rank one points but not in general. Thus our notation here deviates from [CS17, 3]. In the last row of the table, we use $\overline{\mathrm{Igs}^{\circ,b}_{K^p}}$ to denote the canonical compactification of the stratum on $\mathrm{Igs}^\circ_{K^p}$ labelled by b towards Bun_G, c.f. the notation in 9.42.

	$[b_0]$	$[b_1]$
slopes	$-\frac{1}{2}, -\frac{1}{2}$	$0, -1$
isocrystal	$(\breve{\mathbb{Q}}_p^2, F = \begin{pmatrix} 0 & 1 \\ p^{-1} & 0 \end{pmatrix})$	$(\breve{\mathbb{Q}}_p^2, F = \mathrm{diag}\{1, p^{-1}\})$
isogeny class of p-divisible groups	$\mathbb{X}$	$\mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$
vector bundle $\mathscr{E}_b$	$\mathcal{O}(\frac{1}{2})$	$\mathcal{O} \oplus \mathcal{O}(1)$
$\tilde{G}_b = \underline{\mathrm{Aut}}(\mathscr{E}_b)$	$\underline{D}_p^\times$	$\begin{pmatrix} \underline{\mathbb{Q}}_p^\times & \mathcal{BC}(\mathcal{O}(1)) \\ 0 & \underline{\mathbb{Q}}_p^\times \end{pmatrix}$
$\mathbb{P}^{1,b}$	$\Omega := \mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{Q}_p)$	$\mathbb{P}^1(\mathbb{Q}_p)$
$S_{K,k}^b$	supersingular locus	ordinary locus
$\mathcal{S}_{K^p}^{\circ,b}$	the residue discs of the supersingular points (open)	complement of $\mathcal{S}_{K^p}^{\circ,b_0}$ (closed)
Ig^b	the profinite set $D^\times \backslash D_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^p)/K^p$ considered as a k-scheme	a $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$-torsor over the perfection of the ordinary locus
$\overline{\mathrm{Igs}}_{K^p}^{\circ,b}$	$[\underline{D}^\times \backslash \underline{\mathrm{GL}_2(\mathbb{A}_f^p)}/\underline{K^p}]$	$[\overline{\mathrm{Ig}^{b,\diamond}}/\tilde{G}_b]$

2. Diamonds and v-stacks

The objects in consideration will be stacks on the v-site of perfectoid spaces in characteristic p. To be able to work in such a set-up, we recall the definitions and a few properties of perfectoid spaces, pro-étale and v-topology, diamonds and small v-stacks, following closely [SW20, 6-9,17] and [Sch18, 3,5-9,18].

2.1. **Perfectoid spaces.**

Definition 2.1. A topological ring R is called Tate, if it contains an open and bounded subring $R_0 \subset R$ and a topologically nilpotent unit (a pseudo-uniformizer) $\varpi \in R$. A Tate ring R is perfectoid if it is complete, uniform (i.e. the set of power-bounded elements $R^\circ \subset R$ is bounded), and there exists a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p \mid p$ in R° and the Frobenius map

$$\Phi : R^\circ/\varpi \to R^\circ/\varpi^p : x \mapsto x^p$$

is an isomorphism.

Definition 2.2. ([Sch18, 3.9-3.11]) Let R be a perfectoid Tate ring. The tilt of R is the topological ring

$$R^\flat = \varprojlim_{x \mapsto x^p} R,$$

with the inverse limit topology, the pointwise multiplication and the addition given by

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) = (z^{(0)}, z^{(1)}, \dots),$$

where

$$z^{(i)} = \lim_{n \to \infty} (x^{(i+n)} + y^{(i+n)})^{p^n} \in R.$$

This is in fact a perfectoid $\mathbb{F}_p$-algebra, whose subset of power-bounded elements is given by

$$R^{\flat\circ} = \varprojlim_{x \mapsto x^p} R^\circ \cong \varprojlim_\Phi R^\circ/\varpi,$$

where $\varpi^p \mid p \in R^\circ$ is a pseudo-uniformizer of R. Any preimage $\varpi^\flat$ of ϖ under $R^{\flat\circ} = \varprojlim_\Phi R^\circ/\varpi^p \to R^\circ/\varpi^p$ is an element of $R^{\flat\circ}$ which is a pseudo-uniformizer of $R^\flat$, and $R^\flat = R^{\flat\circ}[1/\varpi^\flat]$.

The projection to the zeroth coordinate defines a multiplicative map

$$R^\flat = \varprojlim_{x \mapsto x^p} R \to R : f \mapsto f^\sharp.$$

This induces a ring isomorphism $R^{\flat\circ}/\varpi^\flat \cong R^\circ/\varpi$ and an inclusion preserving bijection between the set of open and integrally closed subrings of $R^{\flat\circ}$ and R° (for both, the set of such subrings is bijective to that of their quotients by the pseudo-uniformizer). Explicitly, $R^+ \subset R^\circ$ corresponds to $R^{\flat+} := \varprojlim_{x \mapsto x^p} R^+$. We also have $R^{\flat+}/\varpi^\flat \cong R^+/\varpi^\flat$.

By an affinoid (perfectoid) Tate ring, we mean a pair of the form (R, R^+), where R is a (perfectoid) Tate ring, and $R^+ \subset R^\circ$ is an open bounded and integrally closed subring. A morphism $(R, R^+) \to (R', R'^+)$ between affinoid Tate rings is a map of topological rings $R \to R'$, carrying R^+ into R'^+. The tilt of an affinoid perfectoid Tate ring (R, R^+) is the affinoid perfectoid Tate ring $(R^\flat, R^{\flat+})$.

By considering Huber's adic spaces attached to affinoid perfectoid Tate rings, we have the notion of affinoid perfectoid spaces and their tilts. This construction is compatible with taking rational open subsets and hence globalizes.

Definition/Proposition 2.3. A perfectoid space X is an adic space covered by open subspaces which are isomorphic to affinoid perfectoid spaces, i.e. of the form $\mathrm{Spa}(R, R^+)$, where (R, R^+) is an affinoid perfectoid Tate ring. By tilting its rational open subsets and gluing, one can functorially construct a perfectoid space $X^\flat$ in characteristic p. Moreover, there is a homeomorphism $|X| \cong |X^\flat|$ that is compatible with passing to rational open subsets. We call $X^\flat$ the *tilt* of X.

Example 2.4. (geometric points) Let C be a complete algebraically closed non-archimedean field of characteristic zero or p and $C^+ \subset C$ an open and bounded valuation subring. Then $\mathrm{Spa}(C, C^+)$ is a perfectoid space. We call a perfectoid space of such form a geometric point. If $C^+ = \mathcal{O}_C$ is the ring of integers of C, we say that it is a rank one geometric point.

Definition 2.5. Let X be a perfectoid space in characteristic p. An *untilt* of X is a pair $(X^\sharp, \iota)$, consisting of a perfectoid space $X^\sharp$ and an isomorphism $\iota : X^{\sharp\flat} \cong X$. To simplify notation, we sometimes drop ι and simply write $X^\sharp$ for an untilt.

Definition 2.6. (morphisms of perfectoid spaces) Let $f : Y \to X$ be a morphism of perfectoid spaces.

(i) f is quasi-compact (resp. quasi-separated) if the map on the underlying toplogical spaces $|f| : |Y| \to |X|$ is quasi-compact (resp. quasi-separated).

(ii) f is an injection if for all perfectoid spaces Z, the map $f_* : \mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$ is injective.

(iii) f is an (resp. closed or open) immersion if it is an injection and $|f|$ is a locally closed (resp. closed or open) immersion.

(iv) f is separated if the diagonal $\Delta_f : Y \to Y \times_X Y$ is a closed immersion.

2.2. **Pro-étale and v-topology.**

Definition 2.7. Let $f : Y \to X$ be a morphism of perfectoid spaces.

(i) f is called étale if for any $y \in Y$, there is an open neighbourhood $V \subset Y$ of y, an affinoid perfectoid open $U = \mathrm{Spa}(R, R^+) \subset X$, such that $f(V) \subset U$ and $f|_V$ factors as $f : V \to W = \mathrm{Spa}(S, S^+) \to U$, with the first arrow being an open immersion and second arrow being a finite étale morphism, i.e. induced by a finite étale ring map $R \to S$.

(ii) f is called pro-étale if for any $y \in Y$, there is an affinoid perfectoid open neighbourhood $V \subset Y$ of y, an affinoid perfectoid open $U = \mathrm{Spa}(R, R^+) \subset X$, such that $f(V) \subset U$ and $f|_V$ can be written as a limit of étale maps $V_i \to U$ along a small cofiltered index category I, from affinoid perfectoid spaces $V_i, i \in I$.

Definition 2.8. Let Perf be the category of perfectoid spaces in characteristic p.

(i) The pro-étale topology on Perf is the Grothendieck topology for which a collection of jointly surjective morphisms $\{f_i : Y_i \to X\}_{i \in I}$ is a covering, if all f_i are pro-étale, and for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i, i \in J$, such that

$U = \bigcup_{i \in J} f_i(V_i)$. The category Perf, endowed with this topology, is called the big pro-étale site.

(ii) The v-topology on Perf is the Grothendieck topology where a collection of jointly surjective morphisms $\{f_i : Y_i \to X\}_{i \in I}$ is a covering, if for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i, i \in J$, such that $U = \bigcup_{i \in J} f_i(V_i)$. The category Perf, endowed with this topology, is called the v-site.[4]

These definitions work the same way for Perfd, the category of all perfectoid spaces. It is proven in [Sch18, 8.6,8.7] that the big pro-étale site, as well as the v-site, is subcanonical, i.e. the functor $\mathrm{Hom}(-, X)$ for X being a perfectoid space is a sheaf on the big pro-étale and the v-site of Perfd. Similarly for $X \in$ Perf. We will sometimes not distinguish a perfectoid space and the v-sheaf represented by it, and this is justified here.

One has also the small pro-étale site $X_{\text{pro-ét}}$ of a perfectoid space X, whose underlying category has objects pro-étale morphisms $Y \to X$ for $Y \in$ Perfd, and the coverings are jointly surjective morphisms with the same condition on quasicompactness as in the big pro-étale site.

Example 2.9. (product of points, c.f. [Gle22, 1.2]) Let $S = \mathrm{Spa}(A, A^+)$ be an affinoid perfectoid space in Perf with a pseudo-uniformizer $\varpi \in A^+$. For any point $x : (A, A^+) \to (K, K^+)$, let ϖ_x be the image of ϖ in $k(x)$, $k(x)^+$ the ϖ_x-adic completion of K^+ and $k(x) := k(x)^+[\frac{1}{\varpi_x}]$ the completed residue field. Define $R^+ := \prod_{x \in |S|} k(x)^+$, with a pseudo-uniformizer $\varpi' := (\varpi_x)$, and $R := R^+[\frac{1}{\varpi'}]$. Then $\tilde{S} := \mathrm{Spa}(R, R^+)$ is perfectoid and $\tilde{S} \to S$ is a v-cover.

More generally we call an affinoid perfectoid space a product of (geometric) points if it is of the shape $\mathrm{Spa}(R, R^+)$, where $R^+ = \prod_i K_i^+$ and $R = R^+[\frac{1}{\varpi}]$, with each (K_i, K_i^+) being an (algebraically closed) affinoid perfectoid field, $\varpi_i \in K_i$ a pseudo-uniformizer. Each $s_i := \mathrm{Spa}(K_i, K_i^+)$ is called a principal component of S.

A product of points is an example of a *totally disconnected* perfectoid space. By using geometric points in the above construction, one gets a v-cover by a *strictly totally disconnected* perfectoid space. These spaces are important as they provide a basis of v-topology and are structurally simple. More precisely,

Definition 2.10. A perfectoid space X is called (strictly) totally disconnected if it is quasi-compact quasi-separated and every (étale) open cover of it splits.

Proposition 2.11. *([Sch18, 1.15]) A perfectoid space X is (strictly) totally disconnected if and only if it is affinoid, and every connected component of X is of the form $\mathrm{Spa}(K, K^+)$ for K being a perfectoid field (resp. an algebraically closed perfectoid field) with an open and bounded valuation subring K^+.*

One can define and study stacks in this context.

---

[4]To avoid using universe, one first takes cutoff cardinals and then takes a limit over all possible cutoffs to define the category of small sheaves on this site, as discussed in [Sch18, 4,8]. We ignore this issue here.

Definition 2.12. A v-stack F is a contravariant 2-functor from the v-site Perf to the 2-category of groupoids (whose objects are groupoids and morphisms are functors), satisfying descent for v-covers, i.e. for a v-cover $Y \to X$, the natural functor

$$F(X) \to F(Y/X),$$

is an equivalence of categories. Here $F(Y/X)$ is the category of descent data, i.e. the objects are couples (s, α), with $s \in F(Y)$ and $\alpha : p_1^* s \cong p_2^* s$, satisfying the cocycle condition $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$, where $p_1, p_2 : Y \times_X Y \rightrightarrows Y$, $p_{12}, p_{23}, p_{13} : Y \times_X Y \times_X Y \rightrightarrows Y \times_X Y$ are the projections.

We will work exclusively with the following class of v-stacks that are more geometric in nature, in the sense that, using charts of perfectoid spaces, one can define underlying topological spaces for them.

Definition 2.13. A small v-stack is a v-stack X on Perf admitting a presentation

$$R = Y \times_X Y \rightrightarrows Y \to X,$$

with Y being the v-sheaf represented by some perfectoid space (not necessarily in characteristic p), and R is a small v-sheaf, i.e. a v-sheaf admitting a surjection (of v-sheaves) from a perfectoid space.

For a small v-stack X with presentation $R \rightrightarrows Y$, where Y is a perfectoid space and R is a small v-sheaf admitting a surjection from a perfectoid space $\tilde{R} \to R$, its underlying topological space is the quotient space

$$|X| = |Y|/|\tilde{R}|.$$

As a set, this is in bijection to

$$\{\mathrm{Spa}(K, K^+) \xrightarrow{s} Y\}/ \sim,$$

where $\mathrm{Spa}(K, K^+)$ runs through all affinoid perfectoid fields, and the equivalence relation is defined by $s_1 \sim s_2$ if there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spa}(K_3, K_3^+) & \longrightarrow\!\!\!\!\rightarrow & \mathrm{Spa}(K_1, K_1^+) \\
\downarrow & \searrow^{s_3} & \downarrow{\scriptstyle s_1} \\
\mathrm{Spa}(K_2, K_2^+) & \xrightarrow{\quad s_2 \quad} & X,
\end{array}
$$

for some third affinoid perfectoid field (K_3, K_3^+).

The topological space $|X|$ is independent of the choice of presentation [Sch18, 12.7,12.8].

Example 2.14. For T a topological space, we denote by $\underline{T}$ the v-sheaf on Perf of continuous homomorphisms into T, i.e.

$$S \mapsto \mathrm{Hom}_{cts}(|S|, T).$$

Let $X \in \mathrm{Perf}$ be a perfectoid space in characteristic p, with an action by a topological group G, one can consider the v-sheaf theoretic coequalizer $[X/\underline{G}]$ of the projection and action maps

$$X \times \underline{G} \rightrightarrows X.$$

This is v-stack. In particular if G is a locally profinite group, then $X \times \underline{G}$ is representable by a perfectoid space.[5] The projection and action maps

$$X \times \underline{G} \rightrightarrows X$$

are pro-étale. If G acts freely on X, the v-stack $[X/\underline{G}]$ is in fact a diamond (see below) and in particular a small v-stack.

We take fiber product of v-stacks as the 2-fiber product of categories fibered in groupoids over Perf.

Definition/Proposition 2.15. (fiber product of v-stacks) Given a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ of small v-stacks. The fiber product $X \times_Z Y$ is the presheaf of groupoids that sends $S \in$ Perf to the groupoid whose objects are triples

$$(x, y, \varphi : f(x) \cong g(y))$$

and morphisms between (x, y, φ) and (x', y', φ') are pairs of maps $(x \xrightarrow{\alpha} x', y \xrightarrow{\beta} y')$ such that $\varphi' \circ f(\alpha) = g(\beta) \circ \varphi$. This is again a small v-stack by [Sch18, 12.10].

For universal properties satisfied by a fiber product, see [Ols16, 3.4.13].

Definition 2.16. A v-stack X is quasi-compact if there is a surjection of v-stacks from an affinoid perfectoid space to X. In particular, if X is quasi-compact, then it is small and its underlying topological space $|X|$ is quasi-compact.

Definition 2.17. (morphism of v-stacks, c.f.[Sch18, 10.7]) Let $f : Y \to X$ be a morphism of v-stacks.

(i) f is 0-truncated if for all $S \in$ Perf, the map of groupoids $f(S) : Y(S) \to X(S)$ is faithful, or equivalently the diagonal map $\Delta_f : Y \to Y \times_X Y$ is an injection.

(ii) f is quasi-compact if for any affinoid perfectoid space S mapping to X, the fiber product $Y \times_X S$ is quasi-compact.

(iii) f is quasi-separated if the diagonal, which is 0-truncated, is quasi-compact quasi-separated (qcqs).

(iv) f is an open (resp. closed) immersion if for every (totally disconnected) perfectoid space T mapping to X, the pullback $Y \times_X T \to T$ is represented by an open (closed) immersion.

(v) f is separated if the diagonal $\Delta_{Y/X} : Y \to Y \times_X Y$ is a closed immersion (hence f is automatically 0-truncated).

(vi) f is partially proper if it is separated and for every diagram

$$
\begin{array}{ccc}
\mathrm{Spa}(R, R^\circ) & \longrightarrow & Y \\
\downarrow & \nearrow & \downarrow f \\
\mathrm{Spa}(R, R^+) & \longrightarrow & X
\end{array}
$$

where R is any perfectoid Tate ring with an open and integrally closed subring $R^+ \subset R$, there exists a unique dotted arrow making it commute.

---

[5]More generally, if X is a perfectoid space and G is a locally profinite group, then for any $\underline{G}$-torsor $f : \tilde{X} \to X$, the v-sheaf $\tilde{X}$ is representable by a perfectoid space, and f by a pro-étale morphism, c.f. [Sch18, 10.13].

We give a criterion for small v-stacks to be qcqs. The proof is adapted from the proof of [SW20, 21.2.1]

Proposition 2.18. *Let X be a small v-sheaf. Let Y be a small v-stack on the slice category* $\mathrm{Perf}_{/X}$, *such that the structure map to X has quasi-separated diagonal. If for any product of geometric points $S \in \mathrm{Perf}_{/X}$ with principal components s_i, $i \in I$, the restriction*

$$res : Y(S) \to \prod_{i \in I} Y(s_i)$$

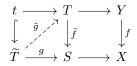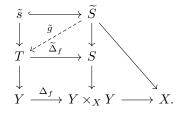*is an equivalence of groupoids, then f is qcqs. The converse implication holds if f is representable in diamonds.*

Proof. We first prove quasi-compactness assuming f is quasi-separated. Take any affinoid perfectoid space S with a map to X and denote by T the fiber product $Y \times_X S$. It suffices to show T is quasi-compact. We fix a representative $\mathrm{Spa}(C_t, C_t^+)$ for each $t \in |T|$ (recall that t is an equivalence class of maps) and choose a pseudo-uniformizer ϖ on S. The map $t \to T \to S$ pulls ϖ back to a pseudo-uniformizer $\varpi_t \in C_t^+$. Define $R^+ = \prod_{t \in |T|} C_t^+$, $\varpi = (\varpi_t)$ and $R = R^+[1/\varpi]$. Then $\widetilde{T} := \mathrm{Spa}(R, R^+)$ is a product of geometric points and the collection of maps $t \to S$ determines a unique map $g : \widetilde{T} \to S$. Hence we obtain commutative diagrams

$$
\begin{array}{ccccc}
t & \longrightarrow & T & \longrightarrow & Y \\
\downarrow & \nearrow^{\tilde{g}} & \downarrow^{\tilde{f}} & & \downarrow^{f} \\
\widetilde{T} & \xrightarrow{g} & S & \longrightarrow & X
\end{array}
$$

By assumption, the outer commutative squares give a unique (up to automorphisms) map $\widetilde{T} \to Y$, and hence a unique (up to automorphisms) dotted arrow $\tilde{g}$ by universal property of T. By construction, it is surjective on topological spaces. As g is qcqs and $\tilde{f}$ is quasi-separated, $\tilde{g}$ is qcqs by cancellation. This shows that it is in fact a surjection of v-stacks and T is quasi-compact as wished.

Now for a general f, we take an affinoid perfectoid space S with a map to $Y \times_X Y$ and consider the pullback T of the diagonal. Note that the map $\widetilde{\Delta}_f : T \to S$ is a quasi-separated map satisfying the condition in the proposition. Indeed, for any product of points $\widetilde{S}$ with a map to S, assume we have commutative diagrams for all principal components $\tilde{s} \in \widetilde{S}$

$$
\begin{array}{ccc}
\tilde{s} & \lhook\joinrel\longrightarrow & \widetilde{S} \\
\downarrow & \tilde{g}^{\nearrow} & \downarrow \\
T & \xrightarrow{\widetilde{\Delta}_f} & S \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\Delta_f} Y \times_X Y & \longrightarrow X.
\end{array}
$$

Then by assumption the outer commutative diagrams determine a unique (up to automorphisms) map $\widetilde{S} \to Y$. By uniqueness, its composition with Δ_f agrees with $\widetilde{S} \to S \to Y \times_X Y$ up to a natural transform. This induces a unique (up to automorphisms) map $\tilde{g}$ by the universal property. It makes the diagram commute, up to an automorphism in T in the upper left triangle. Hence we can apply the argument in the first paragraph to $\widetilde{\Delta}_f$ and deduce that it is quasi-compact. Since

this works for any S mapping to $Y \times_X Y$, it shows Δ_f is quasi-compact and hence f is quasi-separated. Now apply the first paragraph again we see that f is qcqs.

Conversely, if f is qcqs and representable in diamonds, assume for some product of geometric points $S \in \mathrm{Perf}_{/X}$ with principal components s_i, $i \in I$, we are given lifts of $s_i \to X$ to Y. Consider the fiber product $T = Y \times_X S$. This is a spatial diamond. Each s_i maps to T by the universal property. Take a pro-étale surjection $\widetilde{T} \twoheadrightarrow T$ from an affinoid perfectoid space. The maps $s_i \to T$ lift to $\widetilde{T}$, which determines a section $S \to \widetilde{T}$. Composing with the projection to T, we get a unique (up to automorphisms) section $\tilde{g} : S \to T$. This constructs an inverse to $res :$ $Y(S) \to \prod_{i \in I} Y(s_i)$.   □

Remark 2.19. For a map f between small v-stacks with quasi-separated diagonal, one can show f is qcqs by testing the above criterion on any pullback of f to an affinoid perfectoid space.

2.3. **Diamonds.**

Definition 2.20. A diamond is a pro-étale sheaf $\mathcal{D}$ on Perf that can be written as X/R with X, R being representable by perfectoid spaces and $R \subset X \times X$ an equivalence relation, such that the two projections $s, t : R \to X$ are pro-étale.

It was proven in [Sch18, 11.9] that diamonds are (small) v-sheaves. In particular, we can talk about the property of a diamond (resp. a map between diamonds) being quasi-compact or quasi-separated in a topos theoretic sense, see [Sch18, section 8].

Here is an example of a diamond that we will encounter later.

Example 2.21. (Spd$E$) Let $E/\mathbb{Q}_p$ be a finite extension. Joining all p-power roots of unity and then taking completion, one gets the perfectoid field E^{cycl}. Define

$$\mathrm{Spd}E := \mathrm{coeq}(\mathrm{Spa}(E^{\mathrm{cycl}})^{\flat} \times \underline{\mathrm{Gal}(E^{\mathrm{cycl}}/E)} \rightrightarrows \mathrm{Spa}(E^{\mathrm{cycl}})^{\flat}).$$

This is a diamond: in fact, as $\mathrm{Gal}(E^{\mathrm{cycl}}/E)$ is a profinite group which acts freely on

$$\mathrm{Hom}((E^{\mathrm{cycl}})^{\flat}, K),$$

for any perfectoid affinoid field (K, K^+), the map

$$\mathrm{Spa}(E^{\mathrm{cycl}})^{\flat} \times \underline{\mathrm{Gal}(E^{\mathrm{cycl}}/E)} \to \mathrm{Spa}(E^{\mathrm{cycl}})^{\flat} \times \mathrm{Spa}(E^{\mathrm{cycl}})^{\flat}$$

induced by the first projection and the action map is injective, making the former an equivalence relation on $\mathrm{Spa}(E^{\mathrm{cycl}})^{\flat}$. Its post-compositions with the two projections are pro-étale. Since the product $\mathrm{Spa}(E^{\mathrm{cycl}})^{\flat} \times \underline{\mathrm{Gal}(E^{\mathrm{cycl}}/E)}$, being copies of $\mathrm{Spa}(E^{\mathrm{cycl}})^{\flat}$, is perfectoid, $\mathrm{Spd}E$ is a diamond.

In this example, the underlying topological space of the diamond $\mathrm{Spd}E$ is just a point.

The following theorem describes the category of perfectoid spaces over $\mathbb{Q}_p$ in terms of those in characteristic p in aid of diamonds. This explains why for most purposes, it suffices to work with Perf instead of Perfd.

Theorem 2.22. [SW20, 8.4.2] *The category of perfectoid spaces over $\mathbb{Q}_p$ is equivalent to the category of perfectoid spaces X of characteristic p with a structure morphism $X \to \mathrm{Spd}\mathbb{Q}_p$ as sheaves on* Perf.

2.3.1. *Diamonds attached to adic spaces.*

Definition 2.23. (The functor "$\diamond$") Let X be an analytic adic space over $\mathrm{Spa}\mathbb{Z}_p$. Define a presheaf $X^\diamond$ on Perf by:

$$T \mapsto X^\diamond(T) = \{(T^\sharp, T^\sharp \to X)\}/\sim,$$

where $T^\sharp$ is an untilt of T, $T^\sharp \to X$ is a map of adic spaces and the equivalence relation is given by isomorphisms of such pairs.

According to [SW20, 10.1.5], the presheaf $X^\diamond$ is a diamond. And in particular, if X is perfectoid, then $X^\diamond$ is represented by $X^\flat$.

We denote $X^\diamond$ by $\mathrm{Spd}(R, R^+)$ if $X = \mathrm{Spa}(R, R^+)$ and suppress the integral sub-ring if it is the ring of power bounded elements, i.e. we write $\mathrm{Spd}R$ for $\mathrm{Spa}(R, R^\circ)^\diamond$.

In general for any pre-adic space (in the sense of [SW20, Appendix to lecture 3]) over $\mathrm{Spa}\mathbb{Z}_p$, the same functor as in the exhibited formula above (where "a map of adic spaces" is replaced by "a map of pre-adic spaces") is not necessarily representable by a diamond, but it always defines a v-sheaf [SW20, 18.1.1]. This encompasses the important case of formal schemes over $\mathrm{Spf}\mathbb{Z}_p$. For schemes over $\mathbb{Z}_p$, there are two different ways of attaching v-sheaves to it constructed in [AGLR22, 2.2], according to whether we want to view the test object as a ringed space with sheaf of rings given by the structure or the integral structure sheaf. We record these constructions below.

Example 2.24. For an affine p-adic formal scheme $\mathfrak{X} = \mathrm{Spf}A$ over $\mathrm{Spf}\mathbb{Z}_p$, the v-sheaf $\mathfrak{X}^\diamond$ is the v-sheaf attached to the pre-adic space $\mathrm{Spa}(A, A)$. This construction is compatible with localization and hence globalizes and defines a functor from p-adic formal schemes to v-sheaves.

Example 2.25. The v-sheaf $\mathrm{Spd}\mathbb{F}_p$ is **not** representable by diamonds. This is in fact the trivial functor, sending any test object to a point.

Definition 2.26. [AGLR22, 2.10] Let A be a $\mathbb{Z}_p$-algebra and $X = \mathrm{Spec}(A)$.
(1) The *small diamond* functor $X^\diamond$ of X is the v-sheaf on Perf

$$S \mapsto \{(S^\sharp, f : A \to \mathcal{O}_{S^\sharp}^+(S^\sharp))\},$$

where $S^\sharp$ is an untilt of S and f is a ring homomorphism.
(2) The *big diamond* functor $X^\lozenge$ of X is the v-sheaf on Perf

$$S \mapsto \{(S^\sharp, f : A \to \mathcal{O}_{S^\sharp}(S^\sharp))\},$$

where $S^\sharp$ is an untilt of S and f is a ring homomorphism.

Remark 2.27. On proper schemes, the big and small diamond functors agree.

3. p-DIVISIBLE GROUPS

We review some results about p-divisible groups, following [Mes72, I.2], [CS17, 4.1], [SW13], [AB23].

3.1. **Basic definitions.** Let S be a scheme.

Definition 3.1. A sheaf of abelian groups $\mathcal{G}$ on the fpqc site of S is said to be p^∞-torsion if it is the colimit of its p^n-torsion points (denoted by $\mathcal{G}[p^n]$). It is p-divisible if multiplication by p on $\mathcal{G}$ is an epimorphism.

Definition 3.2. Let h be an integer ≥ 0. A p-divisible group $\mathcal{G}$ over S of height h is a fpqc sheaf of abelian groups on S, which is p^∞-torsion, p-divisible and each $\mathcal{G}[p^n]$ is representable by a finite locally free group scheme of order p^{nh}. Morphisms between p-divisible groups are morphisms of sheaves of groups on S_{fpqc}.

The dual p-divisible group $\mathcal{G}^\vee$ of $\mathcal{G}$ is the fpqc sheaf $T \mapsto \varinjlim_n \mathcal{G}[p^n]^\vee(T)$ over S, where $\mathcal{G}[p^n]^\vee$ is the Cartier dual of $\mathcal{G}[p^n]$ and the transition maps are the duals of multiplication by p. This is clearly a p-divisible group.

Definition 3.3. An isogeny between two p-divisible groups is a surjection of fpqc sheaves whose kernel is representable by a finite locally free group scheme.

For two p-divisible groups $\mathcal{G}, \mathcal{G}'$ on a scheme S, we write $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{G}')$ for the sheaf of isogenies between them.

Definition 3.4. Let $\mathcal{G}, \mathcal{G}'$ be two p-divisible groups over a scheme S. A quasi-isogeny is a global section ρ of the sheaf $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{G}') \otimes \mathbb{Q}$ such that Zariski locally on S, $p^n \rho$ is an isogeny for some integer n.

Definition 3.5. A polarization on a p-divisible group $\mathcal{G}$ is a quasi-isogeny

$$\lambda : \mathcal{G} \to \mathcal{G}^\vee,$$

such that the Cartier dual of λ equals $-\lambda$. It is called a principal polarization if it is an isomorphism.

Example 3.6.   (1) $\mathbb{Q}_p/\mathbb{Z}_p$ is a p-divisible group of height one.
  (2) $\mu_{p^\infty} := \varinjlim_n \mathbb{G}_m[p^n]$, where the transition maps are inclusions, is a p-divisible group of height one.
  (3) Let A/S be a d-dimensional abelian scheme. Then the colimit of its p-power torsion points $A[p^\infty] := \varinjlim_n A[p^n]$ is a p-divisible group of height $2d$.

In the above examples, $\mathbb{Q}_p/\mathbb{Z}_p$ and μ_{p^∞} are dual to each other and $A[p^\infty]$ is dual to the p-divisible group of the dual abelian variety $A^\vee$. The pairing between them (or rather the duality pairings on $A^\vee[p^n] \times A[p^n]$ for each n) is called the Weil pairing. In particular when A is principally polarized, $A[p^\infty]$ is self-dual via the principal polarization.

Remark 3.7. Note that a polarization λ on an abelian variety induces a polarization on its p-divisible group. Although slightly confusingly, on the abelian variety λ agrees with its dual isogeny, yet on the p-divisible group it is the inverse of its dual. This is a consequence of the expression of the Weil pairing as a commutator of two translation operators on the sheaf $([p^n] \times id)^* \mathscr{P}_A$ on $A \times A^\vee$, for each integer n, where $[p^n]$ denotes the multiplication by p^n map and $\mathscr{P}_A$ denotes the Poincaré bundle on $A \times A^\vee$. More precisely, one can show using the see-saw principle that

$$([p^n] \times id)^* \mathscr{P}_A \cong (id \times [p^n])^* \mathscr{P}_{A^\vee}.$$

Upon identifying A with its double dual, this means that on $\mathscr{L} := ([p^n] \times id)^* \mathscr{P}_A$ there are actions of both $A[p^n] \times \{0\}$ and $\{0\} \times A^\vee[p^n]$, covering their translation actions on $A \times A^\vee$. The two actions don't commute. Given $(x, \nu) \in A \times A^\vee$, denote by $T_{(x,\nu)}$ the translation by (x, ν) on $A \times A^\vee$. Then the isomorphism obtained by composing the descent data along $[p^n] \times id$, respectively $id \times [p^n]$

$$T_{(x,\nu)}^* \mathscr{L} \xrightarrow{\sim} T_{(0,\nu)}^* \mathscr{L} \xrightarrow{\sim} \mathscr{L} \xrightarrow{\sim} T_{(x,0)}^* \mathscr{L} \xrightarrow{\sim} T_{(x,\nu)}^* \mathscr{L}$$

gives an element in $H^0(A \times A^\vee, \mathcal{O}_{A \times A^\vee}^*)$. One can identify this element with the value of the Weil pairing between x and ν by restricting to $A \times \{0\}$. We denote the pairing by e_A. Identify A with its double dual and perform the same construction on $A^\vee \times A^{\vee\vee}$. The above expression of the Weil pairing as a commutator shows that

$$e_A(x, \nu) = e_{A^\vee}(\nu, x)^{-1}.$$

Now a polarization $A \to A^\vee$ pulls e_A back to an anti-symmetric pairing on its p-divisible group, which explains the minus sign on the Cartier dual. For details, see [Oda69, 1.3].

In terms of Galois representations, one consider

Definition 3.8. The fpqc sheaf $T_p \mathcal{G} = \varprojlim_n \mathcal{G}[p^n]$ (where the transition maps are multiplication by p) on S is called the (integral) Tate module of $\mathcal{G}$. It is a sheaf of $\mathbb{Z}_p$-modules and can be identified with the sheaf $\mathcal{H}om(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G})$, taken in the category of sheaves of abelian groups over S_{fpqc}.

Being an inverse limit of schemes affine over S along affine transition maps, $T_p \mathcal{G}$ is representable by a scheme, affine over S. It is flat over S as being defined by a filtered colimit of flat $\mathcal{O}_S$-algebras.

When the base S is the spectrum of a p-adically complete $\mathbb{Z}_p$-algebra R, which is the main case of interest for us, we will more often consider a p-divisible group as an fpqc sheaf on $\mathrm{Nilp}_R^{\mathrm{op}}$, which is the opposite category of R-algebras on which p is nilpotent, sending $A \in \mathrm{Nilp}_R^{\mathrm{op}}$ to $\varprojlim_i \varinjlim_n \mathcal{G}[p^n](A/p^i)$.

Denote by $e_\mathcal{G}$ the zero section of $\mathcal{G}$. We discuss the formal Lie group attached to a p-divisible group, using which we can define the Lie algebra of the latter.

Definition 3.9. The formal completion $\hat{\mathcal{G}}$ of $\mathcal{G}$ is the fpqc sheaf on $\mathrm{Nilp}_R^{\mathrm{op}}$:

$$A \mapsto \varinjlim_k \{x \in \mathcal{G}(A) \mid x = e_\mathcal{G} \text{ in } A/I, \text{ for an ideal } I \subset A, \text{ such that } I^{k+1} = 0\}.$$

Proposition 3.10. $\hat{\mathcal{G}}$ *is a formal Lie (group) variety in the sense of* [Mes72, II.1.1.4]. *It is hence represented by an affine formal scheme over S, which is Zariski locally on S isomorphic to*

$$\mathrm{Spf}(\mathcal{O}_S[[X_1, ...X_d]]),$$

for some integer $d \geq 0$.

Proof. This is [Mes72, II.3.3.18] or [SW13, 3.1.2]. $\square$

We call d the dimension of the p-divisible group $\mathcal{G}$ relative to S.

Definition 3.11. The fpqc sheaf of $\mathcal{O}_S$-modules

$$Lie\mathcal{G} := Lie\hat{\mathcal{G}}$$

is defined to be the dual of the (Zariski) locally free $\mathcal{O}_S$-module of rank d

$$\omega_\mathcal{G} := e_{\hat{\mathcal{G}}}^* \Omega^1_{\hat{\mathcal{G}}/S}.$$

It is called the Lie algebra of $\mathcal{G}$. We use straight letters $Lie\mathcal{G}$ to denote its global sections. This is a finite projective R-module.

Let R be as above. When $\mathcal{G}$ is connected, $\mathcal{G} = \hat{\mathcal{G}}$ and is hence (pro-)representable by a formal scheme. In general it is not representable, but one can nevertheless define its adic generic fiber $\mathcal{G}_\eta^{\mathrm{ad}}$ as the sheafification of the presheaf on the category of complete affinoid rings over $\mathrm{Spa}(R[1/p], R)$, in the analytic topology where coverings are generated by rational opens:

$$(A, A^+) \mapsto \varinjlim_{A_0 \subset A^+} \mathcal{G}(A_0),$$

where the colimit runs over open and bounded subalgebras of A^+, c.f. [SW13, 2.2.2].

3.2. **Classification over $\mathcal{O}_C$.** Let $C/\mathbb{Q}_p$ be a complete algebraically closed non-archimedean field with ring of integers $\mathcal{O}_C$. We recall Scholze-Weinstein's classification of p-divisible groups over $\mathcal{O}_C$ in terms of the Hodge-Tate filtration on their Tate modules [SW13, Theorem B].

Let $\mathcal{G}$ be a p-divisible group over $\mathcal{O}_C$. Recall the Hodge-Tate exact sequence (due to Fargues) as in [SW20, 12.1.1].

Theorem 3.12. *There is a natural short exact sequence:*

$$0 \to Lie\mathcal{G} \otimes_{\mathcal{O}_C} C(1) \xrightarrow{\alpha_{G^*}^*(1)} T_p\mathcal{G}(\mathcal{O}_C) \otimes_{\mathbb{Z}_p} C \xrightarrow{\alpha_G} (Lie\mathcal{G}^\vee)^* \otimes_{\mathcal{O}_C} C \to 0.$$

Here to define α_G, we view a section f of $T_p\mathcal{G}$ as a homomorphism $\mathbb{Q}_p/\mathbb{Z}_p \to \mathcal{G}$. Then the Lie algebra functor applied to its dual $f^\vee : \mathcal{G}^\vee \to \mu_{p^\infty}$ gives $Lie(f^\vee) :$ $Lie\mathcal{G}^\vee \to Lie\mu_{p^\infty}$. By picking a coordinate of $\mathbb{G}_m$, say t, the $\mathcal{O}_C$-linear dual $(Lie\mu_{p^\infty})^*$ is naturally trivialized and is isomorphic to $\mathcal{O}_C \frac{dt}{t}$. Hence α_G is defined as $f \mapsto (Lie f^\vee)^*(\frac{dt}{t})$.

Let $\{(T, W)\}$ be the category of pairs consisting of a finite free $\mathbb{Z}_p$-module T and $W \subset T \otimes_{\mathbb{Z}_p} C(-1)$ is a sub-C-vector space. A morphism between two such pairs is a pair of morphisms between the $\mathbb{Z}_p$-modules and the sub-vector spaces, compatible with each other. The dual of (T, W) is the pair $(T^*(1), W^\perp)$, with $*$ being the usual vector space dual, (1) the Tate twist and $\perp$ the orthogonal complement (with respect to the natural pairing between $T \otimes_{\mathbb{Z}_p} C$ and $T^* \otimes_{\mathbb{Z}_p} C$). Then we have

Theorem 3.13. [SW13, Theorem B, 5.2.1] *The category of p-divisible groups over $\mathcal{O}_C$ is equivalent to the above category $\{(T, W)\}$ via:*

$$\Psi : \mathcal{G} \mapsto (T_p\mathcal{G}(\mathcal{O}_C), Lie\mathcal{G} \otimes_{\mathcal{O}_C} C),$$

where $Lie\mathcal{G} \otimes_{\mathcal{O}_C} C$ is viewed as a subspace of $T_p\mathcal{G}(\mathcal{O}_C) \otimes_{\mathbb{Z}_p} C(-1)$ via the Hodge-Tate filtration $\alpha_{G^*}^*$. This equivalence is compatible with duality.

For convenience of the reader, we record the proof outline below, under the assumption that C is spherically complete and the norm map $C \to \mathbb{R}_{\geq 0}$ is surjective. The general case follows from a less direct descent argument involving Rapoport-Zink spaces, see [SW13, 6.2].

Proof. Given (T, W) in the target category, we define the p-divisible group

$$\mathcal{G}' := T(-1) \otimes_{\mathbb{Z}_p} \mu_{p^\infty}.$$

Now we have the following diagram

$$
\begin{array}{ccc}
 & & W \otimes_C \mathbb{G}_a \\
 & & \downarrow \\
\mathcal{G}'^{\mathrm{ad}}_\eta & \longrightarrow & T(-1) \otimes_{\mathbb{Z}_p} \mathbb{G}_a,
\end{array}
$$

where $\mathbb{G}_a$ is the sheafification of the functor $(A, A^+) \mapsto A$ on complete affinoid $(C, \mathcal{O}_C)$-algebras; the vertical arrow is induced by the given inclusion $W \hookrightarrow T \otimes C(-1)$, while the horizontal arrow is the logarithm on $\mathcal{G}'^{\mathrm{ad}}_\eta$. Note that since $\mathrm{Lie}\mu_{p^\infty} \cong \mathcal{O}_C$, $\mathrm{Lie}\mathcal{G}'$ is naturally (in T) identified with $T(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$. Hence the logarithm takes the form

$$\mathcal{G}'^{\mathrm{ad}}_\eta \to \mathrm{Lie}\mathcal{G}' \otimes_{\mathcal{O}_C} \mathbb{G}_a \cong T(-1) \otimes_{\mathbb{Z}_p} \mathbb{G}_a.$$

We denote the sheaf theoretic fiber product by $\mathcal{G}^{\mathrm{ad}}_\eta$.

Using results from [Far18], one shows under the assumption of C being spherically complete with surjective norm map $C \to \mathbb{R}_{\geq 0}$, that the formal scheme

$$\mathcal{G} := \coprod_Y \mathrm{Spf} H^0(Y, \mathcal{O}_Y^+),$$

defines a p-divisible group over $\mathcal{O}_C$, where Y runs over connected components of $\mathcal{G}^{\mathrm{ad}}_\eta$.

It remains to check that $\Phi : (T, W) \mapsto \mathcal{G}$ gives an inverse to Ψ.

The composition $\Phi \circ \Psi$ is easily seen to be naturally isomorphic to the identity functor, since the diagram

$$
\begin{array}{ccc}
\mathcal{G}^{\mathrm{ad}}_\eta & \xrightarrow{\quad \log \quad} & \mathrm{Lie}\mathcal{G} \otimes_{\mathcal{O}_C} \mathbb{G}_a \\
\downarrow & & \downarrow{\scriptstyle \alpha^*_{\mathcal{G}^*}(-1)} \\
(T_p\mathcal{G}(\mathcal{O}_C)(-1) \otimes \mu_{p^\infty})^{\mathrm{ad}}_\eta & \xrightarrow{\quad \log \quad} & T_p\mathcal{G}(\mathcal{O}_C)(-1) \otimes_{\mathbb{Z}_p} \mathbb{G}_a,
\end{array}
$$

is cartesian, where the left vertical map is obtained by viewing $T_p\mathcal{G}^*$ as $\mathcal{H}om(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G}^*)$, and hence there is a canonical evaluation

$$T_p\mathcal{G}^*(\mathcal{O}_C) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathcal{G}^*,$$

dualizing and taking adic generic fiber of which gives the desired map.

For $\Psi \circ \Phi$, given (T, W) as in the target category, assume we have constructed $\mathcal{G}$ such that

$$
\begin{array}{ccc}
\mathcal{G}_\eta^{\mathrm{ad}} & \longrightarrow & W \otimes_C \mathbb{G}_a \\
\downarrow & & \downarrow \\
\mathcal{G}_\eta'^{\mathrm{ad}} & \xrightarrow{\log_{\mathcal{G}'}} & T(-1) \otimes_{\mathbb{Z}_p} \mathbb{G}_a,
\end{array}
$$

is cartesian. We have to show that $(T', W') := (T_p \mathcal{G}(\mathcal{O}_C), \mathrm{Lie}\mathcal{G} \otimes C)$ is naturally isomorphic to (T, W).

As this is a pullback diagram, we have identification of the kernels of the two horizontal maps. The kernel of $\log_{\mathcal{G}'}$ is $\mathcal{G}_\eta'^{\mathrm{ad}}[p^\infty]$, so the kernel of the top horizontal map is p^∞-torsion and is therefore contained in $\mathcal{G}_\eta^{\mathrm{ad}}[p^\infty]$. While $W \otimes_C \mathbb{G}_a$ is torsion-free, $\mathcal{G}_\eta^{\mathrm{ad}}[p^\infty]$ is also contained in the kernel. Hence the left vertical map in the diagram is an isomorphism on p^∞-torsion points. This means in particular we have compatibly

$$
\mathcal{G}_\eta^{\mathrm{ad}}[p^n] \xrightarrow{\cong} \mathcal{G}_\eta'^{\mathrm{ad}}[p^n].
$$

Passing to the tilde limit, we get c.f. [SW13, 3.3.2]

$$
(T_p \mathcal{G})_\eta^{\mathrm{ad}} \sim \varprojlim \mathcal{G}_\eta^{\mathrm{ad}}[p^n] \xrightarrow{\cong} \varprojlim \mathcal{G}_\eta'^{\mathrm{ad}}[p^n] \sim (T_p \mathcal{G}')_\eta^{\mathrm{ad}}.
$$

This induces an isomorphism on their $(C, \mathcal{O}_C)$-points and hence

$$
T' = T_p \mathcal{G}(\mathcal{O}_C) = (T_p \mathcal{G})_\eta^{\mathrm{ad}}(C, \mathcal{O}_C) \cong (T_p \mathcal{G}')_\eta^{\mathrm{ad}}(C, \mathcal{O}_C) = T_p \mathcal{G}'(\mathcal{O}_C) = T.
$$

We identify T' and T using the above isomorphism. Now to see that $\mathrm{Lie}\mathcal{G}$ in $T(-1) \otimes C$ agrees with W, assume this the opposite. Then their intersection in $T(-1) \otimes C$ would be a vector subspace of strictly smaller dimension. But the pullback of $\mathcal{G}_\eta'^{\mathrm{ad}}$ along $\mathrm{Lie}\mathcal{G} \cap W$ would still be $\mathcal{G}_\eta^{\mathrm{ad}}$. This is absurd.

Hence one has a natural isomorphism $\Psi \circ \Phi$ with the identity functor, induced by the left vertical arrow in the cartesian diagram used to define Φ. □

3.3. **Complements.** We record below some descent properties of p-divisible groups.

Lemma 3.14. *Given a cartesian diagram of rings*

$$
\begin{array}{ccc}
R & \longrightarrow & R_2 \\
\downarrow & & \downarrow \\
R_1 & \longrightarrow\!\!\!\!\rightarrow & R_3
\end{array}
$$

such that $R_1 \to R_3$ is surjective (a Milnor square), the corresponding diagram of categories of finite projective modules over these rings is 2-cartesian, i.e. the category of finite projective modules over R is equivalent to that of "gluing triples"

$$
(M_1, M_2, \alpha : M_1 \otimes_{R_1} R_3 \xrightarrow{\sim} M_2 \otimes_{R_1} R_3),
$$

where M_i is a finite projective module over R_i for $i = 1, 2$, and α is an isomorphism between their base changes.

Proof. Given a finite projective module over R, one can construct a gluing triple by base changing to R_i, $i = 1, 2, 3$ and the isomorphism α is the identity. Conversely,

given a gluing triple, one can get an R-module by taking the kernel of the difference map

$$M_1 \oplus M_2 \xrightarrow{\alpha - id} M_2 \otimes_{R_1} R_3.$$

That this gives the desired equivalence follows from [Mil72, 2.1-2.3]. $\qquad\square$

Example 3.15. Assume C is a complete algebraically closed non-archimedean field with ring of integers $\mathcal{O}_C$ and C^+ is a bounded valuation subring of $\mathcal{O}_C$. Denote by k the residue field of $\mathcal{O}_C$ and by $\overline{C^+}$ the image of C^+ in k. Let $\varpi \in C^+$ be a pseudo-uniformizer of C. Then taking $R = C^+$, $R_1 = \mathcal{O}_C$, $R_2 = \overline{C^+}$ and $R_3 = k$ gives a Milnor square. Similarly, taking $R = C^+/\varpi \cdot \mathcal{O}_C$, $R_1 = \mathcal{O}_C/\varpi$, $R_2 = \overline{C^+}$ and $R_3 = k$ gives a Milnor square. These two examples will be used later in the proof of proposition 8.13.

Proposition 3.16. *Let A be a ring. Denote the category of p-divisible groups on* $\mathrm{Spec}(A)$ *by $BT(A)$ ("BT" stands for Barsotti-Tate). Given a Milnor square as in lemma 3.14, we have $BT(R)$ is the 2-cartesian product of $BT(R_1)$ and $BT(R_2)$ over $BT(R_3)$.*

Proof. We have a functor

$$BT(R) \to BT(R_1) \times_{BT(R_3)} BT(R_2)$$

by base changes. Given two p-divisible groups $\mathcal{G}, \mathcal{G}'$. We have

$$\mathrm{Hom}_R(\mathcal{G}, \mathcal{G}') = \varprojlim_n \mathrm{Hom}_R(\mathcal{G}[p^n], \mathcal{G}'[p^n]).$$

For each n, $\mathrm{Hom}_R(\mathcal{G}[p^n], \mathcal{G}'[p^n])$ are given by maps between the R-modules $\mathcal{O}(G'[p^n])$ and $\mathcal{O}(G[p^n])$, respecting the Hopf algebra structures on both sides. Since the Hopf algebra structures are given by morphisms of R modules, we conclude by lemma 3.14 that giving such a map is equivalent to giving a pair of maps on the restrictions of $G[p^n]$ and $G'[p^n]$ to R_1 and R_2, identical on R_3. Passing to the inverse limit, this shows full-faithfulness. Essential surjectivity follows from a similar reasoning. Namely given a gluing triple of p-divisible groups, restricting to p^n-torsion points for each n, we can first recover the ring of functions $\mathcal{O}(\mathcal{G}[p^n])$ as an R-module by lemma 3.14 and then endow it with a Hopf algebra structure. And full-faithfulness ensures that this will define a p-divisible group which restricts to the correct thing. $\qquad\square$

Lemma 3.17. *For $R = \prod_i V_i$ being a product of valuation rings (or fields) and n be an integer, the category of rank n projective modules over R is equivalent to the collection of those over each V_i.*

Proof. We have a functor from rank n projective modules on R to those on each V_i by base changes. Conversely, given a collection of rank n projective modules M_i over each V_i, we can take the product $M := \prod_i M_i$. Since each M_i is necessarily free and we can pick a basis $e_{i1}, \ldots, e_{in}$ of it, the product M is also free and trivialized by $(e_{i1})_i, \ldots, (e_{in})_i$. This gives a functor in the opposite direction. To check that these two functors are inverse to each other, we only need to check the composition $M \mapsto (M_i) \mapsto \prod_i M_i$ gives a module that is naturally isomorphic to M, which is clear. $\qquad\square$

Corollary 3.18. *The category of p-divisible groups of a fixed height over R is equivalent to the collection of those over each V_i.*

Proof. Using 3.18, one can again reason by first truncating the p-divisible groups and then saying that the Hopf algebra structure on the rings of functions of each p^n-torsion subgroup is defined using maps between the underlying (finite projective) modules.   □

3.4. **Dieudonné modules.** Here we recall some Dieudonné theory following the work of Anschütz and Le Bras[AB23], in particular the classification of p-divisible groups by their (prismatic) Dieudonné modules. Later in this article we will only need the results over certain semiperfect and perfectoid rings, which are also covered by [Lau18] and [SW20, Appendix to Lecture 17].

Definition 3.19. (c.f. [BMS19, 4.10, 4.20][AB23, 3.3.1, 3.3.5].) A ring R is called quasi-regular semiperfectoid, if it is p-complete with bounded p^∞-torsion, the cotangent complex $L_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$ and that there exists a surjection $S \to R$ from a perfectoid ring S.

Example 3.20. ([AB23, 3.3.6]) Any integral perfectoid ring (c.f. Definition 6.2), or any p-complete bounded p^∞-torsion quotient of a perfectoid ring by a finite regular sequence, is quasi-regular semiperfectoid. In particular, if (R, R^+) is a perfectoid Tate ring with $\varpi \in R^+$ a pseudo-uniformizer of R, then both R^+ and R^+/ϖ are quasi-regular semiperfectoid.

For a p-divisible group G over a quasi-syntomic ring R, Anschütz and Le Bras have defined its prismatic Dieudonné crystal as a sheaf on the small quasi-syntomic site of R. For R being quasi-regular semiperfectoid, giving this crystal is equivalent to giving the evaluation of its associated sheaf on the prismatic site of R at the initial prism $(\mathbb{A}_R, I)$, which is called the *prismatic Dieudonné module* of G. It is a finite locally free $\mathbb{A}_R$ module and is equipped with an endomorphism $\varphi_M : \varphi^* M \to M$, admissible in the sense of [AB23, 4.1.9] (where φ is the Frobenius on $\mathbb{A}_R$). We refer the readers to [AB23, 4] for the precise constructions. The so-defined prismatic Dieudonné module is contravariant in G. To keep consistent with [SW20], we use the **covariant** prismatic Dieudonné module, which is obtained by applying $\mathrm{Hom}_{\mathbb{A}_R}(-, \mathbb{A}_R)$ to the contravariant one. We will denote this covariant prismatic Dieudonné module of G by $M_{\mathbb{A}}(G)$.

For our purposes, it is crucial to have the following theorem, especially in the special cases discussed below.

Theorem 3.21. *(c.f. [AB23, 4.6.10, 4.1.12]) Let R be a quasi-regular semiperfectoid ring. The prismatic Dieudonné module functor sending a p-divisible group G over R to its (covariant) prismatic Dieudonné module $M_{\mathbb{A}}(G)$ is an equivalence between the category of p-divisible groups over R and that of admissible Dieudonné modules over R.*

Example 3.22. Let R be an integral perfectoid ring, then
$$(\mathbb{A}_R, I) = (W(R^\flat), \ker(\theta \circ \varphi_R^{-1})),$$
where θ is Fontaine's theta map. In this case an admissible prismatic Dieudonné module is the same as a minuscule Breuil-Kisin-Fargues module with a leg at $V(\ker(\theta \circ \varphi_R^{-1}))$ (see definition 11.6), and the construction of the covariant Dieudonné module agrees with the construction in [SW20, 17.5.2], c.f. [AB23, 4.3.6].

Example 3.23. Let R be quasi-regular semiperfectoid and $pR = 0$ (e.g. the integral subring in a Tate perfectoid ring modulo a pseudo-uniformizer), then

$$(\mathbb{A}_R, I) = (A_{\mathrm{cris}}(R), (p)).$$

In this case the covariant Dieudonné module agrees with the (naive dual of the contravariant) crystalline Dieudonné module of Berthelot, Breen and Messing. ([AB23, 4.3.3]. See also [Lau18] for the equivalence between p-divisible groups and their Dieudonné modules in this case.)

4. Serre-Tate theory

Following [CS19, 2.4] and [Kat81, 1.2.1], we recall below two lifting theorems regarding the deformation of p-divisible groups and abelian schemes, due to Serre-Tate, Messing and Drinfeld. This is the main input of the fiber product description of the Shimura variety. In fact the cartesian diagram to establish is a simple translation of this Serre-Tate theory into a geometric relation between the moduli spaces of the relevant objects.

Theorem 4.1 (c.f.[CS19, 2.4.1]). *Let $S' \twoheadrightarrow S$ be a surjection of rings in which p is nilpotent, with nilpotent kernel $I \subset S'$.*

(1) *The functor $\mathcal{G}_{S'} \mapsto \mathcal{G}_{S'} \times_{S'} S$ from p-divisible groups up to isogeny over S' to p-divisible groups up to isogeny over S is an equivalence of categories.*

(2) *The functor $A_{S'} \mapsto A_{S'} \times_{S'} S$ from abelian schemes up to p-power isogeny over S' to abelian schemes up to p-power isogeny over S is an equivalence of categories.*

Theorem 4.2 (c.f.[CS19, 2.4.2]). *(Serre-Tate) Let $S' \twoheadrightarrow S$ be a surjection of rings in which p is nilpotent, with nilpotent kernel $I \subset S'$. The functor*

$$A_{S'} \mapsto (A_S, A_{S'}[p^\infty], id)$$

is an equivalence of categories between the category of abelian schemes over S' and the category of triples consisting of an abelian scheme A_S over S, a p-divisible group $\mathcal{G}_{S'}$ over S' and an isomorphism $\rho : A_S[p^\infty] \to \mathcal{G}_{S'} \times_{S'} S$.

Proof. Let i denote the inclusion $\mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(S')$ and fix N such that $I^N = 0$.

For full-faithfulness of the displayed functor, assume we are given two abelian schemes A_0, B_0 over S, with lifitngs A, B over S', we view them as fpqc abelian sheaves over $\mathrm{Spec}(S)$, respectively $\mathrm{Spec}(S')$. Denote by K_B the kernel of the reduction map $B \to i_* B_0$. By applying $\mathrm{Hom}(A, -)$ to $K_B \hookrightarrow B \twoheadrightarrow i_* B_0$, we get

$$\mathrm{Hom}_{S'}(A, K_B) \to \mathrm{Hom}_{S'}(A, B) \xrightarrow{res} \mathrm{Hom}_S(A_0, B_0) \to \mathrm{Ext}^1_{S'}(A, K_B).$$

To compute $\mathrm{Hom}_{S'}(A, K_B), \mathrm{Ext}^1_{S'}(A, K_B)$, we investigate the sequence of multiplication by p^N on A:

$$(*) : 0 \to A[p^N] \to A \xrightarrow{p^N} A \to 0,$$

applying $\mathrm{Hom}(-, K_B)$ to which induces

$$0 \to \mathrm{Hom}_{S'}(A, K_B) \xrightarrow{p^N} \mathrm{Hom}_{S'}(A, K_B) \to \mathrm{Hom}_{S'}(A[p^N], K_B)$$

$$\to \mathrm{Ext}^1_{S'}(A, K_B) \xrightarrow{p^N} \mathrm{Ext}^1_{S'}(A, K_B) \to$$

But K_B is killed by p^N, so the two arrows labelled by p^N are zeros. Consequently $\mathrm{Hom}_{S'}(A, K_B) = 0$ and $\mathrm{Hom}_{S'}(A[p^N], K_B) \cong \mathrm{Ext}^1_{S'}(A, K_B)$. The first equation implies that res is injective, which proves the faithfulness. The second equation implies that for any map $f_0 \in \mathrm{Hom}_S(A_0, B_0)$, $f_0 \circ p^N$ is always liftable to some $f' : A \to B$. Applying $\mathrm{Hom}(-, B)$ to the sequence $(\ast)$, we see that f_0 is liftable if f' annihilates $A[p^N]$. But in the proof of fullness, we are given $f[p^\infty] : A[p^\infty] \to B[p^\infty]$ lifting $f_0[p^\infty]$, so by part (1) of 4.1 it is necessary that f' induces $p^N f[p^\infty]$ on $A[p^\infty]$. In particular it annihilates $A[p^N]$ as wished.

For essential surjectivity, assume we are given an abelian scheme A_0/S, a p-divisible group $\mathcal{G}/S'$ and an isomorphism $A_0[p^\infty] \cong \mathcal{G} \times_{S'} S$. We construct a lift A/S' of A_0, with $A[p^\infty] \cong \mathcal{G}$ lifting the above isomorphism. For this, we first pick an arbitrary lift A' of A_0 over S', which exists up to p-power isogeny according to theorem 4.1. Without loss of generality, we may assume having an actual isogeny (instead of a quasi-isogeny) $A'_S := A' \times_{S'} S \to A_0$. The induced isogeny $A'_S[p^\infty] \to A_0[p^\infty]$ lifts to an isogeny $A'[p^\infty] \to \mathcal{G}$ over S', again by 4.1. Taking quotient of A' by the kernel of this isogeny, one gets the abelian scheme A as desired.

$\square$

5. Shimura varieties

5.1. **Generalities.** We recall the definition of general Shimura data and Shimura varieties, although we will soon specialize to certain PEL-type Shimura varieties.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ be the algebraic group over $\mathbb{R}$ whose functor of points on $\mathbb{R}$-algebras is

$$R \mapsto (R \otimes_\mathbb{R} \mathbb{C})^\times.$$

Definition 5.1. ([Del79, 2.1.1]) A Shimura datum is a pair (G, X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_\mathbb{R}$ satisfying the following axioms:

- For all $h \in X$, only weights $(-1, 1), (0, 0), (1, -1)$ can appear in the adjoint action of $h_\mathbb{C}$ on the Lie algebra $\mathfrak{g}_\mathbb{C}$.
- The adjoint action of $h(i)$ induces a Cartan involution on the adjoint group of $G_\mathbb{R}$.
- The adjoint group of G does not admit a factor H defined over $\mathbb{Q}$ such that the projection of h on H is trivial.

As explained by Deligne, the second axiom ensures that X has a complex structure and that the stabilizer of each h is compact modulo center. The third condition ensures that $\tilde{G}(\mathbb{Q})$ is dense in $\tilde{G}(\mathbb{A}_f)$, where $\tilde{G}$ is the universal cover of the derived group G^{der}.

Definition 5.2. A morphism between Shimura data (G, X) and (G', X') is a homomorphism $G \to G'$ of algebraic groups over $\mathbb{Q}$ sending X to X'.

Let (G, X) be a Shimura datum and $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Consider the double coset space

$$X_K := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K,$$

where $G(\mathbb{Q})$ acts diagonally on $X \times G(\mathbb{A}_f)$ from the left via embeddings $G(\mathbb{Q}) \hookrightarrow G(\mathbb{R})$, $G(\mathbb{Q}) \hookrightarrow G(\mathbb{Q}_p)$ and K acts on $G(\mathbb{A}_f)$ from the right by multiplication. Then X_K is a complex manifold (if K is small enough) and by the work of Borel and Baily-Borel, it is the complex analytification of some complex algebraic variety $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$.

The adelic group $G(\mathbb{A}_f)$ acts on the inverse system $\{\mathrm{Sh}_K(G, X)_{\mathbb{C}}\}_K$ via isomorphisms:

$$\gamma_g : \mathrm{Sh}_K(G, X)_{\mathbb{C}} \to \mathrm{Sh}_{g^{-1}Kg}(G, X)_{\mathbb{C}},$$

which send a double coset $G(\mathbb{Q}) \cdot (h, g_1)K$ to $G(\mathbb{Q}) \cdot (h, g_1 g)(g^{-1}Kg)$. We refer to this action as the *Hecke action*.

For any $h \in X$, giving h is equivalent to giving a bigrading on the complexification $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$ of any $\mathbb{R}$-representation $G_{\mathbb{R}} \to GL(W)$, where the degree (p, q)-piece is the subspace of $W_{\mathbb{C}}$ on which $h(z)$ acts by $z^p \bar{z}^q$. Projecting to the first entry of the grading gives a Hodge cocharacter $\nu_h : \mathbb{G}_{m, \mathbb{C}} \to G_{\mathbb{C}}$. Let $[\nu_h]$ denote its $G(\mathbb{C})$-conjugacy class. Then the *reflex field* $E(G, X)$ of the Shimura datum (G, X) is the subfield of $\mathbb{C}$ fixed by the stabilizer of $[\nu_h]$ in $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. It is proven by Deligne in [Del79] that the system $\{\mathrm{Sh}_K(G, X)_{\mathbb{C}}\}_K$ with Hecke action has a unique form $\{\mathrm{Sh}_K(G, X)\}_K$ with Hecke action defined over $E(G, X)$ subjecting to the conditions that all special points (i.e. points on X_K represented by some (h, g) such that h factors through a $\mathbb{Q}$-torus of G) are algebraic, and that the Galois action on the special points is normalized as in [Del79, 2.2.4]. We call the inverse limit $\varprojlim_K \mathrm{Sh}_K(G, X)$, which is represented by a scheme $\mathrm{Sh}(G, X)/E(G, X)$, the *(canonical model of) Shimura variety attached to the Shimura datum* (G, X), and $\mathrm{Sh}_K(G, X)$ the *Shimura variety at level* K.

The assignment $(G, X) \mapsto \mathrm{Sh}(G, X)$ is functorial in the following sense

Proposition 5.3 (c.f. [Mil17, 5.16], [Del71, 1.15, 5.4]). *A morphism of Shimura data* $(G, X) \to (G', X')$ *induces a map*

$$\mathrm{Sh}(G, X)_{\mathbb{C}} \to \mathrm{Sh}(G', X')_{\mathbb{C}},$$

which is compatible with the Hecke action. If $G \hookrightarrow G'$ *is a closed immersion, then for any compact open subgroup* K *of* $G(\mathbb{A}_f)$, *there exists some compact open subgroup* K' *of* $G'(\mathbb{A}_f)$, *such that*

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} \to \mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$$

is a closed immersion.

Moreover, this map is defined over the composite $E(G, X) \cdot E(G', X')$ *of the reflex fields.*

5.2. **PEL-type Shimura varieties.** Now we introduce the PEL-type Shimura data and their associated Shimura varieties this thesis mainly concerns, following Kottwitz [Kot92, 5], c.f. [Lan13].

5.2.1. *Global PEL setup.*

Definition 5.4. A global PEL-datum is a tuple

$$(B, *, V, (\cdot, \cdot), h),$$

where

- B is a finite-dimensional semisimple $\mathbb{Q}$-algebra.
- $*$ is a positive involution on B, i.e. over $\mathbb{R}$, $*$ satisfies that $\mathrm{tr}_{B_\mathbb{R}/\mathbb{R}}(xx^*) > 0$ for all $0 \neq x \in B_\mathbb{R}$.
- V is a finite left B-module.
- $(\cdot, \cdot)$ is a nondegenerate $\mathbb{Q}$-valued alternating form on V such that $(bv, w) = (v, b^*w)$ for all $v, w \in V$ and $b \in B$. In particular, the induced involution on $\mathrm{End}(V)$ that sends an endomorphism to its adjoint with respect to $(\cdot, \cdot)$ extends $*$ on $B \subset \mathrm{End}(V)$.

  Let $G/\mathbb{Q}$ be the algebraic group determined by the functor:

  $$R \mapsto \{x \in \mathrm{End}_{B \otimes R}(V \otimes R) \mid xx^* \in R^\times\}.$$

- $h : \mathbb{S} \to G_\mathbb{R}$ is a homomorphism, such that $h(\bar{z}) = h(z)^*$ for any $z \in \mathbb{C}$, the symmetric real-valued bilinear form $(v, h(i)w)$ on $V_\mathbb{R}$ is positive-definite, and the induced Hodge structure on $V_\mathbb{R}$ is of type $(1, 0)$, $(0, 1)$.[6]

Let X be the $G(\mathbb{R})$-conjugacy class of h. Then the pair (G, X) is a Shimura datum. Let $V_\mathbb{C} \cong V_1 \oplus V_0$ be the $B_\mathbb{C}$-module decomposition induced by h such that $h(z)$ acts on V_1 (resp. V_0) by z (resp. $\bar{z}$). Let E_0 be the field of definition of the complex representation V_1 of B, i.e.

$$E_0 = \mathbb{Q}[\{\mathrm{tr}(b \mid V_1)\}_{b \in B}].$$

Then the reflex field $E(G, X) = E_0$.

If the $\mathbb{Q}$-algebra B is simple, then its center F is a field and $F^+ := F^{*=id}$ is a totally real subfield. Let $G_1/\mathbb{Q}$ be the closed subgroup of G defined by

$$R \mapsto \{x \in \mathrm{End}_{B \otimes R}(V \otimes R) \mid xx^* = id\}.$$

Then it is the restriction of scalar of some group G_0/F^+ from F^+ to $\mathbb{Q}$. According to the type of the extension F/F^+ and G_0, the PEL-datum falls into three families, cases A, C and D, where respectively F/F^+ is a complex quadratic extension, G_0 is an inner form of the quasi-split unitary group over F^+ (of type A_{n-1}); $F = F^+$ is totally real, G_0 is a symplectic group in $2n$ variables; and $F = F^+$, G_0 is an orthogonal group of $2n$ variables. Here n is half of the positive integer $[F : F^+](\dim_F \mathrm{End}_B(V))^{\frac{1}{2}}$, which is forced to be even by the existence of h.

In general the semisimple $\mathbb{Q}$-algebra B decomposes into a product of simple algebras. According to [Lan13, 1.2.1.11], the involution $*$ leaves stable each simple factor. Hence the symplectic B-module $(V, (\cdot, \cdot))$ decomposes accordingly. Up to similitude factors, G is the product of groups as G_1 above.

From now on, we will make the following additional assumptions on the PEL-datum and on the prime p:

Assumption 5.5.

(1) (Type AC) In the decomposition of B into simple factors, no factor of type D appears.

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[6]This means $V_\mathbb{C}$ decomposes into a direct sum of two subspaces where the $h(z)$ action is by z and $\bar{z}$ respectively, for all $z \in \mathbb{C}$.

(2) $B_{\mathbb{Q}_p}$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$. There exists a $*$-invariant $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B \subset B$, whose p-adic completion is a maximal $\mathbb{Z}_p$-order of $B_{\mathbb{Q}_p}$. There exists a $\mathbb{Z}_{(p)}$-lattice $\Lambda_0 \subset V$, stable under the $\mathcal{O}_B$-action and self-dual with respect to $(\cdot, \cdot)$.

Remark 5.6. Assumption (1) can be rephrased as follows: by Wedderburn's theorem, each simple factor of B is a matrix algebra for some division algebra. Therefore over $\mathbb{R}$, B is a product of matrix algebras of the form $M_n(\mathbb{R})$ (type C), $M_n(\mathbb{C})$ (type A) or $M_n(\mathbb{H})$ (type D), where $\mathbb{H}$ is the Hamilton quaternions.[7] As algebra with positive involution, $(B_{\mathbb{R}}, *_{\mathbb{R}})$ is isomorphic to products of these matrix algebras with standard involutions, i.e. on $M_n(\mathbb{R})$, $*$ sends an element x to its transpose x^τ; on $M_n(\mathbb{C})$ and $M_n(\mathbb{H})$, $*$ sends x to its conjugate transpose $\bar{x}^\tau$. Then we assume that no factors of $M_n(\mathbb{H})$ will appear in the decomposition of $B_{\mathbb{R}}$.

Alternatively, we can assume the algebraic group G to be connected. This automatically excludes type D.

Remark 5.7. The Hasse principle holds for groups of type C. For type A, G_0 above is the inner form of a quasi-split unitary group over F^+, determined by the quadratic extension F/F^+. The Hasse principle holds if the Hermitian space giving rise to the quasi-split unitary group has even dimension over F; otherwise it can fail but this failure comes from the failure of the Hasse principle for the center of G, c.f. [Kot92, 7]. But for type D, the Hasse principle fails in a more essential way. The reason we put assumption (1) is to ensure that the moduli problem we will consider below will be a union of copies of Shimura varieties given by the PEL-datum. Involving type D factors destroys this feature due to failure of the Hasse principle, c.f. [Lan15, A.7.2]. Our assumption (2) on the prime p ensures a smooth integral structure at p. In particular, hyperspecial subgroups exist, or equivalently the group $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified extension of $\mathbb{Q}_p$, c.f. below.

Fix $\mathcal{O}_B$ and a self-dual $\mathcal{O}_B$-lattice $\Lambda_0 \subset V$ as in part (2) of assumption 5.5. These determine a connected reductive group $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ with generic fiber G as

$$G_{\mathbb{Z}_{(p)}}(R) = \{x \in \mathrm{End}_{\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} R}(\Lambda_0 \otimes_{\mathbb{Z}_{(p)}} R) \mid xx^* \in R^\times\}.$$

5.2.2. *Moduli interpretation.* Let $(B, *, V, (\cdot, \cdot), h)$ be a global PEL-datum satisfying assumption 5.5, $(\mathcal{O}_B, *, \Lambda_0, (\cdot, \cdot), h)$ its integral model at p, and $G_{\mathbb{Z}_{(p)}}$ as above. Let Λ be the p-adic completion of Λ_0. We fix the hyperspecial maximal compact open subgroup $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. Let $K^p \subset G(\mathbb{A}_f^p)$ be a compact open subgroup and $K = K_p K^p$.

We can define a moduli stack of polarized abelian varieties with endomorphism by $\mathcal{O}_B$ at level K, over the localization of $\mathcal{O}_{E_0}$ at some prime above p. By what we explained in remark 5.7, its generic fiber will be a finite disjoint union of copies of the Shimura variety determined by the given PEL-datum. The number of copies agrees with the cardinality of the set of locally trivial elements in $H^1(G, \mathbb{Q})$ c.f.[Kot92, 8]. We ignore this difference below.

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[7]In terms of the classification of the group G itself, it will have absolute root system of type C, A, D accordingly.

Definition 5.8. Let S be a scheme over $\mathcal{O}_{E_0} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p. An abelian scheme with G-structure at level K over S is a quadruple $\mathcal{A} = (A, \iota, \lambda, \bar{\eta})$ where:

- A is an abelian scheme of dimension $g = \frac{1}{2}\dim_{\mathbb{Q}} V$ over S;
- $\iota : \mathcal{O}_B \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is an $\mathcal{O}_B$-action, satisfying the Kottwitz condition that
$$\det\nolimits_{\mathcal{O}_S}(\iota(b) \mid Lie(A)) = \det(b \mid V_1),$$
  for all $b \in \mathcal{O}_B$, where $V_{\mathbb{C}} \cong V_1 \oplus V_0$ is the decomposition such that $h(z)$ acts on V_1 (resp. V_0) by z (resp. $\bar{z}$); [8]
- $\lambda : A \to A^{\vee}$ is a prime-to-p quasi-isogeny, symmetric with respect to the double duality $A \cong A^{\vee\vee}$, such that for some natural number n, $n\lambda$ is induced by an ample line bundle on A (hence pointwise a polarization) and whose Rosati involution on $\mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is compatible with $*$ on $\mathcal{O}_B$ via ι;
- $\bar{\eta}$ is a K^p-orbit of a chosen trivialization η of the locally constant pro-étale[9] sheaf $\underline{H_1}(A, \mathbb{A}_f^p)$ on S, under the action of $\underline{G(\mathbb{A}_f^p)}$. Namely, the sheaf
$$\underline{\mathrm{Isom}}_G(\underline{H_1}(A, \mathbb{A}_f^p), \underline{V_{\mathbb{A}_f^p}}),$$
  whose sections are $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$-module isomorphisms that preserve $(\cdot, \cdot)$ up to a scalar in $\underline{\mathbb{A}_f^{p,\times}}$, is a $\underline{G(\mathbb{A}_f^p)}$-torsor on $S_{\mathrm{pro-ét}}$. Choose one section η of it on a trivializing cover $\tilde{S} \to S$ and look at the $\underline{G(\mathbb{A}_f^p)}(\tilde{S})$-action on it. Then $\bar{\eta}$ is its orbit under the subgroup $\underline{K^p}(\tilde{S})$. We further require that $\bar{\eta}$ is invariant under the action of the covering group of $\tilde{S} \to S$.

Now we can define the moduli problem.

Definition 5.9. Let S_K^{pre} be the presheaf of groupoids on the big étale site of schemes over $\mathcal{O}_{E_0} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, whose value on S is the groupoid of abelian schemes over S with G-structure at level K, and an isomorphism between $(A, \iota, \lambda, \bar{\eta})$ and $(A', \iota', \lambda', \bar{\eta}')$ is a prime-to-p quasi-isogeny $f : A \to A'$, such that $f^{\vee} \circ \lambda' \circ f = c\lambda$, for some $c \in \underline{\mathbb{Z}_{(p)}^{\times}}(S)$, where $f^{\vee} : A'^{\vee} \to A^{\vee}$ is the dual quasi-isogeny, f commutes with the action of $\mathcal{O}_B$ on A and A' via ι, ι', and $\bar{\eta} = \bar{\eta}' \circ f_*$.

This moduli problem is a Deligne-Mumford stack. For small enough K^p, it is representable by a smooth quasi-projective scheme. We will always be working in such situations. Let E be the completion of E_0 at some prime $\mathfrak{p}$ above p and $\mathcal{O}_E$ its ring of integers. We base change the moduli functor to $\mathcal{O}_E$ and denote the representing scheme by S_K. Its p-adic completion is denoted by $\mathscr{S}_K$. This is a formal scheme over $\mathrm{Spf}\,\mathcal{O}_E$.

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[8]In other words, the determinant of any element $b \in \mathcal{O}_B$ acting on the Lie algebra $Lie(A)$ (as a free $\mathcal{O}_S$-module) of A agrees with the determinant of it acting on V_1. This makes sense as the decomposition $V_{\mathbb{C}} \cong V_1 \oplus V_0$ is defined over E_0 and the determinant of b lies in $\mathcal{O}_{E_0} \otimes \mathbb{Z}_{(p)}$. As remarked by [Kot92, 5], for a point $s \in S$ with residue field K/E_0, this condition ensures that $\mathrm{Lie}(A_s) \cong V_{1,K}$ as B_K-modules, where $V_K \cong V_{1,K} \oplus V_{2,K}$ is a K-vector space decomposition whose base change to $\mathbb{C}$ is the above. In this way, $V_{\mathbb{C}}$ can be identified with the Betti homology $H_1(A_{\mathbb{C}}, \mathbb{C})$ preserving the Hodge structures.

[9]in the sense of [BS15]

Remark 5.10. Let S be an $\mathcal{O}_E$-scheme. We call a tuple $(\mathcal{G}, \iota, \lambda)$ a p-divisible group with G-structure over S, where

- $\mathcal{G}$ is a p-divisible group over S,
- $\iota : \mathcal{O}_B \otimes \mathbb{Z}_p \to \mathrm{End}(\mathcal{G})$ is a $\mathbb{Z}_p$-linear map satisfying the Kottwitz condition
$$\mathrm{det}_{\mathcal{O}_S}(\iota(b) \mid \mathrm{Lie}(\mathcal{G})) = \det(b \mid V_1 \otimes_{E_0} E),$$
- $\lambda : \mathcal{G} \to \mathcal{G}^\vee$ is a polarization, satisfying for any $b \in \mathcal{O}_B \otimes \mathbb{Z}_p$, $\lambda^{-1}\iota(b)^\vee \lambda = \iota(b^*)$.

An isomorphism (resp. quasi-isogeny) between $(\mathcal{G}, \iota, \lambda)$ and $(\mathcal{G}', \iota', \lambda')$ is an $\mathcal{O}_B \otimes \mathbb{Z}_p$-linear isomorphism (resp. quasi-isogeny) $f : \mathcal{G} \to \mathcal{G}'$ such that $f^\vee \circ \lambda' \circ f = c\lambda$ for some $c \in \underline{\mathbb{Z}}_p^\times(S)$ (resp. $\underline{\mathbb{Q}}_p^\times(S)$).

Taking the p-divisible group of an abelian scheme defines a functor from abelian schemes with G-structure up to (prime-to-p) quasi-isogenies to p-divisible groups with G-structure up to (isomorphisms) quasi-isogenies.

Here are some examples.

Example 5.11. (Siegel Shimura varieties.) $B = \mathbb{Q}, * = id, F = F^+ = \mathbb{Q}, V = \mathbb{Q}^{2n}$, $(\cdot, \cdot)$ given by the matrix
$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$
Then $*$ extends to $\mathrm{End}(V)$, sending a matrix A to $J^{-1}A^\tau J$. The corresponding Shimura data is given by

- $G = \mathrm{GSp}_{2n}/\mathbb{Q}$,
- $X = \{A \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{Im}(A) > 0 \text{ or } \mathrm{Im}(A) < 0\}$,
- $h : \mathbb{S} \to G_\mathbb{R}, a + bi \mapsto aI + bJ$,

where $\mathrm{Sym}_n(\mathbb{C})$ denotes the set of n-by-n symmetric complex matrices, on which $G(\mathbb{R})$ acts via Möbius transform. The reflex field is $\mathbb{Q}$.

This is a moduli problem of type C_n. The corresponding Shimura variety is not compact and is called a Siegel modular variety. In the specific case $n = 1$, $G = \mathrm{GL}_2$, the Shimura variety is the modular curve.

Example 5.12. (Non-compact unitary Shimura varieties appeared in [CS19, 2.1].) Let F be a CM field with totally real subfield $F^+ \subset F$ and $n \geq 1$ be an integer. Then we can take $B = F$, $* =$CM conjugation, $V = F^{2n}$, with alternating form
$$(\cdot, \cdot) : V \times V \to \mathbb{Q}$$
$$((x_i), (y_i)) \mapsto \mathrm{tr}_{F/\mathbb{Q}}(\Sigma_{i=1}^n (x_i \bar{y}_{2n+1-i} - x_{2n+1-i}\bar{y}_i)).$$
The reductive group G is a unitary similitude group and $G_\mathbb{R} \cong \mathrm{GU}(n,n)^{[F^+:\mathbb{Q}]}$.
$$X = \coprod_{\tau: F^+ \to \mathbb{R}} X_{\tau,+} \sqcup \coprod_{\tau: F^+ \to \mathbb{R}} X_{\tau,-},$$
where $X_{\tau,+}$ (resp. $X_{\tau,-}$) is the space of positive (negative) definite n-dimensional subspaces of $V \otimes_{F^+} \mathbb{R} \cong \mathbb{C}^{2n}$, each being isomorphic to the Hermitian upper (lower) half-space
$$\mathcal{H}_{n,n} = \{A \in \mathrm{Herm}_n(\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} : \mathrm{Im}(A) > 0\}$$
$$(\mathcal{H}_{n,n}^- = \{A \in \mathrm{Herm}_n(\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} : \mathrm{Im}(A) < 0\}),$$
where $\mathrm{Herm}_n(\mathbb{C})$ is the set of n-by-n Hermitian matrices (c.f. [Lan16, 3.2.5]), and

$$h = \prod_{\tau:F^+ \to \mathbb{R}} h_\tau : \mathbb{S} \to G_\mathbb{R}, z \mapsto (\mathrm{diag}\{z \cdot I_n, \overline{z} \cdot I_n\}_\tau)_{\tau:F^+ \to \mathbb{R}}.$$

This is a moduli problem of type A, and the corresponding (unitary) Shimura variety is not compact, since the group G is quasi-split and has rationally defined parabolic subgroups. In the specific case $n = 1$, $G_\mathbb{R} = \mathrm{GU}(1,1)$, the Shimura variety is one dimensional and we call it a unitary Shimura curve attached to the imaginary quadratic field F.

5.2.3. *Minimal compactifications.* Let $K = K_p K^p$ with K_p being hyperspecial as before, the smooth quasi-projective scheme S_K over $\mathcal{O}_E$ has a good minimal compactification, whose properties we summarize below. For more details, we refer to [Lan13, 7.2.4] and [LS18, 2.1.2], c.f. [CS19, 2.5.8].

Theorem 5.13. *There exists a flat, projective, normal scheme $S_K^*/\mathrm{Spec}(\mathcal{O}_E)$, together with a set-theoretic partition into locally closed subschemes*

$$S_K^* = \coprod_Z S_{K,Z},$$

where the index set is endowed with a partial order such that the incidence relations among strata are determined by this partial order. There is a unique dense open stratum that is isomorphic to S_K.

If the level K is principal, i.e. it is the kernel of the reduction by N map on $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, for some integer N coprime to p, then each $S_{K,Z}$ is a PEL-type Shimura variety.

Remark 5.14. Over E we can add level at p and the same statements hold. In this case, the scheme $S_{K,E}$ is understood as solution to the moduli problem of abelian schemes with G-structures at level K, where the level $\bar{\eta}$ is a K-orbit of trivializations of $\underline{H}_1(A, \mathbb{A}_f)$ under the action of $\underline{G(\mathbb{A}_f)}$.

Remark 5.15. As in the literature, we will refer to elements in the index set as *cusp labels at level K*. In general, a cusp label is a tuple of the shape $(Z, (X, Y, \phi, \varphi_{-2}, \varphi_0))$ consisting of the following data. c.f. [Lan13, 5.2.7.1, 5.4.1.3]

(1) Z is a $\mathcal{O}_B \otimes_\mathbb{Z} \hat{\mathbb{Z}}$-stable split two step filtration

$$0 = Z_{-3} \subset Z_{-2} \subset Z_{-1} \subset Z_0 = \Lambda_0 \otimes_\mathbb{Z} \hat{\mathbb{Z}},$$

such that each graded piece is isomorphic to $M \otimes_{\mathbb{Z}_{(p)}} \hat{\mathbb{Z}}$ for some finitely generated $\mathcal{O}_B$-torsionfree $\mathcal{O}_B$-module (an $\mathcal{O}_B$-lattice) M, and that Z_{-2} and Z_{-1} are annihilators of each other under the pairing $(\cdot, \cdot)$ induced from Λ_0;

(2) X and Y are $\mathcal{O}_B$-lattices of the same $\mathcal{O}_B$-multi-rank[10] and $\phi : Y \to X$ is an $\mathcal{O}_B$-linear injection;

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[10]Assume $B \cong \prod_i B_i$ is a decomposition of B into simple $\mathbb{Q}$-algebras, then each finite B-module M decomposes as $\prod_i M_i^{m_i}$, with M_i being the unique simple left B_i-module. Then the vector (m_i) is called the B-multi-rank of M. And for an $\mathcal{O}_B$-lattice M, its $\mathcal{O}_B$-multi-rank is the B-multi-rank of $M \otimes \mathbb{Q}$, c.f.[Lan13, 1.2.1.21].

(3) Denote $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ by R, then $\varphi_{-2} : \mathrm{Gr}_{-2}^Z \cong \mathrm{Hom}_R(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}, R(1))$ and $\varphi_0 : \mathrm{Gr}_0^Z \cong Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ are isomorphisms such that the induced pairing

$$(\cdot, \cdot)_{20} : \mathrm{Gr}_{-2}^Z \times \mathrm{Gr}_0^Z \to R(1)$$

is the pullback under $(\varphi_{-2}, \varphi_0)$ of the pairing:

$$\mathrm{Hom}_R(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}, R(1)) \times (Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) \xrightarrow{id \times \phi} \mathrm{Hom}_R(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}, R(1)) \times (X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) \xrightarrow{ev} R(1),$$

where the last map is the evaluation pairing.

There is an action of the group $K_p G(\mathbb{A}_f^p)$ on $\Lambda_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, inducing an action on the set of cusp labels. A cusp label at level K is a K-orbit of cusp labels under this action.

Remark 5.16. Given Z, a cusp label at level K, assume $\mathrm{Gr}_{-1}^Z \cong M \otimes_{\mathbb{Z}_{(p)}} \hat{\mathbb{Z}}$ for some $\mathcal{O}_B$-lattice M. Then the stratum $S_{K,Z}$ is attached to the integral PEL Shimura datum $(\mathcal{O}_B, *, M, (\cdot, \cdot)_{11}, h_{-1})$ (see [Lan13, 5.1.2.2] for the definition of h_{-1}). For an abelian variety corresponding to a C-point of $S_{K,E}$ for some complete algebraically closed non-archimedean field C, it has semistable reduction over the ring of integers $\mathcal{O}_C \subset C$ and hence an attached Raynaud extension. This is an extension

$$0 \to T \to E \to B \to 0$$

of a smaller dimensional polarized abelian scheme B by a torus T, both equipped with $\mathcal{O}_B$-endomorphism. Then $S_{K,Z}$ is a parameter space for such B's. In fact X is obtained from the character group[11] of T (tensored up to $\mathbb{Z}_{(p)}$), Y from that of the dual Raynaud extension, and the filtered pieces of Z are obtained by taking the Tate module of T and that of E. For more details, see [Lan13, 3.3, 4.2], c.f. [CS19, 2.5.1] in the principally polarized case.

5.2.4. *Shimura variety as v-sheaves.* Let K, the scheme $S_K / \mathcal{O}_E$ and its formal completion $\mathscr{S}_K$ over $\mathrm{Spf}(\mathcal{O}_E)$ be as before. Write $\mathrm{Spa}(E)$ for $\mathrm{Spa}(E, \mathcal{O}_E)$. We define below variants of the Shimura variety as v-sheaves that will be used later.

Definition 5.17. The *adic Shimura variety* at level K is the diamond $\mathcal{S}_K$ over $\mathrm{Spd}\,E$ attached to $S_{K,E}$ using the big diamond functor c.f. 2.26, namely $\mathcal{S}_K = S_{K,E}^\diamond$.

By analytifying the universal abelian scheme over $S_{K,E}$ and passing to diamonds, we obtain a proper map of diamonds

$$\pi : \mathcal{A}^\diamond \to \mathcal{S}_K.$$

Consider $\mathbb{Z}_p$ with the p-adic topology as a profinite topological space. Let $\underline{\mathbb{Z}}_p$ be the v-sheaf on $\mathcal{A}^\diamond$ attached to the topological space $\mathbb{Z}_p$. We call the sheaf of $\underline{\mathbb{Z}}_p$-modules on $\mathcal{S}_K$

$$T_p \mathcal{A} := \underline{\mathrm{Hom}}_{\underline{\mathbb{Z}}_p}(R^1 \pi_* \underline{\mathbb{Z}}_p, \underline{\mathbb{Z}}_p)$$

the Tate module of the universal object.

Definition 5.18. The *Shimura variety with infinite level at p* is the diamond $\mathcal{S}_{K^p}$ over $\mathcal{S}_K$ of $\mathcal{O}_B$-linear trivializations of $T_p \mathcal{A}$, which preserve the alternating paring $(\cdot, \cdot)$ up to a constant in $\underline{\mathbb{Z}}_p^\times$, namely

$$\mathcal{S}_{K^p} = \underline{\mathrm{Isom}}_G(T_p \mathcal{A}, \underline{\Lambda}) \to \mathcal{S}_K.$$

---

[11]In [CS19] page 22, X is said to be the cocharacter group instead of the character group and this is a slight inconsistency with the explanations there.

Remark 5.19. Alternatively this is the limit in the category of diamond of $\mathcal{S}_K$'s for K_p running through compact open subgroups of $G(\mathbb{Q}_p)$. It is in fact representable by a perfectoid space, by the work of [Sch15].

Definition 5.20. The *good reduction locus* of the adic Shimura variety at level K is the diamond $\mathcal{S}_K^\circ$ attached to the adic generic fiber of the formal scheme $\mathscr{S}_K$, i.e.

$$(\mathscr{S}_K^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O}_E, \mathcal{O}_E)} \mathrm{Spa}(E))^\diamond.$$

This is a spatial diamond over $\mathrm{Spd}E$.

For convenience of later arguments, we note that the diamond $\mathcal{S}_K^\circ$ still has a moduli interpretation in the following sense:

Lemma 5.21. *The diamond good reduction locus $\mathcal{S}_K^\circ$ is the sheafification with respect to the analytic topology of the presheaf*

$$\mathrm{Perf} \to \mathrm{Sets}$$

$$S = \mathrm{Spa}(R, R^+) \mapsto \{(S^\sharp, \mathrm{Spf}R^{\sharp+} \to \mathscr{S}_K)\},$$

where $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S over E.

Proof. Combine [SW20, 10.1.5] and [SW13, 2.2.2]. $\qquad\qquad\qquad\qquad\square$

Definition 5.22. The *good reduction locus with infinite level at p* is the diamond $\mathcal{S}_{K^p}^\circ$ over $\mathrm{Spd}E$, obtained by pulling back $\mathcal{S}_K^\circ$ to $\mathcal{S}_{K^p}$.

Definition 5.23. The *(v-sheaf) integral model* of the Shimura variety at level K is the v-sheaf $\mathscr{S}_K^\diamond$ attached to the formal scheme $\mathscr{S}_K$, c.f. example 2.24. Alternatively this is the small diamond functor applied to the $\mathcal{O}_E$-scheme S_K, c.f. 2.26 for the construction and [AGLR22, 2.11] for this identification.

As explained in remark 5.14, over E, we have minimal compactifications $S_{K_pK^p,E}^*$ for Shimura varieties with deepening levels at p.

Definition 5.24. The *minimal compactification with infinite level at p* is the diamond $\mathcal{S}_{K^p}^*$ over $\mathrm{Spd}E$:

$$\mathcal{S}_{K^p}^* := \varprojlim_{K_p} S_{K_pK^p,E}^{*,\diamond},$$

where the limit is taken over compact open subgroups $K_p \subset G(\mathbb{Q}_p)$.

6. B_{dR}^+-AFFINE GRASSMANNIAN AND HODGE-TATE PERIOD MAP

In this and next section we introduce the B_{dR}^+-affine Grassmannian and the stack Bun_G of G-torsors on the Fargues-Fontaine curve. Since these local theories work uniformly for general reductive groups over any local field, we present the relevant material also in this generality (though we focus on the mix-characteristic case). However, for our purpose of proving conjecture 1.1 in the global PEL-setup in section 5, we only need the results for $G_{\mathbb{Q}_p}$. Hence in subsection 6.2 below about the Hodge-Tate period map and from section 8 on, we switch back to this situation. This shall, hopefully, not cause too much confusion.

6.1. B_{dR}^+-affine Grassmannian.

Fix a finite extension $F/\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$, uniformizer π and finite residue field $\mathbb{F}$ of cardinality q.

6.1.1. *The ring B_{dR}^+.*

Definition 6.1. We consider the functor[12]

$$W_{\mathcal{O}_F} : \{\text{perfect } \mathbb{F}\text{-algebras}\} \to \begin{Bmatrix} \pi\text{-torsionfree } \pi\text{-adically complete} \\ \mathcal{O}_F\text{-algebras} \end{Bmatrix}$$

$$R \mapsto W(R)\hat{\otimes}_{W(\mathbb{F})}\mathcal{O}_F,$$

where $W(\cdot)$ is the functor of p-typical Witt vectors. For any perfect $\mathbb{F}$-algebra R the unique lift of the q-Frobenius on $W_{\mathcal{O}_F}(R)$ is called the Frobenius endomorphism and will be denoted by φ_R.

The functor $W_{\mathcal{O}_F}(\cdot)$ is left adjoint to the tilting functor

$$\flat : A \mapsto \varprojlim_{x\mapsto x^q} A/\pi,$$

and we call the adjunction counit $\theta : W_{\mathcal{O}_F}(A^\flat) \to A$ Fontaine's theta map.

Definition 6.2. (c.f.[SW20, 17.5.1],[BMS18, 3.5], [BS19, 3.10]) An $\mathcal{O}_F$-algebra A is called integral perfectoid if it is of the form $W_{\mathcal{O}_F}(R)/I$ for some perfect $\mathbb{F}$-algebra R and a principal ideal I, such that $W_{\mathcal{O}_F}$ is I-adically complete and I is generated by some d satisfying $\frac{\varphi(d)-d^q}{\pi} \in W_{\mathcal{O}_F}(R)^\times$ (i.e. d is "distinguished").

Remark 6.3. In this case one has necessarily $R \cong A^\flat$ and I identifies with the kernel of θ under this isomorphism. What's more, it follows from a direct computation that any generator of I is distinguished. In what follows, we will often denote such a (chosen) generator by ξ.

This is related to the perfectoid Tate rings defined in section 2 by the following lemma.

Lemma 6.4. [BMS18, 3.20,3.21] *If R is a perfectoid Tate ring as in 2.1 with a ring of integral elements R^+, then R^+ is an integral perfectoid $\mathbb{Z}_p$-algebra. Conversely if A is an integral perfectoid $\mathbb{Z}_p$-algebra, which is ϖ-adically complete for some non-zero-divisor ϖ such that $\varpi^p \mid p$, then $A[1/\varpi]$ endowed with the ϖ-adic topology is a perfectoid Tate ring in the sense of 2.1.*

Definition 6.5. Let R be a perfect $\mathbb{F}$-algebra. An untilt of R over $\mathcal{O}_F$ is a pair (A, ι) of an integral perfectoid $\mathcal{O}_F$-algebra A and an isomorphism $\iota : A^\flat \cong R$.

As in 2.5, we will often denote an untilt of R by $R^\sharp$ and omit ι from the notations.

Definition 6.6. Given an integral perfectoid $\mathcal{O}_F$-algebra $R^{\sharp+}$ with tilt R^+, let ξ be a generator of $\ker(\theta)$. Assume $R^\sharp := R^{\sharp+}[1/\pi] \neq 0$. The ring $B_{\mathrm{dR}}^+(R^\sharp)$, resp. $B_{\mathrm{dR}}(R^\sharp)$, is defined to be the ξ-adic completion of $W_{\mathcal{O}_F}(R^+)[1/\pi]$, resp. $B_{\mathrm{dR}}^+(R^\sharp)[1/\xi]$. As the notation suggests, they only depend on the perfectoid Tate ring $R^\sharp$, not on the integral subring $R^{\sharp+}$.

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[12]For the sake of simplified formulas, we restrict the source of the usual functor of ramified Witt vectors to perfect $\mathbb{F}$-algebras.

Remark 6.7. For $R = C$ being a complete algebraically closed nonarchimedean field, the ring $B_{\mathrm{dR}}^+(C^\sharp)$ is by Cohen structure theorem isomorphic to the formal power series ring $C^\sharp[[\xi]]$, as a noetherian complete regular local ring.

6.1.2. *The mixed-characteristic affine Grassmannian.* Let $F, \mathcal{O}_F, \pi, \mathbb{F}$ be as before and G be a reductive group over F.

Definition 6.8. (c.f. [SW20, 19.1]) The B_{dR}^+-affine Grassmannian Gr_G associated with G is the v-sheaf on $\mathrm{Perf}/\mathrm{Spd}F$ sending $S = \mathrm{Spa}(R, R^+)$ to the set of isomorphism classes of pairs $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a G-torsor over $B_{\mathrm{dR}}^+(R^\sharp)$ and α is a trivialization of $\mathcal{F}$ over $B_{\mathrm{dR}}(R^\sharp)$, where $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is the untilt of S over $\mathrm{Spa}F$ determined by the structure map $S \to \mathrm{Spd}F$.

Equivalently, this is the étale sheafification[13] of the presheaf sending $S \to \mathrm{Spd}F$ to the set $G(B_{\mathrm{dR}}(R^\sharp))/G(B_{\mathrm{dR}}^+(R^\sharp))$.

Remark 6.9. The v-sheaf Gr_G identifies with the functor on $\mathrm{Perf}_{/\mathrm{Spd}F}$ sending S to the set of isomorphism classes of meromorphic modifications of the trivial G-bundle $\mathscr{E}_1$ over X_S, the relative Fargues-Fontaine curve attached to S and F:

$$i : \mathscr{E}_1 \dashrightarrow \mathscr{E}$$

which restricts to an isomorphism outside $S^\sharp$, the untilt of S determined by the structure map to $\mathrm{Spd}F$. Here two modifications $i : \mathscr{E}_1 \dashrightarrow \mathscr{E}$ and $i' : \mathscr{E}_1 \dashrightarrow \mathscr{E}'$ are called isomorphic if there exists an isomorphism $g : \mathscr{E} \to \mathscr{E}'$ such that $i' = g \circ i$. This point of view will be explained in more detail in the next section.

Proposition 6.10. *([SW20, 19.1.4, 19.2.4]) The B_{dR}^+-affine Grassmannian Gr_G is partially proper and is a union of spatial diamonds.*

The construction of Gr_G is functorial with respect to closed embeddings of reductive groups.

Proposition 6.11. *Let $H \hookrightarrow G$ be a closed embedding of reductive groups over F. Then the induced map*

$$\mathrm{Gr}_H \to \mathrm{Gr}_G$$

is a closed embedding.

Proof. Being a closed immersion can be checked v-locally. Hence one can base change to $\mathrm{Spd}C$ for some complete algebraically closed extension of F and use [SW20, 19.1.5]. □

Over an algebraically closed non-archimedean extension C/F, fixing a split torus and a Borel $T \subset B \subset G_C$, we have the Cartan decomposition

$$G(B_{\mathrm{dR}}(C)) = \coprod_{\mu \in X_*^+(T)} G(B_{\mathrm{dR}}^+(C)) \cdot \xi^\mu \cdot G(B_{\mathrm{dR}}^+(C)).$$

where $X_*^+(T)$ is the set of dominant cocharacters of T. This defines a Bruhat stratification on $\mathrm{Gr}_{G,C}$, the base change of Gr_G to $\mathrm{Spd}C$, by the following.

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[13]According to [KY23], analytic sheafification suffices.

Definition 6.12. Let $\mu \in X_*^+(T)$. Then $\mathrm{Gr}_{G,C,\mu}$ (respectively $\mathrm{Gr}_{G,C,\leq\mu}$) is the subfunctor of $\mathrm{Gr}_{G,C}$ sending $S \to \mathrm{Spd}C$ to the set of maps from S to $\mathrm{Gr}_{G,C}$ such that for any geometric point $x = \mathrm{Spa}(\tilde{C}, \tilde{C}^+)$ of S, the corresponding $\mathrm{Spa}(\tilde{C}, \tilde{C}^+)$-point of $\mathrm{Gr}_{G,C}$ lies in the coset

$$G(B_{\mathrm{dR}}^+(\tilde{C})) \cdot \xi^\mu \cdot G(B_{\mathrm{dR}}^+(\tilde{C}))$$

(respectively in the union of cosets labelled by some $\lambda \leq \mu$ in the Bruhat order on $X_*^+(T)$). If the $G(C)$-conjugacy class of μ is defined over some field E with $C/E/F$, then so is $\mathrm{Gr}_{G,\mu}$.

6.2. **Hodge-Tate period map in the PEL setup.** Let $(B, *, V, (\cdot,\cdot), h)$ be a global PEL-datum satisfying assumption 5.5, and $G/\mathbb{Q}$, ν_h, E_0, E determined by it as in section 5. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ over E_0, where $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$ containing E. Choose a maximal torus and a Borel $T \subset B \subset G_{\overline{\mathbb{Q}}_p}$. Let μ be a dominant cocharacter representing the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of ν_h^{-1}.

We consider the B_{dR}^+-affine Grassmannian Gr_G attached to the group $G_{\mathbb{Q}_p}$, considered as a locally spatial diamond over $\mathrm{Spd}E$. The conjugacy class $[\mu]$ determines a Schubert cell $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,E}$. Since μ is minuscule it equals $\mathrm{Gr}_{G,\leq\mu}$ and is proper due to [SW20, 19.2.3]. Here and later on, we will drop the subscript $\mathbb{Q}_p$ from $G_{\mathbb{Q}_p}$ when it is clear from the context that the situation is local at p.

Remark 6.13. Let C/E be a complete algebraically closed nonarchimedean field. Using theorem 3.12 and a Bialynicki-Birula isomorphism [SW20, 19.4.2], we can interpret $\mathrm{Spd}C$-valued points of $\mathrm{Gr}_{G,\mu}$ as parametrizing p-divisible groups over $\mathcal{O}_C$ with additional structures as follows:

Let $\mathcal{F}l_{G,\mu}$ be the analytification of the partial flag variety G/P_μ over E, with P_μ being the maximal parabolic subgroup of G such that for any $g \in P_\mu$, the limit

$$\varinjlim_{t \to 0} \mu(t)^{-1} g \mu(t)$$

exists. The Bialynicki-Birula isomorphism identifies $\mathrm{Gr}_{G,\mu}$ with $\mathcal{F}l_{G,\mu}^\diamond$. Giving a $\mathrm{Spa}C$-point of $\mathcal{F}l_{G,\mu}$ is equivalent to giving a B-equivariant filtration on $V \otimes_{\mathbb{Q}} C$ by a maximal isotropic subspace with respect to the pairing $(\cdot,\cdot)$:

$$W \subset V \otimes_{\mathbb{Q}} C.$$

Now according to theorem 3.12, this filtration, together with the self-dual lattice $\Lambda \subset V_{\mathbb{Q}_p}$ defines a p-divisible group $\mathcal{G}$ with trivialized Tate module $T_p\mathcal{G} \cong \Lambda$. The p-divisible group $\mathcal{G}$ is equipped with a polarization $\mathcal{G} \to \mathcal{G}^*$ coming from $(\cdot,\cdot) : (\Lambda, W) \to (\Lambda^*(1), W^\perp), (t, w) \mapsto ((\cdot, t), (\cdot, w))$, an $\mathcal{O}_B$-endomorphism coming from the $\mathcal{O}_B$-module structures on (Λ, W), and an infinite level structure coming from the trivialization $T_p\mathcal{G} \cong \Lambda$.

Let K^p be a compact open subgroup of $G(\mathbb{A}_f^p)$ and $\mathcal{S}_{K^p}^\circ/\mathrm{Spd}E$ the good reduction locus of the diamond Shimura variety with infinite level at p. The Hodge-Tate period map of [Sch15] and [CS17, 2.1.3] restricted to $\mathcal{S}_{K^p}^\circ$ can be rewritten as below.

Theorem 6.14. *There exists a $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map of diamonds over $\mathrm{Spd}E$*

$$\pi_{HT}^\circ : \mathcal{S}_{K^p}^\circ \to \mathrm{Gr}_G,$$

with image lying in the Schubert cell $\mathrm{Gr}_{G,\mu}$. *It is also equivariant with respect to the natural* $G(\mathbb{A}_f^p)$-*action on the inverse system* $\{\mathcal{S}_{K^p}^{\circ}\}_{K^p}$ *and the trivial action on the target.*

Proof. View $\mathcal{S}_{K^p}^{\circ}$ as the analytic sheafification of the presheaf $\mathcal{S}_{K^p}^{\circ,\mathrm{pre}}$:

$$S = \mathrm{Spa}(R, R^+) \mapsto \{(S^\sharp, \mathfrak{A}, \beta)\},$$

where $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S over $\mathrm{Spa}E$, $\mathfrak{A}$ is a formal abelian scheme with G-structures over $\mathrm{Spf}R^{\sharp+}$, and $\beta \in \underline{\mathrm{Isom}}_G(T_p\mathcal{A}, \underline{\Lambda})(S)$ is a trivialization of the Tate module of the generic fiber $\mathcal{A}$ of $\mathfrak{A}^\diamond$.

Given an S-point $(S^\sharp, \mathfrak{A}, \beta)$ of $\mathcal{S}_{K^p}^{\circ,\mathrm{pre}}$, write T for $T_p\mathcal{A}(S)$. This is a finite projective module over the ring $C^0(\mathrm{Spec}(R^+), \mathbb{Z}_p)$ of continuous $\mathbb{Z}_p$-valued maps on $\mathrm{Spec}(R^+)$. Consider the prismatic Dieudonné module

$$(M := M(\mathfrak{A}[p^\infty]), \varphi_M).$$

By compatibility of M with the crystalline Dieudonné module of $\mathfrak{A}[p^\infty] \times_{R^{\sharp+}} R^{\sharp+}/p$ [SW20, 17.5.2] and étale-crystalline comparison (base changed to $B_{\mathrm{dR}}(R^\sharp)$), we have a natural comparison isomorphism

$$c : T \otimes_{C^0(\mathrm{Spec}(R^+), \mathbb{Z}_p)} B_{\mathrm{dR}}(R^\sharp) \cong M \otimes_{W(R^+)} B_{\mathrm{dR}}(R^\sharp),$$

compatible with the G-structures.

Let $\mathcal{F}$ be the étale sheaf of symplectic similitude $\mathcal{O}_B$-linear trivializations on $X := \mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$

$$\underline{\mathrm{Isom}}_G(M \otimes_{W(R^+)} \mathcal{O}_X, \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X).$$

This is a $G_{\mathbb{Q}_p}$-torsor by [SW20, 21.6.4, 21.6.5], c.f.[RZ96, 3.16]. And it is trivialized over $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$ by the section $\alpha := (\beta \otimes id) \circ c^{-1}$. The pair $(\mathcal{F}, \alpha)$ defines an S-valued point of Gr_G. This induces a map of sheaves

$$\pi_{HT}^\circ : \mathcal{S}_{K^p}^\circ \to \mathrm{Gr}_G.$$

To see that the image lies in $\mathrm{Gr}_{G,\mu}$, we can post-compose with the closed immersion $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}(\Lambda)}$ and assume $S = s := \mathrm{Spa}(C, C^+)$ is a point. The image is determined by the relative position of the $B_{\mathrm{dR}}^+(C^\sharp)$-lattices $M_0 := M \otimes_{W(C^+)} B_{\mathrm{dR}}^+(C^\sharp)$ and $M := T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+(C^\sharp)$, where the latter is trivialized by β. Since (M, φ_M) comes from a p-divisible group, we have $M \subset M_0 \subset \xi^{-1}M$ and the image of M_0 in $\xi^{-1}M/M \cong T \otimes_{\mathbb{Z}_p} C^\sharp(-1)$ agrees with the Lie algebra $\mathrm{Lie}(\mathfrak{A}[p^\infty]) \otimes_{\mathcal{O}_{C^\sharp}} C^\sharp$, c.f. [SW20, 14.8]. Hence the position of M_0 relative to M is measured by some minuscule cocharacter λ whose weight on the standard representation $\mathrm{GL}(\Lambda)$ is $0, -1$.

Assume that the fiber $\mathcal{A}_s$ algebraizes[14] to some abelian variety A_s over $C^\sharp$. Then using the isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$, we can go through a chain of comparison theorems: between p-adic étale and de Rham homologies [Sch13, 1.6], analytic and algebraic de Rham homologies [ABC20, 32.2.2] (first p-adic analytic with algebraic,

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[14]One can use the techniques developed in [Con06] to prove algebraization, but we don't pursue it here. For us, since λ is constant on a connected component of $\mathcal{S}_{K^p}^\circ$, it suffices to know that there always exists one such point.

then algebraic with complex analytic), and de Rham and Betti homologies [15], to get an isomorphism

$$T \otimes C^\sharp \cong H_1(A_{s,\mathbb{C}}, \mathbb{C})$$

that preserves the Hodge filtrations. On the right hand side, $\nu_{h'}$ has weight 1 and 0 for some representative $\nu_{h'}$ corresponding to $A_{s,\mathbb{C}}$ in the $G(\mathbb{C})$-conjugacy class $[\nu_h]$. The weight spaces correspond respectively to weight -1 and 0 subspaces of λ. This shows that $[\lambda] = [\nu_{h^{-1}}] = [\mu]$.

To check Hecke equivariance, by qcqsness of both $\mathcal{S}_{K^p}^\circ$ and $\mathrm{Gr}_{G,\mu}$ and properness of the latter, we may assume $S = s := \mathrm{Spa}C$ is a rank one geometric point. For $g \in G(\mathbb{Q}_p)$, there exists some $N \gg 0$, such that $p^N \Lambda \subset g\Lambda \subset p^{-N}\Lambda$. Denote by K the image of $g\Lambda$ in the quotient

$$\overline{\beta} : p^{-N}\Lambda/p^N\Lambda \xrightarrow{\sim} \mathcal{A}_s[p^{2N}].$$

Then g-action sends $(\mathcal{A}_s, \beta : T_p\mathcal{A}_s \cong \Lambda)$ to

$$(\mathcal{A}_s' := \mathcal{A}_s/K, T_p\mathcal{A}_s' \cong g\Lambda).$$

This agrees with the g-action on $\mathrm{Gr}_{G,\mu}$ which sends a point (Λ, W) as in remark 6.13 to $(g\Lambda, W)$. Away from p, the Hecke action conjugates $\mathcal{S}_{K^p}^\circ$ to $\mathcal{S}_{g^{-1}K^pg}^\circ$ for some $g \in G(\mathbb{A}_f)$. Let K' be $K^p \cap g^{-1}K^pg$, then the composition $\mathcal{S}_{K'}^\circ \to \mathcal{S}_{K^p}^\circ \xrightarrow{\pi_{HT}^\circ} \mathrm{Gr}_G$ is the Hodge-Tate period map on $\mathcal{S}_{K'}^\circ$, similarly for $\mathcal{S}_{g^{-1}K^pg}^\circ$. □

7. STACK OF G-BUNDLES ON THE FARGUES-FONTAINE CURVE

Here we recall the relative Fargues-Fontaine curve and the classifying stack of G-torsors Bun_G following [CS17, 3.2] and [FS21, II.1].

The setup is as in section 6.1: we fix a finite extension $F/\mathbb{Q}_p$, $\pi \in \mathcal{O}_F$ a uniformizer and assume its residue field $\mathbb{F}$ has cardinality q. We write $\mathrm{Perf}_\mathbb{F}$ for the slice category of perfectoid spaces in characteristic p over $\mathrm{Spd}\mathbb{F}$. $W_{\mathcal{O}_F}(\cdot)$ denotes the ramified Witt vectors. For R an $\mathbb{F}$-algebra, $[\cdot] : R \to W_{\mathcal{O}_F}(R)$ is the Teichmüller lift, i.e. the unique multiplicative lift of the identity on R and $\varphi := \varphi_R$ is the Frobenius on $W_{\mathcal{O}_F}(R)$ lifting the q-th power Frobenius on R. We use $V(\cdot)$ to denote the vanishing locus of a function on a topological space.

7.1. **Fargues-Fontaine curve and vector bundles.**

Definition 7.1 (The Fargues-Fontaine curve). For $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_\mathbb{F}$ an affinoid perfectoid with a pseudo-uniformizer $\varpi \in R^+$, the relative Fargues-Fontaine curve over S has the following incarnations:

- (adic space) Denote by Y_S the adic space $\mathrm{Spa}(W_{\mathcal{O}_F}(R^+))\backslash V(\pi \cdot [\varpi])$, then the adic Fargues-Fontaine curve is the quotient $X_S := Y_S/\varphi^\mathbb{Z}$;
- (diamond) The diamond Fargues-Fontaine curve attached to X_S has the formula $X_S^\diamond \cong (S^\diamond \times \mathrm{Spd}F)/\varphi^\mathbb{Z} \times id$;

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[15]Also, use invariance of étale cohomology under algebraically closed field extensions, and a rigid GAGA theorem [Con06, 3.2.6] to identify the Hodge cohomologies.

- (scheme) The line bundle $\mathcal{O}_{Y_S}$ with the linearization $\pi^{-1}\varphi : \mathcal{O}_{Y_S} \cong \mathcal{O}_{Y_S}$ descends to an ample line bundle $\mathcal{O}_{X_S}(1)$ on X_S. Define $\mathcal{O}_{X_S}(n) := \mathcal{O}_{X_S}(1)^{\otimes n}$ and $P := \bigoplus_{n \geq 0} H^0(X_S, \mathcal{O}_{X_S}(n))$, then

$$X_S^{\mathrm{alg}} := \mathrm{Proj}(P),$$

defines the algebraic Fargues-Fontaine curve, with a natural morphism of locally ringed spaces:

$$X_S \to X_S^{\mathrm{alg}}.$$

This globalizes to a construction of a relative Fargues-Fontaine curve X_S (and hence $X_S^{\diamond}$, X_S^{alg}) for a general $S \in \mathrm{Perf}_{\mathbb{F}}$, not necessarily affinoid.

Remark 7.2. Alternatively, if we write $B_{\mathrm{cris},S}^+$ for the crystalline period ring $A_{\mathrm{cris}}(R^+)[1/\pi]$, then[16]

$$P \cong \bigoplus_{d \geq 0} (B_{\mathrm{cris},S}^+)^{\varphi = \pi^d}.$$

Later we will use this relation to attach vector bundles on the Fargues-Fontaine curve to p-divisible groups via their crystalline Dieudonné modules.

Remark 7.3. To introduce a few notation for later use, we denote by $\mathcal{Y}(S)$ the punctured spectrum $\mathrm{Spa}(W(R^+), W(R^+)) \setminus \{[\varpi] = \pi = 0\}$. For $I = [a,b]$ an interval in $[0,\infty]$ with $a, b \in \mathbb{Q} \cup \{\infty\}$, denote by $\mathcal{Y}_I(S)$ the open subspace of $\mathcal{Y}(S)$ where

$$|\pi|^b \leq |[\varpi]| \leq |\pi|^a.$$

In this notation, the Y_S above is $\mathcal{Y}_{(0,\infty)}(S)$ and the space

$$\mathrm{Spa}(W_{\mathcal{O}_F}(R^+), W_{\mathcal{O}_F}(R^+)) \setminus V([\varpi])$$

is $\mathcal{Y}_{[0,\infty)}(S)$.[17] It is proven in [SW20, 11.2.1] that $\mathcal{Y}_{[0,\infty)}(S)$ is an adic space. It is covered by rational subsets of the form $\{|\pi| \leq |[\varpi^{\frac{1}{p^n}}]|\}, n = 1, 2, \ldots$. Each is represented by an affinoid sousperfectoid space[18] $\mathrm{Spa}(R_n, R_n^+)$, where R_n^+ is the $[\varpi]$-adic completion of $W_{\mathcal{O}_F}(R^+)[\pi/[\varpi^{\frac{1}{p^n}}]]$ and R_n is R_n^+ inverting $[\varpi]$. As R_n has a presentation

$$\{\Sigma_{i \geq 0}[r_i] \left(\frac{p}{[\varpi^{1/p^n}]} \right)^i \mid r_i \in R, r_i \to 0\},$$

which depends only on R and not on R^+, the category of vector bundles over $\mathrm{Spa}(R_n, R_n^+)$ and hence that over $\mathcal{Y}_{[0,\infty)}(S)$ (by [KL15, 2.7.7]) is independent of the choice of R^+ in R. This is also stated in [PR21, 2.1.1]. In particular, the category of vector bundles over Y_S, or X_S, is independent of the choice of R^+ in R.

A GAGA type of result holds in relating the adic and algebraic curve:

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[16]This reduces to the case where S is a geometric point, where one can argue as in [Ans19, 10.15], c.f. [FF18, 5.2.9].

[17]In [SW20, 11], Y_S is alternatively denoted $S \dot{\times} \mathrm{Spa} F$. Similarly $\mathrm{Spa}(W_{\mathcal{O}_F}(R^+)) \setminus V([\varpi])$ is denoted $S \dot{\times} \mathrm{Spa} \mathcal{O}_F$. This is to indicate the analogy between taking the Witt vectors and the fiber product construction $S \times_{\mathrm{Spa}\mathbb{F}} \mathrm{Spa}\mathbb{F}((t))$, respectively $S \times_{\mathrm{Spa}\mathbb{F}} \mathrm{Spa}\mathbb{F}[[t]]$ in equal characteristics.

[18]i.e. locally the adic spectrum of a complete Tate F-algebra R that admits a split injection of topological R-modules into a perfectoid Tate ring

Theorem 7.4. *(GAGA,[KL15, 8.7.7], [FS21, II.2.7]) Pulling back along the morphism $X_S \to X_S^{alg}$ induces an equivalence of categories between vector bundles on X_S and X_S^{alg}.*

Cartier divisors on X_S classify Frobenius orbits of untilts of S. More precisely, fix any untilt $S^\sharp$ over F of S. It is locally of the form $\mathrm{Spa}(R^\sharp, R^{\sharp+})$. Each kernel of the surjections $W_{\mathcal{O}_F}(R^+) \cong W_{\mathcal{O}_F}(R^{\sharp\flat+}) \to R^{\sharp+}$ is a principal ideal generated by an element of the form $\pi - a[\varpi]$ for some $a \in W_{\mathcal{O}_F}(R^+)$. The induced maps

$$\mathrm{Spa}(R^\sharp, R^{\sharp+}) \to Y_{\mathrm{Spa}(R,R^+)}$$

glue and define a closed Cartier divisor $S^\sharp \hookrightarrow Y_S$, which maps to a closed Cartier divisor $S^\sharp \hookrightarrow X_S$. It is cut out by a global section of $\mathcal{O}_{X_S}(1)$. Hence by GAGA there is a corresponding global section of $\mathcal{O}(1)$ on X_S^{alg}, which cuts out a closed Cartier divisor $S^{\sharp,alg} \hookrightarrow X_S^{alg}$.

For S being affinoid, the algebraic curve X_S^{alg} is covered by two principal affine charts $X_S^{alg} \backslash V(f_i)$, $i = 1, 2$ for any two linearly independent $f_1, f_2 \in H^0(X_S^{alg}, \mathcal{O}(1))$. In particular if $S^\sharp$ is cut out by $\xi \in H^0(X_S^{alg}, \mathcal{O}(1))$, choose $t \in H^0(X_S^{alg}, \mathcal{O}(1))$ linearly independent to ξ, then $S^{\sharp,alg} \hookrightarrow X_S^{alg}$ is defined by

$$(P[1/t])_0 \twoheadrightarrow R^\sharp,$$

where $(\cdot)_0$ means taking degree zero part of the graded ring. While the completion of $(P[\frac{1}{t}])_0$ along ξ is $B_{\mathrm{dR}}^+(R^\sharp)$. This relation to the de Rham period ring, combined with the Beauville-Laszlo lemma below, leads to an interpretation of the B_{dR}^+-affine Grassmaniann $\mathrm{Gr}_{\mathrm{GL}_n}$ as parametrizing modifications of the trivial rank n bundle on X_S.

Lemma 7.5. *(Beauville-Laszlo) Let R be a commutative ring, $f \in R$ a non-zero divisor and $\hat{R} := \varprojlim_n R/f^n$ is the f-adic completion of R. The category of R-modules in which f is a non-zero-divisor is equivalent to the category of triples $(M_1, M_2, \alpha : M_1[\frac{1}{f}] \to M_2 \otimes_R \hat{R})$, where M_1 is an $\hat{R}$-module in which f is a non-zero-divisor, M_2 is an $R[\frac{1}{f}]$-module and α is an isomorphism.*

Remark 7.6. It seems that following the argument in [BBT13] and using adic spaces in place of Berkovich spaces, this equivalence globalizes and generalizes to higher codimensional situations: for a scheme X and a closed subscheme Y, denote the completion of X along Y by $\hat{X}$ and name the maps $j : U := X \backslash Y \hookrightarrow X$, $i : \hat{X} \to X$. Then there is an equivalence between the category of coherent sheaves on X, and the category of triples $(\mathcal{F}_Y, \mathcal{F}_U, \alpha : \mathcal{F}_Y \to i^*j_*\mathcal{F}_U)$, where $\mathcal{F}_Y$ is a coherent sheaf on Y, $\mathcal{F}_U$ a coherent sheaf on U, and α an isomorphism. This has been worked out by Robin Louis' Bonn master thesis, under the assumption that X is locally noetherian.

Now for any $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}}$, an S-point of the B_{dR}^+-affine Grassmannian for GL_n/F amounts to a triple of an untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ over F, a rank n vector bundle $\mathcal{F}$ over $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$, and a trivialization of $\mathcal{F}$ over $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$. Via Beauville-Laszlo, this triple defines a new bundle on X_S^{alg} by glueing the trivial rank n bundle on $X_S^{alg} \backslash \mathrm{Spec}(R^\sharp)$ and $\mathcal{F}$, along the trivialization. This corresponds to a rank n vector bundle on the adic curve X_S by GAGA.

7.2. **The stack of G-bundles.** Let G be a reductive group over F and X be a F-scheme or a sousperfectoid space over $\mathrm{Spa}F$. Denote by $\mathrm{Rep}_F(G)$ the exact symmetric monoidal category of finite dimensional algebraic F-representations of G and by $\mathrm{Bun}(X)$ that of vector bundles on X.

Definition/Proposition 7.7. ([SW20, 19.5.1, 19.5.2]) A G-bundle (or a Tannakian G-torsor) on X is an exact tensor functor

$$\mathrm{Rep}_F(G) \to \mathrm{Bun}(X).$$

Equivalently this is an étale sheaf on X with an action of G that is étale locally isomorphic to G.

The relative Fargues-Fontaine curve X_S for a perfectoid space $S \in \mathrm{Perf}_{\mathbb{F}}$ is sousperfectoid by [SW20, Proof of 11.2.1]. Hence one can apply the above and talk about G-torsors on X_S in the sense of exact tensor functors. On the other hand, post-composing with the exact tensor equivalence $\mathrm{Bun}(X_S^{\mathrm{alg}}) \cong \mathrm{Bun}(X_S)$, one sees that GAGA extends to an equivalence between categories of G-torsors on X_S and X_S^{alg}.

Definition/Proposition 7.8. ([FS21, II.2.1,III.1.2,III.1.3]) The pre-stack on $\mathrm{Perf}_{\mathbb{F}}$ sending a perfectoid space $S \in \mathrm{Perf}_{\mathbb{F}}$ to the groupoid of G-torsors on X_S is a small v-stack, denoted by Bun_G.

Using Tannakian G-torsors, the interpretation of the B_{dR}^+-affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ as parametrizing modifications of the trivial rank n vector bundle generalizes to any other reductive group G/F:

For $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}}$ with an untilt $S^\sharp$ over F, viewed as a closed Cartier divisor on X_S, the equivalence between $\mathrm{Bun}(X_S^{\mathrm{alg}})$ and the 2-fiber product

$$\mathrm{Bun}(X_S^{\mathrm{alg}} \backslash S^{\sharp,\mathrm{alg}}) \times_{\mathrm{Bun}(\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp)))} \mathrm{Bun}(\mathrm{Spf}(B_{\mathrm{dR}}^+(R^\sharp)))$$

is exact and symmetric monoidal. Hence there is an equivalence between categories of exact tensor functors from $\mathrm{Rep}_F(G)$ to one and to the other.

Therefore given an S-valued point of Gr_G over $\mathrm{Spd}F$, i.e. a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a G-torsor over $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$ and α is a trivialization of it over $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$, one can glue the trivial G-torsor on $X_S^{\mathrm{alg}} \backslash S^{\sharp,\mathrm{alg}}$ with $\mathcal{F}$ via α to get new G-torsor on X_S^{alg}. This defines the "Beauville-Laszlo uniformization" morphism between small v-stacks:

$$BL : \mathrm{Gr}_G \to \mathrm{Bun}_G.$$

Remark 7.9. In the above interpretation of Gr_G as a moduli space of modifications of G-torsors, the initial G-torsor to modify can be any G-torsor, not necessarily the trivial one.

We have the following result.

Proposition 7.10. *([FS21, III.3.1]) The "Beauville-Laszlo" morphism is surjective as a map of pro-étale-stacks.*

7.3. **Stratification.** Let $F, \mathcal{O}_F$ and G over F be as before. Following [FS21], [Ans22a], we review the Newton (or Harder-Narasimhan) stratification of Bun_G which is labelled by the Kottwitz set $B(G)$. The original study of the Kottwitz set in the setup of isocrystals is due to Kottwitz [Kot85], c.f. [RR96].

Fix an algebraically closed field k containing the residue field $\mathbb{F}$ of $\mathcal{O}_F$. Let L be the fraction field of $W_{\mathcal{O}_F}(k)$ and σ be the Frobenius on L. Fix an algebraic closure $\bar{F}$ of F containing L.

Definition 7.11. [Ans22a, 5.2] The Kottwitz category $\mathcal{B}(G)$ is the groupoid whose objects are elements in $G(L)$ and the set of isomorphisms between $b, b' \in G(L)$ is
$$\{c \in G(L) \mid cb\sigma(c)^{-1} = b'\}.$$
Composition of morphisms is defined by multiplication in $G(L)$. The *Kottwitz set* $B(G)$ is the set of isomorphism classes of objects in this category. This is in bijection to the set of σ-conjugacy classes in $G(L)$.

Remark 7.12. According to Kottwitz [Kot85], $B(G)$ is invariant under passing to algebraically closed extensions of k.

For each perfectoid space S over k, the pullback $\tilde{\mathscr{E}}_1$ of the trivial G-bundle $\mathscr{E}_1$ on X_S to Y_S is equipped with a natural descent datum
$$\alpha : \varphi_S^* \tilde{\mathscr{E}}_1 \xrightarrow{\sim} \tilde{\mathscr{E}}_1.$$
Twist α with the automorphism $b \in G(L) \subset \mathrm{Aut}(\tilde{\mathscr{E}}_1)$. The descent datum $(\mathscr{E}_1, b^{-1}\alpha\varphi^* b)$ gives rise to a new G-bundle $\mathscr{E}_b$ on X_S. This assignment
$$b \mapsto \mathscr{E}_b \in \mathrm{Bun}_G(S)$$
is functorial with respect to pullback along maps $S' \to S$ over $\mathrm{Spd}k$. In this way we obtain a functor
$$\mathcal{B}(G) \to \mathrm{Bun}_G(\mathrm{Spd}k),$$
where the target category is understood by v-descent of G-torsors on the Fargues-Fontaine curve, namely, by taking any v-cover of $\mathrm{Spd}k$ by a perfectoid space S and considering the category of G-torsors on X_S with descent data. We have the following theorem of Anschütz:

Theorem 7.13. *([Ans22a, 5.3]) The functor $\mathcal{B}(G) \to \mathrm{Bun}_G(\mathrm{Spd}k)$ is an equivalence of categories.*

The Kottwitz set $B(G)$ can be endowed with a partial order recording the degeneration relations of G-isocrystals in families. Equip it with the topology defined by the opposite of this partial order. Then a result of Viehmann shows that the above equivalence is compatible with the topologies on the set of objects on both sides.

More precisely, fix $T \subset B \subset G_{\bar{F}}$, where T is a maximal torus and B is a Borel. Let $X_*(T)$ be the cocharacter lattice of T. It has an action by the Weyl group W and the absolute Galois group Γ of F. We write $X_*(T)^\Gamma$ for the Galois invariants. Denote by $\pi_1(G)$ the algebraic fundamental group of G, i.e. the quotient of $X_*(T)$ by the lattice generated by the coroots. It is also equipped with a Γ-action and we write $\pi_1(G)_\Gamma$ for the Γ-coinvariants. Then Kottwitz defined the *Newton map*
$$\nu_G : B(G) \to (X_*(T)_{\mathbb{Q}}/W)^\Gamma$$

and the *Kottwitz map*

$$\kappa_G : B(G) \to \pi_1(G)_\Gamma$$

which satisfy certain characterizing properties, c.f. [RR96, 1.8, 1.15]. In particular

$$\nu_G \times \kappa_G : B(G) \to (X_*(T)_{\mathbb{Q}}/W)^\Gamma \times \pi_1(G)_\Gamma$$

is injective. Using this, for $[b]$ and $[b'] \in B(G)$, we say $[b] \leq [b']$ if $\kappa_G([b]) = \kappa_G([b'])$ and $\nu_G([b]) \leq \nu_G([b'])$ in the Bruhat order, i.e. choosing a dominant cocharacter to represent the W-orbit of each, then the difference $\nu_G([b']) - \nu_G([b])$ is a sum of positive coroots with non-negative coefficients. Equip $B(G)$ with the topology such that $\{[b]\} \in \overline{\{[b']\}}$ if and only if $[b] \geq [b']$. We have

Theorem 7.14. *([Vie21, 1.1]) The equivalence in 7.13 induces a homeomorphism*

$$B(G) \cong |\mathrm{Bun}_G|.$$

Now we can define locally closed substacks of $\mathrm{Bun}_{G,\mathrm{Spd}k}$.

Definition/Proposition 7.15. [FS21, III.0.2(v), III.5.3] For any $[b] \in B(G)$, define the substack Bun_G^b of $\mathrm{Bun}_{G,\mathrm{Spd}k}$ to be

$$\mathrm{Bun}_G \times_{|\mathrm{Bun}_G|} \{[b]\}.$$

It can be identified with the classifying stack of $\tilde{G}_b$-torsors, for the v-sheaf of groups

$$\tilde{G}_b : S \mapsto \mathrm{Aut}_{X_S}(\mathscr{E}_b).$$

Remark 7.16. If the element $[b] \in B(G)$ is *basic*, i.e. maximal under generalization, then the group $\tilde{G}_b$ agrees with the v-sheaf attached to the locally profinite group $G_b(F)$, where G_b is an inner form of G defined by

$$G_b(R) = \{g \in G(L \otimes_F R) \mid g = b\sigma(g)b^{-1}\},$$

for any F-algebra R.

Let us describe the image of the Beauville-Laszlo map on a Schubert cell $\mathrm{Gr}_{G,\mu}$ for some dominant cocharacter μ, under the homeomorphism $|\mathrm{Bun}_G| \cong B(G)$.

Note that μ defines an element $\bar{\mu} \in (X_*(T)_{\mathbb{Q}}/W)^\Gamma$ by averaging its Galois conjugates, i.e.

$$\bar{\mu} := \frac{1}{[E' : F]} \sum_{\gamma \in \mathrm{Gal}(E'/F)} \gamma(\mu)$$

for a large enough Galois extension E'/F over which μ is defined. Also, let $\mu^\flat$ be the image of μ in $\pi_1(G)_\Gamma$.

Definition 7.17. The subset $B(G, \mu) \subset B(G)$ of μ-admissible elements is

$$\{[b] \in B(G) \mid \nu_G([b]) \leq \bar{\mu}, \kappa([b]) = \mu^\flat\}.$$

Proposition 7.18. *The map of topological spaces*

$$|\mathrm{Gr}_{G,\mu}| \xrightarrow{|BL|} |\mathrm{Bun}_G| \to B(G)$$

has image $B(G, \mu)$.

Proof. The image lies in $B(G, \mu)$ is [CS17, 3.5.3], except that our reductive group is defined over a finite extension of $F/\mathbb{Q}_p$ and that our convention on the Cartan decomposition on Gr_G differs from theirs by a minus sign. Note that the proof of their lemma 3.5.4, 3.5.5 applies to our situation upon replacing the Fargues-Fontaine curve considered there by the one attached to the ramified Witt vectors for F, and the different sign convention eliminates the minus sign on μ^{-1} from their statement. The surjectivity is due to Rapoport [Rap18, A.9], c.f. [CS17, remark 3.5.8]. □

8. A PEL-type Igusa stack and the rational conjecture

Let us go back to the global PEL setup as in section 5. Fix the level subgroup $K^p \subset G(\mathbb{A}_f^p)$. Let $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ and $K = K_p K^p$. Let $E/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ be as before, which is the p-adic field over which our adic Shimura variety is defined. S_K over $\mathcal{O}_E$ is the schematic Shimura variety at level K. Let Bun_G and Gr_G be those for the group $G_{\mathbb{Q}_p}$. We construct the PEL type Igusa stack at level K^p over $\mathrm{Spd}\mathbb{F}_q$ and discuss part (1) of conjecture 1.1 on the good reduction locus. The word "rational" in the title is in contrast to the integral model in section 11.

8.1. **Construction of the Igusa stack.**

Definition 8.1. Equip the slice category $\mathrm{Perf}/\mathrm{Spd}\mathbb{F}_q$ with the v-topology. Let $\mathrm{Igs} := \mathrm{Igs}^{\circ}_{K^p}$ be the stackification of $\mathrm{Igs}^{\mathrm{pre}}$, the category fibered in groupoids over $\mathrm{Perf}/\mathrm{Spd}\mathbb{F}_q$ determined by:

$$T = \mathrm{Spa}(R, R^+) \mapsto \mathrm{Igs}^{\mathrm{pre}}(T),$$

where objects in $\mathrm{Igs}^{\mathrm{pre}}(T)$ are quadruples $(A_0, \iota, \lambda, \bar{\eta})$ of abelian schemes with G-structure at level K over R^+/ϖ, or R^+/ϖ-points of S_K (where ϖ is any pseudo-uniformizer of R^+). Isomorphisms between two objects $\mathcal{A}_0 = (A_0, \iota, \lambda, \bar{\eta}), \mathcal{A}_0' = (A_0', \iota', \lambda', \bar{\eta}')$ are quasi-isogenies preserving the G-structures, i.e.

$\mathrm{Hom}_{\mathrm{Igs}^{\mathrm{pre}}(T)}(\mathcal{A}_0, \mathcal{A}_0') =$

$$\left\{ \rho \in (\mathrm{Hom}_{R^+/\varpi}(A_0, A_0') \otimes \mathbb{Q})^{\times} \middle| \begin{array}{l} \rho \circ \iota(b) = \iota'(b) \circ \rho, \text{ for any } b \in \mathcal{O}_B \\ \rho^{\vee} \circ \lambda' \circ \rho = c\lambda, \text{ for} \\ \text{some } c \in \underline{\mathbb{Q}^{\times}}(\mathrm{Spec}(R^+/\varpi)) \\ \bar{\eta} = \bar{\eta}' \circ \rho_* \end{array} \right\}.$$

Remark 8.2. For a different choice of pseudo-uniformizer $\varpi' \in R^+$ (without loss of generality $\varpi \in \varpi' R^+$), the base change along $R^+/\varpi \twoheadrightarrow R^+/\varpi'$ induces an equivalence between $\mathrm{Igs}^{\mathrm{pre}}_{\varpi}(T)$ and $\mathrm{Igs}^{\mathrm{pre}}_{\varpi'}(T)$ by Serre-Tate lifting, see 4.1. So the functoriality of $\mathrm{Igs}^{\mathrm{pre}}$ is ensured by composing with this equivalence, even if a map $\mathrm{Spa}(R_1, R_1^+) \to \mathrm{Spa}(R_2, R_2^+)$ doesn't necessarily map the chosen pseudo-uniformizer of R_2^+ to the one of R_1^+.

Using the moduli interpretation of the good reduction locus $\mathcal{S}_K^{\circ}$, we get immediately the following:

Proposition 8.3. *Sending an isomorphism class of abelian schemes with G-structure to its reduction (modulo a pseudo-uniformizer on the base) defines a map of v-stacks*

$$\text{red} : \mathcal{S}_K^\circ \to \text{Igs}.$$

Construction. View $\mathcal{S}_K^\circ$ as the sheafification of the presheaf on $\text{Perf}_{\mathbb{F}_q}$

$$\mathcal{S}_K^{\circ,\text{pre}} : S = \text{Spa}(R, R^+) \mapsto \{(S^\sharp, \mathscr{S}_K(\text{Spf}R^{\sharp+}))\},$$

where $S^\sharp = \text{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S over E.

Choose a pseudo-uniformizer $\varpi \in R^+$. An S-point of $\mathcal{S}_K^{\circ,\text{pre}}$ gives a formal abelian scheme $\mathfrak{A}/\text{Spf}R^{\sharp+}$ with G-structure at level K. As $R^+/\varpi \cong R^{\sharp+\flat}/\varpi \cong R^{\sharp+}/\varpi^\sharp$, $\mathcal{A}_0 := \mathfrak{A} \times_{R^{\sharp+}} R^{\sharp+}/\varpi^\sharp$ is an object of $\text{Igs}^{\text{pre}}(S)$. Sending $(S^\sharp, \mathfrak{A})$ to $\mathcal{A}_0$ defines a map $\mathcal{S}_K^{\circ,\text{pre}} \to \text{Igs}^{\text{pre}}$, and hence induces the desired map. $\qquad\square$

Remark 8.4. By allowing the untilt $S^\sharp$ to lie over $\text{Spa}\mathcal{O}_E$, the construction above extends to a map on the v-sheaf integral model

$$\text{red} : \mathscr{S}_K^\diamond \to \text{Igs}.$$

Proposition 8.5. *There is a morphism of v-stacks*

$$\overline{\pi}_{HT}^\circ : \text{Igs} \to \text{Bun}_G.$$

(The notation is justified by 8.13.)

Construction. It again suffices to construct the map on Igs^{pre}. As Bun_G is a v-stack, this will necessarily factor through the v-stackification and give the desired map.

For $S = \text{Spa}(R, R^+) \in \text{Perf}$ with a chosen pseudo-uniformizer $\varpi \in R^+$, $\text{Igs}^{\text{pre}}(S)$ is the groupoid of quadruples $\mathcal{A}_0 := (A_0, \iota, \lambda, \bar\eta)$, where A_0 is an abelian scheme over R^+/ϖ; ι is a morphism $\mathcal{O}_B \to \text{End}(A_0) \otimes \mathbb{Z}_{(p)}$; λ is a polarization $A_0 \to A_0^\vee$ whose degree is prime to p and $\bar\eta$ is a K^p-level structure, with morphisms being quasi-isogenies compatible with the G-structures. The p-divisible group of $\mathcal{A}_0$ is defined up to isogeny, and is equipped with $\mathcal{O}_B$-endomorphism and polarization induced by those on A_0. Its rational crystalline Dieudonné module $M[p^{-1}] := M_{crys}(\mathcal{A}_0[p^\infty])[p^{-1}]$ is a well-defined (up to unique isomorphism) finite projective $B_{\text{cris}}^+(R^+/\varpi)$-module. This is equipped with a G-structure, i.e. an $B \otimes_{\mathbb{Q}} B_{\text{cris}}^+(R^+/\varpi)$-module structure and a symplectic pairing, by full-faithfulness of the crystalline Dieudonné module functor [SW13, Theorem A]. Now the graded module $\bigoplus_{d \geq 0}(M[1/p])^{\varphi = p^{d+1}}$ defines a vector bundle $\mathscr{E}(\mathcal{A}_0)$ with G-structure on X_S, using the description of the algebraic curve remarked in 7.2, as well as GAGA theorem for the curve.

Consider the sheaf on X_S of trivializations of $\mathscr{E}(\mathcal{A}_0)$ as a symplectic similitude $\mathcal{O}_B \otimes \mathcal{O}_{X_S}$-module

$$\underline{\text{Isom}}_G(\mathscr{E}(\mathcal{A}_0), \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S}).$$

We claim that this is a G-torsor and hence an object in $\text{Bun}_G(S)$. For this we first give a different description of $\mathscr{E}(\mathcal{A}_0)$ in terms of lifts.

Lemma 8.6. *Let $S^\sharp = \text{Spa}(R^\sharp, R^{\sharp+})$ be any untilt of $S = \text{Spa}(R, R^+)$ over $\text{Spa}E$. Assume $\mathfrak{A}$ is a formal abelian scheme (without G-structures) over $\text{Spf}R^{\sharp+}$ such that*

$$\mathfrak{A} \times_{R^{\sharp+}} R^+/\varpi \cong A_0.$$

Denote the prismatic Dieudonné module of its p-divisible group by $(\tilde{M}, \varphi_{\tilde{M}})$. Then the restriction of $(\tilde{M}, \varphi_{\tilde{M}})$ to $\mathcal{Y}_{[r,\infty)}(S)$ for $r \gg 0$ descends to a vector bundle $\mathscr{E}$ on X_S that is naturally (with respect to isomorphisms $\mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$ and base change in S) isomorphic to $\mathscr{E}(\mathcal{A}_0)$.

Proof. Let $\mathcal{A}$ be the adic generic fiber of $\mathfrak{A}$ and $T_p\mathcal{A}$ be its Tate module, considered as a $\underline{\mathbb{Z}}_p$-local system on S. For any $f \in T_p\mathcal{A}(S)$, by viewing it of a homomorphism $\mathbb{Q}_p/\mathbb{Z}_p \to \mathfrak{A}[p^\infty]$ as p-divisible groups over $\mathrm{Spf}R^{\sharp+}$, one gets a map between the covariant Dieudonné modules $W(R^+) \cong M_{\mathbb{\Delta}}(\mathbb{Q}_p/\mathbb{Z}_p) \to \tilde{M}$. This gives rise to a natural Frobenius equivariant evaluation map

$$T_p\mathcal{A}(S) \otimes W(R^+) \to \tilde{M},$$

where the tensor is over the global sections of the sheaf $\underline{\mathbb{Z}}_p$ on S. Base change to $A_{cris}(R^+)$, one has a similar map. Upon identifying $\tilde{M} \otimes_{W(R^+)} A_{cris}(R^+)$ with the crystalline Dieudonné module of A_0 (c.f.[SW20, 17.5.2]), we see that both $\mathscr{E}$ and $\mathscr{E}(\mathcal{A}_0)$ are naturally the modification of $T_p\mathcal{A} \otimes \mathcal{O}_{X_S}$ by the $B_{\mathrm{dR}}^+(R^\sharp)$-lattice $\tilde{M} \otimes_{W(R^+)} B_{\mathrm{dR}}^+(R^\sharp)$ along the de Rham comparison isomorphism

$$T_p\mathcal{A}(S) \otimes B_{\mathrm{dR}}(R^\sharp) \cong \tilde{M} \otimes_{W(R^+)} B_{\mathrm{dR}}(R^\sharp).$$

Hence we have an identification $\mathscr{E} \cong \mathscr{E}(\mathcal{A}_0)$ depending only on the isomorphism $\mathfrak{A} \times_{R^{\sharp+}} R^+/\varpi \cong A_0$ and is functorial with respect to isomorphisms between lifts and base change in S. □

Remark 8.7. From the proof we see that $\mathscr{E}(\mathcal{A}_0)$ sits in the short exact sequence

$$0 \to T_p\mathcal{A} \otimes_{\underline{\mathbb{Z}}_p} \mathcal{O}_{X_S} \to \mathscr{E}(\mathcal{A}_0) \to i_*Lie(\mathcal{A}) \to 0,$$

where $i : S^\sharp \to X_S$ is defined by the closed immersion of the Cartier divisor $S^\sharp$.

Construction continued. Choose an arbitrary untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ over $\mathrm{Spa}E$. Using the formal smoothness of the Shimura variety $\mathscr{S}_K$, a formal abelian scheme with G-structure $\mathfrak{A}$ over $R^{\sharp+}$ lifting $\mathcal{A}_0$ compatible with the G-structures exists. We can now apply the description of $\mathscr{E}(\mathcal{A}_0)$ in the lemma above. In particular, since both $T_p\mathcal{A} \otimes_{\underline{\mathbb{Z}}_p} \mathcal{O}_{X_S}$ and the completion of $\mathscr{E}(A_0)$ at $S^\sharp$ are étale locally on X_S isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S}$, resp. its completion at $S^\sharp$, as symplectic similitude $\mathcal{O}_B \otimes \mathcal{O}_{X_S}$-modules, the sheaf

$$\underline{\mathrm{Isom}}_G(\mathscr{E}(\mathcal{A}_0), \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S})$$

is an étale G-torsor on X_S. (For the statement about the completion at $S^\sharp$, note that $i^*\mathscr{E}(\mathcal{A}_0)$ is the Lie algebra of the universal extension of $\mathfrak{A}[p^\infty]$ with p inverted. By [RZ96, 3.23 c] c.f. [SW20, 21.6.5], the Kottwitz condition ensures that $i^*\mathscr{E}(\mathcal{A}_0)$ is étale locally on $\mathrm{Spec}(R^\sharp)$ isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}$ as polarized $\mathcal{O}_B \otimes \mathcal{O}$-modules. This property lifts to $B_{\mathrm{dR}}^+(R^\sharp)$, which is complete along (ξ).)

On the other hand, again by full-faithfulness of the Dieudonné module functor, any quasi-isogeny $\mathcal{A}_0 \to \mathcal{A}_0'$ over R^+/ϖ that is compatible with the G-structures, induces an isomorphism of $M[p^{-1}] \cong M'[p^{-1}]$, also compatible with the G-structures, and hence an isomorphism of the associated G-bundles $\mathscr{E}(\mathcal{A}_0) \cong \mathscr{E}(\mathcal{A}_0')$. This is a morphism in the groupoid $\mathrm{Bun}_G(S)$. Everything is functorial in S and hence the above defines a morphism $\mathrm{Igs}^{\mathrm{pre}} \to \mathrm{Bun}_G$. □

Remark 8.8. Here and later we no longer distinguish the vector bundle $\mathscr{E}(\mathcal{A}_0)$ with G-structure and its attached G-torsor. To get back $\mathscr{E}(\mathcal{A}_0)$ from its attached G-torsor, one can perform a pushout along the standard representation of $G_{\mathbb{Q}_p}$ on $V_{\mathbb{Q}_p}$.

We can describe the image of the above map under the homeomorphism from 7.14

$$|\mathrm{Bun}_G| \cong B(G).$$

Proposition 8.9. *The image of*

$$|\mathrm{Igs}| \to |\mathrm{Bun}_G| \cong B(G)$$

is the subset $B(G, \mu)$ of μ-admissible elements.

Proof. This follows from 7.18 and commutativity of the diagram in 8.13 below. Note that π_{HT}° is surjective on topological spaces, because this is known for the Hodge-Tate map π_{HT}^* on the minimal compactification[Sch15, IV.1.1(i)] while the good reduction locus of any fiber of π_{HT}^* is non-empty. $\qquad\square$

8.2. **A fiber product formula.** We first record two lemmas.

Lemma 8.10. *Let V be a valuation ring with fraction field K. Let A be an abelian scheme over V and A_K its generic fiber. Assume $j : G_K \to A_K$ is a finite sub-group scheme. Then there exists a finite sub-group scheme G of A, flat over V, such that its generic fiber agrees with G_K.*

Proof. The map j is quasi-compact. Let Z be the schematic image of j, i.e. defined by the quasi-coherent ideal $\mathcal{I} := \ker(\mathcal{O}_A \to j_*\mathcal{O}_{G_K})$. We use $\mathcal{O}_Z$ to denote the quotient $\mathcal{O}_A/\mathcal{I}$, considered as an $\mathcal{O}_A$-module. We have

$$\mathcal{O}_A \twoheadrightarrow \mathcal{O}_Z \hookrightarrow j_*\mathcal{O}_{G_K}.$$

Since $j_*\mathcal{O}_{G_K}$ is torsionfree, so is the submodule $\mathcal{O}_Z$, which implies that it is flat over V (as V is a valuation ring). It suffices to show that Z can be endowed with a group scheme structure, or equivalently it is equipped with morphisms $m_Z : Z \times_{\mathrm{Spec}(V)} Z \to Z$ (multiplication), $i_Z : Z \to Z$ (inverse) and $e_Z : \mathrm{Spec}(V) \to Z$ (identity section), satisfying the group axioms.

For this, note that $\mathcal{O}_Z(Z)$ is finitely generated over a valuation ring, so it is a projective V-module.[19] Hence the surjection $\mathcal{O}_A \to \mathcal{O}_Z$ splits as a V-module homomorphism. Define m_Z^* to be the composition

$$\mathcal{O}_Z \to \mathcal{O}_A \xrightarrow{m_A^*} m_*\mathcal{O}_{A\times A} \twoheadrightarrow m_*\mathcal{O}_{Z\times Z}.$$

This map, a priori a V-module homomorphism, is in fact a V-algebra homomorphism, because of the commutativity of the following diagram:

$$
\begin{array}{ccccc}
\mathcal{O}_A & \overset{\longleftarrow}{\longrightarrow} & \mathcal{O}_Z & \hookrightarrow & j_*\mathcal{O}_{G_K} \\
\downarrow{\scriptstyle m^*} & & \downarrow{\scriptstyle m_Z^*} & & \downarrow{\scriptstyle m^*} \\
\mathcal{O}_{A\times A} & \longrightarrow & \mathcal{O}_{Z\times Z} & \hookrightarrow & (j\times j)_*\mathcal{O}_{G_K\times G_K}.
\end{array}
$$

---

[19]c.f. the last sentence of [Cou07], definition 2 and proposition 4(iii) of [Hir78]. Note that 4(iii) is easy to prove by choosing M to be C in definition 2.

Namely, when postcomposed with the injection $\mathcal{O}_{Z \times Z} \hookrightarrow (j \times j)_* \mathcal{O}_{G_K \times G_K}$, m_Z^* agrees with $\mathcal{O}_Z \hookrightarrow j_* \mathcal{O}_{G_K} \xrightarrow{m^*} (j \times j)_* \mathcal{O}_{G_K \times G_K}$, which is a V-algebra homomorphism. This defines the multiplication morphpism.

The inverse map on A preserves G_K and hence also preserves its schematic image. Therefore we can restrict the inverse map on A to Z to get i_Z.

The identity section on G_K extends to a section $e_Z : \mathrm{Spec}(V) \to Z$ by properness of Z over V. By uniqueness, it is the same as the identity section of A.

Now since m_Z, i_Z, e_Z are the restriction of the corresponding morphisms on A, they satisfy the desired group axioms. This finishes the proof of the lemma. $\qquad\square$

Lemma 8.11. *Let S be an $\mathcal{O}_E$-scheme, $\mathcal{A} = (A/S, \iota_A, \lambda_A, \bar{\eta})$ an abelian scheme with G-structure at level K, i.e. an S-point of the Shimura variety S_K, and $\mathcal{H} = (\mathcal{H}/S, \iota_H, \lambda_H)$ a p-divisible group with G-structure. Assume*

$$\rho : \mathcal{A}[p^\infty] \to \mathcal{H}$$

is an $\mathcal{O}_B$-linear isogeny preserving the polarization up to a scalar in $\underline{\mathbb{Q}}_p^\times(S)$. Then $A' := A/\ker\rho$ can be uniquely (up to isomorphism) promoted to $(A', \iota_A', \lambda_A', \bar{\eta}')$, an abelian scheme with G-structure at level K, such that the induced map

$$\rho' : A'[p^\infty] \to \mathcal{H}$$

is an isomorphism of p-divisible groups with G-structure and that the quotient map $\pi : A \to A'$ is a G-isogeny preserving the K^p-levels.

Proof. Clearly A' inherits a K^p-level structure $\bar{\eta}$ from $\mathcal{A}$. The condition that π preserves the G- and level structures forces $\bar{\eta}'$ to be $\bar{\eta}$, ι_A' to be $\pi \iota_A \pi^{-1}$ and λ_A' to be $(\pi^\vee)^{-1}(d \cdot \lambda_A)\pi^{-1}$ for some $d \in \underline{\mathbb{Q}}^\times(S)$. Also, for each $b \in \mathcal{O}_B$, $\iota_A'(b) := \pi \iota_A(b)\pi^{-1}$ is indeed prime-to-p, because on the p-divisible groups $\iota_A'(b)[p^\infty] = \iota_H(b)$ is an isomorphism. To fix d, assume $c \in \underline{\mathbb{Q}}_p^\times(S)$ is a constant such that

$$c \cdot \lambda_A[p^\infty] = \rho^\vee \circ \lambda_H \circ \rho.$$

Then ρ' preserves the polarization up to $c \cdot d^{-1} \in \underline{\mathbb{Q}}_p^\times(S)$. The condition that ρ' is an isomorphism of p-divisible groups with G-structure requires this section to be in $\underline{\mathbb{Z}}_p^\times(S)$. Let $v_p : \underline{\mathbb{Q}}_p \to \underline{\mathbb{Z}}$ be the p-adic valuation. We must then require $v_p(d) = v_p(c)$ as sections of $\underline{\mathbb{Z}}(S)$. This fixes d up to a unit in $\underline{\mathbb{Z}}_{(p)}(S)$, which shows uniqueness. It also ensures that λ_A' is prime-to-p, as the induced polarization on H is principal. Hence $(A', \iota_A', \lambda_A', \bar{\eta}')$ is indeed an abelian scheme with G-structure at level K. $\qquad\square$

Remark 8.12. If ρ is only a quasi-isogeny, assume that $p^N\rho$ is an actual isogeny for some $N \gg 0$. Then

$$\mathcal{A}' := (A/\ker(p^N\rho), \pi\iota_A\pi^{-1}, (\pi^\vee)^{-1}(c^{-1} \cdot \lambda_A)\pi^{-1}, \bar{\eta})$$

is an abelian scheme with G-structure at level K by the same reasoning as above. Note that even though p^N, and hence $p^N\rho$, itself might not preserve the G-structures, the conjugation $p^{-N}\rho p^N$ does.

Let $\mathcal{S}^{\circ}_{K^p}$ over $\mathrm{Spd}E$ be the good reduction locus with infinite level at p as before. It maps to Igs by composing the projection $\mathcal{S}^{\circ}_{K^p} \to \mathcal{S}^{\circ}_K$ with the reduction map $\mathcal{S}^{\circ}_K \to \mathrm{Igs}$. Let π°_{HT} be the Hodge-Tate map on it as in section 6.2. The following is the main result of this section.

Theorem 8.13. *The diagram of small v-stacks on* $\mathrm{Perf}_{\mathbb{F}_q}$

$$
\begin{array}{ccc}
\mathcal{S}^{\circ}_{K^p} & \xrightarrow{\ \pi^{\circ}_{HT}\ } & \mathrm{Gr}_{G,\mu} \\
\downarrow{\scriptstyle \mathrm{red}} & & \downarrow{\scriptstyle BL} \\
\mathrm{Igs} & \xrightarrow{\ \overline{\pi}^{\circ}_{HT}\ } & \mathrm{Bun}_G
\end{array}
$$

is 2-cartesian.

Proof. It is convenient to observe that the stackification procedure in defining Igs will not cause much trouble: since stackification commutes with 2-fiber product [Sta23, 04Y1], if we can show $\mathcal{S}^{\circ}_{K^p}$ is the v-stackification of $\mathrm{Igs}^{\mathrm{pre}} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}$, we also have $\mathcal{S}^{\circ}_{K^p} \cong \mathrm{Igs} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}$. Hence it suffices to show

$$\mathcal{S}^{\circ,\mathrm{pre}}_{K^p} \cong \mathrm{Igs}^{\mathrm{pre}} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}$$

on a basis of the v-topology, c.f. 5.21 for the definition of $\mathcal{S}^{\circ,\mathrm{pre}}_{K^p}$.

We quickly check 2-commutativity. For any $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$, an S-point of $\mathcal{S}^{\circ,\mathrm{pre}}_{K^p}$ is a tuple consisting of an untilt $S^{\sharp} = \mathrm{Spa}(R^{\sharp}, R^{\sharp+})$, a formal abelian scheme $\mathfrak{A}$ over $R^{\sharp+}$, with endomorphism ι, polarization λ, K^p-level structure $\bar{\eta}$ and a G-trivialization α of the Tate module of its generic fiber. Write A_0 for the reduction $\mathfrak{A} \times_{R^{\sharp+}} R^+/\varpi$. Lemma 8.6 (also c.f. the construction of the Hodge-Tate map) shows that along both ways in the diagram, the tuple is sent to a point of Bun_G that is naturally (with respect to automorphisms of $\mathfrak{A}$ and base change in S) identified with the vector bundle with G-structure glued from $T_p\mathcal{A} \otimes \mathcal{O}_{X_S}$ and

$$M_{\mathrm{crys}}(A_0[p^{\infty}]) \otimes_{\mathrm{A}_{\mathrm{cris}}(R^+)} B^+_{\mathrm{dR}}(R^{\sharp})$$

along de Rham comparison isomorphism. Hence the diagram is 2-commutative.

Note also that $\mathrm{Igs}^{\mathrm{pre}} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}(S)$ is discrete and hence the fiber product is 0-truncated: indeed, this groupoid is equivalent to the groupoid whose objects are tuples

$$(\mathcal{A}_0, y \in \mathrm{Gr}_{G,\mu}(S), \phi : \mathscr{E}(\mathcal{A}_0) \cong \mathscr{E}(y)),$$

where $\mathcal{A}_0$ over R^+/ϖ is an abelian scheme with G-structure up to quasi-isogenies, y is an S-point of $\mathrm{Gr}_{G,\mu}$, and ϕ is an isomorphism of their attached G-bundles on X_S. An automorphism is a self G-quasi-isogeny $f : \mathcal{A}_0 \to \mathcal{A}_0$ in $\mathrm{Igs}^{\mathrm{pre}}(S)$ such that $\mathscr{E}(f) = id$. But this means f is the identity on $\mathcal{A}_0[p^{\infty}]$ and hence is prime-to-p, i.e. an automorphism in $S_K(R^+/\varpi)$. By representability of S_K, f must be the identity.

Now the result follows from proposition 8.14 below. $\square$

Assume S is a product of points, i.e $R^+ = \prod_{i \in I} C_i^+$, where each C_i is complete algebraically closed with a pseudo-uniformizer ϖ_i, $\varpi = (\varpi_i)$ and $R = R^+[1/\varpi]$. We denote by k_i the residue field of C_i and $\bar{C}_i^+$ the image of C_i^+ in k_i. Later whenever we put a subscript i to a morphism on S, we mean its restriction along $\mathrm{Spa}(C_i, C_i^+) \to S$.

Proposition 8.14. *Let S be as above, the map*

$$F : \mathcal{S}_{K^p}^{\circ;pre}(S) \to (\mathrm{Igs}^{pre} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu})(S)$$

is a natural (in S) bijection.

Proof. Assume we are given an S-point of the fiber product, i.e. a tuple

$$(\mathcal{A}_0, y \in \mathrm{Gr}_{G,\mu}(S), \phi : \mathscr{E}(\mathcal{A}_0) \cong \mathscr{E}(y))$$

as above. The projection of y to $\mathrm{Spd}E$ selects an untilt $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$. Out of this datum, we would like to construct a formal abelian scheme $\mathfrak{A}$ with G-structure at level K over $\mathrm{Spf}(R^{\sharp+})$ and a trivialization of its Tate module. The idea is to apply Serre-Tate theory c.f.4.2, which requires us to obtain a p-divisible group $\mathcal{G}/R^{\sharp+}$ from (y, ϕ), lifting $\mathcal{A}_0[p^\infty]$ up to isomorphism (up to modifying $\mathcal{A}_0$ in its isogeny class).

We first deal with the case where $S = \mathrm{Spa}(C, C^+)$ is a geometric point. By remark 6.13 and full-faithfulness of $\mathscr{E}(\cdot)$ on p-divisible groups up to isogeny over $\mathcal{O}_C/\varpi$ [SW13, 5.1.4(ii)], in this case the tuple simplifies to

$$(\mathcal{A}_0, (S^\sharp, \mathcal{H}, \alpha), \rho : \mathcal{A}_0[p^\infty] \times_{C^+/\varpi} \mathcal{O}_C/\varpi \dashrightarrow \mathcal{H} \times_{\mathcal{O}_{C^\sharp}} \mathcal{O}_C/\varpi),$$

where $S^\sharp = \mathrm{Spa}(C^\sharp, C^{\sharp+})$ is an untilt of S over $\mathrm{Spa}E$, $\mathcal{H}$ is a p-divisible group with G-structure over $\mathcal{O}_{C^\sharp}$, $\alpha : T_p\mathcal{H} \cong \underline{\Lambda}$ is a symplectic similitude $\mathcal{O}_B$-linear trivialization, and ρ is a quasi-isogeny preserving the G-structures.

We may assume $p^n \cdot \rho$ is an actual isogeny, for some integer n. Apply lemma 8.10 to the valuation ring $\overline{C^+}$ with fraction field k, the abelian scheme $\mathcal{B} := \mathcal{A}_0 \times_{C^+/\varpi} \overline{C^+}$ and the finite group scheme $\ker(p^n \cdot \rho_k)$, where ρ_k is the base change of ρ to k. It shows that the closure of $\ker(p^n \cdot \rho_k)$ in $\mathcal{B}$ is a finite flat subgroup scheme. We can therefore take the quotient of $\mathcal{B}$ by it to obtain a new abelian scheme $\mathcal{B}'$, equipped with G-structure by lemma 8.11,[20] whose p-divisible group agrees with $\mathcal{H}$ when base changed to k.

Now using lemma 3.16, we can glue $\mathcal{B}'[p^\infty]$ and $\mathcal{H}$ to get a p-divisible group $\mathcal{G}$ over $C^{\sharp+}$, with a trivialization $\alpha : T_p\mathcal{G} = T_p\mathcal{H} \cong \Lambda$. Also, applying the full-faithfulness part of 3.16 to $p^n\rho$ and $\mathcal{B}[p^\infty] \to \mathcal{B}'[p^\infty]$, we get an isogeny compatible with the G-structures

$$\mathcal{A}_0[p^\infty] \times_{C^+/\varpi} (C^+/\varpi \cdot \mathcal{O}_C) \to \mathcal{G} \times_{C^{\sharp+}} (C^+/\varpi \cdot \mathcal{O}_C).$$

By theorem 4.1, this lifts to a G-isogeny

$$\mathcal{A}_0[p^\infty] \to \mathcal{G} \times_{C^{\sharp+}} C^+/\varpi.$$

Now if S is a general product of points, by restricting to each $s_i = \mathrm{Spa}(C_i, C_i^+)$, the above construction gives a p-divisible group $\mathcal{G}_i/C_i^{\sharp+}$ with G-structure, trivialization $\alpha_i : T_p\mathcal{G}_i \cong \Lambda$, and a G-quasi-isogeny

$$\rho_i : \mathcal{A}_0[p^\infty] \times_{R^+/\varpi} C_i^+/\varpi_i \dashrightarrow \mathcal{G}_i \times_{C_i^{\sharp+}} C_i^+/\varpi_i,$$

such that $p^{n_i}\rho_i$ is an actual isogeny for some integer n_i. By lemma 3.18, we can take the product of $\mathcal{G}_i$'s to get a p-divisible group $\mathcal{G}/R^{\sharp+}$ with G-structure. There

---

[20]Note that although we are not exactly in the situation of the lemma, since we have the quasi-isogeny on p-divisible groups only over k not over $\overline{C^+}$, but this suffices to fix the constant d.

is a unique trivialization $\alpha : T_p\mathcal{G} \cong \underline{\Lambda}$ that restricts to the α_i's, by properness of the diagonal of $[*/\underline{K}_p] \to * := \mathrm{Spd}\mathbb{F}_q$, c.f.2.18. In order to get a G-quasi-isogeny

$$\mathcal{A}_0[p^\infty] \dashrightarrow \mathcal{G}_0 := \mathcal{G} \times_{R^{\sharp+}} R^+/\varpi,$$

which restricts to the ρ_i's, we have to show that the n_i's can be commonly bounded by a large enough integer N.

Note that $(\mathcal{G}, \alpha)$ defines an S-point of $\mathrm{Gr}_{G,\mu}$ by taking the prismatic Dieudonné module $M_{\underline{\mathbb{\Delta}}}(\mathcal{G})$, tensoring it up to $B_{\mathrm{dR}}^+(R^\sharp)$ and comparing with

$$T_p\mathcal{G} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+(R^\sharp) \xrightarrow{\alpha \otimes id} \Lambda \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+(R^\sharp),$$

c.f. the construction of the Hodge-Tate map in 6.14. Since this S-point agrees with the original point y on each s_i, by properness of $\mathrm{Gr}_{G,\mu}$ it is so on the whole S. In particular, the vector bundle $\mathscr{E}(\mathcal{G}) := \mathscr{E}(\mathcal{G}_0)$ attached to $\mathcal{G}$ is isomorphic to $\mathscr{E}(y)$ and the isomorphism ϕ can be viewed as an element in $\mathrm{Hom}(\mathscr{E}(\mathcal{A}_0[p^\infty]), \mathscr{E}(\mathcal{G}))$.

On the other hand, there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathcal{A}_0[p^\infty], \mathcal{G}_0) & \xrightarrow{\;\;\mathscr{E}(\cdot)\;\;} & \mathrm{Hom}(\mathscr{E}(\mathcal{A}_0[p^\infty]), \mathscr{E}(\mathcal{G})) \\
\downarrow & & \downarrow \\
\prod_i(\mathrm{Hom}(\mathcal{A}_{0,i}[p^\infty], \mathcal{G}_{0,i})[1/p]) & \xrightarrow[\sim]{\mathscr{E}(\cdot)} & \prod_i \mathrm{Hom}(\mathscr{E}(\mathcal{A}_{0,i}[p^\infty]), \mathscr{E}(\mathcal{G}_i)).
\end{array}
$$

If we set (M, φ) and (M', φ') to be the crystalline Dieudonné module of $\mathcal{A}_0$ and $\mathcal{G}_0$, then using the classification of p-divisible groups over quasi-regular semiperfect rings, c.f. example 3.23, the top row can be identified with

$$(M^\vee \otimes M')^{\varphi \otimes \varphi'=1} \hookrightarrow (M^\vee \otimes M'[1/p])^{\varphi \otimes \varphi'=1}.$$

It is then clear that for a large enough integer N, $p^N\phi$, and hence $p^N(\phi_i)_{i \in I}$, lies in the image of $\mathscr{E}(\cdot)$, so this N serves as the desired upper bound.

To conclude, by taking product we get a quasi-isogeny

$$\rho = p^{-N}(\prod_i p^N \rho_i) : \mathcal{A}_0[p^\infty] \dashrightarrow \mathcal{G}_0,$$

which is compatible with G-structures and maps to ϕ under $\mathscr{E}$. Using 8.11, we can modify $\mathcal{A}_0$ with ρ inside its isogeny class to get a new abelian scheme that satisfies the condition of the Serre-Tate theorem. More precisely, there exists a unique (up to isomorphism) abelian scheme $\mathcal{A}_0'$ with G-structure over R^+/ϖ with an isomorphism $\mathcal{A}_0 \to \mathcal{A}_0'$ in $\mathrm{Igs}^{\mathrm{pre}}(S)$, which identifies $\mathcal{A}_0'$ $\mathcal{O}_B$-linearly with $\mathcal{G}_0$, preserving the polarization up to a scalar in $\underline{\mathbb{Z}}_p^\times(S)$.

Now upon modifying the polarization on $\mathcal{G}$ by a section in $\underline{\mathbb{Z}}_p^\times(S)$, which does not change its isomorphism class, ρ' preserves the polarization on the nose and therefore by Serre-Tate theorem 4.2, the triple $(\mathcal{A}_0', \mathcal{G}, \rho')$ gives rise to a formal abelian scheme $\mathfrak{A}$ over $R^{\sharp+}$ with G-structure. Its p-divisible group agrees with $\mathcal{G}$ and its reduction to R^+/ϖ is isomorphic to $\mathcal{A}_0$ in $\mathrm{Igs}(S)$. Lifting the K^p-level structure $\bar{\eta}_0$ of $\mathcal{A}_0'$ to $\mathfrak{A}$ is automatic since $(R^{\sharp+}, (\varpi))$ is a Henselian pair. And $\alpha : T_p\mathfrak{A}_\eta = T_p\mathcal{G} \cong \underline{\Lambda}$ gives the level structure at p. This defines an S-point of $\mathcal{S}_{K^p}^{\circ,\mathrm{pre}}$.

To conclude the proof of bijectivity, we are left to check the assignment

$$(x, y, \phi) \mapsto (\mathfrak{A}, \alpha)$$

defines an inverse of F. This is direct and is left to the reader. $\square$

As a corollary, we have the diagram with various levels at p. In particular, to compare with the integral model at hyperspecial level in section 11, we have:

Corollary 8.15. *Denote the quotient stack* $[\mathrm{Gr}_{G,\mu}/K_p]$ *by* Gr_{G,μ,K_p}. *The diagram at level* $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, *i.e.*

$$
\begin{array}{ccc}
\mathcal{S}_K^{\circ} & \xrightarrow{\pi_{HT,K_p}^{\circ}} & \mathrm{Gr}_{G,\mu,K_p} \\
\downarrow{\scriptstyle\mathrm{red}} & & \downarrow{\scriptstyle BL} \\
\mathrm{Igs} & \xrightarrow{\bar{\pi}_{HT}^{\circ}} & \mathrm{Bun}_G,
\end{array}
$$

is 2-cartesian.

Theorem 8.13 leads to some direct corollaries about the geometric properties of Igs.

Corollary 8.16. *The v-stack* Igs *is small.*

Proof. [21] The Beauville-Laszlo map $\mathrm{Gr}_G \to \mathrm{Bun}_G$ is a surjective map of v-stacks (even as pro-étale stacks), c.f. 7.10. Hence $\mathcal{S}_{K^p}^{\circ} \to \mathrm{Igs}$ is surjective. We know that $\mathcal{S}_{K^p}^{\circ}$ is representable by a perfectoid space as remarked in [CS19] before proposition 2.6.4. As $\mathcal{S}_{K^p}^{\circ} \times_{\mathrm{Igs}} \mathcal{S}_{K^p}^{\circ}$ is the fiber product of $\mathcal{S}_{K^p}^{\circ}$ and $\mathrm{Gr}_{G,\mu} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}$ over $\mathrm{Gr}_{G,\mu}$, which is a small v-sheaf, this shows Igs is small. $\square$

Corollary 8.17. *The map* $\bar{\pi}_{HT}^{\circ} : \mathrm{Igs} \to \mathrm{Bun}_G$ *is compactifiable in the sense of* [Sch18, 22.2].

Proof. This follows from proposition 8.13 above, [Sch18, 22.3(iii)] and that the map π_{HT}° is compactifiable. $\square$

Corollary 8.18. *The map* $\bar{\pi}_{HT}^{\circ} : \mathrm{Igs} \to \mathrm{Bun}_G$ *is separated, representable in spatial diamonds and of finite dim. trg.(see* [Sch18, 21.7] *for the definition of the latter.)*

Proof. Representability can be checked by pulling back to $\mathrm{Gr}_{G,\mu}$, c.f. [Sch18, 13.4, 10.11], where π_{HT}° is representable in spatial diamonds. For the claim on finite transcendental dimension, it suffices to check on geometric points of Bun_G in the image of Igs, which lifts to $\mathrm{Gr}_{G,\mu}$. But each fiber of π_{HT}° has finite dim. trg., since up to a canonical compactification, it agrees with an Igusa variety c.f.9.27 later, which is the diamond attached to some perfect scheme with finite dimension over the base field. $\square$

Corollary 8.19. *The small v-stack* Igs *is an Artin v-stack in the sense of* [FS21, IV].

---

[21]One might want to argue following [FS21, III.1.3], i.e. show if $S_i = \mathrm{Spa}(R_i, R_i^+), i \in I$ is an ω_1-cofiltered inverse system of affinoid perfectoid spaces with inverse limit $S = \mathrm{Spa}(R, R^+)$, then

$$\mathrm{Igs}(S) = \varinjlim \mathrm{Igs}(S_i).$$

This holds for the prestack $\mathrm{Igs}^{\mathrm{pre}}$, as an abelian scheme or a quasi-isogeny between abelian schemes is finitely presented. But the v-stackification complicates the situation for Igs.

Proof. This follows from [FS21, IV.1.8(iii)], the fact that Bun_G is an Artin v-stack [FS21, IV.1.19], and corollary 8.18. $\qquad\square$

8.3. **Sheaf theoretic implications.** A sheaf theoretic implication of the above geometric properties is the following result. Here we use the derived category $D_{\text{ét}}(\cdot)$ of a small v-stack in the sense of [Sch18, Definition 14.13].

Proposition 8.20. *For any ring of coefficients Λ such that $n\Lambda = 0$ for some n prime to p. We have a natural base change equivalence*

$$BL^* R\bar{\pi}^{\circ}_{HT,*} \cong R\pi^{\circ}_{HT,*}\mathrm{red}^*$$

of functors $D_{\text{ét}}(\mathrm{Igs}, \Lambda) \to D_{\text{ét}}(\mathrm{Gr}_{G,\mu}, \Lambda)$. In particular, the complex $R\pi^{\circ}_{HT,*}\Lambda$ on $\mathrm{Gr}_{G,\mu}$ descends to the complex $R\bar{\pi}^{\circ}_{HT,*}\Lambda$ on Bun_G. The same statement holds true for $R\bar{\pi}^{\circ}_{HT,!}$, resp. $R\pi^{\circ}_{HT,!}$ in place of $R\bar{\pi}^{\circ}_{HT,*}$, resp. $R\pi^{\circ}_{HT,*}$.

Proof. The statement for usual pushforward follows from Corollary 8.18 and the qcqs base change of [Sch18, 17.6] (see [Sch18, Section 21] for a discussion on bounding the cohomological dimension of a map by its dim. trg..). The statement for the pushforward with proper supports follows from Corollary 8.18, Corollary 8.17 and [Sch18, 22.8]. $\qquad\square$

Recall the notion of universal locally acyclic (ULA) sheaves from [FS21]. We denote by $D^{\mathrm{ULA}}(\mathrm{Bun}_G, \Lambda)$ the full subcategory of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ consisting of objects that are universal locally acyclic for the structure map $\mathrm{Bun}_G \to *$. By [FS21, V.7.1], $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ lies in this subcategory means that for any $b \in B(G)$, its restriction along

$$i_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$$

is "admissible". More precisely, let $D(G_b(\mathbb{Q}_p), \Lambda)$ be the derived category of smooth representations of $G_b(\mathbb{Q}_p)$ with Λ-coefficient (see remark 7.16 for the definition of G_b). Then by [FS21, V.2.2], there is an equivalence $D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(\mathbb{Q}_p), \Lambda)$. Under this equivalence, $i_b^* A$ is identified with a complex of $G_b(\mathbb{Q}_p)$-representations, such that for any open pro-p subgroup K of $G_b(\mathbb{Q}_p)$, the K-invariants of this complex is a perfect complex of Λ-modules. We denote the full subcategory of such complexes in $D(G_b(\mathbb{Q}_p), \Lambda)$ by $D^{\mathrm{adm}}(G_b(\mathbb{Q}_p), \Lambda)$.

Due to this relation with admissible representations, universal locally acyclic sheaves on Bun_G are important in applications to representation theory. For the complex $R\bar{\pi}^{\circ}_{HT,*}\Lambda$, we prove below that it is universally locally acyclic with respect to $\mathrm{Bun}_G \to *$. As we will refer to the description of the Newton stratification on the Igusa stack in remark 9.42, the reader might skim section 9.4 quickly for the relevant statements. We denote the restriction of $\bar{\pi}^{\circ}_{HT}$ to $\mathrm{Igs}^b := \mathrm{Igs}_{K^p}^{\circ,b}$ by $\bar{\pi}^{\circ,b}_{HT}$, and base change everything to $\mathrm{Spd}\overline{\mathbb{F}}_p$.

Proposition 8.21. *The complex $R\bar{\pi}^{\circ}_{HT,*}\Lambda$ lies in $D^{\mathrm{ULA}}(\mathrm{Bun}_G, \Lambda)$.*

Proof. It suffices to check $i_b^* R\bar{\pi}^{\circ}_{HT,*}\Lambda \cong R\bar{\pi}^{\circ,b}_{HT,*}\Lambda$ lies in $D^{\mathrm{adm}}(G_b(\mathbb{Q}_p), \Lambda)$ under the equivalence

$$D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda) = D_{\text{ét}}([*/\widetilde{G}_b], \Lambda) \cong D_{\text{ét}}([*/G_b(\mathbb{Q}_p)], \Lambda) \cong D(G_b(\mathbb{Q}_p), \Lambda),$$

where the equivalence in the middle is induced by pullback along the section

$$\pi_{\mathrm{unip}} : [*/G_b(\mathbb{Q}_p)] \to [*/\widetilde{G}_b].$$

By remark 9.42, up to canonical compactification, $\mathrm{Igs}^b \cong [\mathrm{Ig}^{b,\diamond}/\widetilde{G}_b]$. Then by qcqs base change (along π_{unip}) and the fact that canonical compactification doesn't change the étale cohomology considered here (c.f. [CS17, 4.4.2]), $\pi_{\mathrm{unip}}^* R\overline{\pi}_{HT,*}^{\circ,b}\Lambda$ is identified with

$$R\Gamma(\mathrm{Ig}^{b,\diamond}, \Lambda)^{\mathrm{sm}} := \varinjlim_K R\Gamma(\mathrm{Ig}^{b,\diamond}, \Lambda)^K \in D(G_b(\mathbb{Q}_p), \Lambda),$$

where the K runs over all open pro-p subgroups of $G_b(\mathbb{Q}_p)$, and $G_b(\mathbb{Q}_p)$ acts on $\mathrm{Ig}^{b,\diamond}$ via the embedding $G_b(\mathbb{Q}_p) \hookrightarrow \widetilde{G}_b$. Now for any open pro-$p$ subgroup K of $G_b(\mathbb{Q}_p)$, we have

$$(R\Gamma(\mathrm{Ig}^{b,\diamond}, \Lambda)^{\mathrm{sm}})^K \cong R\Gamma(\mathrm{Ig}_K^{b,\diamond}, \Lambda),$$

where $\mathrm{Ig}_K^{b,\diamond}$ is the diamond attached to the quotient $[\mathrm{Ig}^b/K]$, which is representable by a finite étale cover of the perfect central leaf $\mathscr{C}_{\mathrm{perf}}^{\mathbb{X}_b}$. Now use [Sch18, 27.2] and the fact that $\mathscr{C}_{\mathrm{perf}}^{\mathbb{X}_b}$ is the perfection of a separated scheme of finite type over an algebraically closed field (Λ is torsion coprime to p), we see that $R\Gamma(\mathrm{Ig}_K^{b,\diamond}, \Lambda)$ is a perfect complex of Λ-modules as desired. $\qquad\square$

The above statement combined with [HL23, 4.15] implies the following result on the structure of the complex $R\overline{\pi}_{HT,*}^{\circ}\Lambda$. Here we take Λ to be $\overline{\mathbb{F}}_l$ with $l \neq p$. We let

$$\phi : W_{\mathbb{Q}_p} \to {}^L G(\overline{\mathbb{F}}_l)$$

be a semi-simple L-parameter and write

$$(-)_\phi : D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \to D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$$

for the idempotent localization functor as in Definition A.1 of *loc. cit.*. Note that for any $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$, any Schur irreducible subquotient of A_ϕ has Fargues-Scholze L-parameter equal to ϕ. We let $B(G)_{\mathrm{un}} := \mathrm{Im}(B(T) \to B(G))$ be the unramified elements of $B(G)$, where $T \subset G$ is a maximal torus. This is also the set of $b \in B(G)$ whose σ-centralizer G_b is quasi-split, c.f. [Ham22, 2.12], [XZ17, 4.2.1], and hence independent of the choice of T.

Proposition 8.22. *If ϕ is induced from a toral parameter*

$$\phi_T : W_{\mathbb{Q}_p} \to {}^L T(\overline{\mathbb{F}}_l),$$

which is generic in the sense of [Ham22, 1.4]. *Assuming* [HL23, Assumption 4.4], *we have*

$$(R\overline{\pi}_{HT,*}^{\circ}\Lambda)_\phi \cong \bigoplus_{b \in B(G)_{\mathrm{un}}} (R\overline{\pi}_{HT,*}^{\circ,b}\Lambda)_\phi.$$

Proof. Combine 8.21 and [HL23, 4.15]. $\qquad\square$

Remark 8.23. [HL23, Assumption 4.4] is an assumption on the properties of the Fargues-Scholze local Langlands correspondence and is verified for the cases listed in Table (22) of *loc. cit.*.

9. Minimal compactification

The goal of this section is to construct a minimal compactification of the Igusa stack Igs := $\mathrm{Igs}_{K^p}^{\circ}$ under the assumption that the boundary of the minimal compactification of the Shimura variety has codimension at least two. Below is some notation that will be used only in this section.

Notation 9.1.
- For the pair of structure sheaves $(\mathcal{O}, \mathcal{O}^+)$ on the v-site of an untilted small v-stack, we use the shorthand $(\mathcal{O}, \mathcal{O}^+)(S)$ to denote their global sections on a test object S, which namely means the pair $(\mathcal{O}(S), \mathcal{O}^+(S))$.
- A superscript $()^a$ is a shorthand for "almost", c.f. subsection 9.1.3 below.
- For an adic space (resp. diamond, scheme, formal scheme) X, we use X_τ, $\tau = \mathrm{Zar}$, an, ét, or v, to denote the Zariski, analytic, étale or v-site on X.

9.1. **Basic constructions.** We define the affinization of untilted small v-stacks, and then review the canonical compactification of maps between v-stacks, as well as some almost mathematics as needed.

9.1.1. *Structure sheaf on untilted small v-stacks.*

Definition 9.2. Let X be a small v-stack over $\mathrm{Spd}\mathbb{Z}_p$. Then for any $Y \in \mathrm{Perf}$ with a map $Y \to X$, the composition

$$Y \to X \to \mathrm{Spd}\mathbb{Z}_p$$

determines an untilt $Y^\sharp \to \mathrm{Spa}\mathbb{Z}_p$. The sheaves

$$\mathcal{O}_{X^\sharp}, \mathcal{O}_{X^\sharp}^+$$

are defined to be the unique sheaves which restrict to $\mathcal{O}_{Y^\sharp}, \mathcal{O}_{Y^\sharp}^+$, for all such $Y \to X$.

Remark 9.3. This definition is justified by v-descent of the structure and integral structure sheaf $\mathcal{O}$, $\mathcal{O}^+$. c.f. [Sch18, 8.7] and [Man22, 3.2.1]. The structure morphism $X \to \mathrm{Spd}\mathbb{Z}_p$ is called an untilt of X in the latter, and a small v-stack equipped with a morphism towards $\mathrm{Spd}\mathbb{Z}_p$ is called untilted. When the untilt plays a role in the discussion, we will denote the untilted stack $X^\sharp$. Otherwise we will drop $\sharp$ from notations.

Definition 9.4. For any untilted small v-stack X with structure sheaves $\mathcal{O}_{X^\sharp}$, $\mathcal{O}_{X^\sharp}^+$, we define a sheaf

$$X_0 := (S \mapsto \mathrm{Hom}((\mathcal{O}_{X^\sharp}, \mathcal{O}_{X^\sharp}^+)(X), (\mathcal{O}_{S^\sharp}, \mathcal{O}_{S^\sharp}^+)(S^\sharp))).$$

Here $S^\sharp$ is the untilt of S determined by the structure map to $\mathrm{Spd}\mathbb{Z}_p$; for Hom we take continuous ring homomorphisms from $\mathcal{O}_{X^\sharp}(X)$ to $\mathcal{O}_{S^\sharp}(S^\sharp)$ that maps $\mathcal{O}_{X^\sharp}^+(X)$ into $\mathcal{O}_{S^\sharp}^+(S^\sharp)$. This defines a functor from the category of untilted small v-stacks to that of v-sheaves.

Remark 9.5. The ring $\mathcal{O}_{X^\sharp}(X)$ is computed through covers by perfectoid spaces and it is equipped with a limit topology from its expression as an equalizer. We caution the reader that for a general small v-stack, the global sections $(\mathcal{O}_{X^\sharp}, \mathcal{O}_{X^\sharp}^+)(X)$ will not have the preferable properties like being affinoid perfectoid, or Tate etc.

To perform the above construction, it is not necessary to work with untilted structure sheaves. We could have used global sections of the usual structure sheaf, namely the one that restricts to the structure sheaf on a test object, instead of that of its untilt. The two constructions agree on affinoid perfectoids by tilting equivalence, but differ in general since tilting is not exact.

Lemma 9.6. *There is a map $X \to X_0$ such that for any affinoid perfectoid space $Y = \mathrm{Spa}(R, R^+)$ over $\mathrm{Spd}\mathbb{Z}_p$, any map $X \to Y$ factors uniquely through $X \to X_0$. We call $X \to X_0$ the* **affinization** *of X.*

Proof. For any $S \in \mathrm{Perf}_{\mathbb{Z}_p}$ mapping to X, by taking the induced map on the global sections of the structure sheaves, we obtain a map $S \to X_0$. This defines the desired map $X \to X_0$. Any map $X \to Y$ as above induces a homomorphism

$$(R^\sharp, R^{\sharp+}) \to (\mathcal{O}_{X^\sharp}, \mathcal{O}_{X^\sharp}^+)(X),$$

thus giving a map $X_0 \to Y$ and the composition $X \to X_0 \to Y$ is the given map. $\square$

9.1.2. *Canonical compactification of maps of v-stacks.*

Definition 9.7. (c.f.[Sch18, 18.6]) Let $f : X \to Y$ be a separated map of v-stacks. The functor on totally disconnected perfectoid spaces sending $\mathrm{Spa}(R, R^+)$ to

$$X(R, R^\circ) \times_{Y(R, R^\circ)} Y(R, R^+)$$

extends to a v-stack $\overline{X}^{/Y}$. There is a factorization of f as

$$X \to \overline{X}^{/Y} \xrightarrow{\overline{f}^{/Y}} Y$$

with $\overline{f}^{/Y}$ being partially proper. For any partially proper map $Z \to Y$ of v-stacks, any map $X \to Z$ factors uniquely through $X \to \overline{X}^{/Y}$.

Proposition 9.8. *([Sch18, 18.7, 18.8]) The construction $f \mapsto \overline{f}^{/Y}$ is functorial in f.*

9.1.3. *The almost setup.* Let R be a perfectoid Tate ring with subring of power-bounded elements R°. We denote the category of R°-modules by R°-mod.

Definition/Proposition 9.9. ([Sch18, 3.21,3.23]) An R°-module M is almost zero if $\varpi M = 0$ for all pseudo-uniformizers ϖ. Such modules form a thick Serre subcategory of R°-mod. The category $R^{\circ a}$-mod of almost R°-modules is the quotient of R°-mod by the subcategory of almost zero modules.

Remark 9.10. Similarly one can define almost R^+-modules for any ring of integral elements $R^+ \subset R^\circ$ and the forgetful functor from $R^{\circ a}$-mod to R^{+a}-mod is an equivalence.

Proposition 9.11. *([Sch18, 3.24]) Let (R, R^+) be a Huber pair with R being a perfectoid Tate ring and let X be $\mathrm{Spa}(R, R^+)$. Then the R^+-module $H^i(X, \mathcal{O}_X^+)$ is almost zero for $i > 0$ and $H^0(X, \mathcal{O}_X^+) = R^+$.*

9.2. **Igusa varieties.** In the construction of the minimal compactification of the Igusa stack, we need to know certain geometric properties of the fibers of the Hodge-Tate period map. Since they are identified with Igusa varieties up to canonical compactifications (c.f. 9.27 below), we collect below some relevant facts about Igusa varieties in our PEL setup, based on the work of Caraiani-Scholze [CS17], [CS19].

9.2.1. *Igusa varieties.* Let $E/\mathbb{Q}_p$, $\mathbb{F}_q$, $S_K/\mathcal{O}_E$ with K_p hyperspecial and μ be as in section 5. Fix an algebraically closed field k containing $\mathbb{F}_q$. Denote by $S_{K,k}$ the base change of S_K to k. Consider the Kottwitz set $B(G)$ for $G_{\mathbb{Q}_p}$. Using 8.9 and [SW13, 5.1.4] (or a classification of their attached isocrystals with additional structures), the isogeny classes of p-divisible groups over k with the corresponding G-structure are in bijection to the set $B(G, \mu)$ of μ-admissible elements. For any $[b] \in B(G, \mu)$ we fix a p-divisible group $\mathbb{X}_b$ representing the corresponding isogeny class. Write $\tilde{\mathbb{X}}_b$ for its universal cover.

Definition 9.12. We let $\underline{\mathrm{Aut}}_G(\mathbb{X}_b)$ be the fpqc sheaf of groups on (the opposite category of) k-algebras

$$R \to \mathrm{Aut}_G(\mathbb{X}_b \times_k R)$$

i.e. $\mathcal{O}_B$-linear automorphisms of $\mathbb{X}_b \times_k R$ that preserve the polarization up to an element in $\underline{\mathrm{Aut}}(\mu_{p^\infty})(R) = \underline{\mathbb{Z}_p^\times}(R)$. Here $\underline{\mathbb{Z}_p^\times}$ is considered as a sheaf on k-alg$^{\mathrm{op}}$ by taking continuous maps from $\mathrm{Spec}(R)$ to the profinite group $\mathbb{Z}_p^\times$.

Definition/Proposition 9.13. ([CS19, 2.3.1], [Oor09]) The central leaf $\mathscr{C}^{\mathbb{X}_b} \subset S_{K,k}$ is the smooth locally closed subscheme over which the p-divisible group of the universal abelian scheme is geometric fiberwise isomorphic to $\mathbb{X}_b$.

Definition/Proposition 9.14. ([CS19, 2.2.6], c.f. [CS19, 2.3.2]) The perfect Igusa variety Ig^b is the $\underline{\mathrm{Aut}}_G(\mathbb{X}_b)$-torsor over $\mathscr{C}^{\mathbb{X}_b}$ parametrizing isomorphisms $A[p^\infty] \xrightarrow{\sim} \mathbb{X}_b$, compatible with the G-structures. It is a perfect k-scheme and is (up to isomorphism) independent of the choice of $\mathbb{X}_b$ in its isogeny class. What's more, Ig^b is a $\Gamma_{\mathbb{X}_b} := \mathrm{Aut}_G(\mathbb{X}_b)$-torsor over the perfection $\mathscr{C}^{\mathbb{X}_b}_{\mathrm{perf}}$ of $\mathscr{C}^{\mathbb{X}_b}$.

Remark 9.15. Our notation here is slightly different from that in [CS19], where they use Fraktur letters for the perfect Igusa varieties to distinguish them with the (pro-)Igusa varieties. Since we don't deal with the latter here, we simply use roman letters.

It is manifest from the definition that $\underline{\mathrm{Aut}}_G(\mathbb{X}_b)$ acts on the Igusa variety Ig^b. However it has an alternative moduli interpretation, which allows the action of the larger group of self-quasi-isogenies of $\mathbb{X}_b$, or equivalently, the automorphism group of the universal cover $\tilde{\mathbb{X}}_b$.

Definition 9.16. Let $\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)$ be the fpqc sheaf on $\mathrm{Nilp}^{\mathrm{op}}_{W(k)}$, the opposite category of $W(k)$-algebras on which p is nilpotent,

$$R \to \mathrm{Aut}_G(\tilde{\mathbb{X}}_b \times_k R)$$

i.e. B-linear automorphisms of $\tilde{\mathbb{X}}_b \times_k R$ [22] that preserve the polarization up to an element in $\underline{\mathrm{Aut}}(\tilde{\mu}_{p^\infty})(R) = \underline{\mathbb{Q}_p^\times}(R)$. Here $\underline{\mathbb{Q}_p^\times}$ is considered as a sheaf by taking continuous maps from $\mathrm{Spf}(R)$ to the locally profinite group $\mathbb{Q}_p^\times$. By [CS17, 4.2.10], this is (pro-)representable by a formal group scheme over $\mathrm{Spf}\, W(k)$.

Proposition 9.17. *([CS17, 4.3.4-5], [CT21, 4.2.2]) For a k-algebra R, $\mathrm{Ig}^b(R)$ can be identified with the set of isomorphism classes of pairs (A, ρ), where $A \in S_K(R)$ is an abelian scheme with G-structures, considered up to isogeny and ρ is a quasi-isogeny*

$$\rho : A[p^\infty] \to \mathbb{X}_b \times_k R$$

respecting the G-structures. In particular, $\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)$ acts on Ig^b.

9.2.2. *Partial minimal compactifications.* Using the theory of well-positioned subsets due to Boxer [Box15], Lan and Stroh [LS18], one can construct well-behaved partial minimal compactification of the central leaf $\mathscr{C}^{\mathbb{X}_b}$ inside the minimal compactification $S_{K,k}^*$ of $S_{K,k}$.

Definition 9.18. Let Y be the complement of $\mathscr{C}^{\mathbb{X}_b}$ in its closure in $S_{K,k}$. Then the *partial minimal compactification* $\mathscr{C}^{\mathbb{X}_b,*}$ of $\mathscr{C}^{\mathbb{X}_b}$ is $\overline{\mathscr{C}^{\mathbb{X}_b}}\backslash\overline{Y}$, i.e. the closure of $\mathscr{C}^{\mathbb{X}_b}$ in $S_{K,k}^*$, deleting the closure of Y. This is a normal scheme over k.

Definition 9.19. The partial minimal compactification $\mathrm{Ig}^{b,*}$ is the normalization of $\mathscr{C}^{\mathbb{X}_b,*}$ in Ig^b.

Proposition 9.20. *Consider the set-theoretic partition of $S_K^* = \coprod_Z S_Z$ according to cusp labels at level K as in section 5. For simplicity, we assume K is principal. Then for a cusp label $Z = (Z, (X, Y, \phi, \varphi_{-2}, \varphi_0))$ representing a cusp label at level K, the intersection*

$$\mathscr{C}_Z^\natural := \mathscr{C}^{\mathbb{X}_b,*} \times_{S_{K,k}^*} S_{Z,k}$$

is a central leaf for the smaller dimensional Shimura variety $S_{Z,k}$. This intersection is non-empty if and only if the p-divisible group with G-structure $\mathbb{X}_b$ admits a decomposition

$$\mathbb{X}_b \cong \mathrm{Hom}(X, \mu_{p^\infty}) \oplus \mathbb{X}_Z \oplus Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p).$$

In this case, $\mathscr{C}_Z^\natural$ is attached to the p-divisible group $\mathbb{X}_Z$.

Proof. Combine theorem 2.3.2 and proposition 3.4.2 of [LS18]. c.f. [CS19, 3.1.4] in the principally polarized case. □

The following result is crucial to our construction of the minimal compactification of Igs.

Proposition 9.21. *The partial minimal compactifications $\mathrm{Ig}^{b,*}$ and $\mathscr{C}^{\mathbb{X}_b,*}$ are affine.*

Proof. Note that the proof in [CS19, 3.3.2, 3.3.4] works verbatim, as the results they cited hold in the generality of PEL type Shimura varieties of type AC with good reduction at p. □

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[22]Note that since the universal cover lifts uniquely to $W(k)$, this base change makes sense. We can equally use R/p instead of R in the definition.

9.2.3. *Dimension.* Let $G/\mathbb{Q}$ be the algebraic group defined by the global PEL-datum as before. The dimension of a central leaf labelled by $[b] \in B(G_{\mathbb{Q}_p})$ is computed by [Ham15, 7.8(2)]. The formula agrees with that (relative over $\mathrm{Spf}\,W(k)$) of the automorphism group scheme $\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)$ (c.f. [CS17, 4.2.11]), i.e. we have:

Lemma 9.22. *The dimension of the central leaf $\mathscr{C}^{\mathbb{X}_b}$ agrees with the (formal) dimension of the formal group scheme $\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)$. Both of them are computed by the formula $\langle 2\rho, \nu_G(b) \rangle$, where ρ is the half sum of the (absolute) positive roots of $G_{\mathbb{Q}_p}$ and $\nu_G(b)$ is the Newton point of b.*

Proof. Combine [CS17, 4.2.11] and [Ham15, 7.8(2)]. $\square$

We use this to prove the following proposition, which will be used in constructing the minimal compactification of the Igusa stack.

Proposition 9.23. *If the boundary of the minimal compactification S_K^* of the Shimura variety has codimension at least two, then so does the boundary of the partial minimal compactification of any central leaf $\mathscr{C}^{\mathbb{X}_b, *} \subset S_{K,k}^*$.*

Proof. We may without loss of generality assume the $\mathbb{Q}$-algebra with positive involution $(B, *)$ in the global PEL-datum defining S_K is simple. Assume $\mathbb{X}_b$ admits a decomposition

$$\mathbb{X}_b \cong \mathrm{Hom}(X, \mu_{p^\infty}) \oplus \mathbb{X}_Z \oplus Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p)$$

for some cusp label Z, c.f. 9.20. We compare the dimension of $\mathscr{C}^{\mathbb{X}_b}$ with that of $\mathscr{C}_Z^\natural$. Using 9.22, it suffices to compare the dimension of $\underline{\mathrm{Aut}}_G(\mathbb{X}_b)$ and that of $\underline{\mathrm{Aut}}_G(\mathbb{X}_Z)$ (passing to universal cover doesn't change the dimension). In fact we only have to exclude the case where the difference between their dimensions is one, under our assumption on codimension of the boundary of S_K^*.

But $\underline{\mathrm{Aut}}_G(\mathbb{X}_b)$ is a closed subgroup of $\underline{\mathrm{Aut}}(\mathbb{X}_b)$, which is of the form

$$\begin{pmatrix} \dfrac{\mathrm{Aut}(\mathrm{Hom}(X, \mu_{p^\infty}))}{\mathcal{H}_{\mathbb{X}_Z, \mathrm{Hom}(X, \mu_{p^\infty})}} & \\ \mathcal{H}_{Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p), \mathrm{Hom}(X, \mu_{p^\infty})} & \mathcal{H}_{Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{X}_Z} & \underline{\mathrm{Aut}}(Y \otimes \mathbb{Q}_p/\mathbb{Z}_p). \end{pmatrix}$$

Here we use $\mathcal{H}_{\mathbb{X}_Z, \mathrm{Hom}(X, \mu_{p^\infty})}$ etc. to denote the internal Hom p-divisible groups of [CS17, 4.1.6]. Hence $\underline{\mathrm{Aut}}_G(\mathbb{X}_b)$ admits a 2-step filtration

$$U_2 \subset U_1 \subset U_0 = \underline{\mathrm{Aut}}_G(\mathbb{X}_b),$$

such that

$$U_2 \subset \mathcal{H}_{Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p), \mathrm{Hom}(X, \mu_{p^\infty})},$$

$$U_1/U_2 \subset \mathcal{H}_{\mathbb{X}_Z, \mathrm{Hom}(X, \mu_{p^\infty})} \times \mathcal{H}_{Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{X}_Z}$$

are closed subgroups.

Decompose $(B, *)_{\mathbb{Q}_p}$ into simple factors, which fall into three possible cases, c.f. [Ham15, 4.5]. By checking case by case, we see that both U_2 and U_1/U_2 will be of positive dimensions unless $\mathbb{X}_Z = 0$. Hence we are reduced to consider this situation. In this case the degeneration is towards a 0-dimensional cusp, and the central leaf $\mathscr{C}^{\mathbb{X}_b}$ agrees with the μ-ordinary locus of S_K, which is dense by [Wed99]. This means that the Shimura variety itself is 1-dimensional and has a 0-dimensional cusp, which is excluded by our assumption. $\square$

For a scheme X, write $X_0 := \operatorname{Spec}(\mathcal{O}_X(X))$ for its affinization.

Corollary 9.24. *If the boundary of the minimal compactification S_K^* of the Shimura variety has codimension at least two. Then $(\operatorname{Ig}^b)_0 \cong \operatorname{Ig}^{b,*}$.*

Proof. The statement is true for the corresponding central leaves, i.e. $(\mathscr{C}^b)_0 \cong \mathscr{C}^{b,*}$. This is because the latter is normal, noetherian with boundary codimension at least two (proposition 9.23 above). Therefore the algebraic Hartogs' extension lemma applies. It follows immediately that for their perfections, we also have $(\mathscr{C}^b_{\mathrm{perf}})_0 \cong \mathscr{C}^{b,*}_{\mathrm{perf}}$.

Now for the Igusa varieties, we use that $\operatorname{Ig}^b \xrightarrow{q} \mathscr{C}^b_{\mathrm{perf}}$ is pro-finite étale, Galois under the group $\Gamma_{\mathbb{X}_b} := \operatorname{Aut}(\mathbb{X}_b)$ (see 9.14). For any normal compact open subgroup K_b of $\Gamma_{\mathbb{X}_b}$, the map

$$\operatorname{Ig}^b_{K_b} := \operatorname{Ig}^b/K_b \xrightarrow{q_{K_b}} \mathscr{C}^b_{\mathrm{perf}}$$

is a finite Galois cover under $\Gamma_{\mathbb{X}_b}/K_b$. We define its "partial canonical compactification" to be the normalization of $\mathscr{C}^{b,*}_{\mathrm{perf}}$ in $\operatorname{Ig}^b_{K_b}$ and denote it by

$$\operatorname{Ig}^{b,*}_{K_b} \xrightarrow{\overline{q_{K_b}}} \mathscr{C}^b_{\mathrm{perf}}.$$

We have a commutative diagram

$$
\begin{array}{ccccc}
\operatorname{Ig}^b & \longrightarrow & \operatorname{Ig}^b_{K_b} & \xrightarrow{q_{K_b}} & \mathscr{C}^b_{\mathrm{perf}} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{K_b}} & & \downarrow{\scriptstyle f_0} \\
\operatorname{Ig}^{b,*} & \longrightarrow & \operatorname{Ig}^{b,*}_{K_b} & \xrightarrow{\overline{q_{K_b}}} & \mathscr{C}^{b,*}_{\mathrm{perf}},
\end{array}
$$

and $\operatorname{Ig}^{b,*}$ equals $\varprojlim_{K_b} \operatorname{Ig}^{b,*}_{K_b}$. We denote its projection to $\mathscr{C}^{b,*}_{\mathrm{perf}}$ by $\bar{q}$.

At each K_b-level, we have

$$(\overline{q_{K_b}})_* f_{K_b,*} \mathcal{O}_{\operatorname{Ig}^b_{K_b}} \cong f_{0,*} q_{K_b,*} \mathcal{O}_{\operatorname{Ig}^b_{K_b}}$$

is a finite $\mathcal{O}_{\mathscr{C}^{b,*}_{\mathrm{perf}}}$-algebra, because $q_{K_b,*} \mathcal{O}_{\operatorname{Ig}^b_{K_b}}$ is a finite étale $\mathcal{O}_{\mathscr{C}^b_{\mathrm{perf}}}$-algebra, while $f_{0,*} \mathcal{O}_{\mathscr{C}^b_{\mathrm{perf}}} \cong \mathcal{O}_{\mathscr{C}^{b,*}_{\mathrm{perf}}}$. Hence this is also the normalization of $\mathcal{O}_{\mathscr{C}^{b,*}_{\mathrm{perf}}}$ in it, and we have

$$(\overline{q_{K_b}})_* f_{K_b,*} \mathcal{O}_{\operatorname{Ig}^b_{K_b}} \cong (\overline{q_{K_b}})_* \mathcal{O}_{\operatorname{Ig}^{b,*}_{K_b}}.$$

Therefore we can compute that

$$
\begin{aligned}
\Gamma(\operatorname{Ig}^b, \mathcal{O}_{\operatorname{Ig}^b}) &= \varinjlim_{K_b} \Gamma(\mathscr{C}^b_{\mathrm{perf}}, q_{K_b,*} \mathcal{O}_{\operatorname{Ig}^b_{K_b}}) \\
&= \varinjlim_{K_b} \Gamma(\mathscr{C}^{b,*}_{\mathrm{perf}}, f_{0,*} q_{K_b,*} \mathcal{O}_{\operatorname{Ig}^b_{K_b}}) \\
&= \varinjlim_{K_b} \Gamma(\mathscr{C}^{b,*}_{\mathrm{perf}}, \overline{q_{K_b,*}} f_{K_b,*} \mathcal{O}_{\operatorname{Ig}^b_{K_b}}) \\
&= \varinjlim_{K_b} \Gamma(\mathscr{C}^{b,*}_{\mathrm{perf}}, \overline{q_{K_b,*}} \mathcal{O}_{\operatorname{Ig}^{b,*}_{K_b}}) = \Gamma(\operatorname{Ig}^{b,*}, \mathcal{O}_{\operatorname{Ig}^{b,*}}).
\end{aligned}
$$

This is what we want to prove. $\qquad\square$

From the proof of proposition 9.23 we see that the boundary of the partial minimal compactification of a central leaf having codimension at least two is a very mild condition. In fact when the PEL datum of type AC is simple, this happens only if the corresponding Shimura variety is a non-compact curve. Below we classify such Shimura varieties with a central leaf whose partial minimal compactification has boundary codimension one.[23] A quick observation is that the condition on dimension and existence of cusps already forces the group G to be quasi-split over $\mathbb{Q}$ with absolute root system of type A_1.

Proposition 9.25. *Let $(B, *, V, (\cdot, \cdot), h)$ be a global PEL-datum satisfying assumption 5.5 such that the $\mathbb{Q}$-algebra with positive involution $(B, *)$ is simple. If the boundary of the partial minimal compactification of a central leaf on the attached Shimura variety at hyperspecial level has codimension one, then the central leaf must be the ordinary locus and the Shimura variety is either the modular curve, or a unitary Shimura curve attached to an imaginary quadratic extension of $\mathbb{Q}$ as in example 5.12.*

Proof. We see in the proof of 9.23 that for the codimension-one situtation to happen, the p-divisible group $\mathbb{X}_b$ is ordinary and admit an $\mathcal{O}_B$-linear decomposition

$$\mathbb{X}_b \cong \mathrm{Hom}(X, \mu_{p^\infty}) \oplus Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p).$$

What's more, the dimension of $U_2 = U_1$ should be one. Since each simple factor of B is stable under the involution $*$ [Lan13, 1.2.11], B itself is simple. By Wedderburn's structure theorem, such a B is a matrix algebra over some division $\mathbb{Q}$-algebra D. Hence under Morita equivalence, we may assume $B = D$.

Let F be the center of D, d the degree of D over F, and n the dimension of V over D. Then $\mathrm{End}_D(V) \cong M_n(D)$ and in particular if we base change to $\mathbb{C}$, we have by definition[24]

$$G_\mathbb{C} = \{g \in \prod_{F \hookrightarrow \mathbb{C}} M_{n \cdot d^2} \mid gg^* \in \mathbb{G}_{m,\mathbb{C}}\}.$$

Up to identifying the similitude factors, this will be a product of $\mathrm{GL}_{n \cdot d^2} \times \mathbb{G}_m$ and $\mathrm{GSp}_{n \cdot d^2}$'s, with the number of each factor depending on the shape of the involution $*$. Now the constraint that the root datum is of type A_1 requires $n \cdot d^2$ to be two and therefore we have necessarily $n = 2$ and $d = 1$.

According to whether $\dim_\mathbb{Q} F$ is one or two, and whether $*_B$ is trivial, we have the following cases:

- Case I: $B \cong F \cong \mathbb{Q}$, $*_B$ is trivial, and $V = \mathbb{Q}^2$, equipped with the standard symplectic form. In this case $G = \mathrm{GL}_2$, the Shimura variety is the modular curve;

- Case II: $B \cong F$ is a quadratic extension of $\mathbb{Q}$, $*_B = \overline{(\cdot)}$ is the nontrivial automorphism of F over $\mathbb{Q}$, and $V = F^2$, equipped with the symplectic form $\mathrm{tr}_{F/\mathbb{Q}}\langle \cdot, \cdot \rangle$, where $\langle x, y \rangle = \bar{x}^\tau \cdot y$ is a skew Hermitian form. For $*_B$ to be positive, F must be imaginary quadratic. In this case G is a quasi-split $\mathbb{Q}$-form of $\mathrm{GL}_2 \times \mathbb{G}_m$ that splits over F. It is a unitary similitude group

---

[23]As these cases are well understood, one can probably directly construct minimal compactifications of the Igusa stacks for them.

[24]Recall that the involution on $\mathrm{End}_B(V)$ induced by $(\cdot, \cdot)$ extending that on B is still denoted $*$. We will write $*_B$ for the restriction to B in case confusion might arise.

with signature $(1,1)$ at infinity. The corresponding Shimura variety is a unitary Shimura curve as in example 5.12;

- Case III: $B \cong F$ is a quadratic extension of $\mathbb{Q}$, $*_B$ is trivial, and $V = F^2$ is equipped with the standard symplectic form. For $*_B$ to be positive, F must be real quadratic. But in this case, G is a form of $(\mathrm{GL}_2 \times \mathrm{GL}_2)/\mathbb{G}_m$, which is excluded by the root system constraint.

$\square$

Remark 9.26. That the group $G/\mathbb{Q}$ comes from a global PEL-datum is crucial in this classification. Otherwise there are exotic examples where the group is quasi-split over $\mathbb{Q}$ with type A_1 absolute root system, but does not fall in any of the above cases. For example, one can compute quasi-split outer forms of $G := \mathrm{GL}_2 \times \mathbb{G}_m$ over $\mathbb{Q}$, which amounts to representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into the outer automorphism group of $G_{\overline{\mathbb{Q}}}$. The latter is in bijection to the automorphism group of the root datum

$$\mathcal{R} := (\mathbb{Z}^3, \{\pm\alpha\}, \mathbb{Z}^3, \{\pm\alpha^\vee\}),$$

where $\alpha = e_1 - e_2$ and $\alpha^\vee = e_1^\vee - e_2^\vee$ for a standard basis e_i, $i = 1, 2, 3$. This group is of the form

$$\left\{ \begin{bmatrix} a & a-1 & b \\ a-1 & a & b \\ c & c & d \end{bmatrix} \in \mathrm{GL}_3(\mathbb{Z}) \middle| \det = \pm 1 \right\},$$

which is large due to the high rank of the character lattice. We can also compute its finite order elements. There can be only elements of order 4 and they are of the form $d = 1 - 2a$, $\det = \pm 1$ (among those order 2 elements has additionally $c = \frac{2a(1-a)}{b}$). As an example, one can take the automorphism of $\mathcal{R}$ that fixes $\pm\alpha$ and rotates the plane perpendicular to it by $\pi/2$ radians. Using these one can construct quasi-split $\mathbb{Q}$-forms of $\mathrm{GL}_2 \times \mathbb{G}_m$ that splits only over a degree four extension of $\mathbb{Q}$, and are hence different from all cases in the above classification. To all of them, there shall be attached Shimura varieties, since the relevant axioms of Shimura data concern only the infinte place, while over $\mathbb{R}$ the groups are either $\mathrm{GU}(1,1)$ or $\mathrm{GL}_2 \times \mathbb{G}_m$.

9.2.4. *Fibers of the Hodge-Tate map.* Up to a canonical compactification, fibers of the Hodge-Tate map (resp. its minimal compactification) can be identified with Igusa varieties (resp. their partial minimal compactifications). More precisely, let

$$x : \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Gr}_{G,\mu}$$

be a rank one geometric point with an untilt $\mathrm{Spa}(C^\sharp, \mathcal{O}_{C^\sharp})$ determined by the structure map to $\mathrm{Spd}E$. It determines a p-divisible group $\mathcal{G}_x$ over $\mathcal{O}_{C^\sharp}$ with trivialized Tate module. Write k for the residue field of C. Assume $\mathcal{G}_x \times_{\mathcal{O}_{C^\sharp}} k$ lies in the isogeny class labelled by $b \in B(G, \mu)$. We have the perfect Igusa variety Ig^b as before, which admits a canonical lift to $W(k)$ and hence to $\mathcal{O}_{C^\sharp}$. We set our notation for various Igusa varieties as below:

- Ig^b: the perfect scheme over k;
- $\mathrm{Ig}^b_{\mathcal{O}_C}$: the lift of Ig^b to $\mathrm{Spf}\mathcal{O}_{C^\sharp}$, viewed as a formal scheme;
- Ig^b_C: the adic generic fiber of $\mathrm{Ig}^b_{\mathcal{O}_C}$.

Here for the latter two spaces we are suppressing the $\sharp$ symbols from notation. We use similar notations $\mathrm{Ig}^{b,*}$, $\mathrm{Ig}^{b,*}_{\mathcal{O}_C}$, $\mathrm{Ig}^{b,*}_C$ for the partial minimal compactifications. Note that since Ig^b and $\mathrm{Ig}^{b,*}$ are perfect, Ig^b_C and $\mathrm{Ig}^{b,*}_C$ are perfectoid spaces. The latter is affinoid perfectoid by 9.21. Later we will not distinguish $\mathrm{Ig}^{b,(*)}_C$ and its attached diamond.

Theorem 9.27. *Fixing a quasi-isogeny*

$$\mathcal{G}_x \times_{\mathcal{O}_{C^\sharp}} \mathcal{O}_{C^\sharp}/p \dashrightarrow \mathbb{X}_b \times_k \mathcal{O}_{C^\sharp}/p,$$

there are natural (with respect to complete algebraically closed extensions of C) open immersions

$$\mathrm{Ig}^b_C \hookrightarrow (\pi^\circ_{HT})^{-1}(x),\ \mathrm{Ig}^{b,*}_C \hookrightarrow (\pi^*_{HT})^{-1}(x)$$

inducing isomorphisms on their canonical compactifications towards x.

Proof. The statement for the fiber on the good reduction locus follows from the argument in [CS17, 4.4.4]. The statement on the minimal compactification is proven in [CS19, 4.5.1] for certain unitary Shimura varieties, but their argument work verbatim in our situation, as checked by Santos in her Imperial College London PhD thesis [San23, 4.3.12, 4.3.16].[25]   □

9.2.5. *Torsion in the first cohomology of the integral structure sheaf.* We record a torsion (almost) vanishing result in the first cohomology of $\mathcal{O}^+$ on the Igusa variety $(\mathrm{Ig}^b_C)_{an}$ that will used later. This follows from a general torsion vanishing phenomenon for the first Witt vector cohomology of perfect schemes. It is interesting to ask what happens in higher degrees.

Proposition 9.28. *Let X be a perfect scheme in characteristic p. Denote by $W(X)$ the canonical lift of X to characteristic zero using the p-typical Witt vectors. Then the cohomology $H^1(W(X), \mathcal{O})$ of the structure sheaf on the Zariski site of $W(X)$ is p-torsionfree.*

Proof. Consider the short exact sequence

$$0 \to W\mathcal{O}_X \xrightarrow{\cdot p} W\mathcal{O}_X \to \mathcal{O}_X \to 0$$

on the perfect scheme X, where $W\mathcal{O}_X$ is the sheaf on X sending an open U to the ring of Witt vectors of $\mathcal{O}_X(U)$. This is exact by perfectness of X. Note that the last surjection has a multiplicative section given by the Teichmüller lift. This gives us surjectivity on global sections

$$H^0(X, W\mathcal{O}_X) = W(H^0(X, \mathcal{O}_X)) \twoheadrightarrow H^0(X, \mathcal{O}_X).$$

Take the cohomology long exact sequence and we get

$$H^1(W(X), \mathcal{O})[p] = H^1(X, W\mathcal{O}_X)[p] = 0.$$

□

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[25]Although in the statement of 4.3.12 there, it is said that the partial toroidal compactification of the Igusa variety is affinoid. This is wrong (it might be a typo) but does not affect the other arguments.

Remark 9.29 (Question)**.** Assume X comes from taking perfection of a smooth quasi-affine scheme of finite type over a perfect field k, do we always have that $H^i(X, W\mathcal{O}_X)$, $i > 0$ has bounded p-torsion?

The example below is provided by O. Gabber, which shows that torsion can appear in degree 2 and in fact higher even degrees. By taking product of such and replacing Γ by $\mathbb{Z}/p^n$ with increasing n's, we see that without the finite type assumption, the answer to the question is no.

Example 9.30. Let Γ be the cyclic group of order p, acting trivially on $\mathbb{Z}$. Its group cohomologies with integer coefficients are

$$H^i(\Gamma, \mathbb{Z}) = \begin{cases} \mathbb{Z}, \ i = 0 \\ 0, \ i \text{ odd} \\ \mathbb{Z}/p, i \geq 2, \text{even.} \end{cases}$$

Let V be a finite dimensional faithful k-representation of Γ. We define Y' to be the spectrum of the symmetric algebra on the dual V^* of V. It has an induced Γ-action. Upon replacing V with some tensor power of it, we may assume this action is free on an open U whose complement has codimension $m \geq 2$. Consider the quotient Y'/Γ. Let Z be the complement of U/Γ in Y'/Γ. We may choose homogenous polynomials $h_1, \cdots, h_{\dim Z}$ in $\mathrm{Sym}^\bullet V^*$, such that they form a regular sequence on Z and that the vanishing locus $V(h_1, \cdots, h_{\dim Z})$ is smooth. Now $Z \cap V(h_1, \cdots, h_{\dim Z})$ agrees set-theoretically with the origin. Let C be the algebra $\mathrm{Sym}^\bullet V^*/(h_1, \cdots, h_{\dim Z})$, Y be the perfection of its punctured spectrum. Then Γ acts freely on Y and the quotient $X := Y/\Gamma$ is a smooth, quasi-affine perfect scheme of finite type over k. We can use a Hochschild-Serre spectral sequence to compute the cohomology of X. In particular, we have for any $p \leq m - 1$

$$H^p(\Gamma, H^0(Y, W\mathcal{O}_Y)) \hookrightarrow H^p(X, W\mathcal{O}_X).$$

Since $H^0(Y, \mathcal{O}_Y) = C_{\mathrm{perf}}$ has k as a retract, $H^p(X, W\mathcal{O}_X)$ has $H^p(\Gamma, W(k))$ as a direct summand and hence can be torsion if p is even.

For $x = \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Gr}_{G,\mu}$ as in subsection 9.2.4, let Ig^b be the perfect Igusa variety over k, which deforms to the flat formal scheme $\mathfrak{X} := \mathrm{Ig}^b_{\mathcal{O}_C}$ over $\mathrm{Spf}\mathcal{O}_{C^\sharp}$ with (perfectoid) adic generic fiber Ig^b_C as in 9.2.4. As a corollary to 9.28, we have the following.

Corollary 9.31. *The ϖ-torsion in $H^1_{an}(\mathrm{Ig}^b_C, \mathcal{O}^+)$ is almost zero.*

Proof. Take an affine open cover of Ig^b, lift it using the Witt vector functor and base change to $\mathrm{Spf}\mathcal{O}_{C^\sharp}$. This gives us an affine open cover $\mathfrak{U} = \{\mathfrak{U}_i\}$ of $\mathfrak{X}$, whose adic generic fiber $\{\mathcal{U} = \mathcal{U}_i\}$ is an open cover of Ig^b_C by affinoid perfectoids. Now by almost acyclicity of $\mathcal{O}^+$ on each $\mathcal{U}_i$, we can compute using Cech cohomology that

$$H^1_{\mathrm{an}}(\mathrm{Ig}^b_C, \mathcal{O}^+) =^a \check{H}^1(\mathcal{U}, \mathcal{O}^+) = \check{H}^1(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) = H^1_{\mathrm{Zar}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

Apply proposition 9.28 to Ig^b and tensor it up to $\mathcal{O}_{C^\sharp}$. We see by flat base change that $H^1_{\mathrm{Zar}}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is p- and hence ϖ-torsionfree. This implies the statement we want. $\qquad\square$

9.3. **Compactification of the Igusa stack.** Fix an untilt $\mathrm{Bun}_G \to \mathrm{Spd}E$ and denote Bun_G with this structure map by $\mathrm{Bun}_G^\sharp$. (Hence any small v-stack over Bun_G are also untilted.) Write $\mathcal{O}$, $\mathcal{O}^+$ for the structure sheaves on $\mathrm{Bun}_G^\sharp$. Let $\mathrm{Igs} := \mathrm{Igs}_{K^p}^\circ$ be the Igusa stack at level K^p constructed in section 8, which is equipped with the 0-truncated map

$$\overline{\pi}_{HT}^\circ : \mathrm{Igs} \to \mathrm{Bun}_G$$

descending the Hodge-Tate period map

$$\pi_{HT}^\circ : \mathcal{S}_{K^p}^\circ \to \mathrm{Gr}_{G,\mu}.$$

We now combine the results from 9.1 and 9.2 and set out to construct a minimal compactification of Igs, which extends the cartesian diagram in 8.13 to the minimal compactification $\mathcal{S}_{K^p}^*$ of the Shimura variety.

In terms of notation, for any strictly totally disconnected perfectoid space $T \in$ Perf with a map $T \to \mathrm{Bun}_G^\sharp$, we write Igs_T for the fiber product $\mathrm{Igs} \times_{\mathrm{Bun}_G} T$.

Lemma 9.32. *For any strictly totally disconnected perfectoid space $T \in$ Perf with a map to $\mathrm{Bun}_G^\sharp$, the v-sheaf Igs_T is representable by a qcqs perfectoid space. The global sections $(\mathcal{O}, \mathcal{O}^+)(\mathrm{Igs}_T)$ form a Huber pair with $\mathcal{O}(\mathrm{Igs}_T)$ being a perfectoid Tate ring. In particular, the affinization $(\mathrm{Igs}_T)_0$ is representable by the v-sheaf attached to an affinoid perfectoid space over $\mathrm{Spa}E$.*

Proof. Without loss of generality, we assume $T \to \mathrm{Bun}_G^\sharp$ lies in the image of $\overline{\pi}_{HT}^\circ$. Since T is strictly totally disconnected, by pro-étale surjectivity of the Beauville-Laszlo map and description of the image of $\overline{\pi}_{HT}^\circ$ in 8.9, the map $T \to \mathrm{Bun}_G^\sharp$ lifts to $\mathrm{Gr}_{G,\mu}$ and we fix such a lift. We also assume that the structure map $T \to \mathrm{Bun}_G^\sharp \to \mathrm{Spd}E$ factors through $\mathrm{Spd}K$ for some perfectoid field K, with a chosen pseudo-uniformizer ϖ.

By 8.13 we have

$$\mathrm{Igs}_T \cong \mathrm{Igs} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu} \times_{\mathrm{Gr}_{G,\mu}} T \cong \mathcal{S}_{K^p}^\circ \times_{\mathrm{Gr}_{G,\mu}} T.$$

To show this is representable by a perfectoid space, we note that by [CS19, 2.6.2] the minimal compactification $\mathcal{S}_{K^p}^*$ is representable by a perfectoid space, and the Hodge-Tate map π_{HT}^* on $\mathcal{S}_{K^p}^*$ is affinoid, in the sense that the standard open cover of $\mathrm{Gr}_{G,\mu}$ obtained from a Plücker embedding has affinoid perfectoid preimages (c.f.[Sch15, IV.1.1(i)][26]). In particular, for any connected component s of T, the fiber $\mathcal{S}_{K^p}^* \times_{\mathrm{Gr}_{G,\mu}} s$ is affinoid perfectoid. Therefore we can apply [Sch18, 11.27] to the spatial diamond $\mathcal{S}_{K^p}^* \times_{\mathrm{Gr}_{G,\mu}} T$ and conclude that it is affinoid perfectoid. Since $\mathcal{S}_{K^p}^\circ \times_{\mathrm{Gr}_{G,\mu}} T$ is a quasi-compact open subspace, it is also a perfectoid space and is qcqs.

For the second statement, since Igs_T is qcqs, we can take a finite (analytic) open cover $\{U_i\}$ of Igs_T by affinoid perfectoids with affinoid perfectoid intersections $\{U_{ij}\}$. The ring of global sections of the structure sheaf $\mathcal{O}(\mathrm{Igs}_T)$ has a pseudo-uniformizer ϖ, and $\mathcal{O}^+(\mathrm{Igs}_T) = \mathcal{O}(\mathrm{Igs}_T) \cap \prod_i \mathcal{O}^+(U_i)$ is a ring of definition, which is ϖ-adically complete. By [Sch12, 5.6], to show $\mathcal{O}(\mathrm{Igs}_T)$ is perfectoid, it suffices to show $\mathcal{O}^+(\mathrm{Igs}_T)$ is a perfectoid $K^{\circ a}$-algebra. Since it is a submodule of $\prod_i \mathcal{O}^+(U_i)$,

---

[26]In fact $\mathcal{S}_{K^p}^*$ is integral over the perfectoid space $\mathcal{X}_{\overline{K^p}}^*$ appearing in [Sch15, IV.1.1], so implicitly we also used [BS19, 1.17(1)].

where each $\mathcal{O}^+(U_i)$ is a perfectoid $K^{\circ a}$-module, it is almost flat over K°. By [Sch12, 5.10], it then suffices to show $\mathcal{O}^+(\mathrm{Igs}_T)/\varpi$ is a perfectoid $K^{\circ a}/\varpi$-algebra.

Lemma 9.35 below shows that $\mathcal{O}^+(\mathrm{Igs}_T)/\varpi$ is almost isomorphic to the global section of the sheaf $\mathcal{O}^+/\varpi$ on $\mathrm{Igs}_{T,\mathrm{an}}$. This allows us to translate the problem to characteristic p. Namely, we can consider the tilted structure sheaf $\mathcal{O}^\flat$ (which restricts to the usual structure sheaf on each perfectoid space over $\mathrm{Bun}_G^\sharp$ instead that of its untilt) and reduce the problem to showing perfectoidness of $\mathcal{O}^\flat(\mathrm{Igs}_T)$. This can be computed as the equalizer

$$\mathcal{O}^\flat(\mathrm{Igs}_T) = \mathrm{eq}(\prod_i \mathcal{O}^\flat(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}^\flat(U_{ij})),$$

which is a perfect complete Tate ring and hence perfectoid. $\qquad\square$

Lemma 9.33. *For any strictly totally disconnected perfectoid space T, with a pseudo-uniformizer $\varpi \in \mathcal{O}_T^+(T)$ and any sheaf of $\mathcal{O}_T^+$-modules $\mathcal{F}$ on T_{an}, the value of the sheaf quotient $\mathcal{F}/\varpi$ on a qcqs open $U \subset T$ agrees with $\mathcal{F}(U)/\varpi$.*

Proof. Any qcqs open $U \subset T$ is totally disconnected, so every analytic open cover of it splits and it follows from [Sch18, 7.2] that $H^1_{\mathrm{an}}(U, \mathcal{F}) = 0$. Combined with the short exact sequence of sheaves on U_{an}

$$0 \to \mathcal{F} \xrightarrow{\cdot\varpi} \mathcal{F} \to \mathcal{F}/\varpi \to 0,$$

we have

$$\mathcal{F}(U)/\varpi \cong (\mathcal{F}/\varpi)(U).$$

$\qquad\square$

Remark 9.34. This holds also on $T_{\mathrm{ét}}$ following the same argument.

Lemma 9.35. *Let $T \in \mathrm{Perf}$ be strictly totally disconnected with a map to $\mathrm{Bun}_G^\sharp$, which lies in the image of $\overline{\pi}_{HT}^\circ$. The natural map*

$$\mathcal{O}^+(\mathrm{Igs}_{T,v})/\varpi = \mathcal{O}^+(\mathrm{Igs}_{T,an})/\varpi \to (\mathcal{O}^+/\varpi)(\mathrm{Igs}_{T,an})$$

is an almost isomorphism.

Proof. Denote the projection $\mathrm{Igs}_{T,\mathrm{an}} \to T_{\mathrm{an}}$ by π. We'll show that the natural map

$$(\pi_*\mathcal{O}^+)/\varpi \to \pi_*(\mathcal{O}^+/\varpi)$$

is an almost isomorphism of sheaves on T_{an}. Once this is done, we can take the global sections of both sides and apply lemma 9.33 to $\mathcal{F} = \pi_*\mathcal{O}^+$ to get the desired statement. Fix a lift of $T \to \mathrm{Bun}_G^\sharp$ to $\mathrm{Gr}_{G,\mu}$.

We check on stalks, i.e. by pulling back to each connected component $s : \mathrm{Spa}(C, C^+) \to T$, where C is some complete algebraically closed field with an open and bounded valuation subring C^+. We compute that

$$s^*(\pi_*\mathcal{O}^+)/\varpi = \varinjlim_{s \in U} \mathcal{O}^+(\mathrm{Igs}_U)/\varpi = \mathcal{O}^+(\mathrm{Igs}_s)/\varpi,$$

$$s^*\pi_*(\mathcal{O}^+/\varpi) = \varinjlim_{s \in U}(\mathcal{O}^+/\varpi)(\mathrm{Igs}_U) = (\mathcal{O}^+/\varpi)(\mathrm{Igs}_s).$$

This reduces us to the case when $T = s = \mathrm{Spa}(C, C^+)$ is a geometric point. Here the last equality in the first line follows from the following argument: as the fiber

$\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} s$ is Zariski closed in the affinoid perfectoid space $\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} T$, we have a surjection

$$\mathcal{O}(\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} T) \twoheadrightarrow \mathcal{O}(\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} s).$$

Later (in the proof of 9.38) we will see that under our assumption on the codimension of the boundary of $\mathcal{S}^*_K$, we have

$$\mathcal{O}(\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} s) \cong \mathcal{O}(\mathcal{S}^\circ_{K^p} \times_{\mathrm{Gr}_G} s) = \mathcal{O}(\mathrm{Igs}_s).$$

Since the surjection

$$\mathcal{O}(\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} T) \twoheadrightarrow \mathcal{O}(\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} s) \cong \mathcal{O}(\mathrm{Igs}_s)$$

factors through the restriction to $\mathcal{O}(\mathrm{Igs}_T)$, we have surjectivity of

$$\mathcal{O}(\mathrm{Igs}_T) \to \mathcal{O}(\mathrm{Igs}_s).$$

In particular, any $f \in \mathcal{O}^+(\mathrm{Igs}_s) \subset \mathcal{O}(\mathrm{Igs}_s)$ can be lifted to some $\tilde{f} \in \mathcal{O}(\mathrm{Igs}_T)$. Now the locus $X := \{|\tilde{f}| \leq 1\}$ is a rational open of Igs_T containing Igs_s. By writing s as the intersection of shrinking open and closed quasi-compact open subsets $U \subset T$ and taking complements, we have

$$\mathrm{Igs}_T \backslash X \subset \bigcup_U (\mathrm{Igs}_T \backslash \mathrm{Igs}_U)$$

But the left hand side is quasi-compact, so there must exist some U, such that $\mathrm{Igs}_U \subset X$ and $\tilde{f} \in \mathcal{O}^+(\mathrm{Igs}_U)$. This implies that $\varinjlim_{s \in U} \mathcal{O}^+(\mathrm{Igs}_U) \to \mathcal{O}^+(\mathrm{Igs}_s)$ (and hence also the map modulo ϖ) is surjective. The injectivity is clear: if $f \in \mathcal{O}^+(\mathrm{Igs}_U)$ for some U is mapped to 0 in $\mathcal{O}^+(\mathrm{Igs}_s)/\varpi$, then Igs_s lies in the rational open $\{|f| \leq |\varpi|\} \subset \mathrm{Igs}_T$. Using a similar quasi-compactness argument as above, f is ϖ-divisible in the colimit.

In the case $T = s$ is a geometric point, its lift to $\mathrm{Gr}_{G,\mu}$ determines an Igusa variety Ig^b_C. We know by 9.27 that the inclusion $j : \mathrm{Ig}^b_C \to \mathrm{Igs}_s$ induces an isomorphism of their canonical compactifications towards s and hence $j_*\mathcal{O} = \mathcal{O}$ and $\mathcal{O}^+ \to j_*\mathcal{O}^+$ is an almost isomorphism. We may therefore replace Igs_s by Ig^b_C in the above, and what we need to show becomes that the natural map

$$\mathcal{O}^+(\mathrm{Ig}^b_C)/\varpi \to (\mathcal{O}^+/\varpi)((\mathrm{Ig}^b_C)_{an})$$

is an almost isomorphism. This map is injective with cokernel measured by the ϖ-torsion in $H^1_{an}(\mathrm{Ig}^b_C, \mathcal{O}^+)$, which follows from the cohomology long exact sequence attached to the short exact sequence on $(\mathrm{Ig}^b_C)_{an}$:

$$0 \to \mathcal{O}^+ \xrightarrow{\cdot \varpi} \mathcal{O}^+ \to \mathcal{O}^+/\varpi \to 0.$$

In other words, we are only left to show this ϖ-torsion (almost) vanishes. For this, use corollary 9.31.   □

Now we are ready for the construction.

Definition/Proposition 9.36. The functor on strictly totally disconnected perfectoid spaces over $\mathrm{Bun}^\sharp_G$

$$T \mapsto \mathrm{Hom}_T(T, \overline{(T \times_{\mathrm{Bun}_G} \mathrm{Igs})_0}^{/T}),$$

where $^{/T}$ denotes the canonical compactification towards T, is a sheaf for the v-topology, and hence extends to a v-stack $\mathrm{Igs}^*_{K^p}$ with a 0-truncated map to Bun_G. We call it the *minimal compactification* of Igs. It contains Igs as an open substack.

Remark 9.37. In this section we will drop the subscript K^p from $\mathrm{Igs}^*_{K^p}$ to lighten the notation. The name "minimal compactification" is justified by theorem 9.38 below. Here choosing an untilt of Bun_G is not necessary if we construct affinization using the tilted structure sheaf, c.f. remark 9.5. In fact, as we have seen in the proof of 9.32, for Igs_T the two constructions of affinization agree and in particular, the construction of Igs^* does not depend on the choice of the untilt. Yet using whichever construction as definition, to show that the association $T \mapsto \mathrm{Igs}^*_T$ defines a v-sheaf on $\mathrm{Perf}_{/\mathrm{Bun}_G}$, the comparison in lemma 9.35 will be crucial. Since we have used lemma 9.28 to prove this and hence have to refer to untilts, it seems cleaner to do it globally.

Proof. We write Igs_T for the fiber product $T \times_{\mathrm{Bun}_G} \mathrm{Igs}$ and let $\varpi \in \mathcal{O}(\mathrm{Igs}_T)$ be a pseudo-uniformizer. For any $T' \xrightarrow{f} T$ between strictly totally disconnected perfectoid spaces over Bun_G, the map

$$\mathrm{Igs}_{T'} \to \mathrm{Igs}_T \to (\mathrm{Igs}_T)_0$$

factors through $(\mathrm{Igs}_{T'})_0$ by lemma 9.6 and lemma 9.32. This gives rise to a natural map

$$(\mathrm{Igs}_{T'})_0 \to (\mathrm{Igs}_T)_0 \times_T T'.$$

We need to show that it induces an isomorphism on their canonical compactifications towards T'.

Consider the map

$$\mathcal{O}^+(\mathrm{Igs}_T) \hat{\otimes}_{\mathcal{O}^+(T)} \mathcal{O}^+(T') \to \mathcal{O}^+(\mathrm{Igs}_{T'}).$$

Modulo ϖ, the left hand side becomes

$$\mathcal{O}^+(\mathrm{Igs}_T)/\varpi \otimes_{\mathcal{O}^+(T)/\varpi} \mathcal{O}^+(T')/\varpi \overset{9.35}{=\!^a} (\mathcal{O}^+/\varpi)(\mathrm{Igs}_{T,\mathrm{an}}) \otimes_{\mathcal{O}^+(T)/\varpi} \mathcal{O}^+(T')/\varpi.$$

To compare it with $\mathcal{O}^+(\mathrm{Igs}_{T'})/\varpi$, we take a finite (analytic) open cover $\{U_i\}$ of Igs_T by affinoid perfectoids. Then by the sheaf condition plus flatness of the map $\mathcal{O}^+(T)/\varpi \to \mathcal{O}^+(T')/\varpi$ ([Sch18, 7.23]), we compute that

$$
\begin{aligned}
&(\mathcal{O}^+/\varpi)(\mathrm{Igs}_{T,\mathrm{an}}) \otimes_{\mathcal{O}^+(T)/\varpi} \mathcal{O}^+(T')/\varpi \\
&=^a \mathrm{eq}(\prod_i (\mathcal{O}^+/\varpi)(U_i) \rightrightarrows \prod_{i,j} (\mathcal{O}^+/\varpi)(U_{ij})) \otimes_{\mathcal{O}^+(T)/\varpi} \mathcal{O}^+(T')/\varpi \\
&=^a \mathrm{eq}(\prod_i \mathcal{O}^+(U_i)/\varpi \rightrightarrows \prod_{i,j} \mathcal{O}^+(U_{ij})/\varpi) \otimes_{\mathcal{O}^+(T)/\varpi} \mathcal{O}^+(T')/\varpi \\
&=^a \mathrm{eq}(\prod_i (\mathcal{O}^+(U_i) \hat{\otimes}_{\mathcal{O}^+(T)} \mathcal{O}^+(T'))/\varpi \rightrightarrows \prod_{i,j} (\mathcal{O}^+(U_{ij}) \hat{\otimes}_{\mathcal{O}^+(T)} \mathcal{O}^+(T'))/\varpi) \\
&=^a \mathrm{eq}(\prod_i \mathcal{O}^+(U_i \times_T T')/\varpi \rightrightarrows \prod_{i,j} \mathcal{O}^+(U_{ij} \times_T T')/\varpi) \\
&=^a (\mathcal{O}^+/\varpi)(\mathrm{Igs}_{T',\mathrm{an}}) \\
&\overset{9.35}{=\!^a} \mathcal{O}^+(\mathrm{Igs}_{T'})/\varpi.
\end{aligned}
$$

Apply the above argument to all ϖ^n and pass to the inverse limit. We conclude by ϖ-adically completeness of both sides that

$$\mathcal{O}^+(\mathrm{Igs}_T)\hat{\otimes}_{\mathcal{O}^+(T)}\mathcal{O}^+(T') \to \mathcal{O}^+(\mathrm{Igs}_{T'})$$

is an almost isomorphism. Invert ϖ and we get

$$\mathcal{O}(\mathrm{Igs}_T)\hat{\otimes}_{\mathcal{O}(T)}\mathcal{O}(T') \cong \mathcal{O}(\mathrm{Igs}_{T'}).$$

The left hand side is the global section of the structure sheaf[27] on $(\mathrm{Igs}_T)_0 \times_T T'$ and the right hand side that on $(\mathrm{Igs}_{T'})_0$. Since the canonical compactification depends only on the structure sheaf and not on the integral structure sheaf, we have

$$\overline{(\mathrm{Igs}_{T'})_0}^{/T'} \cong \overline{(\mathrm{Igs}_T)_0 \times_T T'}^{/T'} \cong \overline{(\mathrm{Igs}_T)_0}^{/T} \times_T T'$$

as wished. As totally disconnected affinoid perfectoids form a basis of the v-topology on $\mathrm{Perf}/\mathrm{Bun}_G^\sharp$, the construction in 9.36 defines a v-stack $\mathrm{Igs}^* := \mathrm{Igs}^*_{K^p}$ on Perf with a 0-truncated structure morphism to Bun_G.

We also obtain a map $\mathrm{Igs} \to \mathrm{Igs}^*$ by the functorial-in-T maps

$$\mathrm{Igs}_T \to (\mathrm{Igs}_T)_0 \to \mathrm{Igs}^*_T.$$

That this is an open immersion follows from theorem 9.38 below.   □

Theorem 9.38. *The fiber product $\mathrm{Igs}^* \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu}$ is isomorphic to the minimal compactification $\mathcal{S}^*_{K^p}$ and under this identification, the pullback to $\mathrm{Gr}_{G,\mu}$ of the structure morphism $\mathrm{Igs}^* \to \mathrm{Bun}_G$ is the Hodge-Tate period map on $\mathcal{S}^*_{K^p}$.*

Proof. We take any strictly totally disconnected perfectoid space $T = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$, with a map $T \to \mathrm{Gr}_{G,\mu}$. We obtain a map

$$\mathrm{Igs}^*_T \cong \overline{(\mathcal{S}^\circ_{K^p} \times_{\mathrm{Gr}_{G,\mu}} T)_0}^{/T} \to \mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_{G,\mu}} T.$$

This is the unique map induced by the inclusion of the good reduction locus into the minimal compactification of the Shimura variety:

$$\mathcal{S}^\circ_{K^p} \times_{\mathrm{Gr}_{G,\mu}} T \hookrightarrow \mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_{G,\mu}} T,$$

which factors uniquely through Igs^*_T since the target is affinoid perfectoid and is partially proper over T.

To see that the above map is an isomorphism, since it is proper (by [Sch18, 18.8(vi)]), it suffices to check it is bijective on rank one geometric points. In particular, it suffices to prove in the case $T = \mathrm{Spa}(C, \mathcal{O}_C) =: x$, where C is a complete algebraically closed field and $\mathcal{O}_C$ its ring of integers. The point x defines maps $\mathrm{Ig}^b_C \to \mathcal{S}^\circ_{K^p}$, $\mathrm{Ig}^{b,*}_C \to \mathcal{S}^*_{K^p}$ with images lying in the fiber over x.

To lighten the notation, we let

$$\mathcal{F}^\circ := \mathcal{S}^\circ_{K^p} \times_{\mathrm{Gr}_G} x,$$

$$\mathcal{F}^* := \mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_G} x,$$

and use $\overline{(\cdot)}$ to denote the canonical compactification towards x. We therefore need to compare $\overline{(\mathcal{F}^\circ)_0}$ with $\mathcal{F}^*$.

---

[27]All morphisms are ϖ-adic, so the fiber product exists and on structure sheaves it is given by the completed tensor product, while on the integral structure sheaves it is taking the integral closure of the tensor product in the structure sheaf, followed by a completion.

There is a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Ig}_C^b & \longrightarrow & (\mathrm{Ig}_C^b)_0 & \xrightarrow{\;\iota\;} & \mathrm{Ig}_C^{b,*} \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle \bar{j}} \\
\mathcal{F}^\circ & \longrightarrow & (\mathcal{F}^\circ)_0 & \xrightarrow{\;(*)\;} & \overline{\mathrm{Ig}_C^{b,*}} \cong \mathcal{F}^*,
\end{array}
$$

where the outer square consists of the natural inclusions in 9.27 and the horizontal maps factor through the middle column as the targets are affinoid.

It suffices to show that $(*)$ induces natural bijections on the values on rank one geometric points. This, together with the fact that $(\mathcal{F}^\circ)_0$ is quasi-compact, separated while $\mathcal{F}^*$ is proper, would imply that $\mathcal{F}^*$ is the canonical compactification of $(\mathcal{F}^\circ)_0$ by [CS19, 4.4.2].

Since $\bar{j}$ induces natural bijections on rank one geometric points, it suffices to prove the following two claims:

(1) The map j_0 induces natural bijections on rank one geometric points.

(2) The map ι is an isomorphism.

Proof of claim (1). Since the inclusion $j : \mathrm{Ig}_C^b \to \mathcal{F}^\circ$ induces an isomorphism of their canonical compactifications, we have $j_* \mathcal{O}_{\mathrm{Ig}_C^b} \cong \mathcal{O}_{\mathcal{F}^\circ}$. In particular on global sections

$$
\mathcal{O}_{\mathrm{Ig}_C^b}(\mathrm{Ig}_C^b) \cong \mathcal{O}_{\mathcal{F}^\circ}(\mathcal{F}^\circ).
$$

This proves (1), because points on $(\mathrm{Ig}_C^b)_0$ are in canonical bijection to equivalence classes of maps from $(\mathcal{O}, \mathcal{O}^+)(\mathrm{Ig}_C^b)$ to affinoid perfectoid fields and the subset of rank one points depends only on $\mathcal{O}(\mathrm{Ig}_C^b)$, similarly for $(\mathcal{F}^\circ)_0$. $\square$

Proof of claim (2). Since both sides are affinoid, it suffices to identify the global sections of their structure sheaves. As the global section of $\mathcal{O}$ is obtained from that of $\mathcal{O}^+$ by inverting p, it again suffices to show ι^* induces an almost isomorphism

$$
\mathcal{O}^+(\mathrm{Ig}_C^b) \cong \mathcal{O}^+(\mathrm{Ig}_C^{b,*}).
$$

So we only have to compare the global sections of the structure sheaves on the formal Igusa varieties $\mathrm{Ig}_{\mathcal{O}_C}^b$ and $\mathrm{Ig}_{\mathcal{O}_C}^{b,*}$, which via a Cech cohomology computation as in 9.31, reduces to compare that of their special fibers. The result follows from corollary 9.24 and our assumption on the codimension of the boundary strata being at least two. $\square$

This concludes the proof of proposition 9.38. $\square$

Corollary 9.39. $\mathrm{Igs} \to \mathrm{Igs}^*$ *is an open immersion.*

Proof. Being an open immersion can be checked v-locally by [Sch18, 10.11]. In particular, we can check the statement by pulling back to $\mathrm{Gr}_{G,\mu}$, where this map becomes the open immersion of the good reduction locus into the canonical compactification. $\square$

9.4. **Newton stratification.** Fix an algebraically closed field k containing the residue field $\mathbb{F}_q$ of E. Let $B(G)$ be the Kottwitz set for $G = G_{\mathbb{Q}_p}$ and $B(G, \mu)$ be the subset of μ-admissible elements. We have discussed in section 7 the Newton stratification on $\mathrm{Bun}_{G,\mathrm{Spd}k}$ labelled by $B(G)$. Now we can pull it back to define the Newton stratification on the Igusa stack. We explain this for the minimal compactification. As before, we fix the level subgroup K^p and write Igs^* for $\mathrm{Igs}^*_{K^p}$. For each $[b] \in B(G, \mu)$, we also fix a choice of a representative $\mathbb{X}_b$ of the corresponding isogeny class of p-divisible groups over k and use them to define the perfect Igusa varieties Ig^b over k.

Definition/Proposition 9.40. For any $[b] \in B(G, \mu)$,

$$\mathrm{Igs}^{*,b} := \mathrm{Igs}^* \times_{|\mathrm{Bun}_G|} \{[b]\}$$

is a locally closed substack of $\mathrm{Igs}^*_{\mathrm{Spd}k}$. It is isomorphic to the quotient stack

$$[\overline{\mathrm{Ig}^{b,*,\diamond}}/\tilde{G}_b],$$

where $\mathrm{Ig}^{b,*,\diamond}$ is the v-sheaf attached to the perfect Igusa variety over k using the small diamond functor and the overline denotes its canonical compactification towards $\mathrm{Spd}k$.

Proof. The element $[b]$ defines a surjective map $x_b : \mathrm{Spd}k \to \mathrm{Bun}_G^b$, whose fiber is the v-sheaf of groups $\tilde{G}_b$. Take any complete algebraically closed nonarchimedean field C containing k (without loss of generality k is also the residue field). The structure map $\mathrm{Spa}C \to \mathrm{Spd}k$ is a v-cover. Pick a lift of the point $\mathrm{Spa}C \to \mathrm{Spd}k \to \mathrm{Bun}_G$ to $\mathrm{Gr}_{G,\mu}$. Then by 9.38 and 9.27, we have the following cartesian diagram

$$
\begin{array}{ccc}
\overline{\mathrm{Ig}_C^{b,*}} & \longrightarrow & \mathrm{Spa}C \\
\downarrow & & \downarrow \\
\mathrm{Igs}^{*,b} & \longrightarrow & \mathrm{Bun}_G^b
\end{array}
$$

Via descent along $\mathrm{Spa}C \to \mathrm{Spd}k$, we see that

$$
\begin{array}{ccc}
\overline{\mathrm{Ig}^{b,*,\diamond}} & \longrightarrow & \mathrm{Spd}k \\
\downarrow & & \downarrow \\
\mathrm{Igs}^{*,b} & \longrightarrow & \mathrm{Bun}_G^b.
\end{array}
$$

is cartesian. This identifies $\mathrm{Igs}^{*,b}$ with $[\overline{\mathrm{Ig}^{b,*,\diamond}}/\tilde{G}_b]$. $\square$

Remark 9.41. This identification doesn't depend on the choice of the lift $x : \mathrm{Spa}C \to \mathrm{Gr}_{G,\mu}$. A priori it does, but this dependence is eliminated by fixing an isomorphism $\mathrm{Ig}_{C,x}^{b,*} \cong \mathrm{Ig}_{C,x'}^{b,*}$ for any different lift x'. Assume x and x' correspond to p-divisible groups $\mathcal{H}$ and $\mathcal{H}'$ over $\mathcal{O}_{C^\sharp}$. Then such an isomorphism amounts to choosing a quasi-isogeny between $\mathcal{H}_{\mathcal{O}_{C^\sharp}/p}$ and $\mathcal{H}'_{\mathcal{O}_{C^\sharp}/p}$. But this is a choice we have already made when mapping Ig_C^b into the fiber $\pi^{-1}(x)$, respectively $\pi^{-1}(x')$, where we have to fix a quasi-isogeny between $\mathcal{H}_{\mathcal{O}_{C^\sharp}/p}$, respectively $\mathcal{H}'_{\mathcal{O}_{C^\sharp}/p}$, and the fixed representative $\mathbb{X}_b \times_k \mathcal{O}_{C^\sharp}/p$.

Remark 9.42. Similarly, we have for the canonical compactification $\overline{\mathrm{Igs}^{\circ}_{K^p}}$ of the good reduction locus towards Bun_G, the stratum

$$\overline{\mathrm{Igs}^{\circ,b}_{K^p}} := \overline{\mathrm{Igs}^{\circ}}_{K^p} \times_{|\mathrm{Bun}_G|} \{[b]\}$$

can be identified with $[\overline{\mathrm{Ig}^{b,\diamond}}/\tilde{G}_b]$.

In particular, we have the following corollary.

Corollary 9.43. *The action of* $\tilde{G}_b \cong \underline{\mathrm{Aut}}(\tilde{\mathbb{X}}_b)^{\diamond}$ *on* $\mathrm{Ig}^{b,\diamond}$ *using the moduli interpretation extends to* $\mathrm{Ig}^{b,*,\diamond}$.

10. Hecke action

This short section is devoted to the part of conjecture 1.1 regarding the away-from-p Hecke action on the Igusa stack.

We adopt the notation from section 5 and fix the level at p to be $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$. The adelic group $G(\mathbb{A}_f^p)$-acts on the inverse system $\{S_{K_p K^p}\}_{K^p}$ as follows: for any $g \in G(\mathbb{A}_f^p)$, there is a map between $\mathcal{O}_E$-schemes

$$\gamma_g : S_{K_p K^p} \to S_{K_p g^{-1} K^p g}$$

sending a tuple $(A, \iota, \lambda, \bar{\eta} = K^p \cdot \eta)$ to $(A, \iota, \lambda, g^{-1}\bar{\eta} = g^{-1}K^p g \cdot g^{-1}\eta)$. By Lan [Lan13, 7.2.5], this action extends to the system of minimal compactifications $\{S^*_{K_p K^p}\}_{K^p}$. By taking attached formal schemes, adic spaces or diamonds, we have a $G(\mathbb{A}_f^p)$-action on all these variants of Shimura varieties.

Similarly, since the action is simply on the away-from-p level structures, we can define an action of $G(\mathbb{A}_f^p)$ on the system $\{\mathrm{Igs}^{\circ}_{K^p}\}_{K^p}$ where K^p runs through compact open subgroups of $G(\mathbb{A}_f^p)$: for any $g \in G(\mathbb{A}_f^p)$, define a map of fibered categories

$$\mathrm{Igs}^{\mathrm{pre},\circ}_{K^p} \to \mathrm{Igs}^{\mathrm{pre},\circ}_{g^{-1} K^p g}$$

which sends a tuple $(A_0, \iota, \lambda, \bar{\eta})$ to $(A_0, \iota, \lambda, g^{-1}\bar{\eta})$ and a quasi-isogeny between two tuples to the same quasi-isogeny. This induces a map of v-stacks

$$\bar{\gamma}^{\circ}_g : \mathrm{Igs}^{\circ}_{K^p} \to \mathrm{Igs}^{\circ}_{g^{-1} K^p g}.$$

Under the assumption that the minimal compactification boundary of the Shimura variety has codimension at least two, c.f. our classification in 9.25, the Hecke action extends to the minimal compactifications $\{\mathrm{Igs}^*_{K^p}\}_{K^p}$, which is clear from the formula of $\mathrm{Igs}^*_{K^p}$ given in definition 9.36. By checking on the moduli problems, we have

Proposition 10.1. *Let* $K = K_p K^p$ *with* $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ *as before. For any* $g \in G(\mathbb{A}_f^p)$, *the following diagrams of v-stacks over* Bun_G *commute (on the nose, as all maps to* Bun_G *are 0-truncated)*

$$
\begin{array}{ccc}
\mathcal{S}^*_K & \xrightarrow{\gamma^*_g} & \mathcal{S}^*_{g^{-1}Kg} \\
\downarrow & & \downarrow \\
\mathrm{Igs}^*_{K^p} & \xrightarrow{\bar{\gamma}^*_g} & \mathrm{Igs}^*_{g^{-1}K^p g}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Igs}^*_{K^p} & & \\
\bar{\gamma}^*_g \downarrow & \xrightarrow{\pi^*_{HT}} & \\
\mathrm{Igs}^*_{g^{-1}K^p g} & \xrightarrow{\pi^*_{HT}} & \mathrm{Bun}_G
\end{array}
$$

Proof. The right diagram is commutative by construction. For the commutativity of the left diagram, it suffices to pullback the right diagram along $BL : \mathrm{Gr}_{G,\mu} \to \mathrm{Bun}_G$. When restricted to the good reduction locus of the Shimura variety and the open substack $\mathrm{Igs}^\circ_{K^p}$, it is clear that $\bar{\gamma}^\circ_g$ pulls back to γ_g. But as we have shown in 9.38, on a strictly totally disconnected test object T, we have

$$\mathcal{S}^*_{K^p} \times_{\mathrm{Gr}_{G,\mu}} T \cong \overline{(\mathcal{S}^\circ_{K^p} \times_{Gr_G} T)_0}^{/T},$$

so the map γ^*_g is uniquely determined by its restriction to $\mathcal{S}^\circ_{K^p}$. Hence it must agree with the pullback of $\bar{\gamma}^*_g$. $\qquad\square$

11. Integral model

Consider the formal integral model $\mathscr{S}_K$ over $\mathrm{Spf}\mathcal{O}_E$ of the Shimura variety at hyperspecial level and its attached v-sheaf $\mathscr{S}^\diamond_K$ over $\mathrm{Spd}\mathcal{O}_E$ as in section 5. We have the reduction map

$$\mathrm{red} : \mathscr{S}^\diamond_K \to \mathrm{Igs}^\circ_{K^p}$$

from remark 8.4. The goal of this section is to extend the fiber product structure on $\mathcal{S}^\circ_K$ to this v-sheaf integral model, substituting the left vertical map in the cartesian diagram in 8.15 by the above map. In the integral model diagram the map $\bar{\pi}^\circ_{HT} : \mathrm{Igs}^\circ_{K^p} \to \mathrm{Bun}_G$ remains unchanged, but the minuscule Schubert cell Gr_{G,μ,K_p} will be replaced by a moduli stack of shtukas with extra structures. Again the first two subsections are purely local and hence works for general reductive groups. Starting from section 11.3 we switch back to the global PEL-setup.

We begin by recollecting some definitions. See 7.3 for the notation $\mathcal{Y}, \mathcal{Y}_I$.

11.1. $\mathcal{G}$-torsors, shtukas and BKF-modules.
Let $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ be a reductive group scheme. We denote by $\mathcal{G}$ the adic space whose functor of point takes any adic space S over $\mathrm{Spa}\mathbb{Z}_p$ to $G_{\mathbb{Z}_p}(\mathcal{O}_S(S))$. Assume $G_{\mathbb{Z}_p} = \mathrm{Spec}(A)$, then $\mathcal{G} = \mathrm{Spa}(A, A^+)$, where A^+ is the integral closure of $\mathbb{Z}_p$ in A, equipped with discrete topology. By abuse of notation, we will also write $\mathcal{G}(\mathbb{Z}_p)$ for $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ and the notation can be understood as the $\mathrm{Spa}\mathbb{Z}_p$ points of $\mathcal{G}$. We have the notion of $\mathcal{G}$-torsors on sousperfectoid analytic adic spaces, c.f. definition 7.7.

Definition 11.1. ([SW20, 19.5.2]) Let X be a sousperfectoid analytic adic space over $\mathbb{Z}_p$. A $\mathcal{G}$-torsor $\mathcal{P}$ is an étale sheaf on X with a $\mathcal{G}$-action which is étale locally $\mathcal{G}$-equivariantly isomorphic to $\mathcal{G}$. Equivalently, this is an exact tensor functor

$$\mathrm{Rep}G_{\mathbb{Z}_p} \to \mathrm{Bun}(X),$$

from the exact symmetric monoidal category of algebraic representations of $G_{\mathbb{Z}_p}$ on finite free $\mathbb{Z}_p$-modules, to that of vector bundles on X.

Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}/\mathrm{Spd}\mathbb{Z}_p$ be an affinoid perfectoid space of characteristic p with a pseudo-uniformizer $\varpi \in R^+$. Let $S^\sharp$ be the untilt over $\mathbb{Z}_p$ defined by the structure map $S \to \mathrm{Spd}\mathbb{Z}_p$. We view it as a closed Cartier divisor on the analytic adic space $\mathcal{Y}_{[0,\infty)}(S)$. Write φ_S for the Frobenius on $\mathcal{Y}_{[0,\infty)}(S)$.

Definition 11.2. ([SW20, 11.4.1]) A shtuka over S with one leg at $S^\sharp$ is a pair $(\mathcal{P}, \varphi_{\mathcal{P}})$, where $\mathcal{P}$ is a vector bundle on $\mathcal{Y}_{[0,\infty)}(S)$; and $\varphi_{\mathcal{P}}$ is an isomorphism

$$\varphi_S^* \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S)\backslash S^\sharp} \cong \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S)\backslash S^\sharp},$$

which is meromorphic along $S^\sharp$ in the sense of [SW20, 5.3.5]. A map between two shtukas $(\mathcal{P}, \varphi_{\mathcal{P}})$ and $(\mathcal{P}', \varphi_{\mathcal{P}'})$ is a map of vector bundles $f : \mathcal{P} \to \mathcal{P}'$ such that $\varphi_{\mathcal{P}'} \circ \varphi_S^* f = f \circ \varphi_{\mathcal{P}}$.

Definition 11.3. ([SW20, 23.1]) A $\mathcal{G}$-shtuka over S with one leg at $S^\sharp$ is a pair $(\mathcal{P}, \varphi_{\mathcal{P}})$, where $\mathcal{P}$ is a $\mathcal{G}$-torsor on $\mathcal{Y}_{[0,\infty)}(S)$; and $\varphi_{\mathcal{P}}$ is an isomorphism

$$\varphi_S^* \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S)\backslash S^\sharp} \cong \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S)\backslash S^\sharp},$$

which is meromorphic along $S^\sharp$. A map between two $\mathcal{G}$-shtukas $(\mathcal{P}, \varphi_{\mathcal{P}})$ and $(\mathcal{P}', \varphi_{\mathcal{P}'})$ is a map of $\mathcal{G}$-torsors $f : \mathcal{P} \to \mathcal{P}'$ such that $\varphi_{\mathcal{P}'} \circ \varphi_S^* f = f \circ \varphi_{\mathcal{P}}$.

We have a notion of boundedness: recall from [SW20, 20.3.1] that the mixed characteristic Beilinson-Drinfeld affine Grassmannian attached to $\mathcal{G}$ is the small v-sheaf $\mathrm{Gr}_{\mathcal{G}}$ over $\mathrm{Spd}\mathbb{Z}_p$ sending $S \in \mathrm{Perf}$ to the set of triples consisting of an untilt $S^\sharp$, a $\mathcal{G}$-torsor $\mathcal{F}$ on $\mathcal{Y}_{[0,\infty)}(S)$ and a trivialization α of $\mathcal{F}|_{\mathcal{Y}_{[0,\infty)}(S)\backslash S^\sharp}$, meromorphic along $S^\sharp$. Or equivalently this sends S to the set of triples $\{(S^\sharp, \mathcal{F}, \alpha)\}$, where $S^\sharp$ is an untilt, $\mathcal{F}$ is a $G_{\mathbb{Z}_p}$-torsor on $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$ and α is a trivialization over $\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))$, where if $R^\sharp$ has characteristic p, $B_{\mathrm{dR}}^+(R^\sharp)$ is defined to be $W(R^\sharp)$ and $B_{\mathrm{dR}}(R^\sharp)$ is $W(R^\sharp)[1/p]$.

Fix a maximal torus and a Borel $T \subset B \subset G_{\overline{\mathbb{Q}}_p}$. Let $\mu : \mathbb{G}_{m\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ be a dominant cocharacter, whose $G(\overline{\mathbb{Q}}_p)$-conjugacy class is defined over some finite extension E over $\mathbb{Q}_p$. (Later we will take μ to be the dominant cocharacter and E the local field determined by the global PEL-datum, so we don't introduce new notation.) We have as in [SW20, 20.3.5], a Schubert variety $\mathrm{Gr}_{\mathcal{G},\mathcal{O}_E,\leq\mu} \subset \mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}$ over $\mathrm{Spd}\mathcal{O}_E$, which is the subfunctor where geometric-pointwise on a test object S, the relative position of $\mathcal{F}$ and the trivial $\mathcal{G}$-torsor is bounded by μ, i.e. if the point is of characteristic p, the pair $(\mathcal{F}, \alpha)$ lies in the closure of the Schubert cell labelled by μ in the Witt vector affine Grassmannian; otherwise in that of the B_{dR}^+-affine Grassmannian.

Definition 11.4. Given $S \in \mathrm{Perf}$ with an untilt $S^\sharp$ over $\mathrm{Spa}\mathcal{O}_E$, we say a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_{\mathcal{P}})$ over S with one leg at $S^\sharp$ is *bounded by* μ, if geometric-pointwise on S, the relative position of $\varphi_S^* \mathcal{P}$ and $\mathcal{P}$, completed along $S^\sharp$ is bounded by μ. Here the boundedness is understood in the following sense: for any geometric point $\bar{s} \in S$, choose a trivialization $\mathcal{G} \cong (\varphi_S^* \mathcal{P})_{\bar{s}^\sharp}$ of the stalk of the $\mathcal{G}$-torsor $\varphi_S^* \mathcal{P}$ at $\bar{s}^\sharp$, [28] so that the pair $(\mathcal{P}_{\bar{s}^\sharp}, \varphi_{\mathcal{P},\bar{s}^\sharp})$ defines an $\bar{s}$-point of $\mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}$. Then this point lies in $\mathrm{Gr}_{\mathcal{G},\mathcal{O}_E,\leq\mu}$.

Remark 11.5. If $G_{\mathbb{Z}_p}$ is a smooth parahoric group scheme, we can also define $\mathcal{G}$-shtukas. When the cocharacter μ is minuscule, we define boundedness by μ, by requiring pointwise on a base S, upon choosing trivializations, the point of $\mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}$ given by $(\mathcal{P}_{\bar{s}^\sharp}, \varphi_{\mathcal{P},\bar{s}^\sharp})$ lies in the local model $\mathbb{M}_{\mathcal{G},\mu}$. (Recall that this is the v-sheaf

---

[28]This is possible by smoothness of the group scheme $G_{\mathbb{Z}_p}$ and henselianness of the completion of $\mathcal{O}_{\mathcal{Y}_{[0,\infty)}(S)}$ at $\bar{s}^\sharp$.

closure of the Schubert cell $\mathrm{Gr}_{G_{\mathbb{Q}_p},\mu}$ on the generic fiber.) For more details, the reader is referred to [PR21, section 2.4.4].

As objects with more favorable algebraic properties, we introduce Breuil-Kisin-Fargues modules with $\mathcal{G}$-structure and discuss their relation to $\mathcal{G}$-shtukas.

Definition 11.6. Let S, $S^\sharp$ be as above. Write φ_S for the Frobenius on $W(R^+)$ and ξ is a chosen generator of the kernel of Fontaine's map $\theta : W(R^+) \to R^{\sharp+}$. A *Breuil-Kisin-Fargues module (BKF-module)*, respectively a $\mathcal{G}$-BKF-module, over S with a leg at $S^\sharp$ is a pair (M, φ_M) consisting of a finite projective $W(R^+)$-module M, respectively a $G_{\mathbb{Z}_p}$-torsor M over $\mathrm{Spec}(W(R^+))$, together with an isomorphism

$$\varphi_M : (\varphi_S^* M)[1/\xi] \xrightarrow{\sim} M[1/\xi].$$

A $\mathcal{G}$-BKF module (M, φ_M) defines a $\mathcal{G}$-shtuka by the following construction: restrict (M, φ_M) to the punctured spectrum $\mathrm{Spec}(W(R^+)) \backslash V(I)$, where I is ideal $I = (p, [\varpi])$. Using Tannakian formalism and the equivalence between the exact tensor category of vector bundles on $\mathrm{Spec}(W(R^+)) \backslash V(I)$ and that on $\mathcal{Y}(S)$ due to Kedlaya (c.f. [PR21, 2.1.5 a)]), this gives a $\mathcal{G}$-torsor with meromorphic Frobenius on the latter. Further restricting to $\mathcal{Y}_{[0,\infty)}(S)$, one obtains a $\mathcal{G}$-shtuka. In the rest of this subsection, we will show that when S is a product of rank one geometric points, all $\mathcal{G}$-shtukas come from $\mathcal{G}$-BKF-modules in such a way. For this we need the following key input from the recent work of I. Gleason and A. Ivanov [GI23].

Theorem 11.7. ([GI23, Theorem 8.6], c.f. [GI23, Corollary 1.9]) *Let S be a product of rank one geometric points with untilt $S^\sharp$, then the tensor category of shtukas over S with a leg at $S^\sharp$ is equivalent to that of vector bundles with meromorphic Frobenius over $\mathcal{Y}(S) = \mathcal{Y}_{[0,\infty]}(S)$.*

Corollary 11.8. *Let S be a product of rank one geometric points with untilt $S^\sharp$, then the category of $\mathcal{G}$-shtukas over S with a leg at $S^\sharp$ is equivalent to that of $\mathcal{G}$-torsors with meromorphic Frobenius over $\mathcal{Y}(S)$.*

Proof. Combine lemma 11.7 and Tannakian formalism: since the locus where $p = 0$ has nothing to do with the extension we are interested in, we can restrict a given shtuka to $\mathcal{Y}_{(0,\infty)}(S)$ and consider the statement there. But $\mathcal{G}$-torsors on $\mathcal{Y}_{(0,\infty]}(S)$ are the same as exact tensor functors from the category of finite dimensional algebraic $\mathbb{Q}_p$-representations of $G_{\mathbb{Q}_p}$ to vector bundles on $\mathcal{Y}_{(0,\infty]}(S)$. Since the representation category is semi-simple, the condition on exactness is vacuous and the tensor equivalence for vector bundles implies the equivalence for $\mathcal{G}$-torsors. □

Lemma 11.9. *Let $S = \mathrm{Spa}(R, R^\circ)$ in Perf be a product of rank one geometric points. Then $G_{\mathbb{Z}_p}$-torsors on $\mathrm{Spec}(R)$ are trivial.*

Proof. By [Ked19, 1.4.2], pullback along the map of locally ringed spaces

$$S \to \mathrm{Spec}(R)$$

defines an exact equivalence between the category of vector bundles on both sides. It is direct to check that this equivalence is symmetric monoidal. Hence using Tannakian formalism, we obtain an equivalence between the category of $\mathcal{G}$-torsors on S and that of $G_{\mathbb{Z}_p}$-torsors on $\mathrm{Spec}(R)$. Since S is strictly totally disconnected, étale $\mathcal{G}$-torsors on S are trivial, so are $G_{\mathbb{Z}_p}$-torsors on $\mathrm{Spec}(R)$. □

Proposition 11.10. *Let $S = \mathrm{Spa}(R, R^\circ)$ in* Perf *be a product of rank one geometric points with an untilt $S^\sharp$ over $\mathbb{Z}_p$. Then pullback along*

$$\mathcal{Y}_{[0,\infty)}(S) \to \mathrm{Spa}(W(R^\circ), W(R^\circ)) \to \mathrm{Spec}(W(R^\circ)),$$

defines an equivalence of categories between $\mathcal{G}$-BKF-modules over S with a leg at $S^\sharp$ and $\mathcal{G}$-shtukas over S with a leg at $S^\sharp$.

If $R^\circ = \prod_{i \in I} \mathcal{O}_{C_i}$, where $\mathcal{O}_{C_i}$ is the ring of integers in some complete algebraically closed non-archimedean field C_i, let $s_i := \mathrm{Spa}(C_i, \mathcal{O}_{C_i})$. Then the inverse equivalence is given by the following construction: given a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_{\mathcal{P}})$, restrict it to $\mathcal{Y}_{[0,\infty)}(s_i)$ for each i. The restriction $(\mathcal{P}_i, \varphi_{\mathcal{P}_i})$ extends uniquely to a $\mathcal{G}$-BKF-module (M_i, φ_{M_i}) by [PR21, 2.4.6]. Their product (M, φ_M) is the desired $\mathcal{G}$-BKF module.

Proof. The restriction functor from $\mathcal{G}$-BKF-modules to $\mathcal{G}$-torsors on $\mathcal{Y}_{[0,\infty]}(S)$ is fully faithful, by combining Tannakian formalism with the following results: the restriction of $G_{\mathbb{Z}_p}$-torsors from $\mathrm{Spec}(W(R^\circ))$ to the punctured spectrum

$$\mathrm{Spec}(W(R^\circ))\backslash V(I)$$

is fully faithful, c.f. [Ans22b, 8.4], and the pullback functor on the category of vector bundles along the map of locally ringed spaces

$$\mathcal{Y}(S) \to \mathrm{Spec}(W(R^\circ))\backslash V(I)$$

is an exact tensor equivalence due to Kedlaya, c.f. [PR21, 2.1.5].

To show essential surjectivity, given a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_{\mathcal{P}})$ over S with one leg at $S^\sharp$, we can first apply lemma 11.8 to extend it to a $\mathcal{G}$-torsor with meromorphic Frobenius $(\tilde{\mathcal{P}}, \varphi_{\tilde{\mathcal{P}}})$ over $\mathcal{Y}(S)$. It suffices to show that the underlying $\mathcal{G}$-torsor of $\tilde{\mathcal{P}}$ is trivial, so that we can take the trivial extension to get a $\mathcal{G}$-BKF-module. For this, we adapt an argument from [Ans22b, 9.2], which deals with the case of S being a point.

By GAGA and Beauville-Laszlo gluing, we can view $\tilde{P}$ as a $G_{\mathbb{Z}_p}$-torsor on the scheme

$$\mathrm{Spec}(W(R^\circ))\backslash V(I)$$

and it is glued from a $G_{\mathbb{Z}_p}$-torsor on $\mathrm{Spec}(W(R^\circ)[1/p])$ and one on $\mathrm{Spf}(W(R))$ over $W(R)[1/p]$. But $G_{\mathbb{Z}_p}$-torsors on both are trivial: the statement over $W(R)$ follows from 11.9, smoothness of $G_{\mathbb{Z}_p}$ and henselianness of the pair

$$(W(R), \ker(W(R) \twoheadrightarrow R)),$$

while the statement over $W(R^\circ)[1/p]$ is proven in [Ans22b, 11.5].

This means $\tilde{P}$ can be described by an element of the double coset

$$G_{\mathbb{Z}_p}(W(R))\backslash G_{\mathbb{Q}_p}(W(R)[1/p])/G_{\mathbb{Q}_p}(W(R^\circ)[1/p]).$$

By comparing with the presheaf of the Witt vector affine Grassmannian $\mathrm{Gr}^W_{G_{\mathbb{Z}_p}}$ ([SW20, 20.3.3]), the above set (in fact its étale sheafification on $\mathrm{Spec}(R)$, which doesn't change its value) measures R-points of $\mathrm{Gr}^W_{G_{\mathbb{Z}_p}}$ that don't come from R°-points. But given any R-point of $\mathrm{Gr}^W_{G_{\mathbb{Z}_p}}$, its restriction to C_i extends uniquely to an $\mathcal{O}_{C_i}$-point, since $\mathrm{Gr}^W_{G_{\mathbb{Z}_p}}$ is ind-proper (see [Zhu17, 1.5.2]). The collection of these defines uniquely a $\mathrm{Spec}(R^\circ)$-point of $\mathrm{Gr}^W_{G_{\mathbb{Z}_p}}$ as below: assume the original R-point

lies in some proper subscheme X. Take a finite affine open cover $\{X_j\}$, $j = 1, \ldots, n$ of X. We obtain a finite partition

$$I = \coprod_{j=1}^{n} I_j$$

of the index set I, where $i \in I_j$ if the image of $\mathrm{Spec}(\mathcal{O}_{C_i})$ lies in X_j. Define $R_j^{\circ} := \prod_{i \in I_j} \mathcal{O}_{C_i}$. Since X_j is affine, the collection of maps $\mathrm{Spec}(\mathcal{O}_{C_i}) \to X_j$, $i \in I_j$ determines a unique map

$$\mathrm{Spec}(R_j^{\circ}) \to X_j.$$

As $\{\mathrm{Spec}(R_j^{\circ})\}$, $j = 1, \ldots, n$ is a cover of $\mathrm{Spec}(R^{\circ})$ by open-and-closed subspaces, we get a unique map

$$\mathrm{Spec}(R^{\circ}) \to \mathrm{Gr}_{G_{\mathbb{Z}_p}}^{W}.$$

What's more, by checking for each j we see that when restricted along $\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(R^{\circ})$, we get back the given R-point. This shows that the double coset we considered is a singleton and $\tilde{\mathcal{P}}$ is trivial as desired.

To see that the inverse equivalence takes the stated form, assume $(\mathcal{P}, \varphi_{\mathcal{P}})$ extends to $(M', \varphi_{M'})$. Then the restriction $(M_i', \varphi_{M_i'})$ to each s_i extends the restriction $(\mathcal{P}_i, \varphi_{\mathcal{P}_i})$ of the given $\mathcal{G}$-shtuka. By uniqueness, $(M_i', \varphi_{M_i'}) = (M_i, \varphi_{M_i})$ and hence $(M', \varphi_{M'}) = (M, \varphi_M)$.   □

Remark 11.11. Over a product of rank one geometric points S as in the proposition, by going through the equivalences

$$\{\mathcal{G}\text{-shtukas over } S\} \Leftrightarrow \{\mathcal{G}\text{-BKF-modules over } S\}$$

$$\Leftrightarrow \prod_{i}{}' \{\mathcal{G}\text{-BKF-modules over } s_i\}$$

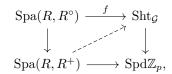$$\Leftrightarrow \prod_{i}{}' \{\mathcal{G}\text{-shtukas over } s_i\},$$

we see that a $\mathcal{G}$-shtuka over S is uniquely determined by its restriction to each s_i. Here we use $\prod_i{}'$ to mean the full subcategory of the product category in which an object is a collection of $\mathcal{G}$-BKF-modules, respectively $\mathcal{G}$-shtukas, for all $i \in I$, that are commonly bounded by some cocharacter μ.[29] If a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_{\mathcal{P}})$ is sent to the collection $(\mathcal{P}_i, \varphi_{\mathcal{P}_i})$ under the above composition of equivalences, we will call $(\mathcal{P}, \varphi_{\mathcal{P}})$ the product of $(\mathcal{P}_i, \varphi_{\mathcal{P}_i})$'s.

11.2. **Moduli of $\mathcal{G}$-shtukas.** We now define the moduli stack of $\mathcal{G}$-shtukas and record some of its geometric properties.

Definition 11.12. Let $\mathrm{Sht}_{\mathcal{G}}$ be the presheaf of groupoids on the v-site of $\mathrm{Perf}/\mathrm{Spd}\mathbb{Z}_p$ sending S to the groupoid of $\mathcal{G}$-shtukas over S with a leg at $S^{\sharp}$, where $S^{\sharp}$ is the untilt of S over $\mathrm{Spa}\mathbb{Z}_p$ determined by the structure map $S \to \mathrm{Spd}\mathbb{Z}_p$. This is a v-stack by [SW20, 19.5.3].

---

[29] We define boundedness of $\mathcal{G}$-BKF-modules by that of their attached $\mathcal{G}$-shtukas.

Proposition 11.13. *The structure map* $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Spd}\mathbb{Z}_p$ *is quasi-separated. For any perfectoid Tate algebra R with an open bounded integrally closed subring R^+ and any commutative diagram with solid arrows*

$$
\begin{array}{ccc}
\mathrm{Spa}(R, R^\circ) & \xrightarrow{\quad f \quad} & \mathrm{Sht}_{\mathcal{G}} \\
\downarrow & & \downarrow \\
\mathrm{Spa}(R, R^+) & \longrightarrow & \mathrm{Spd}\mathbb{Z}_p,
\end{array}
$$

there is a unique (up to isomorphism) dotted arrow making the whole diagram commute up to a natural transform given by an automorphism of f.

Proof. We first check that the diagonal map is quasi-separated.

For this, we need to show that the diagonal map from $\mathrm{Sht}_{\mathcal{G}}$ to the inertia stack is quasi-compact. This is equivalent to saying that over an affinoid perfectoid test object T with untilt $T^\sharp$, assume we are given a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_{\mathcal{P}})$ over T with one leg at $T^\sharp$ together with an automorphism $g \in \mathcal{G}(\mathcal{Y}_{[0,\infty)}(T))$. Then the locus on T where g equals the neutral element e is quasi-compact. But one can show this is a closed condition by checking that the locus where g and e don't agree is open: consider the projection

$$
\pi : |\mathcal{Y}_{[0,\infty)}(T)| \to |T|
$$

and view g, e as sections of the sheaf $\pi_* \mathcal{G}$ on T. If for some $t \in |T|$, the restrictions g_t, e_t to the fiber $\pi^{-1}(t)$ are not equal, we write t as the intersection of quasi-compact open and closed subsets $U_i \subset T$, $i \in I$ for some index set I. It suffices to show there exists some i, $g_u \neq e_u$ for all $u \in U_i$.

Assuming not, then for each i, there exists some $t_i \in U_i$, such that $g_{t_i} = e_{t_i}$. We form a product of points S using the t_i's. By construction there is a map $f : S \to T$ determined by the $t_i \to T$'s. Since the image is quasi-compact, it must contain t. This is because otherwise the image lies in $\bigcup_i T \backslash U_i$ and hence $T \backslash U_i$ for some fixed i, which contradicts the choice of t_i's. Now $f^* g$ and $f^* e$ agreeing on all t_i's forces them to agree on the set $|S|^{\mathrm{cld}}$ of closed points in S by density of the t_i's in $|S|^{\mathrm{cld}}$. Yet $f^{-1}(t) = \bigcap_i f^{-1}(U_i)$ must contain some closed point. (Otherwise $|S|^{\mathrm{cld}} \subset \bigcup_i S \backslash f^{-1}(U_i)$. By quasi-compactness of $|S|^{\mathrm{cld}} \approx \pi_0(S)$, there would exist some i such that
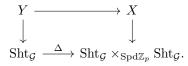
$$
\{t_i\} \subset |S|^{\mathrm{cld}} \subset S \backslash f^{-1}(U_i).
$$

This would be a contradiction to the choices of t_i's.) Therefore choose any closed point $x \in f^{-1}(t)$ we have

$$
g_t = f^* g_x = f^* e_x = e_t.
$$

This contradicts the choice of t. Hence we cannot choose this sequence of t_i's, and there must exist some U_i such that $g_u \neq e_u$ for all $u \in U_i$. This finishes the proof of quasi-separatedness.

For quasi-compactness, take any map from an affinoid perfectoid space X (without loss of generality of the form $\mathrm{Spa}(R, R^\circ)$ with a pseudo-uniformizer $\varpi \in R$) to $\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}\mathbb{Z}_p} \mathrm{Sht}_{\mathcal{G}}$ and consider the cartesian diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Sht}_{\mathcal{G}} & \xrightarrow{\ \Delta\ } & \mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}\mathbb{Z}_p} \mathrm{Sht}_{\mathcal{G}}.
\end{array}
$$

We need to show Y is quasi-compact. For this, we combine remark 11.11 and criterion 2.18. For readers' convenience, we spell out the argument. Fix a representative $y = \mathrm{Spa}(C_y, C_y^+)$ for each geometric point of Y, the composition $y \to Y \to X$ determines a map of affinoid rings $(R, R^\circ) \to (C_y, C_y^+)$ such that the image of ϖ is some pseudo-uniformizer ϖ_y. We can form a product of points S by letting

$$A^+ := \prod_{y \in |Y|} C_y^+, \ \varpi_A := (\varpi_y), \ A := A^+[\frac{1}{\varpi_A}],$$

and taking S to be $\mathrm{Spa}(A, A^+)$. This comes equipped with a map $S \to X$ induced by $R^+ \to \prod_{y \in |Y|} C_y^+$. It suffices to show this map factors through a surjection of v-sheaves $S \to Y$. The map factoring through Y amounts to saying that the two $\mathcal{G}$-shtukas $(\mathcal{P}, \varphi_{\mathcal{P}})$, $(\mathcal{P}', \varphi_{\mathcal{P}'})$ on S obtained by pulling back along $S \to X \to \mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}\mathbb{Z}_p} \mathrm{Sht}_{\mathcal{G}}$ are isomorphic. But it follows from the construction that $(\mathcal{P}_y, \varphi_y) \cong (\mathcal{P}'_y, \varphi_{\mathcal{P}'_y})$ for each $y \in |Y|$. By remark 11.11, the product of these isomorphisms gives an isomorphism $(\mathcal{P}, \varphi_{\mathcal{P}}) \cong (\mathcal{P}', \varphi_{\mathcal{P}'})$. This implies the factorization $S \to Y$. Also, since $S \to Y \to X$ is qcqs and the second map is quasi-separated by quasi-separatedness of Δ, $S \to Y$ is qcqs by cancellation. It is surjective on topological spaces by construction and hence a surjection of v-sheaves by [SW20, 17.4.9]. This finishes the proof of quasi-compactness.
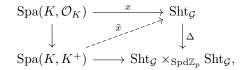
For the last claim, use Tannakian formalism and the tensor exact equivalence for restriction of vector bundles along

$$\mathcal{Y}_{[0,\infty)}(R, R^\circ) \to \mathcal{Y}_{[0,\infty)}(R, R^+).$$

The latter is the content [PR21, 2.1.1], c.f. remark 7.3. $\square$

Corollary 11.14. *The diagonal $\Delta : \mathrm{Sht}_{\mathcal{G}} \to \mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}\mathbb{Z}_p} \mathrm{Sht}_{\mathcal{G}}$ is proper.*

Proof. Use the characterization of properness by [Sch18, 18.3]. Since the diagonal is 0-truncated, qcqs, it suffices to check the valuative criterion, i.e. for any perfectoid field K, with ring of integers $\mathcal{O}_K$ and open bounded valuation subring $K^+ \subset K$ and any commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spa}(K, \mathcal{O}_K) & \xrightarrow{\quad x \quad} & \mathrm{Sht}_{\mathcal{G}} \\
\downarrow & {\scriptstyle \tilde{x}}\nearrow & \downarrow{\scriptstyle \Delta} \\
\mathrm{Spa}(K, K^+) & \xrightarrow{\quad\quad} & \mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}\mathbb{Z}_p} \mathrm{Sht}_{\mathcal{G}},
\end{array}
$$

there should be a unique dotted arrow making the diagram commute. Post-composing with the structure maps to $\mathrm{Spd}\mathbb{Z}_p$, we can obtain a unique (up to isomorphism) $\tilde{x}$ from the valuative criterion for $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Spd}\mathbb{Z}_p$, making the upper left triangle commute up to an automorphism of x. But the commutativity of the lower right triangle rigidifies the situation, i.e. we can modify $\tilde{x}$ with a unique isomorphism making the diagram (of sheaves over $\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Spd}\mathbb{Z}_p} \mathrm{Sht}_{\mathcal{G}}$) commute. $\square$

Let K_p be $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$. To relate $\mathrm{Sht}_{\mathcal{G}}$ to Bun_G, a quick observation is that the generic fiber of $\mathrm{Sht}_{\mathcal{G}}$ can be identified with the quotient $\mathrm{Gr}_{G,K_p} := [\mathrm{Gr}_G / K_p]$, by rephrasing the moduli interpretation of Gr_{G,K_p} as below. Here Gr_G and Bun_G in the proposition below is for $G_{\mathbb{Q}_p}$.

Lemma 11.15. *The value of* $\mathrm{Gr}_{G,K_p}/\mathrm{Spd}\mathbb{Q}_p$ *on an affinoid perfectoid* $S \in \mathrm{Perf}$, *is the groupoid of isomorphism classes of tuples*

$$(S^\sharp, \mathscr{E}_0, \mathscr{E}, \mathbb{T}, \alpha : \mathscr{E}_0 \dashrightarrow \mathscr{E}),$$

where $S^\sharp$ *is an untilt of* S *over* $\mathrm{Spa}\mathbb{Q}_p$; $\mathscr{E}_0$, $\mathscr{E}$ *are* G-torsors on the relative Fargues-Fontaine curve X_S, with $\mathscr{E}_0$ being geometric pointwise on S trivial; $\mathbb{T}$ is a pro-étale $\underline{K}_p$-torsor over S such that*

$$\mathscr{E}_0 = \mathbb{T} \times^{\underline{K}_p} (\mathcal{G} \times X_S);$$

and α *is an isomorphism over* $X_S \backslash S^\sharp$, *memorphic along the closed divisor* $S^\sharp$.

Proof. Note that one can $\underline{K}_p$-equivariantly identify the moduli interpretation of Gr_G with the v-sheaf of trivializations $\underline{\mathrm{Isom}}_S(\mathbb{T}, \underline{K}_p)$ over the above moduli problem. $\qquad\square$

Proposition 11.16. *The generic fiber* $\mathrm{Sht}_{\mathcal{G},\mathbb{Q}_p}$ *can be identified with* $\mathrm{Gr}_{G,K_p}/\mathrm{Spd}\mathbb{Q}_p$. *In particular, the Beauville-Laszlo map on* Gr_G *factors through a map*

$$\mathrm{Sht}_{\mathcal{G},\mathbb{Q}_p} \to \mathrm{Bun}_G.$$

Proof. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ be an affinoid perfectoid. Then an S-point of $\mathrm{Sht}_{\mathcal{G},\mathbb{Q}_p}$ gives an untilt $S^\sharp$ of S over $\mathrm{Spa}\mathbb{Q}_p$ and a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_\mathcal{P})$ over S with one leg at $S^\sharp$. Restrict $(\mathcal{P}, \varphi_\mathcal{P})$ to $\mathcal{Y}_{(r,\infty)}(S)$ for r large enough such that $\mathcal{Y}_{(r,\infty)}(S)$ doesn't meet $S^\sharp$. This descends to a $\mathcal{G}$-torsor $\mathscr{E}$ on X_S. Similarly the restriction of $(\mathcal{P}, \varphi_\mathcal{P})$ to $\mathcal{Y}_{(0,\epsilon]}(S)$ for some ϵ such that $\mathcal{Y}_{(0,\epsilon]}(S)$ doesn't meet $S^\sharp$, descends to a $\mathcal{G}$-torsor $\mathscr{E}_0$ on X_S. By [SW20, 22.6.1, 23.3.1], as its pullback to $\mathcal{Y}_{(0,\epsilon]}(S)$ extends φ_S^{-1}-equivariantly over the locus $p = 0$, $\mathscr{E}_0$ is geometric pointwise on S trivial and there is a pro-étale $\underline{K}_p$-torsor $\mathbb{T}$ on S such that

$$\mathscr{E}_0 = \mathbb{T} \times^{\underline{K}_p} (\mathcal{G} \times X_S).$$

Furthermore, for r as above and $n \in \mathbb{N}$ large enough, $\varphi_\mathcal{P}^n$ induces an isomorphism

$$\alpha : \mathscr{E}_0|_{X_S \backslash S^\sharp} \cong (\varphi_S^n)^* \mathcal{P}|_{\mathcal{Y}_{(r,pr)}} \xrightarrow{\varphi_\mathcal{P}^n} \mathcal{P}|_{\mathcal{Y}_{(r,pr)}} \cong \mathscr{E}|_{X_S \backslash S^\sharp}.$$

The tuple $(S^\sharp, \mathscr{E}_0, \mathscr{E}, \mathbb{T}, \alpha)$ is an object of $\mathrm{Gr}_{G,K_p}(S)$. Also the assignment $(S^\sharp, \mathcal{P}, \varphi_\mathcal{P})$ to $(S^\sharp, \mathscr{E}_0, \mathscr{E}, \mathbb{T}, \alpha)$ sends isomorphisms to isomorphisms.

To go back, pull back $\mathscr{E}_0$ to $\mathcal{Y}_{(0,\infty)}(S)$. By [SW20, 22.6] and the existence of the $\underline{K}_p$-torsor $\mathbb{T}$, this extends to a $\mathcal{G}$-shtuka $\mathcal{P}'$ with no legs. On the other hand, the completion of $\mathscr{E}$ along $S^\sharp \subset X_S$ gives a $B_{\mathrm{dR}}^+(R^\sharp)$-lattice. We use α to modify $\mathcal{P}'$ by this lattice at $\varphi_S^n(S^\sharp)$ for all $n \geq 1$. By doing so we obtain a new $\mathcal{G}$-torsor $\mathcal{P}$, together with a meromorphic map

$$\varphi_\mathcal{P} : \varphi_S^* \mathcal{P} \dashrightarrow \mathcal{P},$$

which is $\varphi_{\mathcal{P}'}$ at $\mathcal{Y}_{[0,\infty)}(S) \backslash \varphi_S^n(S^\sharp)$, $n \geq 0$, identity at $\varphi_S^n(S^\sharp)$, $n \geq 1$ and α^{-1} at the leg. The pair $(\mathcal{P}, \varphi_\mathcal{P})$ is a $\mathcal{G}$-shtuka over S with one leg at $S^\sharp$. Clearly these two constructions are inverse to each other. $\qquad\square$

Proposition 11.17. *There is a map*

$$BL_{K_p}^{\mathrm{int}} : \mathrm{Sht}_{\mathcal{G}} \to \mathrm{Bun}_G$$

extending the map $\mathrm{Sht}_{\mathcal{G},\mathbb{Q}_p} \to \mathrm{Bun}_G$ *induced by the Beauville-Laszlo map on* Gr_G.

Construction. The association $(\mathcal{P}, \varphi_{\mathcal{P}})$ to $\mathscr{E}$ in the proof above doesn't rely on the untilt being in characteristic zero. Namely, for $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ with an untilt $S^\sharp$ over $\mathrm{Spa}\mathbb{Z}_p$, given a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_{\mathcal{P}})$ over S with a leg at $S^\sharp$, restrict $\mathcal{P}$ to $\mathcal{Y}_{[r,\infty)}(S)$ for large enough r such that $S^\sharp$ does not lie in $\mathcal{Y}_{[r,\infty)}(S)$. Then the restriction $\mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)}$ with descent datum provided by $\varphi_{\mathcal{P}}$ descends to X_S, defining an S-point of Bun_G. It is clear that if the leg is not in characteristic p, then this is the map induced by the Beauville-Laszlo map. $\qquad\square$

We can define bounded substacks of $\mathrm{Sht}_{\mathcal{G}}$. Fix $T \subset B \subset G_{\overline{\mathbb{Q}}_p}^\circ$ and let $\mu \in X_*(T)$ be a dominant cocharacter defined over a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$.

Definition 11.18. $\mathrm{Sht}_{\mathcal{G}, \leq \mu}$ is the closed substack of $\mathrm{Sht}_{\mathcal{G}, \mathcal{O}_E}$ where the $\mathcal{G}$-shtukas are bounded by μ. We write $\mathrm{Sht}_{\mathcal{G}, \mu}$ for $\mathrm{Sht}_{\mathcal{G}, \leq \mu}$ if μ is minuscule.

Theorem 11.19. *Let μ and E be as above. The structure map*

$$\mathrm{Sht}_{\mathcal{G}, \leq \mu} \to \mathrm{Spd}\mathcal{O}_E$$

is qcqs, with proper diagonal, and for any perfectoid Tate algebra R with an open bounded integrally closed subring R^+ and any commutative diagram with solid arrows

$$
\begin{array}{ccc}
\mathrm{Spa}(R, R^\circ) & \xrightarrow{\ f\ } & \mathrm{Sht}_{\mathcal{G}, \leq \mu} \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Spa}(R, R^+) & \longrightarrow & \mathrm{Spd}\mathcal{O}_E,
\end{array}
$$

there is a unique (up to isomorphism) dotted arrow making the whole diagram commute up to a natural transform given by an automorphism of f.

Proof. We only have to prove quasi-compactness; the rest is proposition 11.13 and corollary 11.14. But this follows from the qcqsness criterion 2.18 and remark 11.11. $\qquad\square$

11.3. **The crystalline period map.** Go back to our PEL-setup. In particular, $G_{\mathbb{Z}_p}$ is the reductive group determined by the quadruple $(\mathcal{O}_{B_{\mathbb{Q}_p}}, *, \Lambda, (\cdot, \cdot))$; E is the completion of the global reflex field at a fixed prime above p, and μ is a dominant representative in the conjugacy class of the inverse of the Hodge cocharacter. We show that the universal formal abelian scheme over the integral model of the Shimura variety gives rise to a $\mathcal{G}$-shtuka with one leg bounded by μ. This defines a map $\mathscr{S}_K^\diamond \to \mathrm{Sht}_{\mathcal{G}, \mu}$. We call it the crystalline period map.

Proposition 11.20. *Assume $p \neq 2$, then there is a map of small v-stacks over $\mathrm{Spd}\mathcal{O}_E$*

$$\pi_{crys} : \mathscr{S}_K^\diamond \to \mathrm{Sht}_{\mathcal{G}, \mu}.$$

whose base change to $\mathrm{Spd}E$ is the Hodge-Tate period map $\overline{\pi}_{HT}^\circ$ at level K_p. For $p = 2$, the same statement is true if [RZ96, 3.16] holds.

Construction. View $\mathscr{S}_K^\diamond / \mathrm{Spd}\mathcal{O}_E$ as the sheafification of the presheaf on Perf

$$S = \mathrm{Spa}(R, R^+) \mapsto \mathscr{S}_K(\mathrm{Spf}(R^{\sharp+})),$$

where $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is the untilt of S over $\mathcal{O}_E$ determined by the structure morphism to $\mathrm{Spd}\mathcal{O}_E$.

For $S = \mathrm{Spa}(R, R^+)$ as above, we denote by φ the Frobenius on $W(R^+)$, ξ a generator of the kernel of Fontaine's theta map and write X for $\mathrm{Spec}(W(R^+))$. Assume we have a map $\mathrm{Spf}(R^{\sharp+}) \to \mathscr{S}_K$, denote the pullback of the universal formal abelian scheme by $\mathfrak{A}$. Its prismatic Dieudonné module is a BKF-module (M, φ_M) over S with a leg at $\varphi(S^\sharp)$, equipped with an alternating form $(\cdot, \cdot)$ and an $\mathcal{O}_B$-action. Let $\tilde{M}$ be the coherent sheaf on $X_{\text{ét}}$ attached to the $W(R^+)$-module M. Consider the sheaf on $X_{\text{ét}}$

$$\mathcal{P} : T \mapsto \{g \in \mathrm{Isom}_{\mathcal{O}_B}(\tilde{M}_T, \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_T) | g^*(\cdot, \cdot) = c(g)(\cdot, \cdot), c(g) \in \mathcal{O}_T(T)^\times\}.$$

We have to show that this is a $G_{\mathbb{Z}_p}$-torsor, i.e. $\tilde{M}$ is étale locally on X isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_X$ as polarized $\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_X$-modules.

Since both M and $\Lambda \otimes_{\mathbb{Z}_p} W(R^+)$ are ξ-adically complete and ξ-adically separated, we can check this after modulo ξ. (Here we use the smoothness part of [RZ96, 3.16] and hence have to exclude $p = 2$.) But $M \otimes_{W(R^+), \theta} R^{\sharp+}$ agrees with the de Rham homology $H_{1,\mathrm{dR}}(\mathfrak{A}/R^{\sharp+})$ of the formal scheme $\mathfrak{A}$ over $R^{\sharp+}$. For any chosen pseudo-uniformizer $\varpi \in R^{\sharp+}$ of $R^\sharp$ and each integer n, denote the reduction of $\mathfrak{A}$ modulo ϖ^n by A_n and $R_n := R^{\sharp+}/\varpi^n$. We have

$$H_{1,\mathrm{dR}}(\mathfrak{A}/R^{\sharp+}) \cong \varprojlim_n H_{1,\mathrm{dR}}(A_n/R_n).$$

Since the Hodge filtration on $H_{1,\mathrm{dR}}(A_n/R_n)$ is $\mathcal{O}_B$-linear with graded pieces given by the Lie algebra of the abelian scheme A_n and the dual of the Lie algebra of $A_n^\vee$, it is implied by the Kottwitz condition that each $H_{1,\mathrm{dR}}(A_n/R_n)$ is isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} R_n$ as polarized $\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} R_n$-modules. Passing to the limit we have the desired statement.

Now restrict $\varphi^*\mathcal{P}$ to $\mathcal{Y}_{[0,\infty)}(S)$ and equip it with the Frobenius semi-linear endomorphism coming from $\varphi_M \circ \varphi$, we obtain a $\mathcal{G}$-shtuka over S with a leg at $S^\sharp$. It is bounded by μ because of the shape of the Hodge-filtration as explained in 6.14. This induces a map of v-stacks

$$\mathscr{S}_K^\diamond \to \mathrm{Sht}_{\mathcal{G}, \mu}.$$

Compare with the construction in 6.14 and use 11.16, we see that when restricted to the generic fiber of $\mathscr{S}_K^\diamond$ this is the Hodge-Tate period map. $\square$

Remark 11.21. In [PR21, 4.5], Pappas-Rapoport showed in the more general case of Hodge type Shimura varieties at parahoric level, the universal $\mathcal{G}$-shtuka over the generic fiber of the Shimura variety extends over the integral model.

Proposition 11.22. *The map π_{crys} is qcqs.*

Proof. Since the composition

$$\mathscr{S}_K^\diamond \xrightarrow{\pi_{\mathrm{crys}}} \mathrm{Sht}_{\mathcal{G}, \mu} \to \mathrm{Spd}\mathcal{O}_E$$

is qcqs and the second map is quasi-separated by theorem 11.19, the first map is qcqs by cancellation. $\square$

11.4. **Integral model of the cartesian diagram.**

Theorem 11.23. *The following diagram of small v-stacks on* $\mathrm{Perf}/\mathrm{Spd}\mathcal{O}_E$ *is 2-cartesian.*

$$
\begin{array}{ccc}
\mathscr{S}_K^\diamond & \xrightarrow{\;\pi_{crys}\;} & \mathrm{Sht}_{\mathcal{G},\mu} \\
\big\downarrow{\scriptstyle \mathrm{red}} & & \big\downarrow{\scriptstyle BL_{K_p}^{\mathrm{int}}} \\
\mathrm{Igs}_{K^p}^\circ & \xrightarrow{\;\overline{\pi}_{HT}^\circ\;} & \mathrm{Bun}_G
\end{array}
$$

Also, when base changed to $\mathrm{Spd}E$, *it identifies with the diagram in corollary 8.15.*

Proof. The last statement is clear. We only need to show that the diagram is 2-cartesian. For convenience we denote the fiber product $\mathrm{Igs}_{K^p}^\circ \times_{\mathrm{Bun}_G} \mathrm{Sht}_{\mathcal{G},\mu}$ by F. The diagram commutes up to a natural isomorphism (comparison between prismatic and crystalline Dieudonné modules). Hence there is a unique map $\mathscr{S}_K^\diamond \to F$ by the universal property. We need to show that this is an isomorphism.

We know from 8.18 that the map $\overline{\pi}_{HT}^\circ$ is qcqs, and hence its base change $F \to \mathrm{Sht}_{\mathcal{G},\mu}$ is also qcqs. Since the composition

$$
\mathscr{S}_K^\diamond \to F \to \mathrm{Sht}_{\mathcal{G},\mu}
$$

is naturally isomorphic to π_{crys} and is quasi-separated, the map $\mathscr{S}_K^\diamond \to F$ is qcqs by cancellation. In particular, that this is an isomorphism can be checked on geometric points. On a geometric point $s := \mathrm{Spa}(C, C^+)$, a $\mathcal{G}$-shtuka over s with a leg at $s^\sharp$ can be uniquely extended to a $\mathcal{G}$-BKF-module over $\mathcal{O}_C$ with one leg at $\mathcal{O}_{C^\sharp}$. Using Dieudonné theory, it is the same as a p-divisible group with G-structure over $\mathcal{O}_{C^\sharp}$. One can now argue as in 8.13. $\qquad\square$

11.5. **Newton stratification.** Parallel to the discussion in section 7 and section 9, we discuss the Newton stratification on the cartesian diagram in theorem 11.23. This recovers the almost product formula on Newton strata of PEL-type Shimura varieties due to Mantovan [Man05, Proposition 11], c.f. [CS17, 4.3], as well as the p-adic uniformization of Rapoport and Zink [RZ96, 6.30].

Fix an algebraically closed field k containing $\mathbb{F}_q$. Let $B(G)$ be the Kottwitz set for $G_{\mathbb{Q}_p}$. Consider the absolute version of the cartesian diagram in 11.23 on $\mathrm{Perf}_k := \mathrm{Perf}_{/\mathrm{Spd}k}$. Namely, forget the structure morphism $\mathscr{S}_K^\diamond \to \mathrm{Spd}\mathcal{O}_E$, and view it as the v-sheaf

$$
S \mapsto \{(S^\sharp, S^\sharp \to \mathscr{S}_K^{\mathrm{ad}}) \mid S^\sharp \text{ is an untilt of } S\},
$$

similarly for $\mathrm{Sht}_{\mathcal{G},\mu}$.

The Newton stratification on $\mathrm{Bun}_{G,\mathrm{Spd}k}$ pulls back to stratifications on all of $\mathrm{Igs}_{K_p}^\circ$, $\mathrm{Sht}_{\mathcal{G},\mu}$ and $\mathscr{S}_K^\diamond$. For an element $[b] \in B(G)$, we label the corresponding strata by a superscript b (on $\mathrm{Igs}_{K_p}^\circ$ and $\mathscr{S}_K^\diamond$ this will be empty unless $[b] \in B(G,\mu)$, c.f. 8.9). Then as a corollary to theorem 11.23, we have

Corollary 11.24. *The following diagram on* Perf_k *is 2-cartesian.*

$$
\begin{array}{ccc}
\mathscr{S}_K^{\diamond,b} & \xrightarrow{\;\pi_{\mathrm{crys}}^b\;} & \mathrm{Sht}_{\mathcal{G},\mu}^b \\
\big\downarrow & & \big\downarrow{\scriptstyle BL_{K_p}^{\mathrm{int},b}} \\
\mathrm{Igs}_{K^p}^{\circ,b} & \xrightarrow{\;\overline{\pi}_{HT}^{\circ,b}\;} & \mathrm{Bun}_G^b
\end{array}
$$

In the remaining part of this section, we describe the strata and explain the relation to Mantovan's formula and Rapoport-Zink uniformization.

Denote by $\mathcal{D}$ the intergal (local) PEL Shimura datum $(\mathcal{O}_B \otimes \mathbb{Z}_p, *, \Lambda, (\cdot, \cdot), \mu, [b])$, where the first five entries are determined by the integral global PEL-datum and $[b] \in B(G, \mu)$. Fix a representative $\mathbb{X}_b$ of the isogeny class of p-divisible groups with G-structure over k labelled by $[b]$. This defines a formal scheme $\mathfrak{M}_{\mathcal{D}}$ (the Rapoport-Zink space attached to $\mathcal{D}$, c.f. [RZ96, 3.21]) (pro)-representing the following deformation functor of p-divisible groups:

$$\mathrm{Nilp}_{W(k)}^{\mathrm{op}} \to \mathrm{Sets}$$

$$R \mapsto \{(\mathcal{H}, f)\}/\sim,$$

where $\mathcal{H}$ is a p-divisible group over R with principal polarization and $\mathcal{O}_B \otimes \mathbb{Z}_p$-action (satisfying the Kottwitz condition and compatibility with $*$, c.f. 5.10), and $f : \mathcal{H} \times_R R/p \dashrightarrow \mathbb{X}_b \times_k R/p$ is an $B_{\mathbb{Q}_p}$-linear quasi-isogeny, preserving the polarization up to a scalar in $\underline{\mathbb{Q}}_p^\times(\mathrm{Spec}(R/p))$. The equivalence relation is given by isomorphisms of such pairs. The formal group scheme $\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)$ of self-quasi-isogenies compatible with the extra structures on $\mathbb{X}_b$, representing the functor

$$\mathrm{Nilp}_{W(k)}^{\mathrm{op}} \to \mathrm{Sets} : R \mapsto \mathrm{Aut}_G(\tilde{\mathbb{X}}_b \times_k R/p)$$

acts on $\mathfrak{M}_{\mathcal{D}}$ by composing with the quasi-isogeny f. We consider their attached diamonds and view them as defined absolutely over $\mathrm{Spd}k$.

Lemma 11.25. *The locally closed substack $\mathrm{Sht}_{\mathcal{G},\mu}^b$ over $\mathrm{Spd}k$ of $\mathrm{Sht}_{\mathcal{G},\mu}$ is isomorphic to the quotient stack*

$$\left[\mathfrak{M}_{\mathcal{D}}^\diamond / \underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)^\diamond \right]$$

and the quotient map $\mathfrak{M}_{\mathcal{D}}^\diamond \to \mathrm{Sht}_{\mathcal{G},\mu}^b$ identifies with the tautological $\tilde{G}_b$-torsor coming from the (integral) Beauville-Laszlo map $\mathrm{Sht}_{\mathcal{G},\mu}^b \to \mathrm{Bun}_G^b \cong [*/\tilde{G}_b]$.

Proof. Consider the $\tilde{G}_b$-torsor $\mathcal{M}_{(\mathcal{G},b,\mu)}^{\mathrm{int}}$ over $\mathrm{Sht}_{\mathcal{G},\mu}^b$, which parametrizes for a $\mathcal{G}$-shtuka $(\mathcal{P}, \varphi_{\mathcal{P}})$ trivializations

$$\iota_r : \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \cong \tilde{\mathcal{E}}_b,$$

where $r \in (0, \infty)$ is large enough such that $S^\sharp$ doesn't intersect $\mathcal{Y}_{[r,\infty)}(S)$, and $\tilde{\mathcal{E}}_b$ is the pullback of $\mathcal{E}_b$ (c.f. section 7.3) to $\mathcal{Y}_{[r,\infty)}(S)$. This is the integral local Shimura variety of [SW20, 25.1] and is isomorphic to $\mathfrak{M}_{\mathcal{D}}^\diamond$ as a v-sheaf by [SW20, 25.1.3]. Through Dieudonné theory $\tilde{G}_b$ is isomorphic to $\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)^\diamond$. The last statement follows directly from the definition of the (integral) Beauville-Laszlo map in 11.17. $\qquad\square$

Let Ig^b be the perfect Igusa variety over k defined using $\mathbb{X}_b$ and $\overline{\mathrm{Ig}^{b,\diamond}}$ the canonical compactification of its attached v-sheaf towards $\mathrm{Spd}k$. Let $\overline{\mathscr{S}_K^{\diamond,b}}$ be the canonical compactification of $\mathscr{S}_K^{\diamond,b}$ towards $\mathrm{Sht}_{\mathcal{G},\mu}^b$. Combine the above with the description of the Newton strata of the Igusa stack in 9.40, we have the following formula of Newton strata on the Shimura variety.

Corollary 11.26. *(Mantovan's formula, c.f.* [Man05, Proposition 11]*,* [CS17, 4.3]*)*

$$\overline{\mathscr{S}_K^{\diamond,b}} \cong [\overline{\mathrm{Ig}^{b,\diamond}}/\tilde{G}_b] \times_{[*/\tilde{G}_b]} [\mathfrak{M}_{\mathcal{D}}^{\diamond}/\tilde{G}_b]$$

$$\cong [(\overline{\mathrm{Ig}^{b,\diamond}} \times \mathfrak{M}_{\mathcal{D}}^{\diamond})/\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)^{\diamond}],$$

where in the second line, we take quotient by the diagonal action of $\underline{\mathrm{Aut}}_G(\tilde{\mathbb{X}}_b)^{\diamond}$.

Choose any k-point x of the fiber $\mathscr{S}_{K,k}$ of the Shimura variety over k, which is an abelian variety A over k with $\mathcal{O}_B$-endomorphism ι, polarization λ, and K^p-level structure $\bar{\eta}$. Consider the algebraic group I_x over $\mathbb{Q}$ whose value on a $\mathbb{Q}$-algebra R is

$$\{g \in \mathrm{End}_{\mathcal{O}_B}(A) \otimes_{\mathbb{Z}} R \mid gg^* \in R^{\times} \cdot \mathrm{id}_A\},$$

where $*$ denotes the Rosati involution induced by λ. Namely, we take $\mathcal{O}_B$-linear self-quasi-isogenies of A that preserve the polarization up to a scalar. Note that for a prime $l \neq p$,

$$\mathrm{End}_{\mathcal{O}_B}(A) \otimes \mathbb{Q}_l \hookrightarrow \mathrm{End}_B(V_l(A)) \cong \mathrm{End}_B(V \otimes_{\mathbb{Q}} \mathbb{Q}_l),$$

so we have $I_x(\mathbb{Q}_l) \hookrightarrow G(\mathbb{Q}_l)$. We underline the topological groups $I_x(\mathbb{Q})$, $G(\mathbb{A}_f^p)$, K^p to denote their attached v-sheaves. Consider the v-sheaf theoretic double quotient

$$[\underline{I_x(\mathbb{Q})}\backslash\underline{G(\mathbb{A}_f^p)}/\underline{K^p}],$$

where $\underline{I_x(\mathbb{Q})}$ acts on $\underline{G(\mathbb{A}_f^p)}$ from the left via diagonal embedding

$$I_x(\mathbb{Q}) \hookrightarrow I_x(\mathbb{A}_f^p) \hookrightarrow G(\mathbb{A}_f^p)$$

and $\underline{K^p}$ acts from the right by the regular action.

We define a "uniformization map" of small v-stacks

$$\Theta_x : [\underline{I_x(\mathbb{Q})}\backslash\underline{G(\mathbb{A}_f^p)}/\underline{K^p}] \to \mathrm{Igs}_{K^p}^{\circ},$$

as below. Assume we are given a totally disconnected test object $S = \mathrm{Spa}(R, R^+)$ with chosen pseudo-uniformizer $\varpi \in R^+$. Write R_0 for R^+/ϖ. On the level of presheaves of groupoids, Θ_x sends a point of the left hand side, represented by the section

$$g \in \underline{G(\mathbb{A}_f^p)}(S) = \mathrm{Maps}(\pi_0(S), G(\mathbb{A}_f^p)) = \mathrm{Maps}(\mathrm{Spec}(R_0), G(\mathbb{A}_f^p))$$

to the tuple

$$(A_{R_0}, \iota_{R_0}, \lambda_{R_0}, \overline{g^{-1}\eta_{R_0}}),$$

where $A_{R_0} := A \times_k R_0$ and ι_{R_0}, λ_{R_0}, $\overline{\eta_{R_0}}$ are the corresponding base changes. This sends automorphisms to automorphisms and induces the desired map of v-stacks.

Assume the p-divisible group of A is in the isogeny class labelled by $[b] \in B(G, \mu)$. Then the image of the above map lies in the substack $\mathrm{Igs}_{K^p}^{\circ,b}$. If furthermore the element $[b]$ is basic, c.f. remark 7.16, then the group I_x is an inner form of G, agreeing with G at all places but p and infinity (c.f. [RZ96, 6.30]). In this case, by the proof of *loc. cit.*, the set of isogeny classes of abelian varieties over k with G-structure is finite and is bijective to the Hasse kernel $\ker^1(\mathbb{Q}, I_x)$ (and hence to $\ker^1(\mathbb{Q}, G)$) of the map

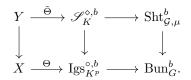$$\mathrm{res} : H^1(\mathbb{Q}, I_x) \to \prod_p H^1(\mathbb{Q}_p, I_x).$$

For each such isogeny class, we fix a point $x_i \in \mathscr{S}_K(k)$ lying in it, and define

$$\Theta = \coprod_{i \in \ker^1(\mathbb{Q},G)} \Theta_{x_i} : \coprod_{i \in \ker^1(\mathbb{Q},G)} [\underline{I_{x_i}(\mathbb{Q})} \backslash \underline{G(\mathbb{A}_f^p)}/\underline{K^p}] \to \mathrm{Igs}_{K^p}^{\circ,b}.$$

We have the following reformulation of Rapoport-Zink uniformization:

Proposition 11.27. *(c.f. [RZ96, 6.30])* *For* $[b]$ *basic, the uniformization map* Θ *is an isomorphism of small v-stacks over* $\mathrm{Spd}k$.

Proof. This can be checked v-locally. Hence we may pull back along the v-cover $\mathscr{S}_K^{\diamond,b} \to \mathrm{Igs}_{K^p}^{\circ,b}$. To simplify notation, let us denote the source of Θ by X and the pullback by Y. Consider the following diagram where all squares are cartesian

$$\begin{array}{ccccc} Y & \xrightarrow{\tilde{\Theta}} & \mathscr{S}_K^{\diamond,b} & \longrightarrow & \mathrm{Sht}_{\mathcal{G},\mu}^b \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\Theta} & \mathrm{Igs}_{K^p}^{\circ,b} & \longrightarrow & \mathrm{Bun}_G^b. \end{array}$$

By a direct computation, the tautological $\tilde{G}_b$-torsor above X corresponding to the map

$$X \xrightarrow{\Theta} \mathrm{Igs}_{K^p}^b \to \mathrm{Bun}_G^b \cong [*/\tilde{G}_b]$$

can be identified with

$$X' := \coprod_{i \in \ker^1(\mathbb{Q},G)} [\underline{I_{x_i}(\mathbb{Q})} \backslash \tilde{G}_b \times \underline{G(\mathbb{A}_f^p)}/\underline{K^p}],$$

where for each i, $\underline{I_{x_i}(\mathbb{Q})}$ acts diagonally on the two middle terms: on $\tilde{G}_b \cong \underline{\mathrm{Aut}}(\widetilde{\mathbb{X}}_b)$ via the map

$$I_{x_i}(\mathbb{Q}) \to \mathrm{Aut}(\widetilde{A_{x_i}[p^\infty]}) \cong \mathrm{Aut}(\widetilde{\mathbb{X}}_b),$$

and on $\underline{G(\mathbb{A}_f^p)}$ as explained earlier; while $\underline{K^p}$ acts via the right regular action on $\underline{G(\mathbb{A}_f^p)}$. The $\widetilde{G}_b$-action on X' is the right regular action on $\widetilde{G}_b$ itself.

Identify $\mathrm{Sht}_{\mathcal{G},\mu}^b$ with $[\mathfrak{M}_{\mathcal{D}}^\diamond/\widetilde{G}_b]$. Then using the outer cartesian square we can compute Y to be the product of X' with $\mathfrak{M}_{\mathcal{D}}^\diamond$ quotienting by the diagonal action of $\widetilde{G}_b$. This simplifies to the following formula

$$\coprod_{i \in \ker^1(\mathbb{Q},G)} [\underline{I_{x_i}(\mathbb{Q})} \backslash (\mathfrak{M}_{\mathcal{D}}^\diamond \times \underline{G(\mathbb{A}_f^p)}/\underline{K^p})],$$

where $I_{x_i}(\mathbb{Q})$ acts diagonally on the middle terms and the action on $\mathfrak{M}_{\mathcal{D}}^\diamond$ is via its map to $\mathrm{Aut}(\widetilde{\mathbb{X}}_b)$. The map $\tilde{\Theta}$ agrees with the p-adic uniformization map of [RZ96, Theorem 6.30] by comparing their construction of the map in [RZ96, (6.3)] with our construction, c.f. the proof of 11.23. Now it follows from [RZ96, 6.30] that $\tilde{\Theta}$ is an isomorphism, except that we need to identify $\mathscr{S}_K^{\diamond,b}$ with the v-sheaf attached to the completion of S_K (as a $W(k)$-scheme) along $T \subset S_{K,k}$, the closed subscheme where the universal p-divisible group is geometric fiberwise of isogeny class b. But this is the case, because both are exactly the open sub-v-sheaf of $\mathscr{S}_K^\diamond$, obtained from sheafifying the presheaf

$$S = \mathrm{Spa}(R,R^+) \mapsto \left\{ (S^\sharp, f \in \mathscr{S}_K(R^{\sharp+})) \,\middle|\, \begin{array}{l} f^*\mathfrak{A} \times_{R^{\sharp+}} R^+/\varpi \text{ is geometric-} \\ \text{pointwise of isogeny class } [b] \end{array} \right\},$$

where $S^\sharp$ is an untilt of S over $\mathcal{O}_E$ and $\mathfrak{A}$ is the universal formal abelian scheme on $\mathscr{S}_K$. □

Corollary 11.28. *If $[b]$ is basic, then the Igusa variety $\mathrm{Ig}^{b,\diamond}$ as a v-sheaf on Perf_k is isomorphic to*

$$\coprod_{i\in\ker^1(\mathbb{Q},G)} [\underline{I_{x_i}(\mathbb{Q})}\backslash\underline{G_b(\mathbb{Q}_p)}\times\underline{G(\mathbb{A}_f^p)}/\underline{K^p}].$$

Proof. Combine the description of $\widetilde{G}_b$ in remark 7.16, description of the strata of the Igusa stack in 9.40, 11.27 above and the identification of the tautological $\widetilde{G}_b$-torsor in the proof of 11.27. (In this case partial minimal and canonical compactifications are unnecessary.) □

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